COMPENDIUM KANGAROO UK

Senior Kangaroo

2015 - 2025

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Toomates Coolección vol. 39.3

Toomates Coolección

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Libros de texto para ESO y bachillerato (en español y en catalán):

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Evaluación diagnóstica y pruebas de acceso:

ACM6EP ACM4 CFGS PAP

Competiciones matemáticas:

Canguro: España Cataluña Francia USA Reino Unido Austria
USA: Mathcounts AMC 8 10 12 AIME USAJMO USAMO TSTST TST ELMO Putnam HMMT
España: OME OMEFL OMEEX OMC OMEA OMEM OMA CDP OFEM
Europa: OMI Arquimede BMO BalkanMO JBMO OPM OMP OMJ ONM
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Otros materiales:

Pizzazz! (Book A B C D E Pre-Algebra Algebra) , REOIM , Llibre3r , Excalibur

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- 2022-2025
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Cangur (Cataluña)
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Concurso de Primavera (Madrid)
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Kanguru (Austria)
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Giochi di Archimede (Italia)
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Las pruebas AMC 8, AMC 10 y AMC 12 USA también siguen el formato de respuesta multiopción, pero con una dificultad mucho más elevada que las anteriores:

AMC (USA)
- AMC 8
  - http://www.toomates.net/biblioteca/CompendiumAMC8.pdf
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- AMC 12
  - http://www.toomates.net/biblioteca/CompendiumAMC8.pdf

Tabla de correspondencia Canguro/Cangur/Kangaroo/Kangourou.

	ESPAÑA			UK (England & Wales)		USA		FRANCIA	
EDAD	CURSO	CANGURO	CANGUR (Catalunya)	YEAR	KANGAROO	GRADE	KANGAROO	Curso	KANGOUROU
6/7	1º Prim.			2		1th			
7/8	2º Prim.			3		2nd	Felix		
8/9	3º Prim.			4		3th		CE2	
9/10	4º Prim.			5		4th	Ecolier	CM1	
10/11	5º Prim.		P5	6		5th		CM2	**E** Écoliers
11/12	6º Prim.		P6	7		6th	Benjamin	6ème	
12/13	1º ESO	N1	E1	8		7th		5ème	**B** Benjamins
13/14	2º ESO	N2	E2	9	Grey	8th	Cadet	4ème	
14/15	3º ESO	N3	E3	10		9th		3ème	**C** Cadets
15/16	4º ESO	N4	E4	11	Pink	10th	Junior	2ème	**J**uniors: Lycées G. et T. **É**tudiants: TS, Bac+
16/17	1º BAT	N5	B1	12		11th		1ème	
17/18	2º BAT	N6	B2	13		12th	Student	T	

Índice.

Todas las soluciones desarrolladas se presentan después de su correspondiente bloque de enunciados.

Fuente.

https://ukmt.org.uk/

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 27th November 2015

Organised by the United Kingdom Mathematics Trust

The Senior Kangaroo paper allows students in the UK to test themselves on questions set for the best school-aged mathematicians from across Europe and beyond.

RULES AND GUIDELINES (to be read before starting):

1. Do not open the paper until the Invigilator tells you to do so.
2. Time allowed: **1 hour**.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
4. **Use B or HB pencil only** to complete your personal details and record your answers on the machine-readable Answer Sheet provided. **All answers are written using three digits, from 000 to 999.** For example, if you think the answer to a question is 42, write 042 at the top of the answer grid and then code your answer by putting solid black pencil lines through the 0, the 4 and the 2 beneath.

   Please note that the machine that reads your Answer Sheet will only see the solid black lines through the numbers beneath, not the written digits above. You must ensure that you code your answers or you will not receive any marks. There are further instructions and examples on the Answer Sheet.
5. The paper contains 20 questions. Five marks will be awarded for each correct answer. There is no penalty for giving an incorrect answer.
6. The questions on this paper challenge you **to think**, not to guess. Though you will not lose marks for getting answers wrong, you will undoubtedly get more marks, and more satisfaction, by doing a few questions carefully than by guessing lots of answers.

Enquiries about the Senior Kangaroo should be sent to:

Maths Challenges Office, School of Maths Satellite,

University of Leeds, Leeds, LS2 9JT

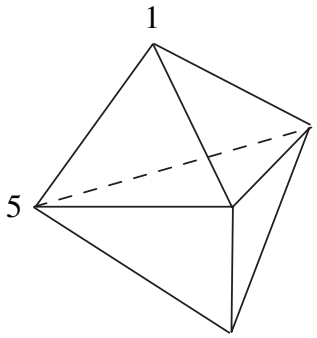
Tel. 0113 343 2339

www.ukmt.org.uk

1. In a pile of 200 coins, 2% are gold coins and the rest are silver. Simple Simon removes one silver coin every day until the pile contains 20% gold coins. How many silver coins does Simon remove?

2. The value of the expression $1 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{5}}}$ is $\dfrac{a}{b}$, where a and b are integers whose only common factor is 1. What is the value of $a + b$?

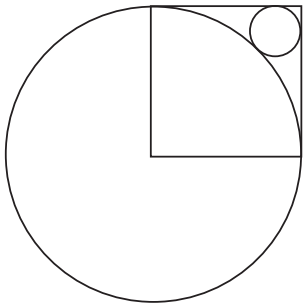
3. The diagram shows a solid with six triangular faces and five vertices. Andrew wants to write an integer at each of the vertices so that the sum of the numbers at the three vertices of each face is the same. He has already written the numbers 1 and 5 as shown.



What is the sum of the other three numbers he will write?

4. A box contains two white socks, three blue socks and four grey socks. Rachel knows that three of the socks have holes in, but does not know what colour these socks are. She takes one sock at a time from the box without looking. How many socks must she take for her to be certain she has a pair of socks of the same colour without holes?

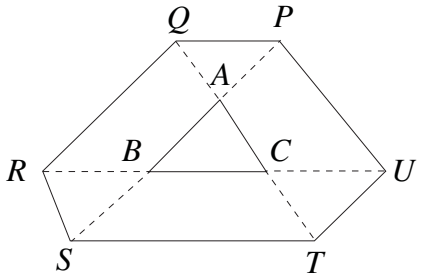
5. The diagram shows two circles and a square with sides of length 10 cm. One vertex of the square is at the centre of the large circle and two sides of the square are tangents to both circles. The small circle touches the large circle. The radius of the small circle is $(a - b\sqrt{2})$ cm.



What is the value of $a + b$?

6. The median of a set of five positive integers is one more than the mode and one less than the mean. What is the largest possible value of the range of the five integers?

7. The diagram shows a triangle ABC with area 12 cm^2. The sides of the triangle are extended to points P, Q, R, S, T and U as shown so that $PA = AB = BS$, $QA = AC = CT$ and $RB = BC = CU$.



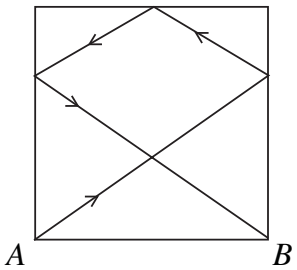
What is the area (in cm^2) of hexagon $PQRSTU$?

8. A mob of 2015 kangaroos contains only red and grey kangaroos. One grey kangaroo is taller than exactly one red kangaroo, one grey kangaroo is taller than exactly three red kangaroos, one grey kangaroo is taller than exactly five red kangaroos and so on with each successive grey kangaroo being taller than exactly two more red kangaroos than the previous grey kangaroo. The final grey kangaroo is taller than all the red kangaroos. How many grey kangaroos are in the mob?

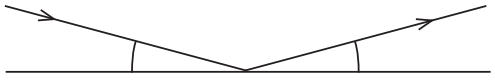
9. A large rectangle is divided into four identical smaller rectangles by slicing parallel to one of its sides. The perimeter of the large rectangle is 18 metres more than the perimeter of each of the smaller rectangles. The area of the large rectangle is 18 m^2 more than the area of each of the smaller rectangles. What is the perimeter in metres of the large rectangle?

10. Katherine and James are jogging in the same direction around a pond. They start at the same time and from the same place and each jogs at a constant speed. Katherine, the faster jogger, takes 3 minutes to complete one lap and first overtakes James 8 minutes after starting. How many seconds does it take James to complete one lap?

11. A ball is propelled from corner A of a *square* snooker table of side 2 metres. After bouncing off three cushions as shown, the ball goes into a pocket at B. The total distance travelled by the ball is $\sqrt{k}$ metres. What is the value of k?



(Note that when the ball bounces off a cushion, the angle its path makes with the cushion as it approaches the point of impact is equal to the angle its path makes with the cushion as it moves away from the point of impact as shown in the diagram below.)



12. Chris planned a 210 km bike ride. However, he rode 5 km/h faster than he planned and finished his ride 1 hour earlier than he planned. His average speed for the ride was x km/h. What is the value of x?

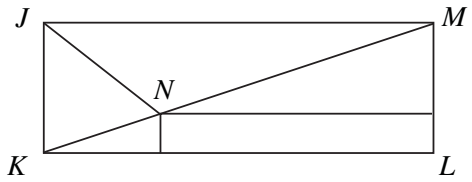
13. Twenty-five people who always tell the truth or always lie are standing in a queue. The man at the front of the queue says that everyone behind him always lies. Everyone else says that the person immediately in front of them always lies. How many people in the queue always lie?

14. Four problems were attempted by 100 contestants in a Mathematics competition. The first problem was solved by 90 contestants, the second by 85 contestants, the third by 80 contestants and the fourth by 75 contestants. What is the smallest possible number of contestants who solved all four problems?

15. The 5-digit number '$XX4XY$' is exactly divisible by 165. What is the value of $X + Y$?

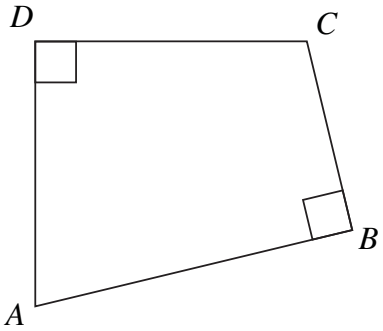
16. How many 10-digit numbers are there whose digits are all 1, 2 or 3 and in which adjacent digits differ by 1?

17. In rectangle $JKLM$, the bisector of angle KJM cuts the diagonal KM at point N as shown. The distances between N and sides LM and KL are 8 cm and 1 cm respectively. The length of KL is $(a + \sqrt{b})$ cm. What is the value of $a + b$?



18. Numbers a, b and c are such that $\dfrac{a}{b+c} = \dfrac{b}{c+a} = \dfrac{c}{a+b} = k$.
How many possible values of k are there?

19. In quadrilateral $ABCD$, $\angle ABC = \angle ADC = 90°$, $AD = DC$ and $AB + BC = 20$ cm.



What is the area in cm^2 of quadrilateral $ABCD$?

20. The number $N = 3^{16} - 1$ has a divisor of 193. It also has some divisors between 75 and 85 inclusive. What is the sum of these divisors?

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 27th November 2015

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

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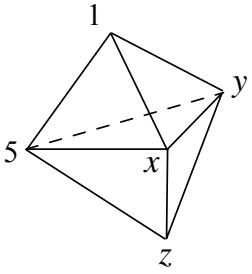
1. **180** The number of gold coins in the original pile is $0.02 \times 200 = 4$. These form 20% of the final pile. Therefore there are $4 \times 5 = 20$ coins left. Hence the number of silver coins Simon removes is $200 - 20 = 180$.

2. **28** The expression can be simplified in stages as follows:

$$1+\frac{1}{1+\dfrac{1}{1+\dfrac{1}{5}}}=1+\frac{1}{1+\dfrac{1}{\left(\dfrac{6}{5}\right)}}=1+\frac{1}{1+\dfrac{5}{6}}=1+\frac{1}{\left(\dfrac{11}{6}\right)}=1+\frac{6}{11}=\frac{17}{11}=\frac{a}{b}.$$

Hence the value of $a + b$ is $17 + 11 = 28$.

3. **11** Let the three missing integers be x, y and z, as shown.



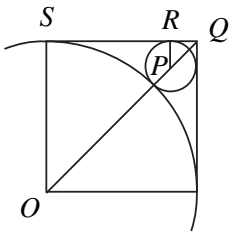
Consider the 'top' three faces. Since the sum of the three numbers at the vertices of each face is the same, we have

$$1 + 5 + x = 1 + x + y = 1 + 5 + y$$

and hence $x = y = 5$. Therefore the sum of the numbers on a face is equal to $5 + 5 + 1 = 11$. But $x + y + z$ is equal to the sum of the numbers on a face. Hence the sum of the other three numbers that Andrew will write is 11.

4. **7** The first six socks Rachel takes out could consist of three different coloured socks and the three socks with holes in, in which case she would not have a pair of socks the same colour without holes in. However, whatever colour her next sock is, she must then complete a pair. Hence she must take seven socks to be certain of getting a pair of socks the same colour without holes in.

5. **50** Let O and P be the centres of the large and small circles respectively and label points Q and S as shown in the diagram. Let the radius of the small circle be r cm. Draw line PR so that R is on QS and PR is parallel to OS. Draw in line OQ. Since triangle OQS is right-angled and isosceles, $OQ^2 = 10^2 + 10^2$ by Pythagoras. Hence $OQ = 10\sqrt{2}$ cm. Similarly, since triangle PQR is right-angled and isosceles, $PQ = r\sqrt{2}$ cm. Note that angle OQS = angle $PQS = 45°$ so OPQ is a straight line. Therefore $10\sqrt{2} = 10 + r + r\sqrt{2}$. This has solution

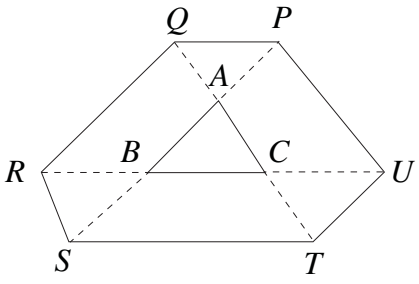


$$r = \frac{10(\sqrt{2}-1)}{\sqrt{2}+1} = \frac{10(\sqrt{2}-1)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{10(2+1-2\sqrt{2})}{2-1} = 30 - 20\sqrt{2}.$$

Hence the radius of the small circle is $(30 - 20\sqrt{2})$ cm and the value of $a + b$ is $30 + 20 = 50$.

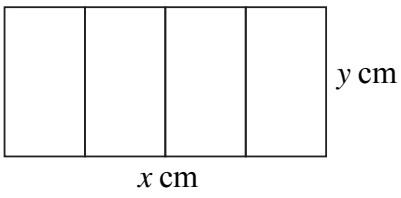
6. **7** Let the five integers be p, q, r, s and t with $p \leqslant q \leqslant r \leqslant s \leqslant t$. The median of the list is r and, since the mode is one less than the median, $p = q = r - 1$ and $r < s < t$. The mean is one more than the median and hence the total of the five integers is $5(r + 1)$. Therefore $r - 1 + r - 1 + r + s + t = 5r + 5$ and hence $s + t = 2r + 7$. Since the smallest possible value of s is $r + 1$, the maximum value of t is $r + 6$. Hence the largest possible value of the range of the five integers is $r + 6 - (r - 1) = 7$.

7. **156** Consider triangles ABC and AST. Angles CAB and TAS are equal because they are the same angle, $SA = 2BA$ and $TA = 2CA$. Hence triangles ABC and AST are similar. The ratio of their sides is 1 : 2 and hence the ratio of their areas is $1^2 : 2^2 = 1 : 4$. Therefore the area of triangle AST is 4×12 cm^2 $= 48$ cm^2 and hence the area of $BSTC$ is $(48 - 12)$ cm^2 $= 36$ cm^2. In a similar way, it can be shown that each of the areas of $CUPA$ and $AQRB$ is also 36 cm^2. Next consider triangles ABC and APQ. Angles BAC and PAQ are equal using vertically opposite angles, $AB = AP$ and $AC = AQ$. Hence triangles ABC and APQ are congruent (SAS) and so the area of triangle APQ is 12 cm^2. In a similar way, it can be shown that the each of areas of triangles BRS and CTU is also 12 cm^2. Hence the total area of hexagon $PQRSTU$ in cm^2 is $(3 \times 36 + 4 \times 12) = 156$.



8. **672** The first grey kangaroo has only one red kangaroo smaller than itself. Apart from that, each grey kangaroo can be grouped with two red kangaroos whose heights lie between its height and that of the previous grey kangaroo. The number of such groups is $(2015 - 2)/3 = 671$. Hence there are 672 grey kangaroos in the mob.

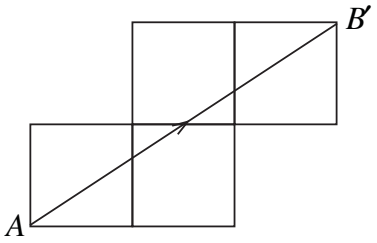
9. **28** Let the length of the original rectangle be x metres and let the height be y metres. Without losing any generality, assume the rectangle is sliced parallel to the height, as shown.



The information in the question tells us that $2x + 2y = 2\left(\frac{x}{4}\right) + 2y + 18$ and that $xy = \left(\frac{x}{4}\right)y + 18$. From the first equation, we have $\frac{3x}{2} = 18$ which has solution $x = 12$. Substitute this value into the second equation to obtain $12y = 3y + 18$, which has solution $y = 2$. Hence the perimeter of the large rectangle in metres is $2 \times 12 + 2 \times 2 = 28$.

10. 288 Katherine catches James after 8 minutes when she has jogged $\frac{8}{3}$ laps. In that time, James will have jogged one lap fewer so will have jogged $\frac{5}{3}$ laps. Therefore, James jogs $\frac{5}{3}$ laps in 8 minutes which is the same as 480 seconds. Hence he will jog $\frac{1}{3}$ of a lap in 96 seconds and so he jogs a whole lap in 288 seconds.

11. 52 A solution can be obtained by reflecting the square repeatedly in the cushion the ball strikes. The path of the ball is then represented by the line AB' in the diagram.



The length of the path can be calculated using Pythagoras Theorem. We have $(AB')^2 = (3 \times 2)^2 + (2 \times 2)^2$. Therefore $(AB')^2 = 36 + 16 = 52$ and so $AB' = \sqrt{52}$ metres and hence the value of k is 52.

12. 35 Chris's time for the ride when he rode at an average speed of x km/h was $\frac{210}{x}$ hours. His planned speed was $(x - 5)$ km/h when his time would have been $\frac{210}{x-5}$ hours. The question tells us that he completed the ride 1 hour earlier than planned, so $\frac{210}{x-5} - \frac{210}{x} = 1$.

Therefore $210x - 210(x - 5) = x(x - 5)$ and hence $1050 = x^2 - 5x$. Thus $x^2 - 5x - 1050 = 0$ and hence $(x - 35)(x + 30) = 0$. Therefore, since x is positive, $x = 35$.

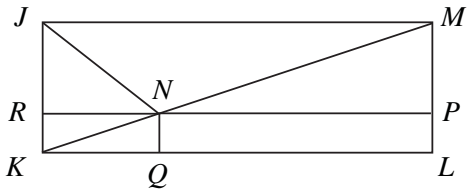
13. 13 Assume the man at the front of the queue is telling the truth and that everyone behind him always lies. However, then the person in third place in the queue would be telling the truth when he says that the person in second place always lies. This contradicts the original assumption and so the man at the front of the queue is lying. In this case, the man in second place is telling the truth, the man in third place is lying etc. Hence, every other person, starting with the first, is lying and so there are $1 + \frac{1}{2} \times 24 = 13$ people in the queue who always lie.

14. 30 The smallest number of contestants solving all four problems correctly occurs when the contestants who fail to solve individual problems are all distinct. In that case, the number failing to solve some question is $10 + 15 + 20 + 25 = 70$ and the number solving them all is $100 - 70 = 30$.

15. 14 First note that $165 = 3 \times 5 \times 11$. Hence, for '$XX4XY$' to be exactly divisible by 165, it must be exactly divisible by 3, 5 and 11. A number is divisible by 3 if and only if the sum of its digits is divisible by 3 so $3X + 4 + Y$ is divisible by 3 and hence $4 + Y$ is divisible by 3. A number is divisible by 5 if and only if its last digit is 5 or 0 so $Y = 5$ or 0. Since $4 + Y$ is divisible by 3 then $Y = 5$. A number is divisible by 11 if and only if the sum of its digits with alternating signs is divisible by 11 so $X - X + 4 - X + Y$ is divisible by 11. Hence $9 - X$ is divisible by 11 and so $X = 9$. Hence the value of $X + Y$ is $9 + 5 = 14$.

16. 64 Since adjacent digits differ by 1, each time the number has a digit that is a 1 or a 3, there is only one choice for the next digit as it must be a 2 whereas each time the number has a digit that is a 2, there are two choices for the next digit, namely 1 or 3. Consider all 10-digit numbers starting in a 1. There is only one choice for the second digit since it must be a 2, then two choices for the third digit, then one for the fourth etc. Altogether there are $1 \times 2 \times 1 \times 2 \times 1 \times 2 \times 1 \times 2 \times 1 = 16$ such numbers. Similarly there are 16 such numbers starting in 3. However, if we consider numbers starting in 2, there are two choices for the second digit then only one choice for the third then two for the fourth etc. Altogether there are $2 \times 1 \times 2 \times 1 \times 2 \times 1 \times 2 \times 1 \times 2 = 32$ such numbers. Hence there are $16 + 16 + 32 = 64$ such numbers with the required property.

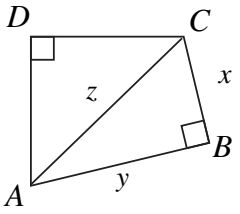
17. 16



Let points P and Q be the points where the perpendiculars from N to ML and KL meet the lines and extend line PN so it meets JK at R, as shown in the diagram. Since JN is the bisector of angle MJK, angle $NJR = 45°$. Since angle JRN is 90°, triangle JRN is isosceles and $JR = RN$. Let the length of RN be x cm. Hence the lengths of JR and PM are also x cm. Observe that triangles NKQ and MNP are similar since they have the same angles. Therefore $\frac{1}{x} = \frac{x}{8}$ and so $x = \sqrt{8}$ since x is positive. The length of KL is equal to the sum of the lengths of NP and NR. Therefore, the length of KL is $\left(8 + \sqrt{8}\right)$ cm. Hence, the value of $a + b$ is 16.

18. 2 Consider the equation $\frac{a}{b+c} = \frac{b}{c+a}$. Multiply each side by $(b + c)(c + a)$ to get $a^2 + ac = b^2 + bc$ and so $a^2 - b^2 + ac - bc = 0$. Therefore $(a - b)(a + b + c) = 0$. Hence $a = b$ or $a + b + c = 0$. Similarly, if we consider the equations $\frac{b}{c+a} = \frac{c}{a+b}$ and $\frac{c}{a+b} = \frac{a}{b+c}$, then $b = c$ or $a + b + c = 0$ and $c = a$ or $a + b + c = 0$ respectively. Therefore, the possible values of k when all three equations are satisfied simultaneously occur when $a = b = c$, giving $k = \frac{1}{2}$, or when $a + b + c = 0$, giving $k = -1$. Hence there are two possible values of k.

19. 100 Let the lengths of BC, AB and AC be x, y and z centimetres respectively. Let the area of ΔACD be U cm^2 and let the area of ΔABC be V cm^2. Note that ΔACD is one quarter of the square which has AC as an edge. Hence $U = \frac{1}{4}z^2$. Next, using Pythagoras, $z^2 = x^2 + y^2 = (x + y)^2 - 2xy = 20^2 - 4V$. Hence $U = \frac{1}{4}(400 - 4V) = 100 - V$. Therefore the area in cm^2 of $ABCD$ is $U + V = 100$.



(Note: *Since the answer to the problem is independent of x and y, one could observe that the given properties of quadrilateral ABCD are satisfied by a square of side* 10 cm *which has area* 100 cm^2 *and conclude that this is therefore the required answer.*)

20. 247 First factorise N twice using the difference of two squares i.e. $N = 3^{16} - 1 = \left(3^8 - 1\right)\left(3^8 + 1\right) = \left(3^4 - 1\right)\left(3^4 + 1\right)\left(3^8 + 1\right) = 80 \times 82 \times \left(3^8 + 1\right)$. This shows that both 80 and 82 are divisors of N in the required range. The question tells us that 193 is a divisor of N and, since 193 is prime, it must be a divisor of $3^8 + 1 = 81 \times 81 + 1 = 6562$. Now observe that $6562 = 2 \times 3281$ and that $3281 \div 193 = 17$. Therefore $N = 80 \times 82 \times 2 \times 17 \times 193$ or $N = \left(2^4 \times 5\right) \times \left(2 \times 41\right) \times 2 \times 17 \times 193$.

Next consider the integers from 75 to 85 inclusive to see which could be divisors of N. Because N has no prime factors of 3 or 7, we know that 75, 77, 78, 81 and 84 are not divisors of N while the initial argument established that 80 and 82 are divisors of N. Both 79 and 83 are prime and $76 = 4 \times 19$ so, since N does not have a prime factor of 79, 83 or 19, these must also be excluded. This only leaves 85 to be considered. Note that $85 = 5 \times 17$ and both 5 and 17 are prime factors of N so 85 is a divisor of N. Hence the divisors of N in the required range are 80, 82 and 85 with sum 247.

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 2nd December 2016

Organised by the United Kingdom Mathematics Trust

The Senior Kangaroo paper allows students in the UK to test themselves on questions set for the best school-aged mathematicians from across Europe and beyond.

RULES AND GUIDELINES (to be read before starting):

1. Do not open the paper until the Invigilator tells you to do so.
2. Time allowed: **1 hour**.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
4. **Use B or HB pencil only** to complete your personal details and record your answers on the machine-readable Answer Sheet provided. **All answers are written using three digits, from 000 to 999.** For example, if you think the answer to a question is 42, write 042 at the top of the answer grid and then code your answer by putting solid black pencil lines through the 0, the 4 and the 2 beneath.

   Please note that the machine that reads your Answer Sheet will only see the solid black lines through the numbers beneath, not the written digits above. You must ensure that you code your answers or you will not receive any marks. There are further instructions and examples on the Answer Sheet.
5. The paper contains 20 questions. Five marks will be awarded for each correct answer. There is no penalty for giving an incorrect answer.
6. The questions on this paper challenge you **to think**, not to guess. Though you will not lose marks for getting answers wrong, you will undoubtedly get more marks, and more satisfaction, by doing a few questions carefully than by guessing lots of answers.

Enquiries about the Senior Kangaroo should be sent to:

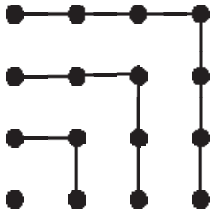
Maths Challenges Office, School of Maths Satellite,

University of Leeds, Leeds, LS2 9JT

Tel. 0113 343 2339

www.ukmt.org.uk

1. Using this picture we can observe that
$1 + 3 + 5 + 7 = 4 \times 4$.
What is the value of
$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21$?



2. Both rows of the following grid have the same sum. What is the value of * ?

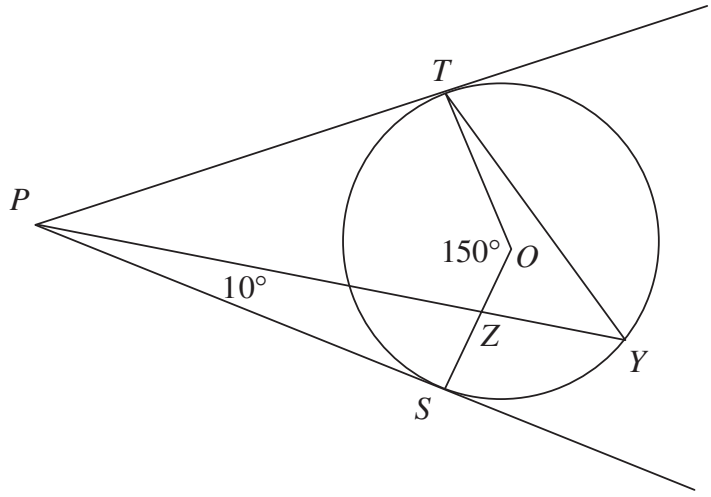
1	2	3	4	5	6	7	8	9	10	1050
11	12	13	14	15	16	17	18	19	20	*

3. Andrew has two containers for carrying water. The containers are cubes without tops and have base areas of 4 dm^2 and 36 dm^2 respectively. Andrew has to completely fill the larger cube with pond water, which must be carried from the pond using the smaller cube. What is the smallest number of visits Andrew has to make to the pond with the smaller cube?

4. How many four-digit numbers formed only of odd digits are divisible by five?

5. The notation $|x|$ is used to denote the absolute value of a number, regardless of sign. For example, $|7| = |-7| = 7$.
The graphs $y = |2x| - 3$ and $y = |x|$ are drawn on the same set of axes. What is the area enclosed by them?

6.



In the diagram, PT and PS are tangents to a circle with centre O. The point Y lies on the circumference of the circle; and the point Z is where the line PY meets the radius OS.

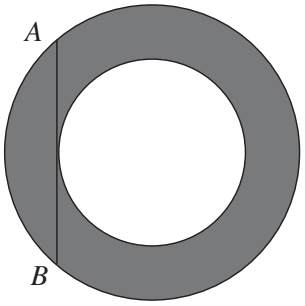
Also, $\angle SPZ = 10°$ and $\angle TOS = 150°$.

How many degrees are there in the sum of $\angle PTY$ and $\angle PYT$?

7. Bav is counting the edges on a particular prism. The prism has more than 310 edges, it has fewer than 320 edges and its number of edges is odd. How many edges does the prism have?

8. The real numbers x, y and z are a solution (x, y, z) of the equation $(x^2 - 9)^2 + (y^2 - 4)^2 + (z^2 - 1)^2 = 0$. How many different possible values are there for $x + y + z$?

9. The diagram shows two concentric circles. Chord AB of the larger circle is tangential to the smaller circle.
The length of AB is 32 cm and the area of the shaded region is $k\pi\,\text{cm}^2$.
What is the value of k?



10. Consider the expression $1 * 2 * 3 * 4 * 5 * 6$.
Each star in the expression is to be replaced with either '+' or '×'.
N is the largest possible value of the expression. What is the largest prime factor of N?

11. Stephanie enjoys swimming. She goes for a swim on a particular date if, and only if, the day, month (where January is replaced by '01' through to December by '12') and year are all of the same parity (that is they are all odd, or all are even). On how many days will she go for a swim in the two-year period between January 1st of one year and December 31st of the following year inclusive?

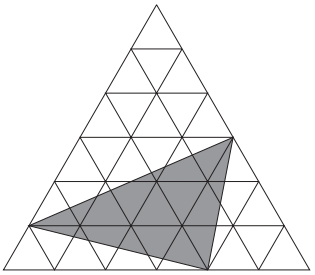
12. Delia is joining three vertices of a square to make four right-angled triangles.
She can create four triangles doing this, as shown.

How many right-angled triangles can Delia make by joining three vertices of a regular polygon with 18 sides?

13. This year, 2016, can be written as the sum of two positive integers p and q where $2p = 5q$ (as $2016 = 1440 + 576$). How many years between 2000 and 3000 inclusive have this property?

14. The lengths of the sides of a triangle are the integers 13, x, y. It is given that $xy = 105$. What is the length of the perimeter of the triangle?

15. The large equilateral triangle shown consists of 36 smaller equilateral triangles. Each of the smaller equilateral triangles has area 10 cm^2.
The area of the shaded triangle is K cm^2. Find K.



16. A function $f(x)$ has the property that, for all positive x, $3f(x) + 7f\left(\frac{2016}{x}\right) = 2x$.
What is the value of $f(8)$?

17. Students in a class take turns to practise their arithmetic skills. Initially a board contains the integers from 1 to 10 inclusive, each written ten times. On each turn a student first deletes two of the integers and then writes on the board the number that is one more than the sum of those two deleted integers. Turns are taken until there is only one number remaining on the board. Assuming no student makes a mistake, what is the remaining number?

18. The sum of the squares of four consecutive positive integers is equal to the sum of the squares of the next three consecutive integers. What is the square of the smallest of these integers?

19. Erin lists all three-digit primes that are 21 less than a square. What is the mean of the numbers in Erin's list?

20. A barcode of the type shown in the two examples is composed of alternate strips of black and white, where the leftmost and rightmost strips are always black. Each strip (of either colour) has a width of 1 or 2. The total width of the barcode is 12. The barcodes are always read from left to right. How many distinct barcodes are possible?

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

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SOLUTIONS

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1. **121** The sum $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21$ has eleven terms. Therefore the value of the required sum is $11 \times 11 = 121$.

2. **950**

1	2	3	4	5	6	7	8	9	10	1050
11	12	13	14	15	16	17	18	19	20	*

We observe that in all but the rightmost column the value in the second row is ten larger than the value in the first row. There are 10 such columns. Therefore the sum of the leftmost ten elements of the second row is 100 more than the corresponding sum in the first row. To achieve the same total in each row, * will need to be 100 less than the value above it. Therefore * = 950.

3. **27** We first observe that any pair of cubes are mathematically similar. These cubes' surface areas are in the ratio 1:9, so that their lengths are in ratio 1:3 and that their volumes are in ratio 1:27.

Therefore Andrew may fill the larger cube in 27 visits, provided the smaller cube is completely filled on each occasion.

4. **125** The number will be of the form '*abcd*' where *a*, *b* and *c* are any odd digits and $d = 5$. Hence there are 5, 5, 5 and 1 possibilities for *a*, *b*, *c* and *d* respectively. Therefore there are $5 \times 5 \times 5 \times 1 = 125$ such numbers.

5. **9** The enclosed area is a concave quadrilateral with vertices at $(-3, 3)$, $(0, 0)$, $(3, 3)$ and $(0, -3)$. Considering this as two conjoined congruent triangles we find the area as $2 \times \frac{1}{2} \times 3 \times 3 = 9$.

6. **160** The tangent-radius property gives $\angle PSO = \angle PTO = 90°$. From the angle sum of quadrilateral *PTOS* we may conclude that $\angle TPS = 30°$ and therefore that $\angle TPY = 20°$. By considering the angle sum of triangle *PTY* we conclude that the required total is 160°.

7. **315** Suppose that the cross-section of the prism is an N-gon with N edges. The prism will have N edges in each of its 'end' faces and a further N edges connecting corresponding vertices of the end faces. Therefore the number of edges is $3N$ and hence is a multiple of 3. The only multiples of 3 in the given range are 312, 315 and 318. Since we know the total is odd, the prism has 315 edges.

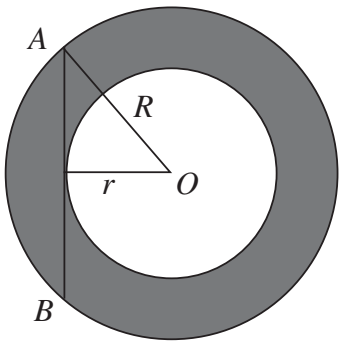
8. **7** Since squares of real numbers are non-negative, the sum can only be 0 if each expression in brackets is zero. Therefore the solutions of the equation are $x = \pm 3$, $y = \pm 2$ and $z = \pm 1$. We observe that the maximum and minimum values for $x + y + z$ are 6 and -6, and that since $x + y + z$ is the sum of one even and two odd numbers, that $x + y + z$ itself will be even.

It suffices to show that each even total between +6 and −6 can be attained.

$$(+3) + (+2) + (+1) = +6 \qquad (+3) + (+2) + (-1) = +4$$
$$(+3) + (-2) + (+1) = +2 \qquad (+3) + (-2) + (-1) = 0$$
$$(-3) + (+2) + (-1) = -2 \qquad (-3) + (-2) + (+1) = -4$$
$$(-3) + (-2) + (-1) = -6$$

Hence there are seven possible values for $x + y + z$.

9. **256** Let the radii of the larger and smaller circles be R and r respectively. Draw radius OA of the larger circle and drop the perpendicular from O to AB. By the tangent-radius property this perpendicular will be a radius of the smaller circle.



Now the area of the shaded region = area of larger circle − area of smaller circle.

The area of the shaded region = $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$.

But $R^2 - r^2 = 16^2 = 256$ (by Pythagoras' theorem), hence the area of the shaded region = 256π and therefore $k = 256$.

10. **103** Note that $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$. We observe that if any multiplication sign, other than the first, is replaced by an addition sign then each remaining product is at most 360. Therefore we retain each multiplication sign except the first which may be replaced by an addition sign to obtain a maximal value of 721.

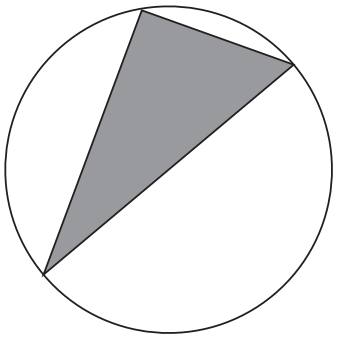
The prime factors of 721 are 7 and 103, of which 103 is the largest.

11. **183** We first observe that exactly one odd year and exactly one even year are under consideration.

In an odd year we need only consider odd months. January, March, May and July each has 16 odd days while September and November has 15. Therefore the number of days Stephanie will swim in the odd year is $4 \times 16 + 2 \times 15 = 94$.

In an even year we need only consider even months. April, June, August, October and December has 15 even days and February has 14 (regardless of whether or not it is a leap year). Therefore the number of days Stephanie will swim in the even year is $5 \times 15 + 14 = 89$. Hence she will swim for $94 + 89 = 183$ days over the two years.

12. 144 The regular 18-gon has a circumcircle, that is, a circle passing through all of its vertices. This is also the circumcircle of each right-angled triangle formed. In order for one of these triangle's angles to be a right angle, the opposite side needs to be a diameter, There are 9 possible choices of diameter. For each choice of diameter, there are 8 vertices on each side for the right angle, making 16 choices overall. For each choice of diameter there are 16 choices for the third vertex of the right-angled triangle.



13. 143 For a year Y to be expressible as as the sum of two positive integers p and q where $2p = 5q$ we require $p + q = Y$ and $2p = 5q$. From the first of these, it follows that $2p + 2q = 2Y$ and hence $5q + 2q = 2Y$.

Therefore $7q = 2Y$ from which it follows that Y is also divisible by 7 (since 2 and 7 are coprime).

We observe that $q = \frac{2Y}{7}$ will be an integer less than Y for all Y that are multiples of 7.

Then $p = Y - q$ will also be an integer.

We now must count all the multiples of 7 between 2000 and 3000 inclusive.

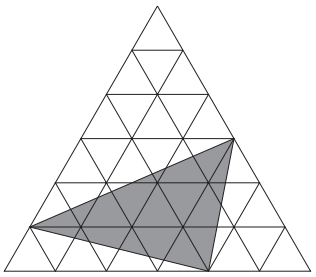
Since $1995 = 285 \times 7$ and $2996 = 428 \times 7$ there are $428 - 285 = 143$ multiples of 7 between 2000 and 3000 and hence there are 143 such years.

14. 35 Assume, without loss of generality, that $x \leqslant y$. Since x, y are positive integers and $xy = 105$, the possible values of (x, y) are $(1, 105)$, $(3, 35)$, $(5, 21)$, $(7, 15)$.

Since we require $13 + x > y$ for the triangle to exist, we may eliminate the first three of these possibilities, leaving only $(7, 15)$ and conclude that the perimeter is $13 + 7 + 15 = 35$.

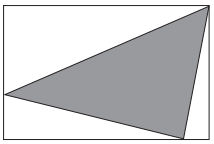
15. 110 For each small equilateral triangle, let the length of each side be x and the perpendicular height be h.

We may trap the shaded triangle in a rectangle as shown, where one vertex is coincident with one of the vertices of the rectangle and the other two vertices lie on sides of the rectangle.



The rectangle has width $4x$ and height $3h$. Therefore the rectangle's area is $12xh$.

The three additional (unshaded) right-angled triangles in the rectangle have areas $\frac{1}{2} \times 4x \times 2h = 4xh$, $\frac{1}{2} \times \frac{1}{2}x \times 3h = \frac{3}{4}xh$ and $\frac{1}{2} \times \frac{7}{2}x \times h = \frac{7}{4}xh$.
Therefore their total area is $4xh + \frac{3}{4}xh + \frac{7}{4}xh = \frac{13}{2}xh$.



Therefore $K = 12xh - \frac{13}{2}xh = \frac{11}{2}xh$.

Each of the 36 smaller equilateral triangles has area $\frac{1}{2}xh$ so we know that $\frac{1}{2}xh = 10$ and therefore that $xh = 20$.

Therefore $K = \frac{11}{2} \times 20 = 110$.

16. 87 The function $f(x)$ has the property that $3f(x) + 7f\left(\frac{2016}{x}\right) = 2x$. First observe that $\frac{2016}{8} = 252$. Therefore $3f(8) + 7f(252) = 16$ and $3f(252) + 7f(8) = 2 \times 252$.

Let $f(8) = V$ and $f(252) = W$. Therefore $3V + 7W = 16$ and $3W + 7V = 504$.

When these equations are solved simultaneously, we obtain $V = 87$ and $W = -35$ so that $f(8) = 87$.

17. 649 We observe that the total of all integers on the board at the start of the process is $10(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) = 550$.

On each turn this total is increased by 1. Since we start with one hundred integers on the board and at each turn this number of integers is decreased by one, then 99 turns will be required to complete the process. Therefore the total of all integers on the board will increase by 99 over the course of the process. Hence the remaining number will be $550 + 99 = 649$.

18. 441 Let the smallest number be x. Therefore

$x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2 = (x+4)^2 + (x+5)^2 + (x+6)^2$ and hence

$x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 + x^2 + 6x + 9 = x^2 + 8x + 16 + x^2 + 10x + 25 + x^2 + 12x + 36$.

This can be rewritten as $4x^2 + 12x + 14 = 3x^2 + 30x + 77$ or $x^2 - 18x - 63 = 0$.

Hence $(x - 21)(x + 3) = 0$, which has solutions $x = 21$ and $x = -3$. The question tells us that x is positive and therefore $x = 21$.

The square of the smallest of these integers is therefore $21^2 = 441$.

19. 421 When an odd number is subtracted from an odd square, an even (and hence composite) number is obtained. Similarly, when a multiple of 3 (or 7) is subtracted from a square of a multiple of 3 (or 7), a multiple of 3 (or 7) is obtained which is also composite. Therefore we need only consider three-digit squares that are neither odd nor a multiple of 3 (or 7). Hence the only squares we need to consider are $16^2 = 256$, $20^2 = 400$, $22^2 = 484$ and $26^2 = 676$ which yield differences of 235, 379, 463 and 655 respectively. It is easy to see that 235 and 655 are multiples of 5 and hence composite. Therefore only 379 and 463 remain as possible primes satisfying the given condition. After checking divisibilty by 11, 13, 17 and 19 for both, both are indeed seen to be prime and their mean is 421.

20. 116 Any code will start with a black strip and a white strip followed by a shorter barcode. Let $C(m)$ be the number of distinct barcodes of width m.

Those codes which start with BW will be followed by a code of width $m - 2$; so there will be $C(m - 2)$ of these. Likewise, there will be $C(m - 3)$ codes starting BBW, the same number starting BWW, and $C(m - 4)$ starting BBWW; and that exhausts the possibilites. So it follows that $C(m) = C(m - 2) + 2C(m - 3) + C(m - 4)$.

When $m \leqslant 4$, it is simple to list all possible barcodes; namely B, BB, BWB and BBWB, BWBB, BWWB. Therefore $C(1) = C(2) = C(3) = 1$ and $C(4) = 3$. We can now calculate $C(m)$ for $m > 4$.

Thus $C(5) = C(3) + 2C(2) + C(1) = 1 + 2 + 1 = 4$, and continuing like this, we get $C(6) = 6$, $C(7) = 11$, $C(8) = 17$, $C(9) = 27$, $C(10) = 45$, $C(11) = 72$, $C(12) = 116$.

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1. Do not open the paper until the Invigilator tells you to do so.
2. Time allowed: **1 hour**.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
4. **Use B or HB pencil only** to complete your personal details and record your answers on the machine-readable Answer Sheet provided. **All answers are written using three digits, from 000 to 999.** For example, if you think the answer to a question is 42, write 042 at the top of the answer grid and then code your answer by putting solid black pencil lines through the 0, the 4 and the 2 beneath.

   Please note that the machine that reads your Answer Sheet will only see the solid black lines through the numbers beneath, not the written digits above. You must ensure that you code your answers or you will not receive any marks. There are further instructions and examples on the Answer Sheet.
5. The paper contains 20 questions. Five marks will be awarded for each correct answer. There is no penalty for giving an incorrect answer.
6. The questions on this paper challenge you **to think**, not to guess. Though you will not lose marks for getting answers wrong, you will undoubtedly get more marks, and more satisfaction, by doing a few questions carefully than by guessing lots of answers.

Enquiries about the Senior Kangaroo should be sent to:

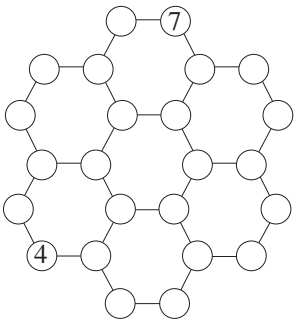
Maths Challenges Office, School of Maths Satellite,

University of Leeds, Leeds, LS2 9JT

Tel. 0113 343 2339

www.ukmt.org.uk

1. An integer is to be written in each circle of the network shown. The integers must be written so that the sum of the numbers at the end of each line segment is the same. Two of the integers have already been written. What is the total of all the integers in the completed diagram?



2. Three sportsmen called Primus, Secundus and Tertius take part in a race every day. Primus wears the number '1' on his shirt, Secundus wears '2' and Tertius wears '3'.

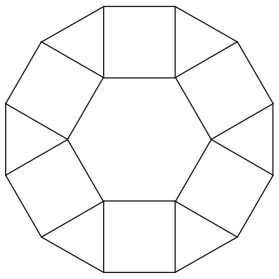
   On Saturday Primus wins, Secundus is second and Tertius is third. Using their shirt numbers this result is recorded as '123'.

   On Sunday Primus starts the race in the lead with Secundus in second. During Sunday's race Primus and Secundus change places exactly 9 times, Secundus and Tertius change places exactly 10 times while Primus and Tertius change places exactly 11 times.

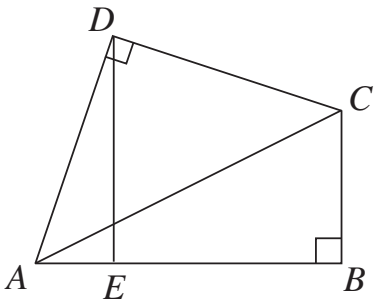
   How will Sunday's result be recorded?

3. All three-digit positive integers whose digit sum is 5 are listed in ascending order. What is the median of this list?

4. The figure shows a shape consisting of a regular hexagon of side 18 cm, six triangles and six squares. The outer perimeter of the shape is P cm. What is the value of P?



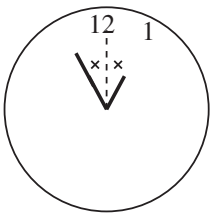
5. The figure shows a quadrilateral $ABCD$ in which $AD = DC$ and $\angle ADC = \angle ABC = 90°$. The point E is the foot of the perpendicular from D to AB. The length DE is 25. What is the area of quadrilateral $ABCD$?



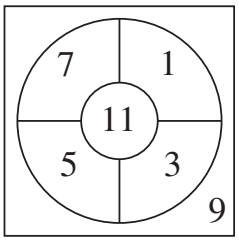
6. Winnie wrote all the integers from 1 to 2017 inclusive on a board. She then erased all the integers that are a multiple of 3. Next she reinstated all those integers that are a multiple of 6. Finally she erased all integers then on the board which are a multiple of 27. Of the 2017 integers that began in the list, how many are now missing?

7. Three rectangles are placed mutually adjacent and without gaps or overlaps to form a larger rectangle. One of the three rectangles has dimensions 70 by 110. Another of the rectangles has dimensions 40 by 80. What is the maximum perimeter of the third rectangle?

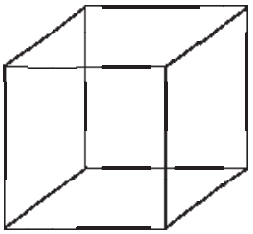
8. Priti is learning a new language called Tedio. During her one hour lesson, which started at midday, she looks at the clock and notices that the hour hand and the minute hand make exactly the same angle with the vertical, as shown in the diagram. How many whole seconds remain until the end of the lesson?



9. Robin shoots three arrows at a target. He earns points for each shot as shown in the figure. However, if any of his arrows miss the target or if any two of his arrows hit adjacent regions of the target, he scores a total of zero. How many different scores can he obtain?

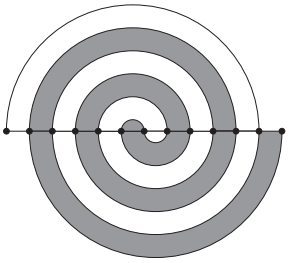


10. At each of the vertices of a cube sits a Bunchkin. Two Bunchkins are said to be adjacent if and only if they sit at either end of one of the cube's edges. Each Bunchkin is either a 'truther', who always tells the truth, or a 'liar', who always lies. All eight Bunchkins say 'I am adjacent to exactly two liars'. What is the maximum number of Bunchkins who are telling the truth?



11. An infinite arithmetic progression of positive integers contains the terms 7, 11, 15, 71, 75 and 79. The first term in the progression is 7. Kim writes down all the possible values of the one-hundredth term in the progression. What is the sum of the numbers Kim writes down?

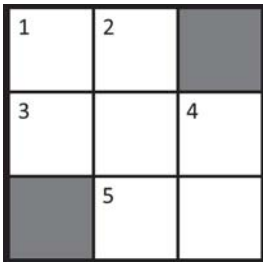
12. The pattern shown in the diagram is constructed using semicircles. Each semicircle has a diameter that lies on the horizontal axis shown and has one of the black dots at either end. The distance between each pair of adjacent black dots is 1 cm. The area, in cm^2, of the pattern that is shaded in grey is $\frac{1}{8}k\pi$. What is the value of k?



13. In the expression $\dfrac{k.a.n.g.a.r.o.o}{g.a.m.e}$, different letters stand for different non-zero digits but the same letter always stands for the same digit. What is the smallest possible integer value of the expression?

14. The set S is given by $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$. A non-empty subset T of S has the property that it contains no pair of integers that share a common factor other than 1. How many distinct possibilities are there for T?

15. Each square in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are fulfilled. The digits used are not necessarily distinct.



ACROSS

1. A square
3. The answer to this Kangaroo question
5. A square

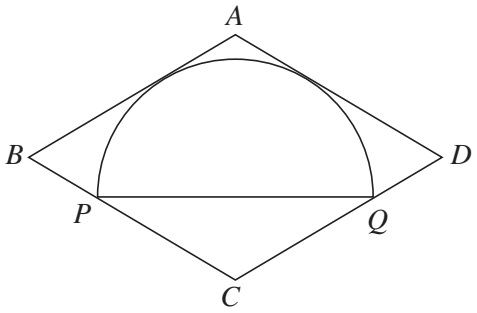
DOWN

1. 4 down minus eleven
2. One less than a cube
4. The highest common factor of 1 down and 4 down is greater than one

16. The curve $x^2 + y^2 = 25$ is drawn. Points on the curve whose x-coordinate and y-coordinate are both integers are marked with crosses. All of those crosses are joined in turn to create a convex polygon P. What is the area of P?

17. Matthew writes a list of all three-digit squares backwards. For example, in his list Matthew writes the three-digit square '625' as '526'. Norma looks at Matthew's list and notices that some of the numbers are prime numbers. What is the mean of those prime numbers in Matthew's list?

18. The diagram shows a semicircle with diameter PQ inscribed in a rhombus $ABCD$. The rhombus is tangent to the arc of the semicircle in two places. Points P and Q lie on sides BC and CD of the rhombus respectively. The line of symmetry of the semicircle is coincident with the diagonal AC of the rhombus. It is given that $\angle CBA = 60°$. The semicircle has radius 10. The area of the rhombus can be written in the form $a\sqrt{b}$ where a and b are integers and b is prime. What is the value of $ab + a + b$?



19. The sequence of functions $F_1(x)$, $F_2(x)$, … satisfies the following conditions:

$$F_1(x) = x, \qquad F_{n+1}(x) = \frac{1}{1 - F_n(x)}.$$

The integer C is a three-digit cube such that $F_C(C) = C$.
What is the largest possible value of C?

20. Let a, b and c be positive integers such that $a^2 = 2b^3 = 3c^5$. What is the minimum possible number of factors of abc (including 1 and abc)?

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 1st December 2017

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

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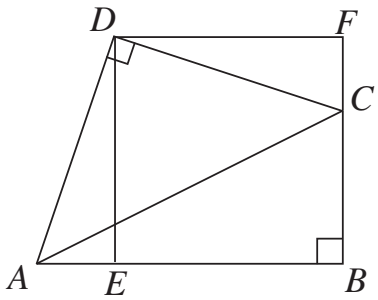
1. **132** Each circle immediately adjacent to the initial '4' must contain the same integer, x say, in order for the sum of those numbers at the end of each line segment to be the same. Those circles immediately adjacent to those with the integer x must contain the integer 4 to preserve the sum of those numbers at the end of each line segment. Continuation of this pattern throughout the network eventually yields that the circle marked with a '7' must contain the integer x. Therefore $x = 7$. The completed network contains twelve '4's and twelve '7's with a total of $12 \times 4 + 12 \times 7 = 132$.

2. **231** At the start of the race, the runners are in order 123. By the end of the race Primus and Tertius have exchanged places 11 times, so Tertius ends the race ahead of Primus. Also, since Secundus and Tertius have exchanged places 10 times, Secundus ends ahead of Tertius. So the result of the race is 231.

3. **221** In ascending order the list is 104, 113, 122, 131, 140, 203, 212, 221, 230, 302, 311, 320, 401, 410, 500. The median of this list is 221.

4. **216** Each of the squares has a side-length of 18 cm. Therefore each of the triangles has two sides of length 18 cm. Hence the triangles are isosceles. Let the angle contained by the two 18 cm sides of these triangle be $x°$. The interior angles of a square and a regular hexagon are 90° and 120° respectively. By considering angles at a point we have $x + 90 + 90 + 120 = 360$. Therefore $x = 60$ and the triangles are equilateral. All twelve outer edges of the figure are 18 cm in length. Therefore $P = 12 \times 18 = 216$.

5. **625**



Draw a line through D that is parallel to AB. Let F be the intersection of that line with BC extended, as shown in the diagram.

Now $\angle EDC + \angle CDF = \angle EDC + \angle ADE = 90°$

Therefore $\angle CDF = \angle ADE = 90° - \angle ECD$

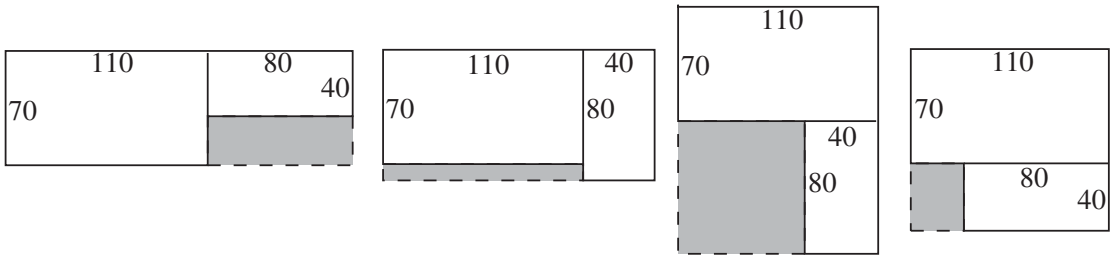
Also $\angle DFC = \angle DEA = 90°$

Therefore triangles ADE and CDF are similar because they have the same set of angles. Because $AD = CD$ they must also be congruent.

Therefore the area of quadrilateral $ABCD$ is equal to the area of rectangle $FDEB$. By the congruent triangles ADE and CDF we know that $DF = DE = 25$. Therefore the required area is $25 \times 25 = 625$.

6. **373** Winnie begins with 2017 integers. There are $2016 \div 3 = 672$ multiples of three which are erased. This leaves $2017 - 672 = 1345$ integers. There are $2016 \div 6 = 336$ multiples of six which are reinstated. This leaves $1345 + 336 = 1681$ integers. The only multiples of twenty-seven that then remain in the list are those that are multiples of six. Winnie therefore erases all the multiples of fifty-four. There are $1998 \div 54 = 37$ multiples of fifty-four which are erased. This leaves $1681 - 37 = 1644$ integers. Of the 2017 integers she began with $2017 - 1644 = 373$ are now missing.

7. **300**



There are essentially four different configurations as shown in the diagram. The perimeters of the third rectangle in these configurations are 220, 240, 300 and 140 respectively. Therefore the maximum possible perimeter of the third rectangle is 300.

8. **276** Let the angle that each hand makes with the vertical be x degrees and let the current time be s seconds after midday.

In one complete hour the hour hand will turn 30°. There are $60 \times 60 = 3600$ seconds in an hour so it takes $3600 \div 30 = 120$ seconds for the hour hand to turn one degree. Therefore $s = 120x$.

In one complete hour the minute hand will turn 360°. There are 3600 seconds in an hour so it takes $3600 \div 360 = 10$ seconds for the minute hand to turn 1°. But the minute hand has turned clockwise through an angle of $(360 - x)°$. Therefore $s = 10(360 - x)$. Equating the two expressions we have obtained for s we obtain the equation $120x = 10(360 - x)$. The solution to this equation is $x = \frac{360}{13}$. Therefore the number of seconds elapsed since midday is $120 \times \frac{360}{13} = \frac{43200}{13} = 3323\frac{1}{13}$. The number of whole seconds remaining is $3600 - 3324 = 276$.

9. **013** Robin could score a total of zero either by missing the target with all three arrows or if any two of his arrows hit adjacent regions.

Robin could score totals of 3, 9, 15, 21, 27 or 33 if all three of his arrows hit regions 1, 3, 5, 7, 9 or 11 respectively.

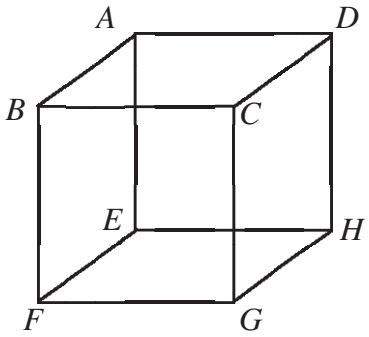
The only scores Robin can obtain from his three arrows hitting non-adjacent regions are $1 + 1 + 5 = 7$, $1 + 5 + 5 = 11$, $3 + 3 + 7 = 13$, $3 + 7 + 7 = 17$, $9 + 9 + 11 = 29$, $9 + 11 + 11 = 31$.

Robin's set of possible score is therefore $\{0, 3, 7, 9, 11, 13, 15, 17, 21, 27, 29, 31, 33\}$.

Hence Robin can obtain 13 different possible scores.

10. **004** Suppose that there is a truther at A. There must be two liars and one truther adjacent to A. Let us suppose, without loss of generality, that B is a truther and D and E are liars. Since B is a truther and is adjacent to A, then C and F are liars. This shows that there cannot be more than 4 truthers.

If we now suppose that G and H are both truthers, then each of the Bunchkins' statements fits the conditions. So 4 is the maximum possible number of Bunchkins.



11. 714 The difference between any two terms is either 4 or is a multiple of 4. So the term-to-term difference in the progression must be a divisor of 4. Since all the terms of the progression are integers the only feasible differences are 1, 2 and 4.

If the term-to-term difference is 1 then the one-hundredth term will be $7 + 99 \times 1 = 106$.

If the term-to-term difference is 2 then the one-hundredth term will be $7 + 99 \times 2 = 205$.

If the term-to-term difference is 4 then the one-hundredth term will be $7 + 99 \times 4 = 403$.

The sum of these numbers is 714.

12. 121 Take each shaded semi-annulus that is below the line and reflect it in the line then move it one centimetre to the left. A shaded semicircle of diameter 11cm is obtained. Therefore the whole shaded area is $\frac{121}{8}\pi$. Hence the value of k is 121.

13. 002 The expression may be simplified to $\dfrac{k{\cdot}a{\cdot}n{\cdot}r{\cdot}o{\cdot}o}{m{\cdot}e}$.

The smallest possible numerator is $5 \times 4 \times 3 \times 2 \times 1 \times 1 = 120$.

The largest possible denominator is $9 \times 8 = 72$.

The smallest possible value of the expression, whether integer or not, is therefore $\dfrac{120}{72} = 1\dfrac{2}{3}$.

We may obtain a value of 2 for the expression via $\dfrac{6 \times 4 \times 3 \times 2 \times 1}{9 \times 8}$.

Therefore the smallest possible integer value of the expression is 2.

14. 027 There are six subsets consisting of one element:

{1}, {2}, {3}, {4}, {5}, {6}.

There are eleven subsets consisting of two elements:

{1, 2}, {1,3}, {1,4}, {1, 5}, {1, 6}; {2, 3}, {2, 5}; {3, 4}, {3, 5}: {4, 5}; {5, 6}.

There are eight subsets consisting of three elements:

{1, 2, 3}, {1, 2, 5}; {1, 3, 4}, {1, 3, 5}; {1, 4, 5}; {1, 5, 6}; {2, 3, 5}; {3, 4, 5}.

There are two subsets consisting of four elements:

{1, 2, 3, 5}; {1, 3, 4, 5}.

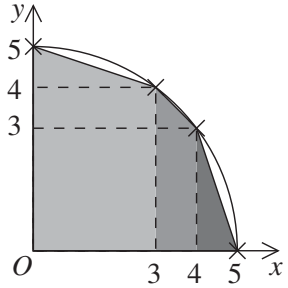
Hence the answer is $6 + 11 + 8 + 2 = 27$.

15. 829 The possible answers for 1 across are 16, 25, 36, 49, 64 and 81.

The possible answers for 2 down are 124, 215, 342, 511, 728, 999.

By considering the last digit of 1 across (which must be the same as the first digit of 2 down) we see that the only possible pairs of answers for 1 across and 2 down are (25, 511), (49, 999) and (81, 124). The pair (49, 999) would leave no possible answer for 5 across, so may be disregarded. The pair (25, 511) gives an answer of 16 for 5 across and thence 36 for 4 down and 25 for 1 down. However these answers contradict the clue for 4 down so this case may be disregarded. (81, 124) gives an answer of 49 for 5 across and hence 99 for 4 down and 88 for 1 down. These answers satisfy all the conditions in the clues and therefore the answer to this Kangaroo question is 829.

16. 074 The curve $x^2 + y^2 = 25$ is a circle of radius 5 centred at the origin.



The polygon P has vertices at coordinates (0,5), (3,4), (4,3), (5,0), (4,–3), (3,–4), (0,–5), (–3, –4), (–4, –3), (–5, 0), (–4, 3), (–3, 4).

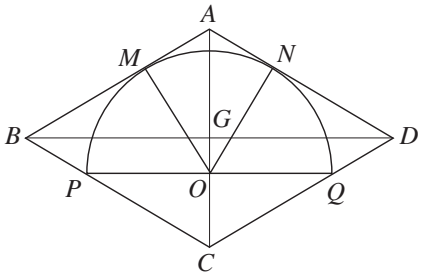
We may find the area of that part of P in the upper-right quadrant by splitting it into two trapezia and a triangle as shown in the diagram. This area is

$$\left(\tfrac{1}{2}(5 + 4) \times 3\right) + \left(\tfrac{1}{2}(4 + 3) \times 1\right) + \left(\tfrac{1}{2} \times 3 \times 1\right) = \tfrac{37}{2}.$$

Therefore the area of P is $4 \times \frac{37}{2} = 74$.

17. 447 Three-digit squares beginning with 2, 4, 5, 6 or 8 may be disregarded since on reversal these will be divisible by two or five. The residual three-digit squares are 100, 121, 144, 169, 196, 324, 361, 729, 784, 900 and 961. We may disregard 144, 324, 729 and 961 since these are divisible by three and will remain so on reversal. We may also disregard 169, 961, 100 and 121 since these each form a square on reversal. This leaves 196, 361 and 784 which on reversal form 163, 487 and 691. Each of these numbers is prime and they have a mean of $\frac{1}{3}(163 + 487 + 691) = 447$.

18. 603 Let O be the centre of the semicircle and let M and N be the feet of the perpendiculars drawn from O to AB and AD respectively. Let G be the intersection of the diagonals of the rhombus.



$PO = 10$ and $\angle OPC = 30°$. So $OC = 10\tan 30° = \dfrac{10}{\sqrt{3}}$.

$MO = 10$ and $\angle OAM = 60°$. So $AO = \dfrac{10}{\sin 60°} = \dfrac{20}{\sqrt{3}}$.

Therefore $AC = \dfrac{10}{\sqrt{3}} + \dfrac{20}{\sqrt{3}} = \dfrac{30}{\sqrt{3}} = 10\sqrt{3}$.

Hence $AG = 5\sqrt{3}$ and $\angle GBA = 30°$. So $BG = \dfrac{5\sqrt{3}}{\tan 30°} = 15$.

Therefore the area of triangle BGA is $\frac{1}{2} \times 15 \times 5\sqrt{3} = \frac{75}{2}\sqrt{3}$.

So the area of the rhombus is $4 \times \frac{75}{2}\sqrt{3} = 150\sqrt{3}$.

Therefore $a = 150$ and $b = 3$, so $ab + a + b = 450 + 150 + 3 = 603$.

19. 343
$$F_1(x) = x;\ F_2(x) = \frac{1}{1-x};\ F_3(x) = \frac{1}{1-\frac{1}{1-x}} = 1 - \frac{1}{x};\ F_4(x) = \frac{1}{1-\left(1-\frac{1}{x}\right)} = x.$$

Hence we have, $F_1(x) = F_4(x) = F_7(x) = \ldots = F_{3k-2}(x)$.

Therefore we are required to solve $F_{3k-2}(3k - 2) = 3k - 2$ where $3k - 2$ is a three-digit cube (given). The cubes are 125, 216, 343, 512, 729. The only one of the format $3k - 2$ for some positive integer k is 343.

20. 077 It is clear that each of a, b and c must have prime factors including 2 and 3 and, since we are seeking a minimal number of factors of abc, these must be the only prime factors.

Let $a = 2^p3^q$, $b = 2^r3^s$ and $c = 2^v3^w$ where p, q, r, s, v and w are positive integers or zero. Since $a^2 = 2b^3 = 3c^5$ we have $2^{2p}3^{2q} = 2^{3r+1}3^{3s} = 2^{5v}3^{5w+1}$.

Considering indices of 2 we have $2p = 3r + 1 = 5v$.

The smallest values of (p, r, v) which satisfy this equation are (5, 3, 2).

Considering indices of 3 we have $2q = 3s = 5w + 1$.

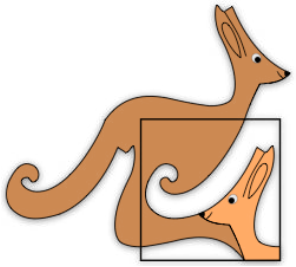
The smallest values of (q, s, w) which satisfy this equation are (3, 2, 1).

Using these values, $abc = (2^53^3) \times (2^33^2) \times (2^23^1) = 2^{10}3^6$.

Any factors of abc will be of the form 2^y3^z where $y \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $z \in \{0, 1, 2, 3, 4, 5, 6\}$.

There are 11 possibilities for y and 7 possibilities for z. Hence abc has $11 \times 7 = 77$ factors.

United Kingdom Mathematics Trust



Senior Kangaroo

Friday 30 November 2018

Organised by the United Kingdom Mathematics Trust

a member of the Association Kangourou sans Frontières

Supported by Overleaf

England & Wales:	*Year* 13 *or below*
Scotland:	*S6 or below*
Northern Ireland:	*Year* 14 *or below*

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **60 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil** to record your answer to each problem as a three-digit number from 000 to 999.
   Pay close attention to the example on the Answer Sheet that shows how to code your answers.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:**
   5 marks are awarded for each correct answer;
   There is no penalty for giving an incorrect answer.
7. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

Enquiries about the Senior Kangaroo should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1. My age is a two-digit number that is a power of 5. My cousin's age is a two-digit number that is a power of 2. The sum of the digits of our ages is an odd number.

What is the product of the digits of our ages?

2. Let K be the largest integer for which $n^{200} < 5^{300}$. What is the value of $10K$?

3. In triangle ABC, we are given that $AC = 5\sqrt{2}$, $BC = 5$ and $\angle BAC = 30°$.

What is the largest possible size in degrees of $\angle ABC$?

4. In a list of five numbers, the first number is 60 and the last number is 300. The product of the first three numbers is 810 000, the product of the three in the middle is 2 430 000 and the product of the last three numbers is 8 100 000.

Which number is third in the list?

5. Rachel and Steven play games of chess. If either wins two consecutive games s/he is declared the champion.

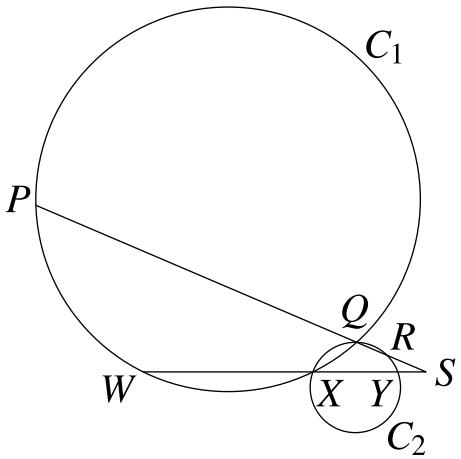
The probability that Rachel will win any given game is 0.6.

The probability that Steven will win any given game is 0.3.

There is a 0.1 probability that any given game is drawn.

The probability that neither is the champion after at most three games is P. Find the value of $1000P$.

6. The line segments $PQRS$ and $WXYS$ intersect circle C_1 at points P, Q, W and X.



The line segments intersect circle C_2 at points Q, R, X and Y. The lengths QR, RS and XY are 7, 9 and 18 respectively. The length WX is six times the length YS.

What is the sum of the lengths of PS and WS?

7. The volume of a cube in cubic metres and its surface area in square metres is numerically equal to four-thirds of the sum of the lengths of its edges in metres.

What is the total volume in cubic metres of twenty-seven such cubes?

8. An integer x satisfies the inequality $x^2 \leq 729 \leq -x^3$. P and Q are possible values of x. What is the maximum possible value of $10(P - Q)$?

9. The two science classes 7A and 7B each consist of a number of boys and a number of girls. Each class has exactly 30 students.

The girls in 7A have a mean score of 48. The overall mean across both classes is 60.

The mean score across all the girls of both classes is also 60.

The 5 girls in 7B have a mean score that is double that of the 15 boys in 7A.

The mean score of the boys in 7B is μ. What is the value of 10μ?

10. The function SPF(n) denotes the sum of the prime factors of n, where the prime factors are not necessarily distinct. For example, $120 = 2^3 \times 3 \times 5$, so SPF(120) $= 2 + 2 + 2 + 3 + 5 = 14$.

Find the value of SPF($2^{22} - 4$).

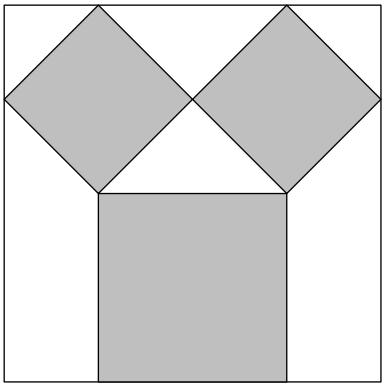
11. A sequence $U_1, U_2, U_3, \ldots$ is defined as follows:

- $U_1 = 2$;
- if U_n is prime then U_{n+1} is the smallest positive integer not yet in the sequence;
- if U_n is not prime then U_{n+1} is the smallest prime not yet in the sequence.

The integer k is the smallest such that $U_{k+1} - U_k > 10$.

What is the value of $k \times U_k$?

12. The diagram shows a 16 metre by 16 metre wall. Three grey squares are painted on the wall as shown.



The two smaller grey squares are equal in size and each makes an angle of 45° with the edge of the wall. The grey squares cover a total area of B metres squared.

What is the value of B?

13. A nine-digit number is odd. The sum of its digits is 10. The product of the digits of the number is non-zero. The number is divisible by seven.

When rounded to three significant figures, how many millions is the number equal to?

14. A square $ABCD$ has side 40 units. Point F is the midpoint of side AD. Point G lies on CF such that $3CG = 2GF$.

What is the area of triangle BCG?

15. In the sequence $20, 18, 2, 20, -18, \ldots$ the first two terms a_1 and a_2 are 20 and 18 respectively. The third term is found by subtracting the second from the first, $a_3 = a_1 - a_2$. The fourth is the sum of the two preceding elements, $a_4 = a_2 + a_3$. Then $a_5 = a_3 - a_4$, $a_6 = a_4 + a_5$, and so on.

What is the sum of the first 2018 terms of this sequence?

16. A right-angled triangle has sides of integer length. One of its sides has length 20. Toni writes down a list of all the different possible hypotenuses of such triangles.

What is the sum of all the numbers in Toni's list?

17. Sarah chooses two numbers a and b from the set $\{1, 2, 3, \ldots, 26\}$. The product ab is equal to the sum of the remaining 24 numbers.

What is the difference between a and b?

18. How many zeros are there at the end of $\dfrac{2018!}{30! \times 11!}$?

19. Shan solves the simultaneous equations

$$xy = 15 \text{ and } (2x - y)^4 = 1$$

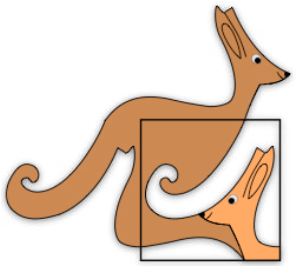
where x and y are real numbers. She calculates z, the sum of the squares of all the y-values in her solutions.

What is the value of z?

20. Determine the value of the integer y given that $y = 3x^2$ and

$$\frac{2x}{5} = \cfrac{1}{1 - \cfrac{2}{3 + \cfrac{1}{4 - \cfrac{5}{6 - x}}}}$$

United Kingdom Mathematics Trust



Senior Kangaroo 2018

Organised by the United Kingdom Mathematics Trust

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Solutions

Enquiries about the Senior Kangaroo should be sent to:

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☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1. My age is a two-digit number that is a power of 5. My cousin's age is a two-digit number that is a power of 2. The sum of the digits of our ages is an odd number.

What is the product of the digits of our ages?

SOLUTION 240

My age must be 25. My cousin's age is a two-digit power of two, so is 16, 32 or 64. The sums of the digits of our ages would then be 14, 12 and 17 respectively. Since this sum must be odd my cousin's age must be 64. Therefore the product of the digits of our ages is $2 \times 5 \times 6 \times 4 = 240$.

2. Let K be the largest integer for which $n^{200} < 5^{300}$. What is the value of $10K$?

SOLUTION 110

We may rewrite the inequality as $(n^2)^{100} < (5^3)^{100} = 125^{100}$. It follows that $n^2 < 5^3 = 125$. The maximum integer value of n is therefore 11. Therefore $K = 11$ and $10K = 110$.

3. In triangle ABC, we are given that $AC = 5\sqrt{2}$, $BC = 5$ and $\angle BAC = 30°$.

What is the largest possible size in degrees of $\angle ABC$?

SOLUTION 135

Let $\angle ABC = \theta$. By the sine rule, $\dfrac{\sin\theta}{5\sqrt{2}} = \dfrac{\sin 30}{5}$, which simplifies to $\dfrac{0.5}{5}$. And so $\sin\theta = \dfrac{0.5 \times 5\sqrt{2}}{5} = \dfrac{\sqrt{2}}{2}$. Therefore $\theta = 45, 135$ and so the largest possible value of θ is 135.

4. In a list of five numbers, the first number is 60 and the last number is 300. The product of the first three numbers is 810 000, the product of the three in the middle is 2 430 000 and the product of the last three numbers is 8 100 000.

Which number is third in the list?

SOLUTION 150

Let the list be 60, x, y, z, 300.

(i) Since $60xy = 810\,000$ we find that $xy = 13\,500$.

(ii) Since $xyz = 2\,430\,000$ and $xy = 13\,500$, we find that $z = 180$.

(iii) Since $300yz = 8\,100\,000$ and $z = 180$, we find that $y = 150$.

We may also find $x = \frac{13\,500}{150} = 90$. As y is the middle number in the list the answer to the problem is 150.

5. Rachel and Steven play games of chess. If either wins two consecutive games s/he is declared the champion.

The probability that Rachel will win any given game is 0.6.

The probability that Steven will win any given game is 0.3.

There is a 0.1 probability that any given game is drawn.

The probability that neither is the champion after at most three games is P. Find the value of $1000P$.

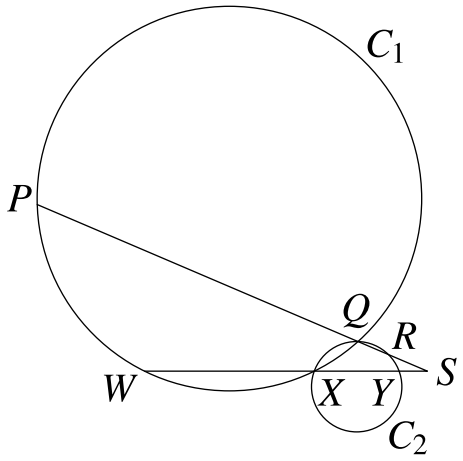
SOLUTION 343

We use R to denote the event 'Rachel wins a game'; S to denote 'Steven wins a game'; and D to denote 'a game is drawn'. We use $\overline{R}$ to denote the event 'Rachel does not win a game', with $\overline{S}$ defined similarly.

- Event R, D has probability $0.6 \times 0.1 = 0.06$.
- Event $R, S, \overline{S}$ has probability $0.6 \times 0.3 \times 0.7 = 0.126$.
- Event $D, R, \overline{R}$ has probability $0.1 \times 0.6 \times 0.4 = 0.024$.
- Event $D, S, \overline{S}$ has probability $0.1 \times 0.3 \times 0.7 = 0.021$.
- Event S, D has probability $0.3 \times 0.1 = 0.03$.
- Event $S, R, \overline{R}$ has probability $0.3 \times 0.6 \times 0.4 = 0.072$.
- Event D, D has probability $0.1 \times 0.1 = 0.01$.

Therefore $P = 0.06 + 0.126 + 0.024 + 0.021 + 0.03 + 0.072 + 0.01 = 0.343$; hence $1000P = 343$.

6. The line segments $PQRS$ and $WXYS$ intersect circle C_1 at points P, Q, W and X.



The line segments intersect circle C_2 at points Q, R, X and Y. The lengths QR, RS and XY are 7, 9 and 18 respectively. The length WX is six times the length YS.

What is the sum of the lengths of PS and WS?

SOLUTION 150

Use the intersecting chords theorem on each circle. Let a be the length of SY. Then, in circle $QXYR$, $a(a + 18) = 9(9 + 7)$ giving solutions of $a = 6, -24$. Since $a > 0$ we conclude $a = 6$. In circle $PWXQ$, $24(24 + 6 \times 6) = 16(16 + z)$. The solution is $z = 74$.

Therefore $PS + WS = 74 + 7 + 9 + 36 + 18 + 6 = 150$.

7. The volume of a cube in cubic metres and its surface area in square metres is numerically equal to four-thirds of the sum of the lengths of its edges in metres.

What is the total volume in cubic metres of twenty-seven such cubes?

SOLUTION 216

Let each of the twelve edges of the cube have length x metres. Then $x^3 + 6x^2 = \frac{4}{3} \times 12x$. This simplifies to $x^3 + 6x^2 - 16x = 0$ or $x(x - 2)(x + 8)$ which has solutions $x = -8, 0, 2$. However x must be positive and so $x = 2$. Then 27 such cubes have a volume of $27 \times 2^3 = 27 \times 8 = 216$.

8. An integer x satisfies the inequality $x^2 \leq 729 \leq -x^3$. P and Q are possible values of x. What is the maximum possible value of $10(P - Q)$?

SOLUTION 180

We observe that $729 = 3^6$. First consider $x^2 \leq 729$. This has solution $-27 \leq x \leq 27$. Now consider $729 \leq -x^3$. This may be rearranged to give $x^3 \leq -729$ with solution $x \leq -9$.

These inequalities are simultaneously satisfied when $-27 \leq x \leq -9$. The maximum value of $P - Q$ is therefore $-9 - (-27) = 18$. So the answer to the question is $10 \times 18 = 180$.

9. The two science classes 7A and 7B each consist of a number of boys and a number of girls. Each class has exactly 30 students.

The girls in 7A have a mean score of 48. The overall mean across both classes is 60.

The mean score across all the girls of both classes is also 60.

The 5 girls in 7B have a mean score that is double that of the 15 boys in 7A.

The mean score of the boys in 7B is μ. What is the value of 10μ?

SOLUTION 672

In 7A there are 15 boys and 15 girls; in 7B there are 25 boys and 5 girls.

All the girls have a total score of $20 \times 60 = 1200$.

Girls in 7A have a total score of $15 \times 48 = 720$.

Hence girls in 7B have a total score of $1200 - 720 = 480$ and a mean of 96.

All pupils have a total score of $60 \times 60 = 3600$.

Hence the 40 boys have a total score of 2400 and a mean of 60.

The 15 boys in 7A have a mean of $\frac{96}{2} = 48$ and total score of 720.

Hence the 25 boys in 7B have a total score of $2400 - 720 = 1680$ and a mean of $\frac{1680}{25}$.

Hence the required number is $\frac{1680}{25} \times 10 = 672$.

10. The function SPF(n) denotes the sum of the prime factors of n, where the prime factors are not necessarily distinct. For example, $120 = 2^3 \times 3 \times 5$, so SPF(120) = 2+2+2+3+5 = 14.

Find the value of SPF($2^{22} - 4$).

SOLUTION 100

Write $2^{22} - 4$ as a product of primes:

$$\begin{aligned} 2^{22} - 4 &= 4 \times (2^{20} - 1) \\ &= 2^2 \times (2^{10} - 1)(2^{10} + 1) \\ &= 2^2 \times (2^5 - 1)(2^5 + 1)(1024 + 1) \\ &= 2^2 \times (32 - 1)(32 + 1)(1025) \\ &= 2^2 \times 31 \times 33 \times 5 \times 205 \\ &= 2^2 \times 31 \times 3 \times 11 \times 5 \times 5 \times 41 \\ &= 2^2 \times 3 \times 5^2 \times 11 \times 31 \times 41. \end{aligned}$$

Therefore $SPF(2^{22} - 4) = 2 + 2 + 3 + 5 + 5 + 11 + 31 + 41 = 100$.

11. A sequence $U_1, U_2, U_3, \ldots$ is defined as follows:

- $U_1 = 2$;
- if U_n is prime then U_{n+1} is the smallest positive integer not yet in the sequence;
- if U_n is not prime then U_{n+1} is the smallest prime not yet in the sequence.

The integer k is the smallest such that $U_{k+1} - U_k > 10$.

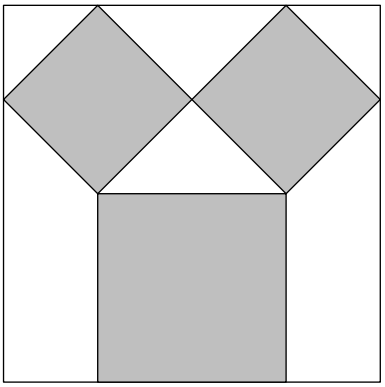
What is the value of $k \times U_k$?

SOLUTION 270

The sequence is 2, 1, 3, 4, 5, 6, 7, 8, 11, 9, 13, 10, 17, 12, 19, 14, 23, 15, 29, ... so that $U_{18} = 15$ and $U_{19} = 29$. These are the first two consecutive terms with a difference greater than 10.

Therefore $k = 18$ and $k \times U_k = 18 \times 15 = 270$.

12. The diagram shows a 16 metre by 16 metre wall. Three grey squares are painted on the wall as shown.



The two smaller grey squares are equal in size and each makes an angle of 45° with the edge of the wall. The grey squares cover a total area of B metres squared.

What is the value of B?

SOLUTION 128

The wall has a width of 16 metres so the diagonal of each smaller grey square is 8 metres. Let the side-length of each smaller grey square be x metres. Then, by Pythagoras' Theorem, $x^2 + x^2 = 8^2$, giving $x = \sqrt{32}$. Therefore each smaller grey square has an area of $32\,\text{m}^2$.

The side-length of the larger grey square is equal to the length of the diagonal of one of the smaller grey squares. Therefore the larger grey square has area $8^2 = 64\,\text{m}^2$.

Hence the total area covered by the grey squares, B, is $32 + 32 + 64 = 128\,\text{m}^2$.

13. A nine-digit number is odd. The sum of its digits is 10. The product of the digits of the number is non-zero. The number is divisible by seven.

When rounded to three significant figures, how many millions is the number equal to?

SOLUTION 112

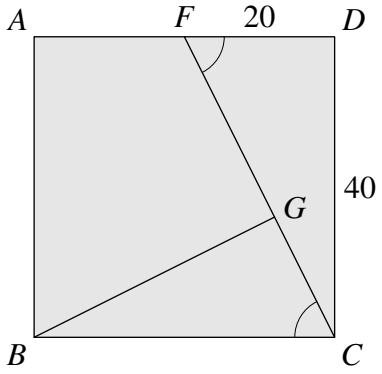
None of the digits in the number may be zero since we know that their product is non-zero. As they sum to 10, we know the digits must be eight 1s and one 2, in some order. It remains to check the eight possible odd integers for divisibility by seven. Of those, only 112 111 111 has zero remainder when divided by seven, and to three significant figure this number is 112 million. Therefore the answer is 112.

14. A square $ABCD$ has side 40 units. Point F is the midpoint of side AD. Point G lies on CF such that $3CG = 2GF$.

What is the area of triangle BCG?

SOLUTION 320

Start by drawing a diagram.



By Pythagoras' Theorem, $FC = 20\sqrt{5}$ and hence $CG = 8\sqrt{5}$. Now, $\sin\angle BCG = \sin\angle CFD = \frac{40}{20\sqrt{5}} = \frac{2}{\sqrt{5}}$.

Hence the area of triangle BCG is $\frac{1}{2} \times 40 \times 8\sqrt{5} \times \frac{2}{\sqrt{5}} = 320$.

15. In the sequence $20, 18, 2, 20, -18, \ldots$ the first two terms a_1 and a_2 are 20 and 18 respectively. The third term is found by subtracting the second from the first, $a_3 = a_1 - a_2$. The fourth is the sum of the two preceding elements, $a_4 = a_2 + a_3$. Then $a_5 = a_3 - a_4$, $a_6 = a_4 + a_5$, and so on.

What is the sum of the first 2018 terms of this sequence?

SOLUTION 038

The sequence is $20, 18, 2, 20, -18, 2, -20, -18, -2, -20, 18, -2, 20, 18, \ldots$. This is periodic and will repeat every twelve terms. The sum of the first twelve terms is 0. Note also that $2018 = 12\times168+2$. Therefore the first 2018 terms will consist of 168 cycles of the first twelve terms with zero sum, followed by $a_{2017} = 20$ and $a_{2018} = 18$. Therefore the sum of the first 2018 terms is $168 \times 0 + 20 + 18 = 38$.

16. A right-angled triangle has sides of integer length. One of its sides has length 20. Toni writes down a list of all the different possible hypotenuses of such triangles.

What is the sum of all the numbers in Toni's list?

SOLUTION 227

Consider a hypotenuse of length 20. Let the shortest side be of length a. For integer side-lengths we require $20^2 - a^2$ to be a square. We observe that of $20^2 - 1^2, 20^2 - 2^2, 20^2 - 3^2, \ldots, 20^2 - 19^2$ only $20^2 - 12^2 = 256$ and $20^2 - 16^2 = 144$ are squares. Therefore a hypotenuse of length 20 is possible.

Now we consider that one of the shorter two sides has length 20. Let the hypotenuse and the other shorter side be of lengths h and b respectively. By Pythagoras' Theorem $20^2 = h^2 - b^2$, yielding $(h - b)(h + b) = 400$. We now consider factor pairs $(m, n), m \leq n$ of 400 to find $(h - b, h + b)$.

- $(m, n) = (1, 400)$ gives $(h, b) = (200.5, 199.5)$ which are non-integers.
- $(m, n) = (2, 200)$ gives $(h, b) = (101, 99)$ with hypotenuse 101.
- $(m, n) = (4, 100)$ gives $(h, b) = (52, 48)$ with hypotenuse 52.
- $(m, n) = (5, 80)$ gives $(h, b) = (42.5, 37.5)$ which are non-integers.
- $(m, n) = (8, 50)$ gives $(h, b) = (29, 21)$ with hypotenuse 29.
- $(m, n) = (10, 40)$ gives $(h, b) = (25, 15)$ with hypotenuse 25.
- $(m, n) = (16, 25)$ gives $(h, b) = (20.5, 4.5)$ which are non-integers.
- $(m, n) = (20, 20)$ gives $(h, b) = (40, 0)$ which is a degenerate case, that is it includes a side of zero length.

Therefore the possible hypotenuses are 101, 52, 29, 25 and 20 with sum 227.

17. Sarah chooses two numbers a and b from the set $\{1, 2, 3, \ldots, 26\}$. The product ab is equal to the sum of the remaining 24 numbers.

What is the difference between a and b?

SOLUTION 006

The sum of the numbers in the set is $\frac{1}{2} \times 26 \times 27 = 351$. Numbers a and b will satisfy the equation $351 - a - b = ab$.

Therefore $352 = ab + a + b + 1 = (a + 1)(b + 1)$. We now search for a factor pair $(a + 1, b + 1)$ of 352 with $a, b \leq 26$ and $a \leq b$. The only such pair $(a + 1, b + 1)$ is $(16, 22)$. Therefore a and b are 15 and 21 respectively and their difference is 6.

18. How many zeros are there at the end of $\dfrac{2018!}{30! \times 11!}$?

SOLUTION 493

Each zero at the end of $N!$ must be generated by a (2×5) in its prime factorisation. In general $N! = 2^a \times 5^b \times K$. For all values of N, a will be greater than b. We therefore determine the value of b in each of 2018!, 30! and 11!. In $2018! = 1 \times 2 \times \ldots \times 2018$ there are

- 403 multiples of five;
- 80 multiples of $5^2 = 25$, each contributing one additional five;
- 16 multiples of $5^3 = 125$, each contributing one further additional five; and
- 3 multiples of $5^4 = 625$, each contributing one yet further additional five.

Therefore, for 2018!, $b = 403 + 80 + 16 + 3 = 502$.

In 30! there are 6 multiples of five and 1 multiple of 25. So, for 30!, $b = 6 + 1 = 7$.

In 11! there are 2 multiples of five. So, for 11!, $b = 2$.

Therefore the power of five in the fraction $\dfrac{2018!}{30! \times 11!}$ is $502 - 7 - 2 = 493$ and so there are 493 zeros at the end of the number.

19. Shan solves the simultaneous equations

$$xy = 15 \text{ and } (2x - y)^4 = 1$$

where x and y are real numbers. She calculates z, the sum of the squares of all the y-values in her solutions.

What is the value of z?

SOLUTION 122

From $(2x - y)^4 = 1$ we know $(2x - y)^2 = \pm 1$. Since any squared quantity must be non-negative, we know $(2x - y)^2 = 1$ from which $2x - y = \pm 1$.

Consider the case $2x - y = 1$. Multiplying by y gives $2xy - y^2 = y$, but $xy = 15$ and so $30 - y^2 = y$. Therefore $y = -6, 5$.

Consider the case $2x - y = -1$. Multiplying by y gives $2xy - y^2 = -y$, but $xy = 15$ and so $30 - y^2 = -y$. Therefore $y = -5, 6$.

Hence $z = (-6)^2 + 5^2 + (-5)^2 + 6^2 = 122$.

20. Determine the value of the integer y given that $y = 3x^2$ and

$$\frac{2x}{5} = \cfrac{1}{1 - \cfrac{2}{3 + \cfrac{1}{4 - \cfrac{5}{6 - x}}}}$$

SOLUTION 147

Note first that

$$4 - \frac{5}{6-x} = \frac{24 - 4x - 5}{6 - x} = \frac{19 - 4x}{6 - x} = F, \text{ say.}$$

Then,

$$3 + \frac{1}{F} = 3 + \frac{6 - x}{19 - 4x} = \frac{57 - 12x + 6 - x}{19 - 4x} = \frac{63 - 13x}{19 - 4x} = G, \text{ say.}$$

Then,

$$1 - \frac{2}{G} = 1 - \frac{2 \times (19 - 4x)}{63 - 13x} = \frac{63 - 13x - (38 - 8x)}{63 - 13x} = \frac{25 - 5x}{63 - 13x} = H, \text{ say.}$$

So we get the equation $\frac{2x}{5} = \frac{1}{H} = \frac{63 - 13x}{25 - 5x}$ which simplifies to $2x^2 - 23x + 63 = 0$. This has solutions $x = 4.5$ and $x = 7$; the corresponding values of y are 60.75 and 147 respectively. Therefore $y = 147$, since we are told y is an integer.

United Kingdom
Mathematics Trust

Senior Kangaroo

Friday 29 November 2019

a member of the Association Kangourou sans Frontières

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England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **60 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil** to record your answer to each problem as a three-digit number from 000 to 999.
   Pay close attention to the example on the Answer Sheet that shows how to code your answers.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:**
   5 marks are awarded for each correct answer;
   There is no penalty for giving an incorrect answer.
7. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

Enquiries about the Senior Kangaroo should be sent to:

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☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1. What is the sum of all the factors of 144?

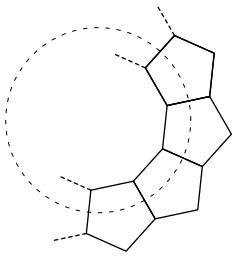
2. When I noticed that $2^4 = 4^2$, I tried to find other pairs of numbers with this property. Trying 2 and 16, I realised that 2^{16} is larger than 16^2. How many times larger is 2^{16}?

3. The two diagonals of a quadrilateral are perpendicular. The lengths of the diagonals are 14 and 30. What is the area of the quadrilateral?

4. The integer n satisfies the inequality $n + (n + 1) + (n + 2) + \cdots + (n + 20) > 2019$.

What is the minimum possible value of n?

5. Identical regular pentagons are arranged in a ring. The partially completed ring is shown in the diagram. Each of the regular pentagons has a perimeter of 65. The regular polygon formed as the inner boundary of the ring has a perimeter of P. What is the value of P?

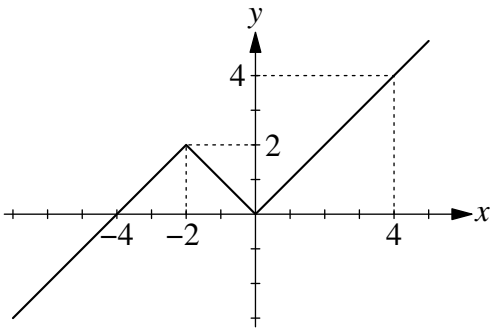


6. For natural numbers a and b we are given that $2019 = a^2 - b^2$. It is known that $a < 1000$. What is the value of a?

7. How many positive? integers n exist such that both $\frac{n+1}{3}$ and $3n + 1$ are three-digit integers?

8. The function $J(x)$ is defined by:

$$J(x) = \begin{cases} 4 + x & \text{for } x \leq -2, \\ -x & \text{for } -2 < x \leq 0, \\ x & \text{for } x > 0. \end{cases}$$



How many distinct real solutions has the equation $J(J(J(x))) = 0$?

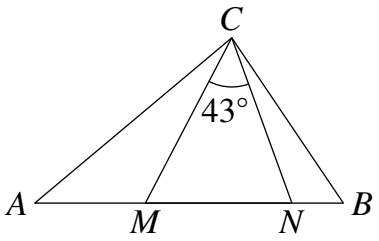
9. What is the smallest three-digit number K which can be written as $K = a^b + b^a$, where both a and b are one-digit positive integers?

10. What is the value of $\sqrt{13 + \sqrt{28 + \sqrt{281}}} \times \sqrt{13 - \sqrt{28 + \sqrt{281}}} \times \sqrt{141 + \sqrt{281}}$?

11. In the triangle ABC the points M and N lie on the side AB such that $AN = AC$ and $BM = BC$.

We know that $\angle MCN = 43°$.

Find the size in degrees of $\angle ACB$.



12. What is the value of $A^2 + B^3 + C^5$, given that:

$A = \sqrt[3]{16\sqrt{2}}$

$B = \sqrt{9\sqrt[3]{9}}$

$C = [(\sqrt[5]{2})^2]^2$

13. The real numbers a and b, where $a > b$, are solutions to the equation $3^{2x} - 10 \times 3^{x+1} + 81 = 0$. What is the value of $20a^2 + 18b^2$?

14. A number N is the product of three distinct primes. How many distinct factors does N^5 have?

15. Five Bunchkins sit in a horizontal field. No three of the Bunchkins are sitting in a straight line. Each Bunchkin knows the four distances between her and each of the others. Each Bunchkin calculates and then announces the total of these distances. These totals are 17, 43, 56, 66 and 76. A straight line is painted joining each pair of Bunchkins. What is the total length of paint required?

16. The real numbers x and y satisfy the equations:

$$xy - x = 180 \quad \text{and} \quad y + xy = 208.$$

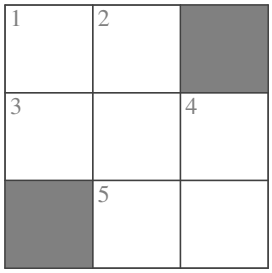
Let the two solutions be (x_1, y_1) and (x_2, y_2).
What is the value of $x_1 + 10y_1 + x_2 + 10y_2$?

17. In triangle ABC, $\angle BAC$ is 120°. The length of AB is 123. The point M is the midpoint of side BC. The line segments AB and AM are perpendicular.
What is the length of side AC?

18. An integer is said to be *chunky* if it consists only of non-zero digits by which it is divisible when written in base 10.
For example, the number 936 is Chunky since it is divisible by 9, 3 and 6.
How many chunky integers are there between 13 and 113?

19. The square $ABCD$ has sides of length 105. The point M is the midpoint of side BC. The point N is the midpoint of BM. The lines BD and AM meet at the point P. The lines BD and AN meet at the point Q.
What is the area of triangle APQ?

20. Each square in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are fulfilled. The digits used are not necessarily distinct.
What is the answer to 3 ACROSS?



ACROSS
1. A composite factor of 1001
3. Not a palindrome
5. pq^3 where p, q prime and $p \neq q$

DOWN
1. One more than a prime, one less than a prime
2. A multiple of 9
4. p^3q using the same p, q as 5 ACROSS

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

Enquiries about the Senior Kangaroo should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1. What is the sum of all the factors of 144?

SOLUTION **403**

The factor pairs of 144 are 1, 144; 2, 72; 3, 48; 4, 36; 6, 24; 8, 18; 9, 16 and 12 (squared). Their sum is 403.

2. When I noticed that $2^4 = 4^2$, I tried to find other pairs of numbers with this property. Trying 2 and 16, I realised that 2^{16} is larger than 16^2. How many times larger is 2^{16}?

SOLUTION **256**

$\frac{2^{16}}{16^2}=\frac{2^{16}}{(2^4)^2}=\frac{2^{16}}{2^8}=2^8=256$

3. The two diagonals of a quadrilateral are perpendicular. The lengths of the diagonals are 14 and 30. What is the area of the quadrilateral?

SOLUTION **210**

Label the quadrilateral $ABCD$ and let $AC = 14$ and $BD = 30$.
Let M be the intersection of AC and BD.
Let $AM = a$, $BM = b$, $CM = c$ and $DM = d$.
Then the sum of the areas is $\frac{1}{2}\times(ab + ad + cb + cd) = \frac{1}{2}\times(a + c)\times(b + d) = \frac{1}{2}\times 14\times 30 = 210$.

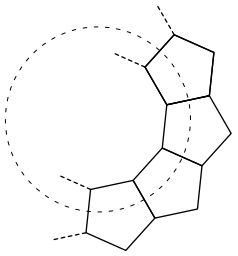
4. The integer n satisfies the inequality $n + (n + 1) + (n + 2) + \cdots + (n + 20) > 2019$.

What is the minimum possible value of n?

SOLUTION **087**

We solve the inequality $n + (n + 1) + (n + 2) + \ldots + (n + 20) > 2019$
Therefore, $21n + 210 > 2019$, i.e. $21n > 1809$ and $7n > 603$.
Therefore, $n > \frac{603}{7} = 86.1\ldots$, so n must be at least 87.

5. Identical regular pentagons are arranged in a ring. The partially completed ring is shown in the diagram. Each of the regular pentagons has a perimeter of 65. The regular polygon formed as the inner boundary of the ring has a perimeter of P. What is the value of P?



SOLUTION **130**

Let the regular N-gon at the centre of the figure have interior angles of size x degrees.
The interior angle of a pentagon is 108°. By angles at a point we have $x + 2 \times 108 = 360$, so $x = 144$.
The exterior angle of the N-gon is $180 - 144 = 36$. Therefore, the N-gon has $\frac{360}{36} = 10$ sides.
As each side has length $\frac{65}{5} = 13$, the perimeter is $10 \times 13 = 130$.

6. For natural numbers a and b we are given that $2019 = a^2 - b^2$. It is known that $a < 1000$. What is the value of a?

SOLUTION **338**

We can write $2019 = (a + b)(a - b)$. The integers $a + b$ and $a - b$ must be a factor pair of 2019.
There are two such factor pairs: $2019, 1$ and $673, 3$. These yield $(a, b) = (1010, 1009)$ and $(a, b) = (338, 335)$ respectively. As the answer must be at most 999, we conclude that $a = 338$.

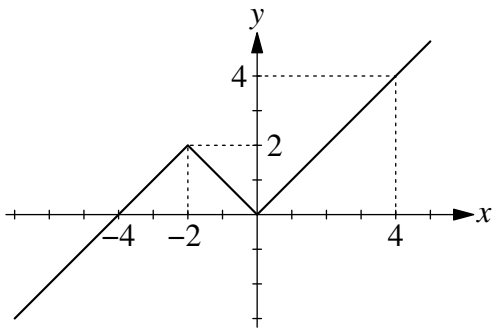
7. How many positive integers n exist such that both $\frac{n+1}{3}$ and $3n + 1$ are three-digit integers?

SOLUTION **012**

For $\frac{n+1}{3}$ to be a three-figure integer we require $99 < \frac{n+1}{3} < 999$.
This simplifies to $297 < n + 1 < 2997$, that is $296 < n < 2996$.
For $3n + 1$ to be a three-figure integer we require $99 < 3n + 1 < 999$.
This simplifies to $98 < 3n < 998$, that is $\frac{98}{3} < n < \frac{998}{3}$.
These inequalities are simultaneously solved when $296 < n < \frac{998}{3} = 332\frac{2}{3}$.
For every integer value of n between 297 and 332 it is clear that $3n + 1$ will be a three-figure integer. However, $\frac{n+1}{3}$ will only be an integer for those values of $n + 1$ which are divisible by 3.
These are 299, 302, 305, 308, 311, 314, 317, 320, 323, 326, 329 and 332. There are 12 numbers in this list.

8. The function $J(x)$ is defined by:

$$J(x) = \begin{cases} 4 + x & \text{for } x \leq -2, \\ -x & \text{for } -2 < x \leq 0, \\ x & \text{for } x > 0. \end{cases}$$



How many distinct real solutions has the equation $J(J(J(x))) = 0$?

SOLUTION **004**

The only solutions to $J(x) = 0$ are $x = 0, -4$.
Since $J(0) = 0$, both will also be solutions of $J(J(J(x))) = 0$.
Any solution to $J(x) = -4$ will also be a solution to $J(J(x)) = 0$. The only solution to $J(x) = -4$ is $x = -8$. Since $J(x) = 0$, $x = -8$ is also a solution of $J(J(J(x))) = 0$.
Any solution to $J(x) = -8$ will also be a solution to $J(J(J(x))) = 0$. The only solution to $J(x) = -8$ is $x = -12$.
Therefore, there are four distinct solutions, $x = 0, -4, -8$ and -12.

9. What is the smallest three-digit number K which can be written as $K = a^b + b^a$, where both a and b are one-digit positive integers?

SOLUTION **100**

As the problem is symmetrical in a, b we assume $a \leq b$ without loss of generality.
If $a = 1$ then the maximum value of $a^b + b^a$ is $1^9 + 9^1 = 1 + 9 = 10$. This is not a three-digit number, so cannot be a value of K.
If $a = 2$ then possible values for K include $2^9 + 9^2 = 512 + 81 = 593$, $2^8 + 8^2 = 256 + 64 = 320$, $2^7 + 7^2 = 128 + 49 = 177$ and $2^6 + 6^2 = 64 + 36 = 100$.
As 100 can be attained then 100 is the smallest three-digit number K.

10. What is the value of $\sqrt{13+\sqrt{28+\sqrt{281}}} \times \sqrt{13-\sqrt{28+\sqrt{281}}} \times \sqrt{141+\sqrt{281}}$?

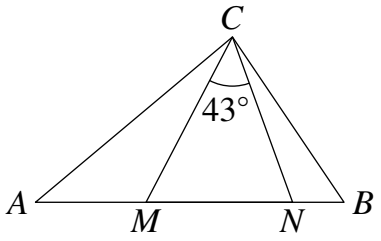
SOLUTION **140**

$$\sqrt{13+\sqrt{28+\sqrt{281}}} \times \sqrt{13-\sqrt{28+\sqrt{281}}} \times \sqrt{141+\sqrt{281}}$$
$$= \sqrt{13^2 - \left(\sqrt{28+\sqrt{281}}\right)^2} \times \sqrt{141+\sqrt{281}}$$
$$= \sqrt{169 - \left(28+\sqrt{281}\right)} \times \sqrt{141+\sqrt{281}}$$
$$= \sqrt{141-\sqrt{281}} \times \sqrt{141+\sqrt{281}} = \sqrt{\left(141-\sqrt{281}\right) \times \left(141+\sqrt{281}\right)}$$
$$= \sqrt{141^2 - 281} = \sqrt{141^2 - 282 + 1} = \sqrt{(141-1)^2} = \sqrt{140^2} = 140$$

11. In the triangle ABC the points M and N lie on the side AB such that $AN = AC$ and $BM = BC$.

We know that $\angle MCN = 43°$.

Find the size in degrees of $\angle ACB$.



SOLUTION **094**

Let $\angle ACM = x°$ and $\angle BCN = y°$.
Using the base angles property of isosceles triangles ACN and BCM, we have $\angle ANC = 43 + x$ and $\angle BMC = 43 + y$.
In triangle CMN, $43 + (43 + x) + (43 + y) = 180$.
Therefore, $\angle ACB = x + 43 + y = 94$.

12. What is the value of $A^2 + B^3 + C^5$, given that:

$A = \sqrt[3]{16\sqrt{2}}$

$B = \sqrt{9\sqrt[3]{9}}$

$C = [(\sqrt[5]{2})^2]^2$

SOLUTION **105**

$A^2 = \left(\sqrt[3]{16 \times \sqrt{2}}\right)^2 = \left(2^4 \times 2^{\frac{1}{2}}\right)^{\frac{2}{3}} = \left(2^{\frac{9}{2}}\right)^{\frac{2}{3}} = 2^{\frac{18}{6}} = 2^3 = 8$

$B^3 = \sqrt{9 \times \sqrt[3]{9}}^3 = \left(9 \times 9^{\frac{1}{3}}\right)^{\frac{3}{2}} = \left(9^{\frac{4}{3}}\right)^{\frac{3}{2}} = 9^{\frac{12}{6}} = 9^2 = 81$

$C^5 = \left(\left(\sqrt[5]{2}^2\right)^2\right)^5 = \left(2^{\frac{1}{5}}\right)^{2\times2\times5} = 2^{\frac{20}{5}} = 2^4 = 16$

$A^2 + B^3 + C^5 = 8 + 81 + 16 = 105$

13. The real numbers a and b, where $a > b$, are solutions to the equation $3^{2x} - 10 \times 3^{x+1} + 81 = 0$. What is the value of $20a^2 + 18b^2$?

SOLUTION **198**

In the equation $3^{2x} - 10 \times 3^{x+1} + 81 = 0$, replace 3^x with y. The equation becomes $y^2 - 10 \times 3 \times y + 81 = 0$. This factorises as $(y - 3)(y - 27) = 0$ with solutions $y = 3, 27$. This means $3^x = 3$ or $3^x = 27$. The x-values are $1, 3$ respectively, so $a = 3$ and $b = 1$. The value of $20a^2 + 18b^2 = 20 \times 9 + 18 \times 1 = 198$.

14. A number N is the product of three distinct primes. How many distinct factors does N^5 have?

SOLUTION **216**

Let the three distinct prime factors of N be p, q and r. Therefore, $N^5 = p^5 \times q^5 \times r^5$. Each factor of N^5 may be written as $p^a \times q^b \times r^c$, where $a, b, c \in \{0, 1, 2, 3, 4, 5\}$. Since there are 6 choices for the value of each of a, b, c there are $6 \times 6 \times 6 = 216$ distinct factors of N^5.

15. Five Bunchkins sit in a horizontal field. No three of the Bunchkins are sitting in a straight line. Each Bunchkin knows the four distances between her and each of the others. Each Bunchkin calculates and then announces the total of these distances. These totals are 17, 43, 56, 66 and 76. A straight line is painted joining each pair of Bunchkins. What is the total length of paint required?

SOLUTION **129**

Each line's length will be announced twice; once by each of the two Bunchkins at its ends. By adding up the total of the numbers announced we will therefore include the length of each line exactly twice.
The total length of paint required is $\frac{1}{2} \times (17 + 43 + 56 + 66 + 76) = \frac{258}{2} = 129$.

16. The real numbers x and y satisfy the equations:

$$xy - x = 180 \quad \text{and} \quad y + xy = 208.$$

Let the two solutions be (x_1, y_1) and (x_2, y_2).
What is the value of $x_1 + 10y_1 + x_2 + 10y_2$?

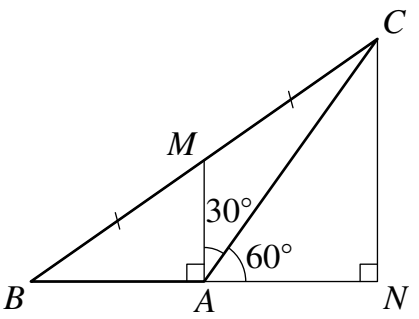
SOLUTION **317**

Subtracting the two equations yields $y + x = 28$. Substituting $y = 28 - x$ into $xy - x = 180$ leads to the quadratic equation $0 = x^2 - 27x + 180$. This has solutions 12, 15. The solution sets are therefore (15, 13) and (12, 16).
The value of $x_1 + 10y_1 + x_2 + 10y_2$ is $15 + 10 \times 13 + 12 + 10 \times 16 = 317$.

17. In triangle ABC, $\angle BAC$ is 120°. The length of AB is 123. The point M is the midpoint of side BC. The line segments AB and AM are perpendicular.
What is the length of side AC?

SOLUTION **246**

Extend the line BA. Draw a line through C, parallel to MA, meeting the extended line BA at point N. By the intercept theorem, $BA = AN = 123$, because $BM = MC$. In triangle NAC, $\cos 60 = \frac{1}{2} = \frac{123}{AC}$. Therefore, $AC = 246$.



18. An integer is said to be *chunky* if it consists only of non-zero digits by which it is divisible when written in base 10.
For example, the number 936 is Chunky since it is divisible by 9, 3 and 6.
How many chunky integers are there between 13 and 113?

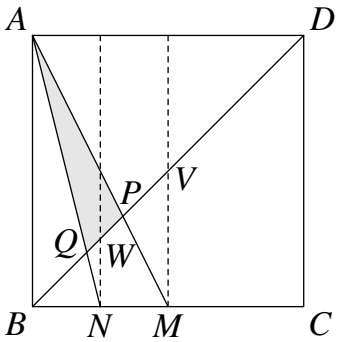
SOLUTION **014**

For a two-digit number $N =$ "ab" we may write $N = 10a + b$.
If N is Chunky, then N will be divisible by a and therefore $b = N - 10a$ will also be divisible by a.
It is therefore sufficient to check only those two-digit numbers which have a units digit divisible by their tens digit. Following checking, 15, 22, 24, 33, 36, 44, 48, 55, 66, 77, 88 and 99 are the only two-digit Chunky numbers (excluding 11 and 12, which are not under consideration). Of those three-digit numbers under consideration, only 111 and 112 are Chunky. The answer is 14.

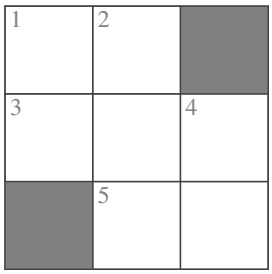
19. The square $ABCD$ has sides of length 105. The point M is the midpoint of side BC. The point N is the midpoint of BM. The lines BD and AM meet at the point P. The lines BD and AN meet at the point Q.
What is the area of triangle APQ?

SOLUTION **735**

Let V be the centre of the square $ABCD$. Let W be the intersection between BD and the line through N parallel to AB.
Triangles APB and MPV are similar, with $BP : PV = AB : MV = \frac{1}{2}$. Therefore, $BP = \frac{2}{3} \times BV = \frac{2}{3} \times \frac{1}{2} \times 105\sqrt{2} = 35\sqrt{2}$.
Similarly, triangles AQB and NQW are similar, with $BQ : QW = AB : NW = \frac{1}{4}$. Therefore, $BQ = \frac{4}{5} \times BW = \frac{4}{5} \times \frac{1}{4} \times 105\sqrt{2} = 21\sqrt{2}$.
The area of APQ is $\frac{1}{2} \times QP \times VA = \frac{1}{2} \times \left(35\sqrt{2} - 21\sqrt{2}\right) \times \frac{1}{2} \times 105\sqrt{2} = \frac{1}{2} \times 14\sqrt{2} \times \frac{1}{2} \times 105\sqrt{2} = 735$.



20. Each square in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are fulfilled. The digits used are not necessarily distinct. What is the answer to 3 ACROSS?



ACROSS
1. A composite factor of 1001
3. Not a palindrome
5. pq^3 where p, q prime and $p \neq q$

DOWN
1. One more than a prime, one less than a prime
2. A multiple of 9
4. p^3q using the same p, q as 5 ACROSS

SOLUTION **295**

1 Across may be either 77 or 91. The only possibility for 1 Down with 7 or 9 as its first digit is 72. So 1 Across is 77 and 1 Down is 72.

In the clues for 5 Across and 4 Down we see that p, q must be 2, 3 in some order, since if any larger prime were used then pq^3 and qp^3 would not both be two-digit. Therefore, 5 Across and 4 Down are $3 \times 2^3 = 24$ and $2 \times 3^3 = 54$ in some order. We know that 3 Across is not a palindrome (so may not end in a 2). Therefore, 5 Across is 24 and 4 Down is 54.

The only three-digit multiples of 9 beginning with a 7 are 702 and 792. As every digit in the completed crossnumber must be non-zero we have 2 Down is 792 and 3 Across is 295.

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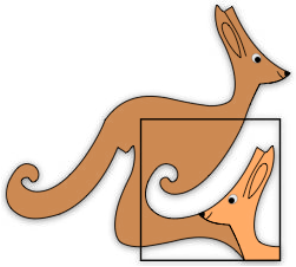


The Andrew Jobbings Senior Kangaroo

25 – 26 November 2020

Organised by the United Kingdom Mathematics Trust

a member of the Association Kangourou sans Frontières

[XTX] MARKETS Overleaf

England & Wales: *Year* 13 *or below*
Scotland: *S*6 *or below*
Northern Ireland: *Year* 14 *or below*

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **60 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil** to record your answer to each problem as a three-digit number from 000 to 999.
   Pay close attention to the example on the Answer Sheet that shows how to code your answers.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:**
   5 marks are awarded for each correct answer;
   There is no penalty for giving an incorrect answer.
7. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

Enquiries about The Andrew Jobbings Senior Kangaroo should be sent to:

*UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS*2 9*JT*

☎ 0113 365 1121 enquiry@ukmt.org.uk www.ukmt.org.uk

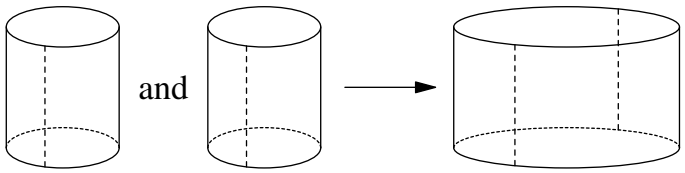
1. What is the difference between the greatest and the least of the following five quantities?

$20 + 20$ $\quad 20 \times 20$ $\quad 202 + 0$ $\quad (2^0)^{(2^0)}$ $\quad 20 + 2 + 0$

2. The positive integers x and y satisfy the equation $yx^2 + xy^2 = 70$. What is the value of $x^4 + y^4$?

3. How many distinct integer solutions (x, y) are there to the equation $5^1 + 4^2 + 3^3 + 2^4 = x^y$?

4. Two identical cylindrical sheets are cut open along the dotted lines and glued together to form one bigger cylindrical sheet, as shown. The smaller sheets each enclose a volume of 100. What volume is enclosed by the larger sheet?



5. Let $a^b = \frac{1}{8}$. What is the value of a^{-3b}?

6. For what value of x does the expression $x^2 - 600x + 369$ take its minimum value?

7. Margot writes the numbers 1,2,3,4,5,6,7 and 8 in the top row of a table, as shown. In the second row she plans to write the same set of numbers, in any order.
Each number in the third row is obtained by finding the sum of the two numbers above it.
In how many different ways can Margot complete row 2 so that every entry in row 3 is even?

1	2	3	4	5	6	7	8

8. The number $(2^{222})^5 \times (5^{555})^2$ is Q digits long. What is the largest prime factor of Q?

9. The radii of two concentric circles are in the ratio 1 : 3.
AC is a diameter of the larger circle. BC is a chord of the larger circle and is tangent to the smaller circle. AB has length 140.
What is the radius of the larger circle?

10. What is the smallest 3-digit positive integer N such that $2^N + 1$ is a multiple of 5?

11. A circle is drawn inside a regular hexagon so that it touches all six sides of the hexagon.
The area of the circle is $\pi \times 64\sqrt{3}$. What is the area of the hexagon?

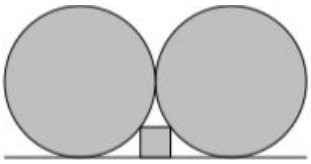
12. What is the value of $\sqrt{20212020 \times 20202021 - 20212021 \times 20202020}$?

13. How many ordered triples of positive integers (x, y, z) satisfy $(x^y)^z = 1024$?

14. Let a, b, c and d be distinct positive integers such that $a+b$, $a+c$ and $a+d$ are all odd and are all square. Let L be the least possible value of $a+b+c+d$. What is the value of $10L$?

15. On an island, kangaroos are always either grey or red. One day, the number of grey kangaroos increased by 28% while the number of red kangaroos decreased by 28%. The ratios of the two types of kangaroos were exactly reversed. By what percentage did the total number of kangaroos change?

16. A square fits snugly between the horizontal line and two touching circles of radius 1000, as shown. The line is tangent to the circles.
What is the side-length of the square?

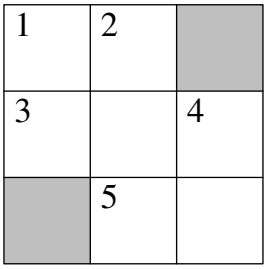


17. How many solutions does equation $|||x-1|-1|-1|=1$ have?
The modulus function $|x|$ evaluates the absolute value of a number; for example $|6| = |-6| = 6$.

18. The operation $\diamond$ is defined on two positive whole numbers as the number of distinct prime factors of the product of the two numbers. For example $8 \diamond 15 = 3$.
What is the cube of the value of $(720 \diamond 1001)$?

19. A random number generator gives outputs of 1, 2, 3, 4 and 5 with equal probability.
The values of a, b and c are each chosen by running the generator once.
The probability that $a \times b + c$ is even can be written as a fraction in its lowest terms as $\frac{N}{D}$.
What is the value of $10N + D$?

20. Each square in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are fulfilled. The digits used are not necessarily distinct.
What is the answer to 3 ACROSS?



ACROSS
1. A multiple of 7
3. The answer to this Question
5. More than 10

DOWN
1. A multiple of a square of an odd prime; neither a square nor a cube
2. The internal angle of a regular polygon; the exterior angle is between 10^o and 20^o
4. A proper factor of 5 ACROSS but not a proper factor of 1 DOWN

United Kingdom
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Senior Kangaroo

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

Enquiries about the Senior Kangaroo should be sent to:

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1. What is the difference between the greatest and the least of the following five quantities?

$20 + 20$ $\quad 20 \times 20$ $\quad 202 + 0$ $\quad (2^0)^{(2^0)}$ $\quad 20 + 2 + 0$

SOLUTION **399**

The five quantities are 40, 400, 202, $1^1 = 1$ and 22.
The difference between the greatest and least of these is $400 - 1 = 399$.

2. The positive integers x and y satisfy the equation $yx^2 + xy^2 = 70$. What is the value of $x^4 + y^4$?

SOLUTION **641**

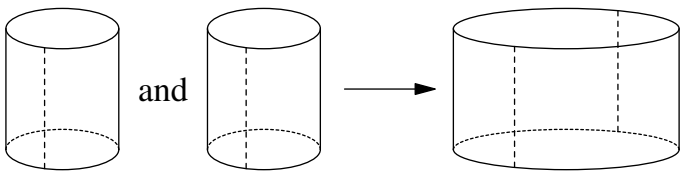
We may assume that $x \leq y$ as the expressions $yx^2 + xy^2$ and $x^4 + y^4$ are symmetrical in x and y.
If $x \geq 4$ (and so $y \geq 4$) then $yx^2 + xy^2 \geq 2 \times 4^3 = 128 > 70$. So x must be one of 1, 2, 3.
When $x = 2$ we obtain the equation $4y + 2y^2 = 70$ which has the integer solution $y = 5$.
The analogous equations in y obtained for $x = 1, 3$ do not have integer solutions.
Therefore, $x = 2$, $y = 5$ and $x^4 + y^4 = 2^4 + 5^4 = 16 + 625 = 641$.

3. How many distinct integer solutions (x, y) are there to the equation $5^1 + 4^2 + 3^3 + 2^4 = x^y$?

SOLUTION **006**

The sum $5^1 + 4^2 + 3^3 + 2^4$ is equal to $5 + 16 + 27 + 16 = 64 = 2^6$.
The positive integer solutions (x, y) to the equation $x^y = 64$ are $(64, 1)$, $(8, 2)$, $(4, 3)$ and $(2, 6)$.
There are also two further solutions involving negative integers, namely $(-8, 2)$ and $(-2, 6)$.
Therefore there are six distinct integer solutions.

4. Two identical cylindrical sheets are cut open along the dotted lines and glued together to form one bigger cylindrical sheet, as shown. The smaller sheets each enclose a volume of 100. What volume is enclosed by the larger sheet?



SOLUTION **400**

Since the circumferences of the smaller cylinder and the larger cylinder are in the ratio 1 : 2, the radii of their cross-sections are also in the ratio 1 : 2. Therefore the areas of their cross-sections are in the ratio $1^2 : 2^2 = 1 : 4$. As the cylinders have the same perpendicular height, the volumes they enclose will also be in the ratio 1 : 4.
Therefore the larger cylinder encloses a volume of $4 \times 100 = 400$.

5. Let $a^b = \frac{1}{8}$. What is the value of a^{-3b}?

SOLUTION **512**

We are given that $a^b = \frac{1}{8}$. Taking the reciprocal we get $a^{-b} = 8$; and then cubing gives $a^{-3b} = 8^3 = 512$.

6. For what value of x does the expression $x^2 - 600x + 369$ take its minimum value?

SOLUTION **300**

Complete the square on $x^2 - 600x + 369$ to obtain $(x - 300)^2 - 300^2 + 369$.
The minimum value of this expression will occur when $(x - 300)^2 = 0$. This is when $x = 300$.

7. Margot writes the numbers 1,2,3,4,5,6,7 and 8 in the top row of a table, as shown. In the second row she plans to write the same set of numbers, in any order.
Each number in the third row is obtained by finding the sum of the two numbers above it.
In how many different ways can Margot complete row 2 so that every entry in row 3 is even?

1	2	3	4	5	6	7	8

SOLUTION **576**

For an entry in row 3 to be even we need the corresponding entry in row 2 to have the same parity as the entry in row 1 (that is: both are odd or both are even).
Columns 1, 3, 5 and 7 must therefore have odd entries in row 2. There are four odd numbers to arrange in these cells, with $4 \times 3 \times 2 \times 1 = 24$ ways to arrange these.
Similarly, columns 2, 4, 6 and 8 must have even entries in row 2. There are four even numbers to arrange in these cells, with $4 \times 3 \times 2 \times 1 = 24$ ways to arrange these.
Therefore there are $24 \times 24 = 576$ ways in which Margot can complete the table in this way.

8. The number $(2^{222})^5 \times (5^{555})^2$ is Q digits long. What is the largest prime factor of Q?

SOLUTION **101**

The number $(2^{222})^5 \times (5^{555})^2 = 2^{1110} \times 5^{1110} = 10^{1110}$.
The number 10^{1110} is 1111 digits long, so $Q = 1111 = 11 \times 101$.
Both 11 and 101 are prime.
Hence the largest prime factor of 1111 is 101.

9. The radii of two concentric circles are in the ratio 1 : 3.
AC is a diameter of the larger circle. BC is a chord of the larger circle and is tangent to the smaller circle. AB has length 140.
What is the radius of the larger circle?

SOLUTION **210**

Let O be the centre of the circles, let T be the point where the chord BC meets the smaller circle and let r be the radius of the smaller circle.
$\angle CTO = 90°$ by the tangent-radius theorem.
$\angle CBA = 90°$ by the angle in a semicircle theorem.
Therefore, $\triangle CTO$ is similar to $\triangle CBA$ since both are right-angled and they share $\angle OCT$.
Now, $OT : AB = CO : CA = 1 : 2$.
However, $AB = 140$, so $r = 70$ and the radius of the larger circle is $3 \times 70 = 210$.

10. What is the smallest 3-digit positive integer N such that $2^N + 1$ is a multiple of 5?

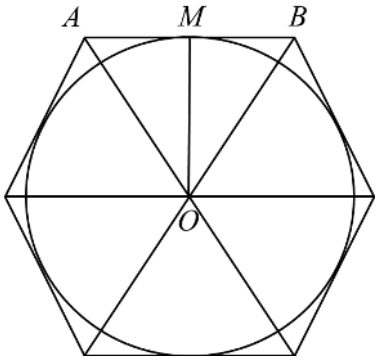
SOLUTION **102**

The powers of 2 (namely $2, 4, 8, 16, 32, 64, \ldots$) have units digits which follow a sequence $2, 4, 8, 6, \ldots$ which repeat every four terms. We may calculate that 2^{100}, 2^{101} and 2^{102} have units digits of 6, 2 and 4 respectively. Therefore the first 3-digit power of 2 which is one less than a multiple of 5 is 2^{102}.

11. A circle is drawn inside a regular hexagon so that it touches all six sides of the hexagon. The area of the circle is $\pi \times 64\sqrt{3}$. What is the area of the hexagon?

SOLUTION **384**

Let O be the centre of the hexagon.
Let AB be an edge of the hexagon with midpoint M.
For the circle we have $\pi r^2 = \pi \times 64\sqrt{3}$.
Therefore $r^2 = 64\sqrt{3}$.
In $\triangle OMB$, $OB = \frac{OM}{cos30°} = \frac{2r}{\sqrt{3}}$.
The area of the equilateral $\triangle OAB$ is
$\frac{1}{2} \times AB \times OM = \frac{1}{2} \times OB \times OM$, since $AB = OB$.
Therefore, area $\triangle OAB = \frac{1}{2} \times \frac{2r}{\sqrt{3}} \times r = \frac{r^2}{\sqrt{3}}$
The area of the hexagon is $6 \times \frac{r^2}{\sqrt{3}} = 6 \times \frac{64\sqrt{3}}{\sqrt{3}} = 6 \times 64 = 384$.



12. What is the value of $\sqrt{20212020 \times 20202021 - 20212021 \times 20202020}$?

SOLUTION **100**

Let $y = 20202020$. We may express the quantity under the square-root as
$(y + 10000) \times (y + 1) - (y + 10001) \times y = y^2 + 10001y + 10000 - y^2 - 10001y = 10000$.
The square root of 10000 is 100.

13. How many ordered triples of positive integers (x, y, z) satisfy $(x^y)^z = 1024$?

SOLUTION **009**

Note that $1024 = 2^{10}$, so we require yz to be a factor of 10.
Hence the solutions are $(1024, 1, 1)$, $(32, 2, 1)$, $(32, 1, 2)$, $(4, 5, 1)$, $(4, 1, 5)$, $(2, 10, 1)$, $(2, 1, 10)$, $(2, 5, 2)$, $(2, 2, 5)$.

14. Let a, b, c and d be distinct positive integers such that $a + b$, $a + c$ and $a + d$ are all odd and are all square. Let L be the least possible value of $a + b + c + d$. What is the value of $10L$?

SOLUTION **670**

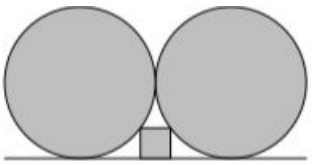
The numbers a, b, c and d are distinct, so $a + b$, $a + c$ and $a + d$ must also be distinct.
The smallest three odd squares which may be formed in this way are 9, 25 and 49.
Therefore, $(a + b) + (a + c) + (a + d) = 9 + 25 + 49 = 83$.
We may write $L = a + b + c + d = 83 - 2a$. So we may minimise L by maximising a.
Since $a + b = 9$, the largest possible value for a is 8. This means that $L = 83 - 2 \times 8 = 67$.
We must check that a solution exists. The values $a = 8$, $b = 1$, $c = 17$ and $d = 41$ satisfy all the conditions provided.
If one started with a different set of odd squares, then at least one would be 81 or larger. However, $a + b + c + d$ is greater than each of the odd squares involved. Hence 67 is indeed the least possible value.
Therefore L is indeed 67 and $10L = 670$.

15. On an island, kangaroos are always either grey or red. One day, the number of grey kangaroos increased by 28% while the number of red kangaroos decreased by 28%. The ratios of the two types of kangaroos were exactly reversed. By what percentage did the total number of kangaroos change?

SOLUTION **004**

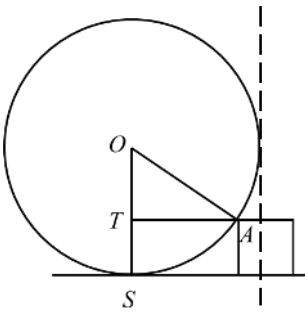
Let the initial populations of grey and red kangaroos be G and R respectively.
After the change, the new populations are $1.28G$ and $0.72R$ respectively.
As the ratios are reversed, we have $\frac{1.28G}{0.72R} = \frac{R}{G}$.
Therefore $R^2 = \frac{128}{72} \times G^2 = \frac{16G^2}{9}$ and hence $R = \frac{4G}{3}$.
The ratio of new population:old population is
$(1.28G + 0.72R) : (G + R) = (1.28 + 0.72\frac{R}{G}) : (1 + \frac{R}{G}) = (1.28 + 0.72 \times \frac{4}{3} : (1 + \frac{4}{3})$
$= (3.84 + 2.88) : (4 + 3) = 6.72 : 7 = 0.96 : 1.$
Therefore there is a reduction of 4% in the population.

16. A square fits snugly between the horizontal line and two touching circles of radius 1000, as shown. The line is tangent to the circles.
What is the side-length of the square?



SOLUTION **400**

Let O be the centre of the left circle, A be the top-left vertex of the square, S be the point at which the left circle meets the tangent and T be the foot of the perpendicular from A to OS.



Let the square have side $2x$.
For simplicity, we let r denote the radius 1000 of the circles.
The line AT has length $r - \frac{1}{2} \times 2x = r - x$, since the common tangent to the circles (shown) is a line of symmetry of the square.
Considering $\triangle OAT$, we have $r^2 = (r - x)^2 + (r - 2x)^2$.
This leads to the quadratic $0 = (5x - r)(x - r)$, which has solutions $x = r$ and $x = 0.2r$.
Since $x < r$, we have $x = 0.2r = 0.2 \times 1000 = 200$.
Therefore the square has side-length $2 \times 200 = 400$.

17. How many solutions does equation $|||x-1|-1|-1| = 1$ have?
The modulus function $|x|$ evaluates the absolute value of a number; for example $|6| = |-6| = 6$.

SOLUTION **004**

Since $|||x-1|-1|-1| = 1$ we have $||x-1|-1|-1 = \pm 1$.
Therefore $||x-1|-1| = \pm 1 + 1 = 0$ or 2.
Continuing in this way, $|x-1|-1 = \pm 0$ or ± 2, so $|x-1| = 3, 1$ or -1, (of which -1 is not possible). Therefore $x - 1 = \pm 3$ or ± 1, so $x = -2, 0, 2$ or 4.

18. The operation $\diamond$ is defined on two positive whole numbers as the number of distinct prime factors of the product of the two numbers. For example $8 \diamond 15 = 3$.
What is the cube of the value of $(720 \diamond 1001)$?

SOLUTION **216**

We have the prime factorisations $720 = 2^4 \times 3^2 \times 5$ and $1001 = 7 \times 11 \times 13$.
Therefore 720×1001 will have 6 distinct prime factors (namely 2, 3, 5, 7, 11 and 13).
The cube of $(720 \diamond 1001)$ is $6^3 = 216$.

19. A random number generator gives outputs of 1, 2, 3, 4 and 5 with equal probability.
The values of a, b and c are each chosen by running the generator once.
The probability that $a \times b + c$ is even can be written as a fraction in its lowest terms as $\frac{N}{D}$.
What is the value of $10N + D$?

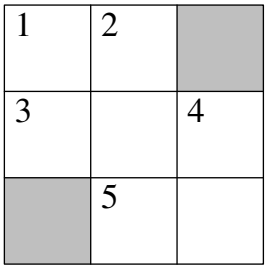
SOLUTION **715**

The expression $a \times b + c$ is even if ab and c are either both odd, or both even.
The product ab is odd only if a and b are both odd. This occurs with probability $\frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$.
Therefore the probability that both ab and c are odd is $\frac{9}{25} \times \frac{3}{5} = \frac{27}{125}$.
The product ab is even with probability $1 - \frac{9}{25} = \frac{16}{25}$.
Therefore the probability that both ab and c are even is $\frac{16}{25} \times \frac{2}{5} = \frac{32}{125}$.
The total probability that $ab + c$ is even is $\frac{27}{125} + \frac{32}{125} = \frac{59}{125}$.
Therefore $N = 59$, $D = 125$ and $10N + D = 10 \times 59 + 125 = 715$.

20. Each square in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are fulfilled. The digits used are not necessarily distinct. What is the answer to 3 ACROSS?

ACROSS	DOWN
1. A multiple of 7	1. A multiple of a square of an odd prime; neither a square nor a cube
3. The answer to this Question	2. The internal angle of a regular polygon; the exterior angle is between 10^o and 20^o
5. More than 10	4. A proper factor of 5 ACROSS but not a proper factor of 1 DOWN

SOLUTION **961**



2 DOWN has possible answers of 162, 165 and 168.
1 ACROSS must end with a '1' so can only be 21 or 91.
1 DOWN must start with a '2' or a '9'. The squares of odd primes are 9, 25, 49. The only multiples of these which are neither square nor cube and which start '2' or '9' are 90 (which cannot be correct since 3 ACROSS cannot start with a 0), 98 and 99. Hence 1 ACROSS is 91.
5 ACROSS could start with a '2', '5' or '8'.
However, the first digit cannot be '2' since none of 21 . . . 29 (for 4 DOWN) has a proper factor which is two digits long and shares the same units digit.
Similarly, the first digit cannot be '5' since none of 51 . . . 59 (for 4 DOWN) has a proper factor which is two digits long and shares the same units digit. Hence 2 DOWN is 168.
Of 81 . . . 89 the only number to have a proper factor which is two digits long and shares the same units digit is 84 with the factor 14. So 5 ACROSS is 84 and 4 DOWN is 14.
For 1 DOWN we now know that 14 cannot be a factor, so this is 99. Therefore the answer to 3 ACROSS is 961.

United Kingdom
Mathematics Trust

Andrew Jobbings Senior Kangaroo

Thursday 2 December 2021

a member of the Association Kangourou sans Frontières

supported by [XTX] MARKETS Overleaf

England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **60 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil** to record your answer to each problem as a three-digit number from 000 to 999.
   Pay close attention to the example on the Answer Sheet that shows how to code your answers.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:**
   5 marks are awarded for each correct answer;
   There is no penalty for giving an incorrect answer.
7. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

Enquiries about the Andrew Jobbings Senior Kangaroo should be sent to:

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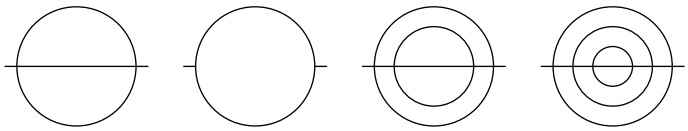
☎ 0113 365 1121 challenges@ukmt.org.uk www.ukmt.org.uk

1. Adil was born in 2015. His younger sister Bav was born in 2018. What is the minimum number of days by which Adil is older than Bav?

2. The total T is obtained as the sum of the integers from 2006 to 2036 inclusive. What is the sum of all the prime factors of T?

3. How many of the figures shown can be drawn with one continuous line without drawing a segment twice?

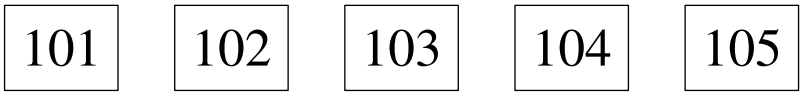


4. On each side of a right-angled triangle, a semicircle is drawn with that side as a diameter. The areas of the three semicircles are x^2, $3x$ and 180 where x^2 and $3x$ are both less than 180. What is the area of the smallest semicircle ?

5. $T = \sqrt{(2021 + 2021) + (2021 - 2021) + (2021 \times 2021) + (2021 \div 2021)}$.
What is the largest prime factor of T?

6. Into how many regions do the x-axis and the graphs of $y = 2 - x^2$ and $y = x^2 - 1$ split the plane?

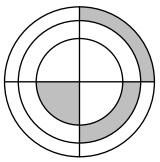
7. Five cards have the numbers 101, 102, 103, 104 and 105 on their fronts.



On the reverse, each card has one of five different positive integers: a, b, c, d and e respectively.
We know that $c = be$, $a + b = d$ and $e - d = a$.
Frankie picks up the card which has the largest integer on its reverse. What number is on the front of Frankie's card?

8. The geometric mean of a set of n positive numbers is defined as the n-th root of the product of those numbers.
Yasmeen writes down a set of four numbers which have a geometric mean of 2048.
Zak writes down a set of four numbers which have a geometric mean of 8.
What is the geometric mean of the combined set of the eight numbers written by Yasmeen and Zak?

9. In the figure shown there are three concentric circles and two perpendicular diameters. The three shaded regions have equal area. The radius of the small circle is 2. The product of the three radii is Y.
What is the value of Y^2?



10. A dealer bought two cars. He sold the first one for 40% more than he paid for it and the second one for 60% more than he paid for it. The total sum he received for the two cars was 54% more than the total sum he paid for them. When written in its lowest terms, the ratio of the prices the dealer paid for the first and the second car was $a : b$. What is the value of $11a + 13b$?

11. Billie has a die with the numbers 1, 2, 3, 4, 5 and 6 on its six faces.
Niles has a die which has the numbers 4, 4, 4, 5, 5 and 5 on its six faces.
When Billie and Niles roll their dice the one with the larger number wins. If the two numbers are equal it is a draw.
The probability that Niles wins, when written as a fraction in its lowest terms, is $\frac{p}{q}$. What is the value of $7p + 11q$?

12. There are 2021 balls in a crate. The balls are numbered from 1 to 2021. Erica works out the digit sum for each ball. For example, the digit sum of 2021 is 5, since $2 + 0 + 2 + 1 = 5$. Erica notes that balls with equal digit sums have the same colour and balls with different digit sums have different colours.
How many different colours of balls are there in the crate?

13. A multiplication table of the numbers 1 to 10 is shown. What is the sum of all the odd products in the complete table?

×	1	2	3	⋯	10
1	1	2	3	⋯	10
2	2	4	6	⋯	20
⋮	⋮				⋮
10	10	20	30	⋯	100

14. The graph of $(x^2 + y^2 - 2x)^2 = 2(x^2 + y^2)^2$ meets the x-axis in p different places and meets the y-axis in q different places.
What is the value of $100p + 100q$?

15. Which is the lowest numbered statement which is true?
Statement 201: "Statement 203 is true".
Statement 202: "Statement 201 is true".
Statement 203: "Statement 206 is false".
Statement 204: "Statement 202 is false".
Statement 205: "None of the statements 201, 202, 203 or 204 are true".
Statement 206: "1 + 1 = 2".

16. A polygon is said to be *friendly* if it is regular and it also has angles that when measured in degrees are either integers or *half-integers* (i.e. have a decimal part of exactly 0.5). How many different friendly polygons are there?

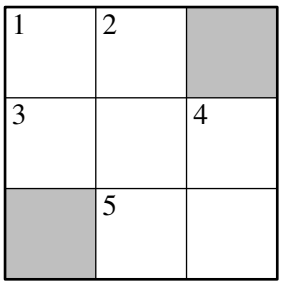
17. Find the value of R, given that the numbers Q and R are defined as:

$Q = 202^1 + 20^{21} + 2^{021}$;
R is the remainder when Q is divided by 1000.

18. Some years, like 2022, have the property that they use at most two different digits. How many years between 1 and 9999, inclusive, have this property?

19. The function $f(x)$ is defined as $f(x) = \frac{x-1}{x+1}$.
The equation $f(x^2) \times f(x) = 0.72$ has two solutions a and b, where $a > b$.
What is the value of $19a + 7b$?

20. Each cell in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are satisfied. The digits used are not necessarily distinct.
What is the answer to 2 DOWN?



ACROSS

1. A prime which is the sum of two squares
3. Twice the answer to 2 DOWN

DOWN

1. $p \times q$, where p, q are prime and $q = p + 4$
4. 60% of 5 ACROSS

United Kingdom
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Andrew Jobbings Senior Kangaroo

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

Enquiries about the Andrew Jobbings Senior Kangaroo should be sent to:

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1. Adil was born in 2015. His younger sister Bav was born in 2018. What is the minimum number of days by which Adil is older than Bav?

SOLUTION **732**

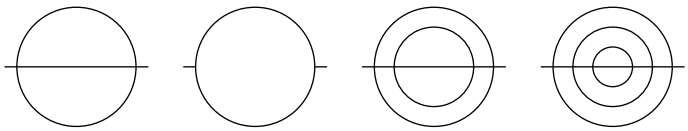
The closest the birth dates can be are 31st December 2015 and 1st January 2018. Since 2016 was a leap year, the number of days these are apart is $1 + 366 + 365 = 732$.

2. The total T is obtained as the sum of the integers from 2006 to 2036 inclusive. What is the sum of all the prime factors of T?

SOLUTION **121**

We can write the sum as $(2021 - 15) + (2021 - 14) + \ldots + (2021 - 1) + 2021 + (2021 + 1) + \ldots + (2021 + 14) + (2021 + 15) = 31 \times 2021 = 31 \times 43 \times 47$. The sum of all the prime factors of T is $31 + 43 + 47 = 121$.

3. How many of the figures shown can be drawn with one continuous line without drawing a segment twice?



SOLUTION **003**

All are possible except for the second figure. Each of the small lines must either start or end the continuous line. But then only one of the semicircles in the second figure can be included.

4. On each side of a right-angled triangle, a semicircle is drawn with that side as a diameter. The areas of the three semicircles are x^2, $3x$ and 180 where x^2 and $3x$ are both less than 180. What is the area of the smallest semicircle ?

SOLUTION **036**

The areas of a square of side $2r$ and its inscribed circle are in the ratio $4r^2 : \pi r^2 = 4 : \pi$. Therefore, the area of the square on each side of the triangle and the area of the corresponding semicircle are in the ratio $4 : \frac{\pi}{2} = 8 : \pi$. This is independent of side length and we therefore may use the Pythagoras' equation on the areas of the semicircles. This gives $x^2 + 3x = 180$. The solutions to this equation are $x = -15$ (which is impossible since this would give an area of $3 \times -15 = -45$) and $x = 12$, meaning the required area is 36.

5. $T = \sqrt{(2021 + 2021) + (2021 - 2021) + (2021 \times 2021) + (2021 \div 2021)}$.
What is the largest prime factor of T?

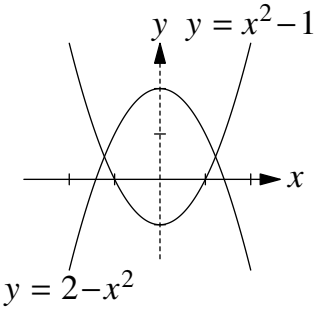
SOLUTION **337**

Let $x = 2021$. Then $T = \sqrt{2x + x^2 + 1} = x + 1 = 2022$.
Since $2022 = 2 \times 3 \times 337$, the largest prime factor of T is 337.

6. Into how many regions do the x-axis and the graphs of $y = 2 - x^2$ and $y = x^2 - 1$ split the plane?

SOLUTION **010**

As can be seen in the diagram the two graphs and the x-axis split the plane into ten regions.



7. Five cards have the numbers 101, 102, 103, 104 and 105 on their fronts.

101	102	103	104	105

On the reverse, each card has one of five different positive integers: a, b, c, d and e respectively.
We know that $c = be$, $a + b = d$ and $e - d = a$.
Frankie picks up the card which has the largest integer on its reverse. What number is on the front of Frankie's card?

SOLUTION **103**

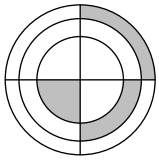
Since $e - d = a$ we have $e = a + d$. Therefore, $e > a$ and $e > d$.
Since $c = be$ we have $c > e$ and $c > b$, for if either of e, b equalled 1 then c would equal the other.
Therefore, $c > e > a$, $c > e > d$ and $c > b$.
The largest integer is c and the correct answer is 103.

8. The geometric mean of a set of n positive numbers is defined as the n-th root of the product of those numbers.
Yasmeen writes down a set of four numbers which have a geometric mean of 2048.
Zak writes down a set of four numbers which have a geometric mean of 8.
What is the geometric mean of the combined set of the eight numbers written by Yasmeen and Zak?

SOLUTION **128**

Let Yasmeen's numbers be y_1, y_2, y_3 and y_4; and Zak's be z_1, z_2, z_3 and z_4.
We are given that $\sqrt[4]{y_1y_2y_3y_4} = 2048$ and $\sqrt[4]{z_1z_2z_3z_4} = 8$.
Therefore $\sqrt[4]{y_1y_2y_3y_4z_1z_2z_3z_4} = \sqrt[4]{y_1y_2y_3y_4} \times \sqrt[4]{z_1z_2z_3z_4} = 2048 \times 8 = 2^{11} \times 2^3 = 2^{14}$.
We now take the square-root of both sides getting $\sqrt[8]{x_1x_2x_3x_4y_1y_2y_3y_4} = 2^7 = 128$.

9. In the figure shown there are three concentric circles and two perpendicular diameters. The three shaded regions have equal area. The radius of the small circle is 2. The product of the three radii is Y.
What is the value of Y^2?



SOLUTION **384**

We note that the central circle, the inner hoop and the outer hoop are also equal in area.
Let the radii of the middle circle and outer circle be a and b respectively.
The area of the central circle is 4π.
The area of the inner hoop is $\pi a^2 - 4\pi$. Therefore, $\pi a^2 - 4\pi = 4\pi$. This simplifies to $a^2 = 8$.
The area of the outer hoop is $\pi b^2 - \pi a^2 = \pi b^2 - 8\pi$.
Therefore, $\pi b^2 - 8\pi = 4\pi$. This simplifies to $b^2 = 12$.
The product of the radii, Y, is $2 \times a \times b$. Therefore, $Y^2 = 4 \times a^2 \times b^2 = 4 \times 8 \times 12 = 384$.

10. A dealer bought two cars. He sold the first one for 40% more than he paid for it and the second one for 60% more than he paid for it. The total sum he received for the two cars was 54% more that the total sum he paid for them. When written in its lowest terms, the ratio of the prices the dealer paid for the first and the second car was $a : b$.
What is the value of $11a + 13b$?

SOLUTION **124**

Let the prices the dealer paid for the first and second cars be X and Y respectively. He sells the cars for $1.4X$ and $1.6Y$.
Since he received 54% more than he paid, we have $1.54(X + Y) = 1.4X + 1.6Y$.
Therefore, $0.14X = 0.06Y$ which gives us the ratio $X : Y = 6 : 14 = 3 : 7$.
Therefore $a = 3$ and $b = 7$ and $11a + 13b = 11 \times 3 + 13 \times 7 = 33 + 91 = 124$.

11. Billie has a die with the numbers 1, 2, 3, 4, 5 and 6 on its six faces.
Niles has a die which has the numbers 4, 4, 4, 5, 5 and 5 on its six faces.
When Billie and Niles roll their dice the one with the larger number wins. If the two numbers are equal it is a draw.
The probability that Niles wins, when written as a fraction in its lowest terms, is $\frac{p}{q}$.
What is the value of $7p + 11q$?

SOLUTION **181**

Niles rolls a 4 with probability $\frac{1}{2}$, and then wins if Billie rolls a 1, 2 or 3, with probability $\frac{1}{2}$.
Niles rolls a 5 with probability $\frac{1}{2}$, and then wins if Billie rolls a 1, 2, 3 or 4, with probability $\frac{2}{3}$.
Therefore, the probability of Niles winning is $\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} = \frac{1}{4} + \frac{1}{3} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$.
Therefore $p = 7$, $q = 12$ and $7p + 11q = 7 \times 7 + 11 \times 12 = 181$.

12. There are 2021 balls in a crate. The balls are numbered from 1 to 2021. Erica works out the digit sum for each ball. For example, the digit sum of 2021 is 5, since $2 + 0 + 2 + 1 = 5$.
Erica notes that balls with equal digit sums have the same colour and balls with different digit sums have different colours.
How many different colours of balls are there in the crate?

SOLUTION **028**

The largest possible digit sum is that of 1999 which is 28. The smallest is that of 1, which is 1. Each of the 9s in 1999 can be replaced by any of 0, 1, . . ., 8. So all digit sums between 1 and 28 can be achieved. Therefore there are 28 different digit sums and colours.

13. A multiplication table of the numbers 1 to 10 is shown.
What is the sum of all the odd products in the complete table?

×	1	2	3	⋯	10
1	1	2	3	⋯	10
2	2	4	6	⋯	20
⋮	⋮				⋮
10	10	20	30	⋯	100

SOLUTION **625**

The odd numbers in the table are all the products of two of the odd numbers 1, 3, 5, 7, 9.
So their sum is $(1 + 3 + 5 + 7 + 9)(1 + 3 + 5 + 7 + 9) = 25^2 = 625$.

14. The graph of $(x^2 + y^2 - 2x)^2 = 2(x^2 + y^2)^2$ meets the x-axis in p different places and meets the y-axis in q different places.
What is the value of $100p + 100q$?

SOLUTION **400**

The graph of $(x^2 + y^2 - 2x)^2 = 2(x^2 + y^2)^2$ meets the x-axis when $y = 0$.
i.e. $(x^2 - 2x)^2 = 2(x^2)^2$
i.e. $x^4 - 4x^3 + 4x^2 = 2x^4$
i.e. $0 = x^4 + 4x^3 - 4x^2$
i.e. $x = 0$ or $0 = x^2 + 4x - 4$
i.e. $x = 0$ or $x = -2 \pm 2\sqrt{2}$.
Therefore there are three points where the graph meets the x-axis, so $p = 3$.

The graph of $(x^2 + y^2 - 2x)^2 = 2(x^2 + y^2)^2$ meets the y-axis when $x = 0$,
i.e. $(y^2)^2 = 2(y^2)^2$
i.e. $y^4 = 2y^4$, with only one solution of $y = 0$.
Therefore there is only one point where the graph meets the y-axis, so $q = 1$.

The value of $100p + 100q$ is $100 \times 3 + 100 \times 1 = 400$.

15. Which is the lowest numbered statement which is true?
Statement 201: "Statement 203 is true".
Statement 202: "Statement 201 is true".
Statement 203: "Statement 206 is false".
Statement 204: "Statement 202 is false".
Statement 205: "None of the statements 201, 202, 203 or 204 are true".
Statement 206: "1 + 1 = 2".

SOLUTION **204**

Statement 206 is clearly true, so Statement 203 must be false. In turn Statement 201 must be false and therefore Statement 202 must also be false.
Statement 204 is therefore true. Therefore, Statement 204 is the lowest numbered true statement.

16. A polygon is said to be *friendly* if it is regular and it also has angles that when measured in degrees are either integers or *half-integers* (i.e. have a decimal part of exactly 0.5). How many different friendly polygons are there?

SOLUTION **028**

The interior angle will be an integer or half-integer precisely when the exterior angle is an integer or a half-integer respectively. A regular polygon of n sides has exterior angle $\frac{360}{n}$. This is either an integer or a half-integer if, and only if, $\frac{720}{n}$ is an integer. Therefore, n must be a factor of 720. The factors of 720 are 720, 360, 240, 180, 144, 120, 90, 80, 72, 60, 48, 45, 40, 36, 30, 24, 20, 18, 16, 15, 12, 10, 9, 8, 6, 5, 4 and 3 (ignoring 2 and 1 neither of which is a valid number of sides for a polygon). There are 28 numbers in this list, so there are 28 different friendly polygons.

17. Find the value of R, given that the numbers Q and R are defined as:

$Q = 202^1 + 20^{21} + 2^{021}$;
R is the remainder when Q is divided by 1000.

SOLUTION **354**

The term 20^{21} is divisible by 10^{21} and therefore is divisible by 1000, meaning it will not contribute to the remainder on division by 1000.
The term $2^{021} = 2^{21} = 2^{10} \times 2^{10} \times 2 = 1024 \times 1024 \times 2 = (1000 + 24) \times (1000 + 24) \times 2$.
When multiplied out, all terms involving a 1000 can be ignored.
So this contributes $24 \times 24 \times 2 = 1152$, or equivalently 152, to the remainder.
Hence the total remainder is $R = 202^1 + 152 = 354$.

18. Next year, 2022, has the property that it may be written using at most two different digits, namely 2 and 0. How many such years will there be between 1 and 9999 inclusive?

SOLUTION **927**

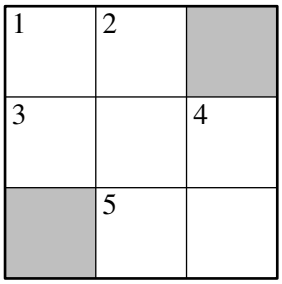
All of the 99 years 1 to 99 may be written using at most two different digits.
Three-digit years (i.e. those between 100 and 999 inclusive) can be written using only one different digit if they are of the form 'aaa', for example 111 or 222. There are 9 such years.
Three-digit years can be written using exactly two different digits a and b if they have any of the three following forms: 'aab', 'aba' or 'abb', where a,b are distinct digits and $a \neq 0$. There are 9 possibilities for a and for each there are then 9 possibilities for b. Therefore there are $9 \times 9 \times 3 = 243$ such years.
Four-digit years (i.e. those between 1000 and 9999 inclusive) can be written using only one different digit if they are of the form 'aaaa', for example 3333 or 4444. There are 9 such years.
Four-digit years can be written using exactly two different digits a and b if they have any of the seven following forms: 'aaab', 'aaba', 'abaa', 'aabb', 'abab', 'abba' or 'abbb', where a,b are distinct digits and $a \neq 0$. There are 9 possibilities for a and for each there are then 9 possibilities for b. Therefore there are $9 \times 9 \times 7 = 567$ such years.
Therefore, the total number of such years is $99 + 9 + 243 + 9 + 567 = 927$.

19. The function $f(x)$ is defined as $f(x) = \frac{x-1}{x+1}$.
The equation $f(x^2) \times f(x) = 0.72$ has two solutions a and b, where $a > b$.
What is the value of $19a + 7b$?

SOLUTION **134**

We are required to solve $\frac{x^2-1}{x^2+1} \times \frac{x-1}{x+1} = \frac{18}{25}$, which simplifies to $\frac{(x-1)(x-1)}{x^2+1} = \frac{18}{25}$.
This leads to the quadratic equation $7x^2 - 50x + 7 = 0$ which factorises as $(7x - 1)(x - 7) = 0$.
Therefore, $a = 7$ and $b = \frac{1}{7}$ and $19a + 7b = 19 \times 7 + 7 \times \frac{1}{7} = 133 + 1 = 134$.

20. Each cell in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are satisfied. The digits used are not necessarily distinct. What is the answer to 2 DOWN?



ACROSS

1. A prime which is the sum of two squares
3. Twice the answer to 2 DOWN

DOWN

1. $p \times q$, where p, q are prime and $q = p + 4$
4. 60% of 5 ACROSS

SOLUTION **397**

The only two-digit pq with $q = p + 4$ are $77 = 7 \times 11$ and $21 = 3 \times 7$. The latter of these is not possible, since 3 ACROSS cannot begin with a 1 as it is twice a three-digit number.
So the answer to 1 DOWN is 77.
We now know the first digit of 1 ACROSS is 7, meaning that as primes 71, 73 and 79 are possible. Of these, the only one that can be expressed as the sum of two squares is $73 = 64 + 9$.
The clue for 4 DOWN tells us that $5\times$ 4 DOWN = $3\times$ 5 ACROSS. Hence 5 ACROSS is a multiple of 5 and 4 DOWN is a multiple of 3. The first of these shows that the bottom right entry is 5 (since 0 is not allowed); and the second then shows that 4 DOWN is a multiple of 15.
By the clue for 3 ACROSS, the first digit of 4 DOWN is even. It cannot be 0, 2, 6, or 8 because 4 DOWN is a multiple of 3. Therefore 4 DOWN = 45 and 5 ACROSS = 75.
Finally, given the first digits of 2 DOWN and 3 ACROSS and the clue for 3 ACROSS, we see that the middle digit must be at least 5 and we then can check quickly that 9 is the only possible value.
Hence 397 is the solution.

United Kingdom
Mathematics Trust

ANDREW JOBBINGS SENIOR KANGAROO

Wednesday 16 November 2022

a member of the Association Kangourou sans Frontières

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England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

INSTRUCTIONS

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **60 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil** to record your answer to each problem as a three-digit number from 000 to 999.
   Pay close attention to the example on the Answer Sheet that shows how to code your answers.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:**
   5 marks are awarded for each correct answer;
   There is no penalty for giving an incorrect answer.
7. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

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challenges@ukmt.org.uk

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1. The number x satisfies $x^2 - 4x + 2 = 0$. What is the value of $x + \frac{2}{x}$?

2. The sum of the ages of Tom and John is 23, the sum of the ages of John and Alex is 24 and the sum of the ages of Tom and Alex is 25. What is the sum of the squares of their three ages?

3. Zia conducted a study of the number of coins each member of her club had in their pocket. To her surprise, nobody had zero coins and exactly seven people had only one coin each. Everybody else had at least two coins. She drew a pie chart of her results, and to her surprise found each sector had an angle which was an integer ten degrees larger than its immediate predecessor. The smallest sector in the pie chart had an angle of 35^o. How many people are in Zia's club?

4. The rectangles S_1 and S_2 are drawn on a set of coordinate axes. S_1 has vertices at $(0, y)$; (x, y); $(x, 4)$; and $(0, 4)$, where $x < 9$ and $y < 4$. S_2 has vertices at $(x, 0)$; $(9, 0)$; $(9, y)$; and (x, y). The rectangles are equal in area. What is the value of $\frac{360x}{y}$?

5. Paul is painting a wall. He knows the area of the wall is 1920 square metres, correct to the nearest ten. He uses tins of paint, each of which can cover 18 square metres, correct to the nearest integer.
He needs to paint the wall completely, and still have at least half a tin of paint left over for any minor repairs.
What is the smallest number of tins he must buy to be sure of having enough paint?

6. A cube is dissected into 6 pyramids by connecting a given point in the interior of the cube with each vertex of the cube, so that each face of the cube forms the base of a pyramid. The volumes of five of these pyramids are 200, 500, 1000, 1100 and 1400. What is the volume of the sixth pyramid?

7. The positive integer N has exactly six distinct (positive) factors including 1 and N. The product of five of these is 6075. What is the value of the sixth factor?

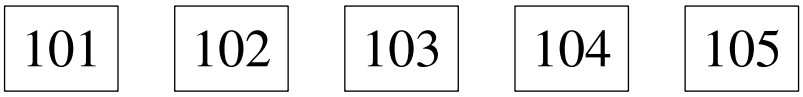
8. In a right-angled triangle ABC (with right angle at A) the bisectors of the acute angles intersect at point P. The distance from P to the hypotenuse is $\sqrt{80000}$. What is the distance from P to A?

9. Using the 24-hour clock, the time 16:11:22 on the date 16/11/22 (i.e. the 16th of November 2022) has hours, minutes and seconds equal to date, month and (two-digit) year respectively. Let S be the number of seconds which will elapse after that date and time until this phenomenon next occurs.
What is the value of $\sqrt{S}$?

10. The equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ both have real roots. It is known that the sum of the squares of the roots of the first equation is equal to the sum of the squares of the roots of the second one, and that $a \neq b$. Find the value of $500 + 10(a + b)$.

11. A quadrilateral contains an inscribed circle (i.e. a circle tangent to the four sides of the quadrilateral). The ratio of the perimeter of the quadrilateral to that of the circle is $4 : 3$. The ratio of the area of the quadrilateral to that of the circle, expressed in its lowest terms, is $a : b$. What is the value of $100a + 49b$?

12. Five cards have the numbers 101, 102, 103, 104 and 105 on their fronts.

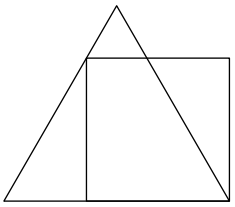


On the reverse, each card has one of five different positive integers: a, b, c, d and e respectively. We know that $a + 2 = b - 2 = 2c = \frac{d}{2} = e^2$.
Gina picks up the card which has the largest integer on its reverse. What number is on the front of Gina's card?

13. A sequence is given by $x_1 = 2$, $x_{n+1} = \frac{{x_n}^{x_n}}{32^{n-1}}$. What is $\frac{x_4}{128}$?

14. A class consists of pupils who each wear either a red tie or a green tie. There are two more pupils wearing green ties than wearing red ties. Two pupils are chosen at random. The probability of selecting two pupils of different tie-colour is exactly treble the probability of selecting two pupils who both wear red ties. Given that R is the number of pupils wearing a red tie, and that $R > 0$, determine $R^3 + R^2 + R$.

15. The perimeter of the square in the figure is 40. The perimeter of the larger equilateral triangle in the figure is $a + b\sqrt{p}$, where p is a prime number. What is the value of $7a + 5b + 3p$?



16. On the island of Friends and Foes, every citizen is either a Friend (who always tells the truth) or a Foe (who always lies). Seven citizens are sitting in a circle. Each declares "I am sitting between two Foes". How many Friends are there in the circle?

17. What is the smallest three-digit positive integer which can be written in the form pq^2r, where p, q and r are distinct primes?

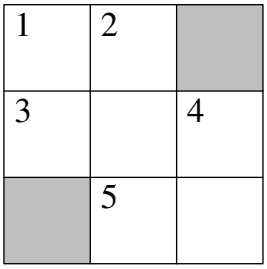
18. The number $\dfrac{20! \times 22!}{16! \times 11!}$ has N prime factors, which are not necessarily distinct.

What is the value of $N(N - 2)$?

19. How many different real number solutions are there to the following equation?

$$\left(x^2 - 8x + 15\right)^{\left(x^5 - 5x^3 + 4x\right)} = 1$$

20. Each cell in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are satisfied. The digits used are not necessarily distinct.
Determine T, the sum of all six answers to the clues. The answer to this question is $0.5 \times T$.



ACROSS:
1. Two less than a prime.
3. Exactly 100 more than its largest proper factor.
5. A multiple of 13.

DOWN:
1: A fourth power.
2: A cube.
4: Not prime, not square, not even.

United Kingdom
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Andrew Jobbings Senior Kangaroo

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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challenges@ukmt.org.uk

1. The number x satisfies $x^2 - 4x + 2 = 0$. What is the value of $x + \frac{2}{x}$?

SOLUTION **004**

As $x \neq 0$ we may divide $x^2 - 4x + 2 = 0$ by x to give $x - 4 + \frac{2}{x} = 0$, and then $x + \frac{2}{x} = 4$.

2. The sum of the ages of Tom and John is 23, the sum of the ages of John and Alex is 24 and the sum of the ages of Tom and Alex is 25. What is the sum of the squares of their three ages?

SOLUTION **434**

Let the ages of Tom, John and Alex be T, J and A respectively. We have $T + J = 23$ and $T + A = 25$. Subtracting these equations we have $A - J = 2$. Solving this, and $J + A = 24$ simultaneously, we have $J = 11$, $A = 13$ and hence $T = 12$. Therefore, the sums of the squares of their ages is $11^2 + 12^2 + 13^2 = 434$.

3. Zia conducted a study of the number of coins each member of her club had in their pocket. To her surprise, nobody had zero coins and exactly seven people had only one coin each. Everybody else had at least two coins. She drew a pie chart of her results, and to her surprise found each sector had an angle which was an integer ten degrees larger than its immediate predecessor. The smallest sector in the pie chart had an angle of 35^o. How many people are in Zia's club?

SOLUTION **72**

The sector angles, measured in degrees, are 35, 45, 55, 65, 75, and 85, which sum to 360.
Since the smallest sector represents a frequency of seven, we know that five degrees corresponds to one member of the club.
Therefore, the number of people in Zia's club is $360 \div 5 = 72$.

4. The rectangles S_1 and S_2 are drawn on a set of coordinate axes. S_1 has vertices at $(0, y)$; (x, y); $(x, 4)$; and $(0, 4)$, where $x < 9$ and $y < 4$. S_2 has vertices at $(x, 0)$; $(9, 0)$; $(9, y)$; and (x, y). The rectangles are equal in area. What is the value of $\frac{360x}{y}$?

SOLUTION **810**

Rectangles S_1 and S_2 have areas $x(4 - y)$ and $y(9 - x)$ respectively.
Equating these, $4x - xy = 9y - xy$.
Therefore, $4x = 9y$, which yields $360x = 810y$ giving the result of $\frac{360x}{y} = 810$.

5. Paul is painting a wall. He knows the area of the wall is 1920 square metres, correct to the nearest ten. He uses tins of paint, each of which can cover 18 square metres, correct to the nearest integer.
He needs to paint the wall completely, and still have at least half a tin of paint left over for any minor repairs.
What is the smallest number of tins he must buy to be sure of having enough paint?

SOLUTION **111**

The number of tins required to certainly cover the wall will be the maximum possible size of the wall divided by the minimum possible amount of paint per tin, i.e. $\frac{1925}{17.5} = \frac{19250}{175} = \frac{770}{7} = 110$ (exactly). Because the calculation uses an exact number of tins, and as he requires a surplus of half a tin for any touch-ups he should buy one additional tin. Therefore, he should buy 111 tins of paint.

6. A cube is dissected into 6 pyramids by connecting a given point in the interior of the cube with each vertex of the cube, so that each face of the cube forms the base of a pyramid. The volumes of five of these pyramids are 200, 500, 1000, 1100 and 1400. What is the volume of the sixth pyramid?

SOLUTION **600**

Each of the cube's faces forms the base of one of the six pyramids. Any two pyramids formed from opposite faces will have a combined volume of $\frac{1}{3} \times$ area of cube face $\times$ distance between opposite faces $= \frac{1}{3} \times s^3$, where s is the cube's side-length.
Pairing the known pyramid volumes we see that $200 + 1400 = 500 + 1100 = 1600$. The missing pyramid therefore has volume $1600 - 1000 = 600$.

7. The positive integer N has exactly six distinct (positive) factors including 1 and N. The product of five of these is 6075. What is the value of the sixth factor?

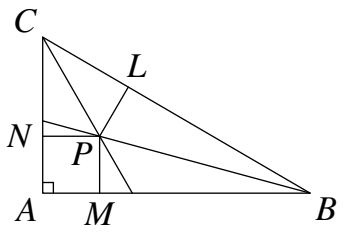
SOLUTION **015**

We determine that $6075 = 3^5 \times 5^2$. This means that 1, 3, 5 and 15 must be factors of N.
Since 3^5 is a factor of 6075, 9 is also a factor of N and so is $9 \times 5 = 45$.
$1 \times 3 \times 5 \times 9 \times 15 \times 45 = 3^6 \times 5^3$. Therefore, the missing factor is $3^6 \times 5^3 \div \left(3^5 \times 5^2\right) = 15$.

8. In a right-angled triangle ABC (with right angle at A) the bisectors of the acute angles intersect at point P. The distance from P to the hypotenuse is $\sqrt{80000}$. What is the distance from P to A?

SOLUTION **400**

Drop the perpendiculars from P to AB, meeting at M; and from P to AC, meeting at N; and from P to BC, meeting at L. Triangles BLP and BMP are congruent, since both have a right angle and one-half of the bisected angle at A, and share edge AP.



Therefore PM is $\sqrt{80000}$ in length. A similar argument is used to prove PN is also $\sqrt{80000}$. The distance AP is then found by Pythagoras' equation in triangle APM: $AP = \sqrt{80000 + 80000} = \sqrt{160000} = 400$.

9. Using the 24-hour clock, the time 16:11:22 on the date 16/11/22 (i.e. the 16th of November 2022) has hours, minutes and seconds equal to date, month and (two-digit) year respectively. Let S be the number of seconds which will elapse after that date and time until this phenomenon next occurs.
What is the value of $\sqrt{S}$?

SOLUTION **300**

The next such time and date will be 17:11:22 on 17/11/22. This is one day and one hour later.
The number of seconds which will elapse is $25 \times 60 \times 60 = 90000$.
Therefore, $\sqrt{S} = 5 \times 60 = 300$.

10. The equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ both have real roots. It is known that the sum of the squares of the roots of the first equation is equal to the sum of the squares of the roots of the second one, and that $a \neq b$. Find the value of $500 + 10(a + b)$.

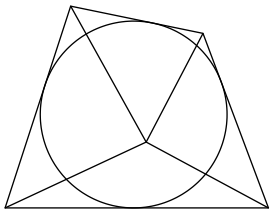
SOLUTION **480**

Let p, q be the roots of the first equation, and r, s the roots of the second.
Then $(x - p)(x - q) = x^2 + ax + b$. After expanding the brackets and comparing coefficients we have $a = -p - q$ and $b = pq$.
A similar analysis of the second equation yields $b = -r - s$ and $a = rs$.
Now, $(-p - q)^2 = p^2 + q^2 + 2pq = p^2 + q^2 + 2b$.
Similarly, $(-r - s)^2 = r^2 + s^2 + 2rs = r^2 + s^2 + 2a$.
Therefore, $(-p - q)^2 - 2b = (-r - s)^2 - 2a$ (since $p^2 + q^2 = r^2 + s^2$).
Therefore, $a^2 - 2b = b^2 - 2a$ and hence $a^2 - b^2 = 2b - 2a = 2(b - a)$.
Therefore $(a - b)(a + b) = -2(a - b)$ and hence $a + b = -2$, since we know that $a - b \neq 0$.
We conclude that $500 + 10(a + b) = 500 - 20 = 480$.

11. A quadrilateral contains an inscribed circle (i.e. a circle tangent to the four sides of the quadrilateral). The ratio of the perimeter of the quadrilateral to that of the circle is $4 : 3$. The ratio of the area of the quadrilateral to that of the circle, expressed in its lowest terms, is $a : b$. What is the value of $100a + 49b$?

SOLUTION **547**

Let the radius of the circle be 3. Then its circumference is 6π and the quadrilateral's perimeter is 8π. The circle's area is 9π. We can determine the quadrilateral's area by dividing it into four triangles, as shown in the example. Each triangle has area $\frac{1}{2} \times 3 \times s$, where s is the side shared by the triangle and the quadrilateral.



By summing the triangles' areas we find the quadrilateral's area, as $\frac{1}{2} \times 3\times$ (the quadrilateral's perimeter), this is $\frac{1}{2} \times 3 \times 8\pi = 12\pi$. The required ratio of areas is $12\pi : 9\pi = 4 : 3$. Therefore $a = 4$, $b = 3$, and $100a + 49b = 547$.

12. Five cards have the numbers 101, 102, 103, 104 and 105 on their fronts.

101	102	103	104	105

On the reverse, each card has one of five different positive integers: a, b, c, d and e respectively. We know that $a + 2 = b - 2 = 2c = \frac{d}{2} = e^2$.
Gina picks up the card which has the largest integer on its reverse. What number is on the front of Gina's card?

SOLUTION **104**

We know that $d = 4c$, $d = 2a + 4$, and $d = 2e^2$.
So we can quickly see that $d > c$, $d > a$ and $d > e$.
Also, $d = 2b - 4 > b$ for $b > 4$. We can eliminate the $b \leq 4$ case, for b cannot be odd (for then c would not be an integer); b cannot be 4 (for then a would be zero); and b cannot be 2 (for then c would be zero). Therefore, $d > b$.
Therefore d is the largest of a, b, c, d and e, so Gina has 104 on the front of her card.

13. A sequence is given by $x_1 = 2$, $x_{n+1} = \frac{{x_n}^{x_n}}{32^{n-1}}$. What is $\frac{x_4}{128}$?

SOLUTION **128**

We determine each term in sequence.
$x_2 = \frac{2^2}{32^0} = \frac{4}{1} = 4$; $x_3 = \frac{4^4}{32^1} = \frac{2^8}{2^5} = 2^3 = 8$; $x_4 = \frac{8^8}{32^2} = \frac{2^{24}}{2^{10}} = 2^{14}$.

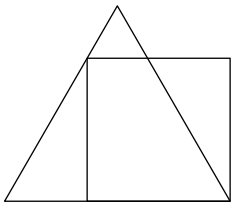
Therefore, $\frac{x_4}{128} = 2^{14} \div 2^7 = 2^7 = 128$.

14. A class consists of pupils who each wear either a red tie or a green tie. There are two more pupils wearing green ties than wearing red ties. Two pupils are chosen at random. The probability of selecting two pupils of different tie-colour is exactly treble the probability of selecting two pupils who both wear red ties. Given that R is the number of pupils wearing a red tie, and that $R > 0$, determine $R^3 + R^2 + R$.

SOLUTION **399**

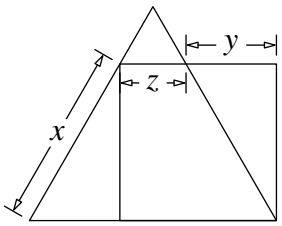
The number of pupils wearing a green tie is $R + 2$, and the total number of pupils in the class is $R + R + 2 = 2R + 2$. Let the probability of selecting two red-tie wearing pupils be q.
Then $\frac{R}{2R+2} \times \frac{R-1}{2R+1} = q$.
Also, $\frac{R}{2R+2} \times \frac{R+2}{2R+1} + \frac{R+2}{2R+2} \times \frac{R}{2R+1} = 3q$.
Eliminating q we have $3\left(\frac{R}{2R+2} \times \frac{R-1}{2R+1}\right) = \frac{R}{2R+2} \times \frac{R+2}{2R+1} + \frac{R+2}{2R+2} \times \frac{R}{2R+1}$.
Therefore, $3R(R-1) = 2R(R+2)$ and hence $3R^2 - 3R = 2R^2 + 4R$.
This quadratic equation simplifies to $R^2 - 7R = 0$ and has solutions $R = 0$ and $R = 7$.
We are given that $R > 0$, so $R = 7$ and $R^3 + R^2 + R = 343 + 49 + 7 = 399$.

15. The perimeter of the square in the figure is 40. The perimeter of the larger equilateral triangle in the figure is $a + b\sqrt{p}$, where p is a prime number. What is the value of $7a + 5b + 3p$?



SOLUTION **269**

Define lengths x, y and z as shown in the diagram, so that the larger equilateral triangle has side length $x + z$.



We have $x = \frac{10}{\sin 60} = \frac{20}{\sqrt{3}}$.
Also, $y = \frac{10}{\tan 60} = \frac{10}{\sqrt{3}}$ and hence $z = 10 - \frac{10}{\sqrt{3}}$.
The perimeter of the larger equilateral triangle is

$$3(x+z) = 3\left(\frac{20}{\sqrt{3}} + 10 - \frac{10}{\sqrt{3}}\right) = 3\left(\frac{10}{\sqrt{3}} + 10\right) = 30 + 10\sqrt{3}.$$

Therefore, $a = 30$, $b = 10$ and $c = 3$ and so $7a + 5b + 3p = 210 + 50 + 9 = 269$.

16. On the island of Friends and Foes, every citizen is either a Friend (who always tells the truth) or a Foe (who always lies). Seven citizens are sitting in a circle. Each declares "I am sitting between two Foes". How many Friends are there in the circle?

SOLUTION **003**

If every citizen is a foe then all are telling the truth, which is a contradiction. Therefore, there is at least one friend. Suppose the citizens are labelled ABCDEFG and sat in the same order, with A (a friend) sat next to B and G. Then B and G are both foes. We now consider C and F.
If C is a foe, then D must be a friend (else C would speak the truth). This makes E a foe, and F must be a friend since F is sitting between two foes, so speaks the truth. In this scenario there are three friends, A, D and F.
If F is a foe, then E must be a friend (else F would speak the truth). This makes D a foe, and C must be a friend since C is sitting between two foes, so speaks the truth. In this scenario there are three friends, A, C and E.
If C and F are both friends then D and E must both be foes. In this scenario there are three friends, A, C and F.
Therefore there are three friends.

17. What is the smallest three-digit positive integer which can be written in the form pq^2r, where p, q and r are distinct primes?

SOLUTION **126**

We consider multiples of q^2, for increasing values of q.
For $q = 2$, we consider numbers of the form $4k$, where k is odd (since neither p nor r can then be 2).
$100 = 2^2 \times 5^2$; $108 = 2^2 \times 3^3$; $116 = 2^2 \times 29$; $124 = 2^2 \times 31$; $132 = 2^2 \times 3 \times 11$, so 132 is the first multiple of 2^2 of the required form.
For $q = 3$ we consider multiples of 9.
$108 = 2^2 \times 3^3$; $117 = 3^2 \times 13$; $126 = 2 \times 3^2 \times 7$, so 126 is the first multiple of 3^2 of the required form.
For $q = 5$ we consider multiples of 25.
$100 = 2^2 \times 5^2$; $125 = 5^3$; and subsequent multiples are larger than 126.
For $q = 7$ we consider multiples of 49. However, the smallest three-digit multiple, 147, is already larger than 126.
For $q = 11$ we consider multiples of 121. However, $121 = 11^2$ and subsequent multiples are larger than 126.
For all other possible values of q, q^2 is larger than 126. Therefore, 126 is the smallest such number.

18. The number $\dfrac{20! \times 22!}{16! \times 11!}$ has N prime factors, which are not necessarily distinct.

What is the value of $N(N-2)$?

SOLUTION **960**

We can rewrite the fraction as $\frac{20!}{11!} \times \frac{22!}{16!}$.
By writing each factorial out as a product, and cancelling like factors top and bottom of each fraction, this simplifies to
$20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 22 \times 21 \times\ 20 \times\ 19 \times 18 \times 17$.
We can count how many non-distinct prime factors each number in the above product has, and find the sum.
Taking the numbers in turn, $N = 3+1+3+1+4+2+2+1+3+2+2+3+1+3+1 = 32$.
Therefore, $N(N-2) = 32 \times 30 = 960$.

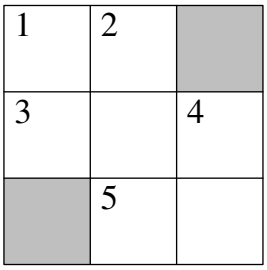
19. How many different real number solutions are there to the following equation?

$$\left(x^2 - 8x + 15\right)^{\left(x^5 - 5x^3 + 4x\right)} = 1$$

SOLUTION **008**

Some solutions occur when $x^5 - 5x^3 + 4x = 0$, provided $x^2 - 8x + 15 \neq 0$ simultaneously. Therefore we consider $x(x^4 - 5x^2 + 4) = 0$, i.e. $x(x^2-1)(x^2-4) = 0$ i.e. $x(x-1)(x+1)(x-2)(x+2) = 0$.
The solutions to this are -2, -1, 0, 1 and 2, none of which makes $x^2 - 8x + 15 = 0$, so all five of these are solutions.
Some solutions occur when $x^2 - 8x + 15 = 1$. This has two distinct solutions $x = 4 \pm \sqrt{2}$.
Some solutions may occur when $x^2 - 8x + 15 = -1$. This is equivalent to $x^2 - 8x + 16 = 0$ which has exactly one solution, namely $x = 4$. This will only solve the original equation if the index $x^5 - 5x^3 + 4x$ is even when $x = 4$. At $x = 4$, the index has a value of $4^5 - 5 \times 4^3 + 4 \times 4 = 720$, which is even. Therefore $x = 4$ is a solution.
Therefore, there are $5 + 2 + 1 = 8$ different solutions.

20. Each cell in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are satisfied. The digits used are not necessarily distinct. Determine T, the sum of all six answers to the clues. The answer to this question is $0.5 \times T$.



ACROSS:
1. Two less than a prime.
3. Exactly 100 more than its largest proper factor.
5. A multiple of 13.

DOWN:
1: A fourth power.
2: A cube.
4: Not prime, not square, not even.

SOLUTION **582**

1 DOWN is either 16 or 81. If 3 ACROSS begins with a six then its largest proper factor is at most 349, which violates the clue. Therefore 3 ACROSS begins with a one, and 1 DOWN must be 81.
1 ACROSS is either 81 or 87. If 2 DOWN begins with a one then it is 125 and 5 ACROSS must be 52. But this would mean 4 DOWN ended with a two, which violates its clue. Therefore 1 ACROSS must be 87, in turn meaning 2 DOWN is 729 and 5 ACROSS is 91.
4 DOWN ends in a one so can only be 21, 51 or 91. This means 3 ACROSS has possible answers of 122, 125 and 129. Of these, only 125 satisfies the clue.
The total number of all the answers in the grid is $87 + 125 + 91 + 81 + 729 + 51 = 1164$. The required quantity is $1164 \div 2 = 582$.

UK Maths Trust

Andrew Jobbings Senior Kangaroo

Wednesday 15 November 2023

a member of the Association Kangourou sans Frontières

supported by Overleaf

England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

Instructions

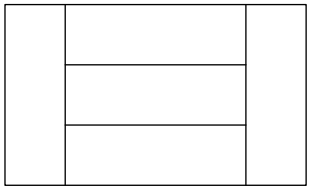
1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **60 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil** to record your answer to each problem as a three-digit number from 000 to 999.
   Pay close attention to the example on the Answer Sheet that shows how to code your answers.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:**
   5 marks are awarded for each correct answer;
   There is no penalty for giving an incorrect answer.
7. **Your Answer Sheet will be read by a machine.** Do not write or doodle on the sheet except to mark your chosen options. The machine will read all black pencil markings even ifthey are in the wrong places. If you mark the sheet in the wrong place, doodle, or leave bits of eraser stuck to the page, the machine will interpret the mark in its own way, or reject the answer sheet.
8. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

Enquiries about the Andrew Jobbings Senior Kangaroo should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. A particular flag is in the shape of a rectangle divided into five smaller congruent rectangles as shown. When written in its lowest terms, the ratio of the side lengths of the smaller rectangle is $\lambda : 1$, where $\lambda < 1$.
What is the value of 360λ?



2. The integers m and n satisfy the equation $3^m \times n = 7! + 8! + 9!$.
What is the smallest possible value for n?

3. How many of the integers from 2^{10} to 2^{18} inclusive are divisible by 2^9?

4. Months of the year are usually labelled numerically by '01' for January, '02' for February, and so on, through to '12' for December. Lydia notices that during January, the number of letters in the name of the month is greater than the month's numerical label (i.e. $7 > 1$).
For how many days during 2024 will the date have that property?

5. The number C is defined as the sum of all the positive integers n such that $n - 6$ is the second largest factor of n. What is the value of $11C$?

6. Five cards have the numbers 101, 102, 103, 104 and 105 on their fronts.

101	102	103	104	105

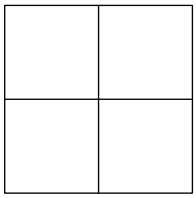
On the reverse, each card has a statement printed as follows:
101: The statement on card 102 is false
102: Exactly two of these cards have true statements
103: Four of these cards have false statements
104: The statement on card 101 is false
105: The statements on cards 102 and 104 are both false
What is the total of the numbers shown on the front of the cards with TRUE statements?

7. A container in the shape of a rectangular box is partially filled with $120\,\text{m}^3$ of water. The depth of the water is either 2 m, 3 m or 5 m, depending on which side of the box is on the ground.
What is the volume of the container in m^3?

8. The product of all the factors of 5^{15} is 5^P.
What is the value of P?

9. What is the lowest prime number that is thirteen more than a cube?

10. The smallest four two-digit primes are written in different squares of a 2×2 table.
The sums of the numbers in each row and column are calculated.
Two of these sums are 24 and 28.
The other two sums are c and d, where $c < d$.
What is the value of $5c + 7d$?



11. The town of Coppersville uses some special coins. Edward has five 5 p coins, four 4 p coins, three 3 p coins, two 2 p coins and one 1 p coin. The smallest amount he cannot pay with his coins alone is W p.
What is the value of W?

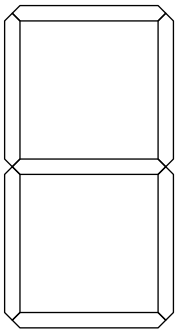
12. How many positive integers are there whose digits do not include 0, and whose digits have sum 6?

13. On a square $ABCD$ a line segment BE is drawn such that the point E lies on the side CD.
The perimeter of triangle BCE is three-quarters of the perimeter of the square $ABCD$.
The ratio of lengths $CE : CD$ is $\lambda : 1$.
What is the value of $960 \times \lambda$?

14. Three different numbers are chosen at random from the list 1, 3, 5, 7, 9, 11, 13, 15, 17, 19.
The probability that one of them is the mean of the other two is p.
What is the value of $\frac{120}{p}$?

15. Kiran is designing a game which involves a bag of twenty-one marbles. Some of the marbles are blue, the rest are red. To play the game, two marbles are drawn out. The game is won if at least one red marble is drawn. To ensure the probability of the game being won is exactly one-half, Kiran uses B blue marbles and R red marbles.
What is the value of $B^2 + R^2$?

16. Stephen's calculator displays only one digit, as shown in the diagram. Unfortunately, the calculator is broken. Each time he switches it on, each of the seven bars will either illuminate (show up) or not, with probability 0.5. The resultant display correctly shows one of the ten digits 0 - 9 with probability $\frac{a}{b}$.
Given that $\frac{a}{b}$ is written in its lowest terms, what is the value of $9a + 2b$?

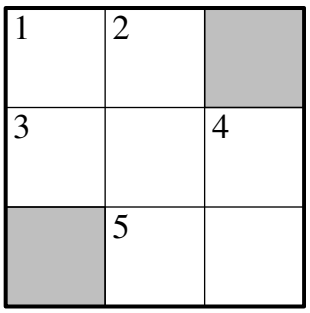


17. The integer part of $\sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72}}}}}}$ is A.
What is the value of $50A$?

18. The number 2024 may be split into its first two digits and its last two digits to form the numbers 20 and 24. The highest common factor of these numbers, HCF(20,24) is equal to 4. Similarly, 2025 may be split, and HCF(20,25)=5.
For how many remaining years this century (i.e. after 2025 and up to and including 2099) will this highest common factor be equal to one?

19. Next year's 'leap day', 29th February 2024, may be written in 'ddmmyy' format as the number 290224. What is the sum of all the two-digit factors of 290224?

20. Each cell in this cross-number can be filled with a non-zero digit so that all of the conditions in the clues are satisfied.



The digits used are not necessarily distinct.

ACROSS
1. Four less than a factor of 105.
3. One more than a palindrome.
5. The square-root of the answer to this Kangaroo question.

DOWN
1. Two less than a square.
2. Four hundred less than a cube.
4. Six less than the sum of the answers to two of the other clues.

What is the square of the answer to 5 ACROSS?

UK Maths Trust

Andrew Jobbings Senior Kangaroo

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. A particular flag is in the shape of a rectangle divided into five smaller congruent rectangles as shown. When written in its lowest terms, the ratio of the side lengths of the smaller rectangle is $\lambda : 1$, where $\lambda < 1$.
What is the value of 360λ?

SOLUTION **120**

Let a, b be the side lengths of the smaller rectangle, with $a < b$.
We observe that $b = 3a$, so $\lambda = \frac{1}{3}$.
The value of 360λ is $360 \times \frac{1}{3} = 120$.

2. The integers m and n satisfy the equation $3^m \times n = 7! + 8! + 9!$.
What is the smallest possible value for n?

SOLUTION **560**

$7! + 8! + 9! = 7!(1 + 8 + 8 \times 9) = 7! \times 81 =$
$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 3^4 = 1 \times 2 \times 3 \times 4 \times 5 \times 2 \times 3 \times 7 \times 3^4 = 1 \times 2 \times 4 \times 5 \times 2 \times 7 \times 3^6$.
The smallest value of n corresponds with $m = 6$, and is $1 \times 2 \times 4 \times 5 \times 2 \times 7 = 560$.

3. How many of the integers from 2^{10} to 2^{18} inclusive are divisible by 2^9?

SOLUTION **511**

We observe that $2^{10} = 2 \times 2^9$, and $2^{18} = 512 \times 2^9$. Therefore, on multiplying 2^9 by any integer between 2 and 512 inclusive, we obtain an integer between 2^{10} and 2^{18} inclusive. There are $512 - 1 = 511$ such integers.

4. Months of the year are usually labelled numerically by '01' for January, '02' for February, and so on, through to '12' for December. Lydia notices that during January, the number of letters in the name of the month is greater than the month's numerical label (i.e. $7 > 1$).
For how many days during 2024 will the date have that property?

SOLUTION **121**

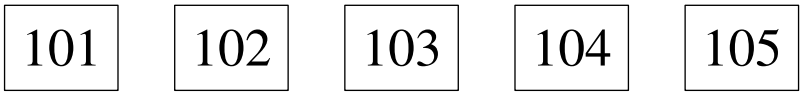
The only months that have the property described are January ('01'), February ('02'), March ('03'), and April ('04'). During 2024, these will have $31 + 29 + 31 + 30 = 121$ days in total.

5. The number C is defined as the sum of all the positive integers n such that $n - 6$ is the second largest factor of n. What is the value of $11C$?

SOLUTION **308**

Any integer is at most double its second largest factor. Therefore, we need only test the integers up to and including $n = 12$ (which is double $n - 6 = 6$) to see if each has $n - 6$ as its second largest factor. Of these, only 7, 9 and 12 do. These have total $C = 7 + 9 + 12 = 28$, so $11C = 11 \times 28 = 308$.

6. Five cards have the numbers 101, 102, 103, 104 and 105 on their fronts.



On the reverse, each card has a statement printed as follows:
101: The statement on card 102 is false
102: Exactly two of these cards have true statements
103: Four of these cards have false statements
104: The statement on card 101 is false
105: The statements on cards 102 and 104 are both false
What is the total of the numbers shown on the front of the cards with TRUE statements?

SOLUTION **206**

If card 101 is true then both card 102 is false and card 104 is false. So card 105 is true and in turn card 103 is false. This means we have exactly two true cards but that simultaneously card 102 is false. This is a contradiction.
So card 101 must be false. This means both card 104 is true and card 102 is true. This means card 105 is false and card 103 is false. This is two true cards.
The sum of the numbers on the two true cards is $102 + 104 = 206$.

7. A container in the shape of a rectangular box is partially filled with $120\,\text{m}^3$ of water. The depth of the water is either 2 m, 3 m or 5 m, depending on which side of the box is on the ground.
What is the volume of the container in m^3?

SOLUTION **240**

Let the dimensions of the box in metres be x, y, z, with $x < y < z$. We know $5xy = 3xz = 2yz = 120$. Therefore, $5xy \times 3xz \times 2yz = 120^3$, giving equation $30x^2y^2z^2 = 120^3$.
We deduce that $x^2y^2z^2 = \frac{120\times120\times120}{30} = 4 \times 120 \times 120$.
Therefore, the volume, in m^3, of the container is $xyz = 2 \times 120 = 240$.

8. The product of all the factors of 5^{15} is 5^P.
What is the value of P?

SOLUTION **120**

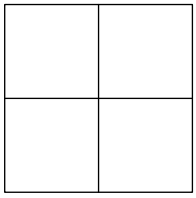
The factors of 5^{15} are $5^0, 5^1, 5^2, \cdots, 5^{15}$.
Their product is $5^{1+2+\cdots+15} = 5^{120}$.

9. What is the lowest prime number that is thirteen more than a cube?

SOLUTION **229**

We consider the cube numbers in sequence until we find one that is thirteen less than a prime.
$1 + 13 = 14$, $8 + 13 = 21$, $27 + 13 = 40$, $64 + 13 = 77$, $125 + 13 = 138$, none of which are prime.
$216 + 13 = 229$, which is prime, and therefore 229 is the answer.

10. The smallest four two-digit primes are written in different squares of a 2×2 table.
The sums of the numbers in each row and column are calculated.
Two of these sums are 24 and 28.
The other two sums are c and d, where $c < d$.
What is the value of $5c + 7d$?



SOLUTION **412**

The only way of obtaining sums of 24 and 28 are using $11 + 13 = 24$ and $11 + 17 = 28$. One possible configuration is shown in the diagram, with other two sums $c = 13 + 19 = 32$ and $d = 17 + 19 = 36$. The value of $5c + 7d$ is $5 \times 32 + 7 \times 36 = 412$.

11	13
17	19

11. The town of Coppersville uses some special coins. Edward has five 5 p coins, four 4 p coins, three 3 p coins, two 2 p coins and one 1 p coin. The smallest amount he cannot pay with his coins alone is W p.
What is the value of W?

SOLUTION **056**

Edward has $5 \times 5 + 4 \times 4 + 3 \times 3 + 2 \times 2 + 1 \times 1$ =55 p in total. He can pay amounts from 1 p to 5 p using the 1 p and 2 p coins only. He can pay amounts from 6 p to 14 p by adding one, two, or three 3 p coins to his payments for amounts 1 p to 5 p. He can pay amounts from 15 p to 30 p by adding one, two, three, or four 4 p coins to his payments for amounts 1 p to 14 p. He can pay amounts from 31 p to 55 p by adding one, two, three, four, or five 5 p coins to his payments for amounts 1 p to 30 p. Therefore, he can pay any amount up to and including 55 p, but cannot pay 56 p.

12. How many positive integers are there whose digits do not include 0, and whose digits have sum 6?

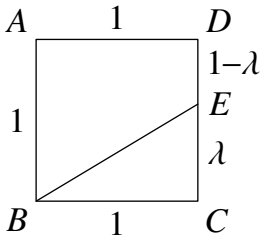
SOLUTION **032**

Integers with digit sum 6 may be grouped according to their digits:
111111;
21111, 12111, 11211, 11121, 11112;
3111, 1311, 1131, 1113;
411, 141, 114;
15, 51;
6;
2211, 2121, 2112, 1221, 1212, 1122;
222;
321, 312, 231, 213, 132, 123;
33;
42, 24.
There are 32 such integers.

13. On a square $ABCD$ a line segment BE is drawn such that the point E lies on the side CD. The perimeter of triangle BCE is three-quarters of the perimeter of the square $ABCD$. The ratio of lengths $CE : CD$ is $\lambda : 1$.
What is the value of $960 \times \lambda$?

SOLUTION **720**

Let the square's side have length 1. The triangle's hypotenuse has length $\sqrt{1+\lambda^2}$, so the triangle's perimeter is $1 + \lambda + \sqrt{1+\lambda^2} = 3$. Therefore, $\sqrt{1+\lambda^2} = 2 - \lambda$ from which we obtain the equation $1 + \lambda^2 = 4 - 4\lambda + \lambda^2$.
Solving this equation we find $\lambda = \frac{3}{4}$, so $960\lambda = 720$.



14. Three different numbers are chosen at random from the list 1, 3, 5, 7, 9, 11, 13, 15, 17, 19. The probability that one of them is the mean of the other two is p.
What is the value of $\frac{120}{p}$?

SOLUTION **720**

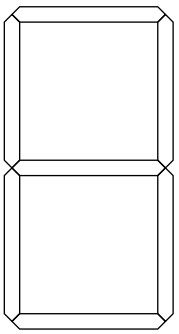
There are $\frac{10\times9\times8}{3\times2\times1} = 120$ equally likely choices of three numbers.
It is not possible for 1 or 19 to be the mean.
For 3 to be the mean there is only one possible selection, namely $1, 3, 5$.
Similarly 17 can only be the mean if $15, 17, 19$ is selected.
5 is the mean of both $1, 9$ and $3, 7$; while 15 is the mean of both $11, 19$ and $13, 17$.
7 is the mean of $1, 13$; $3, 11$; and $5, 9$; while 13 is the mean of $7, 19$; $9, 17$; and $11, 15$.
Finally, 9 is the mean of $1, 17$; $3, 15$; $5, 13$; and $7, 11$; while 11 is the mean of $3, 19$; $5, 17$; $7, 15$; and $9, 13$.
There are 20 such selections in total, so $p = \frac{20}{120} = \frac{1}{6}$. Therefore, $\frac{120}{p} = 120 \times 6 = 720$.

15. Kiran is designing a game which involves a bag of twenty-one marbles. Some of the marbles are blue, the rest are red. To play the game, two marbles are drawn out. The game is won if at least one red marble is drawn. To ensure the probability of the game being won is exactly one-half, Kiran uses B blue marbles and R red marbles.
What is the value of $B^2 + R^2$?

SOLUTION **261**

The probability of selecting blue then blue is 0.5.
Therefore, $\frac{B}{21} \times \frac{B-1}{20} = 0.5$.
This leads to the quadratic equation $B^2 - B - 210 = 0$.
This factorises as $(B - 15)(B + 14) = 0$ with $B = 15$ as the only positive solution.
This means $B = 15$ and $R = 6$, so $B^2 + R^2 = 225 + 36 = 261$.

16. Stephen's calculator displays only one digit, as shown in the diagram. Unfortunately, the calculator is broken. Each time he switches it on, each of the seven bars will either illuminate (show up) or not, with probability 0.5. The resultant display correctly shows one of the ten digits 0 - 9 with probability $\frac{a}{b}$.
Given that $\frac{a}{b}$ is written in its lowest terms, what is the value of $9a + 2b$?



SOLUTION **173**

As each bar illuminates with probability 0.5 then each of the 2^7 possible displays occur with equal likelihood. There are ten displays which correctly show a digit (namely, those which show the digits 0-9). The probability of obtaining one of these displays is $\frac{10}{2^7} = \frac{10}{128}$ which is $\frac{5}{64}$ in lowest terms. So $a = 5$ and $b = 64$. The value of $9a + 2b$ is $9 \times 5 + 2 \times 64 = 45 + 128 = 173$.

17. The integer part of $\sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72}}}}}}$ is A.
What is the value of $50A$?

SOLUTION **400**

The expression is clearly greater than $\sqrt{64}$, so $A \geq 8$.
By replacing the rightmost '72' in the expression with 81, we obtain the slightly larger expression

$$\sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{81}}}}}} = \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{81}}}}} =$$

$$\sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{81}}}} = \sqrt{72 + \sqrt{72 + \sqrt{81}}} = \sqrt{72 + \sqrt{81}} = \sqrt{81} = 9.$$

Therefore, $A < 9$. Since $8 \leq A < 9$ we know $A = 8$ and $50A = 400$.

18. The number 2024 may be split into its first two digits and its last two digits to form the numbers 20 and 24. The highest common factor of these numbers, HCF(20,24) is equal to 4. Similarly, 2025 may be split, and HCF(20,25)=5.
For how many remaining years this century (i.e. after 2025 and up to and including 2099) will this highest common factor be equal to one?

SOLUTION **030**

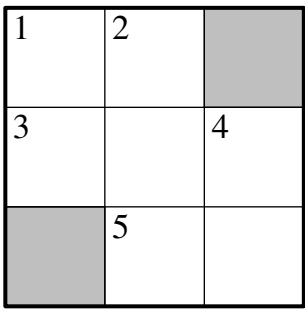
Since $20 = 2 \times 2 \times 5$ any year with last two digits 'YY' that form neither an even number nor a multiple of five will have $HCF(20, YY) = 1$.
That is, any year ending in 1, 3, 7, or 9.
This decade there are two such years remaining (namely 2027 and 2029), while the remaining seven decades of this century have four such years each.
Therefore, there are $2 + 7 \times 4 = 30$ such years.

19. Next year's 'leap day', 29th February 2024, may be written in 'ddmmyy' format as the number 290224. What is the sum of all the two-digit factors of 290224?

SOLUTION **397**

The prime factorisation of 290224 is $290224 = 2 \times 2 \times 2 \times 2 \times 11 \times 17 \times 97$.
Since the product of any two of 11, $16 = 2^4$, 17 and 97 will not be a two-digit number, we need only consider 16; as well as 11, 17, 97 and any multiples of each of these by the powers of two.
The only such two-digit multiples are 22, 44, 88; 34, 68.
The sum of these numbers is $11 + 16 + 17 + 97 + 22 + 44 + 88 + 34 + 68 = 397$.

20. Each cell in this cross-number can be filled with a non-zero digit so that all of the conditions in the clues are satisfied.



The digits used are not necessarily distinct.

ACROSS
1. Four less than a factor of 105.
3. One more than a palindrome.
5. The square-root of the answer to this Kangaroo question.

DOWN
1. Two less than a square.
2. Four hundred less than a cube.
4. Six less than the sum of the answers to two of the other clues.

What is the square of the answer to 5 ACROSS?

SOLUTION **841**

The two-digit factors of 105 are 15, 21 and 35, so 1 ACROSS is one of 11, 17 or 31.
However, as 2 DOWN is 400 less than a cube it cannot start with a 7, so 1 ACROSS is either 11 or 31.
If 1 ACROSS is 11 then 1 DOWN is 14, 2 DOWN is 112 and 3 ACROSS is 415. This means 4 DOWN is at least 50 which is too large to satisfy its clue, since 5 ACROSS is at most 29.
So 1 ACROSS is 31. This means 1 DOWN is 34, 2 DOWN is 112 and 3 ACROSS is 415.
Checking each of the digits 1 to 9 in turn for the units digit of 4 DOWN yields 9 as the only possibility, with $59 = 31 + 34 - 6$.
Therefore, 5 ACROSS is 29 and its square is 841.

UK Maths Trust

Andrew Jobbings Senior Kangaroo

Wednesday 20 November 2024

a member of the Association Kangourou sans Frontières

proudly sponsored by [XTX] MARKETS

England & Wales: Year 13 or below | *Scotland: S6 or below* | *Northern Ireland: Year 14 or below*

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **60 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil** to record your answer to each problem as a three-digit number from 000 to 999.
   Pay close attention to the instructions on the Answer Sheet that shows how to code your answers.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:**
   5 marks are awarded for each correct answer.
   There is no penalty for giving an incorrect answer.
7. **Your Answer Sheet will be read by a machine.** Do not write or doodle on the sheet except to mark your chosen options. The machine will read all black pencil markings even if they are in the wrong places. If you mark the sheet in the wrong place, doodle, or leave bits of eraser stuck to the page, the machine will interpret the mark in its own way, or reject the answer sheet.
8. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

Enquiries about the Andrew Jobbings Senior Kangaroo should be sent to:

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1. The seventh day of an unknown month in an unknown year is a Tuesday. The nth day of the following month is the first Tuesday of that month. What is the product of all the possible values of n?

2. Each of the following expressions has a different value. Which value is greatest?

$$2 - 0 \times 2 + 4 + 11 \times 20$$
$$2 + 0 \times 2 \times 4 + 11 \times 20$$
$$2 \times 0 + 2 \times 4 + 11 \times 20$$
$$2 \times (0 + 2 + 4) + 11 \times 20$$

3. Kevin begins with eleven stones and a mallet. Every time he hits a stone with the mallet it breaks into exactly eight pieces. Kevin keeps hitting stones with his mallet, but stops before he has 1000 stones. What is the largest number of stones he could end up with?

4. The number 35 can be written as the sum of four distinct primes in two different ways: $35 = 2 + 3 + 13 + 17$ and $35 = 2 + 3 + 11 + 19$.
What is the *smallest* number that can be written as the sum of four distinct primes in two different ways?

5. What is the value of $|\sqrt{123} - 456| + |\sqrt{123} + 456|$?
The modulus function, $|x|$ is given by $|x| = x$ for $x \geq 0$, and $|x| = -x$ for $x < 0$.

6. The equation $x^2 - 70x = 2024$ has solutions a, b with $b > a$. What is the value of $10b - a^2$?

7. The triangle ABC is such that $AB = 7$, $BC = 3$, and $CA = 5$.
Furthermore, we are given that $\angle ACB = 120°$.
The area of triangle ABC is M. What is the value of M^2, correct to the nearest integer?

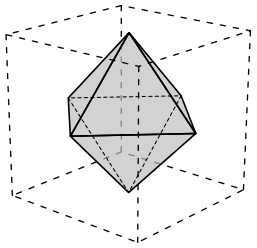
8. At a football match between Glasgow Belles and Doncaster City, five supporters each made a prediction:

1. The game will end in a draw;
2. Glasgow will score;
3. Glasgow will win;
4. Glasgow will not lose;
5. Exactly five goals will be scored.

What is the maximum number of these predictions that could simultaneously come true?

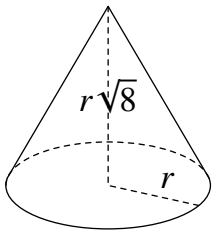
9. The number $N = 24^{2024} + 24^{2025}$ has some factors between 1 and 25.
What is the sum of all those integers between 1 and 25 which are *not* factors of N?

10. An octahedron is inscribed in a cube of side-length 12. The vertices of the octahedron are at the centre of the faces of the cube.
The surface area of the octahedron is $a \times \sqrt{b}$, where b is a prime number. What is the value of $a + 10b$?



11. The number $L = \frac{24!}{20!}$ is one less than a square, S^2.
What is the value of S?

12. A solid right circular cone whose base has radius r metres has total surface area (including the circular base) of A metres squared and a volume of V metres cubed.
The perpendicular height of the cone is $r\sqrt{8}$ metres.
Given that $\frac{V}{A} = \sqrt{5000}$, what is the value of r?



13. Wil goes on a run. Her pedometer is unfortunately rather basic, and records a distance of 40 km, to the nearest integer, and an average speed of 10 km/h, to the nearest integer.
Wil works out her greatest possible and least possible times, and expresses the difference of these times as a fraction in its lowest terms, $\frac{a}{b}$.
What is the value of $a + 2b$?

14. The function f has the property that $f(x + y) = f(x) \times f(y)$ for all values of x and y.
Given that $f(x) > 0$ for all values of x, and also that $f(24) = 4096$, what is $f(18)$?

15. Four friends named Ada, Brin, Chris and Debbie are each carrying a different amount of money. These amounts are £111, £222, £333 and £444.

Ada says: I am neither carrying the most nor the least.

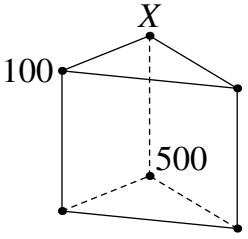
Brin says: I am not carrying the least.

Chris says: I am carrying the most.

Debbie says: I am carrying the least.

Exactly one of the friends is lying. How many letters does the lying friend have in her/his name?

16. The prism in the picture is formed of two triangles and three squares. The six vertices are assigned labels 100, 200, 300, 400, 500, and 600 in such a way that the sum of the four labels of the vertices of each square is the same for all three squares. Labels 100 and 500 are shown.
What number is at the vertex labelled X?



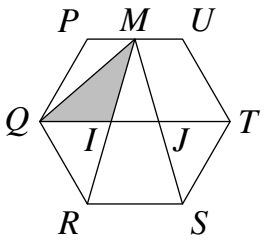
17. The parabola $y = x^2 - 10x + 24$ has two distinct x-intercepts and a y-intercept.
The circle which goes through these three intercepts meets the parabola at a fourth point, P.
What is the distance of P from the origin?

18. Zarah has a bag containing exactly four blue counters and exactly four red counters, and a large supply, outside the bag, of extra blue and red counters.
She may, on any given *move*, do one of four different things:

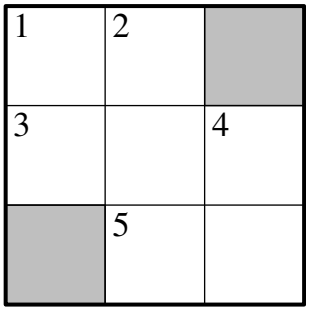
- Remove a blue counter from the bag;
- Remove a red counter from the bag;
- Add a blue counter to the bag;
- Add a red counter to the bag.

At each move, she selects one of those four options at random and with equal probability.
The probability that after four consecutive moves, Zarah's bag again contains exactly four blue counters and exactly four red counters is $\frac{V}{W}$ when written in its lowest terms.
What is the value of $10V + W$?

19. A regular hexagon $PQRSTU$ is shown. The midpoint of UP is M. Point I is the intersection between line segments RM and QT, and point J is the intersection between line segments SM and QT. The area of triangle RMS is 920. What is the area of triangle QMI?



20. Each cell in this cross-number can be filled with a non-zero digit so that all of the conditions in the clues are satisfied.
The digits used are not necessarily distinct.



ACROSS
1. A prime.
3. The answer to this Kangaroo question.
5. Shares a two-digit common factor with 1 DOWN.

DOWN
1. The sum of two different sixth powers.
2. 101 less than a square number.
4. The sum of all the digits in this crossnumber, once correctly completed.

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. The seventh day of an unknown month in an unknown year is a Tuesday. The nth day of the following month is the first Tuesday of that month. What is the product of all the possible values of n?

SOLUTION **840**

The seventh is a Tuesday, so the twenty-eighth will also be a Tuesday. For a 28, 29, 30 or 31 day month, the first Tuesday of the next month will then fall on the 7th, 6th, 5th or 4th respectively. The product of these dates is $7 \times 6 \times 5 \times 4 = 840$.

2. Each of the following expressions has a different value. Which value is greatest?

$$2 - 0 \times 2 + 4 + 11 \times 20$$
$$2 + 0 \times 2 \times 4 + 11 \times 20$$
$$2 \times 0 + 2 \times 4 + 11 \times 20$$
$$2 \times (0 + 2 + 4) + 11 \times 20$$

SOLUTION **232**

The values are:
$2 - 0 \times 2 + 4 + 11 \times 20 = 2 - 0 + 4 + 220 = 226$
$2 + 0 \times 2 \times 4 + 11 \times 20 = 2 + 0 + 220 = 222$
$2 \times 0 + 2 \times 4 + 11 \times 20 = 0 + 8 + 220 = 228$
$2 \times (0 + 2 + 4) + 11 \times 20 = 12 + 220 = 232$

3. Kevin begins with eleven stones and a mallet. Every time he hits a stone with the mallet it breaks into exactly eight pieces. Kevin keeps hitting stones with his mallet, but stops before he has 1000 stones. What is the largest number of stones he could end up with?

SOLUTION **998**

Every time Kevin hits a stone he destroys one stone and creates eight new ones, that is he increases his number of stones by $8 - 1 = 7$. He starts with 11 stones, which has a remainder of four when divided by 7. Repeatedly adding 7 will not change this remainder, so we find the largest three-digit number with this remainder. Since $994 = 7 \times 142$, the answer is 998.

4. The number 35 can be written as the sum of four distinct primes in two different ways: $35 = 2 + 3 + 13 + 17$ and $35 = 2 + 3 + 11 + 19$.
What is the *smallest* number that can be written as the sum of four distinct primes in two different ways?

SOLUTION **023**

The sum of the smallest four odd primes is $3 + 5 + 7 + 11 = 26$. So any such sum that is less than 26 must include 2.
If 2 is to be included in the sum, then the sum must be an odd number.
The smallest four primes are 2, 3, 5, and 7, and $2 + 3 + 5 + 7 = 17$, which is the only way to write 17.
The next possible such sum (using the three smallest primes and the fifth smallest prime) is $2 + 3 + 5 + 11 = 21$, and this is the only way to write 21 as such a sum.
We now consider 23. Because $2 + 3 + 7 + 11 = 2 + 3 + 5 + 13 = 23$ we see that 23 is the smallest such sum.

5. What is the value of $|\sqrt{123} - 456| + |\sqrt{123} + 456|$?
The modulus function, $|x|$ is given by $|x| = x$ for $x \geq 0$, and $|x| = -x$ for $x < 0$.

SOLUTION **912**

We observe that $\sqrt{123} < \sqrt{144} = 12 < 456$. Therefore, $|\sqrt{123} - 456| = 456 - \sqrt{123}$.
The value of $|\sqrt{123} - 456| + |\sqrt{123} + 456|$ is $456 - \sqrt{123} + \sqrt{123} + 456 = 456 + 456 = 912$.

6. The equation $x^2 - 70x = 2024$ has solutions a, b with $b > a$. What is the value of $10b - a^2$?

SOLUTION **436**

We can solve this equation by completing the square: $x^2 - 70x = (x - 35)^2 - 1225 = 2024$.
Therefore $(x - 35)^2 = 3249$ and $x - 35 = \pm 57$, that is $x = 22$ or $x = 92$.
We have $b = 92$, $a = 22$, and $10b - a^2 = 920 - 484 = 436$.

7. The triangle ABC is such that $AB = 7$, $BC = 3$, and $CA = 5$.
Furthermore, we are given that $\angle ACB = 120°$.
The area of triangle ABC is M. What is the value of M^2, correct to the nearest integer?

SOLUTION **042**

There are a multitude of different methods for solving this problem including Heron's formula, or use of the $\frac{1}{2}ab\sin C$ formula, for those students who are aware of them.
Another approach is to observe the exterior angle at C is 60^o and complete an equilateral triangle ACD, where D lies on CB projected. The perpendicular from A to CD extended has length $\sqrt{5^2 - (2.5)^2} = \sqrt{18.75}$. Since this altitude is also perpendicular to BC, the required area is $\frac{1}{2} \times 3 \times \sqrt{18.75} = \frac{1}{2} \times 3 \times \sqrt{6.25}\sqrt{3} = \frac{1}{2} \times 3 \times\ 2.5 \times \sqrt{3} = \frac{15\sqrt{3}}{4}$.
Therefore, $M^2 = \frac{15^2 \times 3}{4^2} = \frac{675}{16} = 42 + \frac{3}{16}$. This is 42 correct to the nearest integer.

8. At a football match between Glasgow Belles and Doncaster City, five supporters each made a prediction:

1. The game will end in a draw;
2. Glasgow will score;
3. Glasgow will win;
4. Glasgow will not lose;
5. Exactly five goals will be scored.

What is the maximum number of these predictions that could simultaneously come true?

SOLUTION **004**

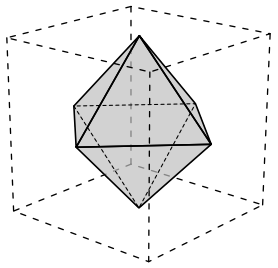
We observe that predictions (1) and (5) cannot simultaneously occur. The scenario where Glasgow Belles win 4-1 satisfies predictions (2), (3), (4) and (5) and therefore the maximum number that could simultaneously happen is four.

9. The number $N = 24^{2024} + 24^{2025}$ has some factors between 1 and 25.
What is the sum of all those integers between 1 and 25 which are *not* factors of N?

SOLUTION **147**

We re-write $N = 24^{2024} + 24^{2025}$ as $N = 24^{2024} \times (1 + 24) = 24^{2024} \times 25 = 2^{3\times 2024} \times 3^{2024} \times 5^2$.
Any integer between 1 and 25 which can be expressed as a product of powers of 2, 3, and 5 (for example, $20 = 2^2 \times 5$), is a factor of N.
There are only nine integers which cannot be expressed in such a way.
These integers have sum $7 + 11 + 13 + 14 + 17 + 19 + 21 + 22 + 23 = 147$.

10. An octahedron is inscribed in a cube of side-length 12. The vertices of the octahedron are at the centre of the faces of the cube.
The surface area of the octahedron is $a \times \sqrt{b}$, where b is a prime number. What is the value of $a + 10b$?



SOLUTION **174**

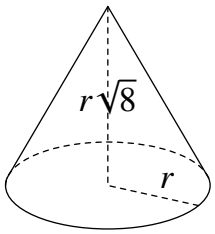
By Pythagoras, the distance between the midpoints of two adjacent faces of the cube is $\sqrt{6^2 + 6^2} = 6 \times \sqrt{2}$.
Each face of the octahedron is an equilateral triangle with area
$\frac{1}{2} \times 6\sqrt{2} \times 6\sqrt{2} \times \sin 60 = \frac{1}{2} \times 6 \times 6 \times 2 \times \frac{\sqrt{3}}{2} = 18\sqrt{3}$.
Therefore, the octahedron has surface area $8 \times 18\sqrt{3} = 144\sqrt{3}$.
This is in the form $a + 10b$ with $a = 144$, $b = 3$, and $a + 10b = 144 + 30 = 174$.

11. The number $L = \frac{24!}{20!}$ is one less than a square, S^2.
What is the value of S?

SOLUTION **505**

We observe that $L = \frac{24!}{20!} = 24 \times 23 \times 22 \times 21 = (24 \times 21) \times (23 \times 22) = 504 \times 506 = (505 - 1) \times (505 + 1) = 505^2 - 1$.

12. A solid right circular cone whose base has radius r metres has total surface area (including the circular base) of A metres squared and a volume of V metres cubed.
The perpendicular height of the cone is $r\sqrt{8}$ metres.
Given that $\frac{V}{A} = \sqrt{5000}$, what is the value of r?



SOLUTION **300**

The total surface area of the cone is given by the formula $\pi rl + \pi r^2$, where l is the slant height of the cone.
By Pythagoras, $l = \sqrt{8r^2 + r^2} = 3r$.
We have $A = \pi \times r \times 3r + \pi \times r^2 = 4\pi r^2$.
The volume of the cone is given by the formula $\frac{1}{3}\pi r^2 h$, where h is the perpendicular height of the cone.
We have $V = \frac{1}{3}\pi r^2 \times r\sqrt{8} = \frac{\pi r^3 \sqrt{8}}{3}$.
Since $\frac{V}{A} = \sqrt{5000}$, $\frac{\pi r^3 \sqrt{8}}{3} \times \frac{1}{4\pi r^2} = \sqrt{5000}$.
Therefore, $\frac{r\sqrt{8}}{12} = \sqrt{5000}$. Hence, $r = 12 \times \frac{\sqrt{5000}}{\sqrt{8}} = 12 \times \sqrt{\frac{5000}{8}} = 12 \times \sqrt{625} = 12 \times 25 = 300$.

13. Wil goes on a run. Her pedometer is unfortunately rather basic, and records a distance of 40 km, to the nearest integer, and an average speed of 10 km/h, to the nearest integer. Wil works out her greatest possible and least possible times, and expresses the difference of these times as a fraction in its lowest terms, $\frac{a}{b}$.
What is the value of $a + 2b$?

SOLUTION **998**

Her greatest and least times are $\frac{40.5}{9.5}$ and $\frac{39.5}{10.5}$ respectively. These have a difference of $\frac{405}{95} - \frac{395}{105} = \frac{81}{19} - \frac{79}{21} = \frac{81\times21-79\times19}{19\times21} = \frac{1701-1501}{399} = \frac{200}{399}$.
Therefore, $a = 200$, $b = 399$, and $a + 2b = 998$.

14. The function f has the property that $f(x + y) = f(x) \times f(y)$ for all values of x and y. Given that $f(x) > 0$ for all values of x, and also that $f(24) = 4096$, what is $f(18)$?

SOLUTION **512**

$4096 = f(24) = f(12 + 12) = f(12) \times f(12)$, so $f(12) = \sqrt{4096} = 64$.
Therefore, $64 = f(12) = f(6 + 6) = f(6) \times f(6)$, so $f(6) = \sqrt{64} = 8$.
Hence, $f(18) = f(12 + 6) = f(12) \times f(6) = 64 \times 8 = 512$.

15. Four friends named Ada, Brin, Chris and Debbie are each carrying a different amount of money. These amounts are £111, £222, £333 and £444.

Ada says: I am neither carrying the most nor the least.

Brin says: I am not carrying the least.

Chris says: I am carrying the most.

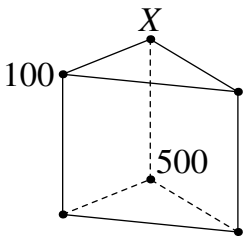
Debbie says: I am carrying the least.

Exactly one of the friends is lying. How many letters does the lying friend have in her/his name?

SOLUTION **005**

If Ada is lying then either Chris or Debbie must also be lying, meaning more than one friend is lying. So Ada must be telling the truth.
If Brin is lying then Debbie is also lying, so Brin must be telling the truth.
If Debbie is lying then none of Debbie, Brin or Ada are carrying the least, so Chris would be carrying the least. This would mean Chris would also be lying, so Debbie must be telling the truth.
So Chris must be lying and Chris is not carrying the most. This is consistent with Brin carrying most, then Chris and Ada in some order, then Debbie.
Chris has five letters in his name, so the answer is 005.

16. The prism in the picture is formed of two triangles and three squares. The six vertices are assigned labels 100, 200, 300, 400, 500, and 600 in such a way that the sum of the four labels of the vertices of each square is the same for all three squares. Labels 100 and 500 are shown.
What number is at the vertex labelled X?



SOLUTION **200**

For the front square, label the vertices 100, a, b and c, in anticlockwise order. Then $100 + a + b + c = X + 500 + b + c$. Therefore, $a - X = 400$ from which it follows at once that $a = 600$ and $X = 200$ (with b and c being 300 and 400 in some order).

17. The parabola $y = x^2 - 10x + 24$ has two distinct x-intercepts and a y-intercept.
The circle which goes through these three intercepts meets the parabola at a fourth point, P.
What is the distance of P from the origin?

SOLUTION **026**

By solving the equation $0 = x^2 - 10x + 24$, we determine that x-intercepts are at 4 and 6, while the y-intercept is at 24.
The parabola has a line of symmetry equidistant from its roots, i.e. with equation $x = 5$. Since $(4, 0)$ and $(6, 0)$ are at opposite ends of a chord to the circle, the line of symmetry $x = 5$ is a diameter of the circle. We may therefore obtain the coordinates of P by reflecting the parabola's y-intercept in the line $x = 5$.
So P is at $(10, 24)$ and is a distance $\sqrt{10^2 + 24^2} = 26$ from the origin.

18. Zarah has a bag containing exactly four blue counters and exactly four red counters, and a large supply, outside the bag, of extra blue and red counters.
She may, on any given *move*, do one of four different things:

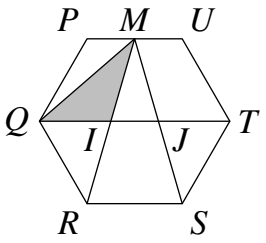
- Remove a blue counter from the bag;
- Remove a red counter from the bag;
- Add a blue counter to the bag;
- Add a red counter to the bag.

At each move, she selects one of those four options at random and with equal probability. The probability that after four consecutive moves, Zarah's bag again contains exactly four blue counters and exactly four red counters is $\frac{V}{W}$ when written in its lowest terms. What is the value of $10V + W$?

SOLUTION **154**

There is a one in four chance that Zarah's second move is the opposite of her first. In this case, Zarah will need her fourth move to be the opposite of her third, which also has a one in four chance.
There is a one in four chance that Zarah's second move is identical to her first. In this case, both her third and fourth moves will need to be the exact opposite to her first and second to return to the original state, with a one in four chance for each of the third and fourth moves being such.
There is a two in four chance that Zarah's second move is neither opposite nor identical to her first (i.e. the first and second moves involve different coloured counters). In this case there is a one in two chance that her third move will be opposite to either of her first or her second moves, followed by a one in four chance that her fourth move is opposite to whichever of her first or second moves was not *undone* during move three.
The respective probabilities of these three cases are $\frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$, $\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4} = \frac{1}{64}$, and $\frac{2}{4}\times\frac{1}{2}\times\frac{1}{4} = \frac{1}{16}$, with sum $\frac{1}{16} + \frac{1}{16} + \frac{1}{64} = \frac{9}{64}$.
Therefore, $V = 9$, $W = 64$, and $10V + W = 90 + 64 = 154$.

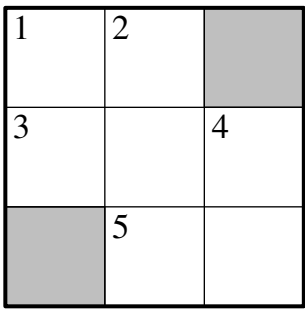
19. A regular hexagon $PQRSTU$ is shown. The midpoint of UP is M. Point I is the intersection between line segments RM and QT, and point J is the intersection between line segments SM and QT.
The area of triangle RMS is 920. What is the area of triangle QMI?



SOLUTION **345**

Let the side-length of the hexagon be $2x$.
The perpendicular height of triangle RMS is $\frac{2\times920}{2x} = \frac{920}{x}$.
By symmetry, the perpendicular height of triangle QMI is $\frac{1}{2} \times \frac{920}{x} = \frac{460}{x}$.
Triangles MIJ and MRS are similar, and IJ therefore has length $\frac{1}{2} \times 2x = x$.
The diagonals PS, QT and RU would divide the original hexagon into six congruent equilateral triangles, from which it is easy to conclude that $QT = 2x + 2x = 4x$.
Therefore, $QI + JT = QT - IJ = 4x - x = 3x$, and by symmetry, $QI = JT = \frac{3x}{2}$.
So the area of triangle QMI is $\frac{1}{2} \times \frac{3x}{2} \times \frac{460}{x} = 345$.

20. Each cell in this cross-number can be filled with a non-zero digit so that all of the conditions in the clues are satisfied. The digits used are not necessarily distinct.



ACROSS
1. A prime.
3. The answer to this Kangaroo question.
5. Shares a two-digit common factor with 1 DOWN.

DOWN
1. The sum of two different sixth powers.
2. 101 less than a square number.
4. The sum of all the digits in this crossnumber, once correctly completed.

SOLUTION **594**

The only possibility for 1 DOWN that is is a two-digit number is 65.
We then know that 1 ACROSS is either 61 or 67.
If 1 ACROSS is 61 then 2 DOWN is 124, 155, or 188.
We now consider 5 ACROSS. The only two-digit factors of 65 (1 DOWN) are 13 and 65.
So 2 DOWN cannot be 124 since there are no multiples of 13 or 65 between 41 and 49.
Also, 2 DOWN cannot be 188 since there are no multiples of 13 or 65 between 81 and 89.
If 2 DOWN were 155 then 5 ACROSS would be 52 (the only multiple of 13 or 65 between 51 and 59).
At this point the *known* digits in the completed crossnumber would sum to 24. The maximum digit-sum for the completed crossnumber is therefore 24 + 9 = 33, meaning 8 is the only case to check (since we know 4 DOWN must end with a 2). However, this yields a contradiction since the overall digit-sum is not 82.
So 2 DOWN cannot be 155, so 1 ACROSS cannot be 61 and must be 67.
2 DOWN must be 799 (observe that while 740 is also 101 less than a square and has the correct hundreds digit, we are told in the question that all cells are filled with *non-zero* digits).
5 ACROSS is then 91 (the only multiple of 13 or 65 between 91 and 99).
At this point the *known* digits in the completed crossnumber would sum to 37. The maximum digit-sum for the completed crossnumber is therefore 37 + 9 = 46, and since we already know that in this case 4 DOWN ends with a 1 we now merely need to verify that if 4 DOWN = 41 that all the clues are satisfied.
The digit-sum in this case is indeed 41 and the answer to 3 ACROSS and this question is 594.

UK Maths Trust

Andrew Jobbings Senior Kangaroo

Wednesday 19 November 2025

a member of the Association Kangourou sans Frontières

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England & Wales: Year 13 or below | *Scotland: S6 or below* | *Northern Ireland: Year 14 or below*

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **60 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank or lined paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil** to record your answer to each problem as a three-digit number from 000 to 999. For example, if your answer is a single digit like '4' please input '004'. Further instructions can be found on the Answer Sheet.
5. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules:**
   5 marks are awarded for each correct answer;
   There is no penalty for giving an incorrect answer.
7. **Your Answer Sheet will be read by a machine.** Do not write or doodle on the sheet except to mark your chosen options. The machine will read all black pencil markings even if they are in the wrong places. If you mark the sheet in the wrong place, doodle, or leave bits of eraser stuck to the page, the machine will interpret the mark in its own way, or reject the answer sheet.
8. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.

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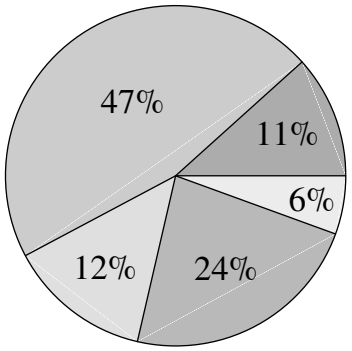
1. The product $2 \times 3 \times 5 \times 7 \times 11 \times 13$ has remainder r when divided by 100.

What is the value of $200 - r$?

2. The pie chart shows how exactly 2000 children in a particular town get to school. Note that the percentages in this pie chart are exact, without any rounding.
Approximately twice as many use a bicycle as use the bus. Roughly the same number come by car as walk. The rest come by train.

How many children use the train?



3. In a sum of five three-digit numbers 'ABC' + 'BCD' + 'CDE' + 'DEA' + 'EAB' = 2664, the letters A, B, C, D and E represent digits, which are not necessarily distinct.

What is the value of $10(A + B + C + D + E)$?

4. Kitty claims that for every positive integer n, the product $(5n + 1)(6n - 1)(n^3 - 2)$ is not divisible by 7.

What is the smallest three-digit value of n which is a counterexample to Kitty's claim?

5. The integers a, b, and c are such that $1 \leq a \leq b \leq c$ and $abc = 2^{16}$.

What is the largest possible value of b?

6. Three fair dice are rolled. The first dice has three red faces, two white faces and one blue face. The second dice has two red faces, two white faces and two blue faces. The third dice has one red face, two white faces and three blue faces. The probability that, following rolling, all three dice will show the same colour is $\frac{K}{1080}$.

What is the value of K?

7. What is the value of $\dfrac{(2025 + 20)^2 + (2025 + 25)^2 - (2025 - 20)^2 - (2025 - 25)^2}{1000}$,

rounded up to the next integer?

8. The first four digits of a 128-digit number are 2025.

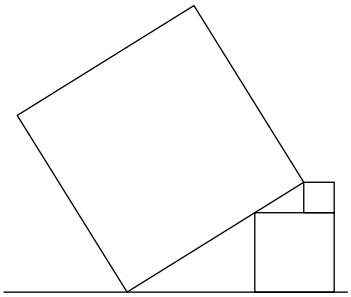
How many digits does the square of this number have?

9. Nineteen integers, which are not necessarily distinct, are arranged in a circle such that the sum of the integers in any nine consecutive positions always has the same value, 2025.

What is the value of the largest of the nineteen integers?

10. Three squares are shown in the picture. The area of the smallest square is 9, and the area of the middle-sized square is 81.

What is the area of the largest square?

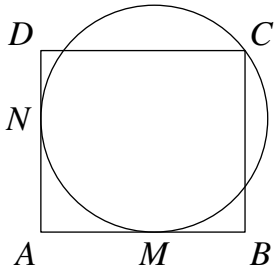


11. The sequence k_n is defined by $k_1 = 19$, $k_2 = 11$ and $k_{n+2} = k_n + k_{n+1}$ for $n \geq 1$.

How many of the first 2025 terms in the sequence are even?

12. A rectangle $ABCD$ and a circle are shown in the diagram.
Sides AB and AD of the rectangle are tangent to the circle at points M and N respectively.
The circle passes through C.
The length of AM is 15 and the length of MB is 12.

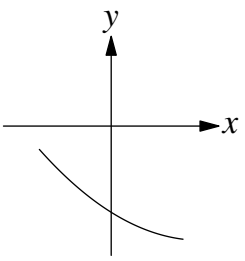
What is the area of the rectangle?



13. Part of the parabola with equation $y = ax^2 + bx + c$ is shown in the diagram.

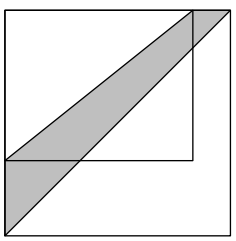
How many of the following expressions have positive values?

a	b	c	ab	bc	ca	$b+c$	abc



14. The length of one side of a rectangular park was increased by 20%, while the length of the other side was increased by 50%. After the changes the park is now square in shape. The shaded area shown between the two diagonals in the diagram is 210.

What was the original area of the park before the changes were made?



15. The simultaneous equations (1) and (2) have at least two distinct (x, y) solutions.

$$2x + 5y = a - b \qquad (1)$$
$$4x + 10y = a + b \qquad (2)$$

The simultaneous equations (3) and (4) also have at least two distinct (x, y) solutions.

$$3x - 2y = a - b + 4 \qquad (3)$$
$$9x - 6y = a + 4b - 1 \qquad (4)$$

What is the value of $10a + b$?

16. The six interior angles of a convex non-regular hexagon, measured in degrees, are a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$, and $a + 5d$. All six angles are integer multiples of $10°$.

What is the value of $7a + 11d$?

17. The navigation app on my smartphone gives distances in both miles and kilometres. When planning a journey, I notice the distance in miles, m, and the distance in kilometres, k, are such that $m = k - 438$.

How long does my planned journey in miles work out to be if you use the approximation 1 mile = 1.6 kilometres?

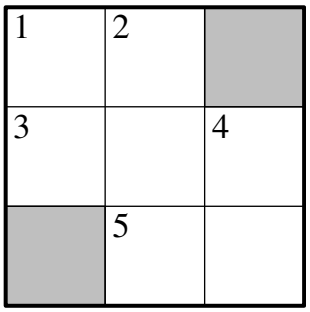
18. A teacher works out the number of different ways, W, she can choose two students from Class A. She also works out the number of different ways, V, she can choose two students from Class B. Class A has four more students in it than class B. She notices that $W = 3V$.

How many students are there in class A?

19. An *almost number* is defined as an integer which is divisible by exactly eight elements of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

What is the largest *almost number* below 1000?

20. Each cell in this cross-number can be filled with a non-zero digit so that all of the conditions in the clues are satisfied. The digits used are not necessarily distinct.



What is the sum of the answers to the clues for

1 ACROSS, 3 ACROSS and 5 ACROSS?

ACROSS
1. A factor of 1221.
3. A multiple of 17.
5. A prime.

DOWN
1. A factor of 1221 different from 1 ACROSS.
2. A cube.
4. A prime.

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. The product $2 \times 3 \times 5 \times 7 \times 11 \times 13$ has remainder r when divided by 100.

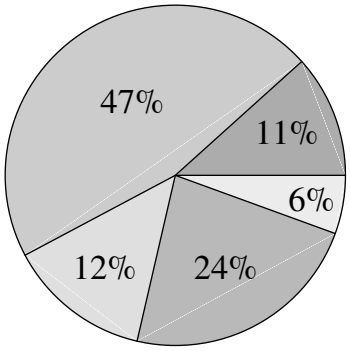
What is the value of $200 - r$?

SOLUTION **170**

The product is $30 \times 1001 = 30030$, so $r = 30$ and $200 - r = 170$.

2. The pie chart shows how exactly 2000 children in a particular town get to school. Note that the percentages in this pie chart are exact, without any rounding. Approximately twice as many use a bicycle as use the bus. Roughly the same number come by car as walk. The rest come by train.

How many children use the train?



SOLUTION **120**

The only two sectors of approximately the same size are those marked 12% and 11%, so these must represent 'car' and 'walk', in some order. Of the remaining sectors, those marked 47% and 24% are the only ones in an approximate ratio of 2 : 1, so these must be 'bicycle' and 'bus' respectively. Therefore the sector marked 6% represents the children who use the train. Therefore, the train is used by 6%, which is 120 children.

3. In a sum of five three-digit numbers 'ABC'+'BCD'+'CDE'+'DEA'+'EAB' = 2664, the letters A, B, C, D and E represent digits, which are not necessarily distinct.

What is the value of $10(A + B + C + D + E)$?

SOLUTION **240**

Writing 'ABC' as $100A + 10B + C$, and similarly for the other numbers, the sum of the five numbers is $111(A + B + C + D + E)$, which is equal to 2664.
It follows that $A + B + C + D + E = \frac{2664}{111} = 24$, so that $10(A + B + C + D + E) = 240$.

4. Kitty claims that for every positive integer n, the product $(5n+1)(6n-1)(n^3-2)$ is not divisible by 7.

What is the smallest three-digit value of n which is a counterexample to Kitty's claim?

SOLUTION **102**

We consider the remainder of each of the three constituent brackets $(5n+1)$, $(6n-1)$, and (n^3-2) when each is divided by 7.
100 has remainder 2 when divided by 7, so the brackets will have remainders of $5\times 2+1=11$ (remainder 4), $6\times 2-1=11$ (remainder 4), and $2^3-2=6$ respectively. As none of these are zero, the product will not be divisible by 7.
Similarly, 101 has remainder 3 when divided by 7, so the brackets will have remainders of $5\times 3+1=16$ (remainder 2), $6\times 3-1=17$ (remainder 3), and $3^3-2=25$ (remainder 4) respectively. As none of these are zero, the product will not be divisible by 7.
However, 102 has remainder 4 when divided by 7. The first bracket then has remainder $5\times 4+1=21$, so is divisible by 7.

5. The integers a, b, and c are such that $1\le a\le b\le c$ and $abc=2^{16}$.

What is the largest possible value of b?

SOLUTION **256**

The case $a=1$ has $b=c=2^8$ as a possible solution, so the largest possible value of b is at least $2^8=256$.
Were $b>2^8$, then $c>2^8$ also, and $abc>2^{8+8}=2^{16}$.
So 256 is the largest possible value of b.

6. Three fair dice are rolled. The first dice has three red faces, two white faces and one blue face. The second dice has two red faces, two white faces and two blue faces. The third dice has one red face, two white faces and three blue faces. The probability that, following rolling, all three dice will show the same colour is $\frac{K}{1080}$.

What is the value of K?

SOLUTION **100**

The probability of obtaining three reds is $\frac{3}{6}\times\frac{2}{6}\times\frac{1}{6}=\frac{6}{216}$.
The probability of obtaining three whites is $\frac{2}{6}\times\frac{2}{6}\times\frac{2}{6}=\frac{8}{216}$.
The probability of obtaining three blues is $\frac{1}{6}\times\frac{2}{6}\times\frac{3}{6}=\frac{6}{216}$.
So, the probability of obtaining three of the same colour is $\frac{6}{216}+\frac{8}{216}+\frac{6}{216}=\frac{20}{216}=\frac{100}{1080}$.

7. What is the value of $\dfrac{(2025+20)^2+(2025+25)^2-(2025-20)^2-(2025-25)^2}{1000}$, rounded up to the next integer?

SOLUTION **365**

The numerator may be written as

$$\begin{aligned}2045^2-2005^2+2050^2-2000^2 &= (2045-2005)(2045+2005)+(2050-2000)(2050+2000)\\ &= 40\times 4050+50\times 4050\\ &= 90\times 4050 = 9\times 40500.\end{aligned}$$

The expression is therefore equal to $\frac{9\times 40500}{1000} = 9\times 40.5 = 364.5$.
Rounded up to the next integer, this is 365.

8. The first four digits of a 128-digit number are 2025.

How many digits does the square of this number have?

SOLUTION **255**

We recall that an m-digit number may be written in standard form as $a\times 10^{m-1}$, where $1\le a<10$.
In standard form, the 128-digit number, N, is such that $2.025\times 10^{127}\le N<2.026\times 10^{127}$.
Therefore, $2.025^2\times 10^{254}\le N^2<2.026^2\times 10^{254}$.
Since $1<2.025^2<2.026^2<10$, this is in standard form and will have length $254+1=255$ digits.

9. Nineteen integers, which are not necessarily distinct, are arranged in a circle such that the sum of the integers in any nine consecutive positions always has the same value, 2025.

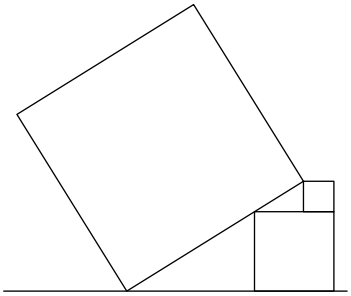
What is the value of the largest of the nineteen integers?

SOLUTION **225**

Label the integers in turn c_1, c_2, c_3, etc., with c_{19} adjacent to c_1.
Hence $c_1+c_2+c_3+c_4+c_5+c_6+c_7+c_8+c_9 = c_2+c_3+c_4+c_5+c_6+c_7+c_8+c_9+c_{10}$, from which we conclude that $c_1=c_{10}$.
A similar argument can be used to show $c_2=c_{11}$, $c_3=c_{12}$, $c_4=c_{13}$, $c_5=c_{14}$, $c_6=c_{15}$, $c_7=c_{16}$, $c_8=c_{17}$, $c_9=c_{18}$, $c_{10}=c_{19}$, $c_{11}=c_1$, $c_{12}=c_2$, $c_{13}=c_3$, $c_{14}=c_4$, $c_{15}=c_5$, $c_{16}=c_6$, $c_{17}=c_7$, $c_{18}=c_8$, and $c_{19}=c_9$.
We can conclude that all nineteen integers are equal to each other.
The largest integer is (and indeed all of the integers are) $\frac{2025}{9}=225$.

10. Three squares are shown in the picture. The area of the smallest square is 9, and the area of the middle-sized square is 81.

What is the area of the largest square?



SOLUTION **720**

The smaller right-angled triangle has hypotenuse $\sqrt{3^2 + 6^2} = \sqrt{45}$.
The larger right-angled triangle (with height 9), is similar to the smaller right-angled triangle (with height 3), so has hypotenuse $\frac{9}{3} \times \sqrt{45} = 3 \times \sqrt{45}$.
Hence the side of the largest square is $\sqrt{45} + 3 \times \sqrt{45} = 4 \times \sqrt{45}$.
Therefore the largest square has area $4^2 \times 45 = 16 \times 45 = 720$.

11. The sequence k_n is defined by $k_1 = 19$, $k_2 = 11$ and $k_{n+2} = k_n + k_{n+1}$ for $n \geq 1$.

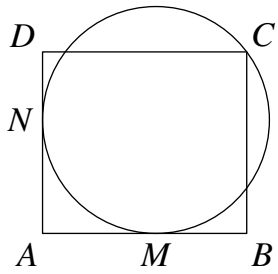
How many of the first 2025 terms in the sequence are even?

SOLUTION **675**

When we evaluate the terms of the sequence in order, we obtain $k_3 = 19 + 11 = 30$, $k_4 = 11 + 30 = 41$, $k_5 = 30 + 41 = 71$, $k_6 = 41 + 71 = 112$, etc.
We see that the pattern 'odd, odd, even, odd, odd, even' will recur indefinitely. Since 2025 is a multiple of 3, exactly one-third of the first 2025 terms will be even.
The number of terms in the sequence that are even is $\frac{2025}{3} = 675$.

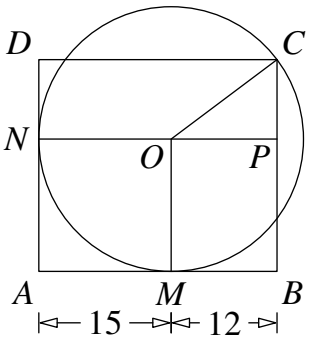
12. A rectangle $ABCD$ and a circle are shown in the diagram. Sides AB and AD of the rectangle are tangent to the circle at points M and N respectively.
The circle passes through C.
The length of AM is 15 and the length of MB is 12.

What is the area of the rectangle?



SOLUTION **648**

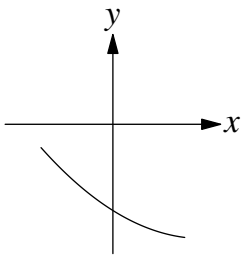
Let O be the centre of the circle and consider the radii OC, OM, and ON as shown, with NO extended to meet BC at P.
Since $OP = MB = 12$ and $OC = ON = MA = 15$, the right-angled triangle COP has side lengths that form a 9–12–15 Pythagorean triple.
Now, $AB = AM + MB = 15 + 12 = 27$, and $BC = BP + PC = MO + PC = 15 + 9 = 24$.
Therefore the area of the rectangle is $27 \times 24 = 648$.



13. Part of the parabola with equation $y = ax^2 + bx + c$ is shown in the diagram.

How many of the following expressions have positive values?

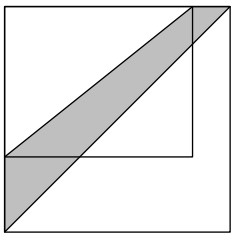
a	b	c	ab	bc	ca	$b+c$	abc



SOLUTION **003**

As the parabola is "U"-shaped we know that $a > 0$. The y-intercept is negative, so $c < 0$. Completing the square, we obtain $y = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c$. Since the parabola's minimum occurs for a positive value of x, and this must correspond with $\left(x + \frac{b}{2a}\right) = 0$, we know that $\frac{b}{2a} < 0$. As $a > 0$, we can conclude $b < 0$.
It follows that $ab < 0$, $ac < 0$, $bc > 0$, $b + c < 0$, and $abc > 0$.
Therefore, there are three positive quantities: a, bc, and abc.

14. The length of one side of a rectangular park was increased by 20%, while the length of the other side was increased by 50%. After the changes the park is now square in shape. The shaded area shown between the two diagonals in the diagram is 210.



What was the original area of the park before the changes were made?

SOLUTION **525**

Let the original rectangle have longer side of length x, and shorter side of length y.
The area of the original rectangle is xy and the area of the square is $1.2x \times 1.5y = 1.8xy$.
The shaded area is $0.5 \times 1.8xy - 0.5 \times xy = 210$.
Therefore, $0.4 \times xy = 210$, and hence the original area of the park was $xy = \frac{210}{0.4} = \frac{2100}{4} = 525$.

15. The simultaneous equations (1) and (2) have at least two distinct (x, y) solutions.

$$2x + 5y = a - b \qquad (1)$$
$$4x + 10y = a + b \qquad (2)$$

The simultaneous equations (3) and (4) also have at least two distinct (x, y) solutions.

$$3x - 2y = a - b + 4 \qquad (3)$$
$$9x - 6y = a + 4b - 1 \qquad (4)$$

What is the value of $10a + b$?

SOLUTION **403**

Equations (1) and (2) can have more than one distinct solutions only if one equation is a multiple of the other. Therefore, $a + b = 2(a - b)$, from which we can conclude that $3b = a$.
Similarly, equations (3) and (4) can have more than one distinct solutions only if one equation is a multiple of the other. Therefore, $a + 4b - 1 = 3(a - b + 4)$.
Solving $3b = a$ and $a + 4b - 1 = 3(a - b + 4)$ simultaneously yields $a = 39$ and $b = 13$. The value of $10a + b$ is $10 \times 39 + 13 = 403$.

16. The six interior angles of a convex non-regular hexagon, measured in degrees, are a, $a+d$, $a+2d$, $a+3d$, $a+4d$, and $a+5d$. All six angles are integer multiples of $10°$.

What is the value of $7a+11d$?

SOLUTION **710**

When you consider the interior angle sum of the hexagon, you obtain $6a+15d=720$. Since both a and $a+d$ are multiples of 10, it follows that d is also a multiple of 10.
We know that d cannot be 0 as the hexagon is not regular.
If $d=10$ then $6a+150=720$ and $a=95$, which isn't a multiple of 10.
If $d=20$ then $6a+300=720$ and $a=70$.
If $d=30$ then $6a+450=720$ and $a=45$, which isn't a multiple of 10.
If $d\geq 40$ then $a+5d\geq 200$, making the hexagon non-convex.
Therefore $a=70$, $d=20$, and hence $7a+11d=7\times 70+11\times 20=490+220=710$.

17. The navigation app on my smartphone gives distances in both miles and kilometres. When planning a journey, I notice the distance in miles, m, and the distance in kilometres, k, are such that $m=k-438$.

How long does my planned journey in miles work out to be if you use the approximation 1 mile = 1.6 kilometres?

SOLUTION **730**

Substituting $k=1.6m$ into the equation gives $m=1.6m-438$.
This has solution $m=730$, which is the length of the planned journey.

18. A teacher works out the number of different ways, W, she can choose two students from Class A. She also works out the number of different ways, V, she can choose two students from Class B. Class A has four more students in it than class B. She notices that $W=3V$.

How many students are there in class A?

SOLUTION **010**

Let the number of students in class A be P.
The number of ways, W, of choosing two students from class A is $\frac{P\times(P-1)}{2}$.
The number of ways, V, of choosing two students from class B is $\frac{(P-4)\times(P-5)}{2}$.
Since $W=3V$, then $\frac{P\times(P-1)}{2}=3\times\frac{(P-4)\times(P-5)}{2}$.
Simplifying this gives the equation $P^2-13P+30=0$. This has solutions $P=10$ or $P=3$.
Only in the $P=10$ case does class B have a positive number of students. Hence there are ten students in class A.

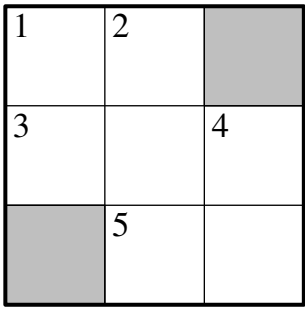
19. An *almost number* is defined as an integer which is divisible by exactly eight elements of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

What is the largest *almost number* below 1000?

SOLUTION **840**

By considering the prime factorisation of those composite elements of the set ($4 = 2^2$, $6 = 2 \times 3$, $8 = 2^3$, and $9 = 3^2$), we can find the lowest common multiple of the set of all nine numbers. This is $2^3 \times 3^2 \times 5 \times 7 = 2520$.
Almost numbers are numbers (or multiples of numbers) obtained on dividing 2520 by 2, 3, 5 or 7. These are, respectively, 1260, 840, 504 and 360, or their multiples. The only multiple of any of these which is less than 1000 is 720.
The largest three-digit almost number is 840.

20. Each cell in this cross-number can be filled with a non-zero digit so that all of the conditions in the clues are satisfied. The digits used are not necessarily distinct.



What is the sum of the answers to the clues for

1 ACROSS, 3 ACROSS and 5 ACROSS?

ACROSS
1. A factor of 1221.
3. A multiple of 17.
5. A prime.

DOWN
1. A factor of 1221 different from 1 ACROSS.
2. A cube.
4. A prime.

SOLUTION **457**

$1221 = 3 \times 11 \times 37$, so its only two-digit factors are 11, 33, and 37. Since the solutions to 1 ACROSS and 1 DOWN are distinct and share a tens digit, they must be 33 and 37 in some order.
Consider the solution to 1 ACROSS to be 33. Then 1 DOWN is 37 and 2 DOWN is 343, since this is the only three-digit cube starting with a 3. The only multiple of 17 that then fits the clue for 3 ACROSS is 748. 4 DOWN must be either 83 – with 5 ACROSS being 33 (which is composite and therefore does not satisfy the clue), or 89 – with 5 ACROSS being 39 (which is also composite and therefore does not satisfy the clue). So 1 ACROSS cannot be 33.
So the solution to 1 ACROSS must be 37. Then 1 DOWN is 33 and 2 DOWN is 729, since this is the only three-digit cube starting with a 7. . The only multiple of 17 that then fits the clue for 3 ACROSS is 323. 4 DOWN must either be 31 - with 5 ACROSS being 91 (which is composite and therefore does not satisfy the clue), or 37 - with 5 ACROSS being 97 which is indeed prime. This completes the crossnumber.
The sum of the answers to the clues for 1 ACROSS, 3 ACROSS and 5 ACROSS is $37+323+97 = 457$.

