COMPENDIUM KANGAROO UK

Senior Mathematical Challenge

2012 - 2025

Gerard Romo Garrido

Toomates Cooleccion vol. 39.4

Toomates Coolección

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El proyecto Toomates tiene como objetivo la promoción y difusión entre el profesorado y el colectivo de estudiantes de unos materiales didácticos libres, gratuitos y de calidad, que fuerce a las empresas comerciales a competir ofreciendo alternativas de pago atractivas aumentando la calidad de los materiales que ofrecen, (que son muy mediocres) y no mediante retorcidas técnicas comerciales.
 Todos los libros se ofrecen en dos versiones: En formato "**pdf**" para una cómoda lectura y en el formato "**doc**" de MSWord para permitir y facilitar su edición y generar versiones parcial o totalmente modificadas. **¡Libérate de la tiranía y mediocridad de los productos comerciales! Crea, utiliza y comparte tus propios materiales didácticos.**

Problem Solving (en español):

Geometría Axiomática Problemas de Geometría Vol. 1 Vol. 2 Vol. 3
Introducción a la Geometría Álgebra Teoría de números Vol. 1 Vol. 2 Combinatoria
Probabilidad Trigonometría Desigualdades Números complejos Calculus & Precalculus

Libros de texto para ESO y bachillerato (en español y en catalán):

Cálculo infinitesimal ESP CAT Álgebra Lineal ESP CAT Geometría Lineal ESP CAT
Números Complejos ESP CAT Combinatoria y probabilidad ESP CAT Estadística ESP CAT
Programación Lineal ESP CAT Álgebra ESP CAT Trigonometría ESP CAT
Geometria analítica ESP CAT Funciones ESP CAT Números (Preálgebra) ESP CAT
Proporcionalidad ESP CAT Medidas geométricas ESP CAT Mates amb Excel

PAU españolas:

Cataluña TEC Cataluña CCSS Valencia Galicia País Vasco Baleares

Reválidas internacionales:

Portugal Italia Francia Rumanía Hungría Polonia Pearson Edexcel Int A Level
China-Gaokao China-Zhongkao Corea-Suneung Israel-Bagrut Cambridge Int A Level
Cambridge IGCSE AQA GCSE International Baccalaureate (IB) Pearson Edexcel IGCSE

Evaluación diagnóstica y pruebas de acceso:

ACM6EP ACM4 CFGS PAP

Competiciones matemáticas:

Canguro: España Cataluña Francia USA Reino Unido Austria
USA: Mathcounts AMC 8 10 12 AIME USAJMO USAMO TSTST TST ELMO Putnam HMMT
España: OME OMEFL OMEEX OMC OMEA OMEM OMA CDP OFEM
Europa: OMI Arquimede BMO BalkanMO JBMO OPM OMP OMJ ONM
Internacional: IMO IGO SMT INMO CMO EGMO KMO KJMO AMC Australia
AHSME: Book 1 2 3 4 5 6 7 8 9

Otros materiales:

Pizzazz! (Book A B C D E Pre-Algebra Algebra) , REOIM , Llibre3r , Excalibur

¡Genera tus propias versiones de este documento! Siempre que es posible se ofrecen las versiones editables "MS Word" de todos los materiales para facilitar su edición.

¡Ayuda a mejorar! Envía cualquier duda, observación, comentario o sugerencia a **toomates@gmail.com**
¡No utilices una versión anticuada! Todos estos libros se revisan y amplían constantemente. Descarga totalmente gratis la última versión de estos documentos en los correspondientes enlaces superiores, en los que siempre encontrarás la versión más actualizada.
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Visita mi **Canal de Youtube**: https://www.youtube.com/c/GerardRomo
Visita mi **blog:** https://toomatesbloc.blogspot.com/

Versión de este documento: **07/04/2026**

Colección Competiciones Canguro y similares.

Canguro (España)
- 2000-2021
  - http://www.toomates.net/biblioteca/Canguro2.pdf
- 2022-2025
  - http://www.toomates.net/biblioteca/Canguro.pdf

Cangur (Cataluña)
- 1999-2015
  - http://www.toomates.net/biblioteca/Cangur2.pdf
- 2016-2024
  - http://www.toomates.net/biblioteca/Cangur.pdf

Concurso de Primavera (Madrid)
- http://www.toomates.net/biblioteca/CompendiumCDP.pdf

Kangourou (Francia)
- http://www.toomates.net/biblioteca/CompendiumKangourou.pdf

Kangaroo (USA)
- http://www.toomates.net/biblioteca/CompendiumKangaroo.pdf

Kangaroo (Reino Unido)
- Kangaroo Pink & Grey
  - http://www.toomates.net/biblioteca/KangarooUK.pdf
- Kangaroo Junior
  - http://www.toomates.net/biblioteca/KangarooUK2.pdf
- Kangaroo Senior
  - http://www.toomates.net/biblioteca/KangarooUK3.pdf
- Kangaroo Senior Mathematical Challenge
  - http://www.toomates.net/biblioteca/KangarooUK4.pdf

Kanguru (Austria)
- http://www.toomates.net/biblioteca/CompendiumKanguru.pdf

Australian Mathematics Competition (Australia)
- http://www.toomates.net/biblioteca/CompendiumAMC.pdf

Giochi di Archimede (Italia)
- http://www.toomates.net/biblioteca/CompendiumArchimede.pdf

Las pruebas AMC 8, AMC 10 y AMC 12 USA también siguen el formato de respuesta multiopción, pero con una dificultad mucho más elevada que las anteriores:

AMC (USA)
- AMC 8
  - http://www.toomates.net/biblioteca/CompendiumAMC8.pdf
- AMC 10
  - http://www.toomates.net/biblioteca/CompendiumAMC10.pdf
- AMC 12
  - http://www.toomates.net/biblioteca/CompendiumAMC8.pdf

Tabla de correspondencia Canguro/Cangur/Kangaroo/Kangourou.

	ESPAÑA			UK (England & Wales)		USA		FRANCIA	
EDAD	CURSO	CANGURO	CANGUR (Catalunya)	YEAR	KANGAROO	GRADE	KANGAROO	Curso	KANGOUROU
6/7	1º Prim.			2		1th			
7/8	2º Prim.			3		2nd	Felix		
8/9	3º Prim.			4		3th		CE2	
9/10	4º Prim.			5		4th	Ecolier	CM1	
10/11	5º Prim.		P5	6		5th		CM2	**E** Écoliers
11/12	6º Prim.		P6	7		6th	Benjamin	6ème	
12/13	1º ESO	N1	E1	8		7th		5ème	**B** Benjamins
13/14	2º ESO	N2	E2	9	Grey	8th	Cadet	4ème	
14/15	3º ESO	N3	E3	10		9th		3ème	**C** Cadets
15/16	4º ESO	N4	E4	11	Pink	10th	Junior	2ème	**J**uniors: Lycées G. et T. **É**tudiants: TS, Bac+
16/17	1º BAT	N5	B1	12		11th		1ème	
17/18	2º BAT	N6	B2	13		12th	Student	T	

Índice.

Todas las soluciones desarrolladas se presentan después de su correspondiente bloque de enunciados.

Fuente.

https://ukmt.org.uk/

UK SENIOR MATHEMATICAL CHALLENGE

Tuesday 6 November 2012

Organised by the **United Kingdom Mathematics Trust**

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making financial sense of the future

RULES AND GUIDELINES (to be read before starting)

1. Do not open the question paper until the invigilator tells you to do so.
2. **Use B or HB pencil only**. Mark *at most one* of the options A, B, C, D, E on the Answer Sheet for each question. Do not mark more than one option.
3. Time allowed: **90 minutes**.
   No answers or personal details may be entered on the Answer Sheet after the 90 minutes are over.
4. The use of rough paper is allowed.
   Calculators, measuring instruments and squared paper are forbidden.
5. Candidates must be full-time students at secondary school or FE college, and must be in Year 13 or below (England & Wales); S6 or below (Scotland); Year 14 or below (Northern Ireland).
6. There are twenty-five questions. Each question is followed by five options marked A, B, C, D, E. Only one of these is correct. Enter the letter A-E corresponding to the correct answer in the corresponding box on the Answer Sheet.
7. **Scoring rules**: all candidates start out with 25 marks;
   0 marks are awarded for each question left unanswered;
   4 marks are awarded for each correct answer;
   1 mark is deducted for each incorrect answer.
8. **Guessing**: Remember that there is a penalty for wrong answers. Note also that later questions are deliberately intended to be harder than earlier questions. You are thus advised to concentrate first on solving as many as possible of the first 15-20 questions. Only then should you try later questions.

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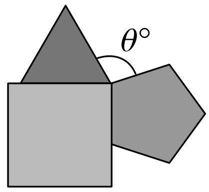
http://www.ukmt.org.uk

1. Which of the following cannot be written as the sum of two prime numbers?

A 5 B 7 C 9 D 10 E 11

2. The diagram shows an equilateral triangle, a square and a regular pentagon which all share a common vertex. What is the value of θ?

A 98 B 102 C 106 D 110 E 112



3. The price of my favourite soft drink has gone up by leaps and bounds over the past ten years. In four of those years it has leapt up by 5p each year, whilst in the other six years it has bounded up by 2p each year. The drink cost 70p in 2002. How much does it cost now?

A £0.77 B £0.90 C £0.92 D £1.02 E £1.05

4. According to one astronomer, there are one hundred thousand million galaxies in the universe, each containing one hundred thousand million stars. How many stars is that altogether?

A 10^{13} B 10^{22} C 10^{100} D 10^{120} E 10^{121}

5. All six digits of three 2-digit numbers are different. What is the largest possible sum of three such numbers?

A 237 B 246 C 255 D 264 E 273

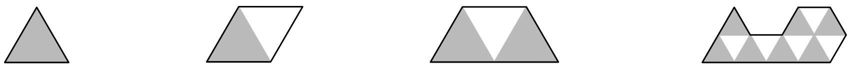
6. What is the sum of the digits of the largest 4-digit palindromic number which is divisible by 15? [Palindromic numbers read the same backwards and forwards, e.g. 7227.]

A 18 B 20 C 24 D 30 E 36

7. Given that $x + y + z = 1$, $x + y - z = 2$ and $x - y - z = 3$, what is the value of xyz?

A -2 B $-\frac{1}{2}$ C 0 D $\frac{1}{2}$ E 2

8. The diagrams below show four types of tile, each of which is made up of one or more equilateral triangles. For how many of these types of tile can we place three identical copies of the tile together, without gaps or overlaps, to make an equilateral triangle?



A 0 B 1 C 2 D 3 E 4

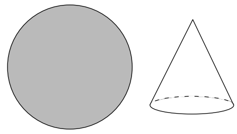
9. Pierre said, "Just one of us is telling the truth". Qadr said, "What Pierre says is not true". Ratna said, "What Qadr says is not true". Sven said, "What Ratna says is not true". Tanya said, "What Sven says is not true".
How many of them were telling the truth?

A 0 B 1 C 2 D 3 E 4

10. Let N be the smallest positive integer whose digits add up to 2012. What is the first digit of $N + 1$?

A 2 B 3 C 4 D 5 E 6

11. Coco is making clown hats from a circular piece of cardboard. The circumference of the base of each hat equals its slant height, which in turn is equal to the radius of the piece of cardboard. What is the maximum number of hats that Coco can make from the piece of cardboard?



A 3 B 4 C 5 D 6 E 7

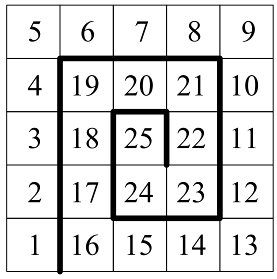
12. The number 3 can be expressed as the sum of one or more positive integers in four different ways:

$$3; \qquad 1 + 2; \qquad 2 + 1; \qquad 1 + 1 + 1.$$

In how many ways can the number 5 be so expressed?

A 8 B 10 C 12 D 14 E 16

13. A cube is placed with one face on square 1 in the maze shown, so that it completely covers the square with no overlap. The upper face of the cube is covered in wet paint. The cube is then 'rolled' around the maze, rotating about an edge each time, until it reaches square 25. It leaves paint on all of the squares on which the painted face lands, but on no others. The cube is removed on reaching the square 25. What is the sum of the numbers on the squares which are now marked with paint?



A 78 B 80 C 82 D 169 E 625

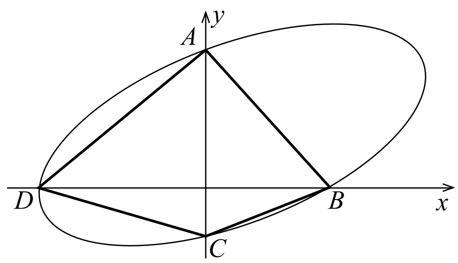
14. Six students who share a house all speak exactly two languages. Helga speaks only English and German; Ina speaks only German and Spanish; Jean-Pierre speaks only French and Spanish; Karim speaks only German and French; Lionel speaks only French and English whilst Mary speaks only Spanish and English. If two of the students are chosen at random, what is the probability that they speak a common language?

A $\frac{1}{2}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{4}{5}$ E $\frac{5}{6}$

15. Professor Rosseforp runs to work every day. On Thursday he ran 10% faster than his usual average speed. As a result, his journey time was reduced by x minutes. How many minutes did the journey take on Wednesday?

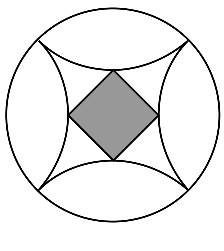
A $11x$ B $10x$ C $9x$ D $8x$ E $5x$

16. The diagram shows the ellipse whose equation is $x^2 + y^2 - xy + x - 4y = 12$. The curve cuts the y-axis at points A and C and cuts the x-axis at points B and D. What is the area of the inscribed quadrilateral $ABCD$?



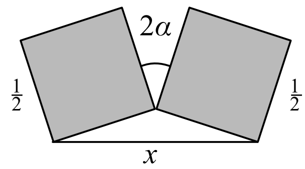
A 28 B 36 C 42 D 48 E 56

17. The diagram shows a pattern found on a floor tile in the cathedral in Spoleto, Umbria. A circle of radius 1 surrounds four quarter circles, also of radius 1, which enclose a square. The pattern has four axes of symmetry. What is the side length of the square?



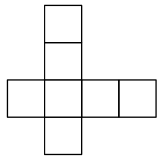
A $\dfrac{1}{\sqrt{2}}$ B $2-\sqrt{2}$ C $\dfrac{1}{\sqrt{3}}$ D $\dfrac{1}{2}$ E $\sqrt{2}-1$

18. The diagram shows two squares, with sides of length $\frac{1}{2}$, inclined at an angle 2α to one another. What is the value of x?



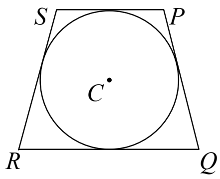
A $\cos\alpha$ B $\dfrac{1}{\cos\alpha}$ C $\sin\alpha$ D $\dfrac{1}{\sin\alpha}$ E $\tan\alpha$

19. The numbers 2, 3, 4, 5, 6, 7, 8 are to be placed, one per square, in the diagram shown so that the sum of the four numbers in the horizontal row equals 21 and the sum of the four numbers in the vertical column also equals 21. In how many different ways can this be done?



A 0 B 2 C 36 D 48 E 72

20. In trapezium $PQRS$, $SR = PQ = 25$cm and SP is parallel to RQ. All four sides of $PQRS$ are tangent to a circle with centre C. The area of the trapezium is 600cm^2. What is the radius of the circle?

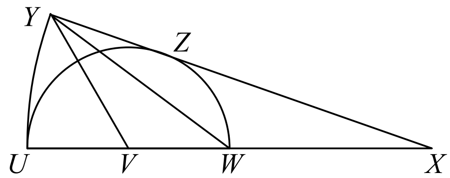


A 7.5cm B 8cm C 9cm D 10cm E 12cm

21. Which of the following numbers does *not* have a square root in the form $x + y\sqrt{2}$, where x and y are positive integers?

A $17 + 12\sqrt{2}$ B $22 + 12\sqrt{2}$ C $38 + 12\sqrt{2}$ D $54 + 12\sqrt{2}$ E $73 + 12\sqrt{2}$

22. A semicircle of radius r is drawn with centre V and diameter UW. The line UW is then extended to the point X, such that UW and WX are of equal length. An arc of the circle with centre X and radius $4r$ is then drawn so that the line XY is a tangent to the semicircle at Z, as shown. What, in terms of r, is the area of triangle YVW?

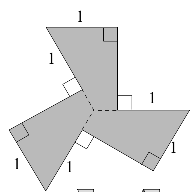


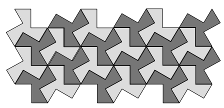
A $\dfrac{4r^2}{9}$ B $\dfrac{2r^2}{3}$ C r^2 D $\dfrac{4r^2}{3}$ E $2r^2$

23. Tom and Geri have a competition. Initially, each player has one attempt at hitting a target. If one player hits the target and the other does not then the successful player wins. If both players hit the target, or if both players miss the target, then each has another attempt, with the same rules applying. If the probability of Tom hitting the target is always $\frac{4}{5}$ and the probability of Geri hitting the target is always $\frac{2}{3}$, what is the probability that Tom wins the competition?

A $\dfrac{4}{15}$ B $\dfrac{8}{15}$ C $\dfrac{2}{3}$ D $\dfrac{4}{5}$ E $\dfrac{13}{15}$

24. The top diagram on the right shows a shape that tiles the plane, as shown in the lower diagram. The tile has nine sides, six of which have length 1. It may be divided into three congruent quadrilaterals as shown. What is the area of the tile?





A $\dfrac{1 + 2\sqrt{3}}{2}$ B $\dfrac{4\sqrt{3}}{3}$ C $\sqrt{6}$ D $\dfrac{3 + 4\sqrt{3}}{4}$ E $\dfrac{3\sqrt{3}}{2}$

25. How many distinct pairs (x, y) of real numbers satisfy the equation $(x + y)^2 = (x + 4)(y - 4)$?

A 0 B 1 C 2 D 3 E 4

UK SENIOR MATHEMATICAL CHALLENGE

November 6th 2012

EXTENDED SOLUTIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some ***Extension Problems*** for further investigations. (The short solutions have been appended and appear at the end of this document.)

The Senior Mathematical Challenge (SMC) is a multiple choice contest, in which you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad and similar competitions).

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments,

either by e-mail to: enquiry@ukmt.co.uk

or by post to: SMC Solutions, UKMT Maths Challenges Office, School of Mathematics, University of Leeds, Leeds LS2 9JT.

Quick Marking Guide

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
E	B	D	B	C	C	D	C	C	E	D	E	B	D	A	A	B	A	E	E	D	B	C	B	B

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1. Which of the following cannot be written as the sum of two prime numbers?

A 5 B 7 C 9 D 10 E 11

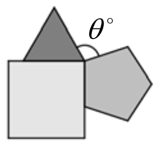
Solution: **E**

If the odd number 11 is the sum of two prime numbers, it must be the sum of the only even prime, 2, and an odd prime. But $11 = 2 + 9$ and 9 is not prime. So 11 is not the sum of two prime numbers. On the other hand, $5 = 2 + 3$, $7 = 2 + 5$, $9 = 2 + 7$ and $10 = 3 + 7$. Therefore, these are all the sum of two prime numbers.

Extension Problem

1.1 In 1742 the German Mathematician Christian Goldbach (1690-1764) made the conjecture that every even number greater than 2 is the sum of two primes. The *Goldbach Conjecture*, as it is now called, has, so far, not been proved, but no counterexample to it has been found. Investigate this conjecture by finding the even number in the range from 4 to 100 which can be expressed as the sum of two prime numbers in the largest number of ways.

2. The diagram shows an equilateral triangle, a square and a regular pentagon which all share a common vertex. What is the value of θ?



A 98 B 102 C 106 D 110 E 112

Solution: **B**

The interior angles of an equilateral triangle, a square and a regular pentagon are 60°, 90° and 108°, respectively. Hence $\theta = 360 - (60 + 90 + 108) = 360 - 258 = 102$.

3. The price of my favourite soft drink has gone up in leaps and bounds over the past ten years. In four of those years it has leapt up by 5p each year, whilst in the other six years it has bounded up by 2p each year. The drink cost 70p in 2002. How much does it cost now?

A £0.77 B £0.90 C £0.92 D £1.02 E £1.05

Solution: **D**

The cost has risen by 5p four times, and by 2p six times. So the total price rise in pence has been $(4 \times 5 + 6 \times 2) = 32$. Therefore the price now is $(70 + 32)\text{p} = 102\text{p} = £1.02$.

4. According to one astronomer, there are one hundred thousand million galaxies in the universe, each containing one hundred thousand million stars. How many stars is that altogether?

A 10^{13} B 10^{22} C 10^{100} D 10^{120} E 10^{121}

Solution: **B**

One hundred thousand is 10^5 and one million is 10^6. So one hundred thousand million is $10^5 \times 10^6 = 10^{5+6} = 10^{11}$. Therefore the total number of stars is $10^{11} \times 10^{11} = 10^{11+11} = 10^{22}$.

5. All six digits of three 2-digit numbers are different. What is the largest possible sum of three such numbers?

A 237 B 246 C 255 D 264 E 273

Solution: **C**

Suppose that the six digits that are used are *a, b, c, d, e* and *f* and that we use them to make the three 2-digit numbers '*ad*', '*be*' and '*cf*'. The sum of these numbers is $(10a + d) + (10b + e) + (10c + f)$, that is $10(a + b + c) + (d + e + f)$. To make this as large as possible, we need to choose *a*, *b* and *c so* that $a + b + c$ is as large as possible, and then *d*, *e* and *f* from the remaining digits so that $d + e + f$ is large as possible. Clearly, this is achieved by choosing *a*, *b*, *c* to be 7, 8, 9, in some order, and then *d*, *e, f* to be 4, 5, 6 in some order. So the largest possible sum is $10(7+8+9)+(4+5+6)=255$. [For example, the three 2-digit numbers could be 74, 85 and 96.]

6. What is the sum of the digits of the largest 4-digit palindromic number which is divisible by 15? [Palindromic numbers read the same backwards and forwards, e.g. 7227.]

A 18 B 20 C 24 D 30 E 36

Solution: **C**

A 4-digit palindromic number has the form deed where d and e are digits. This number is divisible by 15 if and only if it is divisible by both 3 and 5. It will be divisible by 5 if and if only if d is 0 or 5, but the first digit of a number cannot be 0, so d is 5, and the number has the form 5ee5.

A number is divisible by 3 if and only if the sum of its digits is divisible by 3. So 5ee5 is divisible by 3 if and only if $10 + 2\text{e}$ is divisible by 3, that is, when e is 7, 4 or 1. So the largest 4-digit palindromic number which is divisible by 15 is 5775. The sum of the digits of 5775 is $5 + 7 + 7 + 5 = 24$.

Extension problems

6.1 Find the largest 4-digit palindromic number which is a multiple of 45.

6.2 Find the largest 5-digit and 6-digit palindromic numbers which are multiples of 15.

6.3 The argument above uses the fact that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3. Explain why this test works.

6.4 Find a similar test for divisibility by 9.

6.5 Find a test, in terms of its digits, to decide when a positive integer is divisible by 11.

7. Given that $x + y + z = 1$, $x + y - z = 2$ and $x - y - z = 3$, what is the value of *xyz*?

A -2 B $-\frac{1}{2}$ C 0 D $\frac{1}{2}$ E 2

Solution: **D**

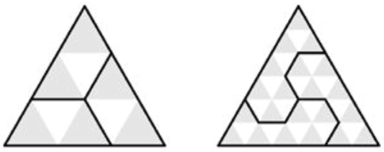
Adding the equations, $x + y + z = 1$ and $x - y - z = 3$, gives $2x = 4$, from which it follows that $x = 2$. Adding the equations $x + y + z = 1$ and $x + y - z = 2$, gives $2(x + y) = 3$. Hence $x + y = \frac{3}{2}$. Therefore, as $x = 2$, it follows that $y = -\frac{1}{2}$. Since $x + y + z = 1$, it now follows that $z = 1 - (x + y) = 1 - \frac{3}{2} = -\frac{1}{2}$. Therefore, $xyz = 2 \times -\frac{1}{2} \times -\frac{1}{2} = \frac{1}{2}$.

8. The diagrams below show four types of tile, each of which is made up of one or more equilateral triangles. For how many of these types of tile can we place three identical copies of the tile together, without gaps or overlaps, to make an equilateral triangle?

A 0 B 1 C 2 D 3 E 4

Solution: **C**

An equilateral triangle may be divided into 1, 4, 9, 16 ,… , that is, a square number, of equilateral triangles of the same size. Three tiles, each made up of k smaller equilateral triangles of the same size, contain altogether $3k$ of the smaller triangles. So, if it is possible to place three copies of the tile to make an equilateral triangle, then $3k$ must be a square number. For the above tiles k takes the values 1, 2, 3 and 12, respectively. Of these, only for $k = 3$ and $k = 12$ do we have that $3k$ is a square. So the only possible cases where three of the above tiles can be rearranged to make an equilateral triangle are the two tiles on the right. We see from the diagrams above that in both cases we can use three of the tiles to make an equilateral triangle. So there are 2 such cases.



Extension problem

8.1 In the argument above we have stated, *but not proved*, that if an equilateral triangle is divided into smaller equilateral triangles of the same size, then the number of the smaller triangles is always a perfect square. Prove that this is correct.

8.2 Determine a necessary and sufficient condition for a positive integer, k, to be such that $3k$ is a square. [Ask your teacher if you are not sure what is meant by a *necessary and sufficient condition*.]

8.3 Is it possible to find a tile which is made up of k equilateral triangles of the same size, where $3k$ is a square, but three copies of the tile cannot be placed together, without gaps or overlaps, to make an equilateral triangle?

9. Pierre said, "Just one of us is telling the truth". Qadr said, "What Pierre says is not true". Ratna said, "What Qadr says is not true". Sven said, "What Ratna says is not true". Tanya said, "What Sven says is not true".

How many of them were telling the truth?

A 0 B 1 C 2 D 3 E 4

Solution: **C**

If Pierre is telling the truth, everyone else is not telling the truth. But, also in this case, what Qadr said is not true, and hence Ratna is telling the truth. So we have a contradiction.

So we deduce that Pierre is not telling the truth. So Qadr is telling the truth. Hence Ratna is not telling the truth. So Sven is also telling the truth, and hence Tanya is not telling the truth. So Qadr and Sven are telling the truth and the other three are not telling the truth.

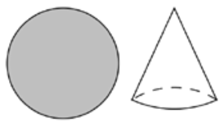
10.	Let N be the smallest positive integer whose digits add up to 2012. What is the first digit of $N+1$? A 2 B 3 C 4 D 5 E 6

Solution: **E**

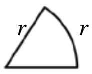
If N is the smallest positive integer whose digits add up to 2012, it will have the smallest possible number of digits among numbers whose digits add up to 2012. This means that, as far as possible, each digit should be a 9. So N will have the form $d\underbrace{999...999}_{k}$, where k is a positive integer and d is a non-zero digit. Now, $2012 = 223 \times 9 + 5$. So $k = 223$ and $d = 5$. Hence $N = 5\underbrace{999...999}_{223}$.

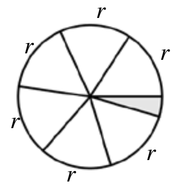
Therefore $N + 1 = 6\underbrace{000...000}_{223}$. So the first digit of $N+1$ is a 6.

11.	Coco is making clown hats from a circular piece of cardboard. The circumference of the base of each hat equals its slant height, which in turn is equal to the radius of the piece of cardboard. What is the maximum number of hats that Coco can make from the piece of cardboard? A 3 B 4 C 5 D 6 E 7



Solution: **D**

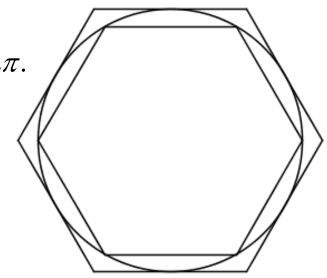




Let the radius of the circle be r. This must also be the slant height of each hat. If we flatten out a hat we get a sector of a circle in which the circular part of the boundary has length r, as shown in the diagram on the left. The circumference of a circle of radius r has length $2\pi r$. Since $3 < \pi < 3.5$, it follows that $6r < 2\pi r < 7r$. So Coco can cut out 6 sectors from the circular piece of card as shown, each of which can be made into a hat. The area of card needed for each hat is $\frac{1}{2}r^2$ and the area of the circle is πr^2. As $\pi r^2 < 7(\frac{1}{2}r^2)$, Coco cannot make 7 hats. So the maximum number of hats that Coco can make is 6.

Extension Problems

In this solution we have used the fact that $3 < \pi < 3.5$. How do we know that this is true? Here we ask you to show this, starting with the definition of π as the ratio of the circumference of a circle to its diameter (and making one geometrical assumption).



11.1 Suppose we have circle with diameter 1. Its radius is therefore 2π. We consider a regular hexagon with its vertices on the circle, and a regular hexagon which touches the circle, as shown in the diagram. Use the assumption that the circumference of the circle lies between the perimeters of the two hexagons to show that $3 < \pi < 2\sqrt{3}$. Deduce that $3 < \pi < 3.5$.

11.2 The method of approximating the circumference of a circle by a regular polygon was discovered independently in more than one culture. The Greek mathematician Archimedes who lived from 287BC to 212BC used regular polygons with 96 sides to obtain the approximation $3\frac{10}{71} < \pi < 3\frac{1}{7}$. The Chinese mathematician Liu Hui, who lived around 260CE, obtained the approximation $3\frac{4407}{62500} < \pi < 3\frac{8919}{62500}$. Use the approximate value $\pi \approx 3.14159$ to estimate the percentage errors in these approximations.

[Today, using iterative methods that converge very rapidly, and powerful computers, π has been calculated to billions of decimal places.]

12. The number 3 can be expressed as the sum of one or more positive integers in four different ways:

$$3; \quad 1+2; \quad 2+1; \quad 1+1+1.$$

In how many ways can the number 5 be so expressed?

A 8 B 10 C 12 D 14 E 16

Solution: **E**

By systematically listing all the possibilities, we see that the number 5 may be written as a sum of one or more positive integers in the following 16 ways:

$$5;\ 1+4;\ 4+1;\ 2+3;\ 3+2;\ 3+1+1;\ 1+3+1;\ 1+1+3;\ 1+2+2;\ 2+1+2;$$
$$2+2+1;\ 1+1+1+2;\ 1+1+2+1;\ 1+2+1+1;\ 2+1+1+1;\ 1+1+1+1+1.$$

Extension Problems

12.1 In how many ways can the number 4 be expressed as the sum of one or more positive integers?

12.2 To generalize these results we need a better method than listing all the possibilities. From the cases $n = 3, 4$ and 5, it seems reason to guess that a positive integer n can be expressed as the sum of one or more positive integers in 2^{n-1} ways. Can you prove that this conjecture is correct?

12.3 We can also consider the number of different ways of expressing a positive integer, n, as the sum of positive integers *when the order does not matter*. For example we count $1+1+2$ as being the same as $1+2+1$. These arrangements, where the order does not matter, are called *partitions*. You can see that there are just 7 different partitions of 5, namely, 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$ and $1+1+1+1+1$. Count the number of different partitions of n, for $1 \le n \le 8$. [There is no simple formula for the number of different partitions of n.]

13. A cube is placed with one face on square 1 in the maze shown, so that it completely covers the square with no overlap. The upper face of the cube is covered in wet paint. The cube is 'rolled' around the maze, rotating about an edge each time, until it reaches square 25. It leaves paint on all of the squares on which the painted face lands, but on no others. The cube is removed on reaching the square 25. What is the sum of the numbers on the squares which are now marked with paint?

5	6	7	8	9
4	19	20	21	10
3	18	25	22	11
2	17	24	23	12
1	16	15	14	13

A 78 B 80 C 82 D 169 E 625

Solution: **B**

We track the position of the side of the cube which is covered in wet paint. We imagine that the maze is horizontal, and that we are looking at it from the side with the squares which are marked with the numbers 1, 16, 15, 14 and 13. We use T, B, F, S, L and R for the Top, Bottom, Front, Stern (back), Left and Right sides of the cube, as we look at it, respectively.

T	S	**B**	L	T
F	S	**B**	L	F
B	S	F	L	**B**
S	S	**B**	L	S
T	S	**B**	L	T

Initially, the wet paint is on Top. We see from the diagram that the wet paint is on the bottom of the cube when the cube is on the squares labelled 3, 7, 11, 15, 20 and 24. The sum of these numbers is $3+7+11+15+20+24=80$.

14. Six students who share a house all speak exactly two languages. Helga speaks only English and German; Ina speaks only German and Spanish; Jean-Pierre speaks only French and Spanish; Karim speaks only German and French; Lionel speaks only French and English whilst Mary speaks only Spanish and English. If two of the students are chosen at random, what is the probability that they speak a common language?

A $\frac{1}{2}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{4}{5}$ E $\frac{5}{6}$

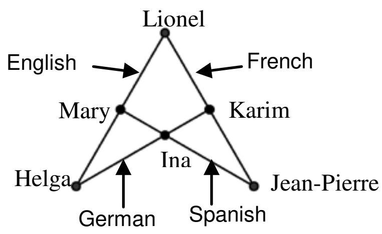
Solution: **D**

We can select two of the six students in 15 ways, as shown in the table. We have put a tick (✓) in each box corresponding to a pair of students who speak a common language, and a cross (✗) in each box corresponding to a pair of students who do not speak a common language.

	Helga E & G	Ina G & S	J-P F & S	Karim G & F	Lionel F & E
Ina G & S	✓				
J-P F & S	✗	✓			
Karim G & F	✓	✓	✓		
Lionel F & E	✓	✗	✓	✓	
Mary S & E	✓	✓	✓	✗	✓

We see that there are 12 ticks in the 15 boxes. Therefore, if two students are chosen at random, the probability that they speak a common language is $\frac{12}{15} = \frac{4}{5}$ ..

Alternative method
Note that each of the students has a common language with 4 out of 5 of the other students. It therefore follows that if two students are chosen at random, the probability that they speak a common language is $\frac{4}{5}$.



Note: We may represent the four languages by lines and the students by points which are on the lines corresponding to the languages they speak, as shown. This makes it easy to see that for each student, there is just one other student with whom they have no common language.

15. Professor Rosseforp runs to work every day. On Thursday he ran 10% faster than his usual average speed. As a result, his journey time was reduced by x minutes. How many minutes did the journey take on Wednesday?

A $11x$ B $10x$ C $9x$ D $8x$ E $5x$

Solution: **A**

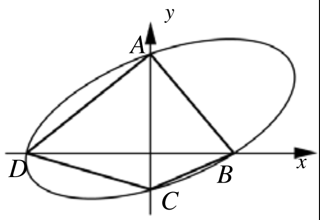
Suppose that the distance that Professor Rosseforp runs is d and his usual average speed (in terms of the units used for distance and minutes) is s. So, on Wednesday his journey takes him $\frac{d}{s}$ minutes. On Thursday his average speed was $\frac{110}{100}s = \frac{11}{10}s$ and so the journey takes him $\frac{d}{\frac{11}{10}s} = \frac{10d}{11s}$ minutes. We therefore have $\frac{d}{s} - \frac{10d}{11s} = x$. This gives $\frac{d}{11s} = x$ and hence $\frac{d}{s} = 11x$. So on Wednesday his journey took him $11x$ minutes.

Extension Problem

15.1 Now consider the general case where Professor Rosseforp runs the same distance $p\%$ slower on Monday than on Tuesday, and $q\%$ faster on Wednesday than on Tuesday. Suppose that his run on Wednesday took him x minutes less than his run on Monday. How many minutes did his run take on Tuesday?

16. The diagram shows the ellipse whose equation is $x^2 + y^2 - xy + x - 4y = 12$. The curve cuts the y-axis at points A and C and cuts the x-axis at points B and D.

What is the area of the inscribed quadrilateral $ABCD$?



A 28 B 36 C 42 D 48 E 56

Solution: **A**

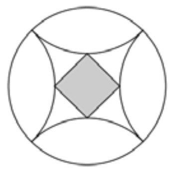
The graph cuts the y-axis at points where $x = 0$, and hence where $y^2 - 4y = 12$, that is, where $y^2 - 4y - 12 = 0$, that is, $(y+2)(y-6) = 0$, giving $y = -2$ and $y = 6$. Hence A is the point (0,6), C is the point $(0,-2)$ and so AC has length 8. Similarly, B and D are points where $y = 0$ and hence $x^2 + x - 12 = 0$, that is, $(x+4)(x-3) = 0$, giving $x = -4$ and $x = 3$. So B is the point $(3,0)$, D is the point $(-4,0)$, and so BD has length 7. The area of the inscribed quadrilateral $ABCD$ is therefore $\frac{1}{2}AC.BD = \frac{1}{2}(8 \times 7) = 28$.

Extension Problems

16.1 The solution above takes it for granted that the area of the quadrilateral $ABCD$ is half the product of the lengths of its diagonals. Give an example to show that this is *not true* for all quadrilaterals.

16.2 What is the special property of $ABCD$ that makes it correct that in this case the area is half the product of the length of the diagonals? Give a proof to show that your answer is correct.

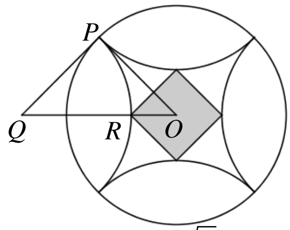
17. The diagram shows a pattern found on a floor tile in the cathedral in Spoleto, Umbria. A circle of radius 1 surrounds four quarter circles, also of radius 1, which enclose a square. The pattern has four axes of symmetry. What is the side length of the square?



A $\frac{1}{\sqrt{2}}$ B $2-\sqrt{2}$ C $\frac{1}{\sqrt{3}}$ D $\frac{1}{2}$ E $\sqrt{2}-1$

Solution: **B**

Let O be the centre of the circle, let P be one of the points where two of the quarter circles meet, let Q be the centre of one of these quarter circles, and let R be the vertex of the square that lies on OQ, as shown.

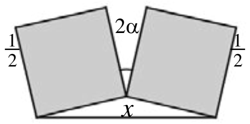


Then in the triangle OPQ there is a right angle at P and $OP = QP = 1$, and therefore, by Pythagoras' Theorem, $OQ = \sqrt{2}$. Since $QR = 1$, it follows that $OR = \sqrt{2} - 1$. It follows that the diagonal of the square has length $2(\sqrt{2} - 1)$. Using, Pythagoras' Theorem again, it follows that the side length of the square is $\frac{1}{\sqrt{2}}\left(2(\sqrt{2}-1)\right) = 2 - \sqrt{2}$.

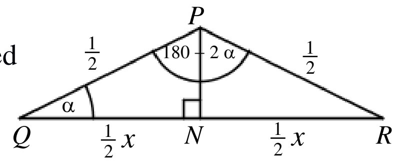
Extension Problem

17.1 In the proof above we claim, by Pythagoras' Theorem, that if the diagonal of a square has length x, then its side length is $\frac{1}{\sqrt{2}}x$. Show that this is indeed a consequence of Pythagoras' Theorem.

18. The diagram shows two squares, with sides of length $\frac{1}{2}$, inclined at an angle 2α to one another. What is the value of *x?* A $\cos\alpha$ B $\dfrac{1}{\cos\alpha}$ C $\sin\alpha$ D $\dfrac{1}{\sin\alpha}$ E $\tan\alpha$

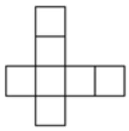


Solution: **A**

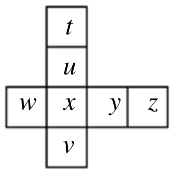


Consider the triangle at the bottom of the diagram. We have labelled the points *P*, *Q* and *R*, as shown. We let *N* be the point where the perpendicular from *P* to *QR* meets *QR*. . Now, $\angle QPR = 360 - 90 - 2\alpha - 90 = 180 - 2\alpha$ **.** In the triangle *PQR* we have $PQ = PR = \frac{1}{2}$ as they are sides of the squares. It follows that the triangle *PQR* is isosceles and hence $\angle PQN = \angle PRN = \alpha$. So the triangles *PQN* and *PRN* are congruent. Hence $QN = NR = \frac{1}{2}x$. Therefore, from the triangle *PQN* we have $\cos\alpha = \dfrac{QN}{QP} = \dfrac{\frac{1}{2}x}{\frac{1}{2}} = x$ .That is, $x = \cos\alpha$.

19. The numbers 2, 3, 4, 5, 6, 7, 8 are to be placed, one per square, in the diagram shown so that the sum of the four numbers in the horizontal row equals 21 and the sum of the four numbers in the vertical column also equals 21. In how many different ways can this be done? A 0 B 2 C 36 D 48 E 72

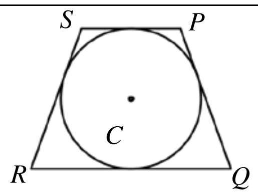


Solution: **E**

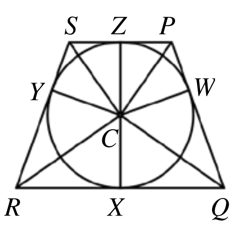


Let the numbers in the squares be *t*, *u*, *v*, *w*, *x*, *y* and *z*, as shown. We have $(w + x + y + z) + (t + u + x + v) = 21 + 21 = 42$, that is $(t + u + v + w + x + y + z) + x = 42$. But $t + u + v + w + x + y + z =$ $2 + 3 + 4 + 5 + 6 + 7 + 8 = 35$ and hence $x = 7$. So *t*, *u*, *v* are three numbers chosen from 2, 3, 4, 5, 6, 8 which add up to 14. It can be checked that there are just two choices for these numbers 2, 4, 8 and 3, 5, 6. If the numbers in the horizontal row are 2, 4, 8 and 7, there are 3 choices for *w* and then 2 choices for *y* and then 1 choice for *z*, making $3 \times 2 \times 1$ choices altogether, and then similarly 6 choices for *t, u, v* , making $6 \times 6 = 36$ combinations. Likewise there are 36 combinations where the numbers in the horizontal row are 3, 5, 6 and 7. This makes $36 + 36 = 72$ different ways altogether.

20. In trapezium *PQRS*, $SR = PQ = 25$ cm and *SP* is parallel to *RQ*. All four sides of *PQRS* are tangent to a circle with centre *C*. The area of the trapezium is 600 cm^2. What is the radius of the circle? A 7.5cm B 8cm C 9cm D 10cm E 12cm



Solution: **E**



Let the radius of circle be *r* cm, and let the points where the trapezium touches the circle be *W*, *X*, *Y* and *Z* as shown. As the radii *CW, CX, CY* and *CZ* are perpendicular to the edges of the trapezium *PQ*, *QR*, *RS* and *SP*, respectively, the area of the trapezium, which is the sum of the areas of the triangles *CPQ, CQR, CRS* and *CSP*, is

$$\tfrac{1}{2}rPQ + \tfrac{1}{2}rQR + \tfrac{1}{2}rRS + \tfrac{1}{2}SP = \tfrac{1}{2}r(PQ + QR + RS + SP).$$

Now using the property that the two tangents to a circle from a given point have equal length, it follows that $SP + QR = SZ + PZ + QX + RX = SY + PW + QW + RY = (PW + QW) + (RY + SY)$ $= PQ + RS = 25 + 25 = 50$ cm. Hence $PQ + QR + RS + SP = 50 + 50 = 100$ cm. Therefore, as the area of the trapezium of 600 cm^2, we have that $\frac{1}{2}r(100) = 600$ and hence $r = 12$.

21. Which of the following numbers does *not* have a square root in the form $x + y\sqrt{2}$, where *x* and *y* are positive integers?

A $17+12\sqrt{2}$ B $22+12\sqrt{2}$ C $38+12\sqrt{2}$ D $54+12\sqrt{2}$ E $73+12\sqrt{2}$

Solution: **D**

Suppose $a+12\sqrt{2}=(x+y\sqrt{2})^2$, where *a*, *x* and *y* are positive integers. It follows that $a+12\sqrt{2}=x^2+2xy\sqrt{2}+2y^2=(x^2+2y^2)+2xy\sqrt{2}$. Therefore $xy=6$ and $x^2+2y^2=a$. Since *x* and *y* are positive integers, the only possibilities for *x* and *y* are $x=1, y=6$; $x=2, y=3$; $x=3, y=2$ and $x=6, y=1$. Therefore the only possibilities for *a* are $1^2+2(6^2)=73$; $2^2+2(3^2)=22$; $3^2+2(2^2)=17$ and $6^2+2(1^2)=38$. Therefore options A, B, C and E are numbers which have square roots of the required form, but not option D.

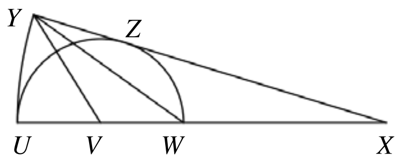
Extension Problems

21.1 Which numbers of the form $a+24\sqrt{2}$, where *a* is a positive integer, have square roots of the form $x+y\sqrt{2}$, where *x* and *y* are positive integers?

21.2 Which numbers of the form $a+24\sqrt{3}$, where *a* is a positive integer, have square roots of the form $x+y\sqrt{3}$, where *x* and *y* are positive integers.

21.2 Give an example of a number of the form $a+b\sqrt{5}$, where *a* and *b* are positive integers, which *does not* have a square root of the form $x+y\sqrt{5}$, where *x* and *y* are positive integers.

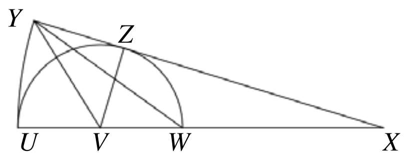
22. A semicircle of radius *r* is drawn with centre *V* and diameter *UW*. The line *UW* is then extended to the point *X*, such that the *UW* and *WX* are of equal length. An arc of the circle with centre *X* and radius 4*r* is then drawn so that the line *XY* is a tangent to the semicircle at *Z*, as shown. What, in terms of *r*, is the area of triangle *YVW?*



A $\frac{4r^2}{9}$ B $\frac{2r^2}{3}$ C r^2 D $\frac{4r^2}{3}$ E $2r^2$

Solution: **B**

Since $UW=WX$, we have $WX=UW=2r$. As *V* is the centre of the circle with *UW* as diameter, $VW=VZ=r$. Hence $VX=VW+WX=3r$. The triangles *YVX* and *VVW* have the same height and $VW=\frac{1}{3}VX$. Therefore $\text{area}(\Delta YVW)=\frac{1}{3}\text{area}(\Delta YVX)$. As *VZ* is a radius of the semicircle, and *XY* is a tangent to the semicircle at *Z*, the lines *VZ* and *XY* are perpendicular. Therefore $\text{area}(\Delta YVX)=\frac{1}{2}VZ.XY=\frac{1}{2}(r.4r)=2r^2$. Hence, $\text{area}(\Delta YVW)=\frac{1}{3}\text{area}(\Delta YVX)=\frac{1}{3}(2r^2)=\frac{2}{3}r^2$.



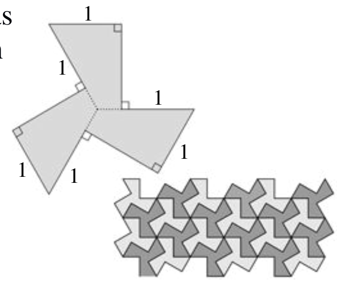
23. Tom and Geri have a competition. Initially, each player has one attempt at hitting a target. If one player hits the target and the other does not then the successful player wins. If both players hit the target, or if both players miss the target, then each has another attempt, with the same rules applying. If the probability of Tom hitting the target is always $\frac{4}{5}$ and the probability of Geri hitting the target is always $\frac{2}{3}$, what is the probability that Tom wins the competition?

A $\dfrac{4}{15}$ B $\dfrac{8}{15}$ C $\dfrac{2}{3}$ D $\dfrac{4}{5}$ E $\dfrac{13}{15}$

Solution: **C**

Let the probability that Tom wins the competition be *p*. The probability that initially Tom hits and Geri misses is $\frac{4}{5}\times\frac{1}{3}=\frac{4}{15}$. The probability that initially they both hit is $\frac{4}{5}\times\frac{2}{3}=\frac{8}{15}$ and that they both miss is $\frac{1}{5}\times\frac{1}{3}=\frac{1}{15}$. So the probability that either they both hit or both miss is $\frac{8}{15}+\frac{1}{15}=\frac{9}{15}=\frac{3}{5}$. If they both hit or both miss the competition is in the same position as it was initially. So Tom's probability of winning is then *p*. Therefore, $p=\frac{4}{15}+\frac{3}{5}p$. So $\frac{2}{5}p=\frac{4}{15}$ and hence $p=\frac{2}{3}$.

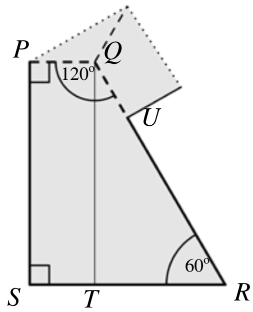
24. The top diagram on the right shows a shape that tiles the plane, as shown in the lower diagram. The tile has nine sides, six of which have length 1. It may be divided into three congruent quadrilaterals as shown. What is the area of the tile?



A $\dfrac{1+2\sqrt{3}}{2}$ B $\dfrac{4\sqrt{3}}{3}$ C $\sqrt{6}$ D $\dfrac{3+4\sqrt{3}}{4}$ E $\dfrac{3\sqrt{3}}{2}$

Solution: **B**

The tile is made up of three congruent quadrilaterals one of which, *PQRS*, is shown in the diagram on the right. We let *T* be the foot of the perpendicular from *Q* to *SR* and we let *U* be the point shown. Then $SR=UR=1$.



Also *PQ*=*QU*. Let their common length be *x*. Hence $QR=1+x$ and $TR=1-x$. Because the three congruent quadrilaterals fit together at *Q*, we have $\angle PQR=120^0$. As *PQ* and *SR* are both perpendicular to *PS*, they are parallel and hence $\angle QRT=60^0$. Hence, from the right-angled triangle *QRT* we have $\dfrac{TR}{QR}=\sin 60^0$, that is, $\dfrac{1-x}{1+x}=\dfrac{1}{2}$, and therefore $2-2x=1+x$. Hence $3x=1$ and so $x=\frac{1}{3}$. Therefore $TR=\frac{2}{3}$ and $QR=\frac{4}{3}$. Now $QT/QR=\cos 60^0$ and hence $QT=QR\cos 60^0$ $=\frac{4}{3}\cos 60^0=\frac{4}{3}(\frac{\sqrt{3}}{2})=\frac{2\sqrt{3}}{3}$. [We could also use Pythagoras' Theorem applied to the triangle *QRT* to deduce this length.]

We can now work out the area of the quadrilateral *PQRS* in two different ways. Most straightforwardly, area(*PQRS*) = area(*PQTS*) + area(*QRT*) $=\frac{2\sqrt{3}}{3}\times\frac{1}{3}+\frac{1}{2}(\frac{2\sqrt{3}}{3}\times\frac{2}{3})=\frac{4\sqrt{3}}{9}$. Alternatively, as *PQ* is parallel to *SR*, *PQRS* is a trapezium. We now use the fact that the area of a trapezium is its height multiplied by the average length of its parallel sides. Hence, *PQRS* has area. $\frac{2\sqrt{3}}{3}\times\frac{1}{2}(\frac{1}{3}+1)$ $=\frac{2\sqrt{3}}{3}\times\frac{2}{3}=\frac{4\sqrt{3}}{9}$. The tile is made up of three copies of *PQRS*. So its area is $3\times\frac{4\sqrt{3}}{9}=\frac{4\sqrt{3}}{3}$.

25. How many distinct pairs (x, y) of real numbers satisfy the equation $(x+y)^2 = (x+4)(y-4)$?

A 0 B 1 C 2 D 3 E 4

Solution: **B**

Method 1. The most straightforward method is to rewrite the given equation as a quadratic in *x*, whose coefficients involve *y*, and then use the "$b^2 - 4ac \geq 0$" condition for a quadratic, $ax^2 + bx + c = 0$, to have real number solutions.

Now, $(x+y)^2 = (x+4)(y-4) \Leftrightarrow x^2 + 2xy + y^2 = xy - 4x + 4y - 16$

$$\Leftrightarrow x^2 + xy + 4x + y^2 - 4y + 16 = 0 \Leftrightarrow x^2 + (y+4)x + (y^2 - 4y + 16) = 0.$$

So here $a = 1$, $b = y + 4$ and $c = y^2 - 4y + 16$. We see that $b^2 - 4ac = (y+4)^2 - 4(y^2 - 4y + 16)$ $= -3y^2 + 24y - 48 = -3(y^2 - 8y + 16) = -3(y-4)^2$. So there is a real number solution for *x* if and only if $-3(y-4)^2 \geq 0$. Now, as for all real numbers *y*, $(y-4)^2 \geq 0$, it follows that $-3(y-4)^2 \geq 0 \Leftrightarrow y - 4 = 0 \Leftrightarrow y = 4$. When $y = 4$ the quadratic equation becomes $x^2 + 8x + 16 = 0$, that is $(x+4)^2 = 0$, which has just the one solution $x = -4$. So $(-4,4)$ is the only pair of real numbers, (x, y), for which $(x+y)^2 = (x+4)(y-4)$.

Method 2. We can simplify the algebra by making the substitution $w = x + 4$ and $z = y - 4$. Then $x + y = (w-4) + (z+4) = w + z$, and the equation becomes $(w+z)^2 = wz$.

Now, $(w+z)^2 = wz \Leftrightarrow w^2 + 2wz + z^2 = wz \Leftrightarrow w^2 + wz + z^2 = 0 \Leftrightarrow (w + \tfrac{1}{2}z)^2 + (\tfrac{\sqrt{3}}{2}z)^2 = 0$. The sum of the squares of two real numbers is zero if and only if each real number is 0. So the only real number solution of $(w + \tfrac{1}{2}z)^2 + (\tfrac{\sqrt{3}}{2}z)^2 = 0$ is $w + \tfrac{1}{2}z = \tfrac{\sqrt{3}}{2}z = 0$. This is equivalent to $w = z = 0$ and hence to $x = -4$ and $y = 4$. So again we deduce that there is just this one solution.

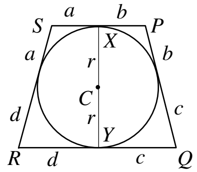
Extension Problems

25.1 It is possible to use the "$b^2 \geq 4ac$" criterion to show that $(0,0)$ is the only pair of real numbers that satisfy the equation $w^2 + wz + z^2 = 0$. Check this.

25.2 Show that the "$b^2 \geq 4ac$" criterion is correct by proving that for all real numbers, *a*, *b*, *c*, with $a \neq 0$, the quadratic equation $ax^2 + bx + c = 0$ has a real number solution if and only if $b^2 \geq 4ac$.

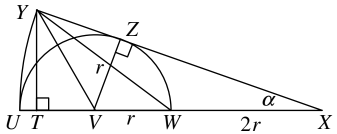
25.3 [For those who know about complex numbers.] If we allow the possibility that *x* and *y* are complex numbers, then the equation $(x+y)^2 = (x+4)(y-4)$ has more than one solution. Check that $x = 6$, $y = -1 + 5\sqrt{3}i$ is one solution of this equation. How many more can you find?

20. E The two tangents drawn from a point outside a circle to that circle are equal in length. This theorem has been used to mark four pairs of equal line segments on the diagram. In the circle the diameter, XY, has been marked. It is also a perpendicular height of the trapezium. We are given that $SR = PQ = 25$ cm so we can deduce that $(a + d) + (b + c) = 25 + 25 = 50$. The area of trapezium $PQRS = \frac{1}{2}(SP + QR) \times XY = 600\ \text{cm}^2$. Therefore $\frac{1}{2}(a + b + c + d) \times 2r = 600$. So $\frac{1}{2} \times 50 \times 2r = 600$, i.e. $r = 12$.



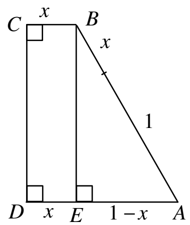
21. D $(x + y\sqrt{2})^2 = x^2 + 2xy\sqrt{2} + 2y^2$. Note that all of the alternatives given are of the form $a + 12\sqrt{2}$ so we need $xy = 6$. The only ordered pairs (x, y) of positive integers which satisfy this are (1, 6), (2, 3), (3, 2), (6, 1). For these, the values of $x^2 + 2y^2$ are 73, 22, 17, 38 respectively. So the required number is $54 + 12\sqrt{2}$.

22. B Let the perpendicular from Y meet UV at T and let $\angle ZXV = \alpha$. Note that $\angle VZX = 90°$ as a tangent to a circle is perpendicular to the radius at the point of contact. Therefore $\sin\alpha = \frac{r}{3r} = \frac{1}{3}$. Consider triangle YTX: $\sin\alpha = \frac{YT}{YX}$. So $YT = YX\sin\alpha = \frac{4r}{3}$. So the area of triangle $YVW = \frac{1}{2} \times VW \times YT = \frac{1}{2} \times r \times \frac{4r}{3} = \frac{2r^2}{3}$.



23. C Tom wins after one attempt each if he hits the target and Geri misses. The probability of this happening is $\frac{4}{5} \times \frac{1}{3} = \frac{4}{15}$. Similarly the probability that Geri wins after one attempt is $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$. So the probability that both competitors will have at least one more attempt is $1 - \frac{4}{15} - \frac{2}{15} = \frac{3}{5}$.
Therefore the probability that Tom wins after two attempts each is $\frac{3}{5} \times \frac{4}{15}$. The probability that neither Tom nor Geri wins after two attempts each is $\frac{3}{5} \times \frac{3}{5}$. So the probability that Tom wins after three attempts each is $\left(\frac{3}{5}\right)^2 \times \frac{4}{15}$ and, more generally, the probability that he wins after n attempts each is $\left(\frac{3}{5}\right)^{n-1} \times \frac{4}{15}$.
Therefore the probability that Tom wins is $\frac{4}{15} + \left(\frac{3}{5}\right) \times \frac{4}{15} + \left(\frac{3}{5}\right)^2 \times \frac{4}{15} + \left(\frac{3}{5}\right)^3 \times \frac{4}{15} + \ldots$.
This is the sum to infinity of a geometric series with first term $\frac{4}{15}$ and common ratio $\frac{3}{5}$. Its value is $\frac{4}{15} \div \left(1 - \frac{3}{5}\right) = \frac{2}{3}$.

24. B The diagram shows one of the three quadrilaterals making up the tile, labelled and with a line BE inserted. Note that it is a trapezium. As three quadrilaterals fit together, it may be deduced that $\angle ABC = 360° \div 3 = 120°$, so $\angle BAD = 60°$. It may also be deduced that the length of AB is $1 + x$, where x is the length of BC. Now $\cos\angle BAD = \cos 60° = \frac{1}{2} = \frac{1-x}{1+x}$. So $1 + x = 2 - 2x$, i.e. $x = \frac{1}{3}$. The area of $ABCD$ is $\frac{1}{2}(AD + BC) \times CD = \frac{1}{2}\left(1 + \frac{1}{3}\right) \times \frac{4}{3}\sin 60°$ $= \frac{2}{3} \times \frac{4}{3} \times \frac{\sqrt{3}}{2} = \frac{4\sqrt{3}}{9}$. So the area of the tile is $3 \times \frac{4\sqrt{3}}{9} = \frac{4\sqrt{3}}{3}$.



25. B Starting with $(x + y)^2 = (x + 4)(y - 4)$ and expanding both sides gives $x^2 + 2xy + y^2 = xy - 4x + 4y - 16$, i.e. $x^2 + (y + 4)x + y^2 - 4y + 16 = 0$.
To eliminate the xy term we let $z = x + \frac{1}{2}y$ and then replace x by $z - \frac{1}{2}y$. The equation above becomes $z^2 + 4\left(z - \frac{1}{2}y\right) + \frac{3}{4}y^2 - 4y + 16 = 0$. However,

$$z^2 + 4\left(z - \tfrac{1}{2}y\right) + \tfrac{3}{4}y^2 - 4y + 16 = (z + 2)^2 + \tfrac{3}{4}y^2 - 6y + 12$$

$$= (z + 2)^2 + \tfrac{3}{4}\left(y^2 - 8y + 16\right) = (z + 2)^2 + \tfrac{3}{4}(y + 4)^2.$$

So the only real solution is when $z = -2$ and $y = 4$; i.e. $x = -4$ and $y = 4$.

1.	E
2.	B
3.	D
4.	B
5.	C
6.	C
7.	D
8.	C
9.	C
10.	E
11.	D
12.	E
13.	B
14.	D
15.	A
16.	A
17.	B
18.	A
19.	E
20.	E
21.	D
22.	B
23.	C
24.	B
25.	B

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**
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SOLUTIONS

Keep these solutions secure until after the test on

TUESDAY 6 NOVEMBER 2012

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. **E** If an odd number is written as the sum of two prime numbers then one of those primes is 2, since 2 is the only even prime. However, 9 is not prime so 11 cannot be written as the sum of two primes. Note that $5 = 2 + 3$; $7 = 2 + 5$; $9 = 2 + 7$; $10 = 3 + 7$, so 11 is the only alternative which is not the sum of two primes.

2. **B** The interior angles of an equilateral triangle, square, regular pentagon are 60°, 90°, 108° respectively. The sum of the angles at a point is 360°. So $\theta = 360 - (60 + 90 + 108) = 102$.

3. **D** The cost now is $(70 + 4 \times 5 + 6 \times 2)$p = £1.02.

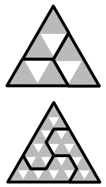
4. **B** One hundred thousand million is $10^2 \times 10^3 \times 10^6 = 10^{11}$. So the number of stars is $10^{11} \times 10^{11} = 10^{22}$.

5. **C** Let the required addition be '*ab*' + '*cd*' + '*ef*', where a, b, c, d, e, f are single, distinct digits. To make this sum as large as possible, we need a, c, e (the tens digits) as large as possible; so they must be 7, 8, 9 in some order. Then we need b, d, f as large as possible, so 4, 5, 6 in some order. Hence the largest sum is $10(7 + 8 + 9) + (4 + 5 + 6) = 10 \times 24 + 15 = 255$.

6. **C** In order to be a multiple of 15, a number must be a multiple both of 3 and of 5. So its units digit must be 0 or 5. However, the units digit must also equal the thousands digit and this cannot be 0, so the required number is of the form '5*aa*5'. The largest such four-digit numbers are 5995, 5885, 5775. Their digit sums are 28, 26, 24 respectively. In order to be a multiple of 3, the digit sum of a number must also be a multiple of 3, so 5775 is the required number. The sum of its digits is 24.

7. **D** Add the first and third equations: $2x = 4$, so $x = 2$. Add the first two equations: $2x + 2y = 3$, so $y = -\frac{1}{2}$. Substitute for x and y in the first equation: $2 + \left(-\frac{1}{2}\right) + z = 1$ so $z = -\frac{1}{2}$. Therefore $xyz = 2 \times \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) = \frac{1}{2}$.

8. **C** If an equilateral triangle is split into a number of smaller identical equilateral triangles then there must be one small triangle in the top row, three small triangles in the row below, five small triangles in the row below that and so on. So the total number of small triangles is 4 or 9 or 16 etc. These are all squares and it is left to the reader to prove that the sum of the first n odd numbers is n^2. So, for three copies of a given tile to form an equilateral triangle, the number of triangles which comprise the tile must be one third of a square number.

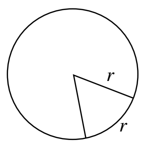


Only the tiles made up of three equilateral triangles and twelve equilateral triangles satisfy this condition. However, it is still necessary to show that three copies of these tiles can indeed make equilateral triangles. The diagrams above show how they can do this.

9. **C** If Pierre is telling the truth then Qadr is not telling the truth. However, this means that Ratna is telling the truth, so this leads to a contradiction as Pierre stated that just one person is telling the truth. So Pierre is not telling the truth, which means that Qadr is telling the truth, but Ratna is not telling the truth. This in turn means that Sven is telling the truth, but Tanya is not. So only Qadr and Sven are telling the truth.

10. **E** It can be deduced that N must consist of at least 224 digits since the largest 223-digit positive integer consists of 223 nines and has a digit sum of 2007. It is possible to find 224-digit positive integers which have a digit sum of 2012.
The largest of these is 99 999 ...999 995 and the smallest is 59 999 ...999 999.
So N = 59 999… 999 999 and $N + 1$ = 60 000… 000 000 (223 zeros).

11. **D** Let the radius of the circular piece of cardboard be r. The diagram shows a sector of the circle which would make one hat, with the minor arc shown becoming the circumference of the base of the hat. The circumference of the circle is $2\pi r$. Now $6r < 2\pi r < 7r$. This shows that we can cut out 6 hats in this fashion and also shows that the area of cardboard unused in cutting out *any* 6 hats is less than the area of a single hat. Hence there is no possibility that more than 6 hats could be cut out.



12. **E** Two different ways of expressing 5 are 1 + 4 and 4 + 1. In the following list these are denoted as {1, 4: two ways}. The list of all possible ways is {5: one way}, {2, 3: two ways}, {1, 4: two ways}, {1, 2, 2: three ways}, {1, 1, 3: three ways}, {1, 1, 1, 2: four ways}, {1, 1, 1, 1, 1: one way}. So in total there are 16 ways.
{*Different expressions of a positive integer in the above form are known as 'partitions'. It may be shown that the number of distinct compositions of a positive integer n is* 2^{n-1}.}

13. **B** The table below shows the position of the face marked with paint when the base of the cube is on the 25 squares. Code: T - top, B - base; F - front; H - hidden (rear); L - left; R - right.

1	2	**3**	4	5	6	**7**	8	9	10	**11**	12	13	14	**15**	16	17	18	19	**20**	21	22	23	**24**	25
T	H	**B**	F	T	R	**B**	L	T	F	**B**	H	T	L	**B**	R	R	R	R	**B**	L	L	L	**B**	F

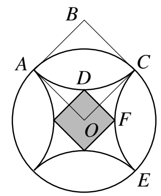
So the required sum is $3 + 7 + 11 + 15 + 20 + 24 = 80$.

14. **D** Note that each student has a language in common with exactly four of the other five students. For instance, Jean-Pierre has a language in common with each of Ina, Karim, Lionel and Mary. Only Helga does not have a language in common with Jean-Pierre. So whichever two students are chosen, the probability that they have a language in common is 4/5.

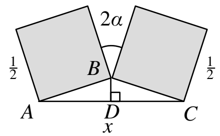
15. **A** Let Professor Rosseforp's usual journey take t minutes at an average speed of v metres/minute. Then the distance to work is vt metres. On Thursday his speed increased by 10%, i.e. it was $11v/10$ metres/minute. The time taken was $(t - x)$ minutes.
Therefore $11v/10 \times (t - x) = vt$. So $11(t - x) = 10t$, i.e. $t = 11x$.

16. **A** At points A and C, $x = 0$. So $y^2 - 4y = 12$, i.e. $(y - 6)(y + 2) = 0$, i.e. $y = 6$ or $y = -2$. So C is $(0, -2)$ and A is $(0, 6)$. At points B and D, $y = 0$. So $x^2 + x = 12$, i.e. $(x - 3)(x + 4) = 0$, i.e. $x = 3$ or $x = -4$. So D is $(-4, 0)$ and B is $(3, 0)$.
Therefore the areas of triangles DAB and DBC are $\frac{1}{2} \times 7 \times 6 = 21$ and $\frac{1}{2} \times 7 \times 2 = 7$.
So $ABCD$ has area 28. {*It is left to the reader to prove that area ABCD* $= \frac{1}{2}BD \times AC$.}

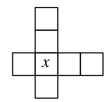
17. **B** In the diagram, B is the centre of the quarter-circle arc AC; D is the point where the central square touches arc AC; F is the point where the central square touches arc CE; O is the centre of the circle.
As both the circle and arc AC have radius 1, $OABC$ is a square of side 1.
By Pythagoras' Theorem: $OB^2 = 1^2 + 1^2$. So $OB = \sqrt{2}$. Therefore $OD = OB - DB = \sqrt{2} - 1$. By a similar argument, $OF = \sqrt{2} - 1$. Now $DF^2 = OD^2 + OF^2 = 2 \times OD^2$ since $OD = OF$. So the side of the square is $\sqrt{2} \times OD = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$.



18. **A** In the diagram, D is the midpoint of AC. Triangle ABC is isosceles since $AB = BC = \frac{1}{2}$. Therefore, BD bisects $\angle ABC$ and BD is perpendicular to AC. The angles at a point total 360°, so $\angle ABC = 360° - 2 \times 90° - 2\alpha = 180° - 2\alpha$. Therefore $\angle ABD = \angle CBD = 90° - \alpha$. So $\angle BAD = \angle BCD = \alpha$.
Therefore $x = AC = 2 \times AD = 2 \times AB \cos\alpha = 2 \times \frac{1}{2}\cos\alpha = \cos\alpha$.



19. **E** Note that the number represented by x appears in both the horizontal row and the vertical column. Note also that $2 + 3 + 4 + 5 + 6 + 7 + 8 = 35$. Since the numbers in the row and those in the column have sum 21, we deduce that $x = 2 \times 21 - 35 = 7$.



We now need two disjoint sets of three numbers chosen from 2, 3, 4, 5, 6, 8 so that the numbers in both sets total 14. The only possibilities are {2, 4, 8} and {3, 5, 6}. We have six choices of which number to put in the top space in the vertical line, then two for the next space down and one for the bottom space. That leaves three choices for the first space in the horizontal line, two for the next space and one for the final space. So the total number of ways is $6 \times 2 \times 1 \times 3 \times 2 \times 1 = 72$.

UK SENIOR MATHEMATICAL CHALLENGE

Thursday 7 November 2013

Organised by the **United Kingdom Mathematics Trust**

and supported by

Institute and Faculty of Actuaries

RULES AND GUIDELINES (to be read before starting)

1. Do not open the question paper until the invigilator tells you to do so.
2. **Use B or HB pencil only**. Mark *at most one* of the options A, B, C, D, E on the Answer Sheet for each question. Do not mark more than one option.
3. Time allowed: **90 minutes**.
   No answers or personal details may be entered on the Answer Sheet after the 90 minutes are over.
4. The use of rough paper is allowed.
   Calculators, measuring instruments and squared paper are forbidden.
5. Candidates must be full-time students at secondary school or FE college, and must be in Year 13 or below (England & Wales); S6 or below (Scotland); Year 14 or below (Northern Ireland).
6. There are twenty-five questions. Each question is followed by five options marked A, B, C, D, E. Only one of these is correct. Enter the letter A-E corresponding to the correct answer in the corresponding box on the Answer Sheet.
7. **Scoring rules**: all candidates start out with 25 marks;
   0 marks are awarded for each question left unanswered;
   4 marks are awarded for each correct answer;
   1 mark is deducted for each incorrect answer.
8. **Guessing**: Remember that there is a penalty for wrong answers. Note also that later questions are deliberately intended to be harder than earlier questions. You are thus advised to concentrate first on solving as many as possible of the first 15-20 questions. Only then should you try later questions.

1. Which of these is the largest number?

A $2 + 0 + 1 + 3$ B $2 \times 0 + 1 + 3$ C $2 + 0 \times 1 + 3$ D $2 + 0 + 1 \times 3$ E $2 \times 0 \times 1 \times 3$

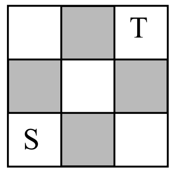
2. Little John claims he is 2m 8cm and 3mm tall. What is this height in metres?

A 2.83m B 2.803m C 2.083m D 2.0803m E 2.0083m

3. What is the 'tens' digit of $2013^2 - 2013$?

A 0 B 1 C 4 D 5 E 6

4. A route on the 3×3 board shown consists of a number of steps. Each step is from one square to an adjacent square of a different colour. How many different routes are there from square S to square T which pass through every other square exactly once?



A 0 B 1 C 2 D 3 E 4

5. The numbers x and y satisfy the equations $x(y + 2) = 100$ and $y(x + 2) = 60$. What is the value of $x - y$?

A 60 B 50 C 40 D 30 E 20

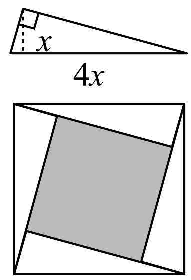
6. Rebecca went swimming yesterday. After a while she had covered one fifth of her intended distance. After swimming six more lengths of the pool, she had covered one quarter of her intended distance. How many lengths of the pool did she intend to complete?

A 40 B 72 C 80 D 100 E 120

7. In a 'ninety nine' shop, all items cost a number of pounds and 99 pence. Susanna spent £65.76. How many items did she buy?

A 23 B 24 C 65 D 66 E 76

8. The right-angled triangle shown has a base which is 4 times its height. Four such triangles are placed so that their hypotenuses form the boundary of a large square as shown.
What is the side-length of the shaded square in the diagram?



A $2x$ B $2\sqrt{2}x$ C $3x$ D $2\sqrt{3}x$ E $\sqrt{15}x$

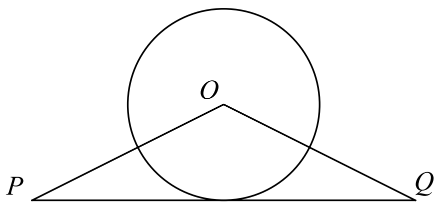
9. According to a headline, 'Glaciers in the French Alps have lost a quarter of their area in the past 40 years'. What is the approximate percentage reduction in the length of the side of a square when it loses one quarter of its area, thereby becoming a smaller square?

A 13% B 25% C 38% D 50% E 65%

10. Frank's teacher asks him to write down five integers such that the median is one more than the mean, and the mode is one greater than the median. Frank is also told that the median is 10. What is the smallest possible integer that he could include in his list?

A 3 B 4 C 5 D 6 E 7

11. The diagram shows a circle with centre O and a triangle OPQ. Side PQ is a tangent to the circle. The area of the circle is equal to the area of the triangle. What is the ratio of the length of PQ to the circumference of the circle?



A $1:1$ B $2:3$ C $2:\pi$ D $3:2$ E $\pi:2$

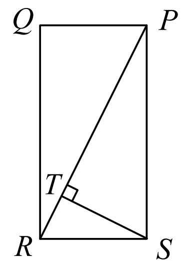
12. As a special treat, Sammy is allowed to eat five sweets from his very large jar which contains many sweets of each of three flavours − Lemon, Orange and Strawberry. He wants to eat his five sweets in such a way that no two consecutive sweets have the same flavour. In how many ways can he do this?

A 32 B 48 C 72 D 108 E 162

13. Two entrants in a school's sponsored run adopt different tactics. Angus walks for half the time and runs for the other half, whilst Bruce walks for half the distance and runs for the other half. Both competitors walk at 3mph and run at 6mph. Angus takes 40 minutes to complete the course. How many minutes does Bruce take?

A 30 B 35 C 40 D 45 E 50

14. The diagram shows a rectangle $PQRS$ in which $PQ : QR = 1 : 2$. The point T on PR is such that ST is perpendicular to PR. What is the ratio of the area of the triangle RST to the area of the rectangle $PQRS$?



A $1:4\sqrt{2}$ B $1:6$ C $1:8$ D $1:10$ E $1:12$

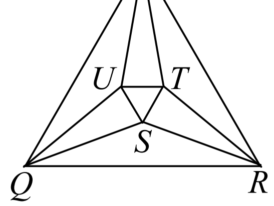
15. For how many positive integers n is $4^n - 1$ a prime number?

A 0 B 1 C 2 D 3 E infinitely many

16. Andrew states that every composite number of the form $8n + 3$, where n is an integer, has a prime factor of the same form. Which of these numbers is an example showing that Andrew's statement is false?

A 19 B 33 C 85 D 91 E 99

17. The equilateral triangle PQR has side-length 1. The lines PT and PU trisect the angle RPQ, the lines RS and RT trisect the angle QRP and the lines QS and QU trisect the angle PQR. What is the side-length of the equilateral triangle STU?

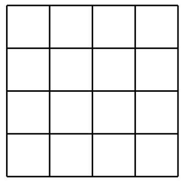


A $\dfrac{\cos 80°}{\cos 20°}$ B $\frac{1}{3}\cos 20°$ C $\cos^2 20°$ D $\frac{1}{6}$ E $\cos 20° \cos 80°$

18. The numbers 2, 3, 12, 14, 15, 20, 21 may be divided into two sets so that the product of the numbers in each set is the same. What is this product?

A 420 B 1260 C 2520 D 6720 E 6350400

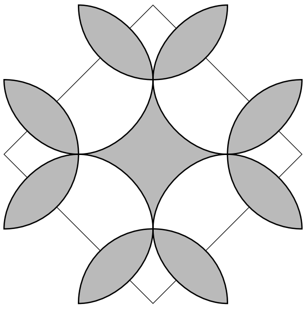
19. The 16 small squares shown in the diagram each have a side length of 1 unit. How many pairs of vertices are there in the diagram whose distance apart is an integer number of units?



A 40 B 64 C 108 D 132 E 16

20. The ratio of two positive numbers equals the ratio of their sum to their difference. What is this ratio?

A $(1+\sqrt{3}):2$ B $\sqrt{2}:1$ C $(1+\sqrt{5}):2$ D $(2+\sqrt{2}):1$ E $(1+\sqrt{2}):1$

21. 

The shaded design shown in the diagram is made by drawing eight circular arcs, all with the same radius. The centres of four arcs are the vertices of the square; the centres of the four touching arcs are the midpoints of the sides of the square. The diagonals of the square have length 1.

What is the total length of the border of the shaded design?

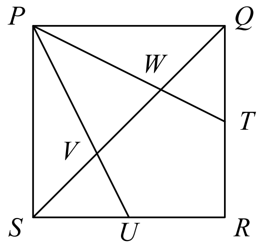
A 2π B $\frac{5\pi}{2}$ C 3π D $\frac{7\pi}{2}$ E 4π

22. Consider numbers of the form $10n + 1$, where n is a positive integer. We shall call such a number 'grime' if it cannot be expressed as the product of two smaller numbers, possibly equal, both of which are of the form $10k + 1$, where k is a positive integer.

How many 'grime numbers' are there in the sequence 11, 21, 31, 41, ..., 981, 991?

A 0 B 8 C 87 D 92 E 99

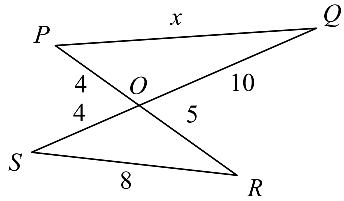
23. $PQRS$ is a square. The points T and U are the midpoints of QR and RS respectively. The line QS cuts PT and PU at W and V respectively. What fraction of the area of the square $PQRS$ is the area of the pentagon $RTWVU$?



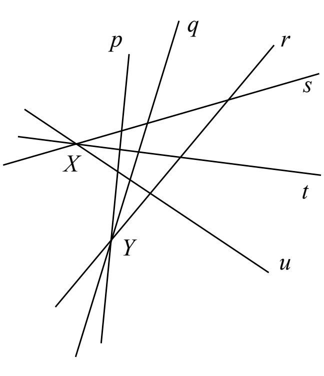
A $\frac{1}{3}$ B $\frac{2}{5}$ C $\frac{3}{7}$ D $\frac{5}{12}$ E $\frac{4}{15}$

24. The diagram shows two straight lines PR and QS crossing at O.

What is the value of x?



A $7\sqrt{2}$ B $2\sqrt{29}$ C $14\sqrt{2}$ D $7(1+\sqrt{13})$ E $9\sqrt{2}$

25. 

Challengeborough's underground train network consists of six lines, p, q, r, s, t, u, as shown. Wherever two lines meet there is a station which enables passengers to change lines. On each line, each train stops at every station.

Jessica wants to travel from station X to station Y. She does not want to use any line more than once, nor return to station X after leaving it, nor leave station Y having reached it.

How many different routes, satisfying these conditions, can she choose?

A 9 B 36 C 41 D 81 E 720

Senior Mathematical Challenge

Thursday 7 November 2013

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries

Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct.Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

UKMT, School of Mathematics Satellite,
University of Leeds, Leeds LS2 9JT

☎ *0113 343 2339* enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	C	D	C	E	E	B	B	A	B	A	B	D	D	B	D	A	C	C	E	B	C	A	E	D

1. Which of these is the largest number?

A $2+0+1+3$ B $2\times0+1+3$ C $2+0\times1+3$ D $2+0+1\times3$ E $2\times0\times1\times3$

Solution

A We calculate the value of each of the given options in turn.

(a) $2+0+1+3=6$

(b) $2\times0+1+3=0+1+3=4$

(c) $2+0\times1+3=2+0+3=5$

(d) $2+0+1\times3=2+0+3=5$

(e) $2\times0\times1\times3=0$

So option A gives the largest number.

Remarks

You may have obtained the wrong answer if you interpreted $2+0+1\times3$ to mean $((2+0)+1)\times3$ rather than $2+0+(1\times3)$. It is a standard convention (sometimes known as BIDMAS or BODMAS) that in evaluating an expression such as $2+0+1\times3$, the multiplications are carried out before the additions. We do not just carry out the operations from left to right. So in calculating $2+0+1\times3$, the multiplication 1×3 is done before the additions.

A decent calculator will produce the correct answer 5 if you press the keys

[2] [+] [0] [+] [1] [×] [3] [=]

in this order. If your calculator produces a different answer, you should replace it!

2. Little John claims he is 2 m 8 cm and 3 mm tall.

What is this height in metres?

A 2.83 m B 2.803 m C 2.083 m D 2.0803 m E 2.0083 m

Solution

C One metre is 100 centimetres. So 1 cm = 0.01 m and 8 cm = 0.08 m. Similarly, one metre is 1000 millimetres. So 1 mm = 0.001 m and 3 mm = 0.003 m. Therefore Little John's height is 2 m + 0.08 m + 0.003 m = 2.083 m.

3. What is the 'tens' digit of $2013^2 - 2013$?

A 0 B 1 C 4 D 5 E 6

Solution

D The 'tens' digit of $2013^2 - 2013$ is the same as that of $13^2 - 13$. Since $13^2 - 13 = 169 - 13 = 156$, the 'tens' digit of $2013^2 - 2013$ is 5.

Remarks

Our comment that the 'tens' digit of $2013^2 - 2013$ is the same as that of $13^2 - 13$ uses the fact that $2013^2 - 2013 = (2000 + 13)^2 - 2013 = (2000^2 + 2 \times 2000 \times 13 + 13^2) - (2000 + 13) = 4000000 + 52000 - 2000 + 13^2 - 13$. It is now clear that only the last two terms, that is, $13^2 - 13$, can have any effect on the 'tens' and 'units' digits of the answer.

A more sophisticated way to say this is to use the language and notation of *modular arithmetic*, which you may already have met. Using this notation we write $a \equiv b \pmod{n}$ to mean that a and b have the same remainder when divided by n. For example, $2013 \equiv 13 \pmod{100}$ and $156 \equiv 56 \pmod{100}$. Then we can say that $2013^2 - 2013 \equiv 13^2 - 13 \pmod{100}$ and $13^2 - 13 \equiv 56 \pmod{100}$. It follows that $2013^2 - 2013 \equiv 56 \pmod{100}$. Thus $2013^2 - 2013$ has remainder 56 when divided by 100. So its last two digits are 5 and 6. In particular its 'tens' digit is 5 and its 'units' digit is 6.

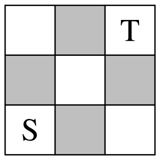
For investigation

3.1 Find the 'tens' digits of (a) $2014^2 - 2014$ and (b) $2013^3 - 2013^2$.

3.2 Find the 'tens' and 'units' digits of (a) 2011^{2011} and (b) 2013^{2013}.

4. A route on the 3 × 3 board shown consists of a number of steps. Each step is from one square to an adjacent square of a different colour.

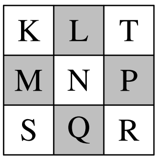
How many different routes are there from square S to square T which pass through every other square exactly once?

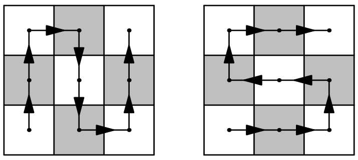


A 0 B 1 C 2 D 3 E 4

SOLUTION

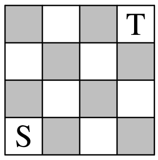
C For convenience, label the other squares as in the left-hand figure. The first move of a route from S to T must be either S→M or S→Q. It is easy to see that a route that visits all the squares must include both the sequence M→K→L and the sequence Q→R→P. Hence, we see that there are just two routes that meet all the required conditions; these are shown in two figures on the the right.





FOR INVESTIGATION

4.1 Consider the analogous problem for a 4 × 4 board. How many different routes are there from square S to square T which pass through every other square exactly once?



4.2 Now consider the analogous problem for a 5 × 5 board.

4.3 Now consider the general case of an $n \times n$ board.

NOTE

By considering the problem with a 4 × 4 board, you should see that the case where n is even is not difficult. However, the case where n is odd is seemingly much more difficult, and we don't know a general formula for this case. Please let us know if you manage to make any progress with this.

5. The numbers x and y satisfy the equations $x(y + 2) = 100$ and $y(x + 2) = 60$.

What is the value of $x - y$?

A 60 B 50 C 40 D 30 E 20

SOLUTION

E The two equations expand to give $xy + 2x = 100$ and $xy + 2y = 60$. It follows that $(xy + 2x) - (xy + 2y) = 100 - 60$. That is, $2x - 2y = 40$. Hence $2(x - y) = 40$ and so $x - y = 20$.

FOR INVESTIGATION

5.1 The wording of the question *implies* that there are numbers x and y which satisfy the equations $x(y+2) = 100$ and $y(x+2) = 60$. Check that this is correct by finding real numbers x and y which satisfy both the equations $x(y+2) = 100$ and $y(x+2) = 60$.

5.2 Show that there are no real numbers x and y such that $x(y+2) = -100$ and $y(x+2) = -80$.

NOTE

There are *complex* number solutions of these equations, and, if x and y are complex numbers which satisfy these equations, then $x - y = -10$. [If you don't know what complex numbers are, ask your teacher.]

6. Rebecca went swimming yesterday. After a while she had covered one fifth of her intended distance. After swimming six more lengths of the pool, she had covered one quarter of her intended distance.

How many lengths of the pool did she intend to complete?

A 40 B 72 C 80 D 100 E 120

SOLUTION

E We have $\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. So the six additional lengths make up $\frac{1}{20}$ of Rebecca's intended distance. So the number of lengths she intended to complete was $20 \times 6 = 120$.

7. In a 'ninety nine' shop all items cost a number of pounds and 99 pence. Susanna spent £65.76.

How many items did she buy?

A 23 B 24 C 65 D 66 E 76

SOLUTION

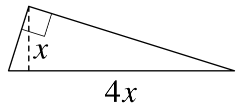
B Let k be the number of items that Susanna bought. The cost of these is a whole number of pounds and $99k$ pence, that is, a whole number of pounds less k pence. Susanna spent £65.76, that is, a whole number of pounds less 24 pence. It follows that k pence is a whole number of pounds plus 24 pence. So k is 24 or 124 or 224 or . . . . . However, since each item costs at least 99 pence and Susanna spent £65.76 pence, she bought at most 66 items. So k is 24.

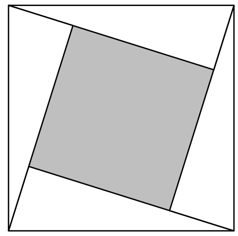
FOR INVESTIGATION

7.1 Is it possible to spend £20.76 in a 'ninety nine' shop?

7.2 For which non-negative integers m and n with $n < 100$ is it possible to spend m pounds and n pence in a 'ninety-nine' shop?

8. The right-angled triangle shown has a base which is 4 times its height. Four such triangles are placed so that their hypotenuses form the boundary of a large square as shown.





What is the side length of the shaded square in the diagram?

A $2x$ B $2\sqrt{2}x$ C $3x$ D $2\sqrt{3}x$ E $\sqrt{15}x$

Solution

B The side length of the large square is $4x$ and hence the area of this square is $16x^2$. Each triangle has base $4x$ and height x and hence has area $\frac{1}{2}(4x \times x) = 2x^2$. So the total area of these four triangles is $8x^2$. Therefore the area of the shaded square is $16x^2 - 8x^2 = 8x^2$. Therefore the side length of the shaded square is $\sqrt{8x^2} = \sqrt{8}x = 2\sqrt{2}x$.

9. According to a headline 'Glaciers in the French Alps have lost a quarter of their area in the past 40 years'.

What is the approximate percentage reduction in the length of the side of a square when it loses one quarter of its area, thereby becoming a smaller square?

A 13% B 25% C 38% D 50% E 65%

Solution

A Suppose that a square of side length 1, and hence area 1, has side length x when it loses one quarter of its area. Then $x^2 = \frac{3}{4}$ and so $x = \frac{\sqrt{3}}{2}$. Now $1.7^2 = 2.89$ and so $1.7 < \sqrt{3}$. Hence $0.85 < \frac{\sqrt{3}}{2}$. So the length of the side of the smaller square is at least 85% of its original value. Therefore the reduction in its length is less than 15%. So, of the given options, it must be that 13% is correct.

For investigation

9.1 The above solution is good enough in the context of the SMC. However, if you couldn't assume that one of the options must be correct, then, to show that the approximate percentage reduction is 13%, you would need to show that the percentage reduction lies between 12.5% and 13.5%. That is, you would need to show that $0.865 < \frac{\sqrt{3}}{2} < 0.875$, or, equivalently, that $1.73 < \sqrt{3} < 1.75$.

You could do this by doing a calculation to show that $1.73^2 < 3 < 1.75^2$. Alternatively, you could calculate $\sqrt{3}$ to the appropriate number of decimal places. This is easy with a calculator, but what could you do if you don't have a calculator? There is an old fashioned method for calculating square roots by hand which resembles long division. This was taught in schools until about fifty years ago. See if you can find out what this method is, perhaps using the internet.

Then use this method to calculate $\sqrt{3}$ to three decimal places.

9.2 The sequence defined by

$$x_1 = 1, \quad x_{n+1} = \frac{x_n + 3}{x_n + 1}$$

gives better and better approximations to $\sqrt{3}$. (We say that it *converges* to $\sqrt{3}$). The first three terms are $x_1 = 1$, $x_2 = \frac{1+3}{1+1} = 2$, and $x_3 = \frac{2+3}{2+1} = \frac{5}{3}$.

(a) Find the values of x_4 and x_5.

(b) Show that if the sequence converges, then it converges to $\sqrt{3}$.

(c) Find a similar rule for a sequence that converges to $\sqrt{5}$.

9.3 The *Generalized Binomial Theorem*, discovered by Isaac Newton, tells us that if $|x| < 1$, then

$$(1 + x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!}x^3 + \cdots$$

Use the first *three* terms of this series with $x = -\frac{1}{4}$ and $\alpha = \frac{1}{2}$ to obtain an approximation to $\frac{\sqrt{3}}{2}$.

9.4 What method does your calculator use to evaluate square roots?

10. Frank's teacher asks him to write down five integers such that the median is one more than the mean, and the mode is one greater than the median. Frank is also told that the median is 10.

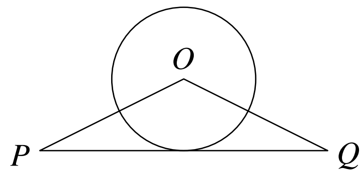
What is the smallest possible integer that he could include in his list?

A 3 B 4 C 5 D 6 E 7

Solution

B The median of the five numbers is 10, hence the mean is 9 and the mode is 11. For the mode to be 11, Frank's list must include more 11s than any other integer. It cannot include three or more 11s, since then the median would be 11. So two of the integers are 11, and the other three integers are all different. For the median to be 10, 10 must be one of the integers, and the list must include two integers that are less than 10 and two that are greater than 10. Hence the five integers are a, b, 10, 11, 11 where a, b are two different integers both less than 10. Say $a < b$. Since the mean is 9, we have $a + b + 10 + 11 + 11 = 5 \times 9 = 45$ and therefore $a + b = 13$. The largest value that b can take is 9, and hence the smallest possible value for a is 4.

11. The diagram shows a circle with centre O and a triangle OPQ. Side PQ is a tangent to the circle. The area of the circle is equal to the area of the triangle.



What is the ratio of the length of PQ to the circumference of the circle?

A $1:1$ B $2:3$ C $2:\pi$ D $3:2$ E $\pi:2$

Solution

A Suppose that the circle has radius r and that PQ has length x. The height of the triangle OPQ is the length of the perpendicular from O to PQ. Since PQ is a tangent to the circle, this perpendicular is a radius of the circle and so has length r. Therefore the area of triangle OPQ is $\frac{1}{2}xr$. The area of the circle is πr^2. Since these areas are equal $\frac{1}{2}xr = \pi r^2$, and hence $x = 2\pi r$. So the length of PQ is the same as the circumference of the circle. So the ratio of their lengths is $1:1$.

12. As a special treat, Sammy is allowed to eat five sweets from his very large jar which contains many sweets of each of three flavours – Lemon, Orange and Strawberry. He wants to eat his five sweets in such a way that no two consecutive sweets have the same flavour.

In how many ways can he do this?

A 32 B 48 C 72 D 108 E 162

Solution

B Sammy has a choice of 3 flavours for the first sweet that he eats. Each of the other sweets he eats cannot be the same flavour as the sweet he has just eaten. So he has a choice of 2 flavours for each of these four sweets. So the total number of ways that he can make his choices is $3 \times 2 \times 2 \times 2 \times 2 = 48$.

For investigation

12.1 What would the answer be if Sammy had sweets with four different flavours in his jar?

12.2 Find a formula for the number of ways if Sammy is allowed to eat k sweets and has sweets of n different flavours in his jar.

13. Two entrants in a school's sponsored run adopt different tactics. Angus walks for half the time and runs for the other half, whilst Bruce walks for half the distance and runs for the other half. Both competitors walk at 3 mph and run at 6 mph. Angus takes 40 minutes to complete the course.

How many minutes does Bruce take?

A 30 B 35 C 40 D 45 E 50

Solution

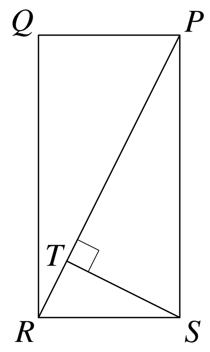
D Angus walks for 20 minutes at 3 mph and runs for 20 minutes at 6 mph. 20 minutes is one-third of an hour. So the number of miles that Angus covers is $3 \times \frac{1}{3} + 6 \times \frac{1}{3} = 6$.

Bruce covers the same distance. So Bruce walks $\frac{1}{2} \times 3$ miles at 3 mph which takes him 30 minutes and runs the same distance at 6 mph which takes him 15 minutes. So altogether it takes Bruce 45 minutes to finish the course.

14. The diagram shows a rectangle $PQRS$ in which $PQ : QR = 1 : 2$. The point T on PR is such that ST is perpendicular to PR.

What is the ratio of the area of the triangle RST to the area of the rectangle $PQRS$?

A $1 : 4\sqrt{2}$ B $1 : 6$ C $1 : 8$ D $1 : 10$
E $1 : 12$



Solution

D We can suppose that we have chosen units so that the length of RS is 1. So the length of PS is 2. It follows from Pythagoras' theorem that the length of PR is $\sqrt{1^2 + 2^2}$, that is, $\sqrt{5}$.

The triangles RST and PRS are both right-angled and the angle at R is common to both triangles. Therefore the triangles are similar. Hence the ratio of their areas is the ratio of the squares of the lengths of corresponding sides. The lengths of their hypotenuses are in the ratio $1 : \sqrt{5}$. Hence, the ratio of the area of triangle RST to the area of triangle PRS is $1^2 : \sqrt{5}^2$, that is, $1 : 5$. The area of triangle PRS is half the area of the rectangle. Hence the ratio of the area of triangle RST to the area of the rectangle is $1 : 10$.

For investigation

14.1 The solution shows that the ratio of the area of triangle RST to the area of the rectangle $PQRS$ is $1 : 10$. Is it possible to prove this by dissecting the rectangle $PQRS$ into 10 triangles each congruent to the triangle RST?

15. For how many positive integers n is $4^n - 1$ a prime number?

A 0 B 1 C 2 D 3 E infinitely many

SOLUTION

B We give two methods for answering this question. Both methods rely on the following property of prime numbers: a positive integer p is a prime number if, and only if, the only way p can be expressed as a product of two positive integers, m and n, is where exactly one of m and n is 1.

METHOD 1

The first method uses the fact that when n is a positive integer $4^n - 1$ is the difference of two squares and so may be factorized as $4^n - 1 = (2^n)^2 - 1^2 = (2^n + 1)(2^n - 1)$. So $4^n - 1$ is not a prime, unless one of the factors $2^n + 1$ and $2^n - 1$ is equal to 1. Now $2^n + 1$ cannot be equal to 1 and $2^n - 1 = 1$ if, and only if, $n = 1$. When $n = 1$, we have $4^n - 1 = 3$ and so $4^n - 1$ is prime. So there is just one positive integer n, namely 1, for which $4^n - 1$ is a prime number.

METHOD 2

The second method uses the fact when n is a positive integer $x - 1$ is a factor of $x^n - 1$, since, for $n \geq 2$,

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

Hence, putting $x = 4$, we deduce that, for each positive integer n, 3 is a factor of $4^n - 1$. Therefore $4^n - 1$ is not prime except when $n = 1$ and $4^n - 1 = 3$. So there is just one positive integer n for which $4^n - 1$ is a prime number.

FOR INVESTIGATION

15.1 For how many positive integers n is $5^n - 1$ a prime number?

15.2 For how many positive integers n is $6^n - 1$ a prime number?

15.3 Prove the identity used in method 2, that is, that for all integers $n \geq 2$,

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

16. Andrew states that every composite number of the form $8n + 3$, where n is an integer, has a prime factor of the same form.

Which of these numbers is an example showing that Andrew's statement is false?

A 19 B 33 C 85 D 91 E 99

Solution

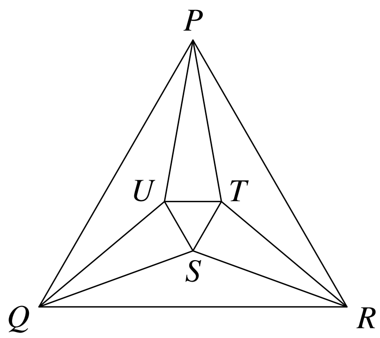
D To show that Andrew's statement is wrong we need to find a composite number of the form $8n + 3$ which does not have any prime factors of this form. This rules out 19 which is not composite, and both 33 and 85 which are not of the form $8n + 3$. The number 99 is of the form $8n + 3$, but it has two prime factors 3 and 11 which are also of this form. So 99 won't do either. This leaves 91. We see that $91 = 8 \times 11 + 3$, and so it is of the form $8n + 3$, but $91 = 7 \times 13$ and neither of its prime factors, 7 and 13, is of this form. So the number 91 shows that Andrew's statement is false.

17. The equilateral triangle PQR has side length 1. The lines PT and PU trisect the angle RPQ, the lines RS and RT trisect the angle QRP and the lines QS and QU trisect the angle PQR.

What is the side length of the equilateral triangle STU?

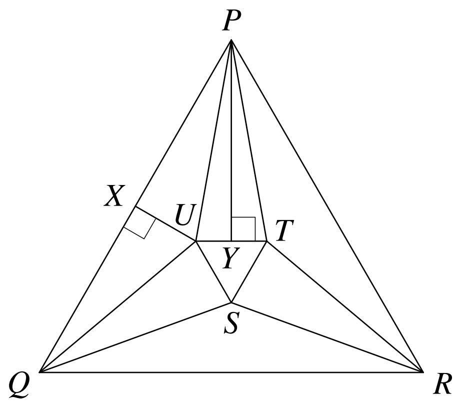
A $\dfrac{\cos 80°}{\cos 20°}$ B $\frac{1}{3}\cos 20°$ C $\cos^2 20°$

D $\frac{1}{6}$ E $\cos 20° \cos 80°$



Solution

A Since the triangle PQR is equilateral, all its three angles are 60°. So the trisectors divide these into three 20° angles. In particular $\angle QPU = 20°$ and $\angle UPT = 20°$. Because of the symmetry of the figure, $QU = PU = PT$. So the triangles PQU and PUT are isosceles. In these triangles, we let X be the foot of the perpendicular from U to PQ, and Y be the foot of perpendicular from P to UT.



In the right-angled triangle PXU, $\angle XPU = 20°$, and so $\frac{PX}{PU} = \cos 20°$. Since the length of PX is $\frac{1}{2}$ it follows that PU has length $\frac{1}{2} \div \cos 20°$. In the right-angled triangle PYU, $\angle UPY = 10°$ and therefore $\angle YUP = 80°$.

Therefore, from triangle PYU, $\cos 80° = \frac{UY}{PU}$ and hence UY has length $PU \cos 80°$, that is,

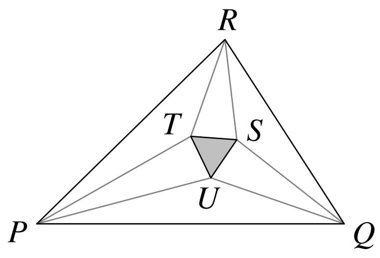
$$\frac{1}{2} \times \frac{\cos 80°}{\cos 20°}.$$

Since the length of UY is half the length of UT, it follows that the length of UT is $\dfrac{\cos 80°}{\cos 20°}$.

REMARKS

In this problem, it is clear that, because the triangle PQR is equilateral, so also is the triangle STU. It is a remarkable fact, discovered by Frank Morley (1860–1937), that:

In every triangle PQR, the triangle STU formed by the points where the trisectors of the angles of PQR meet is an equilateral triangle.



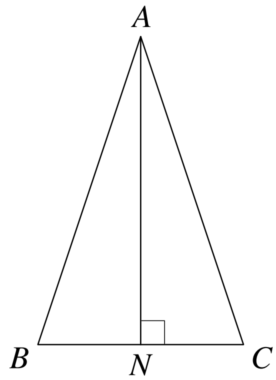
The triangle STU is called the *Morley triangle* of the triangle PQR.

No very easy proofs of this theorem are known.

A proof may be found on pages 186–187 of *The Geometry of the Triangle* by Gerry Leversha, UKMT, 2013. This book may be ordered from the UKMT website. The virtue of the proof in this book is that it uses purely geometrical ideas. Other proofs use trigonometry and complex numbers.

FOR INVESTIGATION

17.1 In the solution above we have assumed that X and Y are the midpoints of PQ and UT, respectively, and that $\angle UPY = \angle TPY$. These facts both follow from the fact that if a triangle ABC is isosceles with $AB = AC$, and N is the foot of the perpendicular from A to BC, then the triangles ABN and ACN are congruent.



Can you prove this?

Note that it then follows that $BN = CN$ and $\angle BAN = \angle CAN$. So we can deduce that X, Y are the midpoints of PQ and UT, and that $\angle UPY = \angle TPY$, as required.

17.2 An alternative method is to apply the Sine Rule to the triangles PQU and PUT. This gives

$$\frac{PU}{\sin 20^\circ} = \frac{PQ}{\sin 140^\circ} \quad \text{and} \quad \frac{UT}{\sin 20^\circ} = \frac{PU}{\sin 80^\circ}.$$

Hence, since PQ has length 1, it follows that

$$UT = PU \times \frac{\sin 20^\circ}{\sin 80^\circ} = \frac{\sin 20^\circ}{\sin 140^\circ} \times \frac{\sin 20^\circ}{\sin 80^\circ} = \frac{\sin^2 20^\circ}{\sin 140^\circ \sin 80^\circ}.$$

Show, *without using a calculator*, that this is the same as the previous answer $\dfrac{\cos 80^\circ}{\cos 20^\circ}$.

18. The numbers 2, 3, 12, 14, 15, 20, 21 may be divided into two sets so that the product of the numbers in each set is the same.

What is this product?

A 420 B 1260 C 2520 D 6720 E 6 350 400

Solution

C If all the numbers 2, 3, 12,14, 15, 20, 21 are multiplied the result will be the square of the common product of the two sets with the same product. Now

$$\begin{aligned} 2 \times 3 \times 12 \times 14 \times 15 \times 20 \times 21 \\ &= 2 \times 3 \times (2^2 \times 3) \times (2 \times 7) \times (3 \times 5) \times (2^2 \times 5) \times 3 \times 7 \\ &= 2^6 \times 3^4 \times 5^2 \times 7^2 \\ &= (2^3 \times 3^2 \times 5 \times 7)^2. \end{aligned}$$

Therefore the common product is $2^3 \times 3^2 \times 5 \times 7 = 2520$.

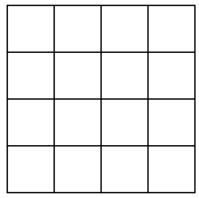
For investigation

18.1 This solution only shows that *if the common product exists* then it equals 2520. A complete solution should also show that it is possible to split the numbers 2, 3, 12, 14, 15, 20, 21 into two sets, each of whose products in 2520. Show that this is possible, and in just one way.

18.2 Devise some other problems of the same type as this one.

19. The 16 small squares shown in the diagram each have a side length of 1 unit.

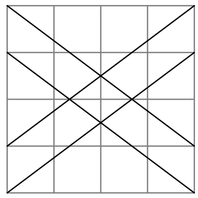
How many pairs of vertices are there in the diagram whose distance apart is an integer number of units?

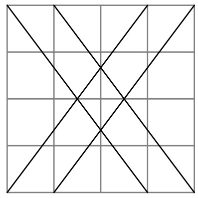


A 40 B 64 C 108 D 132 E 16

Solution

C Each pair of distinct vertices taken from the same row or from the same column is an integer number of units (1, 2, 3 or 4 units) apart. There are 5 vertices in each row and each column, and so 2 vertices may be chosen from the same row or the same column in $\binom{5}{2} = 10$ ways. Since there are 5 rows and 5 columns this gives $50 + 50 = 100$ pairs of vertices that are an integer number of units apart.





In addition, since $3^2 + 4^2 = 5^2$, it follows from Pythagoras' theorem that the opposite vertices of a 3×4 rectangle are 5 units apart. This gives a further 8 pairs of vertices (forming the end points of the 8 diagonals shown in the diagrams) which are an integer number of units apart.

There are no other *Pythagorean triples*, that is, positive integers m, n, q such that $m^2 + n^2 = q^2$, where m and n are both less than 5. So we have found all the pairs of vertices that are an integer number of units apart. So there are altogether $100 + 8 = 108$ pairs of such vertices.

20. The ratio of two positive numbers equals the ratio of their sum to their difference.

What is this ratio?

A $(1 + \sqrt{3}) : 2$ B $\sqrt{2} : 1$ C $(1 + \sqrt{5}) : 2$ D $(2 + \sqrt{2}) : 1$ E $(1 + \sqrt{2}) : 1$

Solution

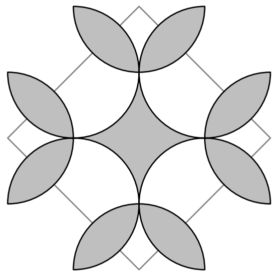
E Let the two numbers be x and y and suppose that $x : y = k : 1$, so that $x = ky$. Since the ratio $x + y : x - y$ is also $k : 1$, it follows that $x + y = k(x - y)$ and so $ky + y = k(ky - y)$. Since y is positive, we may divide through by y to obtain $k + 1 = k(k - 1)$.

It follows that $k^2 - 2k - 1 = 0$, and so, using the standard formula for the roots of a quadratic equation, we get

$$k = \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2}.$$

Since x and y are both positive $k > 0$, and it follows that $k = 1 + \sqrt{2}$.

21. The shaded design shown in the diagram is made by drawing eight circular arcs, all with the same radius. The centres of four arcs are the vertices of the square; the centres of the four touching arcs are the midpoints of the sides of the square. The diagonals of the square have length 1.

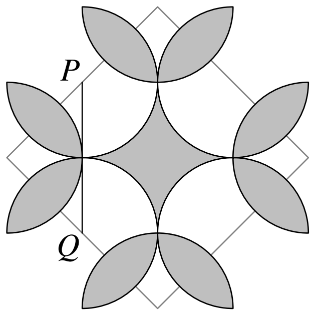


What is the total length of the border of the shaded design?

A 2π B $\frac{5}{2}\pi$ C 3π D $\frac{7}{2}\pi$ E 4π

Solution

B The arcs whose centres are the vertices of the square are semi-circular arcs, and the arcs whose centres are the midpoints of the sides of the square form three-quarter circles. Let all these arcs have radius r. Then the length of the border is



$$4 \times \left(\tfrac{1}{2} \times 2\pi r\right) + 4 \times \left(\tfrac{3}{4} \times 2\pi r\right) = 10\pi r.$$

Consider the line PQ joining the midpoints of two adjacent sides of the square as shown. Clearly the length of PQ is half of the length of the diagonal of the square. So PQ has length $\frac{1}{2}$. PQ joins the midpoints of two touching arcs and so its length is twice the radius of these arcs. Hence $r = \frac{1}{4}$. Therefore the length of the border of the shaded design is $10\pi \times \frac{1}{4} = \frac{5}{2}\pi$.

For investigation

21.1 The above solution assumes that the arcs whose centres are the vertices of the square are semi-circles, and the arcs whose centres are the midpoints of the sides of the square form three-quarter circles. Give arguments to prove that these assumptions are correct.

22. Consider numbers of the form $10n + 1$, where n is a positive integer. We shall call such a number 'grime' if it cannot be expressed as the product of two smaller numbers, possibly equal, both of which are of the form $10k + 1$, where k is a positive integer.

How many 'grime numbers' are there in the sequence $11, 21, 31, 41, \ldots, 981, 991$?

A 0 B 8 C 87 D 92 E 99

Solution

C Instead of checking each of the 99 numbers in the sequence $11, 21, \ldots, 981, 991$ in turn to see whether or not they are 'grime numbers', it is easier to count the numbers in this sequence that are not 'grime numbers'. A number is not a 'grime number' if, and only if, it is a product, say rs, where both r and s are numbers of the form $10k + 1$, where k is a positive integer. We can assume that $r \leq s$, and since we are only interested in numbers rs such that $rs \leq 991$, we can also assume that $r \leq 31$, because $32^2 > 991$.

We see that the only products of this form which are not greater than 991 are the twelve numbers

$$11 \times 11, \quad 11 \times 21, \quad 11 \times 31, \quad 11 \times 41, \quad 11 \times 51, \quad 11 \times 61,$$
$$11 \times 71, \quad 11 \times 81, \quad 21 \times 21, \quad 21 \times 31, \quad 21 \times 41 \quad \text{and} \quad 31 \times 31.$$

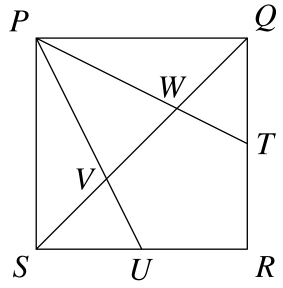
It can be seen, without calculating their values, that all these products are different as they have different prime factorisations.

So there are 12 numbers in the sequence $11, 21, \ldots, 981, 991$ that are not 'grime numbers'. Hence there are $99 - 12 = 87$ 'grime numbers' in this sequence.

23. $PQRS$ is a square. The points T and U are the midpoints of QR and RS respectively. The line QS cuts PT and PU at W and V respectively.

What fraction of the total area of the square $PQRS$ is the area of the pentagon $RTWVU$?

A $\frac{1}{3}$ B $\frac{2}{5}$ C $\frac{3}{7}$ D $\frac{5}{12}$ E $\frac{4}{15}$



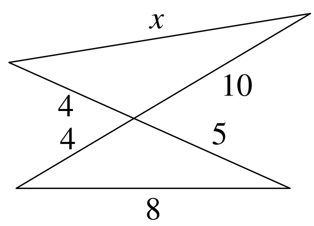
Solution

A The lines PQ and SR are parallel. Hence $\angle USV = \angle PQV$, since they are alternate angles. Similarly $\angle SUV = \angle PQV$. It follows that the triangles USV and QPV are similar. Now $SU : PQ = 1 : 2$ and so the heights of these triangles are in the same ratio. So the height of triangle USV is $\frac{1}{3}$ of the side-length of the square. The base of this triangle is $\frac{1}{2}$ of the side of the square. Hence the area of this triangle is $\frac{1}{2}\left(\frac{1}{2} \times \frac{1}{3}\right) = \frac{1}{12}$ of the area of the square.

Similarly the area of triangle QWT is $\frac{1}{12}$ of the area of the square. The area of triangle PQS is $\frac{1}{2}$ of the area of the square. The area of the pentagon $RTWVU$ is the area of the square minus the total areas of the triangles USV, QWT and PQS, so its area, as a fraction of the area of the square $PQRS$, is $1 - \frac{1}{12} - \frac{1}{12} - \frac{1}{2} = \frac{1}{3}$.

24. The diagram shows two straight lines PR and QS crossing at O.

What is the value of x?



A $7\sqrt{2}$ B $2\sqrt{29}$ C $14\sqrt{2}$
D $7\left(1+\sqrt{13}\right)$ E $9\sqrt{2}$

Solution

E Let $\angle SOR = \theta$. Applying the Cosine Rule to the triangle ROS, we obtain

$$8^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \times \cos\theta,$$

from which it follows that

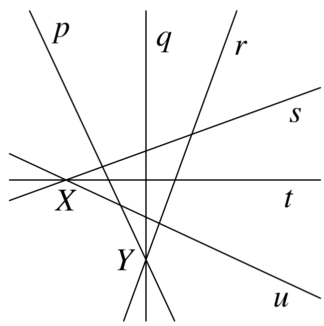
$$\cos\theta = \frac{4^2 + 5^2 - 8^2}{2 \times 4 \times 5} = -\frac{23}{40}.$$

We also have that $\angle QOP = \angle SOR = \theta$, since they are vertically opposite angles. Hence, applying the Cosine Rule to triangle POQ, we get

$$\begin{aligned} x^2 &= 4^2 + 10^2 - 2 \times 4 \times 10 \times \cos\theta \\ &= 16 + 100 - 80\left(-\frac{23}{40}\right) \\ &= 116 + 46 = 162. \end{aligned}$$

Therefore $x = \sqrt{162} = \sqrt{81 \times 2} = 9\sqrt{2}$.

25. Challengeborough's underground train network consists of six lines p, q, r, s, t and u, as shown. Wherever two lines meet there is a station which enables passengers to change lines. On each line, each train stops at every station.



Jessica wants to travel from station X to station Y. She does not want to use any line more than once, nor return to station X after leaving it, nor leave station Y after reaching it.

How many different routes, satisfying these conditions, can she choose?

A 9 B 36 C 41 D 81 E 720

Solution

D A route is specified by giving the sequence of lines that Jessica travels on. We call the lines s, t and u the X-lines and the lines p, q and r the Y-lines. It follows from the layout of the network and Jessica's conditions, that she can change trains between any X-line and any Y-line and *vice versa*, but she cannot change between two X-lines, or between two Y-lines. So her route from X to Y is given by a sequence of lines starting with an X-line, alternating between X-lines and Y-lines, and ending with a Y-line. So it will consist of an even number of lines. Since Jessica does not wish to use any line more than once, a possible route for Jessica consists of 2, 4 or 6 lines. We count these routes according to the number of lines involved.

2 lines A route of 2 lines will be of the form g, h, where g is an X-line and h is a Y-line. There are 3 choices for g and 3 choices for h, and hence $3 \times 3 = 9$ routes of this form.

4 lines A route of 4 lines will be of the form g, h, i, j, where g and i are two different X-lines and h and j are two different Y-lines. Since there are 3 choices for g and then 2 choices for i and likewise for h and j, there are $3 \times 3 \times 2 \times 2 = 36$ routes of this form.

6 lines A route of 6 lines will have the form g, h, i, j, k, l, where g, i and k are X-lines and h, j and l are Y-lines. As before there are 3 choices for g and then 2 choices for i, leaving just one choice for k, and likewise for the choices of h, j and l. So there are $3 \times 3 \times 2 \times 2 \times 1 \times 1 = 36$ routes of this form.

So, altogether, there are $9 + 36 + 36 = 81$ routes that satisfy Jessica's conditions.

For investigation

25.1 How many different routes would there be if there were four lines passing through X and four lines passing through Y, but otherwise the conditions are the same?

25.2 Can you find an expression which gives the number of different routes if there are m lines passing through X and n lines passing through Y, but otherwise the conditions are the same?

UK SENIOR MATHEMATICAL CHALLENGE

Thursday 5 November 2015

Organised by the **United Kingdom Mathematics Trust**

and supported by

Institute and Faculty of Actuaries

RULES AND GUIDELINES (to be read before starting)

1. Do not open the question paper until the invigilator tells you to do so.
2. **Use B or HB pencil only**. Mark *at most one* of the options A, B, C, D, E on the Answer Sheet for each question. Do not mark more than one option.
3. Time allowed: **90 minutes**.
   No answers or personal details may be entered on the Answer Sheet after the 90 minutes are over.
4. The use of rough paper is allowed.
   Calculators, measuring instruments and squared paper are forbidden.
5. Candidates must be full-time students at secondary school or FE college, and must be in Year 13 or below (England & Wales); S6 or below (Scotland); Year 14 or below (Northern Ireland).
6. There are twenty-five questions. Each question is followed by five options marked A, B, C, D, E. Only one of these is correct. Enter the letter A-E corresponding to the correct answer in the corresponding box on the Answer Sheet.
7. **Scoring rules**: all candidates start out with 25 marks;
   0 marks are awarded for each question left unanswered;
   4 marks are awarded for each correct answer;
   1 mark is deducted for each incorrect answer.
8. **Guessing**: Remember that there is a penalty for wrong answers. Note also that later questions are deliberately intended to be harder than earlier questions. You are thus advised to concentrate first on solving as many as possible of the first 15-20 questions. Only then should you try later questions.

1. What is $2015^2 - 2016 \times 2014$?

A -2015 B -1 C 0 D 1 E 2015

2. What is the sum of all the solutions of the equation $6x = \frac{150}{x}$?

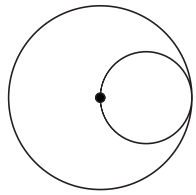
A 0 B 5 C 6 D 25 E 156

3. When Louise had her first car, 50 litres of petrol cost £40. When she filled up the other day, she noticed that 40 litres of petrol cost £50.

By approximately what percentage has the cost of petrol increased over this time?

A 50% B 56% C 67% D 75% E 80%

4. In the diagram, the smaller circle touches the larger circle and also passes through its centre. What fraction of the area of the larger circle is outside the smaller circle?

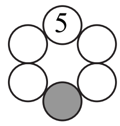


A $\frac{2}{3}$ B $\frac{3}{4}$ C $\frac{4}{5}$ D $\frac{5}{6}$ E $\frac{6}{7}$

5. The integer n is the mean of the three numbers 17, 23 and $2n$. What is the sum of the digits of n?

A 4 B 5 C 6 D 7 E 8

6. The numbers 5, 6, 7, 8, 9, 10 are to be placed, one in each of the circles in the diagram, so that the sum of the numbers in each pair of touching circles is a prime number. The number 5 is placed in the top circle.



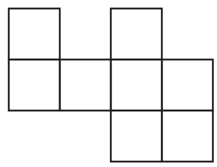
Which number is placed in the shaded circle?

A 6 B 7 C 8 D 9 E 10

7. Which of the following has the largest value?

A $\frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{4}\right)}$ B $\frac{1}{\left(\frac{\left(\frac{2}{3}\right)}{4}\right)}$ C $\frac{\left(\frac{\left(\frac{1}{2}\right)}{3}\right)}{4}$ D $\frac{1}{\left(\frac{2}{\left(\frac{3}{4}\right)}\right)}$ E $\frac{\left(\frac{1}{\left(\frac{2}{3}\right)}\right)}{4}$

8. The diagram shows eight small squares. Six of these squares are to be shaded so that the shaded squares form the net of a cube.



In how many different ways can this be done?

A 10 B 8 C 7 D 6 E 4

9. Four different straight lines are drawn on a flat piece of paper. The number of points where two or more lines intersect is counted.

Which of the following could **not** be the number of such points?

A 1 B 2 C 3 D 4 E 5

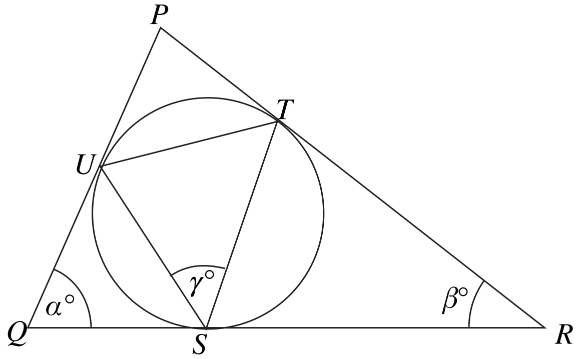
10. The positive integer n is between 1 and 20. Milly adds up all the integers from 1 to n inclusive. Billy adds up all the integers from $n + 1$ to 20 inclusive. Their totals are the same. What is the value of n?

A 11 B 12 C 13 D 14 E 15

11. Rahid has a large number of cubic building blocks. Each block has sides of length 4 cm, 6 cm or 10 cm. Rahid makes little towers built from three blocks stacked on top of each other. How many different heights of tower can he make?

A 6 B 8 C 9 D 12 E 27

12. A circle touches the sides of triangle PQR at the points S, T and U as shown. Also $\angle PQR = \alpha°$, $\angle PRQ = \beta°$ and $\angle TSU = \gamma°$. Which of the following gives γ in terms of α and β?

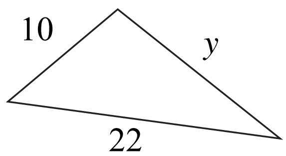


A $\frac{1}{2}(\alpha + \beta)$ B $180 - \frac{1}{2}(\alpha + \beta)$
C $180 - (\alpha + \beta)$ D $\alpha + \beta$
E $\frac{1}{3}(\alpha + \beta)$

13. The Knave of Hearts tells only the truth on Mondays, Tuesdays, Wednesdays and Thursdays. He tells only lies on all the other days. The Knave of Diamonds tells only the truth on Fridays, Saturdays, Sundays and Mondays. He tells only lies on all the other days. On one day last week, they both said, "Yesterday I told lies."
On which day of the week was that?

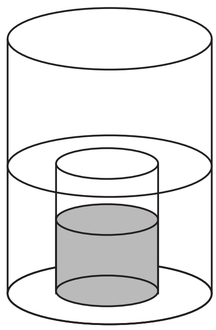
A Sunday B Monday C Tuesday D Thursday E Friday

14. The triangle shown has an area of 88 square units. What is the value of y?



A 17.6 B $2\sqrt{46}$ C $6\sqrt{10}$ D $13\sqrt{2}$ E $8\sqrt{5}$

15. Two vases are cylindrical in shape. The larger vase has diameter 20 cm. The smaller vase has diameter 10 cm and height 16 cm. The larger vase is partially filled with water. Then the empty smaller vase, with the open end at the top, is slowly pushed down into the water, which flows over its rim. When the smaller vase is pushed right down, it is half full of water.
What was the original depth of the water in the larger vase?

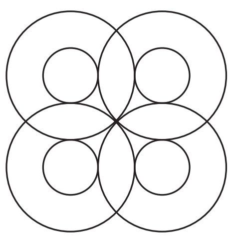


A 10 cm B 12 cm C 14 cm D 16 cm E 18 cm

16. Fnargs are either red or blue and have 2, 3 or 4 heads. A group of six Fnargs consisting of one of each possible form is made to line up such that no immediate neighbours are the same colour nor have the same number of heads. How many ways are there of lining them up from left to right?

A 12 B 24 C 60 D 120 E 720

17. The diagram shows eight circles of two different sizes. The circles are arranged in concentric pairs so that the centres form a square. Each larger circle touches one other larger circle and two smaller circles. The larger circles have radius 1. What is the radius of each smaller circle?

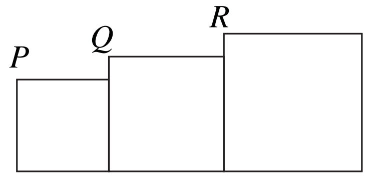


A $\frac{1}{3}$ B $\frac{2}{5}$ C $\sqrt{2} - 1$ D $\frac{1}{2}$ E $\frac{1}{2}\sqrt{2}$

18. What is the largest integer k whose square k^2 is a factor of 10!? $[10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1.]$

A 6 B 256 C 360 D 720 E 5040

19. Three squares are arranged as shown so that their bases lie on a straight line. Also, the corners P, Q and R lie on a straight line. The middle square has sides that are 8 cm longer than the sides of the smallest square. The largest square has sides of length 50 cm.



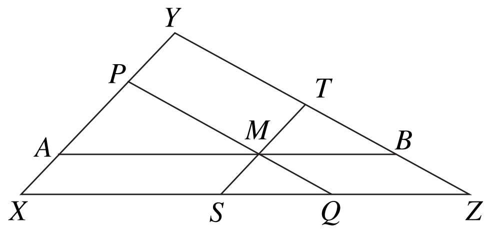
There are two possible values for the length (in cm) of the sides of the smallest square. Which of the following are they?

A 2, 32 B 4, 42 C 4, 34 D 32, 40 E 34, 42

20. A square ink pad has sides of length 1 cm. It is covered in black ink and carefully placed in the middle of a piece of white paper. The square pad is then rotated 180° about one of its corners so that all of the pad remains in contact with the paper throughout the turn. The pad is then removed from the paper. What area of paper, in cm^2, is coloured black?

A $\pi + 2$ B $2\pi - 1$ C 4 D $2\pi - 2$ E $\pi + 1$

21. The diagram shows a triangle XYZ. The sides XY, YZ and XZ have lengths 2, 3 and 4 respectively. The lines AMB, PMQ and SMT are drawn parallel to the sides of triangle XYZ so that AP, QS and BT are of equal length. What is the length of AP?



A $\frac{10}{11}$ B $\frac{11}{12}$ C $\frac{12}{13}$ D $\frac{13}{14}$ E $\frac{14}{15}$

22. Let $f(x) = x + \sqrt{x^2 + 1} + \dfrac{1}{x - \sqrt{x^2 + 1}}$. What is the value of $f(2015)$?

A -1 B 0 C 1 D $\sqrt{2016}$ E 2015

23. Given four different non-zero digits, it is possible to form 24 different four-digit numbers containing each of these four digits. What is the largest prime factor of the sum of the 24 numbers?

A 23 B 93 C 97 D 101 E 113

24. Peter has 25 cards, each printed with a different integer from 1 to 25. He wishes to place N cards in a single row so that the numbers on every adjacent pair of cards have a prime factor in common.

What is the largest value of N for which this is possible?

A 16 B 18 C 20 D 22 E 24

25. A function, defined on the set of positive integers, is such that $f(xy) = f(x) + f(y)$ for all x and y. It is known that $f(10) = 14$ and $f(40) = 20$. What is the value of $f(500)$?

A 29 B 30 C 39 D 48 E 50

1.	**D**
2.	**A**
3.	**B**
4.	**B**
5.	**A**
6.	**E**
7.	**B**
8.	**D**
9.	**B**
10.	**D**
11.	**C**
12.	**A**
13.	**E**
14.	**E**
15.	**C**
16.	**A**
17.	**C**
18.	**D**
19.	**A**
20.	**E**
21.	**C**
22.	**B**
23.	**D**
24.	**C**
25.	**C**

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**
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SOLUTIONS

Keep these solutions secure until after the test on

THURSDAY 5 NOVEMBER 2015

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' at the end of a solution.

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

The UKMT is a registered charity.

1. **D** The expression $2015^2 - 2016 \times 2014$ can be written as $2015^2 - (2015 + 1)(2015 - 1)$ which simplifies, using the difference of two squares, to $2015^2 - \left(2015^2 - 1\right) = 1$.

2. **A** Rearranging $6x = \dfrac{150}{x}$ gives $x^2 = \dfrac{150}{6}$, so $x^2 = 25$. This has two solutions, $x = 5$ and $x = -5$. Therefore the sum of the solutions is $5 + (-5) = 0$.

3. **B** When 50 litres of petrol cost £40, 1 litre cost $\dfrac{£40}{50}$ which is 80 pence. More recently, 1 litre cost $\dfrac{£50}{40} = 125$ pence. The percentage increase is then $\dfrac{\text{actual increase}}{\text{original price}} \times 100$ which is $\dfrac{45}{80} \times 100 = \dfrac{450}{8} = 56.25$. So the approximate increase is 56%.

4. **B** Let the radius of the smaller circle be r and so the radius of the larger circle is $2r$. The area of the smaller circle is then πr^2 and the area of the larger circle is $\pi \times (2r)^2$ which is $4\pi r^2$. The fraction of the larger circle which is outside the smaller circle is then $\dfrac{4\pi r^2 - \pi r^2}{4\pi r^2} = \dfrac{3\pi r^2}{4\pi r^2} = \dfrac{3}{4}$.

5. **A** The mean of 17, 23 and $2n$ is given to be n, so $\dfrac{17 + 23 + 2n}{3} = n$ which gives $40 + 2n = 3n$. As n is then 40, the sum of the digits of n is 4.

6. **E** The prime numbers which are the sums of pairs of numbers in touching circles are all odd as they are greater than 2. This means that any two adjacent circles in the diagram must be filled with one odd number and one even number. The number 10 may not be placed on either side of 5, since $10 + 5 = 15 = 3 \times 5$. So either side of the 5 must be 6 and 8. Below 6 and 8 must be 7 and 9 respectively leaving 10 to be placed in the shaded circle at the bottom.

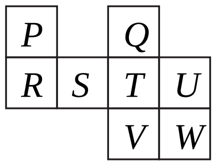
7. **B** Evaluating each option gives

A $\dfrac{\left(\frac{1}{2}\right)}{\left(\frac{3}{4}\right)} = \dfrac{1}{2} \times \dfrac{4}{3} = \dfrac{2}{3}$ B $\dfrac{1}{\left(\frac{\left(\frac{2}{3}\right)}{4}\right)} = \dfrac{1}{\left(\frac{2}{12}\right)} = \dfrac{12}{2} = 6$ C $\dfrac{\frac{\left(\frac{1}{2}\right)}{3}}{4} = \dfrac{\left(\frac{1}{6}\right)}{4} = \dfrac{1}{24}$

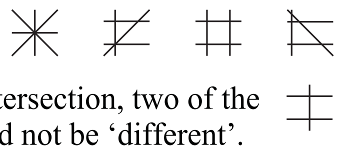
D $\dfrac{1}{\frac{2}{\left(\frac{3}{4}\right)}} = \dfrac{1}{\left(\frac{8}{3}\right)} = \dfrac{3}{8}$ E $\dfrac{\left(\frac{1}{\left(\frac{2}{3}\right)}\right)}{4} = \dfrac{\left(\frac{3}{2}\right)}{4} = \dfrac{3}{8}$.

So B has the largest answer.

8. **D** Let the squares in the diagram be labelled as shown. Each of the nets formed from six squares must contain all of R, S and T. The net must also include one of P and Q (but not both as they will fold into the same position), and any two of U, V and W. This therefore gives $2 \times 3 = 6$ different ways.



9. **B** Possible configurations of four different straight lines drawn in a plane are shown here to give 1, 3, 4 and 5 points of intersection respectively. In order to have exactly 2 points of intersection, two of the straight lines would need to lie in the same position and so would not be 'different'.



10. **D** The total of the numbers from 1 to 20 is $\frac{1}{2} \times 20 \times (20 + 1) = 210$. If Milly and Billy have totals which are equal, their totals must each be 105. Milly's total, of the numbers from 1 to n, is $\frac{1}{2}n(n + 1)$ so $\frac{1}{2}n(n + 1) = 105$ which gives $n^2 + n = 210$. Therefore $n^2 + n - 210 = 0$ which factorises to give $(n + 15)(n - 14) = 0$. As n is a positive integer, $n = 14$.

11. **C** There are several different ways to count systematically the number of towers that Rahid can build. Here is one way.

	All blocks the same size			Exactly two blocks the same size						All blocks of different sizes
	10 10 10	6 6 6	4 4 4	4 10 10	6 10 10	4 6 6	10 6 6	6 4 4	10 4 4	4 6 10
Total height	30	18	12	24	26	16	22	14	18	20

So there are nine different heights of tower (as the height of 18cm can be made from $6 + 6 + 6$ or $10 + 4 + 4$).

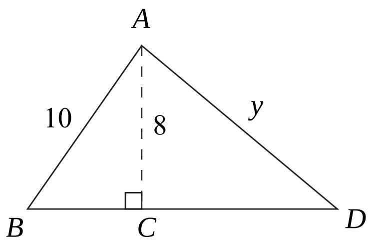
12. **A** Each of the three sides of triangle PQR is a tangent to the circle. Two tangents to a circle which meet at a point are of equal length. So QU and QS are of equal length. Similarly $RT = RS$. This means that $\angle QUS = \angle QSU = \frac{1}{2}(180 - \alpha)$ and also $\angle RTS = \angle RST = \frac{1}{2}(180 - \beta)$. At S we can consider the sum of the three angles, so $\frac{1}{2}(180 - \alpha) + \gamma + \frac{1}{2}(180 - \beta) = 180$. Simplifying gives $90 - \frac{1}{2}\alpha + \gamma + 90 - \frac{1}{2}\beta = 180$ and so $\gamma = \frac{1}{2}(\alpha + \beta)$.

13. **E**

Knave of	Mon	Tue	Wed	Thu	Fri	Sat	Sun	Mon
Hearts	T	T	T	T	L	L	L	T
Diamonds	T	L	L	L	T	T	T	T

When a knave says "Yesterday I told lies" it could be that today he is telling the truth and he did indeed tell lies yesterday. In the table, this is a T preceded by an L.
It could also be that today he is lying, in which case he was in fact telling the truth yesterday. In the table, this is an L preceded by a T. The only day when one or the other of these options applies to each knave is Friday.

14. **E**



Let the vertices of the triangle be labelled A, B and D as shown. Let the point where the perpendicular from A meets BD be labelled C. The area of triangle ABD is given as 88. As BD is 22, AC must be 8. Considering triangle ABC and using Pythagoras' Theorem gives $BC = 6$. The remainder of the base CD is then $22 - 6 = 16$. Considering triangle ACD and using Pythagoras' Theorem again gives $y^2 = 8^2 + 16^2 = 8^2(1^2 + 2^2) = 8^2 \times 5$. So $y = 8\sqrt{5}$.

15. **C** Let the original water level in the larger vase be h cm. The volume of water at the start is then $\pi \times 10^2 \times h$ cm^3. The volume of water completely within the vase is constant, but when the smaller vase is pushed down, some of the water moves into it. In the end the depth of the water in the larger vase is the same as the height of the smaller vase itself, which is 16 cm. We are given that the final depth of water in the smaller vase is 8 cm. So the total volume of water is then $\pi \times 10^2 \times 16$ cm^3 less the gap in the top half of the smaller vase. So $\pi \times 10^2 \times h = \pi \times 10^2 \times 16 - \pi \times 5^2 \times 8$, giving $100\pi h = 1600\pi - 200\pi$ and therefore $h = 14$.

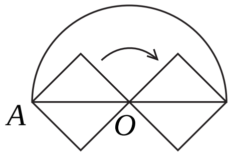
16. **A** Let the six Fnargs in their final positions be denoted by $F_1F_2F_3F_4F_5F_6$. There are six choices for F_1. Once this Fnarg is chosen, the colours of the Fnargs must alternate all along the line and so we need only consider the number of heads. There are $3 - 1 = 2$ choices for F_2 as the number of heads for $F_2 \neq$ the number of heads for F_1. There is only one choice for F_3 as F_3 cannot have the same number of heads as F_2 or F_1 (F_3 and F_1 are the same colour and so have different numbers of heads). There is only one choice for F_4 as it is completely determined by F_3 and F_2, just as F_3 was completely determined by F_2 and F_1. There is only one choice for each of F_5 and F_6 as they are the last of each colour of Fnargs. The total number of ways of lining up the Fnargs is $6 \times 2 \times 1 \times 1 \times 1 \times 1$ which is 12.

17. **C** Let the radius of each of the smaller circles be r and let the centres of the circles be A, B, C and D in order. We are given that $ABCD$ is a square. When two circles touch externally, the distance between their centres equals the sum of their radii. Hence AB and BC have length $r + 1$ and AC has length $1 + 1 = 2$. By Pythagoras' Theorem $(r + 1)^2 + (r + 1)^2 = 2^2$, so $2(r + 1)^2 = 2^2 = 4$ and therefore $(r + 1)^2 = 2$. Square rooting both sides gives $r + 1 = \sqrt{2}$, as we must take the positive root, and so $r = \sqrt{2} - 1$.

18. **D** Expressed as a product of its prime factors, 10! is $2 \times 5 \times 3 \times 3 \times 2 \times 2 \times 2 \times 7 \times 2 \times 3 \times 5 \times 2 \times 2 \times 3 \times 2$ which is $2^8 \times 3^4 \times 5^2 \times 7$. This can be written as $(2^4 \times 3^2 \times 5)^2 \times 7$ so the largest integer k such that k^2 is a factor of 10! is $2^4 \times 3^2 \times 5$ which is 720.

19. **A** Let the length of the side of the smallest square be x cm. So the three squares have sides of lengths x cm, $(x + 8)$ cm and 50 cm respectively. The gradient of PQ is then $\frac{8}{x}$ and the gradient of PR is $\dfrac{50 - x}{x + x + 8}$. As P, Q and R lie on a straight line, $\dfrac{8}{x} = \dfrac{50 - x}{2x + 8}$ so $8(2x + 8) = x(50 - x)$. Expanding gives $16x + 64 = 50x - x^2$ and therefore $x^2 - 34x + 64 = 0$, giving $x = 2$ or 32.

20. **E**



Let the corner of the square about which it is rotated be O and the opposite vertex of the square be A. As the circle is rotated through 180° about O, the vertex A travels along a semicircle whose centre is O. The area coloured black by the ink is then formed from two half squares and a semicircle. The square has side-length 1, so $OA = \sqrt{2}$. The total area of the two half squares and the semicircle is $2 \times (\frac{1}{2} \times 1 \times 1) + \frac{1}{2} \times \pi \times (\sqrt{2})^2$ which is $1 + \pi$.

21. **C** All of the triangles in the diagram are similar as they contain the same angles. The sides of each triangle are therefore in the ratio 2 : 3 : 4. First consider triangle APM. Let $AP = x$, so that $AM = 2x$. Now considering triangle TBM, as $BT = x$, $BM = \frac{4x}{3}$. The quadrilateral $AMSX$ is a parallelogram as AM is parallel to XS and MS is parallel to AX. So $AM = XS = 2x$. Similarly $QZ = BM = \frac{4x}{3}$. Considering the base of triangle XYZ, $XS + SQ + QZ = 4$. So $2x + x + \frac{4x}{3} = 4$ and therefore $x = \frac{12}{13}$.

22. **B** $f(x) = x + \sqrt{x^2 + 1} + \dfrac{1}{x - \sqrt{x^2 + 1}} = \dfrac{\left(x + \sqrt{x^2 + 1}\right)\left(x - \sqrt{x^2 + 1}\right) + 1}{x - \sqrt{x^2 + 1}}$. The numerator is $x^2 - \left(\sqrt{x^2 + 1}\right)^2 + 1 = -1 + 1 = 0$. So $f(x) = 0$. Hence $f(2015) = 0$.

23. **D** Let a four-digit positive integer be expressed as $1000a + 100b + 10c + d$ where a, b, c and d are all different. In the 24 possible permutations of a, b, c and d, each of the four letters appears in each position six times. Adding all 24 numbers together gives $1000(6a + 6b + 6c + 6d) + 100(6a + 6b + 6c + 6d) + 10(6a + 6b + 6c + 6d) + 6a + 6b + 6c + 6d$. The total is therefore $1111 \times 6(a + b + c + d)$ which factorises to $2 \times 3 \times 11 \times 101(a + b + c + d)$. As $a + b + c + d < 101$, the largest prime factor of the sum is 101.

24. **C** There are five cards in Peter's set that are printed with an integer that has no prime factors in common with any other number from 1 to 25. The five numbers are 1 (which has no prime factors) and the primes 13, 17, 19 and 23. These cards cannot be placed anywhere in the row of N cards. One possible row is: 11, 22, 18, 16, 12, 10, 8, 6, 4, 2, 24, 3, 9, 21, 7, 14, 20, 25, 15, 5. So the longest row is of 20 cards.

25. **C** Repeatedly using the rule that $f(xy) = f(x) + f(y)$ allows us to write $f(500)$ as $f(2 \times 2 \times 5 \times 5 \times 5) = f(2) + f(2) + f(5) + f(5) + f(5) = 2f(2) + 3f(5)$. We are given values for $f(40)$ and $f(10)$ and from them we need to calculate the values of $f(2)$ and $f(5)$. Now $f(40)$ can be written as $f(2) + f(2) + f(10)$ so $20 = 2f(2) + 14$ and therefore $f(2) = 3$. Similarly $f(10) = f(2) + f(5)$ so $14 = 3 + f(5)$ giving $f(5) = 11$. So $f(500) = 2f(2) + 3f(5) = 2 \times 3 + 3 \times 11 = 39$.

For reasons of space, these solutions are necessarily brief. There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:
http://www.ukmt.org.uk/

SENIOR MATHEMATICAL CHALLENGE

Thursday 5th November 2015

Organised by the United Kingdom Mathematics Trust

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SOLUTIONS AND INVESTIGATIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some problems for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with all steps explained, and not based on the assumption that one of the given alternatives is correct. In some cases we have added a commentary to indicate the sort of thinking that led to our solution. You should not include commentary of this kind in your written solutions, but we hope that these solutions, without the commentary, provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, UKMT, School of Mathematics Satellite, University of Leeds,
Leeds LS2 9JT

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	A	B	B	A	E	B	D	B	D	C	A	E	E	C	A	C	D	A	E	C	B	D	C	C

1. What is $2015^2 - 2016 \times 2014$?

A -2015 B -1 C 0 D 1 E 2015

Solution **D**

Commentary

This question could be answered by doing two multiplications, to evaluate 2015^2 and 2016×2014, and then one subtraction. It should be clear that this method cannot be what was intended as it would be tedious and take up a lot of time. There must be a better method. The clue is that the product 2016×2014 may be rewritten as $(2015 + 1) \times (2015 - 1)$, and that this last expression is equivalent to the difference of two squares, namely, $2015^2 - 1^2$. This leads to the solution given below.

[See Question 22 for another case where the factorization of the difference of two squares is useful.]

We have

$$\begin{aligned} 2015^2 - 2016 \times 2014 &= 2015^2 - (2015 + 1) \times (2015 - 1) \\ &= 2015^2 - (2015^2 - 1) \\ &= 2015^2 - 2015^2 + 1 \\ &= 1. \end{aligned}$$

2. What is the sum of all the solutions of the equation $6x = \dfrac{150}{x}$?

A 0 B 5 C 6 D 25 E 156

Solution **A**

We have,

$$\begin{aligned} 6x = \frac{150}{x} &\Leftrightarrow 6x^2 = 150 \\ &\Leftrightarrow x^2 = 25 \\ &\Leftrightarrow x = -5 \text{ or } x = 5. \end{aligned}$$

Therefore the equation has the two solutions -5 and 5. It follows that the sum of all the solutions of the equation is $-5 + 5 = 0$.

Note that here the symbol $\Leftrightarrow$ stands for *if, and only if*.

3. When Louise had her first car, 50 litres of petrol cost £40. When she filled up the other day, she noticed that 40 litres of petrol cost £50.

By approximately what percentage has the cost of petrol increased over this time?

A 50% B 56% C 67% D 75% E 80%

Solution **B**

We note first that £40 is 40×100 pence, that is, 4000 pence. Therefore when Louise had her first car, the cost of petrol, in pence per litre, was

$$\frac{4000}{50} = 80.$$

Similarly, when she filled up the other day, the cost of petrol, in pence per litre, was

$$\frac{5000}{40} = 125.$$

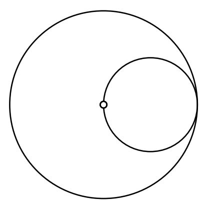
It follows that the cost of petrol has increased by $(125 - 80)$ pence per litre, that is, by 45 pence per litre. As a percentage of the original price this increase is

$$\begin{aligned}\frac{45}{80} \times 100 &= \frac{4500}{80} \\ &= \frac{450}{8} \\ &= 56\tfrac{1}{4}.\end{aligned}$$

Therefore the increase in the cost of petrol over the given period is approximately 56%.

4. In the diagram, the smaller circle touches the larger circle and also passes through its centre.

What fraction of the area of the larger circle is outside the smaller circle?



A $\frac{2}{3}$ B $\frac{3}{4}$ C $\frac{4}{5}$ D $\frac{5}{6}$ E $\frac{6}{7}$

Solution **B**

Because the smaller circle touches the larger circle and passes through its centre, the diameter of the smaller circle is half that of the larger circle. It follows that the radius of the smaller circle is half that of the larger circle.

The area of a circle with radius r is πr^2. Therefore, the area of a circle with radius $\frac{1}{2}r$ is $\pi(\frac{1}{2}r)^2 = \frac{1}{4}\pi r^2$. It follows that the area of the smaller circle is $\frac{1}{4}$ of the area of the larger circle. Therefore $\frac{1}{4}$ of the area of the larger circle is inside the smaller circle. Hence $\frac{3}{4}$ of the area of the larger circle is outside the smaller circle.

5. The integer n is the mean of the three numbers 17, 23 and $2n$.

What is the sum of the digits of n?

A 4 B 5 C 6 D 7 E 8

SOLUTION **A**

Because the mean of 17, 23 and $2n$ is n,

$$\frac{17 + 23 + 2n}{3} = n,$$

and therefore,

$$17 + 23 + 2n = 3n,$$

that is,

$$40 + 2n = 3n,$$

from which it follows that

$$n = 40.$$

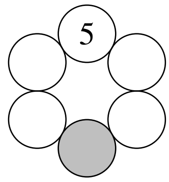
Therefore the sum of the digits of n is $4 + 0 = 4$.

FOR INVESTIGATION

5.1 The integer n is the mean of the four numbers 11, 13, 19 and $3n$. What is the value of n?

5.2 Let k be a positive integer, with $k \geq 2$. Show that if n is the mean of the k numbers, $a_1, a_2, \ldots, a_{k-1}$ and $(k-1)n$, then $n = a_1 + a_2 + \cdots + a_{k-1}$.

6. The numbers 5, 6, 7, 8, 9, 10 are to be placed, one in each of the circles in the diagram, so that the sum of the numbers in each pair of touching circles is a prime number. The number 5 is placed in the top circle.



Which number is placed in the shaded circle?

A 6 B 7 C 8 D 9 E 10

Solution **E**

The primes that are the sum of two numbers chosen from 5, 6, 7, 8, 9 and 10 are all greater than 2, and therefore must be odd primes. So if they are the sum of two integers, one of these must be odd and the other even.

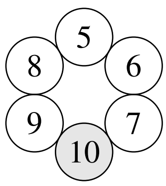
It follows that of the two numbers in any pair of touching circles, one must be odd and one must be even. So the numbers must be arranged in the circles so that they are alternately odd and even. Therefore the number in the shaded circle must be even, that is, 6, 8 or 10.

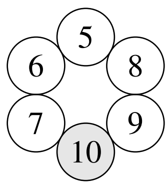
We also deduce from this that the only possible positions for the even numbers not in the shaded circle are the circles adjacent to the top circle which contains the number 5.

Now the number 10 cannot be adjacent to the top circle because $5 + 10 = 15$ which is not a prime. Therefore 10 must be the number in the bottom circle.

> **Commentary**
>
> In the context of the SMC it is adequate to stop here. For a full solution we need to show that it is possible to put the numbers 6, 7, 8, 9 and 10 in the circles in a way that meets the requirement that the sum of the numbers in each pair of touching circles is prime.

To complete the solution we show that there actually is an arrangement with the number 10 in the bottom circle. It is easy to see that the two arrangements shown in the figures below both meet the requirement that the sum of the numbers in each pair of touching circles is a prime. Note that the only difference between them is that in the figure on the left the numbers 5, 6, 7, 10, 9, 8 go round clockwise, whereas in the figure on the right they go round anticlockwise.





For investigation

6.1 Show that the two arrangements of the numbers given in the above solution are the only arrangements that meet the requirements of the question.

7. Which of the following has the largest value?

A $\dfrac{\left(\frac{1}{2}\right)}{\left(\frac{3}{4}\right)}$ B $\dfrac{1}{\left(\dfrac{\left(\frac{2}{3}\right)}{4}\right)}$ C $\dfrac{\left(\dfrac{\left(\frac{1}{2}\right)}{3}\right)}{4}$ D $\dfrac{1}{\left(\dfrac{2}{\left(\frac{3}{4}\right)}\right)}$ E $\dfrac{\left(\dfrac{1}{\left(\frac{2}{3}\right)}\right)}{4}$

Solution **B**

Commentary

The natural method here is to simplify each fraction to the form $\dfrac{p}{q}$, where p and q are positive integers, and then to look and see which of these simplified fractions has the largest value.

However, if you adopt this method, you will see that just one of the fractions is greater than 1.

Now, when x and y are positive numbers,

$$\frac{x}{y} > 1 \text{ if, and only if } x > y.$$

It follows that we can answer this question by evaluating the numerator x and the denominator y for each option in turn, and showing that in only one case we have $x > y$. This saves a little work.

In option A the numerator $\frac{1}{2}$ is smaller than the denominator $\frac{3}{4}$ and so the value of the fraction is less than 1.

In option B the numerator is 1 and the denominator is

$$\frac{\left(\frac{2}{3}\right)}{4} = \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}.$$

Since the numerator is greater than the denominator, the value of this fraction is greater than 1.

In option C the numerator is

$$\frac{\left(\frac{1}{2}\right)}{3} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

and the denominator is 4. Since the numerator is less than the denominator, the value of this fraction is less than 1.

In option D the numerator is 1 and the denominator is

$$\frac{2}{\left(\frac{3}{4}\right)} = 2 \div \frac{3}{4} = 2 \times \frac{4}{3} = \frac{8}{3}.$$

Since the numerator is less than the denominator, the value of this fraction is less than 1.

Finally, in option E the numerator is

$$\frac{1}{\left(\frac{2}{3}\right)} = 1 \div \frac{2}{3} = 1 \times \frac{3}{2} = \frac{3}{2}$$

and the denominator is 4. Again, the numerator is less than the denominator, and so the value of this fraction is also less than 1.

It follows that option B has the largest value.

For investigation

7.1 Arrange the fractions given by the options in Question 7 in order of size.

7.2 The fractions given as options in Question 7 are all built up using 1, 2, 3 and 4 as single digits. There are other possibilities.

Which fraction built up using 1, 2, 3 and 4 as single digits has the largest value?

[The phrase "as single digits" means that the digits cannot be placed next to each other, so that, for example,

$$\frac{432}{1}$$

is not allowed.]

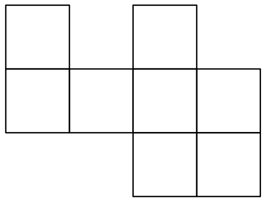
7.3 Which fraction built up using 1, 2, 3, 4 and 5 as single digits has the largest value?

7.4 Simplify the fraction

$$\frac{\left(\frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{4}\right)}\right)}{\left(\frac{\left(\frac{5}{6}\right)}{\left(\frac{7}{8}\right)}\right)}$$

by writing it in the form $\frac{p}{q}$, where p and q are positive integers which have no common factor other than 1.

8. The diagram shows eight small squares. Six of these squares are to be shaded so that the shaded squares form the net of a cube.



In how many different ways can this be done?

A 10   B 8   C 7   D 6   E 4

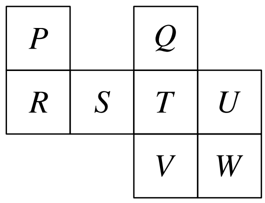
Solution **D**

Commentary

Answering this question as we do to comes down to checking a list of grids obtained from the given diagram by ignoring two of the squares to see in how many cases they form the net of a cube. There are 28 different ways to delete two of the squares in the diagram. Rather a lot to check! So we begin by thinking of a way to cut down the number of grids that we need to check.

When it comes to checking grids, under the conditions of the SMC you will probably have to use your visual imagination to decide whether they could be folded to make cubes. Outside the SMC you could do a practical experiment.

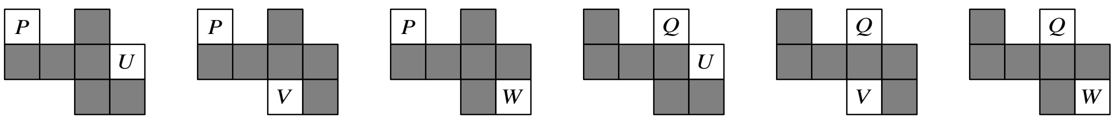
We label the squares as shown in the figure.



We note first that the net of a cube cannot use both the squares P and Q as these must fold to make the same face of any cube.

Each net includes the square T, as otherwise it would not be a connected set of squares. However, no net of a cube can use all four of the squares T, U, V and W. Therefore each net must leave out one of the squares U, V, W.

It follows that there are only six ways in which we might produce the net of the cube by not using a pair of squares. These involve not using one the following pairs of squares: P and U, P and V, P and W, Q and U, Q and V, Q and W. The six grids obtained by ignoring each of these pairs of squares in turn are shown in the figure below.



You should be able to see that each of these is the net of a cube. Therefore there are six different ways in which six of the squares can be used to form the net of a cube.

For investigation

8.1 Are there any other nets of cubes other than those shown in the figure above?

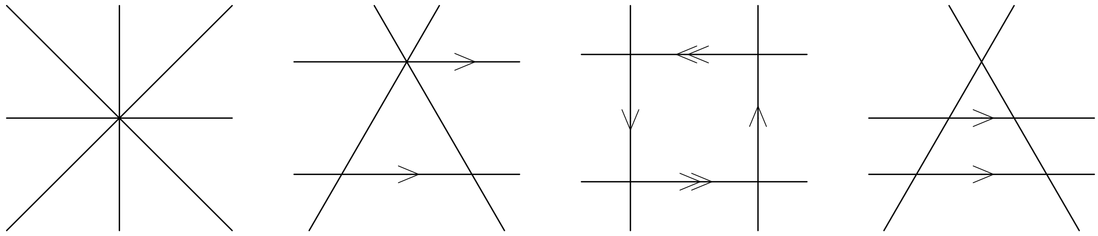
9. Four different straight lines are drawn on a flat piece of paper. The number of points where two or more lines intersect is counted.

Which of the following could *not* be the number of such points?

A 1 B 2 C 3 D 4 E 5

SOLUTION **B**

Each of the figures below shows four straight lines drawn on a flat piece of paper with 1, 3, 4 and 5 points, respectively, where two or more lines intersect.



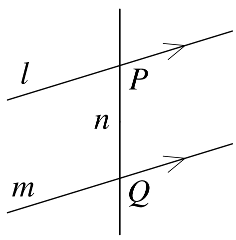
It follows that these numbers of intersections are possible. We deduce from this that the correct option is B.

COMMENTARY

In the context of the SMC, when four of the given options have been eliminated, it is safe to conclude that the remaining option is correct. However, for a complete solution we need to give an argument to show that it is not possible to draw four straight lines so that there are exactly two points of intersection. We now give an argument to prove this.

We show that the attempt to draw four straight lines on a flat piece of paper so that there are just two intersection points is bound to fail.

Suppose that we aim to draw four straight lines so that there are only two points of intersection. Let these points be P and Q.



For P to be a point of intersection, there must be at least two lines through P. Therefore we need to draw at least one line, say l, through P other than the line through P and Q.

Similarly we need to draw at least one line, say m, through Q other than the line through P and Q.

The lines l and m must be parallel since otherwise there would be a third point of intersection.

We can now add the line through P and Q, say n, without creating another intersection point. However, any other line through P would not be parallel to m and hence would create a third intersection point. So we cannot draw such a line. Similarly we cannot draw another line through Q without creating a third intersection point.

So, having drawn the lines l and m, the only line we can add without creating a third intersection point is n. It follows that we cannot draw four straight lines in such a way that there are exactly two intersection points.

For investigation

9.1 Show that four straight lines can be drawn on a flat piece of paper with exactly six points of intersection.

9.2 Show that it is not possible to draw four straight lines on a flat piece of paper with more than six intersection points.

9.3 What are the possibilities for the number of intersection points when five straight lines are drawn on a flat piece of paper?

9.4 For $n \geq 6$, what are the possibilities for the number of intersection points when n straight lines are drawn on a flat piece of paper?

10. The positive integer n is between 1 and 20. Milly adds up all the integers from 1 to n inclusive. Billy adds up all the integers from $n + 1$ to 20 inclusive. Their totals are the same.

What is the value of n?

A 11 B 12 C 13 D 14 E 15

Solution **D**

> Commentary
>
> In the solution below we make use of the formula
>
> $$1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1)$$
>
> for the sum of the positive integers from 1 to n inclusive.
>
> If you are not familiar with this formula, see Problems 10.3 and 10.4 below.

Because the total that Milly gets is the same as the total that Billy gets, each of their totals is half the sum of the positive integers from 1 to 20 inclusive. We can obtain this total by putting $n = 20$ in the formula $\frac{1}{2}n(n+1)$. This gives $\frac{1}{2} \times 20 \times 21 = 210$. Therefore the total of the numbers that Milly adds up is half of 210, that is, 105. So we need to find the positive integer n such that $1 + 2 + 3 + \cdots + n = 105$.

Method 1

We find the value of n by trying each of the given options in turn until we find the correct value.

To test whether option A is correct, we need to see if the sum of the integers from 1 to 11 inclusive is equal to 105. By putting $n = 11$ in the formula $\frac{1}{2}n(n+1)$, we see that

$$1 + 2 + 3 + \cdots + 11 = \tfrac{1}{2} \times 11 \times 12 = 66.$$

So option A is not the correct answer.

We now test the other options in turn. We have

$$1 + 2 + \cdots + 12 = (1 + 2 + \cdots + 11) + 12 = 66 + 12 = 78.$$
$$1 + 2 + \cdots + 13 = (1 + 2 + \cdots + 12) + 13 = 78 + 13 = 91.$$
$$1 + 2 + \cdots + 14 = (1 + 2 + \cdots + 13) + 14 = 91 + 14 = 105.$$

Therefore $n = 14$. So option D is correct.

In the context of the SMC we can stop here.

However, a complete solution should explain why there is no other value of n that works. See Problem 10.1, below.

Method 2

To find the smallest positive integer n such that the sum of the integers from 1 to n, inclusive, is 105, we solve the equation $\frac{1}{2}n(n+1) = 105$. We have

$$\begin{aligned}\tfrac{1}{2}n(n+1) = 105 &\Leftrightarrow n(n+1) = 210\\ &\Leftrightarrow n^2 + n = 210\\ &\Leftrightarrow n^2 + n - 210 = 0\\ &\Leftrightarrow (n+15)(n-14) = 0\\ &\Leftrightarrow n = -15 \text{ or } n = 14.\end{aligned}$$

Because n is a positive integer, we deduce that $n = 14$.

Note that, as in the solution to Question 2, here the symbol $\Leftrightarrow$ stands for *if, and only if*.

For investigation

10.1 Prove that there is exactly one positive integer n that satisfies the equation

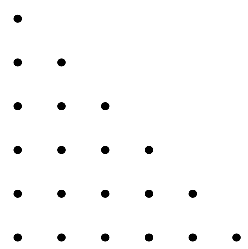
$$\tfrac{1}{2}n(n+1) = 105.$$

10.2 Show that for each positive integer m the equation

$$\tfrac{1}{2}n(n+1) = m$$

has at most one positive integer solution.

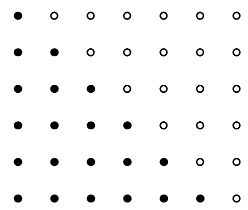
10.3 The numbers obtained by adding up all the positive integers from 1 to n inclusive are called the *triangular numbers*. This is because they correspond to the number of dots in a triangular array. For example, the figure on the right shows an array of dots corresponding to the sum $1 + 2 + 3 + 4 + 5 + 6$.



The notation T_n is often used for the n-th triangular number. That is,

$$T_n = 1 + 2 + 3 + \cdots + n.$$

In the figure on the right we have put together two triangular arrays of dots corresponding to the sum $1 + 2 + 3 + 4 + 5 + 6$. The two triangular arrays together form a rectangle with 6 rows and 7 columns. The rectangle therefore contains 6×7 dots. So the figure illustrates the fact that $2T_6 = 6 \times 7$, and hence that $T_6 = \frac{1}{2}(6 \times 7)$.



Generalize the above to give a proof of the formula $T_n = \frac{1}{2}n(n+1)$. for all positive integers n.

10.4 We can express the sum of the first n positive integers more succinctly using the *sigma* notation. That is, we write $\sum_{k=1}^{n} k$ for the sum $1 + 2 + 3 + \cdots + n$, and, more generally, $\sum_{k=1}^{n} s_k$ for the sum $s_1 + s_2 + s_3 + \cdots + s_n$. We use this notation to give an algebraic proof of the formula for the triangular numbers.

We begin with the observation that

$$(k+1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1.$$

Therefore summing for k going from 1 to n, we obtain

$$\begin{aligned}\sum_{k=1}^{n}((k+1)^2 - k^2) &= \sum_{k=1}^{n}(2k+1) \\ &= 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.\end{aligned}$$

That is,

$$\sum_{k=1}^{n}((k+1)^2 - k^2) = 2T_n + n.$$

Show how the left hand side of the above equation simplifies. Then rearrange the equation to obtain the formula for T_n.

10.5 The solution to Question 10 shows that the equation $T_m = \frac{1}{2}T_n$ has the positive integer solution $m = 14$, $n = 20$.

Show that if $m = a$, $n = b$ is a solution of the equation $T_m = \frac{1}{2}T_n$, then so also is $m = 3a + 2b + 2$ and $n = 4a + 3b + 3$.

Deduce that the equation $T_m = \frac{1}{2}T_n$, has infinitely many positive integer solutions. [Note that another way of putting this is to say that there are infinitely many positive integers T such that both T and $2T$ are triangular numbers.]

11. Rahid has a large number of cubic building blocks. Each block has sides of length 4 cm, 6 cm or 10 cm. Rahid makes little towers built from three blocks stacked on top of each other.

How many different heights of tower can he make?

A 6 B 8 C 9 D 12 E 27

Solution **C**

Commentary

The most straightforward approach here is to list all the possible ways of choosing three of the blocks. For a full solution it is important to do this in a systematic way that makes it clear that every possible case occurs in the list, and no case is listed more than once. We have set out the table in the solution below in a way that we hope makes it clear that we have achieved this.

In the table below we list all combinations of three blocks each of side length 4 cm, 6 cm or 10 cm. In the last column we have give the height of the tower that the given blocks make.

number of 4 cm blocks	number of 6 cm blocks	number of 10 cm blocks	height of tower
3	0	0	12 cm
0	3	0	18 cm
0	0	3	30 cm
2	1	0	14 cm
2	0	1	18 cm
1	2	0	16 cm
0	2	1	22 cm
1	0	2	24 cm
0	1	2	26 cm
1	1	1	20 cm

We see that there are ten ways in which Rahid can choose three blocks with which to make a little tower. However, two of them make towers of the same height, namely 18 cm. So there are 9 different heights of tower that Rahid can make.

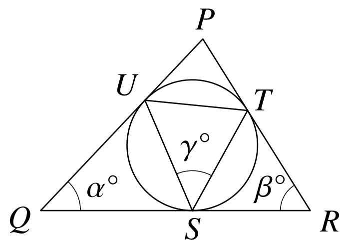
For investigation

11.1 How many different heights of tower can be built using four of the blocks?

12. A circle touches the sides of triangle PQR at the points S, T and U as shown. Also $\angle PQR = \alpha°$, $\angle PRQ = \beta°$ and $\angle TSU = \gamma°$.

Which of the following gives γ in terms of α and β?

A $\frac{1}{2}(\alpha + \beta)$ B $180 - \frac{1}{2}(\alpha + \beta)$
C $180 - (\alpha + \beta)$ D $\alpha + \beta$
E $\frac{1}{3}(\alpha + \beta)$



SOLUTION **A**

METHOD 1

From the given fact that the circle touches the sides of the triangle, we deduce that the lines PQ, QR and RP are tangents to the circle. The two tangents from a given point to a circle are of equal length. In particular, $QS = QU$. Therefore the triangle QSU is isosceles, and hence $\angle QSU = \angle QUS$.

Because the sum of the angles in a triangle is 180°, from the triangle QSU we deduce that $\alpha° + \angle QSU + \angle QUS = 180°$.

Since $\angle QSU = \angle QUS$, it follows that $\alpha° + 2\angle QSU = 180°$. We can rearrange this equation to give

$$\angle QSU = \tfrac{1}{2}(180 - \alpha)°.$$

In a similar way, we have that the tangents RS and RT are of equal length. Hence the triangle RTS is isosceles and therefore

$$\angle RST = \tfrac{1}{2}(180 - \beta)°.$$

Because QSR is a straight line, the angles at S on this line have sum 180°, that is,

$$\angle QSU + \angle RST + \gamma° = 180°.$$

Substituting in this equation the expressions for $\angle QSU$ and $\angle RST$ we have already found, we deduce that

$$\tfrac{1}{2}(180 - \alpha)° + \tfrac{1}{2}(180 - \beta)° + \gamma° = 180°.$$

We can rearrange this equation as

$$180° - \tfrac{1}{2}(\alpha + \beta)° + \gamma° = 180°,$$

from which it follows that

$$\gamma = \tfrac{1}{2}(\alpha + \beta).$$

METHOD 2

As in method 1, PQ and PT are tangents to the circle.

By the *Alternate Angle Theorem*, $\angle PUT = \angle PTU = \gamma°$. Therefore because the angles in the triangle PUT have sum 180°, we have $\angle UPT + \gamma° + \gamma° = 180°$, and therefore

$$\angle UPT = 180° - 2\gamma°.$$

Therefore, as the angles in the triangle PQR have sum 180°,

$$\alpha° + \beta° + (180° - 2\gamma°) = 180°.$$

We can rearrange this last equation to give

$$2\gamma° = \alpha° + \beta°,$$

and it follows that

$$\gamma = \tfrac{1}{2}(\alpha + \beta).$$

For investigation

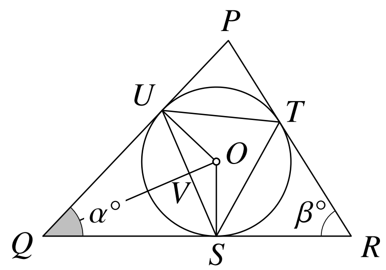
12.1 In Method 1 we used the fact that *the two tangents from a given point to a circle are of equal length.* Find a proof of this (that is, either devise your own proof, look in a book or on the internet, or ask your teacher).

12.2 In Method 2 we have used the *Alternate Segment Theorem* which says that the angle between a tangent to a circle and a chord is equal to the angle subtended by the chord at the circumference of the circle. Find a proof of this theorem.

12.3 Here we consider a third method.

We let O be the centre of the circle. In the figure we have added the lines joining O to the points Q, S, and U. We let V be the point where QO meets US.

Now proceed as follows.



(i) Prove that the triangles QOS and QOU are congruent.
(ii) Deduce that $\angle VQS = \angle VQU = \frac{1}{2}\alpha°$.
(iii) Prove that the triangles QVS and QVU are congruent.
(iv) Deduce that $\angle QVS = 90°$.
(v) Deduce that $\angle USO = \frac{1}{2}\alpha°$.
(vi) Show that, similarly, $\angle TSO = \frac{1}{2}\beta°$.
(vii) Conclude that $\gamma = \frac{1}{2}(\alpha + \beta)$.

13. The Knave of Hearts tells only the truth on Mondays, Tuesdays, Wednesdays and Thursdays. He tells only lies on all the other days. The Knave of Diamonds tells only the truth on Fridays, Saturdays, Sundays and Mondays. He tells only lies on all the other days. On one day last week, they both said, "Yesterday I told lies."

On which day of the week was that?

A Sunday B Monday C Tuesday D Thursday E Friday

SOLUTION **E**

A day on which one of the Knaves says "Yesterday I told lies." is either a day when he is telling the truth but the previous day he was lying, or else a day on which he is telling lies and the previous day he was telling the truth.

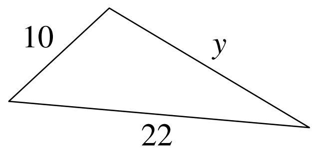
So for the Knave of Hearts that day would have to be a Monday or a Friday. Likewise, for the Knave of Diamonds that day would have to be a Friday or a Tuesday.

Therefore a day on which they both said "Yesterday I told lies." is a Friday.

14. The triangle shown has an area of 88 square units.

What is the value of y?

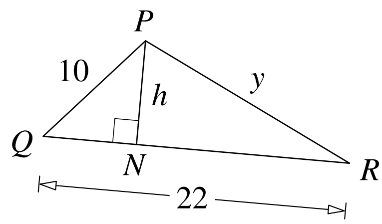
A 17.6 B $2\sqrt{46}$ C $6\sqrt{10}$ D $13\sqrt{2}$ E $8\sqrt{5}$



Solution **E**

Method 1

We label the vertices of the triangle as shown in the figure. We let N be the point where the perpendicular from P to QR meets QR, and we let h be the length of PN.



The triangle has area 88. So, from the formula area $= \frac{1}{2}(\text{base} \times \text{height})$ for a triangle, we have

$$88 = \tfrac{1}{2}(22 \times h),$$

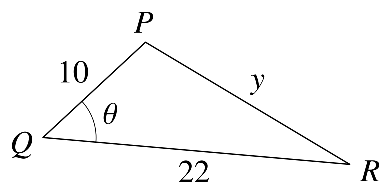
that is, $88 = 11h$. It follows that $h = 8$.

By Pythagoras' Theorem applied to the right-angled triangle QNP, we have $QN^2 + h^2 = 10^2$. Therefore $QN^2 = 10^2 - h^2 = 10^2 - 8^2 = 100 - 64 = 36$. Since QN is a length and therefore positive, we deduce that $QN = 6$. [Alternatively, you could just note that PQN is a right angled triangle with sides in the ratio 5 : 4 : 3.]

It follows that $NR = QR - QN = 22 - 6 = 16$. Now, applying Pythagoras' Theorem to the right-angled triangle PNR, we have $NR^2 + h^2 = y^2$, that is, $16^2 + 8^2 = y^2$. Hence $y^2 = 256 + 64 = 320$. We deduce that $y = \sqrt{320} = \sqrt{64 \times 5} = 8\sqrt{5}$.

Method 2

We use the fact that the area of a triangle is $\frac{1}{2}ab\sin\theta$, where a and b are the lengths of two of the sides and θ is the angle between these two sides, together with the *Cosine Rule*.



We label the vertices of the triangle as shown in the figure and let $\angle PQR = \theta$. It follows that $88 = \frac{1}{2}(22 \times 10)\sin\theta$. That is, $88 = 110\sin\theta$. Hence

$$\sin\theta = \frac{88}{110} = \frac{4}{5}.$$

From the identity $\cos^2\theta + \sin^2\theta = 1$, it follow that

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25} = \left(\frac{3}{5}\right)^2,$$

and therefore $\cos\theta = \pm\frac{3}{5}$.

From the diagram in the question, we see that $\angle PQR$ is an acute angle, that is, $0 < \theta < 90°$. Therefore $\cos\theta > 0$ and it follows that $\cos\theta = \frac{3}{5}$.

We can now deduce, using the Cosine Rule, that

$$\begin{aligned} y^2 &= PQ^2 + QR^2 - 2PQ.QR\cos\theta \\ &= 10^2 + 22^2 - 2 \times 10 \times 22\cos\theta \\ &= 100 + 484 - 2 \times 10 \times 22 \times \frac{3}{5} \\ &= 100 + 484 - 264 \\ &= 320. \end{aligned}$$

Hence $y = \sqrt{320} = 8\sqrt{5}$.

Commentary

In the problems below we cover a third method for answering Question 14. This method uses *Heron's Formula* for the area of a triangle.

In Method 1 we used a formula for the area of a triangle in terms of the height of the triangle. In Method 2 we used a formula which involves the angle between two sides. The advantage of Heron's Formula is that it gives the area of a triangle just in terms of the lengths of the sides of the triangle. However, as you will see if you tackle the problems below, this method involves some quite complicated algebra.

Heron's Formula: The area, A, of a triangle whose sides have lengths a, b and c is given by

$$A = \tfrac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

This formula may also be written as

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$, is half of the length of the perimeter of the triangle.

For investigation

14.1 Use Heron's Formula to find the third side, y, of an acute-angled triangle with sides 10 and 22, and area 88.

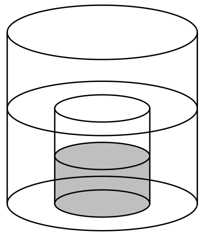
14.2 Deduce Heron's Formula from the formula $A = \frac{1}{2}ah$, where h is the height of the triangle. [Hint: in the figure for Method 1, let $QN = x$. Then use Pythagoras' Theorem, applied to triangles PNQ and PNR, to obtain an expression for x in terms of a, b and c.]

14.3 Deduce Heron's Formula from the formula $A = \frac{1}{2}ab\sin\theta$. [Hint: use the Cosine Formula to obtain an expression for $\cos\theta$ in terms of a, b and c.]

Note

Heron of Alexandria lived in the first century of the current era. He was a noted geometer and writer on mechanics. He invented many machines including a steam turbine. His proof of his formula for the area of a triangle is to be found in his book *Metrica*.

15. Two vases are cylindrical in shape. The larger vase has diameter 20 cm. The smaller vase has diameter 10 cm and height 16 cm. The larger vase is partially filled with water. Then the empty smaller vase, with the open end at the top, is slowly pushed down into the water, which flows over its rim. When the smaller vase is pushed right down, it is half full of water.



What was the original depth of the water in the larger vase?

A 10 cm B 12 cm C 14 cm D 16 cm E 18 cm

SOLUTION **C**

We use the formula $V = \pi r^2 h$ for the volume V of a cylinder of radius r and height h. Expressed in terms of the diameter d of the cylinder, where $r = \frac{1}{2}d$, this becomes $V = \frac{1}{4}\pi d^2 h$.

When the smaller vase is pushed right down and the water finishes flowing over the rim, the water in the larger vase comes to the top of the smaller cylinder and so has the same depth as height of the smaller cylinder, that is, 16 cm.

So the volume of water in the larger vase is that of a cylinder with diameter 20 cm and height 16 cm, less the volume of the empty half of the smaller vase.

The volume of a cylinder with diameter 20 cm and height 16 cm is, $\frac{1}{4}\pi \times 20^2 \times 16\,\text{cm}^3 = 1600\pi\,\text{cm}^3$. The volume of the empty half of the smaller cylinder is $\frac{1}{4}\pi \times 10^2 \times 8\,\text{cm}^3 = 200\pi\,\text{cm}^3$.

It follows that the volume of the water in the cylinder is, in cm^3,

$$1600\pi - 200\pi = 1400\pi.$$

Now suppose that the original depth of the water in the cylinder is x cm. It follows that the volume of the water in the cylinder is, in cm^3,

$$\tfrac{1}{4}\pi \times 20^2 \times x = 100\pi x.$$

Since these two expressions for the volume of the water must be the same,

$$100\pi x = 1400\pi,$$

from which it follows that $x = 14$.

Therefore the original depth of the water in the cylinder was 14 cm.

16. Fnargs are either red or blue and have 2, 3 or 4 heads. A group of six Fnargs consisting of one of each possible form is made to line up such that no immediate neighbours are the same colour nor have the same number of heads.

How many ways are there of lining them up from left to right?

A 12 B 24 C 60 D 120 E 720

Solution **A**

We let R2, R3, R4 be red Fnargs with 2, 3 and 4 heads, respectively, and B2, B3, B4 be blue Fnargs with 2, 3 and 4 heads, respectively. We need to count the number of ways of lining up R2, R3, R4, B2, B3, B4 so that no two Fnargs that are next to each other have the same colour or the same number of heads.

Suppose that a row of these six Fnargs begins with R2 at the left hand end. The second Fnarg in the row must be blue and have 3 or 4 heads. So the row either begins R2, B3 or R2, B4.

If the row begins R2, B3 the third Fnarg must be red and cannot have 3 heads. Since R2 is already in the line up, the third Fnarg must be R4. The fourth Fnarg must be blue and cannot have 4 heads. Since B3 is already in the row, this fourth Fnarg must be B2. This leaves just R3 and B4 to be placed. So, as the colours must alternate, the fifth and sixth Fnargs in the row must be R3 and B4 in this order from left to right. Hence the line up must be

R2, B3, R4, B2, R3, B4.

A similar argument shows that if the row begins R2, B4, then the line up must be as above but with 3 and 4 interchanged, that is,

R2, B4, R3, B2, R4, B3.

So there are just two ways to complete a row that begins with R2.

A similar argument shows that whichever Fnarg begins a row, there are just two ways to complete the row.

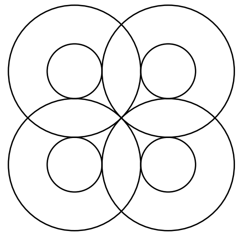
Since the row may begin with any of the six Fnargs, the total number of ways to line them up is $6 \times 2 = 12$.

For investigation

16.1 As a result of a genetic modification there are now Fnargs with 2, 3, 4 or 5 heads, but still only red or blue. In how many ways can we line up eight Fnargs consisting of one of each possible form so that two adjacent Fnargs have neither the same colour nor the same number of heads?

16.2 As a result of a further genetic modification there are now red, blue and green Fnargs, each with 2, 3, 4 or 5 heads. In how many ways can we line up twelve Fnargs consisting of one of each possible form so that two adjacent Fnargs have neither the same colour nor the same number of heads?

17. The diagram shows eight circles of two different sizes. The circles are arranged in concentric pairs so that the centres form a square. Each larger circle touches one other larger circle and two smaller circles. The larger circles have radius 1.

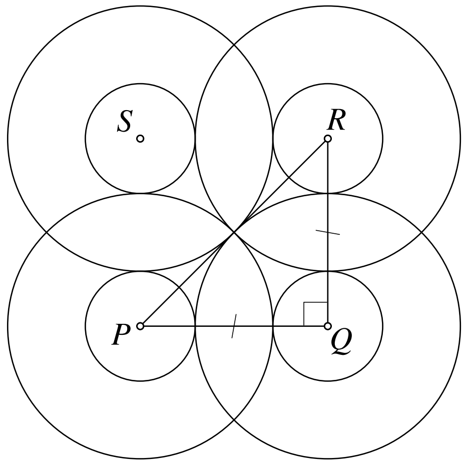


What is the radius of each smaller circle?

A $\frac{1}{3}$ B $\frac{2}{5}$ C $\sqrt{2}-1$ D $\frac{1}{2}$ E $\frac{1}{2}\sqrt{2}$

Solution **C**

Let the centres of the circles be P, Q, R and S, as shown. Let the radius of the smaller circles be r.



Then the length of PQ is the sum of the radius of a larger circle and the radius of a smaller circle. That is, $PQ = 1 + r$. Similarly $QR = 1 + r$. Also, PR is the sum of the radii of two of the large circles. That is, $PR = 1 + 1 = 2$.

Because $PQRS$ is a square, $\angle PQR$ is a right angle.

Therefore, applying Pythagoras' theorem to the triangle PQR, we have

$$(1+r)^2 + (1+r)^2 = 2^2.$$

It follows that

$$2(1+r)^2 = 4$$

and hence

$$(1+r)^2 = 2.$$

Therefore

$$1 + r = \pm\sqrt{2}$$

and hence

$$r = \pm\sqrt{2} - 1.$$

Because a radius must be positive, we deduce that the radius of each smaller circle is

$$\sqrt{2} - 1.$$

.

18. What is the largest integer k whose square k^2 is a factor of 10!?

[$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.]

A 6 B 256 C 360 D 720 E 5040

SOLUTION **D**

COMMENTARY

The most straightforward way to answer this question would be to first evaluate 10!, and then try out the squares of the given options, starting with the largest, until you find a square that is a factor of 10!

If you do the multiplication you will find that $10! = 3\,628\,800$. If you now test the options, you will find that 5040^2 is not a factor of 10!, but that 720^2 is.

However, this approach has several disadvantages. In the SMC calculators are not allowed, and this method requires a lot of arithmetic. Outside the context of SMC, we do not have only five options to test. We would have to start with the largest integer, k, such that $k^2 \leq 10!$, and check whether k^2 is a factor of 10!. If not we would need to check whether $(k-1)^2$ is factor of 10!, and so on, until we find the largest integer whose square is a factor of 10! Also, this method gives no insight into the problem and if 10 were replaced by a much larger number, it would not be feasible to use this method.

A better method is to work with the prime factorization of 10!, using the fact that when we factorize a square into primes, each prime occurs to an even power. So we begin by factorizing 10! into primes, and then we look for the highest product of even powers of the prime factors that is a factor of 10!

We express 10! as a product of prime numbers as follows.

$$\begin{aligned} 10! &= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= (2 \times 5) \times 3^2 \times 2^3 \times 7 \times (2 \times 3) \times 5 \times 2^2 \times 3 \times 2 \\ &= 2^8 \times 3^4 \times 5^2 \times 7 \\ &= (2^4 \times 3^2 \times 5)^2 \times 7. \end{aligned}$$

We deduce that the square of $2^4 \times 3^2 \times 5$ is a factor of 10! but there is no larger integer k such that k^2 is a factor of 10! So the largest integer k whose square k^2 is a factor of 10! is $2^4 \times 3^2 \times 5 = 16 \times 9 \times 5 = 720$.

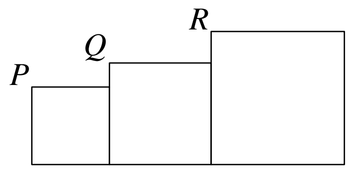
FOR INVESTIGATION

18.1 Which is the largest integer k whose square, k^2, is a factor of 11!?

18.2 Which is the largest integer k whose square, k^2, is a factor of 12!?

18.3 Which is the least integer, n, such that 126^2 is a factor of n!?

19. Three squares are arranged as shown so that their bases lie on a straight line. Also, the corners P, Q and R lie on a straight line. The middle square has sides that are 8 cm longer than the sides of the smallest square. The largest square has sides of length 50 cm. There are two possible values for the length (in cm) of the sides of the smallest square.



Which of the following are they?

A 2, 32 B 4, 42 C 4, 34 D 32, 40 E 34, 42

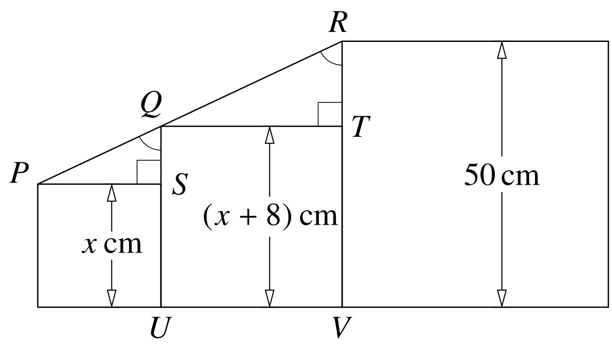
Solution **A**

We let S, T, U and V be the points shown in the figure.

We let the side length of the smallest square be x cm.

It follows that the middle square has sides of length $(x + 8)$ cm

Since the lines QU and RV are both perpendicular to UV they are parallel lines.



Therefore, as PQR is a straight line, the corresponding angles $\angle PQS$ and $\angle QRT$ are equal.

Because the figure is made up of squares, $\angle PSU = \angle QTV = 90°$. Therefore, as angles on a line add up to 180°, we have $\angle PSQ = \angle QTR = 90°$. It follows that the right-angled triangles PSQ and QTR are similar because they have the same angles. Therefore

$$\frac{QS}{PS} = \frac{RT}{QT}.$$

Now QS has length 8 cm, PS has length x cm, RT has length $(50 - (x + 8))$ cm $= (42 - x)$ cm and QT has length $(x + 8)$ cm. It follows that

$$\frac{8}{x} = \frac{42 - x}{x + 8}.$$

Multiplying both sides of this equation by $x(x + 8)$, we see that this last equation is equivalent to

$$8(x + 8) = x(42 - x),$$

that is,

$$8x + 64 = 42x - x^2.$$

This last equation may be rearranged to give

$$x^2 - 34x + 64 = 0.$$

We can now factorize the quadratic in this equation to give

$$(x - 32)(x - 2) = 0.$$

Therefore

$$x = 2 \text{ or } x = 32.$$

We deduce that the possible lengths, in cm, of the sides of the smallest square are 2 and 32.

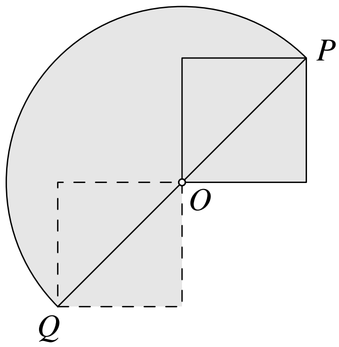
20. A square ink pad has sides of length 1 cm. It is covered in black ink and carefully placed in the middle of a piece of white paper. The square pad is then rotated 180° about one of its corners so that all of the pad remains in contact with the paper throughout the turn. The pad is then removed from the paper.

What area of paper, in cm^2, is coloured black?

A $\pi + 2$ B $2\pi - 1$ C 4 D $2\pi - 2$ E $\pi + 1$

Solution **E**

We let O be the corner of the square about which it rotates. The initial position of the square is shown by solid lines, and its final position by broken lines, so that the corner initially at the position P rotates to the position Q.



The area that is in contact with the paper at some time during the rotation is shown shaded. This is the area that gets coloured black as the square rotates. It is made up of the semicircle with diameter PQ and the two shaded half squares outside this semicircle.

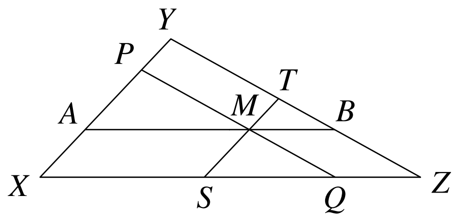
The length of the sides of the squares, in cm, is 1. We let the length of OP be x cm. Then, by Pythagoras' Theorem, $x^2 = 1^2 + 1^2 = 2$. Therefore, the area of the semicircle with diameter PQ is, in cm^2, $\frac{1}{2}\pi x^2 = \frac{1}{2}\pi \times 2 = \pi$.

The area of the two half squares outside the semicircle is equal to the area of one square with side length 1 cm, that is, 1 cm^2.

Therefore, the total area that is coloured black is, in cm^2, $\pi + 1$.

21. The diagram shows a triangle XYZ. The sides XY, YZ and XZ have lengths 2, 3 and 4 respectively. The lines AMB, PMQ and SMT are drawn parallel to the sides of triangle XYZ so that AP, QS and BT are of equal length.

What is the length of AP?



A $\frac{10}{11}$ B $\frac{11}{12}$ C $\frac{12}{13}$ D $\frac{13}{14}$ E $\frac{14}{15}$

SOLUTION **C**

Let the common length of AP, QS and BT be x.

Because AB is parallel to XZ, the corresponding angles, $\angle PAM$ and $\angle YXZ$ are equal. Because PQ is parallel to YZ, the corresponding angles, $\angle APM$ and $\angle XYZ$ are equal.

It follows that the triangles APM and XYZ are similar. Therefore their corresponding sides are in the same ratio. Hence, in particular, $\frac{AP}{XY} = \frac{AM}{XZ}$. That is, $\frac{x}{2} = \frac{AM}{4}$, and therefore $AM = 2x$.

Since AM is parallel to XS, and XA is parallel to SM, the quadrilateral $MAXS$ is a parallelogram. Therefore $XS = AM = 2x$.

Similarly, the triangles MTB and XYZ are similar, and therefore $\frac{TB}{YZ} = \frac{MB}{XZ}$. Hence $\frac{x}{3} = \frac{MB}{4}$. Therefore $MB = \frac{4}{3}x$. Also $MBZQ$ is a parallelogram. Therefore $QZ = MP = \frac{4}{3}x$.

Now, as $XS + SQ + QZ = XZ$, it follows that

$$2x + x + \frac{4}{3}x = 4.$$

That is,

$$\frac{13}{3}x = 4,$$

from which it follows that

$$x = \frac{3}{13} \times 4 = \frac{12}{13}.$$

Therefore the length of AP is $\frac{12}{13}$.

FOR INVESTIGATION

21.1 Find the lengths of XA and PY in terms of x. Then use the fact that XY has length 2 to determine the value of x.

21.2 Similarly, determine the value of x by finding the lengths of YT and BZ, and then use the fact that YZ has length 3.

21.3 Is it true that for every triangle XYZ there is a point M, such that when the lines AMB, PMQ and SMT are drawn parallel to the sides of the triangle, then AP, QS and BT are of equal length ?

22. Let $f(x) = x + \sqrt{x^2+1} + \dfrac{1}{x - \sqrt{x^2+1}}$.

What is the value of $f(2015)$?

A -1 B 0 C 1 D $\sqrt{2016}$ E 2015

Solution **B**

Commentary

At first sight, this looks an impossibly difficult question, as it seems to involve working out the value of the square root $\sqrt{x^2+1}$ for $x = 2015$, *without the use of a calculator*! As this cannot be the intended method, we look for an alternative approach.

The presence of both the terms $x - \sqrt{x^2+1}$ and $x + \sqrt{x^2+1}$ in the expression for $f(x)$ suggests that we can make progress using algebra, and in particular, the *difference of two squares* formula $(a-b)(a+b) = a^2 - b^2$, with $a = x$, and $b = \sqrt{x^2+1}$.

Indeed, if you have the confidence to try this approach, you will see that this question turns out not to be at all difficult.

We have

$$f(x) = x + \sqrt{x^2+1} + \frac{1}{x - \sqrt{x^2+1}}.$$

We now put the two terms involved in $f(x)$ over a common denominator. This gives

$$\begin{aligned} f(x) &= \frac{\left(x - \sqrt{x^2+1}\right)\left(x + \sqrt{x^2+1}\right) + 1}{x - \sqrt{x^2+1}} \\ &= \frac{\left(x^2 - (\sqrt{x^2+1})^2\right) + 1}{x - \sqrt{x^2+1}} \\ &= \frac{\left(x^2 - (x^2+1)\right) + 1}{x - \sqrt{x^2+1}} \\ &= \frac{-1+1}{x - \sqrt{x^2+1}} \\ &= \frac{0}{x - \sqrt{x^2+1}} \\ &= 0. \end{aligned}$$

This holds whatever the value of x. Therefore, in particular, $f(2015) = 0$.

23. Given four different non-zero digits, it is possible to form 24 different four-digit numbers containing each of these four digits.

What is the largest prime factor of the sum of the 24 numbers?

A 23 B 93 C 97 D 101 E 113

SOLUTION **D**

Let the four different non-zero digits be a, b, c and d. If we use them to make 24 different four-digit numbers, then each of a, b, c and d occurs 6 times in the units position, 6 times in the tens position, 6 times in the hundreds position, and 6 times in the thousands position.

It follows that the digits in the units position contribute $6(a+b+c+d)$ to the sum of all 24 numbers, and the digits in the tens position contribute $6(a+b+c+d)\times 10 = 60(a+b+c+d)$ to this sum. Similarly, the digits in the hundreds position contribute $600(a+b+c+d)$ and those in the thousands column contribute $6000(a+b+c+d)$.

Therefore the total sum of all 24 numbers is

$$6000(a+b+c+d)+600(a+b+c+d)+60(a+b+c+d)+6(a+b+c+d) = 6666(a+b+c+d).$$

We can factorize 6666 into primes, as follows,

$$6666 = 6\times 1111 = 2\times 3\times 11\times 101.$$

Since a, b, c, d are four different digits, their sum is at most $9+8+7+6=30$ and so this sum cannot have a prime factor as large as 101. We deduce that the largest prime factor of the sum, $6666(a+b+c+d)$, of all the 24 numbers is 101.

FOR INVESTIGATION

23.1 Given five different non-zero digits, it is possible to form 120 different five-digit numbers containing each of these five digits.

What is the largest prime factor of the sum of the 120 numbers?

23.2 Given six different non-zero digits, how many different six-digit numbers are there which contain each of the given six digits?

What is the largest prime factor of the sum of all these six-digit numbers?

24. Peter has 25 cards, each printed with a different integer from 1 to 25. He wishes to place N cards in a single row so that the numbers on every adjacent pair of cards have a prime factor in common.

What is the largest value of N for which this is possible?

A 16 B 18 C 20 D 22 E 24

Solution **C**

An integer can occur on a card in the row only if it shares a prime factor with at least one other integer on a card in the row. This rules out 1, which has no prime factors, and the primes 13, 17, 19 and 23 which are not factors of any other integers in the range from 1 to 25.

With these five integers excluded, this leaves at most 20 cards that can be in the row. It is possible to arrange all these remaining cards in a row so as to meet the condition that integers on adjacent cards share a prime factor. In fact, there are lots of ways to do this. For example

7, 14, 21, 3, 18, 15, 5, 25, 10, 4, 20, 24, 9, 6, 8, 12, 16, 2, 22, 11.

It follows that the largest value of N with the required property is 20.

For investigation

24.1 There are lots of different ways to arrange the cards carrying the 20 integers

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25

in a row so that the integers on adjacent cards have a prime factor in common. Just one of these arrangements is given in the above solution.

One of these integers must occur at the end, either first or last, in any row of all these integers that meet the given condition. Which is it?

24.2 For which of the integers, n, of the 20 integers listed in Exercise 24.1, is it possible to arrange all 20 cards in a row so that n is on the first card of the row, and so that every pair of adjacent integers in the row have a prime factor in common?

24.3 As we say above, there are lots of ways to arrange the 20 cards in a row so that every pair of adjacent integers has a prime number in common. Counting all the different possible solutions is rather hard. You can try this if you like, but we suggest that instead you consider the case where there are just 12 cards, each printed with a different integer from 1 to 12.

(i) What is the largest value of N such that N of the 12 cards can be placed in a single row so that the numbers on every adjacent pair of cards have a prime factor in common?

(ii) For this value of N how many different ways are there to arrange the N cards in a row so as to meet the required condition?

25. A function, defined on the set of positive integers, is such that $f(xy) = f(x) + f(y)$ for all x and y. It is known that $f(10) = 14$ and $f(40) = 20$.

What is the value of $f(500)$?

A 29 B 30 C 39 D 48 E 50

SOLUTION **C**

COMMENTARY

The factorization of 500 into primes is $500 = 2^2 \times 5^3$. Since the given function satisfies the formula $f(xy) = f(x) + f(y)$, it follows that $f(500) = f(2 \times 2 \times 5 \times 5 \times 5) = f(2) + f(2) + f(5) + f(5) + f(5) = 2f(2) + 3f(5)$.

Therefore we can find the value of $f(500)$ if we can find the values of $f(2)$ and $f(5)$. In the first method given below we go about this directly. The second method is more systematic, but more complicated.

METHOD 1

Using the formula $f(xy) = f(x) + f(y)$ with $x = 4$ and $y = 10$, gives $f(40) = f(4) + f(10)$. Hence $f(4) = f(40) - f(10) = 20 - 14 = 6$. Using the formula with $x = y = 2$, gives $f(4) = f(2) + f(2)$. Therefore $f(2) = \frac{1}{2}f(4) = \frac{1}{2} \times 6 = 3$. Using the formula with $x = 2$ and $y = 5$, gives $f(10) = f(2) + f(5)$ and therefore $f(5) = f(10) - f(2) = 14 - 3 = 11$.

Since $500 = 2^2 \times 5^3$, we can now deduce that $f(500) = 2f(2) + 3f(5) = 2 \times 3 + 3 \times 11 = 6 + 33 = 39$.

METHOD 2

Let $f(2) = a$ and $f(5) = b$. Since $10 = 2 \times 5$, we have $f(10) = f(2) + f(5)$. Since $40 = 2^3 \times 5$, we have $f(40) = 3f(2) + f(5)$.

Therefore, as $f(10) = 14$ and $f(40) = 20$, we obtain the two linear equations

$$a + b = 14 \quad (1)$$
$$3a + b = 20 \quad (2)$$

From equations (1) and (2)

$$(3a + b) - (a + b) = 20 - 14.$$

We deduce that $2a = 6$ and therefore $a = 3$. Hence, by equation (1), we have $b = 11$.

Since $500 = 2^2 \times 5^3$, we deduce that $f(500) = 2f(2) + 3f(5) = 2 \times 3 + 3 \times 11 = 39$.

FOR INVESTIGATION

25.1 Assume that $f(xy) = f(x) + f(y)$ for all positive integers x and y. What is the value of $f(1)$?

25.2 Again, assume that $f(xy) = f(x) + f(y)$ for all positive integers x and y. Show that if the positive integer n has the factorization $n = p^a \times q^b \times r^c$, then $f(n) = af(p) + bf(q) + cf(r)$.

UK SENIOR MATHEMATICAL CHALLENGE

Tuesday 8 November 2016

Organised by the **United Kingdom Mathematics Trust**

and supported by

Institute and Faculty of Actuaries

RULES AND GUIDELINES (to be read before starting)

1. Do not open the question paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**.
   No answers or personal details may be entered on the Answer Sheet after the 90 minutes are over.
3. The use of rough paper is allowed.
   Calculators, measuring instruments and squared paper are forbidden.
4. Candidates must be full-time students at secondary school or FE college, and must be in Year 13 or below (England & Wales); S6 or below (Scotland); Year 14 or below (Northern Ireland).
5. **Use B or HB pencil only**. Mark *at most one* of the options A, B, C, D, E on the Answer Sheet for each question. Do not mark more than one option.
6. **Scoring rules**: all candidates start out with 25 marks;

   0 marks are awarded for each question left unanswered;

   4 marks are awarded for each correct answer;

   1 mark is deducted for each incorrect answer.
7. **Guessing**: Remember that there is a penalty for incorrect answers. Note also that later questions are deliberately intended to be harder than earlier questions. You are thus advised to concentrate first on solving as many as possible of the first 15-20 questions. Only then should you try later questions.

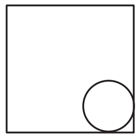
1. How many times does the digit 9 appear in the answer to 987654321×9 ?

A 0 B 1 C 5 D 8 E 9

2. On a Monday, all prices in Isla's shop are 10% more than normal. On Friday all prices in Isla's shop are 10% less than normal. James bought a book on Monday for £5.50. What would be the price of another copy of this book on Friday?

A £5.50 B £5.00 C £4.95 D £4.50 E £4.40

3. The diagram shows a circle with radius 1 that rolls without slipping around the inside of a square with sides of length 5.



The circle rolls once around the square, returning to its starting point.

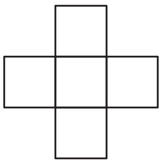
What distance does the centre of the circle travel?

A $16 - 2\pi$ B 12 C $6 + \pi$ D $20 - 2\pi$ E 20

4. Alex draws a scalene triangle. One of the angles is 80°.

Which of the following could be the difference between the other two angles in Alex's triangle?

A 0° B 60° C 80° D 100° E 120°

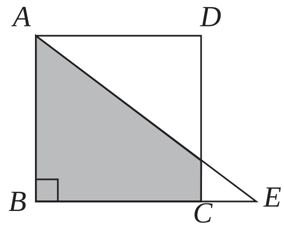
5. 

All the digits 2, 3, 4, 5 and 6 are placed in the grid, one in each cell, to form two three-digit numbers that are squares.

Which digit is placed in the centre of the grid?

A 2 B 3 C 4 D 5 E 6

6. The diagram shows a square $ABCD$ and a right-angled triangle ABE. The length of BC is 3. The length of BE is 4.



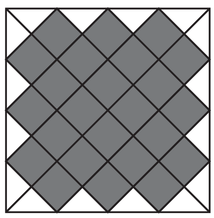
What is the area of the shaded region?

A $5\frac{1}{4}$ B $5\frac{3}{8}$ C $5\frac{1}{2}$ D $5\frac{5}{8}$ E $5\frac{3}{4}$

7. Which of these has the smallest value?

A 2016^{-1} B $2016^{-1/2}$ C 2016^{0} D $2016^{1/2}$ E 2016^{1}

8. Points are drawn on the sides of a square, dividing each side into n equal parts (so, in the example shown, $n = 4$).



The points are joined in the manner indicated, to form several small squares (24 in the example, shown shaded) and some triangles.

How many small squares are formed when $n = 7$?

A 56 B 84 C 140 D 840 E 5040

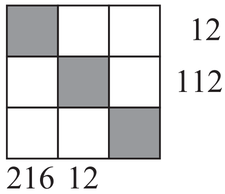
9. A square has vertices at (0, 0), (1, 0), (1, 1) and (0, 1). Graphs of the following equations are drawn on the same set of axes as the square.

$$x^2 + y^2 = 1, \quad y = x + 1, \quad y = -x^2 + 1, \quad y = x, \quad y = \frac{1}{x}$$

How many of the graphs pass through exactly two of the vertices of the square?

A 1 B 2 C 3 D 4 E 5

10. The digits from 1 to 9 are to be written in the nine cells of the 3×3 grid shown, one digit in each cell.
The product of the three digits in the first row is 12.
The product of the three digits in the second row is 112.
The product of the three digits in the first column is 216.
The product of the three digits in the second column is 12.
What is the product of the digits in the shaded cells?



A 24 B 30 C 36 D 48 E 140

11. In the grid below each of the blank squares and the square marked X are to be filled by the mean of the two numbers in its adjacent squares. Which number should go in the square marked X?

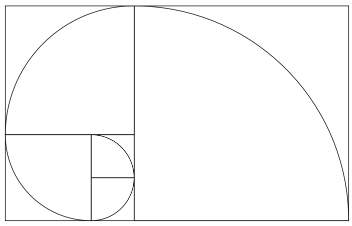
10			X		25

A 15 B 16 C 17 D 18 E 19

12. What is the smallest square that has 2016 as a factor?

A 42^2 B 84^2 C 168^2 D 336^2 E 2016^2

13. Five square tiles are put together side by side. A quarter circle is drawn on each tile to make a continuous curve as shown. Each of the smallest squares has side-length 1.
What is the total length of the curve?

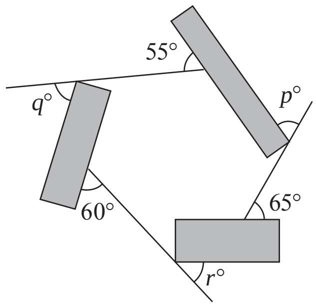


A 6π B 6.5π C 7π D 7.5π E 8π

14. Which of the following values of the positive integer n is a counterexample to the statement: "If n is not prime then $n - 2$ is not prime" ?

A 6 B 11 C 27 D 33 E 51

15. The diagram shows three rectangles and three straight lines.
What is the value of $p + q + r$?



A 135 B 180 C 210 D 225 E 270

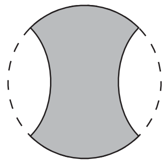
16. For which value of k is $\sqrt{2016} + \sqrt{56}$ equal to 14^k ?

A $\frac{1}{2}$ B $\frac{3}{4}$ C $\frac{5}{4}$ D $\frac{3}{2}$ E $\frac{5}{2}$

17. Aaron has to choose a three-digit code for his bike lock. The digits can be chosen from 1 to 9. To help him remember them, Aaron chooses three different digits in increasing order, for example 278. How many such codes can be chosen?

A 779 B 504 C 168 D 84 E 9

18. The circumference of a circle with radius 1 is divided into four equal arcs. Two of the arcs are ‘turned over’ as shown.
What is the area of the shaded region?

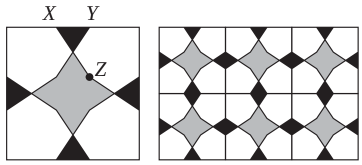


A 1 B $\sqrt{2}$ C $\frac{1}{2}\pi$ D $\sqrt{3}$ E 2

19. Let S be a set of five different positive integers, the largest of which is m. It is impossible to construct a quadrilateral with non-zero area, whose side-lengths are all distinct elements of S. What is the smallest possible value of m?

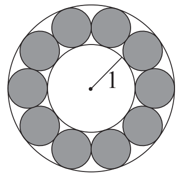
A 2 B 4 C 9 D 11 E 12

20. Michael was walking in Marrakesh when he saw a tiling formed by tessellating the square tile as shown.
The tile has four lines of symmetry and the length of each side is 8 cm. The length of XY is 2 cm. The point Z is such that XZ is a straight line and YZ is parallel to sides of the square.
What is the area of the central grey octagon?



A 6 cm^2 B 7 cm^2 C 8 cm^2 D 9 cm^2 E 10 cm^2

21. The diagram shows ten equal discs that lie between two concentric circles − an inner circle and an outer circle. Each disc touches two neighbouring discs and both circles. The inner circle has radius 1.
What is the radius of the *outer* circle?



A $2 \tan 36°$ B $\dfrac{\sin 36°}{1 - \sin 36°}$ C $\dfrac{1 + \sin 18°}{1 - \sin 18°}$ D $\dfrac{2}{\cos 18°}$ E $\dfrac{9}{5}$

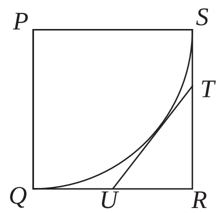
22. Three friends make the following statements.
Ben says, “Exactly one of Dan and Cam is telling the truth.”
Dan says, “Exactly one of Ben and Cam is telling the truth.”
Cam says, “Neither Ben nor Dan is telling the truth.”
Which of the three friends is lying?

A Just Ben B Just Dan C Just Cam D Each of Ben and Cam
E Each of Ben, Cam and Dan

23. A cuboid has sides of lengths 22, 2 and 10. It is contained within a sphere of the smallest possible radius. What is the side-length of the largest cube that will fit inside the same sphere?

A 10 B 11 C 12 D 13 E 14

24. The diagram shows a square $PQRS$. The arc QS is a quarter circle. The point U is the midpoint of QR and the point T lies on SR. The line TU is a tangent to the arc QS. What is the ratio of the length of TR to the length of UR?



A 3 : 2 B 4 : 3 C 5 : 4 D 7 : 6 E 9 : 8

25. Let n be the smallest integer for which $7n$ has 2016 digits.
What is the units digit of n?

A 0 B 1 C 4 D 6 E 8

UK SENIOR MATHEMATICAL CHALLENGE

Tuesday 8 November 2016

Organised by the United Kingdom Mathematics Trust
from the School of Mathematics, University of Leeds

SOLUTIONS LEAFLET

This solutions leaflet for the SMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

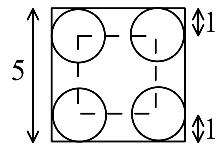
For reasons of space, these solutions are necessarily brief. There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:

http://www.ukmt.org.uk/

1. **B** Since the answer to 987654321×9 is 8 888 888 889, the digit 9 appears once.

2. **D** James bought a book for £5.50, so the normal price would be £5. On Friday, another copy of this book would therefore cost £4.50.

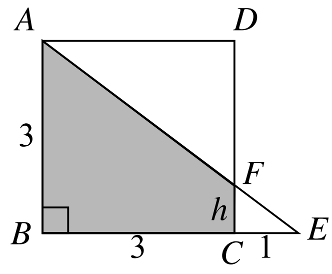
3. **B** As the circle rolls, the centre of the circle moves along four straight lines shown as dashed lines. Each dashed line has length $5 - (1 + 1)$ so the total distance travelled is 4×3 which is 12.



4. **C** One angle in Alex's triangle is 80°. Let $\alpha°$ be the smaller of the other two angles so $(100 - \alpha)°$ is the third angle. The difference between these angles is then $(100 - 2\alpha)°$. Considering each option:
A: $100 - 2\alpha = 0$ gives both α and $100 - \alpha$ to be 50. This triangle is therefore isosceles and not scalene.
B: $100 - 2\alpha = 60$ gives α to be 20 and $100 - \alpha$ to be 80. This is again isosceles.
Option D gives angles of 80, 0 and 100. Option E gives angles of 80, −10 and 110. Neither of these cases forms a triangle.
C: $100 - 2\alpha = 80$ gives α to be 10 and $100 - \alpha$ to be 90. All three angles are different so this is the correct option.

5. **A** From the available digits the squares could be 256, 324 or 625. Since the middle digits must be the same, the centre digit must be 2.

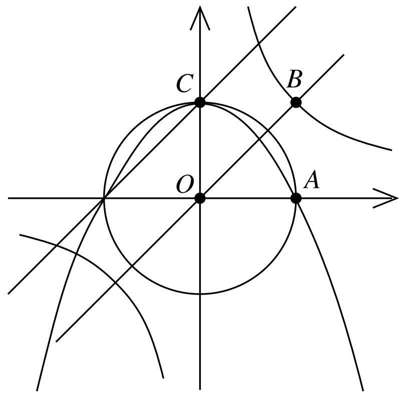
6. **D** Let F be the point of intersection of the lines AE and CD. Let the length of CF be h. Then, using similar triangles, $\frac{CF}{CE} = \frac{BA}{BE}$, so $\frac{h}{1} = \frac{3}{4}$ giving $h = \frac{3}{4}$. The shaded region $ABCF$ is a trapezium, so has area $\frac{1}{2}\left(3 + \frac{3}{4}\right) \times 3 = \frac{45}{8}$ which is $5\frac{5}{8}$.



7. **A** The number 2016^0 has value 1. As $2016 > 1$, $2016^{1/2} < 2016^1$. The values of their reciprocals, $2016^{-1/2}$ and 2016^{-1} are then in the opposite order. So the five options given are in numerical order, with 2016^{-1}, or $\frac{1}{2016}$, being the smallest.

8. **B** One way to count the number of small squares formed is to divide the large square into four quarters along its two diagonals. The number of small squares formed is $4 \times T_{n-1}$, where T_{n-1} is the $(n - 1)$th triangular number. When $n = 7$, this is $4 \times \frac{1}{2}(6 \times 7)$ which is 4×21. So 84 squares are formed.

9. **C** Let $O = (0,0)$, $A = (1,0)$, $B = (1,1)$, $C = (0,1)$ be the vertices of the square. The equation $x^2 + y^2 = 1$ gives a circle passing through A and C. The equation $y = x + 1$ gives a straight line passing only through C. The equation $y = -x^2 + 1$ gives a parabola passing through A and C. The equation $y = x$ gives a straight line passing through O and B. The equation $y = \frac{1}{x}$ gives a rectangular hyperbola which has two branches and passes only through B. So, only $x^2 + y^2 = 1$, $y = x$ and $y = -x^2 + 1$ have graphs passing through exactly two of the vertices of the square.



10. **B** None of the products for the first two rows and first two columns contains a factor of 5, so the bottom right cell must contain the 5. The prime factorisation of 112 is $2^4 \times 7$ and, as 7 is not a factor of 216 or 12, then 7 must be in the right cell of the middle row. The remaining 2^4 must be the product of two different numbers, namely 8 and 2. The 2 must be in the centre cell as 8 is not a factor of 12. The grid is now as shown above. The prime factorisation of 216 is $2^3 \times 3^3$ and the 3^3 must be the product of a 3 and a 9. The 3 must be in the top left cell as the product of the top row is 12 which is not a multiple of 9. Thus, the product of the three shaded cells is $3 \times 2 \times 5$ which is 30. The completed grid is as shown on the right.

8	2	7
		5

3	1	4	12
8	2	7	112
9	6	5	
216	12		

11. **E** For each square to be filled with the mean of the numbers in the adjacent squares, the differences between all five pairs of adjacent numbers must be equal. This common difference is $\left(\frac{25 - 10}{5}\right)$ which is 3. The grid is then

10	13	16	**19**	22	25

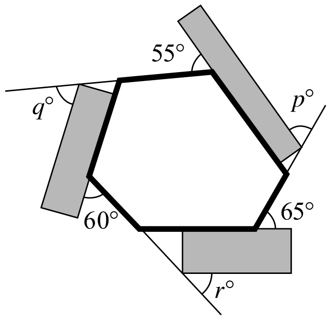
and the 19 is in the desired square.

12. **C** The prime factorisation of 2016 is $2^5 \times 3^2 \times 7$. To create the smallest square which is a multiple of 2016, the powers of each prime must be as small as possible and even, whilst also being at least as big as those in the prime factorisation of 2016. This gives $2^6 \times 3^2 \times 7^2$ which is $(2^3 \times 3 \times 7)^2$ or 168^2.

13. **A** A quarter circle of radius r has length $\frac{2\pi r}{4}$ which is $\frac{\pi r}{2}$. The total length of the curve shown is then $\frac{\pi}{2}(1 + 1 + 2 + 3 + 5)$ which is 6π.

14. D The five options give the values of n to be considered. In option B, 11 is prime so that can be discounted. The options A, C and E are 6, 27 and 51 which are not prime and subtracting 2 from each of these gives 4, 25 and 49 which are also not prime. However in D, $n = 33$ which is not prime but $n - 2 = 31$ is prime.

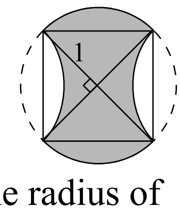
15. B A non-regular hexagon can be drawn on the diagram as shown. Three of the exterior angles of the hexagon are then 55°, 60° and 65°. Since corresponding angles on parallel lines are equal, the other three exterior angles are $p°$, $q°$ and $r°$. The total of the exterior angles of any polygon is 360°. Hence $p + q + r + 55 + 60 + 65 = 360$ and so $p + q + r = 180$.



16. D The expression $\sqrt{2016} + \sqrt{56}$ can be written as $\sqrt{2^5 \times 3^2 \times 7} + \sqrt{2^3 \times 7}$ which is $\sqrt{4^2 \times 3^2 \times 2 \times 7} + \sqrt{2^2 \times 2 \times 7}$. This simplifies to $12\sqrt{14} + 2\sqrt{14}$ which is $14\sqrt{14}$ and, using index notation, this can be written as $14^{3/2}$. Hence $k = \frac{3}{2}$.

17. D One way to count the possible codes is in descending numerical order of the three-digit codes. The list begins: 789; 689, 679, 678; 589, 579, 578, 569, 568, 567; ... . Each initial digit n produces part of the list with the $(8 - n)$th triangular number of possible codes, where $n \leqslant 7$. The total number of possible codes is then the sum of these triangular numbers $1 + 3 + 6 + 10 + 15 + 21 + 28$ including 1 code starting with the digit 7, all the way to 28 codes starting with the digit 1. The total number of codes that Aaron can choose is 84.

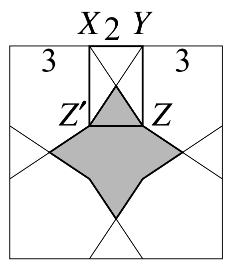
18. E The four arcs are of equal length and their end-points lie on a circle, so the four end-points can be joined to make a square. As two of the arcs are 'turned over', the two unshaded regions inside the square have areas equal to the two shaded regions outside the square.



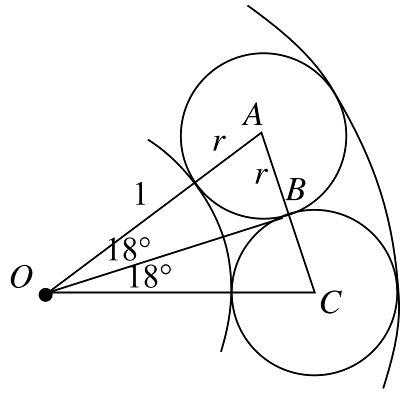
The total shaded area is therefore equal to the area of the square. The radius of the circle is given as 1 so, by Pythagoras' Theorem, the side-length of the square is $\sqrt{1^2 + 1^2} = \sqrt{2}$. So the area of the shaded region is $\sqrt{2} \times \sqrt{2} = 2$.

19. D Let S consist of h, j, k, l, m in ascending order of size. We want m to be as small as possible. Given three side-lengths, there is a quadrilateral with non-zero area with a specified fourth side-length if and only if the fourth side-length is less than the sum of the other three side-lengths. To ensure that j, k, l, m are not the side-lengths of such a quadrilateral, we must have $m \geqslant j + k + l$. Likewise, considering h, j, k, l, we must have $l \geqslant h + j + k$. Since the smallest possible values of h, j and k are 1, 2 and 3 respectively then $l \geqslant 1 + 2 + 3$ so 6 is the smallest value of l. Also $m \geqslant 2 + 3 + 6$ so 11 is the smallest value of m.

20. E Let the point Z' be directly below X, so that $XYZZ'$ is a rectangle. As the length of XY is 2 cm, the distance from Y to the nearest corner of the square is 3 cm. The area of $XYZZ'$ is 2 cm × 3 cm which is 6 cm^2. The diagonals XZ and YZ' split $XYZZ'$ into quarters and each has area $1\frac{1}{2}$ cm^2. The central grey octagon is formed from a square with side $Z'Z$ of length 2 cm together with four triangles, each of area $1\frac{1}{2}$ cm^2. The total area of the shaded octagon is $2 \times 2 + 4 \times 1\frac{1}{2}$ which is 10 cm^2.

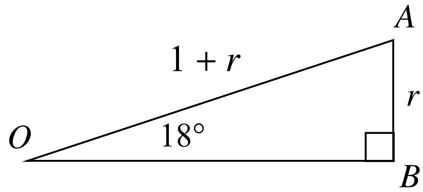


21. C As there are 10 discs, the adjacent lines drawn from the centre of the inner circle to the centre of each disc are separated by an angle of 36°. The line *OB* is a tangent to both the disc with centre *A* and the disc with centre *C*. So the points *A*, *B* and *C* lie on a straight line as angles *OBA* and *OBC* are both 90°.



In the second diagram, from triangle *OAB* we have $\sin 18° = \dfrac{r}{1 + r}$ which rearranges to

$$\frac{\sin 18°}{1 - \sin 18°} = r.$$



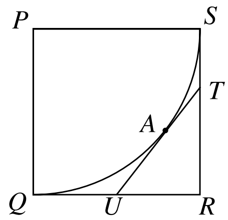
The radius of the outer circle is

$$1 + 2r = 1 + \frac{2\sin 18°}{1 - \sin 18°} = \frac{1 + \sin 18°}{1 - \sin 18°}.$$

22. C Suppose first that Cam is telling the truth, so Ben and Dan are both lying. Then Ben's statement is actually correct, as is Dan's. There is a clear contradiction. So we know that Cam is in fact lying. Therefore at least one of Ben and Dan is telling the truth. If Ben is telling the truth, then we learn that Dan is telling the truth. If Dan is telling the truth then we learn that Ben is telling the truth. So both Ben and Dan are telling the truth. This means that only Cam is lying.

23. E For the cuboid to be contained within a sphere of smallest possible radius, all eight vertices of the cuboid must lie on the sphere. The radius r of the smallest sphere is then half of the length of the body diagonal of the cuboid, so $r = \sqrt{1^2 + 5^2 + 11^2} = \sqrt{147}$. If the largest cube which will fit inside this sphere has side-length $2x$, then $r = \sqrt{x^2 + x^2 + x^2}$. Thus $3x^2 = 147$, so $x^2 = 49$ and so $x = 7$. The side-length of the largest cube is 14.

24. B Let the square have side-length 2, $RT = h$ and let A be the point of contact between TU and the circle. Two tangents to a circle which meet at a point are of equal length. So as $QU = 1$ so does AU. Similarly $TA = TS = 2 - h$. Applying Pythagoras' Theorem to triangle URT gives $1^2 + h^2 = (1 + 2 - h)^2$ so $1 + h^2 = 9 - 6h + h^2$ and therefore $8 - 6h = 0$ which gives $h = \frac{4}{3}$. The required ratio is then 4 : 3.



25. D For n to be the smallest integer for which $7n$ has 2016 digits, $7n$ must start with 1, be followed by 2014 zeros and end with a digit a. When this number is divided by 7, the answer is formed from the repeating sequence of 6 digits 142857. The remainders also form a repeating sequence 3, 2, 6, 4, 5, 1. These sequences are repeated 335 times as 6 × 335 is 2010. The last 4 zeros (to make 2014 zeros in total) and the final a create the last section of the division as shown:

$$\frac{\ldots\ldots\ldots 1\;4\;2\;8}{\ldots\ {}^{1}0\,{}^{3}0\,{}^{2}0\,{}^{6}0\,{}^{4}a}.$$

Finally, $40 + a$ must be divisible by 7 and be as small as possible. So $a = 2$ and as $42 \div 7 = 6$ the units digit of n is 6.

Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries

Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some problems for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with all steps explained, and not based on the assumption that one of the given alternatives is correct. In some cases we have added a commentary to indicate the sort of thinking that led to our solution. You should not include commentary of this kind in your written solutions, but we hope that these solutions, without the commentary, provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

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Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	D	B	C	A	D	A	B	C	B	E	C	A	D	B	D	D	E	D	E	C	C	E	B	D

1. How many times does the digit 9 appear in the answer to $987\,654\,321 \times 9$?

A 0 B 1 C 5 D 8 E 9

Solution **B**

There seems no better method here than just doing the multiplication.

$$9 \times 987\,654\,321 = 8\,888\,888\,889.$$

So the number of times that the digit 9 occurs in the answer is 1.

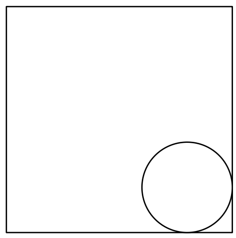
2. On a Monday, all prices in Isla's shop are 10 % more than normal. On Friday all prices in Isla's shop are 10 % less than normal. James bought a book on Monday for £5.50. What would be the price of another copy of this book on Friday?

A £5.50 B £5.00 C £4.95 D £4.50 E £4.40

Solution **D**

The book cost £5.50 on a Monday after prices have risen by 10 %. Therefore the price before this increase was £5. It follows that on a Friday, after prices have been cut by 10 %, the price of the book has fallen by 50p to £4.50.

3. The diagram shows a circle with radius 1 that rolls without slipping around the inside of a square with sides of length 5.



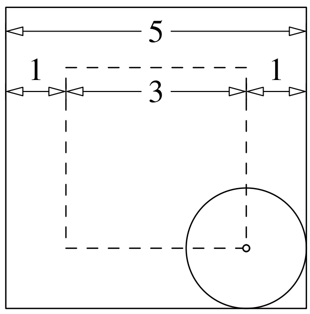
The circle rolls once around the square, returning to its starting point.

What distance does the centre of the circle travel?

A $16 - 2\pi$ B 12 C $6 + \pi$ D $20 - 2\pi$ E 20

Solution **B**

As the circle rolls, its centre is always at a distance 1 from the square. Therefore, as shown in the diagram, the centre traces out a square whose side length is 2 less than the side length of the square. It follows that the centre travels a distance equal to the length of the perimeter of a square with side length 3.



We deduce that the distance that the centre travels is 12.

For investigation

3.1 How far would the centre of the circle travel if it rolls once without slipping around the inside of an equilateral triangle with sides of length 5? What if it rolls *outside* the triangle?

4. Alex draws a scalene triangle. One of the angles is $80°$.

Which of the following could be the difference between the other two angles in Alex's triangle?

A $0°$ B $60°$ C $80°$ D $100°$ E $120°$

Solution **C**

All the side lengths of a scalene triangle are different, therefore all the angles are different.

It follows that the difference between two of the angles cannot be $0°$. This rules out option A.

The sum of the angles in a triangle is $180°$. Therefore, as one of the angles in Alex's triangle is $80°$, the sum of the other two angles is $100°$. It follows that the difference between these angles is less than $100°$. This rules out options D and E.

If the two angles with sum $100°$ have a difference of $60°$, these angles would be $80°$ and $20°$. So the triangle would have two angles of $80°$, which is impossible for a scalene triangle. This rules out option B.

Therefore the correct option is C, since it is the only one we have not eliminated.

Note

In the context of the SMC it is adequate to stop here. Once four of the options have been eliminated, the remaining option must be correct.

However, for a full solution you would need to show that it is possible for the difference between the other two angles to be $80°$. You are asked to do this in Exercise 4.1, below.

For investigation

4.1 What are the angles of a scalene triangle in which one angle is $80°$ and the difference between the other two angles is $80°$?

4.2 Show that, as stated in the solution, if two angles have sum $100°$ and difference $60°$, then the two angles are $80°$ and $20°$.

4.3 The argument used in the solution above is based on the fact that

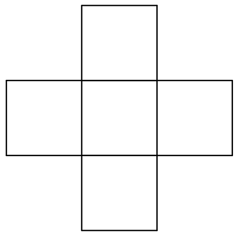
> In a scalene triangle (that is, a triangle where all the side lengths are different) all the angles are different.

Prove that this is correct.

5. All the digits 2, 3, 4, 5 and 6 are placed in the grid, one in each cell, to form two three-digit numbers that are squares.

Which digit is placed in the centre of the grid?

A 2 B 3 C 4 D 5 E 6

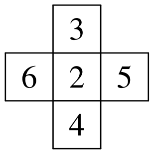


Solution **A**

The three-digit squares use just the digits 2, 3, 4, 5 and 6. The smallest square which uses these digits is $15^2 = 225$ and the largest is $25^2 = 625$. However, the squares that go in the grid cannot repeat digits. Therefore the only squares that we need consider are the three-digit squares that use three of digits 2, 3, 4, 5 and 6, once each. It may be checked (see Exercise 5.1, below) that there are just three squares which meet these conditions. They are $16^2 = 256$, $18^2 = 324$ and $25^2 = 625$.

The digit 2 occurs twice as the tens digit of these three squares. Therefore, the digit that goes in the central square of the grid is 2.

In the context of the SMC we could stop here. However, for a complete solution you would need to add that, as the squares 324 and 625 use each of the digits 3, 4, 5 and 6 just once, it is possible to place the digits in the grid so as to make these squares. The diagram shows one way to do this.



For investigation

5.1 Check that 16^2, 18^2 and 25^2 are the only three-digit squares using just digits from the list 2, 3, 4, 5, 6, without any repeats.

5.2 Arrange the digits 2, 4, 7, 8, 9 in the grid given in the question so as to form two three-digit squares.

5.3 Arrange the digits 1, 2, 3, 4, 5 in the grid given in the question so as to form one three-digit square and one three-digit cube.

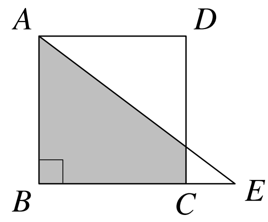
5.4 Which three digits may be used to make three different three-digit squares?

5.5 List all the three-digit squares. Which digits occurs least often and most often in this list?

6. The diagram shows a square $ABCD$ and a right-angled triangle ABE. The length of BC is 3. The length of BE is 4.

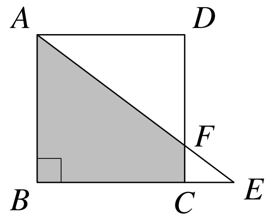
What is the area of the shaded region?

A $5\frac{1}{4}$ B $5\frac{3}{8}$ C $5\frac{1}{2}$ D $5\frac{5}{8}$ E $5\frac{3}{4}$



SOLUTION **D**

We let F be the point where the lines AE and CD meet.



Because $ABCD$ is a square, $\angle BCF = 90°$. It follows, because angles on a line have sum 180°, that $\angle FCE = 90°$.

Hence, in the triangles ABE and FCE we have $\angle ABE = \angle FCE = 90°$. Also, we see that $\angle BEA = \angle CEF$. It follows that these triangles are similar.

Therefore

$$\frac{FC}{AB} = \frac{CE}{BE}. \qquad (1)$$

Because $ABCD$ is a square $AB = BC = 3$. We also have $BE = 4$. Therefore $CE = BE - BC = 4 - 3 = 1$.

It follows from equation (1) that

$$\frac{FC}{3} = \frac{1}{4}.$$

Hence $FC = \frac{3}{4}$.

The shaded region $ABCF$ is a trapezium. The area of a trapezium is given by the formula $\frac{1}{2}h(a+b)$, where a and b are the lengths of the parallel sides, and h is the distance between these sides. It follows that the area of the trapezium $ABCF$ is given by

$$\tfrac{1}{2}BC(AB + FC) = \tfrac{1}{2}(3 \times (3 + \tfrac{3}{4})) = \tfrac{1}{2}(3 \times \tfrac{15}{4}) = \tfrac{45}{8} = 5\tfrac{5}{8}.$$

FOR INVESTIGATION

6.1 The solution above uses the formula $\frac{1}{2}h(a+b)$ for the area of a trapezium. Show that this formula is correct.

6.2 Suppose that BE has length 5. What is the area of the trapezium $ABCF$ in this case?

6.3 Suppose that BE has length x, where $x > 3$. Find a formula for the area of the trapezium $ABCF$ in terms of x. What happens to this area as x gets larger and larger?

7. Which of these has the smallest value?

A 2016^{-1} B $2016^{-1/2}$ C 2016^{0} D $2016^{1/2}$ E 2016^{1}

SOLUTION **A**

METHOD 1

We have $2016^{-1} = \dfrac{1}{2016}$ and $2016^{-1/2} = \dfrac{1}{\sqrt{2016}}$. Now, as $1 < \sqrt{2016} < 2016$, it follows that $\dfrac{1}{2016} < \dfrac{1}{\sqrt{2016}} < 1$, and hence $2016^{-1} < 2016^{-1/2} < 1$.

On the other hand, 2016^0 is equal to 1, and both $2016^{1/2} = \sqrt{2016}$ and $2016^1 = 2016$ are greater than 1.

It follows that, of the given options, 2016^{-1} has the smallest value.

METHOD 2

The function 2016^x may be defined by the equation

$$2016^x = e^{x \ln 2016},$$

where e^x is the exponential function [see the note below] and $\ln 2016$ is the natural logarithm of 2016. (The value of $\ln 2016$ is 7.6089 to 4 decimal places.)

The exponential function is an *increasing function* in the sense that if $x < x'$, then $e^x < e^{x'}$. It follows that 2016^x is also an increasing function, because if $x < x'$, then $x \ln 2016 < x' \ln 2016$ and hence $e^{x \ln 2016} < e^{x' \ln 2016}$.

Therefore, as $-1 < -\frac{1}{2} < 0 < \frac{1}{2} < 1$, we have $2016^{-1} < 2016^{-1/2} < 2016^0 < 2016^{1/2} < 2016^1$.

Hence, of the given options, 2016^{-1} is the smallest.

NOTE

The definition of e^x for a general real number x is quite complicated.

One way to define it is by means of an infinite series, as follows.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots .$$

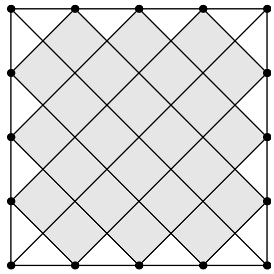
If we adopt the conventions that $x^0 = 1$ and $0! = 1$, this may be rewritten as

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots ,$$

or, using the *sigma notation*, as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

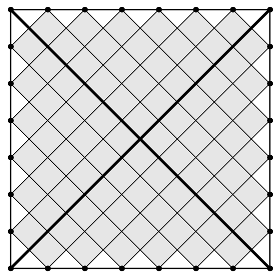
8. Points are drawn on the sides of a square, dividing each side into n equal parts (so, in the example shown, $n = 4$). The points are joined in the manner indicated, to form several small squares (24 in the example, shown shaded) and some triangles.



How many small squares are formed when $n = 7$?

A 56 B 84 C 140 D 840 E 5040

Solution **B**



The main diagonals of the square shown in the question divide the square into four quarters each of which contains $1 + 2 + 3$ small squares. So the number of small squares when $n = 4$ is $4 \times (1+2+3) = 4 \times 6 = 24$, as the question says.

It can be seen from the diagram on the right that, similarly, when $n = 7$ the number of small squares is given by $4 \times (1 + 2 + 3 + 4 + 5 + 6) = 4 \times 21 = 84$.

For investigation

8.1 What fraction of the area of the big square is covered by the small squares in the case when $n = 4$?

8.2 What fraction of the area of the big square is covered by the small squares in the case when $n = 7$?

8.3 Find a formula, in terms of n, for the number of small squares.

8.4 Find a formula, in terms of n, for the fraction of the area of the square that is covered by the small squares. What happens to this fraction as n gets larger and larger?

9. A square has vertices at (0,0), (1,0), (1,1) and (0,1). Graphs of the following equations are drawn on the same set of axes as the square.

$$x^2 + y^2 = 1, \qquad y = x + 1, \qquad y = -x^2 + 1, \qquad y = x, \qquad y = \frac{1}{x}$$

How many of the graphs pass through exactly two of the vertices of the square?

A 1 B 2 C 3 D 4 E 5

Solution **C**

Method 1

One method is to draw the graphs.

The graph of the equation $x^2 + y^2 = 1$ is the circle with centre $(0, 0)$ and radius 1. [By Pythagoras' Theorem, the distance, d of the point with coordinates(x, y) from the point $(0, 0)$ is given by the equation $x^2 + y^2 = d^2$. Therefore the equation $x^2 + y^2 = 1$ is satisfied by those points whose distance from the origin, $(0, 0)$ is 1. These are the points on the circle with centre $(0, 0)$ and radius 1, as shown in the diagram below.]

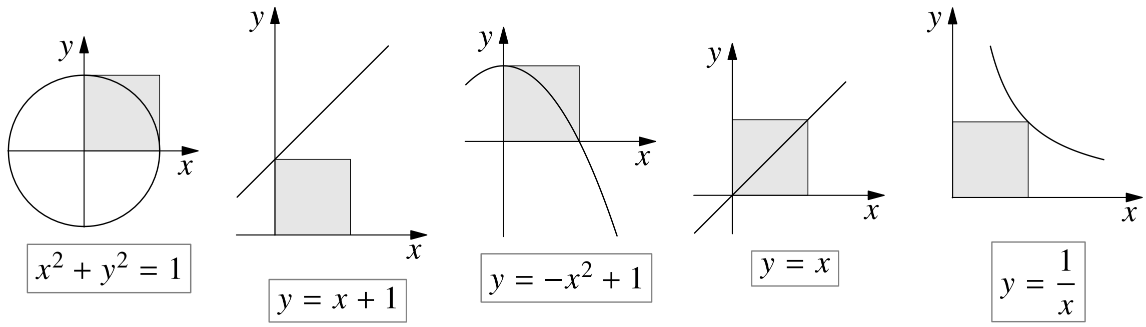
The graph of $y = x + 1$ is the straight line with slope 1 that meets the y-axis at the point $(0, 1)$.

The graph of $y = -x^2 + 1$ is a ∩-shaped parabola which is symmetrical about the y-axis and goes through the point $(0, 1)$.

The graph of $y = x$ is the straight line with gradient 1 which goes through the point $(0, 0)$.

The graph of $y = \frac{1}{x}$ is a rectangular hyperbola with the x-axis and y-axis as its asymptotes. In the diagram below we have just shown part of one arm of this hyperbola.

These graphs are shown in the diagram below.



From these graphs we see that the graphs of $x^2 + y^2 = 1$, $y = -x^2 + 1$ and $y = x$ pass through two vertices of the square, whereas each of the other graphs passes through just one vertex of the square.

Hence the number of graphs that pass through exactly two vertices of the square is 3.

Note

The solution above assumes that we have drawn the graphs correctly. For a more rigorous method, see the solution below.

Method 2

We check whether a given graph passes through a particular vertex of the square, by seeing whether the coordinates of the vertex satisfy the equation of the graph.

For example, when $x = 0$ and $y = 0$ we have $x^2 + y^2 \neq 1$. Therefore the graph of $x^2 + y^2 = 1$ does not go through the vertex $(0, 0)$.

On the other hand, when $x = 0$ and $y = 1$, we have $x^2 + y^2 = 1$. Therefore the graph of $x^2 + y^2 = 1$ does go through the vertex $(0, 1)$.

In the table below we have shown the outcome of all these calculations. We have put 'yes' if the graph goes through the vertex in question, and 'no' if it does not.

vertices	$x^2 + y^2 = 1$	$y = x + 1$	$y = -x^2 + 1$	$y = x$	$y = \frac{1}{x}$
(0,0)	no	no	no	yes	no
(1,0)	yes	no	yes	no	no
(1,1)	no	no	no	yes	yes
(0,1)	yes	yes	yes	no	no

We see that in the case of three of the graphs considered, there are exactly two occurrences of 'yes' in the relevant column. We deduce that the number of graphs that pass through exactly two vertices of the square is 3.

For investigation

9.1 Check that all the entries in the table above are correct.

9.2 Find a graph that goes through all four of the vertices of the square. You should describe the graph geometrically and also find its equation.

9.3 (Harder) Find a graph that goes through exactly three of the vertices of the square. You should describe the graph geometrically and also find its equation.

10. The digits from 1 to 9 are to be written in the nine cells of the 3×3 grid shown, one digit in each cell.

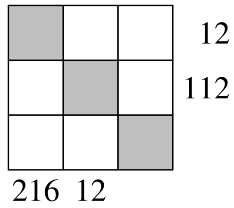
The product of the three digits in the first row is 12.

The product of the three digits in the second row is 112.

The product of the three digits in the first column is 216.

The product of the three digits in the second column is 12.

What is the product of the digits in the shaded cells?

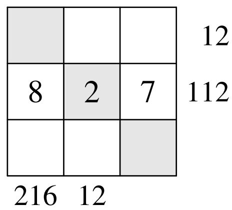


A 24 B 30 C 36 D 48 E 140

Solution **B**

We note first that 112 is divisible by 7, but neither 216 nor 12 is divisible by 7. Therefore, the digit 7 is in the second row but not in the first or second columns. Hence the digit 7 is written in the cell in the second row and third column.

Because $112 = 7 \times 16$, the product of the other two digits in the second row is 16. Since the digits have to be different, these digits are 2 and 8 in some order. Because 8 is not a factor of 12, the digit 8 cannot be in the second column. Therefore 8 is written in the cell in the second row and first column, and 2 is written in the cell in the second row and second column.

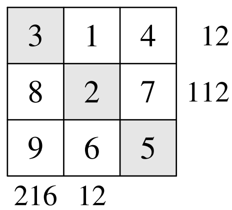


Therefore we know that the second row is as shown in the diagram.

Because $216 = 8 \times 27$, the product of the other two digits in the first column is 27. Hence these digits are 3 and 9. Because 9 is not a factor of 12, the digit 9 cannot be in the first row. Hence 9 is in the first column and third row, and 3 is in the first column and first row.

The product of the three digits in the second column is 12. One of these digits is 2. Hence, the product of the other two digits in the second column is 6. As 2 and 3 are already in the grid, these digits are 1 and 6.

The digit 6 cannot be in the first row, as otherwise the product of the digits in the first row would not be 12. It follows that the digit 1 is in the first row and second column, and 6 is in the third row and second column. The digit in the first row and third column is therefore 4. The remaining digit 5 is in the third row and third column.



Therefore the digits are written as shown in the diagram. The digits in the shaded squares are 3, 2 and 5. The product of these digits is 30.

For investigation

10.1 In the above argument, we showed that the digit in the third row and third column is 5 because it was the only digit we had not yet placed. However, it is possible to see that the digit 5 is in this cell before working out where any of the other digits go. Explain how.

10.2 Invent more puzzles of a similar kind to that in this question.

11. In the grid below each of the blank squares and the square marked X are to be filled by the mean of the two numbers in its adjacent squares.

Which number should go in the square marked X?

10			X		25

A 15 B 16 C 17 D 18 E 19

Solution **E**

We note first that if b is the mean of the numbers a and c, then $b = \frac{1}{2}(a + c)$. Hence $2b = a + c$ and therefore $b - a = c - b$. In other words, the numbers a, b, c form an arithmetic sequence. That is, their differences are equal.

We deduce that, because each number in the grid, other than 10 and 25, is the mean of the two adjacent numbers, the six numbers in the grid form an arithmetic sequence. Suppose that their common difference is d. Then the numbers in the grid will form the sequence $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.

It follows that $10 + 5d = 25$. Hence $d = 3$.

The number which is in the square marked X is $10 + 3d$. Because $d = 3$, this number is 19.

12. Which is the smallest square that has 2016 as a factor?

A 42^2 B 84^2 C 168^2 D 336^2 E 2016^2

Solution **C**

An integer is a square if, and only if, the exponent of each prime in its prime factorization is even.

The prime factorization of 2016 is $2^5 \times 3^2 \times 7$.

If 2016 is a factor of a square, then the exponents of the primes 2, 3 and 7 in the factorization of the square must be no smaller than their exponents in 2016.

Hence the smallest square of which 2016 is a factor has the prime factorization $2^a \times 3^b \times 7^c$, where a, b are c are the smallest even integers which are greater than or equal to 5, 2 and 1, respectively. Therefore $a = 6$, $b = 2$ and $c = 2$. Hence, the smallest square is of which 2016 is a factor is $2^6 \times 3^2 \times 7^2$.

Now

$$2^6 \times 3^2 \times 7^2 = (2^3 \times 3^1 \times 7^1)^2 = (8 \times 3 \times 7)^2 = 168^2.$$

Therefore the smallest square of which 2016 is a factor is 168^2.

For investigation

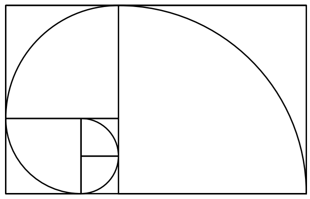
12.1 Check that the prime factorization of 2016 is $2^5 \times 3^2 \times 7$, as given above.

12.2 Explain why *an integer is a square if, and only if, the exponent of each prime in its prime factorization is even.*

12.3 Which is the smallest cube that has 2016 as a factor?

13. Five square tiles are put together side by side. A quarter circle is drawn on each tile to make a continuous curve as shown. Each of the smallest squares has side-length 1.

What is the total length of the curve?



A 6π B 6.5π C 7π D 7.5π E 8π

SOLUTION **A**

The side lengths of the 5 squares are 1, 1, 2, 3 and 5. So the curve is made up of five quarter circles with these radii. The circumference of a circle with radius r is $2\pi r$. Therefore the length of a quarter circle of radius r is $\frac{1}{4}(2\pi r)$, that is, $\frac{1}{2}\pi r$.

Therefore the length of the curve is l, where

$$l = \tfrac{1}{2}\pi 1 + \tfrac{1}{2}\pi 1 + \tfrac{1}{2}\pi 2 + \tfrac{1}{2}\pi 3 + \tfrac{1}{2}\pi 5 = \tfrac{1}{2}\pi(1 + 1 + 2 + 3 + 5) = 6\pi.$$

14. Which of the following values of the positive integer n is a counterexample to the statement: "If n is not prime then $n - 2$ is not prime"?

A 6 B 11 C 27 D 33 E 51

SOLUTION **D**

A *counterexample* to the statement "If n is not prime then $n - 2$ is not prime" is a positive integer n for which this implication is false. An implication of the form "If P then Q" is false if P is true, but Q is false.

Therefore a counterexample to the statement is a positive integer n for which it is true that n is not prime, but false that $n - 2$ is not prime. In other words, a counterexample is a positive integer n such that n is not prime but $n - 2$ is prime.

Since 11 is prime, option B does not provide the required counterexample.

The options 6, 27, 33 and 51 are not primes. However $6 - 2 = 4$, $27 - 2 = 25$ and $51 - 2 = 49$ are also not primes. Therefore none of the options A, C and E provides the required counterexample.

This leaves $n = 33$. We see that 33 is not prime, but $33 - 2 = 31$, is prime. Therefore 33 is a counterexample. Hence the correct option is D.

FOR INVESTIGATION

14.1 Which is the smallest positive integer which provides a counterexample to the statement in the question?

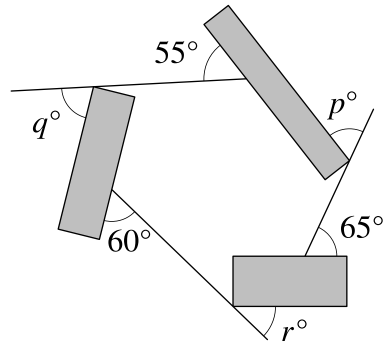
14.2 Find the smallest integer which is greater than 33 and is a counterexample to the statement in the question.

14.3 Are there infinitely many positive integers n that are counterexamples to the statement in the question?

15. The diagram shows three rectangles and three straight lines.

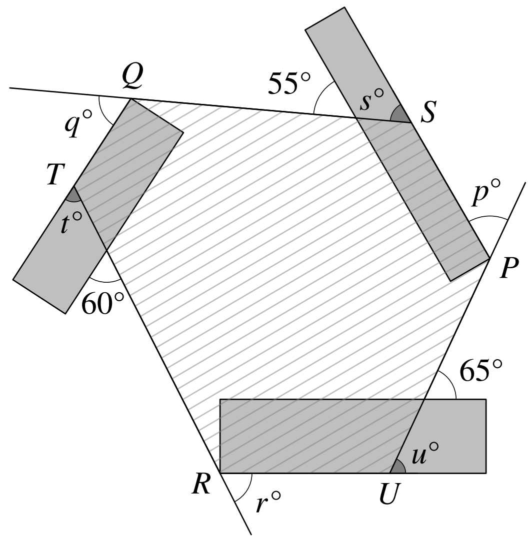
What is the value of $p + q + r$?

A 135 B 180 C 210 D 225 E 270



Solution **B**

Let P, Q and R be the points shown in the diagram where the rectangles touch the straight lines. Let the straight lines when extended meet the rectangles at the points S, T and U, as shown.



Then $PSQTRU$ is a hexagon. The external angles of this hexagon at the vertices P, Q and R are $p°$, $q°$, and $r°$, respectively, as given. Let the external angles at S, T and U be $s°$, $t°$ and $u°$, as shown.

The external angles of a hexagon have sum 360°. Therefore $p + q + r + s + t + u = 360$.

The opposite sides of a rectangle are parallel. When lines are parallel the corresponding angles are equal. Therefore $s = 55$, $t = 60$ and $u = 65$. Hence $s + t + u = 55 + 60 + 65 = 180$.

It follows that $p + q + r = (p + q + r + s + t + u) - (s + t + u) = 360 - 180 = 180$.

For investigation

15.1 Explain why the sum of the external angles of a hexagon, and, indeed, of any polygon, is 360°.

16. For which value of k is $\sqrt{2016} + \sqrt{56}$ equal to 14^k?

A $\frac{1}{2}$ B $\frac{3}{4}$ C $\frac{5}{4}$ D $\frac{3}{2}$ E $\frac{5}{2}$

Solution **D**

We have already seen (in the solution to question 12) that $2016 = 2^5 \times 3^2 \times 7$. It follows that $2016 = (2^4 \times 3^2) \times 2 \times 7 = (2^2 \times 3)^2 \times 2 \times 7 = 12^2 \times 14$. Therefore $\sqrt{2016} = 12\sqrt{14}$.

Also, $\sqrt{56} = \sqrt{2^2 \times 14} = 2\sqrt{14}$. Hence $\sqrt{2016} + \sqrt{56} = 12\sqrt{14} + 2\sqrt{14} = 14\sqrt{14}$.

Now, $14\sqrt{14} = 14^1 \times 14^{1/2} = 14^{1+1/2} = 14^{3/2}$. Therefore $k = \frac{3}{2}$.

17. Aaron has to choose a three-digit code for his bike lock. The digits can be chosen from 1 to 9. To help him remember them, Aaron chooses three different digits in increasing order, for example 278.

How many such codes can be chosen?

A 779 B 504 C 168 D 84 E 9

Solution **D**

Method 1

We count the number of ways in which Aaron can choose three digits in increasing order.

We first consider the case when the first digit Aaron chooses is 1.

If the second digit he chooses is 2, he has 7 remaining choices for the third digit, namely any of the digits from 3 to 9, inclusive. If the second digit Aaron chooses is 3, he has 6 remaining choices for the third digit, namely any of the digits from 4 to 9, inclusive, and so on. Finally, we see that if the second digit Aaron chooses is 8, he has just one choice, 9, for the third digit. He cannot choose 9 as the second digit, as that would leave no choice for the third digit.

Therefore, the number of different codes with first digit 1 that Aaron can choose is $7 + 6 + 5 + 4 + 3 + 2 + 1$, that is, 28.

Similarly, the number of different codes with first digit 2, that Aaron can choose is $6+5+4+3+2+1$, that is, 21.

And so on, until finally, if the first digit Aaron chooses is 7, he has just one choice of code, namely 789.

Therefore the total number of codes that Aaron can choose is $28 + 21 + 15 + 10 + 6 + 3 + 1$, that is, 84.

Method 2

This method assumes some previous knowledge of *binomial coefficients* (see the note below).

Given any three different non-zero digits, there is only one way in which Aaron can use them to make a code, as he wishes to arrange them in increasing order.

Therefore the number of different codes that Aaron can choose is the number of different ways in which Aaron can choose 3 digits from the 9 non-zero digits. This number, '9 choose 3 ', is written $\binom{9}{3}$ or sometimes ${}_9C_3$ or ${}^9C_3$. The number $\binom{9}{3}$ is the coefficient of x^3 in the expansion of $(1 + x)^9$. It is what is called a *binomial coefficient*.

The general formula for the binomial coefficients is given by $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Therefore the number of different codes that Aaron can choose is

$$\binom{9}{3} = \frac{9!}{3!6!} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$$

NOTE

The second method uses knowledge of the formula for the binomial coefficient 'n choose r' which gives the number of ways of choosing r things from a set of n different objects. You may already have met *Pascal's triangle* which is made up of the binomial coefficients.

The branch of mathematics which covers problems of this kind is called *Combinatorics*. The book *Introduction to Combinatorics* by Gerry Leversha and Dominic Rowland, published by the UKMT in 2015, is an excellent introduction to this subject.

If this book is not yet in your school or college library, suggest that it should be acquired. It may be ordered from the UKMT website.

FOR INVESTIGATION

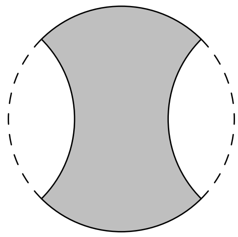
17.1 How many n-digit codes, for $1 \leq n \leq 9$, can Aaron choose when, for each value of n, the code has to consist of n different digits from 1 to 9 which are arranged in increasing order?

17.2 What do you notice about your answers to 17.1?

18. The circumference of a circle with radius 1 is divided into four equal arcs. Two of the arcs are 'turned over' as shown.

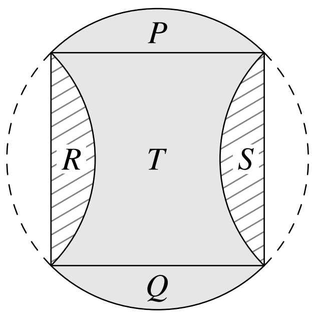
What is the area of the shaded region?

A 1 B $\sqrt{2}$ C $\frac{1}{2}\pi$ D $\sqrt{3}$ E 2



SOLUTION **E**

Because the four arcs are equal, their endpoints form a square. This square is shown in the diagram.



The four segments P, Q, R and S, indicated in the diagram, are congruent. The shaded region is made up of the region T, together with the segments P and Q. It follows that the area of this region is equal to that of the square which is made up of T and the two segments R and S.

The diagonals of the square are diameters of the circle. Therefore they have length 2. It follows that the square has sides of length $\sqrt{2}$. Hence the area of the square is $\sqrt{2} \times \sqrt{2}$, that is, 2.

Therefore the area of the shaded region is 2.

FOR INVESTIGATION

18.1 The above solution uses the fact that because the four arcs are equal, their endpoints form a square. Explain why this follows.

18.2 Explain why it follows from the fact that the diagonals of the square have length 2, that the sides of the square have length $\sqrt{2}$.

19. Let S be a set of five different positive integers, the largest of which is m. It is impossible to construct a quadrilateral with non-zero area, whose side-lengths are all distinct elements of S.

What is the smallest possible value of m?

A 2 B 4 C 9 D 11 E 12

Solution **D**

Let $WXYZ$ be a quadrilateral, with non-zero area. We note first that the shortest route from W to Z is the line segment WZ.

Since $WXYZ$ has non-zero area, the points X and Y are not both on the line segment WZ. Therefore the path made up of the line segments WX, XY and YZ is not the shortest route from W to Z.

It follows that $WX + XY + YZ > WZ$.

Conversely, it may be seen that when the sum of the three smallest of four positive numbers is greater than the largest number of the four, then there is quadrilateral with non-zero area which has these four positive numbers as its side lengths. (You are asked to check this in Exercise 19.1)

Suppose h, j, k, l and m are five positive integers with $h < j < k < l < m$, which have the required property that, whichever four of them are selected, these are not the side lengths of a quadrilateral with non-zero area. Then, whichever four of these numbers are selected, the sum of the three smallest is no greater than the largest.

In Exercise 19.2 you are asked to check that, provided that $h + j + k \leq l$ and $j + k + l \leq m$, the integers h, j, k, l and m satisfy the condition that, whichever four of them are selected, the sum of the three smallest is no greater than the largest.

We wish to choose m to be as small as possible. To achieve this we need h, j, k and l to be as small as possible. The smallest possible values for h, j and k, are $h = 1$, $j = 2$ and $k = 3$. This gives $h + j + k = 1 + 2 + 3 = 6$. Because we need to have $h + j + k \leq l$, the smallest possible value of l is 6. Then $j + k + l = 2 + 3 + 6 = 11$. Because we need to have $j + k + l \leq m$, the smallest possible value of m is 11.

For investigation

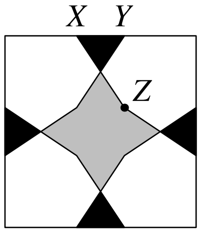
19.1 Show that if p, q, r and s are positive numbers such that $p < q < r < s$ and $p + q + r > s$, then there is a quadrilateral with non-zero area which has these side lengths. [*Hint*: One method is to show that there is a trapezium whose parallel sides have lengths r and s, and whose other two sides have lengths p and q.]

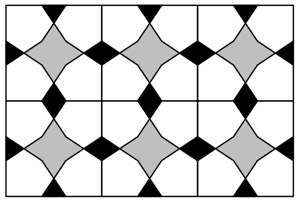
19.2 Suppose that $h < j < k < l < m$ and $h + j + k \leq l$ and $j + k + l \leq m$. Check that *whichever* four of the numbers h, j, k, l, m are selected, the sum of the three smallest is no greater than the largest.

19.3 What is the value of m in the similar problem where S is a set of six different positive integers?

20. Michael was walking in Marrakesh when he saw a tiling formed by tessellating the square tile as shown.

The tile has four lines of symmetry and the length of each side is 8 cm. The length of XY is 2 cm. The point Z is such that XZ is a straight line and YZ is parallel to the sides of the square.



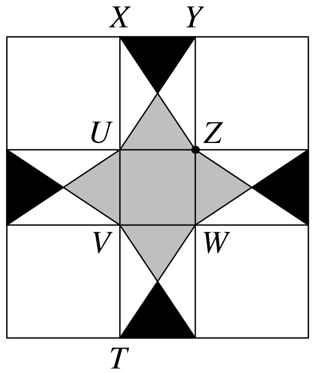


What is the area of the central grey octagon?

A $6\,\text{cm}^2$ B $7\,\text{cm}^2$ C $8\,\text{cm}^2$ D $9\,\text{cm}^2$ E $10\,\text{cm}^2$

SOLUTION **E**

Let T, U, V and W be the points shown in the diagram.



Because the tile has four lines of symmetry and YZ is parallel to the sides of the square, so also is XU, and $XU = YZ$. Therefore $XUZY$ is a rectangle with $UZ = XY = 2$ cm.

Because XZ is a straight line, by the symmetry of the tile, it follows that UY is also a straight line. Therefore XZ and UY are the diagonals of the rectangle $XUZY$ and they divide the rectangle into four triangles with equal areas.

Also, by the symmetry of the tile, $UVWZ$ is a square, XT is a straight line and $XU = VT$. Therefore, because $UV = UZ = 2$ cm and $XT = 8$ cm, it follows that $XU = 3$ cm.

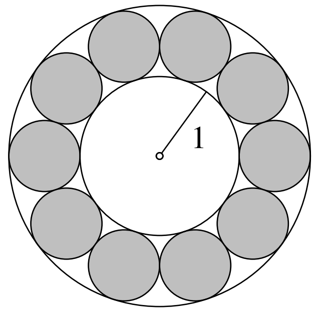
The shaded octagon is made up of the $2\,\text{cm} \times 2\,\text{cm}$ square $UVWZ$ and four congruent triangles. Each of these four triangles has the same area as one quarter of the rectangle $XUZY$. Therefore the area of the shaded octagon equals the area of $UVWZ$ plus the area of the $2\,\text{cm} \times 3\,\text{cm}$ rectangle $XUZY$.

Hence the area of the octagon is $(2 \times 2 + 2 \times 3)\,\text{cm}^2$, that is, $10\,\text{cm}^2$.

FOR INVESTIGATION

20.1 What is the total area of the four white hexagons that form part of each tile?

21. The diagram shows ten equal discs that lie between two concentric circles - an inner circle and an outer circle. Each disc touches two neighbouring discs and both circles. The inner circle has radius 1.



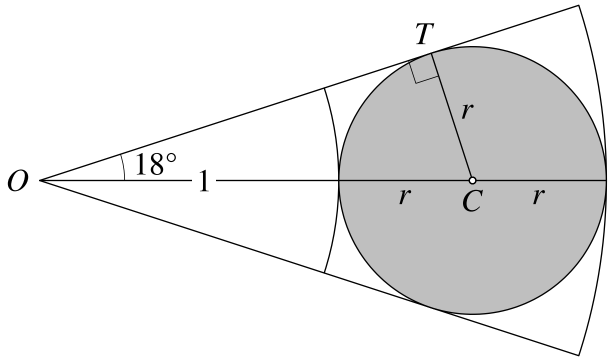
What is the radius of the *outer* circle?

A $2\tan 36°$ B $\dfrac{\sin 36°}{1-\sin 36°}$ C $\dfrac{1+\sin 18°}{1-\sin 18°}$ D $\dfrac{2}{\cos 18°}$ E $\dfrac{9}{5}$

Solution **C**

Let O be the common centre of the inner and outer circles, let C be the centre of one of the discs and let T be the point where one of the tangents from O meets the disc, as shown in the diagram.

Let the radius of each of the discs be r cm. We see from the diagram that the radius of the outer circle is $(1+2r)$ cm.



There are ten equal discs surrounding the inner circle. Therefore the angle between the two tangents from O to one of the discs is $\frac{1}{10}(360°)$, that is, $36°$.

Since $\angle TOC$ is one half of this angle, we have $\angle TOC = 18°$.

The tangent OT is at right angles to the radius CT. Therefore TOC is a right-angled triangle in which the hypotenuse, OC, has length $(1+r)$ cm, and the side opposite the angle of $18°$ has length r cm.

It follows that

$$\sin 18° = \frac{r}{1+r}.$$

From this equation we deduce that

$$(1+r)\sin 18° = r,$$

and hence that

$$\sin 18° = (1-\sin 18°)r.$$

It follows that

$$r = \frac{\sin 18°}{1-\sin 18°}.$$

Therefore the radius of the outer circle is given by

$$1+2r = 1+\frac{2\sin 18°}{1-\sin 18°} = \frac{(1-\sin 18°)+2\sin 18°}{1-\sin 18°} = \frac{1+\sin 18°}{1-\sin 18°}.$$

22. Three friends make the following statements.

Ben says, “Exactly one of Dan and Cam is telling the truth.”

Dan says, “Exactly one of Ben and Cam is telling the truth.”

Cam says, “Neither Ben nor Dan is telling the truth.”

Which of the three friends is lying?

A Just Ben
B Just Dan
C Just Cam
D Each of Ben and Cam
E Each of Ben, Cam and Dan

Solution **C**

If Cam’s statement is true, then both Ben and Dan are lying. But then exactly one of Ben and Cam is telling the truth. So Dan is telling the truth. This is a contradiction.

We deduce that Cam is lying.

Hence at least one of Ben and Dan is telling the truth.

If Ben’s statement is true, then exactly one of Ben and Cam is telling the truth. Hence Dan is telling the truth.

Similarly, if Dan’s statement is true, then Ben is telling the truth.

We deduce that Ben and Dan are telling the truth and that Cam is lying.

For investigation

22.1 Four friends make the following statements.

Ben says, “Exactly one of Cam, Dan and Sam is telling the truth.”

Dan says, “Exactly one of Ben, Cam and Sam is telling the truth.”

Cam says, “Exactly one of Ben, Dan and Sam is telling the truth.”

Sam says, “None of Ben, Dan and Cam is telling the truth.”

What can you deduce about who is telling the truth and who is lying?

23. A cuboid has sides of lengths 22, 2 and 10. It is contained within a sphere of the smallest possible radius.

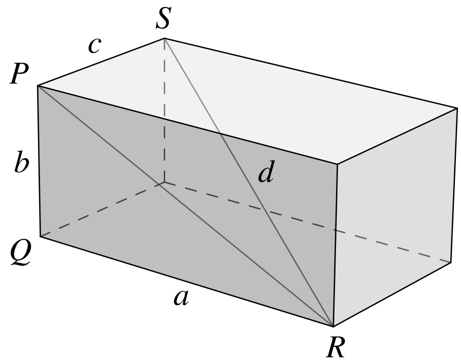
What is the side-length of the largest cube that will fit inside the same sphere?

A 10 B 11 C 12 D 13 E 14

Solution **E**

If a cuboid is contained in a sphere of the smallest possible radius, the vertices of the cuboid will lie on the sphere. It follows that the diagonals of the cuboid will be diameters of the sphere. So we begin by calculating the length of the diagonals of the $22 \times 2 \times 10$ cuboid.

Consider a general cuboid with sides of lengths a, b and c, as shown in the diagram. Let P, Q, R and S be the vertices shown in this diagram. Let d be the length of the diagonal RS.



By Pythagoras' theorem, applied to the right-angled triangle PQR, we have

$$PR^2 = a^2 + b^2.$$

Therefore, by Pythagoras' theorem, applied to the right-angled triangle, SPR, we have

$$d^2 = PR^2 + c^2 = a^2 + b^2 + c^2.$$

It follows that for the cuboid given in the question,

$$d^2 = 22^2 + 2^2 + 10^2 = 484 + 4 + 100 = 588.$$

Therefore the smallest sphere inside which the cuboid will fit has diameter d, where $d = \sqrt{588}$.

Let s be the side length of the largest cube that will fit inside this sphere. By the equation $d^2 = a^2 + b^2 + c^2$ for the diagonal of a cuboid, we have

$$d^2 = s^2 + s^2 + s^2,$$

and hence

$$588 = 3s^2.$$

It follows that $s^2 = 196$, and hence $s = 14$.

For investigation

23.1 Note that, in the above solution, we have

$$3s^2 = 22^2 + 2^2 + 10^2.$$

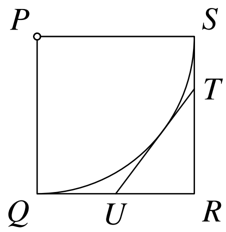
Find examples of three different positive integers a, b and c, with no common factor other than 1, such that there is a positive integer s such that

$$3s^2 = a^2 + b^2 + c^2.$$

24. The diagram shows a square $PQRS$. The arc QS is a quarter circle. The point U is the midpoint of QR and the point T lies on SR. The line TU is a tangent to the arc QS.

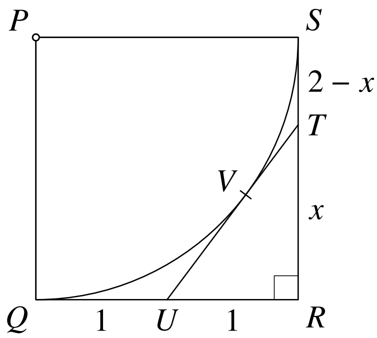
What is the ratio of the length of TR to the length of UR?

A $3:2$ B $4:3$ C $5:4$ D $7:6$ E $9:8$



Solution **B**

It is convenient, because U is the midpoint of QR, to choose units so that the square has sides of length 2. It follows that each of UQ and UR has length 1.



Suppose that TR has length x. Then the length of TS is $2 - x$.

Let V be the point where the line TU is tangent to the arc.

Because the two tangents from a point to a circle have the same length, $UV = UQ = 1$ and $VT = TS = 2 - x$. It follows that $UT = UV + VT = 1 + (2 - x) = 3 - x$.

In the right-angled triangle URT the hypotenuse has length $3 - x$, and the other two sides have lengths 1 and x. Therefore, by Pythagoras' theorem applied to this triangle,

$$1^2 + x^2 = (3 - x)^2.$$

It follows that

$$1 + x^2 = 9 - 6x + x^2,$$

and hence

$$6x = 8.$$

We deduce that $x = \frac{4}{3}$.

Therefore $TR : UR = \frac{4}{3} : 1 = 4 : 3$.

For investigation

24.1 Prove that the two tangents from a point to a circle have the same length.

24.2 In this question the common length of the sides of the square $PQRS$ is not given. Why is it legitimate in answering this question to choose units so that this length is 2?

25. Let n be the smallest integer for which $7n$ has 2016 digits.

What is the units digit of n?

A 0 B 1 C 4 D 6 E 8

SOLUTION **D**

The smallest positive integer which has 2016 digits is the number 10^{2015}, which is written with the digit 1 followed by 2015 copies of the digit 0. It follows that we require n to be the least positive integer for which $7n \geq 10^{2015}$. Therefore n is the least positive integer such that

$$n \geq \frac{10^{2015}}{7}.$$

The decimal expansion of $\frac{1}{7}$ is the recurring decimal $0.\dot{1}4285\dot{7}$, with the block 142857 of six digits repeated for ever after the decimal point.

We have

$$\frac{10^{2015}}{7} = 10^{2015} \times \frac{1}{7}.$$

The effect of multiplying by 10^{2015} is to move the decimal point 2015 places to the right. It follows that the decimal representation of $\frac{10^{2015}}{7}$ consists of the digits 142857 in repeated blocks, with 2015 of these digits coming before the decimal point.

Now $2015 = 6 \times 335 + 5$. Therefore the decimal expansion of $\frac{10^{2015}}{7}$ consists of 335 repetitions of the block 142857 followed by 14285 before the decimal point, with the digit 7 followed by an unending repetition of the block of digits 142857 after the decimal point.

That is,

$$\frac{10^{2015}}{7} = 142857\,142857\ldots142857\,14285\,.\,7\,\dot{1}4285\dot{7}\,.$$

It follows that the least integer n such that $n \geq \frac{10^{2015}}{7}$ is

$$142857\,142857\ldots142857\,14286,$$

whose units digit is 6.

Therefore the units digit of the least positive integer n for which $7n$ has 2016 digits is 6.

FOR INVESTIGATION

25.1 Check that the decimal expansion of $\frac{1}{7}$ is $0.\dot{1}4285\dot{7}$.

25.2 Which is the smallest integer n for which $3n$ has 1000 digits?

25.3 Which is the smallest integer n for which $13n$ has 100 digits?

UK SENIOR MATHEMATICAL CHALLENGE

Tuesday 7 November 2017

Organised by the **United Kingdom Mathematics Trust**

and supported by

Institute and Faculty of Actuaries

RULES AND GUIDELINES (to be read before starting)

1. Do not open the question paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**.
   No answers or personal details may be entered on the Answer Sheet after the 90 minutes are over.
3. The use of rough paper is allowed.
   Calculators, measuring instruments and squared paper are forbidden.
4. Candidates must be full-time students at secondary school or FE college, and must be in Year 13 or below (England & Wales); S6 or below (Scotland); Year 14 or below (Northern Ireland).
5. **Use B or HB pencil only**. Mark *at most one* of the options A, B, C, D, E on the Answer Sheet for each question. Do not mark more than one option.
6. **Scoring rules**: all candidates start out with 25 marks;

   0 marks are awarded for each question left unanswered;

   4 marks are awarded for each correct answer;

   1 mark is deducted for each incorrect answer.
7. **Guessing**: Remember that there is a penalty for incorrect answers. Note also that later questions are deliberately intended to be harder than earlier questions. You are thus advised to concentrate first on solving as many as possible of the first 15-20 questions. Only then should you try later questions.

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1. One of the following numbers is prime. Which is it?

A $2017 - 2$ B $2017 - 1$ C 2017 D $2017 + 1$ E $2017 + 2$

2. Last year, an earthworm from Wigan named Dave wriggled into the record books as the largest found in the UK. Dave was 40 cm long and had a mass of 26 g. What was Dave's mass per unit length?

A 0.6 g/cm B 0.65 g/cm C 0.75 g/cm D 1.6 g/cm E 1.75 g/cm

3. The five integers 2, 5, 6, 9, 14 are arranged into a different order. In the new arrangement, the sum of the first three integers is equal to the sum of the last three integers. What is the middle number in the new arrangement?

A 2 B 5 C 6 D 9 E 14

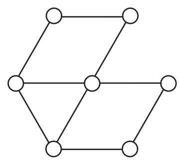
4. Which of the following is equal to $2017 - \frac{1}{2017}$?

A $\frac{2017^2}{2016}$ B $\frac{2016}{2017}$ C $\frac{2018}{2017}$ D $\frac{4059}{2017}$ E $\frac{2018 \times 2016}{2017}$

5. One light-year is nearly 6×10^{12} miles. In 2016, the Hubble Space Telescope set a new cosmic record, observing a galaxy 13.4 thousand million light-years away. Roughly how many miles is that?

A 8×10^{20} B 8×10^{21} C 8×10^{22} D 8×10^{23} E 8×10^{24}

6. The circles in the diagram are to be coloured so that any two circles connected by a line segment have different colours.
What is the smallest number of colours required?



A 2 B 3 C 4 D 5 E 6

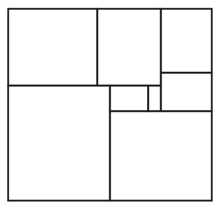
7. The positive integer k satisfies the equation $\sqrt{2} + \sqrt{8} + \sqrt{18} = \sqrt{k}$. What is the value of k?

A 28 B 36 C 72 D 128 E 288

8. When evaluated, which of the following is not an integer?

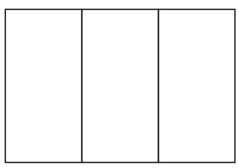
A 1^{-1} B $4^{-\frac{1}{2}}$ C 6^0 D $8^{\frac{2}{3}}$ E $16^{\frac{3}{4}}$

9. The diagram shows an $n \times (n + 1)$ rectangle tiled with $k \times (k + 1)$ rectangles, where n and k are integers and k takes each value from 1 to 8 inclusive.
What is the value of n?



A 16 B 15 C 14 D 13 E 12

10. A rectangle is divided into three smaller congruent rectangles as shown.
Each smaller rectangle is similar to the large rectangle.
In each of these four rectangles, what is the ratio of the length of a longer side to that of a shorter side?

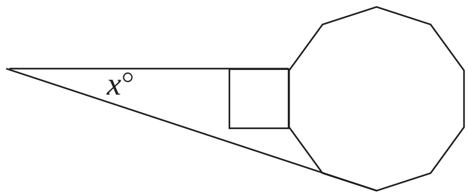


A $2\sqrt{3} : 1$ B $3 : 1$ C $2 : 1$ D $\sqrt{3} : 1$ E $\sqrt{2} : 1$

11. The teenagers Sam and Jo notice the following facts about their ages:
The difference between the squares of their ages is four times the sum of their ages.
The sum of their ages is eight times the difference between their ages.
What is the age of the older of the two?

A 15 B 16 C 17 D 18 E 19

12. The diagram shows a square and a regular decagon that share an edge. One side of the square is extended to meet an extended side of the decagon.
What is the value of x?

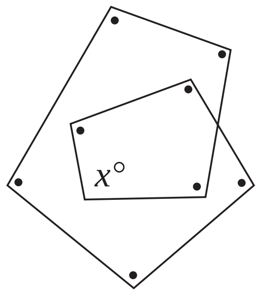


A 15 B 18 C 21 D 24 E 27

13. Isobel: "Josh is innocent" Genotan: "Tegan is guilty"
Josh: "Genotan is guilty" Tegan: "Isobel is innocent"
Only the guilty person is lying; all the others are telling the truth. Who is guilty?

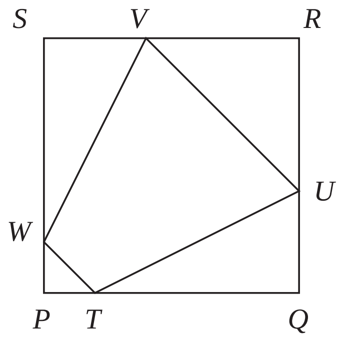
A Isobel B Josh C Genotan D Tegan E More information required

14. In the diagram, all the angles marked • are equal in size to the angle marked $x°$.
What is the value of x?



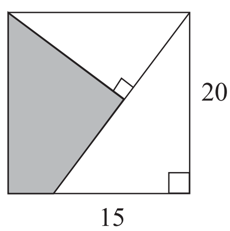
A 100 B 105 C 110 D 115 E 120

15. The diagram shows a square $PQRS$. Points T, U, V and W lie on the edges of the square as shown, such that $PT = 1$, $QU = 2$, $RV = 3$ and $SW = 4$. The area of $TUVW$ is half that of $PQRS$. What is the length of PQ?



A 5 B 6 C 7 D 8 E 9

16. The diagram shows two right-angled triangles inside a square. The perpendicular edges of the larger triangle have lengths 15 and 20.
What is the area of the shaded quadrilateral?



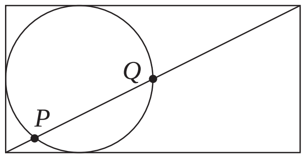
A 142 B 146 C 150 D 154 E 158

17. Amy, Beth and Claire each has some sweets. Amy gives one third of her sweets to Beth. Beth gives one third of all the sweets she now has to Claire. Then Claire gives one third of all the sweets she now has to Amy. All the girls end up having the same number of sweets.
Claire begins with 40 sweets. How many sweets does Beth have originally?
A 20 B 30 C 40 D 50 E 60

18. The arithmetic mean, A, of any two positive numbers x and y is defined to be $A = \frac{1}{2}(x + y)$ and their geometric mean, G, is defined to be $G = \sqrt{xy}$. For two particular values x and y, with $x > y$, the ratio $A : G = 5 : 4$. For these values of x and y, what is the ratio $x : y$?

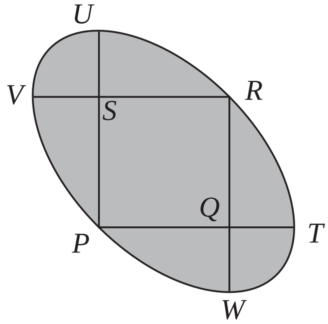
A $5 : 4$ B $2 : 1$ C $5 : 2$ D $7 : 2$ E $4 : 1$

19. The diagram shows a circle of radius 1 touching three sides of a 2×4 rectangle. A diagonal of the rectangle intersects the circle at P and Q, as shown. What is the length of the chord PQ?



A $\sqrt{5}$ B $\frac{4}{\sqrt{5}}$ C $\sqrt{5} - \frac{2}{\sqrt{5}}$ D $\frac{5\sqrt{5}}{6}$ E 2

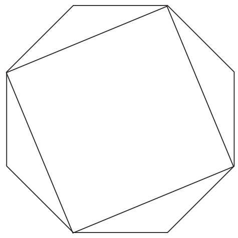
20. The diagram shows a square $PQRS$ with edges of length 1, and four arcs, each of which is a quarter of a circle. Arc TRU has centre P; arc VPW has centre R; arc UV has centre S; and arc WT has centre Q.
What is the length of the perimeter of the shaded region?



A 6 B $(2\sqrt{2} - 1)\pi$ C $(\sqrt{2} - \frac{1}{2})\pi$
D 2π E $(3\sqrt{2} - 2)\pi$

21. How many pairs (x, y) of positive integers satisfy the equation $4^x = y^2 + 15$?
A 0 B 1 C 2 D 4 E an infinite number

22. The diagram shows a regular octagon and a square formed by drawing four diagonals of the octagon. The edges of the square have length 1.
What is the area of the octagon?



A $\frac{\sqrt{6}}{2}$ B $\frac{4}{3}$ C $\frac{7}{5}$ D $\sqrt{2}$ E $\frac{3}{2}$

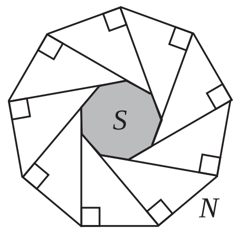
23. The parabola with equation $y = x^2$ is reflected in the line with equation $y = x + 2$. Which of the following is the equation of the reflected parabola?

A $x = y^2 + 4y + 2$ B $x = y^2 + 4y - 2$ C $x = y^2 - 4y + 2$
D $x = y^2 - 4y - 2$ E $x = y^2 + 2$

24. There is a set of straight lines in a plane such that each line intersects exactly ten others. Which of the following could not be the number of lines in that set?

A 11 B 12 C 15 D 16 E 20

25. The diagram shows a regular nonagon N. Moving clockwise around N, at each vertex a line segment is drawn perpendicular to the preceding edge. This produces a smaller nonagon S, shown shaded.
What fraction of the area of N is the area of S?



A $\frac{1 - \cos 40°}{1 + \cos 40°}$ B $\frac{\cos 40°}{1 + \cos 40°}$ C $\frac{\sin 40°}{1 + \sin 40°}$ D $\frac{1 - \sin 40°}{1 + \sin 40°}$ E $\frac{1}{9}$

UK SENIOR MATHEMATICAL CHALLENGE

Tuesday 7 November 2017

For reasons of space, these solutions are necessarily brief.
There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:
http://www.ukmt.org.uk

1. **C** Of the options given, 2015 is a multiple of 5, 2016 and 2018 are even, and 2019 is a multiple of 3. So the prime number must be 2017.

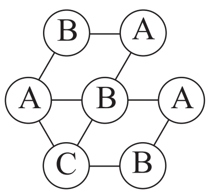
2. **B** Dave's mass per unit length in g/cm is $26 \div 40 = 6.5 \div 10 = 0.65$.

3. **E** In the new arrangement, the sum of the first two numbers and the sum of the last two numbers must be equal. Considering all ten possible pairings, the only two pairs with the same total are 2, 9 and 5, 6. So 14 must be the middle number.

4. **E** By making a common denominator, $2017 - \dfrac{1}{2017} = \dfrac{2017^2 - 1}{2017}$. Then, using the difference of two squares on the numerator, this can be written as $\dfrac{2018 \times 2016}{2017}$.

5. **C** The calculation 13.4 thousand million multiplied by 6×10^{12} gives $13.4 \times 10^9 \times 6 \times 10^{12}$ which is roughly 80×10^{21}. So the distance is roughly 8×10^{22} miles.

6. **B** In the bottom left of the diagram there are three circles connected to each other. These three must be coloured using different colours. Once those have been determined, one possible colouring, using just three colours, is as shown.



7. **C** Simplifying $\sqrt{2} + \sqrt{8} + \sqrt{18}$ gives $\sqrt{2} + 2\sqrt{2} + 3\sqrt{2} = 6\sqrt{2}$. This is the same as $\sqrt{36 \times 2}$ so $\sqrt{72} = \sqrt{k}$ and therefore $k = 72$.

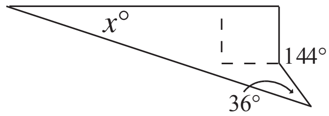
8. **B** Evaluating each option in turn gives $1^{-1} = 1$; $4^{-\frac{1}{2}} = \left(\frac{1}{4}\right)^{\frac{1}{2}} = \frac{1}{2}$; $6^0 = 1$; $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$; $16^{\frac{3}{4}} = \left(16^{\frac{1}{4}}\right)^3 = 2^3 = 8$. Only option B is not an integer.

9. **B** The sum of the areas of the tiles is $(1 \times 2) + (2 \times 3) + (3 \times 4) + \ldots + (8 \times 9)$ $= 2 + 6 + 12 + 20 + 30 + 42 + 56 + 72 = 240 = 15 \times 16$. So the value of n is 15.

10. **D** Let the sides of each of the smaller rectangles be 1 and a with $a > 1$. Then the sides of the larger rectangle are a and 3 with $3 > a$. As the rectangles are similar, $\frac{a}{1} = \frac{3}{a}$. So $a^2 = 3$ and $a = \sqrt{3}$. The ratio of the sides is therefore $\sqrt{3} : 1$.

11. **D** Let the ages of the teenagers be a and b with $a > b$. Then $4(a + b) = a^2 - b^2 = (a + b)(a - b)$ and so $4 = a - b$ as $a + b \neq 0$. Also, $a + b = 8(a - b) = 8 \times 4 = 32$. We now have $a - b = 4$ and $a + b = 32$ so $a = 18$ and $b = 14$. Thus the older of the two is 18.

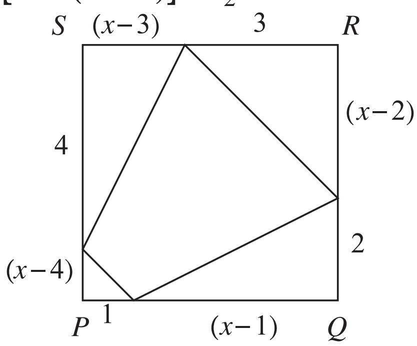
12. **B** Each of the exterior angles of a regular decagon is $360° \div 10 = 36°$ so each interior angle is $180° - 36° = 144°$. The quadrilateral containing x has an angle sum of 360°. So $x + 90 + (360 - 144) + 36 = 360$ and $x = 18$.



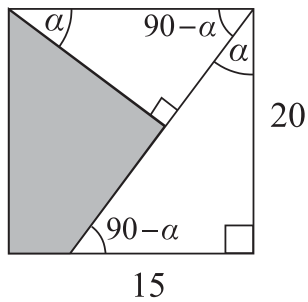
13. **C** Since only one person is lying this must be Genotan or Josh; their statements cannot both be true. If Josh were lying (and therefore guilty) then Genotan's statement would mean that Tegan was also guilty, a contradiction. Hence Genotan is lying and we can check that all four statements are consistent with this.

14. **A** Viewing the diagram as a pentagon inside a hexagon and labelling the unmarked angle in the pentagon as $y°$ gives $4x + y = 540$… (1). For the hexagon, the exterior reflex angle is $y°$ as opposite angles are equal. So the unmarked interior angle is $(360 - y)°$. This gives $5x + 360 - y = 720$, so $5x - y = 360$… (2). Adding equations (1) and (2) gives $9x = 900$ and therefore $x = 100$.

15. **B** Let the length of PQ be x. The sum of the areas of the four right-angled triangles is half the area of the square so $\frac{1}{2}[1 \times (x - 4)] + \frac{1}{2}[4 \times (x - 3)] + \frac{1}{2}[3 \times (x - 2)] + \frac{1}{2}[2 \times (x - 1)] = \frac{1}{2}x^2$. This gives $(x - 4) + 4(x - 3) + 3(x - 2) + 2(x - 1) = x^2$ which simplifies to $10x - 24 = x^2$. Factorising $x^2 - 10x + 24 = 0$ gives $(x - 6)(x - 4) = 0$ so $x = 4$ or 6. For there to be four triangles as shown, $(x - 4) > 0$, so x, the length of PQ, is 6.



16. **D** The area of the shaded quadrilateral is the area of the square less the combined areas of the two right-angled triangles. Using both ‘the angle sum of a triangle is 180°’ and ‘the sum of the angles in the corner of a square is 90°’, those two triangles have the same sized angles so they are similar. By Pythagoras' Theorem, the hypotenuse of the larger triangle has length 25. The lengths of the sides of the smaller triangle are then $\frac{20}{25}$ of the lengths of the sides of the larger triangle. So the shaded area is $20^2 - \frac{1}{2} \times 15 \times 20 - \frac{1}{2} \times 12 \times 16$ which is $400 - 150 - 96 = 154$.



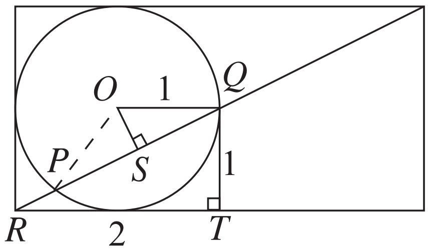
17. **D** Amy's number of sweets is a multiple of 3, so let her start with $3a$ sweets. Once Amy has passed on a of them, Beth's number of sweets must be a multiple of 3. Likewise, after Beth passes on sweets to Claire, Claire's number of sweets must become a multiple of 3. Since $40 = 3 \times 13 + 1$, Beth must pass on $3b - 1$ sweets for some b. So Beth must have had $9b - 3$ sweets after receiving the a sweets from Amy and hence $9b - 3 - a$ to start with. This allows us to fill in the table:

	Amy	Beth	Claire
originally	$3a$	$9b - 3 - a$	40
after Amy's gift to Beth	$2a$	$9b - 3$	40
after Beth's gift to Claire	$2a$	$6b - 2$	$3b + 39$
after Claire's gift to Amy	$2a + b + 13$	$6b - 2$	$2b + 26$

As the girls end up with the same number of sweets, $6b - 2 = 2b + 26$ and so $b = 7$. As $6 \times 7 - 2$ and $2 \times 7 + 26$ both equal 40, the girls each have 40 sweets at the end. Hence Amy's finishing number $2a + 7 + 13 = 40$ so $a = 10$. Then Beth's original number is $9 \times 7 - 10 - 3 = 50$.

18. **E** Given the ratio $A : G$ is $5 : 4$, we have $4A = 5G$, so $4 \times \frac{1}{2}(x + y) = 5\sqrt{xy}$. Squaring gives $4(x + y)^2 = 25xy$ and hence $4x^2 + 8xy + 4y^2 = 25xy$. So $4x^2 - 17xy + 4y^2 = 0$ which factorises to give $(4x - y)(x - 4y) = 0$ so that either $4x = y$ or $x = 4y$. However the first case is impossible as we are given $x > y > 0$. Hence $x = 4y$ and $x : y$ is $4 : 1$.

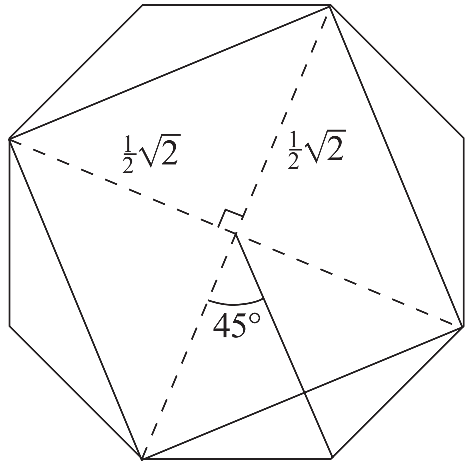
19. **B** Let O be the centre of the circle, R be a corner of the rectangle, T be the point on the rectangle directly below Q and S be the midpoint of PQ. In triangle QRT we have $QT = 1$ and $RT = 2$ and therefore $RQ = \sqrt{5}$, by Pythagoras' Theorem. As RT and OQ are parallel, the right-angled triangles OQS and QRT are similar, so $QS = \frac{2}{\sqrt{5}}$. Therefore the length of chord PQ is $2 \times \frac{2}{\sqrt{5}} = \frac{4}{\sqrt{5}}$.



20. **B** The length of the radius of the quarter circle TRU is the length of PR, which by Pythagoras' Theorem is $\sqrt{2}$. The lengths PU and PR are equal as they are radii of the same circle. So $SU = PU - PS = \sqrt{2} - 1$. Hence the total length of the four quarter circles making up the perimeter of the shaded region is $2 \times \frac{1}{4} \times 2\pi\sqrt{2} + 2 \times \frac{1}{4} \times 2\pi\left(\sqrt{2} - 1\right) = \left(2\sqrt{2} - 1\right)\pi$.

21. **C** The equation $4^x = y^2 + 15$ can be rearranged to give $\left(2^x\right)^2 - y^2 = 15$ which, using the difference of two squares, is $\left(2^x + y\right)\left(2^x - y\right) = 15$. As we are looking for positive integer values then either $2^x + y = 5$ and $2^x - y = 3$ or $2^x + y = 15$ and $2^x - y = 1$. Solving each pair gives either $(x, y) = (2, 1)$ or $(x, y) = (3, 7)$ so there are just two possible pairs.

22. **D** The square can be split into four congruent right-angled triangles by joining each of its vertices to the centre. Each edge of the square has length 1, so the shorter sides of each triangle have length $\frac{1}{2}\sqrt{2}$.

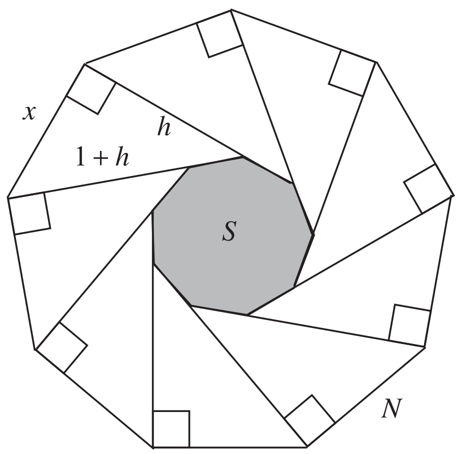


The regular octagon can be split into eight congruent isosceles triangles whose longer sides have length $\frac{1}{2}\sqrt{2}$, by joining all eight vertices to the centre. These sides are separated by an angle of $\frac{1}{8} \times 360° = 45°$. Using the formula 'area = $\frac{1}{2}ab \sin C$', the total area of the octagon is $8 \times \frac{1}{2} \times \frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{2} \sin 45° = 2 \sin 45° = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$.

23. **C** Reflecting a graph in the line $y = x + 2$ is algebraically equivalent to replacing y by $x + 2$ and $x + 2$ by y in the equation of the graph. So $y = x^2$ becomes $(x + 2) = (y - 2)^2$. That is $x + 2 = y^2 - 4y + 4$ which simplifies to $x = y^2 - 4y + 2$.

24. **D** When n non-parallel lines are drawn in a plane, each line intersects every other line exactly once. So each line intersects $(n - 1)$ lines. A line drawn parallel to one of the existing n lines also intersects $(n - 1)$ lines. In order that every line drawn intersects the same number of other lines, each of the original n lines must be part of a set of k parallel lines, for some integer k. Then kn lines will each have $k(n - 1)$ points of intersection. Here $k(n - 1) = 10$ so as $(n - 1)$ is an integer $(n - 1)$ must be 1, 2, 5 or 10 giving $n = 2, 3, 6$ or 11 and $k = 10, 5, 2$ and 1 respectively. Of the options given: A is 11 non-parallel lines ($n = 11, k = 1$); B is 6 sets of 2 parallel lines ($n = 6, k = 2$); C is 3 sets of 5 parallel lines ($n = 3, k = 5$); E is 2 different sets of 10 parallel lines ($n = 2, k = 10$). These are the only four possible solutions to the problem, so 16 could not be the number of lines.

25. **A** Let the side-length of S be 1 and the side-length of N be x. The fraction of the area of N that is the area of S is then $\left(\frac{1}{x}\right)^2 = \frac{1}{x^2}$. Let the length of the hypotenuse of each of the nine triangles be $1 + h$, then the third side of each triangle is h. Using Pythagoras' Theorem, $h^2 + x^2 = (1 + h)^2$ so $h^2 + x^2 = 1 + 2h + h^2$ and the fraction we want, $\frac{1}{x^2}$, is then $\frac{1}{1 + 2h}$. The angle in each triangle which is adjacent to S is also an exterior angle of S, so is $\frac{1}{9} \times 360° = 40°$. We



have $\cos 40° = \dfrac{h}{1 + h}$ which rearranges to give $h = \dfrac{\cos 40°}{1 - \cos 40°}$. Therefore $1 + 2h = \left(\dfrac{1 - \cos 40°}{1 - \cos 40°}\right) + \dfrac{2 \cos 40°}{1 - \cos 40°} = \dfrac{1 + \cos 40°}{1 - \cos 40°}$. So the fraction of the area N which is the area of S is $\dfrac{1}{1 + 2h} = \dfrac{1 - \cos 40°}{1 + \cos 40°}$.

Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries

Solutions and investigations

7 November 2017

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some problems for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	B	E	E	C	B	C	B	B	D	D	B	C	A	B	D	D	E	B	B	C	D	C	D	A

1. One of the following numbers is prime. Which is it?

A $2017 - 2$ B $2017 - 1$ C 2017 D $2017 + 1$ E $2017 + 2$

Solution **C**

We see that:

$2017 - 2 = 2015$ and 2015 is a multiple of 5. So $2017 - 2$ is not prime.

$2017 - 1 = 2016$ and 2016 is a multiple of 2. So $2017 - 1$ is not prime.

$2017 + 1 = 2018$ and 2018 is a multiple of 2. So $2017 + 1$ is not prime.

$2017 + 2 = 2019$ and 2019 is a multiple of 3. So $2017 + 2$ is not prime.

We are told that one of the given options is prime. We may therefore deduce that the remaining option, 2017, is prime.

For investigation

1.1 In the context of the SMC it is safe to assume that the information given in the question is correct. Therefore, having shown that $2017 - 2$, $2017 - 1$, $2017 + 1$ and $2017 + 2$ are not prime, we may deduce that 2017 is prime.

However, without this assumption, if we wish to show that 2017 is prime, we need to check that there are no prime factors of 2017 which are less than 2017.

Which is the largest prime p that we need to check is not a factor of 2017 in order to show that 2017 is prime?

1.2 One way to see that 2019 is a multiple of 3 is by noting that the sum of its digits, $2 + 0 + 1 + 9 = 12$, is a multiple of 3.

Why does this test for divisibility by 3 work?

1.3 (a) Which is the least positive integer n such that either $2017 - n$ or $2017 + n$ is prime?

(b) Which is the least positive integer n such that both $2017 - n$ and $2017 + n$ are prime?

2. Last year, an earthworm from Wigan named Dave wriggled into the record books as the largest found in the UK. Dave was 40 cm long and had a mass of 26 g.

What was Dave's mass per unit length?

A 0.6 g/cm B 0.65 g/cm C 0.75 g/cm D 1.6 g/cm E 1.75 g/cm

Solution **B**

Dave's mass per unit length is $\frac{26}{40}$ g/cm. We have

$$\frac{26}{40} = \frac{26}{10 \times 4} = \frac{2.6}{4} = 0.65.$$

Therefore Dave's mass per unit length is 0.65 g/cm.

3. The five integers 2, 5, 6, 9, 14 are arranged into a different order. In the new arrangement, the sum of the first three integers is equal to the sum of the last three integers.

What is the middle number in the new arrangement?

A 2 B 5 C 6 D 9 E 14

SOLUTION **E**

Let the integers in the new arrangement be in the order p, q, r, s, t. Because the sum of the first three integers is the same as the sum of the last three,

$$p + q + r = r + s + t,$$

and hence

$$p + q = s + t.$$

We therefore see that the pair of integers p, q has the same sum as the pair s, t. Also, the middle number, r, is the one that is not included in either of these pairs.

It is straightforward to check that $2 + 9 = 5 + 6$ and that 2, 9 and 5, 6 are the only two pairs of the given integers with the same sum.

Therefore the middle integer in the new arrangement is 14, as this does not occur in either pair.

FOR INVESTIGATION

3.1 In how many different ways may the integers 2, 5, 6, 9, 14 be arranged into a different order so that the sum of the first three integers is equal to the sum of the last three integers?

3.2 Suppose that the integers 3, 7, 8, 10, 12 are arranged into a different order so that the sum of the first three integers is equal to the sum of the last three. What is the middle number in the new arrangement?

3.3 The integers 3, 6, 9, 12, 15 are to be arranged into a different order so that the sum of the first three integers is equal to the sum of the last three. How many different possibilities are there for the middle number in the new arrangement?

3.4 Five different integers are to be arranged in order so that the sum of the first three integers is the same as the sum of the last three. What is the maximum number of possibilities for the middle number in the new arrangement?

3.5 (a) What is the largest number of integers that may be chosen from the set of all positive integers from 1 to 10, inclusive, so that no two pairs of the chosen integers have the same total?

(b) What is the largest number of integers that may be chosen from the set of all positive integers from 1 to 20, inclusive, so that no two pairs of the chosen integers have the same total?

4. Which of the following is equal to $2017 - \frac{1}{2017}$?

A $\frac{2017^2}{2016}$ B $\frac{2016}{2017}$ C $\frac{2018}{2017}$ D $\frac{4059}{2017}$ E $\frac{2018 \times 2016}{2017}$

SOLUTION **E**

Writing both 2017 and $\frac{1}{2017}$ over a common denominator, we have

$$2017 - \frac{1}{2017} = \frac{2017^2 - 1}{2017}.$$

Now,

$$2017^2 - 1 = 2017^2 - 1^2.$$

Hence, using the standard factorization of the difference of two squares, we have

$$2017 - \frac{1}{2017} = \frac{2017^2 - 1^2}{2017} = \frac{(2017+1)(2017-1)}{2017} = \frac{2018 \times 2016}{2017}.$$

5. One light-year is nearly 6×10^{12} miles. In 2016, the Hubble Space Telescope set a new cosmic record, observing a galaxy 13.4 thousand million light-years away.

Roughly how many miles is that?

A 8×10^{20} B 8×10^{21} C 8×10^{22} D 8×10^{23} E 8×10^{24}

SOLUTION **C**

One thousand million is $1000 \times 1\,000\,000 = 10^3 \times 10^6 = 10^{3+6} = 10^9$. Therefore 13.4 thousand million light-years is 13.4×10^9 light-years. Therefore, because a light-year is nearly 6×10^{12} miles, 13.4 thousand million light-years is approximately

$$(6 \times 10^{12}) \times (13.4 \times 10^9) \text{ light-years.}$$

Now

$$(6 \times 10^{12}) \times (13.4 \times 10^9) = (6 \times 13.4) \times (10^{12} \times 10^9).$$

Now 6×13.4 is approximately 80, therefore, $(6 \times 13.4) \times (10^{12} \times 10^9)$ is approximately

$$80 \times (10^{12} \times 10^9).$$

Finally, we have

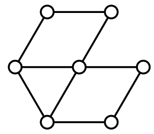
$$80 \times (10^{12} \times 10^9) = 8 \times 10 \times 10^{12+9} = 8 \times 10 \times 10^{21} = 8 \times 10^{22}.$$

Therefore 13.4 thousand million light-years is approximately 8×10^{22} miles.

6. The circles in the diagram are to be coloured so that any two circles connected by a line segment have different colours.

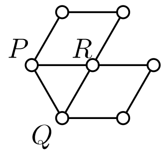
What is the smallest number of colours required?

A 2 B 3 C 4 D 5 E 6



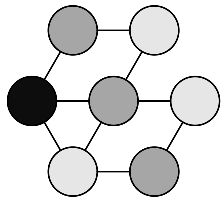
Solution **B**

Each pair of the circles labelled P, Q and R in the figure on the right is connected by a line segment. Therefore these three circles must be coloured using different colours. So at least three colours are needed.



The figure on the right (with the circles enlarged for the sake of clarity) shows one way to colour the circles using three colours so that circles connected by a line segment have different colours.

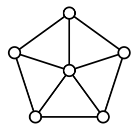
Therefore 3 is the smallest number of colours required.



For investigation

6.1 In how many different ways is it possible to colour the circles in the diagram in the question, using the three colours red, green and blue, so that circles connected by a line segment have different colours?

6.2 What is the smallest number of colours needed to colour the circles in the figure on the right so that circles connected by a line segment have different colours?



6.3 It follows from the *Four Colour Theorem* that any arrangement of circles in the plane, connected by line segments that do not cross one another, may be coloured using at most four colours so that circles connected by a line segment have different colours.

Find an arrangement of circles connected by line segments for which four colours are needed.

What is the smallest number of circles in such an arrangement?

Note

The first proof of the Four Colour Theorem about maps drawn in the plane was published by Kenneth Appel and Wolfgang Haaken in 1977. Their proof reduced the general case to 1482 *unavoidable* configurations which needed to be checked separately. These configurations were generated and checked by a computer program. Since 1977 simpler proofs using a computer have been found. But no-one has yet found a proof which is simple enough for a human being to check it, just using pencil and paper, in a reasonable amount of time.

A good book on the Four Colour theorem is *Four Colours Suffice: How the Map Problem was Solved*, Robin Wilson, 2002.

7. The positive integer k satisfies the equation $\sqrt{2} + \sqrt{8} + \sqrt{18} = \sqrt{k}$.

What is the value of k?

A 28 B 36 C 72 D 128 E 288

Solution **C**

Because $8 = 2^2 \times 2$ and $18 = 3^2 \times 2$, we have $\sqrt{8} = 2\sqrt{2}$ and $\sqrt{18} = 3\sqrt{2}$. Therefore

$$\begin{aligned}\sqrt{2} + \sqrt{8} + \sqrt{18} &= \sqrt{2} + 2\sqrt{2} + 3\sqrt{2}\\ &= 6\sqrt{2}\\ &= \sqrt{6^2 \times 2}\\ &= \sqrt{72}.\end{aligned}$$

Therefore $k = 72$.

8. When evaluated, which of the following is not an integer?

A 1^{-1} B $4^{-\frac{1}{2}}$ C 6^0 D $8^{\frac{2}{3}}$ E $16^{\frac{3}{4}}$

Solution **B**

We have

$$\begin{aligned}1^{-1} &= \frac{1}{1} = 1,\\ 4^{-\frac{1}{2}} &= \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2},\\ 6^0 &= 1,\\ 8^{\frac{2}{3}} &= (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 = 2^2 = 4,\end{aligned}$$

and

$$16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = (\sqrt[4]{16})^3 = 2^3 = 8.$$

We therefore see that only the expression of option B gives a number which is not an integer when it is evaluated.

For investigation

8.1 The standard convention is that for $x \neq 0$, we take 1 as the value of x^0. Why is this convention a sensible one to use?

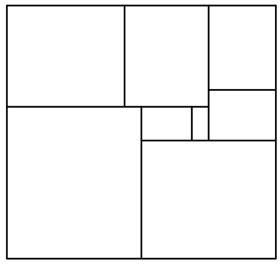
8.2 The standard convention is that 0^0 represents the number 1. Why is this a sensible convention?

8.3 In the answer to Question 8 we have used the fact that when p and q are positive integers, and $x > 0$, the convention is that $x^{\frac{p}{q}}$ means $(\sqrt[q]{x})^p$. Why do we adopt this convention?

9. The diagram shows an $n\times(n+1)$ rectangle tiled with $k\times(k+1)$ rectangles, where n and k are integers and k takes each value from 1 to 8 inclusive.

What is the value of n?

A 16 B 15 C 14 D 13 E 12



SOLUTION **B**

The total area of the rectangles of size $k \times (k+1)$, for $k = 1, 2, 3, 4, 5, 6, 7, 8$, is

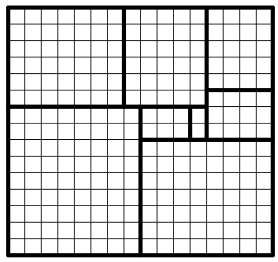
$$\begin{aligned}1\times2+2\times3+3\times4+4\times5+5\times6+6\times7+7\times8+8\times9 &= 2+6+12+20+30+42+56+72\\ &= 240\\ &= 15\times16.\end{aligned}$$

Therefore $n = 15$.

In the context of the SMC the above calculation is sufficient to show that, if the smaller rectangles tile a rectangle of size $n \times (n+1)$, for some integer n, then $n = 15$.

However, for a complete solution it is necessary to show that the eight smaller rectangles can be used to a tile a 15×16 rectangle.

It looks from the figure in the question that this is possible. The figure on the right confirms that the sizes of the rectangles are correct.



Note also that from this figure we can see directly that the large rectangle has size 15×16.

FOR INVESTIGATION

9.1 (a) Find a formula in terms of s for the total area of the rectangles of size $k \times (k+1)$ for all the integer values of k from 1 to s inclusive.

In other words, find a formula for the sum

$$1\times2+2\times3+3\times4+\cdots+s\times(s+1).$$

[Note that using the Σ notation, we may write this sum as

$$\sum_{k=1}^{s} k\times(k+1), \text{ or, suppressing the multiplication sign, as } \sum_{k=1}^{s} k(k+1)].$$

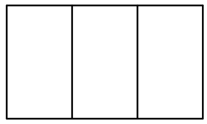
(b) Check that your formula gives the answer 240 when $s = 8$.

9.2 (a) Can you find values of s, other than $s = 8$, such that for some integer n

$$\sum_{k=1}^{s} k(k+1) = n(n+1)?$$

(b) For the values of s that you have found in answer to part (a) is it possible to use the rectangles of size $k \times (k+1)$, where k takes all integer values from 1 to s inclusive, to tile a rectangle of size $n \times (n+1)$?

10. A rectangle is divided into three smaller congruent rectangles as shown.



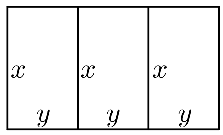
Each smaller rectangle is similar to the large rectangle.

In each of the four rectangles, what is the ratio of the length of a longer side to that of a shorter side?

A $2\sqrt{3} : 1$ B $3 : 1$ C $2 : 1$ D $\sqrt{3} : 1$ E $\sqrt{2} : 1$

Solution **D**

We suppose that the length of the longer sides of the three smaller rectangles is x and the length of their shorter sides is y.



It follows that the longer sides of the large rectangle have length $3y$, and that its shorter sides have length x.

Because the smaller rectangles are similar to the larger rectangle $\frac{x}{y} = \frac{3y}{x}$. Therefore $\frac{x^2}{y^2} = \frac{3}{1}$.

Hence $\frac{x}{y} = \frac{\sqrt{3}}{1}$.

It follows that the ratio of the length of a longer side to that of a shorter side in all the rectangles is $\sqrt{3} : 1$.

For investigation

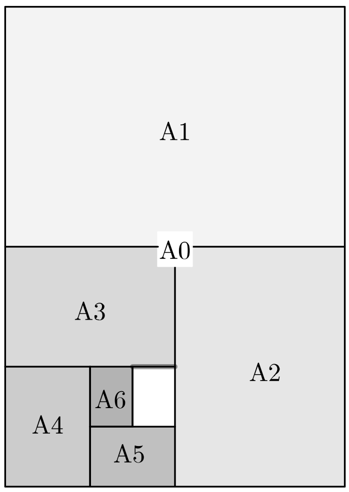
10.1 The A series of paper sizes, (A0, A1, A2, A3, ...), is defined as follows.

The largest size is A0.

Two A1 sized sheets of paper are obtained by cutting an A0 sheet in half along a line parallel to its shorter edges.

Two A2 sized sheets of paper are obtained by cutting an A1 sheet in half in a similar way, and so on, as shown in the figure.

The shapes of all these sheets of paper are similar rectangles.



(a) What is the ratio of the length of a longer side to the length of the shorter side in all these rectangles?

(b) An A0 sheet of paper has area $1\,\text{m}^2$. What are lengths, to the nearest cm, of the longer and shorter sides of an A0 sheet of paper?

(c) The most commonly used of these sizes is A4. What are the lengths, to the nearest cm, of the longer and shorter sides of an A4 sheet of paper?

(d) Standard quality paper weighs $80\,\text{g/m}^2$. What is the weight of one standard quality sheet of A4 paper?

11. The teenagers Sam and Jo notice the following facts about their ages:

The difference between the squares of their ages is four times the sum of their ages.

The sum of their ages is eight times the difference between their ages.

What is the age of the older of the two?

A 15 B 16 C 17 D 18 E 19

Solution **D**

Suppose that the ages of the teenagers are a and b, with $a > b$.

Because the difference between the squares of their ages is four times the sum of their ages

$$a^2 - b^2 = 4(a + b).$$

By factorizing its left hand side, we may rewrite this last equation as

$$(a - b)(a + b) = 4(a + b)$$

Because $a + b \neq 0$, we may divide both sides of this last equation by $a + b$ to give

$$a - b = 4. \qquad (1)$$

Because the sum of their ages is eight times their difference

$$a + b = 8(a - b)$$

Hence, by (1)

$$a + b = 32. \qquad (2)$$

By adding equations (1) and (2), we obtain

$$2a = 36$$

and hence

$$a = 18.$$

Therefore the age of the older of the two teenagers is 18.

For investigation

11.1 What is the age of the younger teenager in Question 11?

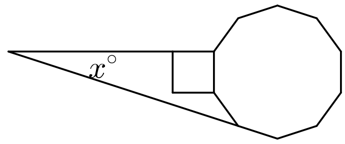
11.2 Suppose that the difference between the squares of the ages of two teenagers is six times the sum of their ages, and the sum of their ages is five times the difference of their ages.

What are their ages in this case?

11.3 Suppose that the difference between the squares of the ages of the two teenagers is k times the sum of their ages, and the sum of their ages in n times the difference of their ages.

Find a formula in terms of k and n for the ages of the teenagers. Check that your formula gives the correct answer to Question 11 and Problems 11.1 and 11.2.

12. The diagram shows a square and a regular decagon that share an edge. One side of the square is extended to meet an extended edge of the decagon.

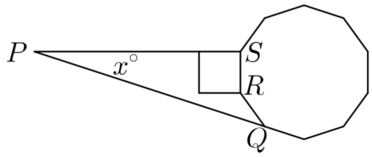


What is the value of x?

A 15 B 18 C 21 D 24 E 27

Solution **B**

We consider the quadrilateral $PQRS$ as shown in the figure on the right. Because $PQRS$ is a quadrilateral its angles have total 360°.



Because it is the exterior angle of a decagon, $\angle RQP = \frac{1}{10} \times 360° = 36°$.

It follows that the interior angle of a decagon is $(180 - 36)° = 144°$. Hence the reflex angle SRQ is $(360 - 144)° = 216°$. The angle PSR is an angle of the square and hence is 90°.

Therefore, we have

$$x° + 36° + 216° + 90° = 360°.$$

It follows that

$$x° = (360 - 342)° = 18°.$$

For investigation

12.1 We have used here the fact that the exterior angle of a regular polygon with n sides is $\frac{1}{n} \times 360°$. Explain why this is true.

12.2 Prove that the sum of the angles of a quadrilateral is 360°.

13. Isobel: "Josh is innocent" Genotan: "Tegan is guilty"

Josh: "Genotan is guilty" Tegan: "Isobel is innocent"

Only the guilty person is lying; all the others are telling the truth.

Who is guilty?

A Isobel B Josh C Genotan D Tegan
E More information required

Solution **C**

There is only one guilty person, so either Genotan or Josh is lying. If Josh is lying, Genotan is innocent and is therefore telling the truth. Hence Tegan is guilty, contradicting the fact that there is just one guilty person. So Josh is not lying. Therefore Genotan is the guilty person.

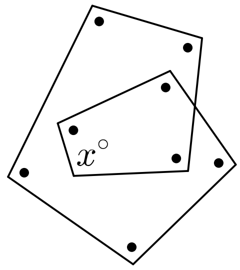
For investigation

13.1 Check that the guilt of Genotan is consistent with all the information given in the question.

14. In the diagram all the angles marked • are equal in size to the angle marked $x°$.

What is the value of x?

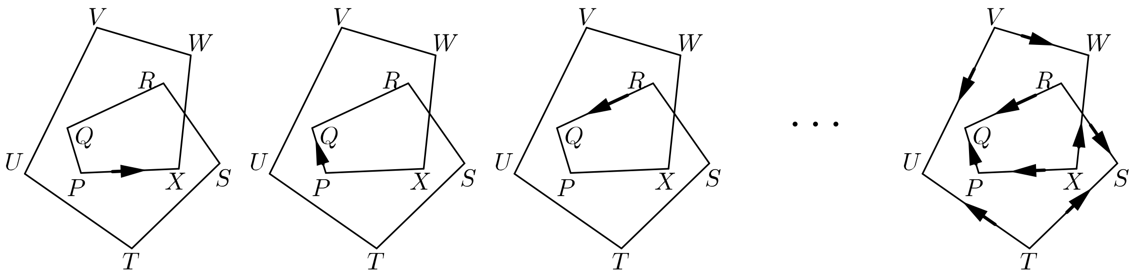
A 100 B 105 C 110 D 115 E 120



SOLUTION **A**

METHOD 1

We label the vertices of the figure as shown below.



We place an arrow lying along PX in the direction shown in the first figure on the left above. We rotate this arrow anticlockwise about the point P until it lies along PQ in the direction shown in the second figure above. The arrow has been turned anticlockwise through the angle $x°$.

Next we rotate the arrow anticlockwise about the point Q until it lies along RQ in the direction shown in the third figure above. As all the angles marked • are $x°$, the arrow has again been turned anticlockwise through $x°$.

We continue this process, rotating the arrow anticlockwise through $x°$ about the points R, S, T, U, V, W and X in turn. The arrow ends up lying along XP in the direction shown in the figure on the right above.

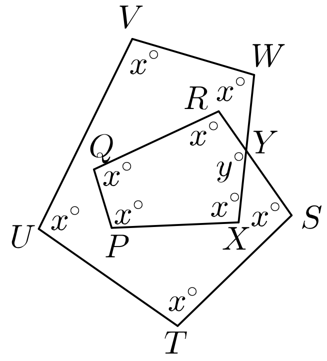
In this figure we have also shown the direction in which the arrow points on all the other edges during this process, apart from its initial position.

It will be seen that in this process the arrow has been turned through $2\frac{1}{2}$ complete revolutions. Therefore the total angle it has turned through is $2\frac{1}{2} \times 360°$, that is, through $900°$. In the process the arrow been rotated 9 times through the angle $x°$.

Therefore $9x = 900$ and hence $x = 100$.

Method 2

We let Y be the point where RS meets WX. We also let $\angle RYX = y°$.



The sum of the angles of a pentagon is 540°. In the pentagon $PXYRQ$, the interior angle at Y is $y°$ and all the other four angles are $x°$. Therefore we have

$$4x + y = 540. \quad (1).$$

The sum of the angles of a hexagon is 720°. In the hexagon $SYWVUT$, the reflex angle at Y is $360° - y°$ and all the other five angles are $x°$.

Therefore $5x + (360 - y) = 720$ and hence

$$5x - y = 360. \quad (2)$$

By adding equations (1) and (2) we deduce that $9x = 900$. Therefore $x = 100$.

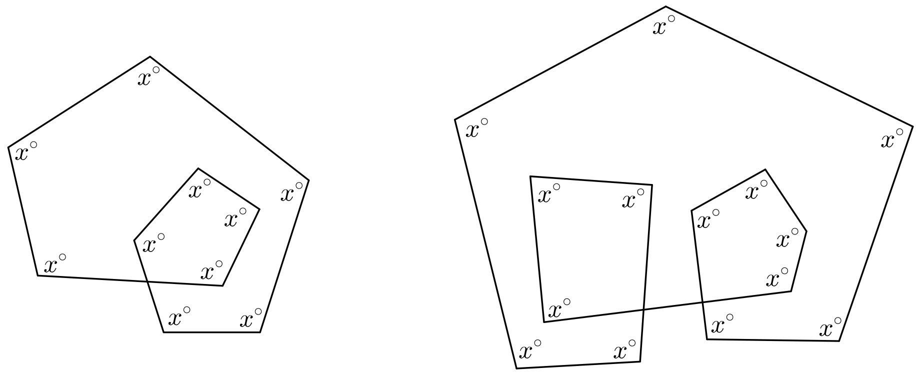
For investigation

14.1 In Method 2 we have used the fact that the sum of the angles of a polygon with n edges is $(n - 2) \times 180°$, in the case $n = 5$ for the pentagon and $n = 6$ for the hexagon.

Prove that this formula is correct.

14.2 In Method 2 why is the reflex angle at Y of the hexagon $SYWVUT$ equal to $360° - y°$?

14.3



(a) In the figure on the left above there are 10 marked angles, all equal to $x°$. What is the value of x?

(b) In the figure on the right above there are 14 marked angles, all equal to $x°$. What is the value of x?

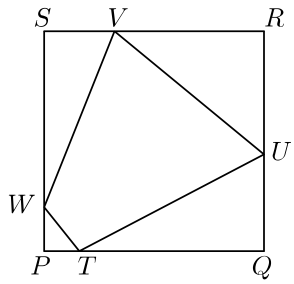
14.4 Generalize the results of Question 14 and Problem 14.3.

15. The diagram shows a square $PQRS$. Points T, U, V and W lie on the edges of the square, as shown, such that $PT = 1$, $QU = 2$, $RV = 3$ and $SW = 4$.

The area of $TUVW$ is half that of $PQRS$.

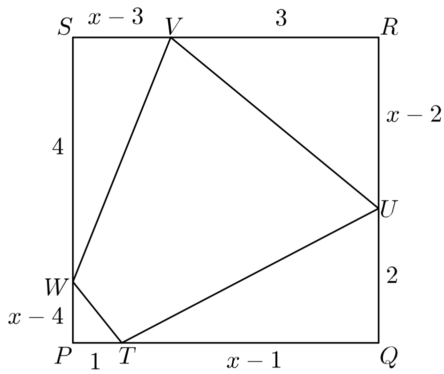
What is the length of PQ?

A 5 B 6 C 7 D 8 E 9



Solution **B**

We let the side length of the square $PQRS$ be x. Then the lengths of TQ, UR, VS and WP are $x-1$, $x-2$, $x-3$ and $x-4$, respectively.



The area of the square $PQRS$ is x^2. The area of $TUVW$ is a half of this. It follows that the sum of the areas of the triangles PTW, TQU, URV and VSW is also a half of the area of the square, and hence this sum equals $\frac{1}{2}x^2$.

The area of a triangle is half the product of its base and its height.

We therefore have

$$\tfrac{1}{2}(1 \times (x-4)) + \tfrac{1}{2}(2 \times (x-1)) + \tfrac{1}{2}(3 \times (x-2)) + \tfrac{1}{2}(4 \times (x-3)) = \tfrac{1}{2}x^2.$$

This equation simplifies to give

$$10x - 24 = x^2,$$

and therefore

$$x^2 - 10x + 24 = 0.$$

The left-hand side of the last equation factorizes to give

$$(x-4)(x-6) = 0.$$

Hence $x = 6$ or $x = 4$.

For there to be four triangles, as shown in the diagram, $x > 4$. We therefore deduce that $x = 6$.

So the length of PQ is 6.

For investigation

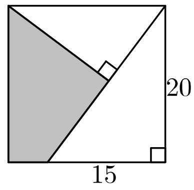
15.1 Suppose that, as in the question $PT = 1$, $QU = 2$, $RV = 3$ and $SW = 4$, but the area of $TUVW$ is two-thirds of the area of $PQRS$.

What is the length of PQ in this case?

16. The diagram shows two right-angled triangles inside a square. The perpendicular edges of the larger triangle have lengths 15 and 20.

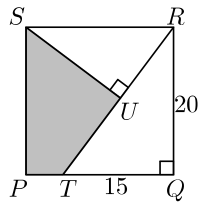
What is the area of the shaded quadrilateral?

A 142 B 146 C 150 D 154 E 158



Solution **D**

We let the vertices of the square be P, Q, R and S, and the points T and U be as shown.



We note first that, by Pythagoras' Theorem, applied to the right-angled triangle TQR,

$$RT^2 = 15^2 + 20^2 = 225 + 400 = 625 = 25^2.$$

It follows that $RT = 25$.

Because SR is parallel to PQ, the alternate angles, $\angle SRU$ and $\angle QTU$ are equal. Therefore, the right-angled triangles SUR and RQT are similar. Therefore their corresponding sides are in proportion. Hence

$$\frac{SR}{RT} = \frac{SU}{RQ} = \frac{RU}{TQ}.$$

Now $SR = RQ = 20$. Hence

$$\frac{20}{25} = \frac{SU}{20} = \frac{RU}{15}.$$

It follows that $SU = 16$ and $RU = 12$.

The area of the shaded quadrilateral is the area of the square $PQRS$ less the areas of the triangles SUR and RQT. It follows that the area of the shaded quadrilateral is

$$20^2 - \tfrac{1}{2}(16 \times 12) - \tfrac{1}{2}(20 \times 15) = 400 - 96 - 150 = 154.$$

Note

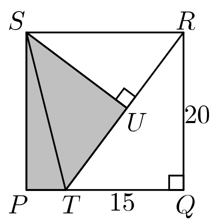
Another way to find the length of RT is to note that the lengths of TQ and QR are in the ratio $3 : 4$, and that therefore the right-angled triangle TQR is a $(3, 4, 5)$ triangle scaled by the factor 5. Hence the length of RT is $5 \times 5 = 25$.

Then, because the triangle RUS is similar to triangle TQR and has a hypotenuse of length 20, it follows that RUS is a $(3, 4, 5)$ triangle scaled by the factor 4. Hence RU has length $4 \times 3 = 12$ and SU has length $4 \times 4 = 16$.

For investigation

16.1 Calculate the area of the triangle SPT and the area of the triangle TUS.

Check that the sum of these areas is 154.



17. Amy, Beth and Claire each has some sweets. Amy gives one third of her sweets to Beth. Beth gives one third of all the sweets she now has to Claire. Then Claire gives one third of all the sweets she now has to Amy. All the girls end up having the same number of sweets.

Claire begins with 40 sweets.

How many sweets does Beth have originally?

A 20 B 30 C 40 D 50 E 60

Solution **D**

We work backwards from the final situation when Amy, Beth and Claire end up with the same number of sweets. We let s be the number of sweets they all end up with.

Claire ends up with s sweets. Therefore, before she gave one third of her sweets to Amy, Claire had t sweets, where $\frac{2}{3}t = s$. This gives $t = \frac{3}{2}s$. We deduce that, before giving one-third of her sweets to Amy, Claire has $\frac{3}{2}s$ sweets and gives one third of these, namely $\frac{1}{2}s$ sweets, to Amy. Hence, before she received these sweets, Amy had $\frac{1}{2}s$ sweets.

Similarly, Beth ends up with s sweets after she has given one third of her sweets to Claire. Hence, before this she had $\frac{3}{2}s$ sweets. As Claire has $\frac{3}{2}s$ sweets after receiving $\frac{1}{2}s$ from Beth, she had s sweets before this.

It follows that after Amy has given one third of her sweets to Beth, Amy has $\frac{1}{2}s$ sweets, Beth has $\frac{3}{2}s$ sweets, and Claire has s sweets.

Therefore, before Amy gives one third of her sweets to Beth, Amy has $\frac{3}{4}s$ sweets. She gives $\frac{1}{4}s$ sweets to Beth. Hence, before receiving these, Beth had $\frac{3}{2}s - \frac{1}{4}s = \frac{5}{4}s$ sweets.

We can sum this up by the following table.

Stage	Amy	Beth	Claire
Final distribution of sweets, after Claire gives one third of her sweets ($\frac{1}{2}s$ sweets) to Amy.	s	s	s
Distribution of sweets after Beth gives one third of her sweets ($\frac{1}{2}s$ sweets) to Claire.	$\frac{1}{2}s$	s	$\frac{3}{2}s$
Distribution of sweets after Amy gives one third her sweets ($\frac{1}{4}s$ sweets) to Beth.	$\frac{1}{2}s$	$\frac{3}{2}s$	s
Initial distribution of sweets.	$\frac{3}{4}s$	$\frac{5}{4}s$	s

We are told that Claire begins with 40 sweets. Therefore $s = 40$. So Beth begins with $\frac{5}{4}(40) = 50$ sweets.

For investigation

17.1 An alternative method is to suppose that, say, Amy begins with a sweets and Beth with b sweets, and then to work out how many sweets they all end up with. Use this method to answer Question 17. [Actually, to avoid fractions, it is better to assume Amy and Beth begin with $27a$ and $9b$ sweets, respectively.]

18. The arithmetic mean, A, of any two positive numbers x and y is defined to be $A = \frac{1}{2}(x+y)$ and their geometric mean, G, is defined to be $G = \sqrt{xy}$.

For two particular values x and y, with $x > y$, the ratio $A : G = 5 : 4$.

For these values of x and y, what is the ratio $x : y$?

A 5 : 4 B 2 : 1 C 5 : 2 D 7 : 2 E 4 : 1

SOLUTION **E**

We are told that

$$\frac{\frac{1}{2}(x+y)}{\sqrt{xy}} = \frac{5}{4}.$$

Therefore

$$2(x+y) = 5\sqrt{xy}.$$

By squaring both sides of this equation we deduce that

$$4(x+y)^2 = 25xy.$$

By expanding the left-hand side of this last equation, we have

$$4(x^2 + 2xy + y^2) = 25xy,$$

or, equivalently,

$$4x^2 + 8xy + 4y^2 = 25xy.$$

Hence

$$4x^2 - 17xy + 4y^2 = 0.$$

The left-hand of this last equation factorizes to give

$$(4x - y)(x - 4y) = 0.$$

Hence

$$4x = y \text{ or } x = 4y.$$

It follows that, because $x > y$,

$$x = 4y.$$

Hence

$$x : y = 4 : 1.$$

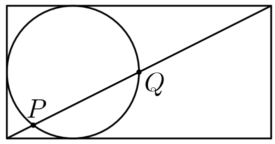
FOR INVESTIGATION

18.1 Suppose that x and y are positive integers, with $x > y$ and $A : G = 5 : 3$.

What is the ratio $x : y$ in this case?

18.2 Do there exist positive integers x and y such that $A < G$, that is, such that $\frac{1}{2}(x+y) < \sqrt{xy}$?

19. The diagram shows a circle of radius 1 touching three sides of a 2×4 rectangle. A diagonal of the rectangle intersects the circle at P and Q, as shown.

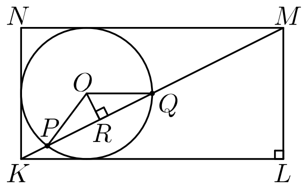


What is the length of the chord PQ?

A $\sqrt{5}$ B $\frac{4}{\sqrt{5}}$ C $\sqrt{5} - \frac{2}{\sqrt{5}}$ D $\frac{5\sqrt{5}}{6}$ E 2

Solution **B**

We let the the vertices of the rectangle be K, L, M, and N as shown in the figure. We let O be the centre of the circle, and R be the point where the perpendicular from O to the chord PQ meets the chord.



We note first that, by Pythagoras' Theorem applied to the right-angled triangle KLM, $KM^2 = KL^2 + LM^2 = 4^2 + 2^2 = 20$.

Therefore $KM = \sqrt{20} = 2\sqrt{5}$.

The radius OQ is parallel to KL. (You are asked to show this in Problem 19.1.) Therefore the alternate angles $\angle OQR$ and$\angle MKL$ are equal. It follows that the right-angled triangles OQR and MKL are similar. In particular,

$$\frac{RQ}{OQ} = \frac{KL}{KM}.$$

It follows that

$$RQ = \frac{KL}{KM} \times OQ = \frac{4}{2\sqrt{5}} \times 1 = \frac{2}{\sqrt{5}}.$$

The right-angled triangles ORQ and ORP are congruent as they share the side OR, and their hypotenuses OQ and OP are equal because they are radii of the same circle. It follows that $PR = RQ = \frac{2}{\sqrt{5}}$. We therefore conclude that

$$PQ = PR + RQ = \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \frac{4}{\sqrt{5}}.$$

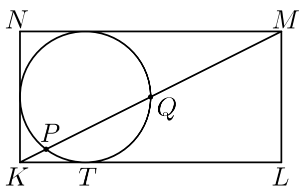
For investigation

19.1 Explain why OQ is parallel to KL.

19.2 Let T be the point where the circle touches the edge KL.

Question 19 may also be solved by making use of the theorem (see Problem 19.3) which tells us that $KT^2 = KP \times KQ$.

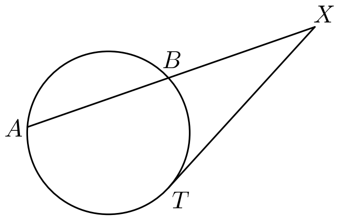
Show how this equation may be used to find the length of the chord PQ.



19.3 The theorem referred to in Problem 19.2 is the theorem which says that

If a chord AB of a circle and the tangent at the point T meet at a point X outside the circle, then

$$AX \times BX = TX^2.$$



Find a proof of this theorem.

You could find a proof for yourself, ask your teacher or look for a proof in a book or on the web.

You will find a discussion of this theorem on page 97 of the geometry book *Crossing the Bridge* by Gerry Leversha, published by UKMT.

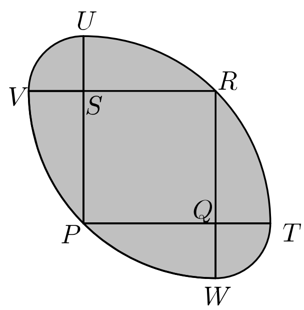
(Go to: http://shop.ukmt.org.uk/ukmt-books/)

20. The diagram shows a square $PQRS$ with edges of length 1, and four arcs, each of which is a quarter of a circle. Arc TRU has centre P; arc VPW has centre R; arc UV has centre S; and arc WT has centre Q.

What is the length of the perimeter of the shaded region?

A 6 B $(2\sqrt{2} - 1)\pi$ C $(\sqrt{2} - \frac{1}{2})\pi$

D 2π E $(3\sqrt{2} - 2)\pi$



SOLUTION **B**

The square $PQRS$ has side length 1, and hence its diagonal PR has length $\sqrt{2}$. So the arc TRU with centre P has radius $\sqrt{2}$. The arc is a quarter of a circle. Hence its length is $\frac{1}{4} \times 2\pi\sqrt{2}$, that is $\frac{1}{2}\sqrt{2}\pi$.

Similarly, the length of the arc VPW is $\frac{1}{2}\sqrt{2}\pi$.

The radius of the arc UV is equal to the length of SU. Because PU has length $\sqrt{2}$ and PS has length 1, the length of SU is $\sqrt{2} - 1$. The arc UV is one quarter of a circle with this radius. Hence the length of this arc is $\frac{1}{4} \times 2\pi(\sqrt{2} - 1)$, that is $\frac{1}{2}(\sqrt{2} - 1)\pi$.

Similarly, the arc WT has length $\frac{1}{2}(\sqrt{2} - 1)\pi$.

Therefore the total length of the perimeter of the shaded region is given by

$$\tfrac{1}{2}\sqrt{2}\pi + \tfrac{1}{2}\sqrt{2}\pi + \tfrac{1}{2}(\sqrt{2} - 1)\pi + \tfrac{1}{2}(\sqrt{2} - 1)\pi = (2\sqrt{2} - 1)\pi.$$

FOR INVESTIGATION

20.1 What is the area of the shaded region?

21. How many pairs (x, y) of positive integers satisfy the equation $4^x = y^2 + 15$?

A 0 B 1 C 2 D 4
E an infinite number

SOLUTION **C**

The equation $4^x = y^2 + 15$ may be rearranged as $4^x - y^2 = 15$. Now $4^x = (2^2)^x = (2^x)^2$. Hence $4^x - y^2$ may be factorized, using the standard factorization of the difference of two squares. This enables us to rewrite the equation as

$$(2^x - y)(2^x + y) = 15.$$

It follows that, for (x, y) to be a pair of positive integers that are solutions of the original equation, $2^x - y$ and $2^x + y$ must be positive integers whose product is 15, and with $2^x - y < 2^x + y$.

The only possibilities are therefore that either

$$2^x - y = 1 \quad \text{and} \quad 2^x + y = 15,$$

or

$$2^x - y = 3 \quad \text{and} \quad 2^x + y = 5.$$

In the first case $2^x = 8$ and $y = 7$, giving $x = 3$ and $y = 7$.

In the second case $2^x = 4$ and $y = 1$, giving $x = 2$ and $y = 1$.

Therefore there are just two pairs of positive integers that satisfy the equation $4^x = y^2 + 15$, namely, $(3, 7)$ and $(2, 1)$.

FOR INVESTIGATION

21.1 (a) Check that if $2^x - y = 1$ and $2^x + y = 15$, then $2^x = 8$ and $y = 7$.

(b) Check that if $2^x - y = 3$ and $2^x + y = 5$, then $2^x = 4$ and $y = 1$.

21.2 How many pairs of positive integers (x, y) are there which satisfy the equation $4^x = y^2 + 31$?

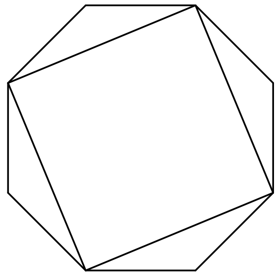
21.3 How many pairs of positive integers (x, y) are there which satisfy the equation $4^x = y^2 + 55$?

21.4 How many pairs of positive integers (x, y) are there which satisfy the equation $4^x = y^2 + 35$?

21.5 What can you say in general about those integers k for which there is at least one pair of positive integers (x, y) which satisfy the equation $4^x = y^2 + k$?

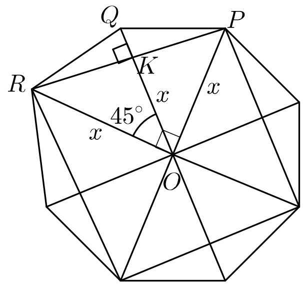
22. The diagram shows a regular octagon and a square formed by drawing four diagonals of the octagon. The edges of the square have length 1.

What is the area of the octagon?



A $\frac{\sqrt{6}}{2}$ B $\frac{4}{3}$ C $\frac{7}{5}$ D $\sqrt{2}$ E $\frac{3}{2}$

Solution **D**



Let O be the centre of the regular octagon, and let P, Q and R be adjacent vertices of the octagon as shown in the figure on the right. Let K be the point where OQ meets PR.

Let x be distance of O from the vertices of the octagon.

Since the edges of the square have length 1, $PR = 1$. By Pythagoras' Theorem applied to the right-angled triangle PRO, we have $x^2 + x^2 = 1^2$. Therefore $x^2 = \frac{1}{2}$ and hence $x = \frac{1}{\sqrt{2}}$.

The triangle ROQ has a base OQ of length x, that is $\frac{1}{\sqrt{2}}$, and height RK of length $\frac{1}{2}$. Therefore the area of the triangle ROQ is $\frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{2}$, which equals $\frac{1}{4\sqrt{2}}$.

The octagon is made up of 8 triangles each congruent to triangle ROQ.

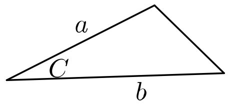
Therefore the area of the octagon is given by $8 \times \frac{1}{4\sqrt{2}} = \sqrt{2}$.

For investigation

22.1 The solution above assumes that $\angle POR = 90°$, $\angle RKO = 90°$ and $RK = \frac{1}{2}$. Explain why these statements are true.

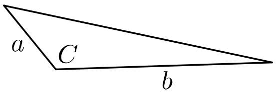
22.2 An alternative method for finding the area of the triangle ROQ is to use the "$\frac{1}{2}ab\sin C$" formula for the area of a triangle. Show how this also gives $\frac{1}{4\sqrt{2}}$ for the area of triangle ROQ.

22.3 (a) Show how the formula $\frac{1}{2}ab\sin C$ for the area of a triangle which has sides of lengths a and b, with included angle C, may be deduced from the fact that the area of a triangle is given by the formula



$$\text{area} = \tfrac{1}{2}(\text{base} \times \text{height}).$$

(b) Does your argument cover the case where C is an obtuse angle as well as the case where it is an acute angle?



23. The parabola with equation $y = x^2$ is reflected in the line with equation $y = x + 2$.

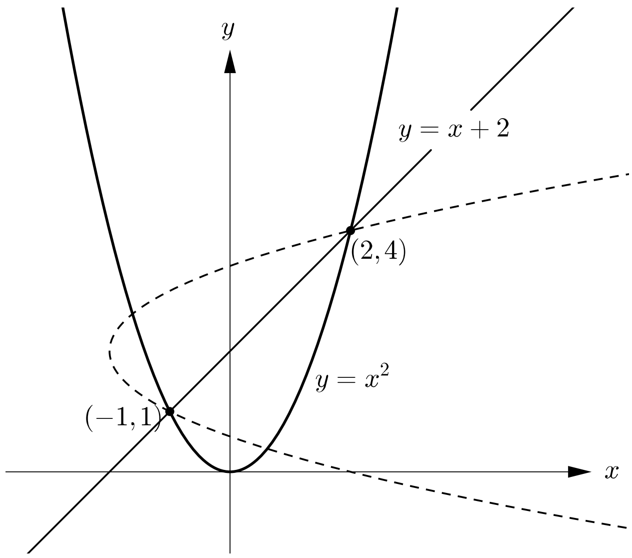
Which of the following is the equation of the reflected parabola?

A $x = y^2 + 4y + 2$ B $x = y^2 + 4y - 2$ C $x = y^2 - 4y + 2$
D $x = y^2 - 4y - 2$ E $x = y^2 + 2$

Solution **C**

Method 1

The parabola with equation $y = x^2$ meets the line with equation $y = x + 2$ where $x^2 = x + 2$. This last equation may be rearranged as $x^2 - x - 2 = 0$ and then factorized as $(x + 1)(x - 2) = 0$. So its solutions are -1 and 2. For $x = -1$, we have $y = 1$ and for $x = 2$ we have $y = 4$. Therefore the parabola meets the line at the points with coordinates $(-1, 1)$ and $(2, 4)$ as shown in the figure on the right.



These two points remain where they are when they are reflected in the line. Therefore the reflected parabola also goes through these two points.

We can now test each equation given as an option to see if it is the equation of a curve which goes through the points $(-1, 1)$ and $(2, 4)$.

For example, because $-1 \neq 1^2 + 4 \times 1 + 2$ the coordinates of the point $(-1, 1)$ do not satisfy the equation $x = y^2 + 4y + 2$. It follows that $x = y^2 + 4y + 2$ is not the equation of a curve which goes through $(-1, 1)$. Therefore it is not the equation of the reflected parabola.

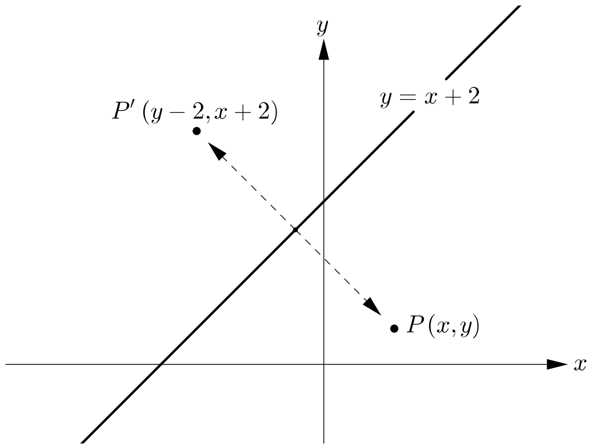
Because $-1 = 1^2 - 4 \times 1 + 2$ and $2 = 4^2 - 4 \times 4 + 2$, it follows that both the points $(-1, 1)$ and $(2, 4)$ lie on the curve with the equation $x = y^2 - 4y + 2$. It can be checked that these points do not lie on the curves given by any of the other equations.

Therefore, in the context of the SMC, we can conclude that the equation of the reflected parabola is $x = y^2 - 4y + 2$.

If we were not given the equations in the options we would need to calculate the equation of the reflected parabola. We adopt this approach in the second method.

Method 2

We leave it to the reader to check that when the point P with coordinates (x, y) is reflected in the line with equation $y = x + 2$, its image point P' has the coordinates $(y - 2, x + 2)$.



We put $x' = y - 2$ and $y' = x + 2$, so that $x = y' - 2$ and $y = x' + 2$

The point with coordinates (x', y') lies on the image under reflection of the parabola if, and only if, the point (x, y) lies on the parabola, that is, if, and only if, $y = x^2$. Therefore the condition that the point (x', y') lies on the image of the parabola is expressed by the equation $(x' + 2) = (y' - 2)^2$.

On expansion this equation may be written as $x' + 2 = y'^2 - 4y' + 4$. Rearranging this equation, and dropping the dashes, we deduce that the equation of the reflected parabola is

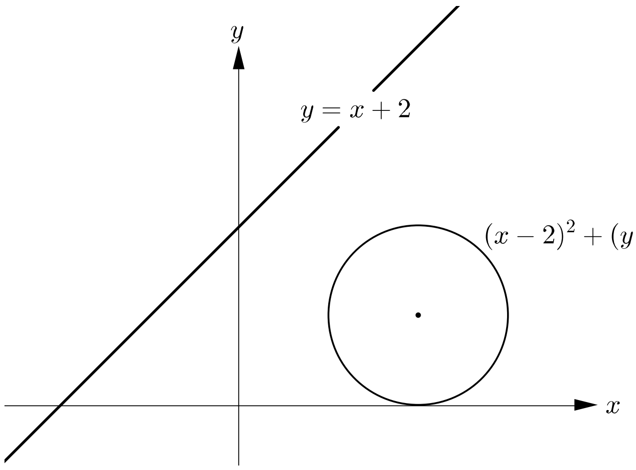
$$x = y^2 - 4y + 2.$$

For investigation

23.1 Check that neither of the points $(-1, 1)$ and $(2, 4)$ lies on any of the curves with equations $x = y^2 + 4y + 2$, $x = y^2 + 4y - 2$, $x = y^2 - 4y - 2$, and $x = y^2 + 2$.

23.2 Show that when the point with coordinates (x, y) is reflected in the line with equation $y = x + 2$, its image is the point with coordinates $(y - 2, x + 2)$.

23.3



Find the equation of the circle that is obtained when the circle with centre $(2, 1)$ and radius 1 is reflected in the line with the equation $y = x + 2$.

[Note that the circle with centre $(2, 1)$ and radius 1 has the equation $(x-2)^2 + (y-1)^2 = 1$.]

23.4 Find the coordinates of the point that is obtained when the point with coordinates (x, y) is reflected in the line with equation $y = mx + c$.

24. There is a set of straight lines in the plane such that each line intersects exactly ten others.

Which of the following could not be the number of lines in that set?

A 11 B 12 C 15 D 16 E 20

Solution **D**

It is convenient in this question to regard a line as being parallel to itself.

Suppose that there are l lines in the given set. Each line intersects all the lines in the set except those that are parallel to it. Therefore, as each line in the given set intersects 10 lines, each line in the set is parallel to $l - 10$ lines. We let $k = l - 10$.

Then $l = nk$, for some positive integer n. The nk lines in the set form n subsets each containing k parallel lines, and such that lines in different subsets are not parallel.

Each line intersects all but k of these nk lines. So each line intersects $nk - k$, that is, $(n-1)k$ lines.

We are given that $(n-1)k = 10$. Therefore $n - 1$ and k are positive integers whose product is 10. All the possible combinations of values of $n - 1$ and k with product 10 are shown in the table below. The third and fourth columns give the corresponding values of n and nk, the latter number being the total number of lines in the set.

$n-1$	k	n	nk
1	10	2	20
2	5	3	15
5	2	6	12
10	1	11	11

We see from this table that the only possibilities for the number of lines in a set of lines in which each line intersects 10 other lines are 11, 12, 15 and 20. In particular, 16 could not be the number of lines in the set.

For investigation

24.1 Consider a set of lines in the plane such that each line intersects exactly 30 others. List the possibilities for the number of lines in this set.

24.2 Consider a set of lines in the plane such that each line intersects m others. What is the smallest possible number of lines in the set?

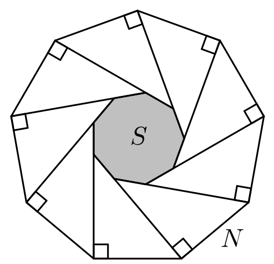
24.3 Consider a set of lines in the plane such that each line intersects m others. Show that the number of different possibilities for the number of lines in the set is equal to the number of different factors of m.

24.4 Consider a set of lines in the plane such that each line intersects 323 others. What is the largest possible number of lines in the set? [Note that $323 = 17 \times 19$].

24.5 Consider a set of lines in the plane such that each line intersects 360 others. What is the largest possible number of lines in the set?

24.6 Find a general method, in terms of m, for calculating the maximum possible number of lines in a set which is such that each line intersects m others.

25. The diagram shows a regular nonagon N. Moving clockwise around N, at each vertex a line segment is drawn perpendicular to the preceding edge. This produces a smaller nonagon S, shown shaded.



What fraction of the area of N is the area of S?

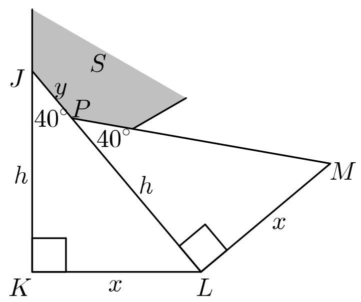
A $\dfrac{1-\cos 40°}{1+\cos 40°}$ B $\dfrac{\cos 40°}{1+\cos 40°}$ C $\dfrac{\sin 40°}{1+\sin 40°}$

D $\dfrac{1-\sin 40°}{1+\sin 40°}$ E $\dfrac{1}{9}$

SOLUTION **A**

We let the side length of the larger regular nonagon N be x and the side length of the smaller regular nonagon S be y. Because S and N are similar figures, their areas are in the ratio $y^2 : x^2$. Therefore the required fraction, giving the area of S in terms of the area of N, is $\dfrac{y^2}{x^2}$.

The area between the nonagons is divided into nine triangles. Two of these adjacent triangles JKL and PLM are shown in the figure on the right.



The triangles JKL and PLM have right angles at K and L, respectively. The sides KL and LM each have length x as they are sides of the regular nonagon N. The angles at J and P in these triangles are exterior angles of S and therefore they are each $\frac{1}{9} \times 360°$, that is, $40°$.

In the right-angled triangles JKL and PLM the angles are equal and $KL = LM$. Therefore the triangles are congruent. Hence $JK = PL$. We let h be the common length of JK and PL.

In the right-angled triangle JKL, the hypotenuse JL has length $y + h$. Therefore, applying Pythagoras' Theorem to this triangle, we have $x^2 + h^2 = (y + h)^2$. On expansion, this gives $x^2 + h^2 = y^2 + 2yh + h^2$, and hence $x^2 = y^2 + 2hy$. It follows that $\dfrac{x^2}{y^2} = \dfrac{y+2h}{y}$ and hence

$$\frac{y^2}{x^2} = \frac{y}{y+2h}. \quad (1)$$

From the triangle JKL we have $\dfrac{h}{y+h} = \cos 40°$. Hence $y + h = \dfrac{h}{\cos 40°}$ and therefore

$$y = \frac{h}{\cos 40°} - h \quad \text{and} \quad y + 2h = \frac{h}{\cos 40°} + h. \quad (2)$$

Substituting from (2) into the right-hand side of equation (1), we may now deduce that the area of S, as a fraction of the area of N, is

$$\frac{y^2}{x^2} = \frac{\dfrac{h}{\cos 40°} - h}{\dfrac{h}{\cos 40°} + h} = \frac{h - h\cos 40°}{h + h\cos 40°} = \frac{1-\cos 40°}{1+\cos 40°}.$$

United Kingdom Mathematics Trust

Senior Mathematical Challenge

Tuesday 6th November 2018

Organised by the United Kingdom Mathematics Trust

Supported by Institute and Faculty of Actuaries

Candidates must be full-time students at secondary school or FE college.
England & Wales: *Year* 13 *or below*
Scotland: *S*6 *or below*
Northern Ireland: *Year* 14 *or below*

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil.** Mark at most one of the options A, B, C, D, E on the Answer Sheet for each question. Do not mark more than one option.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:** all candidates start with 25 marks;
   0 marks are awarded for each question left unanswered;
   4 marks are awarded for each correct answer;
   1 mark is deducted for each incorrect answer (to discourage guessing).
7. Your Answer Sheet will be read by a machine. **Do not write or doodle on the sheet except to mark your chosen options.** The machine will read all black pencil markings even if they are in the wrong places. If you mark the sheet in the wrong place, or leave bits of eraser stuck to the page, the machine will interpret the mark in its own way.
8. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

Enquiries about the Senior Mathematical Challenge should be sent to:

*UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS*2 9*JT*

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1. When the following are evaluated, how many of the answers are odd numbers?

$$1^2,\ 2^3,\ 3^4,\ 4^5,\ 5^6$$

A 1 B 2 C 3 D 4 E 5

2. The positive integer 2018 is the product of two primes.

What is the sum of these two primes?

A 1001 B 1010 C 1011 D 1100 E 1101

3. Which of the following shows the digit 6 after it has been rotated clockwise through 135°?

A  B  C  D  E 

4. Which of the following is not a multiple of 5?

A $2019^2 - 2014^2$ B $2019^2 \times 10^2$ C $2020^2 \div 101^2$ D $2010^2 - 2005^2$ E $2015^2 \div 5^2$

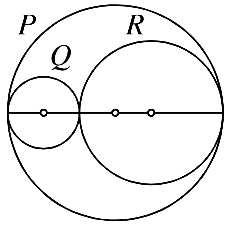
5. Which of the following numbers is the largest?

A $\frac{397}{101}$ B $\frac{487}{121}$ C $\frac{596}{153}$ D $\frac{678}{173}$ E $\frac{796}{203}$

6. Which of the following is equal to $25 \times 15 \times 9 \times 5.4 \times 3.24$?

A 3^9 B 3^{10} C 3^{11} D 3^{14} E 3^{17}

7. The circles P, Q and R are all tangent to each other. Their centres all lie on a diameter of P, as shown in the figure.



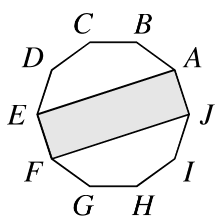
What is the value of $\dfrac{\text{circumference of } Q + \text{circumference of } R}{\text{circumference of } P}$?

A 1 B $\frac{1}{2}$ C $\frac{1}{3}$ D $\frac{1}{4}$ E more information needed

8. What are the last two digits of 7^{2018}?

A 07 B 49 C 43 D 01 E 18

9. The diagram shows a rectangle $AEFJ$ inside a regular decagon $ABCDEFGHIJ$.



What is the ratio of the area of the rectangle to the area of the decagon?

A 2 : 5 B 1 : 4 C 3 : 5 D 3 : 10 E 3 : 20

10. On a training ride, Laura averages speeds of 12 km/h for 5 minutes, then 15 km/h for 10 minutes and finally 18 km/h for 15 minutes.

What was her average speed over the whole ride?

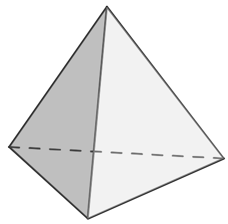
A 13 km/h B 14 km/h C 15 km/h D 16 km/h E 17 km/h

11. How many of the following four equations has a graph that does *not* pass through the origin?

$$y = x^4 + 1 \qquad y = x^4 + x \qquad y = x^4 + x^2 \qquad y = x^4 + x^3$$

A 0 B 1 C 2 D 3 E 4

12. A regular tetrahedron is a polyhedron with four faces, each of which is an equilateral triangle, as shown. A solid regular tetrahedron is cut into two pieces by a single plane cut.



Which of the following could *not* be the shape of the section formed by the cut?

A a pentagon B a square
C a rectangle that is not a square D a trapezium
E a triangle that is not equilateral

13. The lines $y = x$ and $y = mx - 4$ intersect at the point P.

What is the sum of the positive integer values of m for which the coordinates of P are also positive integers?

A 3 B 5 C 7 D 8 E 10

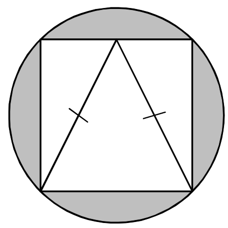
14. The following twelve integers are written in ascending order:

$$1, x, x, x, y, y, y, y, y, 8, 9, 11.$$

The mean of these twelve integers is 7. What is the median?

A 6 B 7 C 7.5 D 8 E 9

15. A square is inscribed in a circle of radius 1. An isosceles triangle is inscribed in the square as shown.



What is the ratio of the area of this triangle to the area of the shaded region?

A $\pi : \sqrt{2}$ B $\pi : 1$ C $1 : 4$ D $1 : \pi - 2$ E $2 : \pi$

16. The numbers p, q, r and s satisfy the following equations:

$$p + 2q + 3r + 4s = k \qquad 4p = 3q = 2r = s.$$

What is the smallest value of k for which p, q, r and s are all positive integers?

A 20 B 24 C 25 D 77 E 154

17. Bethany has 11 pound coins and some 20p coins and some 50p coins in her purse. The mean value of the coins is 52 pence.

Which could not be the number of coins in the purse?

A 35 B 40 C 50 D 65 E 95

18. P, Q and R are the three angles of a triangle, when each has been rounded to the nearest degree.

Which of the following is the complete list of possible values of $P + Q + R$?

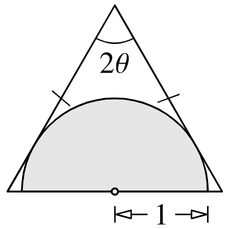
A 179°, 180° or 181° B 180°, 181° or 182° C 178°, 179° or 180° D 180°
E 178°, 179°, 180°, 181° or 182°

19. How many pairs of numbers (m, n) are there such that the following statement is true?

'A regular m-sided polygon has an exterior angle of size $n°$ and
a regular n-sided polygon has an exterior angle of size $m°$.'

A 24 B 22 C 20 D 18 E 16

20. The diagram shows a semicircle of radius 1 inside an isosceles triangle. The diameter of the semicircle lies along the 'base' of the triangle, and the angle of the triangle opposite the 'base' is equal to 2θ. Each of the two equal sides of the triangle is tangent to the semicircle.



What is the area of the triangle?

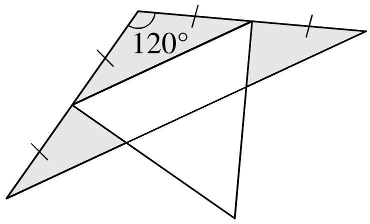
A $\frac{1}{2}\tan 2\theta$ B $\sin\theta\cos\theta$ C $\sin\theta + \cos\theta$ D $\frac{1}{2}\cos 2\theta$
E $\dfrac{1}{\sin\theta\cos\theta}$

21. The graph of $y = \dfrac{1}{x}$ is reflected in the line $y = 1$. The resulting image is reflected in the line $y = -x$.

What is the equation of the final graph?

A $y = \dfrac{-1}{(x+2)}$ B $y = \dfrac{1}{(x-1)}$ C $y = \dfrac{1}{(x-2)}$ D $y = \dfrac{-1}{(x-1)}$ E $y = \dfrac{-1}{(x-2)}$

22. The diagram shows two overlapping triangles; an isosceles triangle with an angle of 120° and an equilateral triangle with area 36. Two of the vertices of the equilateral triangle are midpoints of the equal sides of the isosceles triangle.



What is the total area of the shaded regions (inside the isosceles triangle but outside the equilateral triangle)?

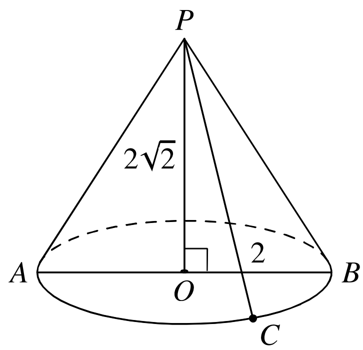
A 24 B 26 C 28 D 30 E 32

23. For particular real numbers a and b, the function f is defined by $f(x) = ax + b$, and is such that $f\big(f\big(f(x)\big)\big) = 27x - 52$.

Which of the following formulas defines the function g such that, for all values of x, $g\big(f(x)\big) = x$?

A $\frac{1}{3}x - 4$ B $\frac{1}{3}x + \frac{4}{3}$ C $4x - 3$ D $\frac{1}{3}x - \frac{4}{3}$ E $3x - 4$

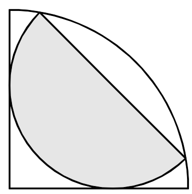
24. The diagram shows a circle with centre O which lies in a horizontal plane. The diameter AB has length 4. Point P lies vertically above O and $PO = 2\sqrt{2}$. Point C lies on the semicircular arc AB such that the ratio of the lengths of the arcs AC and CB is 2 : 1.



What is the shortest distance from A to PC?

A $\sqrt{2}$ B $\sqrt{3}$ C 2 D $2\sqrt{2}$ E 3

25. A semicircle is inscribed in a quarter circle as shown.



What fraction of the quarter circle is shaded?

A $\frac{1}{3}$ B $\frac{1}{\sqrt{3}}$ C $\frac{2}{3}$ D $\frac{\sqrt{3}}{2}$ E $\frac{1}{\sqrt{2}}$

United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Tuesday 6th November 2018

For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:

www.ukmt.org.uk

1. **C** The values of the expressions are 1, 8, 81, 1024 and 15 625 respectively, so there are three odd numbers. Alternatively, note also that positive integer powers of odd and even numbers are themselves odd and even respectively.

2. **C** As 2018 is the product of the primes 2 and 1009, the sum of these primes is 1011.

3. **D** Option D shows the image after the rotation of 135° clockwise.

4. **E** Option A can be expressed using the difference of two squares as $2019^2 - 2014^2 = (2019 + 2014)(2019 - 2014) = 4033 \times 5$, so this answer is a multiple of 5.

Option B is $2019^2 \times 10^2$ which is $2019^2 \times 10 \times 2 \times 5$, which again is a multiple of 5.

Option C is $\frac{2020^2}{101^2}$ which equals $\left(\frac{2020}{101}\right)^2$ and this simplifies to 20^2 which is 80×5, so the answer is a multiple of 5.

Option D is $2010^2 - 2005^2$ which equals $(2010 + 2005)(2010 - 2005)$, so 4015×5 which again is a multiple of 5.

However, option E is $\frac{2015^2}{5^2}$ which equals $\left(\frac{2015}{5}\right)^2$ and this is 403^2. As the final digit of 403^2 is 9, this is not a multiple of 5.

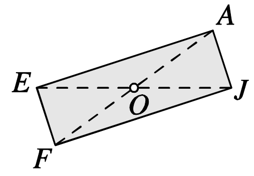
5. **B** Only option B is greater than four, so this is the largest.

6. **B** The expression $25 \times 15 \times 9 \times 5.4 \times 3.24 = 5^2 \times 3 \times 5 \times 3^2 \times \frac{54}{10} \times \frac{324}{100}$ which factorises further to $5^2 \times 3 \times 5 \times 3^2 \times \frac{2\times3^3}{2\times5} \times \frac{2^2\times3^4}{2^2\times5^2}$. All the factors of 2 and 5 in the numerator and denominator cancel to leave only the product of powers of 3. This is $3^1 \times 3^2 \times 3^3 \times 3^4$, so the expression is equal to 3^{10}.

7. **A** Let the radius of the circle Q be q and the radius of the circle R be r. The radius of the largest circle, P, can therefore be written as $(q + r)$. Using 'circumference = $2\pi \times$ radius', the required expression becomes $\dfrac{2\pi q + 2\pi r}{2\pi(q + r)}$ which equals $\dfrac{2\pi(q + r)}{2\pi(q + r)}$ and so has value 1.

8. **B** The first four terms of the geometric sequence 7^n are 7, 49, 343 and 2401. Evaluating 7^5 by considering 2401×7 shows that the tens and units digits are '07' as in the first term. Hence a cyclical pattern for the tens and units digits is formed using '07', '49', '43' and '01'. As $2018 = 504 \times 4 + 2$, then 7^{2018} must end in the second value of the cycle, so '49'.

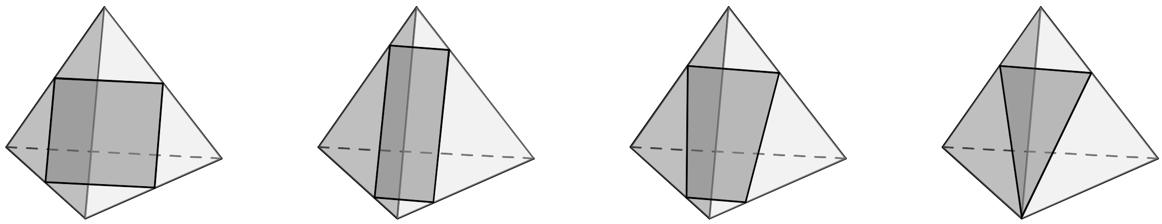
9. A Let O be the centre of the regular decagon. The decagon can be split into ten congruent isosceles triangles each with a vertex at O. Triangles AOJ and EOF are two of these ten isosceles triangles. As O is also the centre of the rectangle and the diagonals of a rectangle split its interior into four equal areas, the triangles EOA and FOJ each have the same area as triangle AOJ. The required ratio is then 4 : 10 which simplifies to 2 : 5.



10. D In order to calculate Laura's average speed over the whole journey we can first calculate the total distance travelled and the total time taken. Using "distance in km = speed in km/h × time in h', the total distance travelled, in km, is $12 \times \frac{5}{60} + 15 \times \frac{10}{60} + 18 \times \frac{15}{60}$ which is $1 + 2.5 + 4.5$, so 8. The total time taken is 5 + 10 + 15 minutes, so $\frac{1}{2}$ hour. Laura's average speed is then $\frac{8}{1/2} = 16\,\text{km/h}$.

11. B For each equation, consider $x = 0$. Then, $y = x^4 + 1$ becomes $y = 0^4 + 1 = 1$ and so the graph with this equation passes through $(0, 1)$ rather than $(0, 0)$. For the other three equations, using $x = 0$ shows that $y = 0$ too and so their graphs all pass through the origin. Hence only one graph does not pass through the origin.

12. A When the regular tetrahedron is cut by a single plane cut, each of its four faces is cut at most once. The line along which each face is cut becomes an edge of the newly formed section. Therefore the pentagon, with five edges, cannot be formed. Each of the other four options is possible as the following examples show.

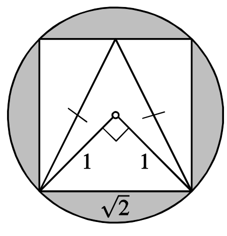


A square   *A rectangle that is not a square*   *A trapezium*   *A triangle that is not equilateral*

13. E Since $y = x$ and $y = mx - 4$, it follows that $x = mx - 4$ and so $4 = mx - x = (m - 1)x$; therefore $x = \frac{4}{m-1}$. In order for x to be a positive integer, $m - 1$ must be a positive factor of 4 therefore $m - 1$ can equal 1, 2 or 4 and so m can equal 2, 3 or 5. The sum of these values of m is then 10.

14. D The mean of the twelve integers is given to be 7. Therefore $1 + 3x + 5y + 8 + 9 + 11 = 12 \times 7$ so $3x + 5y = 55$. As the twelve integers are written in ascending order, the median value is half way between the 6th and 7th values, each of which is y, so the median is y. Rearranging our equation gives $y = \dfrac{55 - 3x}{5}$ and hence $3x$ must be a multiple of 5 so that the numerator can be divided exactly by 5. To ensure that the twelve integers are in ascending order the minimum value of x is 1 and the maximum value of x is 8. The only value in this interval which makes $3x$ a multiple of 5 is $x = 5$, in which case $y = \dfrac{55 - 15}{5} = 8$.

15. D The circle has radius 1, so, using Pythagoras' Theorem, the length of the side of the square is $\sqrt{2}$. The area of the shaded region is then $\pi \times 1^2 - \sqrt{2}^2$ which equals $\pi - 2$. The base of the original isosceles triangle is also one of the sides of the square and so has length $\sqrt{2}$. Its perpendicular height is also $\sqrt{2}$ as it is parallel to two of the other sides of the square. The area of the triangle is then $\frac{1}{2} \times \sqrt{2} \times \sqrt{2}$ which is 1. The required ratio is then $1 : \pi - 2$.



16. D Considering the first set of equations $4p = 3q = 2r = s$, we can see that $p < q < r < s$. Also, s must be a multiple of 2, 3 and 4. The smallest such multiple is 12 which gives $p = 3$, $q = 4$, $r = 6$ and $s = 12$. In the equation $p + 2q + 3r + 4s = k$ we then have $3 + 2 \times 4 + 3 \times 6 + 4 \times 12 = k$, so $k = 77$.

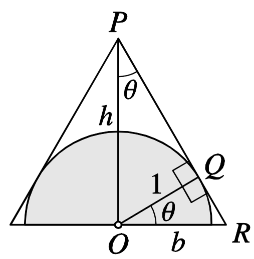
17. B Let the number of 20p coins and 50p coins Bethany has be x and y respectively. Her total number of coins is then $11 + x + y$. As the mean value of her coins is 52p, the total value of her coins is both $52 \times (11 + x + y)$ and $100 \times 11 + 20x + 50y$. The equation formed is $52(11 + x + y) = 1100 + 20x + 50y$ which becomes $572 + 52x + 52y = 1100 + 20x + 50y$ and this simplifies to $32x + 2y = 528$ or $16x + y = 264$. By rearranging we can see that $11 + x + y = 275 - 15x$. So Bethany's total number of coins is of the form $275 - 15x$, or $5 + 15(18 - x)$ which gives values that are 5 more than multiples of 15. Of the given options the only one which is not 5 more than a multiple of 15 is 40. We can also see that Bethany must have a large purse!

18. A Let the three angles P, Q and R be written in the form '$p.a$', '$q.b$' and '$r.c$' respectively, where p, q, and r are positive integers and a, b and c each represent the full decimal part of the angle. The sum of the decimal parts of the angles falls into one of three cases: (i) $a + b + c = 0$, so $p + q + r = 180$; (ii) $a + b + c = 1$, so $p + q + r = 179$; (iii) $a + b + c = 2$, so $p + q + r = 178$.

In case (i) no rounding is necessary as all three angles are integers so $P + Q + R = 180$. In case (ii), none (e.g. 'p.4', 'q.4', 'r.2'), one (e.g. 'p.5', 'q.4', 'r.1') or two (e.g. 'p.5', 'q.5', 'r.0') could be large enough for the respective integer parts to 'round up'. So $P + Q + R$ could be $179 + 0$, $179 + 1$ or $179 + 2$. In case (iii) either two (e.g. 'p.9', 'q.9', 'r.2') or three (e.g. 'p.7', 'q.7', 'r.6') of a, b and c could be large enough for the respective integer parts to 'round up'. So the total of $P + Q + R$ could be $178 + 2$ or $178 + 3$. The complete list of possible values of $P + Q + R$ is then 179, 180, 181.

19. C If the exterior angle of an m-sided polygon is $n°$ then its number of sides, m, is $\frac{360}{n}$. Similarly if the exterior angle of an n-sided polygon is $m°$, then its number of sides, n, is $\frac{360}{m}$. Both n and m must be positive integers greater than two (or no polygon can be formed) and $n \times m$ must be 360. The value of m can therefore be any of the twenty numbers 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90 or 120 and the corresponding value of n will be $\frac{360}{m}$.

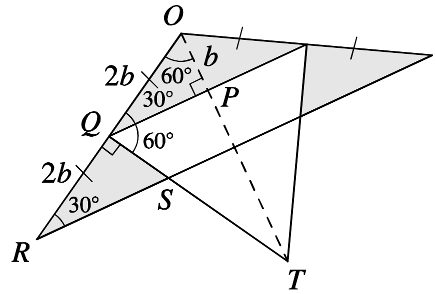
20. E Let O be the centre of the circle, P be the vertex at the top of the triangle, R the vertex at one end of the 'base' and Q be the point where PR is tangent to the semicircle, as shown. Let OP have length h and OR have length b. The line OQ is a radius of the semicircle so has length 1 and is perpendicular to PR.



As OP and OR are perpendicular, triangles ROQ and OPQ are similar and so angle ROQ and angle OPQ are equal. Triangles ROQ and OPQ can be used to find expressions for the lengths of the base and height of the original isosceles triangle. Considering triangle ROQ gives $\cos\theta = \frac{1}{b}$ so $b = \frac{1}{\cos\theta}$. Considering the triangle OPQ gives $\sin\theta = \frac{1}{h}$ so $h = \frac{1}{\sin\theta}$. Then the area of the isosceles triangle is $\frac{1}{2} \times 2b \times h = \frac{1}{\cos\theta} \times \frac{1}{\sin\theta} = \frac{1}{\sin\theta\cos\theta}$.

21. A After reflecting the graph of $y = \frac{1}{x}$ in the line $y = 1$ we have a graph whose equation is $y = 1 + (1 - \frac{1}{x})$, so $y = -\frac{1}{x} + 2$. The second reflection is in the line $y = -x$. Here, y is replaced by $-x$ and $-x$ is replaced by y. Hence the equation $y = -\frac{1}{x} + 2$ becomes $-x = -\frac{1}{-y} + 2$ which rearranges to $-(x + 2) = \frac{1}{y}$ and then to $y = \dfrac{-1}{(x + 2)}$.

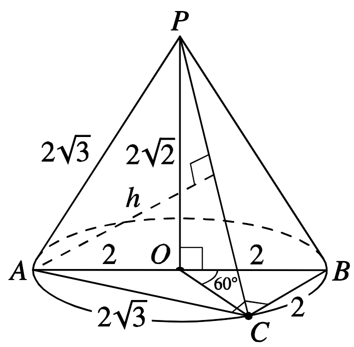
22. C As two vertices of the equilateral triangle are given to be midpoints of the equal sides of the isosceles triangle, OT is a line of symmetry. Let P be the point where OT intersects an edge of the equilateral triangle, as shown. By using the line of symmetry through O, P and T, we can see the shaded region as four grey triangles. As we are given an angle of 120° at O, and angles of an equilateral triangle are each 60°, the four grey triangles each have angles 30°, 60°, 90° and sides whose lengths are in the ratios $1 : \sqrt{3} : 2$.



Let $OP = b$. Then $OQ = QR = 2b$. Also $QP = \sqrt{3}b$ and $QS = \frac{2b}{\sqrt{3}}$ using similar triangles. The total shaded area is then $2(\frac{1}{2} \times b \times \sqrt{3}b + \frac{1}{2} \times \frac{2b}{\sqrt{3}} \times 2b) = \sqrt{3}b^2 + \frac{4b^2}{\sqrt{3}} = \frac{7b^2}{\sqrt{3}}$. Triangle QPT also has sides in the ratios $1 : \sqrt{3} : 2$ and as $QP = \sqrt{3}b$, $PT = 3b$. The area of the equilateral triangle is then $\frac{1}{2} \times 2\sqrt{3}b \times 3b = 3\sqrt{3}b^2$ which we are told is 36. So $3\sqrt{3}b^2 = 36$ and therefore $\frac{b^2}{\sqrt{3}} = 4$. Hence the area of the shaded region is $\frac{7b^2}{\sqrt{3}} = 7 \times 4 = 28$.

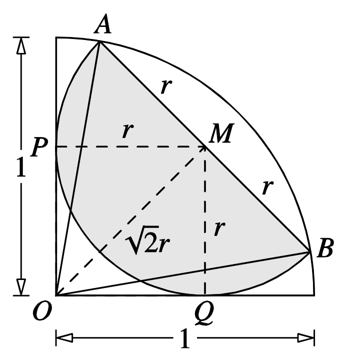
23. B As $f(x) = ax + b$, $f(f(f(x))) = a(a(ax + b) + b) + b$ which expands to give $a^3x + a^2b + ab + b$ and this can be written as $a^3x + b(a^2 + a + 1)$. We therefore have $a^3x + b(a^2 + a + 1) = 27x - 52$. Equating coefficients of x gives $a^3 = 27$, so $a = 3$. Equating the constants gives $b(3^2 + 3 + 1) = -52$, so $b = -4$. So the function f is given by $f(x) = 3x - 4$. As we are given that $g(f(x)) = x$, the function g uses inverse operations to undo the two steps in the function f, in order to return to x. As the operations in f are 'multiply by 3' then 'subtract 4', the operations in g are 'add 4' then 'divide by 3'. Therefore the function g is given by $g(x) = \frac{x+4}{3} = \frac{1}{3}x + \frac{4}{3}$.

24. E As points A and C are both on the circular base with centre O, and P is directly above O, we have $PA = PC$. Applying Pythagoras' Theorem to triangle OPA gives $PA^2 = 2^2 + (2\sqrt{2})^2 = 12$, so $PA = 2\sqrt{3} = PC$. Now we consider the circular base. Since the arcs AC and CB are in the ratio $2 : 1$, then $\angle AOC : \angle COB$ is also $2 : 1$, giving $\angle COB = 60°$. Triangle BOC is therefore equilateral, as $OB = OC$, giving us that $BC = 2$. Triangle ACB is right-angled, as angles in a semicircle are 90°. Applying Pythagoras' Theorem to triangle ACB gives $AC^2 + 2^2 = 4^2$, so $AC = 2\sqrt{3}$.



As $PA = PC = AC = 2\sqrt{3}$, triangle PAC is now shown to be equilateral. The shortest distance h from A to line PC is therefore the line of symmetry of that equilateral triangle. So $h^2 + \sqrt{3}^2 = (2\sqrt{3})^2$ giving $h^2 = 9$ and therefore $h = 3$.

25. C Let O be the centre of the quarter circle, A and B be the ends of the diameter of the semicircle and M be the midpoint of AB. Let P and Q be the points on the straight edges of the quarter circle where the quarter circle is tangent to the semicircle. Let the radius of the quarter circle be 1, so $OA = 1$. Let the radius of the semicircle be r, so $MA = r$. As a tangent to a circle is perpendicular to its radius, $PM = r = QM$, and $OPMQ$ is a square. Using Pythagoras' Theorem on triangle OPM gives $OM = \sqrt{2}r$. Considering triangle OAM gives $(\sqrt{2}r)^2 + r^2 = 1^2$ so $3r^2 = 1$ and $r^2 = \frac{1}{3}$.



The area of the quarter circle is $\frac{1}{4} \times \pi \times 1^2 = \frac{\pi}{4}$. The area of the shaded semicircle is $\frac{1}{2} \times \pi r^2$ which is $\frac{1}{2} \times \pi \times \frac{1}{3}$, so $\frac{\pi}{6}$. The fraction of the quarter circle which is shaded is then $\frac{\pi/6}{\pi/4} = \frac{2}{3}$.

Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

Solutions and investigations

6 November 2018

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some problems for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	C	D	E	B	B	A	B	A	D	B	A	E	D	D	D	B	A	C	E	A	C	B	E	C

1. When the following are evaluated, how many of the answers are odd numbers?

$$1^2,\ 2^3,\ 3^4,\ 4^5,\ 5^6$$

A 1 B 2 C 3 D 4 E 5

Solution **C**

When two odd numbers are multiplied, the result is another odd number. Therefore an odd integer raised to a positive power results in an odd number. Likewise, an even integer raised to a positive power results in an even number.

It follows that, of the given numbers, 1^2, 3^4 and 5^6 are odd numbers, while 2^3 and 4^5 are even.

Therefore 3 of the given expressions evaluate to an odd number.

For investigation

1.1 How many of the numbers n^2, where n takes all the integers values from 1 to 100, inclusive, are odd?

1.2 How many of the numbers n^2, where n takes all the integers values from 1 to 100, inclusive, are

(a) divisible by 3?

(b) divisible by 4?

(c) divisible by 6?

(d) divisible by 12?

1.3 How many of the numbers n^3, where n takes all the integers values from 1 to 100, inclusive, are divisible by 8?

1.4 Prove that an odd positive integer raised to a positive power results in an odd number, and that an even positive integer raised to a positive power results in an even number.

1.5 Let d, n and k be positive integers. Find a criterion for the number n^k to be divisible by d.

2. The positive integer 2018 is the product of two primes.

What is the sum of these two primes?

A 1001 B 1010 C 1011 D 1100 E 1101

Solution **C**

The prime factorization of 2018 is given by $2018 = 2 \times 1009$. The sum of the two prime factors is $2 + 1009 = 1011$.

Note

There is no very quick way to check that 1009 is prime, as this would involve checking that no prime ≤ 29 is a divisor. [Why is this enough?] Fortunately, the question tells us that 2018 is the product of two primes. We can conclude from this that the second factor 1009 is prime.

3. Which of the following shows the digit 6 after it has been rotated clockwise through 135°?

A  B  C  D  E 

Solution **D**

Because $135° = 90° + 45°$, a rotation through 135° clockwise is equivalent to a rotation clockwise through 90°, which is a quarter turn, followed by a rotation clockwise through 45°, which is one-eighth of a complete turn.

Therefore, as the diagram shows, a rotation through 135° clockwise, results in the configuration given as option D.

4. Which of the following is not a multiple of 5?

A $2019^2 - 2014^2$ B $2019^2 \times 10^2$ C $2020^2 \div 101^2$
D $2010^2 - 2005^2$ E $2015^2 \div 5^2$

Solution **E**

Using the factorization of the difference of two squares, we see that $2019^2 - 2014^2 = (2019 - 2014)(2019 + 2014) = 5 \times 4033$, and hence is a multiple of 5.

$2019^2 \times 10^2$ is a multiple of 10^2, that is, a multiple of 100, and hence is a multiple of 5.

$2020^2 \div 101^2 = (20 \times 101)^2 \div 101^2 = 20^2 = 400$ and so is a multiple of 5.

Because 2010^2 and 2005^2 are both multiples of 5, their difference, $2010^2 - 2005^2$, is also a multiple of 5.

However, $2015^2 \div 5^2 = \left(\frac{2015}{5}\right)^2 = 403^2$ and, because 403 is not a multiple of 5, 403^2 is not a multiple of 5.

Hence the correct answer is option E.

For investigation

4.1 Prove the following facts about divisibility by 5 that are used in the above solution.

(a) For all positive integers m and n, if m and n are multiples of 5, then $m - n$ is a multiple of 5.

(b) For all positive integers m, if m is not a multiple of 5, then m^2 is not a multiple of 5.

4.2 Which of the statements in 4.1 are true when '5', is replaced by '6'?

4.3 Which of the statements in 4.1 are true when '5', is replaced by '20'?

4.4 For which integers d is it true that for all positive integers m, if m is not a multiple of d, then m^2 is not a multiple of d?

5. Which of the following numbers is the largest?

A $\frac{397}{101}$ B $\frac{487}{121}$ C $\frac{596}{153}$ D $\frac{678}{173}$ E $\frac{796}{203}$

SOLUTION **B**

COMMENTARY

Without the use of a calculator it is not feasible in the time available to answer this question by calculating the values of these fractions to an appropriate number of decimal places.

It would also not be reasonable to decide the relative sizes of, for example, $\frac{397}{101}$ and $\frac{487}{121}$ by using the fact that

$$\frac{397}{101} < \frac{487}{121} \Leftrightarrow 397 \times 121 < 487 \times 101.$$

Instead, the best approach here is to notice that all five fractions are close to 4, and then to decide which of them are less than 4, and which are greater than 4.

We note that

$$\frac{397}{101} < \frac{404}{101} = 4,$$
$$\frac{487}{121} > \frac{484}{121} = 4,$$
$$\frac{596}{153} < \frac{612}{153} = 4,$$
$$\frac{678}{173} < \frac{692}{173} = 4,$$

and

$$\frac{796}{203} < \frac{812}{203} = 4.$$

These calculations show that the fraction given as option B is the only one that is greater than 4, and hence is the largest of the given numbers.

FOR INVESTIGATION

5.1 Which of the following numbers is the smallest?

$$\frac{527}{105}, \frac{617}{123}, \frac{707}{141}, \frac{803}{161}, \frac{917}{183}.$$

6. Which of the following is equal to $25 \times 15 \times 9 \times 5.4 \times 3.24$?

A 3^9 B 3^{10} C 3^{11} D 3^{14} E 3^{17}

Solution **B**

> Commentary
>
> The last thing we want to do here is to do the multiplications to work out the value of $25 \times 15 \times 9 \times 5.4 \times 3.24$, and then to factorize the answer.
>
> Instead, we write the decimals as fractions, then factorize the individual numbers, and do some cancellation.

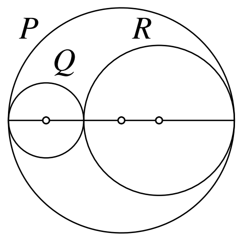
We have $25 = 5^2$, $15 = 3 \times 5$, $9 = 3^2$, $5.4 = \frac{54}{10} = \frac{27}{5} = \frac{3^3}{5}$ and $3.24 = \frac{324}{100} = \frac{81}{25} = \frac{3^4}{5^2}$. Therefore,

$$25 \times 15 \times 9 \times 5.4 \times 3.24 = 5^2 \times (3 \times 5) \times 3^2 \times \frac{3^3}{5} \times \frac{3^4}{5^2}.$$

We can now cancel the factors 5^2 and 5 in the numerator and denominator to deduce that

$$\begin{aligned} 25 \times 15 \times 9 \times 5.4 \times 3.24 &= 3 \times 3^2 \times 3^3 \times 3^4 \\ &= 3^{10}. \end{aligned}$$

7. The circles P, Q and R are all tangent to each other. Their centres all lie on a diameter of P, as shown in the figure.



What is the value of $\dfrac{\text{circumference of } Q + \text{circumference of } R}{\text{circumference of } P}$?

A 1 B $\frac{1}{2}$ C $\frac{1}{3}$ D $\frac{1}{4}$ E more information needed

Solution **A**

Let the radius of the circle Q be q and the radius of the circle R be r. We see that the diameter of the circle P is $2q + 2r$ It follows that the radius of P is $q + r$.

We now use the formula

$$\text{circumference} = 2\pi \times \text{radius},$$

to deduce that

$$\begin{aligned} \frac{\text{circumference of } Q + \text{circumference of } R}{\text{circumference of } P} &= \frac{2\pi q + 2\pi r}{2\pi(q + r)} \\ &= \frac{2\pi(q + r)}{2\pi(q + r)} \\ &= 1 \end{aligned}$$

8. What are the last two digits of 7^{2018}?

A 07 B 49 C 43 D 01 E 18

SOLUTION **B**

COMMENTARY

Clearly, we cannot answer this question by fully evaluating 7^{2018} and then looking to see which are its last two digits.

Instead, we make use of the fact that the last two digits of a product $a \times b$ is determined just by the last two digits of a and b, and then we look for a pattern.

It is convenient to introduce some notation for the last two digits of an integer. There is no standard notation for this (but see 8.1, below). For the purpose of this question we use the notation $[n]$ for the number consisting of the last two digits of the integer n. For example, $[12345] = 45$.

Using this notation we can express the fact we use by the equation

$$[m \times n] = [[m] \times [n]].$$

We then have

$$\begin{aligned}
[7^1] &= [7] = 07 \\
[7^2] &= [49] = 49 \\
[7^3] &= [7^2 \times 7] = [[7^2] \times [7]] = [49 \times 7] = [343] = 43 \\
[7^4] &= [7^3 \times 7] = [[7^3] \times [7]] = [43 \times 7] = [301] = 01 \\
[7^5] &= [7^4 \times 7] = [[7^4] \times [7]] = [01 \times 7] = [7] = 07.
\end{aligned}$$

Each term in the sequence giving the last two digits of 7^n for $n = 1, 2, 3 \ldots$ depends only on the previous term. Hence as 07 has reoccurred we can deduce that the sequence consisting of the values of $[7^n]$ is made up of repeating cycle of length 4, and so is

$$07, \quad 49, \quad 43, \quad 01, \quad 07, \quad 49, \quad 43, \quad 01, \quad \ldots .$$

Because $2018 = 504 \times 4 + 2$ it follows that $[7^{2018}]$ is the second pair of digits in this cycle, namely, 49.

FOR INVESTIGATION

8.1 Show that the formula $[m \times n] = [[m] \times [n]]$ is correct.

NOTE

The number $[m]$ is the remainder when m is divided by 100. If you are familiar with the language of modular arithmetic, you will see that $[m] \equiv m (\text{mod } 100)$.

8.2 What are the last two digits of 9^{2018}?

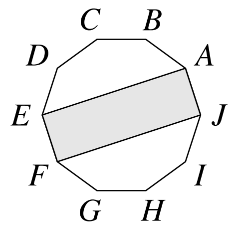
8.3 What are the last two digits of 3^{2018}?

8.4 Prove (by Mathematical Induction) that for every positive integer n, the last two digits of 7^{4n+2} are 49.[To find out about *Mathematical Induction* go to https://nrich.maths.org/4718]

9. The diagram shows a rectangle $AEFJ$ inside a regular decagon $ABCDEFGHIJ$.

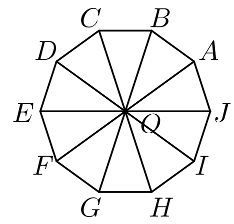
What is the ratio of the area of the rectangle to the area of the decagon?

A 2 : 5 B 1 : 4 C 3 : 5 D 3 : 10 E 3 : 20



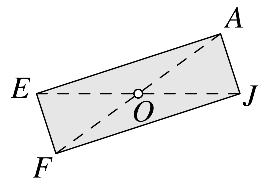
Solution **A**

Let O be the centre of the regular decagon. The decagon is divided into ten congruent isosceles triangles by the radii joining O to each of the vertices of the decagon.The triangles AOJ and EOF are two of these ten congruent triangles.



Because O is the centre of the rectangle $AEFJ$ and the diagonals of a rectangle split its area into four equal areas, the triangles EOA and FOJ each have the same area as triangles AOJ and EOF.

Therefore the area of $AEFJ$ is equal to the area of 4 of the 10 congruent triangles that make up the decagon.



Hence the ratio of the area of $AEFJ$ to the area of the decagon $ABCDEFGHI$ is 4 : 10 which simplifies to 2 : 5.

For investigation

9.1 Prove the following geometrical facts that the above solution tacitly assumes. [A full solution would need to include these proofs.]

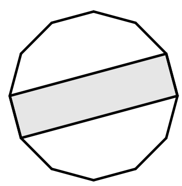
(a) The decagon has a centre, that is, there is circle which goes through all its vertices. (The centre of this circle is the centre of the decagon.)

(b) The centre of the regular decagon is also the centre of the rectangle $AEFJ$.

(c) The triangle EOA has the same area as the triangle AOJ.

9.2 The diagram shows a rectangle whose vertices are two pairs of adjacent vertices of a regular dodecagon.

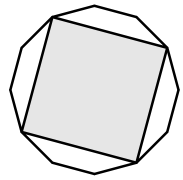
What is the ratio of the area of the rectangle to the area of the dodecagon?



9.3 Generalize the result of the question and the above problem to the case of a rectangle whose vertices are two pairs of adjacent vertices of a regular polygon with $2n$ sides, where n is a positive integer.

9.4 The diagram shows a square inscribed in a regular dodecagon.

What is the ratio of the area of the square to the area of the dodecagon?



10. On a training ride, Laura averages speeds of 12 km/h for 5 minutes, then 15 km/h for 10 minutes and finally 18 km/h for 15 minutes.

What was her average speed over the whole ride?

A 13 km/h B 14 km/h C 15 km/h D 16 km/h E 17 km/h

Solution **D**

To determine Laura's average speed over the whole ride we calculate the total distance she travels, and the total time that she takes.

Because 5 minutes is $\frac{1}{12}$ of an hour, Laura travels $\frac{1}{12} \times 12\,\text{km} = 1\,\text{km}$ when she travels for 5 minutes at 12 km/h.

Because 10 minutes is $\frac{1}{6}$ of an hour, Laura travels $\frac{1}{6} \times 15\,\text{km} = 2.5\,\text{km}$ when she travels for 10 minutes at 15 km/h.

Because 15 minutes is $\frac{1}{4}$ of an hour, Laura travels $\frac{1}{4} \times 18\,\text{km} = 4.5\,\text{km}$ when she travels for 15 minutes at 18 km/h.

Therefore Laura travels at total of $1\,\text{km} + 2.5\,\text{km} + 4.5\,\text{km} = 8\,\text{km}$ in $5 + 10 + 15$ minutes, that is, in 30 minutes, which is half an hour.

Because Laura travels 8 km in half an hour, her average speed is 16 km/h.

For investigation

10.1 Suppose Laura had ridden for a further 20 minutes at 21 km/h. What would then have been her average speed for the whole ride?

10.2 Suppose Laura had extended her training ride by 30 minutes. How fast would she have had to ride in this 30 minutes to make her average speed for the whole hour equal to 20 km/h?

11. How many of the following four equations has a graph that does *not* pass through the origin?

$$y = x^4 + 1 \qquad y = x^4 + x \qquad y = x^4 + x^2 \qquad y = x^4 + x^3$$

A 0 B 1 C 2 D 3 E 4

Solution **B**

The origin has coordinates $(0, 0)$. Thus it is the point where $x = 0$ and $y = 0$. It follows that the graph of an equation passes through the origin if the equation shows that $y = 0$ when $x = 0$.

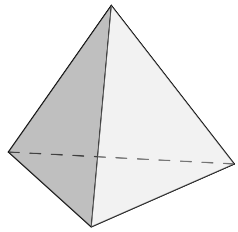
Now, when $x = 0$, the values of $x^4 + 1$, $x^4 + x$, $x^4 + x^2$ and $x^4 + x^3$ are 1, 0, 0 and 0, respectively.

Hence, of the given equations, $y = x^4 + 1$ is the only one whose graph does not pass through the origin.

For investigation

11.1 Sketch the graphs of the four equations given in the question.

12. A regular tetrahedron is a polyhedron with four faces, each of which is an equilateral triangle, as shown. A solid regular tetrahedron is cut into two pieces by a single plane cut.



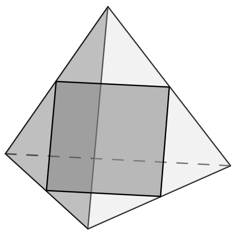
Which of the following could *not* be the shape of the section formed by the cut?

A a pentagon
B a square
C a rectangle that is not a square
D a trapezium
E a triangle that is not equilateral

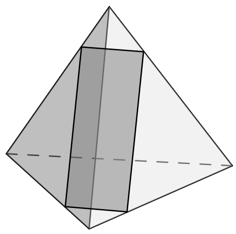
Solution **A**

When the regular tetrahedron is cut by a single plane cut, each of its four faces is cut at most once. The place where each face is cut becomes the edge of the newly formed section. Therefore a pentagon, with five edges, cannot be formed.

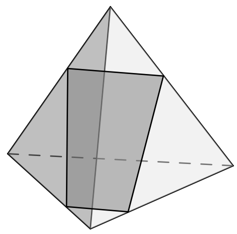
However, each of the other four options is possible, as the following diagrams show.



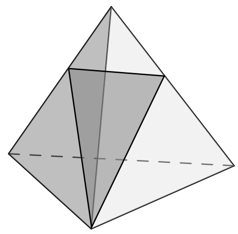
A square



A rectangle that is not a square



A trapezium



A triangle that is not equilateral

13. The lines $y = x$ and $y = mx - 4$ intersect at the point P.

What is the sum of the positive integer values of m for which the coordinates of P are also positive integers?

A 3 B 5 C 7 D 8 E 10

Solution **E**

At the point P we have both $y = x$ and $y = mx - 4$. Therefore at P we have $x = mx - 4$ and hence $4 = mx - x$. This last equation may be written as $4 = (m - 1)x$. If m and x are both positive integers which satisfy this last equation, then $m - 1$ and x are both positive integers which are factors of 4.

Hence the possible values of $m - 1$ are 1, 2 and 4 with corresponding values of 2, 3 and 5 for m and 4, 2 and 1 for x and y.

Hence the sum of the positive integer values of m for which the coordinates of P are positive integers is $2 + 3 + 5 = 10$.

14. The following twelve integers are written in ascending order:

$$1, x, x, x, y, y, y, y, y, 8, 9, 11.$$

The mean of these twelve integers is 7. What is the median?

A 6 B 7 C 7.5 D 8 E 9

Solution **D**

Because the mean of the given integers is 7, we have

$$1 + x + x + x + y + y + y + y + y + 8 + 9 + 11 = 12 \times 7.$$

We may rewrite this last equation as $1 + 3x + 5y + 28 = 84$ and therefore $3x + 5y = 55$.

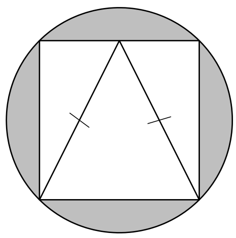
It follows that $3x = 55 - 5y = 5(11 - y)$. Therefore, because x and y are integers, $3x$ is a multiple of 5 and hence x is a multiple of 5.

Because the integers are written in ascending order, $1 \leq x \leq 8$. Hence $x = 5$ and therefore $15 + 5y = 55$. It follows that $5y = 40$ and therefore $y = 8$.

Therefore the twelve integers in ascending order are 1, 5, 5, 5, 8, 8, 8, 8, 8, 8, 9, 11. We now see that the median is 8.

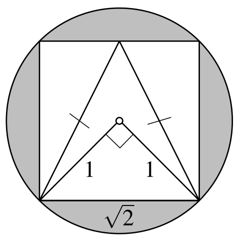
15. A square is inscribed in a circle of radius 1. An isosceles triangle is inscribed in the square as shown.

What is the ratio of the area of this triangle to the area of the shaded region?



A $\pi : \sqrt{2}$ B $\pi : 1$ C $1 : 4$
D $1 : \pi - 2$ E $2 : \pi$

Solution **D**



Let the sides of the square have length s. We see from the diagram that, by Pythagoras' theorem, $s^2 = 1^2 + 1^2$ and hence $s = \sqrt{2}$.

The area of the circle is $\pi \times 1^2$, which equals π. The area of the square is $\sqrt{2} \times \sqrt{2}$, which equals 2. Hence the area of the shaded region is $\pi - 2$.

The base of the isosceles triangle is one of the sides of the square and so has length $\sqrt{2}$. The perpendicular height of the isosceles triangle is also $\sqrt{2}$. The area of the triangle is therefore $\frac{1}{2}(\sqrt{2} \times \sqrt{2})$ which equals 1.

Hence the ratio of the area of the triangle to the area of the shaded region is $1 : \pi - 2$.

For investigation

15.1 Show that the area of the isosceles triangle is half the area of the square.

16. The numbers p, q, r and s satisfy the following equations:

$$p + 2q + 3r + 4s = k \qquad 4p = 3q = 2r = s.$$

What is the smallest value of k for which p, q, r and s are all positive integers?

A 20 B 24 C 25 D 77 E 154

Solution **D**

From the set of equations $4p = 3q = 2r = s$, we have $q = \frac{4}{3}p, r = 2p$ and $s = 4p$.

We require p and q to be positive integers. Therefore, from the equation $q = \frac{4}{3}p$ we deduce that p has to be an integer which is a multiple of 3. We also note that if p is a positive integer, then so also will be r and s.

It also follows that

$$\begin{aligned} k &= p + 2q + 3r + 4s = p + 2(\tfrac{4}{3}p) + 3(2p) + 4(4p) \\ &= p + \tfrac{8}{3}p + 6p + 16p = \tfrac{77}{3}p. \end{aligned}$$

Because p is a positive integer which is a multiple of 3, its smallest value is 3. Therefore the smallest value of k is $\frac{77}{3} \times 3$, which equals 77.

17. Bethany has 11 pound coins and some 20p coins and some 50p coins in her purse. The mean value of the coins is 52 pence.

Which could not be the number of coins in the purse?

A 35 B 40 C 50 D 65 E 95

Solution **B**

We suppose that numbers of 20p and 50p coins that Bethany has are m and n, respectively. (Note that m and n are positive integers.)

Then Bethany has c coins, where $c = 11+m+n$. The total value of these coins is $11+0.20m+0.50n$ pounds, which equals $1100 + 20m + 50n$ pence.

Because the mean value of Bethany's coins is 52 pence, $\dfrac{1100 + 20m + 50n}{11 + m + n} = 52$. It follows that $1100 + 20m + 50n = 52(11 + m + n)$ and hence $1100 + 20m + 50n = 572 + 52m + 52n$. This equation may be rearranged as $2n = 528 - 32m$, which simplifies to $n = 264 - 16m$.

It follows that $c = 11 + m + n = 11 + m + (264 - 16m) = 275 - 15m$. Therefore $275 - c = 15m$. Thus $275 - c$ is an integer multiple of 15.

We now consider the different values for c given by the options in the question.

We have $275 - 35 = 240 = 15 \times 16$, $275 - 50 = 225 = 15 \times 15$, $275 - 65 = 210 = 15 \times 14$ and $275 - 95 = 180 = 15 \times 12$. Hence Bethany could have 35 or 50 or 65 or 95 coins in her purse.

However $275 - 40 = 235$ which is not an integer multiple of 15. Hence Bethany could not have 40 coins in her purse.

18. P, Q and R are the three angles of a triangle, when each has been rounded to the nearest degree.

Which of the following is the complete list of possible values of $P + Q + R$?

A $179°$, $180°$ or $181°$ B $180°$, $181°$ or $182°$ C $178°$, $179°$ or $180°$
D $180°$ E $178°$, $179°$, $180°$, $181°$ or $182°$

Solution **A**

We suppose that the actual angles of the triangle are P', Q' and R', which are rounded to P, Q and R, respectively.

The sum of the angles of a triangle is $180°$ and therefore

$$P' + Q' + R' = 180°.$$

When an angle is rounded up to the nearest degree, it is increased by at most $0.5°$; when it is rounded down to the nearest degree, it is decreased by at most $0.5°$. Therefore,

$$P' - 0.5° \le P \le P' + 0.5°,$$
$$Q' - 0.5° \le Q \le Q' + 0.5°,$$
$$\text{and} \quad R' - 0.5° \le R \le R' + 0.5°.$$

Adding these inequalities gives

$$P' + Q' + R' - 1.5° \le P + Q + R \le P' + Q' + R' + 1.5°.$$

Therefore, as $P' + Q' + R' = 180°$,

$$178.5° \le P + Q + R \le 181.5°.$$

Each of P, Q and R is an integer number of degrees. Hence $P + Q + R$ is also an integer number of degrees. It follows that the only possible values of $P + Q + R$ are $179°$, $180°$ and $181°$.

To complete the solution we show that each of these possible values actually occurs for some triangle. This is shown by the following examples.

$P' = 60.3°$, $Q' = 60.3°$, $R' = 59.4°$ gives $P = 60°$, $Q = 60°$, $R = 59°$. Hence $P + Q + R = 179°$.

$P' = 60°$, $Q' = 60°$, $R' = 60°$ gives $P = 60°$, $Q = 60°$, $R = 60°$. Hence $P + Q + R = 180°$.

$P' = 59.7°$, $Q' = 59.7°$, $R' = 60.6°$ gives $P = 60°$, $Q = 60°$, $R = 61°$. Hence $P + Q + R = 181°$.

We deduce that $179°$, $180°$, $181°$ is a complete list of the possible values of $P + Q + R$.

For investigation

18.1 Give an example of a triangle whose angles are all different non-integer numbers of degrees, but whose rounded angles have sum $180°$.

18.2 P, Q, R and S are the four angles of a quadrilateral, when each has been rounded to the nearest degree. Give a list of all the values that $P + Q + R + S$ can take.

18.3 The angles of a polygon with n vertices are each rounded to the nearest integer. Give, in terms of n, a list of the values that the sum of these rounded angles can take.

19. How many pairs of numbers (m, n) are there such that the following statement is true?

'A regular m-sided polygon has an exterior angle of size $n°$ and
a regular n-sided polygon has an exterior angle of size $m°$.'

A 24 B 22 C 20 D 18 E 16

SOLUTION **C**

We first note that a polygon has at least 3 sides, so we need consider only cases where $m \geq 3$ and $n \geq 3$.

The exterior angle of a regular m-sided polygon is $\left(\frac{360}{m}\right)^\circ$. Hence, the condition for a regular m-sided polygon to have an exterior angle of $n°$ is that $n = \frac{360}{m}$. This condition may be rewritten as $mn = 360$.

Similarly, this is the condition that a regular n-sided polygon has an exterior angle of $m°$.

Therefore the pair of numbers (m, n) satisfies the condition of the question if, and only if m and n are both at least 3 and $mn = 360$.

There are 10 ways of expressing 360 as the product of positive integers which are both at least 3. These are

$$3 \times 120,\ 4 \times 90,\ 5 \times 72,\ 6 \times 60,\ 8 \times 45,\ 9 \times 40,\ 10 \times 36,\ 12 \times 30,\ 15 \times 24 \text{ and } 18 \times 20.$$

Each of these 10 factorizations can be taken in either order to give two pairs (m, n) which meet the required condition. For example, corresponding to the factorization 3×120 we see that (m, n) may be either $(3, 120)$ or $(120, 3)$.

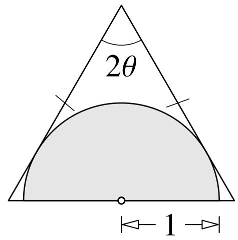
It follows that there are 10×2, that is, 20 pairs of numbers (m, n) meeting the required condition.

FOR INVESTIGATION

19.1 Prove that the exterior angle of a regular polygon with n sides is $\left(\frac{360}{n}\right)^\circ$.

19.2 For how many different values of n is there a regular polygon with n sides whose interior angles are each an integer number of degrees?

20. The diagram shows a semicircle of radius 1 inside an isosceles triangle. The diameter of the semicircle lies along the 'base' of the triangle, and the angle of the triangle opposite the 'base' is equal to 2θ. Each of the two equal sides of the triangle is tangent to the semicircle.



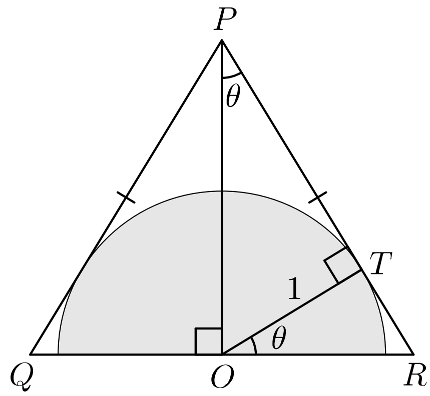
What is the area of the triangle?

A $\frac{1}{2}\tan 2\theta$ B $\sin\theta\cos\theta$ C $\sin\theta + \cos\theta$

D $\frac{1}{2}\cos 2\theta$ E $\dfrac{1}{\sin\theta\cos\theta}$

Solution **E**

Let the vertices of the triangle be P, Q and R, with $PQ = PR$, as shown in the diagram, and let O be the midpoint of QR. We also let T be the point where the semicircle touches PR. In the triangles PQO and PRO, we have $PQ = PR$, $QO = RO$ and the side PO is common. Therefore the triangles are congruent (SSS). Hence the area of the triangle PQR is twice the area of the triangle POR.



It also follows that $\angle QPO = \angle RPO = \theta$. Hence PO is the bisector of $\angle QPR$. Hence O is equidistant from PQ and PR and is therefore the centre of the semicircle.

We can also deduce that $\angle POQ = \angle POR = 90°$.

OT is a radius of the semicircle and therefore has length 1 and is perpendicular to the tangent PR. Hence PTO is a right-angled triangle and therefore

$$\frac{OT}{OP} = \sin\theta$$

and hence

$$OP = \frac{OT}{\sin\theta} = \frac{1}{\sin\theta}.$$

Also, from the triangle POR, we have

$$\frac{OP}{PR} = \cos\theta$$

and hence

$$PR = \frac{OP}{\cos\theta} = \frac{1}{\sin\theta\cos\theta.}$$

If we regard PR as the base of the triangle POR, and OT as its base, we see that

$$\text{area of } POR = \tfrac{1}{2}(PR \times OT) = \frac{1}{2\sin\theta\cos\theta}.$$

It follows that the area of the triangle PQR is $\dfrac{1}{\sin\theta\cos\theta}$.

21. The graph of $y = \frac{1}{x}$ is reflected in the line $y = 1$. The resulting image is reflected in the line $y = -x$.

What is the equation of the final graph?

A $y = \frac{-1}{(x+2)}$ B $y = \frac{1}{(x-1)}$ C $y = \frac{1}{(x-2)}$ D $y = \frac{-1}{(x-1)}$ E $y = \frac{-1}{(x-2)}$

Solution **A**

Commentary

We give two methods for answering this question.

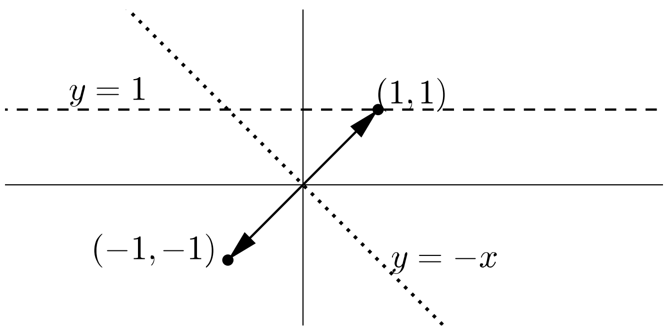
The first quick method is adequate in the context of the SMC where you can assume that one of the options is correct, and you are not required to justify the answer.

The second method uses algebra to calculate the equation of the final graph, and is an example of the kind of answer that you would need to give when fully explained solutions are required.

Method 1

When $x = 1$, we have $\frac{1}{x} = 1$, and therefore the point with coordinates $(1, 1)$ lies on the graph of $y = \frac{1}{x}$.

The point $(1, 1)$ remains fixed when reflected in the line $y = 1$, and then is mapped to the point $(-1, -1)$ after reflection in the line $y = -x$.



It follows that the point $(-1, -1)$ lies on the final graph. It is easy to check that the equation of option A gives $y = -1$ when $x = -1$, but that none of the other equations have this property. We can therefore eliminate options B, C, D and E, and hence conclude that the correct option is A.

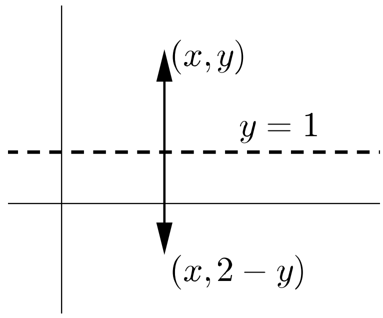
Note

This argument conclusively eliminates the options B, C, D and E but does not *prove* that the equation of option A is the equation of the final graph. It only shows that this is a possibility.

Method 2

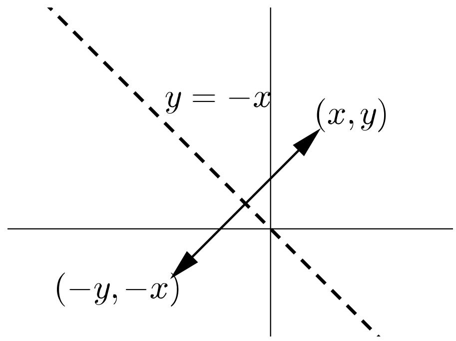
Reflection in the line $y = 1$ interchanges the points with coordinates (x, y) and $(x, 2 - y)$, as shown in the first diagram on the right.

Therefore the image of the graph of $y = \frac{1}{x}$ after this reflection is the graph of $2 - y = \frac{1}{x}$.



Reflection in the line $y = -x$ interchanges the points with coordinates (x, y) and $(-y, -x)$, as shown in the second diagram on the right.

Therefore the image of the graph of $2 - y = \frac{1}{x}$ after this reflection is the graph of $2 - (-x) = \frac{1}{-y}$. This last equation may be rearranged as $y = \frac{-1}{(x + 2)}$.



For investigation

21.1 Check that none of the graphs of $y = \frac{1}{(x-1)}$, $y = \frac{1}{(x-2)}$, $y = \frac{-1}{(x-1)}$ and $y = \frac{-1}{(x-2)}$ goes through the point with coordinates $(-1, -1)$.

21.2 (a) Sketch the graphs of $y = \frac{1}{x}$ and $y = \frac{-1}{(x+2)}$.

(b) Indicate geometrically how reflecting the graph of $y = \frac{1}{x}$ in the line $y = 1$ and then reflecting the resulting graph in the line $y = -x$ produces the graph of $y = \frac{-1}{x+2}$.

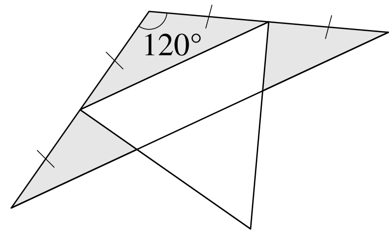
21.3 (a) Show that the image of the point with coordinates (h, k) in the line $y = 1$ is the point with coordinates $(h, 2 - k)$

(b) Let f be some function of x. Show that the image of the graph of $y = f(x)$ after reflection in the line $y = 1$ is the graph of $y = 2 - f(x)$.

21.4 (a) Show that the image of the point with coordinates (h, k) in the line $y = -x$ is the point with coordinates $(-k, -h)$

(b) Let f be some function of x. Show that the image of the graph of $y = f(x)$ after reflection in the line $y = -x$ is the curve whose equation is $x = -f(-y)$.

22. The diagram shows two overlapping triangles; an isosceles triangle with an angle of 120° and an equilateral triangle with area 36. Two of the vertices of the equilateral triangle are midpoints of the equal sides of the isosceles triangle.



What is the total area of the shaded regions (inside the isosceles triangle but outside the equilateral triangle)?

A 24 B 26 C 28 D 30 E 32

Solution **C**

Commentary

This problem may be tackled in many different ways. We first give a full solution. We then sketch two more solutions, leaving the details to the reader. You may well find other methods.

Method 1

We let the vertices of the isosceles triangle with an angle of 120° be P, Q and R, and those of the equilateral triangle be K, L and M, as shown in the diagram below.

We also let N be the midpoint of KM and S and T be the points where QR meets the lines KL and ML, respectively.

We also suppose that the equilateral triangle KLM has sides of length $2s$

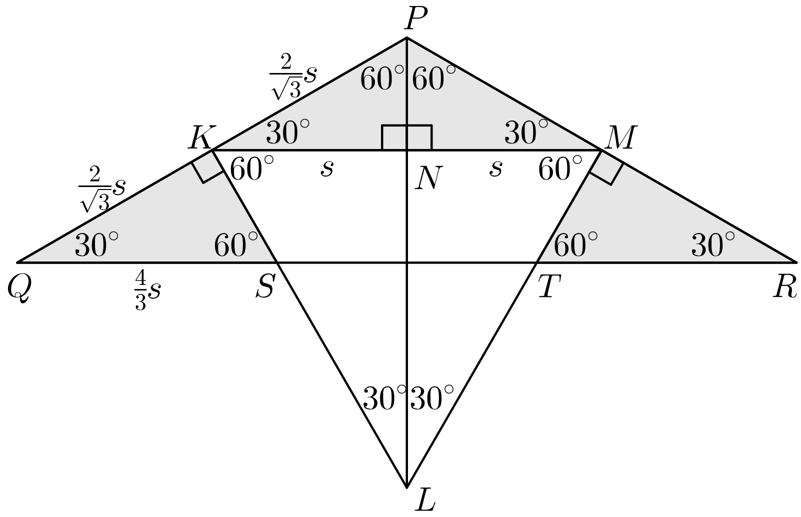


Figure 1

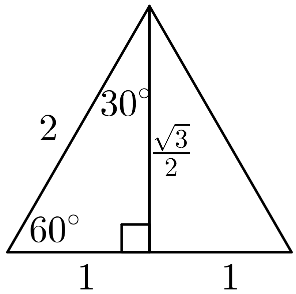
The triangle PQR is isosceles with $PQ = PR$, and K and M are the midpoints of PQ and PR, respectively. Therefore $PK = PM$. Since N is the midpoint of KM we also have $KN = NM$. Therefore the triangles PKN and PMN have sides of the same lengths. Hence they are congruent. Therefore the angles of both these triangles are 90°, 60° and 30°, as shown in the diagram.

Similarly, the triangles KNL and MNL have sides of the same lengths, and so are congruent. It follows that they also have angles of 90°, 60° and 30° as shown.

Because the triangle PQR is isosceles with $PQ = PR$, it follows that $\angle PQR = \angle PRQ$. Therefore, as $\angle QPR = 120°$ and the sum of the angles in a triangle is 180°, it follows that $\angle PQR = \angle PRQ = 30°$.

The angles on the line PQ at K have sum 180°. It follows that $\angle QKS = 90°$. Therefore QKS is yet another triangle with angles 90°, 60° and 30°. Similarly, the same is true of the triangle RMT. Because $QK = RM$, it follows that the triangles QKS and RMT are congruent.

We have therefore seen that KPN, QSK and LKN are similar triangles with angles 90°, 60° and 30°.



By considering a triangle with these angles as half of an equilateral triangle, we see that in such a triangle the hypotenuse and other two sides are in the ratio $2 : \frac{\sqrt{3}}{2} : 1$.

Therefore, as $KN = s$, it follows that $KP = \frac{2}{\sqrt{3}}s$.

Hence $QK = KP = \frac{2}{\sqrt{3}}s$ and therefore $QS = \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}}s = \frac{4}{3}s$.

The ratio of the areas of similar triangles equals the ratio of the squares of their linear dimensions.

We have seen that the similar triangles KPN, QSK and LKN have hypotenuses of lengths $\frac{2}{\sqrt{3}}s$, $\frac{4}{3}s$ and $2s$, respectively. These are in the ratio $\frac{2}{\sqrt{3}} : \frac{4}{3} : 2$. Hence the areas of these triangles are in the ratio $\frac{4}{3} : \frac{16}{9} : 4$, which is equivalent to $\frac{1}{3} : \frac{4}{9} : 1$.

Therefore the areas of the triangles KPN and QSK are, respectively, $\frac{1}{3}$ and $\frac{4}{9}$ of the area of the triangle LKN. Now $\frac{1}{3} + \frac{4}{9} = \frac{7}{9}$. Hence the sum of the areas of the triangles KPN and QSK is $\frac{7}{9}$ of the area of the triangle LKN.

Similarly the sum of the areas of the triangles MPN and TRM is $\frac{7}{9}$ of the area of the triangle LMN.

It follows that the area of the shaded region is $\frac{7}{9}$ of the area of the equilateral triangle. Hence, the shaded area is $\frac{7}{9} \times 36 = 28$.

Method 2

Let O be the midpoint of QR, and join O to K, L and M.

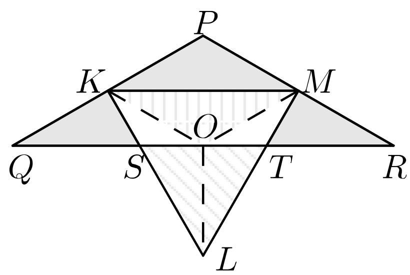


Figure 2

It can be checked that the triangles PKM, KOM, LOK and MOL are all congruent. It follows that the area of the triangle PKM is one third of the area of KLM, and hence the triangle PKM has area 12.

It can also be proved that the length of ST is $\frac{2}{3}$ of the length of KM. Hence the area of the triangle LTS is $\frac{4}{9}$ of the area of KLM, that is 16.

It can also be checked that the triangles KQS, MTR, SLO and LTO are congruent.

Therefore the sum of the areas of the triangles KQS and MTR is equal to the area of the triangle STL, that is, 16.

Hence the shaded area is $12 + 16 = 28$.

Method 3

The third solution we give is more visual. It relies on properties of the diagrams which would need detailed justification if a full solution were required.

Consider the diagram in Figure 3, below, which shows the original shape, outlined by heavy lines, together with two copies of it, rotated by ±120°.

This shows that the hatched area in Figure 3 is equal to one third of the original equilateral triangle, that is, 12. We also see that the large equilateral triangle has area $4 \times 36 = 144$.

Now consider Figure 4, which shows the same diagram, superimposed by several smaller equilateral triangles.

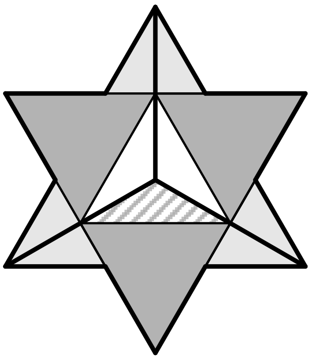


Figure 3

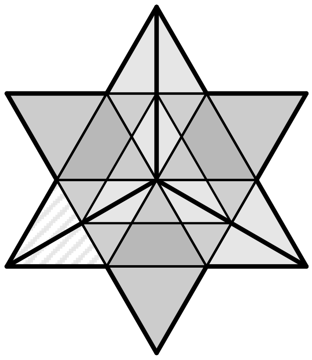


Figure 4

This shows that the hatched area in Figure 4 is equal to one ninth of the large equilateral triangle, that is, 16.

Hence the shaded area in the original figure is $12 + 16 = 28$.

For investigation

22.1 Explain why it follows from the fact that in Figure 1 the triangles PKN and PMN are congruent that the angles in these triangles are 90°, 60° and 30°.

22.2 Show that if in a right-angled triangle the hypotenuse has length 2 and one of the other sides has length 1, then the third side has length $\sqrt{3}$.

22.3 Explain why the ratio of the areas of similar triangles equals the ratio of the squares of their linear dimensions.

22.4 Find the area of the rectangle $STMK$ in Figure 1 on the previous page.

22.5 Prove that in Figure 1 the triangles PQL and PRL are equilateral and hence that $PQLR$ is a rhombus.

22.6 Fill in the missing details in the solutions of *Method* 2 and *Method* 3.

23. For particular real numbers a and b, the function f is defined by $f(x) = ax + b$, and is such that $f\big(f\big(f(x)\big)\big) = 27x - 52$.

Which of the following formulas defines the function g such that, for all values of x, $g\big(f(x)\big) = x$?

A $\frac{1}{3}x - 4$ B $\frac{1}{3}x + \frac{4}{3}$ C $4x - 3$ D $\frac{1}{3}x - \frac{4}{3}$ E $3x - 4$

SOLUTION **B**

Because $f(x) = ax + b$, we have that, for all real numbers x,

$$\begin{aligned} f(f(x)) &= f(ax+b) \\ &= a(ax+b)+b \\ &= a^2x + ab + b. \end{aligned}$$

It follows that, for all real numbers x,

$$\begin{aligned} f(f(f(x))) &= f(a^2x + ab + b) \\ &= a(a^2x + ab + b) + b \\ &= a^3x + a^2b + ab + b \\ &= a^3x + (a^2 + a + 1)b. \end{aligned}$$

Because the expressions $a^3x + (a^2 + a + 1)b$ and $27x - 52$ have the same value for all real numbers x, their coefficients match. Hence $a^3 = 27$ and $(a^2 + a + 1)b = -52$. It follows that $a = 3$ and hence $13b = -52$, giving $b = -4$. Therefore the function f is defined by $f(x) = 3x - 4$.

> COMMENTARY
>
> We now need to find the formula for the function g such that $g(f(x)) = x$. If $f(x) = y$ this implies that $g(y) = x$. Hence the method we use is to start with the equation $y = f(x)$ and then rearrange this so that it has the form $x = g(y)$.

We have

$$\begin{aligned} y = f(x) &\Leftrightarrow y = 3x - 4 \\ &\Leftrightarrow 3x = y + 4 \\ &\Leftrightarrow x = \tfrac{1}{3}y + \tfrac{4}{3}. \end{aligned}$$

It follows that g is given by the formula $g(y) = \frac{1}{3}y + \frac{4}{3}$. Rewriting this formula in terms of x gives $g(x) = \frac{1}{3}x + \frac{4}{3}$.

FOR INVESTIGATION

23.1 In each of the following cases find a formula for the function g such that $g(f(x)) = x$.

(a) $f(x) = 7x - 28$.

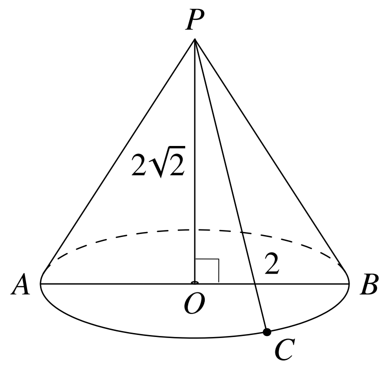
(b) $f(x) = \sqrt{x+3} + 5$.

(c) $f(x) = \dfrac{x}{x+1}$.

24. The diagram shows a circle with centre O which lies in a horizontal plane. The diameter AB has length 4. Point P lies vertically above O and $PO = 2\sqrt{2}$. Point C lies on the semicircular arc AB such that the ratio of the lengths of the arcs AC and CB is $2:1$.

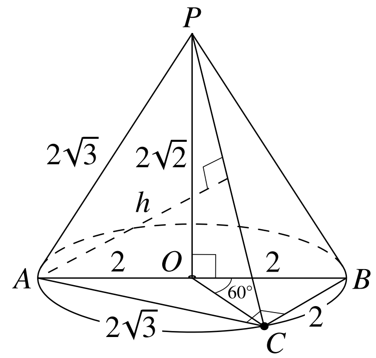
What is the shortest distance from A to PC?

A $\sqrt{2}$ B $\sqrt{3}$ C 2 D $2\sqrt{2}$ E 3



SOLUTION **E**

The triangle POA has a right angle at O. Therefore, by Pythagoras's Theorem, $PA^2 = AO^2 + PO^2 = 2^2 + (2\sqrt{2})^2 = 4 + 8 = 12$. It follows that $PA = \sqrt{12} = 2\sqrt{3}$. Similarly, $PC = 2\sqrt{3}$.



Since arc AC : arc $CB = 2:1$, it follows that $\angle AOC : \angle COB = 2:1$. Hence $\angle COB = 60°$.

Now OC and OB both have length 2 as they are radii of the circle with centre O. Since $\angle COB = 60°$, it follows that the triangle COB is equilateral. Hence $BC = 2$.

We now turn our attention to the triangle ACB. In this triangle $\angle ACB = 90°$ because it is the angle in a semicircle, $AB = 4$ and $BC = 2$. Therefore, by Pythagoras' Theorem, $AB^2 = AC^2 + BC^2$ and hence $4^2 = AC^2 + 2^2$. It follows that $AC^2 = 4^2 - 2^2 = 16 - 4 = 12$. Therefore $AC = \sqrt{12} = 2\sqrt{3}$.

We have therefore shown that PAC is an equilateral triangle in which each side has length $2\sqrt{3}$.

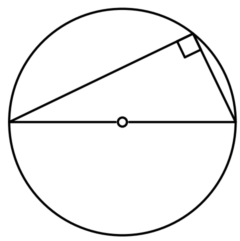
The shortest distance from A to PC is the length, say h, of the perpendicular from A to PC. Thus h is the height of the equilateral triangle PAC.

We have seen in the solution to Question 22 that the height of an equilateral triangle with side length x is $\frac{\sqrt{3}}{2}x$. It follows that

$$h = \frac{\sqrt{3}}{2} \times 2\sqrt{3} = 3.$$

FOR INVESTIGATION

24.1 Prove that the angle in a semicircle is a right angle.



NOTE

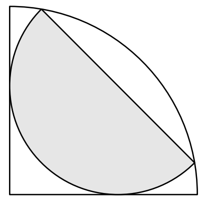
The theorem immediately above is attributed to Thales of Miletus (c624 BCE - c547 BCE), although, as none of his writings has survived, we cannot be sure that he proved this result. For more information about Thales go to the MacTutor History of Mathematics archive:

http://www-history.mcs.st-and.ac.uk

25. A semicircle is inscribed in a quarter circle as shown.

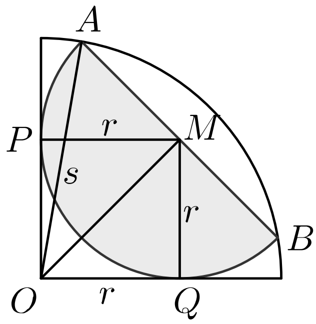
What fraction of the quarter circle is shaded?

A $\frac{1}{3}$ B $\frac{1}{\sqrt{3}}$ C $\frac{2}{3}$ D $\frac{\sqrt{3}}{2}$ E $\frac{1}{\sqrt{2}}$



Solution **C**

Let O be the centre of the quarter circle, A and B be the ends of the diameter of the semicircle and M be the centre of the semicircle, and hence the midpoint of AB.



Let P and Q be the points on the straight edges of the quarter circle where the quarter circle is tangent to the semicircle.

We let the radius of the quarter circle be s and let the radius of the semicircle be r.

Then $OA = s$.

As a tangent to a circle is perpendicular to its radius, $\angle OPM = \angle OQM = 90°$. Because also $\angle POQ = 90°$, all the angle of the quadrilateral $OQMP$ are right angles. Also $PM = r = QM$. Hence $OPMQ$ is a square with side length r.

Applying Pythagoras' theorem to the right-angled triangle OQM gives $OM^2 = r^2 + r^2 = 2r^2$.

The triangle AMO has a right angle at M. Therefore, by Pythagoras' theorem applied to this triangle, we have

$$AO^2 = OM^2 + MA^2$$

and hence

$$\begin{aligned} s^2 &= 2r^2 + r^2 \\ &= 3r^2. \end{aligned}$$

The area of the quarter circle is $\frac{1}{4}\pi s^2$. The area of the shaded semicircle is $\frac{1}{2}\pi r^2$.

Therefore, the fraction of the quarter circle which is shaded is given by

$$\frac{\frac{1}{2}\pi r^2}{\frac{1}{4}\pi s^2} = \frac{2r^2}{s^2} = \frac{2r^2}{3r^2} = \frac{2}{3}.$$

For investigation

25.1 Prove that the triangle AMO has a right angle at M.

25.2 What is the ratio of the perimeter of the quarter circle to the perimeter of the half circle? [In each case the perimeter includes the straight edges.]

United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Thursday 7 November 2019

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries Overleaf

Candidates must be full-time students at secondary school or FE college.
England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil.** Mark at most one of the options A, B, C, D, E on the Answer Sheet for each question. Do not mark more than one option.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:**
   All candidates start with 25 marks;
   0 marks are awarded for each question left unanswered;
   4 marks are awarded for each correct answer;
   1 mark is deducted for each incorrect answer (to discourage guessing).
7. Your Answer Sheet will be read by a machine. **Do not write or doodle on the sheet except to mark your chosen options.** The machine will read all black pencil markings even if they are in the wrong places. If you mark the sheet in the wrong place, or leave bits of eraser stuck to the page, the machine will interpret the mark in its own way.
8. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

Enquiries about the Senior Mathematical Challenge should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1. What is the value of $123^2 - 23^2$?

A 10 000 B 10 409 C 12 323 D 14 600 E 15 658

2. What is the value of $(2019 - (2000 - (10 - 9))) - (2000 - (10 - (9 - 2019)))$?

A 4040 B 40 C −400 D −4002 E −4020

3. Used in measuring the width of a wire, one mil is equal to one thousandth of an inch. An inch is about 2.5 cm.

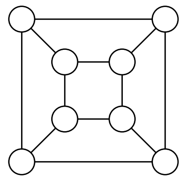
Which of these is approximately equal to one mil?

A $\frac{1}{40}$ mm B $\frac{1}{25}$ mm C $\frac{1}{4}$ mm D 25 mm E 40 mm

4. For how many positive integer values of n is $n^2 + 2n$ prime?

A 0 B 1 C 2 D 3 E more than 3

5. Olive Green wishes to colour all the circles in the diagram so that, for each circle, there is exactly one circle of the same colour joined to it.



What is the smallest number of colours that Olive needs to complete this task?

A 1 B 2 C 3 D 4 E 5

6. Each of the factors of 100 is to be placed in a 3 by 3 grid, one per cell, in such a way that the products of the three numbers in each row, column and diagonal are all equal. The positions of the numbers 1, 2, 50 and x are shown in the diagram.

x	1	50
2		

What is the value of x?

A 4 B 5 C 10 D 20 E 25

7. Lucy is asked to choose p, q, r and s to be the numbers 1, 2, 3 and 4, in some order, so as to make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible.

What is the smallest value Lucy can achieve in this way?

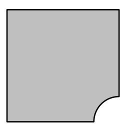
A $\frac{7}{12}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{5}{6}$ E $\frac{11}{12}$

8. The number x is the solution to the equation $3^{(3^x)} = 333$.

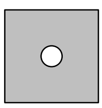
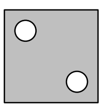
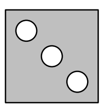
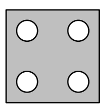
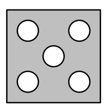
Which of the following is true?

A $0 < x < 1$ B $1 < x < 2$ C $2 < x < 3$ D $3 < x < 4$ E $4 < x < 5$

9. A square of paper is folded in half four times to obtain a smaller square. Then a corner is removed as shown.



Which of the following could be the paper after it is unfolded?

A  B  C  D  E 

10. Which of the following five values of n is a counterexample to the statement in the box below?

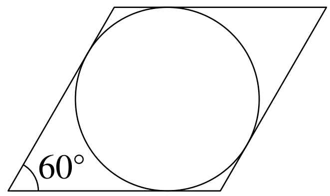
> For a positive integer n, at least one of $6n - 1$ and $6n + 1$ is prime.

A 10 B 19 C 20 D 21 E 30

11. For how many integer values of k is $\sqrt{200 - \sqrt{k}}$ also an integer?

A 11 B 13 C 15 D 17 E 20

12. A circle with radius 1 touches the sides of a rhombus, as shown. Each of the smaller angles between the sides of the rhombus is $60°$.



What is the area of the rhombus?

A 6 B 4 C $2\sqrt{3}$ D $3\sqrt{3}$ E $\frac{8\sqrt{3}}{3}$

13. Anish has a number of small congruent square tiles to use in a mosaic. When he forms the tiles into a square of side n, he has 64 tiles left over. When he tries to form the tiles into a square of side $n + 1$, he has 25 too few.

How many tiles does Anish have?

A 89 B 1935 C 1980 D 2000 E 2019

14. One of the following is the largest square that is a factor of $10!$. Which one?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n - 1) \times n$.

A $(4!)^2$ B $(5!)^2$ C $(6!)^2$ D $(7!)^2$ E $(8!)^2$

15. The highest common factors of all the pairs chosen from the positive integers Q, R and S are three different primes.

What is the smallest possible value of $Q + R + S$?

A 41 B 31 C 30 D 21 E 10

16. The numbers x, y and z satisfy the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$.

What is the mean of x, y and z?

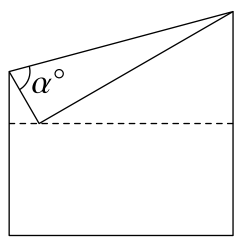
A 10 B 11 C 12 D 13 E 14

17. Jeroen writes a list of 2019 consecutive integers. The sum of his integers is 2019.

What is the product of all the integers in Jeroen's list?

A 2019^2 B $\frac{2019 \times 2020}{2}$ C 2^{2019} D 2019 E 0

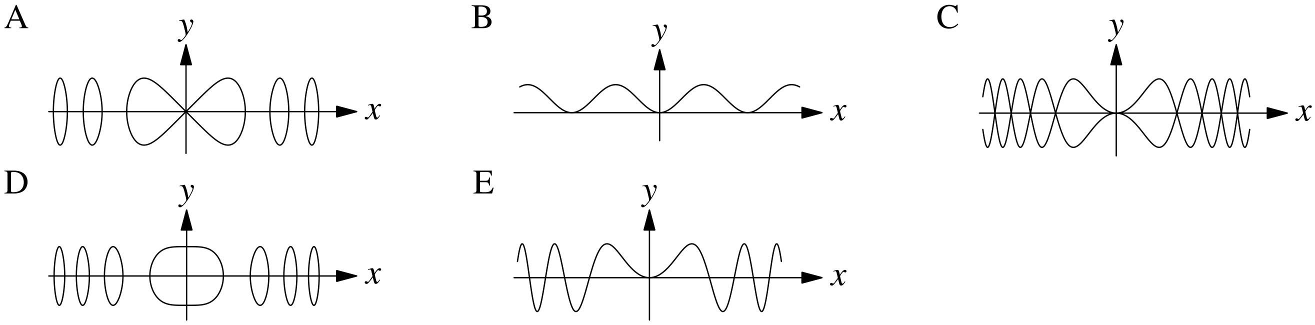
18. Alison folds a square piece of paper in half along the dashed line shown in the diagram. After opening the paper out again, she then folds one of the corners onto the dashed line.



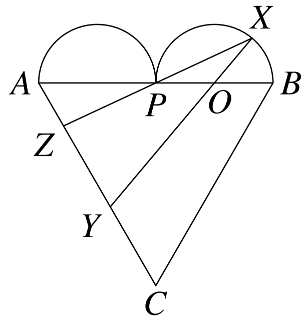
What is the value of α?

A 45 B 60 C 65 D 70 E 75

19. Which of the following could be the graph of $y^2 = \sin(x^2)$?



20. The "heart" shown in the diagram is formed from an equilateral triangle ABC and two congruent semicircles on AB. The two semicircles meet at the point P. The point O is the centre of one of the semicircles. On the semicircle with centre O, lies a point X. The lines XO and XP are extended to meet AC at Y and Z respectively. The lines XY and XZ are of equal length.



What is $\angle ZXY$?

A $20°$ B $25°$ C $30°$ D $40°$ E $45°$

21. In a square garden $PQRT$ of side 10 m, a ladybird sets off from Q and moves along edge QR at 30 cm per minute. At the same time, a spider sets off from R and moves along edge RT at 40 cm per minute.

What will be the shortest distance between them, in metres?

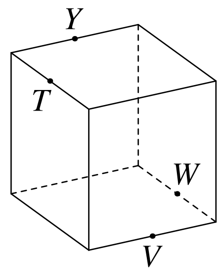
A 5 B 6 C $5\sqrt{2}$ D 8 E 10

22. A function f satisfies the equation $(n - 2019)f(n) - f(2019 - n) = 2019$ for every integer n.

What is the value of $f(2019)$?

A 0 B 1 C 2018×2019 D 2019^2 E 2019×2020

23. The edge-length of the solid cube shown is 2. A single plane cut goes through the points Y, T, V and W which are midpoints of the edges of the cube, as shown.



What is the area of the cross-section?

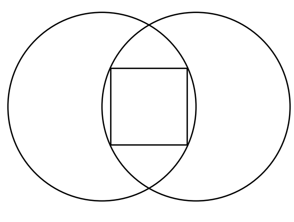
A $\sqrt{3}$ B $3\sqrt{3}$ C 6 D $6\sqrt{2}$ E 8

24. The numbers x, y and z are given by $x = \sqrt{12 - 3\sqrt{7}} - \sqrt{12 + 3\sqrt{7}}$, $y = \sqrt{7 - 4\sqrt{3}} - \sqrt{7 + 4\sqrt{3}}$ and $z = \sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}$.

What is the value of xyz ?

A 1 B -6 C -8 D 18 E 12

25. Two circles of radius 1 are such that the centre of each circle lies on the other circle. A square is inscribed in the space between the circles.



What is the area of the square?

A $2 - \frac{\sqrt{7}}{2}$ B $2 + \frac{\sqrt{7}}{2}$ C $4 - \sqrt{5}$ D 1 E $\frac{\sqrt{5}}{5}$

United Kingdom Mathematics Trust

Senior Mathematical Challenge 2019

Organised by the United Kingdom Mathematics Trust

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Solutions

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	B	A	B	B	D	D	B	D	C	C	E	D	C	B	A	E	E	A	A	D	C	B	E	A

1. What is the value of $123^2 - 23^2$?

A 10 000 B 10 409 C 12 323 D 14 600 E 15 658

Solution **D**

The value of $123^2 - 23^2 = (123 - 23)(123 + 23) = 100 \times 146 = 14600$.

Hints

Rather than direct calculation here, consider the more elegant method of using the difference of two squares $x^2 - y^2 = (x - y)(x + y)$.

2. What is the value of $(2019 - (2000 - (10 - 9))) - (2000 - (10 - (9 - 2019)))$?

A 4040 B 40 C -400 D -4002 E -4020

Solution **B**

The value of $(2019-(2000-(10-9)))-(2000-(10-(9-2019))) = (2019-1999)-(2000-2020) = 20 - (-20)$ which equals 40.

Hints

Work carefully from the innermost brackets to the outermost.

3. Used in measuring the width of a wire, one mil is equal to one thousandth of an inch. An inch is about 2.5 cm.

Which of these is approximately equal to one mil?

A $\frac{1}{40}$ mm B $\frac{1}{25}$ mm C $\frac{1}{4}$ mm D 25 mm E 40 mm

Solution **A**

One mil $= \frac{1}{1000}$ in $\approx \frac{1}{1000} \times 2.5$ cm $= \frac{25}{1000}$ mm $= \frac{1}{40}$ mm.

Hints

Start by writing the words of the statement in symbolic form.

4. For how many positive integer values of n is $n^2 + 2n$ prime?

A 0 B 1 C 2 D 3 E more than 3

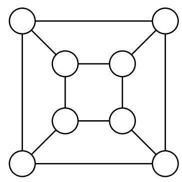
Solution **B**

The expression $n^2 + 2n$ factorises to $n(n + 2)$. For $n(n + 2)$ to be prime, one factor must equal 1 whilst the other must be equal to a prime. This happens when $n = 1$, as $n + 2 = 3$, but not when $n + 2 = 1$ as n would be negative. There is therefore exactly one positive integer value of n which makes $n^2 + 2n$ prime.

Hints

Consider the definition of a prime, in terms of its factors. Try and write the given expression in the same form.

5. Olive Green wishes to colour all the circles in the diagram so that, for each circle, there is exactly one circle of the same colour joined to it.

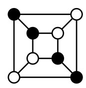


What is the smallest number of colours that Olive needs to complete this task?

A 1 B 2 C 3 D 4 E 5

Solution **B**

Each circle in the diagram is connected to three others, exactly one of which must be filled with the same colour. So, the number of colours required is greater than 1. One possible colouring with just two colours is shown here.



Hints

Consider the symmetry of the diagram.

6. Each of the factors of 100 is to be placed in a 3 by 3 grid, one per cell, in such a way that the products of the three numbers in each row, column and diagonal are all equal. The positions of the numbers 1, 2, 50 and x are shown in the diagram.

x	1	50
2		

What is the value of x?

A 4 B 5 C 10 D 20 E 25

Solution **D**

The product of all the factors of 100 is $1 \times 100 \times 2 \times 50 \times 4 \times 25 \times 5 \times 20 \times 10 = 1\,000\,000\,000$. As there are three rows, each of which has the same 'row product', that row product is 1000. So, considering the top row, $x \times 1 \times 50 = 1000$ and therefore $x = 20$. The completed grid is as shown.

20	1	50
25	10	4
2	100	5

Hints

Start by making a complete list of the factors of 100. Consider the required 'row product'.

7. Lucy is asked to choose p, q, r and s to be the numbers 1, 2, 3 and 4, in some order, so as to make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible.

What is the smallest value Lucy can achieve in this way?

A $\frac{7}{12}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{5}{6}$ E $\frac{11}{12}$

Solution **D**

In order to minimise the value of $\frac{p}{q} + \frac{r}{s}$ we need to make p and r as small as possible and make q and s be as large as possible. Considering $\frac{1}{3} + \frac{2}{4}$ and $\frac{1}{4} + \frac{2}{3}$ and then removing both $\frac{1}{3}$ and $\frac{1}{4}$ from each sum leaves the first with value $\frac{1}{4}$ and the second with value $\frac{1}{3}$. As $\frac{1}{4} < \frac{1}{3}$, the first sum has the smallest value, which is $\frac{5}{6}$.

Hints

Consider the sensible positions in which to put the smaller numbers and the larger numbers.

8. The number x is the solution to the equation $3^{(3^x)} = 333$.

Which of the following is true?

A $0 < x < 1$ B $1 < x < 2$ C $2 < x < 3$ D $3 < x < 4$ E $4 < x < 5$

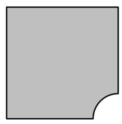
SOLUTION **B**

Considering some integer powers of 3, we have $3^5 = 243$ and $3^6 = 729$. As $243 < 333 < 729$, $3^5 < 3^{(3^x)} < 3^6$ implies that $5 < 3^x < 6$. Rewriting once again to include powers of 3, gives $3^1 < 5 < 3^x < 6 < 3^2$, so $3^1 < 3^x < 3^2$ and finally, $1 < x < 2$.

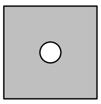
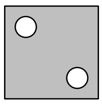
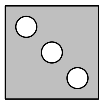
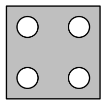
HINTS

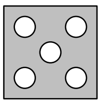
Start by thinking about the size of some small powers of three.

9. A square of paper is folded in half four times to obtain a smaller square. Then a corner is removed as shown.



Which of the following could be the paper after it is unfolded?

A  B  C  D 

E 

SOLUTION **D**

Folding the paper four times gives 2^4 layers. Removing a corner, 16 quarter-circles are formed. Of the given options, only D, with four whole circles could then be possible.

HINTS

We cannot tell from the question or the diagram where the fold lines are, so is there a way to tackle this without that being important?

10. Which of the following five values of n is a counterexample to the statement in the box below?

For a positive integer n, at least one of $6n - 1$ and $6n + 1$ is prime.

A 10 B 19 C 20 D 21 E 30

Solution **C**

To provide a counterexample, we are looking for both the values of $6n - 1$ and $6n + 1$ to be composite for a particular n . When $n = 20$, $6n - 1 = 119 = 7 \times 17$ and $6n + 1 = 121 = 11 \times 11$, which is our counter-example. In each of the other cases, at least one value is prime.

Hints

Consider what properties a positive integer has if it is not prime.

11. For how many integer values of k is $\sqrt{200 - \sqrt{k}}$ also an integer?

A 11 B 13 C 15 D 17 E 20

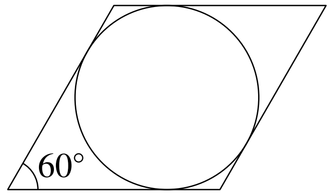
Solution **C**

In order for $\sqrt{200 - \sqrt{k}}$ to be an integer, $200 - \sqrt{k}$ must be a square. As $\sqrt{k} \geq 0$, the smallest possible value of $200 - \sqrt{k}$ which is square is 0, (when $k = 200^2$) and the largest is $14^2 = 196$, (when $k = 4^2$). Counting the squares from 0^2 to 14^2 gives 15 values.

Hints

Consider the smallest and largest values of k.

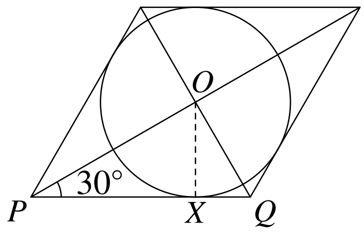
12. A circle with radius 1 touches the sides of a rhombus, as shown. Each of the smaller angles between the sides of the rhombus is 60°.



What is the area of the rhombus?

A 6 B 4 C $2\sqrt{3}$ D $3\sqrt{3}$ E $\frac{8\sqrt{3}}{3}$

SOLUTION **E**



Let the centre of the rhombus and circle be O. Let two of the vertices along an edge of the rhombus be P and Q and let X be the point on PQ where the rhombus is tangent to the circle. In order to relate the radius of the inscribed circle to a useful measurement on the rhombus, we can split the rhombus along its diagonals into four congruent triangles, one of which is POQ. As PQ is tangent to the circle at X, PQ and OX are perpendicular. Triangles OXP and OXQ are then similar 30°, 60°, 90° triangles with $OX = 1$, $XP = \sqrt{3}$ and $XQ = \frac{1}{\sqrt{3}}$. The area of the rhombus is then $4 \times \frac{1}{2} \times (\sqrt{3} + \frac{1}{\sqrt{3}}) \times 1 = 2 \times \frac{(3+1)}{\sqrt{3}} = \frac{8}{3}\sqrt{3}$.

HINTS

Try and split the rhombus into useful right-angled triangles.

13. Anish has a number of small congruent square tiles to use in a mosaic. When he forms the tiles into a square of side n, he has 64 tiles left over. When he tries to form the tiles into a square of side $n + 1$, he has 25 too few.

How many tiles does Anish have?

A 89 B 1935 C 1980 D 2000 E 2019

Solution **D**

Anish has both $n^2 + 64$ and $(n + 1)^2 - 25$ tiles. So, $n^2 + 64 = (n + 1)^2 - 25$ which simplifies to $n^2 + 64 = n^2 + 2n + 1 - 25$ and then $64 = 2n - 24$. So $n = \frac{88}{2} = 44$. Therefore Anish has $44^2 + 64 = 1936 + 64 = 2000$ tiles.

Hints

Start by writing the words of each statement in symbolic form.

14. One of the following is the largest square that is a factor of 10!. Which one?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n - 1) \times n$.

A $(4!)^2$ B $(5!)^2$ C $(6!)^2$ D $(7!)^2$ E $(8!)^2$

Solution **C**

For a square to be a factor of 10!, the prime factors of the square must be present in 10! an even number of times. Writing 10! first as $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ and then as $2 \times 5 \times 3 \times 3 \times 2 \times 2 \times 2 \times 7 \times 2 \times 3 \times 5 \times 2 \times 2 \times 3 \times 2 = 2^8 \times 3^4 \times 5^2 \times 7 = (2^4 \times 3^2 \times 5)^2 \times 7$ we can see that the largest square is $2^4 \times 3^2 \times 5$. Expanding $2^4 \times 3^2 \times 5$ as $(2 \times 3) \times 5 \times (2 \times 2) \times 3 \times 2 \times 1$ shows that it is exactly 6!. So $(6!)^2$ is the largest square which is a factor of 10!.

Hints

Consider how 10! is composed. What does it mean in terms of primes for a square to be a factor of a number?

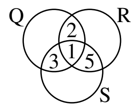
15. The highest common factors of all the pairs chosen from the positive integers Q, R and S are three different primes.

What is the smallest possible value of $Q + R + S$?

A 41 B 31 C 30 D 21 E 10

Solution **B**

For $Q + R + S$ to be as small as possible, we want the highest common factors of the pairs to be as small as possible, and prime. Therefore the highest common factors are 2, 3 and 5 in some order and then Q, R and S are 2×3, 2×5 and 3×5, i.e. 6, 10 and 15, in some order. This gives $Q + R + S = 6 + 15 + 10 = 31$.



Hints

Start by considering which primes to have as the highest common factors.

16. The numbers x, y and z satisfy the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$.

What is the mean of x, y and z?

A 10 B 11 C 12 D 13 E 14

Solution **A**

As $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$, subtracting the first equation from twice the second gives $(10x + 4y - 4z) - (9x + 3y - 5z) = 2 \times 13 - (-4)$. The mean of x, y and z which is $\frac{(x+y+z)}{3}$ is therefore $\frac{30}{3} = 10$.

Hints

Consider the shape of the algebraic expression which will allow you to find the mean of x, y and z. Do you need to know the values of each of x, y and z?

17. Jeroen writes a list of 2019 consecutive integers. The sum of his integers is 2019.

What is the product of all the integers in Jeroen's list?

A 2019^2 B $\frac{2019 \times 2020}{2}$ C 2^{2019} D 2019 E 0

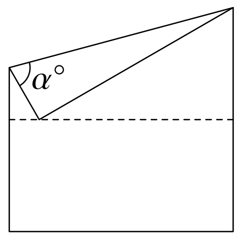
Solution **E**

The sum of the first 2019 positive integers is $\frac{2019 \times 2020}{2}$ which is considerably larger than the required sum of 2019. In order for the sum of Jeroen's 2019 integers to be only 2019, some of the integers must be positive and some must be negative. One of the integers will then be 0, so the product will also be 0. Jeroen's list is $-1008, \ldots - 2, -1, 0, 1, 2, \ldots, 1008, 1009, 1010$.

Hints

Think about what an integer is.

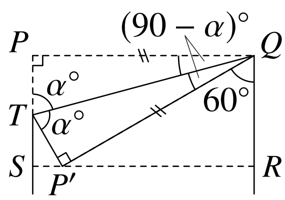
18. Alison folds a square piece of paper in half along the dashed line shown in the diagram. After opening the paper out again, she then folds one of the corners onto the dashed line.



What is the value of α?

A 45 B 60 C 65 D 70 E 75

Solution **E**



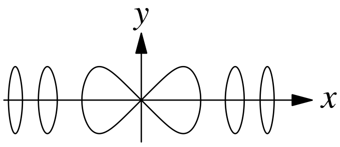
Let P and Q be the vertices on the top edge of the square. Let R and S be the points at the end of the first fold line. Let P' be the position of P on the first fold line after the second fold has been made. Let T be the point on PS which lies on the second fold line. Triangles PQT and $P'QT$ are then congruent so $\angle PQT = \angle P'QT = 90° - \alpha°$. As $PQ = 2QR$ then $P'Q = 2QR$ and so $\angle P'QR = 60°$. Considering angles at Q gives $2(90 - \alpha) + 60 = 90$, so $150 = 2\alpha$ and $\alpha = 75$.

Hints

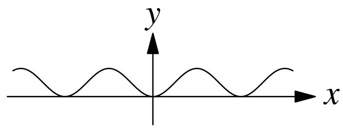
Consider which angles you know in the top half of the diagram. Don't lose sight of the fact that you are dealing with a square.

19. Which of the following could be the graph of $y^2 = \sin(x^2)$?

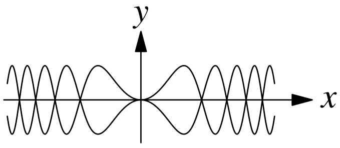
A



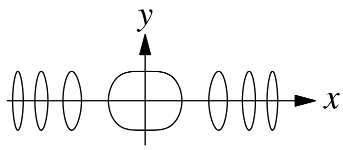
B



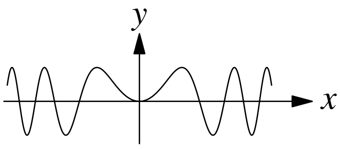
C



D



E



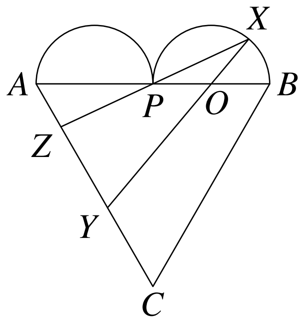
Solution **A**

As $\sin(0^2) = 0^2$ our graph must pass through the origin, so eliminating option D. As $y^2 = \sin(x^2)$, $y = \pm\sqrt{\sin(x^2)}$ and so the x-axis must be a line of symmetry, so eliminating options B and E. For some x-values, $\sin(x^2)$ will be negative and so there will be no corresponding y-value, so eliminating option C, which has y-values for every x-value. The only possible remaining option is then A and it can be checked that the graph is indeed of this form.

Hints

Work to eliminate impossible options.

20. The "heart" shown in the diagram is formed from an equilateral triangle ABC and two congruent semicircles on AB. The two semicircles meet at the point P. The point O is the centre of one of the semicircles. On the semicircle with centre O, lies a point X. The lines XO and XP are extended to meet AC at Y and Z respectively. The lines XY and XZ are of equal length.

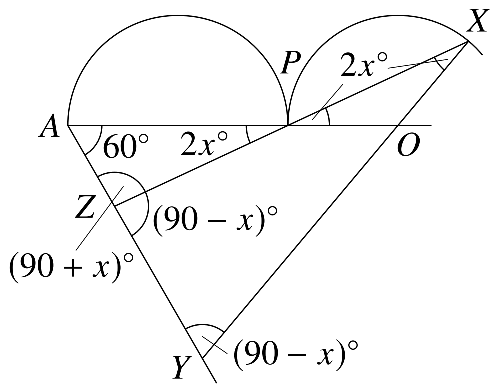


What is $\angle ZXY$?

A 20° B 25° C 30° D 40° E 45°

Solution **A**

Let $\angle ZXY = 2x°$, then the equal angles in isosceles triangle ZXY, are each $\frac{(180-2x)°}{2} = (90-x)°$. We can then find each of the angles inside triangle AZP in terms of x. Considering angles at Z gives $\angle AZP = 180° - (90-x)° = (90+x)°$. Then, as triangle OXP is isosceles, $\angle OPX = \angle OXP = 2x°$. As $\angle ZPA$ is vertically opposite $\angle OPX$, $\angle ZPA$ is also equal to $2x°$. Finally, $\angle OAY = 60°$ as triangle BAC is given to be equilateral. In triangle AZP, $60 + 90 + x + 2x = 180$, so $x = 10$ and $\angle ZXY = 2x° = 20°$.



Hints

Label the angle ZXY as $2x°$ to avoid fractional angles as you 'angle chase' around the diagram.

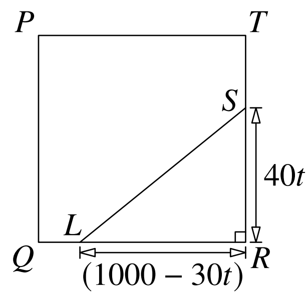
21. In a square garden $PQRT$ of side 10 m, a ladybird sets off from Q and moves along edge QR at 30 cm per minute. At the same time, a spider sets off from R and moves along edge RT at 40 cm per minute.

What will be the shortest distance between them, in metres?

A 5 B 6 C $5\sqrt{2}$ D 8 E 10

Solution **D**

Let L be the position of the ladybird on QR and let S be the position of the spider on RT each after t minutes. The shortest distance between L and S is along the straight line which is the hypotenuse of the right-angled triangle LRS. The distance QL is $30t$ cm, so the distance LR is $(1000 - 30t)$ cm. Also, the distance RS is $40t$ cm. So, $LS^2 = (1000 - 30t)^2 + (40t)^2$ which expands and simplifies to $2500(t^2 - 24t + 400) = 2500((t-12)^2 + 256)$. The distance from L to S is shortest when $t = 12$ and is $\sqrt{2500 \times 256}$ cm $= 800$ cm $= 8$ m.



Hints

Start by drawing a diagram which represents the positions of L and S part way through their journeys. Take care to be consistent with units when labelling the diagram.

22. A function f satisfies the equation $(n-2019)f(n) - f(2019-n) = 2019$ for every integer n.

What is the value of $f(2019)$?

A 0 B 1 C 2018×2019 D 2019^2 E 2019×2020

Solution **C**

At the start, taking $n = 0$ gives $(0-2019)f(0) - f(2019-0) = 2019$. Then $f(2019) = -2019(1 + f(0))$. To find $f(0)$, let $n = 2019$, then $(2019 - 2019)f(2019) - f(2019 - 2019) = 2019$, so $0 - f(0) = 2019$ and $f(0) = -2019$. Then we have $f(2019) = -2019(1 + (-2019)) = 2018 \times 2019$.

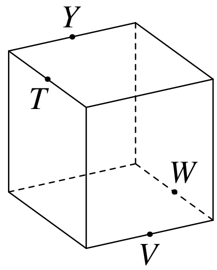
Hints

Think about which values of n will give you useful simplified expressions.

23. The edge-length of the solid cube shown is 2. A single plane cut goes through the points Y, T, V and W which are midpoints of the edges of the cube, as shown.

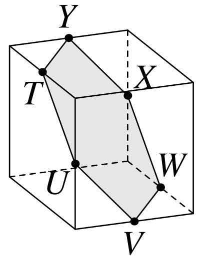
What is the area of the cross-section?

A $\sqrt{3}$ B $3\sqrt{3}$ C 6 D $6\sqrt{2}$ E 8



Solution **B**

When the single plane cut is made through the cube, it passes through points Y, T, V and W and also points U and X which are midpoints of two of the remaining edges of the cube as shown. The cross-section is then a regular hexagon. As the side-length of the cube is 2, the distance between the midpoints of adjacent edges is $\sqrt{2}$. This is the length of each edge of the hexagon.The hexagon can be split into six equilateral triangles and so the area of the hexagon is $6 \times \frac{1}{2} \times \sqrt{2} \times \sqrt{2} \times \sin 60° = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$.



Hints

Try and imagine making the plane cut. What shape is your cross section? Is that shape regular or irregular?

24. The numbers x, y and z are given by $x = \sqrt{12 - 3\sqrt{7}} - \sqrt{12 + 3\sqrt{7}}$, $y = \sqrt{7 - 4\sqrt{3}} - \sqrt{7 + 4\sqrt{3}}$ and $z = \sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}$.

What is the value of xyz ?

A 1 B –6 C –8 D 18 E 12

SOLUTION **E**

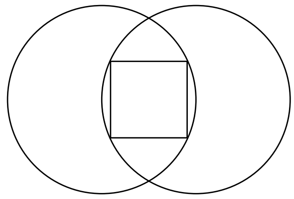
Each of x, y and z has the same form which is $\sqrt{a - b} - \sqrt{a + b}$ for some a and b. Squaring this expression gives $(a - b) - 2\sqrt{a + b} \times \sqrt{a - b} + (a + b) = 2a - 2\sqrt{(a^2 - b^2)}$. Applying this to x with $a = 12$ and $b = 3\sqrt{7}$ gives $x^2 = 24 - 2\sqrt{81} = 6$. Similarly we can calculate that $y^2 = 12$ and $z^2 = 2$. This gives us that $x^2y^2z^2 = 6 \times 12 \times 2 = 144$. From the initial expressions $x < 0$, $y < 0$ and $z > 0$, so $xyz > 0$ and therefore $xyz = 12$.

HINTS

Look for similarities in the form of each of x, y and z. What expression would be easier to calculate than xyz ?

25. Two circles of radius 1 are such that the centre of each circle lies on the other circle. A square is inscribed in the space between the circles.

What is the area of the square?

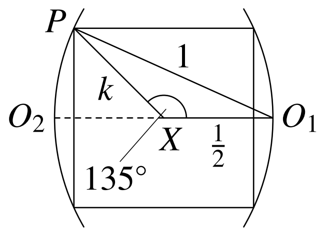


A $2 - \frac{\sqrt{7}}{2}$ B $2 + \frac{\sqrt{7}}{2}$ C $4 - \sqrt{5}$

D 1 E $\frac{\sqrt{5}}{5}$

SOLUTION **A**

Let the centres of the circles and the square be O_1, O_2 and X respectively. Let P be the point on the circle with centre O_1 which is a vertex of the square. Then $O_1P = 1$ and $O_1O_2 = 1$ so $O_1X = \frac{1}{2}$. Let $XP = k$, so the area of the square will be $2k^2$. As O_1X is parallel to the top edge of the square and XP goes from the centre of the square to a vertex, angle $O_1XP = 135°$. Using the cosine rule on triangle O_1XP gives $1^2 = (\frac{1}{2})^2 + k^2 - 2 \times \frac{1}{2} \times k \times \cos 135°$. As $\cos 135° = -\cos 45° = -\frac{1}{\sqrt{2}}$, this simplifies to $4k^2 + 2\sqrt{2}k - 3 = 0$. Since we know $k > 0$, $k = \frac{-1+\sqrt{7}}{2\sqrt{2}}$. So $\sqrt{2}k = \frac{-1+\sqrt{7}}{2}$ and the area of the square is $2k^2 = 2 - \frac{\sqrt{7}}{2}$.



HINTS

Consider a triangle whose vertices are the centre of the square, a vertex of the square and the centre of the circle furthest from that vertex.

United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

7 November 2019

These solutions augment the shorter solutions also available online. For convenience, the shorter solutions are confined to four pages and therefore in many cases omit details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339   enquiry@ukmt.org.uk   www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	B	A	B	B	D	D	B	D	C	C	E	D	C	B	A	E	E	A	A	D	C	B	E	A

1. What is the value of $123^2 - 23^2$?

A 10 000 B 10 409 C 12 323 D 14 600 E 15 658

SOLUTION **D**

COMMENTARY

The SMC is a *no calculator* paper. This should be a clue that there is a better way to answer this question than separately squaring 123 and 23 and then doing a subtraction.

There is no great virtue in being able to evaluate $123^2 - 23^2$ without a calculator, but the underlying method of factorizing the difference of two squares $x^2 - y^2$ as $(x - y)(x + y)$ is very useful and should be remembered.

We evaluate $123^2 - 23^2$ as follows.

$$\begin{aligned} 123^2 - 23^2 &= (123 - 23)(123 + 23) \\ &= 100 \times 146 \\ &= 14\,600. \end{aligned}$$

FOR INVESTIGATION

1.1 Find the values of

(a) $57^2 - 43^2$,

(b) $203^2 - 197^2$,

(c) $2019^2 - 2018^2$.

1.2 Find all the pairs (a, b) of positive integers such that

$$a^2 = b^2 + 2019.$$

1.3 Factorize

$$x^4 - y^4.$$

1.4 Prove that if p and q are both prime numbers, with $p > q > 2$, then $p^4 - q^4$ is divisible by 16.

1.5 Use the fact [see Exercise 10.2] that each prime greater than 3 has the form $6n \pm 1$ for some integer n, to prove that if p and q are primes with $p > q > 3$, then $p^4 - q^4$ is divisible by 48.

1.6 Prove that if p and q are both prime numbers, with $p > q > 5$, then $p^4 - q^4$ is divisible by 240.

2. What is the value of $(2019 - (2000 - (10 - 9))) - (2000 - (10 - (9 - 2019)))$?

A 4040 B 40 C −400 D −4002 E −4020

SOLUTION **B**

We evaluate the expression by first evaluating the innermost brackets and then working outwards.

$$\begin{aligned}(2019 - (2000 - (10 - 9))) - (2000-(10 - (9 - 2019)))\\ &= (2019 - (2000 - 1)) - (2000 - (10 - (-2010)))\\ &= (2019 - (2000 - 1)) - (2000 - (10 + 2010))\\ &= (2019 - 1999) - (2000 - 2020)\\ &= (2019 - 1999) - (-20)\\ &= 20 + 20\\ &= 40.\end{aligned}$$

FOR INVESTIGATION

2.1 What are the values of

(a) $(5 - (4 - (3 - (2 - 1)))) - (1 - (2 - (3 - (4 - 5))))$ and

(b) $(6 - (5 - (4 - (3 - (2 - 1))))) - (1 - (2 - (3 - (4 - (5 - 6)))))$?

2.2 Generalize the results of Exercise 2.1.

3. Used in measuring the width of a wire, one mil is equal to one thousandth of an inch. An inch is about 2.5 cm.

Which of these is approximately equal to one mil?

A $\frac{1}{40}$ mm B $\frac{1}{25}$ mm C $\frac{1}{4}$ mm D 25 mm E 40 mm

SOLUTION **A**

One mil is equal to one thousandth of an inch. An inch is about 2.5 cm, which is the same as 25 mm. Therefore one mil is approximately $\frac{25}{1000}$ mm, which is the same as $\frac{1}{40}$ mm.

FOR INVESTIGATION

3.1 There are twelve inches in one foot, three feet in one yard, and 1760 yards in one mile. Approximately how many metres are there in one mile?

4. For how many positive integer values of n is $n^2 + 2n$ prime?

A 0 B 1 C 2 D 3 E more than 3

Solution **B**

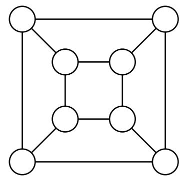
We have $n^2 + 2n = n(n + 2)$. Therefore $n^2 + 2n$ is divisible by n. Hence, for $n^2 + 2n$ to be prime, n can only have the value 1.

When $n = 1$, we have $n^2 + 2n = 3$, which is prime.

Therefore there is just one positive integer value of n for which $n^2 + 2n$ is prime.

5. Olive Green wishes to colour all the circles in the diagram so that, for each circle, there is exactly one circle of the same colour joined to it.

What is the smallest number of colours that Olive needs to complete this task?

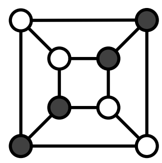


A 1 B 2 C 3 D 4 E 5

Solution **B**

If Olive uses just one colour, then each circle would be joined to three circles with the same colour as it. So one colour is not enough.

However, as the diagram shows, it is possible using just two colours to colour the circles so that each white circle is joined to just one white circle, and each black circle is joined to just one black circle.

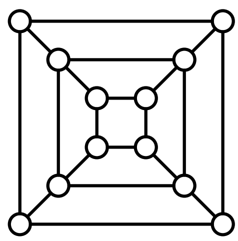


Therefore the smallest number of circles that Olive needs is two.

For investigation

5.1 Cherry Red wishes to colour all the circles in the diagram so that for each circle, there is exactly one circle with the same colour joined to it.

What is the smallest number of colours that Cherry needs to complete this task?



6. Each of the factors of 100 is to be placed in a 3 by 3 grid, one per cell, in such a way that the products of the three numbers in each row, column and diagonal are all equal. The positions of the numbers 1, 2, 50 and x are shown in the diagram.

x	1	50
2		

What is the value of x?

A 4 B 5 C 10 D 20 E 25

SOLUTION **D**

Let P be the common product of the three numbers in each row. Then $P \times P \times P$ is the product of all the numbers in all three rows. Therefore P^3 is the product of all the factors of 100.

Because $100 = 2^2 \times 5^2$, it has the nine factors 1, $2^1 = 2$, $2^2 = 4$, $5^1 = 5$, $2^1 5^1 = 10$, $2^2 5^1 = 20$, $5^2 = 25$, $2^1 5^2 = 50$ and $2^2 5^2 = 100$. The product of these factors is

$$1 \times 2^1 \times 2^2 \times 5^1 \times 2^1 5^1 \times 2^2 5^1 \times 2^1 5^2 \times 5^2 \times 2^2 5^2 = 2^{1+2+1+2+1+2} \times 5^{1+1+1+2+2+2}$$
$$= 2^9 \times 5^9 \ (= 1\,000\,000\,000).$$

It follows that

$$P^3 = 2^9 \times 5^9$$

and therefore $P = 2^3 \times 5^3 = 1000$.

From the first row we have

$$x \times 1 \times 50 = 1000$$

and hence $x = 20$.

NOTE

In the context of the SMC it is not necessary to check that it is possible to complete the grid with all the factors of 100, so as to meet the condition that the product of the three numbers in each row, column and diagonal are all equal. However, you are asked to do this in Exercise 6.1.

Note, also that our solution does not use the position of the factor 2. Exercise 6.1 asks you to show that with 2 in the bottom left-hand cell, there is just one way to complete the grid.

FOR INVESTIGATION

6.1 Show that it is possible to complete the grid with the factors of 100 so as to meet the required condition in just one way.

6.2 How many ways are there to complete the grid if the factor 2 does not have to be in the bottom left-hand cell?

6.3 Suppose that $n = p^2 q^2$, where p and q are different primes. Explain why n has nine factors.

6.4 Suppose that $n = p^a q^b$, where p and q are different primes and a and b are non-negative integers. How many factors does n have?

6.5 Find a general formula for the number of factors of a positive integer in terms of the exponents that occur in its prime factorization.

7. Lucy is asked to choose p, q, r and s to be the numbers 1, 2, 3 and 4, in some order, so as to make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible.

What is the smallest value Lucy can achieve in this way?

A $\frac{7}{12}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{5}{6}$ E $\frac{11}{12}$

Solution **D**

To make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible, q and s need to be as large as possible, and so have the values 3 and 4.

Therefore the expression with the smallest value that Lucy can achieve is either $\frac{1}{3} + \frac{2}{4}$ or $\frac{2}{3} + \frac{1}{4}$.

Now,

$$\begin{aligned}\frac{1}{3} + \frac{2}{4} &= \frac{1}{3} + \frac{1}{4} + \frac{1}{4} \\ &< \frac{1}{3} + \frac{1}{3} + \frac{1}{4} \\ &= \frac{2}{3} + \frac{1}{4}.\end{aligned}$$

It follows that the smallest value that Lucy can achieve is

$$\frac{1}{3} + \frac{2}{4} = \frac{4+6}{12} = \frac{10}{12} = \frac{5}{6}.$$

For investigation

7.1 Lucy is asked to choose p, q, r, s, t and u to be the numbers 1, 2, 3, 4, 5 and 6, in some order, so as to make the value of $\frac{p}{q} + \frac{r}{s} + \frac{t}{u}$ as small as possible.

What is the smallest value Lucy can achieve in this way?

7.2 Explain why the statement in the first line of the above solution that

"to make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible, q and s need to be as large as possible"

is correct.

8. The number x is the solution to the equation $3^{(3^x)} = 333$.

Which of the following is true?

A $0 < x < 1$ B $1 < x < 2$ C $2 < x < 3$ D $3 < x < 4$ E $4 < x < 5$

SOLUTION **B**

The first few powers of 3 are

$$3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ 3^5 = 243 \text{ and } 3^6 = 729.$$

Because $3^{(3^x)} = 333$, it follows that

$$3^5 < 3^{(3^x)} < 3^6$$

and hence

$$5 < 3^x < 6.$$

Therefore

$$3^1 < 3^x < 3^2$$

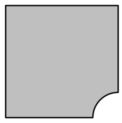
and hence

$$1 < x < 2.$$

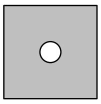
FOR INVESTIGATION

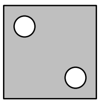
8.1 The number x is the solution to the equation $2^{2^x} = 10^6$. Find the integer n such that $n < x < n + 1$.

9. A square of paper is folded in half four times to obtain a smaller square. Then a corner is removed as shown.



Which of the following could be the paper after it is unfolded?

A


B


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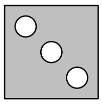

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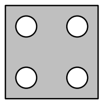

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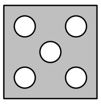


SOLUTION **D**

Each time the square piece of paper is folded in half the number of layers of paper doubles. Therefore, after it has been folded four times, there are 16 layers.

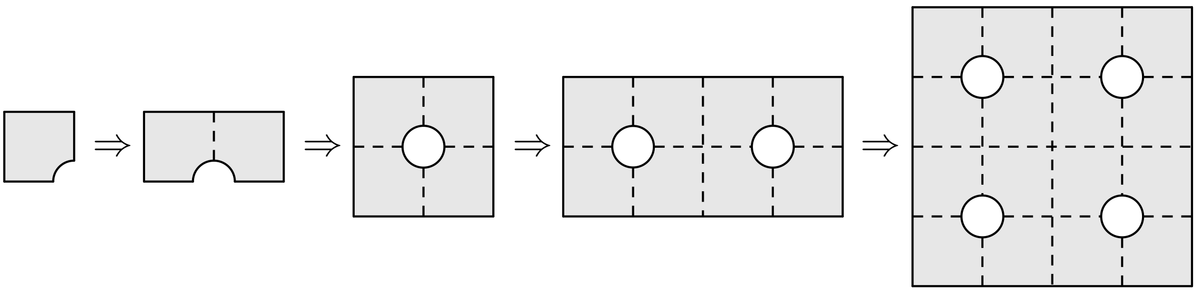
It follows that when the corner is then removed, altogether 16 quarter circles have been removed.

Hence, if these quarter circles come together to make complete circles after the paper has been unfolded, they will make four complete circles.

It follows that of the options given in the question, option D is the only one that is possible.

NOTE

In the context of the SMC it is sufficient to show that option D is the only possibility. For a complete answer it is also necessary to show that the pattern of option D can be achieved. This is shown by the diagram below. This shows that, provided none of the quarter-circles that are removed comes from the edge of paper, the paper will unfold to make the pattern of option D.



FOR INVESTIGATION

9.1 What are the other possibilities for the paper after it has been unfolded?

10. Which of the following five values of n is a counterexample to the statement in the box below?

For a positive integer n, at least one of $6n - 1$ and $6n + 1$ is prime.

A 10 B 19 C 20 D 21 E 30

Solution **C**

A counterexample to the statement in the box is a value of n for which it is not true that at least one of $6n - 1$ and $6n + 1$ is prime. That is, it is a value of n for which neither $6n - 1$ nor $6n + 1$ is prime.

We set out the values of $6n - 1$ and $6n + 1$ for $n = 10,\ 19,\ 20,\ 21$ and 30 in the following table.

n	$6n - 1$	$6n + 1$
10	59	61
19	113	**115**
20	**119**	**121**
21	**165**	167
30	179	181

The values of $6n - 1$ and $6n + 1$ that are not prime are shown in bold.

We therefore see that for $n = 20$, neither $6n - 1$ nor $6n + 1$ is prime. Therefore $n = 20$ provides the required counterexample.

For investigation

10.1 Check that, of the numbers that occur in the table above, 59, 61, 113, 167, 179 and 181 are prime, and that 115, 119, 121 and 165 are not prime.

10.2 Show that if p is a prime number other than 2 or 3, then there is a positive integer n such that p is either equal to $6n - 1$ or $6n + 1$.

11. For how many integer values of k is $\sqrt{200 - \sqrt{k}}$ also an integer?

A 11 B 13 C 15 D 17 E 20

Solution **C**

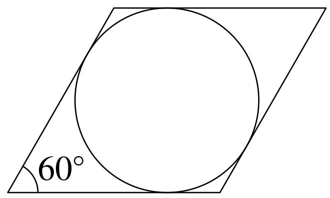
The number $\sqrt{200 - \sqrt{k}}$ is an integer if, and only if, the number $200 - \sqrt{k}$ is a square.

Now $0 \leq 200 - \sqrt{k} \leq 200$. There are 15 squares in this range, namely n^2 for integer values of n with $0 \leq n \leq 14$.

We have that $200 - \sqrt{k} = n^2$, if, and only if $k = (200 - n^2)^2$.

Hence there are 15 integer values of k for which $\sqrt{200 - \sqrt{k}}$ is an integer, namely, $k = (200 - n^2)^2$ for $0 \leq n \leq 14$.

12. A circle with radius 1 touches the sides of a rhombus, as shown. Each of the smaller angles between the sides of the rhombus is $60°$.

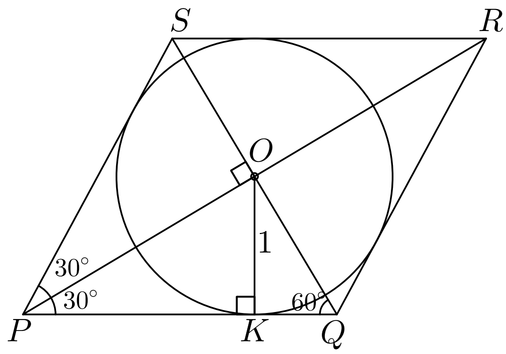


What is the area of the rhombus?

A 6 B 4 C $2\sqrt{3}$ D $3\sqrt{3}$ E $\frac{8\sqrt{3}}{3}$

Solution **E**

Let P, Q, R and S be the vertices of the rhombus.



We leave it to the reader to prove that the diagonals of a rhombus bisect the angles of the rhombus and meet at right angles (see Exercise 12.1).

Let O be the point where the diagonals PR and QS of the rectangle meet.

We also leave it to the reader to prove that the four triangles POQ, QOR, ROS and SOP are congruent (see Exercise 12.1) and that O is the centre of the circle (see Exercise 12.2).

We let K be the point where PQ touches the circle. Then $OK = 1$. Because the radius of a circle is at right angles to the tangent at the point where the radius meets the circle, $\angle PKO = 90°$.

From the right-angled triangle PKO we have

$$\frac{OK}{OP} = \sin 30° = \frac{1}{2},$$

and therefore, since $OK = 1$, it follows that $OP = 2$.

Because $\angle POQ = 90°$, it follows from the triangle POQ that $\angle OQK = 60°$, and hence

$$\frac{OK}{OQ} = \sin 60° = \frac{\sqrt{3}}{2},$$

and therefore $OQ = \dfrac{2}{\sqrt{3}}$.

It now follows that the area of the triangle POQ is given by

$$\frac{1}{2}(OP \times OQ) = \frac{1}{2}\left(2 \times \frac{2}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}.$$

Therefore, as the rhombus is made up of four triangles each congruent to the triangle POQ, the area of the rhombus is

$$4 \times \frac{2}{\sqrt{3}} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}.$$

Note: See Exercise 12.4 for a way to remember the values of $\sin(30°)$ and $\sin(60°)$.

For investigation

12.1 (a) Prove that the diagonals of a rhombus divide the rhombus into four congruent triangles.

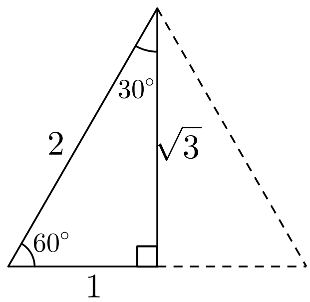
(b) Deduce that the diagonals of a rhombus bisect the angles of the rhombus and meet at right angles.

12.2 Prove that the diagonals of a rhombus meet at the centre of a circle that touches the four sides of the rhombus

12.3 Prove that a radius of a circle is at right angles to the tangent at the point where the radius meets the circle.

12.4 A triangle with angles of 30°, 60° and 90° forms one half of an equilateral triangle, as shown in the diagram. We suppose that the side length of the equilateral triangle is 2.

Then the shortest side of the $30°, 60°, 90°$ triangle is 1.



(a) Use Pythagoras' Theorem to check that the third side of this triangle has length $\sqrt{3}$.

(b) Use this triangle to check that that $\sin(30°) = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$.

(c) Use this triangle to find the values of cos(30°), tan(30°), cos(60°) and tan(60°).

12.5 Use a different triangle to obtain the values of sin(45°), cos(45°) and tan(45°).

13. Anish has a number of small congruent square tiles to use in a mosaic. When he forms the tiles into a square of side n, he has 64 tiles left over. When he tries to form the tiles into a square of side $n + 1$, he has 25 too few.

How many tiles does Anish have?

A 89 B 1935 C 1980 D 2000 E 2019

SOLUTION **D**

Because Anish has 64 tiles left over when he forms a square of side n, he has $n^2 + 64$ tiles.

Because Anish has 25 tiles too few to make a square of side $n + 1$, he has $(n + 1)^2 - 25$ tiles.

Therefore $n^2 + 64 = (n + 1)^2 - 25$. Now

$$\begin{aligned} n^2 + 64 = (n+1)^2 - 25 &\Leftrightarrow n^2 + 64 = (n^2 + 2n + 1) - 25 \\ &\Leftrightarrow 2n = 64 + 25 - 1 \\ &\Leftrightarrow 2n = 88 \\ &\Leftrightarrow n = 44. \end{aligned}$$

Because $n = 44$, the number of tiles that Anish has is given by $44^2 + 64 = 1936 + 64 = 2000$.

FOR INVESTIGATION

13.1 For $n = 44$, check that $(n + 1)^2 - 25$ also equals 2000.

13.2 Anish has exactly enough square 1×1 tiles to form a square of side m. He would need 2019 more tiles to form a square of side $m + 1$.

How many tiles does Anish have?

13.3 Anish has exactly enough $1 \times 1 \times 1$ cubes to form a cube of side m. He would need 397 more cubes to form a cube of side $m + 1$.

How many cubes does Anish have?

14. One of the following is the largest square that is a factor of $10!$. Which one?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$

A $(4!)^2$ B $(5!)^2$ C $(6!)^2$ D $(7!)^2$ E $(8!)^2$

SOLUTION **C**

COMMENTARY

If you are familiar with the values of $n!$ for small values of n, it is not difficult to spot the answer quite quickly, as in Method 1. If not, a more systematic method is to work out the prime factorization of $10!$, as in Method 2.

METHOD 1

We have

$$\begin{aligned} 10! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &= (1 \times 2 \times 3 \times 4 \times 5 \times 6) \times 7 \times (8 \times 9 \times 10) \\ &= 6! \times 7 \times (8 \times 9 \times 10) \\ &= 6! \times 7 \times 720 \\ &= 6! \times 7 \times 6! \\ &= (6!)^2 \times 7. \end{aligned}$$

It follows that $(6!)^2$ is the largest square that is a factor of $10!$.

METHOD 2

We have

$$\begin{aligned} 10! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &= 1 \times 2 \times 3 \times 2^2 \times 5 \times (2 \times 3) \times 7 \times 2^3 \times 3^2 \times (2 \times 5) \\ &= 2^8 \times 3^4 \times 5^2 \times 7 \\ &= (2^4 \times 3^2 \times 5)^2 \times 7. \end{aligned}$$

It follows that $2^4 \times 3^2 \times 5$ is the largest square that is a factor of $10!$. Now $2^4 \times 3^2 \times 5 = 720 = 6!$ and therefore $(6!)^2$ is the largest square that is a factor of $10!$.

FOR INVESTIGATION

14.1 Which is the largest square that is a factor of (a) $12!$, and (b) $14!$?

14.2 List the values of $n!$ for all positive integers n with $n \leq 10$.

15. The highest common factors of all the pairs chosen from the positive integers Q, R and S are three different primes.

What is the smallest possible value of $Q + R + S$?

A 41 B 31 C 30 D 21 E 10

SOLUTION **B**

We use the notation HCF(X, Y) for the highest common factor of the two integers X and Y.

Suppose that HCF$(Q, R) = a$, HCF$(Q, S) = b$ and HCF$(R, S) = c$, where a, b and c are three different primes.

It follows that both a and b are factors of Q. Therefore the smallest possible value of Q is ab. Likewise, the smallest possible values of R and S are ac and bc, respectively.

We seek the smallest possible value of $Q + R + S$, that is, of $ab + ac + bc$, where a, b and c are different primes. To do this we choose the values of a, b and c to be the three smallest primes, that is 2, 3 and 5, in some order.

Because $ab + ac + bc$ is symmetric in a, b and c, the order does not matter. With $a = 2$, $b = 3$ and $c = 5$, we have

$$ab + ac + bc = 2 \times 3 + 2 \times 5 + 3 \times 5 = 6 + 10 + 15 = 31.$$

We deduce that the smallest possible value of $Q + R + S$ is 31.

FOR INVESTIGATION

15.1 Show that if, in the above solution, a, b and c are given the values 2, 3 and 5 in some other order, then the value of $ab + ac + bc$ is again 31.

15.2 Show that if, in the above solution, any of 2, 3 and 5 is replaced by a prime larger than 5, then the resulting value of $Q + R + S$ is greater thsn 31.

15.3 The highest common factors of all the pairs chosen from the positive integers Q, R, S and T are six different primes.

What is the smallest possible value of $Q + R + S + T$?

16. The numbers x, y and z satisfy the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$.

What is the mean of x, y and z?

A 10 B 11 C 12 D 13 E 14

SOLUTION **A**

COMMENTARY

The mean of x, y and z is $\frac{1}{3}(x + y + z)$. Therefore to answer this question we need to find the value of $x + y + z$. We are given just two equations for the three unknowns x, y and z. It follows that if these equations have a solution, they will have an infinite number of solutions.

A systematic method for answering the question would be to use the two equations to find expressions for two of the unknowns in terms of the third unknown. For example, we could find x and y in terms of z, and thus work out $x + y + z$ in terms of z.

However, the wording of the question suggests that $x + y + z$ is independent of z. Thus a good starting point is to try to find a way to use the two equations we are given to find a value for $x + y + z$ without the need to find x and y in terms of z.

We are given that

$$9x + 3y - 5z = -4 \qquad (1)$$

and

$$5x + 2y - 2z = 13 \qquad (2)$$

If we multiply equation (2) by 2, and subtract equation (1) we obtain

$$2(5x + 2y - 2z) - (9x + 3y - 5z) = 2(13) - (-4),$$

that is,

$$10x + 4y - 4z - 9x - 3y + 5z = 26 + 4,$$

that is,

$$x + y + z = 30.$$

We deduce that $\frac{1}{3}(x + y + z) = 10$.

Hence the mean of x, y and z is 10.

FOR INVESTIGATION

16.1 (a) Use the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$ to find expressions for y and z in terms of x.

(b) Use your answers to part (a) to show that for all values of x, we have $x + y + z = 30$.

16.2 The numbers x, y and z satisfy the equations $3x - 5y + 7z = 4$ and $4x - 8y + 10z = 6$. What is the mean of x, y and z?

16.3 You may know that the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$ represent two-dimensional planes in three-dimensional space. What can you deduce about these planes from the fact that the equations imply that $x + y + z = 30$?

17. Jeroen writes a list of 2019 consecutive integers. The sum of his integers is 2019.

What is the product of all the integers in Jeroen's list?

A 2019^2 B $\frac{2019 \times 2020}{2}$ C 2^{2019} D 2019 E 0

SOLUTION **E**

In a list of 2019 consecutive *positive* integers, at least one of them will be greater than or equal to 2019, and therefore the sum of these integers will be greater than 2019. So the integers in Jeroen's list are not all positive.

The sum of 2019 *negative* integers is negative and therefore cannot be equal to 2019. So the integers in Jeroen's list are not all negative.

We deduce that Jeroen's list of consecutive integers includes both negative and positive integers.

Because the integers in the list are consecutive it follows that one of them is 0.

Therefore the product of all the numbers in Jeroen's list is 0.

FOR INVESTIGATION

17.1 Note that we were able to answer this question without finding a list of 2019 consecutive integers with sum 2019. Show that there is just one list of 2019 consecutive integers whose sum is 2019, and find it.

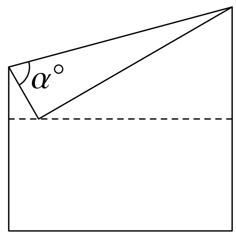
17.2 Find all the lists of at least two consecutive integers whose sum is 2019.

17.3 Investigate which positive integers can be expressed as the sum of two or more consecutive positive integers.

18. Alison folds a square piece of paper in half along the dashed line shown in the diagram. After opening the paper out again, she then folds one of the corners onto the dashed line.

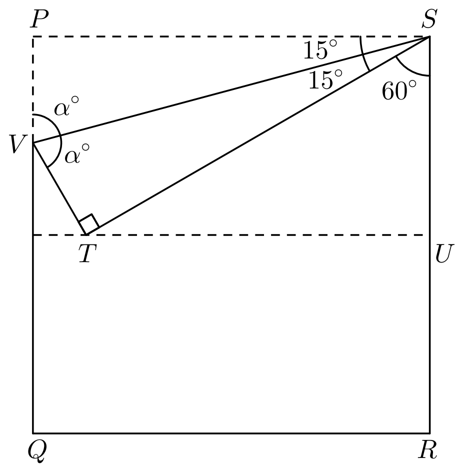
What is the value of α?

A 45 B 60 C 65 D 70 E 75



Solution **E**

Let the vertices of the square be P, Q, R and S, as shown. Let T be the position to which P is folded, and let U and V be the points shown in the diagram.



Because after the fold the triangle PSV coincides with the triangle TSV, these triangles are congruent. In particular $TS = PS$ and $\angle PSV = \angle TSV$.

Therefore SUT is a triangle with a right angle at U and in which $SU = \frac{1}{2}SR = \frac{1}{2}PS = \frac{1}{2}TS$. It follows that [see Exercise 12.4] $\angle TSU = 60°$.

Because $\angle USP = 90°$ and $\angle PSV = \angle TSV$, it follows that $\angle TSV = \frac{1}{2}(90 - 60)° = 15°$.

Therefore, because the sum of the angles in the triangle TSV is 180°, we have

$$\alpha + 15 + 90 = 180,$$

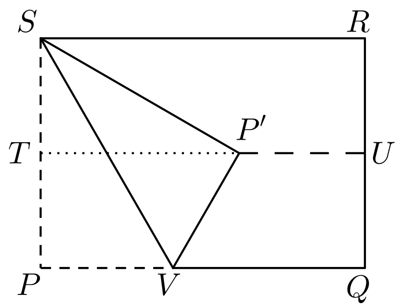
and therefore $\alpha = 75$.

For investigation

18.1 Suppose that $PQRS$ is a rectangular piece of paper, in which PQ is longer than QR.

The paper is folded in half along the line TU, and then unfolded.

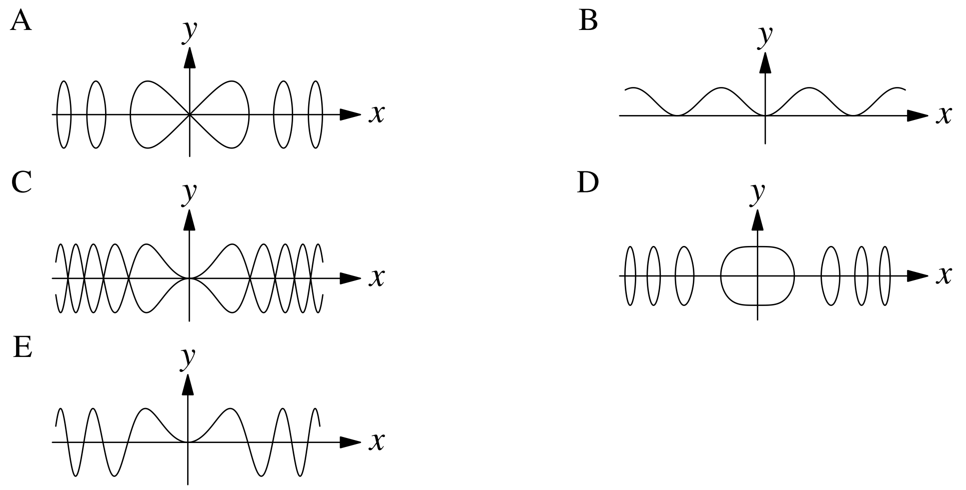
Next, the paper is folded along the line SV through S so that the corner P ends up at the point P' on the first fold line TU.



(a) Prove that $\angle SVP' = 60°$.

(b) Show how a rectangular piece of paper $PQRS$ may be folded to make an equilateral triangle, provided that the ratio $PQ : QR$ is sufficiently large.

19. Which of the following could be the graph of $y^2 = \sin\left(x^2\right)$?



Solution **A**

We have $\sin(0^2) = \sin(0) = 0$. The equation $y^2 = 0$ has only one solution, namely $y = 0$. Therefore the point $(0, 0)$ lies on the graph of $y^2 = \sin(x^2)$, and there is no point on the graph of the form $(0, b)$ with $b \neq 0$. This rules out option D.

If the point with coordinates (a, b) is on the graph, $b^2 = \sin(a^2)$. Hence we also have $(-b)^2 = \sin(a^2)$. Therefore the point with coordinates $(a, -b)$ also lies on the graph.

In other words, the graph of $y^2 = \sin(x^2)$ is symmetric about the x-axis. This rules out the graphs of options B and E.

There are positive values of x for which $\sin(x) < 0$. Suppose, for example, $a > 0$ and $\sin(a) < 0$. Then $\sin((\sqrt{a})^2) < 0$ and so cannot be equal to y^2 for any real number y. That is, there is no value of y for which the point with coordinates $(\sqrt{a}, y)$ lies on the graph.

In other words, the graph is not defined for all values of x. This rules out option C. [Note that this argument could also be used to rule out options B and E.]

This leaves option A as the only possibility for the graph.

For investigation

19.1 For each of the following equations decide which, if any, of the options in this question could be its graph.

(a) $y = \sin(x^2)$,

(b) $y = \cos(x^2)$,

(c) $y^2 = \cos(x^2)$,

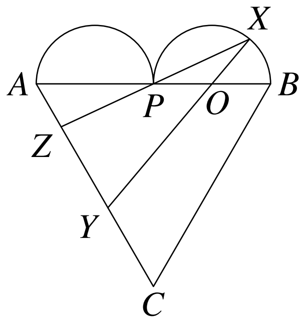
(d) $y = \sin^2(x)$,

(e) $y = \cos^2(x)$,

(f) $y^2 = \sin^2(x^2)$,

(g) $y^2 = \cos^2(x^2)$.

20. The "heart" shown in the diagram is formed from an equilateral triangle ABC and two congruent semicircles on AB. The two semicircles meet at the point P. The point O is the centre of one of the semicircles. On the semicircle with centre O, lies a point X. The lines XO and XP are extended to meet AC at Y and Z respectively. The lines XY and XZ are of equal length.



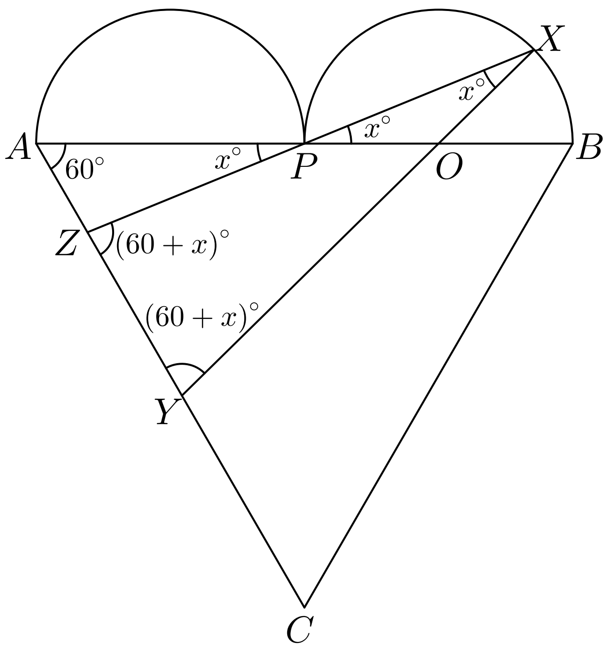
What is $\angle ZXY$?

A 20°   B 25°   C 30°   D 40°   E 45°

Solution **A**

Let $\angle ZXY = x°$.

The triangle POX is isosceles because OP and OX are radii of a semicircle and hence are equal. Therefore, $\angle OPX = \angle OXP = x°$.



Since $\angle APZ$ and $\angle OPX$ are vertically opposite, $\angle APZ = \angle OPX = x°$.

Because it is an angle of an equilateral triangle, $\angle ZAP = 60°$.

It now follows from the External Angle Theorem applied to the triangle AZP that $\angle YZX = \angle ZAP + \angle APZ = (60 + x)°$.

Because $XY = XZ$, the triangle XYZ is isosceles and therefore $\angle ZYX = \angle YZX = (60 + x)°$.

We now apply the fact that the sum of the angles of a triangle is 180° to the triangle XYZ. This gives

$$x + (60 + x) + (60 + x) = 180.$$

Hence

$$3x + 120 = 180.$$

It follows that $x = 20$ and hence $\angle ZXY = 20°$.

For investigation

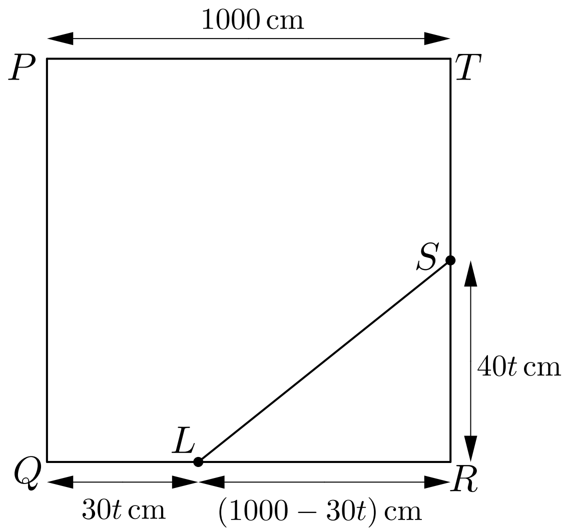
20.1 Prove that if, in the diagram of this question, the triangle ABC is not necessarily equilateral, but is isosceles, with $CB = CA$, and with the lines XY and XZ again of equal length, then $\angle ZXY = \frac{1}{3}\angle ACB$.

21. In a square garden $PQRT$ of side 10 m, a ladybird sets off from Q and moves along edge QR at 30 cm per minute. At the same time, a spider sets off from R and moves along edge RT at 40 cm per minute.

What will be the shortest distance between them, in metres?

A 5 B 6 C $5\sqrt{2}$ D 8 E 10

Solution **D**



Let L be the point that the ladybird reaches after t minutes. Then $QL = 30t$ cm. The side length of the square is 10 m which is the same as 1000 cm. Therefore the length of LR is $(1000 - 30t)$ cm.

Let S be the point that the spider reaches after t minutes. Then $RS = 40t$ cm.

Let the length of LS be x cm. This is the distance between the ladybird and the spider after t minutes.

Because $PQRT$ is a square, $\angle LRS = 90°$. By Pythagoras' Theorem applied to the triangle LRS, we have

$$\begin{aligned}x^2 &= (1000 - 30t)^2 + (40t)^2\\ &= 1\,000\,000 - 60\,000t + 900t^2 + 1600t^2\\ &= 2500t^2 - 60\,000t + 1\,000\,000\\ &= 2500(t^2 - 24t + 400)\\ &= 2500\big((t-12)^2 + 256\big).\end{aligned}$$

Because $(t-12)^2 \geq 0$ for all values of t, it follows that $x^2 \geq 2500 \times 256$ for all values of t. Since $x^2 = 2500 \times 256$ when $t = 12$, it follows that the smallest value of x^2 is 2500×256. Therefore the smallest value taken by x is $\sqrt{2500 \times 256}$, which is equal to 50×16, that is, 800. Hence the shortest distance between the ladybird and the spider is 800 cm which is 8 m.

For investigation

21.1 In the above solution we found the smallest value of the quadratic $t^2 - 24t + 400$ by the method of *completing the square*.

This works for quadratics but not for other functions. The *differential calculus* gives us a general method for finding maximum and minimum values of functions. If you have already met calculus, use it to find the minimum value of $t^2 - 24t + 400$.

21.2 Sketch the graph of $y = t^2 - 24t + 400$.

21.3 Find the minimum value of $2t^2 + 6t - 12$.

22. A function f satisfies the equation $(n-2019)f(n) - f(2019-n) = 2019$ for every integer n.

What is the value of $f(2019)$?

A 0 B 1 C 2018×2019 D 2019^2 E 2019×2020

SOLUTION **C**

Putting $n = 0$ in the equation

$$(n-2019)f(n) - f(2019-n) = 2019 \qquad (1)$$

gives

$$-2019f(0) - f(2019) = 2019 \qquad (2)$$

from which it follows that

$$f(2019) = -2019f(0) - 2019. \qquad (3)$$

Putting $n = 2019$ in equation (1) gives

$$-f(0) = 2019 \qquad (4)$$

and hence

$$f(0) = -2019. \qquad (5)$$

Substituting from (5) in (3) gives

$$\begin{aligned} f(2019) &= -2019 \times -2019 - 2019 \\ &= 2019 \times 2019 - 2019 \\ &= 2019(2019-1) \\ &= 2019 \times 2018. \end{aligned}$$

Therefore, the value of $f(2019)$ is 2018×2019.

FOR INVESTIGATION

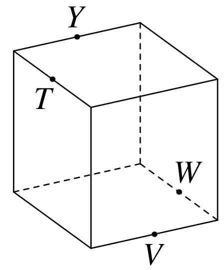
22.1 What is the value of $f(1)$?

22.2 Find the general formula for $f(n)$ in terms of n.

23. The edge-length of the solid cube shown is 2. A single plane cut goes through the points Y, T, V and W which are midpoints of the edges of the cube, as shown.

What is the area of the cross-section?

A $\sqrt{3}$ B $3\sqrt{3}$ C 6 D $6\sqrt{2}$ E 8

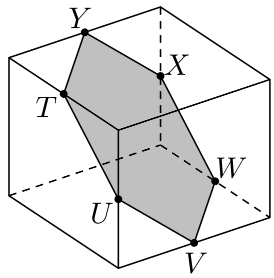


Solution **B**

The plane which goes through the points Y, T, V and W also meets the edges of the cube at the points U and X, as shown.

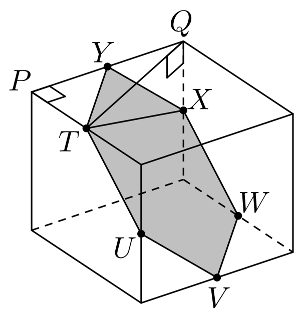
We leave it as an exercise to check that the points U and X are also the midpoints of the edges on which they lie.

The relevant cross-section is therefore the hexagon $TUVWXY$. We first show that this is a regular hexagon.



Let P and Q be the vertices of the cube, as shown. Applying Pythagoras' Theorem to the right-angled triangle TPY, gives $TY^2 = PT^2 + PY^2 = 1^2 + 1^2 = 2$. Therefore $TY = \sqrt{2}$. In a similar way it follows that each edge of the hexagon $TUVWXY$ has length $\sqrt{2}$.

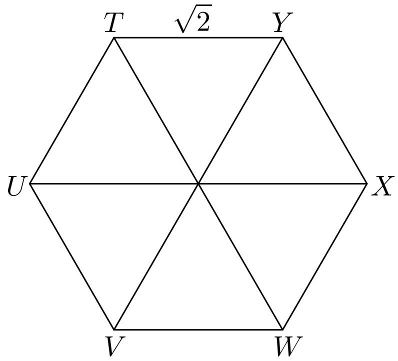
Applying Pythagoras' Theorem to the right-angled triangle TPQ gives $QT^2 = PT^2 + PQ^2 = 1^2 + 2^2 = 5$. The triangle TXQ has a right angle at Q because the top face of the cube is perpendicular to the edge through Q and X. Hence, by Pythagoras' Theorem, $TX^2 = QT^2 + QX^2 = 5 + 1^2 = 6$. Hence $TX = \sqrt{6}$.



It follows that in the triangle TXY we have $TY = YX = \sqrt{2}$ and $TX = \sqrt{6}$. We leave it as an exercise to check that it follows that $\angle TYX = 120°$. It follows similarly that each angle of $TUVWXY$ is $120°$. Hence this hexagon is regular.

The regular hexagon $TUVWXY$ may be divided into six congruent equilateral triangles each with side length $\sqrt{2}$.

We leave it as an exercise to check that the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$. It follows that equilateral triangles with side lengths $\sqrt{2}$ have area $\frac{\sqrt{3}}{4}(\sqrt{2})^2 = \frac{\sqrt{3}}{2}$.



Therefore the area of the cross-section is $6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$.

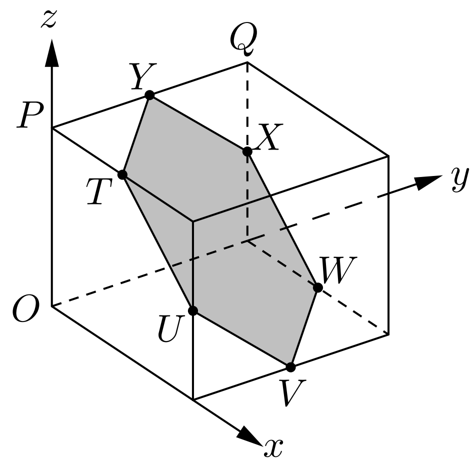
Note: The facts you are asked to check in the above solution are given in as Exercises 23.1, 23.2 and 23.3 below.

For investigation

23.1 Show that a triangle with side lengths $\sqrt{6}$, $\sqrt{2}$ and $\sqrt{2}$, has a largest angle of 120° and two angles each of 30°.

23.2 Show that the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$.

23.3 Let O be the bottom left-hand vertex of the cube as shown.



We let O be the origin of a system of three-dimensional coordinates with x−axis, y−axis and z−axis as shown.

The coordinates in this coordinate system of the point T are $(1, 0, 2)$.

(a) Find the coordinates in this coordinate system of the points Y and V.

(b) The equation of a plane in three-dimensions has the form $ax + by + cz = d$, where a, b, c and d are constants.

Find the equation of the plane that goes through all the points T, Y and V.

(c) Hence, verify that the plane that goes through the points T, Y and V also goes through the point W.

(d) Use the equation of the plane through the points T, Y and V to find the coordinates of the points U and X. Deduce that U and X are the midpoints of the edges on which they lie.

23.4 Why does it not follow just from the fact that each edge of hexagon $TUVWXY$ has length $\sqrt{2}$ that the hexagon is regular?

24. The numbers x, y and z are given by $x = \sqrt{12 - 3\sqrt{7}} - \sqrt{12 + 3\sqrt{7}}$, $y = \sqrt{7 - 4\sqrt{3}} - \sqrt{7 + 4\sqrt{3}}$ and $z = \sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}$.

What is the value of xyz ?

A 1 B –6 C –8 D 18 E 12

Solution **E**

Commentary

A natural first thought on reading this question is that working out the product xyz, without the use of a calculator, will be horribly complicated, and that, given the time constraints of the SMC, it would be sensible to skip this question.

But then you might realize that, because this is an SMC question, there could well be a smarter way to tackle this question.

Because the expressions for x, y and z involve lots of square roots, a smart method might be to work out x^2, y^2 and z^2 separately and hence find $x^2y^2z^2$, which is equal to $(xyz)^2$. It is, perhaps, surprising that this approach works out well.

We have

$$\begin{aligned}
x^2 &= \left(\sqrt{12 - 3\sqrt{7}} - \sqrt{12 + 3\sqrt{7}}\right)^2 \\
&= \left(\sqrt{12 - 3\sqrt{7}}\right)^2 - 2\left(\sqrt{12 - 3\sqrt{7}}\right)\left(\sqrt{12 + 3\sqrt{7}}\right) + \left(\sqrt{12 + 3\sqrt{7}}\right)^2 \\
&= \left(\sqrt{12 - 3\sqrt{7}}\right)^2 - 2\sqrt{\left((12 - 3\sqrt{7})(12 + 3\sqrt{7})\right)} + \left(\sqrt{12 + 3\sqrt{7}}\right)^2 \\
&= \left(\sqrt{12 - 3\sqrt{7}}\right)^2 - 2\left(\sqrt{12^2 - (3\sqrt{7})^2}\right) + \left(\sqrt{12 + 3\sqrt{7}}\right)^2 \\
&= (12 - 3\sqrt{7}) - 2\sqrt{144 - 63} + (12 + 3\sqrt{7}) \\
&= (12 - 3\sqrt{7}) - 2\sqrt{81} + (12 + 3\sqrt{7}) \\
&= 24 - 2 \times 9 \\
&= 6.
\end{aligned}$$

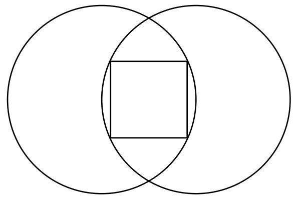
We leave it as an exercise to check that, similarly, $y^2 = 12$ and $z^2 = 2$. It follows that $(xyz)^2 = x^2y^2z^2 = 6 \times 12 \times 2 = 144$. Hence xyz equals either 12 or -12.

Because $12 - 3\sqrt{7} < 12 + 3\sqrt{7}$, we see that $x < 0$. Similarly $y < 0$ and $z > 0$. Hence $xyz > 0$. We conclude that $xyz = 12$.

For investigation

24.1 Check that $y^2 = 12$ and that $z^2 = 2$.

25. Two circles of radius 1 are such that the centre of each circle lies on the other circle. A square is inscribed in the space between the circles.



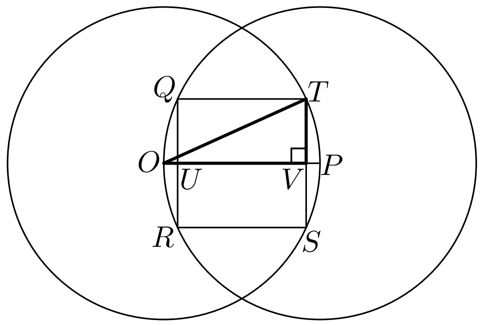
What is the area of the square?

A $2 - \frac{\sqrt{7}}{2}$ B $2 + \frac{\sqrt{7}}{2}$ C $4 - \sqrt{5}$

D 1 E $\frac{\sqrt{5}}{5}$

SOLUTION **A**

We let O and P be the centres of the circles, Q, R, S and T be the vertices of the square, and U and V be the points where the line OP meets the edges of the square, as shown in the diagram.



We let x be the side length of the square.

The point V is the midpoint of ST and therefore $VT = \frac{1}{2}x$.

Because OT and OP are radii of the circle with centre O, we have $OT = OP = 1$.

Because $OP = 1$ and $UV = QT = x$, we have $OU + VP = 1 - x$. Since $OU = VP$, it follows that $OU = \frac{1}{2}(1-x)$. Therefore $OV = OU + UV = \frac{1}{2}(1-x) + x = \frac{1}{2}(1+x)$.

Therefore, applying Pythagoras' Theorem to the right-angled triangle OVT, we obtain $OV^2 + VT^2 = OT^2$. That is, $\left(\frac{1}{2}(1+x)\right)^2 + \left(\frac{1}{2}x\right)^2 = 1^2$. This equation expands to give $\frac{1}{4} + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{4}x^2 = 1$. The last equation may be rearranged to give $2x^2 + 2x - 3 = 0$.

Hence, by the formula for the roots of a quadratic equation, $x = \dfrac{-2 \pm \sqrt{4+24}}{4} = \frac{1}{2}(-1 \pm \sqrt{7})$. Because $x > 0$, we deduce that $x = \frac{1}{2}(-1 + \sqrt{7})$.

Therefore the area of the square is given by

$$x^2 = \left(\tfrac{1}{2}(-1+\sqrt{7})\right)^2 = \tfrac{1}{4}(1 - 2\sqrt{7} + 7) = \tfrac{1}{4}(8 - 2\sqrt{7}) = 2 - \frac{\sqrt{7}}{2}.$$

FOR INVESTIGATION

25.1 Prove that the following claims made in the above solution are correct.

(a) The point V is the midpoint of ST.

(b) $OU = VP$.

(c) $\angle OVT = 90°$.

United Kingdom
Mathematics Trust

Senior Mathematical Challenge

2 – 5 November 2020

Organised by the United Kingdom Mathematics Trust

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Candidates must be full-time students at secondary school or FE college.
England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil.** Mark at most one of the options A, B, C, D, E on the Answer Sheet for each question. Do not mark more than one option.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:**
   All candidates start with 25 marks;
   0 marks are awarded for each question left unanswered;
   4 marks are awarded for each correct answer;
   1 mark is deducted for each incorrect answer (to discourage guessing).
7. Your Answer Sheet will be read by a machine. **Do not write or doodle on the sheet except to mark your chosen options.** The machine will read all black pencil markings even if they are in the wrong places. If you mark the sheet in the wrong place, or leave bits of eraser stuck to the page, the machine will interpret the mark in its own way.
8. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

Enquiries about the Senior Mathematical Challenge should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 365 1121 enquiry@ukmt.org.uk www.ukmt.org.uk

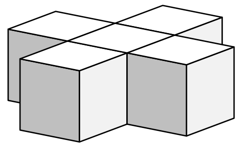
1. What is the value of $\frac{2020}{20 \times 20}$?

A 10.1 B 5.5 C 5.1 D 5.05 E 0.55

2. What is the remainder when 1234×5678 is divided by 5?

A 0 B 1 C 2 D 3 E 4

3. A shape is made from five unit cubes, as shown.



What is the surface area of the shape?

A 22 B 24 C 26 D 28 E 30

4. The numbers p, q, r and s satisfy the equations $p = 2$, $p \times q = 20$, $p \times q \times r = 202$ and $p \times q \times r \times s = 2020$.

What is the value of $p + q + r + s$?

A 32 B 32.1 C 33 D 33.1 E 34

5. What is $\sqrt{123454321}$?

A 1111111 B 111111 C 11111 D 1111 E 111

6. There are fewer than 30 students in the A-level mathematics class. One half of them play the piano, one quarter play hockey and one seventh are in the school play.

How many of the students play hockey?

A 3 B 4 C 5 D 6 E 7

7. Official UK accident statistics showed that there were 225 accidents involving teapots in one year. However, in the following year there were 47 such accidents.

What was the approximate percentage reduction in recorded accidents involving teapots from the first year to the second?

A 50% B 60% C 70% D 80% E 90%

8. What is the largest prime factor of $106^2 - 15^2$?

A 3 B 7 C 11 D 13 E 17

9. In 2018, a racing driver was allowed to use the Drag Reduction System provided that the car was within 1 second of the car ahead. Suppose that two cars were 1 second apart, each travelling at 180 km/h (in the same direction!).

How many metres apart were they?

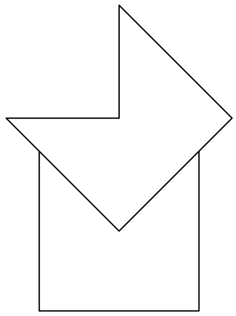
A 100 B 50 C 10 D 5 E 1

10. Six friends Pat, Qasim, Roman, Sam, Tara and Uma, stand in a line for a photograph. There are three people standing between Pat and Qasim, two between Qasim and Roman and one between Roman and Sam. Sam is not at either end of the line.

How many people are standing between Tara and Uma?

A 4 B 3 C 2 D 1 E 0

11. Two congruent pentagons are each formed by removing a right-angled isosceles triangle from a square of side-length 1. The two pentagons are then fitted together as shown.



What is the length of the perimeter of the octagon formed?

A 4 B $4 + 2\sqrt{2}$ C 5 D $6 - 2\sqrt{2}$ E 6

12. A three-piece suit consists of a jacket, a pair of trousers and a waistcoat. Two jackets and three pairs of trousers cost £380. A pair of trousers costs the same as two waistcoats.

What is the cost of a three-piece suit?

A £150 B £190 C £200 D £228 E more information needed

13. The number $16! \div 2^k$ is an odd integer. Note that $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

What is the value of k?

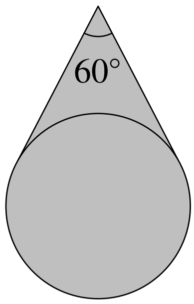
A 9 B 11 C 13 D 15 E 17

14. Diane has five identical blue disks, two identical red disks and one yellow disk. She wants to place them on the grid opposite so that each cell contains exactly one disk. The two red disks are not to be placed in cells that share a common edge.

How many different-looking completed grids can she produce?

A 96 B 108 C 144 D 180 E 216

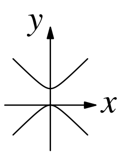
15. The shaded area shown in the diagram consists of the interior of a circle of radius 3 together with the area between the circle and two tangents to the circle. The angle between the tangents at the point where they meet is 60°.

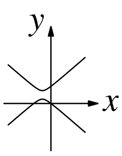


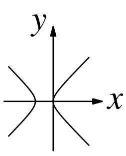
What is the shaded area?

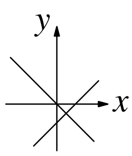
A $6\pi + 9\sqrt{3}$ B $15\sqrt{3}$ C 9π D $9\pi + 4\sqrt{3}$ E $6\pi + \frac{9\sqrt{3}}{4}$

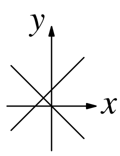
16. Which diagram represents the set of all points (x, y) satisfying $y^2 - 2y = x^2 + 2x$?

A 

B 

C 

D 

E 

17. The positive integers m, n and p satisfy the equation $3m + \dfrac{3}{n + \frac{1}{p}} = 17$.

What is the value of p?

A 2 B 3 C 4 D 6 E 9

18. Two circles C_1 and C_2 have their centres at the point (3,4) and touch a third circle, C_3. The centre of C_3 is at the point (0,0) and its radius is 2.

What is the sum of the radii of the two circles C_1 and C_2?

A 6 B 7 C 8 D 9 E 10

19. The letters p, q, r, s and t represent different positive single-digit numbers such that $p - q = r$ and $r - s = t$.

How many different values could t have?

A 6 B 5 C 4 D 3 E 2

20. The real numbers x and y satisfy the equations $4^y = \dfrac{1}{8(\sqrt{2})^{x+2}}$ and $9^x \times 3^y = 3\sqrt{3}$.

What is the value of 5^{x+y}?

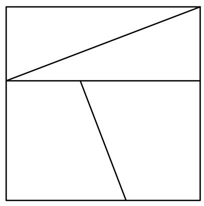
A $5\sqrt{5}$ B 5 C $\sqrt{5}$ D $\frac{1}{5}$ E $\frac{1}{\sqrt{5}}$

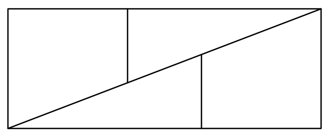
21. When written out in full, the number $(10^{2020} + 2020)^2$ has 4041 digits.

What is the sum of the digits of this 4041-digit number?

A 9 B 17 C 25 D 2048 E 4041

22. A square with perimeter 4 cm can be cut into two congruent right-angled triangles and two congruent trapezia as shown in the first diagram in such a way that the four pieces can be rearranged to form the rectangle shown in the second diagram.





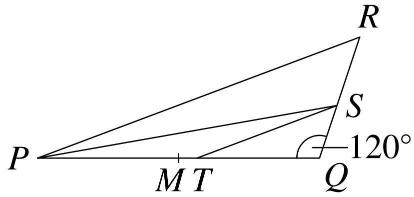
What is the perimeter, in centimetres, of this rectangle?

A $2\sqrt{5}$ B $4\sqrt{2}$ C 5 D $4\sqrt{3}$ E $3\sqrt{7}$

23. A function f satisfies $y^3 f(x) = x^3 f(y)$ and $f(3) \neq 0$. What is the value of $\dfrac{f(20) - f(2)}{f(3)}$?

A 6 B 20 C 216 D 296 E 2023

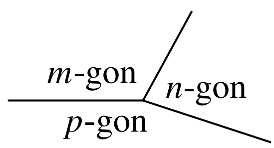
24. In the diagram shown, M is the mid-point of PQ. The line PS bisects $\angle RPQ$ and intersects RQ at S. The line ST is parallel to PR and intersects PQ at T. The length of PQ is 12 and the length of MT is 1. The angle SQT is 120°.



What is the length of SQ?

A 2 B 3 C 3.5 D 4 E 5

25. A regular m-gon, a regular n-gon and a regular p-gon share a vertex and pairwise share edges, as shown in the diagram.



What is the largest possible value of p?

A 6 B 20 C 42 D 50 E 100

United Kingdom
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Senior Mathematical Challenge

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Solutions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	C	A	B	C	E	D	D	B	C	E	B	D	B	A	E	A	E	A	E	C	A	D	B	C

1. What is the value of $\frac{2020}{20 \times 20}$?

A 10.1 B 5.5 C 5.1 D 5.05 E 0.55

SOLUTION **D**

The value of $\frac{2020}{20\times20} = \frac{101}{20} = \frac{50.5}{10} = 5.05$.

2. What is the remainder when 1234×5678 is divided by 5?

A 0 B 1 C 2 D 3 E 4

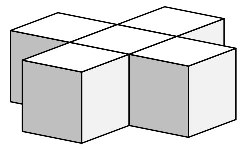
SOLUTION **C**

When the product 1234×5678 is calculated, its units digit is the units digit of 4×8, so 2. This is the remainder when the product is divided by 5.

3. A shape is made from five unit cubes, as shown.

What is the surface area of the shape?

A 22 B 24 C 26 D 28 E 30



SOLUTION **A**

The central cube contributes two faces to the surface area of the shape. The other four cubes contribute five faces each. This gives a surface area of $2 + 4 \times 5 = 22$ faces. The surface area is then 22 as each face has area 1 unit.

4. The numbers p, q, r and s satisfy the equations $p = 2$, $p \times q = 20$, $p \times q \times r = 202$ and $p \times q \times r \times s = 2020$.

What is the value of $p + q + r + s$?

A 32 B 32.1 C 33 D 33.1 E 34

SOLUTION **B**

As $p = 2$, $q = \frac{20}{2} = 10$. Then $20 \times r = 202$ so $r = \frac{202}{20} = 10.1$. Finally, $202 \times s = 2020$ so $s = \frac{2020}{202} = 10$. The value of $p + q + r + s = 2 + 10 + 10.1 + 10 = 32.1$.

5. What is $\sqrt{123454321}$?

A 1111111 B 111111 C 11111 D 1111 E 111

SOLUTION **C**

Consider each option to be in the form $1.11\ldots \times 10^k$. When numbers of this form are squared, they become $1.2\ldots \times 10^{2k}$. As the question asks for $\sqrt{123454321} = \sqrt{1.2\ldots \times 10^8}$, the value of k is 4. Thus the correct option is C, 11111.

6. There are fewer than 30 students in the A-level mathematics class. One half of them play the piano, one quarter play hockey and one seventh are in the school play.

How many of the students play hockey?

A 3 B 4 C 5 D 6 E 7

SOLUTION **E**

The number of students in the maths class must be a multiple of 2, 4 and 7. As there are known to be fewer than 30 students, there must be 28 students in the class, which is the only common multiple of 2, 4 and 7 less than 30. Therefore the number of students who play hockey is $\frac{28}{4} = 7$.

7. Official UK accident statistics showed that there were 225 accidents involving teapots in one year. However, in the following year there were 47 such accidents.

What was the approximate percentage reduction in recorded accidents involving teapots from the first year to the second?

A 50% B 60% C 70% D 80% E 90%

SOLUTION **D**

As a proportion, $\frac{47}{225} \approx \frac{45}{225} = 20\,\%$. So the percentage reduction in reported teapot accidents is approximately 80 %.

8. What is the largest prime factor of $106^2 - 15^2$?

A 3 B 7 C 11 D 13 E 17

SOLUTION **D**

Using the difference of two squares, $106^2 - 15^2 = (106 + 15)(106 - 15) = 121 \times 91$, which when expressed as the product of primes is $11 \times 11 \times 7 \times 13$. The largest of these primes is 13.

9. In 2018, a racing driver was allowed to use the Drag Reduction System provided that the car was within 1 second of the car ahead. Suppose that two cars were 1 second apart, each travelling at 180 km/h (in the same direction!).

How many metres apart were they?

A 100 B 50 C 10 D 5 E 1

Solution **B**

Using distance = speed × time, distance = 180 km/h × 1 second, which written in metres and seconds is $\dfrac{180 \times 1000\,\text{m}}{60 \times 60\,\text{seconds}} \times 1$ second = 50 m.

10. Six friends Pat, Qasim, Roman, Sam, Tara and Uma, stand in a line for a photograph. There are three people standing between Pat and Qasim, two between Qasim and Roman and one between Roman and Sam. Sam is not at either end of the line.

How many people are standing between Tara and Uma?

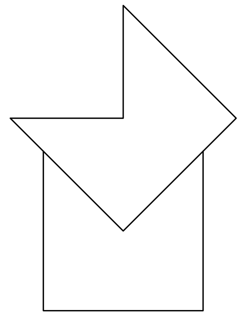
A 4 B 3 C 2 D 1 E 0

Solution **C**

The first piece of information allows us to place P and Q in four possible pairings as shown. In each case, the second piece of information tells us that the position of Roman is uniquely determined and in each case this is next to Pat. As Sam is not at the end of the line, the third piece of information shows us that Sam's position is also uniquely determined and is next to Qasim. The positions which remain for Tara and Uma, in each of the four cases, have two people between them.

P R _ S Q _
or _ P R _ S Q
or Q S _ R P _
or _ Q S _ R P

11. Two congruent pentagons are each formed by removing a right-angled isosceles triangle from a square of side-length 1. The two pentagons are then fitted together as shown.



What is the length of the perimeter of the octagon formed?

A 4 B $4 + 2\sqrt{2}$ C 5 D $6 - 2\sqrt{2}$ E 6

Solution **E**

The perimeter of the octagon is made from four long sides, two medium-length sides and two short sides. The long sides are given to be of length 1. The medium-length sides have length $\frac{1}{\sqrt{2}}$, using Pythagoras' Theorem on the right-angled triangle which was removed from the original square. Therefore the length of each short side is $1 - \frac{1}{\sqrt{2}}$. In total the perimeter has length $4 \times 1 + 2 \times \frac{1}{\sqrt{2}} + 2 \times (1 - \frac{1}{\sqrt{2}}) = 6$.

12. A three-piece suit consists of a jacket, a pair of trousers and a waistcoat. Two jackets and three pairs of trousers cost £380. A pair of trousers costs the same as two waistcoats.

What is the cost of a three-piece suit?

A £150 B £190 C £200 D £228 E more information needed

Solution **B**

In pounds, let the price of a jacket $= j$, the price of a pair of trousers $= t$ and the price of a waistcoat $= w$. We are given that $2j + 3t = 380$ and that $t = 2w$. We want to know the value of $j + t + w$. Rewriting the first equation as $2j + 2t + t = 380$ and substituting $2w$ instead of the final t gives $2j + 2t + 2w = 380$. Dividing by 2 then gives us $j + t + w = 190$ and so the cost of the three-piece suit is £190.

13. The number $16! \div 2^k$ is an odd integer. Note that $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

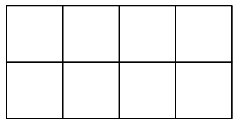
What is the value of k?

A 9 B 11 C 13 D 15 E 17

Solution **D**

When 16! is written out as the product of its primes, $16! = (2^4) \times (3 \times 5) \times (2 \times 7) \times (13) \times (2^2 \times 3) \times (11) \times (2 \times 5) \times (3^2) \times (2^3) \times (7) \times (2 \times 3) \times (5) \times (2^2) \times (3) \times (2)$ in which the power of 2 is $2^{4+1+2+1+3+1+2+1} = 2^{15}$. In order for $16! \div 2^k$ to be an odd integer, 2^k must exactly equal 2^{15}, so $k = 15$.

14. Diane has five identical blue disks, two identical red disks and one yellow disk. She wants to place them on the grid opposite so that each cell contains exactly one disk. The two red disks are not to be placed in cells that share a common edge.

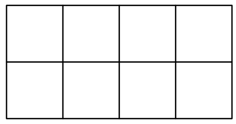


How many different-looking completed grids can she produce?

A 96 B 108 C 144 D 180 E 216

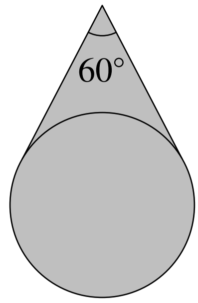
Solution **B**

One counting method is to consider first the position of the two red disks working systematically from left to right. If there is a red disk in either of the two positions in the first column, then there are five possible positions for the second red disk.



If there are no red disks in the first column but one in either of the two positions in the second column then there are three possible positions for the second red disk. If there are no red disks in either of the first two columns but there is one in either of the two positions in the third column then there is exactly one position for the second red disk. This gives a total of $2 \times 5 + 2 \times 3 + 2 \times 1 = 18$ different looking positions for the two red disks. The single yellow disk can then be placed in any of the remaining six squares. The last five positions are filled with the five blue disks. This gives $18 \times 6 \times 1 = 108$ possibilities.

15. The shaded area shown in the diagram consists of the interior of a circle of radius 3 together with the area between the circle and two tangents to the circle. The angle between the tangents at the point where they meet is 60°.

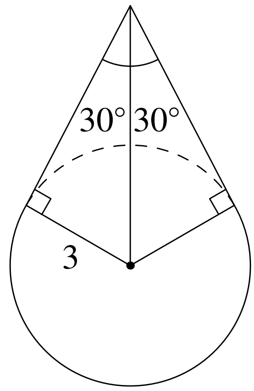


What is the shaded area?

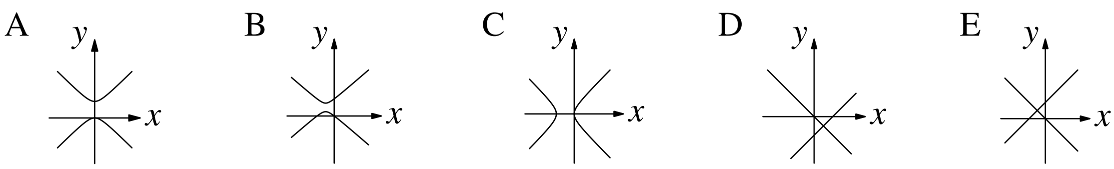
A $6\pi + 9\sqrt{3}$ B $15\sqrt{3}$ C 9π D $9\pi + 4\sqrt{3}$ E $6\pi + \frac{9\sqrt{3}}{4}$

Solution **A**

Drawing three extra lines on the diagram, vertically from the centre of the circle to the top vertex and also from the centre to the points where the tangents touch the circle, allows us to see the shaded area as two triangles and a sector outside the triangles. The tangents are perpendicular to the radii and each triangle has the vertical line as one of its sides, so using 'two sides and a right-angle' the two triangles are congruent. Therefore the triangles have angles of 30°, 60° and 90°. As we are given that the radius is 3, the length of the tangents is $3\sqrt{3}$ and so the area of each triangle is $\frac{1}{2} \times 3 \times 3\sqrt{3} = \frac{9\sqrt{3}}{2}$. The angle between the two radii is 120° so the sector is $\frac{2}{3}$ of the area of the circle of radius 3 which is $\frac{2}{3} \times \pi \times 3^2 = 6\pi$. The total area is then $6\pi + 2 \times \frac{9\sqrt{3}}{2} = 6\pi + 9\sqrt{3}$.



16. Which diagram represents the set of all points (x, y) satisfying $y^2 - 2y = x^2 + 2x$?



Solution **E**

Rearranging $y^2 - 2y = x^2 + 2x$ gives $y^2 - x^2 = 2x + 2y$ then factorising on each side gives $(y + x)(y - x) = 2(x + y)$ and so $(y - x - 2)(y + x) = 0$. Therefore either $y = x + 2$ or $y = -x$. A sketch of those two straight lines is shown in option E.

17. The positive integers m, n and p satisfy the equation $3m + \dfrac{3}{n + \frac{1}{p}} = 17.$

What is the value of p?

A 2 B 3 C 4 D 6 E 9

SOLUTION **A**

As m is a positive integer, in this equation $3m$ could equal 3, 6, 9, 12 or 15 leaving $\dfrac{3}{n + \frac{1}{p}} = 14$, 11, 8, 5 or 2. However $n, p > 1 \implies n + \frac{1}{p} > 1 \implies \dfrac{3}{n + \frac{1}{p}} < 3$. Therefore the only possibility here is that $\dfrac{3}{n + \frac{1}{p}} = 2$. So $\frac{3}{2} = n + \frac{1}{p}$ which is possible only when $n = 1$ and $p = 2$.

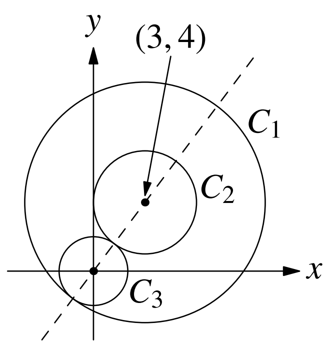
18. Two circles C_1 and C_2 have their centres at the point (3,4) and touch a third circle, C_3. The centre of C_3 is at the point (0,0) and its radius is 2.

What is the sum of the radii of the two circles C_1 and C_2?

A 6 B 7 C 8 D 9 E 10

SOLUTION **E**

The centres of all three circles along with both points where the circles touch each other lie on the same straight line which passes through $(0, 0)$ and $(3, 4)$. Using Pythagoras' theorem the distance between $(0, 0)$ and $(3, 4)$ is 5 units. Circle C_1 has radius $5 + 2 = 7$ and circle C_2 has radius $5 - 2 = 3$ so the sum of the radii of those two circles is $7 + 3 = 10$.



19. The letters p, q, r, s and t represent different positive single-digit numbers such that $p - q = r$ and $r - s = t$.

How many different values could t have?

A 6 B 5 C 4 D 3 E 2

SOLUTION **A**

Combining $p - q = r$ and $r - s = t$ gives $p - q = s + t$ which rearranges to $t = p - (q + s)$. The maximum value of t can be found by maximising p and minimising the sum of q and s, so $t_{max} = 9 - (1 + 2) = 6$. The minimum value of t is 1, for example $t_{min} = 9 - (2 + 6) = 1$. All the values of t from 1 to 6 are possible in a similar way. Therefore t can have six different values.

20. The real numbers x and y satisfy the equations $4^y = \frac{1}{8(\sqrt{2})^{x+2}}$ and $9^x \times 3^y = 3\sqrt{3}$.

What is the value of 5^{x+y}?

A $5\sqrt{5}$ B 5 C $\sqrt{5}$ D $\frac{1}{5}$ E $\frac{1}{\sqrt{5}}$

SOLUTION **E**

Rewriting both sides of the first equation using a base of $\sqrt{2}$ gives $\left(\sqrt{2}^4\right)^y = \frac{1}{\sqrt{2}^6\sqrt{2}^{x+2}}$ so $\sqrt{2}^{4y} = \sqrt{2}^{-(x+8)}$ and therefore $4y = -(x+8)$. Rewriting the second equation with a base of $\sqrt{3}$ gives $\left(\sqrt{3}^4\right)^x \times \left(\sqrt{3}^2\right)^y = \sqrt{3}^3$. Again, employing rules of indices leads to $4x + 2y = 3$. Solving simultaneously the pair of linear equations shows that $x = 2$ and $y = -\frac{5}{2}$ and therefore $x + y = -\frac{1}{2}$. The value of $5^{x+y} = 5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$.

21. When written out in full, the number $(10^{2020} + 2020)^2$ has 4041 digits.

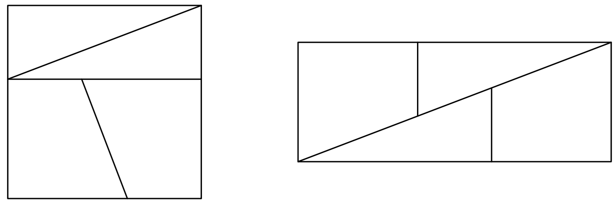
What is the sum of the digits of this 4041-digit number?

A 9 B 17 C 25 D 2048 E 4041

SOLUTION **C**

The expression $(10^{2020} + 2020)^2$ can be written as $(10^{2020} + 2020)(10^{2020} + 2020)$ and expanded to give $10^{4040} + 4040 \times 10^{2020} + 4080400$. The sum of the digits of the 4041-digit number is the sum of the non-zero digits. Since there is no overlap of the positions of the non-zero digits of the three parts of the expanded expression, this sum is $1 + 4 + 4 + 4 + 8 + 4 = 25$.

22. A square with perimeter 4 cm can be cut into two congruent right-angled triangles and two congruent trapezia as shown in the first diagram in such a way that the four pieces can be rearranged to form the rectangle shown in the second diagram.



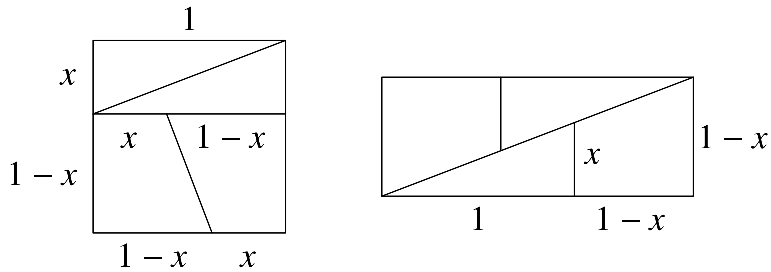
What is the perimeter, in centimetres, of this rectangle?

A $2\sqrt{5}$ B $4\sqrt{2}$ C 5 D $4\sqrt{3}$ E $3\sqrt{7}$

Solution **A**

The perimeter of the square is 4 therefore each side is of length 1 and the area of the square is 1. The square must be cut so that the shortest side of each triangle matches the shorter of the two parallel sides of the trapezium when the pieces are rearranged.

Let each of these sides be of length x and then the remaining perpendicular sides of each trapezium have length $1 - x$ as indicated on the first diagram. The perimeter of the rectangle has length $2 \times 1 + 4(1 - x) = 6 - 4x$.



In order to find the value of x, we can consider the area of the rectangle so $(1 + 1 - x)(1 - x) = 1$. This rearranges to give $x^2 - 3x + 1 = 0$ and therefore, of the two possible solutions, $x = \frac{3-\sqrt{5}}{2}$ as $x < 1$. The perimeter is $6 - 4(\frac{3-\sqrt{5}}{2}) = 2\sqrt{5}$.

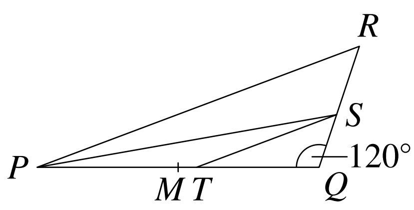
23. A function f satisfies $y^3 f(x) = x^3 f(y)$ and $f(3) \neq 0$. What is the value of $\dfrac{f(20) - f(2)}{f(3)}$?

A 6 B 20 C 216 D 296 E 2023

Solution **D**

Letting $y = 3$ and rearranging gives $\dfrac{f(x)}{f(3)} = \dfrac{x^3}{3^3}$. Therefore $\dfrac{f(20) - f(2)}{f(3)} = \dfrac{f(20)}{f(3)} - \dfrac{f(2)}{f(3)} = \dfrac{20^3}{3^3} - \dfrac{2^3}{3^3} = \dfrac{8000 - 8}{27} = \dfrac{7992}{27} = 296$.

24. In the diagram shown, M is the mid-point of PQ. The line PS bisects $\angle RPQ$ and intersects RQ at S. The line ST is parallel to PR and intersects PQ at T. The length of PQ is 12 and the length of MT is 1. The angle SQT is 120°.



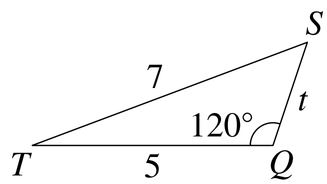
What is the length of SQ?

A 2 B 3 C 3.5 D 4 E 5

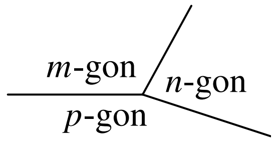
Solution **B**

Triangle PTS can be shown to be isosceles as follows. Let $\angle STQ = 2x°$. Therefore $\angle RPT = 2x°$ as PR and ST are parallel. We can then deduce that $\angle SPT = x°$ as PS bisects $\angle RPQ$. Considering angles at point T, $\angle STP = 180° - 2x°$. Finally, considering angles inside triangle PTS, $\angle PST = 180° - x° - (180° - 2x°) = x° = \angle SPT$. Therefore the length of ST = the length of $PT = \frac{12}{2} + 1 = 7$.

Now considering triangle STQ, we have two known lengths and an angle so we can find the length of SQ using the cosine rule. Let the length of $SQ = t$. So $7^2 = 5^2 + t^2 - 2 \times 5 \times t \times \cos 120°$. As $\cos 120° = -\frac{1}{2}$, this quadratic simplifies to $t^2 + 5t - 24 = 0$ which factorises to $(t + 8)(t - 3) = 0$. As $t > 0$, $t = 3$.



25. A regular m-gon, a regular n-gon and a regular p-gon share a vertex and pairwise share edges, as shown in the diagram.

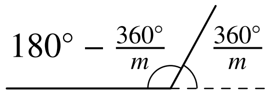


What is the largest possible value of p?

A 6 B 20 C 42 D 50 E 100

Solution **C**

The exterior angle of the regular m-gon is $\frac{360°}{m}$ so the interior angle is $180° - \frac{360°}{m}$. Similarly for the n-gon and p-gon.



Possible interior angles, in order of size, are then 60°, 90°, 108°, 120°, $(180° - \frac{360°}{7})$, 135°, 140°, 144°, ... In order to maximise p, the interior angles of the m-gon and n-gon must have a sum which exceeds 180° by as little as possible. This excess gives the exterior angle of the p-gon, so must fit into 360° an exact number of times. As a starting point, note that when $(m, n) = (4, 4)$ or $(m, n) = (6, 3)$ the interior angles of the m-gon and n-gon sum to exactly 180°. By increasing exactly one of m or n in these pairs, possible candidates for pairings which will give us a sum of interior angles in excess of 180° by a suitable amount are then $(5, 4)$, $(4, 5)$, $(6, 4)$, and $(7, 3)$. The value of p is then found from $p = \frac{360°}{[180° - \frac{360°}{m}] + [180° - \frac{360°}{n}] - 180°}$. Without loss, we can assume $m \geq n$ and therefore discount $(m, n) = (4, 5)$. When $(m, n) = (5, 4)$, $p = \frac{360°}{[180° - \frac{360°}{5}] + [180° - \frac{360°}{4}] - 180°} = \frac{360°}{18°} = 20$. As this is an integer, we don't need to try $(m, n) = (6, 4)$ as that could only give a smaller value of p. When $(m, n) = (7, 3)$, $p = \frac{360°}{[180° - \frac{360°}{7}] + [180° - \frac{360°}{3}] - 180°} = \frac{360°}{60° - \frac{360°}{7}} = \frac{360°}{\frac{60°}{7}} = 42$. As increasing m or n or both any further could only give a smaller value of p, no further pairs need to be tested. Hence the largest value of p is 42.

United Kingdom Mathematics Trust

Senior Mathematical Challenge

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Solutions and investigations

November 2020

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	C	A	B	C	E	D	D	B	C	E	B	D	B	A	E	A	E	A	E	C	A	D	B	C

1. What is the value of $\dfrac{2020}{20 \times 20}$?

A 10.1 B 5.5 C 5.1 D 5.05 E 0.55

SOLUTION **D**

Note: In the absence of a calculator, the best way to tackle this question is to first do some cancelling and then the division. There is more than one way to do this. In our method we twice divide the numerator and denominator by 10.

$$\frac{2020}{20 \times 20} = \frac{202}{2 \times 20} = \frac{20.2}{2 \times 2} = \frac{20.2}{4} = 5.05.$$

FOR INVESTIGATION

1.1 What is the value of the following?

(a) $\dfrac{2020}{2 \times 2}$,

(b) $\dfrac{2020}{200 \times 200}$,

(c) $\dfrac{2020}{2 \times 200}$.

1.2 Find the positive integer n such that $\dfrac{2020}{n \times n} = 126.25$.

1.3 Find all the positive integers n for which $\dfrac{2020}{n \times n}$ is an integer.

2. What is the remainder when 1234×5678 is divided by 5?

A 0 B 1 C 2 D 3 E 4

SOLUTION **C**

The units digit of 1234×5678 is the same as the units digit of 4×8, and hence is 2. Therefore the remainder when 1234×5678 is divided by 5 is 2.

FOR INVESTIGATION

2.1 What is the remainder when 1234×5678 is divided by 3?

2.2 What is the remainder when 1234×5678 is divided by 11?

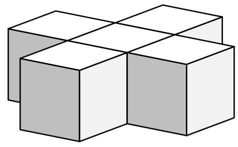
2.3 The integer m has remainder 3 when it is divided by 11. The integer n has remainder 4 when divided by 11.

What is the remainder when mn is divided by 11?

3. A shape is made from five unit cubes, as shown.

What is the surface area of the shape?

A 22 B 24 C 26 D 28 E 30



Solution **A**

The surface of the shape is made up of two faces of the central cube, and five faces of each of the four other cubes.

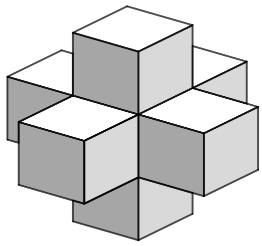
Therefore the surface is made up of $2 + 4 \times 5 = 22$ square faces each of size 1×1.

Hence the surface area of the shape is 22.

For investigation

3.1 The shape shown is made by adding two unit cubes to the shape of this question. It is made from seven unit cubes.

What is the surface area of the shape?



4. The numbers p, q, r and s satisfy the equations $p = 2$, $p \times q = 20$, $p \times q \times r = 202$ and $p \times q \times r \times s = 2020$.

What is the value of $p + q + r + s$?

A 32 B 32.1 C 33 D 33.1 E 34

Solution **B**

We have

$$p = 2,$$
$$q = \frac{p \times q}{p} = \frac{20}{2} = 10,$$
$$r = \frac{p \times q \times r}{p \times q} = \frac{202}{20} = 10.1,$$

and

$$s = \frac{p \times q \times r \times s}{p \times q \times r} = \frac{2020}{202} = 10.$$

Therefore $p + q + r + s = 2 + 10 + 10.1 + 10 = 32.1$.

5. What is $\sqrt{123454321}$?

A 1111111 B 111111 C 11111 D 1111 E 111

Solution **C**

Note: You could not be expected to be able to calculate the value of $\sqrt{123454321}$ without the use of a calculator. So you need to find some other way to select the correct option. We use a method based on the size of the number 123454321.

We have

$$10^8 < 123454321 < 10^{10}.$$

Therefore

$$\sqrt{10^8} < \sqrt{123454321} < \sqrt{10^{10}},$$

that is,

$$10^4 < \sqrt{123454321} < 10^5.$$

The correct option is therefore the only one that is between 10^4 and 10^5. Therefore, of the given options, it is 11111 that equals $\sqrt{123454321}$.

For investigation

5.1 The answer given above assumes that one of the given options is correct.

Verify that $11111 = \sqrt{123454321}$ by checking that $11111^2 = 123454321$.

6. There are fewer than 30 students in the A-level mathematics class. One half of them play the piano, one quarter play hockey and one seventh are in the school play.

How many of the students play hockey?

A 3 B 4 C 5 D 6 E 7

Solution **E**

Because one half of the students play the piano, the number of students is a multiple of 2.

Because one quarter of the students play hockey, the number of students is a multiple of 4.

Because one seventh of the students are in the school play, the number of students is a multiple of 7.

Therefore the number of students is a multiple of 2, 4 and 7. Hence the number of students is a multiple of 28.

Because there are fewer than 30 students in the class, it follows that there are 28 students in the class.

Therefore, because one quarter of the 28 students play hockey, the number of students who play hockey is 7.

7. Official UK accident statistics showed that there were 225 accidents involving teapots in one year. However, in the following year there were 47 such accidents.

What was the approximate percentage reduction in recorded accidents involving teapots from the first year to the second?

A 50 B 60 C 70 D 80 E 90

Solution **D**

The reduction in the number of teapot accidents in the second year was $225 - 47 = 178$.

178 as a percentage of 225 is

$$\frac{178}{225} \times 100 \approx \frac{180}{225} \times 100 = \frac{20}{25} \times 100 = 20 \times 4 = 80.$$

8. What is the largest prime factor of $106^2 - 15^2$?

A 3 B 7 C 11 D 13 E 17

Solution **D**

Note: It is not a good idea to attempt to calculate 106^2 and 15^2, then do a subtraction and finally attempt to factorize the resulting answer.

Instead we make use of the standard factorization of the difference of two squares:

$$x^2 - y^2 = (x - y)(x + y).$$

We have

$$\begin{aligned} 106^2 - 15^2 &= (106 - 15)(106 + 15) \\ &= 91 \times 121 \\ &= 7 \times 13 \times 11 \times 11. \end{aligned}$$

Therefore the prime factorization of $106^2 - 15^2$ is $7 \times 11^2 \times 13$, from which we see that its largest prime factor is 13.

For investigation

8.1 What is the largest prime factor of $300^2 - 3^2$?

9. In 2018, a racing driver was allowed to use the Drag Reduction System provided that the car was within 1 second of the car ahead. Suppose that two cars were 1 second apart, each travelling at 180 km/h (in the same direction!).

How many metres apart were they?

A 100 B 50 C 10 D 5 E 1

SOLUTION **B**

The distance apart of the cars was the distance that a car travelling at 180 km/h travels in 1 second.

There are 1000 metres in one kilometre. Hence there are 180×1000 metres in 180 km.

There are 60×60 seconds in each hour.

It follows that at 180 km/h a car travels 180×1000 metres in 60×60 seconds.

Therefore the number of metres that it travels in 1 second is

$$\frac{180 \times 1000}{60 \times 60} = \frac{3 \times 1000}{60} = \frac{1000}{20} = 50.$$

Therefore when the cars are 1 second apart, they are 50 metres apart.

FOR INVESTIGATION

9.1 The Highway Code gives the following table of typical *stopping distances* in metres for motor vehicles travelling at different velocities.

velocity	stopping distance
32 km/h	12 m
48 km/h	23 m
64 km/h	36 m
80 km/h	53 m
96 km/h	73 m
112 km/h	96 m

For each velocity, how far apart in seconds should two cars travelling in the same direction at that velocity be so that their distance apart is the same as the corresponding stopping distance given in the above table?

9.2 *Note*: The Highway Code adds that "The distances shown are a general guide. The distance will depend on your attention (thinking distance), the road surface, the weather conditions and the condition of your vehicle at the time. "

The Highway Code gives a thinking distance of 12 m for a car travelling at 64 km/h. How much thinking time does that correspond to?

10. Six friends Pat, Qasim, Roman, Sam, Tara and Uma, stand in a line for a photograph. There are three people standing between Pat and Qasim, two between Qasim and Roman and one between Roman and Sam. Sam is not at either end of the line.

How many people are standing between Tara and Uma?

A 4 B 3 C 2 D 1 E 0

SOLUTION **C**

We indicate each of the friends by the first letter of their name, and a person whose name we are not yet sure about by an asterisk (*).

We can assume, without loss of generality, that, from the point of view of the photographer, Qasim is to the right of Pat. Because there are three people standing between Pat and Qasim, the line is either

$$* \; P \; * \; * \; * \; Q \quad \text{or} \quad P \; * \; * \; * \; Q \; * \, .$$

There are two people between Qasim and Roman. Roman cannot be to the right of Quasim, because there is at most one friend to the right of Quasim. Therefore Roman is to the left of Quasim and the line is either

$$* \; P \; R \; * \; * \; Q \quad \text{or} \quad P \; R \; * \; * \; Q \; * \, .$$

There is one person between Roman and Sam. Therefore either Sam is immediately to the left of Pat, or immediately to the left of Qasim.

If Sam is immediately to the left of Pat, the line would be

$$S \; P \; R \; * \; * \; Q.$$

However this is impossible because Sam is not at either end of the line.

Therefore Sam is immediately to the left of Qasim. Hence the line is either

$$* \; P \; R \; * \; S \; Q \quad \text{or} \quad P \; R \; * \; S \; Q \; *$$

with Tara and Uma occupying the two places marked by asterisks.

We see that, however Tara and Uma occupy these two places, the number of people standing between Tara and Uma will be 2.

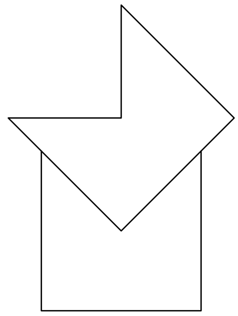
FOR INVESTIGATION

10.1 Specify one additional piece of information that would make it possible to work out the exact order of the six friends from left to right, as seen by the photographer.

11. Two congruent pentagons are each formed by removing a right-angled isosceles triangle from a square of side-length 1. The two pentagons are then fitted together as shown.

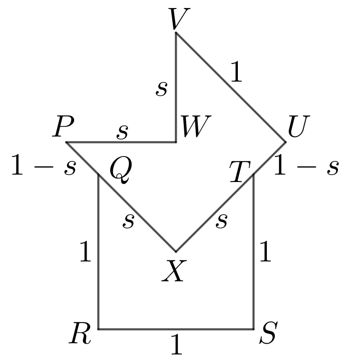
What is the length of the perimeter of the octagon formed?

A 4 B $4 + 2\sqrt{2}$ C 5 D $6 - 2\sqrt{2}$ E 6



Solution **E**

Let P, Q, R, S, T, U, V and W be the vertices of the octagon, as shown in the diagram, and let X be the vertex as shown.



The sides PX, XU, UV, QR, RS and ST all have length 1.

Let PW have length s. Then, because PWV is a right-angled isosceles triangle, VW also has length s.

Because the pentagons $PXUVW$ and $QRSTX$ are congruent, QX and TX also have length s.

It follows that PQ and TU each have length $1 - s$.

We can now deduce the the length of the perimeter of the octagon $PQRSTUVW$ is

$$(1 - s) + 1 + 1 + 1 + (1 - s) + 1 + s + s = 6.$$

Commentary

Note that in order to work out the length of the perimeter we did not need to know the value of s. It is, however, not difficult to find the value of s. You are asked to do this in Problem 11.1.

For investigation

11.1 Find the length of QX.

11.2 Find the area of the octagon $PQRSTUVW$.

12. A three-piece suit consists of a jacket, a pair of trousers and a waistcoat. Two jackets and three pairs of trousers cost £380. A pair of trousers costs the same as two waistcoats.

What is the cost of a three-piece suit?

A £150 B £190 C £200 D £228 E more information needed

SOLUTION **B**

We let the cost of a jacket, a pair of trousers and a waistcoat be £J, £T and £W, respectively.

From the information given in the question we can deduce that

$$2J + 3T = 380 \qquad (1)$$

and

$$T = 2W \qquad (2)$$

Note: In this problem we have three unknowns J, T and W, but only two equations. We don't have enough information to enable us to find the values of J, T and W. However, we can deduce the value of $J + T + W$ which is what we need to know. We give two methods for doing this.

METHOD 1

If we subtract equation (2) from equation (1) we obtain

$$2J + 2T = 380 - 2W.$$

Hence

$$2J + 2T + 2W = 380.$$

Hence, dividing both sides of this last equation by 2, we obtain

$$J + T + W = 190.$$

Therefore the cost of a three piece suit is £190.

METHOD 2

It follows from equation (2) that $W = \frac{1}{2}T$. Hence,

$$\begin{aligned} J + T + W &= J + T + \tfrac{1}{2}T \\ &= J + \tfrac{3}{2}T \\ &= \tfrac{1}{2}(2J + 3T) \\ &= \tfrac{1}{2}(380), \qquad \text{by equation (1),} \\ &= 190. \end{aligned}$$

Therefore the cost of a three-piece suit is £190.

13. The number $16! \div 2^k$ is an odd integer. Note that $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

What is the value of k?

A 9 B 11 C 13 D 15 E 17

SOLUTION **D**

The number k that we seek is such that $16! = 2^k \times q$, where q is an odd integer. Thus k is the power of 2 that occurs in the prime factorization of $16!$.

METHOD 1

To find the highest power of 2 that is a factor of $16!$ we need only consider the even numbers that occur in the product $1 \times 2 \times \cdots \times 15 \times 16$ that gives the value of $16!$

In the following table we give the powers of 2 that divide each of these factors.

factor	2	4	6	8	10	12	14	16
power of 2	1	2	1	3	1	2	1	4

The power of 2 in the prime factorization of 16! is the sum of the powers in the second row of this table. That is, it is $1 + 2 + 1 + 3 + 1 + 2 + 1 + 4 = 15$.

Hence, the required value of k is 15.

METHOD 2

Note:
In this method we use the formula, usually attributed to the French mathematician Adrian-Marie Legendre (1752-1833), for the highest power of a prime p that divides $n!$. This formula makes use of the *floor* function which we first explain.

We define the *floor* of x, written as $\lfloor x \rfloor$, to be the largest integer that is not larger than x.

For example, $\lfloor 2\frac{6}{7} \rfloor = 2$, $\lfloor 4.275 \rfloor = 4$, $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$, $\lfloor 0.43 \rfloor = 0$ and $\lfloor -5.23 \rfloor = -6$.

According to Legendre's formula, the highest power of the prime p that divides $n!$ is given by

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots.$$

At first sight, the formula involves an infinite sum. However, if $p^k > n$, we have $0 < \dfrac{n}{p^k} < 1$ and hence $\left\lfloor \dfrac{n}{p^k} \right\rfloor = 0$. Therefore only a finite number of terms in the above sum are non-zero. The number of non-zero terms in this sum is the largest integer k for which $p^k \leq n$.

By Legendre's formula the highest power of 2 that divides $16!$ is k, where

$$\begin{aligned} k &= \left\lfloor \frac{16}{2} \right\rfloor + \left\lfloor \frac{16}{2^2} \right\rfloor + \left\lfloor \frac{16}{2^3} \right\rfloor + \left\lfloor \frac{16}{2^4} \right\rfloor \\ &= \left\lfloor \frac{16}{2} \right\rfloor + \left\lfloor \frac{16}{4} \right\rfloor + \left\lfloor \frac{16}{8} \right\rfloor + \left\lfloor \frac{16}{16} \right\rfloor \\ &= \lfloor 8 \rfloor + \lfloor 4 \rfloor + \lfloor 2 \rfloor + \lfloor 1 \rfloor \\ &= 8 + 4 + 2 + 1 \\ &= 15. \end{aligned}$$

For investigation

13.1 Why is 15 the only value of k for which $16! \div 2^k$ is an odd integer?

13.2 Use Legendre's formula to find the highest power of 3 that divides $16!$.

13.3 What is the highest power of 2 that divides $100!$?

13.4 What is the highest power of 3 that divides $100!$?

13.5 Find the prime factorization of $100!$

13.6 Find the least positive integer n such that 2^{100} is a factor of $n!$.

13.7 Prove that Legendre's formula is correct.

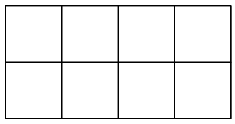
That is, show that the highest power of the prime p that divides $n!$ is given by the sum

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{p^k} \right\rfloor,$$

where k is the largest integer such that $p^k \leq n$.

13.8 Let p be a prime number and let n be a positive integer. Find a formula in terms of p and n for the highest power of p that divides $(p^n)!$.

14. Diane has five identical blue disks, two identical red disks and one yellow disk. She wants to place them on the grid opposite so that each cell contains exactly one disk. The two red disks are not to be placed in cells that share a common edge.



How many different-looking completed grids can she produce?

A 96 B 108 C 144 D 180 E 216

SOLUTION **B**

Note: The key to a problem of this kind is deciding the order in which to consider the placing of the differently coloured disks. In this case it is best to consider first the number of different ways the two red disks may be placed, because they are subject to the condition that they should not be put in cells that share an edge.

In the diagram we have labelled the cells so that we can refer to them.

P	Q	R	S
T	U	V	W

If the first red disk is placed the cell P, then the second red disk may be placed in any one of the 5 cells R, S, U, V and W.

Likewise, if it placed in the any of the cells S, T and Q, there are 5 possible cells for the second red disk.

If the first red disk is placed any one of the cells Q, R, U and V, there are 4 choices for the second red disk.

This gives $5 \times 4 + 4 \times 4 = 36$ ways to place the two red disks, but each possible pair of cells has been counted twice.

Therefore there are $36 \div 2 = 18$ different ways to place the two red disks.

Once the red disks have been placed, there remain 6 cells in which the yellow disk may be placed.

Once the red and yellow disks have been placed, the 5 blue disks must be placed in the remaining 5 empty cells. This may be done in just 1 way.

This gives $18 \times 6 \times 1 = 108$ different-looking ways of filling the grid.

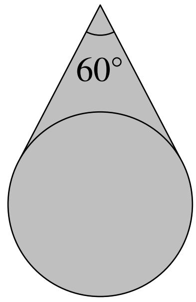
FOR INVESTIGATION

14.1 How many different-looking grids can Diane produce if she has one yellow disk, three identical red disks and four identical blue disks, and there are no restrictions other than that each cell should contain exactly one disk?

14.2 How many different-looking grids can Diane produce if she has one yellow disk, two identical red disks, two identical blue disks and three identical green disks, and there are no restrictions other than that each cell should contain exactly one disk?

14.3 How many different-looking grids can Diane produce if she has one yellow disk, two identical red disks, two identical blue disks and three identical green disks, with each cell containing exactly one disk and the two red disks not in cells that share a common edge?

15. The shaded area shown in the diagram consists of the interior of a circle of radius 3 together with the area between the circle and two tangents to the circle. The angle between the tangents at the point where they meet is 60°.

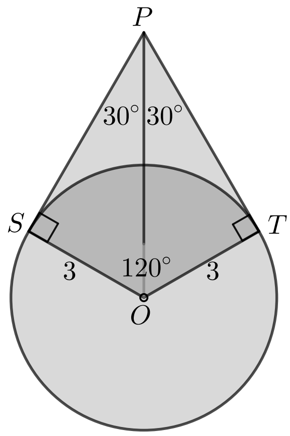


What is the shaded area?

A $6\pi + 9\sqrt{3}$ B $15\sqrt{3}$ C 9π
D $9\pi + 4\sqrt{3}$ E $6\pi + \frac{9\sqrt{3}}{4}$

SOLUTION **A**

Let O be the centre of the circle and let PS and PT be the tangents to the circle, as shown.



In the triangles PSO and PTO we have $\angle PSO = \angle PTO = 90°$ because PS and PT are tangents to the circle; $SO = TO$, because they are radii of the same circle; and the side PO is common.

Therefore, the two triangles are congruent (RHS). Hence $\angle SPO = \angle TPO = 30°$.

It follows that $\dfrac{OS}{PS} = \tan 30° = \dfrac{1}{\sqrt{3}}$.

Hence $PS = OS\sqrt{3} = 3\sqrt{3}$.

Using the formula area $= \frac{1}{2}(\text{base} \times \text{height})$, for the area of a triangle, it follows that the area of the triangle PSO is $\frac{1}{2}(OS \times PS) = \frac{1}{2}(3 \times 3\sqrt{3}) = \frac{1}{2}(9\sqrt{3})$.

This is also the area of the congruent triangle PTO. Therefore the sum of the areas of these two triangles is $9\sqrt{3}$.

The total shaded area is the sum of the areas of these two triangle plus the area of that part of the circle that lies outside the two triangles. Because $\angle SOT = 120°$, the part of the circle outside the two triangles makes up two-thirds of the circle and hence its area is given by

$$\tfrac{2}{3}(\pi \times 3^2) = 6\pi.$$

Hence the shaded area is $6\pi + 9\sqrt{3}$.

FOR INVESTIGATION

15.1 Explain why $\angle SOT = 120°$.

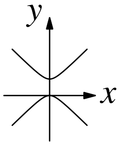
15.2 Explain why it follows that the area of the part of the circle outside the two triangles is two-thirds of the total area of the circle.

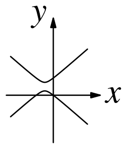
15.3 The solution above uses the fact that $\tan 30° = \dfrac{1}{\sqrt{3}}$.

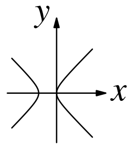
Explain why this is correct.

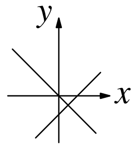
15.4 What is the shaded area in the case where $\angle SPT = 120°$?

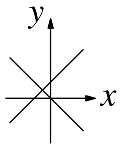
16. Which diagram represents the set of all points (x, y) satisfying $y^2 - 2y = x^2 + 2x$?

A


B


C


D


E


Solution **E**

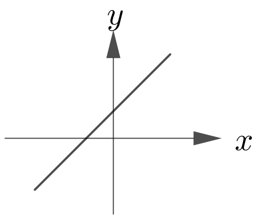
We have

$$\begin{aligned} y^2 - 2y = x^2 + 2x &\Leftrightarrow y^2 - x^2 - 2y - 2x = 0 \\ &\Leftrightarrow (y - x)(y + x) - 2(y + x) = 0 \\ &\Leftrightarrow (y - x - 2)(y + x) = 0 \\ &\Leftrightarrow y - x - 2 = 0 \quad \text{or} \quad y + x = 0 \\ &\Leftrightarrow y = x + 2 \quad \text{or} \quad y = -x. \end{aligned}$$

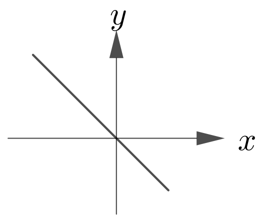
Note: An alternative method is to complete the square on both sides of the equation. See Problem 16.1 below.

We therefore see that the set of points satisfying the equation $y^2 - 2y = x^2 + 2x$ is made up of all the points on the line with the equation $y = x + 2$ together with all the points on the line with the equation $y = -x$.

The line with the equation $y = x + 2$ is the line with slope 1 that goes through the point $(0, 2)$.



$y = x + 2$



$y = -x$

The line with the equation $y = -x$ is the line with slope -1 that goes through the point $(0, 0)$.

The diagram that shows these two lines is that in option E.

For investigation

16.1 Show that adding 1 to both sides of the equation $y^2 - 2y = x^2 + 2x$, provides an alternative way to show that $y = x + 2$ or $y = -x$.

16.2 What are the coordinates of the point where the lines with equations $y = x + 2$ and $y = -x$ meet?

16.3 Draw a diagram to represent the set of all points (x, y) which satisfy the equation $x^2 - 1 = y^2 + 2y$.

16.4 Draw a diagram to represent the set of all points (x, y) which satisfy the equation $x^2y^2 + xy = x^3 + y^3$.

17. The positive integers m, n and p satisfy the equation $3m + \dfrac{3}{n + \frac{1}{p}} = 17.$

What is the value of p?

A 2 B 3 C 4 D 6 E 9

SOLUTION **A**

Because $\dfrac{3}{n + \frac{1}{p}} = 17 - 3m$, where m is an integer, it follows that $\dfrac{3}{n + \frac{1}{p}}$ is an integer.

Because n and p are positive integers, $1 < n + \frac{1}{p}$ and therefore $0 < \dfrac{3}{n + \frac{1}{p}} < 3$. Therefore $\dfrac{3}{n + \frac{1}{p}}$ is either 1 or 2.

If $\dfrac{3}{n + \frac{1}{p}} = 1$, then $3m = 17 - \dfrac{3}{n + \frac{1}{p}} = 17 - 1 = 16$. This implies that $m = \frac{16}{3}$ contradicting the fact that m is an integer. Hence $\dfrac{3}{n + \frac{1}{p}} = 2$. Therefore $n + \frac{1}{p} = \frac{3}{2} = 1 + \frac{1}{2}$. Hence $n = 1$ and $p = 2$. Also $3m = 17 - \dfrac{3}{1 + \frac{1}{2}} = 17 - 2 = 15$ and so $m = 5$. We therefore see that $m = 5$, $n = 1$ and $p = 2$.

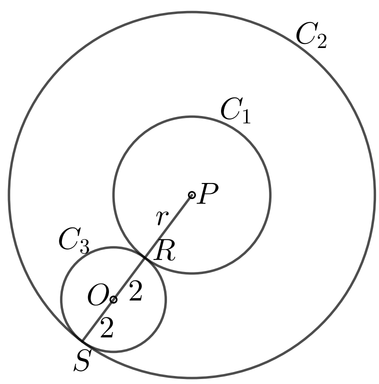
18. Two circles C_1 and C_2 have their centres at the point (3,4) and touch a third circle, C_3. The centre of C_3 is at the point (0,0) and its radius is 2.

What is the sum of the radii of the two circles C_1 and C_2?

A 6 B 7 C 8 D 9 E 10

SOLUTION **E**

Let P be the centre of the circles C_1 and C_2, O be the centre of the circle C_3, and R and S be the points where C_1 and C_2, respectively, touch C_3. We let the radius of C_1 be r.



Note that the points O, P, R and S lie on a straight line.

By Pythagoras' Theorem, the distance between the point P with coordinates $(3, 4)$ and the point O with coordinates $(0, 0)$ is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

Therefore, $r + 2 = 5$ and hence $r = 3$.

The radius of the circle C_2 is $r + 2 + 2 = 3 + 2 + 2 = 7$.

Therefore the sum of the radii of C_1 and C_2 is $3 + 7 = 10$.

FOR INVESTIGATION

18.1 Prove that the points P, R, O and S lie on a straight line.

19. The letters p, q, r, s and t represent different positive single-digit numbers such that $p - q = r$ and $r - s = t$.

How many different values could t have?

A 6 B 5 C 4 D 3 E 2

SOLUTION **A**

Because $r = p - q$, we have $t = r - s = (p - q) - s = p - (q + s)$.

It follows that the maximum value of t occurs when p takes its maximum value, and $q + s$ takes its minimum value.

The maximum value of p is 9.

Because q and s have different values, the minimum value of $q + s$ is $1 + 2 = 3$.

Therefore the maximum possible value of t is $9 - 3 = 6$.

The following table shows that t can take all the positive integer values from 1 to 6.

p	q	r	s	t
9	1	8	2	6
9	1	8	3	5
9	2	7	3	4
9	2	7	4	3
9	3	6	4	2
9	3	6	5	1

Therefore the number of values that t could take is 6.

FOR INVESTIGATION

19.1 In how many different ways is it possible to choose different single-digit values for p, q, r and s so that $t = 1$?

19.2 The letters p, q, r and s represent different positive single-digit numbers such that $p - q = r$ and $q - r = s$. How many different values could s have?

19.3 The letters p, q, r and s represent different positive single-digit numbers such that $p - q = r$ and $p - r = s$. How many different values could s have?

20. The real numbers x and y satisfy the equations $4^y = \dfrac{1}{8(\sqrt{2})^{x+2}}$ and $9^x \times 3^y = 3\sqrt{3}$.

What is the value of 5^{x+y}?

A $5\sqrt{5}$ B 5 C $\sqrt{5}$ D $\frac{1}{5}$ E $\frac{1}{\sqrt{5}}$

Solution **E**

We have $4^y = (2^2)^y = 2^{2y}$, and $\dfrac{1}{8(\sqrt{2})^{x+2}} = \dfrac{1}{2^3(2^{\frac{1}{2}})^{x+2}} = 2^{-(3+\frac{1}{2}(x+2))}$.

Therefore from the equation $4^y = \dfrac{1}{8(\sqrt{2})^{x+2}}$ we deduce that $2y = -(3 + \frac{1}{2}(x+2))$.

Therefore

$$y = -\tfrac{1}{4}x - 2. \qquad (1)$$

Also, $9^x \times 3^y = (3^2)^x \times 3^y = 3^{2x+y}$ and $3\sqrt{3} = (3^1)(3^{\frac{1}{2}}) = 3^{1+\frac{1}{2}} = 3^{\frac{3}{2}}$.

Therefore, from the equation $9^x \times 3^y = 3\sqrt{3}$ we deduce that $2x + y = \frac{3}{2}$.

Therefore

$$y = -2x + \tfrac{3}{2}. \qquad (2)$$

From equations (1) and (2)

$$-\tfrac{1}{4}x - 2 = -2x + \tfrac{3}{2}.$$

This last equation may be rearranged to give

$$\tfrac{7}{4}x = \tfrac{7}{2}.$$

Hence $x = 2$.

Therefore, from equation (1), $y = -\frac{5}{2}$.

It follows that $x + y = 2 + (-\frac{5}{2}) = 2 - \frac{5}{2} = -\frac{1}{2}$.

Therefore $5^{x+y} = 5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$.

For investigation

20.1 The real numbers x and y satisfy the equations

$$(\sqrt{3})^x \times 3^y = \sqrt[5]{27}$$

and

$$8^x \times (\sqrt{2})^y = \sqrt[3]{4}.$$

Find the value of 8^{x+y}.

21. When written out in full, the number $(10^{2020} + 2020)^2$ has 4041 digits.

What is the sum of the digits of this 4041-digit number?

A 9 B 17 C 25 D 2048 E 4041

Solution **C**

Using the expansion $(x + y)^2 = x^2 + 2xy + y^2$, we have

$$\begin{aligned}(10^{2020} + 2020)^2 &= (10^{2020})^2 + 2 \times 10^{2020} \times 2020 + 2020^2 \\ &= 10^{4040} + 4040 \times 10^{2020} + 4080400. \qquad (1)\end{aligned}$$

We now note that when we add the three terms 10^{4040}, 4040×10^{2002} and 408040, no two non-zero digits occur in the same column:

$$\begin{array}{rr} & 100\ldots\ldots\ldots.0000\ldots\ldots\ldots.0000000 \\ + & 4040\ldots\ldots\ldots.0000000 \\ + & 4080400 \\ \hline & 100\ldots\ldots\ldots.4040\ldots\ldots\ldots.4080400 \end{array}$$

It follows that the non-zero digits in the final answer are exactly the non-zero digits in the three terms in (1) above.

Therefore the sum of the digits in the number $(10^{2020} + 2020)^2$ is

$$1 + 4 + 4 + 4 + 8 + 4 = 25.$$

Commentary

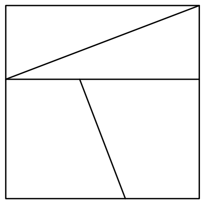
The exact number of 0s in the final answer is not important. What matters is that the non-zero digits in the three terms 10^{4040}, 4040×10^{2020} and 4080400 are in different columns, and so, when they are added, the non-zero digits in the final answer are exactly the non-zero digits in these terms. However, you are asked to work out the number of 0s in Problem 21.1 below.

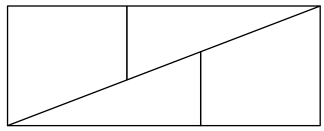
For investigation

21.1 Find the positive integers m and n so that

$$(10^{2020} + 2020)^2 = 1\,\overbrace{000\ldots000}^{m}\,404\,\overbrace{000\ldots000}^{n}\,4080400.$$

22. A square with perimeter 4 cm can be cut into two congruent right-angled triangles and two congruent trapezia as shown in the first diagram in such a way that the four pieces can be rearranged to form the rectangle shown in the second diagram.



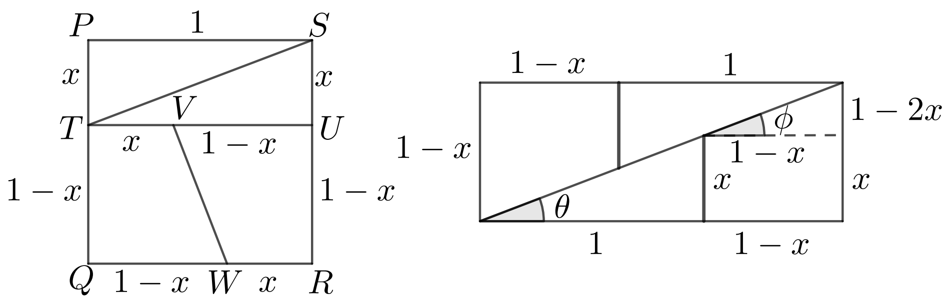


What is the perimeter, in centimetres, of this rectangle?

A $2\sqrt{5}$ B $4\sqrt{2}$ C 5 D $4\sqrt{3}$ E $3\sqrt{7}$

SOLUTION **A**

The square has perimeter 4 cm and hence its sides each have length 1 cm.



Let the vertices in the diagram on the left be labelled as shown.

We let the length of SU be x cm.

For the pieces to fit together as shown on the right, TV and WR must also have length x cm. It follows that the other lengths, in cm, are as shown.

METHOD 1

For the pieces to fit together the angles ϕ and θ must be equal, as shown on the right of the above diagram. Therefore $\tan\phi = \tan\theta$. That is,

$$\frac{1-2x}{1-x} = \frac{x}{1}.$$

Hence

$$1 - 2x = x(1-x).$$

This last equation may be rearranged to give

$$x^2 - 3x + 1 = 0.$$

From the standard formula $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ for the solutions of a quadratic equation, we can deduce that

$$x = \frac{3 \pm \sqrt{5}}{2}.$$

Since $x < 1$ it follows that $x = \dfrac{3 - \sqrt{5}}{2}$.

From the diagram on the right above we see that the perimeter of the rectangle is given by

$$(1 - x) + (2 - x) + (1 - x) + (2 - x) = 6 - 4x = 6 - 4\left(\frac{3 + \sqrt{5}}{2}\right) = 2\sqrt{5}.$$

Method 2

Since they are made up from the same two triangles and two trapezia, the area of the rectangle equals the area of the square.

Therefore, from the diagram above, we have

$$(1 + (1 - x)) \times (1 - x) = 1 \times 1,$$

that is,

$$(2 - x)(1 - x) = 1.$$

Hence

$$2 - 3x + x^2 = 1$$

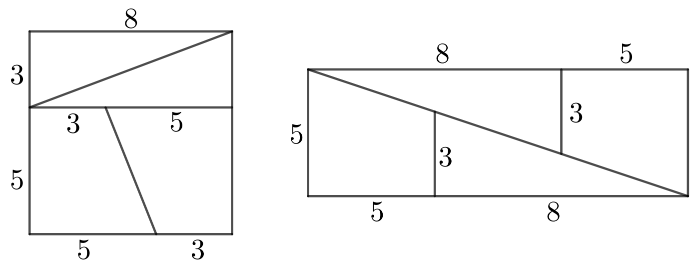
and therefore

$$x^2 - 3x + 1 = 0.$$

Thus, as in Method 1, $x = \dfrac{3 - \sqrt{5}}{2}$, and the perimeter of the rectangle is $2\sqrt{5}$.

For investigation

22.1



The diagram above shows on the left an 8×8 square divided into four pieces which are rearranged to make the 13×5 rectangle shown on the right.

However $8 \times 8 < 13 \times 5$.

Where has the additional area come from?

Note: This puzzle has been attributed to William Hooper in his book *Rational Recreations* of 1774.

23. A function f satisfies $y^3 f(x) = x^3 f(y)$ and $f(3) \neq 0$. What is the value of $\frac{f(20) - f(2)}{f(3)}$?

A 6 B 20 C 216 D 296 E 2023

SOLUTION **D**

By putting $x = 20$ and $y = 3$ in the equation $y^3 f(x) = x^3 f(y)$, we have $27 f(20) = 8000 f(3)$. Hence

$$f(20) = \frac{8000}{27} f(3).$$

By putting $x = 2$ and $y = 3$ in the same equation, we have $27 f(2) = 8 f(3)$. Hence

$$f(2) = \frac{8}{27} f(3).$$

Therefore,

$$\begin{aligned} f(20) - f(2) &= \frac{8000}{27} f(3) - \frac{8}{27} f(3) \\ &= \left(\frac{8000}{27} - \frac{8}{27}\right) f(3) \\ &= \frac{7992}{27} f(3) \\ &= 296 f(3). \end{aligned}$$

Therefore, because $f(3) \neq 0$, it follows that

$$\frac{f(20) - f(2)}{f(3)} = 296.$$

FOR INVESTIGATION

23.1 What is the value of

$$\frac{f(46) - f(23)}{f(23)}?$$

23.2 Show that there are infinitely many positive integers x, y and z which satisfy the equation

$$\frac{f(x) - f(y)}{f(z)} = 7.$$

23.3 Show that for each positive integer n the equation

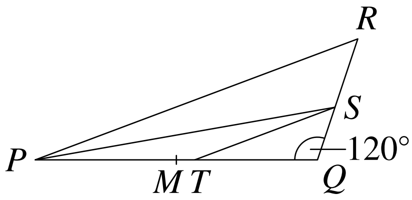
$$\frac{f(x) - f(y)}{f(z)} = n$$

has either infinitely many solutions in which x, y and z are positive integers, or no such solutions.

23.4 Show that a function g satisfies the equation $y^3 g(x) = x^3 g(y)$ for all real numbers x and y if, and only if, there is a constant k such that $g(x) = kx^3$, for all real numbers x.

24. In the diagram shown, M is the mid-point of PQ. The line PS bisects $\angle RPQ$ and intersects RQ at S. The line ST is parallel to PR and intersects PQ at T. The length of PQ is 12 and the length of MT is 1. The angle SQT is 120°.

What is the length of SQ?



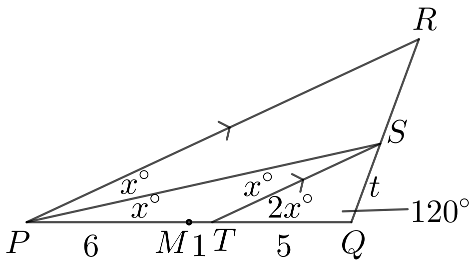
A 2 B 3 C 3.5 D 4 E 5

Solution **B**

Let t be the length of SQ.

Let $\angle QPS = x°$.

Because PQ has length 12 and M is the midpoint of PQ, it follows that MP has length 6. We are given that MT has length 1. Therefore PT has length 7, and TQ has length 5.



Because PS bisects $\angle RPQ$, we have $\angle RPS = \angle QPS = x°$. Therefore $\angle RPQ = 2x°$.

Because ST is parallel to PR, we have $\angle STQ = \angle RPQ = 2x°$.

It follows from the External Angle Theorem [See Problem 24.1], applied to the triangle PTS, that $\angle PST = x°$.

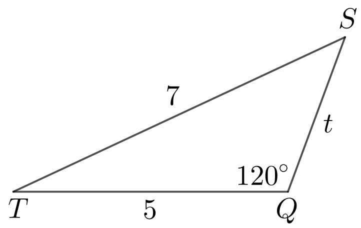
Therefore the triangle PST is isosceles and hence ST has the same length as PT, namely 7.

We now apply the Cosine Rule to the triangle STQ. This gives

$$7^2 = 5^2 + t^2 - 2(5t\cos 120°).$$

Hence, as $\cos 120° = -\frac{1}{2}$,

$$49 = 25 + t^2 + 5t.$$



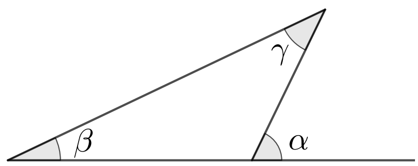
Therefore $t^2 + 5t - 24 = 0$. Hence $(t-3)(t+8) = 0$. Therefore t is either 3 or -8. Since t corresponds to a length, $t > 0$. We deduce that $t = 3$.

For investigation

24.1 The *External Angle Theorem* says that the external angle of a triangle is the sum of the two opposite angles.

In terms of the diagram it says that $\alpha = \beta + \gamma$.

Explain why the External Angle Theorem is true.

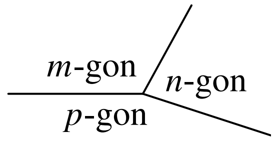


24.2 The solution above uses the fact that $\cos 120° = -\frac{1}{2}$. Explain why this is correct.

25. A regular m-gon, a regular n-gon and a regular p-gon share a vertex and pairwise share edges, as shown in the diagram.

What is the largest possible value of p?

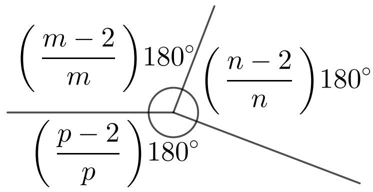
A 6 B 20 C 42 D 50 E 100



SOLUTION **C**

The sum of the internal angles of a polygon with m sides is $(m-2)180°$ [see Problem 25.1]. Hence each internal angle of the regular m-gon is $\left(\frac{m-2}{m}\right)180°$, and similarly for the regular n-gon and the regular p-gon. Therefore, because the sum of the angles at a point is 360°, we have

$$\left(\frac{m-2}{m}\right)180+\left(\frac{n-2}{n}\right)180+\left(\frac{p-2}{p}\right)180=360.$$



It follows that

$$\frac{m-2}{m}+\frac{n-2}{n}+\frac{p-2}{p}=2.$$

We may rewrite this last equation as $\left(1-\frac{2}{m}\right)+\left(1-\frac{2}{n}\right)+\left(1-\frac{2}{p}\right)=2$. Hence

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{p}=\frac{1}{2}. \qquad (1)$$

We note that as p is a positive integer, it follows from (1) that $\frac{1}{m}+\frac{1}{n}<\frac{1}{2}$. (2)

We seek a solution of (1), where m, n and p are positive integers and p is as large as possible.

For p to be as large as possible, $\frac{1}{p}$ needs to be as small as possible. Hence, by (1), $\frac{1}{m}+\frac{1}{n}$ needs to be as large as possible, subject to the inequality (2). Thus m and n need to be as small as possible. Without loss of generality we may assume that $m \leq n$.

By (2), $m \geq 3$. If $m=3$, then $\frac{1}{n}<\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$, and hence $n>6$. When $m=3$ and $n=7$, we have $\frac{1}{m}+\frac{1}{n}=\frac{1}{3}+\frac{1}{7}=\frac{10}{21}$. Hence, by (1), $\frac{1}{p}=\frac{1}{2}-\frac{10}{21}=\frac{1}{42}$ and so $p=42$. We show that this gives the largest possible value of p.

We cannot have $m=n=4$, as this is not compatible with the inequality (2).

If $n>m\geq 4$, we have $\frac{1}{m}+\frac{1}{n}\leq\frac{1}{4}+\frac{1}{5}=\frac{9}{20}<\frac{10}{21}$ and so $\frac{1}{m}+\frac{1}{n}$ is not as large as possible.

Hence the largest possible value of p is given by $m=3$, $n=7$ and $p=42$.

FOR INVESTIGATION

25.1 Prove that the sum of the angles of a polygon with m sides is $(m-2)180°$ and hence that each internal angle of a regular m-gon is $\left(\frac{m-2}{m}\right)180°$.

United Kingdom
Mathematics Trust

Senior Mathematical Challenge

10 – 11 November 2021

Organised by the United Kingdom Mathematics Trust

supported by [XTX] MARKETS Overleaf

Candidates must be full-time students at secondary school or FE college.
England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil.** Mark A, B, C, D, E on the Answer Sheet for each question. Mark only one option, boldly, within the box.
5. Your Answer Sheet will be read by a machine. **Do not write or doodle on the sheet except to mark your chosen options.** The machine will read all markings, including bits of eraser stuck to the page, and interpret the mark in its own way.
6. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
7. **Scoring rules:** All candidates start with 25 marks; 0 marks are awarded for each question left unanswered; 4 marks are awarded for each correct answer; 1 mark is deducted for each incorrect answer (to discourage guessing).
8. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.
9. To accommodate candidates sitting at other times, please do not discuss the paper on the internet until 08:00 BST on Friday 12 November.

Enquiries about the Senior Mathematical Challenge should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 365 1121 challenges@ukmt.org.uk www.ukmt.org.uk

1. Cicely had her 21st birthday in 1939.

When did she have her 100th birthday?

A 2020 B 2019 C 2018 D 2010 E 2008

2. The sequence, formed from the sequence of primes by rounding each to the nearest ten, begins 0, 0, 10, 10, 10, 10, 20, 20, 20, 30, ...

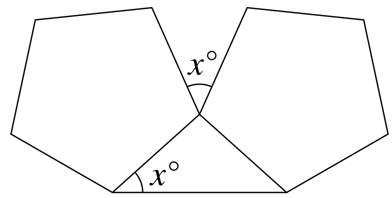
When continued, how many terms in this sequence are equal to 40?

A 1 B 2 C 3 D 4 E 5

3. The diagram shows two congruent regular pentagons and a triangle. The angles marked $x°$ are equal.

What is the value of x?

A 24 B 30 C 36 D 40 E 45



4. The positive integer k is a solution of the equation $(k \div 12) \div (15 \div k) = 20$.

What is the sum of the digits of k?

A 15 B 12 C 9 D 6 E 3

5. The sum of four consecutive primes is itself prime.

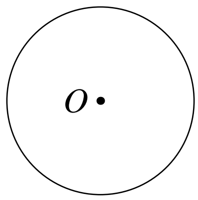
What is the largest of the four primes?

A 37 B 29 C 19 D 13 E 7

6. Three points, P, Q and R are placed on the circumference of a circle with centre O. The arc lengths PQ, QR and RP are in the ratio 1 : 2 : 3.

In what ratio are the areas of the sectors POQ, QOR and ROP?

A 1 : 1 : 1 B 1 : 2 : 3 C $1 : \pi : \pi^2$ D 1 : 4 : 9 E 1 : 8 : 27

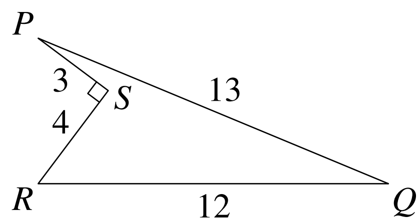


7. Which of these numbers is the largest?

A 2^{5000} B 3^{4000} C 4^{3000} D 5^{2000} E 6^{1000}

8. What is the area of the region inside the quadrilateral $PQRS$?

A 18 B 24 C 36 D 48 E more information needed



9. Alison has a set of ten fridge magnets showing the integers from 0 to 9 inclusive.

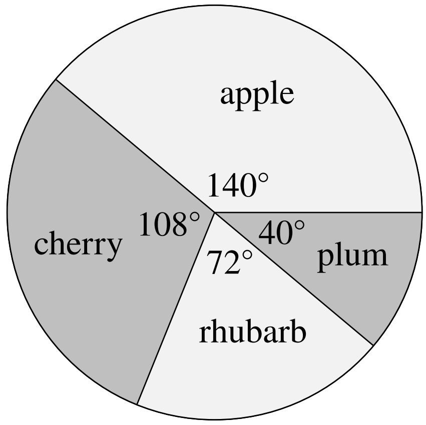
In how many different ways can she split the set into five pairs so that the sum of each pair is a multiple of 5?

A 1 B 2 C 3 D 4 E 5

10. In a survey, people were asked to name their favourite fruit pie. The pie chart shows the outcome. The angles shown are exact with no rounding.

What is the smallest number of people who could have been surveyed?

A 45 B 60 C 80 D 90 E 180



11. Alitta claims that if p is an odd prime then $p^2 - 2$ is also an odd prime.

Which of the following values of p is a counterexample to this claim?

A 3 B 5 C 7 D 9 E 11

12. For how many positive integers N is the remainder 6 when 111 is divided by N?

A 5 B 4 C 3 D 2 E 1

13. Which of these is the mean of the other four?

A $\sqrt{2}$ B $\sqrt{18}$ C $\sqrt{200}$ D $\sqrt{32}$ E $\sqrt{8}$

14. What is the smallest number of rectangles, each measuring 2 cm by 3 cm, which are needed to fit together without overlap to form a rectangle whose sides are in the ratio 5 : 4 ?

A 10 B 15 C 20 D 30 E 60

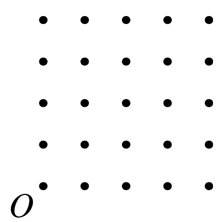
15. Three dice, each showing numbers 1 to 6, are coloured red, blue and yellow respectively. Each of the dice is rolled once. The total of the numbers rolled is 10. In how many different ways can this happen?

A 36 B 30 C 27 D 24 E 21

16. An array of 25 equally spaced dots is drawn in a square grid as shown. Point O is in the bottom left corner. Linda wants to draw a straight line through the diagram which passes through O and exactly one other point.

How many such lines can Linda draw?

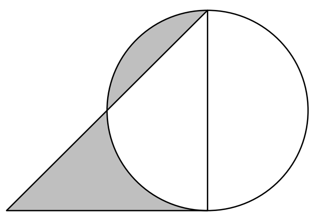
A 4 B 6 C 8 D 12 E 24



17. A circle of radius r and a right-angled isosceles triangle are drawn such that one of the shorter sides of the triangle is a diameter of the circle.

What is the shaded area?

A $\sqrt{2}r$ B r^2 C $2\pi r$ D $\frac{\pi r^2}{4}$
E $(\sqrt{2} - 1)\pi r^2$



18. The number 840 can be written as $\frac{p!}{q!}$, where p and q are positive integers less than 10.

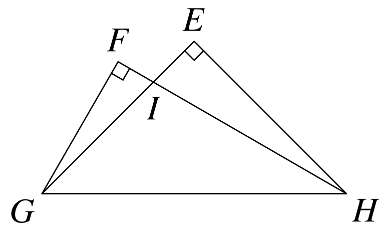
What is the value of $p + q$?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

A 8 B 9 C 10 D 12 E 15

19. The diagram shows two overlapping triangles: triangle FGH with interior angles 60°, 30° and 90° and triangle EGH which is a right-angled isosceles triangle.

What is the ratio of the area of triangle IFG to the area of triangle IEH?



A 1 : 1 B $1 : \sqrt{2}$ C $1 : \sqrt{3}$ D 1 : 2 E 1 : 3

20. Laura and Dina have a running race. Laura runs at constant speed and Dina runs n times as fast where $n > 1$. Laura starts s m in front of Dina.

What distance, in metres, does Dina run before she overtakes Laura?

A $\frac{ns}{n-1}$ B ns C $\frac{s}{n-1}$ D $\frac{ns}{n+1}$ E $\frac{s}{n}$

21. The numbers m and k satisfy the equations $2^m + 2^k = p$ and $2^m - 2^k = q$.

What is the value of 2^{m+k} in terms of p and q?

A $\frac{p^2 - q^2}{4}$ B $\frac{pq}{2}$ C $p + q$ D $\frac{(p-q)^2}{4}$ E $\frac{p+q}{p-q}$

22. A triangle with interior angles 60°, 45° and 75° is inscribed in a circle of radius 2.

What is the area of the triangle?

A $2\sqrt{3}$ B 4 C $6 + \sqrt{3}$ D $6\sqrt{3}$ E $3 + \sqrt{3}$

23. Let x be a real number. What is the minimum value of $\left(x^2 - 4x + 3\right)\left(x^2 + 4x + 3\right)$?

A −16 B −9 C 0 D 9 E 16

24. Saba, Rayan and Derin are cooperating to complete a task. They each work at a constant rate independent of whoever else is working on the task. When all three work together, it takes 5 minutes to complete the task. When Saba is working with Derin, the task takes 7 minutes to complete. When Rayan is working with Derin, the task takes 15 minutes to complete.

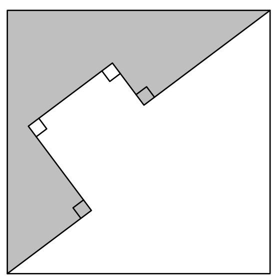
How many minutes does it take for Derin to complete the task on his own?

A 21 B 28 C 35 D 48 E 105

25. Five line segments of length 2, 2, 2, 1 and 3 connect two corners of a square as shown in the diagram.

What is the shaded area?

A 8 B 9 C 10 D 11 E 12



United Kingdom Mathematics Trust

Senior Mathematical Challenge

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Solutions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	C	C	D	E	B	B	B	D	D	E	A	D	D	C	C	B	C	D	A	A	E	A	E	B

1. Cicely had her 21st birthday in 1939.

When did she have her 100th birthday?

A 2020 B 2019 C 2018 D 2010 E 2008

SOLUTION **C**

Since Cicely was 21 in 1939, she was born in 1918 as $1939 - 21 = 1918$. So her 100^{th} birthday would have been in 2018.

2. The sequence, formed from the sequence of primes by rounding each to the nearest ten, begins 0, 0, 10, 10, 10, 10, 20, 20, 20, 30, ...

When continued, how many terms in this sequence are equal to 40?

A 1 B 2 C 3 D 4 E 5

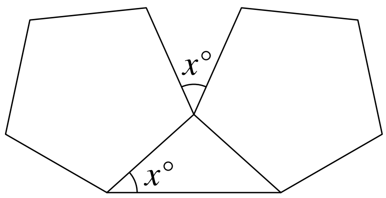
SOLUTION **C**

Terms of 40 in this sequence come from all primes in the interval 35 to 44 inclusive, namely 37, 41 and 43. So three terms in this sequence are equal to 40.

3. The diagram shows two congruent regular pentagons and a triangle. The angles marked $x°$ are equal.

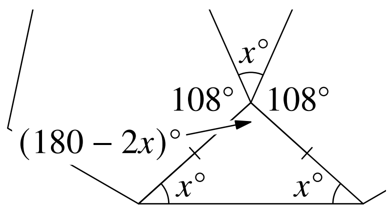
What is the value of x?

A 24 B 30 C 36 D 40 E 45



SOLUTION **C**

Both pentagons are regular and congruent so the triangle is isosceles, with angles $x°$, $x°$ and $(180 - 2x)°$. The interior angles of the pentagons are $108°$. Considering angles around the point where the pentagons meet, we have $108° + x° + 108° + (180 - 2x)° = 360°$, which simplifies to $396° - x° = 360°$ and therefore $x = 36$.



4. The positive integer k is a solution of the equation $(k \div 12) \div (15 \div k) = 20$.

What is the sum of the digits of k?

A 15 B 12 C 9 D 6 E 3

SOLUTION **D**

Re-writing the equation as $\frac{\frac{k}{12}}{\frac{15}{k}} = 20$ and then rearranging gives $\frac{k}{12} = 20 \times \frac{15}{k}$ and so $k^2 = 3600$. As $k > 0$, $k = 60$ so the sum of the digits of k is 6.

5. The sum of four consecutive primes is itself prime.

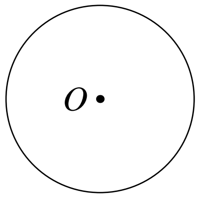
What is the largest of the four primes?

A 37 B 29 C 19 D 13 E 7

SOLUTION **E**

For the sum of four primes to itself be prime, that sum must be odd. This means that one of the primes must be even. The four consecutive primes can only be 2, 3, 5 and 7 and their sum, 17, is indeed a prime. The largest of those four primes is 7.

6. Three points, P, Q and R are placed on the circumference of a circle with centre O. The arc lengths PQ, QR and RP are in the ratio $1:2:3$.

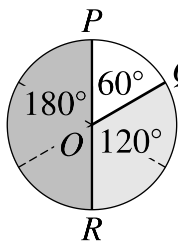


In what ratio are the areas of the sectors POQ, QOR and ROP?

A $1:1:1$ B $1:2:3$ C $1:\pi:\pi^2$ D $1:4:9$ E $1:8:27$

SOLUTION **B**

The arc lengths PQ, QR and RP being in the ratio $1:2:3$ corresponds to the angles at point O also being in the ratio $1:2:3$, so the angles are 60°, 120° and 180° as shown. The ratio of the areas of sectors POQ, QOR and ROP is then also $1:2:3$.



7. Which of these numbers is the largest?

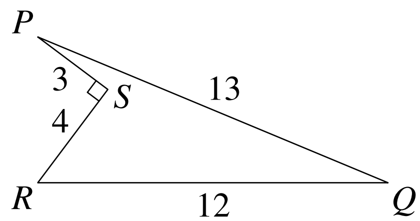
A 2^{5000} B 3^{4000} C 4^{3000} D 5^{2000} E 6^{1000}

SOLUTION **B**

Repeatedly using the rule of indices $a^{bc} = (a^b)^c$ allows us to express each option in the form $(a^b)^{1000}$. Thus $2^{5000} = (2^5)^{1000} = 32^{1000}$, $3^{4000} = (3^4)^{1000} = 81^{1000}$, $4^{3000} = (4^3)^{1000} = 64^{1000}$ and $5^{2000} = (5^2)^{1000} = 25^{1000}$. The last option, 6^{1000}, is already in the required form. Comparing the base numbers 32, 81, 64, 25 and 6 allows us to see that the largest of the numbers given is $81^{1000} = 3^{4000}$.

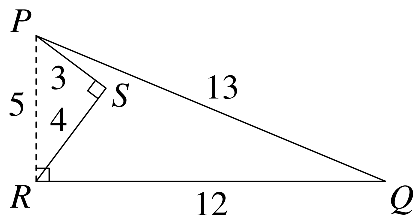
8. What is the area of the region inside the quadrilateral $PQRS$?

A 18 B 24 C 36 D 48
E more information needed



SOLUTION **B**

Joining P to R forms the hypotenuse of right-angled triangle PSR. Using Pythagoras' Theorem, $PR = \sqrt{3^2 + 4^2} = 5$. Considering triangle PRQ and noting that $5^2 + 12^2 = 13^2$ implies, by the converse of Pythagoras' Theorem, that PR is perpendicular to RQ. So the area of the region is $\frac{5 \times 12}{2} - \frac{3 \times 4}{2} = 30 - 6 = 24$.



9. Alison has a set of ten fridge magnets showing the integers from 0 to 9 inclusive.

In how many different ways can she split the set into five pairs so that the sum of each pair is a multiple of 5?

A 1 B 2 C 3 D 4 E 5

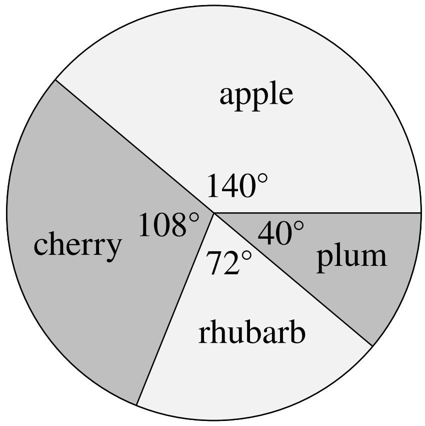
SOLUTION **D**

The numbers 0 and 5 must be paired with one another (one choice). The number 1 must be paired with either 4 or 9 (two choices). The number 6 must be paired with whichever of 4 or 9 is not paired with 1 (one choice). Similarly, the number 2 must be paired with either 3 or 8 (two choices). Lastly, the number 7 must be paired with whichever of 3 or 8 is not paired with 2 (one choice) . The number of possible combinations is then the product of the number of choices which is $1 \times 2 \times 1 \times 2 \times 1 = 4$.

10. In a survey, people were asked to name their favourite fruit pie. The pie chart shows the outcome. The angles shown are exact with no rounding.

What is the smallest number of people who could have been surveyed?

A 45 B 60 C 80 D 90 E 180



Solution **D**

To find the smallest number of people who could have been surveyed, we can find the largest angle that can be used to represent one person's answer in the survey. This angle, in degrees, is the highest common factor of 40, 72, 108 and 140 which is 4. If each person's answer is represented by an angle of 4°, the number of people surveyed is $\frac{360°}{4°} = 90$.

11. Alitta claims that if p is an odd prime then $p^2 - 2$ is also an odd prime.

Which of the following values of p is a counterexample to this claim?

A 3 B 5 C 7 D 9 E 11

Solution **E**

Consider the five options in turn: $3^2-2 = 9-2 = 7$ which is prime; $5^2-2 = 25-2 = 23$ which is prime; $7^2-2 = 49-2 = 47$ which is prime; 9 is not a prime; and $11^2-2 = 121-2 = 119 = 7\times17$. So option E provides a counterexample.

12. For how many positive integers N is the remainder 6 when 111 is divided by N?

A 5 B 4 C 3 D 2 E 1

Solution **A**

In order to have a remainder of 6 when 111 is divided by the positive integer N we must be able to express 111 as $111 = k \times N + 6$ for some positive integer k. Equivalently, $105 = k \times N$ and so N is a factor of 105. The prime factorisation of 105 is $3 \times 5 \times 7$ and so the factors of 105 are 1, 3, 5, 7, 15, 21, 35 and 105. To produce a remainder of 6, the divisor N must be greater than 6 so there are five possible values of N: 7, 15, 21, 35 and 105.

13. Which of these is the mean of the other four?

A $\sqrt{2}$ B $\sqrt{18}$ C $\sqrt{200}$ D $\sqrt{32}$ E $\sqrt{8}$

Solution **D**

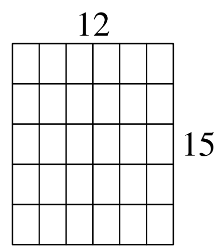
Asking which of the five numbers is the mean of the other four is equivalent to asking for the mean of all five numbers. Each of the options can be rewritten as a multiple of $\sqrt{2}$. A $= \sqrt{2} = 1 \times \sqrt{2}$, B $= \sqrt{18} = 3 \times \sqrt{2}$, C $= \sqrt{200} = 10 \times \sqrt{2}$, D $= \sqrt{32} = 4 \times \sqrt{2}$ and E $= \sqrt{8} = 2 \times \sqrt{2}$. Finding the mean of 1, 3, 10, 4 and 2 gives $\frac{1+3+10+4+2}{5} = \frac{20}{5} = 4$. Therefore the mean is $4\sqrt{2}$ which is $\sqrt{32}$.

14. What is the smallest number of rectangles, each measuring 2 cm by 3 cm, which are needed to fit together without overlap to form a rectangle whose sides are in the ratio 5 : 4 ?

A 10 B 15 C 20 D 30 E 60

Solution **D**

Consider the areas of the small rectangles and the larger rectangle. Each small, 2 cm by 3 cm, rectangle has area $6\,\text{cm}^2$. The larger rectangle has sides in the ratio 4 : 5 so has sides of length $4k$ and $5k$, for some positive integer k, giving an area of $20k^2$. Values of $20k^2$ are 20, 80, 180, 320, ... The smallest of these which is a multiple of 6 is 180, when $k = 3$. The sides of the larger rectangle are then 12 cm and 15 cm. Now we can check to see that the $\frac{180\,\text{cm}^2}{6\,\text{cm}^2} = 30$ small rectangles can be arranged. One possible arrangement of five rows containing six small rectangles is as shown.



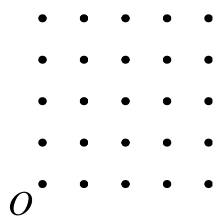
15. Three dice, each showing numbers 1 to 6, are coloured red, blue and yellow respectively. Each of the dice is rolled once. The total of the numbers rolled is 10. In how many different ways can this happen?

A 36 B 30 C 27 D 24 E 21

Solution **C**

To reach a total of 10, each of the dice could show a different number. Alternatively, two of the dice could show the same numbers. However, it is not possible that all three numbers could be the same, as 10 is not a multiple of three. The possible sets of three different numbers are (1, 3, 6), (1, 4, 5) and (2, 3, 5) and there are six ways each could come from the three coloured dice. The sets which include a repeated number are (2, 2, 6), (3, 3, 4) and (4, 4, 2) and there are three ways each of these could come from the coloured dice. In total this gives $3 \times 6 + 3 \times 3 = 27$ ways.

16. An array of 25 equally spaced dots is drawn in a square grid as shown. Point O is in the bottom left corner. Linda wants to draw a straight line through the diagram which passes through O and exactly one other point.

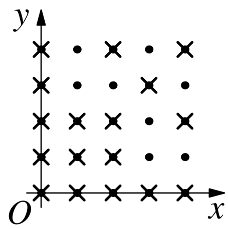


How many such lines can Linda draw?

A 4 B 6 C 8 D 12 E 24

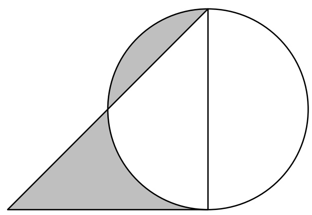
Solution **C**

Overlaying the 25 dots onto a coordinate grid allows us to describe their positions using (x, y). Point O is at $(0, 0)$. Lines through O and any points where either $x = 0$ or $y = 0$ or $y = x$ are not possible as these lines will pass though more than the two allowable points. Above the line $y = x$, Linda can choose to draw lines through O and one of $(1, 3)$, $(1, 4)$, $(2, 3)$ or $(3, 4)$, but not through O and either $(1, 2)$ or $(2, 4)$ as those two points both lie on the same line, $y = 2x$.



By symmetry, there are also four valid points under the line $y = x$ she could choose through which to draw her line, by exchanging the x and y coordinates. This makes a total of eight lines that could be drawn fitting the given criteria.

17. A circle of radius r and a right-angled isosceles triangle are drawn such that one of the shorter sides of the triangle is a diameter of the circle.



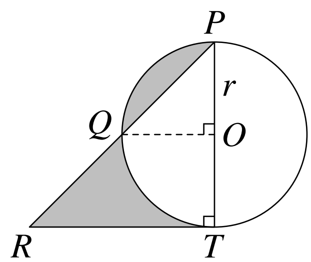
What is the shaded area?

A $\sqrt{2}r$ B r^2 C $2\pi r$ D $\frac{\pi r^2}{4}$

E $(\sqrt{2} - 1)\pi r^2$

Solution **B**

In the right-angled isosceles triangle PRT, $PT = 2r = RT$. As $\angle OPQ = 45°$ and triangle OPQ is isoceles, $\angle POQ = 90°$. Therefore the segment above line PQ can be reflected in a line through O and Q to become a segment below QT which then completes a new shaded triangle, QRT. Triangle QRT has base RT of length $2r$ and perpendicular height OT, of length r. The shaded area is then $\frac{2r \times r}{2} = r^2$.



18. The number 840 can be written as $\frac{p!}{q!}$, where p and q are positive integers less than 10.

What is the value of $p + q$?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

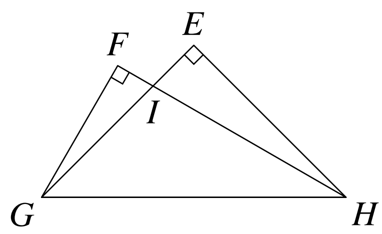
A 8 B 9 C 10 D 12 E 15

SOLUTION **C**

When written as the product of its prime factors, $840 = 2 \times 2 \times 2 \times 3 \times 5 \times 7$. Rearranging the order of the primes gives $840 = 7\times3\times2\times5\times2\times2 = 7\times6\times5\times4 = 7\times6\times5\times4\times\frac{3\times2\times1}{3\times2\times1} = \frac{7!}{3!}$. Therefore $p = 7$ and $q = 3$, so $p + q = 10$. One can check that there are no further possible solutions with p and q less than 10.

19. The diagram shows two overlapping triangles: triangle FGH with interior angles 60°, 30° and 90° and triangle EGH which is a right-angled isosceles triangle.

What is the ratio of the area of triangle IFG to the area of triangle IEH?



A $1 : 1$ B $1 : \sqrt{2}$ C $1 : \sqrt{3}$ D $1 : 2$ E $1 : 3$

SOLUTION **D**

We consider triangles IFG and IEH. Angles GFI and HEI are right angles and angles GIF and HIE are equal since they are vertically opposite. Hence the angles FGI and EHI are equal using 'the angle sum of a triangle is 180°'. Triangles IFG and IEH have the same angles and therefore are similar.

Let FG be of length x. Then GH has length $2x$ as triangle FGH is a 30°, 60°, 90° triangle. Now considering triangle EGH, the length of EH is $\frac{2x}{\sqrt{2}} = \sqrt{2}x$ as this is a 45°, 45°, 90° triangle.

Comparing the corresponding lengths of FG and EH we have a length scale factor of $\sqrt{2}$ and so comparing the areas of triangles IFG and IEH we have an area scale factor of $(\sqrt{2})^2$. So the ratio of the areas is $1 : 2$.

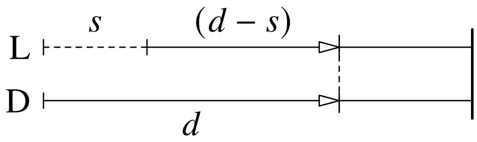
20. Laura and Dina have a running race. Laura runs at constant speed and Dina runs n times as fast where $n > 1$. Laura starts s m in front of Dina.

What distance, in metres, does Dina run before she overtakes Laura?

A $\frac{ns}{n-1}$ B ns C $\frac{s}{n-1}$ D $\frac{ns}{n+1}$ E $\frac{s}{n}$

SOLUTION **A**

Let Laura's speed be v m/s, so Dina's speed is nv m/s. Let the distance Dina runs before she overtakes Laura be d m and the time until that happens be t s.



Using distance = speed × time, for Laura we have $d - s = vt$ and for Dina we have $d = nvt$. Eliminating v and t between these two equations allows us to find d in terms of n and s, so $d = n(d - s)$. This expands and rearranges to $nd - d = ns$ and so $d(n-1) = ns$. Dividing through gives $d = \frac{ns}{n-1}$.

21. The numbers m and k satisfy the equations $2^m + 2^k = p$ and $2^m - 2^k = q$.

What is the value of 2^{m+k} in terms of p and q?

A $\frac{p^2-q^2}{4}$ B $\frac{pq}{2}$ C $p+q$ D $\frac{(p-q)^2}{4}$ E $\frac{p+q}{p-q}$

SOLUTION **A**

Adding the equations $2^m + 2^k = p$ and $2^m - 2^k = q$, gives $2 \times 2^m = p + q$. Therefore $2^m = \frac{p+q}{2}$. Similarly, subtracting the equations gives $2 \times 2^k = p - q$. Therefore $2^k = \frac{p-q}{2}$. As 2^{m+k} is $2^m \times 2^k$, $2^{m+k} = \frac{(p+q)}{2} \times \frac{(p-q)}{2} = \frac{(p^2-q^2)}{4}$.

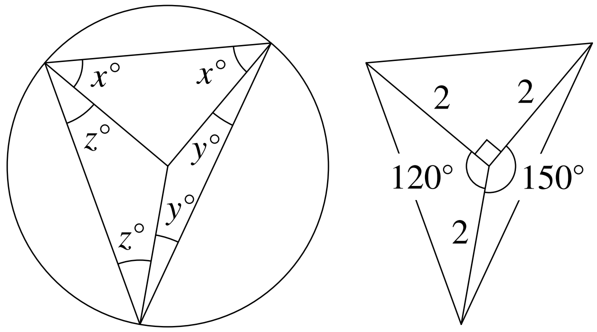
22. A triangle with interior angles 60°, 45° and 75° is inscribed in a circle of radius 2.

What is the area of the triangle?

A $2\sqrt{3}$ B 4 C $6+\sqrt{3}$ D $6\sqrt{3}$ E $3+\sqrt{3}$

Solution **E**

First split the inscribed triangle into three isosceles triangles, each with two vertices on the circle and the third vertex at the centre of the circle. Let the base angles in the isosceles triangles be $x°$, $y°$ and $z°$ as shown here, where $x+y=60$, $y+z=45$ and $x+z=75$.



Adding the equations gives $2(x+y+z)=180$ and therefore $x+y+z=90$. Subtracting the earlier equations from this, one at a time, gives $z=30$, $x=45$, and $y=15$.

The angles of the isosceles triangles at the centre of the circle are then 150°, 120°, and 90°. These angles could also have been found using 'the angle at the centre is twice the angle at the circumference'. Using the formula 'area $=\frac{1}{2}ab\sin C$', with $a=b=2$, gives the total area as $\frac{1}{2}\times 2\times 2\times\sin 150°+\frac{1}{2}\times 2\times 2\times\sin 120°+\frac{1}{2}\times 2\times 2$.

Since $\sin 150°=\sin 30°=\frac{1}{2}$ and $\sin 120°=\sin 60°=\frac{\sqrt{3}}{2}$, the total area $=(2\times\frac{1}{2})+(2\times\frac{\sqrt{3}}{2})+2=1+\sqrt{3}+2=3+\sqrt{3}$.

23. Let x be a real number. What is the minimum value of $(x^2-4x+3)(x^2+4x+3)$?

A -16 B -9 C 0 D 9 E 16

Solution **A**

Expanding the expression gives $(x^2-4x+3)(x^2+4x+3)=x^4-10x^2+9$ which in completed square form is $(x^2-5)^2-16$. This expression has a minimum when $x^2-5=0$ and that minimum value is then $0^2-16=-16$.

24. Saba, Rayan and Derin are cooperating to complete a task. They each work at a constant rate independent of whoever else is working on the task. When all three work together, it takes 5 minutes to complete the task. When Saba is working with Derin, the task takes 7 minutes to complete. When Rayan is working with Derin, the task takes 15 minutes to complete.

How many minutes does it take for Derin to complete the task on his own?

A 21 B 28 C 35 D 48 E 105

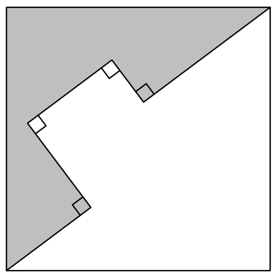
Solution **E**

Let Derin work at d units per minute, let Saba work at s units per minute and let Rayan work at r units per minute. Let there be w units of work to be done to complete the task. Derin's time to complete the task is $\frac{w}{d}$. When all three work together we have (1) $w = 5d + 5s + 5r$. When Saba and Derin work together we have (2) $w = 7d + 7s$. Finally, when Rayan and Derin work together we have (3) $w = 15d + 15r$. Combining (1) and (3) to eliminate d and r gives $2w = 15s$ so $w = \frac{15s}{2}$. Combining this with (2) gives $\frac{s}{2} = 7d$ so $d = \frac{s}{14}$. Therefore $\frac{w}{d} = \frac{15s}{2} \times \frac{14}{s} = 15 \times 7 = 105$. Hence Derin would take 105 minutes.

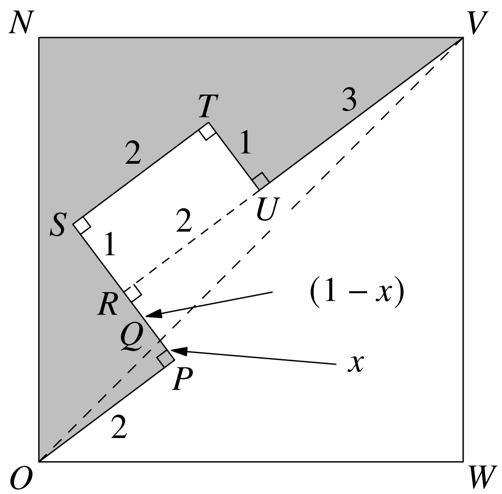
25. Five line segments of length 2, 2, 2, 1 and 3 connect two corners of a square as shown in the diagram.

What is the shaded area?

A 8 B 9 C 10 D 11 E 12



Solution **B**



By identifying similar right-angled triangles, we can first calculate the side-length of the large square. Drawing an extra line RU to complete rectangle $RSTU$ gives $SR = 1$ and $RV = 5$. A straight line from O to V passes through SP at Q. Let $PQ = x$ and therefore $QR = 1 - x$. As $\angle OQP$ and $\angle VQR$ are vertically opposite, they are equal, so triangle OQP and triangle VQR are similar. Therefore $\frac{x}{2} = \frac{1-x}{5}$ which rearranges to give $x = \frac{2}{7}$.

The ratio $PQ : QR = 2 : 5$ and so the ratio $OQ : QV = 2 : 5$. This gives $OV = \frac{7}{2} \times OQ$. Using Pythagoras' Theorem, $OV = \frac{7}{2} \times \sqrt{2^2 + (\frac{2}{7})^2} = 5\sqrt{2}$. So $OW = VW = 5$.

The shaded area is then area of triangle VNO − area of rectangle $RSTU$ − area of triangle VRQ + area of triangle OPQ $= \frac{1}{2} \times 5 \times 5 - 1 \times 2 - \frac{1}{2} \times (2+3) \times \frac{5}{7} + \frac{1}{2} \times 2 \times \frac{2}{7} = 9$.

United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

November 2021

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	C	C	D	E	B	B	B	D	D	E	A	D	D	C	C	B	C	D	A	A	E	A	E	B

1. Cicely had her 21st birthday in 1939.

When did she have her 100th birthday?

A 2020 B 2019 C 2018 D 2010 E 2008

SOLUTION **C**

Cicely had her 21st birthday in 1939. Since 1939 – 21 = 1918, it follows that she was born in 1918.

Now, 1918 + 100 = 2018.

Therefore Cicely's 100th birthday was in 2018.

FOR INVESTIGATION

1.1 The mathematician Augustus De Morgan was born and died in the 19th century. On one birthday he noticed that the square of his age was the same as the year number.

In which year was Augustus De Morgan born?

1.2 Determine for which values of n a person born in year n could have the same experience as Augustus De Morgan if they lived long enough, that is, they would have a birthday on which the square of their age was the same as the year number.

2. The sequence, formed from the sequence of primes by rounding each to the nearest ten, begins 0, 0, 10, 10, 10, 10, 20, 20, 20, 30, ...

When continued, how many terms in this sequence are equal to 40?

A 1 B 2 C 3 D 4 E 5

SOLUTION **C**

The integers that round to 40 are those in the range from 35 to 44.

The primes in this range are 37, 41 and 43.

Therefore there are 3 primes that round to 40.

FOR INVESTIGATION

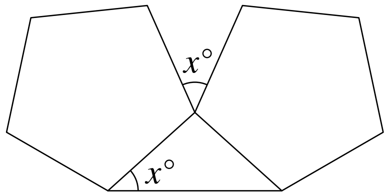
2.1 How many primes are rounded to 50?

2.2 What is the largest number of primes that round to the same multiple of 10?

3. The diagram shows two congruent regular pentagons and a triangle. The angles marked $x°$ are equal.

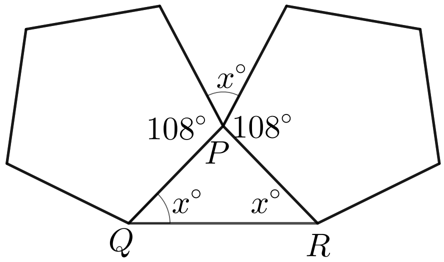
What is the value of x?

A 24 B 30 C 36 D 40 E 45



Solution **C**

Let P, Q and R be the points shown in the diagram.



The interior angles of a regular pentagon are all 108°.

Because the two pentagons are congruent, $PR = PQ$. Therefore $\angle PRQ = \angle PQR = x°$.

Because the sum of the angles in a triangle is 180°, from the triangle PQR, we have $x° + x° + \angle QPR = 180°$. Therefore $\angle QPR = 180° - 2x°$.

The sum of the angles at a point is 360°. Therefore from the angles at the point P, we have

$$(180° - 2x°) + 108° + x° + 108° = 360°.$$

That is

$$396° - x° = 360°.$$

It follows that

$$x° = 396° - 360° = 36°.$$

For investigation

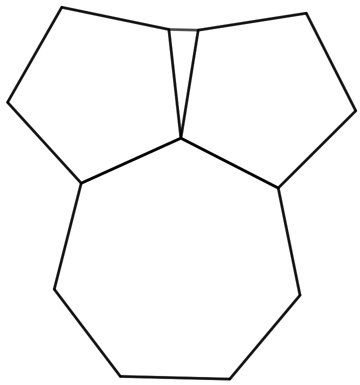
3.1 Prove that the sum of the angles in a triangle is 180°.

3.2 Prove that each interior angle of a regular pentagon is 108°.

3.3 Find a formula in terms of n for the size of the interior angles of a regular polygon with n sides.

3.4 The diagram shows a regular heptagon, two regular pentagons and a triangle.

What are the interior angles of the triangle?



4. The positive integer k is a solution of the equation $(k \div 12) \div (15 \div k) = 20$.

What is the sum of the digits of k?

A 15 B 12 C 9 D 6 E 3

Solution **D**

We have

$$\begin{aligned}(k \div 12) \div (15 \div k) &= \frac{k}{12} \div \frac{15}{k} \\ &= \frac{k}{12} \times \frac{k}{15} \\ &= \frac{k \times k}{12 \times 15} \\ &= \frac{k^2}{180}.\end{aligned}$$

It follows that

$$\begin{aligned}(k \div 12) \div (15 \div k) = 20 &\Leftrightarrow \frac{k^2}{180} = 20 \\ &\Leftrightarrow k^2 = 3600 \\ &\Leftrightarrow k = 60, \text{ as } k > 0.\end{aligned}$$

The sum of the digits of 60 is $6 + 0 = 6$.

For investigation

4.1 Find the solutions of the following equations.

(a) $(x \div 5) \div (5 \div x) = 4$.

(b) $(x \div 2) \div ((x \div 10) \div (x \div 3)) = 15$.

5. The sum of four consecutive primes is itself prime.

What is the largest of the four primes?

A 37 B 29 C 19 D 13 E 7

Solution **E**

The sum of four odd primes is an even number greater than 2, and therefore is not a prime. Therefore the four consecutive primes whose sum is prime includes the only even prime 2.

It follows that the four consecutive primes are 2, 3, 5 and 7. Their sum is 17 which is a prime. The largest of these four consecutive primes is 7.

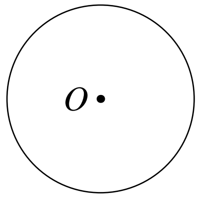
For investigation

5.1 Which is the smallest prime that is the sum of five consecutive primes?

6. Three points, P, Q and R are placed on the circumference of a circle with centre O. The arc lengths PQ, QR and RP are in the ratio $1 : 2 : 3$.

In what ratio are the areas of the sectors POQ, QOR and ROP?

A $1 : 1 : 1$ B $1 : 2 : 3$ C $1 : \pi : \pi^2$
D $1 : 4 : 9$ E $1 : 8 : 27$



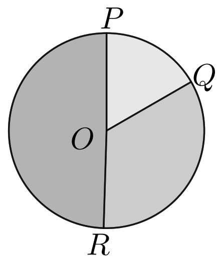
Solution **B**

The key facts that we need to use here are

(a) the ratio of the length of an arc to the length of the circumference is the same as the ratio of the angle that the arc subtends at the centre of the circle to the angle in a complete revolution,

and

(b) the ratio of the area of a sector to the area of the circle is the same as the ratio of the angle in the sector to the angle in a complete revolution.



It follows from (a) that the ratios of the arc lengths of PQ, QR and RP are the same as the ratios of the angles that the arcs subtend at the centre of the circle.

Therefore these angles are in the ratio $1 : 2 : 3$.

Similarly, it follows from (b) that the ratios of the areas of the sectors POQ, QOR and ROP are the same as the ratios of the angles in the sectors.

Therefore the areas of the sectors POQ, QOR and ROP are in the ratio $1 : 2 : 3$.

Commentary

From the basic facts (a) and (b) we can deduce that

(c) The ratio of the length of an arc of a circle to the circumference of a circle is equal to the ratio of the area the arc subtends at the centre of the circle to the area of the circle.

An alternative method would have been to base the solution on this fact.

For investigation

6.1 Suppose that the circle has radius 3 and the arc lengths PQ, QR and RP are in the ratio $2 : 3 : 4$.

What is the area of the sector QOR?

7. Which of these numbers is the largest?

A 2^{5000} B 3^{4000} C 4^{3000} D 5^{2000} E 6^{1000}

SOLUTION **B**

We have

$$2^{5000} = (2^5)^{1000} = 32^{1000},$$
$$3^{4000} = (3^4)^{1000} = 81^{1000},$$
$$4^{3000} = (4^3)^{1000} = 64^{1000},$$

and

$$5^{2000} = (5^2)^{1000} = 25^{1000}.$$

Since $6 < 25 < 32 < 64 < 81$, it follows that $6^{1000} < 25^{1000} < 32^{1000} < 64^{1000} < 81^{1000}$.

Therefore $6^{1000} < 5^{2000} < 2^{5000} < 4^{3000} < 3^{4000}$.

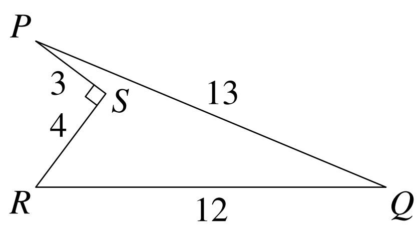
Hence, of the given numbers, it is 3^{4000} that is the largest.

FOR INVESTIGATION

7.1 Which of these numbers is the largest?
(a) 2^{7000}, (b) 3^{6000}, (c) 4^{5000}, (d) 5^{4000}, (e) 6^{3000}.

8. What is the area of the region inside the quadrilateral $PQRS$?

A 18 B 24 C 36 D 48
E more information needed

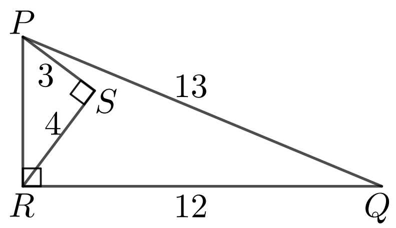


SOLUTION **B**

By Pythagoras' Theorem applied to the right-angled triangle PSR, we have $PR^2 = 3^2 + 4^2 = 9 + 16 = 25$. Therefore $PR = 5$.

It follows that in the triangle PRQ we have

$$PQ^2 = 13^2 = 169 = 25+144 = 5^2+12^2 = PR^2+RQ^2.$$



Therefore, by the converse of Pythagoras' Theorem, $\angle PRQ = 90°$.

Because $\angle PRQ = 90°$, the area of the triangle PRQ is $\frac{1}{2}(RQ \times RP) = \frac{1}{2}(12 \times 5) = 30$. Similarly, the area of the triangle PSR is $\frac{1}{2}(SP \times SR) = \frac{1}{2}(3 \times 4) = 6$.

The area of the quadrilateral $PQRS$ is the area of the triangle PRQ minus the area of the triangle PSR. Hence the area of $PQRS$ is $30 - 6 = 24$.

9. Alison has a set of ten fridge magnets showing the integers from 0 to 9 inclusive.

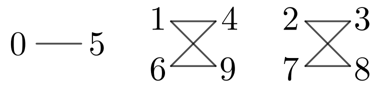
In how many different ways can she split the set into five pairs so that the sum of each pair is a multiple of 5?

A 1 B 2 C 3 D 4 E 5

Solution **D**

The number 0 can only be paired with 5.

The number 1 may be paired with 4 or with 9. If 1 is paired with 4, 6 has to be paired with 9. If 1 is paired with 9, 6 has to be paired with 4.



The number 2 may be paired with 3 or with 8. If 2 is paired with 3, 7 has to be paired with 8. If 2 is paired with 8, 7 has to be paired with 3.

These possibilities are shown in the diagram above. Thus the complete pairing is determined by first the choice which of 4 or 9 to pair with 1, giving two choices, and then the choice of which of 3 or 8 to pair with 2.

These choices are independent. It follows that there are $2 \times 2 = 4$ ways to split the set of numbers into five pairs so that the sum of each pair is a multiple of 5.

For investigation

9.1 List the 4 different pairings that satisfy the condition that the sum of each pair is a multiple of 5.

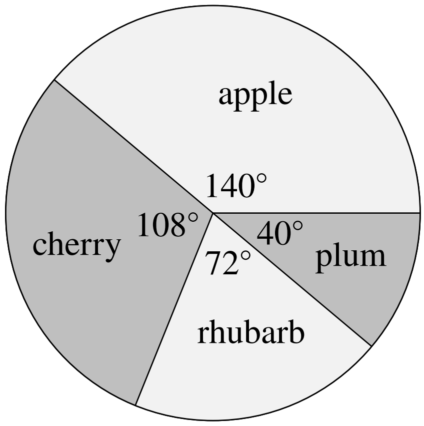
9.2 Bibi has a set of twenty fridge magnets showing the integers from 0 to 19, inclusive. In how many different ways can she split the set into ten pairs so that the sum of each pair is a multiple of 5?

9.3 Chandra has a set of twenty-four fridge magnets showing the integers from 0 to 23, inclusive. In how many different ways can she split the set into twelve pairs so that the sum of each pair is a multiple of 5?

10. In a survey, people were asked to name their favourite fruit pie. The pie chart shows the outcome. The angles shown are exact with no rounding.

What is the smallest number of people who could have been surveyed?

A 45 B 60 C 80 D 90 E 180



Solution **D**

Suppose that p people were surveyed.

The total of the angles is 360°. Therefore the proportion of the people who said that their favourite is apple pie is $\frac{140}{360} = \frac{7}{18}$. Hence the number who chose apple pie was $\frac{7}{18}p$. This is an integer. Therefore p is a multiple of 18.

Similarly, as the proportion who said their favourite is cherry pie is $\frac{108}{360} = \frac{3}{10}$, we deduce that p is a multiple of 10.

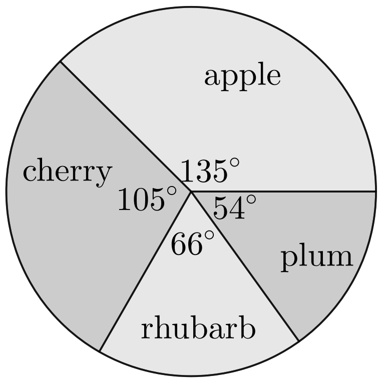
Likewise, as $\frac{72}{360} = \frac{1}{5}$ and $\frac{40}{360} = \frac{1}{9}$, we know that p is also a multiple of 5 and of 9.

Therefore the smallest possible value of p is the least common multiple of 18, 10, 5 and 9, which is 90. Hence the smallest number of people who could have been surveyed is 90.

For investigation

10.1 The results of another survey about people's favourite fruit pies are shown in the pie chart on the right. Again, the angles are exact with no rounding.

What is the smallest number of people who could have been surveyed?



11. Alitta claims that if p is an odd prime then $p^2 - 2$ is also an odd prime.

Which of the following values of p is a counterexample to this claim?

A 3 B 5 C 7 D 9 E 11

Solution **E**

A counterexample to the claim is an odd prime p such that $p^2 - 2$ is *not* an odd prime.

3 is an odd prime, and $3^2 - 2 = 7$ is also an odd prime. So 3 is not a counterexample.

5 is an odd prime, and $5^2 - 2 = 23$ is also an odd prime. So 5 is not a counterexample.

7 is an odd prime, and $7^2 - 2 = 47$ is also an odd prime. So 7 is not a counterexample.

9 is not an odd prime. So 9 is not a counterexample

11 is an odd prime, but $11^2 - 2 = 119 = 7 \times 17$ is not an odd prime. Therefore 11 is a counterexample.

For investigation

11.1 For each of the following statements find a counterexample.

(a) If p is a prime, then $6p + 1$ is also a prime.

(b) If p is an integer with $p > 1$ and $6p + 1$ is a prime, then p is also a prime.

(c) If p is a prime, then $3^p + 20$ is also a prime.

(d) If p is a prime, then there is another prime between p and $p + 10$.

12. For how many positive integers N is the remainder 6 when 111 is divided by N?

A 5 B 4 C 3 D 2 E 1

Solution **A**

The remainder when 111 is divided by N is 6 provided that $111 = QN + 6$, where Q is a non-negative integer and $6 < N$. In other words, N is a factor of $111 - 6$ with $6 < N$.

Now $111 - 6 = 105$. The prime factorization of 105 is $3 \times 5 \times 7$. Therefore the factors of $111 - 6$ are 1, 3, 5, 7, 15, 21, 35 and 105.

Of these 8 factors all but 1, 3 and 5 are greater than 6.

Therefore there are 5 positive integers N which give a remainder 6 when 111 is divided by N.

For investigation

12.1 For how many positive integers N is the remainder 7 when 112 is divided by N?

13. Which of these is the mean of the other four?

A $\sqrt{2}$ B $\sqrt{18}$ C $\sqrt{200}$ D $\sqrt{32}$ E $\sqrt{8}$

SOLUTION **D**

We use the fact that one of the options is the mean of the other four options provided that it is the mean of all five of the options. Your are asked to check this fact in Problem 13.2.

The mean of the five numbers given as options is

$$\begin{aligned}\frac{\sqrt{2}+\sqrt{18}+\sqrt{200}+\sqrt{32}+\sqrt{8}}{5} &= \frac{\sqrt{2}+3\sqrt{2}+10\sqrt{2}+4\sqrt{2}+2\sqrt{2}}{5}\\ &= \left(\frac{1+3+10+4+2}{5}\right)\sqrt{2}\\ &= \left(\frac{20}{5}\right)\sqrt{2}\\ &= 4\sqrt{2}\\ &= \sqrt{32}.\end{aligned}$$

Therefore the correct option is D.

FOR INVESTIGATION

13.1 (a) Find the mean of the primes 5, 7, 11, 13 and 19.

(b) Hence show that this mean is one of these primes.

(c) Check that this mean is also the mean of the other four primes.

13.2 (a) Show that if the number p is the mean of the five numbers p, q, r, s and t, then p is also the mean of the four numbers q, r, s and t.

(b) Show that if the number p is the mean of the four numbers q, r, s and t, then p is also the mean of the five numbers p, q, r, s and t.

13.3 Show that the result of Problem 13.2 generalizes to the case of a set of n numbers, for each integer n with $n \geq 2$. That is, show that given a set of n numbers, a number p in the set is the mean of the other $n-1$ numbers in the set if and only if it is the mean of all the n numbers in the set.

13.4 Which of the seven primes consecutive 7, 11, 13, 17, 19, 23 and 27 is the mean of the other six primes in the list?

13.5 Which of the seven consecutive primes 101, 103, 107, 109, 113, 127 and 131 is the mean of the other six primes in the list?

14. What is the smallest number of rectangles, each measuring 2 cm by 3 cm, which are needed to fit together without overlap to form a rectangle whose sides are in the ratio 5 : 4 ?

A 10 B 15 C 20 D 30 E 60

SOLUTION **D**

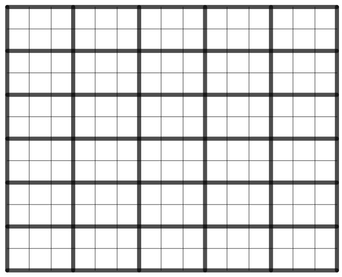
A rectangle whose sides are in the ratio 5 : 4 has dimensions $5k$ cm $\times$ $4k$ cm, for some positive number k. Since we aim to cover this rectangle with $2\,\text{cm} \times 3\,\text{cm}$ rectangles without overlap, k needs to be a positive integer.

The area of the $5k\,\text{cm}\times 4k\,\text{cm}$ rectangle is $(5k \times 4k)\,\text{cm}^2 = 20k^2\,\text{cm}^2$. The $2\,\text{cm}\times 3\,\text{cm}$ rectangles have area $6\,\text{cm}^2$.

It follows that $20k^2$ needs to be a multiple of 6. The least positive integer k for which this is the case is 3. In this case the larger rectangle has dimensions $15\,\text{cm} \times 12\,\text{cm}$ and area $(15 \times 12)\,\text{cm}^2 = 180\,\text{cm}^2$. Therefore the smallest number of $2\,\text{cm} \times 3\,\text{cm}$ rectangles that are needed would be $\dfrac{180}{6} = 30$, provided that it is possible to cover a $15\,\text{cm} \times 12\,\text{cm}$ with 30 $2\,\text{cm} \times 3\,\text{cm}$ rectangles.

To complete the question we need to show that this is possible.

One way in which this can be done is shown in the diagram on the right.



Therefore the smallest number of $2\,\text{cm} \times 3\,\text{cm}$ rectangles that are needed is 30.

FOR INVESTIGATION

14.1 Find other ways to fit together 30 rectangles measuring $2\,\text{cm}\times 3\,\text{cm}$ to make a $15\,\text{cm}\times 12\,\text{cm}$ rectangle.

15. Three dice, each showing numbers 1 to 6 are coloured red, blue and yellow respectively. Each of the dice is rolled once. The total of the numbers rolled is 10. In how many different ways can this happen?

A 36 B 30 C 27 D 24 E 21

Solution **C**

We note first that, because the dice are coloured, two outcomes with total 10, but with different numbers rolled on particular dice, count as being different. For example

red : 6 blue : 3 yellow : 1

counts as being different from the outcome

red : 6 blue : 1 yellow : 3.

With three different numbers there are three choices for the dice which rolls the first number, then two choices for the dice which rolls the second number, leaving just one choice for the dice which rolls the third number. This gives a total of $3 \times 2 \times 1 = 6$ arrangements for the three numbers.

It can be checked that when two of the numbers are the same these can occur in 3 different ways.

It is not possible to have three equal scores with total 10.

To solve this problem we now list in the following table all possible ways a total of 10 may be obtained by throwing three dice. In each row of the table we also give the number of different ways the three numbers in the row may be arranged between the three dice.

scores	no. of ways
6, 3, 1	6
6, 2, 2	3
5, 4, 1	6
5, 3, 2	6
4, 4, 2	3
4, 3, 3	3

Therefore the total number of different ways of achieving a total of 10 is $6 + 3 + 6 + 6 + 3 + 3 = 27$.

For investigation

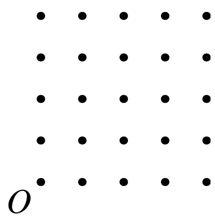
15.1 Check that when two of the numbers are the same they can occur in 3 different ways on the three dice.

15.2 In how many different ways can the total of the numbers rolled be 12?

15.3 For $3 \leq T \leq 18$, calculate the number of different ways in which a total of T can be rolled using the three dice.

What do you notice about the answers?

16. An array of 25 equally spaced dots is drawn in a square grid as shown. Point O is in the bottom left corner. Linda wants to draw a straight line through the diagram which passes through O and exactly one other point.

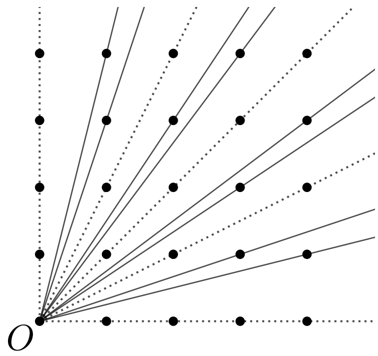


How many such lines can Linda draw?

A 4 B 6 C 8 D 12 E 24

Solution **C**

In the diagram on the right the solid lines go through O and exactly one other point, and the dotted lines go through O and at least two other points.



There is a line through every point so all possible lines have been considered.

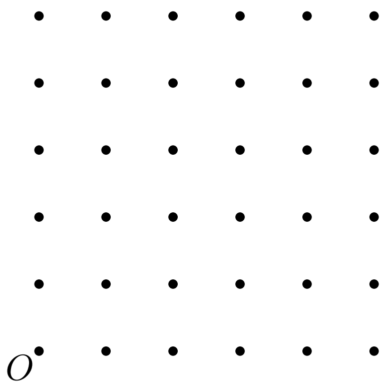
The solid lines are the lines that Linda can draw. We therefore see that the number of lines that Linda can draw is 8.

[*Note*: Because the diagram is symmetric about the bottom-left to top-right diagonal, it was only really necessary to draw half the lines in the diagram.]

For investigation

16.1 An array of 36 equally spaced dots is drawn in a square grid as shown. Mollie wants to draw a straight line which passes through the dot marked O and exactly one other dot.

How many of these lines can Mollie draw?



16.2 Naomi has a piece of paper on which are drawn 400 equally spaced dots in a square 20×20 grid.

Naomi wants to draw a straight line which passes through the bottom left-hand dot and exactly one other dot.

How many of these lines can Naomi draw?

16.3 Olivia has a piece of paper on which are drawn 10 000 equally spaced dots in a square 100×100 grid.

Olivia wants to draw a straight line which passes through the bottom left-hand dot and exactly one other dot.

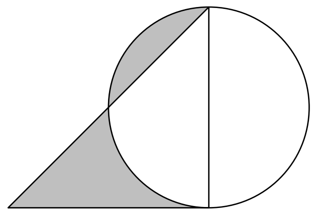
How many of these lines can Olivia draw?

17. A circle of radius r and a right-angled isosceles triangle are drawn such that one of the shorter sides of the triangle is a diameter of the circle.

What is the shaded area?

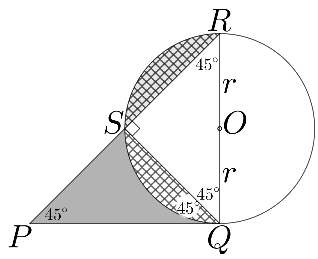
A $\sqrt{2}r$ B r^2 C $2\pi r$ D $\frac{\pi r^2}{4}$

E $(\sqrt{2}-1)\pi r^2$



Solution **B**

Let O be the centre of the circle and let P, Q, R and S be the points as shown in the diagram.



Because PQR is a right-angled isosceles triangle, $\angle PRQ = \angle RPQ = 45°$, and $PQ = QR = 2r$.

Because the angle in a semicircle is a right angle [this is Thales' theorem], $\angle RSQ = 90°$. Therefore, because the sum of the angles in a triangle is 180°, we have $\angle RQS = 45°$.

We therefore have $\angle SRQ = 45° = \angle SQR$. It follows that $SQ = SR$.

Because $SQ = SR$ the segments of the circle cut off by these lines, shown as hatched in the diagram, are congruent. Hence they have the same area.

It follows that the shaded area is the same as the area of the triangle PQS.

PQS is a right-angled isosceles triangle with hypotenuse PQ of length $2r$. Therefore the triangle PQS has area r^2.

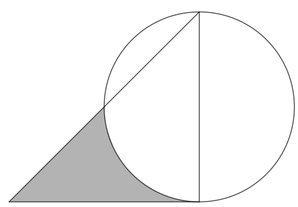
Therefore the shaded area is r^2.

For investigation

17.1 Explain why from the fact that the hypotenuse of the triangle PQS has length $2r$, it follows that the area of the triangle is r^2.

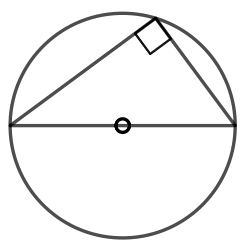
17.2 In the diagram on the right there is a circle of radius r and a right-angled isosceles triangle. One of the shorter sides of the triangle is a diameter of the circle.

What is the shaded area?



17.3 Give a proof of Thales' theorem:

The angle in a semicircle is a right angle.



18. The number 840 can be written as $\frac{p!}{q!}$, where p and q are positive integers less than 10.

What is the value of $p + q$?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

A 8 B 9 C 10 D 12 E 15

SOLUTION **C**

We note first that, as $\frac{p!}{q!} = 840$, it follows that $p! > q!$ and hence $p > q$. Therefore

$$\frac{p!}{q!} = \frac{1 \times 2 \times \cdots \times q \times (q+1) \times \cdots \times p}{1 \times 2 \times \cdots \times q} = (q+1) \times (q+2) \times \cdots \times (p-1) \times p.$$

Thus $840 = \frac{p!}{q!}$ is the product of the consecutive integers $q+1, q+2, \ldots, p-1, p$, where $p \leq 9$.

Since 840 is not a multiple of 9, $p \neq 9$. Since 840 is a multiple of 7, $p \geq 7$.

Now $5 \times 6 \times 7 \times 8 = 1680 > 840$, while $6 \times 7 \times 8 = 336 < 840$. Hence 840 is not the product of consecutive integers of which the largest is 8.

We deduce that $p = 7$. It is now straightforward to check that

$$840 = 4 \times 5 \times 6 \times 7 = \frac{7!}{3!}.$$

Therefore $p = 7$ and $q = 3$. Hence $p + q = 7 + 3 = 10$.

FOR INVESTIGATION

18.1 Find positive integers p and q with $q < p \leq 20$ such that

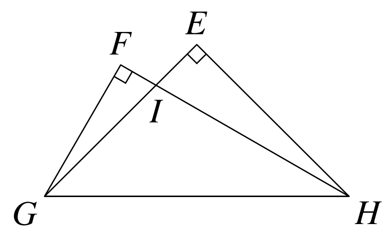
$$\frac{p!}{q!} = 2730.$$

18.2 Is it possible to find positive integers p and q with $q < p \leq 20$ such that

$$\frac{p!}{q!} = 253\,?$$

19. The diagram shows two overlapping triangles: triangle FGH with interior angles 60°, 30° and 90° and triangle EGH which is a right-angled isosceles triangle.

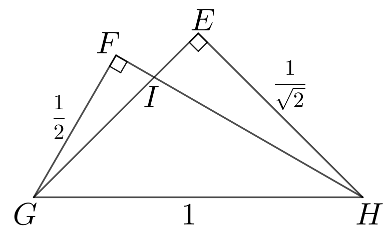
What is the ratio of the area of triangle IFG to the area of triangle IEH?



A $1 : 1$ B $1 : \sqrt{2}$ C $1 : \sqrt{3}$ D $1 : 2$ E $1 : 3$

Solution **D**

We suppose that we have chosen units so that the length of GH is 1.



Because FGH is a 60°, 30°, 90° triangle, it follows that FG has length $\frac{1}{2}$.

Because EGH is a right-angled isosceles triangle it also follows that EH has length $\frac{1}{\sqrt{2}}$.

In the triangles IFG and IEH we have

$$\angle GFI = \angle HEI = 90°$$

and

$$\angle GIF = \angle HIE \text{ (vertically opposite angles).}$$

Because the sum of the angles in both these triangles is 180°, it follows that

$$\angle FGI = \angle EHI.$$

Therefore the triangles IFG and IEH are similar.

The ratio of the areas of similar triangles equals the ratio of the squares of the lengths of corresponding sides. Therefore

$$\begin{aligned}\text{area of } IFG : \text{area of } IEH &= FG^2 : EH^2\\ &= \left(\tfrac{1}{2}\right)^2 : \left(\tfrac{1}{\sqrt{2}}\right)^2\\ &= \tfrac{1}{4} : \tfrac{1}{2}\\ &= 1 : 2.\end{aligned}$$

For investigation

19.1 Explain why, given that GH has length 1, FG has length $\frac{1}{2}$ and EH has length $\frac{1}{\sqrt{2}}$.

19.2 Explain why the ratio of the areas of similar triangles equals the ratio of the squares of the lengths of corresponding sides.

19.3 (a) Given that GH has length 1, find the area of the triangle GIH.

(b) Given that GH has length 1, find the areas of the triangles IFG and IEH.

(c) Hence verify that the ratio of the areas of the triangles IFG and IEH is $1 : 2$.

20. Laura and Dina have a running race. Laura runs at constant speed and Dina runs n times as fast where $n > 1$. Laura starts s m in front of Dina.

What distance, in metres, does Dina run before she overtakes Laura?

A $\frac{ns}{n-1}$ B ns C $\frac{s}{n-1}$ D $\frac{ns}{n+1}$ E $\frac{s}{n}$

SOLUTION **A**

Suppose that Dina has run a distance of d metres when she overtakes Laura. Because Laura has a start of s metres, at this time Laura has run a distance of $d - s$ metres.

Because they have been running for the same amount of time when Dina overtakes Laura, at this time the ratio of the distances they have run is the same as the ratio of their speeds. That is

$$d : d - s = n : 1.$$

It follows that

$$\frac{d}{d-s} = \frac{n}{1}.$$

Hence

$$d = n(d - s).$$

This last equation may be rearranged as

$$ns = d(n - 1).$$

Therefore

$$d = \frac{ns}{n-1}.$$

FOR INVESTIGATION

20.1 Suppose that when Dina overtakes Laura she has run twice as far as Laura.

What is the ratio of Dina's speed to Laura's speed?

21. The numbers m and k satisfy the equations $2^m + 2^k = p$ and $2^m - 2^k = q$.

What is the value of 2^{m+k} in terms of p and q?

A $\frac{p^2 - q^2}{4}$ B $\frac{pq}{2}$ C $p + q$ D $\frac{(p-q)^2}{4}$ E $\frac{p+q}{p-q}$

SOLUTION **A**

We have

$$2^m + 2^k = p \qquad (1)$$

and

$$2^m - 2^k = q. \qquad (2)$$

Adding equations (1) and (2), we obtain

$$2(2^m) = p + q$$

and hence

$$2^m = \frac{p+q}{2}. \qquad (3)$$

Subtracting equations (2) from equation (1), we obtain

$$2(2^k) = p - q$$

and hence

$$2^k = \frac{p-q}{2}. \qquad (4)$$

Therefore, by (3) and (4)

$$\begin{aligned} 2^{m+k} &= 2^m \times 2^k \\ &= \left(\frac{p+q}{2}\right) \times \left(\frac{p-q}{2}\right) \\ &= \frac{(p+q)(p-q)}{4} \\ &= \frac{p^2 - q^2}{4}. \end{aligned}$$

FOR INVESTIGATION

21.1 Use the equations $p = 2^m + 2^k$ and $q = 2^m - 2^k$ to obtain expressions for p^2 and q^2.

Hence deduce that $\frac{p^2 - q^2}{4} = 2^{m+k}$.

21.2 The numbers a and b satisfy the equations

$$2^{a+b} = r$$

and

$$2^{a-b} = s.$$

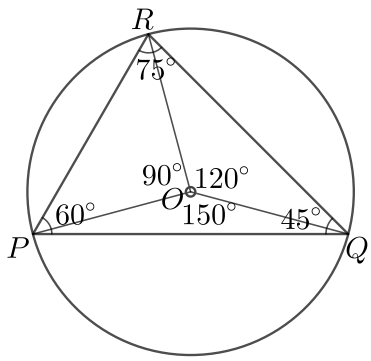
Find $2^a + 2^b$ in terms of r and s.

22. A triangle with interior angles 60°, 45° and 75° is inscribed in a circle of radius 2.

What is the area of the triangle?

A $2\sqrt{3}$ B 4 C $6+\sqrt{3}$ D $6\sqrt{3}$ E $3+\sqrt{3}$

Solution **E**



Let P, Q and R be the vertices of the triangle, let O be the centre of the circle, as shown in the diagram.

The angle subtended by an arc at the centre of a circle is twice the angle subtended at the circumference. Therefore

$$\angle QOR = 120°,$$

$$\angle ROP = 90°$$

and

$$\angle POQ = 150°.$$

We use the notation $[XYZ]$ for the area of a triangle XYZ.

Using the formula $\frac{1}{2}ab\sin\theta$ for the area of a triangle with sides of lengths a and b with included angle θ, we have

$$[QOR] = \frac{1}{2}(OQ \times OR)\sin 120° = \frac{1}{2}(2\times 2)\frac{\sqrt{3}}{2} = \sqrt{3},$$

$$[ROP] = \frac{1}{2}(OR \times OP)\sin 90° = \frac{1}{2}(2\times 2) = 2$$

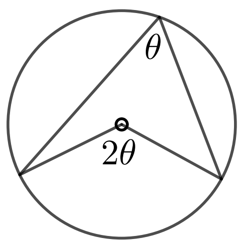
and

$$[POQ] = \frac{1}{2}(OP \times OQ)\sin 150° = \frac{1}{2}(2\times 2)\frac{1}{2} = 1.$$

Therefore

$$[PQR] = [QOR] + [ROP] + [POQ] = \sqrt{3} + 2 + 1 = 3 + \sqrt{3}.$$

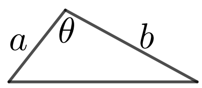
For investigation



22.1 Find a proof that the angle subtended by an arc at the centre of a circle is twice the angle subtended at the circumference. [That is, try and prove this for yourself, or find a proof in a book or on the internet, or ask your teacher.]

22.2 Show that $\sin 120° = \dfrac{\sqrt{3}}{2}$ and $\sin 150° = \dfrac{1}{2}$.

22.3 Show that the area of a triangle with side lengths a and b with included angle θ is $\frac{1}{2}ab\sin\theta$.



23. Let x be a real number. What is the minimum value of $\left(x^2 - 4x + 3\right)\left(x^2 + 4x + 3\right)$?

A -16 B -9 C 0 D 9 E 16

Solution **A**

We have

$$\begin{aligned}\left(x^2 - 4x + 3\right)\left(x^2 + 4x + 3\right) &= \left((x^2 + 3) - 4x\right)\left((x^2 + 3) + 4x\right) \\ &= (x^2 + 3)^2 - (4x)^2 \\ &= x^4 + 6x^2 + 9 - 16x^2 \\ &= x^4 - 10x^2 + 9 \\ &= (x^2 - 5)^2 - 16.\end{aligned}$$

For every real number x, $(x^2 - 5)^2 \geq 0$ and therefore $(x^2 - 5)^2 - 16 \geq -16$.

Now, when $x = \sqrt{5}$, $(x^2 - 5)^2 - 16 = 0^2 - 16 = -16$.

It follows that the minimum value of $\left(x^2 - 4x + 3\right)\left(x^2 + 4x + 3\right)$ is -16.

For investigation

23.1 (a) Find the real numbers a and b for which

$$x^4 - 8x^2 + 12 = (x^2 + a)^2 + b, \text{ for all real numbers } x.$$

(b) Hence find the minimum value of $x^4 - 8x^2 + 12$.

23.2 (a) Find the real numbers a and b for which

$$x^4 + 8x^2 + 12 = (x^2 + a)^2 + b, \text{ for all real numbers } x.$$

(b) Hence find the minimum value of $x^4 + 8x^2 + 12$.

Note

If you have met the *differential calculus*, you will know that the minimum values of polynomials may be found using calculus. In this case, check that using calculus you obtain the minimum value -16 for the polynomial $(x^2 - 4x + 3)(x^2 + 4x + 3)$.

Also, use calculus to solve Problems 23.1 and 23.2.

24. Saba, Rayan and Derin are cooperating to complete a task. They each work at a constant rate independent of whoever else is working on the task. When all three work together, it takes 5 minutes to complete the task. When Saba is working with Derin, the task takes 7 minutes to complete. When Rayan is working with Derin, the task takes 15 minutes to complete.

How many minutes does it take for Derin to complete the task on his own?

A 21 B 28 C 35 D 48 E 105

SOLUTION **E**

Suppose that it takes Saba, Rayan and Derin, working on their own, s, r and d minutes, respectively, to complete the task.

Then in 1 minute Saba completes $\frac{1}{s}$ of the task, Rayan completes $\frac{1}{r}$ of the task, and Derin completes $\frac{1}{d}$ of the task. Hence when all three are working together in 1 minute they complete $\frac{1}{s}+\frac{1}{r}+\frac{1}{d}$ of the task. Since it takes them 5 minutes to complete the task when they all work together,

$$\frac{1}{s}+\frac{1}{r}+\frac{1}{d}=\frac{1}{5}.$$

Similarly, as it takes Saba working with Derin 7 minutes to complete the task,

$$\frac{1}{s}+\frac{1}{d}=\frac{1}{7}.$$

Likewise, as it takes Rayan working with Derin 15 minutes to complete the task,

$$\frac{1}{r}+\frac{1}{d}=\frac{1}{15}.$$

We therefore have

$$\begin{aligned}\frac{1}{d}&=\left(\frac{1}{s}+\frac{1}{d}\right)+\left(\frac{1}{r}+\frac{1}{d}\right)-\left(\frac{1}{s}+\frac{1}{r}+\frac{1}{d}\right)\\&=\frac{1}{7}+\frac{1}{15}-\frac{1}{5}\\&=\frac{15}{105}+\frac{7}{105}-\frac{21}{105}\\&=\frac{1}{105}.\end{aligned}$$

Therefore it takes Derin 105 minutes to complete the task on his own.

FOR INVESTIGATION

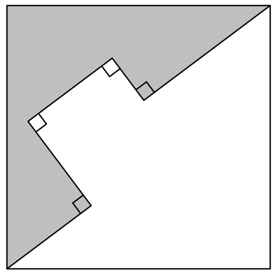
24.1 (a) How many minutes does it take for Rayan to complete the task on his own?

(b) How many minutes does it take for Saba to complete the task on her own?

25. Five line segments of length 2, 2, 2, 1 and 3 connect two corners of a square as shown in the diagram.

What is the shaded area?

A 8 B 9 C 10 D 11 E 12

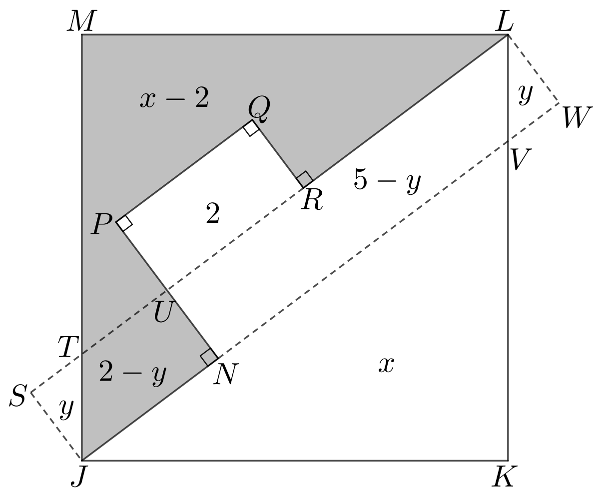


Solution **B**

Commentary

There are many different ways in which this problem may be solved. We give a solution which involves few calculations, but quite a lot of facts about the diagram. You are asked to check these facts in Problems 25.1 and 25.2.

Three other ways of solving the problem are indicated in problems 25.3, 25.4 and 25.5.



We let the points in the diagram be labelled as shown. The points S and W are chosen so that $SJWL$ is a rectangle.

The shaded region is the polygon $JNPQRLM$. We use the notation $[JNPQRLM]$ for the area of this polygon, and similar notation for the areas of other polygons.

In the rectangle $SJWL$ we have $JW = 7$ and $WL = 1$. Therefore, by Pythagoras' Theorem applied to the triangle JWL, we have $JL^2 = 7^2 + 1^2 = 50$. Hence $JL = \sqrt{50} = 5\sqrt{2}$. Since the diagonal of the square $JKLM$ has length $5\sqrt{2}$, it follows that the side length of the square is 5. Hence $[JKLM] = 5^2 = 25$. (You are asked to check all these facts in Problem 25.1.)

The triangles JKV and LMT are congruent. We let x be the common area of these triangles. Also, the triangles JTS and LVW are congruent. We let y be the common area of these triangles. It follows that the areas of the polygons in the diagram are as shown. (You are asked to check all these facts in Problem 25.2.)

We have

$$[JKLM] = [JKV] + [LMT] + [SJWL] - [JTS] - [LVW],$$

and therefore

$$25 = 2x + 7 - 2y.$$

Hence

$$2x - 2y = 25 - 7 = 18,$$

and therefore

$$x - y = 9.$$

It follows that

$$[JNPQRLM] = [TUPQRLM] + [JNUT] = (x - 2) + (2 - y) = x - y = 9.$$

Note: In the problems below, we use the same notation as in the solution above.

FOR INVESTIGATION

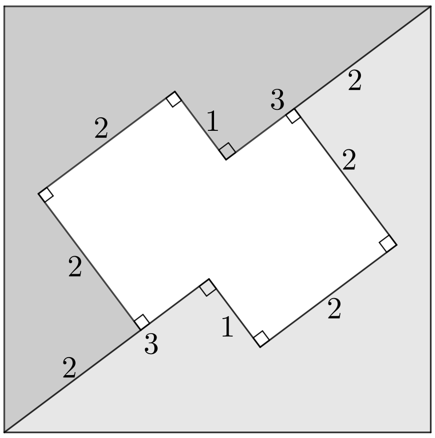
25.1 (a) Explain why in the rectangle $SJWL$ we have $JW = 7$ and $WL = 1$, and hence $JL = 5\sqrt{2}$.

(b) Explain why it follows from the fact that the diagonal JL has length $5\sqrt{2}$ that the square $JKLM$ has side length 5.

25.2 (a) Show the the triangles JKV and LMT are congruent, and that the triangles JTS and LVW are congruent.

(b) Deduce that the areas of the polygons in the diagram are as shown.

25.3



The lightly shaded region in the diagram above has been drawn so that it is congruent to the region whose area we need to find.

(a) Find the area of the unshaded region in the diagram.

(b) Use the fact that the area of the square is 25 to find the area of each of the shaded regions.

25.4 (a) Show that the triangles JKV and LWV are similar.

(b) Deduce that LV has length $\frac{5}{4}$ and that VK has length $\frac{15}{4}$.

(c) Use these lengths to find the values of x and y.

Hence check that $x - y = 9$.

25.5 Let $\angle VJK = \theta$.

(a) By considering the projections of the line segments JN, NP, PQ, QR and RL on JK and on JM, show that both

$$7\cos\theta - \sin\theta = JK$$

and

$$7\sin\theta + \cos\theta = JM.$$

(b) Use the fact that $JK = JM$ to deduce that $\sin\theta = \frac{3}{4}\cos\theta$. Hence find the values of $\cos\theta$ and $\sin\theta$.

(c) Use the values of $\cos\theta$ and $\sin\theta$ to calculate the area of the polygon $JNPQRLM$.

United Kingdom Mathematics Trust

Senior Mathematical Challenge

Tuesday 4 October 2022

Organised by the United Kingdom Mathematics Trust

supported by Overleaf

Candidates must be full-time students at secondary school or FE college.
England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil.** Mark A, B, C, D, E on the Answer Sheet for each question. Mark only one option, boldly, within the box.
5. Your Answer Sheet will be read by a machine. **Do not write or doodle on the sheet except to mark your chosen options.** The machine will read all markings, including bits of eraser stuck to the page, and interpret the mark in its own way.
6. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
7. **Scoring rules:** All candidates start with 25 marks; 0 marks are awarded for each question left unanswered; 4 marks are awarded for each correct answer; 1 mark is deducted for each incorrect answer (to discourage guessing).
8. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.
9. To accommodate candidates sitting at other times, please do not discuss the paper on the internet until 08:00 BST on Wednesday 5 October.

Enquiries about the Senior Mathematical Challenge should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. When the expression $\dfrac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)}$ is simplified, which of the following is obtained?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{5}$ E $\frac{1}{6}$

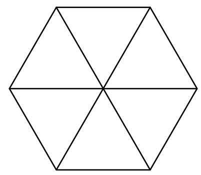
2. What is the smallest prime which is the sum of five different primes?

A 39 B 41 C 43 D 47 E 53

3. The figure shows a regular hexagon.

How many parallelograms are there in the figure?

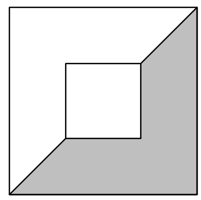
A 2 B 4 C 6 D 8
E more than 8



4. The diagram shows two symmetrically placed squares with sides of length 2 and 5.

What is the ratio of the area of the small square to that of the shaded region?

A 7 : 24 B 1 : 3 C 8 : 25 D 8 : 21 E 2 : 5



5. What is the value of $\dfrac{1}{1.01}+\dfrac{1}{1.1}+\dfrac{1}{1}+\dfrac{1}{11}+\dfrac{1}{101}$?

A 2.9 B 2.99 C 3 D 3.01 E 3.1

6. What is the value of $\dfrac{4^{800}}{8^{400}}$?

A $\frac{1}{2^{400}}$ B $\frac{1}{2^{200}}$ C 1 D 2^{200} E 2^{400}

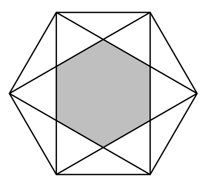
7. In 2021, a first class postage stamp cost 85p and a second class postage stamp cost 66p. In order to spend an exact number of pounds and to buy at least one of each type, what is the smallest total number of stamps that should be purchased?

A 10 B 8 C 7 D 5 E 2

8. In the diagram, the outer hexagon is regular and has an area of 216.

What is the shaded area?

A 108 B 96 C 90 D 84 E 72



9. A light-nanosecond is the distance that a photon can travel at the speed of light in one billionth of a second. The speed of light is $3\times10^8\,\mathrm{ms}^{-1}$.

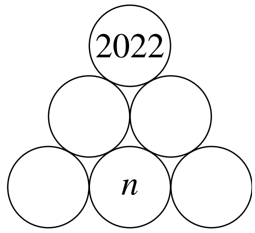
How far is a light-nanosecond?

A 3 cm B 30 cm C 3 m D 30 m E 300 m

10. What is the value of x in the equation $\dfrac{1+2x+3x^2}{3+2x+x^2}=3$?

A -5 B -4 C -3 D -2 E -1

11. In the number triangle shown, each disc is to be filled with a positive integer. Each disc in the top or middle row contains the number which is the product of the two numbers immediately below.



What is the value of n?

A 1 B 2 C 3 D 6 E 33

12. What is the sum of the digits of the integer which is equal to $6666666^2 - 3333333^2$?

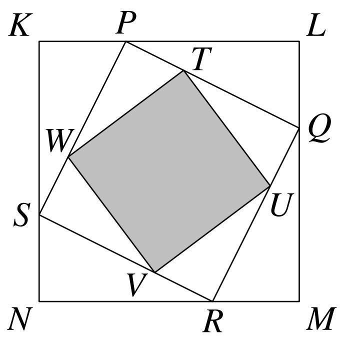
A 27 B 36 C 45 D 54 E 63

13. Three rugs have a combined area of $90\,\text{m}^2$. When they are laid down to cover completely a floor of area $60\,\text{m}^2$, the area which is covered by exactly two layers of rug is $12\,\text{m}^2$.

What is the area of floor covered by exactly three layers of rug?

A $2\,\text{m}^2$ B $6\,\text{m}^2$ C $9\,\text{m}^2$ D $10\,\text{m}^2$ E $12\,\text{m}^2$

14. The diagram shows a square, $KLMN$. A second square $PQRS$ is drawn inside it, as shown in the diagram, where P divides the side KL in the ratio $1 : 2$. Similarly, a third square $TUVW$ is drawn inside $PQRS$ with T dividing PQ in the ratio $1 : 2$.



What fraction of the area of $KLMN$ is shaded?

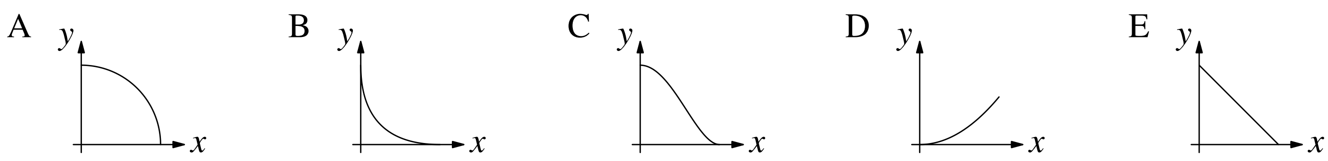
A $\frac{25}{81}$ B $\frac{16}{49}$ C $\frac{4}{9}$ D $\frac{40}{81}$ E $\frac{2}{3}$

15. The hare and the tortoise had a race over 100 m, in which both maintained constant speeds. When the hare reached the finish line, it was 75 m in front of the tortoise. The hare immediately turned around and ran back towards the start line.

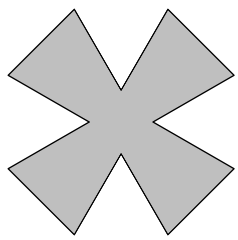
How far from the finish line did the hare and the tortoise meet?

A 54 B 60 C 64 D $66\frac{2}{3}$ E 72

16. Which diagram could be a sketch of the curve $\sqrt{x} + \sqrt{y} = 1$?



17. The shape shown is made by removing four equilateral triangles with side-length 1 from a regular octagon with side-length 1.



What is the area of the shape?

A $2 - 2\sqrt{2} + \sqrt{3}$ B $2 + 2\sqrt{2} - \sqrt{3}$ C $2 + 2\sqrt{2} + \sqrt{3}$
D $3 - 2\sqrt{2} - \sqrt{3}$ E $2 - 2\sqrt{2} - \sqrt{3}$

18. The numbers x and y are such that $3^x + 3^{y+1} = 5\sqrt{3}$ and $3^{x+1} + 3^y = 3\sqrt{3}$.

What is the value of $3^x + 3^y$?

A $\sqrt{3}$ B $2\sqrt{3}$ C $3\sqrt{3}$ D $4\sqrt{3}$ E $5\sqrt{3}$

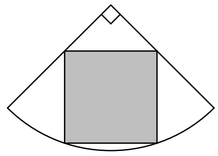
19. How many pairs of real numbers (x, y) satisfy the simultaneous equations $x^2 - y = 2022$ and $y^2 - x = 2022$?

A infinitely many B 1 C 2 D 3 E 4

20. A square is inscribed inside a quadrant of a circle. The circle has radius 10.

What is the area of the square?

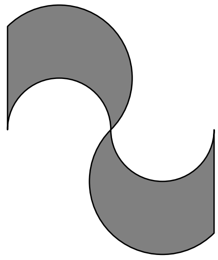
A $25\sqrt{2}$ B 36 C 12π D 40 E $30\sqrt{2}$



21. The perimeter of a logo is created from two vertical straight edges, two small semicircles with horizontal diameters and two large semicircles. Both of the straight edges and the diameters of the small semicircles have length 2. The logo has rotational symmetry as shown.

What is the shaded area?

A 4 B $4 - \pi$ C 8 D $4 + \pi$ E 12



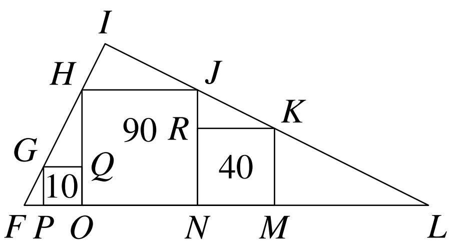
22. How many pairs of integers (x, y) satisfy the equation $\sqrt{x - \sqrt{x + 23}} = 2\sqrt{2} - y$?

A 0 B 1 C 4 D 8 E infinitely many

23. Three squares $GQOP$, $HJNO$ and $RKMN$ have vertices which sit on the sides of triangle FIL as shown. The squares have areas of 10, 90 and 40 respectively.

What is the area of triangle FIL?

A 220.5 B $\frac{21}{5}\sqrt{10}$ C 252 D $\frac{21}{2}\sqrt{10}$ E 441



24. The numbers x, y, p and q are all integers. x and y are variable and p and q are constant and positive. The four integers are related by the equation $xy = px + qy$.

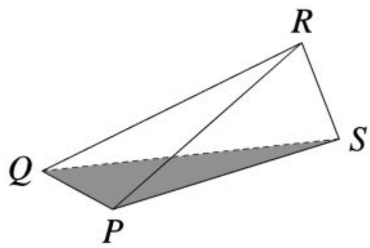
When y takes its maximum possible value, which expression is equal to $y - x$?

A $pq - 1$ B $(p - 1)(q - 1)$ C $(p + 1)(q - 1)$ D $(p - 1)(q + 1)$ E $(p + 1)(q + 1)$

25. A drinks carton is formed by arranging four congruent triangles as shown. $QP = RS = 4$ cm and $PR = PS = QR = QS = 10$ cm.

What is the volume, in cm^3, of the carton?

A $\frac{16}{3}\sqrt{23}$ B $\frac{4}{3}\sqrt{2}$ C $\frac{128}{25}\sqrt{6}$ D $\frac{13}{2}\sqrt{23}$ E $\frac{8}{3}\sqrt{6}$



United Kingdom
Mathematics Trust

Senior Mathematical Challenge

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Solutions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	C	C	D	C	E	C	E	B	D	A	E	C	A	B	B	B	B	E	D	D	B	A	D	A

1. When the expression $\dfrac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)}$ is simplified, which of the following is obtained?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{5}$ E $\frac{1}{6}$

Solution **D**

The expression simplifies to $\dfrac{3\times8\times15\times24}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)}$. Cancelling common factors gives $\frac{1}{5}$.

2. What is the smallest prime which is the sum of five different primes?

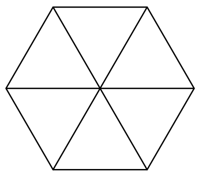
A 39 B 41 C 43 D 47 E 53

Solution **C**

For the sum of five different primes to be prime, each of those five primes must be odd, Listing the primes starting with 3, 5, 7, 11, 13, 17, 19, . . . and working systematically through possible sums gives a smallest sum of $3 + 5 + 7 + 11 + 13 = 39$ which is not prime. However, the next smallest sum $3 + 5 + 7 + 11 + 17 = 43$ which is prime as required.

3. The figure shows a regular hexagon.

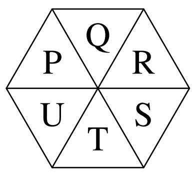
How many parallelograms are there in the figure?



A 2 B 4 C 6 D 8 E more than 8

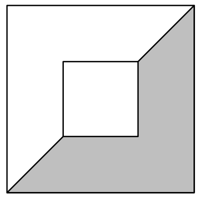
Solution **C**

Each of the possible parallelograms is formed from two adjacent equilateral triangles: P and Q, Q and R, R and S, S and T, T and U and finally U and P. Therefore there are six possible parallelograms.



4. The diagram shows two symmetrically placed squares with sides of length 2 and 5.

What is the ratio of the area of the small square to that of the shaded region?



A 7 : 24 B 1 : 3 C 8 : 25 D 8 : 21 E 2 : 5

Solution **D**

The area of the small square is $2\times 2 = 4$. The area of the shaded region is then $\frac{1}{2}\times 5\times 5-\frac{1}{2}\times 2\times 2 = \frac{25-4}{2} = \frac{21}{2}$. Therefore the ratio of the area of the small square to the area of the shaded region is $4 : \frac{21}{2} = 8 : 21$.

5. What is the value of $\dfrac{1}{1.01}+\dfrac{1}{1.1}+\dfrac{1}{1}+\dfrac{1}{11}+\dfrac{1}{101}$?

A 2.9 B 2.99 C 3 D 3.01 E 3.1

Solution **C**

Rewriting the calculation as $\frac{100}{101}+\frac{10}{11}+\frac{1}{1}+\frac{1}{11}+\frac{1}{101}$ shows that we can reorder the sum to give $\frac{101}{101}+\frac{11}{11}+\frac{1}{1} = 3$.

6. What is the value of $\dfrac{4^{800}}{8^{400}}$?

A $\frac{1}{2^{400}}$ B $\frac{1}{2^{200}}$ C 1 D 2^{200} E 2^{400}

Solution **E**

Rewriting $\dfrac{4^{800}}{8^{400}}$ using a base of 2 gives $\dfrac{(2^2)^{800}}{(2^3)^{400}} = \dfrac{2^{1600}}{2^{1200}} = 2^{400}$ using rules of indices.

7. In 2021, a first class postage stamp cost 85p and a second class postage stamp cost 66p. In order to spend an exact number of pounds and to buy at least one of each type, what is the smallest total number of stamps that should be purchased?

A 10 B 8 C 7 D 5 E 2

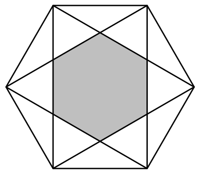
Solution **C**

Consider first the units digits of 85 and 66. Multiples of 5 can only end in 5 or 0. No multiples of 6, an even number, can end in 5. So in order that the units digit of our sum can be 0, each of the multiples of 85 and 66 must individually have units digits of 0. The smallest multiples of 85 and 66 with this property, $2\times 85 = 170$ and $5\times 66 = 330$, have sum 500. So 7 is the smallest number of stamps and they cost £5.

8. In the diagram, the outer hexagon is regular and has an area of 216.

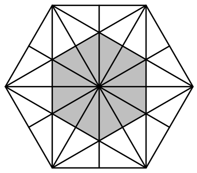
What is the shaded area?

A 108 B 96 C 90 D 84 E 72



SOLUTION **E**

By drawing extra lines from the centre of the outer hexagon to each of its vertices and from the centre to the midpoint of each edge of the outer hexagon, 12 in total, the diagram can be shown to be made of 36 congruent triangles each with angles 30°, 60° and 90°. Twelve of these triangles are shaded giving a shaded area of $\frac{1}{3} \times 216 = 72$.



9. A light-nanosecond is the distance that a photon can travel at the speed of light in one billionth of a second. The speed of light is $3 \times 10^8\,\mathrm{ms}^{-1}$.

How far is a light-nanosecond?

A 3 cm B 30 cm C 3 m D 30 m E 300 m

SOLUTION **B**

Using speed $= \dfrac{\text{distance}}{\text{time}}$ gives $3 \times 10^8 = \dfrac{d}{10^{-9}}$. Therefore $d = 3 \times 10^{-1}\,\mathrm{m} = 0.3\,\mathrm{m} = 30\,\mathrm{cm}$.

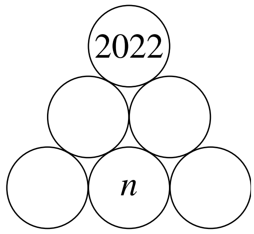
10. What is the value of x in the equation $\dfrac{1 + 2x + 3x^2}{3 + 2x + x^2} = 3$?

A -5 B -4 C -3 D -2 E -1

SOLUTION **D**

Rearranging the equation gives $1 + 2x + 3x^2 = 9 + 6x + 3x^2$ so $1 + 2x = 9 + 6x$ and $4x = -8$. Therefore $x = -2$.

11. In the number triangle shown, each disc is to be filled with a positive integer. Each disc in the top or middle row contains the number which is the product of the two numbers immediately below.



What is the value of n?

A 1 B 2 C 3 D 6 E 33

Solution **A**

When expressed as the product of its prime factors, $2022 = 2 \times 3 \times 337$. However, the integer n must be a factor of each integer in the middle row and so n^2 must be a factor of their product 2022. Therefore $n = 1$.

12. What is the sum of the digits of the integer which is equal to $6666666^2 - 3333333^2$?

A 27 B 36 C 45 D 54 E 63

Solution **E**

Using the difference of two squares, the calculation we are given can be written in the form $6666666^2 - 3333333^2 = (6666666 + 3333333)(6666666 - 3333333) = 9999999 \times 3333333 = 10000000 \times 3333333 - 1 \times 3333333 = 33333330000000 - 3333333 = 33333326666667$. The sum of the digits of this integer is 63.

13. Three rugs have a combined area of $90\,\text{m}^2$. When they are laid down to cover completely a floor of area $60\,\text{m}^2$, the area which is covered by exactly two layers of rug is $12\,\text{m}^2$.

What is the area of floor covered by exactly three layers of rug?

A $2\,\text{m}^2$ B $6\,\text{m}^2$ C $9\,\text{m}^2$ D $10\,\text{m}^2$ E $12\,\text{m}^2$

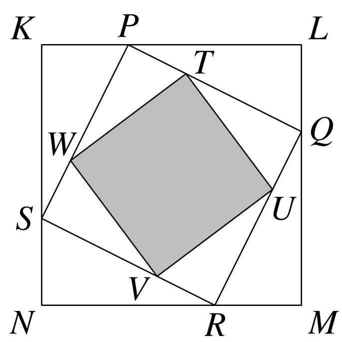
Solution **C**

Let the area of floor covered by exactly one rug be a, the area of floor covered by exactly two rugs be b and the area of floor covered by three rugs be c. Therefore, $a + 2b + 3c = 90$ and $a + b + c = 60$. Subtracting the second equation from the first leaves $b + 2c = 30$ and using $b = 12$ gives $c = 9$.

14. The diagram shows a square, $KLMN$. A second square $PQRS$ is drawn inside it, as shown in the diagram, where P divides the side KL in the ratio $1 : 2$. Similarly, a third square $TUVW$ is drawn inside $PQRS$ with T dividing PQ in the ratio $1 : 2$.

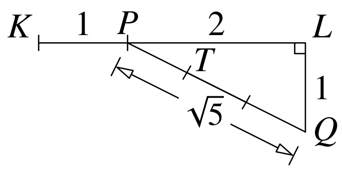
What fraction of the area of $KLMN$ is shaded?

A $\frac{25}{81}$ B $\frac{16}{49}$ C $\frac{4}{9}$ D $\frac{40}{81}$ E $\frac{2}{3}$



SOLUTION **A**

Let KL be 3 units long. Then $KP = 1$, $PL = 2$ and area $KLMN = 3 \times 3 = 9$. Removing four right-angled triangles congruent to PLQ from square $KLMN$ gives area $PQRS = 9 - 4 \times \frac{1}{2} \times 1 \times 2 = 5$. The area of $PQRS$ is $\frac{5}{9}$ of the area of $KLMN$. By the same reasoning the area of $TUVW$ is $\frac{5}{9}$ of the area of $PQRS$.



Combining these proportions gives the shaded area as $\frac{5}{9} \times \frac{5}{9} = \frac{25}{81}$ of the area of $KLMN$.

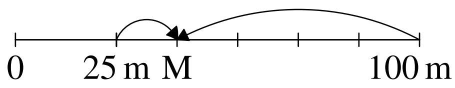
15. The hare and the tortoise had a race over 100 m, in which both maintained constant speeds. When the hare reached the finish line, it was 75 m in front of the tortoise. The hare immediately turned around and ran back towards the start line.

How far from the finish line did the hare and the tortoise meet?

A 54 B 60 C 64 D $66\frac{2}{3}$ E 72

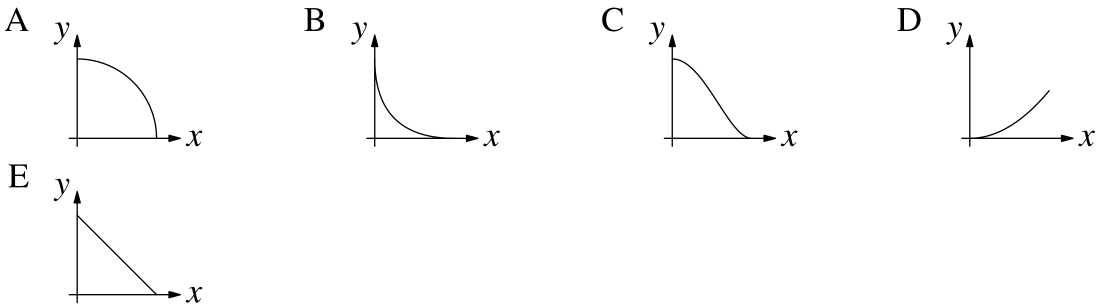
SOLUTION **B**

When the hare and tortoise are moving in the same direction, the hare completes 100 m while the tortoise completes 25 m. After the hare reverses direction and the hare and tortoise are moving towards one another, the hare is still moving four times as fast.



Therefore the meeting point, M, is $\frac{4}{5}$ of 75 m = 60 m away from the finish line.

16. Which diagram could be a sketch of the curve $\sqrt{x} + \sqrt{y} = 1$?



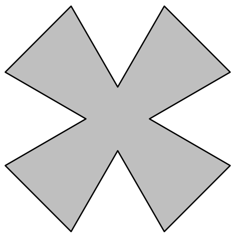
SOLUTION **B**

As x and y are interchangeable in the equation, the graph must be symmetric about the line $y = x$. This excludes options C and D. Substituting $x = 0$ and $x = 1$ into the equation shows that the graph crosses the axes at $(0, 1)$ and $(1, 0)$. Note that in option E the line $x + y = 1$ meets $y = x$ at $(\frac{1}{2}, \frac{1}{2})$ whereas our curve meets $y = x$ at $(\frac{1}{4}, \frac{1}{4})$ and must therefore lie below the straight line shown in option E. The only possible option then is B.

17. The shape shown is made by removing four equilateral triangles with side-length 1 from a regular octagon with side-length 1.

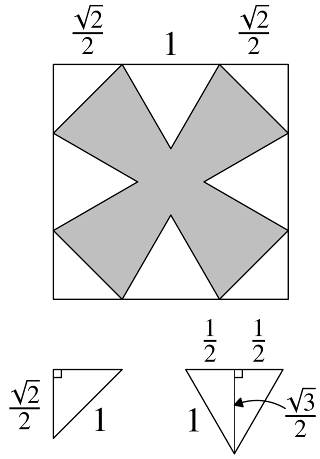
What is the area of the shape?

A $2 - 2\sqrt{2} + \sqrt{3}$ B $2 + 2\sqrt{2} - \sqrt{3}$ C $2 + 2\sqrt{2} + \sqrt{3}$
D $3 - 2\sqrt{2} - \sqrt{3}$ E $2 - 2\sqrt{2} - \sqrt{3}$



SOLUTION **B**

We enclose the regular octagon within a square as shown. Since the side-length of the octagon is 1, the right-angled isosceles triangles in the corners have two short sides of length $\frac{\sqrt{2}}{2}$ and so the square has side-length $1 + \sqrt{2}$. Each of the right-angled triangles has area $\frac{1}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = \frac{1}{4}$. Each of the equilateral triangles which were removed has base 1 and so height $\frac{\sqrt{3}}{2}$. The shaded area can be obtained as the area of the square minus that of the four isosceles corners and the four equilateral triangles; that is $(1 + \sqrt{2})^2 - 4 \times \frac{1}{4} - 4 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = 3 + 2\sqrt{2} - 1 - \sqrt{3} = 2 + 2\sqrt{2} - \sqrt{3}$.



18. The numbers x and y are such that $3^x + 3^{y+1} = 5\sqrt{3}$ and $3^{x+1} + 3^y = 3\sqrt{3}$.

What is the value of $3^x + 3^y$?

A $\sqrt{3}$ B $2\sqrt{3}$ C $3\sqrt{3}$ D $4\sqrt{3}$ E $5\sqrt{3}$

SOLUTION **B**

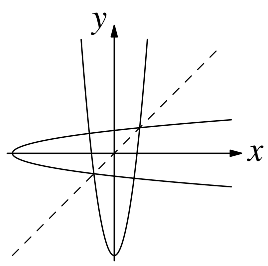
Let $3^x = X$ and $3^y = Y$. The two equations can then be written as $X + 3Y = 5\sqrt{3}$ and $3X + Y = 3\sqrt{3}$. Subtracting three lots of the second equation from the first gives $-8X = -4\sqrt{3}$ so $X = \frac{\sqrt{3}}{2}$. Subtracting three lots of the first equation from the second gives $-8Y = -12\sqrt{3}$ so $Y = \frac{3\sqrt{3}}{2}$. The value of $3^x + 3^y = X + Y = \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} = 2\sqrt{3}$. Alternatively, we could add the two equations giving $4X + 4Y = 8\sqrt{3}$. Dividing by 4, $X + Y = 3^x + 3^y = 2\sqrt{3}$ without knowing the value of either 3^x or 3^y individually.

19. How many pairs of real numbers (x, y) satisfy the simultaneous equations $x^2 - y = 2022$ and $y^2 - x = 2022$?

A infinitely many B 1 C 2 D 3 E 4

SOLUTION **E**

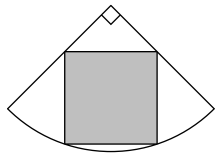
The first equation can be rearranged to the form $y = x^2 - 2022$ which is a translation of $y = x^2$ down 2022 units. The second equation is a reflection of the first, in the line $y = x$. There are four points of intersection of these two parabolas.



20. A square is inscribed inside a quadrant of a circle. The circle has radius 10.

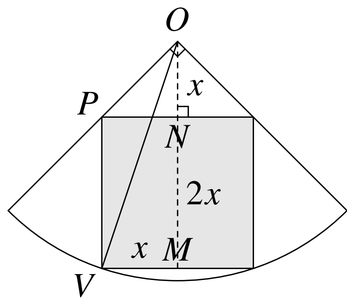
What is the area of the square?

A $25\sqrt{2}$ B 36 C 12π D 40 E $30\sqrt{2}$



SOLUTION **D**

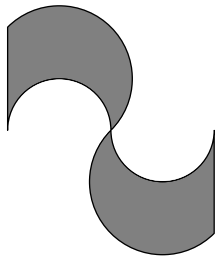
Let O be the centre of the circle, M and N be the midpoints of two sides of the square and V and P be vertices of two sides of the square as shown. Line ONM is a line of symmetry. Let $ON = x$. Therefore $NP = x$ as ON and NP are sides of the right-angled isosceles triangle ONP. Also, $PV = MN = 2x$. Consider right-angled triangle OVM. The radius of the circle is given as 10, therefore $(3x)^2 + x^2 = 10^2$ so $10x^2 = 100$ and $x^2 = 10$. Hence the area of the square is $(2x)^2 = 4x^2 = 4 \times 10 = 40$.



21. The perimeter of a logo is created from two vertical straight edges, two small semicircles with horizontal diameters and two large semicircles. Both of the straight edges and the diameters of the small semicircles have length 2. The logo has rotational symmetry as shown.

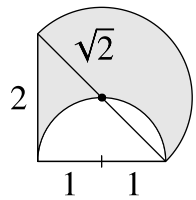
What is the shaded area?

A 4 B $4 - \pi$ C 8 D $4 + \pi$ E 12



SOLUTION **D**

Half the diagram is shown here. In it, the shaded area equals the area of a right-angled isosceles triangle of side-length 2 plus the area of a large semicircle minus the area of a small semicircle of radius 1. Using Pythagoras' Theorem, the diameter of the large semicircle has length $2\sqrt{2}$ and so the radius is $\sqrt{2}$. Therefore the shaded area of the full diagram is $2[\frac{1}{2} \times 2 \times 2 + \frac{1}{2}\pi \times (\sqrt{2})^2 - \frac{1}{2}\pi \times 1^2] = 2(2 + \pi - \frac{1}{2}\pi) = 4 + \pi$.



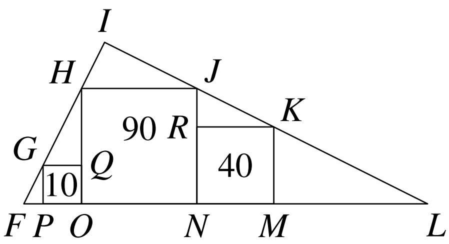
22. How many pairs of integers (x, y) satisfy the equation $\sqrt{x - \sqrt{x + 23}} = 2\sqrt{2} - y$?

A 0 B 1 C 4 D 8 E infinitely many

Solution **B**

Squaring both sides of the equation gives $x - \sqrt{x+23} = 8 - 4\sqrt{2}y + y^2$ which can be rearranged to $\sqrt{x+23} = (x - 8 - y^2) + 4\sqrt{2}y$ [1]. Squaring equation [1] gives $x + 23 = (x - 8 - y^2)^2 + 2(4\sqrt{2}y)(x - 8 - y^2) + 32y^2$ [2]. We are given that both x and y are integers and so the surd component, $2(4\sqrt{2}y)(x - 8 - y^2)$, must equal 0. Therefore either $y = 0$ or $(x - 8 - y^2) = 0$ [3]. Consider first the case $y = 0$. Here, equation [2] reduces to $x + 23 = (x - 8)^2$. This expands to $x^2 - 17x + 41 = 0$ which has no integer solutions as its discriminant is $(-17)^2 - 4 \times 1 \times 41 = 125$, which is not square. Secondly considering $(x - 8 - y^2) = 0$ [3] reduces [1] to $\sqrt{x+23} = 4\sqrt{2}y$ and therefore $x + 23 = 32y^2$. Using [3] again gives $x = 8 + y^2$ and so $31 + y^2 = 32y^2$. Therefore $y^2 = 1$. Hence $y = \pm 1$ and in either case, $x = 8 + 1 = 9$. Because equations have been squared, some solutions could be spurious. Substituting in the original equation, we see that $(9, 1)$ is a solution but $(9, -1)$ is not. Hence there is just one solution.

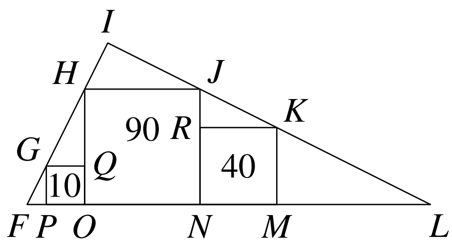
23. Three squares $GQOP$, $HJNO$ and $RKMN$ have vertices which sit on the sides of triangle FIL as shown. The squares have areas of 10, 90 and 40 respectively.



What is the area of triangle FIL?

A 220.5 B $\frac{21}{5}\sqrt{10}$ C 252 D $\frac{21}{2}\sqrt{10}$ E 441

Solution **A**



The lengths of the sides of the three squares are $\sqrt{10}$, $3\sqrt{10}$ and $2\sqrt{10}$ respectively. Therefore $HQ = 2\sqrt{10}$ and $RJ = \sqrt{10}$. In triangle GQH, the gradient of GH is $\frac{2\sqrt{10}}{\sqrt{10}} = 2$. In triangle JRK, the gradient of JK is $\frac{-\sqrt{10}}{2\sqrt{10}} = \frac{-1}{2}$. Therefore lines FI (on which GH lies) and IL (on which JK lies) are perpendicular.

All five right-angled triangles around the edge of the figure and triangle FIL itself are similar as they contain the same angles. They all have sides in the ratio $1 : 2 : \sqrt{5}$. To calculate the area of triangle FIL we need the length IL, as the area of $FIL = \frac{1}{2} \times IL \times \frac{1}{2}IL$. The length IL is made of three sections: $JK = \sqrt{10} \times \sqrt{5}$, $KL = 2JK = 2 \times \sqrt{10} \times \sqrt{5}$ and $IJ = \frac{2}{\sqrt{5}} \times HJ = \frac{2}{\sqrt{5}} \times 3\sqrt{10}$. Therefore $IL = IJ + JK + KL = 6\sqrt{2} + \sqrt{50} + 2\sqrt{50} = 21\sqrt{2}$. Hence the area of triangle $FIL = \frac{1}{2} \times 21\sqrt{2} \times \frac{21}{2}\sqrt{2} = 220.5$.

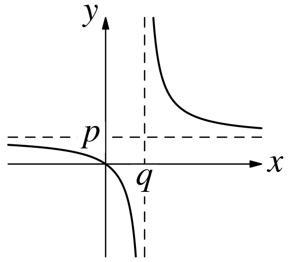
24. The numbers x, y, p and q are all integers. x and y are variable and p and q are constant and positive. The four integers are related by the equation $xy = px + qy$.

When y takes its maximum possible value, which expression is equal to $y - x$?

A $pq - 1$ B $(p-1)(q-1)$ C $(p+1)(q-1)$ D $(p-1)(q+1)$ E $(p+1)(q+1)$

Solution **D**

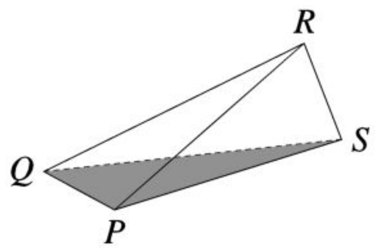
Rearranging $xy = px + qy$ to make y the subject, gives $xy - qy = px$ so $y(x-q) = px$ and therefore $y = \frac{px}{x-q}$ which rearranges to $y = p + \frac{pq}{x-q}$. A sketch of the graph of this function for real values of x and y is shown. As x and y are both integers in this question, y takes its maximum value when $x - q$ is as small as possible therefore $x - q = 1$ so $x = q + 1$. The expression $y - x$ then becomes $\frac{px}{1} - x = (p-1)x = (p-1)(q+1)$.



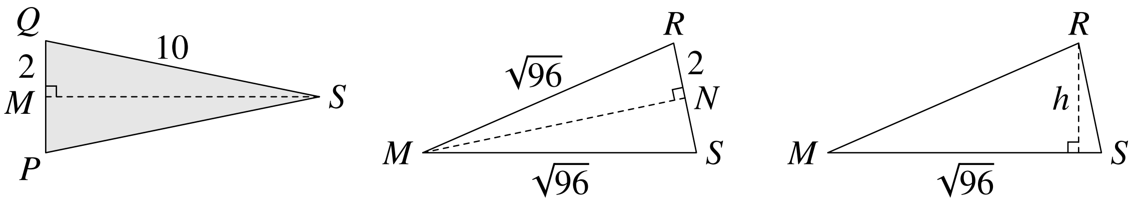
25. A drinks carton is formed by arranging four congruent triangles as shown. $QP = RS = 4$ cm and $PR = PS = QR = QS = 10$ cm.

What is the volume, in cm^3, of the carton?

A $\frac{16}{3}\sqrt{23}$ B $\frac{4}{3}\sqrt{2}$ C $\frac{128}{25}\sqrt{6}$ D $\frac{13}{2}\sqrt{23}$ E $\frac{8}{3}\sqrt{6}$



Solution **A**



Let M be the midpoint of QP. The volume of the carton is $\frac{1}{3} \times$ base area of triangle $PQS \times$ the perpendicular height from R to the plane containing PQS. Triangle PQS is isosceles and $MS = \sqrt{10^2 - 2^2} = \sqrt{96}$. So area of $PQS = \frac{1}{2} \times 4 \times \sqrt{96} = 8\sqrt{6}$. Consider isosceles triangle MRS and let N be the midpoint of RS. $MN = \sqrt{(\sqrt{96})^2 - 2^2} = \sqrt{92}$, so with RS as the base, area of $MRS = \frac{1}{2} \times 4 \times \sqrt{92} = 4\sqrt{23}$. Now with MS as the base, area of $MRS = \frac{1}{2} \times MS \times h$. Therefore $4\sqrt{23} = \frac{1}{2} \times \sqrt{96} \times h$ and $h = \frac{2\sqrt{23}}{\sqrt{6}}$. Finally, the volume $= \frac{1}{3} \times 8\sqrt{6} \times \frac{2\sqrt{23}}{\sqrt{6}} = \frac{16\sqrt{23}}{3}$.

United Kingdom
Mathematics Trust

Senior Mathematical Challenge

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Solutions and investigations

October 4th, 2022

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, UKMT, 84/85 *Pure Offices,* 4100 *Park Approach, Leeds LS*15 8*GB*

☎ 0113 365 1121 challenges@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	C	C	D	C	E	C	E	B	D	A	E	C	A	B	B	B	B	E	D	D	B	A	D	A

1. When the expression $\dfrac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)}{(2\times 3)\times(3\times 4)\times(4\times 5)\times(5\times 6)}$ is simplified, which of the following is obtained?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{5}$ E $\frac{1}{6}$

Solution **D**

Using the standard factorization $n^2-1=(n-1)(n+1)$, we have

$$\begin{aligned}\frac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)}{(2\times 3)\times(3\times 4)\times(4\times 5)\times(5\times 6)} &= \frac{(1\times 3)\times(2\times 4)\times(3\times 5)\times(4\times 6)}{(2\times 3)\times(3\times 4)\times(4\times 5)\times(5\times 6)}\\ &= \frac{1\times 2\times 3\times 4}{2\times 3\times 4\times 5}\\ &= \frac{1}{5}.\end{aligned}$$

For investigation

1.1 Write the following expressions in simplified form.

(a) $\dfrac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)\times(6^2-1)}{(2\times 3)\times(3\times 4)\times(4\times 5)\times(5\times 6)\times(6\times 7)}$,

(b) $\dfrac{(2^2-1^2)\times(3^2-2^2)\times(4^2-3^2)\times(5^2-4^2)}{1\times 3\times 5\times 7}$.

(c) $\dfrac{(3^4-1)\times(4^4-1)\times(5^4-1)\times(6^4-1)}{(2\times 4)\times(3\times 5)\times(4\times 6)\times(5\times 7)}$.

2. What is the smallest prime which is the sum of five different primes?

A 39 B 41 C 43 D 47 E 53

Solution **C**

If five different primes include 2, they consist of 2 and four odd primes. Hence their sum is even and hence is not prime.

Therefore five different primes whose sum is also prime cannot include 2. Hence they are five odd primes.

The sum of the first five odd primes is $3+5+7+11+13=39$, which not prime.

The smallest prime greater than 13 is 17. If we replace 13 by 17 we obtain the sum $3+5+7+11+17=43$, which is prime.

It follows that the smallest prime that is the sum of five different primes is 43.

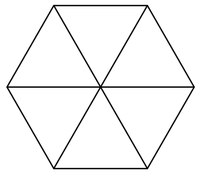
For investigation

2.1 Which is the smallest prime which is the sum of six different primes?

3. The figure shows a regular hexagon.

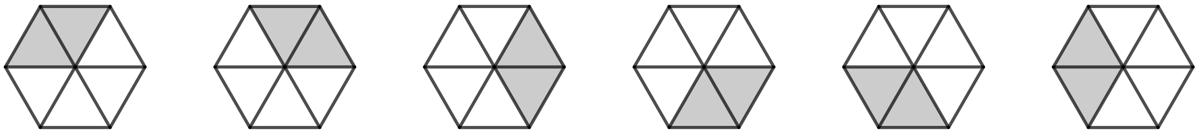
How many parallelograms are there in the figure?

A 2 B 4 C 6 D 8 E more than 8



Solution **C**

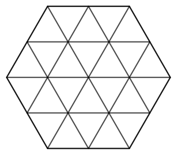
The parallelograms in the figure are made up from pairs of adjacent triangles. Therefore there are six parallelograms in the figure, as shown in the diagrams below.



For investigation

3.1 The figure on the right shows a regular hexagon divided into 24 congruent equilateral triangles.

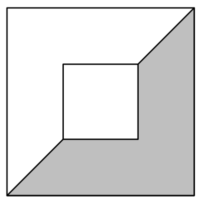
How many parallelograms are there in the figure?



4. The diagram shows two symmetrically placed squares with sides of length 2 and 5.

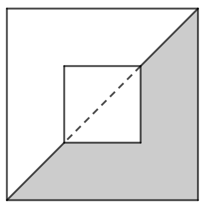
What is the ratio of the area of the small square to that of the shaded region?

A 7 : 24 B 1 : 3 C 8 : 25 D 8 : 21 E 2 : 5



Solution **D**

The small square has area 2^2, that is, 4. The shaded region is made up of half the large square with side length 5, with half the small square with side length 2 removed.



Therefore, the shaded area is $\frac{1}{2}(5^2) - \frac{1}{2}(2^2) = \frac{25}{2} - 2 = \frac{21}{2}$. Hence the ratio of the area of the small square to that of the shaded region is $4 : \frac{21}{2}$, that is, $8 : 21$.

For investigation

4.1 Suppose that the area of the shaded region were one-third of the area of the outer square. What would be the ratio of the area of the inner square to the area of the outer square?

4.2 Suppose that the ratio of the shaded area to the area of the small square is 7 : 18. What is the ratio of the side length of the small square to the side length of the large square?

5. What is the value of $\frac{1}{1.01}+\frac{1}{1.1}+\frac{1}{1}+\frac{1}{11}+\frac{1}{101}$?

A 2.9 B 2.99 C 3 D 3.01 E 3.1

SOLUTION **C**

Note:
The key to an efficient solution is to regroup the terms so as to simplify the arithmetic.

We have

$$\begin{aligned}\frac{1}{1.01}+\frac{1}{1.1}+\frac{1}{1}+\frac{1}{11}+\frac{1}{101} &= \left(\frac{1}{1.01}+\frac{1}{101}\right)+\left(\frac{1}{1.1}+\frac{1}{11}\right)+\frac{1}{1}\\ &= \left(\frac{100}{101}+\frac{1}{101}\right)+\left(\frac{10}{11}+\frac{1}{11}\right)+\frac{1}{1}\\ &= \frac{101}{101}+\frac{11}{11}+\frac{1}{1}\\ &= 1+1+1\\ &= 3.\end{aligned}$$

FOR INVESTIGATION

5.1 What is the value of $\frac{1}{1.24}+\frac{1}{6.2}+\frac{1}{31}$?

6. What is the value of $\frac{4^{800}}{8^{400}}$?

A $\frac{1}{2^{400}}$ B $\frac{1}{2^{200}}$ C 1 D 2^{200} E 2^{400}

SOLUTION **E**

We have

$$\begin{aligned}\frac{4^{800}}{8^{400}} &= \frac{(2^2)^{800}}{(2^3)^{400}}\\ &= \frac{2^{1600}}{2^{1200}}\\ &= 2^{1600-1200}\\ &= 2^{400}.\end{aligned}$$

FOR INVESTIGATION

6.1 For which integer n is $\frac{27^{900}}{9^{2700}} = 3^n$?

7. In 2021, a first class postage stamp cost 85p and a second class postage stamp cost 66p. In order to spend an exact number of pounds and to buy at least one of each type, what is the smallest total number of stamps that should be purchased?

A 10 B 8 C 7 D 5 E 2

Solution **C**

The cost of r first class postage stamps at 85p each, and s second class postage stamps at 66p each is $(85r + 66s)$p.

This is an exact number of pounds provided that

$$85r + 66s = 100t \qquad (1)$$

for some positive integer t.

We therefore seek the solution of (1) in which r, s and t are positive integers with $r + s$ as small as possible.

Equation (1) may be rearranged as

$$85r = 100t - 66s. \qquad (2)$$

Since 100 and 66 are both divisible by 2, it follows from (2) that $85r$ is divisible by 2. Therefore r is divisible by 2. Hence $r \geq 2$.

Equation (1) may also be rearranged as

$$66s = 100t - 85r. \qquad (3)$$

Since 100 and 85 are both divisible by 5, it follows from (3) that $66s$ is divisible by 5. Therefore s is divisible by 5. Hence $s \geq 5$.

We now note that when $r = 2$ and $s = 5$, we have

$$85r + 66s = 85 \times 2 + 66 \times 5 = 170 + 330 = 500.$$

Therefore equation (1) has the solution $r = 2$, $s = 5$ and $t = 5$. Because $r = 2$ and $s = 5$ are the least possible values for r and s, it follows that $r + s$ has the least possible value among the solutions of (1) in which r, s and t are positive integers.

Since, in this case, $r + s = 2 + 5 = 7$, we deduce that the least number of stamps that should be purchased is 7.

For investigation

7.1 Today a first class postage stamp costs 95p, and a second class postage stamp costs 68p. In order to spend an exact number of pounds and to buy at least one of each type, what is the smallest total number of stamps that should be purchased?

7.2 Find the solution of the equation

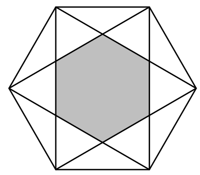
$$45r + 56s = 100t$$

in which r, s and t are positive integers and $r + s$ is as small as possible.

8. In the diagram, the outer hexagon is regular and has an area of 216.

What is the shaded area?

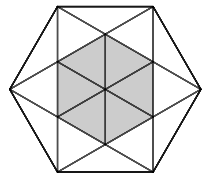
A 108 B 96 C 90 D 84 E 72



Solution **E**

Method 1

In the diagram on the right we have added three lines to the diagram given in the question.



The outer hexagon is now divided into 12 congruent equilateral triangles and 6 congruent triangles with angles 120°, 30° and 30°.

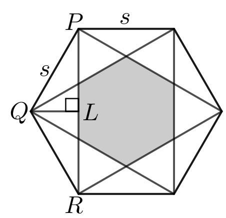
We leave it to the reader to check that each of the triangles with angles 120°, 30°, and 30° has the same area as each of the equilateral triangles (see Problem 8.1).

It follows that the outer hexagon is divided into 18 triangles all with the same area.

The shaded area is made up of 6 of these triangles. Hence its area is $\frac{6}{18}$, that is, $\frac{1}{3}$ of the area of the outer hexagon. Therefore the area of the shaded hexagon is $\frac{1}{3} \times 216 = 72$.

Method 2

We let s be the side length of the outer hexagon, and P, Q and R be three adjacent vertices of this hexagon, as shown.



We let L be the midpoint of PR.

We leave it to the reader to check that PLQ is a right-angled triangle in which $\angle PQL = 60°$ and $PL = \frac{\sqrt{3}}{2}s$ (see Problem 8.2). It follows that $PR = \sqrt{3}s$.

The inner hexagon is regular. The distance between its parallel sides is s. The outer hexagon is regular. The distance between its parallel sides is $\sqrt{3}s$. The ratio of the areas of similar figures is the same as the ratio of the squares of their corresponding lengths. Therefore

$$\text{area of inner hexagon} : \text{area of outer hexagon} = s^2 : (\sqrt{3}s)^2 = s^2 : 3s^2 = 1 : 3.$$

It follows that the area of the shaded hexagon is $\frac{1}{3} \times 216 = 72$.

For investigation

8.1 Show that in the diagram of Method 1, all the 18 triangles into which the outer hexagon is divided have the same area.

8.2 Show that in the diagram of Method 2, the triangle PLQ is right-angled, $\angle PQL = 60°$ and $PL = \frac{\sqrt{3}}{2}s$

9. A light-nanosecond is the distance that a photon can travel at the speed of light in one billionth of a second. The speed of light is $3 \times 10^8\,\text{ms}^{-1}$.

How far is a light-nanosecond?

A 3 cm B 30 cm C 3 m D 30 m E 300 m

Solution **B**

One billionth of a second is $\dfrac{1}{10^9}$ seconds.

Hence, in one billionth of a second light travels

$$\frac{1}{10^9} \times (3 \times 10^8)\,\text{m} = \frac{3}{10}\,\text{m} = 30\,\text{cm}.$$

Therefore a light-nanosecond is 30 cm.

For investigation

9.1 A light-minute is the distance that a photon travels in one minute at the speed of light. The mean distance of the Earth from the Sun is approximately 150 million kilometres.

How many light-minutes is that?

10. What is the value of x in the equation $\dfrac{1 + 2x + 3x^2}{3 + 2x + x^2} = 3$?

A -5 B -4 C -3 D -2 E -1

Solution **D**

We note first that $3 + 2x + x^2 = 2 + (1 + x)^2$. Therefore for all real numbers x, we have $3 + 2x + x^2 \geq 2$. Hence for all real numbers x, it follows that $3 + 2x + x^2 \neq 0$.

Therefore we can multiply both sides of the equation given in the question by $3 + 2x + x^2$. In this way, we have

$$\begin{aligned}\frac{1 + 2x + 3x^2}{3 + 2x + x^2} = 3 &\Leftrightarrow 1 + 2x + 3x^2 = 3(3 + 2x + x^2)\\ &\Leftrightarrow 1 + 2x + 3x^2 = 9 + 6x + 3x^2\\ &\Leftrightarrow 4x + 8 = 0\\ &\Leftrightarrow x = -2.\end{aligned}$$

For investigation

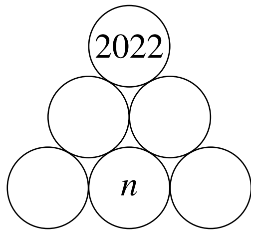
10.1 Find all the solutions of the following equations.

(a) $\dfrac{1 + 3x + 5x^2}{5 + 3x + x^2} = 1.$ (b) $\dfrac{5 + 7x + 2x^2}{7 + 10x + 3x^2} = 1.$

11. In the number triangle shown, each disc is to be filled with a positive integer. Each disc in the top or middle row contains the number which is the product of the two numbers immediately below.

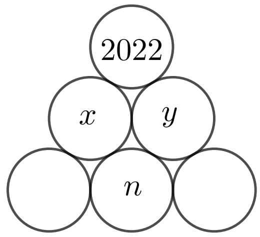
What is the value of n?

A 1 B 2 C 3 D 6 E 33



SOLUTION **A**

We let x and y be the positive integers in the discs in the middle row, as shown.



Since these are the products of the two numbers immediately below, n is a factor of both x and y. Hence n^2 is a factor of xy.

Now $xy = 2022$. It follows that n^2 is a factor of 2022.

The prime factorization of 2022 is

$$2022 = 2 \times 3 \times 337.$$

It follows that 1^2 is the only square of a positive integer that is a factor of 2022.

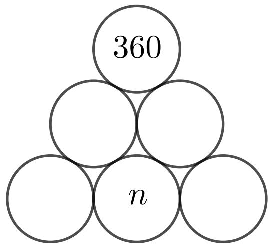
We deduce that $n = 1$.

FOR INVESTIGATION

11.1 Check that $2 \times 3 \times 337$ is the factorization of 2022 into prime factors.

11.2 In the number triangle shown on the right, each disc is to be filled with a positive integer. Each disc in the top or middle row contains the number which is the product of the two numbers immediately below.

What are the possible values of n?



12. What is the sum of the digits of the integer which is equal to $6666666^2 - 3333333^2$?

A 27 B 36 C 45 D 54 E 63

SOLUTION **E**

Using the standard factorization $x^2 - y^2 = (x - y)(x + y)$, we have

$$\begin{aligned} 6666666^2 - 3333333^2 &= (6666666 - 3333333)(6666666 + 3333333) \\ &= 3333333 \times 9999999 \\ &= 3333333 \times (10000000 - 1) \\ &= 3333333 \times 10000000 - 3333333 \times 1 \\ &= 33333330000000 - 3333333 \\ &= 33333326666667. \end{aligned}$$

It may be seen that the number 33333326666667 is written with six 3s followed by one 2, six 6s and one 7. Therefore the sum of its digits is

$$\begin{aligned} (6 \times 3) + 2 + (6 \times 6) + 7 &= 18 + 2 + 36 + 7 \\ &= 63. \end{aligned}$$

FOR INVESTIGATION

12.1 Let $n = 666\,666\,666^2 - 333\,333\,333^2$.

What is the sum of the digits of n?

12.2 (a) Let a be the integer $\overbrace{666\ldots666}^{k}$ which in standard form is written as a string of k 6s, and let b be the integer $\overbrace{333\ldots333}^{k}$ which is written as a string of k 3s.

Find a formula, in terms of k, for the sum of the digits of the integer $a^2 - b^2$.

(b) Find a formula for the sum of the digits of $c^2 - d^2$ where $c = \overbrace{777\ldots777}^{k}$ and $d = \overbrace{222\ldots222}^{k}$.

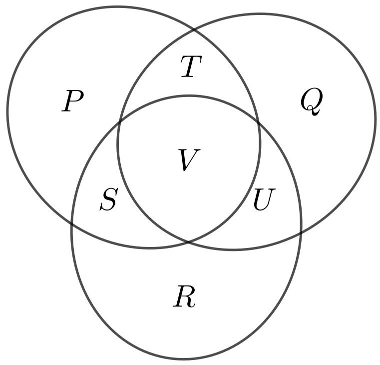
13. Three rugs have a combined area of $90\,\text{m}^2$. When they are laid down to cover completely a floor of area $60\,\text{m}^2$, the area which is covered by exactly two layers of rug is $12\,\text{m}^2$.

What is the area of floor covered by exactly three layers of rug?

A $2\,\text{m}^2$ B $6\,\text{m}^2$ C $9\,\text{m}^2$ D $10\,\text{m}^2$ E $12\,\text{m}^2$

Solution **C**

The diagram shows the three overlapping rugs.



We let the areas of the different regions, in m^2, be P, Q, R, S, T, U and V, as shown.

The three rugs have areas $P+S+T+V$, $Q+T+U+V$ and $R+S+U+V$. Therefore, because the three rugs have a combined area of $90\,\text{m}^2$,

$$(P+S+T+V)+(Q+T+U+V)+(R+S+U+V)=90.$$

This equation may be rearranged to give

$$(P+Q+R)+2(S+T+U)+3V=90. \qquad (1)$$

Because the floor has an area of $60\,\text{m}^2$,

$$(P+Q+R)+(S+T+U)+V=60. \qquad (2)$$

By subtracting equation (2) from equation (1), we obtain

$$(S+T+U)+2V=30. \qquad (3)$$

Because the area covered by exactly two layers of rug is $12\,\text{m}^2$,

$$S+T+U=12. \qquad (4)$$

By subtracting equation (4) from equation (3), we have $2V=18$. Therefore $V=9$.

Therefore the area of floor covered by exactly three layers of rug is $9\,\text{m}^2$.

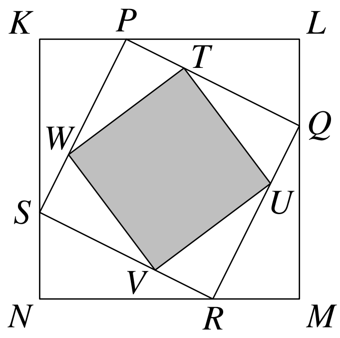
For investigation

13.1 What is the area of the floor that is covered by exactly one rug?

14. The diagram shows a square, $KLMN$. A second square $PQRS$ is drawn inside it, as shown in the diagram, where P divides the side KL in the ratio $1 : 2$. Similarly, a third square $TUVW$ is drawn inside $PQRS$ with T dividing PQ in the ratio $1 : 2$.

What fraction of the area of $KLMN$ is shaded?

A $\frac{25}{81}$ B $\frac{16}{49}$ C $\frac{4}{9}$ D $\frac{40}{81}$ E $\frac{2}{3}$



SOLUTION **A**

Since P divides the side KL in the ratio $1 : 2$, we choose units so that KL has length 3. Hence KP has length 1, and PL has length 2. We note also that it follows that the square $KLMN$ has area 3^2, that is, 9.

We leave it to the reader to show that the triangles PKS, QLP, RMQ and SNR are congruent.

It follows, in particular, that $SK = PL$. Therefore SK has length 2.

Using Pythagoras' Theorem, applied to the right-angled triangle PKS, we have $PS^2 = 1^2 + 2^2 = 5$. Therefore the square $PQRS$ has area 5.

Therefore the area of the square $PQRS$ as a fraction of the area of the square $KLMN$ is $\frac{5}{9}$.

It follows, similarly, that the area of the square $TUVW$ as a fraction of the area of the square $PQRS$ is also $\frac{5}{9}$.

It follows that the area of the shaded square $TUVW$ is $\frac{5}{9} \times \frac{5}{9} \times$ the area of the square $KLMN$.

Since $\frac{5}{9} \times \frac{5}{9} = \frac{25}{81}$, it follows that the fraction of the area of $KLMN$ that is shaded is $\frac{25}{81}$.

FOR INVESTIGATION

14.1 Show that the triangles PKS, QLP, RMQ and SNR are congruent.

14.2 Suppose that P divides the side KL in the ratio $1 : 3$, and that T divides PQ in the ratio $1 : 3$.

In this case, what fraction of the area of $KLMN$ is shaded?

14.3 Suppose that P divides the side KL in the ratio $p : q$, and that T divides PQ in the ratio $r : s$.

In this case, what fraction of the area of $KLMN$ is shaded?

15. The hare and the tortoise had a race over 100 m, in which both maintained constant speeds. When the hare reached the finish line, it was 75 m in front of the tortoise. The hare immediately turned around and ran back towards the start line.

How far from the finish line did the hare and the tortoise meet?

A 54 B 60 C 64 D $66\frac{2}{3}$ E 72

Solution **B**

The tortoise is 75 m behind the hare when the hare has run the full 100 m. Therefore the tortoise runs 25 m in the same time as the hare runs 100 m. Hence the hare runs at four times the speed of the tortoise.

When the hare turns round, the hare and the tortoise are 75 m apart and running towards each other.

Since the hare is running at four times the speed of the tortoise, when they meet the hare has run $\frac{4}{5}$ths of the 75 m they were apart when the hare turned around. The tortoise has run $\frac{1}{5}$th of this distance.

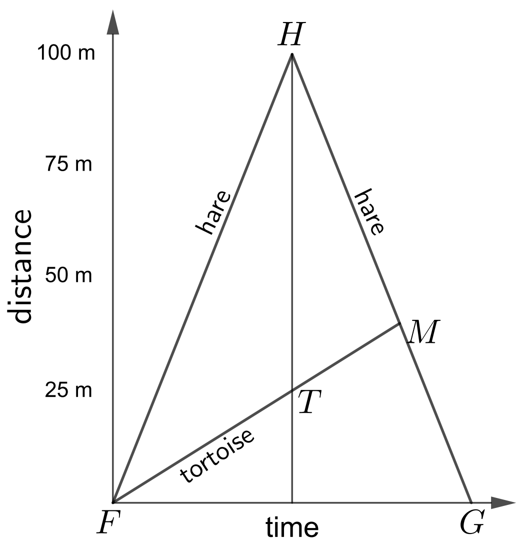
Therefore they meet at a distance $\frac{4}{5} \times 75$ m, that is, 60 m, from the finish line.

For investigation

15.1 The distance-time graph below illustrates the paths of the hare and the tortoise.

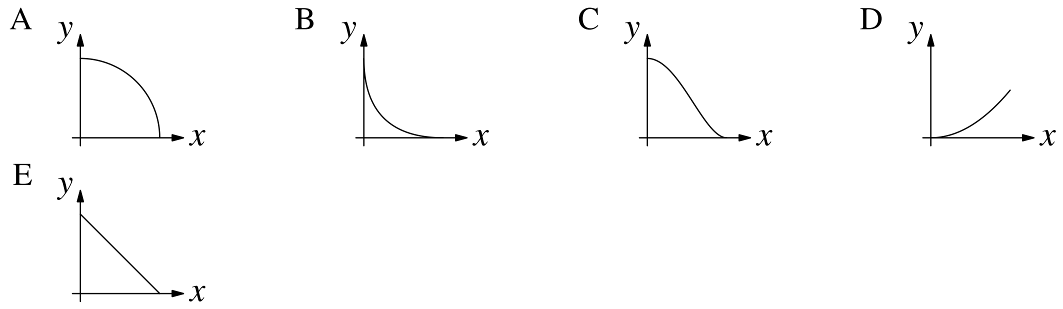
The point H corresponds to the position of the hare when the hare reaches the finish line and turns around. The point T corresponds to the position of the tortoise at this time. The point M corresponds to their common position when they meet. The points F and G are as shown.

Use the geometry of the diagram to work out how far from the finish line are the hare and tortoise when they meet.



[*Hint*: Extend the line FM to meet the line through G which is perpendicular to FG.]

16. Which diagram could be a sketch of the curve $\sqrt{x} + \sqrt{y} = 1$?



Solution **B**

Commentary

In a question of this type you are not expected to sketch the curve. Instead, you need to look for some feature of the equation that enables you quickly to rule out all but one of the diagrams given as options.

In this case it may be seen that the curve corresponding to the equation $\sqrt{x} + \sqrt{y} = 1$ has reflectional symmetry in the line $y = x$, and that the points $(1, 0)$ and $(0, 1)$ lie on the curve.

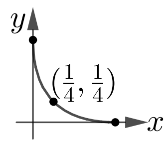
Unfortunately these facts are compatible with any of the options A, B and E being the sketch of the curve. So we need to find another fact to eliminate all but one of these options.

It turns out that one way to do this is to find a point on the curve different from $(1, 0)$ and $(0, 1)$.

Note that $\sqrt{\frac{1}{4}} = \frac{1}{2}$. Hence $\sqrt{\frac{1}{4}} + \sqrt{\frac{1}{4}} = 1$.

It follows that the point $(\frac{1}{4}, \frac{1}{4})$ lies on the curve with equation $\sqrt{x} + \sqrt{y} = 1$.

It may been seen that this rules out all the graphs given as options, other than the graph given in option B.



For investigation

16.1 For each of the following equations determine which, if any, of the diagrams given in this question might be a sketch of all, or part, of the curve corresponding to the equation.

(a) $y = x^3$,

(b) $y = \frac{1}{2}x^2$,

(c) $x^2 + y^2 = 1$,

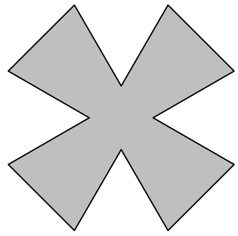
(d) $x + y = 1$,

(e) $x^2 - y^2 = 1$,

(f) $y = \frac{1}{2}(1 + \cos x)$.

17. The shape shown is made by removing four equilateral triangles with side-length 1 from a regular octagon with side-length 1.

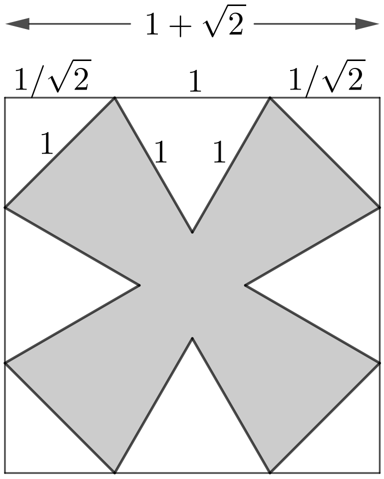
What is the area of the shape?



A $2 - 2\sqrt{2} + \sqrt{3}$ B $2 + 2\sqrt{2} - \sqrt{3}$ C $2 + 2\sqrt{2} + \sqrt{3}$
D $3 - 2\sqrt{2} - \sqrt{3}$ E $2 - 2\sqrt{2} - \sqrt{3}$

Solution **B**

As the diagram on the right shows, a regular octagon with side length 1 may be obtained by cutting four triangular corners from the square.



Each of these corners is an isosceles right-angled triangles with a hypotenuse of length 1. It follows, by Pythagoras' Theorem that the other two sides of these triangles have length $1/\sqrt{2}$.

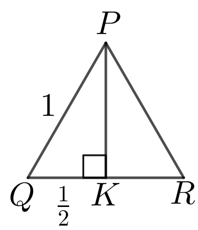
It follows that the side length of the square is given by $1/\sqrt{2} + 1 + 1/\sqrt{2} = 1 + \sqrt{2}$. Therefore the area of the square is $(1 + \sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}$.

The four triangular corners fit together to make a square of side length 1 and hence area 1.

Therefore the area of the octagon is $(3 + 2\sqrt{2}) - 1 = 2 + 2\sqrt{2}$.

From this area we need to subtract the area of the four equilateral triangles that are removed from the octagon to make the shaded shape.

We let PQR be an equilateral triangle with side length 1, and let PK be the perpendicular from P to QR as shown.



It can be checked that K is the midpoint of QR (see Problem 17.1) and hence that QK has length $\frac{1}{2}$. By Pythagoras' Theorem applied to the right-angled triangle PQK we have $QK^2 + PK^2 = PQ^2$. Therefore $PK^2 = PQ^2 - QK^2 = 1 - \frac{1}{2}^2 = \frac{3}{4}$.

Therefore $PK = \frac{1}{2}\sqrt{3}$.

We can now deduce that the area of the triangle PQR is $\frac{1}{2}(QR \times PK) = \frac{1}{2}(1 \times \frac{1}{2}\sqrt{3}) = \frac{1}{4}\sqrt{3}$.

It now follows that the area of the shape is

$$2 + 2\sqrt{2} - 4 \times \tfrac{1}{4}\sqrt{3} = 2 + 2\sqrt{2} - \sqrt{3}.$$

For investigation

17.1 Prove that the triangles PKQ and PKR are congruent. Deduce that the length of QK is $\frac{1}{2}$.

17.2 Use the $\frac{1}{2}ab\sin C$ formula for the area of a triangle to confirm that the area of an equilateral triangle with side length 1 is $\frac{1}{4}\sqrt{3}$.

18. The numbers x and y are such that $3^x + 3^{y+1} = 5\sqrt{3}$ and $3^{x+1} + 3^y = 3\sqrt{3}$.

What is the value of $3^x + 3^y$?

A $\sqrt{3}$ B $2\sqrt{3}$ C $3\sqrt{3}$ D $4\sqrt{3}$ E $5\sqrt{3}$

SOLUTION **B**

Note that $3^{x+1} = 3(3^x)$ and $3^{y+1} = 3(3^y)$.

Therefore the equations given in the question may be rewritten as

$$3^x + 3(3^y) = 5\sqrt{3} \qquad (1)$$

and

$$3(3^x) + 3^y = 3\sqrt{3}. \qquad (2)$$

Adding equations (1) and (2) gives

$$4(3^x + 3^y) = 8\sqrt{3}. \qquad (3)$$

Therefore

$$3^x + 3^y = 2\sqrt{3}. \qquad (4)$$

FOR INVESTIGATION

18.1 (a) Use equations (1) and (2) to find the values of 3^x and 3^y.

(b) Check that the values for 3^x and 3^y that you have found satisfy equation (4).

18.2 (For those who know about logarithms.)

(a) Use your answer to Problem 18.1 (a) to show that $x = \frac{1}{2} - \frac{\ln 2}{\ln 3}$.

(b) Find a similar expression for the value of y.

18.3 Find the values of x and y that satisfy both of the equations

$$4^{x+1} + 5^y = 281$$

and

$$4^x + 5^{y+1} = 189.$$

19. How many pairs of real numbers (x, y) satisfy the simultaneous equations $x^2 - y = 2022$ and $y^2 - x = 2022$?

A infinitely many B 1 C 2 D 3 E 4

SOLUTION **E**

By subtracting the equation $y^2 - x = 2022$ from the equation $x^2 - y = 2022$, we obtain

$$x^2 - y^2 + x - y = 0. \qquad (1)$$

The left hand side of equation (1) factorizes to give

$$(x - y)(x + y + 1) = 0. \qquad (2)$$

It follows that either $x - y = 0$ or $(x + y + 1) = 0$. That is, either $y = x$ or $y = -x - 1$.

When $y = x$ both equations of the question are equivalent to the equation $x^2 - x = 2022$. We can rewrite this equation as

$$x^2 - x - 2022 = 0. \qquad (3)$$

Equation (3) is a quadratic equation of the form $ax^2 + bx + c = 0$, with $a = 1$, $b = -1$ and $c = -2022$. Therefore $b^2 - 4ac = (-1)^2 + 4 \times 2022$ which is greater than 0.

It follows that equation (3) has two distinct real number solutions, say x_1 and x_2. It follows that the two pairs of real numbers (x_1, x_1) and (x_2, x_2) satisfy the simultaneous equations of the question.

It can be checked that when $y = -x - 1$ both equations of the question are equivalent to the equation

$$x^2 + x - 2021 = 0. \qquad (4)$$

Equation (4) is a quadratic equation of the form $ax^2 + bx + c = 0$, with $a = 1$, $b = 1$ and $c = -2021$. Therefore $b^2 - 4ac = 1^2 + 4 \times 2021$ which is greater than 0.

It follows that equation (4) has two distinct real number solutions, say x_3 and x_4. It follows that the two pairs of real numbers $(x_3, -x_3 - 1)$ and $(x_4, -x_4 - 1)$ satisfy the simultaneous equations of the question.

In these last two solutions $y \neq x$ and therefore they are different from the first two solutions.

We can therefore conclude that there are four pairs of real numbers that satisfy the simultaneous equations of the question.

FOR INVESTIGATION

19.1 Check that when $y = -x - 1$ both equations (1) and (2) are equivalent to the equation $x^2 + x - 2021 = 0$.

19.2 Consider the curves corresponding to the equations $x^2 - y = 2022$ and $y^2 - x = 2022$.

(a) What is the geometrical relationship between these two curves?

(b) Sketch the two curves.

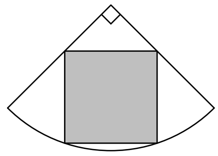
(c) Check that the two curves meet in four distinct points.

19.3 Explain why $b^2 - 4ac > 0$ is the condition for the quadratic equation $ax^2 + bx + c = 0$ to have two distinct real number solutions.

20. A square is inscribed inside a quadrant of a circle. The circle has radius 10.

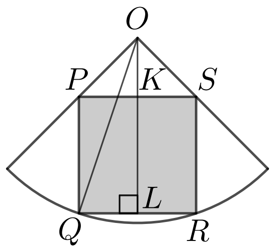
What is the area of the square?

A $25\sqrt{2}$ B 36 C 12π D 40 E $30\sqrt{2}$



Solution **D**

We let O be the centre of the circle, and P, Q, R and S be the vertices of the square, as shown. We let OL be the perpendicular from O to QR, and K be the point where this perpendicular meets PS.



We let s be the side length of the square. It may be checked (see Problem 20.1) that the triangles OLQ and OLR are congruent. It follows that $QL = \frac{1}{2}QR = \frac{1}{2}s = PK$.

It may be checked that OKP is a right-angled isosceles triangle (see Problem 20.2). Therefore $OK = PK = \frac{1}{2}s$.

It follows that in the right-angled triangle OLQ we have $QL = \frac{1}{2}s$, $OL = OK + KL = \frac{1}{2}s + s = \frac{3}{2}s$ and $OQ = 10$.

Therefore, by Pythagoras' Theorem

$$(\tfrac{1}{2}s)^2 + (\tfrac{3}{2}s)^2 = 10^2.$$

Hence

$$\tfrac{1}{4}s^2 + \tfrac{9}{4}s^2 = 100.$$

That is,

$$\tfrac{5}{2}s^2 = 100.$$

Hence

$$s^2 = \tfrac{2}{5} \times 100 = 40.$$

It follows that the area of the square is 40.

For investigation

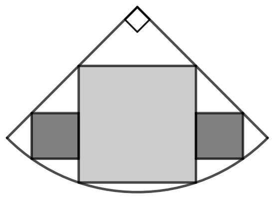
20.1 Show that the triangles OLQ and OLR are congruent.

20.2 Show that OKP is a right-angled isosceles triangle.

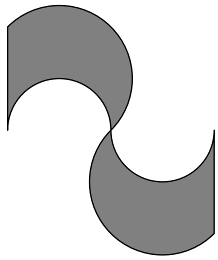
20.3 The diagram on the right shows one larger square and two smaller squares inscribed in the quadrant of a circle.

The circle has radius 10.

Find the side length of the smaller squares.



21. The perimeter of a logo is created from two vertical straight edges, two small semicircles with horizontal diameters and two large semicircles. Both of the straight edges and the diameters of the small semicircles have length 2. The logo has rotational symmetry as shown.

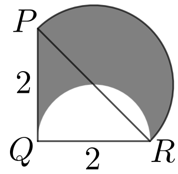


What is the shaded area?

A 4 B $4 - \pi$ C 8 D $4 + \pi$ E 12

Solution **D**

The diagram on the right shows the top half of the shaded region.



This region is made up of the right-angled triangle PQR and the semicircle with diameter PR, but with the semicircle with QR as diameter removed.

The area of the triangle PQR is $\frac{1}{2}(PQ \times QR) = \frac{1}{2}(2 \times 2) = 2$.

By Pythagoras' Theorem, applied to the right-angled triangle PQR, we have $PR^2 = 2^2 + 2^2 = 8$. Therefore $PR = 2\sqrt{2}$.

Hence the semicircle with diameter PR has radius $\sqrt{2}$. Hence the area of this semicircle is $\frac{1}{2}\left(\pi(\sqrt{2})^2\right) = \pi$.

The semicircle with QR as diameter has radius 1, and therefore its area is $\frac{1}{2}(\pi 1^2) = \frac{1}{2}\pi$.

It follows that the area of the shaded region in the diagram above is

$$2 + \pi - \tfrac{1}{2}\pi = 2 + \tfrac{1}{2}\pi.$$

This area is half of the shaded area in the logo.

Therefore the shaded area in the logo is $2 \times (2 + \frac{1}{2}\pi) = 4 + \pi$.

For investigation

21.1 Find the length of the perimeter of the shaded area.

22. How many pairs of integers (x, y) satisfy the equation $\sqrt{x - \sqrt{x+23}} = 2\sqrt{2} - y$?

A 0 B 1 C 4 D 8 E infinitely many

Solution **B**

Suppose that (x, y) is a pair of integers that satisfies the equation given in the question. That is, suppose that x and y are integers such that

$$\sqrt{x - \sqrt{x+23}} = 2\sqrt{2} - y. \quad (1)$$

Squaring both sides of equation (1), we obtain

$$x - \sqrt{x+23} = 8 - 4\sqrt{2}y + y^2. \quad (2)$$

We can rewrite equation (2) as

$$\sqrt{x+23} = 4\sqrt{2}y - (8 + y^2 - x). \quad (3)$$

Since x and y are integers, $8 + y^2 - x$ is an integer. For convenience we put $z = 8 + y^2 - x$. Then equation (3) becomes

$$\sqrt{x+23} = 4\sqrt{2}y - z. \quad (4)$$

By squaring both sides of equation (4), we obtain

$$x + 23 = 32y^2 - 8\sqrt{2}yz + z^2. \quad (5)$$

Equation (5) may be rearranged to give

$$8\sqrt{2}yz = 32y^2 + z^2 - x - 23. \quad (6)$$

Now, if $yz \neq 0$, equation (6) implies that

$$\sqrt{2} = \frac{32y^2 + z^2 - x - 23}{8yz}. \quad (7)$$

Because x, y and z are integers, equation (7) implies that $\sqrt{2}$ is rational. Since $\sqrt{2}$ is irrational (see Problem 22.1), we deduce that $yz = 0$. It follows that $y = 0$ or $z = 0$.

Suppose first that $y = 0$. It then follows from equation (2) that

$$x - \sqrt{x+23} = 8. \quad (8)$$

We leave it as an exercise for the reader to show that equation (8) does not have an integer solution (see Problem 22.2).

Now suppose that $z = 0$. Then

$$8 + y^2 - x = 0. \quad (9)$$

It now follows from equations (3) and (9) that

$$\sqrt{x+23} = 4\sqrt{2}y. \quad (10)$$

Squaring both sides of equation (10), we obtain $x + 23 = 32y^2$ and hence

$$x = 32y^2 - 23. \quad (11)$$

We now use equation (11) to substitute $32y^2 - 23$ for x in equation (9). In this way we obtain

$$8 + y^2 - (32y^2 - 23) = 0. \quad (12).$$

Equation (12) may be rearranged to give

$$31(1 - y^2) = 0. \quad (13)$$

Hence, $1 - y^2 = 0$. Therefore, $y = -1$ or $y = 1$. In either case, it follows from the equation (11) that $x = 9$.

It seems that we have found two pairs of integers $(9, -1)$ and $(9, 1)$ that satisfy equation (1). However, we obtained these solutions by squaring both sides of equations (1), (4) and (10).

Squaring both sides of an equation is an operation that runs the risk of introducing spurious solutions. Therefore we need to check whether or not $(9, -1)$ and $(9, 1)$ are solutions of the original equation.

We leave it to the reader (see Problem 22.3) to check that $(9, 1)$ is a solution of equation (1), but $(9, -1)$ is *not* a solution.

It follows that there is just one pair of integers (x, y), namely $(9, 1)$, that satisfies the equation $\sqrt{x - \sqrt{x + 23}} = 2\sqrt{2} - y$.

For investigation

22.1 The solution above uses the fact that $\sqrt{2}$ is not a rational number. This means that there do not exist integers p and q such that

$$\sqrt{2} = \frac{p}{q}.$$

Find a proof of this fact. That is, find a proof in a book or on the internet, or ask your teacher.

22.2 Show that the equation

$$x - \sqrt{x + 23} = 8$$

does not have an integer solution.

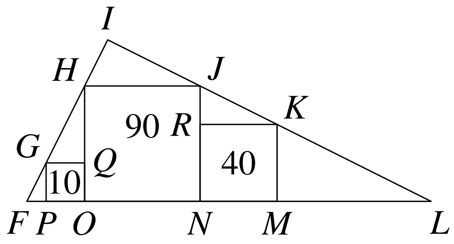
[Hint: rewrite this equation as $x - 8 = \sqrt{x + 23}$. Square both sides of this equation. Then consider the solutions of the quadratic equation that you obtain.]

22.3 Show that $x = 9$, $y = 1$ is a solution of the equation $\sqrt{x - \sqrt{x + 23}} = 2\sqrt{2} - y$, but $x = 9$, $y = -1$ is not a solution.

23. Three squares $GQOP$, $HJNO$ and $RKMN$ have vertices which sit on the sides of triangle FIL as shown. The squares have areas of 10, 90 and 40 respectively.

What is the area of triangle FIL?

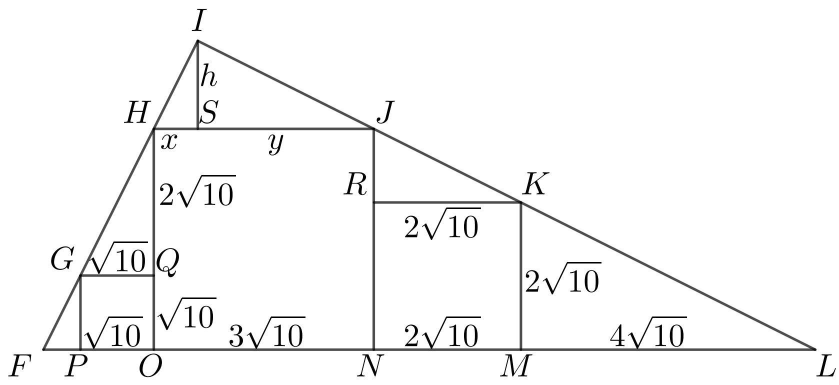
A 220.5 B $\frac{21}{5}\sqrt{10}$ C 252 D $\frac{21}{2}\sqrt{10}$ E 441



Solution **A**

We let S be the foot of the perpendicular from the point I to HJ. We let h, x and y be the lengths of IS, HS and SJ, respectively.

The square $GQOP$ has area 10 and hence side length $\sqrt{10}$.



Similarly, the side length of the square $HJNO$ is $\sqrt{90} = 3\sqrt{10}$ and the side length of the square $RKMN$ is $\sqrt{40} = 2\sqrt{10}$.

We have $GQ = \sqrt{10}$ and $HQ = HO - QO = 3\sqrt{10} - \sqrt{10} = 2\sqrt{10}$. We leave it to the reader to check that the triangles GPF, HQG and ISH are similar. It follows that $FP = \frac{1}{2}\sqrt{10}$ and $h = 2x$.

We also have $JR = JN - RN = 3\sqrt{10} - 2\sqrt{10} = \sqrt{10}$ and $RK = 2\sqrt{10}$. The triangles KML, JRK and ISJ are similar. Therefore $ML = 4\sqrt{10}$ and $h = \frac{1}{2}y$.

Since $h = 2x$ and $h = \frac{1}{2}y$, it follows that $y = 4x$. Also $x + y = 3\sqrt{10}$. Therefore $5x = 3\sqrt{10}$. Hence $x = \frac{3}{5}\sqrt{10}$, $y = \frac{12}{5}\sqrt{10}$ and $h = \frac{6}{5}\sqrt{10}$.

We now have that the base of the triangle FIL is

$$FL = FP + PO + ON + NM + ML = \tfrac{1}{2}\sqrt{10} + \sqrt{10} + 3\sqrt{10} + 2\sqrt{10} + 4\sqrt{10} = \tfrac{21}{2}\sqrt{10}.$$

and the height of the triangle is

$$IS + HO = \tfrac{6}{5}\sqrt{10} + 3\sqrt{10} = \tfrac{21}{5}\sqrt{10}.$$

Hence the area of the triangle FIL is given by

$$\tfrac{1}{2}(\text{base} \times \text{height}) = \tfrac{1}{2}\left(\tfrac{21}{2}\sqrt{10} \times \tfrac{21}{5}\sqrt{10}\right) = \tfrac{1}{2}\left(\tfrac{21^2}{10} \times 10\right) = 220.5.$$

For investigation

23.1 Prove (a) that the triangles GPF, HQG and ISH are similar, and (b) that the triangles KML, JRK and ISJ are similar.

23.2 Show that the side FI of the triangle FIL is perpendicular to the side LI.

24. The numbers x, y, p and q are all integers. x and y are variable and p and q are constant and positive. The four integers are related by the equation $xy = px + qy$.

When y takes its maximum possible value, which expression is equal to $y - x$?

A $pq - 1$ B $(p - 1)(q - 1)$ C $(p + 1)(q - 1)$ D $(p - 1)(q + 1)$ E $(p + 1)(q + 1)$

SOLUTION **D**

Since

$$xy = px + qy,$$

we have

$$xy - qy = px.$$

Therefore

$$(x - q)y = px.$$

Hence

$$y = \frac{px}{x - q} = p + \frac{pq}{x - q}.$$

Since p and q are constant, y attains its maximum value when x has the value that maximises $\frac{pq}{x - q}$. Since $pq > 0$, this is the value of x for which $x - q$ attains its minimum positive value.

Because x and q are integers the minimum positive value of $x - q$ is 1. This occurs for $x = q + 1$. For this value of x, we have $y = p + pq$, which is an integer.

It follows that when y takes its maximum possible value, $y - x = (p + pq) - (q + 1) = p(q + 1) - (q + 1) = (p - 1)(q + 1)$.

FOR INVESTIGATION

24.1 (a) Find all the solutions of the equation

$$xy = 6x + 5y$$

in which x and y are positive integers.

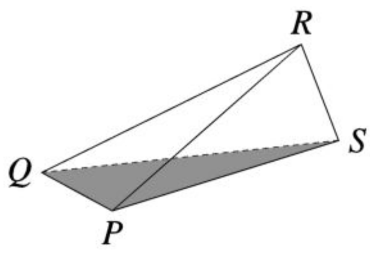
(b) Find the maximum value of y which occurs among these solutions.

(c) Verify that this value of y is $p + pq$ with $p = 6$ and $q = 5$.

25. A drinks carton is formed by arranging four congruent triangles as shown. $QP = RS = 4$ cm and $PR = PS = QR = QS = 10$ cm.

What is the volume, in cm^3, of the carton?

A $\frac{16}{3}\sqrt{23}$ B $\frac{4}{3}\sqrt{2}$ C $\frac{128}{25}\sqrt{6}$
D $\frac{13}{2}\sqrt{23}$ E $\frac{8}{3}\sqrt{6}$



Solution **A**

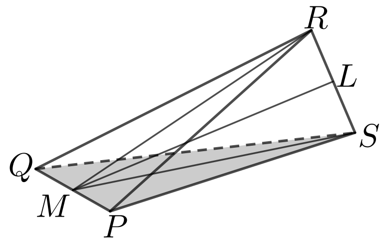
The lengths in this question are given in terms of centimetres. However, for convenience, we will ignore these units in the calculations until we reach the final answer.

The drinks carton is in the shape of a pyramid. We therefore use the fact that the volume of a pyramid is given by the formula

$$\text{volume} = \tfrac{1}{3}(\text{area of base} \times \text{height}).$$

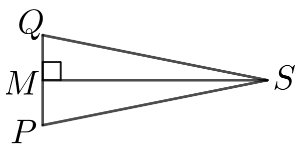
We let M be the midpoint of PQ and let L be the midpoint of RS.

The carton may be thought of as made up of two pyramids, one with the triangle RMS as its base and P as its apex, and the other with RMS as its base and Q as its apex.



Since $PS = QS$, and M is the midpoint of QP, the triangles PMS and QMS are congruent.

It follows that the angles $\angle PMS$ and $\angle QMS$ are equal. Because they are angles on a line, it follows that they are both right angles.



Since $\angle PMS$ is a right angle, PM is the height of the pyramid with base RMS and apex P. Therefore the volume of this pyramid is $\frac{1}{3}(\text{area of } RMS \times PM)$.

Similarly the volume of the pyramid with base RMS and apex Q is $\frac{1}{3}(\text{area of } RMS \times MQ)$.

We let V be the volume of the carton.

We now have that

$$\begin{aligned} V &= \tfrac{1}{3}(\text{area of } RMS \times PM) + \tfrac{1}{3}(\text{area of } RMS \times MQ) \\ &= \tfrac{1}{3}(\text{area of } RMS \times (PM + MQ)) \\ &= \tfrac{1}{3}(\text{area of } RMS \times PQ). \qquad (1) \end{aligned}$$

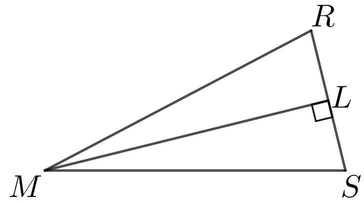
We therefore now calculate the area of the triangle RMS.

Since the triangles PRQ and PSQ are congruent it may be shown that $RM = SM$ (see Problem 25.3).

It follows that the triangles RML and SML are congruent, and hence that ML is at right angles to RS.

Therefore

$$\text{area of } RMS = \tfrac{1}{2}(RS \times ML). \qquad (2)$$



By Pythagoras theorem applied to the right-angled triangle SMP, we have $MS^2 = PS^2 - PM^2 = 10^2 - 2^2 = 100 - 4 = 96$. Therefore, by Pythagoras' Theorem applied to the right-angled triangle SLM, we have $ML^2 = MS^2 - SL^2 = 96 - 2^2 = 92$. Hence $ML = \sqrt{92} = 2\sqrt{23}$.

We can now deduce, by (1) and (2) that

$$\begin{aligned} V &= \tfrac{1}{3}(\tfrac{1}{2}(RS \times ML) \times PQ) \\ &= \tfrac{1}{6}(RS \times ML \times PQ) \\ &= \tfrac{1}{6}(4 \times 2\sqrt{23} \times 4) \\ &= \tfrac{16}{3}\sqrt{23}. \end{aligned}$$

Hence the volume of the carton is $\frac{16}{3}\sqrt{23}\,\text{cm}^3$.

For investigation

25.1 How can it be proved that the volume of a pyramid is given by the formula

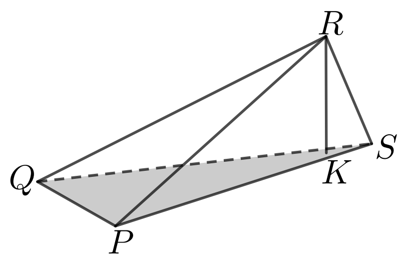
$$\tfrac{1}{3}(\text{area of base} \times \text{height})\,?$$

25.2 Another method for solving this problem is to regard the carton as a pyramid with the triangle PQS as its base and R as its apex. Then the volume of the carton is given

$$\tfrac{1}{3}(\text{area of } PQS \times RK),$$

where K is the foot of the perpendicular from R to the triangle PQS.

Use this method to find the volume of the carton.



25.3 Explain how $RM = SM$ follows from the fact that the triangles PRQ and PSQ are congruent.

UK Maths Trust

Senior Mathematical Challenge

Tuesday 3 October 2023

Organised by the United Kingdom Mathematics Trust

supported by Overleaf

Candidates must be full-time students at secondary school or FE college.
England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**. No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank paper for rough working is allowed; **squared paper, calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil**. Mark at most one of the options, A, B, C, D, or E, on the Answer Sheet for each question. Do not mark more than one option.
5. **Do not expect to finish the whole paper in the time allowed**. The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules**: All candidates start with 25 marks; 0 marks are awarded for each question left unanswered; 4 marks are awarded for each correct answer; 1 mark is deducted for each incorrect answer (to discourage guessing).
7. **Your Answer Sheet will be read by a machine**. Do not write or doodle on the sheet except to mark your chosen options. The machine will read all black pencil markings even if they are in the wrong places. If you mark the sheet in the wrong place, doodle, or leave bits of eraser stuck to the page, the machine will interpret the mark in its own way, or reject the answer sheet.
8. **The questions on this paper are designed to challenge you to think, not to guess**. You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.
9. To accommodate candidates sitting at other times, please do not discuss the paper on the internet until **08:00 BST on Thursday 5 October**.

Enquiries about the Senior Mathematical Challenge should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. What is the value of $\sqrt{\dfrac{2023}{2+0+2+3}}$?

A 13 B 15 C 17 D 19 E 21

2. What is the difference between one-third and 0.333?

A 0 B $\dfrac{3}{1000}$ C $\dfrac{1}{3000}$ D $\dfrac{3}{10000}$ E $\dfrac{1}{30000}$

3. The base of a triangle is increased by 20% and its height is decreased by 15%.

What happens to its area?

A It decreases by 3% B It remains the same C It decreases by 2%
D It increases by 2% E It increases by 5%

4. In 2016, the world record for completing a 5000m three-legged race was 19 minutes and 6 seconds. It was set by Damian Thacker and Luke Symonds in Sheffield.

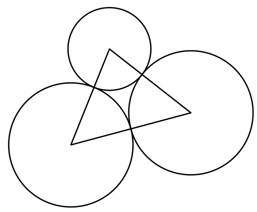
What was their approximate average speed in km/h?

A 10 B 12 C 15 D 18 E 25

5. Three circles with radii 2, 3 and 3 touch each other, as shown in the diagram.

What is the area of the triangle formed by joining the centres of these circles?

A 10 B 12 C 14 D 16 E 18



6. How many lines of three adjacent cells can be chosen from this grid, horizontally, vertically or diagonally, such that the sum of the numbers in the three cells is a multiple of three?

A 30 B 24 C 18 D 12 E 6

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

7. A sequence begins 2023, 2022, 1, ... After the first two terms, each term is the positive difference between the previous two terms.

What is the value of the 25^{th} term?

A 2010 B 2009 C 2008 D 2007 E 2006

8. What is the value of $99(0.\dot{4}\dot{9} - 0.\dot{4})$?

A 5 B 4 C 3 D 2 E 1

9. When completed correctly, the cross number is filled with four three-digit numbers.

What digit is *?

A 0 B 1 C 2
D 4 E 6

Across
1. A square
3. A fourth power

Down
1. Twice a fifth power
2. A cube

1	*	2
	(shaded)	
3		

10. How many of the numbers 6, 7, 8, 9, 10 are factors of the sum $2^{2024} + 2^{2023} + 2^{2022}$?

A 1 B 2 C 3 D 4 E 5

11. Wenlu, Xander, Yasser and Zoe make the following statements:

Wenlu: "Xander is lying." Xander: " Yasser is lying."

Yasser: "Zoe is telling the truth." Zoe: "Wenlu is telling the truth."

What are the possible numbers of people telling the truth?

A 1 or 2 B 1 or 3 C 2 D 2 or 3 E 3

12. The greatest power of 7 which is a factor of 50! is 7^k $(n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n)$.

What is k?

A 4 B 5 C 6 D 7 E 8

13. $PQRST$ is a regular pentagon. The point U lies on ST such that $\angle QPU$ is a right angle.

What is the ratio of the interior angles in triangle PUT?

A 1 : 3 : 6 B 1 : 2 : 4 C 2 : 3 : 4 D 1 : 4 : 8 E 1 : 3 : 5

14. The points P $(d, -d)$ and Q $(12 - d, 2d - 6)$ both lie on the circumference of the same circle whose centre is the origin.

What is the sum of the two possible values of d?

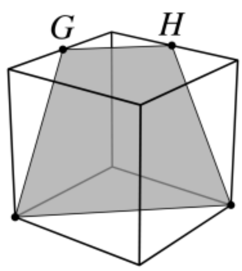
A −16 B −4 C 4 D 8 E 16

15. In Bethany's class of 30 students, twice as many people played basketball as played football. Twice as many played football as played neither.

Which of the following options could have been the number of people who played both?

A 19 B 14 C 9 D 5 E 0

16. G and H are midpoints of two adjacent edges of a cube. A trapezium-shaped cross-section is formed cutting through G, H and two further vertices, as shown. The edge-length of the cube is $2\sqrt{2}$.



What is the area of the trapezium?

A 9 B 8 C $4\sqrt{5}$ D $4\sqrt{3}$ E $4\sqrt{2}$

17. The number $M = 124563987$ is the smallest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number. For example, the 5th and 6th digits of M make the number 63 which is not prime. N is the largest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number.

What are the 5th and 6th digits of N?

A 6 and 3 B 5 and 4 C 5 and 2 D 4 and 8 E 3 and 5

18. How many solutions are there of the equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ with $0° < X < 360°$?

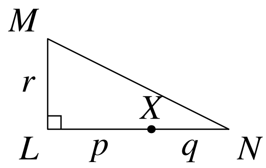
A 1 B 2 C 4 D 6 E 8

19. The expression $\dfrac{7n+12}{2n+3}$ takes integer values for certain integer values of n.

What is the sum of all such integer values of the expression?

A 4 B 8 C 10 D 12 E 14

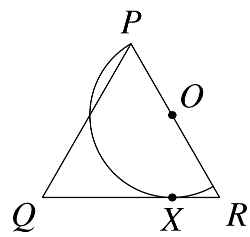
20. Triangle LMN represents a right-angled field with $LM = r$, $LX = p$ and $XN = q$. Jenny and Vicky walk at the same speed in opposite directions along the edge of the field, starting at X at the same time. Their first meeting is at M.



Which of these expressions gives q in terms of p and r?

A $\dfrac{p}{2} + r$ B $\sqrt{p^2 + r^2} + \dfrac{p}{2}$ C $\dfrac{pr}{2p + r}$ D $\dfrac{p}{2}$ E 1

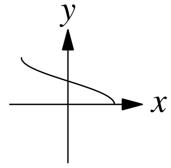
21. Triangle PQR is equilateral. A semicircle with centre O is drawn with its diameter on PR so that one end is at P and the curved edge touches QR at X. The radius of the semicircle is $\sqrt{3}$.

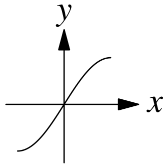


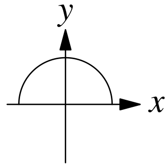
What is the length of QX?

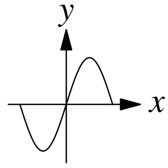
A $\sqrt{3}$ B $2 - \sqrt{3}$ C $2\sqrt{3} - 1$ D $1 + \sqrt{3}$ E $2\sqrt{3}$

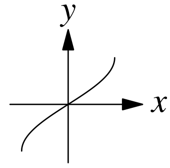
22. Which diagram could be a sketch of the curve $y = \sin(\cos^{-1} x)$?

A


B


C


D


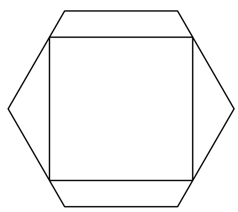
E


23. The length of a rectangular piece of paper is three times its width. The paper is folded so that one vertex lies on top of the opposite vertex, thus forming a pentagonal shape.

What is the area of the pentagon as a fraction of the area of the original rectangle?

A $\dfrac{2}{3}$ B $\dfrac{11}{16}$ C $\dfrac{12}{17}$ D $\dfrac{13}{18}$ E $\dfrac{14}{19}$

24. A square has its vertices on the edges of a regular hexagon. Two of the edges of the square are parallel to two edges of the hexagon, as shown in the diagram. The sides of the hexagon have length 1.



What is the length of the sides of the square?

A $\dfrac{5}{4}$ B $3 - \sqrt{3}$ C $\dfrac{4}{3}$ D $\sqrt{2}$ E $\dfrac{3}{2}$

25. What is the area of the part of the xy-plane within which $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ and $0 \leq x \leq y$?

A $\dfrac{1}{4}$ B $\dfrac{1}{2}$ C 1 D 2 E 4

UK Maths Trust

Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

supported by Overleaf

Solutions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	C	D	C	B	B	D	A	D	B	B	E	A	E	D	A	E	C	E	C	D	C	D	B	A

1. What is the value of $\sqrt{\dfrac{2023}{2+0+2+3}}$?

A 13 B 15 C 17 D 19 E 21

SOLUTION **C**

The prime factorisation of 2023 is $7 \times 17 \times 17$ so $\sqrt{\dfrac{2023}{2+0+2+3}} = \sqrt{\dfrac{2023}{7}} = \sqrt{17^2} = 17$.

2. What is the difference between one-third and 0.333?

A 0 B $\dfrac{3}{1000}$ C $\dfrac{1}{3000}$ D $\dfrac{3}{10000}$ E $\dfrac{1}{30000}$

SOLUTION **C**

The difference between one third and 0.333 is $\dfrac{1}{3} - \dfrac{333}{1000} = \dfrac{1000-999}{3000} = \dfrac{1}{3000}$.

3. The base of a triangle is increased by 20% and its height is decreased by 15%.

What happens to its area?

A It decreases by 3% B It remains the same C It decreases by 2%
D It increases by 2% E It increases by 5%

SOLUTION **D**

The new area = the old area $\times 1.2 \times 0.85$ = the old area $\times 1.02$. This represents a 2% increase.

4. In 2016, the world record for completing a 5000m three-legged race was 19 minutes and 6 seconds. It was set by Damian Thacker and Luke Symonds in Sheffield.

What was their approximate average speed in km/h?

A 10 B 12 C 15 D 18 E 25

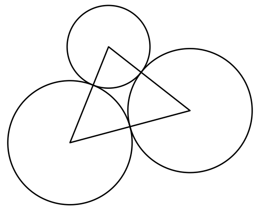
SOLUTION **C**

The world record of 5000 m in 19 minutes and 6 seconds $\simeq$ 5000 m in 20 minutes = 15000 m in 60 minutes = 15000 m in an hour = 15 km/h.

5. Three circles with radii 2, 3 and 3 touch each other, as shown in the diagram.

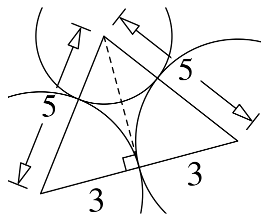
What is the area of the triangle formed by joining the centres of these circles?

A 10 B 12 C 14 D 16 E 18



SOLUTION **B**

The triangle formed by joining the centres of the circles is isosceles, so splitting it along its line of symmetry gives us two right-angled triangles each with a base of 3 and a hypotenuse of 5. Using Pythagoras' Theorem the perpendicular height is 4. The area of the whole triangle is then $\frac{1}{2} \times 6 \times 4 = 12$.



6. How many lines of three adjacent cells can be chosen from this grid, horizontally, vertically or diagonally, such that the sum of the numbers in the three cells is a multiple of three?

A 30 B 24 C 18 D 12 E 6

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

SOLUTION **B**

The sum of any three integers in arithmetic progression is a multiple of 3. For proof of this, if we let the smallest integer be a and the common difference of the sequence be d, then $a + (a + d) + (a + 2d) = 3a + 3d = 3(a + d)$. As a result of the way the grid is filled, all the horizontal, vertical and diagonal lines contain numbers which are in arithmetic progression. Horizontally there are 2 lines of three cells in each of the 4 rows. Here $d = 1$. Vertically, there are again 2 lines in each of the 4 columns. Here $d = 4$. On the diagonals with positive gradient, there are 4 lines, with $d = -3$. On the diagonals with negative gradient there are four lines with $d = 5$. This is a total of $8 + 8 + 4 + 4 = 24$ lines.

7. A sequence begins 2023, 2022, 1, ... After the first two terms, each term is the positive difference between the previous two terms.

What is the value of the 25th term?

A 2010 B 2009 C 2008 D 2007 E 2006

SOLUTION **D**

The sequence begins 2023, 2022, 1, 2021, 2020, 1, 2019, 2018, 1 ... . Let the k^{th} term be u_k. Now consider the sequence $u_1, u_4, u_7, \ldots$, which starts 2023, 2021, 2019, ... Here the terms decrease by two each time. Since $25 = 1 + 8 \times 3$, $u_{25} = u_1 - 8 \times 2 = 2023 - 16 = 2007$.

8. What is the value of $99(0.\dot{4}\dot{9} - 0.\dot{4})$?

A 5 B 4 C 3 D 2 E 1

Solution **A**

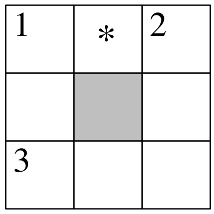
The value of $99(0.\dot{4}\dot{9} - 0.\dot{4}) = 99\left(\frac{49}{99} - \frac{4}{9}\right) = 99\left(\frac{49}{99} - \frac{44}{99}\right) = 99\left(\frac{49-44}{99}\right) = 99 \times \frac{5}{99} = 5$.

9. When completed correctly, the cross number is filled with four three-digit numbers.

What digit is *?

Across
1. A square
3. A fourth power

Down
1. Twice a fifth power
2. A cube



A 0 B 1
C 2 D 4
E 6

Solution **D**

For 1 Down, $2 \times 2^5 = 64$ is too small and $2 \times 4^5 = 2048$ is too big and therefore we must have $2 \times 3^5 = 486$. 3 Across must then start with a 6 and is therefore $5^4 = 625$. 2 Down must then end in a 5 and is therefore $5^3 = 125$. 1 Across is then 4 * 1. The only square of this form is $21^2 = 441$, so * is a 4.

10. How many of the numbers 6, 7, 8, 9, 10 are factors of the sum $2^{2024} + 2^{2023} + 2^{2022}$?

A 1 B 2 C 3 D 4 E 5

Solution **B**

The sum $2^{2024} + 2^{2023} + 2^{2022}$ can be factorised to $2^{2022}(2^2 + 2^1 + 1) = 2^{2022} \times 7$. Hence, of the numbers listed, only 7 and $8 = 2^3$ are factors of 2^{2022}.

11. Wenlu, Xander, Yasser and Zoe make the following statements:

Wenlu: "Xander is lying." Xander: " Yasser is lying."

Yasser: "Zoe is telling the truth." Zoe: "Wenlu is telling the truth."

What are the possible numbers of people telling the truth?

A 1 or 2 B 1 or 3 C 2 D 2 or 3 E 3

SOLUTION **B**

Each of the four people is either telling the truth or lying. Assume first that Wenlu is telling the truth, then Xander is lying, which implies that Yasser is telling the truth which finally implies that Zoe is also telling the truth. In this case 3 people tell the truth. Now assume that Wenlu is lying. Therefore Xander is telling the truth that Yasser is lying and finally Zoe is also lying. In this case only 1 person tells the truth. In both cases, all four statements are consistent with each other.

12. The greatest power of 7 which is a factor of 50! is 7^k $(n! = 1\times2\times3\times4\times\ldots\times(n-1)\times n.)$

What is k?

A 4 B 5 C 6 D 7 E 8

SOLUTION **E**

As $n! = 1\times2\times3\times4\times\ldots\times(n-1)\times n$, factors of 50! which contain a factor of 7 are 7, 14, 21, 28, 35, 42 and 49. The first six of these each contribute a single factor of 7 and 49 contributes two. The greatest power of 7 which is a factor of 50! is then 7^8, so $k = 8$.

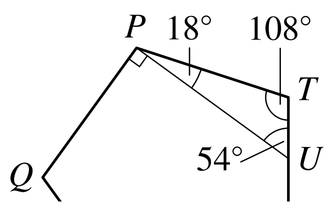
13. $PQRST$ is a regular pentagon. The point U lies on ST such that $\angle QPU$ is a right angle.

What is the ratio of the interior angles in triangle PUT?

A 1 : 3 : 6 B 1 : 2 : 4 C 2 : 3 : 4 D 1 : 4 : 8 E 1 : 3 : 5

SOLUTION **A**

The interior angles of a regular pentagon are $180° - \frac{360°}{5} = 108°$. As $\angle QPU$ is a right angle, $\angle UPT = 108° - 90° = 18°$. As angles in a triangle sum to 180°, $\angle PUT = 180° - (108° + 18°) = 54°$. Therefore $\angle TPU : \angle PUT : \angle UTP = 18 : 54 : 108 = 1 : 3 : 6$.



14. The points P $(d, -d)$ and Q $(12 - d, 2d - 6)$ both lie on the circumference of the same circle whose centre is the origin.

What is the sum of the two possible values of d?

A -16 B -4 C 4 D 8 E 16

Solution **E**

The equation of the circle is $x^2 + y^2 = r^2$. At Q, $(12 - d)^2 + (2d - 6)^2 = r^2$. At P, $d^2 + (-d)^2 = r^2$, so $2d^2 = r^2$. Expanding the first equation and subtracting the second gives $144 - 24d + d^2 + 4d^2 - 24d + 36 - 2d^2 = 0$, which simplifies to $3d^2 - 48d + 180 = 0$. Dividing by 3 and factorising gives $(d - 6)(d - 10) = 0$. Therefore $d = 6$ or $d = 10$ and the sum of these values is 16.

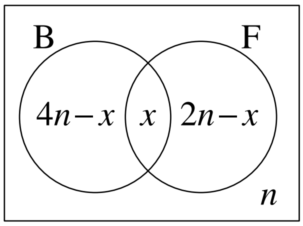
15. In Bethany's class of 30 students, twice as many people played basketball as played football. Twice as many played football as played neither.

Which of the following options could have been the number of people who played both?

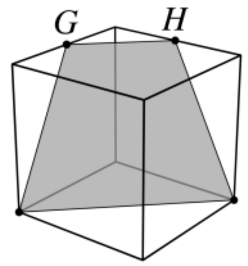
A 19 B 14 C 9 D 5 E 0

Solution **D**

Let the number of people who play both basketball and football be x and the number who play neither be n. A Venn diagram can then be filled as shown. As there are 30 students, $7n - x = 30$. As $x \geq 0$, $7n - 30 \geq 0$ and so $n \geq 5$. From the Venn diagram it can be seen that $2n - x \geq 0$, therefore $2n - (7n - 30) \geq 0$ so $n \leq 6$. So $n = 5$ or 6 and the corresponding values of x are 5 or 12. The only one of these in the listed options is $x = 5$.



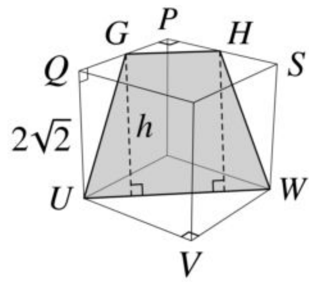
16. G and H are midpoints of two adjacent edges of a cube. A trapezium-shaped cross-section is formed cutting through G, H and two further vertices, as shown. The edge-length of the cube is $2\sqrt{2}$.



What is the area of the trapezium?

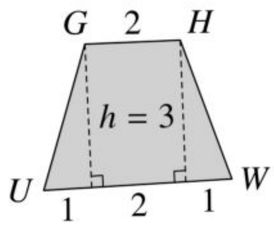
A 9 B 8 C $4\sqrt{5}$ D $4\sqrt{3}$ E $4\sqrt{2}$

SOLUTION **A**



To find the area of the trapezium, we require lengths of GH, UW and the perpendicular distance between them, h, say. In triangle PGH, $PG = PH = \sqrt{2}$ therefore $GH = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = 2$.

Triangle VUW is an enlargement of triangle PGH with scale factor 2, so $UW = 4$. In order to find h we must first find length GU. In triangle QUG, $UG = \sqrt{(2\sqrt{2})^2 + \sqrt{2}^2} = \sqrt{10}$.



From the triangular end of the trapezium, it follows that $1^2 + h^2 = \sqrt{10}^2$ therefore $h = 3$. The area of the trapezium $= \frac{1}{2} \times (4+2) \times 3 = 9$.

17. The number $M = 124563987$ is the smallest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number. For example, the 5th and 6th digits of M make the number 63 which is not prime. N is the largest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number.

What are the 5th and 6th digits of N?

A 6 and 3 B 5 and 4 C 5 and 2 D 4 and 8 E 3 and 5

SOLUTION **E**

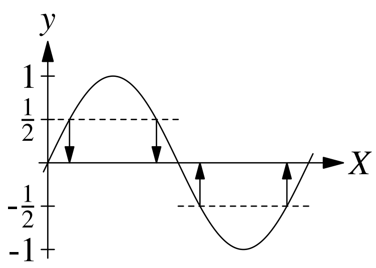
In order to get the largest number, N, we need to make its earlier digits as large as possible, starting 9876 . . . as far as this works. However, since 53, 43, 23 and 13 are all prime, the digit 3 must precede all of 5, 4, 2 and 1. So the latest 3 can come is immediately after 6. Thereafter there are no reasons not to follow numerical order, making $N = 987635421$. Its 5^{th} and 6^{th} digits are 3 and 5.

18. How many solutions are there of the equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ with $0° < X < 360°$?

A 1 B 2 C 4 D 6 E 8

Solution **C**

The equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ factorises to give $(1 + 2\sin X) - 4\sin^2 X(1 + 2\sin X) = 0$ and then to $(1 + 2\sin X)(1 - 4\sin^2 X) = 0$. Fully factorised, we have $(1 + 2\sin X)(1 + 2\sin X)(1 - 2\sin X) = 0$. So $\sin X = -\frac{1}{2}$ or $\sin X = \frac{1}{2}$. For $0° < X < 360°$, there are then four solutions as shown in the diagram.



19. The expression $\dfrac{7n + 12}{2n + 3}$ takes integer values for certain integer values of n.

What is the sum of all such integer values of the expression?

A 4 B 8 C 10 D 12 E 14

Solution **E**

The expression $\dfrac{7n + 12}{2n + 3} \equiv \dfrac{4(2n + 3)}{2n + 3} - \dfrac{n}{2n + 3} \equiv 4 - \dfrac{n}{2n + 3}$. The first expression takes integer values precisely when $\dfrac{n}{2n + 3}$ is an integer.

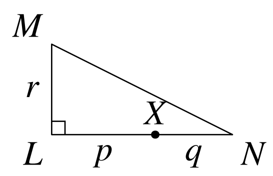
Consider first $n > 0$. When $n > 0$, $2n + 3 > n$, therefore $\dfrac{n}{2n + 3} < 1$ so no integer values of the expression are possible.

Next, consider $n = 0$. In this case, $\dfrac{n}{2n + 3} = \dfrac{0}{0 + 3} = 0$ which is an integer.

When $n < 0$, in order to form an integer, we require $n \leq 2n + 3$, therefore $n \geq -3$.

Possible values of n are then $n = -1$, -2 and -3. The values of $\dfrac{n}{2n + 3}$ in these cases are $\dfrac{-1}{2 \times (-1) + 3} = -1$, $\dfrac{-2}{2 \times (-2) + 3} = 2$ and $\dfrac{-3}{2 \times (-3) + 3} = 1$. Therefore the sum of the integer values of the initial expression is $(4 - 0) + (4 - (-1)) + (4 - 2) + (4 - 1) = 14$.

20. Triangle LMN represents a right-angled field with $LM = r$, $LX = p$ and $XN = q$. Jenny and Vicky walk at the same speed in opposite directions along the edge of the field, starting at X at the same time. Their first meeting is at M.



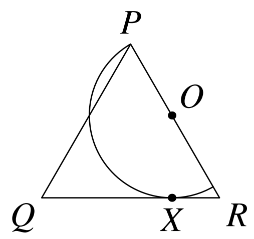
Which of these expressions gives q in terms of p and r?

A $\frac{p}{2} + r$ B $\sqrt{p^2 + r^2} + \frac{p}{2}$ C $\frac{pr}{2p + r}$ D $\frac{p}{2}$ E 1

Solution **C**

Using Pythagoras' Theorem, $NM = \sqrt{(p+q)^2 + r^2}$ so the two journeys have lengths $p + r$ and $q + \sqrt{(p+q)^2 + r^2}$. Equating and rearranging, $p + r - q = \sqrt{(p+q)^2 + r^2}$ and so $(p+r-q)^2 = (p+q)^2 + r^2$. Expanding leads to $p^2 + 2pr - 2pq + r^2 - 2qr + q^2 = p^2 + 2pq + q^2 + r^2$ and therefore $2pr - 2qr = 4pq$. Rearranging to give q in terms of p and r, $pr = q(2p + r)$ so $q = \frac{pr}{2p + r}$.

21. Triangle PQR is equilateral. A semicircle with centre O is drawn with its diameter on PR so that one end is at P and the curved edge touches QR at X. The radius of the semicircle is $\sqrt{3}$.

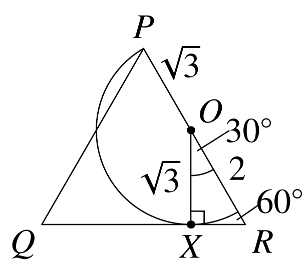


What is the length of QX?

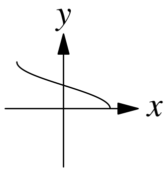
A $\sqrt{3}$ B $2 - \sqrt{3}$ C $2\sqrt{3} - 1$ D $1 + \sqrt{3}$ E $2\sqrt{3}$

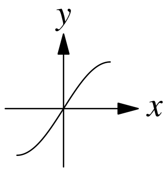
Solution **D**

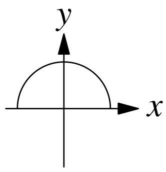
As the semicircle touches QR at X, the radius OX and tangent QR are perpendicular as shown. Triangle OXR is a $30°$, $90°$, $60°$ triangle and OX is given as $\sqrt{3}$. Therefore $XR = 1$ and $OR = 2$. As OP is also a radius of the circle, $OP = \sqrt{3}$ and $PR = QR = 2 + \sqrt{3}$. The length $QX = (2 + \sqrt{3}) - 1 = 1 + \sqrt{3}$.

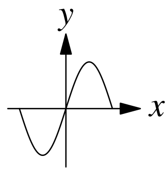


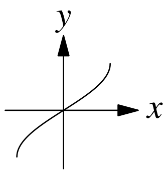
22. Which diagram could be a sketch of the curve $y = \sin(\cos^{-1} x)$?

A


B


C


D


E


Solution **C**

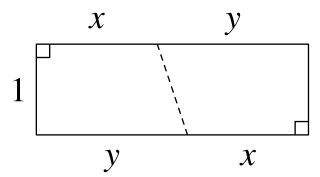
Let $z = (\cos^{-1} x)$. Then $x = \cos z$ and $y = \sin z$ and therefore $x^2 + y^2 = 1$. As z lies between 0° and 180°, x lies between −1 and 1 and y lies between 0 and 1. Hence we get the upper semicircle shown on the graph in option C.

23. The length of a rectangular piece of paper is three times its width. The paper is folded so that one vertex lies on top of the opposite vertex, thus forming a pentagonal shape.

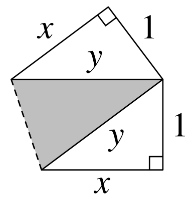
What is the area of the pentagon as a fraction of the area of the original rectangle?

A $\frac{2}{3}$ B $\frac{11}{16}$ C $\frac{12}{17}$ D $\frac{13}{18}$ E $\frac{14}{19}$

Solution **D**



In the first diagram shown, the paper is to be folded so that the bottom left vertex will lie on top of the top right vertex in order to form the desired pentagon. The fold line, shown dotted, must therefore lie on the perpendicular bisector of the line joining the bottom left and top right vertices and so pass through the centre of the rectangle.



Labelling the longest sides of the rectangle with x and y,

$$x + y = 3. \qquad (1)$$

From the folded diagram, we have two right-angled triangles and in each, $1 + x^2 = y^2$. Rearranging and factorising gives

$$1 = (y + x)(y - x). \qquad (2)$$

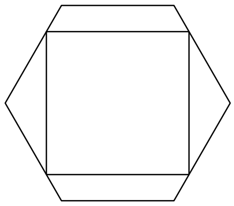
Substituting (1) into (2) gives $1 = 3(y - x)$ and so

$$\tfrac{1}{3} = y - x. \qquad (3)$$

Solving (1) and (3) leads to $y = \frac{5}{3}$ and $x = \frac{4}{3}$. The area of the pentagon $= 1 \times 3 -$ the shaded area. As the shaded area can be viewed as a triangle with base y and therefore perpendicular height 1, the area of the pentagon $= 3 - \frac{1}{2} \times y \times 1 = 3 - \frac{1}{2} \times \frac{5}{3} \times 1 = \frac{13}{6}$.

The area of the pentagon as a fraction of the area of the original rectangle is $\dfrac{\frac{13}{6}}{3} = \dfrac{13}{18}$.

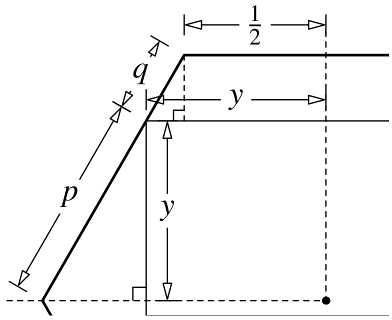
24. A square has its vertices on the edges of a regular hexagon. Two of the edges of the square are parallel to two edges of the hexagon, as shown in the diagram. The sides of the hexagon have length 1.



What is the length of the sides of the square?

A $\frac{5}{4}$ B $3-\sqrt{3}$ C $\frac{4}{3}$ D $\sqrt{2}$ E $\frac{3}{2}$

Solution **B**



Let the square have side-length $2y$.
The two triangles shown, with hypotenuses p and q, have angles 30°, 60° and 90°. As the hexagon has side-length 1,

$$p + q = 1. \qquad (1)$$

From the larger triangle and from the top left of the square,

$$y = \frac{\sqrt{3}p}{2} \quad \text{and} \quad y = \tfrac{1}{2}q + \tfrac{1}{2}. \qquad (2)$$

Equating the two equations in (2) and rearranging gives $\sqrt{3}p - q = 1.$ (3)

Solving (1) and (3) simultaneously gives $(\sqrt{3}+1)p = 2.$

Rearranging and rationalising leads to

$$p = \frac{2}{(\sqrt{3}+1)}\frac{(\sqrt{3}-1)}{(\sqrt{3}-1)} = \sqrt{3} - 1.$$

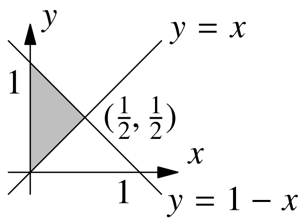
Therefore, the length of the side of the square $2y = \sqrt{3}(\sqrt{3}-1) = 3 - \sqrt{3}$.

25. What is the area of the part of the xy-plane within which $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ and $0 \leq x \leq y$?

A $\frac{1}{4}$ B $\frac{1}{2}$ C 1 D 2 E 4

SOLUTION **A**

Factorising $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ gives $xy^2(x^2 - x - y^2 + y) \geq 0$. Rearranging to $xy^2(y - x - (y^2 - x^2)) \geq 0$ and then factorising gives $xy^2(y - x)(1 - y - x) \geq 0$. As $0 \leq x \leq y$, we know that $x \geq 0$, $y^2 \geq 0$ and $(y - x) \geq 0$ so the fourth factor, $(1 - y - x) \geq 0$. This rearranges to $y \leq 1 - x$. The lines $y = x$ and $y = 1 - x$ meet at $(\frac{1}{2}, \frac{1}{2})$ so the shaded region has area $\frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{4}$.



United Kingdom Mathematics Trust

SENIOR MATHEMATICAL CHALLENGE

Organised by the United Kingdom Mathematics Trust

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SOLUTIONS AND INVESTIGATIONS

October 3rd, 2023

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, challenges@ukmt.org.uk

www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	C	D	C	B	B	D	A	D	B	B	E	A	E	D	A	E	C	E	C	D	C	D	B	A

1. What is the value of $\sqrt{\dfrac{2023}{2+0+2+3}}$?

A 13 B 15 C 17 D 19 E 21

Solution **C**

$2+0+2+3=7$ and $2023 \div 7 = 289 = 17^2$. Therefore

$$\sqrt{\frac{2023}{2+0+2+3}} = \sqrt{17^2} = 17.$$

For investigation

1.1 Which is the smallest integer n with $n > 2023$ which has the property that n is divisible by the sum of its digits?

1.2 Note that since $2023 \div 7 = 17^2$, it follows that $2023 = 7 \times 17^2$ and that 7 and 17 are primes.

Which is the smallest integer n with $n > 2023$ which can be expressed as $p \times q^2$, where p and q are different primes?

1.3 Which is the smallest integer n with $n > 2023$ which can be expressed as $p \times q^2$, where p and q are different primes, and p is the sum of the digits of n?

2. What is the difference between one-third and 0.333?

A 0 B $\dfrac{3}{1000}$ C $\dfrac{1}{3000}$ D $\dfrac{3}{10000}$ E $\dfrac{1}{30000}$

Solution **C**

Expressed as fractions, one-third is $\dfrac{1}{3}$ and 0.333 is $\dfrac{333}{1000}$. Therefore, the difference between one-third and 0.333 is

$$\frac{1}{3} - \frac{333}{1000} = \frac{1000}{3000} - \frac{999}{3000} = \frac{1}{3000}.$$

For investigation

2.1 (a) What is the difference between one-third and 0.3333 ?

(b) What is the least positive k such that the difference between one-third and $0.\overbrace{33\ldots33}^{k}$ is less than 10^{-6} ?

2.2 What is the difference between $\dfrac{22}{7}$ and 3.141 ?

2.3 What is the difference between 1 and $0.\dot{9}$?

3. The base of a triangle is increased by 20% and its height is decreased by 15%.

What happens to its area?

A It decreases by 3% B It remains the same C It decreases by 2%
D It increases by 2% E It increases by 5%

Solution **D**

Suppose the original triangle has base b and height h. The area of this triangle is X, where $X = \frac{1}{2}(b \times h)$.

When the base of the triangle is increased by 20%, its base becomes $b' = \frac{6}{5}b$. When its height is decreased by 15%, its height becomes $h' = \frac{17}{20}h$.

Therefore the area of the changed triangle is X', where $X' = \frac{1}{2}(b' \times h') = \frac{1}{2}(\frac{6}{5}b \times \frac{17}{20}h) = (\frac{6}{5} \times \frac{17}{20})(\frac{1}{2}(b \times h)) = \frac{102}{100}X$.

Therefore the effect of the changes is to increase the area of the triangle by 2%.

For investigation

3.1 The base of a triangle is decreased by 15% and its height is increased by 20%. What happens to its area?

3.2 The base of a triangle is increased by 20% and its height is decreased by 20%. What happens to its area?

3.3 The base of a triangle is increased by 20%. By what percentage should its height be decreased to keep the area unchanged?

4. In 2016, the world record for completing a 5000m three-legged race was 19 minutes and 6 seconds. It was set by Damian Thacker and Luke Symonds in Sheffield.

What was their approximate average speed in km/h?

A 10 B 12 C 15 D 18 E 25

Solution **C**

The average speed running 5000 m in 19 minutes and 6 seconds is approximately the same as running this distance in 20 minutes. So their average speed was approximately $\dfrac{5000}{20}$ metres per minute, that is 250 metres per minute.

A speed of 250 metres per minute is the same as 60×250 metres per hour, that is, 15 000 metres per hour. This is the same as 15 km/h.

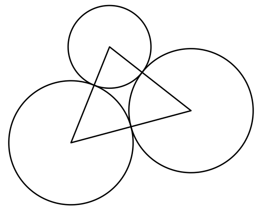
For investigation

4.1 What is the approximation 15 km/h as a percentage of the actual average speed of Damian Thacker and Luke Symonds?

5. Three circles with radii 2, 3 and 3 touch each other, as shown in the diagram.

What is the area of the triangle formed by joining the centres of these circles?

A 10 B 12 C 14 D 16 E 18

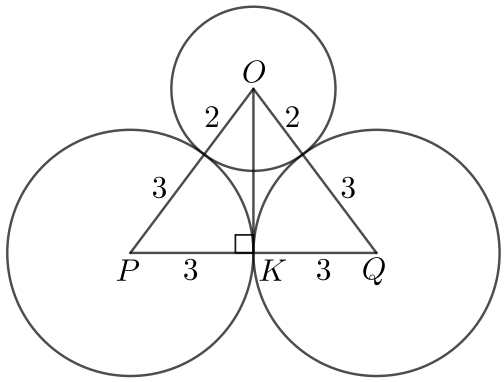


Solution **B**

We let O be the centre of the circle with radius 2, and let P and Q be the centres of the circles with radius 3.

The line joining the centres of touching circles goes through the point where the circles touch. [You are asked to prove this in Problem 5.2.] It follows that both OP and OQ have length $2 + 3 = 5$, and PQ has length $3 + 3 = 6$.

Let K be the midpoint of PQ.



The triangles OPK and OQK are congruent (SSS), and therefore the angles $\angle OKP$ and $\angle OKQ$ are equal, and therefore they are both 90°.

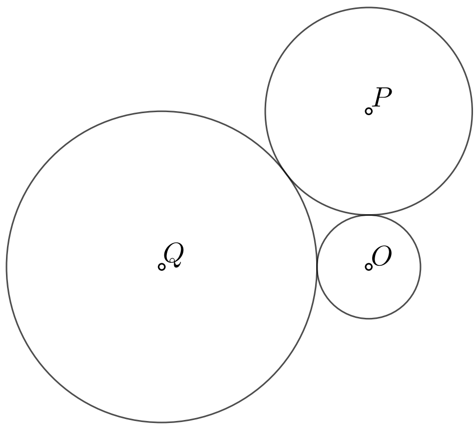
By Pythagoras' Theorem, applied to the triangle OKP, we have $OK^2 = OP^2 - PK^2 = 5^2 - 3^2 = 25 - 9 = 16$. Therefore $OK = 4$.

The triangle OPQ has base PQ of length 6, and height OK of length 4. Therefore the area of this triangle is $\frac{1}{2}(6 \times 4) = 12$.

For investigation

5.1 Three circles with centres O, P and Q with radii 1, 2 and 3, respectively, touch each other as shown.

What is the area of the triangle OPQ ?



5.2 Prove that the line joining the centres of touching circles goes through the point where the circles touch.

6. How many lines of three adjacent cells can be chosen from this grid, horizontally, vertically or diagonally, such that the sum of the numbers in the three cells is a multiple of three?

A 30 B 24 C 18 D 12 E 6

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Solution **B**

Suppose first that in a line of three numbers, adjacent numbers have the same difference. Let n be the first of these numbers, and d be the common difference. Then these numbers are n, $n + d$ and $n + 2d$. The sum of these numbers is $3n + 3d$ and therefore is a multiple of 3.

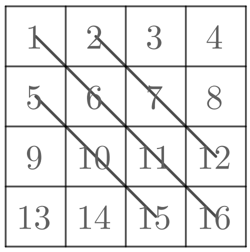
Adjacent numbers in each row have a common difference 1. Therefore, the sum of the numbers in three adjacent cells in the same row is always a multiple of 3.

There are two lines of three adjacent cells in each row, for example 1,2,3 and 2,3,4 in the top row.

Therefore, in the 4 rows there are $4 \times 2 = 8$ lines of three adjacent cells such that the sum of the numbers in these cells is a multiple of 3.

Adjacent numbers in each column differ by 4. Hence, it follows similarly, that there are 8 lines of three adjacent cells in the same column such that the sum of the numbers in these cells is a multiple of 3.

We now consider the diagonals from top left to bottom right. Adjacent numbers in these diagonals each column differ by 5. Therefore the sum of numbers in three adjacent cells in the same diagonal is always a multiple of 3.



One of these diagonals contains 4 numbers. There are 2 lines of three adjacent cells in this diagonal whose sum is a multiple of 3. There are 2 of these diagonals containing three numbers. Therefore, altogether, there are 4 lines of three adjacent numbers on theses diagonals whose sum is a multiple of 3.

Similarly, on the diagonals from top right to bottom left adjacent numbers have a common difference 3, and therefore there are 4 lines of three adjacent numbers on these diagonals whose sum is a multiple of 3.

Hence, in total, there are $8 + 8 + 4 + 4 = 24$ lines of three adjacent cells whose sum is a multiple of 3.

For investigation

6.1 How many lines of three adjacent cells can be chosen from the grid, horizontally, vertically or diagonally, such that the sum of the numbers in the three cells is a multiple of six?

6.2 How many lines of four adjacent cells can be chosen from the grid, horizontally, vertically or diagonally, such that the sum of the numbers in the four cells is a multiple of four?

6.3 In how many ways can three different cells in any position, be chosen from the grid such that sum of the numbers in the three cells is a multiple of three?

7. A sequence begins 2023, 2022, 1, ... After the first two terms, each term is the positive difference between the previous two terms.

What is the value of the 25th term?

A 2010 B 2009 C 2008 D 2007 E 2006

SOLUTION **D**

The sequence begins

$$2023,\ 2022,\ 1,\ 2021,\ 2020,\ 1,\ 2019,\ 2018,\ 1,\ \ldots .$$

From this it seems that, in general, for each non-negative integer k, the terms in positions $3k + 1$, $3k + 2$ and $3k + 3$ are $2023 - 2k$, $2023 - 2k - 1$ and 1. [In fact, this holds only provided $2023 - 2k - 1 \geq 0$, that is, only for $k \leq 1011$. See Problems 7.1 and 7.4.]

Now $25 = 3\times8+1$. Therefore, by putting $k = 8$, we deduce that the 25 th term is $2023-2\times8 = 2007$.

FOR INVESTIGATION

7.1 (a) What are the 3034 th, 3045 th and 3046 th terms of the sequence of this question?

(b) What is the 5000 th term of the sequence of this question?

7.2 A sequence begins

$$2023,\ 2021,\ 2, \ldots$$

After the first two terms, each term is the positive difference between the previous two terms.

What is the 25 th term of the sequence?

7.3 A sequence begins 2023, s, $2023 - s$, ... After the first two terms, each term is the positive difference between the previous two terms.

Which is the positive integer s for which the 25 th term of this sequence is 199?

7.4 We let u_n be the n th term of the sequence of this question.

If you have met the method of *Proof by Mathematical Induction,* use this method to prove that for each non-negative integer k,

$$u_{3k+1} = \begin{cases} 2023 - 2k, & \text{if } k \leq 1011, \\ 1, & \text{otherwise,} \end{cases}$$

$$u_{3k+2} = \begin{cases} 2023 - 2k - 1, & \text{if } k \leq 1011, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$u_{3k+3} = 1.$$

8. What is the value of $99(0.\dot{4}\dot{9} - 0.\dot{4})$?

A 5 B 4 C 3 D 2 E 1

SOLUTION **A**

The standard method for converting a recurrent decimal to a fraction shows that

$$0.\dot{4}\dot{9} = \frac{49}{99} \quad \text{and} \quad 0.\dot{4} = \frac{4}{9}.$$

[Problem 8.1 asks you to check this.]

Therefore

$$\begin{aligned} 99(0.\dot{4}\dot{9} - 0.\dot{4}) &= 99\left(\frac{49}{99} - \frac{4}{9}\right) \\ &= 99\left(\frac{49}{99} - \frac{44}{99}\right) \\ &= 99\left(\frac{5}{99}\right) \\ &= 5. \end{aligned}$$

FOR INVESTIGATION

8.1 Show that $0.\dot{4} = \frac{4}{9}$ and $0.\dot{4}\dot{9} = \frac{44}{99}$.

8.2 Express the recurring decimal $0.\dot{2}3\dot{4}$ as a fraction in its lowest terms.

8.3 Write the solution to the equation

$$x + 0.0\dot{7} = 0.\dot{1}\dot{3}$$

as a recurring decimal.

8.4 Prove that every recurring decimal may be expressed in the form $\frac{p}{q}$ where p and q are integers, with $q > 0$.

9.

Across	**Down**
1. A square	1. Twice a fifth power
3. A fourth power	2. A cube

1	*	2
3		

When completed correctly, the cross number is filled with four three-digit numbers.

What digit is *?

A 0 B 1 C 2 D 4 E 6

Solution **D**

When you are faced with a crossnumber, the best strategy is to look for clues where it is easy to find a unique solution.

Among the three-digit integers there are more squares and cubes than fourth and fifth powers. So the best strategy is to begin with 1 Down and 3 Across.

The first few numbers that are twice fifth powers are $2 \times 1^5 = 2 \times 1 = 2$, $2\times2^5 = 2\times32 = 64$, $2\times3^5 = 2\times243 = 486$ and $2\times4^5 = 2\times1024 = 2048$. We can deduce from this that 486 which is the only three-digit number in this list is the answer for 1 Down.

1 4	4	2 1
8		2
3 6	2	5

It follows that 3 Across is a three-digit fourth power with 6 as its hundreds digit. We have $3^4 = 81$, $4^4 = 256$, $5^4 = 625$ and $6^4 = 1296$. We deduce that 3 Across is 625.

We now see that 2 Down is three-digit cube with units digit 5. Hence 2 Down is is $5^3 = 125$.

Finally, 1 Across is a three-digit square with hundreds digit 4 and units digit 1. Therefore 1 Across is $21^2 = 441$.

We can now deduce that * is 4.

For investigation

9.1 Complete this crossnumber in such a way that no two clues have the same answer.

Across	**Down**
1. $3 \times$ a cube	1. $3 \times$ a square
3. $3 \times$ a square	2. $3 \times$ a square

1		2
3		

10. How many of the numbers 6, 7, 8, 9, 10 are factors of the sum $2^{2024} + 2^{2023} + 2^{2022}$?

A 1 B 2 C 3 D 4 E 5

SOLUTION **B**

For convenience, we put $S = 2^{2024} + 2^{2023} + 2^{2022}$.

We have

$$S = 2^{2024} + 2^{2023} + 2^{2022} = 2^{2022}(2^2 + 2^1 + 1) = 2^{2022}(4 + 2 + 1) = 2^{2022} \times 7.$$

It follows that 7 is a factor of S. Also, since $8 = 2^3$, 8 is a factor of 2^{2022} and hence it is a factor of S. On the other hand, 3 and 5 are neither factors of 2^{2022} nor factors of 7. So they are not factors of S. It follows that 6 and 9 which are multiples of 3 are not factors of S. Similarly, 10 is a multiple of 5 and hence it is not a factor of S.

We therefore see that just two of the numbers 6, 7, 8, 9 and 10 are factors of S, namely 7 and 8.

FOR INVESTIGATION

10.1 Which is the largest prime factor of $2^{2024} + 2^{2023} + 2^{2022} + 2^{2021}$?

11. Wenlu, Xander, Yasser and Zoe make the following statements:

Wenlu: "Xander is lying." Xander: " Yasser is lying."

Yasser: "Zoe is telling the truth." Zoe: "Wenlu is telling the truth."

What are the possible numbers of people telling the truth?

A 1 or 2 B 1 or 3 C 2 D 2 or 3 E 3

SOLUTION **B**

Suppose Wenlu is telling the truth. Then Xander is lying. Therefore Yasser is telling the truth. Hence Zoe is telling the truth, and this agrees with the fact that Wenlu is telling the truth.

Hence it is possible that Wenlu is telling the truth. We have seen that in this case Wenlu, Yasser and Zoe are telling the truth.

Suppose Wenlu is lying. Then Xander is telling the truth. Therefore Yasser is lying. Hence Zoe is lying, and this agrees with the fact that Wenlu is lying.

Hence it is possible that Wenlu is lying. We have seen that in this case only Xander is telling the truth.

Therefore the number of people telling the truth is either 1 or 3.

FOR INVESTIGATION

11.1 Wenlu says "Xander is lying", Xander say "Yasser is telling the truth", Yasser says "Zoe is lying" and Zoe says "Wenlu is telling the truth".

How many of them could be telling the truth?

12. The greatest power of 7 which is a factor of 50! is 7^k $(n! = 1\times2\times3\times4\times\ldots\times(n-1)\times n)$.

What is k?

A 4 B 5 C 6 D 7 E 8

SOLUTION **E**

$50! = 1\times 2\times\cdots\times 50$. Because the seven numbers 7, 14, 21, 28, 35, 42 and 49 are divisible by 7, they each contribute 1 to the power of 7 which is a factor 50! In addition, because 49 is divisible by 7^2, it contributes an additional power of 7.

Therefore the highest power of 7 that divides 50! is $7+1=8$.

FOR INVESTIGATION

12.1 Find the greatest power of 3 which is a factor of 50!

The general formula for the greatest power of a prime p which is a factor of $n!$ is

$$\left\lfloor\frac{n}{p}\right\rfloor+\left\lfloor\frac{n}{p^2}\right\rfloor+\left\lfloor\frac{n}{p^3}\right\rfloor+\ldots. \qquad (1)$$

Here $\lfloor x\rfloor$ is the *integer part* of x which is defined by

$$\lfloor x\rfloor = k \Leftrightarrow k \text{ is the largest integer } \leq x.$$

For example $\left\lfloor\frac{3}{4}\right\rfloor=0$, $\left\lfloor\frac{22}{7}\right\rfloor=3$, and $\lfloor\sqrt{5}\rfloor=2$.

Note that, although the sum in (1) looks infinite, whenever $p^t>n$, we have $0<\frac{n}{p^t}<1$ and therefore $\left\lfloor\frac{n}{p^t}\right\rfloor=0$. Therefore we could replace (1) by the finite sum

$$\left\lfloor\frac{n}{p}\right\rfloor+\left\lfloor\frac{n}{p^2}\right\rfloor+\left\lfloor\frac{n}{p^3}\right\rfloor+\cdots+\left\lfloor\frac{n}{p^s}\right\rfloor \qquad (2)$$

where s is the largest integer such that $p^s\leq n$.

For example, since $7^3<1000$, but $7^4>1000$, it follows from (2) that the greatest power of 7 that is a factor of 1000! is

$$\left\lfloor\frac{1000}{7}\right\rfloor+\left\lfloor\frac{1000}{7^2}\right\rfloor+\left\lfloor\frac{1000}{7^3}\right\rfloor=\left\lfloor\frac{1000}{7}\right\rfloor+\left\lfloor\frac{1000}{49}\right\rfloor+\left\lfloor\frac{1000}{343}\right\rfloor=142+20+2=164.$$

12.2 Find the greatest power of 2 that is a factor of 100!

12.3 Find the greatest power of 11 that is a factor of 1000!

12.4 Find the greatest power of 10 that is a factor of 1000!

12.5 Explain why the formula (1) given above for the greatest power of a prime p that divides $n!$ is correct.

13. $PQRST$ is a regular pentagon. The point U lies on ST such that $\angle QPU$ is a right angle.

What is the ratio of the interior angles in triangle PUT?

A 1 : 3 : 6 B 1 : 2 : 4 C 2 : 3 : 4 D 1 : 4 : 8 E 1 : 3 : 5

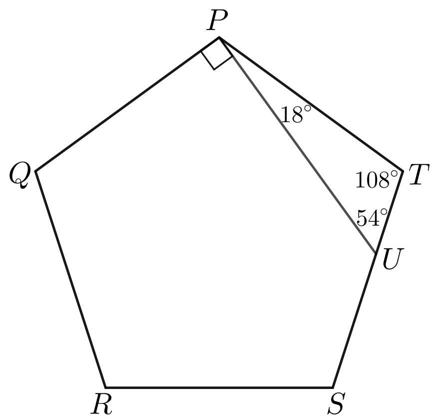
Solution **A**

The interior angles of a regular pentagon are all 108°. [You are asked to prove this in Problem 13.1.] Therefore $\angle PTU = \angle QPT = 108°$.

Hence $\angle UPT = \angle QPT - \angle QPU = 108° - 90° = 18°$.

Because the angles in a triangle have sum 180°, it follows that $\angle TUP = 180° - 108° - 18° = 54°$.

Therefore the ratio of the interior angles in the triangle PUT is 18 : 54 : 108 = 1 : 3 : 6.



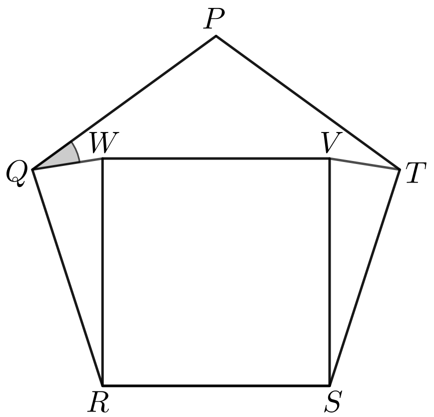
For investigation

13.1 (a) Show that the sum of the interior angles of a polygon with n vertices is $(n-2)180°$.

(b) Deduce that the interior angles of a regular pentagon are equal to 108°.

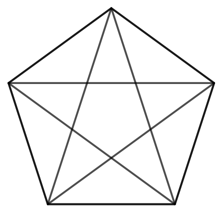
13.2 The regular pentagon $PQRST$ and the square $RSVW$ share the edge RS.

What is $\angle WQP$?



13.3 The diagram shows a regular pentagon and all its diagonals.

Find all the angles in the diagram.



14. The points P $(d, -d)$ and Q $(12 - d, 2d - 6)$ both lie on the circumference of the same circle whose centre is the origin.

What is the sum of the two possible values of d?

A -16 B -4 C 4 D 8 E 16

SOLUTION **E**

Let O $(0, 0)$ be the origin.

Because the points P and Q lie on the same circle with centre O, we have $OP^2 = OQ^2$. That is,

$$d^2 + (-d)^2 = (12 - d)^2 + (2d - 6)^2.$$

Expanding both sides of this equation, we obtain

$$d^2 + d^2 = (144 - 24d + d^2) + (4d^2 - 24d + 36).$$

We can rearrange this equation to obtain

$$3d^2 - 48d + 180 = 0.$$

By dividing both sides of this last equation by 3, it follows that

$$d^2 - 16d + 60 = 0.$$

We can now use the fact that the sum of the roots of the quadratic equation $x^2 + px + q = 0$ is $-p$ to deduce that the sum of the two possible values of d is $-(-16)$, that is, 16.

FOR INVESTIGATION

14.1 (a) Find the two possible values of d by solving the equation $d^2 - 16d + 60 = 0$.

(b) Hence check that the sum of the two possible values of d is 16.

14.2 Find the centre of the circle that goes through the points $(4, -14)$, $(-3, -13)$ and $(-7, -11)$.

14.3 Prove that the sum of the roots of the quadratic equation $x^2 + px + q = 0$ is $-p$.

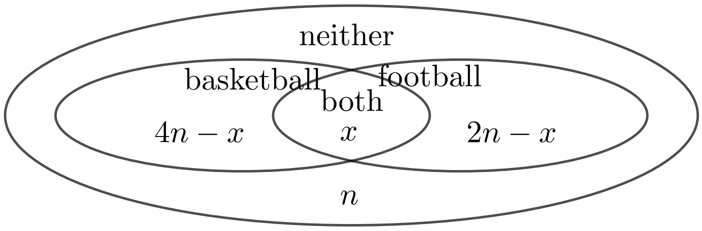
15. In Bethany's class of 30 students, twice as many people played basketball as played football. Twice as many played football as played neither.

Which of the following options could have been the number of people who played both?

A 19 B 14 C 9 D 5 E 0

Solution **D**

Let x be the number of students who played both basketball and football and let n be the number of students who played neither.



Then $2n$ students played football and hence $2 \times 2n = 4n$ students played basketball.

Hence there are $4n - x$ students who played basketball, but not football, and $2n - x$ students who played football but not basketball.

We can deduce from this that the number of students who played basketball or football or both was $(4n - x) + x + (2n - x) = 6n - x$.

Because there are 30 students in the class, the sum of the number of students who played basketball or football or both, and the number who played neither is 30. That is, $(6n - x) + n = 30$. Therefore

$$x = 7n - 30. \qquad (1)$$

The number x cannot be negative. It follows, by (1), that $4 < n$.

The number of students who played football but not basketball is $2n - x$. This number cannot be negative. Hence $x \leq 2n$. Therefore, by (1), $7n - 30 \leq 2n$. Hence $5n \leq 30$ and so $n \leq 6$.

Therefore, the only possible values of n are 5 and 6. It follows, by (1), that the only possible values of x are 5 and 12. Hence, only option D gives a possible number of students who played both.

For investigation

15.1 In Claire's class of 30 students, twice as many play neither cricket nor tennis, as play both.

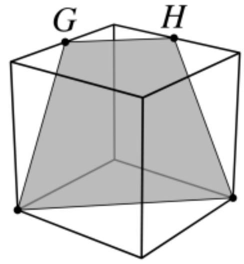
The ratio of those playing cricket to those playing tennis is $7 : 5$.

How many in Claire's class play cricket?

16. G and H are midpoints of two adjacent edges of a cube. A trapezium-shaped cross-section is formed cutting through G, H and two further vertices, as shown. The edge-length of the cube is $2\sqrt{2}$.

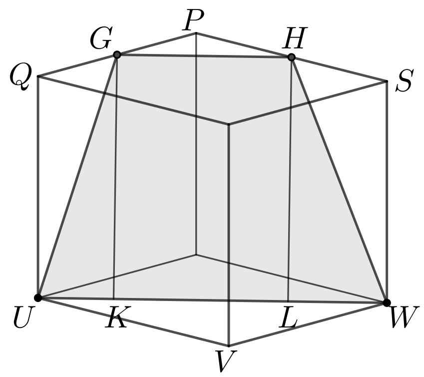
What is the area of the trapezium?

A 9 B 8 C $4\sqrt{5}$ D $4\sqrt{3}$ E $4\sqrt{2}$



Solution **A**

We let P, Q, S, U, V and W be vertices of the cube, as shown in the diagram, and K, L be the feet of the perpendiculars from G, H, respectively, to UW.



We apply Pythagoras' Theorem to the right-angled triangle UVW. This gives $UW^2 = UV^2 + WV^2 = (2\sqrt{2})^2 + (2\sqrt{2})^2 = 8 + 8 = 16$. Hence $UW = 4$.

G is the midpoint of the edge PQ. Hence $PG = \sqrt{2}$. Similarly, $PH = \sqrt{2}$. Therefore, applying Pythagoras' Theorem to the right-angled triangle GPH gives $GH^2 = PG^2 + PH^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 2 + 2 = 4$. Hence $GH = 2$.

Also, applying Pythagoras' Theorem to the right-angled triangle GQU gives $GU^2 = GQ^2 + UQ^2 = (\sqrt{2})^2 + (2\sqrt{2})^2 = 2 + 8 = 10$. Hence $GU = \sqrt{10}$.

Since $GKLH$ is a rectangle, $KL = GH = 2$. Therefore $UK + WL = UW - KL = 4 - 2 = 2$. By symmetry, $UK = WL$. Hence $UK = WL = 1$.

Applying Pythagoras' Theorem to the right-angled triangle GKU, gives $GK^2 = GU^2 - UK^2 = (\sqrt{10})^2 - 1^2 = 10 - 1 = 9$. Hence $GK = 3$.

We now use the formula $\frac{1}{2}(a + b)h$ for the area of a trapezium whose parallel sides have lengths a and b and which has height h. [You are asked to prove this formula in Problem 16.1.]

It follows from this formula that

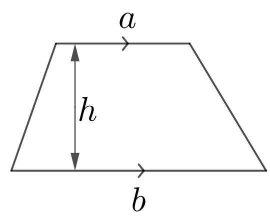
$$\text{area of the trapezium } GUWH = \tfrac{1}{2}(UW + GH)GK = \tfrac{1}{2}(4 + 2) \times 3 = 9.$$

For investigation

16.1 Prove that the formula

$$\tfrac{1}{2}(a + b)h$$

for the area of a trapezium is correct.



17. The number $M = 124563987$ is the smallest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number. For example, the 5th and 6th digits of M make the number 63 which is not prime. N is the largest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number.

What are the 5th and 6th digits of N?

A 6 and 3 B 5 and 4 C 5 and 2 D 4 and 8 E 3 and 5

Solution **E**

For convenience, in this solution by a *subnumber* of a positive number n we mean a number formed from consecutive digits of n. For example, the two-digit subnumbers of 1234 are 12, 23 and 34, and the three-digit subnumbers of 1234 are 123 and 234.

Note that this is not standard mathematical terminology, but has been introduced just for the purposes of this question.

We note first that 63 and 93 are the only numbers formed of two different non-zero digits with 3 as the unit digits that are not primes. It follows that, if the digit 3 is not the first digit of the number N, the digit 3 could only occur in N immediately after either the digit 9 or the digit 6.

We construct N by beginning with the largest digit 9, and then use all the other non-zero digits once each by always choosing the largest digit not yet used subject to the condition that 3 comes immediately after 9 or immediately after 6. In this way we obtain the number 987635421.

We now see that in the number 987635421 none of the two-digit subnumbers is a prime. Any larger number using all the digits must either begin 98765... or 98764..., but in each case the 3 would follow either 1,2,4 or 5, and so produce a two-digit subnumber that is a prime.

Therefore the largest number with the required property is $N = 987635421$. It follows that the 5th and 6th digits of N are 3 and 5.

For investigation

17.1 Find the largest positive integer that uses different non-zero digits, and has the property that all its two-digit subnumbers are prime.

17.2 Find the largest positive integer that uses different non-zero digits, and has the property that all its two-digit subnumbers are divisible by 7.

17.3 Find the largest positive integer that uses different non-zero digits, and has the property that all its three-digit subnumbers are divisible by 3.

17.4 Find the largest positive integer that uses different non-zero digits, and has the property that all its three-digit subnumbers are divisible by 11.

18. How many solutions are there of the equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ with $0° < X < 360°$?

A 1 B 2 C 4 D 6 E 8

SOLUTION **C**

For convenience we put $x = \sin X$.

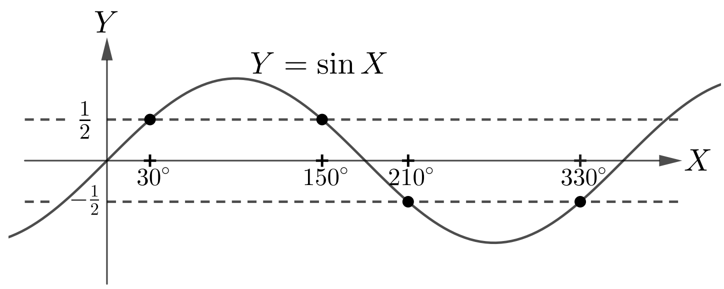
We have

$$\begin{aligned} 1 + 2\sin X - 4\sin^2 X - 8\sin^3 X &= 1 + 2x - 4x^2 - 8x^3 \\ &= 1 + 2x - 4x^2(1 + 2x) \\ &= (1 + 2x)(1 - 4x^2) \\ &= (1 + 2x)(1 + 2x)(1 - 2x), \end{aligned}$$

It follows that

$$\begin{aligned} &1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0 \\ &\Leftrightarrow (1 + 2x)(1 + 2x)(1 - 2x) = 0 \\ &\Leftrightarrow x = -\tfrac{1}{2} \text{ or } x = \tfrac{1}{2} \\ &\Leftrightarrow \sin X = -\tfrac{1}{2} \text{ or } \sin X = \tfrac{1}{2}. \end{aligned}$$

The solutions of the equation $\sin X = -\frac{1}{2}$ with $0° < X < 360°$ are $x = 210°$ and $X = 330°$. The solutions of $\sin X = \frac{1}{2}$ in the same interval are $X = 30°$ and $X = 150°$.



Therefore the equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ has 4 solutions with $0° < X < 360°$.

FOR INVESTIGATION

18.1 Find all the solutions of the equation

$$10\sin^2 X - 8\sin^4 X = 3,$$

with $0° < X < 360°$.

18.2 Find all the solutions of the equation

$$\sin^4 X + \sin^2 X = 3,$$

with $0° < X < 360°$.

19. The expression $\dfrac{7n+12}{2n+3}$ takes integer values for certain integer values of n.

What is the sum of all such integer values of the expression?

A 4 B 8 C 10 D 12 E 14

Solution **E**

We have

$$\frac{7n+12}{2n+3} = \frac{7}{2} + \frac{3}{2(2n+3)} = \frac{1}{2}\left(7 + \frac{3}{2n+3}\right).$$

It follows that $\dfrac{7n+12}{2n+3}$ is an integer provided that $7 + \dfrac{3}{2n+3}$ is an even integer and hence provided that $\dfrac{3}{2n+3}$ is an odd integer.

Now $\dfrac{3}{2n+3}$ is an integer provided that $2n+3$ is a factor of 3.

Therefore the only possible values of $2n+3$ are -3, -1, 1 and 3. In all these cases $\dfrac{3}{2n+3}$ is a factor of 3 and hence is odd.

When $2n+3=-3$, we have $n=-3$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7+\dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7+\dfrac{3}{-3}\right) = \dfrac{1}{2}\left(7-1\right) = 3$.

When $2n+3=-1$, we have $n=-2$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7+\dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7+\dfrac{3}{-1}\right) = \dfrac{1}{2}\left(7-3\right) = 2$.

When $2n+3=1$, we have $n=-1$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7+\dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7+\dfrac{3}{1}\right) = \dfrac{1}{2}\left(7+3\right) = 5$.

When $2n+3=3$, we have $n=0$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7+\dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7+\dfrac{3}{3}\right) = \dfrac{1}{2}\left(7+1\right) = 4$.

We see that n is an integer whenever $2n+3$ is equal to either -3, -1, 1 or 3. Therefore the sum of the integer values of $\dfrac{7n+12}{2n+3}$ that correspond to integer values of n is $3+2+5+4=14$.

For investigation

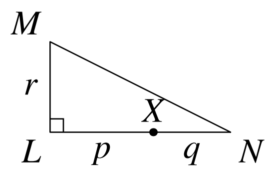
19.1 Check that $\dfrac{7n+12}{2n+3} = \dfrac{7}{2} + \dfrac{3}{2(2n+3)}$.

19.2 What is the sum of all the integer values taken by the expression

$$\frac{6n+5}{2n+9}$$

when n is an integer?

20. Triangle LMN represents a right-angled field with $LM = r$, $LX = p$ and $XN = q$. Jenny and Vicky walk at the same speed in opposite directions along the edge of the field, starting at X at the same time. Their first meeting is at M.



Which of these expressions gives q in terms of p and r?

A $\frac{p}{2} + r$ B $\sqrt{p^2 + r^2} + \frac{p}{2}$ C $\frac{pr}{2p + r}$ D $\frac{p}{2}$ E 1

SOLUTION **C**

Because Jenny and Vicky meet after walking at the same speed, they walk the same distance. Therefore

$$XL + LM = XN + NM. \qquad (1)$$

The question tells us that $XL = p$, $LM = r$ and $XN = q$, To find NM, we apply Pythagoras' Theorem to the right-angled triangle LMN. This gives

$$NM^2 = LN^2 + LM^2 = (p + q)^2 + r^2.$$

It follows that $NM = \sqrt{(p + q)^2 + r^2}$. Substituting these values in equation (1) gives

$$p + r = q + \sqrt{(p + q)^2 + r^2}.$$

Hence

$$p + r - q = \sqrt{(p + q)^2 + r^2}. \qquad (2)$$

By squaring both sides of equation (2) we obtain

$$(p + r - q)^2 = (p + q)^2 + r^2.$$

It follows that

$$p^2 + r^2 + q^2 + 2pr - 2pq - 2rq = p^2 + 2pq + q^2 + r^2. \qquad (3)$$

Equation (3) may be rearranged to give

$$4pq + 2rq = 2pr$$

from which it follows that

$$q(2p + r) = pr.$$

Therefore

$$q = \frac{pr}{2p + r}.$$

FOR INVESTIGATION

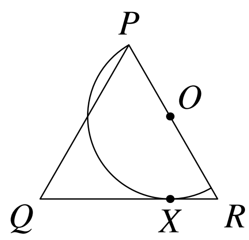
20.1 Check that $(p + r - q)^2 = p^2 + r^2 + q^2 + 2pr - 2pq - 2rq$.

20.2 Suppose that in this problem, $LM = LN$. In this case what is the ratio $LX : XN$?

21. Triangle PQR is equilateral. A semicircle with centre O is drawn with its diameter on PR so that one end is at P and the curved edge touches QR at X. The radius of the semicircle is $\sqrt{3}$.

What is the length of QX?

A $\sqrt{3}$ B $2-\sqrt{3}$ C $2\sqrt{3}-1$
D $1+\sqrt{3}$ E $2\sqrt{3}$

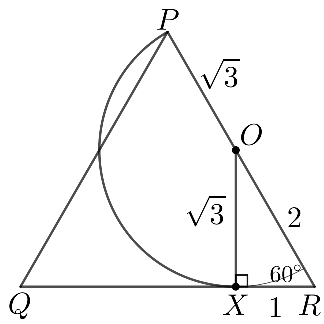


SOLUTION **D**

Because OP and OX are radii of the semicircle, we have $OP = OX = \sqrt{3}$.

Because the semicircle touches QR at X, the line QR is a tangent to the semicircle at X and therefore the radius OX is perpendicular to QR. Therefore OXR is a right-angled triangle.

Because the triangle PQR is equilateral, $\angle ORX = 60°$.



Hence
$$\frac{OX}{OR} = \sin 60° = \frac{\sqrt{3}}{2} \quad \text{and} \quad \frac{XR}{OR} = \cos 60° = \frac{1}{2}.$$
It follows that
$$OR = \frac{2}{\sqrt{3}} \times OX = \frac{2}{\sqrt{3}} \times \sqrt{3} = 2,$$
and, therefore,
$$XR = \frac{1}{2} \times OR = \frac{1}{2} \times 2 = 1.$$
We can now deduce that
$$QR = PR = OR + OP = 2 + \sqrt{3}.$$
Therefore
$$QX = QR - XR = (2+\sqrt{3}) - 1 = 1 + \sqrt{3}.$$

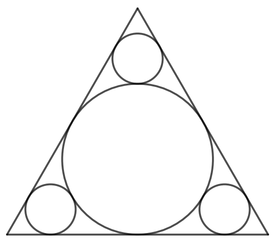
FOR INVESTIGATION

21.1 The diagram shows an equilateral triangle with sides of length 1, and four circles.

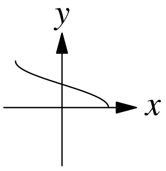
The largest circle touches the three sides of the triangle. [It is the *incircle* of the triangle.]

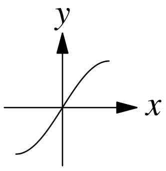
Each of the smaller circles touches two sides of the triangle and the largest circle.

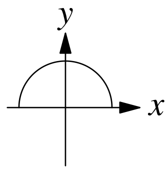
Find the radius of the smaller circles.

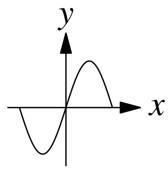


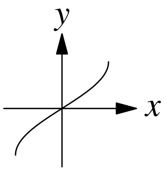
22. Which diagram could be a sketch of the curve $y = \sin(\cos^{-1} x)$?

A


B


C


D


E


SOLUTION **C**

Our first method is to find points that lie on the curve, and to note that these points are on just one of the curves given as options. In the context of the SMC it should be safe to assume that the question setters have not made a mistake and therefore this one curve is the correct one.

However, this is not a fully justified mathematical answer. In the second method we show that the equation of the curve may be written in a more familiar way. This enables us to deduce which is the correct option.

METHOD 1

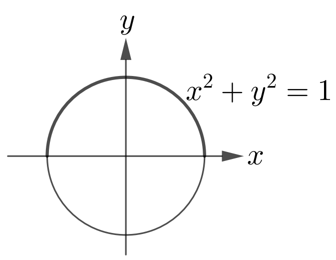
Since $\cos^{-1} 0 = 90°$, when $x = 0$ we have $y = \sin(\cos^{-1} 0) = \sin 90° = 1$. It follows that the point $(0, 1)$ is on the curve. This rules out all the options other than A and C.

Since $\cos^{-1}(-1) = 180°$, when $x = -1$ we have $y = \sin(\cos^{-1}(-1)) = \sin 180° = 0$. It follows that the point $(-1, 0)$ is on the curve. This rules out option A.

We conclude that just option C could be a sketch of the curve.

METHOD 2

If we substitute $\cos^{-1} x$ for θ in the identity $\cos^2\theta + \sin^2\theta = 1$, we obtain $\quad \cos^2(\cos^{-1} x) + \sin^2(\cos^{-1} x) = 1. \quad (1)$



We are given that $y = \sin(\cos^{-1} x)$. Also, because $\cos^{-1}$ is the inverse of the function cos, $x = \cos(\cos^{-1} x)$. Hence it follows from (1) that $x^2 + y^2 = 1$. This is the equation of the circle with centre $(0, 0)$ and radius 1. We leave it to the reader [See Problem 22.1] to work out why the curve is just the top half of this circle.

It follows that just option C could be a sketch of the curve.

FOR INVESTIGATION

22.1 Explain why the curve corresponding to the equation $y = \sin(\cos^{-1} x)$ is just the part of the circle with equation $x^2 + y^2 = 1$ where $y \geq 0$.

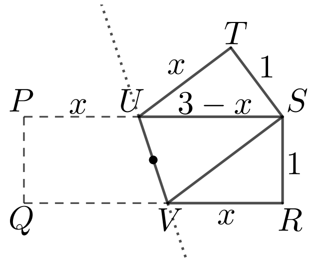
23. The length of a rectangular piece of paper is three times its width. The paper is folded so that one vertex lies on top of the opposite vertex, thus forming a pentagonal shape.

What is the area of the pentagon as a fraction of the area of the original rectangle?

A $\frac{2}{3}$ B $\frac{11}{16}$ C $\frac{12}{17}$ D $\frac{13}{18}$ E $\frac{14}{19}$

Solution **D**

Let P, Q, R and S be the vertices of the rectangular piece of paper. We choose units so that $PQ = 1$ and $PS = 3$.



Because the fold moves Q to S, the fold line goes through the midpoint of QS which is the centre of the rectangle. We let U and V be the points where this fold line meets the edges of the rectangle, as shown in the diagram.

We let x be the length of VR. The rectangle has a rotational symmetry that interchanges U and V. Therefore, $PU = VR = x$. Hence $US = 3 - x$.

We let T be the point where the vertex Q ends up after the fold. The quadrilateral $STUV$ is the reflection of $QPUV$ in the line through U and V. It follows that $TS = 1$, $UT = x$ and $\angle UTS$ is a right angle.

We need to find the area of the pentagon $RSTUV$. This pentagon is made up of the trapezium $RSUV$ and the triangle STU.

It follows from the rotational symmetry of the rectangle that the area of the trapezium $RSUV$ is half the area of the rectangle $PQRS$. Therefore the area of the trapezium is $\frac{1}{2}\left(3 \times 1\right) = \frac{3}{2}$.

The area of the right-angled triangle STU is $\frac{1}{2}\left(1 \times x\right) = \frac{x}{2}$.

By Pythagoras' Theorem applied to the triangle UST, we have $1 + x^2 = (3 - x)^2$. Therefore $1 + x^2 = 9 - 6x + x^2$. It follows that $6x = 8$. Hence $x = \frac{8}{6} = \frac{4}{3}$.

It follows that the area of the pentagon $RSTUV$ is given by $\frac{3}{2} + \frac{4/3}{2} = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$.

The area of rectangle $PQRS$ is 3. Therefore the area of the pentagon as a fraction of the area of the original rectangle is $\frac{13/6}{3} = \frac{13}{18}$.

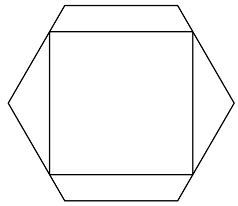
For investigation

23.1 Consider the similar problem with an A4 sheet of paper where the ratio of the length to the width is $\sqrt{2} : 1$. What is the area of the pentagon as a fraction of the area of the original rectangle in this case?

23.2 In a similar problem the length of a rectangular piece of paper is k times its width. After the paper is folded so that one vertex lies on top of the opposite vertex, the area of the pentagon that is formed is 74% of the area of the rectangle. What is the value of k?

24. A square has its vertices on the edges of a regular hexagon. Two of the edges of the square are parallel to two edges of the hexagon, as shown in the diagram. The sides of the hexagon have length 1.

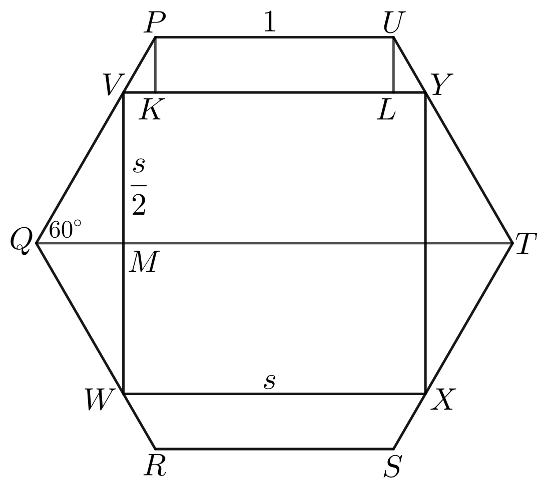
What is the length of the sides of the square?



A $\frac{5}{4}$ B $3-\sqrt{3}$ C $\frac{4}{3}$ D $\sqrt{2}$ E $\frac{3}{2}$

Solution **B**

We label the vertices of the hexagon P, Q, R, S, T and U, and the vertices of the square V, W, X and Y, as shown in the diagram.



Also, we let K and L be the feet of the perpendiculars from P and U, respectively, to VY, and we let M be the point where VW meets QT.

We let the length of the sides of the square be s.

The internal angles of the regular hexagon are 120°. The figure is symmetrical about the line QT. Therefore $\angle VQM = 60°$, $\angle VMQ = 90°$ and $VM = \frac{s}{2}$.

From the right-angled triangle VMQ, we have $\frac{VM}{VQ} = \sin 60° = \frac{\sqrt{3}}{2}$. Therefore $VQ = \frac{2}{\sqrt{3}}VM = \frac{2}{\sqrt{3}} \times \frac{s}{2} = \frac{s}{\sqrt{3}}$.

$KLUP$ is a rectangle. Therefore $KL = PU = 1$. Therefore, $VK + LY = VY - KL = s - 1$. By symmetry, $VK = LY$. Hence $VK = \frac{s-1}{2}$.

$\angle PKV = 90°$ and $\angle PVK = 60°$. Therefore in the triangle PVK we have $\frac{VK}{PV} = \cos 60° = \frac{1}{2}$. Hence $PV = 2VK = s - 1$.

Since $PQ = 1$, we have $PV + VQ = 1$, and hence $s - 1 + \frac{s}{\sqrt{3}} = 1$. Hence $s\left(1 + \frac{1}{\sqrt{3}}\right) = 2$. That is, $s\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right) = 2$. Therefore $s = \frac{2\sqrt{3}}{\sqrt{3}+1} = \frac{2\sqrt{3}(\sqrt{3}-1))}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{6-2\sqrt{3}}{2} = 3-\sqrt{3}$.

For investigation

24.1 Express the value of $\dfrac{\text{area of the square } VWXY}{\text{area of the hexagon } PQRSTU}$ in the form $a + b\sqrt{3}$, where a and b are rational numbers.

25. What is the area of the part of the xy-plane within which $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ and $0 \leq x \leq y$?

A $\frac{1}{4}$ B $\frac{1}{2}$ C 1 D 2 E 4

Solution **A**

We have

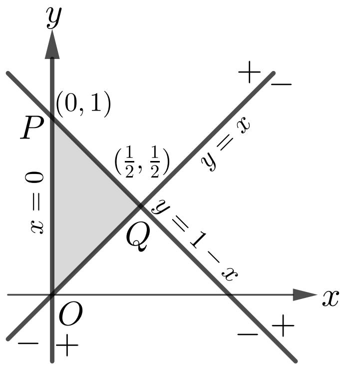
$$\begin{aligned} x^3y^2 - x^2y^2 - xy^4 + xy^3 &= xy^2(x^2 - x - y^2 + y) \\ &= xy^2((x^2 - y^2) - (x - y)) \\ &= xy^2((x - y)(x + y) - (x - y)) \\ &= xy^2(x - y)(x + y - 1). \end{aligned}$$

In the region where $0 \leq x \leq y$, we have $xy^2 \geq 0$ and $x - y \leq 0$. Therefore in this region $xy^2(x - y)(x + y - 1) \geq 0 \Leftrightarrow (x + y - 1) \leq 0$.

Therefore the region of the xy-plane within which we have $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ and $0 \leq x \leq y$ is the region where $x + y - 1 \leq 0$ and $0 \leq x \leq y$.

The line with equation $x = 0$ (the y-axis) divides the plane into the region where $x > 0$ and the region where $x < 0$. In the diagram the symbol + is used to indicate the side of the line where $x > 0$ and – to indicate the side where $x < 0$.

In a similar way the line with equation $y = x$ divides the plane into the region where $y > x$ and the region where $y < x$. Again, we use + and – to mark these two regions.



Also, the line with the equation $x + y - 1 = 0$ (or, equivalently, $y = 1 - x$) divides the plane into the region where $x + y - 1 > 0$ and the region where $x + y - 1 < 0$.

Therefore the region whose area we need to find is the triangle OPQ which is bounded by the lines with equations $x = 0$, $x = y$ and $y = 1 - x$, as shown in the diagram.

The point O is the origin where the lines $x = 0$ and $y = x$ meet. The point P is the point where the lines $x = 0$ and $y = 1 - x$ meet. The co-ordinates of P are $(0, 1)$. The point Q is the point where the lines $y = x$ and $y = 1 - x$ meet. The co-ordinates of Q are $(\frac{1}{2}, \frac{1}{2})$.

The triangle OPQ has a base OP which has length 1. The height of the triangle with this base is the distance of Q from the y-axis which is $\frac{1}{2}$. Therefore, the area of this triangle is

$$\frac{1}{2}\left(1 \times \frac{1}{2}\right) = \frac{1}{4}.$$

For investigation

25.1 What is the area of the part of the xy-plane where $x^2 + y^2 \leq 4$ and $x^2 - y^2 \leq 0$?

25.2 What is the area of the part of the xy-plane where $x^3y^2 - xy^4 \leq 0$, $-2 \leq x \leq 2$ and $-3 \leq y \leq 3$?

UK Maths Trust

SENIOR MATHEMATICAL CHALLENGE

Tuesday 1 October 2024

Organised by the United Kingdom Mathematics Trust

proudly sponsored by [XTX] MARKETS

Candidates must be full-time students at secondary school or FE college.
England & Wales: Year 13 or below | Scotland: S6 or below | Northern Ireland: Year 14 or below

INSTRUCTIONS

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**. No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank paper for rough working is allowed; **squared paper, calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil**. Mark at most one of the options, A, B, C, D, or E, on the Answer Sheet for each question. Do not mark more than one option.
5. **Do not expect to finish the whole paper in the time allowed**. The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules**: All candidates start with 25 marks; 0 marks are awarded for each question left unanswered; 4 marks are awarded for each correct answer; 1 mark is deducted for each incorrect answer (to discourage guessing).
7. **Your Answer Sheet will be read by a machine**. Do not write or doodle on the sheet except to mark your chosen options. The machine will read all black pencil markings even if they are in the wrong places. If you mark the sheet in the wrong place, doodle, or leave bits of eraser stuck to the page, the machine will interpret the mark in its own way, or reject the answer sheet.
8. **The questions on this paper are designed to challenge you to think, not to guess**. You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.
9. To accommodate candidates sitting at other times, please do not discuss the paper on the internet until 08:00 BST on Thursday 3 October, when the solutions video will be released at ukmt.org.uk/competition-papers. Candidates in time zones more than 5 hours ahead of GMT must sit the paper on Wednesday 2 October (as defined locally).

Enquiries about the Senior Mathematical Challenge should be sent to:

challenges@ukmt.org.uk   *www.ukmt.org.uk*

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1. What is two-fifths of the recurring decimal $0.\dot{2}\dot{5}$?

A $0.\dot{1}$ B $0.0\dot{1}$ C $0.\dot{0}\dot{1}$ D 0.10 E $0.\dot{1}\dot{0}$

2. A ***twip*** is a very short unit of length, derived from imperial units, and is equal to approximately 0.000018 metres. A ***league*** is a long unit of length which is equal to approximately 4800 metres.

Roughly how many twips are there in a league?

A 270 000 000 B 27 000 000 C 2 700 000 D 270 000 E 27 000

3. Two standard dice are placed on a table, with one on top of the other, so that only nine of the faces of the dice may be seen. The touching faces have the same number on them. The sum of the numbers on the visible faces is 33.

What is the number on the touching faces?

A 1 B 2 C 3 D 4 E 6

4. The sizes of the three angles in a triangle, in degrees, are x, $7x$ and x^2.

What is the size of the largest angle?

A 10° B 18° C 100° D 120° E 121°

5. When $4^5 \times 5^4$ is correctly calculated, how many digits are there in the answer?

A 4 B 6 C 10 D 16 E 20

6. One face of a solid polyhedron is an octagon.

What is the smallest possible number of edges the solid could have?

A 9 B 10 C 12 D 16 E 24

7. Which is the largest prime factor of $3^8 - 1$?

A 41 B 37 C 31 D 29 E 23

8. In the following expressions, x is non-zero. When one of these expressions is removed, the mean of the remaining four is $11x$.

Which expression is removed?

A $4x$ B $8x$ C $12x$ D $16x$ E $20x$

9. A palindromic number is one where the digits read the same forwards as backwards, such as 123 321.

What is the hundreds digit of the largest six-digit palindromic number that is divisible by 18?

A 9 B 7 C 5 D 3 E 1

10. The prime factorization of 2024 is $2^3 \times 11 \times 23$.

How many two-digit numbers are factors of 2024?

A 2 B 4 C 6 D 7 E 8

11. Which one of the following expressions is a square number for each positive integer n?

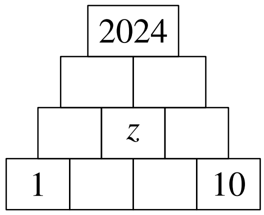
A $n + 1$ B $n(n + 1) + 1$ C $n(n + 1)(n + 2) + 1$
D $n(n + 1)(n + 2)(n + 3) + 1$ E $n(n + 1)(n + 2)(n + 3)(n + 4) + 1$

12. p, q, r and s are two-digit primes which between them use all the non-zero digits except 5.

What is the value of $p + q + r + s$?

A 220 B 210 C 200 D 190
E more information needed

13. The diagram shows a partially completed number pyramid. When correctly completed, the number on any brick above the bottom row should be the sum of the two numbers on the two bricks on which it rests.



What number should appear on the brick marked 'z'?

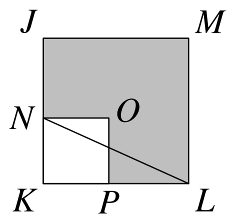
A 176 B 617 C 671 D 716 E 761

14. P, Q, R, S and T are the digits 1, 2, 3, 4 and 5 in some order. 'PRT' and 'QRS' are both three-digit primes.

Which digit is R?

A 1 B 2 C 3 D 4 E 5

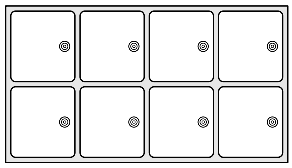
15. The diagram shows two squares, $JKLM$ and $NKPO$.
The length of NL is 10 cm. The shaded region has area 62 cm^2.



What is the length of KN in cm?

A 3 B $\sqrt{18}$ C $\sqrt{19}$ D $\sqrt{22}$ E 5

16. A set of cupboards containing eight identical blue doors is arranged in a 2 by 4 grid as shown. A fussy decorator wishes to paint three of the doors red such that at least one door in each row is painted red and at least two of the four corners are painted red.



How many ways are there to do this?

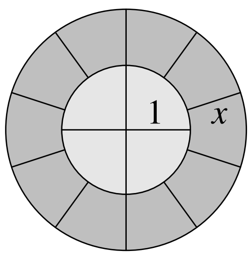
A 12 B 24 C 36 D 40 E 56

17. A bag contains four balls each of which is coloured either red or white. If one ball is drawn at random from the bag but not replaced and then a second ball is drawn at random, the probability that both balls are red is $\frac{1}{2}$.

What is the probability that both balls are white?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{6}$ E 0

18. The diagram shows two concentric circles divided by radial lines into 14 pieces of equal area. The radius of the smaller circle is 1.



What is the length, x, of an outer radial line?

A $\sqrt{14} - 1$ B $\sqrt{14} - 2$ C $\frac{\sqrt{14}}{2} - 1$ D $\frac{\sqrt{14}}{2} - 2$ E $\frac{\sqrt{14} - 1}{2}$

19. Five friends are dealt two cards each from a set of twelve cards. The cards are numbered 1 to 12 inclusive. In turn, the friends declare the sum of the values of their two cards. Paolo scores 4, Quinn scores 11, Romy scores 16, Stephen scores 19 and Thomas scores 20.

Which of the following statements is true?

A Paolo has card 2 B Quinn has card 3 C Romy has card 5 D Stephen has card 7 E Thomas has card 11

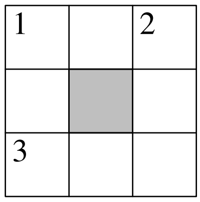
20. Let x and y be positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$. What is the maximum possible value of y?

A 40 B 60 C 240 D 420 E 480

21. The crossnumber is to be filled with eight of the digits 1 to 9, which are each used once.

Across
1. A multiple of 9
3. A square

Down
1. A multiple of 11
2. A multiple of 13 and of 19



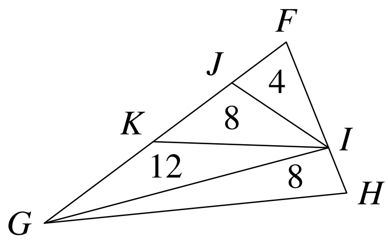
Which digit is not used?

A 9 B 8 C 5 D 3 E 2

22. As shown in the diagram, triangle FGH is divided into four smaller triangles which have areas 4, 8, 12 and 8 respectively.

What is the area of triangle IKH?

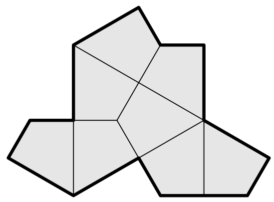
A 4 B 5 C 6 D 7 E 8



23. The plane can be tiled using the 'hat tile' shown here. This tile can be subdivided into eight congruent kites. The area of the hat tile is $8\sqrt{3}$.

What is the perimeter of the hat tile?

A $8 + 12\sqrt{3}$ B $16 + 6\sqrt{3}$ C $8 + 8\sqrt{3}$ D $6 + 8\sqrt{3}$ E $8 + 6\sqrt{3}$



24. A function f satisfies the equation $f(x) + f\left(\frac{1}{1-x}\right) = 24x$ for all real values of x except $x = 0$ and $x = 1$.

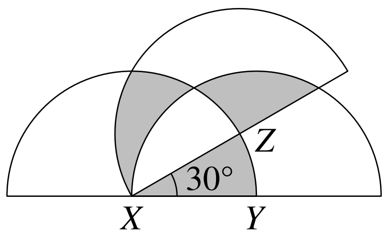
What is the value of $f(3)$?

A 40 B 42 C 45 D 48 E 50

25. Three semicircles, each of area 24, overlap as shown in the diagram. The centres of the arcs are X, Y and Z and $\angle ZXY = 30°$.

What is the total area of the shaded regions?

A 12 B $6\sqrt{3}$ C 15 D 18 E $8\sqrt{3}$



UK Maths Trust

Senior Mathematical Challenge

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Solutions

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
E	A	B	C	B	D	A	D	E	D	D	A	C	B	C	B	E	C	C	D	B	A	E	E	A

1. What is two-fifths of the recurring decimal $0.\dot{2}\dot{5}$?

A $0.\dot{1}$ B $0.0\dot{1}$ C $0.\dot{0}\dot{1}$ D 0.10 E $0.\dot{1}\dot{0}$

Solution **E**

Written as a fraction, $0.\dot{2}\dot{5}$ is $\frac{25}{99}$. One-fifth of the number is $\frac{5}{99}$ so two-fifths $= \frac{10}{99} = 0.\dot{1}\dot{0}$.

2. A ***twip*** is a very short unit of length, derived from imperial units, and is equal to approximately 0.000018 metres. A ***league*** is a long unit of length which is equal to approximately 4800 metres.

Roughly how many twips are there in a league?

A 270 000 000 B 27 000 000 C 2 700 000 D 270 000 E 27 000

Solution **A**

The number of ***twips*** in a ***league*** is $\dfrac{4800\,\text{m}}{0.000018\,\text{m}} = \dfrac{4.8\times10^3}{1.8\times10^{-5}} = \dfrac{8}{3}\times10^8 = 2.\dot{6}\times10^8 \approx$ 270000000.

3. Two standard dice are placed on a table, with one on top of the other, so that only nine of the faces of the dice may be seen. The touching faces have the same number on them. The sum of the numbers on the visible faces is 33.

What is the number on the touching faces?

A 1 B 2 C 3 D 4 E 6

Solution **B**

The numbers on opposite faces of a standard dice sum to 7. On the bottom dice, there are two pairs of opposite faces which are visible. On the top dice, there are again two pairs of opposite faces which are visible along with the number on the top of that dice, n say. Therefore, $(2+2)\times7+n=33$ and so $n=5$. The number on the touching faces is then $7-5=2$.

4. The sizes of the three angles in a triangle, in degrees, are x, $7x$ and x^2.

What is the size of the largest angle?

A $10°$ B $18°$ C $100°$ D $120°$ E $121°$

Solution **C**

As the angle sum of a triangle is $180°$, $x+7x+x^2=180$. Therefore, $x^2+8x-180=0$ which factorises to $(x+18)(x-10)=0$. As $x>0$, $x=10$. The angles are then $10°$, $70°$ and $100°$, so the largest angle is $100°$.

5. When $4^5 \times 5^4$ is correctly calculated, how many digits are there in the answer?

A 4 B 6 C 10 D 16 E 20

Solution **B**

$4^5 \times 5^4 = 2^{10} \times 5^4 = 2^6 \times 2^4 \times 5^4 = 2^6 \times 10^4 = 640000$. Hence the answer has six digits.

6. One face of a solid polyhedron is an octagon.

What is the smallest possible number of edges the solid could have?

A 9 B 10 C 12 D 16 E 24

Solution **D**

The solid with the minimum number of edges must contain an octagonal 'base'. This has 8 edges and 8 vertices. Including one extra edge from each of these vertices, all of which meet at a single vertex that is not on the base, creates an octagonal pyramid. There are $8 + 8 = 16$ edges.

7. Which is the largest prime factor of $3^8 - 1$?

A 41 B 37 C 31 D 29 E 23

Solution **A**

In order to write the expression $3^8 - 1$ as the product of its primes without first calculating its value, we can use the difference of two squares: $3^8 - 1 = (3^4+1)(3^4-1) = 82 \times 80 = 41 \times 2 \times 2^4 \times 5$. Hence the largest prime factor is 41.

8. In the following expressions, x is non-zero. When one of these expressions is removed, the mean of the remaining four is $11x$.

Which expression is removed?

A $4x$ B $8x$ C $12x$ D $16x$ E $20x$

Solution **D**

The mean of four terms is $11x$ so the sum of those four terms is $44x$. As the total of the five options is $4x + 8x + 12x + 16x + 20x = 60x$, the term to exclude from the sum is $60x - 44x = 16x$.

9. A palindromic number is one where the digits read the same forwards as backwards, such as 123 321.

What is the hundreds digit of the largest six-digit palindromic number that is divisible by 18?

A 9 B 7 C 5 D 3 E 1

SOLUTION **E**

For the number we seek to be divisible by 18, it must be divisible by both 2 and 9. To be even, it must have a last digit of 0, 2, 4, 6 or 8. This last digit is also the first digit of the palindromic number, so we choose it to be an 8. To be divisible by 9, the sum of all six digits must be a multiple of 9. As we already have two 8s, the remaining four middle digits, with a maximum sum of $4 \times 9 = 36$, could sum to 2, 11, 20 or 29. The sum of these four digits must itself be even as there are two repeated pairs. So we require the second and third numbers from the left to sum to $\frac{20}{2} = 10$ at the same time as maximising the second digit due to its place value. Hence the number we seek is 891198 and the hundreds digit is a 1.

10. The prime factorization of 2024 is $2^3 \times 11 \times 23$.

How many two-digit numbers are factors of 2024?

A 2 B 4 C 6 D 7 E 8

SOLUTION **D**

As $2024 = 2 \times 2 \times 2 \times 11 \times 23$, the two-digit numbers which are factors of 2024 are either suitable multiples of 11: 11, $11 \times 2 = 22$, $11 \times 2 \times 2 = 44$, $11 \times 2 \times 2 \times 2 = 88$ or multiples of 23: 23, $23 \times 2 = 46$, $23 \times 2 \times 2 = 92$. In total, this gives seven two-digit factors.

11. Which one of the following expressions is a square number for each positive integer n?

A $n + 1$
B $n(n + 1) + 1$
C $n(n + 1)(n + 2) + 1$
D $n(n + 1)(n + 2)(n + 3) + 1$
E $n(n + 1)(n + 2)(n + 3)(n + 4) + 1$

SOLUTION **D**

Choosing n to be 1, say, eliminates options A, B and C as the resulting values 2, 3 and 7 are not square. Similarly, choosing n to be 2 eliminates option E as 721 is not square. Now considering option D, we can see that $n(n + 1)(n + 2)(n + 3) + 1 = n^4 + 6n^3 + 11n^2 + 6n + 1$ which can be written as $(n^2 + 3n + 1)^2$ and is therefore a square for each positive integer n.

12. p, q, r and s are two-digit primes which between them use all the non-zero digits except 5.

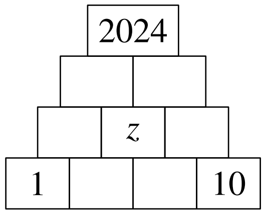
What is the value of $p+q+r+s$?

A 220 B 210 C 200 D 190 E more information needed

Solution **A**

In order to create two-digit primes, the units digits must be 1, 3, 7 and 9 in some order. Therefore the tens digits must be 2, 4, 6 and 8 in some order. We can add together all the tens and all the units without being concerned which combines with which to create the actual primes, so $p+q+r+s = (20+40+60+80)+(1+3+7+9) = 220$. There are in fact four possible ways to assign the digits to create p, q, r and s.

13. The diagram shows a partially completed number pyramid. When correctly completed, the number on any brick above the bottom row should be the sum of the two numbers on the two bricks on which it rests.

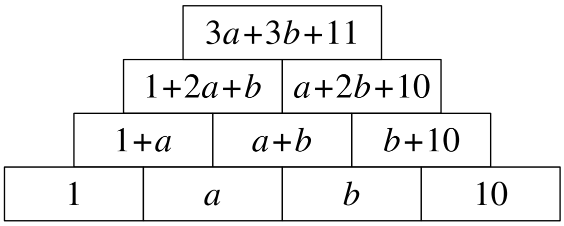


What number should appear on the brick marked 'z'?

A 176 B 617 C 671 D 716 E 761

Solution **C**

Labelling the missing numbers in the bottom row as a and b, the pyramid can be filled as shown. Therefore $3a+3b+11 = 2024$, and so subtracting 11 and dividing by 3 gives $a+b = 671$. Hence 671 appears on the brick marked z.



14. P, Q, R, S and T are the digits 1, 2, 3, 4 and 5 in some order. 'PRT' and 'QRS' are both three-digit primes.

Which digit is R?

A 1 B 2 C 3 D 4 E 5

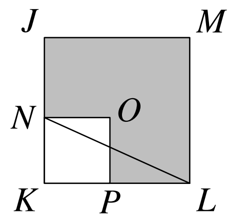
Solution **B**

In order to be primes, the three-digit numbers cannot end in 2, 4 or 5. S and T must therefore be 1 and 3 in either order. We may assume that $S = 3$ so that $T = 1$. Now Q and R cannot be 2 and 4 in either order nor 4 and 5 in either order as then QRS would be a multiple of 3. Therefore Q and R must be 2 and 5 in some order. However $253 = 23 \times 11$. So $QRS = 523$ which is (and in the context of the SMC, must be) a prime. Therefore $R = 2$ and then $PRT = 421$ which is also a prime.

15. The diagram shows two squares, $JKLM$ and $NKPO$. The length of NL is $10\,\text{cm}$. The shaded region has area $62\,\text{cm}^2$.

What is the length of KN in cm?

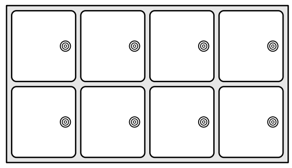
A 3 B $\sqrt{18}$ C $\sqrt{19}$ D $\sqrt{22}$ E 5



Solution **C**

Let $KL = x$ cm and $KN = y$ cm. As the shaded area is 62, $x^2 - y^2 = 62$. Applying Pythagoras' Theorem to triangle NKL gives $x^2 + y^2 = 10^2$. Subtracting the first equation from the second gives $2y^2 = 38$ so $y^2 = 19$ and hence $y = \sqrt{19}$.

16. A set of cupboards containing eight identical blue doors is arranged in a 2 by 4 grid as shown. A fussy decorator wishes to paint three of the doors red such that at least one door in each row is painted red and at least two of the four corners are painted red.

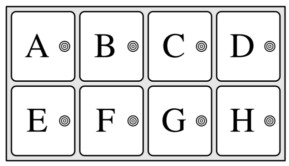


How many ways are there to do this?

A 12 B 24 C 36 D 40 E 56

Solution **B**

Let the doors be labelled A to H as shown. If all three red doors are corners, the one which remains blue can be A, D, E or H. This gives 4 ways. Now suppose that just two corners are red. If those two corners are in different rows, (that is AE, AH, DE or DH) then the third red door is any one of B, C, F or G giving four ways for each of the four cases. This gives another 16 ways.



Finally, suppose the two red corner doors are in the same row. If they are AD then the third door must be F or G and if they are EH the third must be B or C. This gives a further 4 ways. So the total number of ways is $4 + 16 + 4 = 24$.

17. A bag contains four balls each of which is coloured either red or white. If one ball is drawn at random from the bag but not replaced and then a second ball is drawn at random, the probability that both balls are red is $\frac{1}{2}$.

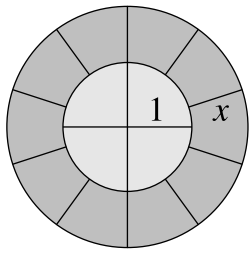
What is the probability that both balls are white?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{6}$ E 0

Solution **E**

We start by finding how many red balls are in the bag. Let this number be n. Therefore the probability of two reds is $\frac{n}{4} \times \frac{(n-1)}{3} = \frac{1}{2}$. This rearranges to give $n^2 - n - 6 = 0$, which factorises to $(n-3)(n+2) = 0$. As $n \geq 0$, $n = 3$. So there are three red balls and only one white. Hence the probability that both balls are white is 0.

18. The diagram shows two concentric circles divided by radial lines into 14 pieces of equal area. The radius of the smaller circle is 1.



What is the length, x, of an outer radial line?

A $\sqrt{14} - 1$ B $\sqrt{14} - 2$ C $\frac{\sqrt{14}}{2} - 1$ D $\frac{\sqrt{14}}{2} - 2$ E $\frac{\sqrt{14} - 1}{2}$

Solution **C**

The area of each of the four central quadrants is $\frac{1}{4} \times \pi \times 1^2 = \frac{\pi}{4}$. Therefore the area enclosed by the outer circle is $\frac{14\pi}{4}$. The radius of the outer circle is $1 + x$, therefore $\pi \times (1+x)^2 = \frac{14\pi}{4}$, which rearranges to $2x^2 + 4x - 5 = 0$. As $x > 0$, the quadratic formula leads us to $x = \frac{\sqrt{14}}{2} - 1$.

19. Five friends are dealt two cards each from a set of twelve cards. The cards are numbered 1 to 12 inclusive. In turn, the friends declare the sum of the values of their two cards. Paolo scores 4, Quinn scores 11, Romy scores 16, Stephen scores 19 and Thomas scores 20.

Which of the following statements is true?

A Paolo has card 2 B Quinn has card 3 C Romy has card 5
D Stephen has card 7 E Thomas has card 11

SOLUTION **C**

In order to find the correct option, we will try to determine which cards are held by which friend. Here is a list of pairs of cards that are feasible for each friend, given their declared totals.

Pablo (4)	Quinn (11)	Romy (16)	Stephen (19)	Thomas (20)
3, 1	10, 1	12, 4	12, 7	12, 8
	9, 2	11, 5	11, 8	11, 9
	8, 3	10, 6	10, 9	
	7, 4	9, 7		
	6, 5			

The total of all the cards 1 to 12 is 78. The cards held by the friends sum to $4+11+16+19+20 = 70$ so the unused cards sum to 8. Paolo's total is 4 so he has 1 and 3. The unused cards must then be 2 and 6.

The only possibilities for Thomas to have 20 are 12, 8 or 11, 9. Suppose that Thomas has 11, 9. Then Stephen must have 12, 7. However, then there is no way for Romy to have 16. Hence Thomas must have 12, 8. Then Stephen has 10, 9, Romy has 11, 5 and Quinn has 7, 4.

Pablo (4)	Quinn (11)	Romy (16)	Stephen (19)	Thomas (20)
3, 1	7, 4	11, 5	10, 9	12, 8

Of the options given, only *C* is true.

20. Let x and y be positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$. What is the maximum possible value of y?

A 40 B 60 C 240 D 420 E 480

SOLUTION **D**

Rearranging $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$ gives $y = \frac{20x}{x-20} = 20 + \frac{400}{x-20}$. In order to maximise y, we require $x - 20$ to be as small as possible. As x is an integer, $x = 21$. Then $y = 20 + \frac{400}{21-20} = 420$.

21.

Across
1. A multiple of 9
3. A square

Down
1. A multiple of 11
2. A multiple of 13 and of 19

1		2
3		

The crossnumber is to be filled with eight of the digits 1 to 9, which are each used once.

Which digit is not used?

A 9 B 8 C 5 D 3 E 2

Solution **B**

We begin where we have least choice. Here that is '2 Down'. The smallest multiple of both 13 and 19 is $13 \times 19 = 247$. The list of all such three-digit multiples is 247, 494, 741 and 988. As digits may not be repeated we have only 247 and 741. The units digit of '2 Down' is also the units digit of '3 Across', 'A square'. As squares do not end in 7, '2 Down' must be 741. Considering '3 Across', three-digit squares which end in 1 come from $11^2, 21^2, 31^2, 19^2$ or 29^2. However, without repeated digits or use of 4 or 7, our only possibilities are $31^2 = 961$ or $19^2 = 361$. Now considering '1 Down' we look for multiples of 11 which end in either 3 or 9. Multiples ending in 3 come from the answers to $11 \times 13, 11 \times 23, \ldots, 11 \times 83$. Given the remaining available digits, this is either $11 \times 23 = 253$ or $11 \times 53 = 583$. Multiples ending in 9 come from the answers to $11 \times 19, 11 \times 29, \ldots, 11 \times 89$. Given the remaining available digits, this can only be $11 \times 49 = 539$. At this stage we have three cases under consideration.

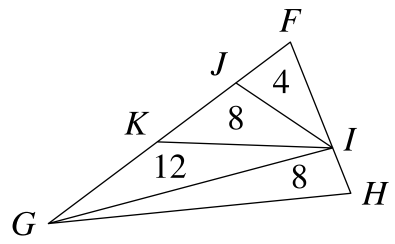
To complete the crossnumber with a multiple of 9 in '1 Across', we require the digits in the top row to sum to a multiple of 9. If '1 Down' were to be either 539 or 583 the middle digit would need to be a 6 so that $5 + 6 + 7 = 18$. However the 6 has already been used. '1 Down' must therefore be the only remaining possibility, 253, and '1 Across' must be 297. The completed grid is as shown. The digit which is not used is 8.

1 2	9	2 7
5		4
3 3	6	1

22. As shown in the diagram, triangle FGH is divided into four smaller triangles which have areas 4, 8, 12 and 8 respectively.

What is the area of triangle IKH?

A 4 B 5 C 6 D 7 E 8



SOLUTION **A**

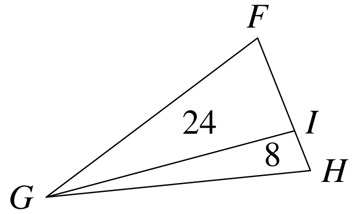


Fig. 1

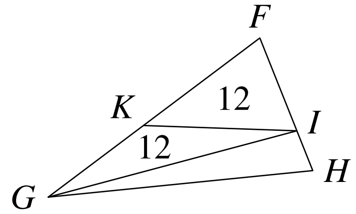


Fig. 2

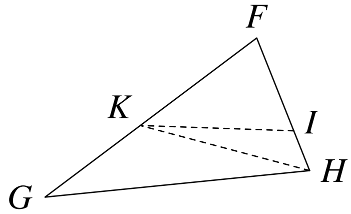


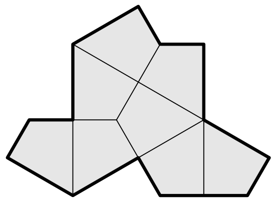
Fig. 3

Using 'area of a triangle = $\frac{1}{2}$ base × perpendicular height' with FH as the base, triangles FGI and FGH have the same perpendicular height. Their areas are therefore in the same proportions as the lengths of their bases and so $IH = \frac{1}{4}FH$.

Now viewing FI as the base of both triangles FGI and FKI, we can deduce that the perpendicular height from FI to K is half the perpendicular height from FI to G.

The area of triangle $IKH = \frac{1}{2}\ IH \times$ the perpendicular height from IH to $K = \frac{1}{2} \times \frac{1}{4}FH \times \frac{1}{2}$ the perpendicular distance from IH to $G = \frac{1}{8} \times$ the area of triangle $FGH = \frac{1}{8} \times 32 = 4$.

23. The plane can be tiled using the 'hat tile' shown here. This tile can be subdivided into eight congruent kites. The area of the hat tile is $8\sqrt{3}$.

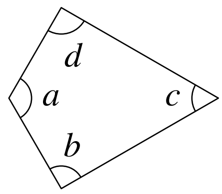


What is the perimeter of the hat tile?

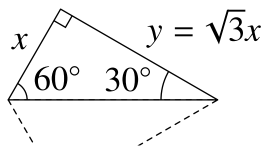
A $8 + 12\sqrt{3}$ B $16 + 6\sqrt{3}$ C $8 + 8\sqrt{3}$ D $6 + 8\sqrt{3}$ E $8 + 6\sqrt{3}$

Solution **E**

On the diagram shown in the question, we can see that each kite has a line of symmetry. Therefore $\angle b = \angle d$. Also, where three kites meet at a point with no gaps, $\angle a = 120°$. Where four kites meet at a point with no gaps, $\angle b = \angle d = 90°$. As the angle sum of a quadrilateral is $360°$, $\angle c = 60°$. Each half-kite is therefore a $30°, 60°, 90°$ triangle, with lengths in the ratio $1 : \sqrt{3} : 2$.



Let the perpendicular lengths be x and y as shown. So $y = \sqrt{3}x$. As the area of the whole hat tile is $8\sqrt{3}$, the area of each kite is $\sqrt{3}$. Therefore $2 \times \frac{xy}{2} = \sqrt{3}$, so $\sqrt{3}x^2 = \sqrt{3}$ and $x = 1$. Therefore $y = \sqrt{3}$. The perimeter of the hat tile is $8x + 6y = 8 \times 1 + 6 \times \sqrt{3} = 8 + 6\sqrt{3}$.



24. A function f satisfies the equation $f(x) + f\left(\frac{1}{1-x}\right) = 24x$ for all real values of x except $x = 0$ and $x = 1$.

What is the value of $f(3)$?

A 40 B 42 C 45 D 48 E 50

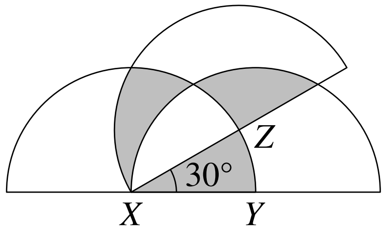
Solution **E**

First let $x = 3$, then $f(3) + f(\frac{1}{1-3}) = 24 \times 3$. Therefore $f(3) + f(-\frac{1}{2}) = 72$ (a). Now let $x = -\frac{1}{2}$, then $f(-\frac{1}{2}) + f(\frac{1}{1-\frac{-1}{2}}) = 24 \times -\frac{1}{2}$. So $f(-\frac{1}{2}) + f(\frac{2}{3}) = -12$ and thus $-f(-\frac{1}{2}) - f(\frac{2}{3}) = 12$ (b). Finally, let $x = \frac{2}{3}$, then $f(\frac{2}{3}) + f(\frac{1}{1-\frac{2}{3}}) = 24 \times \frac{2}{3}$. This simplifies to $f(\frac{2}{3}) + f(3) = 16$ (c). Adding equations (a), (b) and (c) leads to $2 \times f(3) = 72 + 12 + 16$. Therefore $f(3) = 50$.

25. Three semicircles, each of area 24, overlap as shown in the diagram. The centres of the arcs are X, Y and Z and $\angle ZXY = 30°$.

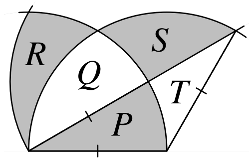
What is the total area of the shaded regions?

A 12 B $6\sqrt{3}$ C 15 D 18 E $8\sqrt{3}$

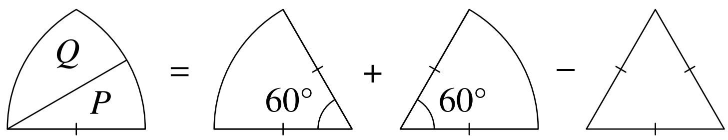


SOLUTION **A**

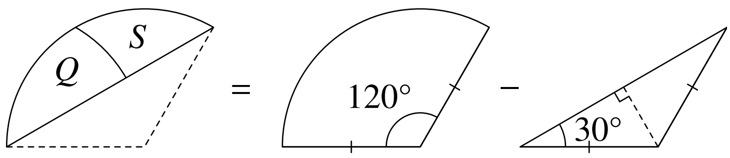
The diagram shows regions P, Q, R, S and T. As $\angle ZXY = 30°$, area of $P = \frac{1}{6}$ of the area of a semicircle $= \frac{1}{6} \times 24 = 4$. Regions $(P + Q)$ and $(Q + R)$ are congruent therefore area of $(P + Q)$ = area of $(Q + R)$ and so area of R = area of $P = 4$. We can also show that the area of $(P + Q)$ is the same as that of $(Q + S)$ as follows:



Region $(P+Q)$ can be deconstructed into two overlapping, congruent 60° sectors minus an equilateral triangle.



Region $(Q + S)$ can be deconstructed into a 120° sector, minus two 30°, 60°, 90° triangles whose total area is that of the equilateral triangle shown in the deconstruction of the region $(P + Q)$.



Therefore area of $(P + Q)$ = area of $(Q + S)$ and so area of P = area of $S = 4$. Hence the total area of the shaded region $= 4 + 4 + 4 = 12$.

United Kingdom Mathematics Trust

Senior Mathematical Challenge

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Solutions and investigations

October 1st, 2024

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
E	A	B	C	B	D	A	D	E	D	D	A	C	B	C	B	E	C	C	D	B	A	E	E	A

1. What is two-fifths of the recurring decimal $0.\dot{2}\dot{5}$?

A $0.\dot{1}$ B $0.0\dot{1}$ C $0.\dot{0}\dot{1}$ D 0.10 E $0.\dot{1}\dot{0}$

SOLUTION **E**

METHOD 1

One-fifth of $0.\dot{2}\dot{5}$ is $0.\dot{0}\dot{5}$. Hence two-fifths of $0.\dot{2}\dot{5}$ is $0.\dot{1}\dot{0}$.

METHOD 2

Since $0.\dot{2}\dot{5} = \frac{25}{99}$, it follows that that two-fifths of $0.\dot{2}\dot{5} = \frac{2}{5} \times \frac{25}{99} = \frac{10}{99} = 0.\dot{1}\dot{0}$.

FOR INVESTIGATION

1.1 Express two-thirds of $0.\dot{0}\dot{6}$ both as a recurring decimal and as a fraction.

1.2 Check that $0.\dot{2}\dot{5} = \frac{25}{99}$.

1.3 Express $\frac{17}{99}$ as a recurring decimal.

1.4 Express the recurring decimal $0.\dot{1}2\dot{3}$ as a fraction.

2. A ***twip*** is a very short unit of length, derived from imperial units, and is equal to approximately 0.000018 metres. A ***league*** is a long unit of length which is equal to approximately 4800 metres.

Roughly how many twips are there in a league?

A 270 000 000 B 27 000 000 C 2 700 000 D 270 000 E 27 000

SOLUTION **A**

The number of twips in a league is $\frac{4800}{0.000018}$. Now

$$\frac{4800}{0.000018} = \frac{4\,800\,000\,000}{18} = \frac{800\,000\,000}{3} \approx \frac{810\,000\,000}{3} = 270\,000\,000.$$

FOR INVESTIGATION

2.1 The length of a cricket pitch is one *chain*, which is 22 yards. A *light year* is the distance that light travels (through a vacuum) in one year. Light travels at around 186 000 miles per second in a vacuum. There are 1760 yards in one mile.

Approximately, how many cricket pitches can be fitted into a light year?

3. Two standard dice are placed on a table, with one on top of the other, so that only nine of the faces of the dice may be seen. The touching faces have the same number on them. The sum of the numbers on the visible faces is 33.

What is the number on the touching faces?

A 1 B 2 C 3 D 4 E 6

Solution **B**

Let x be the number which is on the two touching faces. On a standard dice the numbers on opposite faces have sum 7. Hence the number on the bottom face which touches the table and hence is not visible is $7 - x$. Therefore the total of the numbers on the three faces that are not visible is $x + x + (7 - x) = 7 + x$.

The total of the numbers on all six faces of a standard dice is $1 + 2 + 3 + 4 + 5 + 6 = 21$.

It follows that the sum of the visible numbers is $2 \times 21 - (7 + x) = 35 - x$. Therefore $35 - x = 33$. Hence $x = 2$.

For investigation

3.1 Suppose that three standard dice are placed on a table stacked one above the other. In this situation there are thirteen visible numbers.

What is the highest possible total of the thirteen visible numbers?

4. The sizes of the three angles in a triangle, in degrees, are x, $7x$ and x^2.

What is the size of the largest angle?

A 10° B 18° C 100° D 120° E 121°

Solution **C**

The sum of the angles in a triangle is 180°. Therefore $x + 7x + x^2 = 180$. Hence $x^2 + 8x - 180 = 0$. By factorizing the left-hand side of this equation, we deduce that $(x + 18)(x - 10) = 0$.

This implies that either $x = 10$ or $x = -18$. Because $x°$ is one of the angles of the triangle, $x > 0$. We conclude that $x = 10$.

It follows that the angles of the triangle are 10°, 70° and 100°. Hence the largest angle is 100°.

For investigation

4.1 The angles of a triangle, in degrees, are $x^2 + 9x$, x^2 and $9x$.

What is the size of the smallest angle?

4.2 The angles of a triangle, in degrees, are x^3, $14x^2$ and $9x$, where x is an integer.

Show that this triangle is isosceles.

5. When $4^5 \times 5^4$ is correctly calculated, how many digits are there in the answer?

A 4 B 6 C 10 D 16 E 20

SOLUTION **B**

We have

$$4^5 \times 5^4 = (2^2)^5 \times 5^4 = 2^{10} \times 5^4 = 2^6 \times (2^4 \times 5^4) = 2^6 \times 10^4 = 64 \times 10\,000 = 640\,000.$$

Therefore there are 6 digits in the correct answer.

FOR INVESTIGATION

5.1 How many digits are there in the answer when the number $8^3 \times 5^8$ is calculated correctly?

6. One face of a solid polyhedron is an octagon.

What is the smallest possible number of edges the solid could have?

A 9 B 10 C 12 D 16 E 24

SOLUTION **D**

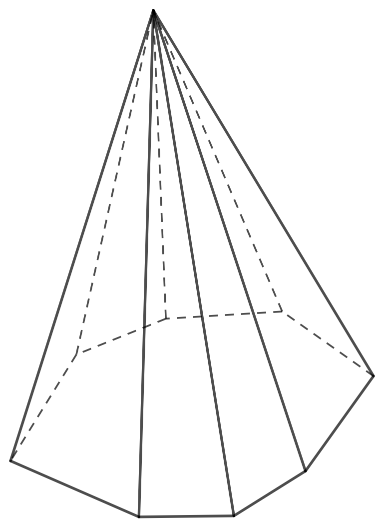
The octagonal face of the polyhedron has 8 edges.

Each vertex of this face must be joined by an edge to a vertex not in this face. So there must be at least 8 more edges.

Hence the polyhedron has at least $8 + 8 = 16$ edges.

If all the vertices of the octagonal face are joined by an edge to the same vertex, we obtain a pyramid with an octagonal base that has 16 edges.

Therefore the smallest number of edges that the solid could have is 16.



FOR INVESTIGATION

6.1 Three of the faces of a polyhedron are squares.

What is the smallest number of vertices that this polyhedron could have?

7. Which is the largest prime factor of $3^8 - 1$?

A 41 B 37 C 31 D 29 E 23

SOLUTION **A**

Using the difference of two squares factorization $x^2 - y^2 = (x - y)(x + y)$, we have

$$\begin{aligned} 3^8 - 1 &= (3^4 - 1)(3^4 + 1) \\ &= (3^2 - 1)(3^2 + 1)(3^4 + 1) \\ &= (3 - 1)(3 + 1)(9 + 1))(81 + 1) \\ &= 2 \times 4 \times 10 \times 82 \\ &= 2 \times 2^2 \times (2 \times 5) \times (2 \times 41) \\ &= 2^5 \times 5 \times 41. \end{aligned}$$

Because 41 is a prime, we deduce that the largest prime factor of $3^8 - 1$ is 41.

FOR INVESTIGATION

7.1 Find the largest prime factor of the integers

(a) $2^8 - 1$, (b) $2^{16} - 1$ and (c) $5^6 - 1$.

8. In the following expressions, x is non-zero. When one of these expressions is removed, the mean of the remaining four is $11x$.

Which expression is removed?

A $4x$ B $8x$ C $12x$ D $16x$ E $20x$

SOLUTION **D**

Because the mean of the remaining four numbers is $11x$, their sum is $4 \times 11x$, that is, $44x$.

The sum of all five of the numbers is $4x + 8x + 12x + 16x + 20x$, that is, $60x$.

Since $60x - 44x = 16x$, the number that is removed is $16x$.

FOR INVESTIGATION

8.1 In the following expressions, the number x is non-zero: x, $5x$, $9x$, $13x$, $17x$. When one of these expressions is removed, the mean of the remaining four is $8x$.

Which expression is removed?

8.2 In the following expressions, the number x is a positive integer: x^2, x, $2x$, $3x$, $4x$. When one of these expressions is removed, the mean of the remaining four is 4.

(a) Which expression is removed?

(b) What is the value of x?

> **9.** A palindromic number is one where the digits read the same forwards as backwards, such as 123 321.
>
> What is the hundreds digit of the largest six-digit palindromic number that is divisible by 18?
>
> A 9 B 7 C 5 D 3 E 1

SOLUTION **E**

Let the six-digit palindromic number be '$pqrrqp$'.

For this number to be divisible by 18, it needs to be divisible by 2 and by 9. To be divisible by 2, the digit p must be even. For the number to be as large as possible, p has to be as large as possible. Therefore we try $p = 8$. This makes the number '$8qrrq8$'.

A number is divisible by 9 precisely when the sum of its digits is divisible by 9. [See Problem 9.4.] Therefore $2(8 + q + r)$ needs to be divisible by 9. Hence $8 + q + r$ needs to be divisible by 9. Again, we want q to be as large as possible . So we try $q = 9$. With this choice for q, we require that $8 + 9 + r$ is divisible by 9. Therefore $8 + r$ must be divisible by 9. Hence $r = 1$.

It follows that the number '$8qrrq8$' is 891198. Therefore this is the largest six-digit palindromic number that is divisible by 18. The hundreds digit of this number is 1.

FOR INVESTIGATION

9.1 Which is the smallest six-digit palindromic number which is divisible by 18?

9.2 Which are the smallest and largest six-digit palindromic numbers which are divisible by 45?

9.3 Which is the largest seven-digit palindromic number which is divisible by 18?

9.4 In the solution above we have used the fact that

> A positive integer is divisible by 9 precisely when the sum of its digits is divisible by 9.

Prove that this is correct.

9.5 One way to answer Problem 9.4 is to prove the following more general fact.

> A positive integer and the sum of its digits have the same remainder when divided by 9.

Prove that this is correct.

9.6 The solution also uses the fact that, for every positive integer n,

$$n \text{ is divisible by } 18 \iff n \text{ is divisible by } 2 \text{ and } n \text{ is divisible by } 9.$$

Explain why this is correct.

9.7 Is it true that, for every positive integer n,

$$n \text{ is divisible by } 24 \iff n \text{ is divisible by } 4 \text{ and } n \text{ is divisible by } 6 \ ?$$

10. The prime factorization of 2024 is $2^3 \times 11 \times 23$.

How many two-digit numbers are factors of 2024?

A 2 B 4 C 6 D 7 E 8

Solution **D**

Written in full we have

$$2024 = 2 \times 2 \times 2 \times 11 \times 23.$$

The factors of 2024 are 1 and the numbers obtained by multiplying together some, or all, of the numbers that occur in this product.

We see from the factorization that 11 and 23 are two-digit factors of 2024.

Since $11 \times 23 = 253$ no product which includes both 11 and 23 produces a two-digit factor of 2024. However, we can obtain two-digit factors of 2024 by multiplying 11 or 23 by 2, 2^2 or 2^3 provided that the resulting product has two digits.

In this way we obtain the two-digit factors

$$\begin{aligned} 2 \times 11 &= 22, \\ 2^2 \times 11 &= 44, \\ 2^3 \times 11 &= 88, \\ 2 \times 23 &= 46 \end{aligned}$$

and

$$2^2 \times 23 = 92.$$

Hence, altogether there are 7 two-digit factors of 2024, namely, 11, 22, 23, 44, 46, 88 and 92.

For investigation

10.1 Find all the three-digit factors of 2024.

10.2 Find all the two-digit factors of 10 000.

11. Which one of the following expressions is a square number for each positive integer n?

A $n + 1$
B $n(n + 1) + 1$
C $n(n + 1)(n + 2) + 1$
D $n(n + 1)(n + 2)(n + 3) + 1$
E $n(n + 1)(n + 2)(n + 3)(n + 4) + 1$

SOLUTION **D**

When $n = 1$, we have

$$\begin{aligned} n + 1 &= 2 \\ n(n + 1) + 1 &= 1 \times 2 + 1 = 3 \\ n(n + 1)(n + 2) + 1 &= 1 \times 2 \times 3 + 1 = 7 \\ n(n + 1)(n + 2)(n + 3) + 1 &= 1 \times 2 \times 3 \times 4 + 1 = 25 \\ \text{and} \quad n(n + 1)(n + 2)(n + 3)(n + 4) + 1 &= 1 \times 2 \times 3 \times 4 \times 5 + 1 = 121. \end{aligned}$$

Of these values, only 25 and 121 are squares. So the correct answer is either D or E. To decide between these we put $n = 2$. This gives

$$\begin{aligned} n(n + 1)(n + 2)(n + 3) + 1 &= 2 \times 3 \times 4 \times 5 + 1 = 121 \\ \text{and} \quad n(n + 1)(n + 2)(n + 3)(n + 4) + 1 &= 2 \times 3 \times 4 \times 4 \times 6 + 1 = 721 \end{aligned}$$

of which only 121 is a square.

In the context of the SMC we can now conclude that the correct option is D.

In the SMC you are entitled to assume that just one of the given options is correct. Therefore the correct answer can be found by eliminating four of the options, as we have done above. However, this would not be an adequate answer if you were required to give a fully explained answer.

In this case we can show that all the values of the polynomial $n(n + 1)(n + 2)(n + 3) + 1$ are squares, as follows.

$$\begin{aligned} n(n + 1)(n + 2)(n + 3) + 1 &= n(n + 3)(n + 1)(n + 2) + 1 \\ &= (n^2 + 3)(n^2 + 3n + 2) + 1 \\ &= ((n^2 + 3n + 1) - 1)((n^2 + 3n + 1) + 1) + 1 \\ &= (n^2 + 3n + 1)^2 - 1 + 1 \\ &= (n^2 + 3n + 1)^2, \end{aligned}$$

and hence is always a square.

It is true that, in general, if a polynomial has a value which is a square for each positive integer n then this polynomial is the square of another polynomial. The proof of this fact is not straightforward. The above calculation shows the truth of this in one particular case.

FOR INVESTIGATION

11.1 Prove that there does not exist a positive integer n for which $n(n + 1) + 1$ is a square.

12. p, q, r and s are two-digit primes which between them use all the non-zero digits except 5.

What is the value of $p + q + r + s$?

A 220 B 210 C 200 D 190
E more information needed

Solution **A**

Of the digits 1, 2, 3, 4, 6, 7, 8 and 9, only 1, 3, 7 and 9 can be the units digit of a two-digit prime. This leaves 2, 4, 6 and 8 as the tens digits of the primes.

Therefore the sum of the units digits of the four primes is $1 + 3 + 7 + 9$, which is equal to 20, and the sum of the tens digits is $2 + 4 + 6 + 8$, which is also 20.

Hence the sum of the four primes is $20 \times 10 + 20 \times 1 = 200 + 20 = 220$.

This answer has been obtained without finding the primes p, q, r and s, or indeed even showing that four primes which use all the non-zero digits except 5, exist. You are asked to check in Problem 12.1 that a solution does exist.

For investigation

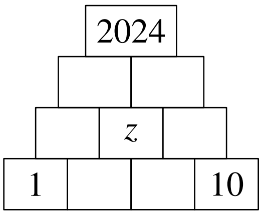
12.1 Find four two-digit primes p, q, r and s which between them use all the non-zero digits except 5.

12.2 How many different sets are there of four two-digit primes which between them use all the digits except 5?

12.3 Find one three-digit prime and three two-digit primes which between them include all the non-zero digits.

12.4 Find two three-digit primes and two two-digit primes which between them use all the ten digits, including 0.

13. The diagram shows a partially completed number pyramid. When correctly completed, the number on any brick above the bottom row should be the sum of the two numbers on the two bricks on which it rests.



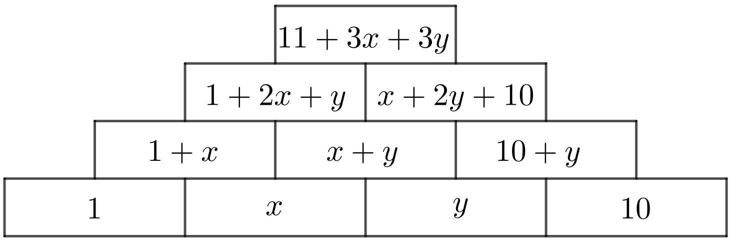
What number should appear on the brick marked 'z'?

A 176 B 617 C 671 D 716 E 761

Solution **C**

We let the numbers that are in the second and third boxes in the bottom row be x and y.

Then the remaining numbers are as shown in the diagram on the right.



Then $z = x + y$ and $11 + 3x + 3y = 2024$.

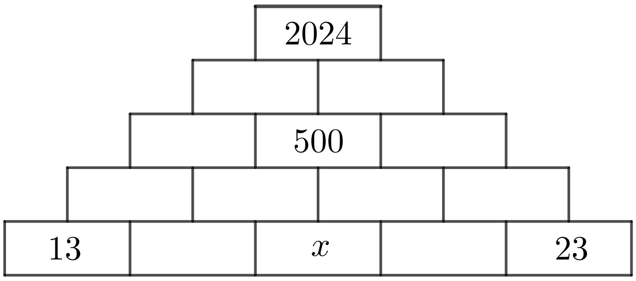
It follows from the second of these equations that $3x + 3y = 2013$. Hence $x + y = 2013 \div 3 = 671$. Hence $z = 671$.

For investigation

13.1 Suppose that the numbers in the bottom row of the number pyramid are w, x, y and z, from left to right. What number is in top box? What do you notice?

13.2 The diagram on the right shows a partially completed number pyramid.

When correctly completed, the number on any brick above the bottom row should be the sum of the two numbers on the two bricks on which it rests.



Which number should appear on the bricked marked with an x ?

13.3 Suppose you are given the partially completed number pyramid as shown in the diagram of Problem 13.2. In this diagram you are given the numbers on four of the bricks.

What is the smallest number of additional bricks on which you would need to know the number in order to work out which number should be on each of the bricks?

14. P, Q, R, S and T are the digits 1, 2, 3, 4 and 5 in some order. 'PRT' and 'QRS' are both three-digit primes.

Which digit is R?

A 1 B 2 C 3 D 4 E 5

SOLUTION **B**

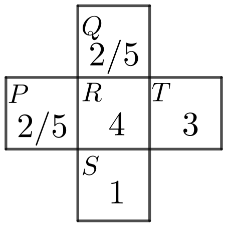
Of the available digits, only 1 and 3 can be the units digit of a three-digit prime. Without loss of generality we can assume that S is 1 and T is 3.

This leaves 2, 4 and 5 for the values of P, Q and R.

Suppose that R were 4. This would leave 2 and 5 for the values of P and Q.

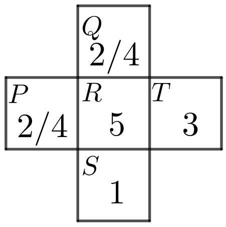
Hence 'QRS' and 'PRT' would either be 241 and 543, or 541 and 243.

Neither of these is possible, as both 543 and 243 are divisible by 3, and hence they are not primes. So R cannot be 4.



Suppose next that R were 5. This would leave 2 and 4 for the values of P and Q. Hence 'QRS' and 'PRT' would either be 251 and 453, or 451 and 253.

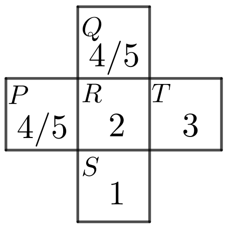
Neither of these is possible, as 453 is divisible by 3, and 253 is divisible by 11 and hence they are not primes. So R cannot be 5.



In the context of the SMC we can conclude that R is 2. [In Problem 14.1 you are asked to check that this is possible.]

FOR INVESTIGATION

14.1 Show that when R is 2, the digits P, Q, S and T may be chosen to be 1, 3, 4 and 5 in some order, so that both 'QRS' and 'PRT' are primes.



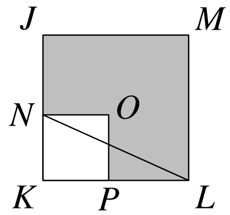
14.2 In how many way can P, Q, R, S and T be chosen to be the digits 2, 3, 4, 5 and 6 in some order so that 'PRT' and 'QRS' are both three-digit primes?

14.3 In how many way can P, Q, R, S and T be chosen to be the digits 3, 4, 5, 6 and 7 in some order so that 'PRT' and 'QRS' are both three-digit primes?

15. The diagram shows two squares, $JKLM$ and $NKPO$.
The length of NL is 10 cm. The shaded region has area 62 cm^2.

What is the length of KN in cm?

A 3 B $\sqrt{18}$ C $\sqrt{19}$ D $\sqrt{22}$ E 5



Solution **C**

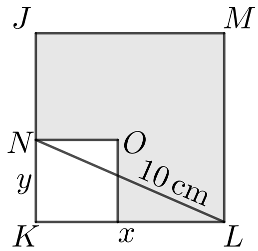
Suppose that the side length of the square $JKLM$ is x cm and the side length of the square $NKPO$ is y cm.

The area of the shaded region is the area of $JKLM$ minus the area of $NKPO$. Therefore

$$x^2 - y^2 = 62. \qquad (1)$$

Also, by Pythagoras' Theorem $KL^2 + NK^2 = NL^2$. That is,

$$x^2 + y^2 = 10^2. \qquad (2)$$



By subtracting equation (1) from equation (2), we obtain

$$2y^2 = 10^2 - 62 = 100 - 62 = 38.$$

Therefore $y^2 = 19$. Hence $y = \sqrt{19}$.

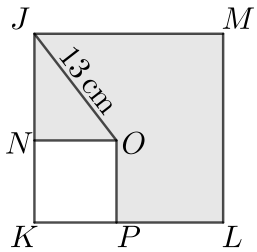
For investigation

15.1 $JKLM$ and $KPON$ are squares.

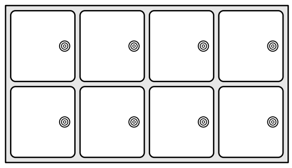
The square $JKLM$ has area 289 cm^2.

JO has length 13 cm.

What are the possibilities for the area of the square $KPON$?



16. A set of cupboards containing eight identical blue doors is arranged in a 2 by 4 grid as shown. A fussy decorator wishes to paint three of the doors red such that at least one door in each row is painted red and at least two of the four corners are painted red.



How many ways are there to do this?

A 12 B 24 C 36 D 40 E 56

Solution **B**

To solve this problem we need to count the number of ways of painting the doors systematically so that each possible arrangement is counted once, and once only.

We label the doors with the letters A, B, C, D, E, F, G and H, as shown in the diagram on the right, so that we can refer to them.

A	B	C	D
E	F	G	H

We know that at least two of the doors A, D, E and H must be painted red. We list the possible cases according to which is the first of these doors that is painted red.

The first row of the diagrams below covers the cases where door A is painted red. Diagram 1 deals with the cases where door D is also red. The third red door must be in the second row. Hence there are four doors which could be the third red door. These are numbered 1, 2, 3 and 4 in the diagram. Diagram 2 deals with the cases where door A is red, door D is not red, but door E is red. In this case there is already a red door in each row. Hence there are five possibilities for the third red door, numbered 1, 2, 3, 4 and 5 in the diagram.

We leave it to the reader to check that all the remaining possibilities are covered in a similar way in diagrams 3 to 6, and that no case is counted twice.

R	$\times$	$\times$	R
1	2	3	4

diagram 1

R	1	2	$\times$
R	3	4	5

diagram 2

R	1	2	$\times$
$\times$	3	4	R

diagram 3

$\times$	1	2	R
R	3	4	5

diagram 4

$\times$	1	2	R
$\times$	3	4	R

diagram 5

$\times$	1	2	$\times$
R	$\times$	$\times$	R

diagram 6

We can calculate the number of possible cases by adding up the numbers corresponding to the six diagrams. Hence there are $4 + 5 + 4 + 5 + 4 + 2 = 24$ ways to paint the doors so as to meet the required conditions.

For investigation

16.1 Check that diagrams 1 to 6 cover all of the possible cases once each.

16.2 How many ways are there to paint four of the doors red, so that there is at least one red door in each row, and at least two of the four corner doors are red?

17. A bag contains four balls each of which is coloured either red or white. If one ball is drawn at random from the bag but not replaced and then a second ball is drawn at random, the probability that both balls are red is $\frac{1}{2}$.

What is the probability that both balls are white?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{6}$ E 0

SOLUTION **E**

Suppose that r of the four balls are red. There are r ways to choose a red ball first, and this leaves $r - 1$ ways to choose a second red ball. So there are are $r \times (r - 1)$ ways in which both balls are red.

With 4 balls in the bag there are $4 \times 3 = 12$ ways to choose first one of the four balls and then one of the three of the remaining balls.

Since the probability that both balls are red is $\frac{1}{2}$,

$$\frac{r(r-1)}{12} = \frac{1}{2}.$$

Hence

$$r(r-1) = 6.$$

Therefore

$$r^2 - r - 6 = 0.$$

That is,

$$(r-3)(r+2) = 0.$$

Hence $r = 3$ or $r = -2$. Since $r \geq 0$, we conclude that $r = 3$.

It follows that there is just one white ball in the bag. Hence the probability that both balls are white is 0.

FOR INVESTIGATION

17.1 The following argument is taken from Lewis Carroll's *Pillow Problems* (1893).

Problem: Suppose a bag contains two counters each of which is red or white with equal probability. Ascertain their colours without taking them out of the bag.

Answer: One counter is white and the other is red.

Full solution: The chances that the counters in the given bag are (a) two white, (b) one white and one red and (c) two red are, respectively, $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$.

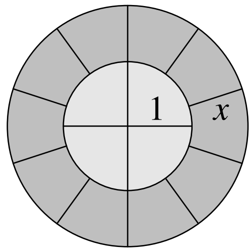
Add a red counter. Then the chances that the counters in the bag are now (a) two white and one red, (b) one white and two red and (c) three red are $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$, as before.

Hence the chance of now drawing a red counter from the bag is $\frac{1}{4} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} + \frac{1}{4} \times 1 = \frac{1}{12} + \frac{1}{3} + \frac{1}{4} = \frac{2}{3}$. Hence the bag must now contain one white and two red counters since with any other combination the chance of drawing a red counter would not be $\frac{2}{3}$.

Hence before the red counter was added the bag contained one white counter and one red counter.

Is this argument correct?

18. The diagram shows two concentric circles divided by radial lines into 14 pieces of equal area. The radius of the smaller circle is 1.



What is the length, x, of an outer radial line?

A $\sqrt{14} - 1$ B $\sqrt{14} - 2$ C $\dfrac{\sqrt{14}}{2} - 1$

D $\dfrac{\sqrt{14}}{2} - 2$ E $\dfrac{\sqrt{14} - 1}{2}$

SOLUTION **C**

The outer circle has radius $1 + x$. Hence its area is $\pi(1 + x)^2$.

The inner circle has radius 1, and hence has area $\pi(1^2) = \pi$. Hence each of the four inner pieces has area $\dfrac{\pi}{4}$.

Because all the 14 pieces have the same area, the area of the outer circle is 14 times the area of one of the inner pieces. Therefore

$$\pi(1 + x)^2 = 14\left(\frac{\pi}{4}\right).$$

This simplifies to give

$$(1 + x)^2 = \frac{14}{4}.$$

Hence, as $1 + x > 0$,

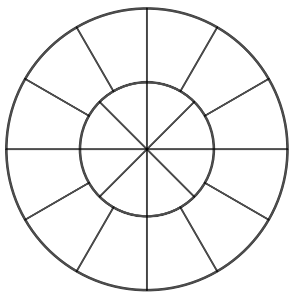
$$1 + x = \sqrt{\frac{14}{4}} = \frac{\sqrt{14}}{2}.$$

We conclude that

$$x = \frac{\sqrt{14}}{2} - 1.$$

FOR INVESTIGATION

18.1 The diagram on the right shows two concentric circles divided by radial lines into 20 pieces of equal area.



The inner circle has radius r.

The outer circle has radius s.

Find the ratio $r : s$.

19. Five friends are dealt two cards each from a set of twelve cards. The cards are numbered 1 to 12 inclusive. In turn, the friends declare the sum of the values of their two cards. Paolo scores 4, Quinn scores 11. Romy scores 16. Stephen scores 19 and Thomas scores 20.

Which of the following statements is true?

A Paolo has card 2
B Quinn has card 3
C Romy has card 5
D Stephen has card 7
E Thomas has card 11

Solution **C**

The numbers on the cards are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.

Since Paolo scored 4, he must have been dealt cards 1 and 3. It follows that both the statements A and B are false.

Because Paolo has cards 1 and 3, cards 2, 4, 5, 6, 7, 8, 9, 10, 11 and 12 are left for Quinn, Romy, Stephen and Thomas.

We now consider the possibility that statement C is true. In this case Romy's cards must be 5 and 11. This leaves cards 2, 4, 6, 7, 8, 9, 10 and 12 as possible cards held by Quinn, Stephen and Thomas.

Since Thomas scores 20 his cards must be 8 and 12. Then, as Stephen scores 19, his cards must be 9 and 10. This leaves 2, 4, 6 and 7. So Quinn's cards are 4 and 7.

We have therefore shown that it is possible for statement C to be true.

In the context of the SMC we can stop here. We can assume that only one of the statements could be true. So having shown that C could be true, we can conclude that C is the correct option.

For a complete answer it would be necessary to check that statements D and E cannot be true. You are asked to do this Problem 19.1.

For investigation

19.1 Determine which cards each of the five friends were dealt.

Hence, show that statements D and E are not true.

20. Let x and y be positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$. What is the maximum possible value of y?

A 40 B 60 C 240 D 420 E 480

SOLUTION **D**

The maximum possible value of y corresponds to the minimum possible value of $\frac{1}{y}$.

Since $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$, the minimum possible value of $\frac{1}{y}$ corresponds to the maximum possible value of $\frac{1}{x}$, and hence to the minimum possible value of x.

Now $\frac{1}{x} = \frac{1}{20} - \frac{1}{y} < \frac{1}{20}$ because $y > 0$. Therefore $x > 20$.

Since x is a positive integer, it follows that the minimum possible value of x is 21.

When $x = 21$, we have $\frac{1}{y} = \frac{1}{20} - \frac{1}{21} = \frac{21 - 20}{20 \times 21} = \frac{1}{420}$ and hence $y = 420$.

Therefore the maximum possible value of y is 420.

FOR INVESTIGATION

20.1 (a) Rearrange the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$ to express y in terms of x.

(b) Use this rearranged equation to show that the maximum possible value of y is 420.

20.2 Let x and y be positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{25}$.

What is the maximum possible value of y?

20.3 Let x, y and z be positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.

In terms of z, what is the maximum possible value of y?

20.4 Find all the the solutions of the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$, where x and y are positive integers with $x < y$.

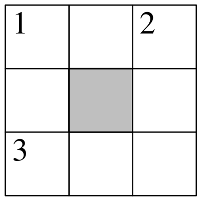
20.5 Find all the solutions of the equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, where x, y and z are positive integers with $x < y < z$.

20.6 Find all the solutions of the equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}$, where x, y and z are positive integers with $x < y < z$.

21.

Across
1. A multiple of 9
3. A square

Down
1. A multiple of 11
2. A multiple of 13 and of 19



The crossnumber is to be filled with eight of the digits 1 to 9, which are each used once.

Which digit is not used?

A 9 B 8 C 5 D 3 E 2

Solution **B**

We begin with 2 Down because it looks as though there are fewer possible answers for this clue than for the other clues.

13 and 19 have no factors in common (other than 1). Hence a multiple of 13 and 19 is also a multiple of 13×19. That is, it is a multiple of 247.

The three-digit multiples of 247 are 247, 494, 741 and 988. 2 Down is not 247, because then 3 Across would have 7 as its units digit which is not possible for a square. 2 Down is neither 494 nor 988 because these numbers contain repeated digits. We may therefore deduce that 2 Down is 741.

It follows that 3 Across is a three-digit square with units digit 1. So the possibilities for 3 Across are $11^2 = 121$, $19^2 = 361$, $21^2 = 441$, $29^2 = 841$ and $31^2 = 961$.

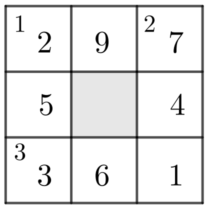
We can rule out 121 and 441 because they have a repeated digit, and 841 because the digit 4 has already been used in 2 Down. This leaves 361 and 961 as possible values for 3 Across.

Suppose 3 Across is 361. Then the digits 2, 5, 8 and 9 have not yet been used.

For 1 Across to be a multiple of 9 its digits must add up to a multiple of 9. The units digit of 1 Across is 7, and its other digits are two of 2, 4, 8 and 9. The only possibilities are that the digits of 1 Across add up to 18, and that 1 Across is either 297 or 927.

Suppose 1 Across is 297. Then 1 Down is $2x3$ where x is either 5 or 8.

Now $253 = 11 \times 23$ and hence is a multiple of 11. Therefore we can complete the crossnumber as shown on the right.



In this completed crossnumber the digit that is not used is 8.

In the context of the SMC where we are entitled to assume that just one of the given options is correct, we can stop here now that we have found one solution. For a complete solution we would need to show that there is no other solution. You are asked to do this in Problem 21.1

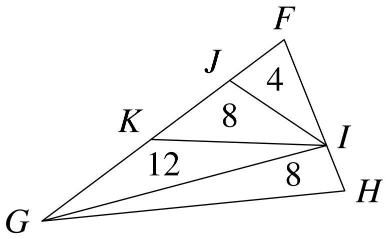
For investigation

21.1 Show that the solution given above is the only solution of the crossnumber.

22. As shown in the diagram, triangle FGH is divided into four smaller triangles which have areas 4, 8, 12 and 8 respectively.

What is the area of triangle IKH?

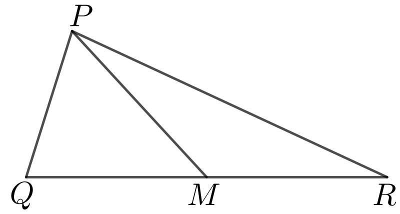
A 4 B 5 C 6 D 7 E 8



Solution **A**

In this question we use the fact that if M is a point on the side QR of the triangle PQR, we have

$$\frac{\text{area } MPQ}{\text{area } MPR} = \frac{MQ}{MR}. \quad (1)$$



[You are asked to prove this in Problem 22.1.]

From the diagram in the question we have

$$\frac{\text{area } IGF}{\text{area } IGH} = \frac{12+8+4}{8} = \frac{24}{8} = 3. \quad (2)$$

It follows from (1) and (2) that

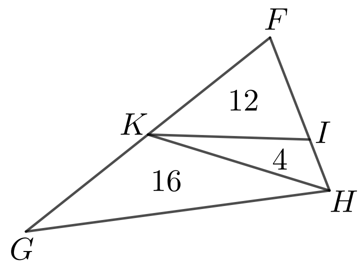
$$\frac{IF}{IH} = 3. \quad (3)$$

Therefore, by (1) and (3),

$$\frac{\text{area } IKF}{\text{area } IKH} = 3.$$

Hence

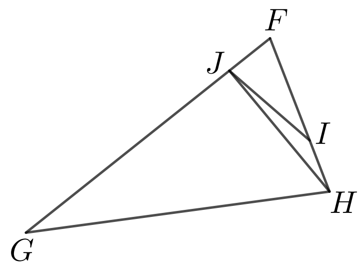
$$\text{area } IKH = \tfrac{1}{3}(\text{area } IKF) = \tfrac{1}{3}(4+8) = \tfrac{1}{3}(12) = 4.$$



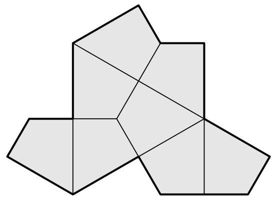
For investigation

22.1 Prove that equation (1) is correct.

22.2 Find the area of the triangle IJH.



23. The plane can be tiled using the 'hat tile' shown here. This tile can be subdivided into eight congruent kites. The area of the tile is $8\sqrt{3}$.

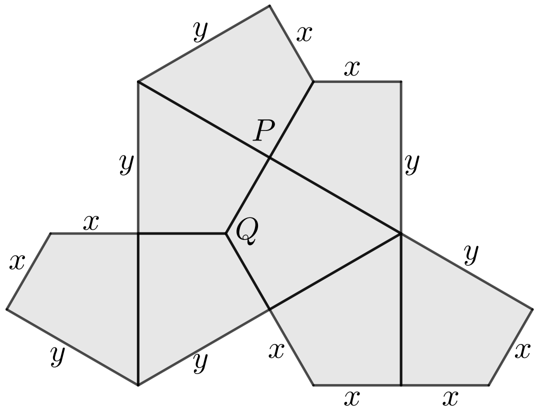


What is the perimeter of the hat tile?

A $8 + 12\sqrt{3}$ B $16 + 6\sqrt{3}$ C $8 + 8\sqrt{3}$
D $6 + 8\sqrt{3}$ E $8 + 6\sqrt{3}$

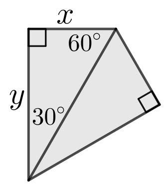
Solution **E**

Since four congruent kites meet at the point P, the angles in the kites at P are each equal to 90°. Since three congruent kites meet at Q, the angles at Q are each 120°.



It follows that the longer diagonal of each kite divides it into two triangles in which two of the angles are 90° and 60° as shown in the second diagram.

We let the lengths of the perpendicular edges of these triangles be x and y, as shown.



It follows that the area of each triangle is $\frac{1}{2}xy$. Hence the area of each kite is $2 \times \frac{1}{2}xy = xy$.

Now $y = x \tan 60° = \sqrt{3}x$. So $xy = \sqrt{3}x^2$.

Since the area of the hat tile is $8\sqrt{3}$, the area of each kite is $\sqrt{3}$ Hence $\sqrt{3}x^2 = \sqrt{3}$. Therefore, as $x > 0$, we deduce that $x = 1$ and hence $y = \sqrt{3}$.

From the first diagram, we see that the perimeter of the hat tile is $8x + 6y$, that is, $8 + 6\sqrt{3}$.

In 1974 Roger Penrose gave an example of a set of six tiles which could be used to cover an infinite plane *aperiodically*, that is, without a repeating pattern. Subsequently Penrose showed that there was a set of two tiles with the same property.

The question of whether there was a single tile which could be used to tile the plane aperiodically remained open for a long time until in November 2022, David Smith, an amateur mathematician from Bridlington in Yorkshire, discovered the hat-tile. He invoked the help of the mathematicians Craig S. Kaplan, Joseph Myers, and Chaim Goodman-Strauss. They were able to prove that the hat-tile can be used to form an aperiodic tiling of the plane. They also showed that the hat-tile can be generalized to an infinite family of tiles with the same aperiodic property.

Joseph Myers is a long-standing UKMT volunteer. He was a member of the UK team for the International Mathematical Olympiad in 1994 and 1995. He is currently a member of the UKMT committee which oversees the British Mathematical Olympiad.

24. A function f satisfies the equation $f(x) + f\left(\frac{1}{1-x}\right) = 24x$ for all real values of x except $x = 0$ and $x = 1$.

What is the value of $f(3)$?

A 40 B 42 C 45 D 48 E 50

SOLUTION **E**

In a question of this type, it is not easy to see which path will lead to a solution.

All we can do is to substitute values of x in the equation $f(x) + f\left(\frac{1}{1-x}\right) = 24x$.

Since we are asked for the value of $f(3)$, we begin by putting $x = 3$. This brings in the value of $f(-\frac{1}{2})$, so we next put $x = -\frac{1}{2}$. Now $f(\frac{2}{3})$ appears. It looks as this might go on indefinitely, but when we put $x = \frac{2}{3}$ we have an "AHA" moment.

We put $x = 3$ in the equation

$$f(x) + f\left(\frac{1}{1-x}\right) = 24x. \qquad (1)$$

given in the question.

This gives

$$f(3) + f(-\tfrac{1}{2}) = 72. \qquad (2)$$

Next, putting $x = -\frac{1}{2}$ in equation (1), we obtain

$$f(-\tfrac{1}{2}) + f(\tfrac{2}{3}) = -12. \qquad (3)$$

Now, putting $x = \frac{2}{3}$, we have

$$f(\tfrac{2}{3}) + f(3) = 16. \qquad (4)$$

By adding equations (2) and (4) we deduce that

$$2f(3) + f(-\tfrac{1}{2}) + f(\tfrac{2}{3}) = 88. \qquad (5)$$

Hence, from equation (3),

$$2f(3) - 12 = 88.$$

Therefore

$$2f(3) = 100.$$

We conclude that

$$f(3) = 50.$$

FOR INVESTIGATION

24.1 What is the value of $f(4)$?

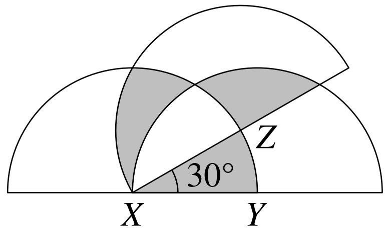
24.2 (a) Find a general formula for $f(x)$, for $x \neq 0, 1$, in terms of x.

(b) Check that your formula produces the correct values for $f(3)$ and $f(4)$.

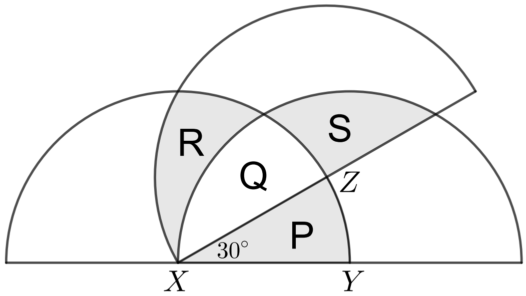
25. Three semicircles, each of area 24, overlap as shown in the diagram. The centres of the arcs are X, Y and Z and $\angle ZXY = 30°$.

What is the total area of the shaded regions?

A 12 B $6\sqrt{3}$ C 15 D 18 E $8\sqrt{3}$



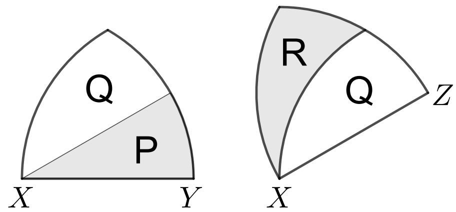
SOLUTION **A**



We let P, Q, R, and S be the areas of the regions as shown in the diagram on the right.

Because 30° is one-sixth of 180°, P is one-sixth of the area of the semicircle with centre X.

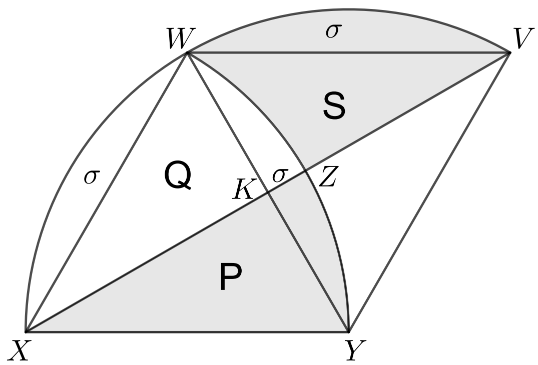
Therefore P = $\frac{1}{6} \times 24 = 4$.



In the two shapes shown in the diagram on the right $XY = XZ$ because they are radii of the same circle, and the other edges are arcs of circles of the same radius with centres X and Y, and X and Z, respectively. Therefore these shapes are congruent.

It follows that P + Q = R + Q.

Hence R = P = 4.



We let V, W and K be the points shown in the diagram on the right.

Because the semicircles have the same area, they have the same radius. Hence $WX = XY = YW$. It follows that WXY is an equilateral triangle, Hence $\angle WYX = 60°$.

Also, we have $XY = VY$. Therefore $\angle XVY = \angle YXV = 30°$. Hence, from the triangle VXY we have $\angle VYW = 180° - 30° - 30° - 60° = 60°$.

It follows that the isosceles triangle WYV is equilateral. Hence $WV = WY$.

These are chords of the same length in circles of the same radius. Therefore they cut off segments of equal area. Let this area be σ. Since $WV = VY = YX = XW$, it follows that $VWXY$ is a rhombus which is divided by the diagonals XV and WY into four congruent triangles, WKY, WKV, YKX and YKV. Let τ be the common area of these triangles.

We have P + Q = $2\sigma + 2\tau$ = Q + S. It follows that S = P = 4. We conclude that the shaded area is P + R + S = 4 + 4 + 4 = 12.

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SMC 2024 - Problem Group Comments

By Mrs Karen Fogden - SMC Chair

We were extremely pleased to receive a large number of entries for the SMC in 2024, once again 121,000 as in 2023, with all candidates sitting their challenge on paper for the first time since 2020. We are very grateful to teachers and others working in schools who make it possible for their students to enjoy preparing for and sitting the challenge and also facilitating discussion afterwards.

The SMC 2024 mean score was similar to the previous year, at 58. The majority of the earlier questions were well answered, and a good proportion of the middle questions were attempted by many candidates. Qn 9, divisibility by 18 caused a noticeable problem with half of candidates leaving this question blank. Qn 19, a logic problem, was popular being attempted by 65% of candidates. As always, the later questions which are designed to be tougher were attempted by fewer participants.

We once again hope that the questions provided an opportunity for interesting and useful discussion after the challenge, especially with the new resource of video solutions, designed to make the after-challenge discussion even more accessible for students individually and within their classrooms. Extended solutions to the SMC questions, with extension problems for further investigation, are available at: https://ukmt.org.uk/competition-papers

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Senior Mathematical Challenge

Thursday 9 October 2025

Organised by the United Kingdom Mathematics Trust

proudly sponsored by XTX MARKETS

Candidates must be full-time students at secondary school or FE college.
England & Wales: Year 13 or below | Scotland: S6 or below | Northern Ireland: Year 14 or below

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**. No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank paper for rough working is allowed; **squared paper, calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil**. Mark at most one of the options, A, B, C, D, or E, on the Answer Sheet for each question. Do not mark more than one option.
5. **Do not expect to finish the whole paper in the time allowed**. The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules**: All candidates start with 25 marks; 0 marks are awarded for each question left unanswered; 4 marks are awarded for each correct answer; 1 mark is deducted for each incorrect answer (to discourage guessing).
7. **Your Answer Sheet will be read by a machine**. Do not write or doodle on the sheet except to mark your chosen options. The machine will read all black pencil markings even if they are in the wrong places. If you mark the sheet in the wrong place, doodle, or leave bits of eraser stuck to the page, the machine will interpret the mark in its own way, or reject the answer sheet.
8. **The questions on this paper are designed to challenge you to think, not to guess**. You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.
9. To accommodate candidates sitting at other times, please do not discuss the paper on the internet until 08:00 BST on Saturday 11th October, when the solutions video will be released at ukmt.org.uk/video-solutions-list. Candidates in time zones more than 5 hours ahead of GMT must sit the paper on Friday 10th October (as defined locally).

Enquiries about the Senior Mathematical Challenge should be sent to:

challenges@ukmt.org.uk   *www.ukmt.org.uk*

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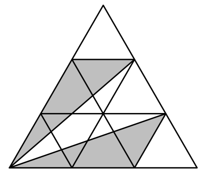
1. Pablo has 100 identical small cubes. He uses some of them to build the largest possible solid cube. How many of the small cubes are left over?

A 16 B 27 C 36 D 73 E 92

2. The diagram shows an equilateral triangle divided into nine smaller equilateral triangles, with two additional lines.



What fraction of the large triangle is shaded?

A $\frac{1}{3}$ B $\frac{1}{4}$ C $\frac{3}{8}$ D $\frac{2}{9}$ E $\frac{4}{9}$

3. What is $25^2 - 24^2 - 23^2 + 22^2$?

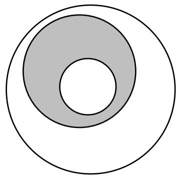
A 0 B 1 C 2 D 3 E 4

4. Sonia writes down three 2-digit numbers whose sum is 46. The first number is prime, the second is square and the third is even.

What is the even number?

A 10 B 12 C 14 D 16 E 18

5. The diagram shows three circles with radii 1, 2 and 3.



What is the ratio of the shaded area to the area of the largest circle?

A 1 : 3 B 1 : 2 C $\sqrt{2} : \sqrt{3}$ D 2 : 3 E 4 : 9

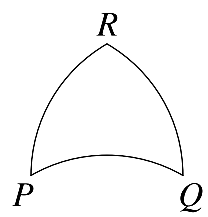
6. For what value of x is $\sqrt{(\sqrt{(\sqrt{x}+1)}+1)}+1=3$?

A 4096 B 64 C 8 D 3 E 0

7. What is the 100th term of the sequence 1, 5, 7, 11, 13, 17, 19, 23, . . . whose terms are consecutive odd numbers but with all the multiples of 3 removed?

A 201 B 203 C 247 D 299 E 301

8. The diagram shows three arcs of circles of radius 1. P is the centre of the circle of which RQ is an arc and Q is the centre of the circle of which PR is an arc.



What is the area of this shape?

A $\frac{\sqrt{3}\pi}{10}$ B $\frac{\pi}{6}$ C $\frac{\pi}{8}$ D $\frac{\sqrt{3}}{4}$ E $\frac{\sqrt{2}\pi}{9}$

9. Fifty squares are drawn side by side in a line. The first and last squares are shaded. Other squares in the line must be shaded such that both these rules apply: (a) no two adjacent squares are shaded and (b) there are no more than three consecutive unshaded squares.

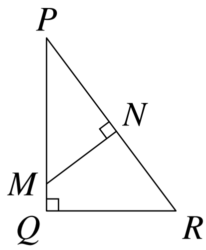
What is the difference between the smallest and largest number of squares that can be shaded?

A 8 B 10 C 11 D 13 E 15

10. How many different squares are factors of 2025?

A 2 B 3 C 4 D 5 E 6

11. The diagram shows a triangle PQR with $\angle PQR = 90°$, $PQ = 20$ and $PR = 25$. Point M lies on PQ, point N lies on PR and PNM is a right-angled triangle whose area is half that of triangle PQR.



What is the length of MN?

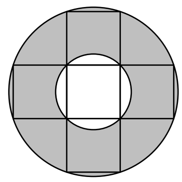
A $6\sqrt{2}$ B $\frac{15}{2}\sqrt{2}$ C $8\sqrt{2}$ D $\frac{17}{2}\sqrt{2}$ E $10\sqrt{2}$

12. Ayain writes down all the 3-digit numbers consisting of three different odd digits.

How many of Ayain's numbers are divisible by 3?

A 42 B 36 C 30 D 24 E 18

13. Five congruent squares, each of side $2a$, are placed edge to edge. Two circles with the same centre are drawn through the vertices as shown.



What is the area of the region between the two circles?

A $2\pi a^2$ B $4\pi a^2$ C $6\pi a^2$ D $8\pi a^2$ E $10\pi a^2$

14. The simultaneous equations $x + \frac{1}{y} = 2$ and $y + \frac{1}{x} = \frac{9}{4}$ have two pairs of real solutions.

What is the difference between the possible values of x?

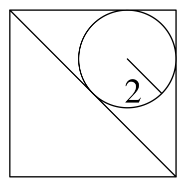
A $\frac{1}{3}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D 1 E $\frac{3}{2}$

15. The integer n is such that $1 < n < 99$. P is the difference between 1 and n, Q is the difference between 99 and n, and R is the difference between P and Q.

For how many values of n is R a prime number?

A 0 B 2 C 4 D 8 E 99

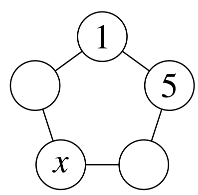
16. The diagram shows a square, one of its diagonals and a circle. The circle touches the diagonal and two sides of the square. The circle has radius 2.



What is the length of the side of the square?

A $4 + 2\sqrt{2}$ B $8 - \sqrt{3}$ C $2 + 2\sqrt{2}$ D $8 - 2\sqrt{2}$ E $4 + 2\sqrt{3}$

17. The circles are filled with five integers so that any integer from 1 to 21 can be made, either by choosing one of the integers or by summing up to 5 adjacent integers.



When 1 and 5 are in the positions shown, what is the value of x?

A 2 B 3 C 7 D 10 E 11

18. The positive integer N has 2025 digits. The first digit is a 3. Every two consecutive digits of N form a number that is divisible by either 17 or 23. The units digit of N could either be p or q.

What is the value of $p + q$?

A 3 B 6 C 7 D 9 E 10

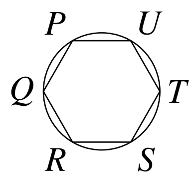
19. A 'complete' football kit consists of a shirt, a pair of shorts and a pair of socks. Three pairs of shorts and one pair of socks together cost the same as two shirts. Seven pairs of shorts and four pairs of socks together cost the same as five shirts. Eden has exactly the right amount of money to buy nine shirts. How many 'complete' football kits could be bought for the same amount of money?

A 3 B 4 C 5 D 6 E 7

20. When $\frac{1}{x} - \frac{1}{y} = 2025$, what is the value of $\frac{x + 2026xy - y}{2y - 2025xy - 2x}$?

A 0 B 1 C $\frac{1}{2}$ D 2026 E $\frac{1}{2025}$

21. A regular hexagon $PQRSTU$ is inscribed in a circle of radius 5. A point X on the circumference of the circle is connected to the vertices of the hexagon to form six chords XP, XQ, XR, XS, XT and XU.



What is the value of $XP^2 + XQ^2 + XR^2 + XS^2 + XT^2 + XU^2$?

A 150 B 216 C 256 D 300 E 360

22. Together, n aardvarks and 12 anteaters eat $n^2 + 20n + 25$ ants. Each animal eats the same whole number of ants.

How many ants does each animal eat?

A 61 B 66 C 71 D 74 E 79

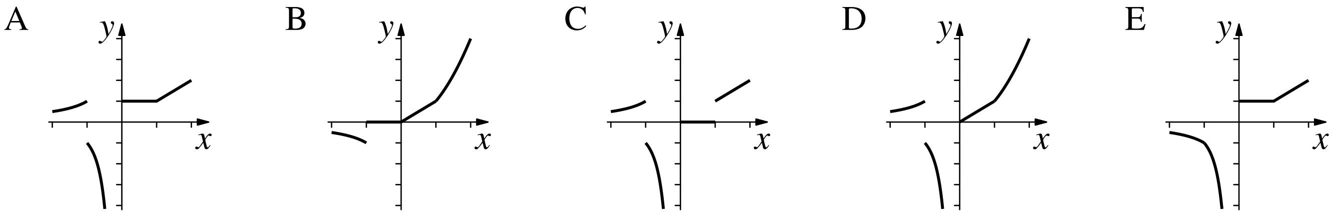
23. Jemima took a series of n tests each with the same maximum mark. After $(n - 2)$ tests her average score was m. She scored full marks in the $(n - 1)$th test, and so raised her average score by 4 marks. In the nth test when she again scored full marks, she increased her average score by 3 marks.

How many tests did Jemima take?

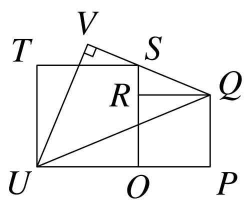
A 6 B 7 C 8 D 9 E 10

24. The output of the function $F(x)$ when applied to a real number x is the greatest integer less than or equal to x. For example, $F(3) = 3$, $F(4.7) = 4$, $F(-2.3) = -3$.

Which of the following is the graph of $y = x^{F(x)}$ for non-zero values of x in the interval $-2 < x < 2$?



25. The diagram shows two squares, $OPQR$ and $OSTU$. Point S lies on QV. Triangle UQV is isosceles with a right angle at V. Square $OPQR$ has area 25.



What is the area of square $OSTU$?

A 36 B 45 C 50 D 54 E 60

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Senior Mathematical Challenge

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Solutions

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	E	E	A	A	B	D	B	C	E	B	D	D	B	B	A	D	C	C	E	D	B	C	A	C

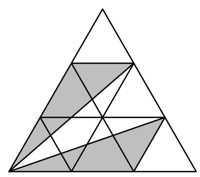
1. Pablo has 100 identical small cubes. He uses some of them to build the largest possible solid cube. How many of the small cubes are left over?

A 16 B 27 C 36 D 73 E 92

Solution **C**

Pablo can use 1, 8, 27 or 64 small cubes to build solid cubes. As he builds the largest solid cube possible, he has $100 - 64 = 36$ small cubes left over.

2. The diagram shows an equilateral triangle divided into nine smaller equilateral triangles, with two additional lines.



What fraction of the large triangle is shaded?

A $\frac{1}{3}$ B $\frac{1}{4}$ C $\frac{3}{8}$ D $\frac{2}{9}$ E $\frac{4}{9}$

Solution **E**

The largest equilateral triangle is made from 9 small equilateral triangles. The two shaded triangular areas can be rearranged to form a parallelogram made from four small equilateral triangles. Therefore, the shaded area is $\frac{4}{9}$ of the large triangle.

3. What is $25^2 - 24^2 - 23^2 + 22^2$?

A 0 B 1 C 2 D 3 E 4

Solution **E**

Rewriting $25^2 - 24^2 - 23^2 + 22^2$ as $25^2 - 24^2 - (23^2 - 22^2)$ and using the difference of two squares twice gives $(25 + 24)(25 - 24) - (23 + 22)(23 - 22) = 49 \times 1 - 45 \times 1 = 4$.

4. Sonia writes down three 2-digit numbers whose sum is 46. The first number is prime, the second is square and the third is even.

What is the even number?

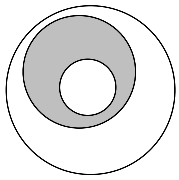
A 10 B 12 C 14 D 16 E 18

Solution **A**

The sum of Sonia's three numbers, 46, is even. Therefore the prime and square are either both even or both odd. As all 2-digit primes are odd, the square must also be odd. There is exactly one odd, 2-digit square smaller than 46, which is 25, so 25 is Sonia's square. The sum of the remaining prime and even number is $46 - 25 = 21$. There is one possible pair of 2-digit integers satisfying prime + even = 21 which is $11 + 10 = 21$. Therefore Sonia's even number is 10.

5. The diagram shows three circles with radii 1, 2 and 3.

What is the ratio of the shaded area to the area of the largest circle?



A $1 : 3$ B $1 : 2$ C $\sqrt{2} : \sqrt{3}$ D $2 : 3$ E $4 : 9$

SOLUTION **A**

The area of the largest circle is $\pi \times 3^2 = 9\pi$. The shaded area is $\pi \times 2^2 - \pi \times 1^2 = 4\pi - \pi = 3\pi$. Therefore the ratio of the shaded area to the area of the largest circle is $3\pi : 9\pi = 1 : 3$.

6. For what value of x is $\sqrt{(\sqrt{(\sqrt{x} + 1)} + 1)} + 1 = 3$?

A 4096 B 64 C 8 D 3 E 0

SOLUTION **B**

In order to solve the equation, the following process is repeated three times: subtract 1 from each side then square. This produces the equation $x = (((3 - 1)^2 - 1)^2 - 1)^2 = 64$.

7. What is the 100th term of the sequence 1, 5, 7, 11, 13, 17, 19, 23, . . . whose terms are consecutive odd numbers but with all the multiples of 3 removed?

A 201 B 203 C 247 D 299 E 301

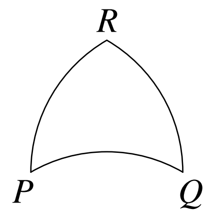
SOLUTION **D**

The sequence can be viewed as alternating between two linear sub-sequences, $1, 7, 13, 19, \ldots$ and $5, 11, 17, 23, \ldots$. Hence the $(2n - 1)$th term of the original sequence is of the form $(6n - 5)$ and the $(2n)$th term is of the form $(6n - 1)$. The 100th term of the original sequence is the 50th term of the second sub-sequence, $6n - 1 = 6 \times 50 - 1 = 299$.

8. The diagram shows three arcs of circles of radius 1. P is the centre of the circle of which RQ is an arc and Q is the centre of the circle of which PR is an arc.

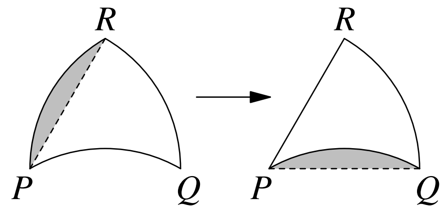
What is the area of this shape?

A $\frac{\sqrt{3}\pi}{10}$ B $\frac{\pi}{6}$ C $\frac{\pi}{8}$ D $\frac{\sqrt{3}}{4}$ E $\frac{\sqrt{2}\pi}{9}$



SOLUTION **B**

By drawing chords PR and PQ and moving the region which lies between chord PR and arc PR to fill the space of the same size between chord PQ and arc PQ we form a sector of radius 1. The angle RPQ is 60°. Therefore the area of the shape is $\frac{1}{6} \times \pi \times 1^2 = \frac{\pi}{6}$.



9. Fifty squares are drawn side by side in a line. The first and last squares are shaded. Other squares in the line must be shaded such that both these rules apply: (a) no two adjacent squares are shaded and (b) there are no more than three consecutive unshaded squares.

What is the difference between the smallest and largest number of squares that can be shaded?

A 8 B 10 C 11 D 13 E 15

SOLUTION **C**

Labelling the squares in the line 1 to 50, we start by shading squares 1 and 50. One possible maximal shading is all of the odd numbered squares 1 to 47 inclusive, but not 49 as 49 and 50 are adjacent and so can't both be shaded. This is 25 shaded squares. One possible minimal shading using the largest permitted gaps is every fourth square from 1 to 45 inclusive namely $1, 5, 9, \ldots, 41, 45$. As before, 49 can't be shaded but either 47 or 48 must be. This is 14 shaded squares. So the difference is $25 - 14 = 11$ shaded squares.

10. How many different squares are factors of 2025?

A 2 B 3 C 4 D 5 E 6

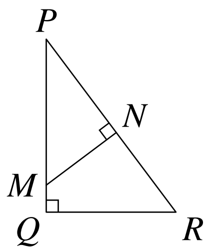
SOLUTION **E**

The prime factorisation of 2025 is $3 \times 3 \times 3 \times 3 \times 5 \times 5$. Therefore squares which are factors of 2025 are 1, $3^2 = 9$, $5^2 = 25$, $(3^2)^2 = 81$, $(3 \times 5)^2 = 225$, $(3^2 \times 5)^2 = 2025$, so 6 in total.

11. The diagram shows a triangle PQR with $\angle PQR = 90°$, $PQ = 20$ and $PR = 25$. Point M lies on PQ, point N lies on PR and PNM is a right-angled triangle whose area is half that of triangle PQR.

What is the length of MN?

A $6\sqrt{2}$ B $\frac{15}{2}\sqrt{2}$ C $8\sqrt{2}$ D $\frac{17}{2}\sqrt{2}$ E $10\sqrt{2}$



Solution **B**

Triangles PQR and PNM are both right-angled and contain $\angle MPN$. The triangles are therefore similar, with shortest sides QR and MN respectively. Using Pythagoras' Theorem, $QR = \sqrt{25^2 - 20^2} = 15$. For triangle PNM to have area which is half that of triangle PQR, the length scale factor must be $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$. The length of MN is then $15 \times \frac{\sqrt{2}}{2} = \frac{15}{2}\sqrt{2}$.

12. Ayain writes down all the 3-digit numbers consisting of three different odd digits.

How many of Ayain's numbers are divisible by 3?

A 42 B 36 C 30 D 24 E 18

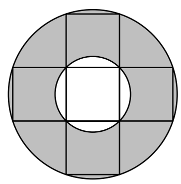
Solution **D**

A 3-digit number is divisible by 3 if, and only if, the sum of its digits is divisible by 3. The digits of Ayain's numbers are three different numbers from the set $1, 3, 5, 7, 9$. Hence each number is obtained from one of the sets $(1, 3, 5)$, $(1, 5, 9)$, $(3, 5, 7)$, or $(5, 7, 9)$. For each possible triple, there are 6 possible orders, e.g. 135, 153, 315, 351, 513, 531. So in all, $4 \times 6 = 24$ of Ayain's numbers are divisible by 3.

13. Five congruent squares, each of side $2a$, are placed edge to edge. Two circles with the same centre are drawn through the vertices as shown.

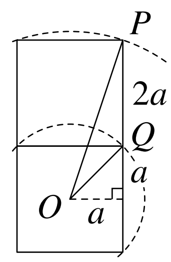
What is the area of the region between the two circles?

A $2\pi a^2$ B $4\pi a^2$ C $6\pi a^2$ D $8\pi a^2$ E $10\pi a^2$



Solution **D**

Let O be the centre of the middle square, Q be a vertex of the middle square and P be a vertex of the square above as shown. Therefore OQ is a radius of the smaller circle and OP is a radius of the larger circle. Using Pythagoras' Theorem twice gives $(OQ)^2 = a^2 + a^2 = 2a^2$ and $(OP)^2 = a^2 + (3a)^2 = 10a^2$. The shaded area is therefore $\pi \times (OP)^2 - \pi \times (OQ)^2 = \pi(10a^2 - 2a^2) = 8\pi a^2$.



14. The simultaneous equations $x + \frac{1}{y} = 2$ and $y + \frac{1}{x} = \frac{9}{4}$ have two pairs of real solutions.

What is the difference between the possible values of x?

A $\frac{1}{3}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D 1 E $\frac{3}{2}$

SOLUTION **B**

First rearrange $x + \frac{1}{y} = 2$ (1) and $y + \frac{1}{x} = \frac{9}{4}$ (2) to $xy + 1 = 2y$ (1′) and $xy + 1 = \frac{9x}{4}$ (2′). Equating the RHS of (1′) and (2′) , gives $2y = \frac{9x}{4}$ and therefore $y = \frac{9x}{8}$. Substituting into (1′), gives $x \times \frac{9x}{8} + 1 = 2 \times \frac{9x}{8}$ which simplifies to $9x^2 - 18x + 8 = 0$ with solutions $x = \frac{4}{3}$ and $x = \frac{2}{3}$. The difference between the values of x is $\frac{4}{3} - \frac{2}{3} = \frac{2}{3}$.

15. The integer n is such that $1 < n < 99$. P is the difference between 1 and n, Q is the difference between 99 and n, and R is the difference between P and Q.

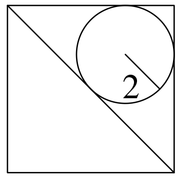
For how many values of n is R a prime number?

A 0 B 2 C 4 D 8 E 99

SOLUTION **B**

We start by writing expressions for P and Q as $P = n - 1$ and $Q = 99 - n$. Note that if $n = 50$, then both P and Q are 49 and so $R = 0$, which is not prime. Now consider values of n for which $1 < n < 50$. In this case, $Q > P$ therefore $R = Q - P = (99 - n) - (n - 1) = 100 - 2n$. In order for this value of R to be prime, $100 - 2n = 2$ and so $n = 49$. However, considering values of n for which $50 < n < 99$ leads to $P > Q$. In this case $R = P - Q = (n - 1) - (99 - n) = 2n - 100$. In order for this value of R to be prime, $2n - 100 = 2$ and therefore $n = 51$. Hence there are two possible values of n which lead to R being prime, $n = 49$ and $n = 51$.

16. The diagram shows a square, one of its diagonals and a circle. The circle touches the diagonal and two sides of the square. The circle has radius 2.

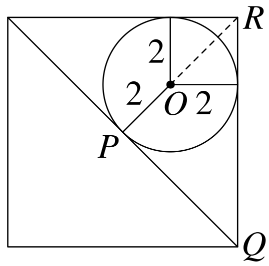


What is the length of the side of the square?

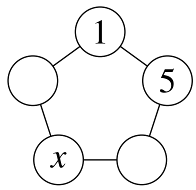
A $4 + 2\sqrt{2}$ B $8 - \sqrt{3}$ C $2 + 2\sqrt{2}$ D $8 - 2\sqrt{2}$ E $4 + 2\sqrt{3}$

SOLUTION **A**

Let the centre of the circle be O, the centre of the square be P and vertices of the square be Q and R as shown. The radius of the circle is 2, so $PR = PO + OR = 2 + 2\sqrt{2}$. As triangle RPQ is right-angled and isosceles its sides are in the ratio $1 : 1 : \sqrt{2}$. The length of the side of the square, QR, is therefore $\sqrt{2}(2 + 2\sqrt{2}) = 2\sqrt{2} + 4 = 4 + 2\sqrt{2}$.



17. The circles are filled with five integers so that any integer from 1 to 21 can be made, either by choosing one of the integers or by summing up to 5 adjacent integers.



When 1 and 5 are in the positions shown, what is the value of x?

A 2 B 3 C 7 D 10 E 11

SOLUTION **D**

We note that there 21 possible sums: 1 sum of all five numbers, 5 sums of four numbers, 5 sums of three adjacent numbers, 5 sums of two adjacent numbers and 5 single numbers. For brevity and without loss we will consider in this last case that a single number is a sum of 1 number. As there are 21 possible sums and we seek to create each total from 1 to 21 inclusive, every sum must occur exactly once; there can be no repeated sums. Given that 1 and 5 are already shown, the remaining numbers must sum to $21 - (1 + 5) = 15$. To create a sum of 2, we must use a 2, as $1 + 1$ would be a repeat. The remaining two integers must have a total of $21 - (1 + 5 + 2) = 13$. The only possible pairs are 6, 7 or 4, 9 or 3, 10. We first discount using a 6 because the adjacent 1 and 5 already have a total of 6. We next discount 4, 9. In this case, as there is no single 3, the 2 would be placed next to the 1 to create a sum of 3. Placing the 4 next to the 5 would create 9 in two ways and placing the 4 next to the 2 would create 6 in two ways. So now we have our five integers, 1, 2, 3, 5 and 10. To create a sum of 4, the 3 must be next to the 1. All the totals from 1 to 6 inclusive are now possible. To create a sum of 7, the 2 must be next to the 5, leaving $x = 10$. All sums from 1 to 21 inclusive are now possible, which is left as an exercise for the reader.

18. The positive integer N has 2025 digits. The first digit is a 3. Every two consecutive digits of N form a number that is divisible by either 17 or 23. The units digit of N could either be p or q.

What is the value of $p + q$?

A 3 B 6 C 7 D 9 E 10

SOLUTION **C**

Listing the 2-digit multiples of 17 and of 23 gives us the numbers which we can use when building the integer N, namely 17, 34, 51, 68, 85 and 23, 46, 69, 92. The first digit is 3, so using 34, the second digit is 4 and using 46, the third digit is 6. When we reach 6, there are two choices, 68 or 69. Choosing 8, the following three digits would be 5,1,7. However as there are no multiples of either 17 or 23 in the 70s we reach an end. This can only happen at the very end of N. If however we choose 69, then a cycle of five digits is formed, ..346923.. . Our positive integer N is either formed from 405 cycles of 34692 (ending in 2) or from 404 cycles of 34692 followed by 34685 (ending in 5). The value of $p + q$ is therefore $2 + 5 = 7$.

19. A 'complete' football kit consists of a shirt, a pair of shorts and a pair of socks. Three pairs of shorts and one pair of socks together cost the same as two shirts. Seven pairs of shorts and four pairs of socks together cost the same as five shirts. Eden has exactly the right amount of money to buy nine shirts. How many 'complete' football kits could be bought for the same amount of money?

A 3 B 4 C 5 D 6 E 7

SOLUTION **C**

Let the cost of a shirt be x, the cost of a pair of shorts be y and the cost of a pair of socks be z. A 'complete' kit then costs $x + y + z$. Eden has $9x$ so we want to write $9x$ in the form $k(x + y + z)$ for some integer k, $1 \leq k \leq 8$. We are given that $2x = 3y + z$ and $5x = 7y + 4z$. We need a linear combination of these two equations such that in the resulting equation, the coefficients of y and z are equal. By inspection, subtracting three times the first equation from twice the second shows that $4x = 5y + 5z$. Then $9x = 5x + 4x = 5x + 5y + 5z = 5(x + y + z)$ and so five complete football kits could be bought for the same amount of money as nine shirts.

20. When $\frac{1}{x} - \frac{1}{y} = 2025$, what is the value of $\frac{x + 2026xy - y}{2y - 2025xy - 2x}$?

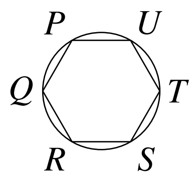
A 0 B 1 C $\frac{1}{2}$ D 2026 E $\frac{1}{2025}$

Solution **E**

Starting with $\frac{x + 2026xy - y}{2y - 2025xy - 2x}$ and dividing both the numerator and denominator of the expression by xy gives

$$\frac{\frac{1}{y} + 2026 - \frac{1}{x}}{\frac{2}{x} - 2025 - \frac{2}{y}} = \frac{-(\frac{1}{x} - \frac{1}{y}) + 2026}{2(\frac{1}{x} - \frac{1}{y}) - 2025} = \frac{-2025 + 2026}{2 \times 2025 - 2025} = \frac{1}{2025}.$$

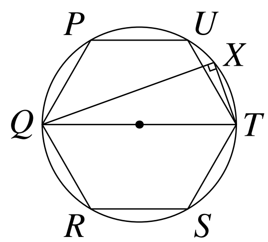
21. A regular hexagon $PQRSTU$ is inscribed in a circle of radius 5. A point X on the circumference of the circle is connected to the vertices of the hexagon to form six chords XP, XQ, XR, XS, XT and XU.



What is the value of $XP^2 + XQ^2 + XR^2 + XS^2 + XT^2 + XU^2$?

A 150 B 216 C 256 D 300 E 360

Solution **D**



Placing X in a general position on the circle and creating a triangle by joining X to points at opposite ends of a diameter, for example Q and T as shown, we can use 'the angle in a semicircle is 90°'. As the angle at X is 90° we can use Pythagoras' Theorem, $XQ^2 + XT^2 = QT^2 = 10^2 = 100$. Similarly, with RU as diameter $XR^2 + XU^2 = RU^2 = 100$ and with SP as diameter $XS^2 + XP^2 = SP^2 = 100$.

Therefore $XP^2 + XQ^2 + XR^2 + XS^2 + XT^2 + XU^2 = 3 \times 100 = 300$. Note that if X is placed at a vertex, there would only be five chords but the total would be the same.

22. Together, n aardvarks and 12 anteaters eat $n^2 + 20n + 25$ ants. Each animal eats the same whole number of ants.

How many ants does each animal eat?

A 61 B 66 C 71 D 74 E 79

SOLUTION **B**

There are $(n + 12)$ animals. The number of ants that each animal eats is $\frac{(n^2 + 20n + 25)}{(n + 12)} = \frac{(n + 12)(n + 8) - 71}{(n + 12)} = (n + 8) - \frac{71}{(n + 12)}$. For the number of ants eaten by each animal to be a whole number, $(n + 12)$ must be a factor of 71. As 71 is prime, the only possibility is that $n + 12 = 71$ therefore $n = 59$. The number of ants eaten by each animal is then $59 + 8 - 1 = 66$.

23. Jemima took a series of n tests each with the same maximum mark. After $(n - 2)$ tests her average score was m. She scored full marks in the $(n - 1)$th test, and so raised her average score by 4 marks. In the nth test when she again scored full marks, she increased her average score by 3 marks.

How many tests did Jemima take?

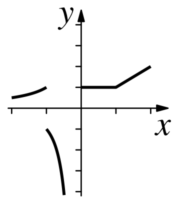
A 6 B 7 C 8 D 9 E 10

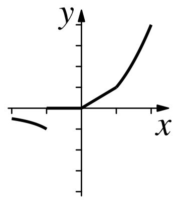
SOLUTION **C**

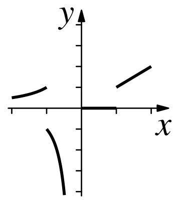
Let f be the maximum mark in each test. We can write two expressions for f. Firstly, f is the difference in Jemima's total score between taking $(n - 1)$ tests and taking $(n - 2)$ tests. So $f = (m + 4)(n - 1) - m(n - 2) = mn + 4n - m - 4 - (mn - 2m) = 4n + m - 4$. Secondly, f is the difference in her total score between taking n tests and taking $(n - 1)$ tests. So $f = (m + 4 + 3)n - (m + 4)(n - 1) = mn + 7n - (mn + 4n - m - 4) = 3n + m + 4$. Equating these two expressions for f gives $4n + m - 4 = 3n + m + 4$ and therefore $n = 8$.

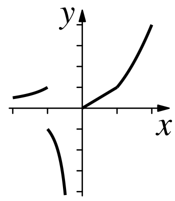
24. The output of the function $F(x)$ when applied to a real number x is the greatest integer less than or equal to x. For example, $F(3) = 3$, $F(4.7) = 4$, $F(-2.3) = -3$.

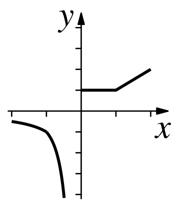
Which of the following is the graph of $y = x^{F(x)}$ for non-zero values of x in the interval $-2 < x < 2$?

A


B


C


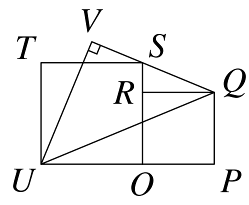
D


E


Solution **A**

First consider $0 < x < 1$. For all values of x in this interval, $F(x) = 0$ and therefore $y = x^{F(x)} = x^0 = 1$. This eliminates options B, C and D. Now consider $-2 < x < -1$. For all values of x in this interval, $F(x) = -2$. Therefore $y = x^{F(x)} = x^{-2} = \frac{1}{x^2} > 0$, as $x^2 > 0$. This eliminates option E. Using a similar approach, we can show that, for $-1 < x < 0$, $F(x) = \frac{1}{x}$ and, for $1 < x < 2$, $F(x) = x$ which are consistent with the graph in the remaining option A.

25. The diagram shows two squares, $OPQR$ and $OSTU$. Point S lies on QV. Triangle UQV is isosceles with a right angle at V. Square $OPQR$ has area 25.

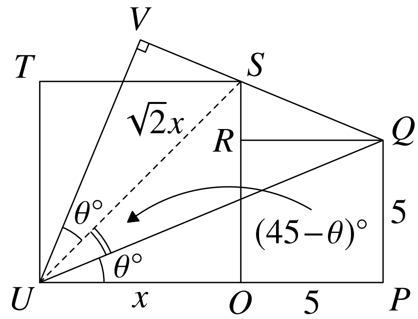


What is the area of square $OSTU$?

A 36 B 45 C 50 D 54 E 60

Solution **C**

Let $OU = x$ and $\angle QUP = \theta°$. As $\angle SUO = 45°$, $\angle SUQ = 45° - \theta°$. As $\angle VUQ = 45°$, $\angle VUS = 45° - (45 - \theta)° = \theta°$. Thus right-angled triangles QUP and VUS are similar. In each triangle, we have enough information to consider $\cos\theta$. In triangle QUP, the adjacent side $UP = x + 5$. Using Pythagoras' Theorem, the hypotenuse $QU = \sqrt{(x+5)^2 + 5^2} = \sqrt{x^2 + 10x + 50}$. In triangle VUS, the adjacent side VU is a shorter side of the right-angled isosceles triangle VUQ, therefore $VU = \frac{1}{\sqrt{2}}QU = \frac{\sqrt{x^2+10x+50}}{\sqrt{2}}$. The hypotenuse SU is also the diagonal of square $OSTU$ therefore $SU = \sqrt{2}x$.



This gives two expressions for $\cos\theta$, namely $\dfrac{x+5}{\sqrt{x^2+10x+50}}$ and $\dfrac{\frac{\sqrt{x^2+10x+50}}{\sqrt{2}}}{\sqrt{2}x}$. Equating these and rearranging gives $2x(x+5) = x^2 + 10x + 50$. Therefore $2x^2 + 10x = x^2 + 10x + 50$ and so $x^2 = 50$. Hence the area of square $OSTU$ is 50.

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A version of this document including each of the questions alongside its solution is also available on the UKMT website:
www.ukmt.org.uk

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1. C Pablo can use 1, 8, 27 or 64 small cubes to build solid cubes. As he builds the largest solid cube possible, he has $100 - 64 = 36$ small cubes left over.

2. E The largest equilateral triangle is made from 9 small equilateral triangles. The two shaded triangular areas can be rearranged to form a parallelogram made from four small equilateral triangles. Therefore, the shaded area is $\frac{4}{9}$ of the large triangle.

3. E Rewriting $25^2 - 24^2 - 23^2 + 22^2$ as $25^2 - 24^2 - (23^2 - 22^2)$ and using the difference of two squares twice gives $(25 + 24)(25 - 24) - (23 + 22)(23 - 22) = 49 \times 1 - 45 \times 1 = 4$.

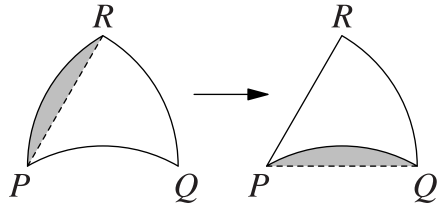
4. A The sum of Sonia's three numbers, 46, is even. Therefore the prime and square are either both even or both odd. As all 2-digit primes are odd, the square must also be odd. There is exactly one odd, 2-digit square smaller than 46, which is 25, so 25 is Sonia's square. The sum of the remaining prime and even number is $46 - 25 = 21$. There is one possible pair of 2-digit integers satisfying prime + even = 21 which is $11 + 10 = 21$. Therefore Sonia's even number is 10.

5. A The area of the largest circle is $\pi \times 3^2 = 9\pi$. The shaded area is $\pi \times 2^2 - \pi \times 1^2 = 4\pi - \pi = 3\pi$. Therefore the ratio of the shaded area to the area of the largest circle is $3\pi : 9\pi = 1 : 3$.

6. B In order to solve the equation, the following process is repeated three times: subtract 1 from each side then square. This produces the equation $x = (((3 - 1)^2 - 1)^2 - 1)^2 = 64$.

7. D The sequence can be viewed as alternating between two linear sub-sequences, $1, 7, 13, 19, \ldots$ and $5, 11, 17, 23, \ldots$. Hence the $(2n - 1)$th term of the original sequence is of the form $(6n - 5)$ and the $(2n)$th term is of the form $(6n - 1)$. The 100th term of the original sequence is the 50th term of the second sub-sequence, $6n - 1 = 6 \times 50 - 1 = 299$.

8. B By drawing chords PR and PQ and moving the region which lies between chord PR and arc PR to fill the space of the same size between chord PQ and arc PQ we form a sector of radius 1. The angle RPQ is $60°$. Therefore the area of the shape is $\frac{1}{6} \times \pi \times 1^2 = \frac{\pi}{6}$.



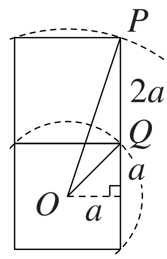
9. C Labelling the squares in the line 1 to 50, we start by shading squares 1 and 50. One possible maximal shading is all of the odd numbered squares 1 to 47 inclusive, but not 49 as 49 and 50 are adjacent and so can't both be shaded. This is 25 shaded squares. One possible minimal shading using the largest permitted gaps is every fourth square from 1 to 45 inclusive namely $1, 5, 9, \ldots, 41, 45$. As before, 49 can't be shaded but either 47 or 48 must be. This is 14 shaded squares. So the difference is $25 - 14 = 11$ shaded squares.

10. E The prime factorisation of 2025 is $3 \times 3 \times 3 \times 3 \times 5 \times 5$. Therefore squares which are factors of 2025 are 1, $3^2 = 9$, $5^2 = 25$, $(3^2)^2 = 81$, $(3 \times 5)^2 = 225$, $(3^2 \times 5)^2 = 2025$, so 6 in total.

11. B Triangles PQR and PNM are both right-angled and contain $\angle MPN$. The triangles are therefore similar, with shortest sides QR and MN respectively. Using Pythagoras' Theorem, $QR = \sqrt{25^2 - 20^2} = 15$. For triangle PNM to have area which is half that of triangle PQR, the length scale factor must be $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$. The length of MN is then $15 \times \frac{\sqrt{2}}{2} = \frac{15}{2}\sqrt{2}$.

12. D A 3-digit number is divisible by 3 if, and only if, the sum of its digits is divisible by 3. The digits of Ayain's numbers are three different numbers from the set $1, 3, 5, 7, 9$. Hence each number is obtained from one of the sets $(1, 3, 5)$, $(1, 5, 9)$, $(3, 5, 7)$, or $(5, 7, 9)$. For each possible triple, there are 6 possible orders, e.g. 135, 153, 315, 351, 513, 531. So in all, $4 \times 6 = 24$ of Ayain's numbers are divisible by 3.

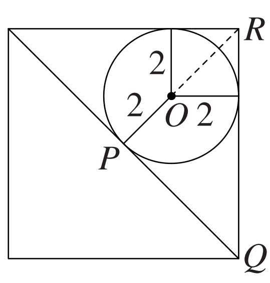
13. D Let O be the centre of the middle square, Q be a vertex of the middle square and P be a vertex of the square above as shown. Therefore OQ is a radius of the smaller circle and OP is a radius of the larger circle. Using Pythagoras' Theorem twice gives $(OQ)^2 = a^2 + a^2 = 2a^2$ and $(OP)^2 = a^2 + (3a)^2 = 10a^2$. The shaded area is therefore $\pi \times (OP)^2 - \pi \times (OQ)^2 = \pi(10a^2 - 2a^2) = 8\pi a^2$.



14. B First rearrange $x + \dfrac{1}{y} = 2$ (1) and $y + \dfrac{1}{x} = \dfrac{9}{4}$ (2) to $xy + 1 = 2y$ (1′) and $xy + 1 = \dfrac{9x}{4}$ (2′). Equating the RHS of (1′) and (2′) , gives $2y = \dfrac{9x}{4}$ and therefore $y = \dfrac{9x}{8}$. Substituting into (1′), gives $x \times \dfrac{9x}{8} + 1 = 2 \times \dfrac{9x}{8}$ which simplifies to $9x^2 - 18x + 8 = 0$ with solutions $x = \dfrac{4}{3}$ and $x = \dfrac{2}{3}$. The difference between the values of x is $\dfrac{4}{3} - \dfrac{2}{3} = \dfrac{2}{3}$.

15. B We start by writing expressions for P and Q as $P = n - 1$ and $Q = 99 - n$. Note that if $n = 50$, then both P and Q are 49 and so $R = 0$, which is not prime. Now consider values of n for which $1 < n < 50$. In this case, $Q > P$ therefore $R = Q - P = (99 - n) - (n - 1) = 100 - 2n$. In order for this value of R to be prime, $100 - 2n = 2$ and so $n = 49$. However, considering values of n for which $50 < n < 99$ leads to $P > Q$. In this case $R = P - Q = (n - 1) - (99 - n) = 2n - 100$. In order for this value of R to be prime, $2n - 100 = 2$ and therefore $n = 51$. Hence there are two possible values of n which lead to R being prime, $n = 49$ and $n = 51$.

16. A Let the centre of the circle be O, the centre of the square be P and vertices of the square be Q and R as shown. The radius of the circle is 2, so $PR = PO + OR = 2 + 2\sqrt{2}$. As triangle RPQ is right-angled and isosceles its sides are in the ratio $1 : 1 : \sqrt{2}$. The length of the side of the square, QR, is therefore $\sqrt{2}(2 + 2\sqrt{2}) = 2\sqrt{2} + 4 = 4 + 2\sqrt{2}$.



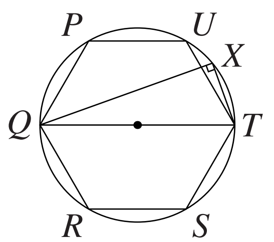
17. D We note that there 21 possible sums: 1 sum of all five numbers, 5 sums of four numbers, 5 sums of three adjacent numbers, 5 sums of two adjacent numbers and 5 single numbers. For brevity and without loss we will consider in this last case that a single number is a sum of 1 number. As there are 21 possible sums and we seek to create each total from 1 to 21 inclusive, every sum must occur exactly once; there can be no repeated sums. Given that 1 and 5 are already shown, the remaining numbers must sum to $21 - (1 + 5) = 15$. To create a sum of 2, we must use a 2, as $1 + 1$ would be a repeat. The remaining two integers must have a total of $21 - (1 + 5 + 2) = 13$. The only possible pairs are 6, 7 or 4, 9 or 3, 10. We first discount using a 6 because the adjacent 1 and 5 already have a total of 6. We next discount 4, 9. In this case, as there is no single 3, the 2 would be placed next to the 1 to create a sum of 3. Placing the 4 next to the 5 would create 9 in two ways and placing the 4 next to the 2 would create 6 in two ways. So now we have our five integers, 1, 2, 3, 5 and 10. To create a sum of 4, the 3 must be next to the 1. All the totals from 1 to 6 inclusive are now possible. To create a sum of 7, the 2 must be next to the 5, leaving $x = 10$. All sums from 1 to 21 inclusive are now possible, which is left as an exercise for the reader.

18. C Listing the 2-digit multiples of 17 and of 23 gives us the numbers which we can use when building the integer N, namely 17, 34, 51, 68, 85 and 23, 46, 69, 92. The first digit is 3, so using 34, the second digit is 4 and using 46, the third digit is 6. When we reach 6, there are two choices, 68 or 69. Choosing 8, the following three digits would be 5,1,7. However as there are no multiples of either 17 or 23 in the 70s we reach an end. This can only happen at the very end of N. If however we choose 69, then a cycle of five digits is formed, ..346923.. . Our positive integer N is either formed from 405 cycles of 34692 (ending in 2) or from 404 cycles of 34692 followed by 34685 (ending in 5). The value of $p + q$ is therefore $2 + 5 = 7$.

19. C Let the cost of a shirt be x, the cost of a pair of shorts be y and the cost of a pair of socks be z. A 'complete' kit then costs $x + y + z$. Eden has $9x$ so we want to write $9x$ in the form $k(x + y + z)$ for some integer k, $1 \leq k \leq 8$. We are given that $2x = 3y + z$ and $5x = 7y + 4z$. We need a linear combination of these two equations such that in the resulting equation, the coefficients of y and z are equal. By inspection, subtracting three times the first equation from twice the second shows that $4x = 5y + 5z$. Then $9x = 5x + 4x = 5x + 5y + 5z = 5(x + y + z)$ and so five complete football kits could be bought for the same amount of money as nine shirts.

20. E Starting with $\dfrac{x + 2026xy - y}{2y - 2025xy - 2x}$ and dividing both the numerator and denominator of the expression by xy gives $\dfrac{\frac{1}{y} + 2026 - \frac{1}{x}}{\frac{2}{x} - 2025 - \frac{2}{y}} = \dfrac{-(\frac{1}{x} - \frac{1}{y}) + 2026}{2(\frac{1}{x} - \frac{1}{y}) - 2025} = \dfrac{-2025 + 2026}{2 \times 2025 - 2025} = \dfrac{1}{2025}$.

21. D Placing X in a general position on the circle and creating a triangle by joining X to points at opposite ends of a diameter, for example Q and T as shown, we can use 'the angle in a semicircle is 90°'. As the angle at X is 90° we can use Pythagoras' Theorem, $XQ^2 + XT^2 = QT^2 = 10^2 = 100$. Similarly, with RU as diameter $XR^2 + XU^2 = RU^2 = 100$ and with SP as diameter $XS^2 + XP^2 = SP^2 = 100$.



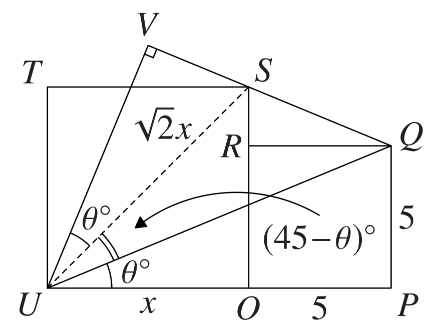
Therefore $XP^2 + XQ^2 + XR^2 + XS^2 + XT^2 + XU^2 = 3 \times 100 = 300$. Note that if X is placed at a vertex, there would only be five chords but the total would be the same.

22. B There are $(n+12)$ animals. The number of ants that each animal eats is $\dfrac{(n^2+20n+25)}{(n+12)} = \dfrac{(n+12)(n+8)-71}{(n+12)} = (n+8) - \dfrac{71}{(n+12)}$. For the number of ants eaten by each animal to be a whole number, $(n+12)$ must be a factor of 71. As 71 is prime, the only possibility is that $n+12=71$ therefore $n=59$. The number of ants eaten by each animal is then $59+8-1=66$.

23. C Let f be the maximum mark in each test. We can write two expressions for f. Firstly, f is the difference in Jemima's total score between taking $(n-1)$ tests and taking $(n-2)$ tests. So $f=(m+4)(n-1)-m(n-2)=mn+4n-m-4-(mn-2m)=4n+m-4$. Secondly, f is the difference in her total score between taking n tests and taking $(n-1)$ tests. So $f=(m+4+3)n-(m+4)(n-1)=mn+7n-(mn+4n-m-4)=3n+m+4$. Equating these two expressions for f gives $4n+m-4=3n+m+4$ and therefore $n=8$.

24. A First consider $0<x<1$. For all values of x in this interval, $F(x)=0$ and therefore $y=x^{F(x)}=x^0=1$. This eliminates options B, C and D. Now consider $-2<x<-1$. For all values of x in this interval, $F(x)=-2$. Therefore $y=x^{F(x)}=x^{-2}=\frac{1}{x^2}>0$, as $x^2>0$. This eliminates option E. Using a similar approach, we can show that, for $-1<x<0$, $F(x)=\frac{1}{x}$ and, for $1<x<2$, $F(x)=x$ which are consistent with the graph in the remaining option A.

25. C Let $OU=x$ and $\angle QUP=\theta°$. As $\angle SUO=45°$, $\angle SUQ=45°-\theta°$. As $\angle VUQ=45°$, $\angle VUS=45°-(45-\theta)°=\theta°$. Thus right-angled triangles QUP and VUS are similar. In each triangle, we have enough information to consider $\cos\theta$. In triangle QUP, the adjacent side $UP=x+5$. Using Pythagoras' Theorem, the hypotenuse $QU=\sqrt{(x+5)^2+5^2}=\sqrt{x^2+10x+50}$. In triangle VUS, the adjacent side VU is a shorter side of the right-angled isosceles triangle VUQ, therefore $VU=\frac{1}{\sqrt{2}}QU=\frac{\sqrt{x^2+10x+50}}{\sqrt{2}}$. The hypotenuse SU is also the diagonal of square $OSTU$ therefore $SU=\sqrt{2}x$.



This gives two expressions for $\cos\theta$, namely $\dfrac{x+5}{\sqrt{x^2+10x+50}}$ and $\dfrac{\frac{\sqrt{x^2+10x+50}}{\sqrt{2}}}{\sqrt{2}x}$. Equating these and rearranging gives $2x(x+5)=x^2+10x+50$. Therefore $2x^2+10x=x^2+10x+50$ and so $x^2=50$. Hence the area of square $OSTU$ is 50.

UK Maths Trust

Senior Mathematical Challenge

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Solutions and investigations

9 October 2025

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given options, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given options is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, challenges@ukmt.org.uk www.ukmt.org.uk

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	E	E	A	A	B	D	B	C	E	B	D	D	B	B	A	D	C	C	E	D	B	C	A	C

1. Pablo has 100 identical small cubes. He uses some of them to build the largest possible solid cube. How many of the small cubes are left over?

A 16 B 27 C 36 D 73 E 92

SOLUTION **C**

We need to find the largest integer n with $n^3 \leq 100$. Then the largest possible solid cube would use n^3 small cubes, leaving Pablo with $100 - n^3$ unused small cubes.

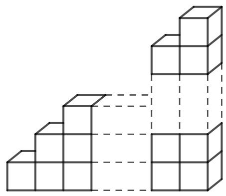
We see that $4^3 = 64$ which is less than 100 but $5^3 = 125$ is not. So the largest possible cube that Pablo could build uses 64 small cubes with $100 - 64 = 36$ left over.

FOR INVESTIGATION

1.1 Pablo next uses some of his 100 small cubes to build a staircase, as shown.

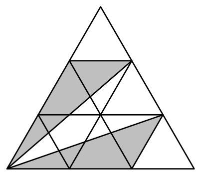
He builds the staircase with largest possible number of steps.

How many of the small cubes are left over?



2. The diagram shows an equilateral triangle divided into nine smaller equilateral triangles, with two additional lines.

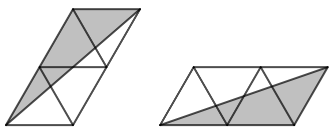
What fraction of the large triangle is shaded?



A $\frac{1}{3}$ B $\frac{1}{4}$ C $\frac{3}{8}$ D $\frac{2}{9}$ E $\frac{4}{9}$

SOLUTION **E**

The shaded region is made up of two triangles. Each of these triangles forms half of a parallelogram which consists of four of the smaller equilateral triangles. Hence each of these triangles has the same area as two of the smaller triangles.



It follows that the shaded region has the same area as four of the smaller triangles. The large equilateral triangle is made up of nine of the smaller triangles.

Therefore the fraction of the large triangle that is shaded is $\frac{4}{9}$.

FOR INVESTIGATION

2.1 The unshaded region in the large equilateral triangle of this question is made up of three triangles. Find the area of each of these triangles as a fraction of the area of the large triangle. Deduce that $\frac{5}{9}$ths of the area of the large triangle is not shaded.

2.2 How many different parallelograms are there in the diagram of this question?

3. What is $25^2 - 24^2 - 23^2 + 22^2$?

A 0 B 1 C 2 D 3 E 4

SOLUTION **E**

You are not allowed to use a calculator in the SMC. So when, as here, you are faced with a question that seems to involve a lot of arithmetic, you should look for a method that doesn't involve a lot of calculation. Often, finding a quick method is the point of the question.

The expression $25^2 - 24^2$ in this question should bring to mind the very useful factorization $x^2 - y^2 = (x - y)(x + y)$.

This is the key to answering this question without a lot of numerical work.

We have

$$\begin{aligned} 25^2 - 24^2 - 23^2 + 22^2 &= (25^2 - 24^2) - (23^2 - 22^2) \\ &= (25 - 24)(25 + 24) - (23 - 22)(23 + 22) \\ &= 1 \times 49 - 1 \times 45 \\ &= 49 - 45 \\ &= 4. \end{aligned}$$

FOR INVESTIGATION

3.1 What is the value of $2025^2 - 2024^2 - 2023^2 + 2022^2$?

3.2 What is the value of $n^2 - (n - 1)^2 - (n - 2)^2 + (n - 3)^2$?

4. Sonia writes down three 2-digit numbers whose sum is 46. The first number is prime, the second is square and the third is even.

What is the even number?

A 10 B 12 C 14 D 16 E 18

SOLUTION **A**

All the 2-digit primes are odd numbers. So Sonia writes down an odd prime, a square and an even number whose sum is the even number 46. It follows that the square is also an odd number because an even number cannot be the sum of two even numbers and an odd number.

The only odd 2-digit square that is less than 46 is 25. We deduce that the square is 25. Therefore the sum of the prime and the even number is $46 - 25$, that is, 21.

The only two 2-digit numbers with sum 21 are 10 and 11. Hence the prime is 11 and the even number is 10.

FOR INVESTIGATION

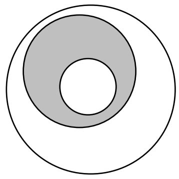
4.1 Sonia next writes down three 2-digit numbers whose sum is 47. The first number is prime, the second is square and the third is even.

How many possibilities are there for the even number?

5. The diagram shows three circles with radii 1, 2 and 3.

What is the ratio of the shaded area to the area of the largest circle?

A 1 : 3 B 1 : 2 C $\sqrt{2} : \sqrt{3}$ D 2 : 3 E 4 : 9



Solution **A**

The circles have areas $\pi(1^2)$, $\pi(2^2)$ and $\pi(3^2)$, that is, π, 4π and 9π, respectively.

It follows that the shaded area is $4\pi - \pi = 3\pi$.

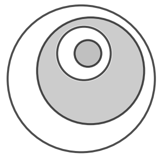
Therefore the ratio of the shaded area to the area of the largest circle is $3\pi : 9\pi = 1 : 3$.

For investigation

5.1

(a) Suppose that the four circles in the diagram have radii 1, 2, 3 and 4. What fraction of the area of the largest circle is shaded?

(b) Find integer values for the radii of the circles shown in the diagram so that half the area of the largest circle is shaded.



6. For what value of x is $\sqrt{(\sqrt{(\sqrt{x} + 1)} + 1)} + 1 = 3$?

A 4096 B 64 C 8 D 3 E 0

Solution **B**

From the given equation, we have $\sqrt{(\sqrt{(\sqrt{x} + 1)} + 1)} = 2.$

Therefore, squaring both sides, $\sqrt{(\sqrt{x} + 1)} + 1 = 4.$

Hence $\sqrt{(\sqrt{x} + 1)} = 3.$

Therefore, squaring again, $\sqrt{x} + 1 = 9.$

Hence $\sqrt{x} = 8.$

Finally, by squaring again, we deduce that $x = 64$. [In Problem 6.2 you are asked to check that squaring three times has not introduced a spurious solution.]

For investigation

6.1 For what value of x is $\sqrt{\sqrt{(\sqrt{(\sqrt{x} + 1)} + 1)} + 1} + 1 = 3$?

6.2 Check that when $x = 64$ the value of $\sqrt{(\sqrt{(\sqrt{x} + 1)} + 1)} + 1$ is 3,

7. What is the 100th term of the sequence 1, 5, 7, 11, 13, 17, 19, 23, ... whose terms are consecutive odd numbers but with all the multiples of 3 removed?

A 201 B 203 C 247 D 299 E 301

Solution **D**

Method 1

The terms of this sequence occur in pairs where each pair consists of the integers 2 less and 2 more than successive odd multiples of 3.

That is, we can rewrite the terms of this sequence as

$$3 - 2, 3 + 2, \quad 9 - 2, 9 + 2, \quad 15 - 2, 15 + 2, \ldots,$$

where $3, 9, 15, \ldots$ is the sequence of odd multiples of 3.

The 100th term of the sequence therefore is 2 more than the 50th odd multiple of 3.

The 50th odd number is $2 \times 50 - 1 = 99$. Hence the 50th odd multiple of 3 is $3 \times 99 = 297$.

Therefore the 100th term of the sequence is $297 + 2 = 299$.

Method 2

Each positive integer can be expressed as either $6n$, $6n + 1$, $6n + 2$, $6n + 3$, $6n + 4$ or $6n + 5$, where n is a non-negative integer. Of these only $6n + 1$ and $6n + 5$ are odd positive integers that are not multiples of 3. Therefore the first 100 numbers in this sequence are the numbers $6n + 1$ and $6n + 5$ for $0 \leq n \leq 49$. Hence the 100th number in the sequence is $6 \times 49 + 5 = 299$.

For investigation

7.1 The sequence

$$1, \; 3, \; 7, \; 9, \; 11, \; 13, \; 17, \; 19 \ldots$$

consists of the positive odd integers that are not multiples of 5, in numerical order.

What is the 100th term of this sequence?

7.2 The sequence

$$1, \; 3, \; 5, \; 9, \; 11, \; 13, \; 15, \; 17 \ldots$$

consists of the positive odd integers that are not multiples of 7, in numerical order.

What is the 100th term of this sequence?

7.3 The sequence

$$7, \; 21, \; 49, \; 63, \; 77, \; 91, \; 119 \ldots$$

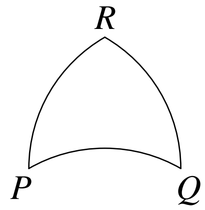
consists of the positive odd integers that are multiples of 7 but not multiples of 5, in numerical order.

What is the 100th term of this sequence?

8. The diagram shows three arcs of circles of radius 1. P is the centre of the circle of which RQ is an arc and Q is the centre of the circle of which PR is an arc.

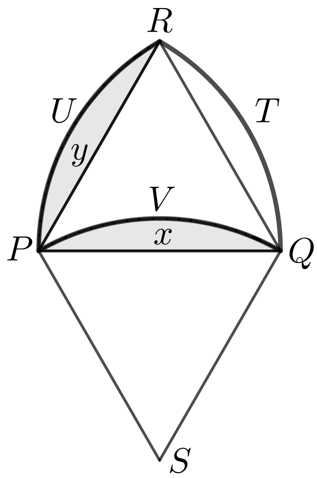
What is the area of this shape?

A $\frac{\sqrt{3}\pi}{10}$ B $\frac{\pi}{6}$ C $\frac{\pi}{8}$ D $\frac{\sqrt{3}}{4}$ E $\frac{\sqrt{2}\pi}{9}$



Solution **B**

In the diagram on the right we have added the labels T, U and V so that we can distinguish between the lines joining P, Q and R and the arcs joining these points. For example we refer to the line joining P and Q as PQ and the arc joining them as PVQ.



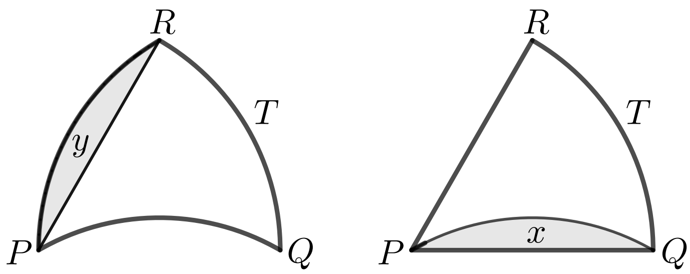
The point S is the centre of the circle that the arc PVQ is part of.

Since PQ, QR and RP all have length 1, the triangle PQR is equilateral. Therefore $\angle RQP = 60°$. Similarly, $\angle QSP = 60°$.

Therefore the regions $RUPQ$ and $QVPS$ are both sectors of circles of radius 1 bounded by arcs that subtend an angle of 60° at the centres of the circles. Hence these two regions are congruent.

The shaded regions, marked x and y in the diagram, are obtained from the regions $RUPQ$ and $QVPS$ by removing the congruent equilateral triangles PQR and PSQ.

It follows that the regions x and y are congruent and hence they have have the same area.



Therefore the shape of the question has the same area as the shape obtained from it by removing the region y and adding the region x.

This new shape is the sector $PQTR$. Because $\angle QPR = 60°$ its area is one sixth of the area of the circle centre P and radius 1, namely $\frac{1}{6}\pi(1^2)$, that is $\frac{\pi}{6}$.

Therefore the area of the shape of this question is also $\frac{\pi}{6}$.

For investigation

8.1 What is the area of the regions marked x and y in the diagram above?

9. Fifty squares are drawn side by side in a line. The first and last squares are shaded. Other squares in the line must be shaded such that both these rules apply: (a) no two adjacent squares are shaded and (b) there are no more than three consecutive unshaded squares.

What is the difference between the smallest and largest number of squares that can be shaded?

A 8 B 10 C 11 D 13 E 15

Solution **C**

We number the squares 1 to 50 in order from left to right so that we may refer to them.

Because of rule (a) at most one of each pair of consecutive squares can be shaded. So at most half, that is, 25, squares can be shaded. We can achieve this by shading all the odd numbered squares from 1 to 47 inclusive. Square 49 cannot be shaded because it is next to square 50 which is shaded. However, when the squares with odd numbers up to 47 and square 50 are shaded we have shaded 25 of the squares, as shown below.

Because of rule (b) in every block of four consecutive squares at least one must be shaded. So at least 12 of the squares from 2 to 49 must be shaded. This means that, together with squares 1 and 50, at least 14 squares must be shaded. One way to achieve this is to shade squares 1, 5, 9 and so on shading every fourth square up to square 45, and also squares 47 and 50, as shown below.

Therefore the largest number of squares that can be shaded is 25 and the smallest number is 14.

The difference between these two numbers is $25 - 14$ which is equal to 11.

For investigation

9.1 (a) In how many different ways is it possible to shade 25 squares, including squares 1 and 50, so that no two adjacent squares are shaded?

(b) In how many different ways is it possible to shade 25 squares so that no two adjacent squares are shaded, when the requirement that squares 1 and 50 must be shaded is dropped?

9.2 In how many different ways is it possible to shade 14 squares, including squares 1 and 50, so that there are no more than three consecutive unshaded squares?

9.3 If the requirement that squares 1 and 50 must be shaded is dropped, what is the smallest number of squares that need to be shaded so that there are no more than three consecutive unshaded squares?

10. How many different squares are factors of 2025?

A 2 B 3 C 4 D 5 E 6

SOLUTION **E**

One method would be to list *all* the factors of 2025 and then to check which of them are squares. This would take rather a lot of time.

It is quicker to make use of the factorization of integers into primes, This can be used to decide whether an integer is a square.

Before discussing the general case, we give the example of the number 72^2.

The prime factorization of 72 is $2^3 \times 3^2$. It follows that $72^2 = (2^3)^2 \times (3^2)^2 = 2^6 \times 3^4$. Note that the exponents of the primes in this factorization are both even numbers.

In general, suppose that $n = p^a \times q^b \times r^c \times \ldots$, where $p, q, r, \ldots$ are the different primes that are factors of n. Then $n^2 = p^{2a} \times q^{2b} \times r^{2c} \times \ldots$. So the exponents of the primes in the prime factorization of a square are always even integers.

Conversely, an integer with a prime factorization of this form is a square.

Thus the test for whether an integer n is a square is that in the prime factorization of n, the exponent of each prime is an even number.

The factorization of 2025 into primes is given by

$$2025 = 3 \times 3 \times 3 \times 3 \times 5 \times 5, \text{ that is, } 3^4 \times 5^2.$$

Therefore the factors of 2025 that are squares are all the integers of the form $3^a \times 5^b$, where a is an even number in the range $0 \leq a \leq 4$ and b is an even integer in the range $0 \leq b \leq 2$.

Hence there are 3 possible values for a, namely, 0, 2 and 4, and 2 possible values, 0 and 2, for b. This gives $2 \times 3 = 6$ choices for the pair a, b.

Therefore 2025 has 6 square factors.

FOR INVESTIGATION

10.1 List the 6 square factors of 2025.

10.2 How many square factors does the number 2025^2 have?

10.3 How many square factors does the number 10! have?
[*Note:* $10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$.]

10.4 Explain why, if the prime factorization of n is $n = p^a \times q^b \times r^c \times \ldots$, where $p, q, r, \ldots$ are distinct primes, then n has $(a+1)(b+1)(c+1)\ldots$ different factors.

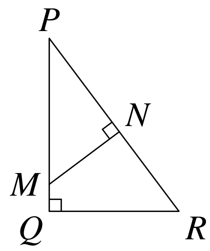
10.5 Suppose that the prime factorization of n is $n = p^a \times q^b \times r^c \times \ldots$, where $p, q, r, \ldots$ are distinct primes.

Give a formula in terms of $a, b, c, \ldots$ for the number of factors of n that are squares.

11. The diagram shows a triangle PQR with $\angle PQR = 90°$, $PQ = 20$ and $PR = 25$. Point M lies on PQ, point N lies on PR and PNM is a right-angled triangle whose area is half that of triangle PQR.

What is the length of MN?

A $6\sqrt{2}$ B $\frac{15}{2}\sqrt{2}$ C $8\sqrt{2}$ D $\frac{17}{2}\sqrt{2}$ E $10\sqrt{2}$



SOLUTION **B**

We note first that, by Pythagoras' Theorem applied to the right-angled triangle PQR,

$$QR^2 = PR^2 - PQ^2 = 25^2 - 20^2 = 625 - 400 = 225 = 15^2.$$

Therefore $QR = 15$.

Note that PQR is the standard right-angled triangle with sides of lengths 3, 4 and 5 scaled up by the factor 5.

If you notice that $PQ : PR = 4 : 5$, you could deduce that $QR = 15$ without the need to do the above calculation.

In the triangles PQR and PNM, we have $\angle PQR = \angle PNM = 90°$. These triangles share the angle at P. Because the angles in a triangle have sum 180°, it follows that $\angle PRQ = \angle NMP$. Therefore these triangles are similar.

The areas of similar triangles are proportional to the squares of the lengths of their corresponding sides. [Problem 11.2 asks you to prove this.]

Therefore since area of PNM : area of $PQR = 1 : 2$, it follows that $MN : QR = 1 : \sqrt{2}$. Therefore $MN = \frac{QR}{\sqrt{2}} = \frac{15}{\sqrt{2}} = \frac{15}{2}\sqrt{2}$.

FOR INVESTIGATION

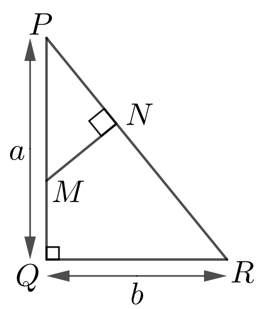
11.1 The diagram on the right shows a triangle PQR in which $\angle PQR = 90°$, $PQ = a$ and $QR = b$. M is a point on PQ and N is the point on PR such that MN is perpendicular to PR.

(a) Suppose that area of $PMN = k\times$ area of PQR, with $0 < k < 1$.

Find the ratio $PN : NR$ in terms of a, b and k.

(b) Suppose now that the point M coincides with the point Q.

Find, in terms of a and b, the ratio $PN : NR$.



11.2 Prove that the areas of similar triangles is proportional to the squares of their corresponding sides.

12. Ayain writes down all the 3-digit numbers consisting of three different odd digits.

How many of Ayain's numbers are divisible by 3?

A 42 B 36 C 30 D 24 E 18

Solution **D**

The test for whether an integer is divisible by 3 is that the sum of its digits is divisible by 3. [You are asked to prove this in Problem 12.4.]

There are four sets made up of three of the odd digits 1, 3, 5, 7, 9, whose sum is a multiple of 3. These are $\{1, 3, 5\}$, $\{1, 5, 9\}$, $\{3, 5, 7\}$ and $\{5, 7, 9\}$.

The three digits of each set can be arranged in six ways to form six three digits numbers. Therefore $4 \times 6 = 24$ of Ayain's numbers are divisible by 3.

For investigation

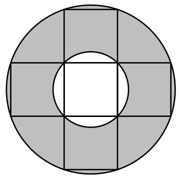
12.1 Make a list, in numerical order of the 24 of Ayain's numbers that are divisible by 3.

12.2 How many of Ayain's numbers are divisible by 5?

12.3 How many of Ayain's numbers are divisible by 11?

12.4 Prove that an integer is divisible by 3 if, and only if, the sum of its digits is divisible by 3.

13. Five congruent squares, each of side $2a$, are placed edge to edge. Two circles with the same centre are drawn through the vertices as shown.

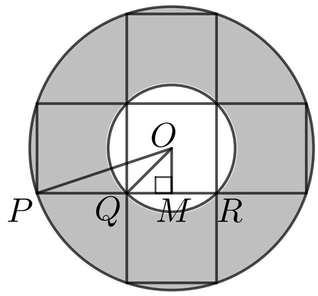


What is the area of the region between the two circles?

A $2\pi a^2$ B $4\pi a^2$ C $6\pi a^2$ D $8\pi a^2$ E $10\pi a^2$

Solution **D**

We let O be the centre of the two circles and P, Q and R be the vertices shown in the diagram.



We also let M be the midpoint of the side QR of the central square.

Then OM is perpendicular to QR and $OM = QM$. [You are asked to prove this in Problem 13.1.]

Since each square has side length $2a$, $OM = QM = a$ and $PM = PQ + QM = 2a + a = 3a$.

Therefore, by applying Pythagoras' Theorem to the right-angled triangle QMO, we have

$$QO^2 = QM^2 + OM^2 = a^2 + a^2 = 2a^2.$$

QO is the radius of the smaller circle. Hence the area of this circle is $\pi(QO^2) = \pi(2a^2) = 2\pi a^2$.

Similarly. by applying Pythagoras' Theorem to the right-angled triangle PMO, we have

$$PO^2 = PM^2 + OM^2 = (3a)^2 + a^2 = 9a^2 + a^2 = 10a^2.$$

Hence the area of the larger circle is $10\pi a^2$.

The area of the region between the circles is the difference of their areas. Hence this area is $10\pi a^2 - 2\pi a^2 = 8\pi a^2$.

For investigation

13.1 Prove that OM is perpendicular to QR and $OM = QM$.

13.2 What is the area of the region inside each of the outer squares but not in the smaller circle?

14. The simultaneous equations $x + \frac{1}{y} = 2$ and $y + \frac{1}{x} = \frac{9}{4}$ have two pairs of real solutions.

What is the difference between the possible values of x?

A $\frac{1}{3}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D 1 E $\frac{3}{2}$

SOLUTION **B**

From the equation $x + \frac{1}{y} = 2$, we have $\frac{1}{y} = 2 - x$ and therefore $y = \frac{1}{2-x}$.

By substituting $\frac{1}{2-x}$ for y in the equation $y + \frac{1}{x} = \frac{9}{4}$, we deduce that

$$\frac{1}{2-x} + \frac{1}{x} = \frac{9}{4}.$$

This last equation may be rearranged to give

$$9x^2 - 18x + 8 = 0. \qquad (1)$$

[You are asked to check this in Problem 14.1.]

The left hand side of this last equation factorizes to give

$$(3x-4)(3x-2) = 0.$$

Therefore either $3x - 4 = 0$ or $3x - 2 = 0$. Hence the solutions of (1) are $x = \frac{4}{3}$ and $x = \frac{2}{3}$.

Therefore the difference between the two possible values of x is $\frac{4}{3} - \frac{2}{3} = \frac{2}{3}$.

FOR INVESTIGATION

14.1 Check that the equation $\frac{1}{2-x} + \frac{1}{x} = \frac{9}{4}$ may be rearranged to give $9x^2 - 18x + 8 = 0$.

14.2 Let x and y be as in this question.

Put $z = xy$.

Find a quadratic equation satisfied by z. Use this equation to find the values of z and hence the values of x and y.

14.3 Find the solution of the following system of simultaneous equations.

$$x + \frac{1}{y} = 2$$
$$y + \frac{1}{z} = 1$$
$$z + \frac{1}{x} = 5.$$

15. The integer n is such that $1 < n < 99$. P is the difference between 1 and n, Q is the difference between 99 and n, and R is the difference between P and Q.

For how many values of n is R a prime number?

A 0 B 2 C 4 D 8 E 99

SOLUTION **B**

For n between 1 and 99, the difference between 1 and n is $n - 1$, and the difference between n and 99 is $99 - n$. Thus $P = n - 1$ and $Q = 99 - n$.

The difference between P and Q depends on which is the larger. You are asked, in Problem 15.1, to check that $P < Q$ for $1 < n < 50$ and $Q < P$ for $50 < n < 99$, while $P = Q$ when $n = 50$.

It follows that when $1 < n < 50$, we have $R = Q - P = (99 - n) - (n - 1) = 100 - 2n$ and that when $50 < n < 99$, we have $R = P - Q = (n - 1) - (99 - n) = 2n - 100$. Also, $R = 0$ when $n = 50$.

When n is an integer both $100 - 2n$ and $2n - 100$ are even integers. The only even prime is 2. It follows that R is prime only when either $100 - 2n = 2$ or $2n - 100 = 2$.

The equation $100 - 2n = 2$ has the single solution $n = 49$.

The equation $2n - 100 = 2$ has the single solution $n = 51$.

Therefore R is prime only when n is either 49 or 51.

Therefore there are just 2 values of n for which R is prime.

FOR INVESTIGATION

15.1 Check that $P < Q$ for $1 < n < 50$ and $Q < P$ for $50 < n < 99$, while $P = Q$ when $n = 50$.

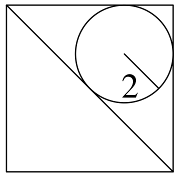
15.2 *Note:* This problem is the version of Question 15 with the restriction that the integer n is between 1 and 99 dropped.

With P, Q and R as in Question 15, for how many integer values of n is R a prime number?

15.3 *Note:* This problem is the version of Question 15 with the restriction that n is an integer dropped.

With P, Q and R as in Question 15, for how many rational numbers, n, where $1 < n < 99$, is R a prime number?

16. The diagram shows a square, one of its diagonals and a circle. The circle touches the diagonal and two sides of the square. The circle has radius 2.

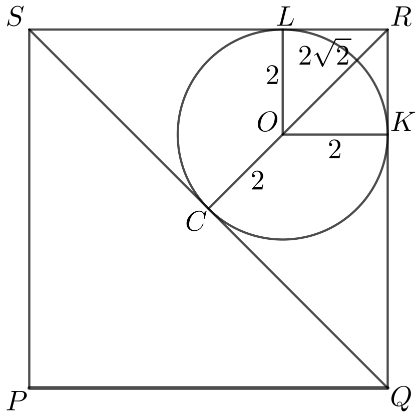


What is the length of the side of the square?

A $4 + 2\sqrt{2}$ B $8 - \sqrt{3}$ C $2 + 2\sqrt{2}$ D $8 - 2\sqrt{2}$ E $4 + 2\sqrt{3}$

SOLUTION **A**

We let O be the centre of the circle, and C be the centre of the square. The other points are labelled as shown in the diagram. Since both the circle and the square are symmetrical about diagonal PR of the square, it follows that the circle touches the diagonal QS at C.



Since the circle touches the sides QR and RS of the square, the radii OK and OL are perpendicular to QR and RS, respectively. Therefore $OKRL$ is a square with side length 2.

By Pythagoras' Theorem applied to the right angled triangle OKR, we have $RO^2 = OK^2 + KR^2 = 2^2 + 2^2 = 8$.

It follows that $OR = \sqrt{8} = 2\sqrt{2}$. Hence $RC = RO + OC = 2\sqrt{2} + 2$.

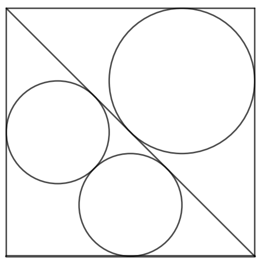
Because C is the centre of the square, $CR = CQ$. Because QS is a tangent to the circle, $\angle RCQ = 90°$. Hence RCQ is a right-angled isosceles triangle. Therefore its sides have lengths in the ratio $1 : 1 : \sqrt{2}$.

It follows that the length of the side of the square is given by

$$QR = \sqrt{2}(2\sqrt{2} + 2) = 4 + 2\sqrt{2}.$$

FOR INVESTIGATION

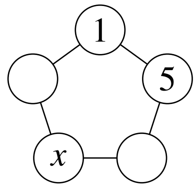
16.1 The diagram shows a square, one of its diagonals and three circles.



The largest circle touches the diagonal and two sides of the square. The smaller circles touch the diagonal, one side of the square and each other, as shown.

What is the ratio of the area of the largest circle to the total area of the two smaller circles?

17. The circles are filled with five integers so that any integer from 1 to 21 can be made, either by choosing one of the integers or by summing up to 5 adjacent integers.



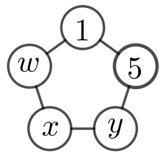
When 1 and 5 are in the positions shown, what is the value of x?

A 2 B 3 C 7 D 10 E 11

Solution **D**

We let w and y be the integers in the other two circles, as shown in the first diagram below.

There are just 21 ways in which we can use these integers to make an integer: each of the 5 integers can be used separately; there are 5 sums of two adjacent integers ($1 + w$, $w + x$, $x + y$, $y + 5$, $5 + 1$); 5 sums of three adjacent integers ($1 + w + x$, $w + x + y$, and so on); 5 sums of four adjacent integers ($1 + w + x + y$, and so on); and the sum, $1 + w + x + y + 5$, of all 5 of the integers.

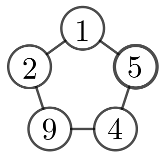


Because all the integers from 1 to 21 can be made, each of the 21 ways described above must have a different outcome. In particular, the integers in the circles must be different. It follow that 2 cannot be made from the sum $1 + 1$. Hence 2 is one of the integers in the circles.

We can also deduce that as the largest sum that can be made is 21, we have $1 + w + x + y + 5 = 21$ and hence $w + x + y = 15$.

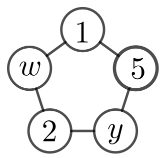
The number 2 must be either be w, x or y. We consider these cases in turn.

Suppose that w is 2. Then 4 cannot be made as $1 + 3$ or $2 + 2$, and must therefore be either x or y. It cannot be x since otherwise, two sums $5 + 1$ and $2 + 4$ would give 6. So 4 would have to be y. Then, as $w + x + y = 15$, we would have that x is 9, as shown on the right. However, in this case 7 cannot be made. We deduce that w is not 2.



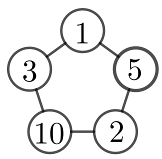
Next suppose that x is 2. Then 3 could not be made as the sum $1 + 2$. Hence either w or y is 3. However in each of these cases 5 would occur both as a single integer and as the sum $2 + 3$, so all the sums would not be different.

It follows that x is not 2. Therefore y is 2.



In this case 3 cannot be made as the sum $1 + 2$. Therefore 3 is either w or x. It cannot be x as 5 could then again be made in more than one way. So w is 3.

Since $w + x + y = 15$, it then follows that x is 10. This case is shown in diagram on the right. You are asked in Problem 17.1 to check that with this arrangement all the integers from 1 to 21 can be made.



For investigation

17.1 Check that, in the situation of the final diagram above, all the integers from 1 to 21 can be made.

17.2 Is there an other way to put five positive integers in the circles so that all the integers from 1 to 21 can be made?

18. The positive integer N has 2025 digits. The first digit is a 3. Every two consecutive digits of N form a number that is divisible by either 17 or 23. The units digit of N could either be p or q.

What is the value of $p + q$?

A 3 B 6 C 7 D 9 E 10

Solution **C**

The two-digit multiples of 17 are 17, 34, 51, 68 and 85.

The two-digit multiples of 23 are 23, 46, 69 and 92.

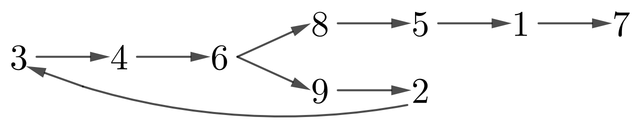
In numerical order, these multiples are 17, 23, 34, 46, 51, 68, 69, 85, 92.

From this we see that in the number N the digit 3 must be followed by 4, and 4 by 6. The digit 6 could be followed by either 8 or 9.

The digit 8 must be followed by 5, 5 by 1 and 1 by 7. The sequence of digits would then end as there is no two-digit multiple of either 17 or 23 with tens digit 7.

The digit 9 must be followed by 2 and 2 by 3, creating a loop.

This is illustrated by the following diagram:



Since $2025 \div 5 = 405$, we see that the 2025 digits of N result either from going 405 times round the loop $3 \to 4 \to 6 \to 9 \to 2$, or from going round this loop 404 times followed by the sequence $3 \to 4 \to 6 \to 8 \to 5$.

Therefore the final digit of N is either 2 or 5.

Hence $p + q = 2 + 5 = 7$.

For investigation

18.1 Which digits *cannot* occur in the number N?

18.2 Consider those positive integers with 2025 digits in which every two consecutive digits form a number that is divisible by either 17 or 23, but which do not necessarily have first digit 3.

(a) Which digits could be the first digit of such an integer?

(b) For each digit that could be the first digit of such a number, determine which digits could be the units digit.

18.3 The positive integer M has 1000 digits.

Every two consecutive digits of M form a number that is divisible by either 13 or 41.

(a) Which digits could be the first digit of M?

(b) Which digits could be the final digit of M?

19. A 'complete' football kit consists of a shirt, a pair of shorts and a pair of socks. Three pairs of shorts and one pair of socks together cost the same as two shirts. Seven pairs of shorts and four pairs of socks together cost the same as five shirts. Eden has exactly the right amount of money to buy nine shirts. How many 'complete' football kits could be bought for the same amount of money?

A 3 B 4 C 5 D 6 E 7

Solution **C**

Let the cost of one shirt, one pair of shorts and one pair of socks be x, y and z pounds, respectively. Thus the cost of a full football kit is $x + y + z$ pounds.

From the information given in the question, we have

$$3y + z = 2x \qquad (1)$$
$$7y + 4z = 5x. \qquad (2)$$

Since $5 \times 2x = 2 \times 5x$, it follows from (1) and (2) that

$$5(3y + z) = 2(7y + 4x).$$

That is,

$$15y + 5z = 14y + 8z,$$

and hence

$$y = 3z. \qquad (3)$$

It follows from (1) and (3) that $9z + z = 2x$ and hence that

$$z = \frac{1}{5}x.$$

Hence, by (3),

$$y = \frac{3}{5}x.$$

It follows that

$$\begin{aligned} x + y + z &= x + \frac{3}{5}x + \frac{1}{5}x \\ &= \frac{9}{5}x. \end{aligned}$$

Therefore the cost of a complete football kit is $\frac{9}{5}x$ pounds.

Eden has exactly enough money to buy 9 shirts, and so has $9x$ pounds.

Therefore Eden has enough money to buy $9x \div \frac{9}{5}x = 5$ complete football kits.

For investigation

19.1 If, in this question, one pair of football socks costs £2.40, what is the cost of a complete football kit?

20. When $\frac{1}{x} - \frac{1}{y} = 2025$, what is the value of $\frac{x + 2026xy - y}{2y - 2025xy - 2x}$?

A 0 B 1 C $\frac{1}{2}$ D 2026 E $\frac{1}{2025}$

SOLUTION **E**

Since $\frac{1}{x} - \frac{1}{y} = 2025$, it follows that $\frac{y - x}{xy} = 2025$, and hence $y - x = 2025xy$.

Therefore

$$\begin{aligned}\frac{x + 2026xy - y}{2y - 2025xy - 2x} &= \frac{2026xy - (y - x)}{2(y - x) - 2025xy} \\ &= \frac{2026xy - 2025xy}{2(2025xy) - 2025xy} \\ &= \frac{xy}{2025xy} \\ &= \frac{1}{2025}.\end{aligned}$$

FOR INVESTIGATION

20.1 Show that for every integer n, if $\frac{1}{x} - \frac{1}{y} = n$, then

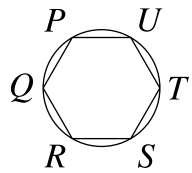
$$\frac{x + (n + 1)xy - y}{2y - nxy - 2x} = \frac{1}{n}.$$

20.2 You are given that

$$\frac{5y + xy - 5x}{3y - xy - 3x} = 2.$$

What is the value of $\frac{1}{x} - \frac{1}{y}$?

21. A regular hexagon $PQRSTU$ is inscribed in a circle of radius 5. A point X on the circumference of the circle is connected to the vertices of the hexagon to form six chords XP, XQ, XR, XS, XT and XU.

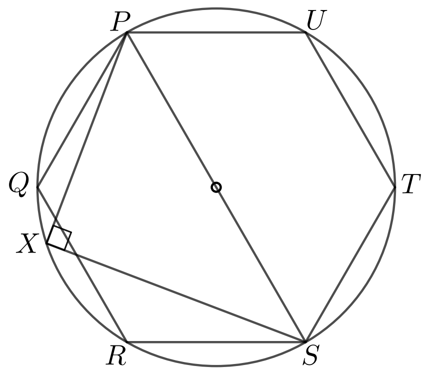


What is the value of $XP^2 + XQ^2 + XR^2 + XS^2 + XT^2 + XU^2$?

A 150 B 216 C 256 D 300 E 360

Solution **D**

The circle has radius 5. Hence it has diameter 10.



PS is a diameter of the circle. Hence, by the *Angle in a Semicircle* theorem (otherwise known as *Thales'* theorem), $\angle PXS = 90°$.

Therefore, by Pythagoras' Theorem

$$XP^2 + XS^2 = PS^2 = 10^2 = 100.$$

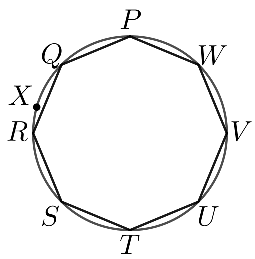
Similarly, $XQ^2 + XT^2 = 100$ and $XR^2 + XU^2 = 100$.

It follows that

$$XP^2 + XQ^2 + XR^2 + XS^2 + XT^2 + XU^2 = 100 + 100 + 100 = 300.$$

For investigation

21.1 A regular octagon $PQRSTUVW$ is inscribed in a circle of radius r.



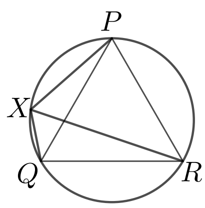
A point X on the circumference of the circle is connected to the vertices of the octagon to form eight chords, XP, XQ, XR, XS, XT, XU, XV and XW.

Find the value of

$$XP^2 + XQ^2 + XR^2 + XS^2 + XT^2 + XU^2 + XV^2 + XW^2$$

in terms of r.

21.2 An equilateral triangle PQR is inscribed in a circle of radius r.



The point X is on the circumference of the circle between P and Q, as shown in the diagram.

X is connected to the vertices of the triangle to form three chords, XP, XQ and XR.

(a) Find the value of $XP^2 + XQ^2 + XR^2$ in terms of r.

(b) Prove that $XP + XQ = XR$. [This is van Schooten's Theorem. Search the web to find a proof. This theorem can be generalized to any regular polygon with an odd number of sides, but the proof of this is not easy.]

22. Together, n aardvarks and 12 anteaters eat $n^2 + 20n + 25$ ants. Each animal eats the same whole number of ants.

How many ants does each animal eat?

A 61 B 66 C 71 D 74 E 79

SOLUTION **B**

There are $n + 12$ animals. They each eat the same number of ants. Therefore the number of ants each of them eats is m, where $m = \dfrac{n^2 + 20n + 25}{n + 12}$.

By division we find that $m = n + 8 - \dfrac{71}{n + 12}$. [You are asked to check this in Problem 22.1.]

Since n is an integer, for m to be an integer $\dfrac{71}{n + 12}$ must be an integer, and hence $n + 12$ needs to be a factor of 71.

71 is prime. Therefore its only factors are 1 and 71, The integer n is positive. Hence $n + 12 > 1$.

It follows that $n + 12 = 71$.

Hence $n = 59$.

Therefore $m = 59 + 8 - \dfrac{71}{71} = 67 - 1 = 66$. So each animal ate 66 ants.

FOR INVESTIGATION

22.1 Check that $\dfrac{n^2 + 20n + 25}{n + 12} = n + 8 - \dfrac{71}{n + 12}$.

22.2 Together n vegans and 11 vegetarians ate $n^2 + n + 3$ carrots. They each ate the same number of carrots.

How many carrots did each person eat?

22.3 For which integer values of n is $7n^3 + 5n^2 + 3n + 1$ divisible by $n + 1$?

23. Jemima took a series of n tests each with the same maximum mark. After $(n-2)$ tests her average score was m. She scored full marks in the $(n-1)$th test, and so raised her average score by 4 marks. In the nth test when she again scored full marks, she increased her average score by 3 marks.

How many tests did Jemima take?

A 6 B 7 C 8 D 9 E 10

SOLUTION **C**

We let M be the maximum mark for each test.

After $(n-2)$ tests Jemima's average mark was m. Therefore she scored a total of $m(n-2)$ marks in these tests.

After getting a maximum mark in the next test, Jemima has a total of $m(n-2)+M$ marks, with an average of $(m+4)$ for the $(n-1)$ tests. Therefore,

$$\frac{m(n-2)+M}{n-1} = m+4,$$

and hence

$$m(n-2)+M = (m+4)(n-1). \qquad (1)$$

After gaining another maximum mark, Jemima has a total of $(m+4)(n-1)+M$ marks with an average of $(m+4)+3=(m+7)$ for the n tests. Therefore,

$$\frac{(m+4)(n-1)+M}{n} = m+7,$$

and hence

$$(m+4)(n-1)+M = (m+7)n. \qquad (2)$$

Subtracting equation (1) from equation (2) gives

$$(m+4)(n-1)-m(n-2) = (m+7)n-(m+4)(n-1). \qquad (3)$$

Expanding both sides of equation (3) gives

$$mn-m+4n-4-mn+2m = mn+7n-mn+m-4n+4. \qquad (4)$$

Most of the terms in equation cancel (4), and so the equation simplifies to give $n=8$.

Therefore Jemima took 8 tests.

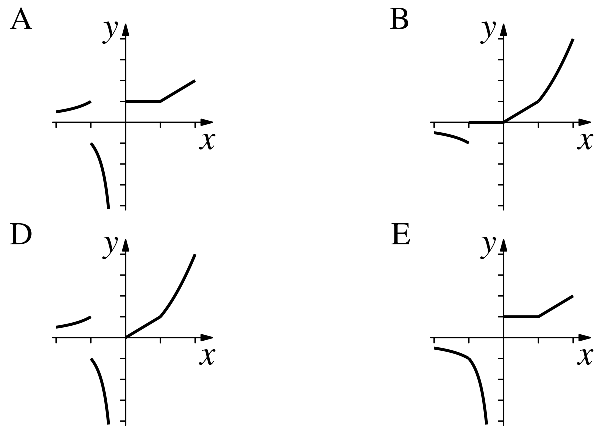
FOR INVESTIGATION

23.1 Jemima took one more test and again scored full marks.

As a result, by how much did her average mark increase?

24. The output of the function $F(x)$ when applied to a real number x is the greatest integer less than or equal to x. For example, $F(3) = 3$, $F(4.7) = 4$, $F(-2.3) = -3$.

Which of the following is the graph of $y = x^{F(x)}$ for non-zero values of x in the interval $-2 < x < 2$?



SOLUTION **A**

$F(x) = 0$ for $0 < x < 1$. Hence $y = x^0 = 1$ for $0 < x < 1$.

Therefore the correct graph of $y = x^{F(x)}$ is horizontal and shows a positive value for y in this range.

This means that options B, C and D are not correct. This leaves options A and E as the only possibilities.

$F(x) = -2$ for $-2 < x < -1$. Hence $y = x^{-2} = \dfrac{1}{x^2}$ for $-2 < x < -1$.

Therefore the correct graph shows a positive value for y in this range.

Hence option E is not correct.

In the context of the SMC, where it can be assumed that one of the given options is correct, we conclude that it is option A that is the graph of $y = x^{F(x)}$.

FOR INVESTIGATION

24.1 (a) Sketch the graph of $y = x^{F(x)}$ in the range $-2 < x < 2$.

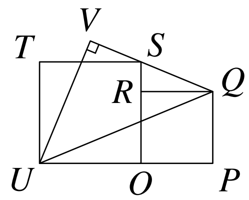
(b) Check that the graph of option A is also correct in the ranges $-1 \le x < 0$ and $1 < x < 2$.

24.2 Sketch the graph of $y = x^{F(x)}$ in the range $-3 < x < 3$.

25. The diagram shows two squares, $OPQR$ and $OSTU$. Point S lies on QV. Triangle UQV is isosceles with a right angle at V. Square $OPQR$ has area 25.

What is the area of square $OSTU$?

A 36 B 45 C 50 D 54 E 60



Solution **C**

This is a hard question because it is not at all clear what approach to use. We give two methods for answering this question. The short solutions use a third approach. Problems 25.2 and 25.3 indicate two more methods, and there are others.

Our first method uses properties of similar triangles.

Our second method uses the addition formula for the tangent function:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Depending on which School Year you are in, you may not yet have met this formula.

The fact that the square $OPQR$ has area 25 is not relevant until the last step. For the earlier part of the answer we have let b be the side length of this square, and only use the fact that $b^2 = 25$ at the end of the solution.

It helps to use a letter for the side length of each square. For example, in equation (3) of Method 1, we can easily see that all the terms represent *squares* of distances. This *dimensional analysis* does not prove that the equation is correct, but a check of this kind can help to detect algebraic slips.

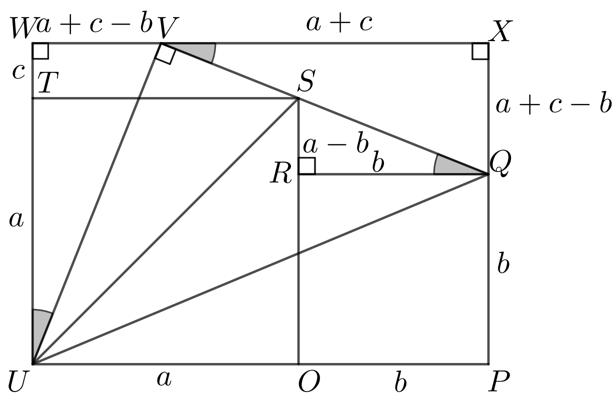
Method 1

We let a and b be the side lengths of the squares $OSTU$ and $OPQR$, respectively.

Let W and X be the points where the line through V parallel to UP meets UT extended, and PQ extended, respectively.

We let c be the length of TW.

$WUPX$ is a rectangle. Therefore $UW = PX$. Hence $a + c = b + QX$. It follows that $QX = a + c - b$.



The angles on the line WX at V have sum 180°, as do the angles in the triangle UVW. Therefore,

$$\angle QVX + 90° + \angle WVU = \angle VUW + 90° + \angle WVU.$$

Hence $\angle QVX = \angle VUW$. Therefore the right-angled triangles VQX and VUW are similar. Since $VQ = VU$, these triangles are congruent. It follows that $WV = QX = a + c - b$ and $VX = UW = a + c$.

Since $WX = UP$, we have $(a + c - b) + (a + c) = a + b$. It follows that $2c = 2b - a$. Hence $c = b - \frac{1}{2}a$.

Therefore

$$QX = a + c - b = a + (b - \tfrac{1}{2}a) - b = \tfrac{1}{2}a \qquad (1)$$

and

$$VX = a + c = a + (b - \tfrac{1}{2}a) = \tfrac{1}{2}a + b \qquad (2)$$

Since RQ is parallel to VX, we have $\angle SQR = \angle QVX$. Therefore the right-angled triangles SRQ and QXV are similar. Hence

$$\frac{SR}{QX} = \frac{QR}{VX}.$$

Therefore, from (1) and (2),

$$\frac{a-b}{\frac{1}{2}a} = \frac{b}{\frac{1}{2}a + b}.$$

Hence

$$(\tfrac{1}{2}a + b)(a - b) = \tfrac{1}{2}ab.$$

This gives

$$\tfrac{1}{2}a^2 - \tfrac{1}{2}ab + ab - b^2 = \tfrac{1}{2}ab. \qquad (3)$$

It follows from equation (3) that

$$a^2 = 2b^2.$$

We deduce that the area of the square $OSTU$ is twice the area of the square $OPQR$. Hence the area of $OSTU$ is 50.

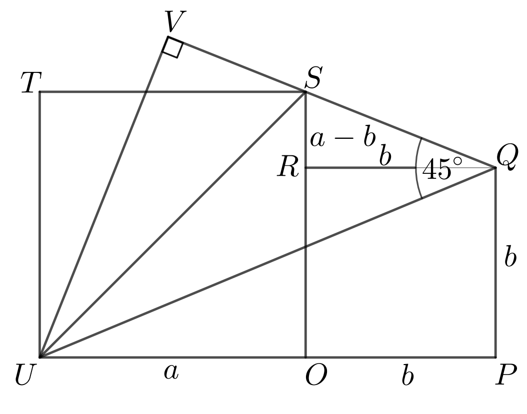
METHOD 2

From the right-angled triangle SRQ, we have

$$\tan(\angle SQR) = \frac{SR}{QR} = \frac{a-b}{b},$$

where a and b are as in Method 1. RQ is parallel to UP. Therefore $\angle UQR = \angle PUQ$. Hence, from the right-angled triangle QUP, we have

$$\tan(\angle UQR) = \tan(\angle PUQ) = \frac{QP}{UP} = \frac{b}{a+b}.$$



Since $\angle VQU = \angle SQR + \angle UQR$, from the addition formula for the tangent function, we obtain

$$\begin{aligned}\tan(\angle VQU) &= \tan(\angle SQR + \angle UQR)\\ &= \frac{\tan(\angle SQR) + \tan(\angle UQR)}{1 - \tan(\angle SQR)\tan(\angle UQR)}\\ &= \frac{\left(\frac{a-b}{b}\right) + \left(\frac{b}{a+b}\right)}{1 - \left(\frac{a-b}{b}\right)\left(\frac{b}{a+b}\right)}.\end{aligned}$$

Hence. multiplying top and bottom of this last fraction by $b(a+b)$, we have

$$\begin{aligned}\tan(\angle VQU) &= \frac{(a-b)(a+b)+b^2}{b(a+b)-(a-b)b}\\ &= \frac{a^2-b^2+b^2}{(ab+b^2)-(ab-b^2)}\\ &= \frac{a^2}{2b^2}.\end{aligned}$$

UVQ is an isosceles right-angled triangle. Therefore $\angle VQU = 45°$, and hence

$$\tan(\angle VQU) = 1.$$

Therefore,

$$\frac{a^2}{2b^2} = 1$$

and hence

$$a^2 = 2b^2.$$

As in Method 1, we conclude that the area of the square $OSTU$ is 50.

[For a version of this method that leads to less complicated algebra, see Problem 25.3.]

For investigation

25.1 Check that from equation (3) of Method 1 it follows that $a^2 = 2b^2$.

25.2 (a) Show that the triangles UPQ and UVS are similar.

(b) Hence find the lengths of UV and VS and hence of SQ in terms of a and b.

(c) Use Pythagoras' Theorem, applied to the triangle SRQ, to obtain a second expression for SQ.

(d) Equate the two expressions for SQ from (b) and (c), and deduce that $a^2 = 2b^2$.

25.3 Let $\angle QUP = \alpha$. Then $\angle SQR = 45° - \alpha$.

(a) Find an expression for $\tan\alpha$ in terms of a and b.

(b) Find an expression for $\tan(45° - \alpha)$ in terms of a and b.

(c) Use the identity

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

to find a second expression for $\tan(45° - \alpha)$.

(d) Equate the two expressions for $\tan(45° - \alpha)$ from (b) and (c) and deduce that $a^2 = 2b^2$.

25.4 Prove that $\angle SQR = \angle UQR$.

25.5 Prove that the line through P and R also goes through V.

UK Maths Trust

SMC 2025 - Problem Group Comments

By Mrs Karen Fogden - SMC Chair

We were extremely pleased to receive a large number of entries for the SMC in 2025, with the number of candidates sitting the challenge rising by 7,000 and all candidates again taking their challenge on paper, having returned to this way of working in 2024. We are very grateful to teachers and others working in schools, including exams officers, who make it possible for their students to enjoy preparing for and sitting the challenge and also facilitating discussion afterwards.

The SMC 2025 mean score of 53 was lower than 58 the previous year. The thresholds for certificates and follow-on rounds were also lower than in 2024, reflecting a slightly tougher paper.

The majority of the earlier questions were well answered; in questions 1 to 6, there were only single-digit percentages of candidates answering incorrectly or leaving questions blank. A good proportion of the middle questions were attempted by many candidates. Qn 10, 'How many different squares are factors of 2025?' caused a surprising problem with the correct answer being the least popular. As question setters, we have been waiting to use the square property of 2025 for many years! We expect that well-prepared candidates should know the prime factorisation of the year, so often incorporated into a question. As always, the later questions that are designed to be tougher were attempted by fewer participants, but Qn 24, a graph problem, was attempted by a good proportion of candidates.

We once again hope that the questions provided an opportunity for interesting and useful discussion after the challenge, especially with the relatively new resource of video solutions, designed to make the after-challenge discussion even more accessible for students both individually and within their classrooms. Extended solutions to the SMC questions, with extension problems for further investigation, are available at: https://ukmt.org.uk/competition-papers.