Dan Brânzei
Ioan Șerdean
Vasile Șerdean

JUNIOR BALKAN MATHEMATICAL OLYMPIADS

www.eplus.ro

EDITURA PLUS

Chapter 1

Junior Balkan Mathematical Olympiads The Problems

1.1 First Junior Balkan Mathematical Olympiad Beograd, Yugoslavia, June, 1997

1. Nine points are given inside a unit square. Prove that three of them are the vertices of a triangle with the area not greater than $\frac{1}{8}$.

Bulgaria

2. Let
$$\frac{x^2+y^2}{x^2-y^2}+\frac{x^2-y^2}{x^2+y^2}=k.$$
Find the value of
$$\frac{x^8+y^8}{x^8-y^8}+\frac{x^8-y^8}{x^8+y^8}$$
in terms of k.

Cyprus

3. Let I be the incenter of the triangle ABC, and let D and E be the midpoints of the sides AB and AC respectively. Lines DE and BI meet at point K and lines DE and CI meet at point L. Prove that
$$AI+BI+CI>BC+KL.$$

Greece

4. Find the triangle ABC so that

$$R\,(b+c)=a\sqrt{bc}.$$

Romania

1.2 Second Junior Balkan Mathematical Olympiad Athens, Greece, June, 1998

5. Prove that the number

$$\underbrace{11\ldots11}_{1997}\underbrace{22\ldots22}_{1998}5$$

is a perfect square.

Yugoslavia

6. Let $ABCDE$ be a pentagon so that

$$AB=AE=CD=1,\ \angle ABC=\angle DEA=90^\circ \text{ and } BC+DE=1.$$

Find the area of the pentagon.

Greece

7. Find all the pairs $(x,\ y)$ of positive integers so that

$$x^y=y^{x-y}.$$

Albania

8. Can one find 16 three digit numbers, using only 3 digits, without having two of them with the same remainder when divided by 16?

Bulgaria

1.3 Third Junior Balkan Mathematical Olympiad Plovdiv, Bulgaria, June, 1999

9. Let $a,\ b,\ c,\ x,\ y$ be real numbers so that:

$$a^3+ax+y=0,\ b^3+bx+y=0 \text{ and } c^3+cx+y=0.$$

Show that if a,b,c are distinct numbers, different from 0, then $a+b+c=0$.

Cyprus

10. Find the greatest common divisor of the numbers

$$A_n=2^{3n}+3^{6n+2}+5^{6n+2}$$

when $n=0,\ 1,\ \ldots,\ 1999$.

Romania

11. Let S be a square of side 20 and let M be a set consisting of the vertices of the square and 1999 arbitrary inner points of S. Prove the existence of a triangle with the area at most equal to $\frac{1}{10}$ and having all the vertices in the set M.

Yugoslavia

12. In a triangle ABC the sides AB and AC are equal. Let D be a point on BC such that $BC>BD>DC>0$. Consider the circumcircles k_1 and k_2 of the triangles ABD and ADC respectively. Let M be the midpoint of $B'C'$, when BB' and CC' are diameters of k_1 and k_2 respectively. Prove that the area of the triangle MBC is constant (with respect to D).

Greece

1.4 Fourth Junior Balkan Mathematical Olympiad Ohrid, Macedonia, June, 2000

13. Let $x,\ y$ be integer numbers so that

$$x^3+y^3+(x+y)^3+30xy=2000.$$

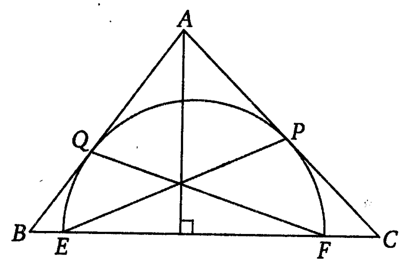
Prove that $x+y=10$.

Romania

14. Find all the positive integers $n,\ n\geq 1$, such that n^2+3^n is a perfect square.

Bulgaria

15. A semicircle of diameter EF, lying on the side BC of the ABC triangle, is tangent to the sides AB and AC in Q and P respectively.



The lines EP and FQ meet at point K.

Prove that K is a point on the altitude from A of the triangle ABC.

Albania

16. At a tennis tournament there were twice as many girls participating than boys. Each pair of players had only one match and there were no draws. The ratio between girl winnings and boy winnings was $\frac{7}{5}$.

How many players took part at the tournament?

Serbia

1.5 Fifth Junior Balkan Mathematical Olympiad Nicosia, Cyprus, June, 2001

17. Find all the positive integers a, b, c such that

$$a^3 + b^3 + c^3 = 2001.$$

Romania

18. Let ABC be a triangle with $\angle ACB = 90^\circ$ and $AC \neq BC$. The points L and H of the segment $[AB]$ are chosen such that $\angle ACL = \angle LCB$, and CH is perpendicular to AB.

a) For every point X (other than C) on the line CL, prove that $\angle XAC \neq \angle XBC$.

b) For every point Y (other than C) on the line CH prove that $\angle YAC \neq \angle YBC$.

Bulgaria

19. Let ABC be an equilateral triangle and let D, E be arbitrary points on the sides $[AB]$ and $[AC]$ respectively. If DF, EG (with $F \in AE$, $G \in AD$) are internal bisectors of the angles of the triangle ADE, prove that the sum of the areas of the triangles DEF and DEG is less than or equal to the area of the area of triangle ABC. Explain when the equality holds.

Greece

20. A convex polygon with 1415 sides has the perimeter of 2001 centimeters. Prove that there exist three vertices of this polygon, which form a triangle having the area less than 1 square centimeter.

Yugoslavia

1.6 Sixth Junior Balkan Mathematical Olympiad Tg. Mures, Romania, June, 2002

21. Let ABC be an isosceles triangle with $AC = BC$ and let P be a point on the arc AB of the circumcircle which does not contain C. The perpendicular from C on PB intersects PB in D. Prove that

$$PA + PB = 2PD.$$

Greece

22. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ of different radii have two common points A and B and their centers O_1 and O_2 are separated by the straight line AB. Let B_1, B_2 the diametrically opposed points of B on these circles respectively. The points M_1 on $\mathcal{C}_1$ and M_2 on $\mathcal{C}_2$ are chosen such that $\angle AO_1M_1 \equiv \angle AO_2M_2$, B_1 is an internal point of $\angle AO_1M_1$ and B is an internal point of $\angle AO_2M_2$. Let M be the midpoint of the segment B_1B_2. Prove that $\angle MM_1B \equiv \angle MM_2B$.

Cyprus

23. Find the positive integers N having the following properties:

i) N has exactly 16 divisors $1 = d_1 < d_2 < \ldots < d_{15} < d_{16} = N$.

ii) the divisor having the index d_5 (that is d_{d_5}) is equal to $(d_2 + d_4)d_6$.

Bulgaria

24. Let a, b, c, be positive numbers. Prove that:

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}.$$

Greece

Chapter 2

Team selection tests

2.1 First team selection test for the second JBMO Iasi, May 27, 1998

25. Let

$$A = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{1997\cdot 1998}$$

and

$$B = \frac{1}{1000\cdot 1998} + \frac{1}{1001\cdot 1997} + \ldots + \frac{1}{1998\cdot 1000}.$$

Prove that $\frac{A}{B}$ is an integer.

26. A rectangle $ABCD$ is given. Let M, N, P, Q be the points on the sides AB, BC, CD, DA respectively. If p is the perimeter of the quadrilateral $MNPQ$, prove that:

i) $p \geq AC + BD$;

ii) If $p = AC + BD$, then area$[MNPQ] \leq \frac{\text{area}[ABCD]}{2}$.

iii) If $p = AC + BD$, then $MP^2 + NQ^2 \geq AC^2$.

27. Let n be a positive integer. Find all the integer numbers that writes as:

$$\frac{1}{a_1} + \frac{2}{a_2} + \ldots + \frac{n}{a_n},$$

for some positive integers a_1, a_2, $\ldots$, a_n.

2.2 Second selection test for the second JBMO Iasi, May 28, 1998

28. Find all the integers x and y so that

$$(x+1)(x+2)(x+3)+x(x+2)(x+3)+x(x+1)(x+3)+x(x+1)(x+2)=y^{2^x}.$$

29. A triangle ABC is given. The points D, E, F, G are chosen on the sides of the triangle such that the quadrilateral $DEFG$ is circumscriptible and $DF \perp EG$. Find the locus of the intersection point $M \in DF \cap EG$, so that $\{D, E, F, G\} \cap \{A, B, C\} \neq \varnothing$.

30. Find the smallest value for n for which there exist the positive integers $x_1, \ldots, x_n$ with

$$x_1^4 + x_2^4 + \ldots + x_n^4 = 1998.$$

2.3 First team selection test for the third JBMO Iasi, May 26, 1999

31. Let n the positive integer. Prove that there is a polynomial P with integer coefficients so that if $a+b+c=0$, then:

$$a^{2n+1} + b^{2n+1} + c^{2n+1} = abc[P(a, b) + P(b, c) + P(c, a)].$$

32. Let ABC be a triangle and let $\bar{x}$, $\bar{y}$, $\bar{z}$ be three arbitrary vectors. For any real number $\lambda > 0$, the points M, N, P are chosen so that:

$$\overline{AM} = \lambda\bar{x},\ \overline{BN} = \lambda\bar{y},\ \overline{CP} = \lambda\bar{z}.$$

Find the locus of the centroid Q of the triangle MNP.

33. Let $A \subset (0, 1)$ be a set of real number having the properties:

a) $\frac{1}{2} \in A$;

b) if $x \in A$, then $\frac{x}{2}$ and $\frac{1}{1+x}$ belong to A.

Prove that the set A contains all the rational numbers from the interval $(0, 1)$.

34. Let D_1, D_2, D_3 be three distinct disks in the plane and let a_{ij} be the area of $D_i \cap D_j$, for all $i, j \in \{1, 2, 3\}$. Prove that if x_1, x_2, x_3 are real numbers, not all of them equal to zero, then:

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 > 0.$$

2.4 Second team selection test for the third JBMO Iasi, May 27, 1999

35. Let A, B, C be the measures (in degrees) of the angles of the ABC triangle. A straight line cuts the ABC triangle in two isosceles triangles. Find the relations between the numbers A, B, C.

36. Find the number of five-digit perfect squares having the last two digits equal.

37. M is the set of all values of the greatest common divisor d of the numbers $A = 2n + 3m + 13$, $B = 3n + 5m + 1$, $C = 6n + 8m - 1$, where m and n are positive integers. Prove that M is the set of all divisors of an integer k.

38. Consider a convex quadrilateral $ABCD$ and let A_1, B_1, C_1, D_1 be the reflection points of A, B, C, D across B, C, D, A respectively.

a) If E and F are the midpoints of the segments BC and AD, and E_1 and F_1 are the midpoints of the segments A_1B_1 and C_1D_1, prove that $EE_1 = FF_1$.

b) The points A, B, C, D are erased. Can you obtain them again, knowing only the location of A_1, B_1, C_1, D_1?

2.5 First team selection tests for the fourth JBMO, Brasov, April 27, 2000.

39. For all the positive integers $k \leq 1999$, let $S_1(k)$ be the sum of all the remainders of the numbers 1, 2, ..., k when divided by 4, and let $S_2(k)$ be the sum of all the remainders of the numbers $k+1$, $k+2$, ..., 2000 when divided by 3. Prove that there is an unique positive integer $m \leq 1999$ so that $S_1(m) = S_2(m)$.

40. Let $S(k)$ be the sum of the digits of a positive integer k in decimal representation. Find all the positive integers n to exist the non-negative integers a and b with

$$S(a) = S(b) = S(a+b) = n.$$

41. For all the numbers $p \in \mathbb{R}$ and $n \in \mathbb{N}^*$ let $A_n(p)$ be the set of integers px where x is a real number and $n-1 < x \leq n$. For a given real number a, find all the real numbers b such that the sets $A_n(a)$ and $A_n(b)$ have the same number of elements for all the positive integers n.

42. Let ABC be a triangle with $\angle BAC = 90^\circ$ and $AB = AC$. The points M and N are given on the side BC such that N lies between the points M and C and

$$BM^2 - MN^2 + NC^2 = 0$$

Prove that $\angle MAN = 45^\circ$.

2.6 Second team selection test for the fourth JBMO, Bucium, Iasi, May 13, 2000

43. Find the integer solution of the equation

$$9^x - 3^x = y^4 + 2y^3 + y^2 + 2y.$$

44. A plane is covered by a net of unit squares. A person walks on the edges, any two consecutive edges being perpendicular, and returns in the initial position after n steps.

a) Prove that 4 divides n.

b) State and prove a reciprocal.

45. Find all the real values of the number a such that

$$x + y + xy > a,$$

for all the real numbers $x > a$ and $y > a$

46. A triangle ABC is given. The points $A' \in (BC)$, $B' \in (CA)$, $C' \in (AB)$ are chosen such that the the lines AA', BB', CC' meet at the point M. Let a, b, c, x, y, z be the areas of the triangles $AB'M$, $BC'M$, $CA'M$, $AC'M$, $BA'M$, $CB'M$ respectively. Prove that:

1° $abc = xyz$;

2° $ab + bc + ca = xy + yz + zx$.

2.7 Third team selection test for the fourth JBMO Bucium, Iasi, May 19, 2000

47. For any integer $n \geq 2$, consider $n-1$ positive real numbers $a_1, a_2, \ldots, a_{n-1}$ having the sum 1, and n real numbers $b_1, b_2, \ldots, b_n$. Prove that

$$b_1^2 + \frac{b_2^2}{a_1} + \frac{b_3^2}{a_2} + \ldots + \frac{b_n^2}{a_{n-1}} \geq 2b_1\left(b_2 + b_3 + \ldots + b_n\right).$$

When does the equality holds?

48. Let $a \geq 0$ be an integer number. Find the number of elements of the set

$$A = \left\{x \mid x \in \mathbb{Z} \text{ and } \frac{2^a}{3x+1} \in \mathbb{Z}\right\}.$$

49. The internal bisectors of the angles A, B, C of the ABC triangle intersect the sides BC, CA, AB at the points D, E, F respectively. The points A', B', C' are the reflections of the points A, B, C with respect to D, E, F. If A, B, C lie respectively on the line segments $B'C'$, $A'C'$, $A'B'$, prove that ABC is an equilateral triangle.

50. Two square of side length 5 are divided into 5 regions each. These 10 regions are colored using the same 5 colors for each square. Overlapping the squares, the sum of the areas of the parts sharing having the same color is computed. Prove that there is a coloring for which this sum is at least 5.

2.8 First team selection test for the fifth JBMO Tg. Mures, April 12, 2001

51. Let ABC be an arbitrary triangle. A circle passes through B and C and intersects the lines AB and AC in D and E respectively. The projection of the points B and E on CD are denoted by B' and E'. The projection of the points D and C on BE are denoted by D' and C'.

Prove that the points B', D', E', C' are on the same circle.

52. Find all the integers n so that the number $\sqrt{\frac{4n-2}{n+5}}$ is rational.

53. 1200 points are given inside a circle centered at the point O so that no two of them lie on a diameter of the circle. Prove that there exist the points M and N on the circle so that $\angle MON = 30°$ and in the interior of the angle $\angle MON$ lie exactly 100 points.

54. Three students write on the blackboard three two-digit squares next to each other. At the end they observe that the 6-digit number obtained is also a square. Find this number.

2.9 Second team selection test for the fifth JBMO Buzau, May 19, 2001

55. Let $ABCD$ be a rectangle. The points $E \in CA$, $F \in AB$, $G \in BC$ are considered so that $DE \perp CA$, $EF \perp AB$, $EG \perp BC$. Find the rational solutions of the equation

$$AC^x = EF^x + EG^x.$$

56. Let A be a non-empty subset of $\mathbb{R}$ so that if x, y are real numbers with $x + y \in A$, then $xy \in A$. Prove that $A = \mathbb{R}$.

57. Let $ABCD$ be a quadrilateral inscribed in the circle $\mathcal{C}(O, R)$. For any point E of the circle we consider its projections K, L, M, N on the lines DA, AB, BC, CD. For some point E, different than A, B, C, D, one observe that the point N is the orthocenter of the triangle KLM.

Prove that this holds for any point E on the circle.

58. Find all the positive integers $a < b < c < d$ with the property that each of them divides the sum of the other three.

2.10 Third team selection test for the fifth JBMO Buzau, May 20, 2001

59. Let n be a non-negative integer. Find all the non-negatives integers a, b, c, d such that

$$a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n.$$

60. The opposite sides of a hexagon $ABCDEF$ are parallel and the diagonals AD, BE and CF are equal. Prove that the hexagon is cyclic.

61. Let $n \geq 2$ be an integer. Find all the integers x so that

$$\sqrt{x + \sqrt{x + \ldots + \sqrt{x}}} < n$$

for any number of radicals.

62. Find the minimal area of a rectangular box of a volume strictly greater than 1000 if the side lengths are integer numbers.

2.11 First team selection test for the sixth JBMO Rm. Valcea, March 21, 2001

63. For a positive number n, let $f(n)$ be the value of

$$f(n) = \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n+1} + \sqrt{2n-1}}.$$

Calculate $f(1) + f(2) + f(3) + \ldots + f(40)$.

64. Let K, n, p be non-negative integers so that p is prime, $K < 1000$ and $\sqrt{K} = n\sqrt{p}$.

a) Prove that if the equation $\sqrt{K + 100x} = (n + x)\sqrt{p}$ has an integer solution different from 0, then $p \mid 10$.

b) In that case find the number of all the positive integer solutions of the equation (that is, when $p = 2$ or $p = 5$).

65. Consider a $1 \times n$ rectangle made out of n tiles. A pavement is a coloring of each of the n tiles with one of the 4 possible color so that no two consecutive tiles have the same color.

i) What is the number of the distinct symmetrical pavements? (a symmetrical pavement is a pavement for which tile symmetrical with respect to the center have the same color).

ii) What is the number of distinct pavements so that in any block of three consecutive tiles no two tiles have the same color?

66. Let $ABCD$ be a parallelogram centered in O. Let M and N be the midpoints of BO and CD. Prove that if the triangles ABC and AMN are similar, then $ABCD$ is a square.

2.12 Second team selection test for the sixth JBMO Bucharest, April 13, 2002

67. A unit square is divided naturally into 9 congruent squares of side $\frac{1}{3}$. The central square is colored. We call this procedure P. For each of the 8 remaining squares apply the procedure P. For each of the next 64 remaining squares apply the procedure P and so on. Prove that after 1000 applications of procedure P the area colored exceeds 0.999.

68. Find all the positive integers a, b, c, d so that

$$a + b + c + d - 3 = ab = cd.$$

69. Let ABC be an isosceles triangle with $AB = AC$ and $\angle BAC = 20°$. Let M be the projection of the point C on the side AB and let N be a point on the side AC so that $CN = \frac{BC}{2}$. Find the measure of the angle AMN.

70. Let $ABCD$ be a unit square. Suppose M, N are two interior points so that no vertex of the square lies on the line MN. Let $s(M, N)$ be the smallest area of a triangle with vertices in the set $\{A, B, C, D, M, N\}$. Find the smallest real number k so that for any points M, N with the mentioned property we have $s(M, N) \leq k$.

2.13 Third team selection test for the sixth JBMO Bucharest, April 14, 2002

71. Let n be an even positive integer and let a, b be positive coprime integers. Find a and b if $a + b$ divide $a^n + b^n$.

72. Let $ABCD$ be a convex quadrilateral and O the point of intersection of its diagonals. The measure of the angle between the two diagonals is m. For any angle xOy of measure m, the area inside the angle that is in the interior of the quadrilateral is constant. Prove that $ABCD$ is a square.

73. An equilateral triangle of side 10 is divided into 100 unit equilateral triangles by lines parallel to the sides of the triangle. Find the number of (not necessarily unit) equilateral triangles in the configuration described above so that the sides of the triangle are parallel to the sides of the initial one.

74. If a, b, $c \in (0, 1)$, prove that

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

2.14 Fourth team selection test for the sixth JBMO Bucharest, June 1, 2002

75. Let a be an integer. Prove that for any real number x such that $x^2 < 3$, the numbers $\sqrt{3-x^2}$ and $\sqrt[3]{a-x^3}$ are not both rational.

76. The last four digits of a perfect square are equal. Prove they are all zero.

77. Consider the circles $\mathcal{C}_1(O_1)$ and $\mathcal{C}_2(O_2)$ such that $\mathcal{C}_1$ passes through the point O_2. Let M be a point on the circle $\mathcal{C}_1$ but not on the line O_1O_2. The tangents from M to $\mathcal{C}_2$ meet again the circle $\mathcal{C}_1$ at the points A and B. Prove that the tangents from A and B to $\mathcal{C}_2$ (not those going through M), meet on $\mathcal{C}_1$.

78. Consider five points in the plane such that any three of them form a triangle of area at least 2. Prove that there are three of them forming a triangle of area at least 3.

2.15 Fifth team selection test for the sixth JBMO Bucharest, June 2, 2002

79. Let $m, n > 1$ be integer numbers. Solve in positive integers the equation

$$x^n + y^n = 2^m.$$

80. Consider $n > 2$ concentric circles and two lines d_1, d_2 which meet at P, a point inside all the circles. The rays determined by P on the line d_1 meet the circles at the points $A_1, A_2, \ldots, A_n$ and $A'_1, A'_2, \ldots, A'_n$ respectively. similarly, the rays determined by P on the line d_2 meet the circles at the points $B_1, B_2, \ldots, B_n$ and $B'_1, B'_2, \ldots, B'_n$ respectively (the points of equal index are on the same circle). Prove that if the small arcs A_1B_1 and A_2B_2 are equal, then all the small arcs A_iB_i and $A'_iB'_i$ are equal for all $i = \overline{1, n}$.

81. Let ABC be a triangle and $a = BC$, $b = CA$, $c = AB$ be the side lengths. On the same side of BC as A consider the points D and E such that $DB = c$, $CE = b$ and the area of $DECB$ is maximal. Let F be the midpoint of DE and let $FB = x$. Prove that $FC = x$ and $4x^3 = (a^2 + b^2 + c^2)x + abc$.

82. Let p, q be two distinct primes. Prove that there are positive integers a, b so that the arithmetic mean of all the divisors of the number $n = p^a \cdot q^b$ is also an integer.

Chapter 3

Short-Listed Problems

3.1 Fourth Junior Balkan Mathematical Olympiad Ohrid, 2000

83. Prove that there are at least 666 positive composite numbers with 2006 digits, having a digit equal to 7 and all the rest equal to 1.

84. Find all the positive perfect cubes that are not divisible by 10 so that the number obtained by erasing the last three digits is also a perfect cube.

85. Find the greatest positive integer x such that 23^{6+x} divides 2000!.

86. Find all the integers written as $\overline{abcd}$ in decimal representation and $\overline{dcba}$ in 7 base.

87. Find all the pairs of integers (m, n) so that the numbers $A = n^2 + 2mn + 3m^2 + 2$, $B = 2n^2 + 3mn + m^2 + 2$, $C = 3n^2 + mn + 2m^2 + 1$ have a common divisor greater than 1.

88. Find all the four-digit numbers so that when decomposed in prime factors have the sum of the prime factors equal to the sum of the exponents.

89. Find all the pairs of integers (m, n) such that the numbers $A = n^2 + 2mn + 3m^2 + 3n$, $B = 2n^2 + 3mn + m^2$, $C = 3n^2 + mn + 2m^2$ are consecutive in some order.

90. Find all the positive integers a, b for which $a^4 + 4b^4$ is a prime number.

91. Find all the triples (x, y, z) of positive integers such that $xy + yz + zx - xyz = 2$.

92. Prove that there are no integers x, y, z so that

$$x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 = 2000.$$

93. Prove that for any integer n one can find integers a and b such that

$$n = \left[a\sqrt{2}\right] + \left[b\sqrt{3}\right].$$

94. Consider a sequence of positive integers x_n such that:

$$\begin{aligned} \text{(A) } x_{2n+1} &= 4x_n + 2n + 2, \\ \text{(B) } x_{3n+2} &= 3x_{n+1} + 6x_n, \end{aligned}$$

for all $n \geq 0$.
Prove that

$$\text{(C) } x_{3n-1} = x_{n+2} - 2x_{n+1} + 10x_n,$$

for all $n \geq 0$.

95. Prove that

$$\sqrt{(1^k + 2^k)(1^k + 2^k + 3^k)\dots(1^k + 2^k + \dots + n^k)}$$
$$\geq 1^k + 2^k + \dots + n^k - \frac{2^{k-1} + 2 \cdot 3^{k-1} + \dots + (n-1)n^{k-1}}{n}.$$

for all integers n, $k \geq 2$.

96. Let m and n be positive integers with $m \leq 2000$ and $k = 3 - \frac{m}{n}$. Find the smallest positive value of k.

97. Let x, y, a, b be positive real numbers such that $x \neq y$, $x \neq 2y$, $y \neq 2x$, $a \neq 3b$ and $\frac{2x-y}{2y-x} = \frac{a+3b}{a-3b}$. Prove that $\frac{x^2+y^2}{x^2-y^2} \geq 1$.

98. Find all the triples (x, y, z) of real number such that

$$2x\sqrt{y-1} + 2y\sqrt{z-1} + 2z\sqrt{x-1} \geq xy + xz + yz.$$

99. A triangle ABC is given. Find all the pairs of points X, Y so that X is on the sides of the triangle, Y is inside the triangle an four non-intersecting segments from the set $\{XY, AX, AY, BX, BY, CX, CY\}$ divide the ABC triangle in four triangles with equal areas.

100. A triangle ABC is given. Find all the segments XY that lies inside the triangle such that XY and five of the segments XA, XB, XC, YA, YB, YC divide the ABC triangle in 5 regions with equal areas. Furthermore, prove that all the segments XY have a common point.

101. Let ABC be a triangle. Find all the triangles XYZ with the vertices inside ABC such that XY, YZ, ZX and six non-intersecting segments from the following AX, AY, AZ, BX, BY, BZ, CX, CY, CZ divide the ABC triangle in seven regions with equal areas.

102. Let ABC be a triangle and let a, b, c be the lengths of the sides BC, CA, AB respectively. Consider a triangle DEF with the side lengths $EF = \sqrt{au}$, $FD = \sqrt{bu}$, $DE = \sqrt{cu}$. Prove that $\angle A > \angle B > \angle C$ implies $\angle A > \angle D > \angle E > \angle F > \angle C$.

103. All the angles of the hexagon $ABCDEF$ are equal. Prove that

$$AB - DE = EF - BC = CD - FA.$$

104. Consider a quadrilateral $ABCD$ with $\angle DAB = 60°$, $\angle ABC = 90°$ and $\angle BCD = 120°$. The diagonals AC and BD intersect at M. If $MB = 1$ and $MD = 2$, find the area of the quadrilateral $ABCD$.

105. A point P is considered inside of an equilateral triangle of the side length 10 so that the distances from P to two of the sides are 1 and 3, respectively. Find the distance from P to the third side.

3.2 Fifth Junior Balkan Mathematical Olympiad Nicosia, 2001

106. Find the positive integers n that are not divisible by 3 if the number $2^{n^2-10} + 2133$ is a perfect cube.

107. Let P_n ($n = 3, 4, 5, 6, 7$) be the set of integers $n^k + n^l + n^m$, where k, l, m are positive integers. Find n so that:

i) In the set P_n there are infinitely many squares.

ii) In the set P_n there are no squares.

108. Find all the three digit numbers $\overline{abc}$ such that the 6003-digit number $\overline{abcabc\dots abc}$ is divisible by 91. ($\overline{abc}$ occurs 2001 times).

109. The discriminant of the equation $x^2 - ax + b = 0$ is the square of a rational number and a and b are integers. Prove that the roots of the equation are integers.

110. Let $x_k = \frac{k(k+1)}{2}$ for all the integers $k \geq 1$. Prove that for any integer $n \geq 10$, between the numbers $A = x_1 + x_2 + \dots + x_{n-1}$ and $B = A + x_n$ there is at least a square.

111. Find all the integers x and y such that $x^3 \pm y^3 = 2001p$, where p is a prime.

112. Prove that there are no positive integers x and y such that

$$x^5 + y^5 + 1 = (x+2)^5 + (y-3)^5.$$

113. Prove that no three points with integer coordinates can be the vertices of an equilateral triangle.

114. Consider a convex quadrilateral $ABCD$ with $AB = CD$ and $\angle BAC = 30°$. If $\angle ADC = 150°$, prove that $\angle BCA = \angle ACD$.

115. A triangle ABC is inscribed in the circle $\mathcal{C}(O, R)$. Let $\alpha < 1$ be the ratio of the radii of the circles tangent to $\mathcal{C}$, and both of the rays $(AB$ and $(AC$. The numbers $\beta < 1$ and $\gamma < 1$ are defined analogously. Prove that $\alpha + \beta + \gamma = 1$.

116. Consider an isosceles triangle ABC with $AB = AC$, and D the foot of the altitude from the vertex A. The point E lies on the side AB such that

$$\angle ACE = \angle ECB = 18^\circ.$$

If $AD = 3$, find the length of the segment CE.

117. Consider the triangle ABC with $\angle A = 90^\circ$ and $\angle B \neq \angle C$. A circle $\mathcal{C}(O, R)$ passes through B and C and intersect the sides AB and AC in D and E, respectively. Let S be the foot of the perpendicular from A to BC and let K be the intersection point of AS with the segment DE. If M the midpoint of BC, prove that $AKOM$ is a parallelogram.

118. At a conference there are n mathematicians. Each of them knows exactly k participants. Find the smallest value of k such that there are at least three mathematicians that are acquainted with the other two.

3.3 Sixth Junior Balkan Mathematical Olympiad Tg Mures, 2002

119. A student plays a computer game. The computer provides him with 2002 positive distinct numbers randomly chosen. The game rules allows him to do the following operations:

– take two of the given numbers, double one of them, add the second number and keep the sum;

– next, choose two other numbers from the remaining ones, double one of them and add the second; then multiply the sum with the previous one and keep the result;

– repeat the above procedure until all the 2002 given numbers are used.

The student wins the game if the last product is maximal. Find, with proof, the winning strategy of the game.

120. All the positive integers are arranged in a triangular array as shown below:

$$\begin{array}{lllll} 1 & 3 & 6 & 10 & 15 \ \dots \\ 2 & 5 & 9 & 14 \ \dots & \\ 4 & 8 & 13 \ \dots & & \\ 7 & 12 \ \dots & & & \\ 11 \ \dots & & & & \end{array}$$

Find the number of the column and the number of the row where 2002 is put.

121. Let a, b, c be positive real numbers such that $abc = \frac{9}{4}$. Prove that the following inequality holds

$$a^3 + b^3 + c^3 > a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

122. (Committee's variant for problem 121). If a, b, c are positive real numbers such that $abc = 2$, then

$$a^3 + b^3 + c^3 \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

When does the equality hold?

123. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}.$$

124. Let a_1, a_2, a_3, a_4, a_5, a_6 be real numbers such that $a_1 \neq 0$, $a_1a_6 + a_3a_4 = 2a_2a_5$ and $a_1a_3 \geq a_2^2$. Show that $a_4a_6 \leq a_5^2$. When does the equality hold?

125. Consider 2002 integers a_i, $i = 1, 2, 3, \dots, 2002$ such that

$$a_1^{-3} + a_2^{-3} + \dots + a_{2002}^{-3} = \frac{1}{2}.$$

Prove that at least three of them are equal.

126. Let G be the centroid of a triangle ABC, and let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB respectively. The parallel line from A_1 to BB_1 meets B_1C_1 in F. Prove that the triangles ABC and FA_1A are similar with the same orientation if and only if the quadrilateral AB_1GC_1 is cyclic.

127. Let ABC be a triangle and let H, I, O be the orthocenter, the incenter and the circumcenter of the triangle, respectively. The line CI meets again the circumcircle at the point L. It is known that $AB = IL$ and $AH = OH$. Find the measure of the angles of the triangle ABC.

128. Let ABC be a triangle of area S and consider the points D, E, F on the lines BC, CA, AB respectively. The perpendicular lines at points D, E, F on the lines BC, CA, AB intersect the circumcircle of the triangle ABC in the pairs of points (D_1, D_2), (E_1, E_2), (F_1, F_2) respectively. Prove that

$$|D_1B \cdot D_1C - D_2B \cdot D_2C| + |E_1C \cdot E_1A - E_2C \cdot E_2A| + |F_1A \cdot F_1B - F_2A \cdot F_2B| > 4S.$$

129. Let ABC be an isosceles triangle such that $AB = AC$ and $\angle A = 20^\circ$. Point D is chosen on the side AC such that $AD = BC$. Find the angle $\angle\, BDC$.

130. Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $BC = CD$. On the sides AB, BC, CD, DA, points K, L, L_1, K_1 are chosen respectively such that KLL_1K_1 is a rectangle. Then, suppose that a rectangle $MNPQ$, is inscribed in the triangle BLK where $M \in KB$, $N \in BL$, P, $Q \in LK$ and, similarly, $M_1N_1P_1Q_1$ is inscribed in the triangle DK_1L_1, where $M_1 \in DK_1$, $N_1 \in DL_1$ and $P_1, Q_1 \in L_1K_1$. Let $2S$, $2S_1$, S_2, S_3 be the areas of the quadrilaterals $ABCD$, KLL_1K_1, $MNPQ$, $M_1N_1P_1Q_1$ respectively. Find the greatest value of $\frac{2S_1+S_2+S_3}{2S}$.

131. Let $A_1, A_2, \ldots, A_{2002}$ be arbitrary points in a plane. Prove that for any unit circle in the plane and for any rectangle inscribed in the circle, there are three vertices M, N, P of the rectangle such that

$$MA_1 + \ldots + MA_{2002} + NA_1 + \ldots + NA_{2002} + PA_1 + \ldots + PA_{2002} \geq 6006.$$

Chapter 4

Training Problems

Test 1

132. Let a, b, c, d be positive real numbers with $a + b + c + d = 1$. Prove that:

$$\frac{bcd}{a+2} + \frac{acd}{b+2} + \frac{abd}{c+2} + \frac{abc}{d+2} < \frac{1}{13}.$$

133. Find all non-empty subsets $A \subset \mathbb{R}^*$ with the properties:

i) A has at most 5 elements;

ii) If $x \in A$ then $\frac{1}{x} \in A$ and $1 - x \in A$.

134. Let ABC be a triangle and let D, E be the points in the exterior of the triangle such that triangles ABD and ACE are isosceles and right-angled at B and C respectively.

Prove that the lines CD and BE meet on the altitude from A in the triangle ABC.

135. Consider a parallelogram $ABCD$ such that $\angle ACB = 80°$ and $\angle ACB = 20°$. A line passing through B meets the line AB at an angle of 20° and intersects the line AC in the point R. A line passing through C meets the line AC at an angle of 30° an intersects the line AB in the point T.

Find the measure of the angle determined by the lines TR and DC.

Test 2

136. Find the cube of the number $N = \sqrt{7\sqrt{3\sqrt{7\sqrt{3\sqrt{7\sqrt{3\ldots}}}}}}$

137. Prove that for any non-negative integer n the number

$$A = 2^n + 3^n + 5^n + 6^n$$

is not a perfect cube.

138. The points A, B, C are the vertices of a triangle with no equal sides. How many points D exist such that the set $\{A, B, C, D\}$ has a symmetry axis?

139. A cyclic quadrilateral $ABCD$ is given. On the rays $(AB$ and $(AD$ the points P and Q are considered so that $AP = CD$ and $AQ = BC$.
The lines PQ and AC meet at point M and N is the midpoint of the segment BD.
Prove that $PM = MQ = CN$.

Test 3

140. Solve in positive integers the equation

$$x^y \cdot y^x + x^y + y^x = 5329.$$

141. Find all the positive integers n for which the number obtained by erasing the last digit is a divisor for n.

142. Prove that a quadrilateral $ABCD$ with

$$\text{area}\,[ABC] \le \text{area}\,[BCD] \le \text{area}\,[CDA] \le \text{area}\,[ABD]$$

is a trapezoid.

143. Inside a rectangle of area 5 are given 9 polygons each of area 1. Prove that there exists 2 of them with the common area not less then $\frac{1}{9}$.

Test 4

144. Prove that for any real numbers a and b there are numbers x, $y \in [0, 1]$ such that

$$|xy - ax - by| \ge \frac{1}{3}.$$

145. Find the greatest number that can be written as a product of some positive integers with the sum 1976.

146. An acute triangle ABC is given. Prove that the internal bisector of angle $\angle BAC$, the altitude from B and the perpendicular bisector of the line segment AB are concurrent if and only if $\angle A = 60°$.

147. The points M, K, L are considered respectively on the sides AB, BC, AC of a triangle ABC. Prove that at least one of the areas of the triangles MAL, KBM or LCK is not less than a quarter of the area of the triangle ABC.

Test 5

148. Find all the integers x, y, z so that $4^x + 4^y + 4^z$ is a square.

149. Find all the primes a, b, c such that

$$ab + bc + ac > abc.$$

150. Five points are given inside of an equilateral triangle of side length 1. Prove that there exist 2 points at a distance less than $\frac{1}{2}$.

151. Let $A_1A_2 \ldots A_n$ be a regular polygon, $n \ge 3$. Find the number of obtuse triangles $A_iA_jA_k$.

Test 6

152. Find all the positive integers x, y, z, t so that $x + y + z = xyzt$.

153. Find all the positive integers n for which the set

$$\{n,\ n+1,\ n+2,\ n+3,\ n+4,\ n+5\}$$

can be decomposed in two disjoint subsets such that the product of elements in these subsets are equal.

154. Prove that in any tetrahedron there is a vertex such that the edges arising from it are the sides of a triangle.

155. Let $ABCD$ be a convex quadrilateral and let E and T be the midpoints of the sides BC and CD respectively. If $AE + AT = 4$, prove that the area of the quadrilateral $ABCD$ is less than 8.

Test 7

156. A number x is formed using the digits 1, 2, 3, 4, 5, 6, 7 once and only once. Rearranging the digits we obtain a number y. Prove that y is not a divisor of x.

157. Let x, y, z be distinct integers such that $xy + yz + xz = 26$.
Prove that $x^2 + y^2 + z^2 \ge 29$.

158. Inside a unit square lies a convex polygon of area greater than $\frac{1}{2}$. Prove that there is a line d parallel with one of the sides of the square that cuts from the polygon a line segment of length greater than or equal to $\frac{1}{2}$.

159. A triangle ABC is considered. The internal bisectors of the angles $\angle ABC$ and $\angle ACB$ intersects the sides AC and AB in the points D and E, respectively. Find the angles of the triangle ABC if $\angle BDE = 24°$ and $\angle CED = 18°$.

Test 8

160. Let $N = \underbrace{44\ldots4}_{2002}\underbrace{88\ldots8}_{2001}9$. Calculate $\sqrt{N}$.

161. Numbers $x_1, x_2, \ldots, x_n$ are chosen from the interval $[2, 4]$ such that

$$x_1 + x_2 + \ldots + x_n = \frac{17n}{6} \text{ and } x_1^2 + x_2^2 + \ldots + x_n^2 = 9n.$$

Prove that 12 divides n.

162. Inside a box of dimensions L, l and h are given $n^3 + 1$ points. Prove that there are two of them at a distance less than $\frac{\sqrt{L^2+l^2+h^2}}{n}$.

163. A point M is given inside a triangle ABC. Let D, E, F be the projections of the point M onto the sides BC, CA, AB respectively. Find the minimum value of the sum

$$\frac{BC}{MD} + \frac{CA}{ME} + \frac{AB}{MF}.$$

Test 9

164. Let $a > b > 0$ be the real numbers such that $a^5 + b^5 = a - b$. Prove that $a^4 + b^4 < 1$.

165. Let $n > 2$ be an integer.. Prove that the number of irreducible fractions from the set $\left\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}\right\}$ is even.

166. In the interior of a unit square are considered 129 points. Prove that there exists a disk of radius $\frac{1}{8}$ that contains at least three points.

167. Find all triangles with integer side lengths so that the semiperimeter has the same value as the area of the triangle.

Test 10

168. Let n and p be positive integers $n > 1$. Prove that the numbers $n - 1$ and $np + 1$ cannot have other divisors than the divisors of $p + 1$.

169. Find a relation between the numbers a, b, c if

$$x + \frac{1}{x} = a, \; y + \frac{1}{y} = b \text{ and } xy + \frac{1}{xy} = c.$$

170. Prove that in any polygon there are two sides with the length ratio greater then or equal to 1 and less then 2.

171. Inside a unit cube 28 points are given. Prove that among them there are two points at a distance not greater than $\frac{\sqrt{3}}{3}$.

Test 11

172. Find the last 5 digits of the number 5^{1981}.

173. Compute the sum

$$S = \frac{2}{3+1} + \frac{2^2}{3^2+1} + \ldots + \frac{2^{n+1}}{3^{2^n}+1}.$$

174. In a tetrahedron all the altitudes are congruent. One of them passes through the orthocenter of the corresponding face. Prove that the tetrahedron is regular.

175. Let $ABCD$ be a parallelogram. On the sides BC and CD points E and F are chosen such that $\frac{EB}{EC} = a$ and $\frac{FC}{FD} = b$. Lines AE and BF intersect in the point M. Find the ratio $\frac{AM}{ME}$.

Test 12

176. Prove that there are at least 2002 rational numbers m so that $\sqrt{m + 2002}$ and $\sqrt{m + 2003}$ are both rational numbers.

177. Let a, b, c be positive real numbers such that $abc > 1$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > a + b + c$. Prove that:

i) All numbers are different than 1.

ii) Only one numbers is less than 1.

178. 5 points are given in a plane, not three of them collinear. Prove that there are 4 among them which are vertices of a convex quadrilateral.

179. Consider a convex hexagon of area S. Prove that there is a triangle determined by three consecutive vertices of the hexagon with an area not greater than $\frac{S}{6}$.

Test 13

180. Find all the positive integers n which are equal to the sum of its digits added to the product of its digits.

181. Consider the sum
$$S=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{99\cdot 100}.$$
Find the sequences of consecutive terms of S that add up to $\frac{1}{6}$.

182. Prove that any polygon with the perimeter 2004 can be covered by a disk of diameter 1002.

183. Prove that there are no triangles in which the incircle divides an internal bisector of an angle in three equal segments.

Test 14

184. Let k, n_1, n_2, $\ldots$, n_k be odd integers. Prove that the numbers of odd numbers among $\frac{n_1+n_2}{2}$, $\frac{n_2+n_3}{2}$, $\ldots$, $\frac{n_k+n_1}{2}$ is odd.

185. Solve in $\mathbb{R}$ the equation:
$$[x\,[x]]=1,$$
($[x]$ denotes the integer part of the number x).

186. Prove that in any triangle the following inequality holds
$$b+c-a<2b\cos\frac{A}{2}.$$

187. A convex polygon with n^2 sides ($n>2$) is decomposed into n convex pentagons. Prove that $n=3$.

Test 15

188. Find the greatest number n such that any subset with $1984-n$ elements of the set $\{1, 2, \ldots, 1984\}$ contains a pair of coprime numbers.

189. Find the real numbers a_1, a_2, $\ldots$, a_{2n+1} so that
$$a_1+a_2+\ldots+a_{2n}+a_{2n+1}=2n+1 \text{ and } |a_1-a_2|=|a_2-a_3|=\ldots=|a_{2n+1}-a_1|.$$

190. Considers $2n+1$ real numbers between 1 and 2^n. Prove that there are three of them which are the side lengths of a triangle.

191. A convex octagon has all the angles congruent and all side lengths rational numbers. Prove that the octagon has a symmetry point.

Test 16

192. Find the sum of the digits of the numbers from 1 to 1,000,000.

193. Find the elements of the set
$$A=\left\{x\in\mathbb{Z}\mid \frac{x^3-3x+2}{2x+1}\in\mathbb{Z}\right\}.$$

194. Prove that 2002 points can be joined two by two with 1001 segments such that no two of them intersect.

195. A triangle ABC with $\angle A=90^\circ$ is given. A square $MNPQ$ is inscribed in the triangle such that M lies on AB, N lies on BC, P lies on BC and Q lies on CA. Likewise, the squares of the sides l_1, l_2, l_3 are inscribed in the right triangles QPC, MBN, AMQ respectively, all having two vertices on the hypotenuses and a vertex on each leg of the triangles. Prove that
$$\frac{1}{l_1^2}+\frac{1}{l_2^2}=\frac{1}{l_3^2}.$$

Test 17

196. Consider n distinct positive integers less than $2n$. Prove that among these numbers there is one equal to n or there are two numbers with the sum equal to $2n$.

197. Let a, b, c be odd integers. Prove that the roots of the equations $ax^2+bx+c=0$ are not rational numbers.

198. Let ABC be a triangle with $\angle A=90^\circ$. Consider the altitude AD and T, E the midpoints of the segments AD and DC respectively. Prove that $\angle ABT=\angle CAE$.

199. Let $ABCD$ be a trapezoid with the middle line equal to the altitude. Prove that the diagonals are perpendicular if and only if the trapezoid is isosceles.

Test 18

200. The sum of 10 distinct non-negative integers is equal to 62. Prove that the product of these numbers is divisible by 60.

201. Let a, b, c be real numbers so that $a+2b+3c=2$ and $2ab+3ac+6bc=1$. Show that $a\in\left[0, \frac{4}{3}\right]$, $b\in\left[0, \frac{2}{3}\right]$ and $c\in\left[0, \frac{4}{9}\right]$.

202. Consider an acute triangle $A_1A_2A_3$ and let H_1, H_2, H_3 be the feet of the altitudes from A_1, A_2, A_3, respectively. If a_1, a_2, a_3 are the lengths of the sides A_2A_3, A_3A_1, A_1A_2 and H is the orthocenter of the triangle, prove that
$$\frac{a_1}{HH_1}+\frac{a_2}{HH_2}+\frac{a_3}{HH_3}=2\left(\frac{a_1}{HA_1}+\frac{a_2}{HA_2}+\frac{a_3}{HA_3}\right).$$

203. Consider a convex quadrilateral $ABCD$ and let M, Q, N, P be the midpoints of the sides AB, BC, CD, AD respectively. Prove that if $2(MN+PQ)=AB+BC+CD+DA$, then $ABCD$ is a parallelogram.

Test 19

204. Let a, b, c be positive real numbers with $\sqrt{ab}+\sqrt{bc}+\sqrt{ac}=1$. Find the minimum value of the expression
$$E=\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}.$$

205. On the faces of a cube are written the numbers from 1 to 6. Prove that the sum of the numbers written on three faces with a common vertex cannot be constant.

206. A triangle ABC with $\angle A=90^\circ$ is given. Let D be the foot of the altitude from A. Prove that
$$BC+AD>AB+AC.$$

207. Consider a trapezoid $ABCD$ with $AB \parallel CD$ and $CD=kAB$ $(k>1)$.

a) Prove that
$$BC^2+AD^2+2kAB^2=AC^2+BD^2.$$

b) If the trapezoid is circumscriptible, prove that
$$(k+1)AB=BC+AD.$$

Test 20

208. The numbers $1, 2, 3, 4, \ldots, 2n$ are divided in two groups each: $a_1<a_2<\ldots<a_n$ and $b_1>b_2>\ldots>b_n$. Prove that
$$|a_1-b_1|+|a_2-b_2|+\ldots+|a_n-b_n|=n^2.$$

209. Let a, b, c, d be real numbers so that
$$\left(a^2+b^2-1\right)\left(c^2+d^2-1\right)>(ac+bd-1)^2.$$
Prove that
$$a^2+b^2>1 \text{ and } c^2+d^2>1.$$

210. Find the location of a point M inside a convex quadrilateral $ABCD$ such that the sum $MA^2+MB^2+MC^2+MD^2$ is minimal.

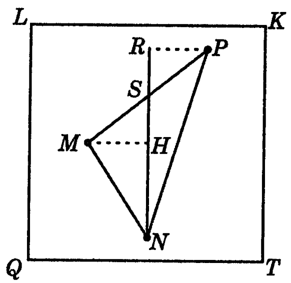
211. A triangle ABC with $AB>AC$ is given. Prove that the length of the median from B is greater than the length of median from C.

Chapter 5

Junior Balkan Mathematical Olympiad Formal Solutions

1. Nine points are given inside a unit square. Prove that three of them are the vertices of a triangle with the area not greater than $\frac{1}{8}$.

Solution. Divide the unit square in 4 equal squares of area $\frac{1}{4}$. By the Pigeonhole principle, three of the nine points are inside or on the sides of a small square. Let M, N, P be the points and let $LKTQ$ be the square of area $\frac{1}{4}$.



Consider the parallel lines from M, N, P to LQ. One of them lies between the other two, intersecting the corresponding side of the triangle. Without loss of generality, let $NS \parallel LQ$ and $S \in [MP]$. Let MH and PR be the perpendicular lines to NS with $H, R \in NS$. Then:

$$\begin{aligned}\text{area}\,[MPN] &= \text{area}\,[MNS]+\text{area}\,[PSN]=\frac{NS\cdot MH+NS\cdot PR}{2}\\ &\le \frac{LQ\cdot LK}{2}=\frac{\text{area}\,[LQTK]}{2}=\frac{1}{8},\end{aligned}$$

as desired.

The equality holds if $NS(MH+PR) = LQ\cdot LK$, hence $NS = LQ$ and $MH+PR = LK$. That is when a side of the triangle is equal to a side of the square and the third vertex lies on the opposite side of the square.

2. Let

$$\frac{x^2+y^2}{x^2-y^2}+\frac{x^2-y^2}{x^2+y^2}=k.$$

Find the value of

$$\frac{x^8+y^8}{x^8-y^8}+\frac{x^8-y^8}{x^8+y^8}$$

in terms of k.

Solution. The equality

$$\frac{x^2+y^2}{x^2-y^2}+\frac{x^2-y^2}{x^2+y^2}=k$$

implies

$$\frac{\left(x^2+y^2\right)^2+\left(x^2-y^2\right)^2}{x^4-y^4}=k,$$

hence

$$\frac{x^4+y^4}{x^4-y^4}=\frac{k}{2},$$

and

$$\left(\frac{x}{y}\right)^4=\frac{k+2}{k-2}.$$

Therefore

$$\begin{aligned}\frac{x^8+y^8}{x^8-y^8}+\frac{x^8-y^8}{x^8+y^8} &= \frac{\left(x^8+y^8\right)^2+\left(x^8-y^8\right)^2}{x^{16}-y^{16}}=\frac{2\left(x^{16}+y^{16}\right)}{x^{16}-y^{16}}\\ &= 2\frac{\left(\frac{k+2}{k-2}\right)^4+1}{\left(\frac{k+2}{k-2}\right)^4-1}=2\frac{(k+2)^4+(k-2)^4}{(k+2)^4-(k-2)^4}.\end{aligned}$$

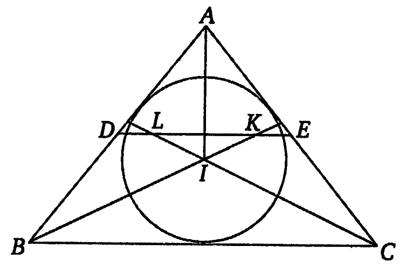
3. Let I be the incenter of the triangle ABC, and let D and E be the midpoints of the sides AB and AC respectively. Lines DE and BI meet at point K and lines DE and CI meet at point L. Prove that

$$AI+BI+CI > BC+KL.$$

Solution. The segment DE is the middle line of the triangle, so

$$DE \parallel BC \tag{1}$$

and $DE = \frac{BC}{2}$.



The rays $[BI$ şi $[CI$ are bisector lines and $DE \parallel BC$,hence triangles DBK and CLE are isosceles and

$$DB = DK \text{ and } EC = EL. \tag{2}$$

Using the triangle inequality in AIB, BIC, CIA, yields

$$AB < AI+BI, \tag{3}$$

$$BC < BI+CI, \tag{4}$$

$$AC < CI+AI. \tag{5}$$

Summing the inequalities yields

$$AB+BC+AC < 2(AI+BI+CI),$$

and consequently

$$\frac{AB+BC+AC}{2} < AI+BI+CI. \tag{6}$$

On the other hand,

$$\begin{aligned}\frac{AB+BC+AC}{2} &= DB+DE+CE = DK+LE+DE\\ &= DE+KL+DE = 2DE+KL = BC+KL. \end{aligned}\tag{7}$$

From (6) and (7) we obtain

$$BC+KL < AI+BI+CI,$$

as desired.

4. Find the triangle ABC so that

$$R\,(b+c) = a\sqrt{bc}.$$

Solution. In a circle the diameter is longer then a chord, so

$$2R \geq a. \tag{1}$$

Using the AM-GM inequality yields

$$\frac{b+c}{2} \geq \sqrt{bc}. \qquad (2)$$

It follows that

$$R(b+c) \geq a\sqrt{bc},$$

with equality when $a = 2R$ and $b = c$, hence the triangle is right and isosceles.

5. Prove that the number

$$\underbrace{11\ldots11}_{1997}\underbrace{22\ldots22}_{1998}5$$

is a perfect square.

Solution.

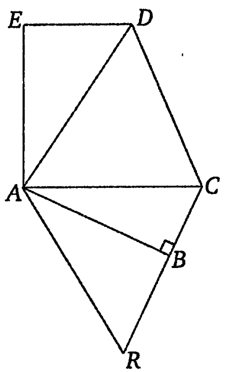
$$\begin{aligned} N &= \underbrace{11\ldots11}_{1997}\cdot 10^{1999} + \underbrace{22\ldots22}_{1998}\cdot 10 + 5 \\ &= \frac{1}{9}\left(10^{1997}-1\right)\cdot 10^{1999} + \frac{2}{9}\left(10^{1998}-1\right)\cdot 10 + 5 \\ &= \frac{1}{9}\left(10^{3996} + 2\cdot 5\cdot 10^{1998} + 25\right) \\ &= \left[\frac{1}{3}\left(10^{1998}+5\right)\right]^2 = \left(\frac{1\overbrace{00\ldots00}^{1997}5}{3}\right)^2 = \underbrace{33\ldots33}_{1997}5^2. \end{aligned}$$

6. Let $ABCDE$ be a pentagon so that

$$AB = AE = CD = 1,\ \angle ABC = \angle DEA = 90^\circ \text{ and } BC + DE = 1.$$

Find the area of the pentagon.

Solution. Consider a point R on the line CB so that $BR = DE$ and $CR = BR + BC = 1$. The triangles ABR and AED are congruent (SAS), hence $AR = AD$. Since $CD = CR$, it follows that ACD and ACR are also congruent triangles.



Therefore

$$\begin{aligned} \text{area}\,[ABCDE] &= \text{area}\,[ABC] + \text{area}\,[ADE] + \text{area}\,[ACD] \\ &= \text{area}\,[ABC] + \text{area}\,[ABR] + \text{area}\,[ACD] \\ &= 2\text{area}\,[ARC] = CR\cdot AB = 1. \end{aligned}$$

7. Find all the pairs (x, y) of positive integers so that

$$x^y = y^{x-y}.$$

Solution. At first, notice that $x = y = 1$ is a solution of the equation. If $x > 1$, then $x > y$, else $y^{x-y} \leq 1 < x^y$.

We may assume that $x > y \geq 2$. The equation rewrites

$$\left(\frac{x}{y}\right)^y = y^{x-2y}, \qquad (1)$$

hence $x - 2y > 0$ and consequently $\frac{x}{y}$ is an integer greater than 2.

The equation (1) is equivalent to

$$\frac{x}{y} = y^{\frac{x}{y}-2}. \qquad (2)$$

Since $y^{\frac{x}{y}-2} \geq 2^{\frac{x}{y}-2}$, it follows that $\frac{x}{y} \geq 2^{\frac{x}{y}-2}$. Inducting on $n \geq 5$ one can prove that $2^{n-2} > n$, hence $\frac{x}{y} \leq 4$ and so $\frac{x}{y} = 3$ or $\frac{x}{y} = 4$.

1° If $\frac{x}{y} = 3$, the relation (2) gives $x = 9$, $y = 3$.

2° If $\frac{x}{y} = 4$, the relation (2) gives $x = 8$, $y = 2$.

The solutions $(x; y)$ are $(1, 1)$, $(9, 3)$, $(8, 2)$.

8. Can one find 16 three digit numbers, using only 3 digits, without having two of them with the same remainder when divided by 16?

Solution. Assume that there are 16 numbers having distinct remainders when divided by 16. Then 8 numbers are odd and 8 numbers are even. Thus the **3** digits cannot have the same parity, so assume that two of them are even (a and b) and one is odd (c). There are 9 odd three digit numbers that can be formed with these digits: $\overline{aac}$, $\overline{abc}$, $\overline{acc}$, $\overline{bac}$, $\overline{bbc}$, $\overline{bcc}$, $\overline{cac}$, $\overline{cbc}$, $\overline{ccc}$.

Let $a_1, a_2, \ldots, a_9$ be the two-digit numbers obtained by erasing the last digit (c) from the above sequence.

The numbers $\overline{a_ik}$ and $\overline{a_jk}$, with $i \neq j$, have different remainders when divided by 16 if and only if 16 is not a divisor of $\overline{a_ik} - \overline{a_jk}$; that is if and only if 8 is not a divisor of $a_i - a_j$.

Among the numbers $a_1, a_2, \ldots, a_9$ there are only three odd numbers $\overline{ac}$, $\overline{bc}$, $\overline{cc}$. Hence among any 8 numbers from $a_1, a_2, \ldots, a_9$ one can find two of them with the same remainder at division by 8, a contradiction.

The same conclusion follows from the case when two digits are odd and only one is even.

9. Let a, b, c, x, y be real numbers so that:

$$a^3 + ax + y = 0,\ b^3 + bx + y = 0 \text{ and } c^3 + cx + y = 0.$$

Show that if a, b, c are distinct numbers, different from 0, then $a + b + c = 0$.

Solution. Subtracting the first two relation yields

$$(a-b)(a^2 + ab + b^2 + x) = 0,$$

and

$$a^2 + ab + b^2 = -x, \tag{1}$$

since $a \neq b$.

Likewise,

$$b^2 + bc + c^2 = -x, \tag{2}$$

since $b \neq c$.

The equalities (1) and (2) imply

$$b(a-c) + (a-c)(a+c) = 0,$$

From $a \neq c$ we get $b + a + c = 0$, as desired.

10. Find the greatest common divisor of the numbers

$$A_n = 2^{3n} + 3^{6n+2} + 5^{6n+2}$$

when $n = 0, 1, \ldots, 1999$.

Solution. We have

$$A_0 = 1 + 9 + 25 = 35 = 5 \cdot 7.$$

Using congruence mod 5, it follows that

$$A_n \equiv 2^{3n} + 3^{6n+2} \equiv 2^{3n} + 9^{3n+1} \equiv 2^{3n} + (-1)^{3n+1} (\bmod 5).$$

For $n = 1$, $A_1 \equiv 9 \not\equiv 0 (\bmod 5)$, hence 5 is not a common divisor.

On the other hand,

$$\begin{aligned} A_n &= 8^n + 9 \cdot 9^{3n} + 25 \cdot 25^{3n} \equiv 1 + 2 \cdot 2^{3n} + 4 \cdot 4^{3n} \\ &\equiv 1 + 2 \cdot 8^n + 4 \cdot 64^n \equiv 1 + 2 \cdot 1^n + 4 \cdot 1^n \equiv 7 \equiv 0 (\bmod 7), \end{aligned}$$

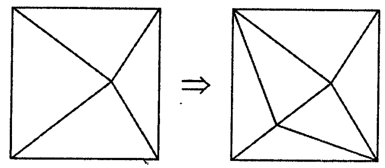
therefore 7 divides A_n, for all integers $n \geq 0$.

Consequently, the greatest common divisor of the numbers $A_0, A_1, \ldots A_{1999}$ is equal to 7.

11. Let S be a square of side 20 and let M be a set consisting of the vertices of the square and 1999 arbitrary inner points of S. Prove the existence of a triangle with the area at most equal to $\frac{1}{10}$ and having all the vertices in the set M.

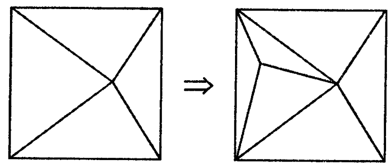
Solution. The main idea is to join the 2003 points so that 4000 triangles with disjoint interiors are formed.

Consider an interior point and join it with the four vertices of the square; four triangles are determined. Choose a second interior point. If it is located inside of a triangle, join it with the vertices of the triangle. The triangle is divided in three triangles, so two more triangles are formed:



4 triangles　　6 = 4 + 2 triangles

If the second point is located on a segment which is a common side of two triangles- both triangles are divided in two small triangles:



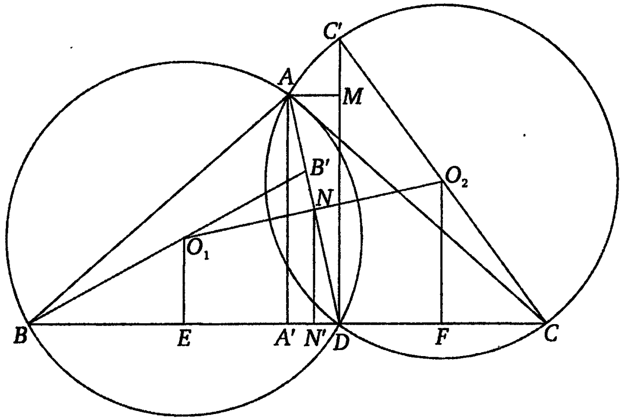
4 triangles　　6 = 4 + 2 triangles

As in the previous case, the number of triangles increases by two.

For any new interior point that is considered the number of triangles increases - as above - with two more triangles. In the end, $4 + 2 \cdot 1998 = 4000$ triangles are obtained. The area of the square is 400, hence there is a triangle with area not greater than $\frac{1}{10}$, as desired.

12. In a triangle ABC the sides AB and AC are equal. Let D be a point on BC such that $BC > BD > DC > 0$. Consider the circumcircles k_1 and k_2 of the triangles ABD and ADC respectively. Let M be the midpoint of $B'C'$, when BB' and CC' are diameters of k_1 and k_2 respectively. Prove that the area of the triangle MBC is constant (with respect to D).

Solution. Using the Sine Law and the equality $\angle BDA + \angle ADC = 180°$ follows that the circles k_1 and k_2 are equal, hence the quadrilateral AO_1DO_2 is a rhombus.



Let N be the intersection point of the diagonals of the rhombus AO_1DO_2 and let E, A', N', F be the projections of the points O_1, A, N, O_2 on the line BC respectively. The segments O_1E and O_2F are the middle lines of the triangles $BB'D$ and CDC' respectively, so we have

$$DM = \frac{DB' + DC'}{2} = \frac{2EO_1 + 2FO_2}{2} = EO_1 + FO_2.$$

Furthermore, since the segment NN' is the middle line in the trapezoid EFO_2O_1 and also in the triangle ADA', it follows that $NN' = \frac{EO_1+FO_2}{2} = \frac{DM}{2}$ and $NN' = \frac{AA'}{2}$.

Consequently, $DM = AA'$ and

$$\text{area}\,[MBC] = \frac{MD \cdot BC}{2} = \frac{AA' \cdot BC}{2} = \text{area}\,[ABC].$$

Therefore the area of the triangle MBC is constant, as desired.

Is the proof still valid if the angle $\angle BAC$ is obtuse?

13. Let x, y be integer numbers so that

$$x^3 + y^3 + (x+y)^3 + 30xy = 2000.$$

Prove that $x + y = 10$.

Solution. We have

$$\begin{aligned} E &= x^3 + y^3 + (x+y)^3 + 30xy - 2000 \\ &= 2(x+y)^3 - 3x^2y - 3xy^2 + 30xy - 2000 \\ &= 2\left[(x+y)^3 - 1000\right] - 3xy(x+y-10) \\ &= (x+y-10)\left[2\left((x+y)^2 + 10(x+y) + 100\right) - 3xy\right] \\ &= (x+y-10)\left(2x^2 + xy + 2y^2 + 20x + 20y + 200\right). \end{aligned}$$

Since

$$\begin{aligned} F &= 2x^2 + xy + 2y^2 + 20x + 20y + 200 \\ &= \left(x^2 + xy + y^2\right) + \left(x^2 + 20x + 100\right) + \left(y^2 + 20y + 100\right) \\ &= \frac{x^2 + y^2 + (x+y)^2}{2} + (x+10)^2 + (y+10)^2 > 0, \end{aligned}$$

it follows that $x + y = 10$.

14. Find all the positive integers n, $n \geq 1$, such that $n^2 + 3^n$ is a perfect square.

Solution. Let m be a positive integer such that

$$m^2 = n^2 + 3^n.$$

Since $(m-n)(m+n) = 3^n$, there is $k \geq 0$ such that $m - n = 3^k$ and $m + n = 3^{n-k}$.

From $m - n < m + n$ follows $k < n - k$, and so $n - 2k \geq 1$.

If $n - 2k = 1$, then $2n = (m+n) - (m-n) = 3^{n-k} - 3^k = 3^k(3^{n-2k} - 1) = 3^k(3^1 - 1) = 2 \cdot 3^k$, so $n = 3^k = 2k + 1$. By induction on $m \geq 2$ one obtains $3^m > 2m + 1$, therefore $k = 0$ or $k = 1$ and consequently $n = 1$ or $n = 3$.

If $n - 2k > 1$, then $n - 2k \geq 2$ and $k \leq n - k - 2$. It follows that $3^k \leq 3^{n-k-2}$, and consequently

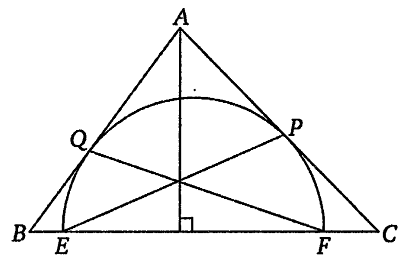
$$\begin{aligned} 2n &= 3^{n-k} - 3^k \geq 3^{n-k} - 3^{n-k-2} = 3^{n-k-2}(3^2 - 1) = 8 \cdot 3^{n-k-2} \\ &\geq 8[1 + 2(n-k-2)] = 16n - 16k - 24, \end{aligned}$$

which implies $8k + 12 \geq 7n$.

On the other hand, $n \geq 2k + 2$, hence $7n \geq 14k + 14$, contradiction.

In conclusion, the only possible values for n are 1 and 3.

15. A semicircle of diameter EF, lying on the side BC of the ABC triangle, is tangent to the sides AB and AC in Q and P respectively.



The lines EP and FQ meet at point K.

Prove that K is a point on the altitude from A of the triangle ABC.

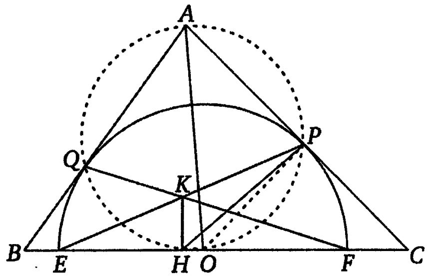
Solution. Let O be the center of the semicircle and let H be the projection of the point K on the side BC. Consider the case when the point O lies on the line segment HF. The angle $\angle EPF$ subtends a diameter of the semicircle, hence it is right, as $\angle KHF$. Consequently, the quadrilateral $KHFP$ is cyclic and

$$\angle KHP = \angle KFP = \overset{\frown}{PQ} = \frac{1}{2}\angle QOP.$$

The triangles AOP and AOQ are right-angled and have equal legs $OP = OQ$, so

$$\angle AOP = \angle AOQ = \frac{1}{2}\angle QOP = \angle KHP.$$

It follows that $\angle PHO \equiv \angle PAO$, hence $APOH$ is a cyclic quadrilateral. Thus the angle AHO is right and $AH \perp BC$, hence $K \in AH$, as desired.



The case $O \in (EH)$ is solved similarly.

If $H = O$ then the triangle ABC is isosceles and the claim is obvious.

16. At a tennis tournament there were twice as many girls participating than boys. Each pair of players had only one match and there were no draws. The ratio between girl winnings and boy winnings was $\frac{7}{5}$.

How many players took part at the tournament?

Solution. Let n be the number of girls, $2n$ the number of boys and $3n$ the total number of the players in the tournament. The total number of matches is $\binom{3n}{2} = \frac{3n(3n-1)}{2}$. The number of matches won by the boys is $\frac{5}{12}\binom{3n}{2} = \frac{5n(3n-1)}{8}$. The matches played between boys are $\binom{2n}{2} = \frac{2n(2n-1)}{2} = n(2n-1)$ and counts as winnings for boys, hence

$$\frac{5n(3n-1)}{8} \geq n(2n-1) \Leftrightarrow 15n - 5 \geq 16n - 8 \Leftrightarrow n \leq 3.$$

Moreover, 8 divides $5n(3n-1)$, hence $n = 3$.

Thus there were 9 players in the tournament.

17. Find all the positive integers a, b, c such that

$$a^3 + b^3 + c^3 = 2001.$$

Solution. Assume without loss of generality that $a \leq b \leq c$.

It is obvious that $1^3 + 10^3 + 10^3 = 2001$. We prove that (1, 10, 10) is the only solution of the equation, except for its permutations.

We start proving a useful

Lemma: Suppose n is an integer. The remainder of n^3 when divided by 9 is 0, 1 or -1.

Indeed, if $n = 3k$, then $9 \mid n^3$ and if $n = 3k \pm 1$, then $n^3 = 27k^3 \pm 27k^2 + 9k \pm 1 = \mathfrak{M}9 \pm 1$.

Since $2001 = 9 \cdot 222 + 3 = \mathfrak{M}9 + 3$, then $a^3 + b^3 + c^3 = 2001$ implies $a^3 = \mathfrak{M}9 + 1$, $b^3 = \mathfrak{M}9 + 1$ and $c^3 = \mathfrak{M}9 + 1$, hence a, b, c are numbers of the form $\mathfrak{M}3 + 1$. We search for a, b, c in the set $\{1, 4, 7, 10, 13, ...\}$.

If $c \geq 3$ then $c^3 \geq 2197 > 2001 = a^3 + b^3 + c^3$, which is false. If $c \leq 7$ then $2001 = a^3 + b^3 + c^3 \leq 3 \cdot 343$ and again is false. Hence $c = 10$ and consequently $a^3 + b^3 = 1001$. If $b < c = 10$ then $a \leq b \leq 7$ and $1001 = a^3 + b^3 \leq 2 \cdot 7^3 = 2 \cdot 343$, a contradiction. Thus $b = 10$ and $a = 1$.

Therefore $(a,\ b,\ c) \in \{(1,\ 10,\ 10),\ (10,\ 1,\ 10),\ (10,\ 10,\ 1)\}$.

18. Let ABC be a triangle with $\angle ACB = 90°$ and $AC \neq BC$. The points L and H of the segment $[AB]$ are chosen such that $\angle ACL = \angle LCB$, and CH is perpendicular to AB.

a) For every point X (other than C) on the line CL, prove that $\angle XAC \neq \angle XBC$.

b) For every point Y (other than C) on the line CH prove that $\angle YAC \neq \angle YBC$.

Solution. a) Assume that there is a point $X \in CL$, $X \neq C$ such that $\angle XAC = \angle XBC$. Then the triangles AXC and BXC are congruent and consequently $AC = BC$, a contradiction.

b) Without loss of generality we may assume that $CA < CB$. Suppose by contradiction that there is a point $Y \in (CH)$ such that $\angle YAC = \angle YBC$. Using the Sine Law, it follows that the circumcircles C_1 and C_2 of the triangles AYC and BYC are equal. Let A' be the reflection point of A across the line CH. Then A' lies on the line AB and on the circle C_2 and $\angle HCA' = \angle HCA = \angle ABC$.

Let O be the center of the circle C_2. Then $\angle COA' = 2\angle CBA' = 2\angle ABC$.

On the other hand, the triangle $OA'C$ is isosceles and

$$2\angle A'CO = 180° - \angle COA' = 180° - 2\angle ABC,$$

which implies $\angle A'CO = 90° - \angle ABC$.

It follows that

$$\angle HCO = \angle HCA' + \angle A'CO = \angle ABC + 90° - \angle ABC = 90°,$$

and consequently $CY \perp OC$. This implies that CY is tangent to $\mathcal{C}_2$, a contradiction.

For any other position of the point Y on the line CH we use the same way of reasoning.

19. Let ABC be an equilateral triangle and let D, E be arbitrary points on the sides $[AB]$ and $[AC]$ respectively. If DF, EG (with $F \in AE$, $G \in AD$) are internal bisectors of the angles of the triangle ADE, prove that the sum of the areas of the triangles DEF and DEG is less than or equal to the area of the area of triangle ABC. Explain when the equality holds.

Solution. Notice that $\angle AGE$ is an external angle of the triangle DGE, so

$$\begin{aligned} \angle AGE &= \angle ADE + \angle GED = \angle ADE + \frac{1}{2}[180^\circ - \angle A - \angle ADE] \\ &= 60^\circ + \frac{1}{2}\angle ADE \end{aligned}$$

and

$$\angle DFE = \angle ADF + \angle A = 60^\circ + \frac{1}{2}\angle ADE,$$

hence

$$\angle AGE \equiv \angle DFE. \tag{1}$$

Let I be the intersection point of the bisectors $(DF$ and $(EG$, and consider the point M on the line segment DE so that $\angle DIM \equiv \angle DIG$. Then $\Delta DIM \equiv \Delta DIG$ and consequently

$$DM = DG,$$

and

$$\angle DMI \equiv \angle DGI. \tag{3}$$

From the relations (1) and (3) follows that $\angle IME \equiv \angle IFE$ and then $\Delta IME \equiv \Delta IFE$, hence

$$ME = FE. \tag{4}$$

The relations (2) and (4) yield

$$DE = DG + EF. \tag{5}$$

Let r be the inradius of the triangle ADE. Using (5), we obtain

$$\text{area}[IDG] + \text{area}[IEF] = \frac{r}{2}(DG + EF) = \frac{1}{2}r \cdot DE = \text{area}[IDE],$$

hence

$$\text{area}[DEG] + \text{area}[DEF] = 3\text{area}[IDE]. \tag{6}$$

We use the fact that if *M is a point on the arc subtended by the chord XY, then the area of the triangle MXY is maximal when M is the midpoint of the arc XY.*

Consequently area$[EDI] \leq$area$[PDE]$, where the triangle PDE is isosceles with $\angle DPE = \angle EID = 120^\circ$.

If O is the circumcenter of the triangle ABC, then the triangles PDE and OBC are similar and area$[PDE] \leq$area$[OBC]$, with equality only when $D = B$ and $E = C$. Then

$$\text{area}[IDE] \leq \text{area}[OBC], \tag{7}$$

and from the relations (6) and (7) follows that

$$\text{area}[DEG] + \text{area}[DEF] \leq 3\text{area}[OBC] = \text{area}[ABC],$$

as desired. The equality holds when $D = B$ and $E = C$.

20. A convex polygon with 1415 sides has the perimeter of 2001 centimeters. Prove that there exist three vertices of this polygon, which form a triangle having the area less than 1 square centimeter.

Solution. Let $A_1, A_2, \ldots, A_{1415}$ be the vertices of a polygon. Suppose by contradiction that all triangles determined by three consecutive vertices of the polygon, namely $A_1A_2A_3$, $A_2A_3A_4$, $\ldots$, $A_{1414}A_{1415}A_1$, $A_{1415}A_1A_2$, have the area greater or equal to 1.

In any triangle ABC holds

$$2\text{area}[ABC] = AB \cdot AC \cdot \sin(\angle BAC) \leq AB \cdot AC,$$

hence $2 \leq 2[A_1A_2A_3] \leq A_1A_2 \cdot A_2A_3$. By the AM-GM inequality we obtain

$$A_1A_2 + A_2A_3 \geq 2\sqrt{A_1A_2 \cdot A_2A_3} \geq 2\sqrt{2},$$

and likewise

$$\begin{aligned} A_2A_3 + A_3A_4 &\geq 2\sqrt{2}, \\ &\ldots \\ A_{1414}A_{1415} + A_{1415}A_1 &\geq 2\sqrt{2}, \\ A_{1415}A_1 + A_1A_2 &\geq 2\sqrt{2}. \end{aligned}$$

Summing these 1415 inequalities, the left-hand side is equal to twice the perimeter of the polygon, hence $2 \cdot 2001 \geq 2\sqrt{2} \cdot 1415$ or $2001^2 > 2 \cdot 1415^2$. This yields $4004001 \geq 4004450$, a contradiction. Thus at least one of the triangle has the area less than 1.

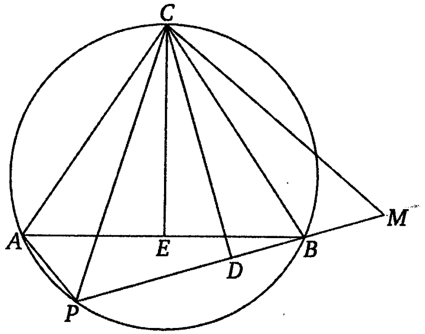
21. Let ABC be an isosceles triangle with $AC = BC$ and let P be a point on the arc AB of the circumcircle which does not contain C. The perpendicular from C on PB intersects PB in D. Prove that

$$PA + PB = 2PD.$$

Solution. Extend the segment PB with the segment $BM = AP$. The triangles BCM and ACP are congruent, hence $CP = CM$ and consequently the triangle CPM is isosceles having the altitude CD. Thus D is the midpoint of PM, hence

$$2PD = PM = PB + BM = PB + PA,$$

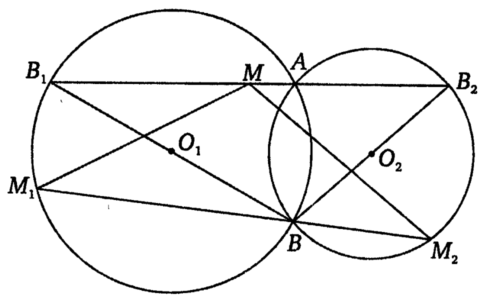
as desired.



An alternative solution can be obtained by using the Ptolemy's theorem.

22. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ of different radii have two common points A and B and their centers O_1 and O_2 are separated by the straight line AB. Let B_1, B_2 the diametrically opposed points of B on these circles respectively. The points M_1 on $\mathcal{C}_1$ and M_2 on $\mathcal{C}_2$ are chosen such that $\angle AO_1M_1 \equiv \angle AO_2M_2$, B_1 is an internal point of $\angle AO_1M_1$ and B is an internal point of $\angle AO_2M_2$. Let M be the midpoint of the segment B_1B_2. Prove that $\angle MM_1B \equiv \angle MM_2B$.

Solution. Notice that $\angle B_1AB = \angle B_2AB = 90°$, hence the points B_1, A, B_2 are collinear.



The condition $\angle AO_1M_1 = \angle AO_2M_2$ implies $\angle ABM_1 = \angle AB_2M_2$, and since both AB_1M_1B and AB_2M_2B are cyclic quadrilaterals, follows that $\angle AB_1M_1 = \angle ABM_2$. Then $\angle AB_1M_1 + \angle AB_2M_2 = \angle ABM_2 + \angle ABM_1 = \angle ABM_1 + \angle AB_1M_1 = 180°$ hence the lines M_1B_1 and M_2B_2 are parallel and the points M_1, B_1 and M_2 are collinear.

It suffices to prove that $MM_1 = MM_2$. For this, notice that $M_1B_1B_2M_2$ is a trapezoid with $\angle B_1MB = 90°$, and the middle line passes through M and is the perpendicular bisector of the line M_1M_2. The claim is obvious.

23. Find the positive integers N having the following properties:

i) N has exactly 16 divisors $1 = d_1 < d_2 < \ldots < d_{15} < d_{16} = N$.

ii) the divisor having the index d_5 (that is d_{d_5}) is equal to $(d_2 + d_4)d_6$.

Solution. First, observe that N has no more than 4 prime distinct divisors. Moreover, $d_2 = 2$, otherwise all the divisors are odd, which contradicts the second condition.

From the hypothesis we will have $2+d_4 \geq d_5 \geq 7$, so $d_4 \geq 5$. Since $d_4 < d_5 \leq 2+d_4$, we should have $d_5 = d_4 + 1$ or $d_5 = d_4 + 2$.

In the first case we have $d_6 = 2 + d_4$, so N has three consecutive divisors. Hence $3 \mid N$ and $d_3 = 3$. It follows that $6 \mid N$ and $d_4 = 6$, implying $d_5 = 7$, $d_6 = 8$, and consequently $4 | N$. Therefore $d_4 = 4$, a contradiction.

It remains the case $d_5 = 2 + d_4$. We consider the following:

i) $4 \mid N$. Since $d_4 \geq 5$, we have $d_3 = 4$, implying $8 \mid N$. From $d_6 \geq 8$ we derive that $8 \in \{d_4, d_5, d_6\}$. All of these cases lead to a contradiction as follows:

If $d_4 = 8$, then $d_5 = 10$, and so $5 \mid N$ and consequently $d_4 = 5$, false.

If $d_5 = 8$, then $d_4 = 6$, and so $3 \mid N$ thus $d_3 = 3$, false.

If $d_6 = 8$, then $d_5 = 7$, $d_4 = 5$, thus $10 \mid N$. On the other hand, $d_7 = (2 + 5)8 = 56 > 10$, a contradiction . Since N is not divisible by 4 we conclude that d_3 is prime.

ii) $3 \mid N$ and consequently $d_3 = 3$. It follows that $6 \mid N$ and since $d_4 \geq 6$, we must have $d_4 = 6$. Thus $d_5 = 8$, implying $4 \mid N$, false.

Therefore 3 does not divide N and we conclude that $d_3 \geq 5$ and $d_4 \geq 7$. Since N and $2 + d_4$ are not multiples of 4, we deduce that d_4 is odd. As $2 + d_4$ and d_4 are not divisible by 3, we obtain $d_4 = 3k + 2$ for some integer k. Actually, as d_4 is odd we have $d_4 = 6l + 5$, for some integer l. Since $d_5 \leq 16$, we find that $7 \leq d_4 \leq 14$. Thus $d_4 = 11$ and $d_5 = 13$.

Then, since $2 \cdot d_3$ is a divisor of N greater than $d_4 = 11$, we obtain $d_3 \geq 6$. Moreover, $d_3 < 11$ and d_3 is prime, hence $d_3 = 7$. Therefore $N = 2 \cdot 7 \cdot 11 \cdot 13 = 2002$.

24. Let a, b, c, be positive numbers. Prove that:

$$\frac{1}{b\,(a+b)} + \frac{1}{c\,(b+c)} + \frac{1}{a\,(c+a)} \geq \frac{27}{2\,(a+b+c)^2}.$$

Solution. By the AM-GM inequality

$$\left(\frac{1}{b\,(a+b)} + \frac{1}{c\,(b+c)} + \frac{1}{a\,(a+c)}\right)^3 \geq \frac{27}{abc\,(a+b)\,(b+c)\,(c+a)}.$$

On the other hand, using the same inequality we infer $\left(\frac{a+b+c}{3}\right)^3 \geq abc$ and

$$\left(\frac{2(a+b+c)}{3}\right)^3 = \left(\frac{(a+b)+(b+c)+(c+a)}{3}\right)^3 \geq (a+b)(b+c)(c+a).$$

Multiplying these inequalities yields

$$\frac{1}{abc(a+b)(b+c)(c+a)} \geq \frac{3^3 \cdot 3^3}{2^3(a+b+c)^6},$$

as needed.

Chapter 6

Team Selection Tests Formal Solutions

25. Let

$$A = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{1997\cdot 1998}$$

and

$$B = \frac{1}{1000\cdot 1998} + \frac{1}{1001\cdot 1997} + \ldots + \frac{1}{1998\cdot 1000}.$$

Prove that $\frac{A}{B}$ is an integer.

Solution. Using the equality

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

we have:

$$\begin{aligned} A &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{1997} - \frac{1}{1998} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{1997} + \frac{1}{1998} - 2\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{1998}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{1997} + \frac{1}{1998} - 1 - \frac{1}{2} - \ldots - \frac{1}{999} \\ &= (1-1) + \left(\frac{1}{2} - \frac{1}{2}\right) + \ldots + \left(\frac{1}{999} - \frac{1}{999}\right) + \frac{1}{1000} + \ldots + \frac{1}{1998} \\ &= \frac{1}{1000} + \frac{1}{1001} + \frac{1}{1002} + \ldots + \frac{1}{1997} + \frac{1}{1998}. \end{aligned}$$

Then

$$2A = \left(\frac{1}{1000} + \frac{1}{1998}\right) + \left(\frac{1}{1001} + \frac{1}{1997}\right) + \ldots + \left(\frac{1}{1998} + \frac{1}{1000}\right)$$

$$= 2998 \cdot \left(\frac{1}{1000 \cdot 1998} + \frac{1}{1001 \cdot 1997} + \dots + \frac{1}{1998 \cdot 1000}\right)$$
$$= 2998 \cdot B,$$

hence $\frac{A}{B} = 1499$ is an integer.

26. A rectangle $ABCD$ is given. Let M, N, P, Q be the points on the sides AB, BC, CD, DA respectively. If p is the perimeter of the quadrilateral $MNPQ$, prove that:

i) $p \geq AC + BD$;

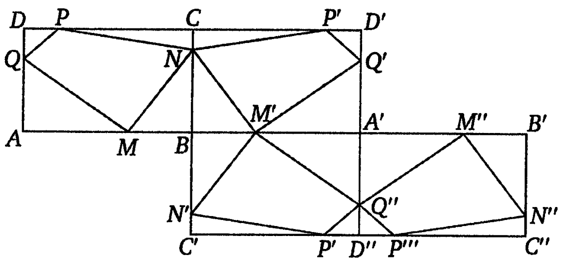
ii) If $p = AC + BD$, then $\text{area}[MNPQ] \leq \frac{\text{area}[ABCD]}{2}$.

iii) If $p = AC + BD$, then $MP^2 + NQ^2 \geq AC^2$.

Solution. i) Reflect $ABCD$ across BC and denote $BA'D'C$ its reflection. Reflect $BA'D'C$ across BA' and let $BC'D''A'$ be its reflection. Finally, consider $A'D''C''B'$ the mirror image of $BC'D''A'$ with respect to $A'D''$. Through this chain of reflections, the points M, N, P, Q map successively into the points: M', M'', N', N'', P', P'', P''', Q', Q''. Then

$$PN+NM+MQ+QP = PN+NM'+M'Q''+Q''P''' \geq PP''' = DD'' = AC+BD.$$

The equality holds when the points P, N, M', Q'', P''' are collinear, that is when $MNPQ$ is a parallelogram with the sides parallel to the diagonals of the rectangle $ABCD$.



ii) If $p = AC + BD$, then

$$PN \parallel BD \parallel QM \text{ and } QP \parallel AC \parallel MN.$$

Let $k = \frac{AM}{AB}$. Then

$$\text{area}[AMQ] = k^2\text{area}[ABD] = \frac{k^2\text{area}[ABCD]}{2}.$$

Furthermore, $\frac{BM}{AB} = 1 - k$ and $\text{area}[MNB] = \frac{(1-k)^2\text{area}[ABCD]}{2}$.

Since $\text{area}[MNPQ] = \text{area}[ABCD] - 2\text{area}[MNB] - 2\text{area}[AMQ]$, we obtain

$$\begin{aligned}\text{area}[MNPQ] &= \text{area}[ABCD]\left[1 - (1-k)^2 - k^2\right] \\ &= \left[\frac{1}{2} - 2\left(k - \frac{1}{2}\right)^2\right]\text{area}[ABCD] \leq \frac{1}{2}\text{area}[ABCD],\end{aligned}$$

as needed. The equality holds when M, N, P, Q are the midpoints of the sides of the rectangle $ABCD$.

iii) We have

$$AC^2 = AD^2 + DC^2 \leq PM^2 + QN^2,$$

with equality when M, N, P, Q are the midpoints of the sides of $ABCD$.

27. Let n be a positive integer. Find all the integer numbers that writes as:

$$\frac{1}{a_1} + \frac{2}{a_2} + \dots + \frac{n}{a_n},$$

for some positive integers $a_1, a_2, \dots, a_n$.

Solution. First, observe that $k = \frac{1}{a_1} + \frac{2}{a_2} + \dots + \frac{n}{a_n}$, then

$$k \geq 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

We prove that any integer $k \in \left\{1, 2, \dots, \frac{n(n+1)}{2}\right\}$ can be written as requested.

For $k = 1$, put $a_1 = a_2 = \dots = a_n = \frac{n(n+1)}{2}$.

For $k = n$, set $a_1 = 1$, $a_2 = 2$, $\dots$, $a_n = n$.

For $1 < k < n$, let $a_{k-1} = 1$ and $a_i = \frac{n(n+1)}{2} - k + 1$ for $i \neq k - 1$.

Thus

$$\frac{1}{a_1} + \frac{2}{a_2} + \dots + \frac{n}{a_n} = \frac{k-1}{1} + \sum_{\substack{i=1 \\ i \neq k-1}}^{n} \frac{i}{a_i} = k - 1 + \frac{\frac{n(n+1)}{2} - k + 1}{\frac{n(n+1)}{2} - k + 1} = k.$$

For $n < k < \frac{n(n+1)}{2}$, write k as

$$k = n + p_1 + p_2 + \dots + p_i,$$

with $1 \leq p_i \leq \dots \leq p_2 \leq p_1 \leq n - 1$.

Setting $a_{p_1+1} = a_{p_2+1} = \dots = a_{p_i+1} = 1$ and else $a_j = j$ we are done.

28. Find all the integers x and y so that

$$(x+1)(x+2)(x+3)+x(x+2)(x+3)+x(x+1)(x+3)+x(x+1)(x+2)=y^{2^x}.$$

Solution. i) If $x \geq 1$, then y^{2^x} is a square. The numbers x, $x+1$, $x+2$, $x+3$, have the form $4k$, $4k+1$, $4k+2$, $4k+3$, not necessarily in this order, hence three summands of the left-hand are divisible by 4 and the fourth is of the form $4k+2$. Consequently, the left-hand side is not a square.

ii) If $x \leq -4$, the left hand side is a negative number, while the right-hand side is positive. It remains to check the cases when $x \in \{-3, -2, -1, 0\}$.
We obtain $(x, y) \in \{(-2, 16), (0, 6)\}$.

29. A triangle ABC is given. The points D, E, F, G are chosen on the sides of the triangle such that the quadrilateral $DEFG$ is circumscriptible and $DF \perp EG$. Find the locus of the intersection point $M \in DF \cap EG$, so that $\{D, E, F, G\} \cap \{A, B, C\} \neq \varnothing$.

Solution. The quadrilateral $DEFG$ is circumscriptible, hence

$$DE+FG=EF+DG,$$

which implies

$$\sqrt{MD^2+ME^2}+\sqrt{MF^2+MG^2}=\sqrt{MF^2+ME^2}+\sqrt{MD^2+MG^2}$$

and

$$\left(MD^2+ME^2\right)\left(MF^2+MG^2\right)=\left(MF^2+ME^2\right)\left(MD^2+MG^2\right).$$

Therefore

$$\left(MD^2-MF^2\right)\left(MG^2-ME^2\right)=0,$$

and consequently $MD=MF$ or $MG=ME$. It follows that one of the diagonals of the quadrilateral $DEFG$ passes through the midpoints of the other diagonals.

Assuming that $D=A$ we consider two cases:

i) if $MG=ME$, then M lies on the bisector of the angle A;

ii) if $MD=MF$, then M lies on the line segments determined by the midpoints of the sides AB and AC.

30. Find the smallest value for n for which there exist the positive integers $x_1, \ldots, x_n$ with

$$x_1^4+x_2^4+\ldots+x_n^4=1998.$$

Solution. Observe that for any integer x we have $x^4=16k$ or $x^4=16k+1$ for some k.

As $1998=16\cdot 124+14$, it follows that $n \geq 14$.

If $n=14$, all the numbers $x_1, x_2, \ldots, x_{14}$ must be odd, so let $x_k^4=16a_k+1$. Then $a_k=\frac{x_k^4-1}{16}$, $k=\overline{1,14}$ hence $a_k \in \{0, 5, 39, 150, \ldots\}$ and $a_1+a_2+\ldots+a_{14}=124$. It follows that $a_k \in \{0, 5, 39\}$ for all $k=\overline{1,14}$, and since $124=5\cdot 24+4$, the number of the terms a_k equal to 39 is 1 or at least 6. A simple analysis show that the claim fails in both cases, hence $n \geq 15$. Any of the equalities

$$\begin{aligned}1998 &= 5^4+5^4+3^4+3^4+3^4+3^4+3^4+3^4+3^4+3^4+3^4+2^4+1^4+1^4+1^4\\&= 5^4+5^4+4^4+3^4+3^4+3^4+3^4+3^4+3^4+1^4+1^4+1^4+1^4+1^4+1^4\end{aligned}$$

prove that $n=15$.

31. Let n the positive integer. Prove that there is a polynomial P with integer coefficients so that if $a+b+c=0$, then:

$$a^{2n+1}+b^{2n+1}+c^{2n+1}=abc[P(a,b)+P(b,c)+P(c,a)].$$

Solution. Observe that for any positive integer n there is a polynomial with integer coefficients $Q_n(a, b)$ so that

$$a^{2n+1}+b^{2n+1}=(a+b)\left[a^{2n}+b^{2n}-abQ_n(a,b)\right]. \tag{*}$$

Since $a+b+c=0$, it follows that

$$a^{2n+1}+b^{2n+1}=-c\left(a^{2n}+b^{2n}\right)+abcQ_n(a,b). \tag{1}$$

Likewise,

$$a^{2n+1}+c^{2n+1}=-b\left(a^{2n}+c^{2n}\right)+abcQ_n(a,c). \tag{2}$$

and

$$b^{2n+1}+c^{2n+1}=-a\left(b^{2n}+c^{2n}\right)+abcQ_n(b,c). \tag{3}$$

Summing the relations (1), (2), (3), yields:

$$\begin{aligned}2\left(a^{2n+1}+b^{2n+1}+c^{2n+1}\right) &= -a^{2n}(b+c)-b^{2n}(c+a)-c^{2n}(a+b)\\&\quad +abc\left[Q_n(a,b)+Q_n(a,c)+Q_n(b,c)\right].\end{aligned}$$

Substituting $b+c, c+a, a+b$ for $-a, -b, -c$ in the right-hand side and cancelling the terms a^{2n+1}, b^{2n+1}, c^{2n+1} we obtain

$$a^{2n+1}+b^{2n+1}+c^{2n+1}=abc\left[Q_n(a,b)+Q_n(b,c)+Q_n(a,c)\right].$$

Therefore the claim holds for the polynomial $P(x, y)=Q_n(x, y)$.

Comment. Identifying the polynomial Q_n from the relation (*) was not an issue. Anyway, notice that $Q_1(a,b)=1$, $Q_2(a,b)=a^2+b^2-ab$ and $Q_{n+1}(a,b)=a^{2n}+b^{2n}-abQ_{n-1}(a,b)$ for $n \geq 2$.

32. Let ABC be a triangle and let $\vec{x}$, $\vec{y}$, $\vec{z}$ be three arbitrary vectors. For any real number $\lambda > 0$, the points M, N, P are chosen so that:

$$\overline{AM} = \lambda\vec{x},\ \overline{BN} = \lambda\vec{y},\ \overline{CP} = \lambda\vec{z}.$$

Find the locus of the centroid Q of the triangle MNP.

Solution. Let G be the centroid of the triangle ABC. We have:

$$\begin{aligned} 3\overrightarrow{GQ} &= \overrightarrow{GM} + \overrightarrow{GN} + \overrightarrow{GP} = \left(\overrightarrow{GA} + \lambda\vec{x}\right) + \left(\overrightarrow{GB} + \lambda\vec{y}\right) + \left(\overrightarrow{GC} + \lambda\vec{z}\right) \\ &= \vec{0} + \lambda\left(\vec{x} + \vec{y} + \vec{z}\right). \end{aligned}$$

Setting $\vec{x} + \vec{y} + \vec{z} = \vec{v}$, the relation $3\overrightarrow{GQ} = \lambda\vec{v}$ shows that if $\vec{v} \neq 0$, then Q lies on the passing through the point G, having the direction of the vector $\vec{v}$. If $\vec{v} = 0$, then the locus of the point Q reduces to the point G.

33. Let $A \subset (0,\,1)$ be a set of real number having the properties:

a) $\frac{1}{2} \in A$;

b) if $x \in A$, then $\frac{x}{2}$ and $\frac{1}{1+x}$ belong to A.

Prove that the set A contains all the rational numbers from the interval $(0,\,1)$.

Solution. If $p < q$ are positive integers with $\frac{p}{q} \in A$, then $\frac{p}{2q} \in A$ and $\frac{q}{p+q} \in A$ using the procedure b).

We prove that any rational numbers from the interval $(0,\,1)$ can be obtained from $\frac{1}{2}$ using the procedure b).

First, observe that $\frac{p}{q} \in A$ if (b′): $\frac{2p}{q} \in A$ and $2p < q$ or (b″): $\frac{q-p}{p} \in A$ and $q < 2p$. $\left(\text{if } q = 2 \text{ then } \frac{p}{q} = \frac{1}{2} \in A\right)$.

Now consider the integers $0 < p < q$.

If $q = 2^k \cdot p$ for some $k > 0$, then $\frac{1}{2} \in A \Rightarrow \frac{1}{2\cdot 2} \in A,\ \frac{1}{2\cdot 4} \in A, \ldots \Rightarrow \frac{1}{2^k} = \frac{p}{q} \in A$.

If else, let $k > 0$ such that $2^k \cdot p < q < 2^{k+1} \cdot p$. Applying successively the procedure (b′), notice that $\frac{2^k \cdot p}{q} \in A$ implies $\frac{p}{q} \in A$ (as needed), and (b″) shows that it suffices to have $\frac{q - 2^k \cdot p}{2^k \cdot p} \in A$. As $q - 2^k \cdot p + 2^k \cdot p = q < p + q$, it follows that after a finite number of steps the number $\frac{p}{q}$ can be obtained from the number $\frac{m}{n}$ with $m < n$ and $m + n < p + q$, hence it can be obtained from $\frac{1}{2}$, as desired.

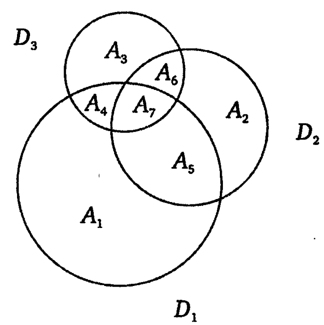
For example $\left(\text{we denote by } \text{"} \leftarrow \text{"} \text{ that } \frac{p}{q} \text{ can be obtained from } \frac{m}{n}\right)$:

$$\frac{2}{11} \xleftarrow{b'} \frac{4}{11} \xleftarrow{b'} \frac{8}{11} \xleftarrow{b''} \frac{3}{8} \xleftarrow{b'} \frac{3}{4} \xleftarrow{b''} \frac{1}{3} \xleftarrow{b'} \frac{2}{3} \xleftarrow{b''} \frac{1}{2}.$$

34. Let D_1, D_2, D_3 be three distinct disks in the plane and let a_{ij} be the area of $D_i \cap D_j$, for all $i,\ j \in \{1,\ 2,\ 3\}$. Prove that if x_1, x_2, x_3 are real numbers, not all of them equal to zero, then:

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 > 0.$$

Solution. Divide $D_1 \cup D_2 \cup D_3$ into 7 regions, as shown below (some of the sets A_1, A_2, …, A_7 can be empty):



Let a_i be the area of the region A_i, $i = \overline{1,7}$. Then $D_1 = A_1 \cup A_5 \cup A_7 \cup A_4$ and $D_1 \cap D_2 = A_5 \cup A_7$, hence $a_{11} = a_1 + a_5 + a_7 + a_4$ and $a_{12} = a_5 + a_7$.

Using the analogous equalities and substituting them in the given expression, we obtain

$$\begin{aligned} E \ = \ & a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4\left(x_1 + x_3\right)^2 + a_5\left(x_1 + x_2\right)^2 + a_6\left(x_2 + x_3\right)^2 \\ & + a_7\left(x_1 + x_2 + x_3\right)^2. \end{aligned}$$

Since $E \geq 0$, it is left to prove that $E = 0$ implies $x_1 = x_2 = x_3 = 0$.

Suppose that $(x_1,\ x_2,\ x_3) \neq (0,\ 0,\ 0)$, and $E = 0$. We prove that

$$a_1a_2a_3 = a_1a_2a_6 = a_1a_2a_4 = a_1a_2a_7 = 0.$$

Indeed,

- if $a_1a_2a_3 \neq 0$, then $x_1 = x_2 = x_3 = 0$.
- if $a_1a_2a_6 \neq 0$, then $x_1 = x_2 = x_2 + x_3 = 0$, so $x_1 = x_2 = x_3 = 0$.
- if $a_1a_2a_4 \neq 0$, then $x_1 = x_2 = x_1 + x_3 = 0$, hence $x_1 = x_2 = x_3 = 0$.
- if $a_1a_2a_7 \neq 0$, then $x_1 = x_2 = x_1 + x_2 + x_3 = 0$ and again $x_1 = x_2 = x_3 = 0$.

Now, if $a_1a_2 \neq 0$ then $a_3 = a_6 = a_4 = a_7 = 0$, hence $D_3 = \varnothing$, a contradiction.

Thus $a_1a_3 = 0$ and likewise $a_2a_3 = a_3a_1 = 0$. Consequently, at least two of the numbers a_1, a_2, a_3 are equal to zero, say $a_1 = a_2 = 0$.

1. If $a_3 \neq 0$, then $E = 0$ implies

$$\begin{aligned} E &= a_4x_1^2 + a_5(x_1+x_2)^2 + a_6x_2^2 + a_7(x_1+x_2)^2 \\ &= a_4x_1^2 + a_6x_2^2 + (a_5+a_7)(x_1+x_2)^2. \end{aligned}$$

If $a_4 \neq 0$ then $a_6 = a_5 = a_7 = 0 \Rightarrow D_2 = \varnothing$, false.

If $a_6 \neq 0$ then $a_4 = a_5 = a_7 = 0 \Rightarrow D_1 = \varnothing$, false.

If $a_4 = a_6 = 0$, as $a_1 = a_2 = 0 \Rightarrow D_1 = D_2 = A_5 \cup A_7$, a contradiction.

2. If $a_3 = 0$, then

$$E = a_4(x_1+x_3)^2 + a_5(x_1+x_2)^2 + a_6(x_2+x_3)^2 + a_7(x_1+x_2+x_3)^2 = 0.$$

Assuming that $a_4a_5a_6 \neq 0$ then $x_1+x_3 = x_1+x_2 = x_2+x_3 = 0$, so $x_1 = x_2 = x_3 = 0$, false. Hence $a_4a_5a_6 = 0$.

If $a_4 = a_5 = 0$ then $D_2 = D_3$, false.

If $a_4 = a_6 = 0$ then $D_1 = D_2$, false.

If $a_4 = a_7 = 0$ then $D_1 = A_5$, $D_3 = A_6$, and $D_2 = A_6 \cup A_5 = D_1 \cup D_2$, a contradiction (a disk cannot be the union of two distinct disks).

Therefore, if $(x_1, x_2, x_3) \neq (0, 0, 0)$ then $E > 0$.

35. Let A, B, C be the measures (in degrees) of the angles of the ABC triangle. A straight line cuts the ABC triangle in two isosceles triangles. Find the relations between the numbers A, B, C.

Solution. The line that cuts the triangle must pass through a vertex, otherwise one of the region is a quadrilateral. Assume that A is the vertex and let D be the intersection of the line with the side BC. The 9 cases are described in the array below.

	$AB = BD$	$AB = AD$	$BD = AD$
$AD = AC$	a)	b)	c)
$AD = DC$	d)	e)	f)
$AC = CD$	g)	h)	i)

We obtain as follows:

a) $B + C = 90°$; b) $B + 4C = 180°$; c) $B = 2C$; d) $C + 4B = 180°$; g) $C = 2B$.

The cases e), f), h), i) cannot occur.

36. Find the number of five-digit perfect squares having the last two digits equal.

Solution. Suppose $n = \overline{abcdd}$ is a perfect square. Then $n = 100\overline{abc} + 11d = \mathfrak{M}4 + 3d$, and since all the squares have the form $\mathfrak{M}4$ or $\mathfrak{M}4+1$ and $d \in \{0, 1, 4, 5, 6, 9\}$ – as the last digit of a square – it follows that $d = 0$ or $d = 4$.

• If $d = 0$, then $n = 100\overline{abc}$ is a square if $\overline{abc}$ is a square.
Hence $\overline{abc} \in \{10^2, 11^2, \ldots, 31^2\}$, so there are 22 numbers.

• If $d = 4$, then $100\overline{abc} + 44 = n = k^2$ implies $k = 2p$ and $\overline{abc} = \frac{p^2-11}{25}$.

1) If $p = 5x$, then $\overline{abc}$ is not an integer, false;

2) If $p = 5x+1$, then $\overline{abc} = \frac{25x^2+10x-10}{25} = x^2 + \frac{2(x-1)}{5} \Rightarrow x \in \{11, 16, 21, 26, 31\}$, so there are 5 solutions.

3) If $p = 5x+2$, then $\overline{abc} = x^2 + \frac{20x-7}{25} \notin \mathbb{N}$, false.

4) If $p = 5x+3$, then $\overline{abc} = x^2 + \frac{30x-2}{25} \notin \mathbb{N}$, false.

5) If $p = 5x+4$ then $\overline{abc} = x^2 + \frac{8x+1}{5}$, hence $x = \mathfrak{M}5 + 3 \Rightarrow x \in \{13, 18, 23, 28\}$, so there are 4 solutions.

Finally, there are $22 + 5 + 4 = 31$ squares.

37. M is the set of all values of the greatest common divisor d of the numbers $A = 2n + 3m + 13$, $B = 3n + 5m + 1$, $C = 6n + 8m - 1$, where m and n are positive integers. Prove that M is the set of all divisors of an integer k.

Solution. If d is a common divisor of the numbers A, B and C, then d divides $E = 3A - C = m + 40$, $F = 2B - C = 2m + 3$ and $G = 2E - F = 77$. We prove that $k = 77$ satisfies the conditions.

Let d' be the greatest common divisor of the numbers E and F. Then $d' = 7u$ for $m = 7p+2$. Moreover, $u = 1$ if $p \neq 11v+5$ and $u = 11$ if $p = 11v+5$. On the other hand, $d' = 11v$ for $m = 11q+4$; furthermore, $v = 1$ for $q \neq 7z+3$ and $v = 7$ for $q = 7z+3$.

The number d' is common divisor of the numbers A, B, C if and only if d' divides A.

For $m = 7p+2$, 7 divides $A = 2n + 21p + 19$ if and only if $n = 7p' + 1$.

For $m = 7(11v+5)$, $A = 2(n+59) + 3 \cdot 77v$ is divisible by 77 if and only if $n = 77t + 18$.

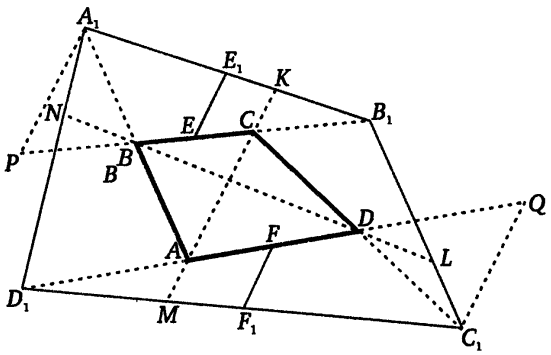
38. Consider a convex quadrilateral $ABCD$ and let A_1, B_1, C_1, D_1 be the reflection points of A, B, C, D across B, C, D, A respectively.

a) If E and F are the midpoints of the segments BC and AD, and E_1 and F_1 are the midpoints of the segments A_1B_1 and C_1D_1, prove that $EE_1 = FF_1$.

b) The points A, B, C, D are erased. Can you obtain them again, knowing only the location of A_1, B_1, C_1, D_1?

Solution. a) Consider P the reflection of C across B, and Q the reflection of A with respect to D. The segments EE_1 and FF_1 are middle lines in the triangles B_1A_1P and C_1D_1Q, hence there are equal to half of $PA_1$and QC_1 respectively. Using the congruences of the triangles $BAC \equiv BA_1P$ and $DAC \equiv DQC_1$ follows that $PA_1 = Q_1C_1 = AC$, hence $EE_1 = FF_1$.

b) Consider K on A_1B_1, and M on C_1D_1 such that $A_1K = 2KB_1$, $C_1M = 2MD_1$. We have $CK \parallel PA_1 \parallel AC$ and $AM \parallel QC_1 \parallel AC$, hence A and C lie on the segment KM. Moreover, $3CK = A_1P = AC = C_1Q = 3AM$, so $\frac{MA}{1} = \frac{AC}{3} = \frac{CK}{1} = \frac{MK}{5}$, and the points A and C are obtained. Likewise, choosing N on D_1A_1 and L on B_1C_1 with $D_1N = 2NA_1$, $B_1L = 2LC_1$, the points B and D are located on NL such that $\frac{NB}{1} = \frac{BD}{3} = \frac{DL}{1} = \frac{NL}{5}$.



Comments. 1° A vectorial approach can be considered. With an origin O we have

$$\overrightarrow{OA_1} = 2\overrightarrow{OB} - \overrightarrow{OA}, \text{ etc.}$$

and the analogous relations. Then

$$\begin{aligned}\overrightarrow{EE_1} &= 2\left(\overrightarrow{OE_1} - \overrightarrow{OE}\right) = \overrightarrow{OA_1} + \overrightarrow{OB_1} - \overrightarrow{OB} - \overrightarrow{OC}\\ &= \left(2\overrightarrow{OB} - \overrightarrow{OA}\right) + \left(2\overrightarrow{OC} - \overrightarrow{OB}\right) - \overrightarrow{OB} - \overrightarrow{OC}\\ &= \overrightarrow{OC} - \overrightarrow{OA},\end{aligned}$$

and likewise $\overrightarrow{F_1F} = \overrightarrow{OC} - \overrightarrow{OA}$, and so on.

2° The problem can be generalized defining A_1, B_1, C_1, D_1 by

$$\overrightarrow{BA_1} = u\overrightarrow{AB},\ \overrightarrow{CB_1} = v\overrightarrow{BC},\ \overrightarrow{DC_1} = t\overrightarrow{CD},\ \overrightarrow{AD_1} = p\overrightarrow{DA},$$

for some numbers u, v, t, p (asking for reconstruction of $ABCD$ when A_1, B_1, C_1, D_1 are given).

We present another solution of the proposed problem, more difficult, but useful for such a generalization.

Let H_a, H_b, H_c, H_d be the homotheties of centers A_1, B_1, C_1, D_1, and magnitudes $\frac{1}{2}$; these transformations map A to B, B to C, C to D, and D to A respectively. Hence $T = H_b \circ H_a$ maps A to C and $T' = H_d \circ H_c$ maps C to A.

The homotheties T and T' have the magnitudes $\frac{1}{4}$ and centers K and M, respectively. The point A is fixed for the homothety $T'' = T' \circ T$, hence T'' has the center A, located on KM. Likewise, $T \circ T'$ have the center C, also located on KM. The points B and C can be obtained similarly.

39. For all the positive integers $k \leq 1999$, let $S_1(k)$ be the sum of all the remainders of the numbers 1, 2, ..., k when divided by 4, and let $S_2(k)$ be the sum of all the remainders of the numbers $k+1$, $k+2$, ..., 2000 when divided by 3. Prove that there is an unique positive integer $m \leq 1999$ so that $S_1(m) = S_2(m)$.

Solution. Let $A_k = \{1, 2, 3, \ldots, k\}$ and $B_k = \{k+1, k+2, \ldots, 2000\}$. From the division of integer we have

$$k = 4q_1 + r_1, \text{ with } r_1 \in \{0, 1, 2, 3\}. \tag{1}$$

If $s_1(k)$ is the sum of the remainders at the division by 4 of the last r_1 elements of A_k, then

$$S_1(k) = 6q_1 + s_1(k), \text{ with } 0 \leq s_1(k) \leq 6. \tag{2}$$

(if $r_1 = 0$, then set $s_1(k) = 0$).

Using again the division of integers there exist the integers q_2, r_2 such that

$$2000 - k = 3q_2 + r_2, \text{ with } r_2 \in \{0, 1, 2\}. \tag{3}$$

If $s_2(k)$ is the sum of the remainders at the division by 3 of the last r_2 elements of B_k, then

$$S_2(k) = 3q_2 + s_2(k), \text{ cu } 0 \leq s_2(k) \leq 3. \tag{4}$$

(again we set $s_2(k) = 0$, if $r_2 = 0$).

As $S_1(k) = S_2(k)$, $s_2(k) - s_1(k) = 3(2q_1 - q_2)$, so $3|2q_1 - q_2| = |s_2(k) - s_1(k)| \leq 6$, and $|2q_1 - q_2| \leq 2$. In other words, $|2q_1 - q_2| \in \{0, 1, 2\}$.

If $2q_1 = q_2$, then (1) and (3) imply $2000 - (r_1 + r_2) = 10q_1$, hence $10 \mid (r_1 + r_2)$. Then $r_1 = r_2 = 0$ and $q_1 = 200$. From (1) follows that $k = 800$, and from (2) and (4) we have $S_1(800) = S_2(800) = 1200$.

Furthermore $S_1(k) \leq S_1(k+1)$, and $S_2(k) \geq S_2(k+1)$ for all $k \in \{1, 2, \ldots, 1998\}$.

Since $S_1(799) = S_1(800)$ and $S_2(799) = S_2(800) + 2 < S_1(800)$, we deduce that $S_1(k) < S_2(k)$ for all $k \in \{1, 2, \ldots, 799\}$. Since $S_1(801) = S_1(800) + 1 > S_2(800) \geq S_2(801)$, we derive that $S_1(k) > S_2(k)$ for all $k \in \{801, 802, \ldots, 1999\}$. Consequently, $S_1(m) = S_2(m)$ if and only if $m = 800$.

40. Let $S(k)$ be the sum of the digits of a positive integer k in decimal representation. Find all the positive integers n to exist the non-negative integers a and b with

$$S(a) = S(b) = S(a+b) = n.$$

Solution. We prove that the required numbers are all multiples of 9.

a) Let n be an integer such that there are positive integers a and b so that

$$S(a) = S(b) - S(a+b)$$

We prove (in two steps) that $9 \mid n$.

i) If k is a positive integer

$$9 \mid (k - S(k)). \tag{1}$$

Indeed,

$$\begin{aligned} k - S(k) &= \overline{a_k a_{k-1} \dots a_1} - (a_s + a_{s-1} + \dots + a_1) \\ &= a_s \cdot 10^{s-1} + a_{s-1} \cdot 10^{s-2} + \dots + a_2 \cdot 10 + a_1 - a_s - a_{s-1} - a_{s-2} - \dots - a_1 \\ &= a_s \left(10^{s-1} - 1\right) + a_{s-1} \left(10^{s-2} - 1\right) + \dots + a_2 (10 - 1), \end{aligned}$$

is divisible by 9, since $9 \mid 10^{k-1} - 1$ for all $t > 0$.

ii) Using the relation (1) we obtain

$$9 \mid a - S(a) \tag{2}$$

$$9 \mid b - S(b), \tag{3}$$

and

$$9 \mid (a+b) - S(a+b). \tag{4}$$

From (2) and (3) follows that

$$9 \mid a + b - (S(a) + S(b)) \tag{5}$$

hence

$$9 \mid S(a) + S(b) - S(a+b) = n + n - n = n, \tag{6}$$

as desired.

b) Conversely, we prove that if $n = 9p$ is a multiple of 9, then integers $a, b > 0$ with $S(a) = S(b) = S(a+b)$ can be found. Indeed, set $a = \underbrace{531531\dots531}_{3p \text{ digits}}$ and $b = \underbrace{171171\dots171}_{3p \text{ digits}}$. Then $a + b = \underbrace{702702\dots702}_{3p \text{ digits}}$, and

$$S(a) = S(b) = S(a+b) = 9p = n,$$

as claimed.

41. For all the numbers $p \in \mathbb{R}$ and $n \in \mathbb{N}^*$ let $A_n(p)$ be the set of integers px where x is a real number and $n - 1 < x \leq n$. For a given real number a, find all the real numbers b such that the sets $A_n(a)$ and $A_n(b)$ have the same number of elements for all the positive integers n.

Solution. Consider $p > 0$ a real number and set $k = px$. Then $n - 1 < x \leq n \Leftrightarrow (n-1)p < k < np$, so $[(n-1)p] + 1 \leq k \leq [np]$. Hence the number of elements of the set $A_n(p)$ is $[np] - [(n-1)p]$.

In the case $p < 0$ we obtain similarly that the number of elements of the set $A_n(p)$ is $[(n-1)p] - [np]$.

Finally, for $p = 0$, the set $A_n(p)$ has one element for all integers $n > 0$.
We prove two useful lemmas.

Lemma 1: Let $f(p) = [np] - [(n-1)p]$, $p \neq 0$. *Then* $f(p) = f(-p)$.

Indeed, this rewrites as $[np] - [-np] = [(n-1)p] + [-(n-1)p]$. Both sides are zero if $p \in \mathbb{Z}$ and -1 if else.

Lemma 2: Let $a, b > 0$ *be real numbers such that* $[na] - [(n-1)a] = [nb] - [(n-1)b]$ *for all the integers* $n > 0$. *Then* $a = b$.

To prove this, observe that $n = 1$ implies $[a] = [b]$ and $n = 2$ yields $[2a] - [a] = [2b] - [b]$, then $[2a] = [2b]$. Inducting on n we obtain $[na] = [nb]$ for all $n > 0$. Suppose by contradiction that $a \neq b$ and assume that $a > b$. Then $a = b + \varepsilon$, $\varepsilon > 0$ and $[na] = [nb] = [nb + n\varepsilon]$. Setting $n > \frac{1}{\varepsilon}$ leads to contradiction, hence $a = b$.

Furthermore, observe that if $a > 0$ and $[na] - [(n-1)a] = 1$ for all $n > 0$, then $a = 1$.

Therefore, for $a \in R \backslash \{-1, 0, 1\}$ we have $b = \pm a$ and for $a \in \{-1, 0, 1\}$ we have $b \in \{-1, 0, 1\}$.

42. Let ABC be a triangle with $\angle BAC = 90^\circ$ and $AB = AC$. The points M and N are given on the side BC such that N lies between the points M and C and

$$BM^2 - MN^2 + NC^2 = 0$$

Prove that $\angle MAN = 45^\circ$.

Solution. By Cosine Law we have

$$MN^2 = AM^2 + AN^2 - 2AM \cdot AN \cdot \cos(\angle MAN). \tag{1}$$

Since $AM^2 = BM^2 + AB^2 - BM \cdot AB\sqrt{2}$ and $AN^2 = NC^2 + AC^2 - NC \cdot AC\sqrt{2}$, it follows that

$$\cos(\angle MAN) = \frac{2AB^2 - AB \cdot BM\sqrt{2} - AB \cdot CN\sqrt{2}}{2AM \cdot AN}. \tag{2}$$

On the other hand,

$$\text{area}[MAN] = \text{area}[ABC] - \text{area}[ABM] - \text{area}[ACN] = \frac{2AB^2 - AB\cdot MB\sqrt{2} - AB\cdot CN\sqrt{2}}{4}.$$

As

$$\text{area}[AMN] = \frac{AM\cdot AN\cdot \sin(\angle MAN)}{2},$$

we obtain

$$\sin(\angle MAN) = \frac{2AB^2 - AB\cdot BM\sqrt{2} - AB\cdot CN\sqrt{2}}{2AM\cdot AN}. \tag{3}$$

The relations (2) and (3) imply $\tan(\angle MAN) = 1$, hence $\angle MAN = 45°$.

43. Find the integer solution of the equation

$$9^x - 3^x = y^4 + 2y^3 + y^2 + 2y.$$

Solution. We have successively

$$4\left((3^x)^2 - 3^x\right) + 1 = 4y^4 + 8y^3 + 4y^2 + 8y + 1,$$

then

$$(2t-1)^2 = 4y^4 + 8y^3 + 4y^2 + 8y + 1,$$

where $3^x = t \geq 1$.

Observe that

$$\left(2y^2 + 2y\right)^2 < E \leq \left(2y^2 + 2y + 1\right)^2.$$

Since $E = (2t-1)^2$ is a square, then

$$E = \left(2y^2 + 2y + 1\right)^2 \Leftrightarrow 4y\,(y-1) = 0.$$

so $y = 0$ or $y = 1$.

If $y = 0$ then $t = 1$ and $x = 0$.

If $y = 1$,then $t = 3$ and $x = 1$.

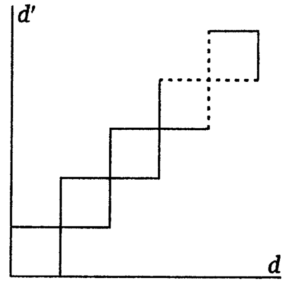
Hence the solutions (x, y) are $(0,\ 0)$ and $(1,\ 1)$.

44. A plane is covered by a net of unit squares. A person walks on the edges, any two consecutive edges being perpendicular, and returns in the initial position after n steps.

a) Prove that 4 divides n.

b) State and prove a reciprocal.

Solution. a) Let d and d' be the horizontal and vertical directions introduced by the sides of the squares. Project the horizontal edges of the path on the line d and observe that the number of unit sides visited from the left to right must be equal to those visited from the right to left. Thus the number of horizontal unit sides of the path is even and the same goes for the vertical sides. Since the person alternates the horizontal sides with the vertical ones, it follows that $n = 2m$. As m is even, then 4 divides m, as claimed.



b) A possible statement: *Let $n > 0$ be a multiple of 4. Then there exists a closed path of length n.*

A simple proof: If $n = 4k$ just go around a unit square for k times!.

45. Find all the real values of the number a such that

$$x + y + xy > a,$$

for all the real numbers $x > a$ and $y > a$

Solution. Set $x = y = a + t$, $t > 0$. Then $a + a^2 + 2t\,(a+1) + t^2 > 0$, for all $t > 0$.

We prove that $a^2 + a \geq 0$. Suppose by contradiction that $a^2 + a < 0$, i.e. $a \in (-1, 0)$.

The equation $t^2 + 2t\,(a+1) + a^2 + a = 0$ has the roots $t_1 = -(a+1) - \sqrt{a+1} < 0$ and $t_2 = -(a+1) + \sqrt{a+1} > 0$.

For $t \in (0, t_2)$ we have $t^2 + 2t\,(a+1) + a^2 + a < 0$, a contradiction. Thus, $a^2 + a \geq 0$ that is $a \in (-\infty, -1] \cup [0, \infty)$.

If $a \geq 0$, then $x,\ y > a$ implies $x + y + xy > 2a + a^2 > a$, so any $a \in [0, \infty)$ satisfies the condition.

If $a < -1$, set $x = \frac{a-1}{2} > a$ and $y = 2 > a$. Then $xy + x + y = a + \frac{a+1}{2} < a$, a contradiction.

If $a = -1$, then $x > -1$, $y > -1$, implies $x + y + xy = (x+1)(y+1) - 1 > -1$, as needed.

Thus, $a \in \{-1\} \cup [0, \infty)$.

46. A triangle ABC is given. The points $A' \in (BC)$, $B' \in (CA)$, $C' \in (AB)$ are chosen such that the the lines AA', BB', CC' meet at the point M. Let a, b, c, x, y, z be the areas of the triangles $AB'M$, $BC'M$, $CA'M$, $AC'M$, $BA'M$, $CB'M$ respectively. Prove that:

1° $abc = xyz$;

2° $ab + bc + ca = xy + yz + zx$.

Solution. We have

$$\begin{aligned} abc &= \frac{AM \cdot MB' \cdot \sin \angle AMB'}{2} \cdot \frac{BM \cdot MC' \cdot \sin \angle BMC'}{2} \cdot \frac{CM \cdot A'M \cdot \sin \angle CMA'}{2} \\ &= \frac{AM \cdot MC' \cdot \sin \angle AMC}{2} \cdot \frac{BM \cdot MA' \cdot \sin \angle BMA'}{2} \cdot \frac{CM \cdot MB' \cdot \sin \angle CMB'}{2} \\ &= xyz \end{aligned}$$

as needed.

b) Notice that

$$\frac{A'B}{A'C} = \frac{\text{area}[MBA']}{\text{area}[MCA']} = \frac{\text{area}[ABA']}{\text{area}[ACA']} = \frac{\text{area}[AMB]}{\text{area}[AMC]}.$$

Hence

$$\frac{y}{c} = \frac{x+b}{z+a},$$

or

$$yz - bc = cx - ay. \quad (1)$$

Likewise,

$$zx - ca = ay - bz \quad (2)$$

and

$$xy - ab = bz - cx. \quad (3)$$

Summing these equalities yields

$$yz - bc + zx - ca + xy - ab = 0,$$

as desired.

47. For any integer $n \geq 2$, consider $n-1$ positive real numbers $a_1, a_2, \ldots, a_{n-1}$ having the sum 1, and n real numbers $b_1, b_2, \ldots, b_n$. Prove that

$$b_1^2 + \frac{b_2^2}{a_1} + \frac{b_3^2}{a_2} + \ldots + \frac{b_n^2}{a_{n-1}} \geq 2b_1 (b_2 + b_3 + \ldots + b_n).$$

When does the equality holds?

Solution. By Cauchy-Schwarz inequality,

$$\left(\frac{b_2^2}{a_1} + \ldots + \frac{b_n^2}{a_{n-1}}\right)(a_1 + \ldots + a_{n-1}) \geq (b_2 + \ldots + b_n)^2.$$

As $a_1 + \ldots + a_{n-1} = 1$, we have $b_1^2 + \frac{b_2^2}{a_1} + \ldots + \frac{b_n^2}{a_{n-1}} \geq b_1^2 + (b_2 + \ldots + b_n)^2$, so it suffices to observe that $b_1^2 + (b_2 + \ldots + b_n)^2 \geq 2b_1 (b_2 + \ldots + b_n)$; indeed, this reduces to $[b_1 - (b_2 + \ldots + b_n)]^2 \geq 0$.

48. Let $a \geq 0$ be an integer number. Find the number of elements of the set

$$A = \left\{x \mid x \in \mathbb{Z} \text{ and } \frac{2^a}{3x+1} \in \mathbb{Z}\right\}.$$

Solution. If $\frac{2^a}{5x+1} \in \mathbb{Z}$, then $3x + 1 = \pm 2$, with $b \in \{0, 1, \ldots a\}$. For b even we have only a solution $x = \frac{2^b-1}{3} \in \mathbb{Z}$. For b odd we also obtain a unique solution $x = \frac{-(2^b+1)}{3} \in \mathbb{Z}$. Hence the set A has $a+1$ elements.

49. The internal bisectors of the angles A, B, C of the ABC triangle intersect the sides BC, CA, AB at the points D, E, F respectively. The points A', B', C' are the reflections of the points A, B, C with respect to D, E, F. If A, B, C lie respectively on the line segments $B'C'$, $A'C'$, $A'B'$, prove that ABC is an equilateral triangle.

Solution. Let $a \geq b \geq c$. Suppose that $a > \max(b, c)$ and draw AX parallel to BC. Since $\frac{AE}{EC} = \frac{c}{a} < 1$ it follows that B' is on the same side of the line AX, as B and C. Similarly, C' lies on the same side of the line AX as B and C. Hence the points A, B', C' cannot be collinear, a contradiction.

If $a = b > c$ then $C' \in AX$, but C' is still on the same side of AX as B and C; thus A, B', C' are not collinear.

Therefore $a = b = c$, as needed.

50. Two square of side length 5 are divided into 5 regions each. These 10 regions are colored using the same 5 colors for each square. Overlapping the squares, the sum of the areas of the parts sharing having the same color is computed. Prove that there is a coloring for which this sum is at least 5.

Solution. Let A_1, A_2, A_3, A_4, A_5 and B_1, B_2, B_3, B_4, B_5 be the regions in which are divided the two squares. Overlapping the squares, we obtain the regions $A_{ij} = A_i \cap B_j$, $i, j \in \{1, 2, 3, 4, 5\}$. The number of coloring for the regions B_i, $i \in \{1, 2, 3, 4, 5\}$ is 5!. Consider a given coloring for the regions $A_i, i \in \{1, 2, 3, 4, 5\}$. For a coloring $k = 1, 2, \ldots, 5!$ of the regions B_i, $i \in \{1, 2, 3, 4, 5\}$, denote by S_k the sum of the areas of the parts A_{ij} having the same color in both colorings.

Then $S_k = \sum\limits_{i,j=1}^{5} a_{ij}\text{area}[A_{ij}]$, where $a_{ij} = 1$, if A_i and B_j have the same color and $a_{ij} = 0$, if else. Consequently, $\sum_{k=1}^{5!} S_k = \sum_{i,j=1}^{5} k_{ij} \cdot \text{area}[A_{ij}]$, where k_{ij} is the number of colorings in which A_i and B_j have the same color. This number is equal to the number of colorings of 4 regions with 4 colors, hence $k_{ij} = 4!$. Then $\sum_{k=1}^{5!} S_k = 4! \sum_{i,j=1}^{5} \text{area}[A_{ij}] = 4! \cdot 5^2 = 25 \cdot 4!$. As $S_1, S_2, \ldots, S_{5!} = 25 \cdot 4!$, there is $n \in \{1, 2, 3, \ldots, 5!\}$ such that $S_n \geq \frac{25 \cdot 4!}{5!} = 5$, as needed.

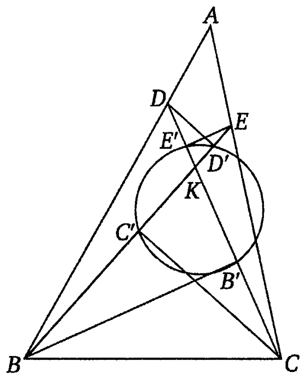
51. Let ABC be an arbitrary triangle. A circle passes through B and C and intersects the lines AB and AC in D and E respectively. The projection of the points B and E on CD are denoted by B' and E'. The projection of the points D and C on BE are denoted by D' and C'.

Prove that the points B', D', E', C' are on the same circle.

Solution. Let K be the intersection point of the lines BE and CD. We consider that the points B', C', D', E' are distinct, otherwise all is clear.

The quadrilaterals $BCED$ and $BDD'B'$ are cyclic, so $\angle BDC = \angle BEC$ and $\angle BDB' = \angle B'D'K$.

Since $CC'E'E$ is also cyclic, $\angle CEC' \equiv \angle KE'C'$. It follows that $\angle B'D'K \equiv \angle KE'C'$, so $B'C'E'D'$ is a cyclic quadrilateral as needed.



An alternative solution uses the power of a point theorem.

52. Find all the integers n so that the number $\sqrt{\frac{4n-2}{n+5}}$ is rational.

Solution. Suppose $\frac{4n-2}{n+5} = \frac{a^2}{b^2}$, where a and b are coprime integers. We obtain

$$n = \frac{2b^2 + 5a^2}{4b^2 - a^2} = -5 + \frac{22b^2}{4b^2 - a^2},\ 4b^2 - a^2 \neq 0.$$

As b^2 and $4b^2 - a^2$ are coprime, it follows that $4b^2 - a^2$ divides 22, so $4b^2 - a^2 \in \{-22, -11, -1, 1, 11, 22\}$.

Observe that $4b^2 - a^2$ has the form $4u$ or $4u+3$, hence $4b^2 - a^2 = -1$ or $4b^2 - a^2 = 11$.

If $4b^2 - a^2 = -1$, then $(2b - a)(2b + a) = -1$ and consequently,

$$\begin{cases} 2b - a = -1 \\ 2b + a = 1 \end{cases}.$$

We obtain $b = 0$, a contradiction.

If $4b^2 - a^2 = 11$, then

$$\begin{cases} 2b - a = 1 \\ 2b + a = 11 \end{cases},$$

from which $a = 5$, $b = 3$ and $n = 13$.

53. 1200 points are given inside a circle centered at the point O so that no two of them lie on a diameter of the circle. Prove that there exist the points M and N on the circle so that $\angle MON = 30°$ and in the interior of the angle $\angle MON$ lie exactly 100 points.

Solution. Using 6 diameters that do not contain any of the given points, divide the interior of the circle into 12 congruent sectors of angle 30°. If one of the sector contains 100 points, we are done. Since it is not possible that all the sectors contain less then 100 points or more than 100 points, we can find a sector S containing less than 100 points and a sector S' containing more than 100 points.

Rotate the sector S towards the sector S'. At each moment at most one point gets in or out of the sector S (note that it is possible that a point gets in at the same moment when another point gets out; in this case the number of points inside S remains constant). The number of moments in which the number of points inside S changes (with a unit!) is finite, hence there exists a moment in which the rotating sector S contains exactly 100 points.

54. Three students write on the blackboard three two-digit squares next to each other. At the end they observe that the 6-digit number obtained is also a square. Find this number.

Solution. Let x, y, z be the three two-digit squares and u^2 the six-digit square. As x, y, z can be 16, 25, 36, 49 or 81 we have

$$161616 \leq u^2 \leq 818181, \text{ hence } 402 \leq u \leq 904.$$

If $u = \overline{abc}$, then $a \geq 4$. It follows that :

i) $a = 4$ and $b \in \{0, 1\}$ or

ii) $a > 4$ and $b = 0$.

i) If $a = 4$ and $b = 0$, then $x = 16$ and $8c \cdot 100 + c^2 = 100y + z$. It follows that $y = 8c$ and $z = c^2$, hence $y = 16$, $c = 2$, $z = 4$ (impossible) or $y = 64$, $c = 8$, $z = 64$, and $u = 408$ so $u^2 = 166464$.

If $a = 4$ and $b = 1$, then $x = 16$, $y = 81$ and $\overline{82c} \cdot c = z$ false.

ii) If $a > 4$ and $b = 0$, then $(200a + c)\,c = 100y + z$, hence $y = 2ac$ and $z = c^2$. Since $a > 4$ and $c \geq 4$, we obtain $y = 64$, $a = 8$, $c = 4$ and $u = 804$, $804^2 = 646416$.

55. Let $ABCD$ be a rectangle. The points $E \in CA$, $F \in AB$, $G \in BC$ are considered so that $DE \perp CA$, $EF \perp AB$, $EG \perp BC$. Find the rational solutions of the equation

$$AC^x = EF^x + EG^x.$$

Solution. In the right triangle ADC we have

$$AC^2 = AD^2 + DC^2,\ AC \cdot AE = AD^2, \tag{1}$$

$$AC \cdot CE = DC^2. \tag{2}$$

The triangles AEF and ACB are similar, hence

$$\frac{EF}{BC} = \frac{AE}{AC}. \tag{3}$$

The relations (1) and (3) implies

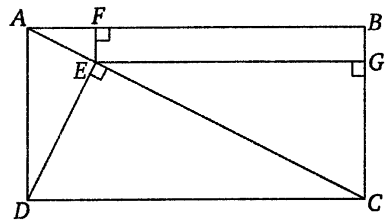
$$EF = \frac{AD^3}{AC^2}. \tag{4}$$

Likewise,

$$EG = \frac{AB^3}{AC^2}. \tag{5}$$

The equation $AC^x = EF^x + EG^x$ rewrites

$$\left(AD^2 + AB^2\right)^{3x} = \left(AD^{3x} + AB^{3x}\right)^2.$$



Observe that $x = \frac{2}{3}$ is a solution. We prove that this solution is unique.

If $AD = AB$, then $\left(2AD^2\right)^{3x} = \left(2AD^{3x}\right)^2$, hence $2^{3x} = 2^2$ and $x = \frac{2}{3}$.

If $AD \neq AB$, let $AD > AB$ and denote $k = \frac{AB}{AD} \in (0,\,1)$. The equation becomes

$$\left(1 + k^2\right)^{3x} = \left(1 + k^{3x}\right)^2.$$

Suppose by contradiction that $x < \frac{2}{3}$. Then $k^{3x} > k^2$, and $1 + k^{3x} > 1 + k^2 > 1$, hence $\left(1 + k^{3x}\right)^2 > \left(1 + k^2\right)^2 > 1$, a contradiction.

Similarly, $x > \frac{2}{3}$ leads to a contradiction.

56. Let A be a non-empty subset of $\mathbb{R}$ so that if x, y are real numbers with $x + y \in A$, then $xy \in A$. Prove that $A = \mathbb{R}$.

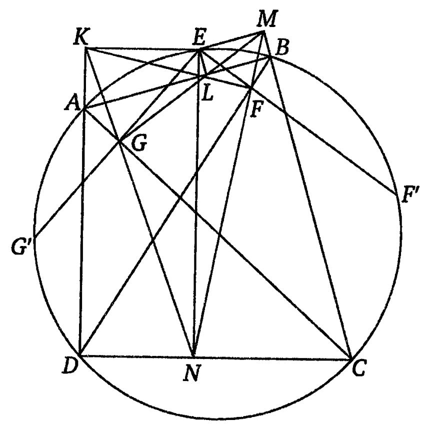
Solution. Let $a \in A$. As $a + 0 = a \in A$, it follows that $0 = a \cdot 0 \in A$. For any real number b we have $0 = b + (-b) \in A$, hence $-b^2 \in A$. Thus $(-\infty,\,0] \subset A$.

Let $c > 0$. Since $-\sqrt{c} + (-\sqrt{c}) < 0$ then $-\sqrt{c} - \sqrt{c} \in A$, therefore $c = -\sqrt{c}(-\sqrt{c}) \in A$.The conclusion follows.

57. Let $ABCD$ be a quadrilateral inscribed in the circle $\mathcal{C}(O,\,R)$. For any point E of the circle we consider its projections K, L, M, N on the lines DA, AB, BC, CD. For some point E, different than A, B, C, D, one observe that the point N is the orthocenter of the triangle KLM.

Prove that this holds for any point E on the circle.

Solution. Let F, G be the projection of E on the diagonals BD and AC respectively. From the Simson's theorem it follows that the point triplets $(K,\,L,\,F)$, $(M,\,N,\,F)$, $(K,\,G,\,N)$, $(M,\,L,\,G)$ are collinear.



The point N is the orthocenter of the triangle KLM if and only if $KL \perp MN$ and $ML \perp KN$. Let F' and G' be the points in which EF and EG intersect the second time the circle. We have $KF \parallel AF'$, $MG \parallel BG'$, $KN \parallel DG'$ and $MN \parallel CF'$. Thus $KL \perp MN$ is equivalent to $AF' \perp CF'$ and then $O \in AC$. Similarly, $ML \perp KN$

if and only if $O \in BD$, hence $ABCD$ is a rectangle. Conversely, if $ABCD$ is a rectangle, one can easily check that N is the orthocenter of the triangle KLM for any position of the point E (do not forget to consider the case $E \in \{A, B, C, D\}$!).

58. Find all the positive integers $a < b < c < d$ with the property that each of them divides the sum of the other three.

Solution. Since $d \mid (a+b+c)$ and $a+b+c < 3d$, it follows that $a+b+c = d$ or $a+b+c = 2d$.

Case i) If $a+b+c = d$, as $a \mid (b+c+d)$, we have $a \mid 2d$ and similarly $b \mid 2d$, $c \mid 2d$. Let $2d = ax = by = cz$, where $2 < z < y < x$. Thus $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}$.

1° If $z = 3$, then $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$. The solutions are

$$(x, y) = \{(42, 7), (24, 8), (18, 9), (15, 10)\},$$

hence

$$(a, b, c, d) \in \{(k, 6k, 14k, 21k), (k, 3k, 8k, 12k), (k, 2k, 6k, 9k), (2k, 3k, 10k, 15k), (k, 3k, 8k, 12k)\},$$

for $k > 0$.

2° If $z = 4$, then $\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$, and

$$(x, y) = \{(20, 5), (12, 6)\}.$$

The solutions are

$$(a, b, c, d) = (k, 4k, 5k, 10k) \text{ and } (a, b, c, d) = (k, 2k, 3k, 6k),$$

for $k > 0$.

3° If $z = 5$, then $\frac{1}{x} + \frac{1}{y} = \frac{3}{10}$, and $(3x - 10)(3y - 10) = 100$.

As $3x - 10 \equiv 2 \pmod 3$, it follows that $3x - 10 = 20$ and $3y - 10 = 5$.Thus $y = 3$, false.

4° If $z \geq 6$ then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$ so there are no solutions.

Case ii) If $a+b+c = 2d$, we obtain $a \mid 3d$, $b \mid 3d$, $c \mid 3d$.

Then $3d = ax = by = cz$, with $x > y > z > 3$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{2}{3}$. Since $x \geq 4$, $y \geq 5$, $z \geq 6$ we have $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{1}{6} + \frac{1}{5} + \frac{1}{4} = \frac{37}{60} < \frac{2}{3}$, so there are no solutions in this case.

59. Let n be a non-negative integer. Find all the non-negatives integers a, b, c, d such that

$$a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n.$$

Solution. For $n = 0$ we have the solutions $(2, 1, 1, 1)$, $(1, 2, 1, 1)$, $(1, 1, 2, 1)$ and $(1, 1, 1, 2)$.

If $n \geq 1$, then $a^2 + b^2 + c^2 + d^2 \equiv 0 \pmod 4$, hence a, b, c, d have the same parity. We consider two cases.

i) If a, b, c, d are odd numbers, set $a = 2x+1$, $b = 2y+1$, $c = 2z+1$, $d = 2t+1$. The equation rewrites

$$4x(x+1) + 4y(y+1) + 4z(z+1) + 4t(t+1) = 4\left(7 \cdot 4^{n-1} - 1\right).$$

Since $n(n+1)$ is a multiple of 2, the left-hand side is divisible by 8, hence $7 \cdot 4^{n-1} - 1$ must be even. Consequently, $n = 1$ and the equation $a^2 + b^2 + c^2 + d^2 = 28$ has the solutions $(5, 1, 1, 1)$, $(1, 3, 3, 3)$ and all their permutations.

ii) If a, b, c, d are even numbers, then setting $a = 2x$, $b = 2y$, $c = 2z$, $d = 2t$ leads to

$$x^2 + y^2 + z^2 + t^2 = 7 \cdot 4^{n-1}$$

so we proceed recursively.

Finally, we obtain the solutions

$$\left(2^{n+1}, 2^n, 2^n, 2^n\right), \left(3 \cdot 2^n, 3 \cdot 2^n, 3 \cdot 2^n, 2^n\right), \left(2^n, 2^n, 2^n, 5 \cdot 2^n\right),$$

and all their permutations.

60. The opposite sides of a hexagon $ABCDEF$ are parallel and the diagonals AD, BE and CF are equal. Prove that the hexagon is cyclic.

Solution. Observe that $ABDE$ is an isosceles trapezoid or rectangle, hence the segments AB and DE have the same perpendicular bisector. Let O and R be the center and the radius of the circumcircle of the triangle ABC. Then O lies on the perpendicular bisectors of the segments AB and BC, hence the perpendicular bisectors of the segments DE and EF also pass through O. Thus O is the circumcenter of the triangle DEF. If $R_1$is the circumradius of DEF , then $R = R_1$, as $ACDF$ is an isosceles trapezoid or rectangle. Therefore $ABCDEF$ is cyclic, as desired.

61. Let $n \geq 2$ be an integer. Find all the integers x so that

$$\sqrt{x + \sqrt{x + \ldots + \sqrt{x}}} < n$$

for any number of radicals.

Solution. Set $u = n^2 - x$. Since $n^2 \geq x$, the integer u is positive. Consequently, $n^2 < u(u+1)$, so $u \geq n$, that is $x \leq n^2 - n$. Inducting on the number of square roots follows that any positive integer $x \leq n^2 - n$ satisfies the claim.

Thus $x = \left\{0, 1, 2, \ldots, n^2 - n\right\}$.

62. Find the minimal area of a rectangular box of a volume strictly greater than 1000 if the side lengths are integer numbers.

Solution. Let x, y, z be the dimensions of the rectangular box, with the volume $V = abc \geq 1001$ and the area $2S = 2(ab + bc + ca)$.

We prove that $S \geq 310$, with equality for $a = 8$, $b = 9$ and $c = 14$. (note that in this case $V = 1008$).

1) $c = 11 \Rightarrow ab \geq 91 \Rightarrow a + b \geq 20$. Then $S = 11(a + b) + ab \geq 311$.

2) $c = 12 \Rightarrow ab \geq 84 \Rightarrow a + b \geq 19 \Rightarrow S = 12(a + b) + ab \geq 312$.

3) $c = 13 \Rightarrow ab \geq 77 \Rightarrow a + b \geq 18 \Rightarrow S = 13 \cdot 18 + 77 = 311$.

4) $c = 11 \Rightarrow b = 9$ and $a = 8$; $\mathbf{S = 310}$.

5) $c = 15 \Rightarrow ab \geq 67 \Rightarrow a + b \geq 17 \Rightarrow S = 15 \cdot 17 + 67 = 321$.

6) $c = \overline{16, 18} \Rightarrow ab \geq 56 \Rightarrow a + b \geq 16$ or $a = 7, b = 8, c = 18$. Then $S \geq 312$ or $S = 326$.

7) $c = \overline{18, 20} \Rightarrow ab \geq 51 \Rightarrow a + b \geq 15 \Rightarrow S \geq 15 \cdot 18 + 51 = 321$.

8) $c \geq 21 \Rightarrow ab \geq 48 \Rightarrow a + b \geq 16 \Rightarrow S \geq 21 \cdot 14 + 48 = 342$.

Therefore, the box with minimal area is $8 \times 9 \times 14$.

63. For a positive number n, let $f(n)$ be the value of

$$f(n) = \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n + 1} + \sqrt{2n - 1}}.$$

Calculate $f(1) + f(2) + f(3) + \ldots + f(40)$.

Solution. From

$$f(n) = \frac{\sqrt{(2n+1)}^2 + \sqrt{(2n-1)}^2 + \sqrt{4n^2 - 1}}{\sqrt{2n + 1} + \sqrt{2n - 1}},$$

it follows that

$$f(n) = \frac{\left(\sqrt{2n+1}\right)^3 - \sqrt{(2n-1)}^3}{2},$$

so

$$f(1) + f(2) + \ldots + f(40)$$

$$= \frac{\left(\sqrt{3^3} - \sqrt{1^3}\right) + \left(\sqrt{5^3} - \sqrt{3^3}\right) + \ldots + \left(\sqrt{81^3} - \sqrt{79^3}\right)}{2}$$

$$= \frac{\sqrt{81^3} - \sqrt{1^3}}{2} = 364.$$

64. Let K, n, p be non-negative integers so that p is prime, $K < 1000$ and $\sqrt{K} = n\sqrt{p}$.

a) Prove that if the equation $\sqrt{K + 100x} = (n + x)\sqrt{p}$ has an integer solution different from 0, then $p \mid 10$.

b) In that case find the number of all the positive integer solutions of the equation (that is, when $p = 2$ or $p = 5$).

Solution. a) By squaring the both sides of the equation we get $K + 100x = n^2p + 2nxp + x^2p$, or $100 = p(2n + x)$.

The conclusion follows from the fact that p is a prime number.

b) If $p = 2$ then $50 = 2n + x$, and $0 \leq n \leq 25$. Since $n^2 = \frac{K}{p} = \frac{K}{2} < 500$, it follows that $n \leq 22$ and we have 23 solutions.

If $p = 5$, then $20 = 2n + x$, and $0 \leq n \leq 10$. Notice that $n^2 = \frac{K}{5} < 200$ for any $n \leq 10$, therefore we have other 11 solutions.

We have 34 solutions in all.

65. Consider a $1 \times n$ rectangle made out of n tiles. A pavement is a coloring of each of the n tiles with one of the 4 possible color so that no two consecutive tiles have the same color.

i) What is the number of the distinct symmetrical pavements? (a symmetrical pavement is a pavement for which tile symmetrical with respect to the center have the same color).

ii) What is the number of distinct pavements so that in any block of three consecutive tiles no two tiles have the same color?

Solution. i) If $n = 2k$ there are no symmetrical pavements (otherwise the k and $k + 1$ must have the same color).

If $n = 2k + 1$ the problem is to count the possible pavements for $k + 1$ squares. There are $4 \cdot \underbrace{3 \cdot 3 \ldots \cdot 3}_{k \text{ times}} = 4 \cdot 3^k$ such pavements.

ii) There are $4 \cdot 3 \cdot \underbrace{2 \cdot 2 \ldots \cdot 2}_{k \text{ times}} = 4 \cdot 3 \cdot 2^{n-2}$ pavements.

66. Let $ABCD$ be a parallelogram centered in O. Let M and N be the midpoints of BO and CD. Prove that if the triangles ABC and AMN are similar, then $ABCD$ is a square.

Solution. From the similarity of the triangles AMN and ABC, we obtain

$$\frac{AM}{AB} = \frac{AN}{AC}. \quad (1)$$

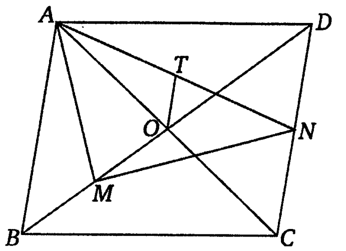
Hence

$$\angle MAN \equiv \angle BAC \text{ and } \angle BAM \equiv \angle CAN \quad (2)$$

The relations (1) and (2) imply the similarity of the triangles BAM and CAN. Hence we obtain the proportions

$$\frac{AN}{AN} = \frac{AB}{AC} = \frac{BM}{CN}, \tag{3}$$

and $\angle ABM \equiv \angle ACN$. The last equality implies that $ABCD$ is a rectangle.



To conclude the proof, notice that $BM = \frac{1}{4}BD = \frac{1}{4}AC$ and $CN = \frac{1}{2}AB$. Hence the last equality in (3) becomes $\frac{AB}{AC} = \frac{AC}{2AB}$, that is $2AB^2 = AC^2 = AB^2 + BC^2$, which proves that $ABCD$ is a square.

67. A unit square is divided naturally into 9 congruent squares of side $\frac{1}{3}$. The central square is colored. We call this procedure P. For each of the 8 remaining squares apply the procedure P. For each of the next 64 remaining squares apply the procedure P and so on. Prove that after 1000 applications of procedure P the area colored exceeds 0.999.

Solution. The first procedure give rise to one colored square of area $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$. After the second procedure we obtain eight more squares of side $\frac{1}{9}$, the colored region increasing by $\frac{8}{9^2}$. In the same manner, the third procedure increases the colored area by $8^2 = 64$ colored squares, each of area $\frac{1}{27}$, that is at this stage the colored area becomes

$$\frac{1}{9} + \frac{8}{9^2} + \frac{8^2}{9^3}$$

We conclude that after 1000 applications of the procedure P, the area of the colored region is

$$\frac{1}{9} + \frac{8}{9^2} + \ldots + \frac{8^{999}}{9^{1000}} = \frac{1}{9}\left(1 + \frac{8}{9} + \ldots + \left(\frac{8}{9}\right)^{999}\right) = 1 - \left(\frac{8}{9}\right)^{1000}.$$

It is left to prove that the last number is greater than 0.001. This easy follows by using a binomial expansion evaluation, that is

$$\left(\frac{9}{8}\right)^{1000} = \left(1 + \frac{1}{8}\right)^{1000} > \binom{1000}{2}\left(\frac{1}{8}\right)^2 > 1000.$$

Therefore

$$1 - \left(\frac{8}{9}\right)^{1000} > 1 - \frac{1}{1000} = 0.999,$$

and the proof is complete.

68. Find all the positive integers a, b, c, d so that

$$a + b + c + d - 3 = ab = cd.$$

Solution. We have $ab + cd = 2(a + b + c + d) - 6$ or

$$(a-2)(b-2) + (c-2)(d-2) = 2. \tag{1}$$

Assuming that a is the smallest number among a, b, c, d, we get $-1 \leq a - 2 \leq 1$.

1° If $a - 2 = 1$, then $b - 2 = c - 2 = d - 2$ and $a = b = c = d = 3$.

2° If $a - 2 = 0$, then $c - 2 = 1$ and $d - 2 = 2$ (or $c - 2 = 2$ and $d - 2 = 1$). It follows that $cd = 12$, $a = 2$, that is $b = 6$.

3° If $a - 2 = -1$, then $a = 1$ and $b + c + d - 2 = b = cd$. Hence $c + d = 2$, implying $c = d = 1$ and $b = 1$.

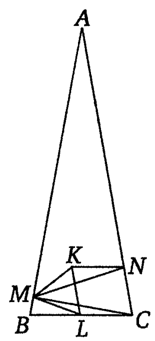
We conclude that the solutions are

$$(a, b, c, d) \in \{(1,1,1,1), (3,3,3,3), (2,6,3,4), (6,2,3,4), (2,6,4,3), (6,2,4,3), (3,4,2,6), (3,4,6,2), (4,3,2,6), (4,3,6,2)\}.$$

69. Let ABC be an isosceles triangle with $AB = AC$ and $\angle BAC = 20°$. Let M be the projection of the point C on the side AB and let N be a point on the side AC so that $CN = \frac{BC}{2}$. Find the measure of the angle AMN.

Solution. Let L be the midpoint of BC. Since ML is a median in the right-angled triangle MBC, it follows that

$$ML = BL = LC = CN.$$



The point K is considered such that $LCNK$ is a rhombus. Notice that

$$\begin{aligned}\angle KLM &= \angle KLB - \angle MLB\\ &= \angle ACB - [180^\circ - 2\angle MBC] = 60^\circ\end{aligned}$$

and $LK = ML$, that is MKL is an equilateral triangle. Hence

$$MK = KL = KN,$$

and

$$\angle MKN = \angle MKL + \angle NKL = 60^\circ + 80^\circ = 140^\circ.$$

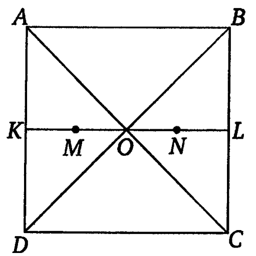
Then

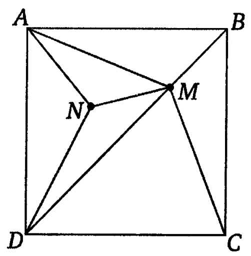
$$\begin{aligned}\angle KMN) &= \angle KNM = 20^\circ,\\ \angle ANM &= 20^\circ + 80^\circ = 100^\circ,\end{aligned}$$

and the required angle $\angle AMN$ equals 60°.

70. Let $ABCD$ be a unit square. Suppose M, N are two interior points so that no vertex of the square lies on the line MN. Let $s(M, N)$ be the smallest area of a triangle with vertices in the set $\{A, B, C, D, M, N\}$. Find the smallest real number k so that for any points M, N with the mentioned property we have $s(M, N) \leq k$.

Solution. Let K and L be the midpoints of AD and BC respectively and let M, N be the midpoints of OK and OL. It is easy to check that $s(M, N) = \frac{1}{8}$, hence $k \geq \frac{1}{8}$.





Observe that for any interior point M of the square we have

$$\text{area}[BMC] + \text{area}[AMD] = \frac{1}{2}.$$

Assume that N is an interior point of the triangle AMD. Therefore

$$\text{area}[AND] + \text{area}[ANM] + \text{area}[DNM] + \text{area}[BMC] = \frac{1}{2}. \tag{1}$$

It follows that one of the triangles involved in the sum above has the area greater than $\frac{1}{8}$, hence k cannot be less than $\frac{1}{8}$. Hence $k = \frac{1}{8}$.

71. Let n be an even positive integer and let a, b be positive coprime integers. Find a and b if $a + b$ divide $a^n + b^n$.

Solution. As n is even, we have

$$a^n - b^n = \left(a^2 - b^2\right)\left(a^{n-2} - a^{n-4}b^2 + \ldots + b^{n-2}\right).$$

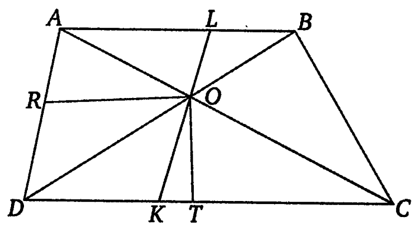
Since $a + b$ is a divisor of $a^2 - b^2$, it follows that $a + b$ is a divisor of $a^n - b^n$. In turn, $a + b$ divides $2a^n = (a^n + b^n) + (a^n - b^n)$, and $2b^n = (a^n + b^n) - (a^n - b^n)$. But a and b are coprime numbers, and so g.c.d. $(2a^n, 2b^n) = 2$. Therefore $a + b$ is a divisor of 2, hence $a = b = 1$.

72. Let $ABCD$ be a convex quadrilateral and O the point of intersection of its diagonals. The measure of the angle between the two diagonals is m. For any angle xOy of measure m, the area inside the angle that is in the interior of the quadrilateral is constant. Prove that $ABCD$ is a square.

Solution. Consider $\angle AOD = m \leq 90^\circ$. As the angles $\angle AOD$ and $\angle BOC$ equal m, we find area$[AOD]$ = area$[BOC]$. It follows

$$AO \cdot DO \cdot \sin m = BO \cdot CO \cdot \sin m,$$

hence $\frac{AO}{CO} = \frac{BO}{DO}$.



Since $\angle AOB = \angle DOC$, the triangles AOB and DOC are similar and AB is parallel to DC.

Draw line KL that contains O such that $\angle AOL = \angle COK = m$ and $L \in (AB)$, $F \in (DC)$. The triangles AOL and COK are similar and have the same area, therefore they are congruent. It follows that $AO = CO$, and in the same way $BO = DO$. Consequently $AD \parallel BC$. Moreover, area$[BOC]$ = area$[COK]$, and since $ABCD$ is a parallelogram, we find area$[BOC]$ = area$[DOC]$. Hence $D = K$ and

$$m = \angle COD = \angle COK = \angle BOC = 90^\circ$$

We have proved that $ABCD$ is a rhombus.

To conclude, consider the bisector lines $[OR$ and $[OT$ of the angles $\angle AOD$ and $\angle DOC$ respectively, where $R \in (AD)$, $T \in (DC)$. It is easy to check that $\angle ROT = \angle AOD = m = 90^\circ$, hence area$[ROT]$ = area$[AOD]$. Thus area$[DOT]$ = area$[AOR]$, that is area$[AOR]$ = area$[DOR]$ = $\frac{1}{2}$area$[AOD]$. It follows that OR is a median in the AOD triangle, that is $AO = DO$, which proves that the rhombus $ABCD$ is a square.

73. An equilateral triangle of side 10 is divided into 100 unit equilateral triangles by lines parallel to the sides of the triangle. Find the number of (not necessarily unit) equilateral triangles in the configuration described above so that the sides of the triangle are parallel to the sides of the initial one.

Solution. We solve the general case, that is to consider the number a_n of equilateral triangles formed by division in n segments. The main idea is to find a recurrence relation for the sequence a_n.

Consider an equilateral triangle with the sides partitioned into $n+1$ equal segments and draw the n parallels to each side of the given triangle. We will count all the triangles with at least one vertex on (BC) ,the remaining ones are triangles counted in a_n.

First, consider the triangles that have two vertices on (BC). When choosing two division points on BC, say M and N with $M \in (BN)$, one counts exactly one triangle, namely that one obtained by drawing parallels from M, N to AB, AC respectively. Hence we add $\frac{(n+2)(n+1)}{2}$ triangles with one side on BC.

Considering the triangles with only one vertex on BC. Observe that for any of the n division points of the segment (BC) we count one triangle of side 1. Then, except for the extreme points of division, we count $n-2$ triangles of side 2, and so on. Hence we add $n+(n-2)+(n-4)+\ldots$ triangles with one vertex on BC. It follows that

$$a_{n+1} = a_n + \frac{(n+2)(n+1)}{2} + n + (n-2) + (n-4) + \ldots$$

Changing n with $n+1$ we have

$$a_{n+2} = a_{n+1} + \frac{(n+3)(n+2)}{2} + (n+1) + (n-1) + (n-3) + \ldots$$

Adding up, we obtain

$$\begin{aligned} a_{n+2} &= a_n + \frac{(n+2)(n+1)}{2} + \frac{(n+3)(n+2)}{2} + \frac{(n+1)(n+2)}{2} \\ &= x_n + \frac{(n+2)(3n+5)}{2}. \end{aligned}$$

It follows that

$$a_{10} = a_8 + \frac{10(3\cdot 8+5)}{2} = a_8 + 145 = \ldots = a_0 + 315 = 315.$$

Therefore, the number of triangles is 315.

74. If a, b, $c \in (0, 1)$, prove that

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

Solution. Observe that $x^{\frac{1}{2}} < x^{\frac{1}{3}}$ for $x \in (0, 1)$. Thus we have

$$\sqrt{abc} < \sqrt[3]{abc},$$

and

$$\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)}.$$

By AM-GM inequality, we get

$$\sqrt{abc} < \sqrt[3]{abc} \le \frac{a+b+c}{3},$$

and

$$\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)} \le \frac{(1-a)+(1-b)+(1-c)}{3}.$$

Summing up, we obtain

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < \frac{a+b+c+1-a+1-b+1-c}{3} = 1,$$

as desired.

75. Let a be an integer. Prove that for any real number x such that $x^2 < 3$, the numbers $\sqrt{3-x^2}$ and $\sqrt[3]{a-x^3}$ are not both rational.

Solution. Suppose by a way of contradiction that $A = \sqrt{3-x^2}$ and $B = \sqrt[3]{a-x^3}$ are both rational numbers. It follows that

$$x^2 = 3 - A^2, \tag{1}$$

and

$$x^3 = a - B^3, \tag{2}$$

hence

$$a - B^3 = \pm\left(3-A^2\right)\sqrt{3-A^2}.$$

We infer that $\sqrt{3-A^2} = k$ has to be rational and $A^2+k^2 = 3$, both A and k being rational numbers..

Let y, z, t be integers with g.c.d.$(y, z, t) = 1$ so that $A = \frac{y}{t}$ and $B = \frac{z}{t}$. Then $y^2+z^2 = 3t^2$, that is 3 is a divisor of y^2+z^2. It is easy to see that 3 has to be a divisor of both y and z. Furthermore 9 is a divisor of $3t^2$, implying that 3 divides t. Since g.c.d.$(y, z, t) = 1$ we get a contradiction.

76. The last four digits of a perfect square are equal. Prove they are all zero.

Solution. Denote by k^2 the perfect square and by a the digit that appears in the last four position. It easily follows that a is one of the numbers 0, 1, 4, 5, 6, 9. Thus $k^2 \equiv a \cdot 1111 \pmod{10^4}$ and consequently $k^2 \equiv a \cdot 1111 \,(\mathrm{mod}\, 16)$.

1° If $a = 0$, we are done.

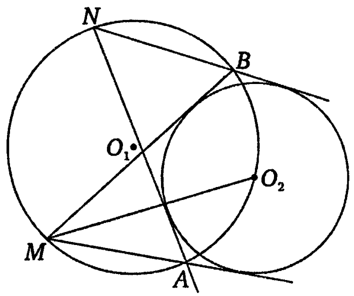
2° Suppose that $a \in \{1, 5, 9\}$. Since $k^2 \equiv 0 \,(\mathrm{mod}\, 8)$, $k^2 \equiv 1 \,(\mathrm{mod}\, 8)$ or $k^2 \equiv 4 \,(\mathrm{mod}\, 8)$ and $1111 \equiv 7 \,(\mathrm{mod}\, 8)$, we obtain $1111 \equiv 7 \,(\mathrm{mod}\, 8)$, $5 \cdot 1111 \equiv 3 \,(\mathrm{mod}\, 8)$ and $9 \cdot 1111 \equiv 7 \,(\mathrm{mod}\, 8)$. Thus the congruence $k^2 \equiv a \cdot 1111 \,(\mathrm{mod}\, 16)$ cannot hold.

3° Suppose $a \in \{4, 6\}$. As $1111 \equiv 7 \,(\mathrm{mod}\, 16)$, $4 \cdot 1111 \equiv 12 \,(\mathrm{mod}\, 16)$ and $6 \cdot 1111 \equiv 10 \,(\mathrm{mod}\, 16)$, we conclude that in this case the congruence $k^2 \equiv a \cdot 1111 \,(\mathrm{mod}\, 16)$ cannot hold.

77. Consider the circles $\mathcal{C}_1(O_1)$ and $\mathcal{C}_2(O_2)$ such that $\mathcal{C}_1$ passes through the point O_2. Let M be a point on the circle $\mathcal{C}_1$ but not on the line O_1O_2. The tangents from M to $\mathcal{C}_2$ meet again the circle $\mathcal{C}_1$ at the points A and B. Prove that the tangents from A and B to $\mathcal{C}_2$ (not those going through M), meet on $\mathcal{C}_1$.

Solution. Since O_2 is at equal distance from the tangents MA and MB, it follows that MO_2 is a bisector line of the angle $\angle AMB$ or of the exterior angle defined by MA and MB.

In the first case we find $\operatorname{arc} O_2A = \operatorname{arc} O_2B$. In the second case using the notation in the figure, we have $\operatorname{arc} BO_2 = \operatorname{arc} M_2 + \operatorname{arc} AM = \operatorname{arc} AO_2$ and $O_2A = O_2B$.



Reflecting the figure with respect to the line O_1O_2, the circles remain fixed, M reflects in N, and A reflects in B. It is obvious that NA, the reflection of MB, is tangent to $\mathcal{C}_2$ and the same is valid for NB. Observe that N is on $\mathcal{C}_1$, proving thus the claim.

78. Consider five points in the plane such that any three of them form a triangle of area at least 2. Prove that there are three of them forming a triangle of area at least 3.

Solution. Denote by A, B, C, D, L the five given points. If the pentagon $ABCDL$ is concave we can suppose that D is located inside the triangle ABC or inside the quadrilateral $ABCD$ (see the figure).

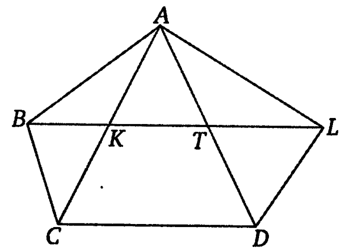
In the first case

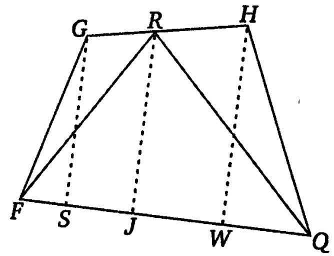
$$\operatorname{area}[ABC] = \operatorname{area}[ABD] + \operatorname{area}[ACD] + \operatorname{area}[BDC] > 6 > 3.$$

In the second case, D is inside one of the triangles ABC, ABL, ACL, BCL. Suppose without loss of generality, that D is inside the triangle BCL.Then:

$$\operatorname{area}[BCL] > \operatorname{area}[CDL] + \operatorname{area}[DCB] > 4 > 3.$$

Consider now the case when $ABCDL$ is a convex pentagon. Let K and T be the intersection points of BL with AC and AD respectively.





The following result will be useful.

Lemma: Let $GHQF$ be a quadrilateral and R a point on the side PQ. Then:

$$\operatorname{area}[FRQ] \geq \min\left(\operatorname{area}[GFQ], \operatorname{area}[HFQ]\right).$$

(The proof consist of simply observing that the distance from R to FQ is bounded up and below by the distance from G and H to FQ).

In our case, suppose that $BK \geq \frac{1}{3}BL$, which implies $BK \geq \frac{1}{2}KL$. Then:

$$\begin{aligned}
\operatorname{area}[BDL] &= \operatorname{area}[BDK] + \operatorname{area}[KDL] \geq \frac{\operatorname{area}[KDL]}{2} + \operatorname{area}[LDK] \\
&= \frac{3}{2}\operatorname{area}[KDL] \geq \min\left(\operatorname{area}[CDL], \operatorname{area}[ADL]\right) > \frac{3}{2} \cdot 2 = 3.
\end{aligned}$$

The case $TL \geq \frac{BL}{3}$ is similar. It is left to consider the case when $KT \geq \frac{BL}{3}$. We have

$$\begin{aligned}
\operatorname{area}[AKT] &= \frac{1}{3}\operatorname{area}[ABL] > \frac{2}{3}, \\
\operatorname{area}[KTD] &\geq \frac{1}{3}\operatorname{area}[BLD] > \frac{2}{3}, \\
\operatorname{area}[KCD] &\geq \min\left(\operatorname{area}[BCD], \operatorname{area}[LCD]\right) > 2.
\end{aligned}$$

Summing up, we conclude

$$\operatorname{area}[ACD] > 2 + \frac{2}{3} + \frac{2}{3} > 3,$$

and the proof is complete.

79. Let m, $n > 1$ be integer numbers. Solve in positive integers the equation

$$x^n + y^n = 2^m.$$

Solution. Let $d = c.g.d.(x, y)$ and $x = da$, $y = db$, where $(a, b) = 1$. It is easy to see that a and b are both odd numbers and $a^n + b^n = 2^k$, for some integer k.

Suppose that n is even. As $a^2 \equiv b^2 \equiv 1 \pmod 8$, we have also $a^n \equiv b^n \equiv 1 \pmod 8$. As $2^k = a^n + b^n \equiv 2 \pmod 8$, we conclude $t = 1$ and $u = v = 1$, thus $x = y = d$. The equation becomes $x^n = 2^{m-1}$ and it has an integer solution if and only if n is a divisor of $m - 1$ and $x = y = 2^{\frac{m-1}{n}}$.

Consider the case when n is odd. From the decomposition

$$a^n + b^n = (a + b)\left(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots + b^{n-1}\right),$$

we easily get $a + b = 2^k = a^n + b^n$. In this case $a = b = 1$, and the proof goes on the line of the previous case.

To conclude, the given equations have solutions if and only if $\frac{m-1}{n}$ is an integer and in this case $x = y = 2^p$.

80. Consider $n > 2$ concentric circles and two lines d_1, d_2 which meet at P, a point inside all the circles. The rays determined by P on the line d_1 meet the circles at the points $A_1, A_2, \ldots, A_n$ and $A'_1, A'_2, \ldots, A'_n$ respectively. similarly, the rays determined by P on the line d_2 meet the circles at the points $B_1, B_2, \ldots, B_n$ and $B'_1, B'_2, \ldots, B'_n$ respectively (the points of equal index are on the same circle). Prove that if the small arcs A_1B_1 and A_2B_2 are equal, then all the small arcs A_iB_i and $A'_iB'_i$ are equal for all $i = \overline{1, n}$.

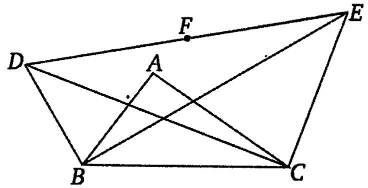
Solution. Let O be the common center of the n circles and $\alpha = \operatorname{arc} A_1B_1 = \operatorname{arc} A_2B_2$ (the arcs are directly orientated). Rotate the figure around the center O by an angle α such that A_1, A_2 become B_1, B_2 respectively. The above rotation R maps lines into lines, that is $R(D_1) = R(D_2)$, since $D_1 = A_1A_2$ and $D_2 = B_1B_2$. Moreover, a point M on a circle C_i with the center O remains on the same circle after rotation. Because $R(A_i)$ lies both on D_2 and on C_i, we get that $R(A_i) = B_i$, hence $\operatorname{arc} A_iB_i = \alpha$.

In the same way we obtain $R(A'_i) = B'_i$ and $\operatorname{arc} A'_iB'_i = \alpha$. This concludes the proof.

81. Let ABC be a triangle and $a = BC$, $b = CA$, $c = AB$ be the side lengths. On the same side of BC as A consider the points D and E such that $DB = c$, $CE = b$ and the area of $DECB$ is maximal. Let F be the midpoint of DE and let $FB = x$. Prove that $FC = x$ and $4x^3 = (a^2 + b^2 + c^2)x + abc$.

Solution. Let $BCED$ be the quadrilateral of maximum area. It is easy to prove

that $\angle DBE = \angle DCE = 90°$.



It follows that $BF = CF = \frac{DE}{2} = x$ and the quadrilateral $DBCE$ is cyclic. By Ptolemy's theorem we have

$$DC \cdot BE = BC \cdot DE + DB \cdot CE.$$

Squaring, we obtain

$$\left(4x^2 - b^2\right)\left(4x^2 - c^2\right) = (2ax + bc)^2,$$

hence

$$16x^4 - 4\left(b^2 + c^2\right)x^2 + (bc)^2 = 4a^2x^2 + 4abcx + b^2c^2.$$

Thus $4x^3 = x\left(a^2 + b^2 + c^2\right) + abc$, as desired.

82. Let p, q be two distinct primes. Prove that there are positive integers a, b so that the arithmetic mean of all the divisors of the number $n = p^a \cdot q^b$ is also an integer.

Solution. The sum of all divisors of n is given by the formula

$$\left(1 + p + p^2 + \ldots + p^a\right)\left(1 + q + q^2 + \ldots + q^b\right),$$

as it can be easily seen by expanding the brackets. The number n has $(a + 1)(b + 1)$ positive divisors and their arithmetic mean is

$$M = \frac{\left(1 + p + p^2 + \ldots + p^a\right)\left(1 + q + q^2 + \ldots + q^b\right)}{(a + 1)(b + 1)}.$$

If p and q are both odd numbers, we can take $a = p$ and $b = q$, and it is easy to see that m is an integer.

If $p = 2$ and q odd, choose again $b = q$ and consider $a + 1 = 1 + q + q^2 + \ldots + q^{q-1}$. Then $m = 1 + 2 + 2^2 + \ldots + 2^a$, and it is an integer.

For p odd and $q = 2$, set $a = p$ and $b = p + p^2 + p^3 + \ldots + p^{p-1}$. The solution is complete.

Chapter 7

Short-Listed Problems Formal Solutions

83. Prove that there are at least 666 positive composite numbers with 2006 digits, having a digit equal to 7 and all the rest equal to 1.

Solution. The given numbers are $n_k = 111...17\underbrace{11...1}_{k\text{ digits}} = \underbrace{111...1}_{2006\text{ digits}} + 6\underbrace{000...0}_{k\text{ digits}} = \frac{1}{9}(10^{2006} - 1) + 6 \cdot 10^k$, $k = \overline{0, 2005}$.

It is obvious that none of these numbers is a multiple of 2, 3, 5 or 11, as 11 divides $\underbrace{111...1}_{2006\text{ digits}}$, but not $6 \cdot 10^k$.

So we are lead to the idea of counting multiples of 7 and 13. We have $9n_k = 100 \cdot 1000^{668} - 1 + 54 \cdot 10^k \equiv 2 \cdot (-1)^{668} - 1 + (-2) \cdot 10^k \equiv 1 - 2 \cdot 10^k (\text{mod } 7)$, hence $7 \mid n_k$ if $10^k \equiv 3^k \equiv 4 (\text{mod } 7)$. This happens for $k = 4, 10, 16, ...2002$ so there are 334 multiples of 7. Furthermore, $9n_k = 7 \cdot (-1)^{668} - 1 + 2 \cdot 10^k = 6 + 2 \cdot 10^k (\text{mod } 13)$, hence $13 \mid n_k$ if $10^k \equiv 10 (\text{mod } 13)$. This happens for $k = 1, 7, 13, 19, ...2005$, so there are 335 multiples of 13. In all we have found 669 non-prime numbers.

84. Find all the positive perfect cubes that are not divisible by 10 so that the number obtained by erasing the last three digits is also a perfect cube.

Solution. We have $(10m + n)^3 = 1000a^3 + b$, where $1 \le n \le 9$ and $b < 1000$.

The equality gives

$$(10m + n)^3 - (10a)^3 \Rightarrow b < 1000,$$

so

$$(10m + n - 10a)\left[(10m + n)^2 + (10m + n) \cdot 10a + 100a^2\right] < 1000.$$

As $(10m + n)^2 + (10m + n) \cdot 10a + 100a^2 > 100$, we obtain $10m + n - 10a < 10$, hence $m = a$.

If $m \geq 2$, then $n\left(300m^2 + 30mn + n^2\right) > 1000$ false.
Then $m = 1$ and$n\left(300 + 30n + n^2\right) < 1000$, hence $n \leq 2$. For $n = 2$, we obtain $12^3 = 1728$ and for $n = 1$ we get $11^3 = 1331$.

85. Find the greatest positive integer x such that 23^{6+x} divides 2000!.

Solution. The number 23 is prime and divides every 23^{rd} number; in all, there are $\left[\frac{2000}{23}\right] = 86$ numbers from 1 to 2000 that are divisible by 23. Among those 86 numbers, three of them, namely $23, 2 \cdot 23$ and $3 \cdot 23^2$, are divisible by 23^3. Hence $23^{89} \mid 2000!$ and $x = 89 - 6 = 83$.

86. Find all the integers written as $\overline{abcd}$ in decimal representation and $\overline{dcba}$ in 7 base.

Solution. We have

$$\overline{abcd}_{(10)} = \overline{dcba}_{(7)} \Leftrightarrow 999a + 93b = 39c + 342d \Leftrightarrow 333a + 31b = 13c + 114d,$$

hence $b \equiv c\,(\text{mod}\,3)$. As $b, c \in \{0, 1, 2, 3, 4, 5, 6\}$, the possibilities are:

i) $b = c$;

ii) $b = c + 3$;

iii) $b + 3 = c$.

In the first case we must have $a = 2a'$, $d = 3d'$, $37a' + b = 19d'$, $d' = 2$; hence $a' = 1$, $a = 2$, $d = 6$, $b = 1$, $c = 1$, and the number $\overline{abcd}$ is 2116.

In the other cases a has to be odd. Considering $a = 1, 3$ or5 we obtain no solutions.

87. Find all the pairs of integers (m, n) so that the numbers $A = n^2 + 2mn + 3m^2 + 2$, $B = 2n^2 + 3mn + m^2 + 2$, $C = 3n^2 + mn + 2m^2 + 1$ have a common divisor greater than 1.

Solution. A common divisor of A, B and C is also a divisor for $D = 2A - B$, $E = 3A - C$, $F = 5E - 7D$, $G = 5D - E$, $H = 18A - 2F - 3E$, $I = nG - mF$ and $126 = 18nI - 5H + 11F = 2 \cdot 3^2 \cdot 7$. Since 2 and 3 do not divide A, B and C, then $d = 7$. It follows that (m, n) is equal to $(7a + 2, 7b + 3)$ or $(7c + 5, 7d + 4)$.

88. Find all the four-digit numbers so that when decomposed in prime factors have the sum of the prime factors equal to the sum of the exponents.

Solution. 1° If the number has at least four prime divisors, then $n \geq 2^{14} \cdot 3 \cdot 5 \cdot 7 > 9999$, a contradiction.

2° If n has 3 prime divisors, these must be 2, 3 or 5. The numbers are

$$2^8 \cdot 3 \cdot 5 = 3840, \quad 2^7 \cdot 3^2 \cdot 5 = 5760, \quad 2^6 \cdot 3^3 \cdot 5 = 8640, \quad \text{and} \quad 2^7 \cdot 3 \cdot 5^2 = 9600.$$

3° If n has 2 prime divisors, at least one of them must be 2 or 3. The numbers

$$2^4 \cdot 5^3 = 2000, \quad 2^3 \cdot 5^4 = 5000, \quad 2^8 \cdot 7 = 1792, \quad 2^7 \cdot 7^2 = 6272$$

satisfy the solutions.

4° If n has only one prime factor, then $5^5 = 3125$.

Therefore there are 9 solutions.

89. Find all the pairs of integers (m, n) such that the numbers $A = n^2 + 2mn + 3m^2 + 3n$, $B = 2n^2 + 3mn + m^2$, $C = 3n^2 + mn + 2m^2$ are consecutive in some order.

Solution. Let $D = \frac{A+B+C}{3} = 2n^2 + 2mn + 2m^2 + n$. We consider the following cases:

1° $D = A$. Then $m^2 + 1 = (n-1)^2$, and consequently $m = 0$, $n = 0$ or $m = 0$, $n = 2$, false.

2° $D = B$. Then $m^2 = mn - n$. All the cases $A = B - 1$, $C = B + 1$ and $A = B + 1$, $C = B - 1$ lead to contradiction.

3° $D = C$. It follows that $n(n - m - 1) = 0$. If $n = 0$ then $m = 1$ or $m = -1$. For $n = m + 1$ we have $B = 6m^2 + 7m + 2$ and B, C, A are positive integers for all integers m.

Thus the pairs (m, n) are $(1, 0)$ and $(m, m + 1)$ for all integers m.

90. Find all the positive integers a, b for which $a^4 + 4b^4$ is a prime number.

Solution. Observe that

$$\begin{aligned} a^4 + 4b^4 &= a^4 + 4b^4 + 4a^2b^2 - 4a^2b^2 = \left(a^2 + 2b^2\right)^2 - 4a^2b^2 \\ &= \left(a^2 + 2b^2 + 2ab\right)\left(a^2 + 2b^2 - 2ab\right) = \left[(a+b)^2 + b^2\right]\left[(a-b)^2 + b^2\right]. \end{aligned}$$

As $(a+b)^2 + b^2 > 1$, then $a^4 + 4b^4 = 5$ can be a prime number only for $(a-b)^2 + b^2 > 1$. Indeed, for $a = b = 1$, $a^4 + 4b^4 = 5$ is prime.

91. Find all the triples (x, y, z) of positive integers such that $xy + yz + zx - xyz = 2$.

Solution. Let $x \leq y \leq z$. We consider the following cases:

1° For $x = 1$, we obtain $y + z = 2$, and then

$$(x, y, z) = (1, 1, 1).$$

2° If $x = 2$, then $2y + 2z - yz = 2$, which gives $(z-2)(y-2) = 2$. The solutions are $z = 4$, $y = 3$ or $z = 3$, $y = 4$. Due to the symmetry of the relations the solutions (x, y, z) are

$$(2, 3, 4), (2, 4, 3), (3, 2, 4), (4, 2, 3), (3, 4, 2), (4, 3, 2).$$

3° If $x \geq 3$, $y \geq 3$, $z \geq 3$ then $xyz \geq 3yz$, $xyz \geq 3xz$, $xyz \geq 3xy$. Thus $xy + xz + yz - xyz \leq 0$, so there are no solutions.

92. Prove that there are no integers x, y, z so that

$$x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 = 2000.$$

Solution. Suppose by way of contradiction that such numbers exist. Assume without loss of generality that x, y, z are non-negative integers.

At first we prove that the numbers are distinct. For this, consider that $y = z$. Then $x^4 - 4x^2y^2 = 2000$,hence x is even.

Setting $x = 2t$ yields $t^2(t^2 - y^2) = 125$. It follows that $t^2 = 25$ and $y^2 = 20$, a contradiction.

Let now $x > y > z$. Since $x^4 + y^4 + z^4$ is odd, at least one of the numbers x, y, z is even and the other two have the same parity. Observe that

$$\begin{aligned} & x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 \\ = & \left(x^2 - y^2\right)^2 - 2\left(x^2 - y^2\right)z^2 + z^4 - 4y^2z^2 \\ = & \left(x^2 - y^2 - z^2 - 2yz\right)\left(x^2 - y^2 - z^2 + 2yz\right) \\ = & (x + y + z)(x - y - z)(x - y + z)(x + y - z), \end{aligned}$$

each of the four factors being even. Since $2000 = 16 \cdot 125 = 2^4 \cdot 125$ we deduce that each factor is divisible by 2, but not by 4. Moreover, the factors are distinct

$$x + y + z > x + y - z > x - y + z > x - y - z.$$

The smallest even divisors of 2000 that are not divisible by 4 are 2, 10, 50, 250. But $2 \cdot 10 \cdot 50 \cdot 250 > 2000$, a contradiction.

93. Prove that for any integer n one can find integers a and b such that

$$n = \left[a\sqrt{2}\right] + \left[b\sqrt{3}\right].$$

Solution. For any integer n, one can find an integer b so that

$$\sqrt{2} + b\sqrt{3} - 2 < n \le \sqrt{2} + b\sqrt{3}.$$

We consider the cases:

1° If $n = \left[\sqrt{2}\right] + \left[b\sqrt{3}\right]$, we are done.

2° If $n = \left[\sqrt{2}\right] + \left[b\sqrt{3}\right] + 1$, then $n = \left[2\sqrt{2}\right] + \left[b\sqrt{3}\right]$.

3° If $n = \left[\sqrt{2}\right] + \left[b\sqrt{3}\right] - 1$, then $n = \left[0\sqrt{2}\right] + \left[b\sqrt{3}\right]$.

94. Consider a sequence of positive integers x_n such that:

$$\begin{aligned} \text{(A) } x_{2n+1} &= 4x_n + 2n + 2, \\ \text{(B) } x_{3n+2} &= 3x_{n+1} + 6x_n, \end{aligned}$$

for all $n \ge 0$.

Prove that

$$\text{(C) } x_{3n-1} = x_{n+2} - 2x_{n+1} + 10x_n,$$

for all $n \ge 0$.

Solution. We have

$$\begin{aligned} x_{6n+5} &= x_{2(3n+2)+1} = 6\left(2x_{n+1} + 4x_n + n + 1\right) = x_{3(2n+1)+2} \\ &= 3\left(x_{2(n+1)} + 8x_n + 4n + 4\right), \end{aligned}$$

hence $x_{2(n+1)} = 4x_{n+1} - 2(n+1)$ or $x_{2n} = 4x_n - 2n$. Setting $n = 0$ yields $x_0 = 0$. Inducting on n we obtain $x_n = n(n+1)$ for all $n \ge 0$.Now the relation (C) is easy to be verified.

95. Prove that

$$\sqrt{\left(1^k + 2^k\right)\left(1^k + 2^k + 3^k\right)\ldots\left(1^k + 2^k + \ldots + n^k\right)}$$
$$\ge 1^k + 2^k + \ldots + n^k - \frac{2^{k-1} + 2 \cdot 3^{k-1} + \ldots + (n-1)n^{k-1}}{n}.$$

for all integers n, $k \ge 2$.

Solution. Use the AM-GM inequality for the expression $S_n^k = 1^k + 2^k + \ldots + n^k$.

96. Let m and n be positive integers with $m \le 2000$ and $k = 3 - \frac{m}{n}$. Find the smallest positive value of k.

Solution. As $k = 3 - \frac{m}{n} = \frac{3n-m}{n} > 0$ for a given number n the minimal value of k is obtained for $3n - m = 1$. Since $m = 3n - 1 \le 2000$, then $n \ge 667$. The smallest value of $k = \frac{3n-(3n-1)}{n} = \frac{1}{n}$ is reached when $n = 667$.

97. Let x, y, a, b be positive real numbers such that $x \ne y$, $x \ne 2y$, $y \ne 2x$, $a \ne 3b$ and $\frac{2x-y}{2y-x} = \frac{a+3b}{a-3b}$. Prove that $\frac{x^2+y^2}{x^2-y^2} \ge 1$.

Solution. We have

$$\frac{2x-y}{2y-x} = \frac{a+3b}{a-3b} \Leftrightarrow \frac{2x-y}{a+3b} = \frac{2y-x}{a-3b} = \frac{2x-y+2y-x}{a+3b+a-3b} = \frac{x+y}{2a} = k;$$

$$\frac{2x-y}{2y-x} = \frac{a+3b}{a-3b} \Leftrightarrow \frac{2x-y}{a+3b} = \frac{2y-x}{a-3b} = \frac{2x-y-2y+x}{a+3b-a+3b} = \frac{x-y}{2b} = k.$$

It follows that $x + y = 2ka$ and $x - y = 2kb$, hence $x = k(a+b)$ and $y = k(a-b)$. Thus

$$\frac{x^2+y^2}{x^2-y^2} = \frac{k^2(a+b)^2 + k^2(a-b)^2}{k^2(a+b)^2 - k^2(a-b)^2} = \frac{a^2+b^2}{2ab} \ge 1.$$

98. Find all the triples (x, y, z) of real number such that

$$2x\sqrt{y-1} + 2y\sqrt{z-1} + 2z\sqrt{x-1} \ge xy + xz + yz.$$

Solution. Obviously x, y, $z \ge 1$. The relation is equivalent to

$$\left(xy - 2x\sqrt{y-1}\right) + \left(yz - 2y\sqrt{z-1}\right) + \left(zx - 2z\sqrt{x-1}\right) \le 0$$

$$\Leftrightarrow x\left(y-1+1-2\sqrt{y-1}\right)+y\left(z-1+1-2\sqrt{z-1}\right)+z\left(x-1+1-2\sqrt{x-1}\right)\le 0$$

$$\Leftrightarrow x\left(\sqrt{y-1}-1\right)^2+y\left(\sqrt{z-1}-1\right)^2+z\left(\sqrt{x-1}-1\right)^2\le 0.$$

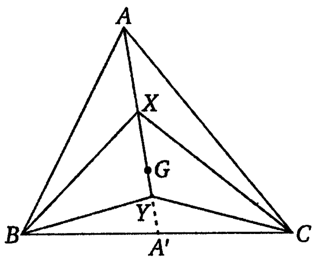
Since x, y, z are positive numbers, it follows that $\sqrt{x-1}-1=0$, $\sqrt{y-1}-1=0$, $\sqrt{z-1}-1=0$, hence $x=y=z=2$.

99. A triangle ABC is given. Find all the pairs of points X, Y so that X is on the sides of the triangle, Y is inside the triangle an four non-intersecting segments from the set $\{XY, AX, AY, BX, BY, CX, CY\}$ divide the ABC triangle in four triangles with equal areas.

Solution. For a point X on the segment (BC), there are three possibilities: $BC=4BX$, $BC=2BX$, $3BC=4BX$, and only a position of the point Y for each case. Thus we have 9 solutions in all, three for each side of the triangle.

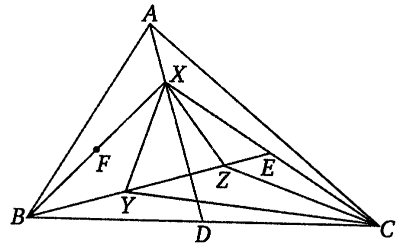
100. A triangle ABC is given. Find all the segments XY that lies inside the triangle such that XY and five of the segments XA, XB, XC, YA, YB, YC divide the ABC triangle in 5 regions with equal areas. Furthermore, prove that all the segments XY have a common point.

Solution. Assume that X is connected with all the vertices A, B and of the triangle. Since area$[ABX]=$area$[ACX]=\frac{S}{5}$, it follows that X is on the median (AA') and $\frac{AX}{AA'}=\frac{2}{5}$. Then Y must be the centroid of the triangle XBC, hence $Y\in AA'$ and the centroid G of the triangle ABC lies on all segments (XY).



101. Let ABC be a triangle. Find all the triangles XYZ with the vertices inside ABC such that XY, YZ, ZX and six non-intersecting segments from the following AX, AY, AZ, BX, BY, BZ, CX, CY, CZ divide the ABC triangle in seven regions with equal areas.

Solution. A point X with area$[ABX]=$area$[ACX]=\frac{\text{area}[ABC]}{7}$ lies on the median (AD) such that $\frac{AX}{AD}=\frac{2}{7}$. In order to divide the triangle BXC into five triangles with equal areas we cannot use the median (XD) (as in the problem 100). For the median (BE) of the triangle we obtain a solution (as shown below)



Another solution will be obtained using the median (CF). In all, there are six triangles XYZ with the desired property.

102. Let ABC be a triangle and let a, b, c be the lengths of the sides BC, CA, AB respectively. Consider a triangle DEF with the side lengths $EF=\sqrt{au}$, $FD=\sqrt{bu}$, $DE=\sqrt{cu}$. Prove that $\angle A>\angle B>\angle C$ implies $\angle A>\angle D>\angle E>\angle F>\angle C$.

Solution. i) We have

$$A>D\Leftrightarrow \cos A<\cos D\Leftrightarrow \frac{b^2+c^2-a^2}{2bc}<\frac{bu+cu-au}{2u\sqrt{bc}}$$

$$\Leftrightarrow b^2+c^2-a^2<(b+c-a)\sqrt{bc}\Leftrightarrow f(a)>0,$$

where

$$f(x)=x^2-x\sqrt{bc}+(b+c)\sqrt{bc}-b^2-c^2.$$

The function $f(x)$ is increasing for $x>\frac{\sqrt{bc}}{2}$ and

$$f(b)=b^2-b\sqrt{bc}+(b+c)\sqrt{bc}-b^2-c^2=c\sqrt{c}\left(\sqrt{b}-\sqrt{c}\right)>0.$$

Thus $f(a)>f(b)>0$ as needed.

ii) From $\sqrt{au}>\sqrt{bu}>\sqrt{cu}$ follows $D>E>F$.

iii) We have $F>C\Leftrightarrow g(c)<0$, where

$$g(x)=x^2-x\sqrt{ab}+ab\sqrt{ab}-a^2-b^2.$$

Since $g(b)<0$, $g(a)>0$, $g(0)<0$, we obtain $g(c)<g(b)<0$, as claimed.

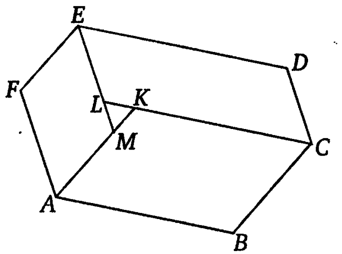
103. All the angles of the hexagon $ABCDEF$ are equal. Prove that

$$AB - DE = EF - BC = CD - FA.$$

Solution. Each angle of the hexagon has the measure of 120°. Consequently, the opposite sides of the hexagon are parallel.

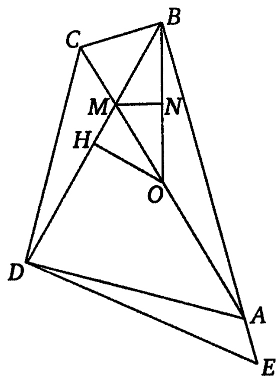
1° If $ABCDEF$ is a regular hexagon, then $AB - DE = EF - BC = CD - FA = 0$, and we are done.

2° If $ABCDEF$ is not regular, construct the parallelograms $ABCK$, $LCDE$, $AMEF$. Then KLM is an isosceles triangle, therefore $AB - DE = EF - BC = CD - FA$, as needed.



104. Consider a quadrilateral $ABCD$ with $\angle DAB = 60°, \angle ABC = 90°$ and $\angle BCD = 120°$. The diagonals AC and BD intersect at M. If $MB = 1$ and $MD = 2$, find the area of the quadrilateral $ABCD$.

Solution. Summing the angles of the quadrilateral $ABCD$ yields $\angle ADC = 90°$. Let O be the midpoint of the segment AC. It follows $AO = BO = CO = DO$, as DO and BO are medians in the right-angled triangles ADC and ABC respectively. The angles $\angle BOC$ and $\angle DOC$ are exterior angles of the triangles BAO and ADO respectively. Thus, $\angle BOC = 2\angle BAO$ and $\angle DOC = 2\angle DAO$.



Then

$$\angle BOD = \angle BOC + \angle DOC = 2\angle BAD = 2 \cdot 60° = 120°,$$

and since $BO = BD$ we obtain

$$\angle OBD = \angle ODB = 30°.$$

Consider the points H on BD and N on OB such that $OH \perp BD$ and $MN \perp OB$. The segment OH is an altitude in the isosceles triangle ODB, hence is also a median and $BH = HD$. We have

$$MH = BH - BM = \frac{BD}{2} - BM = \frac{3}{2} - 1 = \frac{1}{2}.$$

From the right triangle BMN we deduce that $MN = \frac{BM}{2} = \frac{1}{2} = MH$, as $\angle MON = 30°$. Then MON and MOH are congruent triangles and

$$\angle MON = \angle MOH = \frac{\angle BOH}{2} = 30°.$$

Consequently,

$$\angle OBA = \angle OAB = \frac{\angle BOM}{2} = 15°.$$

$$\angle ABD = 15° + 30° = 45°,\ \angle ADB = 75° \text{ and } \angle DAC = 45°.$$

Furthermore, $\angle ACD = 45°$ and $\angle BDC = 15°$, hence $AD = CD$.

Let E be a point on the line AB such that $ED \perp AB$. Then the triangles ADE and CDB are congruent and

$$\begin{aligned}\text{area}[ABCD] &= \text{area}[ABD] + \text{area}[BCD] = \text{area}[EBC] + \text{area}[ADE] \\ &= \text{area}[BDE] = \frac{BD^2}{2} = \frac{3^2}{2} = 4,5.\end{aligned}$$

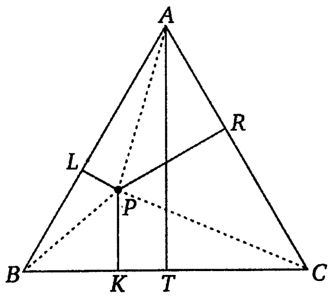
105. A point P is considered inside of an equilateral triangle of the side length 10 so that the distances from P to two of the sides are 1 and 3, respectively. Find the distance from P to the third side.

Solution. Let ABC be the given triangle and let L, K, R be the projections of P on the sides AB, BC and AC respectively. Consider $PL = 1$ and $PR = 3$.

The relation

$$\text{area}[ABC] = \text{area}[PAB] + \text{area}[PBC] + \text{area}[PAC]$$

gives $25\sqrt{3} = 5(PL + PR + PK)$.



Hence $PK = 5\sqrt{3} - 4$.

106. Find the positive integers n that are not divisible by 3 if the number $2^{n^2-10} + 2133$ is a perfect cube.

Solution. Notice that $n^2 - 10 > 3$. Since $3 \nmid n$, then $3 \mid n^2 - 10$. Set $n^2 - 10 = 3k$, and let

$$x^3 = 2^{n^2-10} + 2133 = 2^{3k} + 2133.$$

It follows that

$$\left(x - 2^k\right)\left(x^2 + x \cdot 2^k + 2^{2k}\right) = 3^3 \cdot 79.$$

Since $x^3 > 2133$, we have $x > 12$ and $x^2 + x \cdot 2^k + 2^{2k} > 156$ thus $x - 2^k \in \{1, 3, 3^2\}$.

On the other hand, $x^2 + x \cdot 2^k + 2^{2k} - \left(x - 2^k\right) = 3 \cdot 2^k \cdot x$.

Therefore,

1° If $x - 2^k = 1$, then $3 \cdot 2^k \cdot x = 2132$, false.

2° If $x - 2^k = 3$, then $3 \cdot 2^k \cdot x = 702 = 3^3 \cdot 2 \cdot 13$, false.

3° If $x - 2^k = 3^2$, then $3 \cdot 2^k \cdot x = 2^2 \cdot 3 \cdot 13$ and consequently $k = 2$, $x = 13$, $n = 4$.

107. Let P_n $(n = 3, 4, 5, 6, 7)$ be the set of integers $n^k + n^l + n^m$, where k, l, m are positive integers. Find n so that:

i) In the set P_n there are infinitely many squares.

ii) In the set P_n there are no squares.

Solution. 1° For $n = 3$ consider $k = l = m = 2p + 1$. Then $3^k + 3^l + 3^m = 3^{2p+1} \cdot 3 = 3^{2p+2} = \left(3^{p+1}\right)^2$.

2° Let $n = 4$. As $1 + 2 \cdot 2^x + 2^{2x}$ is a perfect square, then $4^k\left(1 + 2 \cdot 2^x + 2^{2x}\right)$, with $x = 2p + 1$, has the form $4^k + 4^l + 4^m$ and is also a square.

3° Let $n = 5$. Since $5^k \equiv 1 \pmod 4$ we have $5^k + 5^l + 5^m \equiv 3 \pmod 4$, so there are no squares in the set P_5.

4° Let $n = 6$. The last digit of 6^k is 6, so $6^k + 6^l + 6^m$ ends in 8, and is not a square.

5° For $n = 7$, we have $7^{2p} + 7^{2p} + 7^{2p+1} = \left(7^p \cdot 3\right)^2$.

Thus the set P_n contains infinitely many squares for $n \in \{3, 4, 7\}$ and no squares for $n \in \{5, 6\}$.

108. Find all the three digit numbers $\overline{abc}$ such that the 6003-digit number $\overline{abcabc\ldots abc}$ is divisible by 91. ($\overline{abc}$ occurs 2001 times).

Solution. The number is equal to

$$\overline{abc}\left(1 + 10^3 + 10^6 + \ldots + 10^{6000}\right).$$

Since 91 is a divisor of $1001 = 1 + 10^3$ and the sum $1 + 10^3 + 10^6 + \ldots + 10^{6000}$ has 201 terms, it follows that 91 does not divide $1 + 10^3 + 10^6 + \ldots + 10^{6000}$. Thus $\overline{abc}$ is divisible by 91; the numbers are

$$182,\ 273,\ 364,\ 455,\ 546,\ 637,\ 728,\ 819,\ 910.$$

109. The discriminant of the equation $x^2 - ax + b = 0$ is the square of a rational number and a and b are integers. Prove that the roots of the equation are integers.

Solution. The discriminant of the equation is $\Delta = a^2 - 4b = k^2$, where k is rational and the roots are $x_1 = \frac{a-k}{2}$ and $x_2 = \frac{a+k}{2}$. One can easy see that k is an integer. Thus

$$a^2 - k^2 = 4b,$$

where a, b, k are integers.

Observe that a and k have the same parity, otherwise 4 doesn't divide $a^2 - k^2$. The conclusion follows.

110. Let $x_k = \frac{k(k+1)}{2}$ for all the integers $k \geq 1$. Prove that for any integer $n \geq 10$, between the numbers $A = x_1 + x_2 + \ldots + x_{n-1}$ and $B = A + x_n$ there is at least a square.

Solution. We have

$$\begin{aligned} A &= x_1 + x_2 + \ldots + x_{n-1} = \frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2} + \ldots + \frac{(n-1)n}{2} \\ &= \frac{1}{2}\left[\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k\right] = \frac{(n-1)n(n+1)}{6}; \\ B &= A + x_n = \frac{(n-1)n(n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}. \end{aligned}$$

It suffices to prove that $\sqrt{B} - \sqrt{A} > 1$. We have

$$\sqrt{\frac{(n+2)(n+1)n}{6}} - \sqrt{\frac{(n-1)n(n+1)}{6}} > 1$$

$$\Leftrightarrow \sqrt{(n+1)n} > \sqrt{2(n+2)} + \sqrt{2(n-1)}.$$

Since $\sqrt{n(n+1)} > n$ and $2\sqrt{2(n+2)} > \sqrt{2(n+2)} + \sqrt{2(n-1)}$, we only need to prove that $n > 2\sqrt{2(n+2)}$, which is equivalent to $n^2 > 8n + 16$ or $(n-4)^2 > 32$. As $n \geq 10$, the claim holds.

111. Find all the integers x and y such that $x^3 \pm y^3 = 2001p$, where p is a prime.

Solution. a) Consider the case $p \neq 3$.

1° If $x \equiv y \equiv 0 \pmod 3$, then $x^3 \pm y^3 \equiv 0 \pmod{27}$, and $27 \mid 2001p$, false.

2° If $x \equiv y \equiv \pm 1 \pmod 3$, then $x - y \equiv 0 \pmod 3$ and $x^2 \equiv y^2 \equiv xy \equiv 1 \pmod 3$. Since $x^3 - y^3 = (x-y)(x^2+xy+y^2)$,and $x^2+xy+y^2 \equiv 0 \pmod 3$, it follows that $9 \mid x^3 + y^3 = 2001p$, false.

As $n^3 \equiv n \pmod 3$, then $x^3 + y^3 \equiv x + y \equiv \pm 2 \pmod 3$, a contradiction.

3° If $x \equiv y \equiv \pm 1 \pmod 3$, then $x + y \equiv 0 \pmod 3$ and $x^2 \equiv y^2 \equiv -xy \equiv 1 \pmod 3$. Moreover, $x^3 - y^3 \equiv x - y \equiv \pm 2 \pmod 3$, hence $x^3 - y^3 = 2001p$ has no solution. Since $x^3 + y^3 \equiv (x+y)(x^2+y^2-xy)$ and $x^2+y^2-xy \equiv 0 \pmod 3$, we obtain $9 \mid 2001p$, false.

b) If $p = 3$, then $x^3 \pm y^3 = 6003 \equiv 4 \pmod 7$. On the other hand, $x^3 \equiv 0 \pmod 7$ or $x^3 \equiv \pm 1 \pmod 7$, so there are no solutions. Thus, the given equation has no solutions.

112. Prove that there are no positive integers x and y such that

$$x^5 + y^5 + 1 = (x+2)^5 + (y-3)^5.$$

Solution. Notice that $z^5 \equiv z \pmod{10}$, hence $x+y+1 \equiv (x+2)+(y-3) \pmod{10}$, impossible.

113. Prove that no three points with integer coordinates can be the vertices of an equilateral triangle.

Solution. Assume that there are points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ with integer coordinates such that ABC is an equilateral triangle.

Observe that $AB^2 = (x_1 - x_1)^2 + (y_1 - y_2)^2$ is integer, but area$[ABC] = \frac{l^2\sqrt{3}}{4}$ is an irrational number.

On the other hand,

$$\text{area}\,[ABC] = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2),$$

hence area$[ABC]$ is a rational number, a contradiction.

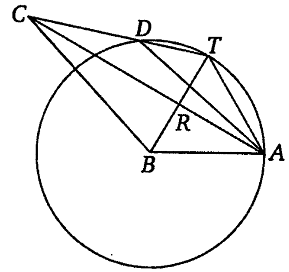
114. Consider a convex quadrilateral $ABCD$ with $AB = CD$ and $\angle BAC = 30°$. If $\angle ADC = 150°$, prove that $\angle BCA = \angle ACD$.

Solution. Let T be the reflection of B across AC and let R be the intersection point of AC and BT. The right angled triangles ABR and ATR are congruent, hence

$$\angle ABT \equiv \angle ATB \tag{1}$$

and

$$\angle BAR \equiv \angle TAR. \tag{2}$$



It follows that ABT is an equilateral triangle, so

$$AB = BT = TA. \tag{3}$$

The circle with the center B and radius BA passes through the points A, T, D. As $\angle ABT = 60°$, then $\angle ADT = 30°$ and

$$\angle TDC = \angle TDA + \angle ADC = 30° + 150° = 180°,$$

hence the points T, D, C are collinear.

From the congruence of the triangles BCR and TCR one can find that $\angle BCR \equiv \angle ACD$, as desired.

115. A triangle ABC is inscribed in the circle $\mathcal{C}(O, R)$. Let $\alpha < 1$ be the ratio of the radii of the circles tangent to C, and both of the rays $(AB$ and $(AC$. The numbers $\beta < 1$ and $\gamma < 1$ are defined analogously. Prove that $\alpha + \beta + \gamma = 1$.

Solution. We have

$$\alpha = \frac{r}{r_a} = \frac{r(p-a)}{S}. \tag{1}$$

and the analogous relations.

Summing up yields $\alpha + \beta + \gamma = \frac{rp}{S} = 1$, as claimed.

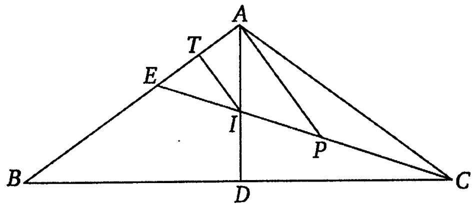
116. Consider an isosceles triangle ABC with $AB = AC$, and D the foot of the altitude from the vertex A. The point E lies on the side AB such that

$$\angle ACE = \angle ECB = 18°.$$

If $AD = 3$, find the length of the segment CE.

Solution. Let I be the intersection point of the bisectors AD and CE, that is I is the incenter of the ABC triangle. It follows that I is equally distanced from BA and BC, hence $ID = IT$,where T is the projection of I to AB.

Consider the point P on the segment CE such that $AP \perp AB$.



Since $\angle ACE = \angle ECB = 18°$, we find that $\angle ACB = 36°$and $\angle DAB = \angle DAC = \angle IEA = 54°$. It follows that the triangle AEJ is isosceles with $AI = EI$, hence T is the midpoint of AE. Furthermore, $\angle IAP = 90° - \angle EAI = 90° - 54° = 36°$ and $\angle EAP = 90° - \angle AEP = 90° - 54° = 36°$, so $AI = IP$. Thus JK is the middle line in the triangle AEP and $IT = \frac{1}{2}AP$. As $\angle PAC = \angle BAC - \angle EAP = 2 \cdot 54° - 90° = 18° = \angle ACP$, we obtain $AP = PC = 2IT = 2ID$.

Now $EC = EI + IP + PC = AI + AI + 2ID = 2(AI + ID) = 2 \cdot AD = 2 \cdot 3 = 6$ and we are done.

117. Consider the triangle ABC with $\angle A = 90°$ and $\angle B \neq \angle C$. A circle $\mathcal{C}(O, R)$ passes through B and C and intersect the sides AB and AC in D and E, respectively. Let S be the foot of the perpendicular from A to BC and let K be the intersection point of AS with the segment DE. If M the midpoint of BC, prove that $AKOM$ is a parallelogram.

Solution. Since $AK \perp BC$ and $OM \perp BC$, we derive

$$AK \parallel OM. \tag{1}$$

The triangles SAB and ABC are right-angled, thus

$$\angle SAB = 90° - \angle SBA = \angle ACB. \tag{2}$$

Since $BDEC$ is cyclic,

$$\angle ACB \equiv \angle ADK, \tag{3}$$

and consequently ADK is isosceles with

$$AK = DK. \tag{4}$$

On the other hand,

$$\angle CAS = 90° - \angle C = \angle B = \angle AEK, \tag{5}$$

hence AEK is isosceles with

$$AK = KE. \tag{6}$$

The relations (4) and (6) show that K is the midpoint of the chord (DE), and consequently

$$OK \perp DE. \tag{7}$$

Since $AM = MB = MC$, we have $\angle MAC \equiv \angle ACM \equiv \angle ADE = 90° - \angle AED$. As $\angle MAC + \angle AED = 90°$, we obtain

$$DE \perp AM. \tag{8}$$

From (7) and (8) we obtain that

$$OK \parallel AM. \tag{9}$$

Recalling (1), we conclude the proof.

118. At a conference there are n mathematicians. Each of them knows exactly k participants. Find the smallest value of k such that there are at least three mathematicians that are acquainted with the other two.

Solution. We prove that $k = \left[\frac{n}{2}\right] + 1$. First we show that $\left[\frac{n}{2}\right] < k$. Indeed, divide the set M of the n mathematicians into subsets A and B having $\left[\frac{n}{2}\right]$ and $n - \left[\frac{n}{2}\right] \geq \left[\frac{n}{2}\right]$ elements, respectively. Any mathematicians from A has $\left[\frac{n}{2}\right]$ acquaintances in M, so we may assume that all of them are in the set B. Likewise, all of the $\left[\frac{n}{2}\right]$ acquaintances of mathematicians from B are in the set A. Now choose three mathematicians from M; two of them are in the same set A or B so they do not know each other. This is a contradiction and consequently $\left[\frac{n}{2}\right] < k$.

It is left to prove that $k = \left[\frac{n}{2}\right] + 1$, then one can choose the mathematicians that are known to each other. Consider a mathematician x from M and let A be the set of his acquaintances. Let $y \in A$ and B the set of his acquaintances. If $A \cap B = \varnothing$, then

$$n = |M| \geq |A \cup B| = |A| + |B| - |A \cap B| = 2\left(\left[\frac{n}{2}\right] + 1\right) > 2\frac{n}{2} = n,$$

a contradiction. Thus $A \cap B \neq \varnothing$ has at least one element z. Then z knows x and y since x and y are also acquainted, we are done.

119. A student plays a computer game. The computer provides him with 2002 positive distinct numbers randomly chosen. The game rules allows him to do the following operations:

– take two of the given numbers, double one of them, add the second number and keep the sum;

– next, choose two other numbers from the remaining ones, double one of them and add the second; then multiply the sum with the previous one and keep the result;

– repeat the above procedure until all the 2002 given numbers are used.

The student wins the game if the last product is maximal. Find, with proof, the winning strategy of the game.

Solution. Let $x_1 < x_2 < \ldots < x_{2002}$ be the given numbers and let A be the maximum value of the product. The number A has the form:

$$A = (2x_{i_1} + x_{i_2})(2x_{i_3} + x_{i_4}) \ldots (2x_{i_{2001}} + x_{i_{2002}}),$$

where $x_{i_1}, x_{i_2}, \ldots, x_{i_{2002}}$ is a permutation of the given numbers.

First remark that if $x > u$, then $2x+u > x+2u$. Hence the product increases when in a pair the greatest number is doubled. It follows that, for all $j = 1, 3, 5, \ldots, 1001$ in P we must have $x_{i_j} > x_{i_{j+1}}$.

Next we prove that P contains the factor $2x_{2002} + x_1$. If else, P contains a factor of the form $(2x_{2002} + u)(2v + x_1)$. Now observe that

$$(2x_{2002} + u)(2v + x_1) < (2x_{2002} + x_1)(2v + u),$$

since this reduces to

$$(x_{2002} - v)(u - x_1) > 0,$$

which is obvious. We have reached a contradiction.

The same argument works for the product $\frac{A}{2x_{2002}+x_1}$, which should be maximal for $x_2, x_3, \ldots, x_{2001}$ and so on. Thus, the maximal value of A is given by the formula

$$A_{\max} = (2x_{2002} + x_1)(2x_{2001} + x_2) \ldots (2x_{1002} + x_{1001}).$$

The winning strategy consists in choosing at each moment the smallest and the greatest number and then doubling the greatest one.

120. All the positive integers are arranged in a triangular array as shown below:

$$\begin{array}{lllll} 1 & 3 & 6 & 10 & 15 \ldots \\ 2 & 5 & 9 & 14 \ldots & \\ 4 & 8 & 13 \ldots & & \\ 7 & 12 \ldots & & & \\ 11 \ldots & & & & \end{array}$$

Find the number of the column and the number of the row where 2002 is put.

Solution. Let i be the number of the column and let j be the number of the row where 2002 is put. It is easy to observe that n^{th} numbers of the first row is equal to $\frac{n(n+1)}{2}$. Since $\frac{62\cdot 63}{2} = 1953$ and $\frac{63\cdot 64}{2} = 2016$ from $1953 < 2002 < 2016$, we conclude that $i + j = 64$. It follows that $j = 2016 - 2002 + 1 = 15$, and $i = 64 - 15 = 49$.

121. Let a, b, c be positive real numbers such that $abc = \frac{9}{4}$. Prove that the following inequality holds

$$a^3 + b^3 + c^3 > a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

Solution. First, notice that $(a-b)^2(a+b) \geq 0$, for any positive integers a and b. This inequality rewrites $(a^2 - ab + b^2 - ab)(a+b) \geq 0$, and consequently

$$a^3 + b^3 > ab(a+b). \tag{1}$$

By the AM-GM inequality we have

$$a^3 + b^3 + c^3 \geq ab(a+b) + c^3 \geq \sqrt{9c^2(a+b)} = 3c\sqrt{a+b}.$$

Likewise,

$$a^3 + b^3 + c^3 \geq 3a\sqrt{b+c} \text{ and } a^3 + b^3 + c^3 \geq 3b\sqrt{c+a}.$$

Summing these inequality yields

$a^3 + b^3 + c^3 \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}$. The equality cannot hold, as this implies $a = b = c = 0$. Hence the inequality is strict, as desired.

122. (Committee's variant for problem 121). If a, b, c are positive real numbers such that $abc = 2$, then

$$a^3 + b^3 + c^3 > a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

When does the equality hold?

Solution. Apply Cauchy-Schwarz inequality gives

$$3(a^2 + b^2 + c^2) \geq (a+b+c)^2, \tag{1}$$

and

$$(a^2 + b^2 + c^2)^2 \leq (a+b+c)(a^3 + b^3 + c^3). \tag{2}$$

These two inequalities combined yield

$$\begin{aligned} a^3 + b^3 + c^3 &\geq \frac{(a^2+b^2+c^2)(a+b+c)}{3} \\ &= \frac{(a^2+b^2+c^2)[(b+c)+(a+c)+(a+b)]}{6} \\ &\geq \frac{\left(a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b}\right)^2}{6}. \end{aligned} \tag{3}$$

Using the AM-GM inequality we obtain

$$\begin{aligned} a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b} &\geq 3\sqrt[3]{abc\left(\sqrt{(a+b)(b+c)(c+a)}\right)} \\ &\geq 3\sqrt[3]{abc\sqrt{8abc}} = 3\cdot\sqrt[3]{8} = 6, \end{aligned}$$

hence

$$\left(a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{a+b}\right)^3 \geq 27\cdot 8,$$

and consequently

$$a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b} \geq 6.$$

Thus

$$\left(a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{a+b}\right)^2 \geq 6\left(a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b}\right). \quad (4)$$

The desired inequality follows from (3) and (4).

123. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2} \geq \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}.$$

Solution. We shall use the inequality

$$\frac{a^3}{b^2} \geq \frac{a^2}{b}+a-b,$$

which is equivalent to the obvious one $(a-b)^2(a+b) \geq 0$. Analogously,

$$\frac{b^3}{c^2} \geq \frac{b^2}{c}+b-c,$$

and

$$\frac{c^3}{a^2} \geq \frac{b^2}{c}+b-c.$$

Adding all three inequalities gives the desired one.

124. Let a_1, a_2, a_3, a_4, a_5, a_6 be real numbers such that $a_1 \neq 0$, $a_1a_6+a_3a_4 = 2a_2a_5$ and $a_1a_3 \geq a_2^2$. Show that $a_4a_6 \leq a_5^2$. When does the equality hold?

Solution. Let $k \geq 0$ such that $a_1a_3 = a_2^2+k$, so

$$a_3 = \frac{a_2^2+k}{a_1}. \quad (1)$$

Multiplying the first given relation by a_4, one has $a_1a_6a_4+a_3a_4^2 = 2a_2a_5a_4$, hence

$$a_6a_4 = \frac{2a_2a_5a_4-a_3a_4^2}{a_1}. \quad (2)$$

From (1) and (2) follows that

$$a_6a_4-a_5^2 = -\frac{(a_1a_5-a_2a_4)+ka_4^2}{2} < 0, \text{ as needed.}$$

The equality holds only if $a_1a_5 = a_2a_4$ and $a_1a_3 = a_2^2$.

125. Consider 2002 integers a_i, $i = 1, 2, 3, \ldots, 2002$ such that

$$a_1^{-3}+a_2^{-3}+\ldots+a_{2002}^{-3} = \frac{1}{2}.$$

Prove that at least three of them are equal.

Solution. It is obvious that $a_k \neq 1$ for all $k = \overline{1, 2002}$. We have

$$\frac{1}{n^3} < \frac{1}{n^3-n} = \frac{1}{2}\left(\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}\right)$$

for all integers $n > 0$.

Assume that in the given sum there are not more then two equal summands. Then

$$\begin{aligned}\frac{1}{2} &= \frac{1}{a_1^3}+\frac{1}{a_2^3}+\ldots+\frac{1}{a_{2002}^3} \leq 2\left(\frac{1}{2^3}+\frac{1}{3^3}+\ldots+\frac{1}{1002^3}\right)\\ &\leq \frac{1}{2}-\frac{1}{1002}+\frac{1}{1003} = \frac{1}{2}-\frac{1}{1002\cdot 1003},\end{aligned}$$

a contradiction. Thus, at least three of the given numbers are equal.

126. Let G be the centroid of a triangle ABC, and let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB respectively. The parallel line from A_1 to BB_1 meets B_1C_1 in F. Prove that the triangles ABC and FA_1A are similar with the same orientation if and only if the quadrilateral AB_1GC_1 is cyclic.

Solution. Extend the segment GA_1 with $A_1D = GA_1$. The quadrilaterals CC_1AF, CBC_1F and $BGCD$ are parallelograms. Due to a homothety, we observe that AB_1GC_1 is cyclic if and only if $ABDC$ is cyclic. In this hypothesis we have $\angle GAB_1 = \angle GCA_1 = \angle GC_1B_1$ and $\angle BAD = \angle BCD$, hence $\angle BAC = \angle DCG$.

On the other hand, $\angle ACB = \angle AB_1C_1 = \angle AGC_1 = \angle CGD$ and $\angle ADC = \angle ABC$. Thus, triangles ABC and CDG are similar. Since the triangles CDG and FA_1A are similar, we obtain that ABC and FA_1A are also similar, as desired.

Conversely, consider that ABC and FA_1A are similar. As FA_1A and CDG are similar triangles we deduce that the triangles ABC and CDG are also similar. Thus $\angle ACB = \angle CDG = \angle AGC_1 = \angle AB_1C_1$ and consequently AB_1GC_1 is a cyclic quadrilateral, as needed.

127. Let ABC be a triangle and let H, I, O be the orthocenter, the incenter and the circumcenter of the triangle, respectively. The line CI meets again the circumcircle at the point L. It is known that $AB = IL$ and $AH = OH$. Find the measure of the angles of the triangle ABC.

Solution. Since $\angle IAL = \angle AIL = \frac{(\angle BAC+\angle BCA)}{2}$, we have $AL = IL = AB = BL$. Hence $\angle AOB = \angle ACB = 120°$ and $\angle ALB = 60°$.

From $\angle AHB = 180° - \angle ACB = 60°$ we obtain that A, O, B, H are on the same circle, hence $\angle AHO = \angle ABO = 30°$. Since $HO = HA$, we deduce that $\angle AOH = \angle HAO = 75°$. As $\angle BAO = \angle HAC = 30°$, we find $\angle HAO = 60° + \angle BAC = 75°$ and $\angle BAC = 15°$. Finally, $\angle ABC = 45°$ and $\angle ACB = 120°$.

128. Let ABC be a triangle of area S and consider the points D, E, F on the lines BC, CA, AB respectively. The perpendicular lines at points D, E, F on the lines BC, CA, AB intersect the circumcircle of the triangle ABC in the pairs of points (D_1, D_2), (E_1, E_2), (F_1, F_2) respectively. Prove that

$$|D_1B \cdot D_1C - D_2B \cdot D_2C| + |E_1C \cdot E_1A - E_2C \cdot E_2A| + |F_1A \cdot F_1B - F_2A \cdot F_2B| > 4S.$$

Solution. We start with a useful result.

Lemma. Suppose AB and D_1D_2 are perpendicular chords in a circle of center O. Then:

$$|\text{area}\,[D_1AB] - \text{area}\,[ABD_2]| = 2\text{area}\,[AOB]\,. \tag{1}$$

Proof. Let D_1' be the reflection of D_1 across AB. Then $\angle BAD_1' = \angle BAD_1 = \angle D_1D_2B = 90° - \angle ABD_2$, hence AD' is perpendicular to BD_2. If BB' is the diameter of the circle, we infer that $B'D_2$ is parallel to AD_1' and AB' is parallel to D_1D_2. Thus, the quadrilateral $AB'D_2D_1'$ is a parallelogram and $D_2D_1' = AB' = 2OO'$, where O' is the projection of O on AB. Consequently,

$$\text{area}\,[ABD_2] - \text{area}\,[ABD_1] = \frac{AB \cdot D_1'D_2}{2} = 2\text{area}\,[AOB]\,,$$

as desired.

Now, apply the lemma successively for the pairs of perpendicular chords $BC \perp D_1D_2$, $CA \perp E_1E_2$ and $AB \perp F_1F_2$. It follows that

$$|D_1B \cdot D_1C - D_2B \cdot D_2C| \geq |D_1B \cdot D_1C - D_2B \cdot D_2C| \cdot |\sin A|$$

$$= |D_1B \cdot D_1C \cdot \sin A - D_2B \cdot D_2C \cdot \sin A| = 2\,|\text{area}\,[BCD_1] - \text{area}\,[BCD_2]|\,.$$

Since $\angle A = \angle BD_1C = 180° - \angle BD_2C$, then $\sin A = \sin \angle BD_1C = \sin \angle BD_2C$.

Therefore, by the lemma we have

$$|D_1B \cdot D_1C - D_2B \cdot D_2C| \geq 4\text{area}\,[BOC]\,. \tag{1}$$

Likewise,

$$|F_1A \cdot F_1B - F_2A \cdot F_2B| \geq 4\text{area}\,[AOB]\,, \tag{2}$$

$$|E_1A \cdot E_1C - E_2C \cdot E_2A| \geq 4\text{area}\,[AOC]\,. \tag{3}$$

Adding (1), (2) and (3) gives the desired result, since the equality holds only if $\sin A = \sin B = \sin C = 1$, which is impossible.

129. Let ABC be an isosceles triangle such that $AB = AC$ and $\angle A = 20°$. Point D is chosen on the side AC such that $AD = BC$. Find the angle $\angle\, BDC$.

Solution. We have $\angle B = \angle C = 80°$. Let K be the point on the side AC such that $\angle CBK = 20°$. Then $\angle CKB = 80° = \angle KCB$, so $BK = BC$. Furthermore,

$\angle ABK = 60°$. Let L be a point of the side AB such that $BL = BK$. The triangle BKL is equilateral, hence $\angle BLK = \angle BKL = 60°$.

Consider a point M on the side AC such that $\angle KML = 40°$. It follows that LMK is an isosceles triangle. As $\angle ALM = 20°$ we also find that ALM is an isosceles triangle and $AM = ML = LK = BK = BC$. Thus M coincides with D. Since $LB = LM$, we find that $\angle LBM = \angle LBM = 10°$. Finally, $\angle BMC = 40° - 10° = 30°$.

130. Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $BC = CD$. On the sides AB, BC, CD, DA, points K, L, L_1, K_1 are chosen respectively such that KLL_1K_1 is a rectangle. Then, suppose that a rectangle $MNPQ$, is inscribed in the triangle BLK where $M \in KB$, $N \in BL$, P, $Q \in LK$ and, similarly, $M_1N_1P_1Q_1$ is inscribed in the triangle DK_1L_1, where $M_1 \in DK_1$, $N_1 \in DL_1$ and $P_1, Q_1 \in L_1K_1$. Let $2S$, $2S_1$, S_2, S_3 be the areas of the quadrilaterals $ABCD$, KLL_1K_1, $MNPQ$, $M_1N_1P_1Q_1$ respectively. Find the greatest value of $\frac{2S_1+S_2+S_3}{2S}$.

Solution. As the quadrilateral $ABCD$ is symmetric with respect to the diagonal AC, it would be enough to consider the triangle ABC which includes half of the rectangle KLL_1K_1 and the rectangle $MNPQ$. Cutting of these parts from the triangle ABC we are left with a triangle BMN, which is similar to the triangle BAC and with two pairs of right-angled triangles which, if adequately connected, can form two triangles which are similar to the triangle ABC. Denote x, y, z and S_1, S_2, S_3 the heights and the areas of the triangle BMN and of the new formed triangles respectively, such that $x + y + z$ is the height of the triangle BAC. Then

$$\frac{s_1}{S} = \frac{x^2}{(x+y+z)^2};\ \frac{s_2}{S} = \frac{y^2}{(x+y+z)^2};\ \frac{s_3}{S} = \frac{z^2}{(x+y+z)^2}.$$

Due to the symmetry of the quadrilateral $ABCD$, maximizing $\frac{2S_1+S_2+S_3}{2S}$ is equivalent to maximizing $\frac{S_1+S_2}{S}$.

We have

$$\frac{S_1+S_2}{S} = \frac{S-(s_1+s_2+s_3)}{S} = 1 - \left(\frac{s_1}{S} + \frac{s_2}{S} + \frac{s_3}{S}\right) = \frac{2\,(xy+yz+zx)}{(x+y+z)^2}.$$

As $(x+y+z)^2 \geq 3\,(xy+yz+zx)$, we obtain

$$\frac{2S_1+S_2+S_3}{2S} = \frac{S_1+S_2}{S} = \frac{2\,(xy+yz+zx)}{(x+y+z)^2} \leq \frac{2}{3},$$

with equality only if $x = y = z$.

131. Let $A_1, A_2, \ldots, A_{2002}$ be arbitrary points in a plane. Prove that for any unit circle in the plane and for any rectangle inscribed in the circle, there are three vertices M, N, P of the rectangle such that

$$MA_1 + \ldots + MA_{2002} + NA_1 + \ldots + NA_{2002} + PA_1 + \ldots + PA_{2002} \geq 6006.$$

Solution. Consider a unit circle in a plane and MN a diameter of this circle. Then

$$2 = MN \le MA_i + NA_i \text{ for all } i \in \{1, 2, \ldots, 2002\},$$

and consequently

$$MA_1 + MA_2 + \ldots + MA_{2002} + NA_1 + NA_2 + \ldots + NA_{2002} \ge 4004.$$

For another diameter P_1P_2 of the same circle we obtain similarly

$$P_1A_1 + P_1A_2 + \ldots + P_1A_{2002} + P_2A_1 + P_2A_2 + \ldots + P_2A_{2002} \ge 4004.$$

The point P is one of the points P_1 or P_2 for which

$$P_1A_1 + P_1A_2 + \ldots + P_1A_{2002} \ge 2002 \text{ or } P_2A_1 + P_2A_2 + \ldots + P_2A_{2002} \ge 2002.$$

Thus M, N, P are the three required points.

Chapter 8

Training Problems Formal Solutions

132. Let a, b, c, d be positive real numbers with $a + b + c + d = 1$. Prove that:

$$\frac{bcd}{a+2} + \frac{acd}{b+2} + \frac{abd}{c+2} + \frac{abc}{d+2} < \frac{1}{13}.$$

Solution. By the AM-GM inequality we have

$$abc \le \left(\frac{a+b+c}{3}\right)^3,$$

hence

$$\frac{abc}{d+2} \le \left(\frac{a+b+c}{3}\right)^3 \frac{1}{d+2} < \left(\frac{a+b+c+d}{3}\right)^3 \frac{1}{d+2}$$
$$= \frac{1}{27}\frac{1}{d+2} < \frac{1}{27 \cdot 2}.$$

Therefore,

$$\frac{bcd}{a+2} + \frac{acd}{b+2} + \frac{abd}{c+2} + \frac{abc}{d+2} < \frac{4}{27 \cdot 2} < \frac{1}{13},$$

as required.

133. Find all non-empty subsets $A \subset \mathbb{R}^*$ with the properties:

i) A has at most 5 elements;

ii) If $x \in A$ then $\frac{1}{x} \in A$ and $1 - x \in A$.

Solution. Let $x \in A$. Hence $\frac{1}{x} \in A$ and $1 - x \in A$. Next, $1 - \frac{1}{x} = \frac{x-1}{x} \in A$ and $\frac{1}{1-x} \in A$. Furthermore, $\frac{1}{\frac{x-1}{x}} = \frac{x}{x-1} \in A$.

Since A has at most 5 elements, two of the numbers $x, \frac{1}{x}1-x, \frac{x-1}{x}, \frac{1}{1-x}$ and $\frac{x}{x-1}$ has to be equal.

Considering all cases yields $x \in \left\{1, -1, 0, 2, \frac{1}{2}\right\}$. The values $x = 0$ and $x = 1$ do not satisfy the second condition.

It is easy to check that $A = \left\{-1, \frac{1}{2}, 2\right\}$ is the only solution.

134. Let ABC be a triangle and let D, E be the points in the exterior of the triangle such that triangles ABD and ACE are isosceles and right-angled at B and C respectively.

Prove that the lines CD and BE meet on the altitude from A in the triangle ABC.

Solution. Let D', F, E' be the projections of the points D, A, E on the line BC. The triangles $DD'B$ and BFA are congruent, since $\angle D' = \angle F = 90°$, $AB = BD$ and $\angle DBD' = 90° - \angle ABF = \angle BAF$. It follows that $DD' = BF$ and $D'B' = FA$. Similarly, $EE' = CF$ and $E'C = FA$.

Denote M and P the intersection points of the line AF with the lines BE and CD respectively.

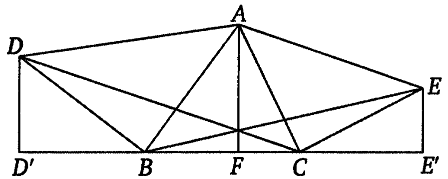
As $MF \parallel EE'$, we have

$$\frac{MF}{EE'} = \frac{BF}{BE'} \text{ hence } MF = \frac{EE' \cdot BF}{BE'} = \frac{FC \cdot BF}{BC + AF}. \tag{1}$$

Since $PF \parallel DD'$, it follows that

$$\frac{PF}{DD'} = \frac{CF}{CD'} \text{ hence } PF = \frac{FC \cdot DD'}{CD'} = \frac{FC \cdot BF}{BC + AF}. \tag{2}$$

The relations (1) and (2) shows that $M = P$ thus CD, BE, AF are concurrent, as desired.



135. Consider a parallelogram $ABCD$ such that $\angle ACB = 80°$ and $\angle ACB = 20°$. A line passing through B meets the line AB at an angle of 20° and intersects the line AC in the point R. A line passing through C meets the line AC at an angle of 30°an intersects the line AB in the point T.

Find the measure of the angle determined by the lines TR and DC.

Solution. Consider the point K on the diagonal AC such that $\angle CBK = 20°$. Hence the triangle BKC is isosceles with $BK = BC$.

Moreover,

$$\angle BCT = 80° - 30° = 50°$$

and

$$\angle BTC = 180° - 50° - \angle ABC = 130° - 80° = 50°,$$

so

$$BC = BT,$$

Now

$$\angle TBK = \angle TBC - \angle CBK = 80° - 20° = 60°$$

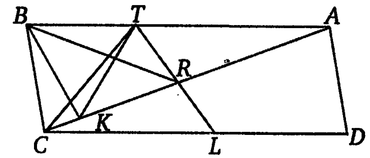
and

$$BK = BT,$$

so we infer that KBT is an equilateral triangle and $BK = TK$.

In the triangle BKR, we have $\angle BKR = 180° - \angle BKC = 100°$ and $\angle KBR = \angle KBT - \angle RBA = 60° - 20° = 40°$, so $\angle BRK = 40°$ and consequently

$$BK = RK.$$



Furthermore, $\angle RKT = \angle RKB - \angle TKB = 100° - 60° = 40°$ and

$$TK = RK,$$

therefore $\angle KTR = \angle KRT = \frac{1}{2}(180° - 40°) = 70°$. Finally, $\angle TLC = \angle TRC - \angle ACD = 70° - 20° = 50°$, so the angle between the lines TR and CD is equal to 50°.

136. Find the cube of the number $N = \sqrt{7\sqrt{3\sqrt{7\sqrt{3\sqrt{7\sqrt{3 \ldots}}}}}}$

Solution. We have $N^4 = 7^2 \cdot 3N$ and $N \neq 0$, hence $N^3 = 147$.

137. Prove that for any non-negative integer n the number

$$A = 2^n + 3^n + 5^n + 6^n$$

is not a perfect cube.

Solution. We will use modular arithmetic. A perfect cube has the form $\mathfrak{M}7$, $\mathfrak{M}7 + 1$ or $\mathfrak{M}7 - 1$, since

$$(7x+1)^3 \equiv (7x+2)^3 \equiv (7x+4)^3 \equiv 1 (\bmod 7),$$

and

$$(7x+3)^3 \equiv (7x+5)^3 \equiv (7x+6)^3 \equiv -1 (\bmod 7).$$

Now observe that

$$\begin{aligned} 2^6 &= 4^3 \equiv 1 (\bmod 7); \\ 3^6 &= 9^3 \equiv 2^3 \equiv 1 (\bmod 7); \\ 5^6 &= (-2)^6 = 2^6 \equiv 1 (\bmod 7); \\ 6^6 &\equiv (-1)^6 \equiv 1 (\bmod 7). \end{aligned}$$

It follows that $2^{6k} \equiv 3^{6k} \equiv 5^{6k} \equiv 6^{6k} \equiv 1 (\bmod 7)$.

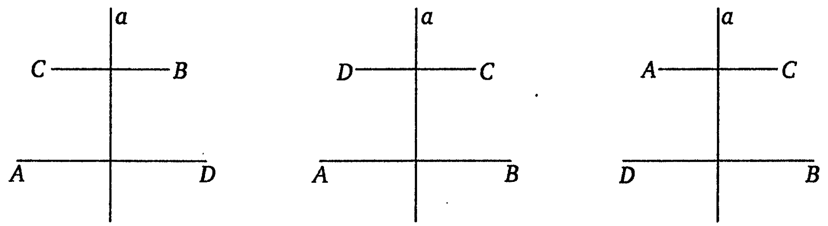
Denote $a_n = 2^n + 3^n + 5^n + 6^n$ for any integers $n \geq 0$. Set $n = 6k + r$, with $r \in \{0, 1, 2, 3, 4, 5, 6\}$. As $2^n \equiv 2^r (\bmod 7)$, $3^n \equiv 3^r (\bmod 7)$, $5^n \equiv 5^r (\bmod 7)$, and $6^n \equiv 6^r (\bmod 7)$ we have $a_n \equiv a_r (\bmod 7)$.

It is easy to observe that $a_0 \equiv a_2 \equiv a_6 \equiv 4 (\bmod 7), a_1 \equiv a_4 \equiv 2 (\bmod 7)$ and $a_3 \equiv 5 (\bmod 7)$. Therefore, a_n is not a perfect cube.

138. The points A, B, C are the vertices of a triangle with no equal sides. How many points D exist such that the set $\{A, B, C, D\}$ has a symmetry axis?

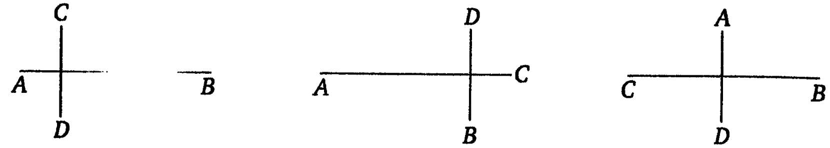
Solution. Let a be the symmetry axis of the set $\{A, B, C, D\}$. We consider two cases:

1. None of the points A, B, C, D lies on the line a. First, consider that D and A are symmetric with respect to the line a. Then B and C are also symmetric with respect to the line a. In other words, D is the reflection of A with respect to the perpendicular bisector of the line segment BC. Thus we have three possibilities to choose such a point D, except for the case when the triangle ABC is right-angled. In this situation we have only 2 solutions, since the reflections across the perpendicular bisectors of the legs of the right-angled triangle produce the same point D.



i) ii) iii)

2. The line a passes through a vertex of the triangle ABC. Suppose that A lies on the line a. The reflection of A across a is obviously the point A. The points B and C are not symmetric with respect to a, since $AB \neq AC$. Because the point D cannot be the symmetric point of both B and C across a, it follows that $B \in a$ or $C \in a$. Consider the case $B \in a$; that is $a = AB$. Now reflect C across AB and find D. Since a can be AC, AB or BC, we infer that there are three possible choices of the point D.

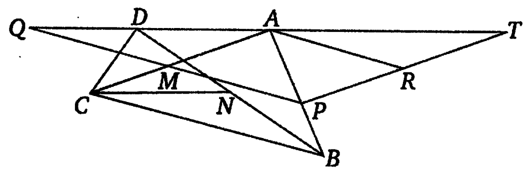


i) ii) iii)

Consequently there are 5 locations for a point D with the desired property if ABC is a right-angled triangle; otherwise there are 6 possibilities.

139. A cyclic quadrilateral $ABCD$ is given. On the rays $(AB$ and $(AD$ the points P and Q are considered so that $AP = CD$ and $AQ = BC$.
The lines PQ and AC meet at point M and N is the midpoint of the segment BD.
Prove that $PM = MQ = CN$.

Solution. Let T be a point on the line AQ such that $AT = AQ = BC$. The quadrilateral $ABCD$ is cyclic, so the angles $\angle DCD$ and $\angle PAT$ are congruent. Since $DC = AP$ and $CB = AT$ we deduce that the triangles DCB and PAT are congruent, hence $\angle PTA = \angle CBD$.



Furthermore, the angles $\angle CBD$ and $\angle CAD$ are congruent since $ABCD$ is cyclic. Then $\angle PTA \equiv \angle CAD$ and consequently the lines PT and AC are parallel. In the triangle QTP, AM is the middle line, so $PM = MQ$.

Let R be the midpoint of the segment PT. The medians AR and CN correspond to the congruent sides of the triangles PAT and DCB, hence they are also congruent. In the triangle TQP, AR is the middle line, so $CN = AR = QM = MP$. Therefore $CN = PM = QM$, as desired.

140. Solve in positive integers the equation

$$x^y \cdot y^x + x^y + y^x = 5329.$$

Solution. The equation is equivalent to

$$(y^x + 1)(y^x + 1) = 5330.$$

Factorizing the number 5330, we obtain

$$1 \cdot 5330 = 5 \cdot 1066 = 10 \cdot 533 = 13 \cdot 410 = 26 \cdot 205 = 41 \cdot 130 = 65 \cdot 82 = 2 \cdot 2665.$$

In the first six cases we find no solutions.
If $(x^y + 1)(y^x + 1) = 65 \cdot 82$; then:

a) $\begin{cases} x^y + 1 = 65 \\ y^x + 1 = 82 \end{cases} \Leftrightarrow \begin{cases} x^y = 64 \\ y^x = 81 \end{cases} \Leftrightarrow \begin{cases} x = 4 \\ y = 3 \end{cases},$

or

b) $\begin{cases} x^y + 1 = 82 \\ y^x + 1 = 65 \end{cases} \Leftrightarrow \begin{cases} x^y = 81 \\ y^x = 64 \end{cases} \Leftrightarrow \begin{cases} x = 3 \\ y = 4 \end{cases}.$

Finally, if $(x^y+1)(y^x+1) = 2 \cdot 2665$ we obtain $x = 1$, $y = 2664$ or $y = 1$, $x = 2664$.
Thus

$$(x, y) \in \{(3, 4), (4, 3), (1, 2664), (2664, 1)\}.$$

141. Find all the positive integers n for which the number obtained by erasing the last digit is a divisor for n.

Solution. Let b be the last digit of the number n and let a be the number obtained from n by erasing the last digit b.Then $n = 10a + b$.

Since a is a divisor of n, we infer that a divides b. Any number n that ends in 0 is therefore a solution. If $b \neq 0$, then a is a digit and n is one of the numbers 11, 12, ..., 19, 22, 24, 26, 28, 33, 36, 39, 44, 48, 55, 56, 77, 88 or 99.

142. Prove that a quadrilateral $ABCD$ with

$$\text{area}\,[ABC] \leq \text{area}\,[BCD] \leq \text{area}\,[CDA] \leq \text{area}\,[ABD]$$

is a trapezoid.

Solution. Let O be the intersection point of the diagonals AC and BD. Since area$[ABC]$ $\leq$area$[BCD]$, we have

$$\text{area}\,[AOB] + \text{area}\,[BOC] \leq \text{area}\,[BOC] + \text{area}\,[DOC],$$

thus

$$\text{area}\,[AOB] \leq \text{area}\,[DOC]. \qquad (1)$$

From area$[CDA]$ $\leq$area$[ABD]$ we deduce similarly that

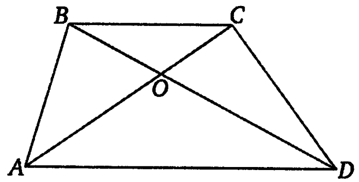
$$\text{area}\,[DOC] \leq \text{area}\,[AOB]. \qquad (2)$$

Therefore

$$\text{area}\,[DOC] = \text{area}\,[AOB] \qquad (3)$$

Adding area$[BOC]$ in both sides of the relation (3) yields

$$\text{area}\,[BCD] = \text{area}\,[CAB].$$



The triangles ABC and DBC have the same area and a common side BC, hence the altitudes from A and D are congruent. It follows that $AD \parallel BC$, as desired.

143. Inside a rectangle of area 5 are given 9 polygons each of area 1. Prove that there exists 2 of them with the common area not less then $\frac{1}{9}$.

Solution. Let $T_1, T_2, \ldots, T_9$ be the nine polygons, each having the area 1. Suppose, by way of contradiction, that any two of the polygons T_i have a common area which is less than $\frac{1}{9}$. Then, the area of the polygon T_2 which is not inside T_1 is greater than $1 - \frac{1}{9} = \frac{8}{9}$.

Furthermore, the area of the polygon T_3 which is not covered by T_1 and T_2 is greater than $1 - \frac{1}{9} - \frac{1}{9} = \frac{7}{9}$.

On this line of reasoning we find in the end that the area of the polygon T_9, which is not included in the union of $T_1, T_2, \ldots, T_8$ is at least $1 - 8\frac{1}{9} = \frac{1}{9}$.

Consequently the area covered by all 9 polygons $T_1, T_2, \ldots, T_9$ is at least $1 + \frac{8}{9} + \frac{7}{9} + \ldots + \frac{2}{9} + \frac{1}{9}$, hence is greater than the area of the rectangle which contains the polygons $T_1, T_2, \ldots, T_9$, a contradiction.

144. Prove that for any real numbers a and b there are numbers $x, y \in [0, 1]$ such that

$$|xy - ax - by| \geq \frac{1}{3}.$$

Solution. Suppose by contradiction that there are real numbers a and b such that

$$|xy - ax - by| < \frac{1}{3}.$$

for any $x, y \in [0, 1]$.
For $x = 0$ and $y = 1$ we obtain $|b| < \frac{1}{3}$.
For $x = 1$ and $y = 0$ we infer that $|a| < \frac{1}{3}$.
Setting $x = 1$ and $y = 1$ yields $|1 - a - b| < \frac{1}{3}$.
Therefore $|1 - a - b| \geq 1 - |a| - |b| > 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$, a contradiction.

145. Find the greatest number that can be written as a product of some positive integers with the sum 1976.

Solution. Let $x_1, x_2, \ldots, x_n$ be the numbers having the sum $x_1+x_2+\ldots+x_n = 1976$ and the maximum value of the product $x_1 \cdot x_2 \cdot \ldots \cdot x_n = p$.

If one of the numbers, say x_1, is equal to 1, then $x_1 + x_2 = 1 + x_2 > x_2 = x_1x_2$. Hence the product $(x_1+x_2)\cdot x_3 \cdot \ldots \cdot x_n$ is greater than $x_1 \cdot x_2 \cdot \ldots \cdot x_n = p$, false. Therefore $x_k \geq 2$ for all k.

If one of the numbers is equal to 4 we can replace him with two numbers 2 without changing the sum or the product.

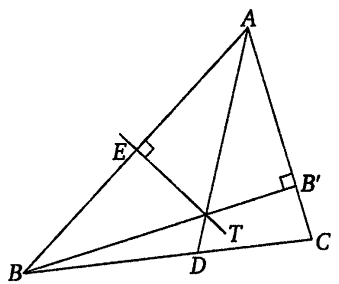
Suppose that $x_k \geq 5$ for some $k = \overline{1,n}$. Then $x_k < 3(x_k-3)$, so replacing the number x_k with the numbers 3 and $x_k - 3$, the sum remains constant while the product increases, contradiction.

Therefore all the numbers are equal to 2 or 3. If there are more than 3 numbers equal to 2, we can replace them by two numbers equal to 3, preserving the sum and increasing the product (as $2\cdot 2\cdot 2 < 3\cdot 3$). Hence at most two terms equal to 2 are allowed. Since $1976 = 3\cdot 658 + 2$ the maximum product is equal to $2\cdot 3^{658}$.

146. An acute triangle ABC is given. Prove that the internal bisector of angle $\angle BAC$, the altitude from B and the perpendicular bisector of the line segment AB are concurrent if and only if $\angle A = 60°$.

Solution. Let AD be the bisector line of the angle $\angle BAC$ and let BB' be the altitude from B. The lines AD and BB' meet at M and E is the midpoint of the side AB.

First, we prove that if AD, BB' and the perpendicular bisector of the segment AB are concurrent, then $\angle A = 60°$. We have that ME is the perpendicular bisector of AB, so $AM = AB$ and $\angle MBA = \angle MAB$. On the other hand, $\angle MAB = \angle MAC$, hence $\angle MBA = \angle MAC$. Moreover, $\angle AMB' = \angle MAB + \angle MBA = 2\angle B'AM$. Summing the angles of the triangle AMB' we obtain $3\angle MAB' + 90° = 180°$, so $\angle MAB' = 30°$, and consequently $\angle A = 60°$, as desired.



Conversely, we prove that if $\angle A = 60°$, then M lies on the perpendicular bisector of the side AB. Since $\angle ABB' = 90° - \angle A = 30°$ and consequently $\angle MAB = \frac{1}{2}\angle A = 30°$, it follows that the triangle MAB is isosceles, hence $MA = MB$.

Therefore EM is the perpendicular bisector of AB, as desired.

147. The points M, K, L are considered respectively on the sides AB, BC, AC of a triangle ABC. Prove that at least one of the areas of the triangles MAL, KBM or LCK is not less than a quarter of the area of the triangle ABC.

Solution. We have

$$\text{area}\,[ABC] = \frac{AB\cdot AC\cdot \sin A}{2}; \quad \text{area}\,[AML] = \frac{AM\cdot AL\cdot \sin A}{2},$$

hence

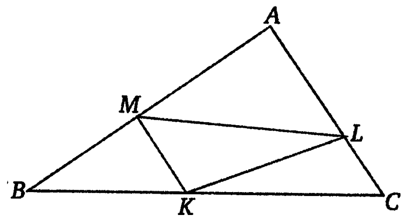
$$\frac{\text{area}\,[AML]}{\text{area}\,[ABC]} = \frac{AM\cdot AL}{AB\cdot AC}. \tag{1}$$

Likewise,

$$\frac{\text{area}\,[BMK]}{\text{area}\,[ABC]} = \frac{BM\cdot BK}{BA\cdot BC} \tag{2}$$

and

$$\frac{\text{area}\,[CLK]}{\text{area}\,[ABC]} = \frac{CL\cdot KC}{CA\cdot CB}. \tag{3}$$



Suppose by contradiction that $\frac{\text{area}[AML]}{\text{area}[ABC]} > \frac{1}{4}$, $\frac{\text{area}[BMK]}{\text{area}[ABC]} > \frac{1}{4}$ and $\frac{\text{area}[CLK]}{\text{area}[ABC]} > \frac{1}{4}$. Multiplying the relations (1), (2) and (3) leads us to

$$\frac{AM\cdot AL}{AB\cdot AC}\cdot\frac{BM\cdot BK}{BA\cdot BC}\cdot\frac{CL\cdot KC}{CA\cdot CB} > \frac{1}{64},$$

then

$$\frac{AL\cdot CL}{AC\cdot AC}\cdot\frac{BM\cdot AM}{AB\cdot AB}\cdot\frac{BK\cdot CK}{BC\cdot BC} > \frac{1}{64}. \tag{4}$$

By $AM-GM$ inequality, $\sqrt{AL\cdot CL} \leq \frac{AL+CL}{2} = \frac{AC}{2}$, hence

$$\frac{AL\cdot CL}{AC\cdot AC} \leq \frac{1}{4}$$

and similarly,

$$\frac{AM\cdot BM}{AB\cdot AB} \leq \frac{1}{4} \text{ and } \frac{BK\cdot CK}{BC\cdot BC} \leq \frac{1}{4}.$$

Multiplying these inequalities we obtain a contradiction with the relation (4).

148. Find all the integers x, y, z so that $4^x + 4^y + 4^z$ is a square.

Solution. Without loss of generality assume that $x \le y \le z$ and let $4^x + 4^y + 4^z = u^2$. Then $2^{2x}\left(1 + 4^{y-x} + 4^{z-x}\right) = u^2$ and so $1 + 4^{y-x} + 4^{z-x} = (1+2a)^2$. It follows that

$$4^{y-x-1} + 4^{z-x-1} = a(a+1) \text{ and then } 4^{y-x-1}\left(1 + 4^{z-y}\right) = a(a+1).$$

We consider two cases.

1° The number a is even. Then $a+1$ is odd, so $4^{y-x-1} = a$ and $1 + 4^{z-y} = a+1$. It follows that $4^{y-x-1} = 4^{z-y}$, hence $y - x - 1 = z - y$. Thus $z = 2y - x - 1$ and

$$4^x + 4^y + 4^z = 4^x + 4^y + 4^{2y-x-1} = \left(2^x + 2^{2y-x-1}\right)^2.$$

2° The number a is odd. Then $a+1$ is even, so $a = 4^{z-y} + 1, a + 1 = 4^{y-x-1}$ and $4^{y-x-1} - 4^{z-y} = 2$. It follows that $2^{2y-2x-3} = 2^{2z-2y-1} + 1$, which is impossible since $2x - 2y - 1 \neq 0$.

149. Find all the primes a, b, c such that

$$ab + bc + ac > abc.$$

Solution. Assume that $a \le b \le c$. If $a \ge 3$ then $ab + bc + ac \le 3bc \le abc$, a contradiction. Since a is prime it is left that $a = 2$.

The inequality becomes $2b + 2c + bc > 2bc$, hence $\frac{1}{c} + \frac{1}{b} > \frac{1}{2}$.

If $b \ge 5$, then $c \ge 5$ and
$$\frac{1}{2} < \frac{1}{b} + \frac{1}{c} < \frac{1}{5} + \frac{1}{5} = \frac{2}{5},$$
false.

Therefore $b \le 5$, that is

1° $b = 2$ and c is any prime;

2° $b = 3$ and c is 3 or 5.

150. Five points are given inside of an equilateral triangle of side length 1. Prove that there exist 2 points at a distance less than $\frac{1}{2}$.

Solution. Divide the triangle into five equilateral triangles of side length $\frac{1}{2}$ by drawing the middle lines. Among the five given points, at least two of them will be in the interior or on the sides of one of these 4 triangles. The distance between them is less than $\frac{1}{2}$, so we are done.

151. Let $A_1A_2 \ldots A_n$ be a regular polygon, $n \ge 3$. Find the number of obtuse triangles $A_iA_jA_k$.

Solution. We consider two cases.

i) The number n is even. We will evaluate the number of triangles with the vertex

A_1, and $\angle A_1 > 90°$. These are the triangles $A_1A_jA_k$ with $j < k$ and $k - j > \frac{n}{2}$, so we have to count the number of pairs (j, k) such that $2 \le j < k \le n$ and $k - j > \frac{n}{2}$. For a number j between 2 and $\frac{n}{2} - 1$, there are $\frac{n}{2} - j$ possible values of the number k. Hence the total number of the pairs is equal to

$$\left(\frac{n}{2} - 2\right) + \left(\frac{n}{2} - 3\right) + \ldots + 1 = \frac{1}{2}\left(\frac{n}{2} - 2\right)\left(\frac{n}{2} - 1\right) = \frac{1}{8}(n-2)(n-4).$$

Therefore, the number of triangles obtuse at A_1 is $\frac{(n-2)(n-4)}{8}$ and the number of obtuse triangles is equal to $\frac{n(n-2)(n-4)}{8}$.

ii) The number n is odd. On the same line of reasoning, we count the triangles $A_1A_jA_k$ obtuse at A_1. That is the number of pairs (j, k) with $2 \le j < k \le n$ and $k - j > \frac{n-1}{2}$, which is

$$\frac{n-3}{2} + \frac{n-5}{2} + \ldots + 1 = \frac{(n-1)(n-3)}{8}.$$

Finally, the total number of obtuse triangles is

$$\frac{n(n-1)(n-3)}{8}.$$

152. Find all the positive integers x, y, z, t so that $x + y + z = xyzt$.

Solution. Assume that $x \le y \le z$ the equation is equivalent to

$$\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = t, \text{ so it is obvious that } t \le 3.$$

We consider three cases.

1) If $t = 3$, then $x = y = z = 2$.

2) If $t = 2$, then $x = 1$. Indeed, if $2 \le x \le y \le z$ then

$$2 = \frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz} \le \frac{3}{4}, \text{ a contradiction.}$$

Moreover $y = 1$, since $2 \le y \le z$ implies

$$2 = \frac{1}{y} + \frac{1}{yz} + \frac{1}{z} \le \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2} = \frac{5}{4}, \text{ false.}$$

It remains $2 = 1 + \frac{2}{z}$, so $z = 2$.

3) If $t = 1$, then $x = 1$, otherwise

$$1 \le \frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz} \le \frac{3}{4},$$

as shown before. If $y \geq 3$, then $z \geq 3$ and

$$1 = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 3} + \frac{1}{3 \cdot 1} = \frac{7}{9}, \text{ false.}$$

It follows that $y \leq 2$. The case $y = 1$ leads to contradiction, so $y = 2$ and

$$1 = \frac{1}{2} + \frac{1}{2z} + \frac{1}{z}.$$

Finally, $z = 3$. The solutions are

$$(x, y, z, t) \in \{(1, 1, 1, 3), (1, 2, 3, 1), (1, 3, 2, 1), (1, 3, 2, 1), (3, 1, 2, 1), (2, 3, 1, 1), (3, 2, 1, 1)\}$$

153. Find all the positive integers n for which the set

$$\{n, n+1, n+2, n+3, n+4, n+5\}$$

can be decomposed in two disjoint subsets such that the product of elements in these subsets are equal.

Solution. We prove that no such numbers $n \geq 0$ exist. For $n = 0$ this is obvious, so assume that $n \geq 1$.

First, observe that if an element of the set $E = \{n, n+1, n+2, n+3, n+4, n+5\}$ is divisible by a prime number p, at least another number must be divisible by p. Among 6 consecutive numbers there is a multiple of 5, so there must be two of them. The only possibility is to have n and $n+5$ divisible by 5, so $n \geq 5$.

Now observe that any element of E is less than any product of two numbers from E.

For this, it suffices to show that $n(n+1) > n+5$ which is obviously for $n \geq 5$. Consequently, the subsets of E must have three elements each and the numbers n and $n+5$ are not in the same subset. We have the following cases.

a) $n(n+1)(n+2)$ and $(n+3)(n+4)(n+5)$;

b) $n(n+1)(n+3)$ and $(n+2)(n+4)(n+5)$;

c) $n(n+1)(n+4)$ and $(n+2)(n+3)(n+5)$;

d) $n(n+2)(n+3)$ and $(n+1)(n+4)(n+5)$;

e) $n(n+2)(n+4)$ and $(n+1)(n+3)(n+5)$;

f) $n(n+3)(n+4)$ and $(n+1)(n+2)(n+5)$.

In the first 5 cases, the product of the elements from the first subset is less than the product of the elements from the second one.

In the last case, the equality $n(n+3)(n+4) = (n+1)(n+2)(n+5)$ leads to $n^2 + 5n + 10 = 0$, which has no integer solution. The proof is complete.

154. Prove that in any tetrahedron there is a vertex such that the edges arising from it are the sides of a triangle.

Solution. Let AB be the greatest edge of the tetrahedron $VABC$. Applying the triangle inequality yields

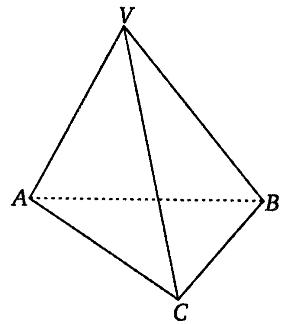
$$AV + BV > AB \tag{1}$$

and

$$AC + BC > AB, \tag{2}$$

hence

$$AV + AC + BC + BV > 2AB. \tag{3}$$



Suppose that AB, AC, AV cannot be the length of the sides of a triangle. Then

$$AB \geq AV + AC. \tag{4}$$

From the inequalities (3) and (4) we infer that $AB < BC + BV$, thus AB, BC and BV are the lengths of a triangle.

155. Let $ABCD$ be a convex quadrilateral and let E and T be the midpoints of the sides BC and CD respectively. If $AE + AT = 4$, prove that the area of the quadrilateral $ABCD$ is less than 8.

Solution. We use the fact that a median divides a triangle in two triangles having the same area. As AE and AT are medians in the triangles ABC and ADC, we have

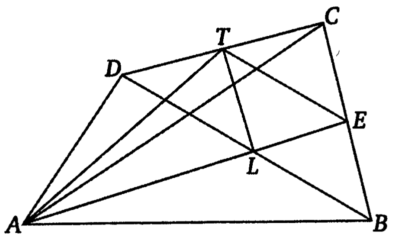
$$\text{area}[ABC] = 2\text{area}[AEC]$$

and

$$\text{area}[ADC] = 2\text{area}[ATC],$$

hence

$$\text{area}[ABC] + \text{area}[ADC] = 2(\text{area}[AEC] + \text{area}[ATC]) = 2\text{area}[AECT]. \tag{1}$$



Let L be the intersection point of the lines AE and BD. Since ET is the middle line of the triangle BCD, then $ET \parallel BD$ and consequently the altitudes from C and L in the triangles CTE and LTE are congruent. Thus

$$\text{area}\,[ECT] = \text{area}\,[LTE]$$

and

$$\begin{aligned}\text{area}\,[AECT] &= \text{area}\,[ATE] + \text{area}\,[CTE] \\ &= \text{area}\,[ATE] + \text{area}\,[LTE] < 2\text{area}\,[AET]\,.\end{aligned} \tag{2}$$

Set $AT = x$, then $AE = 4 - x$ and

$$2area\,[AET] = x\,(4-x)\sin\angle EAT \le x\,(4-x) \le 4 \tag{3}$$

Combining (1), (2) and (3) gives

$$\text{area}\,[ABCD] = 2\text{area}\,[AECT] < 2\cdot 2\text{area}\,[AET] < 4\cdot 2 = 8,$$

as desired.

156. A number x is formed using the digits 1, 2, 3, 4, 5, 6, 7 once and only once. Rearranging the digits we obtain a number y. Prove that y is not a divisor of x.

Solution. The sum of the digits of the numbers x and y is 28, hence x and y have the form $\mathfrak{M}9+1$. If $y = kx$ for some integer k, then $k \in \{2,3,4,5,6\}$. This is impossible, as $\mathfrak{M}9 + 1 \neq \mathfrak{M}9 + k$.

157. Let x, y, z be distinct integers such that $xy + yz + xz = 26$.
Prove that $x^2 + y^2 + z^2 \ge 29$.

Solution. Assume that $x < y < z$. Then $y - x \ge 1$, $z - y \ge 1$, $z - x \ge 2$, hence

$$(x-y)^2 + (y-z)^2 + (z-x)^2 \ge 6.$$

Furthermore,

$$x^2 + y^2 + z^2 - xy - yz - zx \ge 3,$$

and since

$$xy + yz + zx = 26,$$

we obtain

$$x^2 + y^2 + z^2 \ge 29,$$

as needed.

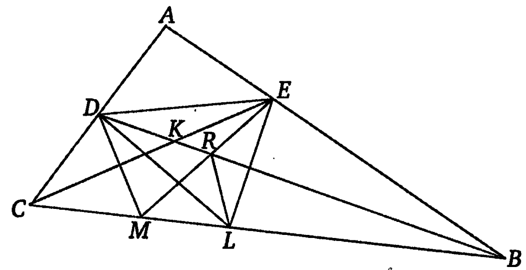
158. Inside a unit square lies a convex polygon of area greater than $\frac{1}{2}$. Prove that there is a line d parallel with one of the sides of the square that cuts from the polygon a line segment of length greater than or equal to $\frac{1}{2}$.

Solution. Draw from all the vertices of the polygon parallel lines to the same side of the square, dividing the polygon in several triangles, trapezoids or rectangles. Assume by contradiction that all the segments determined by these lines and the polygon have the length less than $\frac{1}{2}$. The sum of the altitudes of all the regions created from the polygon is less than 1, hence the area of the polygon is less than $\frac{1}{2}$, a contradiction.

159. A triangle ABC is considered. The internal bisectors of the angles $\angle ABC$ and $\angle ACB$ intersects the sides AC and AB in the points D and E, respectively. Find the angles of the triangle ABC if $\angle BDE = 24°$ and $\angle CED = 18°$.

Solution. Let K be the intersection point of the lines BD and CE. Then

$$\angle KCB + \angle KBC = 24° + 18° = 42°.$$



We can find the measure of the angle A

$$\angle A = 180° - 2\,(\angle KBC + \angle KCB) = 180° - 2\cdot 42° = 96°.$$

Let M be the reflection of D across the line CE and let L be the reflection of E across the line BD. Since CE and BD are angular bisectors of $\angle C$ and $\angle B$, it follows that the points M and L are located on the line BC. Let the line BD meets EM at R. Then

$$\angle ERB = \angle RED + \angle EDR = 2\cdot 18° + 24° = 60°.$$

Consequently,

$$\angle BRL = 60°,\ \angle MRL = 180° - \angle ERL = 60°$$

and

$$\angle LDM = \angle EDM - \angle EDL = 90° - 18° - 2\cdot 24° = 24° = \angle RDL.$$

Thus L is the excenter of the triangle RDM and

$$\angle DMC = \angle RML = \frac{\angle DMC + \angle RML}{2} = \frac{\angle DEM + \angle EDM}{2} = 18^\circ + 36^\circ = 54^\circ.$$

Finally, note that

$$\angle ACB = 180^\circ - \angle DMC \cdot 2 = 72^\circ$$

and

$$\angle ABC = 180^\circ - (96^\circ + 72^\circ) = 12^\circ.$$

160. Let $N = \underbrace{44\ldots4}_{2002}\underbrace{88\ldots8}_{2001}9$. Calculate $\sqrt{N}$.

Solution. We have

$$\begin{aligned} N &= \underbrace{44\ldots4}_{2002}\underbrace{88\ldots8}_{2001}9 = 4\cdot \underbrace{11\ldots1}_{2002}\cdot 10^{2002} + 8\cdot \underbrace{11\ldots1}_{2001}\cdot 10 + 9 \\ &= 4\left(10^{2001} + 10^{2000} + \ldots + 10 + 1\right)\cdot 10^{2002} \\ &\quad +8\cdot\left(10^{2000} + 10^{1999} + \ldots + 10 + 1\right)\cdot 10 + 9 \\ &= 4\cdot\frac{10^{2002}-1}{9}\cdot 10^{2002} + 8\cdot\frac{10^{2001}-1}{9}\cdot 10 + 9 \\ &= \frac{4}{9}\left(10^{4004} - 10^{2002}\right) + \frac{8}{9}\left(10^{2002} - 10\right) + \frac{81}{9} \\ &= \frac{4\cdot 10^{4004} - 4\cdot 10^{2002} + 8\cdot 10^{2002} - 80 + 81}{9} \\ &= \left(\frac{2\cdot 10^{2002}+1}{3}\right)^2, \end{aligned}$$

thus

$$\sqrt{N} = \sqrt{\left(\frac{2\cdot 10^{2002}+1}{3}\right)^2} = \frac{2\cdot 10^{2002}+1}{3} = \underbrace{66\ldots6}_{2001}7.$$

161. Numbers $x_1, x_2, \ldots, x_n$ are chosen from the interval $[2, 4]$ such that

$$x_1 + x_2 + \ldots + x_n = \frac{17n}{6} \text{ and } x_1^2 + x_2^2 + \ldots + x_n^2 = 9n.$$

Prove that 12 divides n.

Solution. As $x_i \in [2, 4]$, we have $0 \le (x_i - 2)(4 - x_i) = 6x_i - x_i^2 - 8$, for all $i = \overline{1, n}$.

Summing these inequalities yields

$$6(x_1 + x_2 + \ldots + x_n) - \left(x_1^2 + x_2^2 + \ldots + x_n^2\right) - 8n \ge 0,$$

with equality when $x_i = 2$ or 4 for all $i = 1, 2, \ldots, n$.

Since

$$x_1 + x_2 + \ldots + x_n = \frac{17n}{6}$$

and

$$x_1^2 + x_2^2 + \ldots + x_n^2 = 9n,$$

note that

$$6\cdot\frac{17n}{6} - 9n - 8n = 0,$$

hence $x_i \in \{2, 4\}$ for all $i = 1, 2, \ldots, n$.

Thus $x_1 + x_2 + \ldots + x_n$ is even and $17n = 6(x_1 + x_2 + \ldots + x_n)$ is divisible by 12 and consequently n is a multiple of 12.

162. Inside a box of dimensions L, l and h are given $n^3 + 1$ points. Prove that there are two of them at a distance less than $\frac{\sqrt{L^2+l^2+h^2}}{n}$.

Solution. Divide each edge of the box into n equal segments, then divide the parallelepiped into n^3 boxes of dimensions $\frac{L}{n}$, $\frac{l}{n}$, $\frac{h}{n}$. By the Pigeonhole principle, at least two of the $n^3 + 1$ given points are inside of such a small box. Then the distance between these two points is less than the diagonal of this box, which is equal to $\sqrt{\frac{L^2+l^2+h^2}{n^2}}$.

163. A point M is given inside a triangle ABC. Let D, E, F be the projections of the point M onto the sides BC, CA, AB respectively. Find the minimum value of the sum

$$\frac{BC}{MD} + \frac{CA}{ME} + \frac{AB}{MF}.$$

Solution. We have

$$\text{area}\,[ABC] = \text{area}\,[MAC] + \text{area}\,[MAB] + \text{area}\,[MBC],$$

so

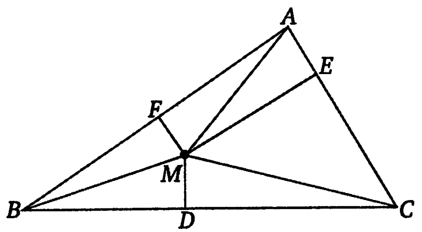
$$2\text{area}\,[ABC] = BC\cdot MD + AC\cdot ME + AB\cdot MF$$

By Cauchy-Schwarz inequality,

$$(BC\cdot MD + AC\cdot ME + AB\cdot MF)\left(\frac{BC}{MD} + \frac{CA}{ME} + \frac{AB}{MF}\right) \ge (AB + AC + BC)^2$$

hence

$$\frac{BC}{MD} + \frac{CA}{ME} + \frac{AB}{MF} \ge \frac{2p^2}{2S} = \frac{2p}{r},$$



where r, p, S denote the inradius, semiperimeter and area of the triangle ABC.

Therefore the minimum value of the sum is equal to $\frac{2p}{r}$ and it is obtained when

$$\frac{BC \cdot MD}{\frac{BC}{MD}} = \frac{AC \cdot ME}{\frac{AC}{ME}} = \frac{AB \cdot MF}{\frac{AB}{MF}}.$$

That is when $MD = MF = ME$; in other words, M is the incenter of the triangle ABC.

164. Let $a > b > 0$ be the real numbers such that $a^5 + b^5 = a - b$. Prove that $a^4 + b^4 < 1$.

Solution. As $a > b > 0$,then $a^5 + b^5 = a - b < a + b$. On the other hand,

$$a^5 + b^5 = (a + b)\left(a^4 + a^3b + a^2b^2 + ab^3 + b^4\right),$$

hence $a^4 + b^4 < a^4 + a^3b + a^2b^2 + ab^3 + b^4 < 1$, as needed.

165. Let $n > 2$ be an integer.. Prove that the number of irreducible fractions from the set $\left\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}\right\}$ is even.

Solution. We prove that if $\frac{k}{n}$ is irreducible, then $\frac{n-k}{n}$ is also irreducible. Indeed, $(k, n) = 1$ implies $(n - k, n) = 1$, as desired.

Moreover, the numbers $\frac{k}{n}$ and $\frac{n-k}{n}$ are distinct, otherwise $\frac{k}{n} = \frac{n-k}{n}$ yields $n = 2k$, and consequently $\frac{k}{n} = \frac{1}{2}$ is a reducible fraction. Thus we can pair the irreducible fraction from the set $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right\}$, proving the claim.

166. In the interior of a unit square are considered 129 points. Prove that there exists a disk of radius $\frac{1}{8}$ that contains at least three points.

Solution. We prove a more general claim:

If $2n^2 + 1$ are given inside a unit square, then there are three points inside a disk of radius $\frac{1}{n}$.

For this, divide the unit square into n^2 squares of side length $\frac{1}{n}$. By the Pigeonhole principle, there are three points inside or on the sides of one of these small squares. As a square of side $\frac{1}{n}$ can be covered by a disk of radius $\frac{1}{n}$, we are done.

167. Find all triangles with integer side lengths so that the semiperimeter has the same value as the area of the triangle.

Solution. Let a, b, c, p, S be the side lengths, the semiperimeter and the area of the triangle. The given condition is $p = S$ and by Heron's formula

$$S^2 = p\,(p - a)\,(p - b)\,(p - c)$$

we derive that

$$S = (S - a)\,(S - b)\,(S - c)\,. \tag{1}$$

Without loss of generality assume that $a \le b \le c$. Set $S - a = x$, $S - b = y$, $S - c = z$. Then $x \ge y \ge z > 0$ are integer numbers and

$$S = S - a + S - b + S - c = x + y + z\,.$$

The relation (1) is equivalent to

$$x + y + z = xyz, \tag{2}$$

and

$$y + z = x\,(yz - 1)\,. \tag{3}$$

Then $y\,(yz - 1) \le x\,(yz - 1) = y + z \le 2y$, hence $yz - 1 \le 2$ and $yz \le 3$.

As $z \le y$, we have $z^2 \le yz \le 3$, thus $z = 1$.

Furthermore,

$$x + y + 1 = xy \text{ and } y = \frac{x + 1}{x - 1} = 1 + \frac{2}{x - 1}.$$

It follows that $x - 1 \in \{1, 2\}$, then $x \in \{2, 3\}$.

1° If $x = 2$, then $y = 3 \ge x$, false.

2° For $x = 3$, we obtain $y = 2$ and $S = x + y + z = 6$.

Finally, $a = S - x = 3$, $b = S - y = 4$ and $c = S - z = 5$.

168. Let n and p be positive integers $n > 1$. Prove that the numbers $n - 1$ and $np + 1$ cannot have other divisors than the divisors of $p + 1$.

Solution. Let d be a common divisor of the numbers $n - 1$ and $np + 1$. Then d divides $n - 1 + np + 1 = n\,(p + 1)$ so $d = ab$, where a divides n and b divides $p + 1$. As $a \mid d$ and $d \mid n - 1$, then $a \mid n - 1$. Hence $a \mid (n, n - 1) = 1$, therefore $a = 1$ and $d = b$ is a divisor of $p + 1$, as claimed.

169. Find a relation between the numbers a, b, c if

$$x + \frac{1}{x} = a,\ \ y + \frac{1}{y} = b \text{ and } xy + \frac{1}{xy} = c.$$

Solution. We have

$$\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right) = \left(xy + \frac{1}{xy}\right) + \frac{x}{y} + \frac{y}{x},$$

hence

$$\frac{x}{y}+\frac{y}{x}=ab-c. \qquad (1)$$

On the other hand,

$$\left(\frac{x}{y}+\frac{y}{x}\right)\left(xy+\frac{1}{xy}\right)=x^2+y^2+\frac{1}{x^2}+\frac{1}{y^2}=\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2-4. \qquad (2)$$

From (1) and (2) we obtain the relation

$$a^2+b^2+c^2-abc=4.$$

170. Prove that in any polygon there are two sides with the length ratio greater then or equal to 1 and less then 2.

Solution. Let $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$ be the side lengths of the polygon. We prove by contradiction that there are two sides with the lengths ratio greater than or equal to 1 and less than 2.

If not, then

$$a_1 \geq 2a_2, a_2 \geq 2a_3, \ldots, a_{n-1} \geq 2a_n.$$

Summing these inequalities, we obtain

$$a_1 \geq a_2+a_3+\ldots+2a_n > a_2+a_3+\ldots+a_n.$$

This is a contradiction, since a_1, a_2, ..., a_n are the side lengths of a polygon.

171. Inside a unit cube 28 points are given. Prove that among them there are two points at a distance not greater than $\frac{\sqrt{3}}{3}$.

Solution. Divide naturally the unit cube in 27 cubes of side length $\frac{1}{3}$. By Pigeonhole Principle, at least two points from the 28 given ones are inside (or on the faces) of a small cube. The distance between these points is not greater than the diagonal of this cube, which is $\frac{\sqrt{3}}{3}$, as needed.

172. Find the last 5 digits of the number 5^{1981}.

Solution. First, we prove that $5^{1981} \equiv 5^5 \pmod{10^5}$. We have

$$\begin{aligned}
5^{1981}-5^5 &= \left(5^{1976}-1\right)5^5 = 5^5\left[\left(5^8\right)^{247}-1\right] = \mathfrak{M}\left[5^5\left(5^8-1\right)\right] \\
&= \mathfrak{M}\left[5^5\left(5^4-1\right)\left(5^4+1\right)\right] = \mathfrak{M}\left[5^5(5-1)(5+1)\left(5^2+1\right)\left(5^4+1\right)\right] \\
&= \mathfrak{M}5^5 2^5 = \mathfrak{M}100{,}000.
\end{aligned}$$

Therefore $5^{1981} = \mathfrak{M}100{,}000+5^5 = \mathfrak{M}100{,}000+3125$, so 03125 are the last 5 digits of the number 5^{1981}.

173. Compute the sum

$$S=\frac{2}{3+1}+\frac{2^2}{3^2+1}+\ldots+\frac{2^{n+1}}{3^{2^n}+1}.$$

Solution. For $x \neq 1$ we have

$$\frac{1}{x+1}=\frac{x-1}{x^2-1}=\frac{1}{2}\left[\frac{1}{x-1}+\frac{1}{x+1}\right]-\frac{1}{x^2-1},$$

so

$$\frac{1}{2(x+1)}=\frac{1}{2(x-1)}-\frac{1}{x^2-1}. \qquad (1)$$

Consequently,

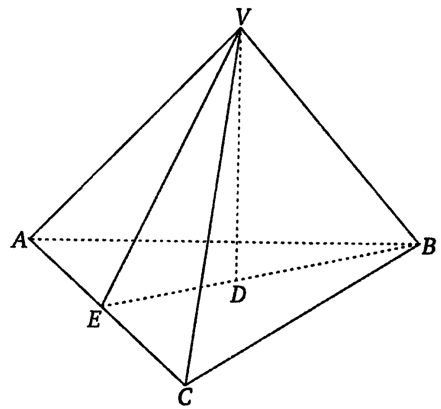
$$\frac{2^{k+1}}{3^{2^k}+1}=\frac{2^{k+1}}{3^{2^k}-1}-\frac{2^{k+2}}{3^{2^{k+1}}-1}. \qquad (2)$$

Using repeatedly the identity (2), we obtain

$$\begin{aligned}
S &= \left(\frac{2}{3-1}-\frac{2^2}{3^2-1}\right)+\left(\frac{2^2}{3^2-1}-\frac{2^3}{3^{2^2}-1}\right)+\ldots+\left(\frac{2^{n+1}}{3^{2^n}-1}-\frac{2^{n+2}}{3^{2^{n+1}}-1}\right) \\
&= \frac{2}{3-1}-\frac{2^{n+2}}{3^{2^{n+1}}-1}=1-\frac{2^{n+2}}{3^{2^{n+1}}-1}.
\end{aligned}$$

174. In a tetrahedron all the altitudes are congruent. One of them passes through the orthocenter of the corresponding face. Prove that the tetrahedron is regular.

Solution. The areas of the faces are equal, since all the altitudes of the tetrahedron have the same length. To fix the notation, let D be the orthocenter of the base ABC and let VD be an altitude of the tetrahedron. Let the lines BD and AC meet at point E. Then BD is an altitude in the triangle VAC. The areas of the triangles VAC and ABC are equal, hence $VE = BE$. The right-angled triangles VEC and BEC are congruent, and so $VA = AB$.



Analogously, we deduce that $CV = AC$, $VB = AB$, $VB = BC$ and $AC = VA$; in other words, all the edges of the tetrahedron have equal lengths, as claimed.

175. Let $ABCD$ be a parallelogram. On the sides BC and CD points E and F are chosen such that $\frac{EB}{EC} = a$ and $\frac{FC}{FD} = b$. Lines AE and BF intersect in the point M. Find the ratio $\frac{AM}{ME}$.

Solution. The parallel from C to BF intersects the line AB at Q. The lines AE and CQ intersects at T. We have

$$\frac{AM}{ME} = \frac{\frac{AM}{MT}}{\frac{ME}{MT}}. \tag{1}$$

Since $MB \parallel CQ$, we have

$$\frac{AM}{MT} = \frac{AB}{BQ} \tag{2}$$

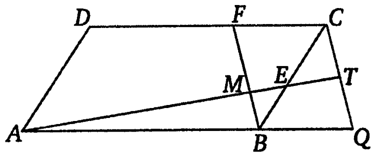
and

$$\frac{ET}{ME} = \frac{EC}{EB}. \tag{3}$$

Furthermore,

$$\frac{AM}{MT} = \frac{AB}{BQ} = \frac{DC}{CF} = \frac{DF+FC}{CF} = \frac{1}{b} + 1 = \frac{b+1}{b};$$

$$\frac{MT}{ME} = \frac{ME+ET}{ME} = \frac{ME}{ME} + \frac{ET}{ME} = 1 + \frac{EC}{EB} = 1 + \frac{1}{a} = \frac{a+1}{a}.$$



The relation (1) gives

$$\frac{AM}{ME} = \frac{AM}{MT} \cdot \frac{MT}{ME} = \frac{b+1}{b} \cdot \frac{a+1}{a} = \frac{(a+1)(b+1)}{ab}.$$

176. Prove that there are at least 2002 rational numbers m so that $\sqrt{m+2002}$ and $\sqrt{m+2003}$ are both rational numbers.

Solution. Set $m = \frac{a^4-2a^2+1}{4a^2} - 2002$ for some integer $a > 0$. Then the numbers

$$\sqrt{m+2002} = \sqrt{\left(\frac{a^2-1}{2a}\right)^2} = \frac{a^2-1}{2a}$$

and

$$\sqrt{m+2003} = \sqrt{\left(\frac{a^2+1}{2a}\right)^2} = \frac{a^2+1}{2a}$$

are both rational numbers.

Thus, there are infinitely many rational numbers that satisfy the condition.

177. Let a, b, c be positive real numbers such that $abc > 1$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > a + b + c$. Prove that:

i) All numbers are different than 1.

ii) Only one numbers is less than 1.

Solution. 1) Assume by contradiction that $a = 1$, then $bc > 1$ and $\frac{1}{b} + \frac{1}{c} > b + c$; it follows that

$$\frac{b+c}{bc} > b + c,$$

and since $b + c > 0$, then $bc < 1$. On the other hand, $bc = abc > 1$, a contradiction.

2) We have

$$\begin{aligned}(a-1)(b-1)(c-1) &= abc + a + b + c - (ab + bc + ac) - 1 \\ &< abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 1 \\ &= abc\left(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1\right) \\ &= -abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1\right) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1\right) \\ &= (1 - abc)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1\right). \end{aligned} \tag{1}$$

We consider the cases:

a) All the numbers are greater than 1. Then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > a + b + c$, a contradiction.

b) All the numbers are less than 1. This is impossible since $abc > 1$.

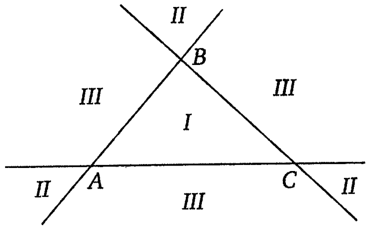
c) Only one number is greater than 1. Suppose $a < 1$, $b < 1$ and $c > 1$, hence $a - 1 < 0$, $b - 1 < 0$, $c - 1 > 0$. Since $abc > 1$, from (1) we infer $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. This is a contradiction, since $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > \frac{1}{a} > 1$.

Consequently, only one number from a, b, c is less than 1.

178. 5 points are given in a plane, not three of them collinear. Prove that there are 4 among them which are vertices of a convex quadrilateral.

Solution. Suppose that the quadrilateral $ABCD$ is concave and D is inside the triangle ABC. The lines AB, BC, AC divide the plane in seven regions. Suppose that the point E is in the region I, namely inside the triangle ABC. Then the line

DE intersect only two sides of the triangle ABC say, AB and AC. It follows that D, E, B, C are the vertices of a convex quadrilateral. Now assume that the point E is in one of the three regions marked with II. Without loss of generality, assume that E is inside the vertical angle of $\angle BAC$.



Then the points A and D are inside the triangle EBC and we solve like in the previous case. Finally, if the point E is in one of the three regions marked with III, the claim is obvious.

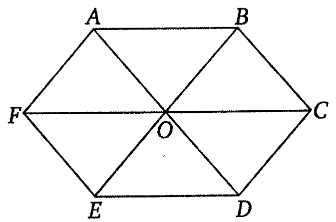
179. Consider a convex hexagon of area S. Prove that there is a triangle determined by three consecutive vertices of the hexagon with an area not greater than $\frac{S}{6}$.

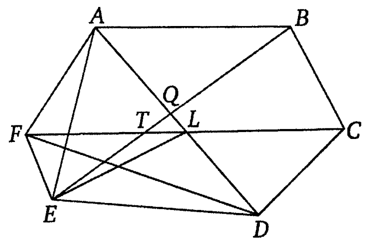
Solution. Let $ABCDEF$ be a hexagon of area S.

i) Assume that the diagonals AD, BE, CF intersect at point O. Then the hexagon is divided in three quadrilaterals $ABOF$, $BCDO$, $ODEF$, one of them having the area not greater than $S/3$.

ii) The diagonals AD, BE, CF are not concurrent. Let Q, L, T be the intersection points of the diagonals AD and BE, AD and BE, BE and CF respectively.

The hexagon is divided in three quadrilaterals $ABTF, BCDQ, EDLF$ and the triangle QLT. Again, one of the quadrilaterals has the area not greater than $\frac{S}{3}$.





Suppose that the quadrilateral $FEDL$ is the one with the area not exceeding $\frac{S}{3}$ (for the first case take $FEDO$ instead of $FEDL$). The diagonal EL divides the quadrilateral in two triangles, one of them (say FEL) with area not greater than $\frac{S}{6}$. Now observe that the distance from L to FE is between the distance from A and D to FE. Consequently, one of the triangles AFE or DFE has the area not exceeding the area of the triangle FEL and furthermore, not greater than $\frac{S}{6}$. This completes the proof.

180. Find all the positive integers n which are equal to the sum of its digits added to the product of its digits.

Solution. Let $\overline{a_1a_2\ldots a_n}$, $a_1 \neq 0$ and $a_2, a_3, \ldots, a_n \in \{0, 1, \ldots, 9\}$, be a number such that

$$\overline{a_1a_2\ldots a_n} = a_1 + a_2 + \ldots + a_n + a_1a_2\ldots a_n.$$

The relation is equivalent to

$$a_1\left(10^{n-1}-1\right) + a_2\left(10^{n-2}-1\right) + \ldots + 9a_{n-1} = a_1a_2\ldots a_n$$

and

$$a_2\left(10^{n-2}-1\right) + \ldots + 9a_{n-1} = a_1\left(a_2a_3\ldots a_n - \underbrace{99\ldots9}_{n-1\text{ digits}}\right).$$

The left-hand side of the equality is non-negative, while the right-hand side is non-positive, hence both are equal to zero. The left-hand side is zero if $n = 0$ or

$$a_2 = a_3 = \ldots = a_{n-1} = 0.$$

For $a_2 = a_3 = \ldots = a_{n-1} = 0$ the left-hand side do not equal zero, hence $n = 2$.

Then $a_1(a_2 - 9) = 0$, so $a_2 = 0$ and $a_1 \in \{1, 2, \ldots, 9\}$.

The numbers are 19, 29, 39, 49, 59, 69, 79, 89, 99.

181. Consider the sum

$$S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{99\cdot 100}.$$

Find the sequences of consecutive terms of S that add up to $\frac{1}{6}$.

Solution. The problem is to find positive integers n and p such that

$$\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(n+p-1)(n+p)} = \frac{1}{6}.$$

We have

$$\frac{1}{6} = \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \ldots + \left(\frac{1}{n+p-1} - \frac{1}{n+p}\right)$$
$$= \frac{1}{n} - \frac{1}{n+p} = \frac{p}{n(n+p)},$$

hence $6p = n(n+p)$. Since $n^2 > 0$, then $6p = n^2 + np > np$ and $n \in \{1, 2, 3, 4, 5\}$.

1) If $n = 1$, then $p = \frac{1}{5}$, false.

2) If $n = 2$, then $p = 1$ and we obtain the term $\frac{1}{2\cdot 3} = \frac{1}{6}$.

3) For $n = 3$ then $p = 3$ and we have $\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} = \frac{1}{6}$.

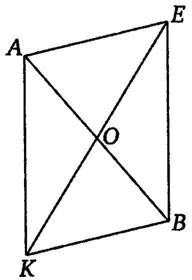
4) For $n = 4$, we find $p = 8$ and $\frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \ldots + \frac{1}{11\cdot 12} = \frac{1}{6}$.

5) For $n = 5$, we have $p = 25$ and $\frac{1}{5\cdot 6} + \frac{1}{6\cdot 7} + \ldots + \frac{1}{29\cdot 30} = \frac{1}{6}$.

182. Prove that any polygon with the perimeter 2004 can be covered by a disk of diameter 1002.

Solution. Let A and B be two points on the sides of the polygon P which divide the perimeter in two equal parts of length 1002. We have $AB < 1002$ and we prove that the disk of radius $501 = \frac{1002}{2}$, centered at the midpoint of the segment AB, will cover the polygon P. Assume by contradiction that there is a point E on the polygon P such that $OE > 501$.

Notice that E is different from A and B, since $OA = OB < 501$.



Let x be the length of the shortest path from A to E, using only the sides of the polygon P and define y similarly for the points B and E. We have $x + y = 501$. Since $AE \leq x$ and $BE \leq y$, then

$$501 = x + y \geq AE + BE.$$

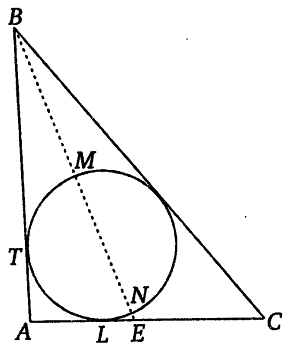
Reflect E across O at point K. As $AKBE$ is a parallelogram, $BE = AK$ and

$$AE + BE = AE + AK \geq EK = 2 \cdot OE > 501,$$

a contradiction.

183. Prove that there are no triangles in which the incircle divides an internal bisector of an angle in three equal segments.

Solution. Assume that there exists a triangle ABC in which the bisector BE of the angle $\angle B$ meet the incircle at M and N such that $BM = MN = NE = x$.



The incircle touches AB and AC at T and L, respectively. From the power of a point theorem we have

$$BT^2 = BM \cdot BN = 2x \cdot x$$

and

$$EL^2 = EN \cdot EM = 2x \cdot x,$$

hence $BT = LE$. On the other hand $AL = AT$, so $AB = AE$.

It follows that $\angle AEB = \angle ABE = \angle EBC$, thus $AC \parallel BC$, a contradiction.

184. Let k, $n_1, n_2, \ldots, n_k$ be odd integers. Prove that the numbers of odd numbers among $\frac{n_1+n_2}{2}, \frac{n_2+n_3}{2}, \ldots, \frac{n_k+n_1}{2}$ is odd.

Solution. The numbers $n_1, n_2, \ldots, n_k$ are odd, hence the numbers $\frac{n_1+n_2}{2}, \frac{n_2+n_3}{2}, \ldots, \frac{n_k+n_1}{2}$ are integers.

The sum

$$\frac{n_1+n_2}{2} + \frac{n_2+n_3}{2} + \ldots + \frac{n_k+n_1}{2} = n_1 + n_2 + \ldots + n_k,$$

is an odd number, having an odd number of odd summands. Consequently, among $\frac{n_1+n_2}{2}, \frac{n_2+n_3}{2}, \ldots, \frac{n_k+n_1}{2}$ there is an odd number of odd numbers.

185. Solve in $\mathbb{R}$ the equation:

$$[x\,[x]] = 1,$$

($[x]$ denotes the integer part of the number x).

Solution. By definition,

$$[x\,[x]] = 1$$

implies

$$1 \leq x\,[x] < 2.$$

We consider the following cases:

a) $x \in (-\infty, -1)$. Then $[x] \leq -2$ and $x\,[x] > 2$, a contradiction.

b) $x = -1 \Rightarrow [x] = -1$. Then $x\,[x] = (-1) \cdot (-1) = 1$ and $[x\,[x]] = 1$, so $x = -1$ is a solution.

c) $x \in (-1, 0)$. We have $[x] = -1$ and $x\,[x] = -x < 1$, false.

d) If $x \in [0, 1)$, then $[x] = 0$ and $x\,[x] = 0 < 1$, so we have no solution in this case.

e) For $x \in [1, 2)$ we obtain $[x] = 1$ and $x\,[x] = [x] = 1$, as needed.

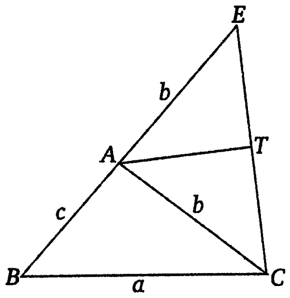
f) Finally, for $x \geq 2$ we have $[x] \geq 2$ and $x\,[x] = 2x \geq 4 \cdot 2$, a contradiction with (1).

Consequently, $x \in \{-1\} \cup [1, 2)$.

186. Prove that in any triangle the following inequality holds

$$b + c - a < 2b \cos \frac{A}{2}.$$

Solution. Consider a triangle ABC with $AB = c$, $BC = a$, $AC = b$. Extend the segment BA with $AE = b$ such that the point A lies on BE. Then AEC is an isosceles triangle and $\angle BEC = \frac{\angle BAC}{2} = \frac{A}{2}$. Let T be the midpoint of EC. As $\angle ATE = 90°$, we have $\cos \angle AET = \cos \frac{A}{2} = \frac{ET}{AE}$, hence $EC = 2ET = AE \cos \frac{A}{2} = 2b \cos \frac{A}{2}$.



On the other hand, from the triangle inequality we have

$$EC + BC > BE.$$

Thus $2b \cos \frac{A}{2} + a > b + c$, as needed.

187. A convex polygon with n^2 sides $(n > 2)$ is decomposed into n convex pentagons. Prove that $n = 3$.

Solution. The sum of the angles of the polygon with m sides is $(m-2)\,180°$. As the sum of the angles of the n pentagons is greater than the sum of the angles of the polygon with n^2 sides, we have

$$n \cdot 3 \cdot 180° > \left(n^2 - 2\right) \cdot 180°.$$

It follows that $2 > n\,(n-3)$, hence $n = 3$, as needed.

188. Find the greatest number n such that any subset with $1984-n$ elements of the set $\{1, 2, \ldots, 1984\}$ contains a pair of coprime numbers.

Solution. First, observe that if $n \geq 992$ then $1984 - n \leq 992$, so we can select $1984 - n$ even numbers from the set $\{1, 2, 3, \ldots, 1984\}$ and there is no pair of coprime numbers. Hence $n \leq 991$.

We prove that $n = 991$ satisfies the condition. For this, divide the set $\{1, 2, 3, \ldots, 1984\}$ in 992 pairs of consecutive numbers

$$\{1,2\}, \{3,4\}, \ldots \{1983, 1984\}$$

Any subset with $1984 - 991 = 993$ elements must contain one of the pairs of consecutive numbers. These are coprime numbers and we are done.

189. Find the real numbers $a_1, a_2, \ldots, a_{2n+1}$ so that

$$a_1 + a_2 + \ldots + a_{2n} + a_{2n+1} = 2n+1 \text{ and } |a_1 - a_2| = |a_2 - a_3| = \ldots = |a_{2n+1} - a_1|.$$

Solution. Let

$$|a_1 - a_2| = |a_2 - a_3| = \ldots = |a_{2n+1} - a_1| = k.$$

Then

$$\begin{aligned} a_1 - a_2 &= \pm k, \\ a_2 - a_3 &= \pm k, \\ &\ldots \\ a_{2n} - a_{2n+1} &= \pm k, \\ a_{2n+1} - a_1 &= \pm k. \end{aligned}$$

Summing these equalities yields $0 = \underbrace{\pm k \pm k \pm \ldots \pm k}_{2n+1} = k\,\underbrace{(\pm 1 \pm 1 \pm \ldots \pm 1)}_{2n+1}$. As $\underbrace{(\pm 1 \pm 1 \pm \ldots \pm 1)}_{2n+1}$ is an odd number, we obtain $k = 0$, hence all numbers $a_1, a_2, \ldots, a_{2n+1}$ are equal. Since $a_1 + a_2 + \ldots + a_{2n+1} = 2n+1$, we find

$$a_1 = a_2 = \ldots = a_{2n+1} = 1.$$

190. Considers $2n + 1$ real numbers between 1 and 2^n. Prove that there are three of them which are the side lengths of a triangle.

Solution. Divide the interval $(1, 2^n)$ into n distinct intervals

$$(1, 2), \left[2, 2^2\right), \left[2^2, 2^3\right), \ldots, \left[2^{n-1}, 2^n\right).$$

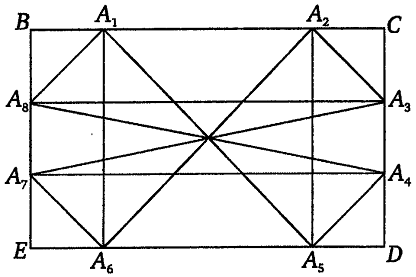
By Pigeonhole principle, there is an interval $\left[2^k, 2^{k+1}\right)$, $k \in \{1, 2, \ldots, n-1\}$ which contains three of the $2n + 1$ given numbers, say a, b, c. Since

$$\begin{aligned} a + b &\geq 2 \cdot 2^k = 2^{k+1} > c, \\ c + b &\geq 2 \cdot 2^k = 2^{k+1} > a, \\ a + c &\geq 2 \cdot 2^k = 2^{k+1} > b, \end{aligned}$$

the conclusion follows.

191. A convex octagon has all the angles congruent and all side lengths rational numbers. Prove that the octagon has a symmetry point.

Solution. Since all the angles of the octagon are equal to $135°$, the exterior angles are equal to $45°$. Let $A_1 A_2 \dots A_8$ be the octagon and let the lines A_1A_2 and A_3A_4 meet at point C. As $\angle CA_2A_3 = \angle CA_3A_2 = 45°$, it follows that the lines A_1A_2 and A_3A_4 are perpendicular. Similarly, the lines A_3A_4 and A_6A_5 are perpendicular at D, the lines A_5A_6 and A_8A_7 are perpendicular at E and finally the lines A_2A_1 and A_7A_8 are perpendicular at B. It is obvious that $BCDE$ is a rectangle.



We prove that $A_1A_2A_5A_6$ is a parallelogram. The points A_1, A_2 and A_5, A_6 lie on the segments BC and DE respectively, hence

$$BC = BA_1 + A_1A_2 + A_2C = A_1A_8 \cos 45° + A_1A_2 + A_2A_3 \cos 45°,$$

and

$$ED = EA_6 + A_6A_5 + A_5D = A_6A_7 \cos 45° + A_5A_6 + A_4A_5 \cos 45°.$$

As $BC = ED$, we have

$$A_1A_2 - A_5A_6 = \cos 45° \left(A_4A_5 + A_6A_7 - A_2A_3 - A_1A_8\right).$$

The numbers $A_1A_2 - A_5A_6$ and $A_4A_5 + A_6A_7 - A_2A_3 - A_1A_8$ are both rational, while $\cos 45° = \frac{\sqrt{2}}{2}$ is not. Consequently, $A_1A_2 - A_5A_6 = 0$, so $A_1A_2A_5A_6$ is a parallelogram.

Let O be the center of the parallelogram. In the same way we prove that $A_2A_3A_6A_7$ is a parallelogram centered at the midpoint of A_2A_6, i.e. at point O. It suffices to observe that $A_3A_4A_7A_8$ is also a parallelogram and therefore O is the center of symmetry for the octagon.

192. Find the sum of the digits of the numbers from 1 to 1,000,000.

Solution. Write the numbers from 0 to 999,999 in a rectangular array as follows:

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 0 & 9 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 1 & 9 \\
0 & 0 & 0 & 0 & 2 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
9 & 9 & 9 & 9 & 9 & 9
\end{array}$$

There are 1,000,000 six-digits numbers, hence 6,000,000 digits are used. In each column every digit is equally represented, as in the units column each digit appears from 10 to 10, in the tens column each digit appears successively in blocks of 10 and so on. Thus each digit appears 600,000 times, so the required sum is

$$600,000 \cdot 45 + 1 = 27,000,001.$$

(do not forget to count 1 from 1,000,000).

193. Find the elements of the set

$$A = \left\{x \in \mathbb{Z} \mid \frac{x^3 - 3x + 2}{2x + 1} \in \mathbb{Z}\right\}.$$

Solution. As x is an integers, so are $2x + 1$ and $x^3 - 3x + 2$. Since $\frac{x^3-3x+2}{2x+1} \in \mathbb{Z}$, then

$$\frac{8x^3 - 24x + 16}{2x + 1} = 4x^2 - 2x - 11 + \frac{27}{2x + 1} \in \mathbb{Z}.$$

It follows that $2x + 1$ divides 27, so

$$2x + 1 \in \{\pm 1, \pm 3, \pm 9, \pm 27\} \quad \text{and} \quad x \in \{-14, -5, -2, -1, 0, 1, 4, 13\}.$$

One can easy check that

$$A = \{-14, -5, -2, -1, 0, 1, 4, 13\}.$$

194. Prove that 2002 points can be joined two by two with 1001 segments such that no two of them intersect.

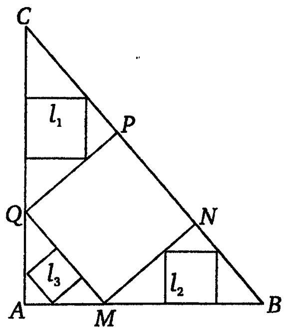
Solution. Let E be the set of all the lines determined by the 2002 given points. Choose a line d which is not perpendicular to any line from E.

Project the 2002 points on the line d and let $B_{k_1}, B_{k_2}, \ldots, B_{k_{2002}}$ be the projections, in this order (notice that the points B_i are distinct due to the choice of d). Label the initial points with $A_1, A_2, \ldots, A_{2002}$ such that B_k is the projection of A_k for all $k = \overline{1, 2002}$. Now the segments $A_1A_2, A_3A_4, \ldots, A_{2001}A_{2002}$ have the required property.

195. A triangle ABC with $\angle A = 90°$ is given. A square $MNPQ$ is inscribed in the triangle such that M lies on AB, N lies on BC, P lies on BC and Q lies on CA. Likewise, the squares of the sides l_1, l_2, l_3 are inscribed in the right triangles QPC, MBN, AMQ respectively, all having two vertices on the hypotenuses and a vertex on each leg of the triangles. Prove that

$$\frac{1}{l_1^2} + \frac{1}{l_2^2} = \frac{1}{l_3^2}.$$

Solution. The triangles QPC, BNM, MAQ are similar to BAC having the ratios equal to the ratios of the inscribed squares.



Hence, if $l = MN$, then

$$\frac{l_1}{l} = \frac{QP}{AB} = \frac{l}{AB} \Rightarrow AB = \frac{l^2}{l_1};$$
$$\frac{l_2}{l} = \frac{MN}{AC} = \frac{l}{AC} \Rightarrow AC = \frac{l^2}{l_2};$$
$$\frac{l_3}{l} = \frac{QM}{BC} = \frac{l}{BC} \Rightarrow BC = \frac{l^2}{l_3}.$$

Thus

$$AB^2 + AC^2 = BC^2 \Leftrightarrow \frac{l^4}{l_1^2} + \frac{l^4}{l_2^2} = \frac{l^4}{l_3^2} \Leftrightarrow \frac{1}{l_1^2} + \frac{1}{l_2^2} = \frac{1}{l_3^2},$$

as claimed.

196. Consider n distinct positive integers less than $2n$. Prove that among these numbers there is one equal to n or there are two numbers with the sum equal to $2n$.

Solution. Let $x_1, x_2, \ldots, x_n$ be n distinct integers from 1 to $2n-1$. Then $2n-x_1$, $2n-x_2, \ldots, 2n-x_n$ are also n distinct integers from 1 to $2n-1$.

The numbers $x_1, x_2, \ldots, x_n, 2n-x_1, 2n-x_2, \ldots, 2n-x_n$ cannot be all distinct, hence there are indices $i, k \in \{1, 2, \ldots, n\}$ such that $a_i = 2n - a_k$.

1) For $i = k$ we obtain $a_i = n$.

2) For $i \neq k$ we have $a_i + a_k = 2n$ and the claim holds.

197. Let a, b, c be odd integers. Prove that the roots of the equations $ax^2+bx+c=0$ are not rational numbers.

Solution. Assume by contradiction that $x = \frac{m}{k}$ is a rational root of the equation $ax^2+bx+c=0$, where m and k are coprime integers. Then

$$a\left(\frac{m}{k}\right)^2 + b\frac{m}{k} + c = 0 \text{ and } am^2 + bmk + ck^2 = 0.$$

The numbers m and k are coprime, so there are not both even. Recall that a, b, c are odd and consider the following cases.

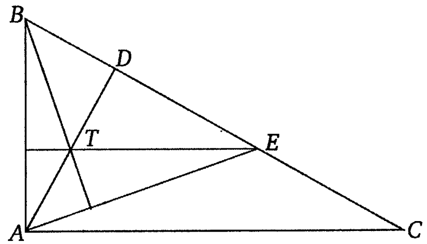
i) If m, k are odd, then am^2, bmk, $(1-abc)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right).ck^2$ are odd, and consequently $0 = am^2+bmk+ck^2$ is odd, a contradiction.

ii) If m is odd and k is even, then am^2 is odd and bmk, ck^2 are even. Again $0 = am^2+bmk+ck^2$ is odd, false.

ii) The case m even and k odd leads also to a contradiction.

198. Let ABC be a triangle with $\angle A = 90°$. Consider the altitude AD and T, E the midpoints of the segments AD and DC respectively. Prove that $\angle ABT = \angle CAE$.

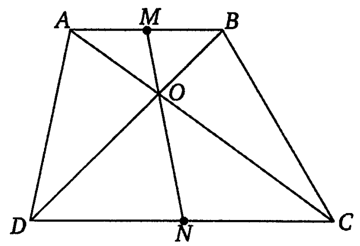
Solution. The segment TE is the middle line of the triangle ADC, hence TE is parallel to AC.



On the other hand $AC \perp AB$, hence ET is an altitude of the triangle ABE. Since AD is also an altitude, it follows that T is the orthocenter of the triangle ABE. Thus $BT \perp AE$ and $\angle ABT = 90° - \angle BAE = \angle CAE$, as desired.

199. Let $ABCD$ be a trapezoid with the middle line equal to the altitude. Prove that the diagonals are perpendicular if and only if the trapezoid is isosceles.

Solution. Let M and N be the midpoints of the bases AB and CD respectively, and let O be the intersection point of the diagonals. As $\frac{AO}{OC} = \frac{AB}{CD} = \frac{2AM}{2NC} = \frac{AM}{NC}$, it follows that N, O, M are collinear.



Suppose that $AC \perp BD$. We prove that $ABCD$ is an isosceles trapezoid. As OM and ON are medians in the right-angled triangles AOB and DOC, we have $MN = MO + ON = \frac{1}{2}AB + \frac{1}{2}CD$, hence MN is equal to the middle line of $ABCD$.

By hypothesis, MN is equal to the altitude of the trapezoid, so $MN \perp AB$ and $ABCD$ is isosceles.

Conversely, suppose that $ABCD$ is isosceles. Then MN is the altitude of the trapezoid, which is equal to the middle line $\frac{1}{2}(AB + CD)$.

Assume by contradiction that angle $\angle AOB$ is acute. Then $OM > \frac{1}{2}AB$ and $ON > \frac{1}{2}CD$. Summing these inequalities, we obtain $MN > \frac{1}{2}(AB + CD)$, a contradiction. Assuming that angle $\angle AOB$ is obtuse we infer that $OM > \frac{1}{2}AB$ and $ON > \frac{1}{2}CD$. Then $MN > \frac{1}{2}(AB + CD)$, a contradiction.

200. The sum of 10 distinct non-negative integers is equal to 62. Prove that the product of these numbers is divisible by 60.

Solution. We prove that among the given numbers, one is divisible by 2, one is divisible by 4 and one is divisible by 5. Indeed if none of them is a multiple of 3, the sum is at least

$$1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + 13 + 14 = 75,$$

false.

Assume that there are no multiples of 4. Then the sum is at least

$$1 + 2 + 3 + 5 + 6 + 7 + 9 + 10 + 11 + 13 = 67,$$

false.

Finally, if among the given numbers none is divisible by 5, then the sum is at least

$$1 + 2 + 3 + 4 + 6 + 7 + 8 + 9 + 11 + 12 = 63,$$

a contradiction.

Thus the product of the number is divisible by $3 \cdot 5 \cdot 4 = 60$.

201. Let a, b, c be real numbers so that $a + 2b + 3c = 2$ and $2ab + 3ac + 6bc = 1$. Show that $a \in \left[0, \frac{4}{3}\right]$, $b \in \left[0, \frac{2}{3}\right]$ and $c \in \left[0, \frac{4}{9}\right]$.

Solution. We have $a + 2b = 2 - 3c$, and

$$2ab = 1 - 3c(a + 2b) = 1 - 3c(2 - 3c) = 1 - 6c + 9c^2 = (3c - 1)^2.$$

The quadratic equation

$$x^2 - (2 - 3c)x + (3c - 1)^2 = 0$$

has the roots a and $2b$, hence $\Delta = (2 - 3c)^2 - 4(3c - 1)^2 = 3c(4 - 9c) \geq 0$, and consequently $c \in \left[0, \frac{4}{9}\right]$.

On the other hand,

$$2b + 3c = 2 - a$$

and

$$2b \cdot 3c = 6bc = 1 - a(2b + 3c) = 1 - a(2 - a) = 1 - 2a + a^2 = (a - 1)^2 = 0$$

The quadratic equation

$$y^2 - (2 - a)y + (a - 1)^2 = 0$$

has the roots $2b$ and $3c$, hence $\Delta = (2 - a)^2 - 4(a - 1)^2 = -a(4 - 3a) \geq 0$, we obtain $a \in \left[0, \frac{4}{3}\right]$.

Finally,

$$a + 3c = 2 - 2b$$

and

$$a \cdot 3c = 1 - 2ab - 6bc = 1 - 2b(a + 3c) = 1 - 2b(2 - 2b) = 1 - 4b + 4b^2 = (2b - 1)^2.$$

The quadratic equation

$$z^2 - (2 - 2b)z + (2b - 1)^2 = 0$$

has the roots a and $3c$, thus $\Delta \geq 0$. It follows that $\Delta = (2 - 2b)^2 - 4(2b - 1)^2 = 2b(4 - 6b) \geq 0$, so $b \in \left[0, \frac{2}{3}\right]$.

202. Consider an acute triangle $A_1A_2A_3$ and let H_1, H_2, H_3 be the feet of the altitudes from A_1, A_2, A_3, respectively. If a_1, a_2, a_3 are the lengths of the sides A_2A_3, A_3A_1, A_1A_2 and H is the orthocenter of the triangle, prove that

$$\frac{a_1}{HH_1} + \frac{a_2}{HH_2} + \frac{a_3}{HH_3} = 2\left(\frac{a_1}{HA_1} + \frac{a_2}{HA_2} + \frac{a_3}{HA_3}\right).$$

Solution. Since $A_1A_2A_3$ is an acute triangle, the orthocenter H lies inside the triangle.

We use the fact that the reflections of H across the sides of the triangle lie on the circumcircle of $A_1A_2A_3$. By the power of the point theorem, the products $HH_1 \cdot HA_1, HH_2 \cdot HA_2, HH_3 \cdot HA_3$ are equal and let k be their common value. Let $S = \text{area}[A_1A_2A_3]$. We have successively

$$\frac{a_1}{HH_1} + \frac{a_2}{HH_2} + \frac{a_3}{HH_3} = \frac{a_1HA_1}{HH_1 \cdot HA_1} + \frac{a_2HA_2}{HH_2 \cdot HA_2} + \frac{a_3HA_3}{HH_3 \cdot HA_3}$$

$$= \frac{a_1HA_1 + a_2HA_2 + a_3HA_3}{k}$$

$$= \frac{a_1(A_1H_1 - HH_1) + a_2(A_2H_2 - HH_2) + a_3(A_3H_3 - HH_3)}{k}$$

$$= \frac{(a_1A_1H_1 + a_2A_2H_2 + a_3A_3H_3) - (a_1HH_1 + a_2HH_2 + a_3HH_3)}{k}$$

$$= \frac{(2S + 2S + 2S) - 2S}{k} = 2 \cdot \frac{2S}{k}$$

$$= 2a\frac{a_1HH_1 + a_2HH_2 + a_3HH_3}{k}$$

$$= 2\left(\frac{a_1HH_1}{HH_1 \cdot HA_1} + \frac{a_2HH_2}{HH_2 \cdot HA_2} + \frac{a_3HH_3}{HH_3 \cdot HA_3}\right)$$

$$= 2\left(\frac{a_1}{HA_1} + \frac{a_2}{HA_2} + \frac{a_3}{HA_3}\right),$$

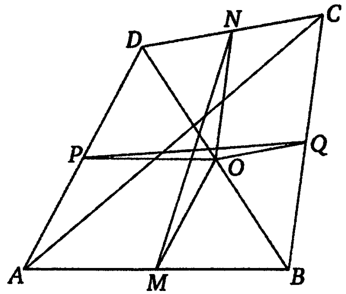
as needed.

203. Consider a convex quadrilateral $ABCD$ and let M, Q, N, P be the midpoints of the sides AB, BC, CD, AD respectively. Prove that if $2(MN + PQ) = AB + BC + CD + DA$, then $ABCD$ is a parallelogram.

Solution. Let O be the midpoint of the diagonal BD. The segments OM and ON are the middle lines in the triangles ABD and BED, hence $MO = \frac{AD}{2}$ and $NO = \frac{BC}{2}$.

By triangle inequality we have $MO + ON \geq MN$, hence

$$\frac{AD + BC}{2} \geq MN. \tag{1}$$



Similarly $PO = \frac{AB}{2}$, $OQ = \frac{DC}{2}$, and from $PO + OQ \geq PQ$, we have

$$\frac{AB + DC}{2} \geq PQ. \tag{2}$$

From (1) and (2) we obtain

$$AB + BC + CD + AD \geq 2(MN + PQ).$$

The equality holds if and only if M, O, N are collinear and P, O, Q are collinear. It follows that $AD \parallel BC$ and $AB \parallel CD$, hence $ABCD$ is a parallelogram.

204. Let a, b, c be positive real numbers with $\sqrt{ab} + \sqrt{bc} + \sqrt{ac} = 1$. Find the minimum value of the expression

$$E = \frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a}.$$

Solution. We have

$$\frac{a^2}{a+b} = \frac{a^2 + ab - ab}{a+b} = \frac{a(a+b)}{a+b} - \frac{ab}{a+b} = a - \frac{ab}{a+b}. \tag{1}$$

As $a + b \geq 2\sqrt{ab}$, it follows that

$$\frac{a^2}{a+b} \geq a - \frac{ab}{2\sqrt{ab}} \geq a - \frac{\sqrt{ab}}{2}, \tag{2}$$

and similarly

$$\frac{b^2}{b+c} \geq b - \frac{\sqrt{bc}}{2}$$

and

$$\frac{c^2}{a+c} \geq c - \frac{\sqrt{ca}}{2}.$$

Summing these inequalities, yields

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \geq a + b + c - \frac{1}{2}\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ac}\right) = a + b + c - \frac{1}{2}. \tag{3}$$

Moreover, by Cauchy-Schwarz inequality we obtain

$$a+b+c \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ac} = 1, \tag{4}$$

hence (3) rewrites

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{a+c} \geq 1-\frac{1}{2}=\frac{1}{2}.$$

Thus the minimum value of E is $\frac{1}{2}$ and it is obtained for $a=b=c=\frac{1}{3}$.

205. On the faces of a cube are written the numbers from 1 to 6. Prove that the sum of the numbers written on three faces with a common vertex cannot be constant.

Solution. Assume that the sum is constant for each vertex of the cube and denote it by S. Each of the numbers 1, 2, 3, 4, 5, 6 appears as summand in four sums one for each vertex of the face having that number. Then the sum of all numbers at the 8 vertices is

$$8S = 4(1+2+3+4+5+6),$$

so $8S = 21\cdot 4$. Since S is an integer, we have reached a contradiction.

206. A triangle ABC with $\angle A = 90°$ is given. Let D be the foot of the altitude from A. Prove that

$$BC+AD > AB+AC.$$

Solution. In the right angled triangle ABC we have

$$AB^2+AC^2=BC^2. \tag{1}$$

Then

$$\begin{aligned} BC+AD > AB+AC &\Leftrightarrow (BC+AD)^2 > (AB+AC)^2 \\ &\Leftrightarrow BC^2+AD^2+2BC\cdot AD > AB^2+AC^2+2AB\cdot AC \\ &\Leftrightarrow AD^2>0, \end{aligned}$$

which is obvious.

207. Consider a trapezoid $ABCD$ with $AB \parallel CD$ and $CD = kAB$ $(k>1)$.

a) Prove that

$$BC^2+AD^2+2kAB^2 = AC^2+BD^2.$$

b) If the trapezoid is circumscriptible, prove that

$$(k+1)AB = BC+AD.$$

Solution. a) Let M and N be the midpoints of the diagonals BD and AC. In the triangle AMC, MN is median, hence

$$4MN^2 = 2\left(AM^2+CM^2\right)-AC^2.$$

The segments AM and CM are also medians in the triangles ABD and CBD, so

$$2AM^2 = AB^2+AD^2-\frac{BD^2}{2}$$

and

$$2CM^2 = CB^2+CD^2-\frac{BD^2}{2}.$$

Thus

$$4MN^2+AC^2+BD^2 = AB^2+BC^2+CD^2+DA^2.$$

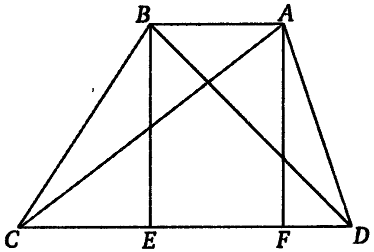
As

$$MN=\frac{CD-AB}{2}=\frac{k-1}{2}AB,$$

then

$$AC^2+BD^2 = AB^2+BC^2+DA^2+k^2AB^2-(k-1)^2AB^2 = BC^2+DA^2+2kAB^2,$$

as desired.



b) The trapezoid is circumscriptible if and only if $AD+BC = AB+CD = AB(k+1)$, as claimed.

208. The numbers 1, 2, 3, 4, …, $2n$ are divided in two groups each: $a_1<a_2<\ldots<a_n$ and $b_1>b_2>\ldots>b_n$. Prove that

$$|a_1-b_1|+|a_2-b_2|+\ldots+|a_n-b_n| = n^2.$$

Solution. We prove that one of the numbers a_i and b_i is less than or equal to n and the other is greater than n, for all $i=\overline{1,n}$. Indeed, assume that $a_i \leq n$ and $b_i \leq n$. Then the numbers $a_1, a_2, \ldots, a_i, b_i, b_{i+1}, \ldots, b_n$ are $n+1$ positive integers less than or equal to n, false. The case $a_i, b_i > n$ leads similarly to a contradiction. Consequently, each of the summands $|a_i-b_i|$ is a difference between a number greater than n and another less than or equal to n. Therefore,

$$\begin{aligned} &|a_1-b_1|+|a_2-b_2|+\ldots+|a_n-b_n| \\ &= (n+1)+(n+2)+\ldots+(n+n)-(1+2+\ldots+n) \\ &= n\cdot n+(1+2+3+\ldots+n)-(1+2+3+\ldots+n) = n^2, \end{aligned}$$

as desired.

209. Let a, b, c, d be real numbers so that

$$\left(a^2+b^2-1\right)\left(c^2+d^2-1\right) > (ac+bd-1)^2.$$

Prove that

$$a^2+b^2>1 \text{ and } c^2+d^2>1.$$

Solution. Assume by contradiction that both numbers $x = 1-a^2-b^2$ and $y = 1-c^2-d^2$ are non-negative.

The inequality

$$\left(a^2+b^2-1\right)\left(c^2+d^2-1\right) > (ac+bd-1)^2$$

is equivalent to

$$4xy > (2ac+2bd-2)^2 = \left(a^2+b^2+x+c^2+d^2+y-2ac-2bd\right)^2.$$

On the other hand

$$\left[(a-c)^2+(b-d)^2+x+y\right]^2 \geq (x+y)^2 = x^2+2xy+y^2.$$

It follows that $4xy > x^2+2xy+y^2$ or $0 > (x-y)^2$, a contradiction.

210. Find the location of a point M inside a convex quadrilateral $ABCD$ such that the sum $MA^2+MB^2+MC^2+MD^2$ is minimal.

Solution. Let E, T, P, R be the midpoints of the sides AB, BC, CD, DA respectively. The segments MT and MR are medians in the triangles BMC and AMD, hence

$$BC^2+4MT^2 = 2\left(MB^2+MC^2\right) \tag{1}$$

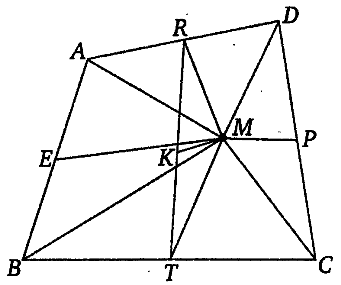
and

$$AD^2+4MR^2 = 2\left(AM^2+MD^2\right). \tag{2}$$

Summing (1) and (2) we obtain

$$AD^2+BC^2+4\left(MT^2+MR^2\right) = 2\left(MA^2+MB^2+MC^2+MD^2\right).$$

As AD^2+BC^2 is constant, the sum $MA^2+MB^2+MC^2+MD^2$ is minimum when MT^2+MR^2 is minimum.



Let K be the midpoint of RT. Since KM is median in the triangle RMT, we have

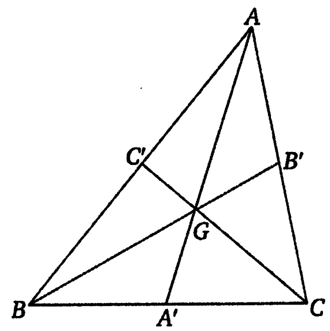
$$RT^2+4KM^2 = 2\left(RM^2+MT^2\right),$$

hence the minimum value of RM^2+MT^2 is obtained when KM is minimum. That is when $M = K$, so M is the midpoint of the segment RT.

Comment: The point K is the centroid of the quadrilateral $ABCD$, located at the intersection of the lines EP and RT. It is easy to prove that $ERPT$ is a parallelogram and K is the center.

211. A triangle ABC with $AB > AC$ is given. Prove that the length of the median from B is greater than the length of median from C.

Solution. Let G be the centroid of the triangle ABC and let A' be the midpoint of the side BC. Obviously, the points A, G, A' are collinear.



The triangles ABA' and ACA' share a common side AA'. Since $BA' = A'C$ and $AB > AC$, it follows that $\angle AA'B > \angle AA'C$.

The triangles GBA' and $GA'C$ have GA' in common, $BA' = CA'$ and $\angle BA'G > \angle GA'C$, hence $BG > GC$. Thus $\frac{2}{3}BB' > \frac{2}{3}CC'$ and $BB' > CC'$, as claimed.