COMPENDIUM USAJMO

USA Junior Mathematical Olympiad

Gerard Romo Garrido

Toomates Coolección vol. 46

Toomates Coolección

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Versión de este documento: **09/07/2023**

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1^{st} United States of America Junior Mathematical Olympiad 2010

Day I 12:30 PM – 5 PM EDT

April 27, 2010

1. A *permutation* of the set of positive integers $[n] = \{1, 2, \ldots, n\}$ is a sequence $(a_1, a_2, \ldots, a_n)$ such that each element of $[n]$ appears precisely one time as a term of the sequence. For example, $(3, 5, 1, 2, 4)$ is a permutation of $[5]$. Let $P(n)$ be the number of permutations of $[n]$ for which ka_k is a perfect square for all $1 \le k \le n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

2. Let $n > 1$ be an integer. Find, with proof, all sequences $x_1, x_2, \ldots, x_{n-1}$ of positive integers with the following three properties:

   (a) $x_1 < x_2 < \cdots < x_{n-1}$;

   (b) $x_i + x_{n-i} = 2n$ for all $i = 1, 2, \ldots, n-1$;

   (c) given any two indices i and j (not necessarily distinct) for which $x_i + x_j < 2n$, there is an index k such that $x_i + x_j = x_k$.

3. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

1^{st} United States of America Junior Mathematical Olympiad 2010

Day II 12:30 PM – 5 PM EDT

April 28, 2010

4. A triangle is called a *parabolic triangle* if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

5. Two permutations $a_1, a_2, \ldots, a_{2010}$ and $b_1, b_2, \ldots, b_{2010}$ of the numbers $1, 2, \ldots, 2010$ are said to *intersect* if $a_k = b_k$ for some value of k in the range $1 \le k \le 2010$. Show that there exist 1006 permutations of the numbers $1, 2, \ldots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

6. Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

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1. **Solution from Andy Niedermier:** Every integer in $[n]$ can be uniquely written in the form $x^2 \cdot q$, where q either 1 or *square free*, that is, a product of distinct primes. Let $\langle q \rangle$ denote the set $\{1^2 \cdot q, 2^2 \cdot q, 3^2 \cdot q, \ldots\} \subseteq [n]$.

   Note that for f to satisfy the square-free property, it must permute $\langle q \rangle$ for every $q = 1, 2, 3, \ldots$. To see this, notice that given an arbitrary square-free q, in order for $q \cdot f(q)$ to be a square, $f(q)$ needs to contribute one of every prime factor in q, after which it can take only even powers of primes. Thus, $f(q)$ is equal to the product of q and some perfect square.

   The number of f that permute the $\langle q \rangle$ is equal to

   $$\prod_{\substack{q \leq n \\ q \text{ is square-free}}} \left\lfloor \sqrt{\frac{n}{q}} \right\rfloor !$$

   For $2010 = 2 \cdot 3 \cdot 5 \cdot 67$ to divide $P(n)$, we simply need $67!$ to appear in this product, which will first happen in $\langle 1 \rangle$ so long as $\sqrt{n/q} \geq 67$ for some n and q. The smallest such n is $67^2 = 4489$.

   This problem was proposed by Andy Niedermier.

2. **Solution from Răzvan Gelca:** There is a unique sequence $2, 4, 6, \ldots, 2n-2$ satisfying the conditions of the problem.

   Note that (b) implies $x_i < 2n$ for all i. We will examine the possible values of x_1.

   If $x_1 = 1$, then (c) implies that all numbers less than $2n$ should be terms of the sequence, which is impossible since the sequence has only $n - 1$ terms.

If $x_1 = 2$, then by (c) the numbers $2, 4, 6, \ldots, 2n-2$ are terms of the sequence, and because the sequence has exactly $n-1$ terms we get $x_i = 2i$, $i = 1, 2, \ldots, n-1$. This sequence satisfies conditions (a) and (b) as well, so it is a solution to the problem.

For $x_1 \geq 3$, we will show that there is no sequences satisfying the conditions of the problem. Assume on the contrary that for some n there is such a sequence with $x_1 \geq 3$. If $n = 2$, the only possibility is $x_1 = 3$, which violates (b). If $n = 3$, then by (a) we have the possibilities $(x_1, x_2) = (3, 4)$, or $(3, 5)$, or $(4, 5)$, all three of which violate (b). Now we assume that $n > 3$. By (c), the numbers

$$x_1,\ 2x_1,\ \ldots,\ \left\lfloor \frac{2n}{x_i} \right\rfloor \cdot x \tag{1}$$

are terms of the sequence, and no other multiples of x_1 are. Because $x_1 \geq 3$, the above accounts for at most $\frac{2n}{3}$ terms of the sequence. For $n > 3$, we have $\frac{2n}{3} < n-1$, and so there must be another term besides the terms in (1). Let x_j be the smallest term of the sequence that does not appear in (1). Then the first j terms of the sequence are

$$x_1,\ x_2 = 2x_1,\ \ldots,\ x_{j-1} = (j-1)x_1,\ x_j, \tag{2}$$

and we have $x_j < jx_1$. Condition (b) implies that the last j terms of the sequence must be

$$\begin{aligned} &x_{n-j} = 2n - x_j,\ x_{n-j+1} = 2n - (j-1)x_1,\ \ldots, \\ &\qquad\qquad\qquad x_{n-2} = 2n - 2x_1,\ x_{n-1} = 2n - x_1. \end{aligned}$$

But then $x_1 + x_{n-j} < x_1 + x_{n-1} = 2n$, hence by condition (c) there exists k such that $x_1 + x_{n-j} = x_k$. On the one hand, we have

$$\begin{aligned} x_k &= x_1 + x_{n-j} = x_1 + 2n - x_j = 2n - (x_j - x_1) \\ &> 2n - (jx_1 - x_1) = 2n - (j-1)x_1 = x_{n-j+1}. \end{aligned}$$

One the other hand, we have

$$x_k = x_1 + x_{n-j} < x_1 + x_{n-j+1} = x_{n-j+2}.$$

This means that x_k is between two consecutive terms x_{n-j+1} and x_{n-j+2}, which is impossible by (a). (In the case $j = 2$, $x_k > x_{n-j+1} = x_{n-1}$, which is also impossible.) We conclude that there is no such sequence with $x_1 \geq 3$.

Remark. This problem comes from the study of Weierstrass gaps in the theory of Riemann surfaces.

Alternate Solution from Richard Stong: Assume that $x_1, x_2, \ldots, x_{n-1}$ is a sequence satisfying the conditions of the problem. By condition (a), the following terms

$$x_1,\ 2x_1,\ x_1 + x_2,\ x_1 + x_3,\ x_1 + x_4, \ldots,\ x_1 + x_{n-2}$$

form an increasing sequence. By condition (c), this new sequence is a subsequence of the original sequence. Because both sequences have exactly $n-1$ terms, these two sequences are identical; that is, $2x_1 = x_2$ and $x_1 + x_j = x_{j+1}$ for $2 \leq j \leq n-2$. It follows that $x_j = jx_1$ for $1 \leq j \leq n-1$. By condition (b), we conclude that $(x_1, x_2, \ldots, x_{n-1}) = (2, 4, \ldots, 2n-2)$.

Remark. The core of the second solution is a result due to Freiman:

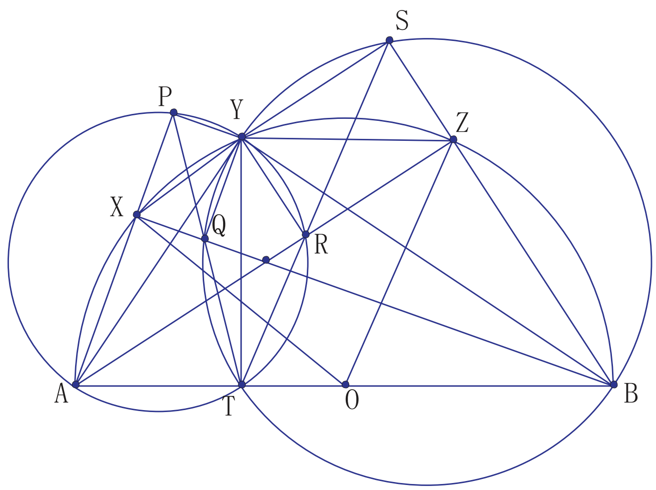
> Let A be a set of positive integers. Then the set $A+A = \{a_1 + a_2 \mid a_1, a_2 \in A\}$ has at least $2|A| - 1$ elements and equality holds if and only if A is a set of an arithmetic progression.

Freiman's theorem and its generalization below are very helpful in proofs of many contest problems, such as, USAMO 2009 problem 2, IMO 2000 problem 1, and IMO 2009 problem 5.

Let A and B be finite nonempty subsets of integers. Then the set $A+B = \{a+b \mid a \in A, b \in B\}$ has at least $|A|+|B|-1$ elements. Equality holds if and only if either A and B are arithmetic progressions with equal difference or $|A|$ or $|B|$ is equal to 1.

This problem was suggested by Răzvan Gelca.

3. **Solution by Titu Andreescu:** Let T be the foot of the perpendicular from Y to line AB. We note the P, Q, T are the feet of the perpendiculars from Y to the sides of triangle ABX. Because Y lies on the circumcircle of triangle ABX, points P, Q, T are collinear, by Simson's theorem. Likewise, points S, R, T are collinear.



We need to show that $\angle XOZ = 2\angle PTS$ or

$$\angle PTS = \frac{\angle XOZ}{2} = \frac{\overset{\frown}{XZ}}{2} = \frac{\overset{\frown}{XY}}{2} + \frac{\overset{\frown}{YZ}}{2}$$
$$= \angle XAY + \angle ZBY = \angle PAY + \angle SBY.$$

Because $\angle PTS = \angle PTY + \angle STY$, it suffices to prove that

$$\angle PTY = \angle PAY \quad \text{and} \quad \angle STY = \angle SBY;$$

that is, to show that quadrilaterals $APYT$ and $BSYT$ are cyclic, which is evident, because $\angle APY = \angle ATY = 90^\circ$ and $\angle BTY = \angle BSY = 90^\circ$.

Alternate Solution from Lenny Ng and Richard Stong: Since YQ, YR are perpendicular to BX, AZ respectively, $\angle RYQ$ is equal to the acute angle between lines BX and AZ, which is $\frac{1}{2}(\widehat{AX} + \widehat{BZ}) = \frac{1}{2}(180^\circ - \widehat{XZ})$ since X, Z lie on the circle with diameter AB. Also, $\angle AXB = \angle AZB = 90^\circ$ and so $PXQY$ and $SZRY$ are rectangles, whence $\angle PQY = 90^\circ - \angle YXB = 90^\circ - \widehat{YB}/2$ and $\angle YRS = 90^\circ - \angle AZY = 90^\circ - \widehat{AY}/2$. Finally, the angle between PQ and RS is

$$\begin{aligned}
\angle PQY + \angle YRS - \angle RYQ &= (90^\circ - \widehat{YB}/2) + (90^\circ - \widehat{AY}/2) - (90^\circ - \widehat{XZ}/2) \\
&= \widehat{XZ}/2 \\
&= (\angle XOZ)/2,
\end{aligned}$$

as desired.

This problem was proposed by Titu Andreescu.

4. **Solution from Zuming Feng:**

Let $A = (a, a^2)$, $B = (b, b^2)$, and $C = (c, c^2)$, with $a < b < c$. We have $\overrightarrow{AB} = [b - a, b^2 - a^2]$ and $\overrightarrow{AC} = [c - a, c^2 - a^2]$. Hence the area of triangle ABC is equal to

$$\begin{aligned}
[ABC] = (2^n m)^2 &= \frac{|(b-a)(c^2-a^2) - (c-a)(b^2-a^2)|}{2} \\
&= \frac{(b-a)(c-a)(c-b)}{2}.
\end{aligned}$$

Setting $b - a = x$ and $c - b = y$ (where both x and y are positive integers), the above equation becomes

$$(2^n m)^2 = \frac{xy(x+y)}{2}. \tag{3}$$

If $n = 0$, then $(m, x, y) = (1, 1, 1)$ is clearly a solution to (3). If $n \geq 1$, it is easy to check that,

$$(m, x, y) = \big((2^{4n-2} - 1, 2^{2n+1}, (2^{2n-1} - 1)^2)\big)$$

satisfies (3).

Alternate Solution from Jacek Fabrykowski:

The beginning is the same up to $(2^n m)^2 = \frac{xy(x+y)}{2}$. If $n = 0$, we take $m = x = y = 1$. If $n = 1$, we take $m = 3$, $x = 1$, $y = 8$. Assume that $n \geq 2$. Let a, b, c be a primitive Pythagorean triple with b even. Let $b = 2^r d$ where d is odd and $r \geq 2$. Let $x = 2^{2k}$, $y = 2^{2k} b$ and $z = 2^{2k} c$ where $k \geq 0$. We let $m = adc$ and $r = 2$ if $n = 3k + 2$, $r = 3$ if $n = 3k + 3$ and $r = 4$ if $n = 3k + 4$.

Assuming that $x = a \cdot 2^s$, $y = b \cdot 2^2$, other triples are possible:

(a) If $n = 3k$, then let $m = 1$ and $x = y = 2^{2k}$.

(b) If $n = 3k + 1$, then take $m = 3$, $x = 2^{2k}$, $y = 2^{2k+3}$.

(c) If $n = 3k + 2$, then take $m = 63$, $x = 49 \cdot 2^{2k}$, and $y = 2^{2k+5}$.

This problem was suggested by Zuming Feng.

5. **Solution from Gregory Galperin:**

Let us create the following 1006 permutations $X_1, ..., X_{1006}$, the first 1006 positions of which are all possible cyclic rotations of the sequence

$1, 2, 3, 4, ..., 1005, 1006$, and the remaining 1004 positions are filled arbitrarily with the remaining numbers $1006, 1007, ..., 2009, 2010$:

$$\begin{aligned}
X_1 &= 1,\ 2,\ 3,\ 4,\ \dots,\ 1005,\ 1006,\ \ *,\ *,\ \dots,\ *\ ; \\
X_2 &= 2,\ 3,\ 4,\ \dots,\ 1005,\ 1006,\ 1,\ \ *,\ *,\ \dots,\ *\ ; \\
X_3 &= 3,\ 4,\ \dots,\ 1005,\ 1006,\ 1,\ 2,\ \ *,\ *,\ \dots,\ *\ ; \\
&\cdots\cdots \\
X_{1006} &= 1006,\ 1,\ 2,\ 3,\ 4,\ \dots,\ 1005,\ \ *,\ *,\ \dots,\ *\ .
\end{aligned}$$

We claim that at least one of these 1006 sequences has the same integer at the same position as the initial (unknown) permutation X.

Suppose not. Then the set of the first (leftmost) integers in the permutation X contains no integers from 1 to 1006. Hence it consists of the 1004 integers in the range from 1007 to 2010 only. By the pigeon-hole principle, some two of the integers from the permutation X must be equal, which is a contradiction: there are not two identical integers in the permutation X.

Consequently, the permutation X has at last one common element with some sequence X_i, $i = 1, \dots 1006$ and we are done.

This problem was proposed by Gregory Galperin.

6. **Solution from Zuming Feng:** The answer is *no*, it is not possible for segments AB, BC, BI, ID, CI, IE to all have integer lengths.

Assume on the contrary that these segments do have integer side lengths. We set $\alpha = \angle ABD = \angle DBC$ and $\beta = \angle ACE = \angle ECB$. Note that I is the incenter of triangle ABC, and so $\angle BAI = \angle CAI = 45°$. Applying the Law of Sines to triangle ABI yields

$$\frac{AB}{BI} = \frac{\sin(45° + \alpha)}{\sin 45°} = \sin\alpha + \cos\alpha,$$

by the addition formula (for the sine function). In particular, we conclude that $s = \sin\alpha + \cos\alpha$ is rational. It is clear that $\alpha + \beta = 45^\circ$. By the subtraction formulas, we have

$$s = \sin(45^\circ - \beta) + \cos(45^\circ - \beta) = \sqrt{2}\cos\beta,$$

from which it follows that $\cos\beta$ is not rational. On the other hand, from right triangle ACE, we have $\cos\beta = AC/EC$, which is rational by assumption. Because $\cos\beta$ cannot not be both rational and irrational, our assumption was wrong and not all the segments AB, BC, BI, ID, CI, IE can have integer lengths.

Alternate Solution from Jacek Fabrykowski: Using notations as introduced in the problem, let $BD = m$, $AD = x$, $DC = y$, $AB = c$, $BC = a$ and $AC = b$. The angle bisector theorem implies

$$\frac{x}{b-x} = \frac{c}{a}$$

and the Pythagorean Theorem yields $m^2 = x^2 + c^2$. Both equations imply that

$$2ac = \frac{(bc)^2}{m^2 - c^2} - a^2 - c^2$$

and since $a^2 = b^2 + c^2$ is rational, a is rational too (observe that to reach this conclusion, we only need to assume that b, c, and m are integers). Therefore, $x = \frac{bc}{a+c}$ is also rational, and so is y. Let now (similarly to the notations above from the solution by Zuming Feng) $\angle ABD = \alpha$ and $\angle ACE = \beta$ where $\alpha + \beta = \pi/4$. It is obvious that $\cos\alpha$ and $\cos\beta$are both rational and the above shows that also $\sin\alpha = x/m$ is rational. On the other hand, $\cos\beta = \cos(\pi/4 - \alpha) = (\sqrt{2}/2)(\sin\alpha + \sin\beta)$, which is a contradiction. The solution shows that a stronger statement holds true: There is no right triangle with both legs and bisectors of acute angles all having integer lengths.

Alternate Solution from Zuming Feng: Prove an even stronger result: there is no such right triangle with AB, AC, IB, IC having rational side lengths. Assume on the contrary, that AB, AC, IB, IC have rational side lengths. Then $BC^2 = AB^2 + AC^2$ is rational. On the other hand, in triangle BIC, $\angle BIC = 135^\circ$. Applying the law of cosines to triangle BIC yields

$$BC^2 = BI^2 + CI^2 - \sqrt{2}BI \cdot CI$$

which is irrational. Because BC^2 cannot be both rational and irrational, we conclude that our assumption was wrong and that not all of the segments AB, AC, IB, IC can have rational lengths.

This problem was proposed by Zuming Feng.

JMO 2010 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2010 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let $P(n)$ be the number of permutations $(a_1, \dots, a_n)$ of the numbers $(1, 2, \dots, n)$ for which ka_k is a perfect square for all $1 \le k \le n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

2. Let $n > 1$ be an integer. Find, with proof, all sequences $x_1, x_2, \dots, x_{n-1}$ of positive integers with the following three properties:
   (a) $x_1 < x_2 < \dots < x_{n-1}$;
   (b) $x_i + x_{n-i} = 2n$ for all $i = 1, 2, \dots, n-1$;
   (c) given any two indices i and j (not necessarily distinct) for which $x_i + x_j < 2n$, there is an index k such that $x_i + x_j = x_k$.

3. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

4. A triangle is called a *parabolic* triangle if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

5. Two permutations $a_1, a_2, \dots, a_{2010}$ and $b_1, b_2, \dots, b_{2010}$ of the numbers $1, 2, \dots, 2010$ are said to intersect if $a_k = b_k$ for some value of k in the range $1 \le k \le 2010$. Show that there exist 1006 permutations of the numbers $1, 2, \dots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

6. Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

§1 Solutions to Day 1

§1.1 JMO 2010/1, proposed by Andy Niedermier

Available online at https://aops.com/community/p1860909.

Problem statement

Let $P(n)$ be the number of permutations $(a_1, \dots, a_n)$ of the numbers $(1, 2, \dots, n)$ for which ka_k is a perfect square for all $1 \le k \le n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

The answer is $n = 4489$.

We begin by giving a complete description of $P(n)$:

Claim — We have

$$P(n) = \prod_{c \text{ squarefree}} \left\lfloor \sqrt{\frac{n}{c}} \right\rfloor !$$

Proof. Every positive integer can be uniquely expressed in the form $c \cdot m^2$ where c is a squarefree integer and m is a perfect square. So we may, for each squarefree positive integer c, define the set

$$S_c = \left\{c \cdot 1^2, c \cdot 2^2, c \cdot 3^2, \dots \right\} \cap \{1, 2, \dots, n\}$$

and each integer from 1 through n will be in exactly one S_c. Note also that

$$|S_c| = \left\lfloor \sqrt{\frac{n}{c}} \right\rfloor .$$

Then, the permutations in the problem are exactly those which send elements of S_c to elements of S_c. In other words,

$$P(n) = \prod_{c \text{ squarefree}} |S_c|! = \prod_{c \text{ squarefree}} \left\lfloor \sqrt{\frac{n}{c}} \right\rfloor ! \qquad \square$$

We want the smallest n such that 2010 divides $P(n)$.

- Note that $P(67^2)$ contains $67!$ as a term, which is divisible by 2010, so 67^2 is a candidate.
- On the other hand, if $n < 67^2$, then no term in the product for $P(n)$ is divisible by the prime 67.

So $n = 67^2 = 4489$ is indeed the minimum.

§1.2 JMO 2010/2, proposed by Răzvan Gelca

Available online at https://aops.com/community/p1860914.

Problem statement

Let $n > 1$ be an integer. Find, with proof, all sequences $x_1, x_2, \dots, x_{n-1}$ of positive integers with the following three properties:

(a) $x_1 < x_2 < \cdots < x_{n-1}$;

(b) $x_i + x_{n-i} = 2n$ for all $i = 1, 2, \dots, n-1$;

(c) given any two indices i and j (not necessarily distinct) for which $x_i + x_j < 2n$, there is an index k such that $x_i + x_j = x_k$.

The answer is $x_k = 2k$ only, which obviously work, so we prove they are the only ones.

Let $x_1 < x_2 < \ldots < x_n$ be any sequence satisfying the conditions. Consider:

$$x_1 + x_1 < x_1 + x_2 < x_1 + x_3 < \cdots < x_1 + x_{n-2}.$$

All these are results of condition (c), since $x_1 + x_{n-2} < x_1 + x_{n-1} = 2n$. So each of these must be a member of the sequence.

However, there are $n-2$ of these terms, and there are exactly $n-2$ terms greater than x_1 in our sequence. Therefore, we get the one-to-one correspondence below:

$$\begin{aligned} x_2 &= x_1 + x_1 \\ x_3 &= x_1 + x_2 \\ &\vdots \\ x_{n-1} &= x_1 + x_{n-2} \end{aligned}$$

It follows that $x_2 = 2x_1$, so that $x_3 = 3x_1$ and so on. Therefore, $x_m = mx_1$. We now solve for x_1 in condition (b) to find that $x_1 = 2$ is the only solution, and the desired conclusion follows.

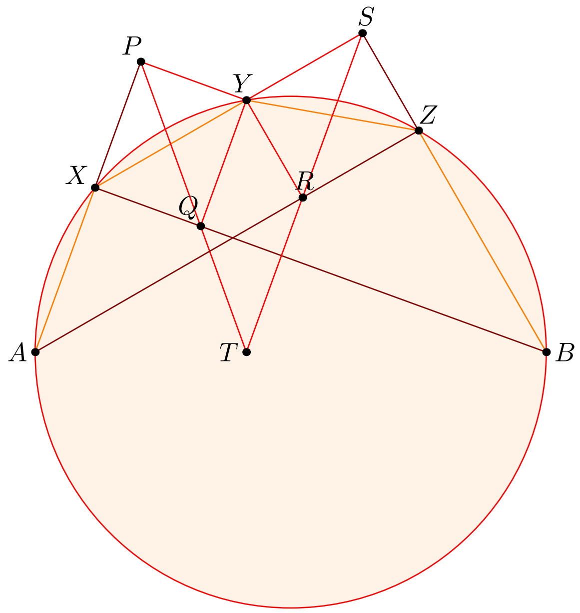
§1.3 JMO 2010/3, proposed by Titu Andreescu

Available online at https://aops.com/community/p1860802.

Problem statement

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Let T be the foot from Y to $\overline{AB}$. Then the Simson line implies that lines PQ and RS meet at T.



Now it's straightforward to see $APYRT$ is cyclic (in the circle with diameter $\overline{AY}$), and therefore

$$\angle RTY = \angle RAY = \angle ZAY.$$

Similarly,

$$\angle YTQ = \angle YBQ = \angle YBX.$$

Summing these gives $\angle RTQ$ is equal to half the measure of arc $\widehat{XZ}$ as needed.

(Of course, one can also just angle chase; the Simson line is not so necessary.)

§2 Solutions to Day 2

§2.1 JMO 2010/4, proposed by Zuming Feng

Available online at https://aops.com/community/p1860772.

Problem statement

A triangle is called a *parabolic* triangle if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

For $n = 0$, take instead $(a, b) = (1, 0)$.

For $n > 0$, consider a triangle with vertices at (a, a^2), $(-a, a^2)$ and (b, b^2). Then the area of this triangle was equal to

$$\frac{1}{2}(2a)\left(b^2 - a^2\right) = a(b^2 - a^2).$$

To make this equal $2^{2n}m^2$, simply pick $a = 2^{2n}$, and then pick b such that $b^2 - m^2 = 2^{4n}$, for example $m = 2^{4n-2} - 1$ and $b = 2^{4n-2} + 1$.

§2.2 JMO 2010/5, proposed by Gregory Galperin

Available online at https://aops.com/community/p1860912.

Problem statement

Two permutations $a_1, a_2, \dots, a_{2010}$ and $b_1, b_2, \dots, b_{2010}$ of the numbers $1, 2, \dots, 2010$ are said to intersect if $a_k = b_k$ for some value of k in the range $1 \le k \le 2010$. Show that there exist 1006 permutations of the numbers $1, 2, \dots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

A valid choice is the following 1006 permutations:

$$\begin{array}{ccccccc|ccccc}
1 & 2 & 3 & \cdots & 1004 & 1005 & 1006 & 1007 & 1008 & \cdots & 2009 & 2010 \\
2 & 3 & 4 & \cdots & 1005 & 1006 & 1 & 1007 & 1008 & \cdots & 2009 & 2010 \\
3 & 4 & 5 & \cdots & 1006 & 1 & 2 & 1007 & 1008 & \cdots & 2009 & 2010 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1004 & 1005 & 1006 & \cdots & 1001 & 1002 & 1003 & 1007 & 1008 & \cdots & 2009 & 2010 \\
1005 & 1006 & 1 & \cdots & 1002 & 1003 & 1004 & 1007 & 1008 & \cdots & 2009 & 2010 \\
1006 & 1 & 2 & \cdots & 1003 & 1004 & 1005 & 1007 & 1008 & \cdots & 2009 & 2010
\end{array}$$

This works. Indeed, any permutation should have one of $\{1, 2, \dots, 1006\}$ somewhere in the first 1006 positions, so one will get an intersection.

> **Remark.** In fact, the last 1004 entries do not matter with this construction, and we chose to leave them as $1007, 1008, \dots, 2010$ only for concreteness.

> **Remark.** Using Hall's marriage lemma one may prove that the result becomes false with 1006 replaced by 1005.

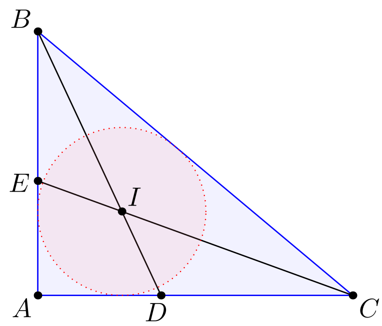
§2.3 JMO 2010/6, proposed by Zuming Feng

Available online at https://aops.com/community/p1860753.

Problem statement

Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

The answer is no. We prove that it is not even possible that AB, AC, CI, IB are all integers.



First, we claim that $\angle BIC = 135°$. To see why, note that

$$\angle IBC + \angle ICB = \frac{\angle B}{2} + \frac{\angle C}{2} = \frac{90°}{2} = 45°.$$

So, $\angle BIC = 180° - (\angle IBC + \angle ICB) = 135°$, as desired.

We now proceed by contradiction. The Pythagorean theorem implies

$$BC^2 = AB^2 + AC^2$$

and so BC^2 is an integer. However, the law of cosines gives

$$\begin{aligned} BC^2 &= BI^2 + CI^2 - 2BI \cdot CI \cos \angle BIC \\ &= BI^2 + CI^2 + BI \cdot CI \cdot \sqrt{2}. \end{aligned}$$

which is irrational, and this produces the desired contradiction.

2^{nd} United States of America Junior Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 27, 2011

JMO 1. Find, with proof, all positive integers n for which $2^n + 12^n + 2011^n$ is a perfect square.

JMO 2. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3\,.$$

JMO 3. For a point $P = (a, a^2)$ in the coordinate plane, let $\ell(P)$ denote the line passing through P with slope $2a$. Consider the set of triangles with vertices of the form $P_1 = (a_1, a_1^2)$, $P_2 = (a_2, a_2^2)$, $P_3 = (a_3, a_3^2)$, such that the intersections of the lines $\ell(P_1), \ell(P_2), \ell(P_3)$ form an equilateral triangle Δ. Find the locus of the center of Δ as $P_1P_2P_3$ ranges over all such triangles.

2^{nd} United States of America Junior Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 28, 2011

JMO 4. A *word* is defined as any finite string of letters. A word is a *palindrome* if it reads the same backwards as forwards. Let a sequence of words $W_0, W_1, W_2, \ldots$ be defined as follows: $W_0 = a$, $W_1 = b$, and for $n \geq 2$, W_n is the word formed by writing W_{n-2} followed by W_{n-1}. Prove that for any $n \geq 1$, the word formed by writing $W_1, W_2, \ldots, W_n$ in succession is a palindrome.

JMO 5. Points A, B, C, D, E lie on circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

JMO 6. Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

2^{nd} United States of America Junior Mathematical Olympiad

1. The answer is $n = 1$. Clearly, $n = 1$ is a solution because $2 + 12 + 2011 = 45^2$. Next we show that there is no other solutions.

   Assume that $n \geq 2$. If n is odd, then $2^n + 12^n + 2011^n$ cannot be a perfect square because it is congruent to 3 modulo 4. If n is even, we can complete our solution in two ways.

   - $2^n + 12^n + 2011^n$ cannot be a perfect square because it is congruent to 2 modulo 3.
   - $2^n + 12^n + 2011^n$ cannot be a perfect square because it is in between two consecutive perfect squares. Indeed, say $n = 2k$, then

     $$(2011^k)^2 < 2^{2k}+12^{2k}+2011^{2k} = 4^k+144^k+2011^{2k} < 1+2\cdot 2011^k+2011^{2k} = (2011^k+1)^2.$$

2. The given condition is equivalent to $a^2 + b^2 + c^2 + ab + bc + ca \leq 2$. We will prove that

   $$\frac{2ab+2}{(a+b)^2} + \frac{2bc+2}{(b+c)^2} + \frac{2ca+2}{(c+a)^2} \geq 6\,.$$

   Indeed, we have

   $$\frac{2ab+2}{(a+b)^2} \geq \frac{2ab+a^2+b^2+c^2+ab+bc+ca}{(a+b)^2} = 1 + \frac{(c+a)(c+b)}{(a+b)^2}.$$

   Adding the last inequality with its cyclic analogous forms yields

   $$\frac{2ab+2}{(a+b)^2} + \frac{2bc+2}{(b+c)^2} + \frac{2ca+2}{(c+a)^2} \geq 3 + \frac{(c+a)(c+b)}{(a+b)^2} + \frac{(a+b)(a+c)}{(b+c)^2} + \frac{(b+c)(b+a)}{(c+a)^2}$$

   Hence it remains to prove that

   $$\frac{(c+a)(c+b)}{(a+b)^2} + \frac{(a+b)(a+c)}{(b+c)^2} + \frac{(b+c)(b+a)}{(c+a)^2} \geq 3.$$

   But this follows directly from the AM–GM inequality. Equality holds if and only if $a+b = b+c = c+a$, which together with the given condition, shows that it occurs if and only if $a = b = c = \frac{1}{\sqrt{3}}$.

OR

Set $2x = a + b$, $2y = b + c$, and $2z = c + a$; that is, $a = z + x - y$, $b = x + y - z$, and $c = y + z - x$. Hence

$$\frac{ab+1}{(a+b)^2} = \frac{(z+x-y)(x+y-z)+1}{4x^2} = \frac{x^2-(y-z)^2+1}{4x^2} = \frac{x^2+2yz+1-y^2-z^2}{4x^2}.$$

On the other hand, the given condition is equivalent to $2a^2+2b^2+2c^2+2ab+2bc+2ca \le 4$ or $(a+b)^2+(b+c)^2+(c+a)^2 \le 4$; that is, $x^2+y^2+z^2 \le 1$ or $1-y^2-z^2 \ge x^2$. It follows that

$$\frac{ab+1}{(a+b)^2} = \frac{x^2+2yz+1-y^2-z^2}{4x^2} \ge \frac{x^2+2yz+x^2}{4x^2} = \frac{1}{2} + \frac{yz}{2x^2}.$$

Likewise, we have

$$\frac{bc+1}{(b+c)^2} = \frac{1}{2} + \frac{zx}{2y^2} \quad \text{and} \quad \frac{ca+1}{(c+a)^2} = \frac{1}{2} + \frac{xy}{2z^2}.$$

Adding the last three inequalities gives

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge \frac{3}{2} + \frac{yz}{2x^2} + \frac{zx}{2y^2} + \frac{xy}{2z^2} \ge 3,$$

by the AM–GM inequality. Equality holds if and only if $x = y = z$ or $a = b = c = \frac{1}{\sqrt{3}}$.

3. For $1 \le i < j \le 3$, solving the system $y = 2x_i x - x_i^2 = 2x_j x - x_j^2$ yields the intersection $\left(\frac{x_i+x_j}{2}, x_i x_j\right)$ of lines ℓ_i and ℓ_j. Hence the center of the equilateral triangle is

$$O = (O_x, O_y) = \left(\frac{x_1+x_2+x_3}{3}, \frac{x_1x_2+x_2x_3+x_3x_1}{3}\right).$$

Let $0^\circ \le \alpha_i < 180^\circ$ be the standard angle formed by lines ℓ_i and the positive x-axis. Without loss of generality, we may assume that $\alpha_1 < \alpha_2 < \alpha_3$. By the given condition, we have $\alpha_2 - \alpha_1 = \alpha_3 - \alpha_2 = 60^\circ$. By the subtraction formulas, we have

$$\tan 60^\circ = \frac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1\tan\alpha_2} = \frac{\tan\alpha_3 - \tan\alpha_2}{1+\tan\alpha_2\tan\alpha_3} \quad \text{and} \quad \tan 120^\circ = \frac{\tan\alpha_3 - \tan\alpha_1}{1+\tan\alpha_3\tan\alpha_1}$$

or

$$\sqrt{3} = \frac{2x_2-2x_1}{1+4x_1x_2} = \frac{2x_3-2x_2}{1+4x_2x_3} \quad \text{and} \quad -\sqrt{3} = \frac{2x_3-2x_1}{1+4x_3x_1}.$$

Therefore,

$$1+4x_1x_2 = \frac{2(x_2-x_1)}{\sqrt{3}}, \quad 1+4x_2x_3 = \frac{2(x_3-x_2)}{\sqrt{3}}, \quad 1+4x_3x_1 = \frac{2(x_1-x_3)}{\sqrt{3}}. \qquad (1)$$

Adding these equations gives $3 + 4(x_1x_2 + x_2x_3 + x_3x_1) = 0$, implying that $O_y = -\frac{1}{4}$; that is, O always lie on the directrix ℓ of the parabola $y = x^2$.

Next we show that G can be any point on ℓ. Solving the first and the equations in (1) for x_2 and x_3 in terms of x_1 gives

$$x_2 = \frac{2x_1 + \sqrt{3}}{2 - 4\sqrt{3}x_1} \quad \text{and} \quad x_3 = \frac{2x_1 - \sqrt{3}}{2 + 4\sqrt{3}x_1},$$

implying that

$$\begin{aligned} x_1 + x_2 + x_3 &= x_1 + \frac{(2x_1 + \sqrt{3})(2 + 4\sqrt{3}x_1) + (2x_1 - \sqrt{3})(2 - 4\sqrt{3}x_1)}{4 - 48x_1^2} \\ &= x_1 + \frac{8x_1}{1 - 12x_1^2} = \frac{12x_1^3 - 9x_1}{12x_1^2 - 1}. \end{aligned}$$

Because lines ℓ_1, ℓ_2, ℓ_3 are evenly spaced with $60°$ between each other, slopes $2x_1, 2x_2, 2x_3$ are symmetric with each other; that is,

$$x_1 + x_2 + x_3 = \frac{12x_i^3 - 9x_i}{12x_i^2 - 1} \quad \text{for } i = 1, 2, 3.$$

Therefore,

$$O_x = \frac{x_1 + x_2 + x_3}{3} = \frac{4x^3 - 3x}{12x^2 - 1},$$

where $-\infty < x < \infty$, because $x = x_i$ for some $i = 1, 2, 3$, and the combined ranges of slopes $2x_i$ are the interval $(-\infty, \infty)$. Because $4x^3 - 3x = O_x(12x^2 - 1)$ is a cubic equation, it has a real root in x for every real number O_x; that is, the range of O_x is the interval $(-\infty, \infty)$. We conclude that the locus of O is line $y = -\frac{1}{4}$.

4. According to the statement of the problem we have

$$W_0 = a, \quad W_1 = b, \quad W_2 = ab, \quad W_3 = bab, \quad W_4 = abbab,$$

and so forth. Let $V_n = W_1W_2 \cdots W_n$, where we place two or more words next to one another to denote the single word obtained by writing all their letters in succession. We find that

$$V_1 = b, \quad V_2 = bab, \quad V_3 = babbab, \quad V_4 = babbababbab.$$

We wish to show that V_n is a palindrome for all positive integers n. The above list shows this to be true for $1 \le n \le 4$; these cases will serve as the base cases for a proof by strong induction.

We use a bar over a word to indicate writing its letters in the reverse order. Thus $\overline{W_4} =$ *babba* and $\overline{V_3} = V_3$ since V_3 is a palindrome. Now assume that the words V_1 through V_n are all palindromes; we will show that V_{n+1} is also a palindrome. By the definition of V_{n+1} and W_{n+1} we have

$$V_{n+1} = V_n W_{n+1} = \overline{V_n}\, W_{n-1} W_n,$$

using the fact that $\overline{V_n} = V_n$ since V_n is a palindrome. But we know that $V_n = V_{n-2} W_{n-1} W_n$, so we may write

$$\overline{V_n}\, W_{n-1} W_n = \overline{W_n}\; \overline{W_{n-1}}\; \overline{V_{n-2}}\, W_{n-1} W_n.$$

The latter word is clearly a palindrome since V_{n-2} reads the same forward as backwards. Hence V_{n+1} is a palindrome, thus completing the proof.

5. Let O be the center of circle ω and let M be the midpoint of $\overline{AC}$. It is clear that $\overline{DE}$ bisects $\overline{AC}$ if and only if E, M, B are collinear. Consequently, it suffices to show that

$$\angle MED = \angle BED. \tag{2}$$

The proof is divided into four parts.

1. Triangle MED is isosceles with $\angle MED = \angle MDE$. (Note that $ACDE$ is an isosceles trapezoid and M is midpoint of the base $\overline{AC}$. The fact that triangle MED is isosceles then follows by the Pythagorean Theorem if nothing more elegant comes to mind.) This fact together with Alternate Interior Angles gives

$$\angle AME = \angle MED = \angle MDE = \angle PMD.$$

2. *Claim.* The circle ω' with diameter $\overline{OP}$ contains points B, D, and M.

   Proof. For each of the cases $X = B$, D, M, it is straightforward to verify that $\overline{OX}$ is perpendicular to $\overline{PX}$. For $X = B$ it is true that $\overline{OBP}$ is a right angle because $\overline{PB}$ is tangent to the circle at B. The same is true for $X = D$. For $X = M$, simply use the fact that if M is the midpoint of any given chord, then $\overline{OM}$ is perpendicular to the chord.

3. Referring to the circle ω', the Inscribed Angle Theorem gives $\angle PBD = \angle PMD$.

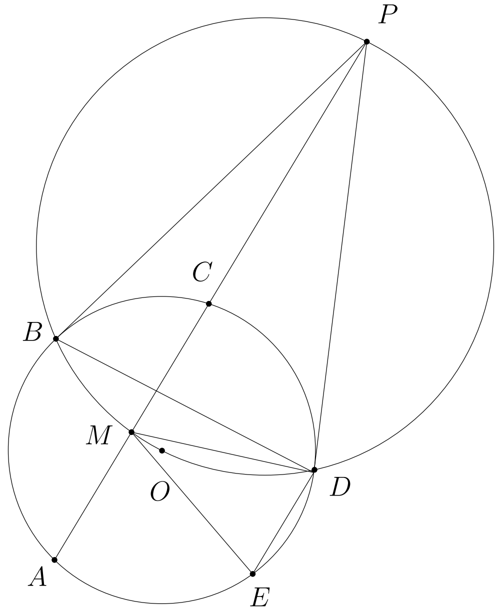
4. Because $\overline{BP}$ is tangent to ω at B,

$$\angle BED = \frac{1}{2}\overset{\frown}{BD} = \angle PBD.$$

Results from step 1 yield

$$\angle BED = \angle PBD = \angle PMD = \angle MED,$$

establishing 2 and completing the proof.



6. The assertion is false, and the smallest n for which it fails is $n = 25$. Given $n \geq 2$, let r be the remainder when 2^n is divided by n. Then $2^n = kn + r$ where k is a positive integer and $0 \leq r < n$. It follows that

$$2^{2^n} = 2^{kn+r} \equiv 2^r \mod 2^n - 1,$$

and $2^r < 2^n - 1$ so 2^r is the remainder when 2^{2^n} is divided by $2^n - 1$. If r is even then 2^r is power of 4. Hence to disprove the assertion, it is enough to find an n for which the corresponding r is odd.

If n is even then so is $r = 2^n - kn$.

If n is an odd prime then $2^n \equiv 2 \pmod{n}$ by Fermat's Little Theorem; hence $r \equiv 2^n \equiv 2$ mod n and $r = 2$.

There remains the case in which n is odd and composite. In the first three instances $n = 9$, 15, 21 there is no contradiction to the assertion:

$$\begin{aligned} n = 9 :& 2^6 \equiv 1 \mod 9 \Rightarrow 2^9 \equiv 2^6 \cdot 2^3 \equiv 8 \mod 9 \\ n = 15 :& 2^4 \equiv 1 \mod 15 \Rightarrow 2^{15} \equiv (2^4)^3 \cdot 2^3 \equiv 8 \mod 15 \\ n = 21 :& 2^6 \equiv 1 \mod 21 \Rightarrow 2^{21} \equiv (2^6)^3 \cdot 2^3 \equiv 8 \mod 21 \end{aligned}$$

However,

$$2^{10} = 1024 \equiv -1 \Rightarrow 2^{20} \equiv 1 \Rightarrow 2^{25} \equiv 2^5 \equiv 7 \mod 25,$$

so 7 is the remainder when 2^{25} is divided by 25 and 2^7 is the remainder when $2^{2^{25}}$ is divided by $2^{25} - 1$.

JMO 2011 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2011 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Find all positive integers n such that $2^n + 12^n + 2011^n$ is a perfect square.

2. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a+b+c)^2 \le 4$. Prove that
$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge 3.$$

3. For a point $P = (a, a^2)$ in the coordinate plane, let $\ell(P)$ denote the line passing through P with slope $2a$. Consider the set of triangles with vertices of the form $P_1 = (a_1, a_1^2)$, $P_2 = (a_2, a_2^2)$, $P_3 = (a_3, a_3^2)$, such that the intersection of the lines $\ell(P_1), \ell(P_2), \ell(P_3)$ form an equilateral triangle Δ. Find the locus of the center of Δ as $P_1P_2P_3$ ranges over all such triangles.

4. A *word* is defined as any finite string of letters. A word is a *palindrome* if it reads the same backwards and forwards. Let a sequence of words $W_0, W_1, W_2, \dots$ be defined as follows: $W_0 = a, W_1 = b$, and for $n \ge 2$, W_n is the word formed by writing W_{n-2} followed by W_{n-1}. Prove that for any $n \ge 1$, the word formed by writing $W_1, W_2, W_3, \dots, W_n$ in succession is a palindrome.

5. Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

6. Consider the assertion that for each positive integer $n \ge 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

§1 Solutions to Day 1

§1.1 JMO 2011/1, proposed by Titu Andreescu

Available online at https://aops.com/community/p2254778.

Problem statement

Find all positive integers n such that $2^n + 12^n + 2011^n$ is a perfect square.

The answer $n = 1$ works, because $2^1 + 12^1 + 2011^1 = 45^2$. We prove it's the only one.

- If $n \geq 2$ is even, then modulo 3 we have $2^n + 12^n + 2011^n \equiv 1 + 0 + 1 \equiv 2 \pmod 3$ so it is not a square.
- If $n \geq 3$ is odd, then modulo 4 we have $2^n + 12^n + 2011^n \equiv 0 + 0 + 3 \equiv 3 \pmod 4$ so it is not a square.

This completes the proof.

§1.2 JMO 2011/2, proposed by Titu Andreescu

Available online at https://aops.com/community/p2254758.

Problem statement

Let a, b, c be positive real numbers such that $a^2+b^2+c^2+(a+b+c)^2 \le 4$. Prove that
$$\frac{ab+1}{(a+b)^2}+\frac{bc+1}{(b+c)^2}+\frac{ca+1}{(c+a)^2} \ge 3.$$

The condition becomes $2 \ge a^2+b^2+c^2+ab+bc+ca$. Therefore,

$$\begin{aligned}
\sum_{\text{cyc}} \frac{2ab+2}{(a+b)^2} &\ge \sum_{\text{cyc}} \frac{2ab+(a^2+b^2+c^2+ab+bc+ca)}{(a+b)^2} \\
&= \sum_{\text{cyc}} \frac{(a+b)^2+(c+a)(c+b)}{(a+b)^2} \\
&= 3+\sum_{\text{cyc}} \frac{(c+a)(c+b)}{(a+b)^2} \\
&\ge 3+3\sqrt[3]{\prod_{\text{cyc}} \frac{(c+a)(c+b)}{(a+b)^2}} = 3+3=6
\end{aligned}$$

with the last line by AM-GM. This completes the proof.

§1.3 JMO 2011/3, proposed by Zuming Feng

Available online at https://aops.com/community/p2254823.

Problem statement

For a point $P = (a, a^2)$ in the coordinate plane, let $\ell(P)$ denote the line passing through P with slope $2a$. Consider the set of triangles with vertices of the form $P_1 = (a_1, a_1^2)$, $P_2 = (a_2, a_2^2)$, $P_3 = (a_3, a_3^2)$, such that the intersection of the lines $\ell(P_1), \ell(P_2), \ell(P_3)$ form an equilateral triangle Δ. Find the locus of the center of Δ as $P_1P_2P_3$ ranges over all such triangles.

The answer is the line $y = -1/4$. I did not find this problem inspiring, so I will not write out most of the boring calculations since most solutions are just going to be "use Cartesian coordinates and grind all the way through".

The "nice" form of the main claim is as follows (which is certainly overkill for the present task, but is too good to resist including):

Claim (Naoki Sato) — In general, the orthocenter of Δ lies on the directrix $y = -1/4$ of the parabola (even if the triangle Δ is not equilateral).

Proof. By writing out the equation $y = 2a_ix - a_i^2$ for $\ell(P_i)$, we find the vertices of the triangle are located at

$$\left(\frac{a_1+a_2}{2}, a_1a_2\right); \quad \left(\frac{a_2+a_3}{2}, a_2a_3\right); \quad \left(\frac{a_3+a_1}{2}, a_3a_1\right).$$

The coordinates of the orthocenter can be checked explicitly to be

$$H = \left(\frac{a_1+a_2+a_3+4a_1a_2a_3}{2}, -\frac{1}{4}\right).$$

An advanced synthetic proof of this fact is given at https://aops.com/community/p2255814. □

This claim already shows that every point lies on $y = -1/4$. We now turn to showing that, even when restricted to equilateral triangles, we can achieve every point on $y = -1/4$. In what follows $a = a_1$, $b = a_2$, $c = a_3$ for legibility.

Claim — Lines $\ell(a)$, $\ell(b)$, $\ell(c)$ form an equilateral triangle if and only if

$$a + b + c = -12abc$$
$$ab + bc + ca = -\frac{3}{4}.$$

Moreover, the x-coordinate of the equilateral triangle is $\frac{1}{3}(a + b + c)$.

Proof. The triangle is equilateral if and only if the centroid and orthocenter coincide, i.e.

$$\left(\frac{a+b+c}{3}, \frac{ab+bc+ca}{3}\right) = G = H = \left(\frac{a+b+c+4abc}{2}, -\frac{1}{4}\right).$$

Setting the x and y coordinates equal, we derive the claimed equations. □

Let λ be any real number. We are tasked to show that

$$P(X) = X^3 - 3\lambda \cdot X^2 - \frac{3}{4}X + \frac{\lambda}{4}$$

has three real roots (with multiplicity); then taking those roots as (a, b, c) yields a valid equilateral-triangle triple whose x-coordinate is exactly λ, be the previous claim.

To prove that, pick the values

$$\begin{aligned} P(-\sqrt{3}/2) &= -2\lambda \\ P(0) &= \tfrac{1}{4}\lambda \\ P(\sqrt{3}/2) &= -2\lambda. \end{aligned}$$

The intermediate value theorem (at least for $\lambda \neq 0$) implies that P should have at least two real roots now, and since P has degree 3, it has all real roots. That's all.

§2 Solutions to Day 2

§2.1 JMO 2011/4, proposed by Gabriel Carroll

Available online at https://aops.com/community/p2254808.

Problem statement

A *word* is defined as any finite string of letters. A word is a *palindrome* if it reads the same backwards and forwards. Let a sequence of words $W_0, W_1, W_2, \dots$ be defined as follows: $W_0 = a, W_1 = b$, and for $n \geq 2$, W_n is the word formed by writing W_{n-2} followed by W_{n-1}. Prove that for any $n \geq 1$, the word formed by writing $W_1, W_2, W_3, \dots, W_n$ in succession is a palindrome.

To aid in following the solution, here are the first several words:

$$\begin{aligned}
W_0 &= a \\
W_1 &= b \\
W_2 &= ab \\
W_3 &= bab \\
W_4 &= abbab \\
W_5 &= bababbab \\
W_6 &= abbabbababbab \\
W_7 &= bababbababbabbababbab
\end{aligned}$$

We prove that $W_1 W_2 \cdots W_n$ is a palindrome by induction on n. The base cases $n = 1, 2, 3, 4$ can be verified by hand.

For the inductive step, we let $\overline{X}$ denote the word X written backwards. Then

$$\begin{aligned}
W_1 W_2 \cdots W_{n-3} W_{n-2} W_{n-1} W_n &\overset{\text{IH}}{=} (\overline{W_{n-1} W_{n-2} W_{n-3}} \cdots \overline{W_2 W_1}) W_n \\
&= (\overline{W_{n-1} W_{n-2} W_{n-3}} \dots \overline{W_2 W_1}) W_{n-2} W_{n-1} \\
&= \overline{W_{n-1} W_{n-2}} (\overline{W_{n-3}} \dots \overline{W_2 W_1}) W_{n-2} W_{n-1}
\end{aligned}$$

with the first equality being by the induction hypothesis. By induction hypothesis again the inner parenthesized term is also a palindrome, and so this completes the proof.

§2.2 JMO 2011/5, proposed by Zuming Feng

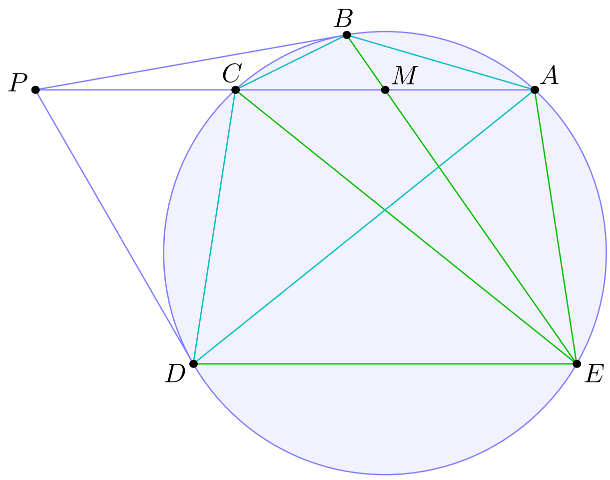
Available online at https://aops.com/community/p2254813.

Problem statement

Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

We present two solutions.

¶ First solution using harmonic bundles Let $M = \overline{BE} \cap \overline{AC}$ and let ∞ be the point at infinity along $\overline{DE} \parallel \overline{AC}$.



Note that $ABCD$ is harmonic, so

$$-1 = (AC;BD) \stackrel{E}{=} (AC;M\infty)$$

implying M is the midpoint of $\overline{AC}$.

¶ Second solution using complex numbers (Cynthia Du) Suppose we let b, d, e be free on unit circle, so $p = \frac{2bd}{b+d}$. Then $d/c = a/e$, and $a + c = p + ac\overline{p}$. Consequently,

$$\begin{aligned} ac &= de \\ \frac{1}{2}(a+c) &= \frac{bd}{b+d} + de \cdot \frac{1}{b+d} = \frac{d(b+e)}{b+d}. \\ \frac{a+c}{2ac} &= \frac{(b+e)}{e(b+d)}. \end{aligned}$$

From here it's easy to see
$$\frac{a+c}{2} + \frac{a+c}{2ac} \cdot be = b + e$$
which is what we wanted to prove.

§2.3 JMO 2011/6, proposed by Sam Vandervelde

Available online at https://aops.com/community/p2254810.

Problem statement

Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

We claim $n = 25$ is a counterexample. Since $2^{25} \equiv 2^0 \pmod{2^{25}-1}$, we have

$$2^{2^{25}} \equiv 2^{2^{25} \bmod 25} \equiv 2^7 \bmod 2^{25} - 1$$

and the right-hand side is actually the remainder, since $0 < 2^7 < 2^{25}$. But 2^7 is not a power of 4.

> **Remark.** Really, the problem is just equivalent for asking 2^n to have odd remainder when divided by n.

3^{rd} United States of America Junior Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 24, 2012

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

JMO 1. Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).

JMO 2. Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, ..., a_n$ with

$$\max(a_1, a_2, ..., a_n) \leq n \cdot \min(a_1, a_2, ..., a_n),$$

there exist three that are the side lengths of an acute triangle.

JMO 3. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geq \frac{2}{3}(a^2 + b^2 + c^2)\,.$$

3rd United States of America Junior Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 25, 2012

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

JMO 4. Let α be an irrational number with $0 < \alpha < 1$, and draw a circle in the plane whose circumference has length 1. Given any integer $n \geq 3$, define a sequence of points P_1, P_2, $\ldots$, P_n as follows. First select any point P_1 on the circle, and for $2 \leq k \leq n$ define P_k as the point on the circle for which the length of arc $P_{k-1}P_k$ is α, when travelling counter-clockwise around the circle from P_{k-1} to P_k. Suppose that P_a and P_b are the nearest adjacent points on either side of P_n. Prove that $a + b \leq n$.

JMO 5. For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \leq k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

JMO 6. Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB, respectively. Prove that A', B', C' are collinear.

3^{rd} United States of America Junior Mathematical Olympiad

Day I, II 12:30 PM – 5 PM EDT

April 24-25, 2012

JMO 1. **Solution 1** We use the following lemma.

Lemma. Given a triangle ABC, X, Y, Z are points on BC, CA, AB respectively. Then three perpendicular lines of BC, CA, AB which go through X, Y, Z respectively are concurrent if and only if $AY^2 + BZ^2 + CX^2 = AZ^2 + BX^2 + CY^2$.

Proof of Lemma. If the lines are concurrent, let P be the point on the three lines. From $BX^2 - CX^2 = (PB^2 - PX^2) - (PC^2 - PX^2) = PB^2 - PC^2$ and so on, we obtain the desired result. Conversely, if $AY^2 + BZ^2 + CX^2 = AZ^2 + BX^2 + CY^2$ holds, let Q be the intersection of perpendicular lines of BC, CA which go through X, Y respectively. Then as we have seen $BX^2 - CX^2 = QB^2 - QC^2$ and $CY^2 - AY^2 = QC^2 - QA^2$ holds. Summing up these equations, we have $AZ^2 - BZ^2 = QA^2 - QB^2$. This implies that QZ and AB are perpendicular, as desired. **End of the Proof**

Let M be the midpoint of SR. We show that $AP^2 + BM^2 + CQ^2 = AQ^2 + BP^2 + CM^2$. Since $AP = AQ$, $CQ^2 = CR \cdot CS$, $BP^2 = BS \cdot BR$, and $BM^2 - CM^2 = (BM + CM)(BM - CM) = BC(BS - RC)$, we have $(AP^2 + BM^2 + CQ^2) - (AQ^2 + BP^2 + CM^2) = BC(BS - RC) - BS \cdot BR + CR \cdot CS = BS \cdot CR - CR \cdot BC = 0$. Thus there exists a point O such that $OP \perp BC$, $OQ \perp AC$, $OM \perp BC$. Then O is the center of a circimcircle of PRS, since the circle is tangent to AB at P. Similarly, O is the center of a circumcircle of QRS, which implies that P, Q, R, S are on a circle.

Solution 2 By the given hypothesis, we have a circle Γ_1 which passes through S and R, and touches AB at P. Similarly, we have a circle Γ_2 which passes through S and R, and touches AC at Q. Suppose that the circles Γ_1 and Γ_2 are different from each other. Then the power of A onto Γ_1 is AP^2, and the power of A onto Γ_2 is AQ^2. This implies that A is on the radical axis of Γ_1 and Γ_2, namely the line BC, which is a contradiction. Hence, we have $\Gamma_1 = \Gamma_2$, so that P, Q, R, S are concyclic, as desired.

Solution 3 We use the same notations as in the Solution 2. Suppose again that $\Gamma_1 \neq \Gamma_2$. Let l be the perpendicular bisector of SR, and consider a circle γ passing through S and R whose center is moving on l. Suppose that initially the center of γ is on the half plane divided by BC in which A does not lie. Moving the center toward A, γ would touch AB and AC, not simultaneously by the hypothesis. Without loss of generality, suppose that γ touches AB at P first, and then touches AC at Q. Note that γ of these situations are Γ_1 and Γ_2 respectively.

We increases the radius of Γ_1, keeping the circle tangent to AB. Then it will touch AC eventually. Let Γ_1' be the circle, which is tangent to AB and AC at P and Q respectively and meets BC at two points S' and R'. Note that on BC, the points are ordered as B, S', S, R, R', C. We have $\angle BPS = \angle PRS$ and $\angle BPS' = \angle PR'S'$, which imply $\angle SPS' = \angle RPR'$. Similarly, we have $\angle SQS' = \angle RQR'$. Without loss of generality, suppose that on the circle Γ_1', the points are ordered as S', P, Q, R'. Let lines PS, PR, QS, QR meet Γ_1' again at T_1, U_1, T_2, U_2 respectively. Then the points on Γ_1' are ordered as $S', T_2, T_1, U_2, U_1, R'$. From $\angle SPS' = \angle RPR'$ we have $\overset{\frown}{S'T_1} = \overset{\frown}{U_1R'}$ and from $\angle SQS' = \angle RQR'$, we have $\overset{\frown}{S'T_2} = \overset{\frown}{U_2R'}$. However, we have $\overset{\frown}{S'T_2} < \overset{\frown}{S'T_1} = \overset{\frown}{U_1R'} < \overset{\frown}{U_2R'}$, which leads us to a contradiction. Hence, we have $\Gamma_1 = \Gamma_2$, as desired.

Solution 4 Let Γ_3 be the circle tangent to AB and AC at P and Q respectively. Inverse the plane around P. We denote by X' the image of any point or any set X via the inversion. A', P, B' are collinear in this order, and the image of AC is a circle $(AC)'$ passing through A' and P. Then Γ_3' is a line which is tangent to $(AC)'$ and parallel to $A'P$. Note that the tangency point is Q'. Γ_1' is a line parallel to $A'P$. Finally, B', S', R' are on a circle passing through P, and S', R' are on Γ_1'.

Suppose $\Gamma_1 \neq \Gamma_3$. Then clearly we have $\Gamma_1' \neq \Gamma_3'$. Note that Q' is on the perpendicular bisector l of $A'P$. Since $PB'R'S'$ is cyclic and PB' and $R'S'$ are parallel, it is an isosceles trapezoid. Now we consider Γ_2'. This circle should be tangent to Γ_1' at Q', so the center of Γ_2' must lie on l. However, Since Γ_2' passes through R' and S', the center must lie on the perpendicular bisector of $R'S'$ which is the same as the one of PB'. Since A' and B' lie on the different ray centered on P, this is impossible. Therefore, we have $\Gamma_1 = \Gamma_3$, on which P, Q, R, S lie.

Solution 5 In the case that $AB = AC$, suppose $\alpha = \angle BPS > \angle CQR = \beta$. Let R' be a point on BC such that $BS = R'C$. We then have that two triangles BPS and CQR' are congruent. Hence, $\angle CQR' = \alpha > \beta = \angle CQR$, so that R lies between R' and C. However, then we have $\beta = \angle QSC = \angle PR'S > \angle PRS = \alpha$, contradiction. Hence we have $\alpha = \beta$, so the trapezoid $PQRS$ is isosceles, as desired.

Now suppose $AB \neq AC$, and PQ and BC meet at X. Without loss of generality, suppose $B > C$ so that B lies between X and C. Let $AP = AQ = t, XB = x, BS = y, RC = z$. To deduce x, we apply Menelaus' theorem to the triangle ABC and a line XPQ to obtain $\frac{AQ}{QC}\frac{CX}{XB}\frac{BP}{PA} = 1$. This yields $x = \frac{c-t}{b-c}a$.

From the hypothesis, we have $(c-t)^2 = y(a-z)$ and $(b-t)^2 = z(a-y)$. From these results, we have $(c-t)^2 - (b-t)^2 = (y-z)a$, so that $y - z = \frac{(c-b)(b+c-2t)}{a}$. Hence, we obtain

$$\begin{aligned} XS \cdot XR &= (x+y)(x+a-z) = x^2 + (a+y-z)x + (c-t)^2 \\ &= x^2 + (a + \frac{(c-b)(b+c-2t)}{a})x + (c-t)^2 \\ &= \frac{(c-t)^2}{(b-c)^2}a^2 + \frac{c-t}{b-c}a^2 + (t-c)(b+c-2t) + (c-t)^2 \\ &= \frac{(b-t)(c-t)}{(b-c)^2}a^2 + (t-c)(b-t) = \frac{(b-t)(c-t)}{(b-c)^2}(a^2 - (b-c)^2) \\ &= \frac{(b-t)(c-t)}{(b-c)^2}(a-b+c)(a+b-c). \end{aligned}$$

On the other hand, since $\angle APQ = \frac{\pi - A}{2}$, we have $\angle PXB = \frac{B-C}{2}$. Applying the Sine theorem to the triangle XPB, we have $\frac{x}{\sin\frac{\pi-A}{2}} = \frac{XP}{\sin B} \Leftrightarrow XP = x\frac{\sin B}{\cos\frac{A}{2}}$. From Menelaus' theorem again, we have $\frac{QX}{XP}\frac{PB}{BA}\frac{AC}{CQ} = 1$, or equivalently $XQ = XP\frac{c}{c-t}\frac{b-t}{b}$. Hence, we have

$$\begin{aligned}
XP \cdot XQ &= x^2 \frac{\sin^2 B}{\cos^2\frac{A}{2}} \frac{c(b-t)}{b(c-t)} \\
&= \frac{(c-t)^2}{(b-c)^2} a^2 \frac{(\frac{b}{2R})^2}{\frac{(a+b+c)(-a+b+c)}{4bc}} \frac{c(b-t)}{b(c-t)} \\
&= \frac{(b-t)(c-t)}{(b-c)^2} \frac{a^2b^2c^2}{R^2(a+b+c)(-a+b+c)} \\
&= \frac{(b-t)(c-t)}{(b-c)^2} \frac{16R^2S^2}{R^2(a+b+c)(-a+b+c)} \\
&= \frac{(b-t)(c-t)}{(b-c)^2}(a-b+c)(a+b-c),
\end{aligned}$$

where R is the circumradius of the triangle ABC and S is the area of the triangle ABC. Since we have now that $XP \cdot XQ = XS \cdot XR$, the four points are concyclic, as desired.

Comment. It is a degenerated version of the following statement: if $ABCDEF$ is a convex hexagon and $ABCD, CDEF$, and $EFAB$ are cyclic quadrilaterals, then $ABCDEF$ is a cyclic hexagon. This can be easily verified by the similar idea to the First and Second solution.

This problem and solution were suggested by Sungyoon Kim and Inseok Seo.

JMO 2. First we prove that any $n \geq 13$ is a solution of the problem. Suppose that $a_1, a_2, ..., a_n$ satisfy $\max(a_1, a_2, ..., a_n) \leq n \cdot \min(a_1, a_2, ..., a_n)$, and that we cannot find three that are the side-lengths of an acute triangle. We may assume that $a_1 \leq a_2 \leq ... \leq a_n$. Then $a_{i+2}^2 \geq a_i^2 + a_{i+1}^2$ for all $i \leq n-2$. Let (F_n) be the Fibonacci sequence, with $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. It is easy to check that $F_n < n^2$ for $n \leq 11$, $F_{12} = 12^2$ and $F_n > n^2$ for $n > 12$ (the last inequality follows by an immediate induction, while the first one can be checked by hand). The inequality $a_{i+2}^2 \geq a_i^2 + a_{i+1}^2$ and the fact that $a_1 \leq a_2 \leq ... \leq a_n$ imply that $a_i^2 \geq F_i \cdot a_1^2$ for all $i \leq n$. Hence, if $n \geq 13$, we obtain $a_n^2 > n^2 \cdot a_1^2$, contradicting the hypothesis. This shows that any $n \geq 13$ is a solution of the problem.

By taking $a_i = \sqrt{F_i}$ for $1 \leq i \leq n$, we have $\max(a_1, a_2, ..., a_n) \leq n \cdot \min(a_1, a_2, ..., a_n)$, for any $n < 13$, but it is easy to see that no three a_i's can be the side-lengths of an acute triangle. Hence the answer to the problem is: all $n \geq 13$.

This problem and solution were suggested by Titu Andreescu.

JMO 3. **Solution 1:** Recall the following form of Cauchy-Schwarz inequality,

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \ldots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{y_1 + y_2 + \ldots + y_n}.$$

It also follows from the Cauchy-Schwarz inequality that $x_1^2+x_2^2+x_3^2 \geq x_1x_2+x_2x_3+x_3x_1$. From these two inequalities, deduce that

$$\begin{aligned} \frac{a^3}{5a+b}+\frac{b^3}{5b+c}+\frac{c^3}{5c+a} &= \frac{a^4}{5a^2+ab}+\frac{b^4}{5b^2+bc}+\frac{c^4}{5c^2+ca} \\ &\geq \frac{(a^2+b^2+c^2)^2}{5(a^2+b^2+c^2)+(ab+bc+ca)} \\ &\geq \frac{1}{6}(a^2+b^2+c^2). \end{aligned}$$

The equality holds if and only if $a=b=c$.

This problem and solution were suggested by Titu Andreescu.

Solution 2: Note that

$$\begin{aligned} 0 &\leq (41a+83b)(a-b)^2 \\ &= 41a^3+a^2b-125ab^2+83b^3, \end{aligned}$$

which is equivalent to

$$(5a+b)\left(-a^2+25b^2\right) \leq 36\left(a^3+3b^3\right).$$

Hence,

$$\frac{a^3+3b^3}{5a+b} \geq -\frac{1}{36}a^2+\frac{25}{36}b^2.$$

Adding this with two other analogous inequalities completes the proof.

Discovery: The solution can be discovered naturally. We start with guessing

$$\frac{a^3+3b^3}{5a+b} \geq ta^2+\left(\frac{2}{3}-t\right)b^2,$$

and rewrite it into

$$(1-5t)a^3-ta^2b-5\left(\frac{2}{3}-t\right)ab^2+\left(\frac{7}{3}+t\right)b^3 \geq 0.$$

Wishing $(a-b)^2$ to be a factor, we use synthetic division to write the left-hand side as

$$(a-b)^2\left[(1-5t)a+(2-11t)b\right]-\left(\frac{1}{3}+12t\right)b^3,$$

and get $t=-1/36$ by setting the remainder equal to 0.

This solution was suggested by Titu Andreescu and independently by Li Zhou, Polk State College, Winter Haven, FL.

Solution 3:

It is convenient to use the shorthand notation $\sum_{\text{cyc}} *$ to denote the sum of the three expressions obtained from $*$ by cyclically permuting the variables a, b, c. For instance,

$$\sum_{\text{cyc}} a^4 b = a^4 b + b^4 c + c^4 a.$$

In this notation, by clearing denominators, we may rewrite the desired inequality as

$$0 \leq \sum_{\text{cyc}} (190a^4 b + 35a^3 b^2 + 38ab^4 - 35a^2 b^3 - 168a^3 bc - 60a^2 b^2 c). \tag{1}$$

It is tempting to attempt to prove this using Muirhead's inequality, but this fails because we are working with cyclic sums rather than symmetric sums. For instance, it is not true that

$$\sum_{\text{cyc}} a^4 b \geq \sum_{\text{cyc}} a^3 b^2$$

(e.g., take $(a, b, c) = (10, 7, 1)$) even though Muirhead's inequality does imply the corresponding inequality for symmetric sums.

One must instead keep in mind not the statement of Muirhead's inequality but its underlying intuition: one should use "less mixed" monomials to dominate "more mixed" monomials. We will see two key techniques for realizing this intuition in the following argument. (Note that the breakdown we will give is in no way unique; there is some flexibility in the choice of how to separate (1) into tractable pieces.)

We first use what one might call a "sum of squares" argument: writing down cyclic sums of manifestly nonnegative expressions in order to match a few of the terms in (1). For instance, the following inequalities are all valid:

$$0 \leq \sum_{\text{cyc}} 84a^2 b(a-c)^2 = \sum_{\text{cyc}} (84a^4 b - 168a^3 bc + 84a^2 b^2 c), \tag{2}$$

$$0 \leq \sum_{\text{cyc}} \frac{35}{2} ab^2 (a-b)^2 = \sum_{\text{cyc}} \left(\frac{35}{2} a^3 b^2 - 35a^2 b^3 + \frac{35}{2} ab^4 \right), \tag{3}$$

$$0 \leq \sum_{\text{cyc}} \frac{35}{2} ab^2 (a-c)^2 = \sum_{\text{cyc}} \left(\frac{35}{2} a^3 b^2 - 35a^2 b^2 c + \frac{35}{2} ab^2 c^2 \right), \tag{4}$$

and these completely account for the summands $35a^3 b^2, -35a^2 b^3, -168a^3 bc$ in (1). We would like to add (2), (3), (4), and one more true inequality to get (1); that final inequality then would have to be

$$0 \leq \sum_{\text{cyc}} \left(\frac{177}{2} a^4 b + 38ab^4 - \frac{253}{2} a^2 b^2 c \right). \tag{5}$$

This inequality does not immediately present itself as a sum of squares, so we resort to a second technique: the weighted arithmetic-geometric mean inequality. This inequality implies that for any nonnegative real numbers u, v, w adding up to 1,

$$\sum_{\text{cyc}} a^4 b = \sum_{\text{cyc}} (ua^4 b + vb^4 c + wc^4 a) \geq \sum_{\text{cyc}} a^{4u+w} b^{u+4v} c^{v+4w}.$$

We may then deduce that

$$\sum_{\text{cyc}} a^4 b \geq \sum_{\text{cyc}} a^2 b^2 c \tag{6}$$

by solving the linear equations

$$4u + w = 2, u + 4v = 2, v + 4w = 1$$

and discovering that the unique real solution

$$(u, v, w) = \left(\frac{6}{13}, \frac{5}{13}, \frac{2}{13}\right)$$

consists of nonnegative real numbers. (It is not necessary to check separately that the three numbers add up to 1, because adding the three given equations together gives $5(u+v+w) = 5$.) By switching a and b, we also obtain the valid inequality

$$\sum_{\text{cyc}} ab^4 \geq \sum_{\text{cyc}} a^2 b^2 c. \tag{7}$$

Adding 177/2 times (6) by 177/2 plus 38 times (7) then gives (5), so this inequality is also valid. As noted earlier, we may then add (5) to (2), (3), (4) to obtain the desired inequality (1).

This solution was adapted and refined by Kiran Kedlaya from several students' solutions.

JMO 4. Observe that since α is irrational no two of the points will coincide. It will be useful to define the auxiliary point P_0 such that the length of arc P_0P_1 is α, when travelling counter-clockwise around the circle from P_0 to P_1. We begin by noting that for any $n \geq 3$, if $a + b = n$ then P_0 lies on the arc from P_a to P_b containing P_n. For if we travel back (clockwise) around the circle through a distance of $b\alpha$ from P_n then we reach P_a. The same translation must map P_b to P_0, and since P_n is situated between P_a and P_b, we deduce that P_0 must be also.

The claim is clearly true for $n = 3$. Now suppose to the contrary that for some value of n we have $a + b > n$ and consider the minimal such counterexample. If in fact $a + b > n + 1$, then we may translate the three points P_a, P_b, and P_n clockwise around the circle through a distance α to find points P_{a-1} and P_{b-1} adjacent to P_{n-1} on either side. But then we would have $(a - 1) + (b - 1) > (n - 1)$ for this trio of points, which contradicts our assumption that n was the minimal counterexample.

Therefore we must have $a + b = n + 1$. Again we translate points P_a, P_b, and P_n clockwise around the circle through a distance α to obtain points P_{a-1} and P_{b-1} adjacent to P_{n-1} on either side with $(a-1)+(b-1) = (n-1)$. By our earlier observation this implies that P_0 lies on the arc from P_{a-1} to P_{b-1} containing P_{n-1}. But now translating forward again, we conclude that P_1 lies on the arc from P_a to P_b containing P_n, contradicting the fact that P_a and P_b were the nearest adjacent points to P_n on either side. This completes the proof.

This problem and solution were suggested by Sam Vandervelde.

JMO 5. For simplicity, we will define $g(n)$ to be $n \pmod{2012}$. Note that $g(ak)+g(a(2012-k))$ is either 0 or 2012; it is 0 exactly when 2012 divides ak. This means that for $1 \le k \le 1005$, the number of elements i in $\{k, 2012-k\}$ such that $ai \pmod{2012} > bi \pmod{2012}$ is

$$\begin{cases} 0 & \text{if } g(ak) = 0 \text{ or } g(ak) = g(bk); \\ 2 & \text{if } g(bk) = 0 \text{ and } g(ak) \neq 0; \\ 1 & \text{otherwise.} \end{cases}$$

Let $T = \{1, 2, \ldots, 1005\}$. Note that the condition $g(ak) = g(bk)$ is equivalent to $g((a-b)k) = 0$. We will try to choose a, b so as to maximize the number of numbers k in T such that the first of the three cases occurs. From the prime factorization $2012 = 2 \cdot 2 \cdot 503$, the proper divisors of 2012 are 1, 2, 4, 503, and 1006. We shall choose a and $a-b$ to be multiples of some of these numbers. It is not hard to verify that we can choose a to be a multiple of 1006 and $a-b$ to be a multiple of 4. We will take $a = 1006$ and $b = 1002$.

With this choice of a and b, the second of the three cases (i.e. $g(bk) = 0$ and $g(ak) \neq 0$) never occurs, hence minimizing the number of elements i in $T - \{1006\}$ such that $ai \pmod{2012} > bi \pmod{2012}$. Moreover, $g(1006a) = 0$, meaning that $g(1006a) > g(1006b)$ does not hold. This means that our choice of a and b minimizes $f(a, b)$.

Note that $g(1006k) = 0$ occurs for 502 values in T, and $g(1006k) = g(1002k)$ occurs for 1 value in T. No value in T satisfies both condition. Hence $S = 1005 - 502 - 1 = 502$.

Note: Similarly, we can solve the problem in which 2012 is replaced by any positive integer $n \ge 3$. The answer is

$$\begin{cases} \dfrac{n}{2}\left(1 - \dfrac{1}{p}\right) & \text{if } n = p^k \text{ for some prime } p; \\ \dfrac{n}{2}\left(1 - \dfrac{1}{p_1}\right)\left(1 - \dfrac{1}{p_2}\right) & \text{otherwise, where } p_1 \text{ and } p_2 \text{ are the two smallest prime divisors of } n. \end{cases}$$

It is worth noting that the answer depends on no more than two prime divisors of n. Hence it might be interesting to ask the question for a value of n with at least three distinct prime divisors, or for all n.

This problem and solution were suggested by Warut Suksompong.

JMO 6. **Solution 1:** The proof is split into two cases.

Case 1: P is on the circumcircle of ABC. Then P is the Miquel point of A', B', C' with respect to ABC. Indeed, because $\angle A'B'C' = \angle CBA = \angle CPA = \angle A'PC'$, points P, A', B', C' are concyclic, and the same can be said for P, A, B', C' and P, A', B', C. Hence $\angle CA'B' = \angle CPB' = \angle BPC' = \angle BA'C'$, so $A'B'C'$ are collinear.

Case 2: P is not on the circumcircle of ABC. Let Q be isogonal conjugate of P with respect to ABC (which is not degenerate).

Claim. Let Q' be the isogonal conjugate of P with respect to $AB'C'$. Then $Q = Q'$.

Proof of the claim. Note that

$$\angle BQC = \angle BAC + \angle CPB \quad \text{(because } P \text{ and } Q \text{ are isogonal conjugates in } ABC\text{)}$$

$$
\begin{aligned}
&= \angle C'AB' + \angle B'PC' \\
&= \angle C'Q'B' \quad \text{(because } P \text{ and } Q \text{ are isogonal conjugates in } AB'C'\text{)}.
\end{aligned}
$$

Let X, Y, Z denote the reflections of P in sides BC, CA, AB, respectively, and let X' denote P's reflection in side $B'C'$ of triangle $AB'C'$. Then $\angle ZXY = \angle BQC$ (because QC is orthogonal to XY and QB is orthogonal to XZ), whereas $\angle ZX'Y' = \angle C'Q'B'$ because $Q'B'$ is orthogonal to $X'Y$ and $Q'C'$ is orthogonal to $X'Z$ and $Q'C'$ is orthogonal to $X'Z$, so since $\angle C'Q'B' = \angle BQC$, we get $\angle ZXY = \angle ZX'Y$. It follows that X, Y, Z, X' are concyclic. The center of the XYZ-circle is Q while the center of the $X'Y'Z$-circle is Q'. Thus $Q = Q'$.

Note. We have made use of the well-known fact that the circumcenter of the triangle determined by the reflections of a point across the sidelines of another given triangle is precisely the isogonal conjugate of the point with respect to that triangle. For a proof see R. A. Johnson, *Advanced Euclidean Geometry*, 1929 ed., reprinted by Dover, 2007.

Similar arguments show that Q is also the isogonal point of P with respect to triangles $A'BC'$ and $A'B'C$. Therefore,

$$
\begin{aligned}
\angle BC'A' &= \angle AC'A' = \angle AC'P + \angle PC'Q + \angle QC'A' \\
&= \angle QC'B' + \angle PC'Q + \angle BC'P \\
&= \angle BC'B' = \angle AC'B'.
\end{aligned}
$$

This means that A', B', C' are collinear. ■

This problem and solution were suggested by Titu Andreescu and Cosmin Pohoata.

Solution 2: It's easy to see (say, by law of sines) that

$$
\frac{AC'}{BC'} = \frac{AP\sin\angle APC'}{BP\sin\angle BPC'}, \quad \frac{BA'}{CA'} = \frac{BP\sin\angle BPA'}{CP\sin\angle CPA'}, \quad \frac{CB'}{AB'} = \frac{CP\sin\angle CPB'}{AP\sin\angle APB'}.
$$

The construction of A', B', C' by reflections implies that

$$
\sin\angle APC' = \sin\angle CPA', \quad \sin\angle BPC' = \sin\angle CPB', \quad \sin\angle BPC' = \sin\angle CPB'.
$$

Hence,

$$
\frac{AC'}{BC'} \cdot \frac{BA'}{CA'} \cdot \frac{CB'}{AB'} = 1,
$$

and the proof is complete by Menelaus' theorem.

This second solution was suggested by Li Zhou, Polk State College, Winter Haven FL.

JMO 2012 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2012 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

2. Find all integers $n \ge 3$ such that among any n positive real numbers a_1, a_2, ..., a_n with
$$\max(a_1, a_2, \dots, a_n) \le n \cdot \min(a_1, a_2, \dots, a_n),$$
there exist three that are the side lengths of an acute triangle.

3. For $a, b, c > 0$ prove that
$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \ge \frac{2}{3}(a^2 + b^2 + c^2).$$

4. Let α be an irrational number with $0 < \alpha < 1$, and draw a circle in the plane whose circumference has length 1. Given any integer $n \ge 3$, define a sequence of points $P_1, P_2, \dots, P_n$ as follows. First select any point P_1 on the circle, and for $2 \le k \le n$ define P_k as the point on the circle for which the length of arc $P_{k-1}P_k$ is α, when travelling counterclockwise around the circle from P_{k-1} to P_k. Suppose that P_a and P_b are the nearest adjacent points on either side of P_n. Prove that $a + b \le n$.

5. For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \le k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

6. Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

§1 Solutions to Day 1

§1.1 JMO 2012/1, proposed by Sungyoon Kim, Inseok Seo

Available online at https://aops.com/community/p2669111.

Problem statement

Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Assume for contradiction that (PRS) and (QRS) are distinct. Then $\overline{RS}$ is the radical axis of these two circles. However, $\overline{AP}$ is tangent to (PRS) and $\overline{AQ}$ is tangent to (QRS), so point A has equal power to both circles, which is impossible since A does not lie on line BC.

§1.2 JMO 2012/2, proposed by Titu Andreescu

Available online at https://aops.com/community/p2669112.

Problem statement

Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \dots, a_n$ with

$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$

there exist three that are the side lengths of an acute triangle.

The answer is all $n \geq 13$.

Define (F_n) as the sequence of Fibonacci numbers, by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. We will find that Fibonacci numbers show up naturally when we work through the main proof, so we will isolate the following calculation now to make the subsequent solution easier to read.

Claim — For positive integers m, we have $F_m \leq m^2$ if and only if $m \leq 12$.

Proof. A table of the first 14 Fibonacci numbers is given below.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
1	1	2	3	5	8	13	21	34	55	89	144	233	377

By examining the table, we see that $F_m \leq m^2$ is true for $m = 1, 2, \dots 12$, and in fact $F_{12} = 12^2 = 144$. However, $F_m > m^2$ for $m = 13$ and $m = 14$.

Now it remains to prove that $F_m > m^2$ for $m \geq 15$. The proof is by induction with base cases $m = 13$ and $m = 14$ being checked already. For the inductive step, if $m \geq 15$ then we have

$$\begin{aligned} F_m &= F_{m-1} + F_{m-2} > (m-1)^2 + (m-2)^2 \\ &= 2m^2 - 6m + 5 = m^2 + (m-1)(m-5) > m^2 \end{aligned}$$

as desired. □

We now proceed to the main problem. The hypothesis $\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n)$ will be denoted by (†).

Proof that all $n \geq 13$ have the property. We first show now that every $n \geq 13$ has the desired property. Suppose for contradiction that no three numbers are the sides of an acute triangle. Assume without loss of generality (by sorting the numbers) that $a_1 \leq a_2 \leq \dots \leq a_n$. Then since a_{i-1}, a_i, a_{i+1} are not the sides of an acute triangle for each $i \geq 2$, we have that $a_{i+1}^2 \geq a_i^2 + a_{i-1}^2$; writing this out gives

$$\begin{aligned} a_3^2 &\geq a_2^2 + a_1^2 \geq 2a_1^2 \\ a_4^2 &\geq a_3^2 + a_2^2 \geq 2a_1^2 + a_1^2 = 3a_1^2 \\ a_5^2 &\geq a_4^2 + a_3^2 \geq 3a_1^2 + 2a_1^2 = 5a_1^2 \\ a_6^2 &\geq a_5^2 + a_4^2 \geq 5a_1^2 + 3a_1^2 = 8a_1^2 \end{aligned}$$

and so on. The Fibonacci numbers appear naturally and by induction, we conclude that $a_i^2 \geq F_i a_1^2$. In particular, $a_n^2 \geq F_n a_1^2$.

However, we know $\max(a_1, \dots, a_n) = a_n$ and $\min(a_1, \dots, a_n) = a_1$, so (†) reads $a_n \le n \cdot a_1$. Therefore we have $F_n \le n^2$, and so $n \le 12$, contradiction!

Proof that no $n \le 12$ have the property. Assume that $n \le 12$. The above calculation also suggests a way to pick the counterexample: we choose $a_i = \sqrt{F_i}$ for every i. Then $\min(a_1, \dots, a_n) = a_1 = 1$ and $\max(a_1, \dots, a_n) = \sqrt{F_n}$, so (†) is true as long as $n \le 12$. And indeed no three numbers form the sides of an acute triangle: if $i < j < k$, then $a_k^2 = F_k = F_{k-1} + F_{k-2} \ge F_j + F_i = a_j^2 + a_i^2$.

§1.3 JMO 2012/3, proposed by Titu Andreescu

Available online at https://aops.com/community/p2669114.

Problem statement

For $a, b, c > 0$ prove that

$$\frac{a^3+3b^3}{5a+b}+\frac{b^3+3c^3}{5b+c}+\frac{c^3+3a^3}{5c+a} \ge \frac{2}{3}(a^2+b^2+c^2).$$

Apply Titu lemma to get

$$\sum_{\text{cyc}} \frac{a^3}{5a+b} = \sum_{\text{cyc}} \frac{a^4}{5a^2+ab} \ge \frac{(a^2+b^2+c^2)^2}{\sum_{\text{cyc}}(5a^2+ab)} \ge \frac{a^2+b^2+c^2}{6}$$

where the last step follows from the identity $\sum_{\text{cyc}}(5a^2+ab) \le 6(a^2+b^2+c^2)$.

Similarly,

$$\sum_{\text{cyc}} \frac{b^3}{5a+b} = \sum_{\text{cyc}} \frac{b^4}{5ab+b^2} \ge \frac{(a^2+b^2+c^2)^2}{\sum_{\text{cyc}}(5ab+b^2)} \ge \frac{a^2+b^2+c^2}{6}$$

using the fact that $\sum_{\text{cyc}} 5ab+b^2 \le 6(a^2+b^2+c^2)$.

Therefore, adding the first display to three times the second display implies the result.

§2 Solutions to Day 2

§2.1 JMO 2012/4, proposed by Sam Vandervelde

Available online at https://aops.com/community/p2669956.

Problem statement

Let α be an irrational number with $0 < \alpha < 1$, and draw a circle in the plane whose circumference has length 1. Given any integer $n \geq 3$, define a sequence of points $P_1, P_2, \dots, P_n$ as follows. First select any point P_1 on the circle, and for $2 \leq k \leq n$ define P_k as the point on the circle for which the length of arc $P_{k-1}P_k$ is α, when travelling counterclockwise around the circle from P_{k-1} to P_k. Suppose that P_a and P_b are the nearest adjacent points on either side of P_n. Prove that $a + b \leq n$.

No points coincide since α is irrational.

Assume for contradiction that $n < a + b < 2n$. Then

$$\overline{P_nP_{a+b-n}} \parallel \overline{P_aP_b}.$$

This is an obvious contradiction since then P_{a+b-n} is contained in the arc $\widehat{P_aP_b}$ of the circle through P_n.

§2.2 JMO 2012/5, proposed by Warut Suksompong

Available online at https://aops.com/community/p2669967.

Problem statement

For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \le k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

The answer is $S = 502$ (not 503!).

Claim — If $\gcd(k, 2012) = 1$, then necessarily either k or $2012 - k$ will counts towards S.

Proof. First note that both ak, bk are nonzero modulo 2012. Note also that $ak \not\equiv bk \pmod{2012}$.

So if r_a is the remainder of $ak \pmod{2012}$, then $2012 - r_a$ is the remainder of $a(2012-k) \pmod{2012}$ Similarly we can consider r_b and $2012 - r_b$. As mentioned already, we have $r_a \neq r_b$. So either $r_a > r_b$ or $2012 - r_a > 2012 - r_b$. □

This implies $S \ge \frac{1}{2}\varphi(2012) = 502$.

But this can actually be achieved by taking $a = 4$ and $b = 1010$, since

- If k is even, then $ak \equiv bk \pmod{2012}$ so no even k counts towards S; and
- If $k \equiv 0 \pmod{503}$, then $ak \equiv 0 \pmod{2012}$ so no such k counts towards S.

This gives the final answer $S \ge 502$.

Remark. A similar proof works with 2012 replaced by any n and will give an answer of $\frac{1}{2}\varphi(n)$. For composite n, one uses the Chinese remainder theorem to pick distinct a and b not divisible by n such that $\operatorname{lcm}(a - b, a) = n$.

§2.3 JMO 2012/6, proposed by Titu Andreescu, Cosmin Pohoata

Available online at https://aops.com/community/p2669960.

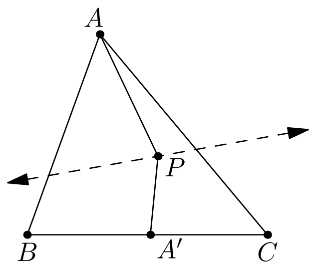
Problem statement

Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

We present three solutions.

¶ First solution (complex numbers) Let $p = 0$ and set γ as the real line. Then A' is the intersection of bc and $p\overline{a}$. So, we get

$$a' = \frac{\overline{a}(\overline{b}c - b\overline{c})}{(\overline{b} - \overline{c})\overline{a} - (b - c)a}.$$



Note that

$$\overline{a}' = \frac{a(b\overline{c} - \overline{b}c)}{(b - c)a - (\overline{b} - \overline{c})\overline{a}}.$$

Thus it suffices to prove

$$0 = \det \begin{bmatrix} \frac{\overline{a}(\overline{b}c - b\overline{c})}{(\overline{b}-\overline{c})\overline{a}-(b-c)a} & \frac{a(b\overline{c}-\overline{b}c)}{(b-c)a-(\overline{b}-\overline{c})\overline{a}} & 1 \\ \frac{\overline{b}(\overline{c}a-c\overline{a})}{(\overline{c}-\overline{a})\overline{b}-(c-a)b} & \frac{b(c\overline{a}-\overline{c}a)}{(c-a)b-(\overline{c}-\overline{a})\overline{b}} & 1 \\ \frac{\overline{c}(\overline{a}b-a\overline{b})}{(\overline{a}-\overline{b})\overline{c}-(a-b)c} & \frac{c(a\overline{b}-\overline{a}b)}{(a-b)c-(\overline{a}-\overline{b})\overline{c}} & 1 \end{bmatrix}.$$

This is equivalent to

$$0 = \det \begin{bmatrix} \overline{a}(\overline{b}c - b\overline{c}) & a(\overline{b}c - b\overline{c}) & (\overline{b} - \overline{c})\overline{a} - (b - c)a \\ \overline{b}(\overline{c}a - c\overline{a}) & b(\overline{c}a - c\overline{a}) & (\overline{c} - \overline{a})\overline{b} - (c - a)b \\ \overline{c}(\overline{a}b - a\overline{b}) & c(\overline{a}b - a\overline{b}) & (\overline{a} - \overline{b})\overline{c} - (a - b)c \end{bmatrix}.$$

This determinant has the property that the rows sum to zero, and we're done.

Remark. Alternatively, if you don't notice that you could just blindly expand:

$$\begin{aligned} &\sum_{\text{cyc}} ((\overline{b} - \overline{c})\overline{a} - (b - c)a) \cdot -\det \begin{bmatrix} b & \overline{b} \\ c & \overline{c} \end{bmatrix} (\overline{c}a - c\overline{a})\left(\overline{a}b - a\overline{b}\right) \\ &= (\overline{b}c - c\overline{b})(\overline{c}a - c\overline{a})(\overline{a}b - a\overline{b}) \sum_{\text{cyc}} \left(ab - ac + \overline{ca} - \overline{ba}\right) = 0. \end{aligned}$$

¶ Second solution (Desargues involution) We let $C'' = \overline{A'B'} \cap \overline{AB}$. Consider complete quadrilateral $ABCA'B'C''C$. We see that there is an involutive pairing τ at P swapping (PA, PA'), (PB, PB'), (PC, PC''). From the first two, we see τ coincides with reflection about ℓ, hence conclude $C'' = C$.

¶ Third solution (barycentric), by Catherine Xu We will perform barycentric coordinates on the triangle PCC', with $P = (1, 0, 0)$, $C' = (0, 1, 0)$, and $C = (0, 0, 1)$. Set $a = CC'$, $b = CP$, $c = C'P$ as usual. Since A, B, C' are collinear, we will define $A = (p : k : q)$ and $B = (p : \ell : q)$.

Claim — Line γ is the angle bisector of $\angle APA'$, $\angle BPB'$, and $\angle CPC'$.

Proof. Since $A'P$ is the reflection of AP across γ, etc. □

Thus B' is the intersection of the isogonal of B with respect to $\angle P$ with the line CA; that is,

$$B' = \left(\frac{p}{k} \frac{b^2}{\ell} : \frac{b^2}{\ell} : \frac{c^2}{q} \right).$$

Analogously, A' is the intersection of the isogonal of A with respect to $\angle P$ with the line CB; that is,

$$A' = \left(\frac{p}{\ell} \frac{b^2}{k} : \frac{b^2}{k} : \frac{c^2}{q} \right).$$

The ratio of the first to third coordinate in these two points is both $b^2pq : c^2k\ell$, so it follows A', B', and C' are collinear.

4^{th} United States of America Junior Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 30, 2013

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

JMO 1. Are there integers a and b such that a^5b+3 and ab^5+3 are both perfect cubes of integers?

JMO 2. Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be *adjacent* if their cells share a common side. (Note that two numbers in cells that share only a corner are not adjacent.) The filling is called a *garden* if it satisfies the following two conditions:

(i) The difference between any two adjacent numbers is either 0 or 1.

(ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0.

Determine the number of distinct gardens in terms of m and n.

JMO 3. In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let $\omega_A, \omega_B, \omega_C$ denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects $\omega_A, \omega_B, \omega_C$ again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

4^{th} United States of America Junior Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

May 1, 2013

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

JMO 4. Let $f(n)$ be the number of ways to write n as a sum of powers of 2, where we keep track of the order of the summation. For example, $f(4) = 6$ because 4 can be written as 4, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Find the smallest n greater than 2013 for which $f(n)$ is odd.

JMO 5. Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter XY. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line XY. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$

JMO 6. Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

4^{th} United States of America Junior Mathematical Olympiad

Day I, II 12:30 PM – 5 PM EDT

April 30 - May 1, 2013

JMO 1. The answer is negative. Modulo 9, a cube is 0 or ± 1. Assuming that one of a^5b+3 and ab^5+3 is 0 mod 9, it follows that at least one of the numbers a and b, say a, is divisible by 3, hence a^5b+3 is 3 mod 27, not a perfect cube. If a^5b+3 and ab^5+3 are both perfect cubes of the form ± 1 mod 9, then a^5b and ab^5 are both 7 or 5 mod 9, and so their product, $(ab)^6$, is -1, -2, or 4 mod 9. But $(ab)^6$ is the square of a perfect cube not divisible by 3, so is precisely 1 mod 9, a contradiction.

This problem and solution were suggested by Titu Andreescu.

JMO 2. Answer: $2^{mn}-1$.

First note that if $m=n=1$, then condition (ii) is vacuously satisfied, so the one cell must contain 0. Henceforth, we assume that $m>1$ or $n>1$, so that every cell has at least one adjacent cell.

We define the distance between two cells to be $|x_1-x_2|+|y_1-y_2|$, where (x_1,y_1), (x_2,y_2) are the centers of the respective cells. In particular, two cells are adjacent if and only if the distance between them is 1.

By condition (ii), the smallest value among the cells of any given garden must be 0. In particular, a garden has at least one zero.

We construct an explicit bijection between the set of nonempty subsets of the mn cells in the array filled with 0 and the set of all possible gardens. Given a subset of the mn cells filled with zeroes, fill every cell in the array with the value of the distance to the nearest cell filled with a zero. This filling of the cells is well-defined and satisfies both properties (i) and (ii). Given two different subsets of cells filled with zeroes, the filling of all cells with minimum distances must necessarily be different, so the function is injective (or one-to-one).

Let an arbitrary garden be given and suppose that a cell in that garden contains an integer $k\geq 1$. By condition (ii), it has an adjacent cell with a smaller integer. Since the difference is either 0 or 1, the difference must be 1. Thus, a cell assigned k will have an adjacent cell assigned $k-1$. We draw a line segment between the two center points of these two cells. Repeating this procedure, we can find a path from k to a 0-cell. We call such a path a *garden path.* There may be more than one garden path from a given cell, but all such paths will have length k.

Suppose that for some cell C assigned k there is a path of length $n<k$ from C to a 0-cell D. Let the numbers in the cells the path goes through be $a_0=k, a_1, \ldots, a_n=0$. Now $a_i-a_{i+1}\leq 1$, so

$$k=\sum_{i=0}^{n-1}(a_i-a_{i+1})\leq n<k,$$

a contradiction. Thus, the nearest 0-cell to C has distance $\geq k$ from C. By the previous paragraph, there exists a path from C to a 0-cell with distance k. Therefore, the distance to the nearest 0-cell is exactly k. The mapping is surjective (or onto).

Therefore, each garden is uniquely determined by the position of zeros. Consequently, we just need to count the number of ways to put zeros in mn cells, subject to the condition that there is at least one zero. This is clearly $2^{mn} - 1$.

This problem and solution were suggested by Sungyoon Kim.

JMO 3. **First Solution:** Assume that ω_B and ω_C intersect again at another point S (other than P). (The degenerate case of ω_B and ω_C being tangent at P can be dealt similarly.) Because $BPSR$ and $CPSQ$ are cyclic, we have $\angle RSP = 180° - \angle PBR$ and $\angle PSQ = 180° - \angle QCP$. Hence, we obtain

$$\angle QSR = 360° - \angle RSP - \angle PSQ = \angle PBR + \angle QCP = \angle CBA + \angle ACB = 180° - \angle BAC;$$

from which it follows that $ARSQ$ is cyclic; that is, $\omega_A, \omega_B, \omega_C$ meet at S. (This is Miquel's theorem.)

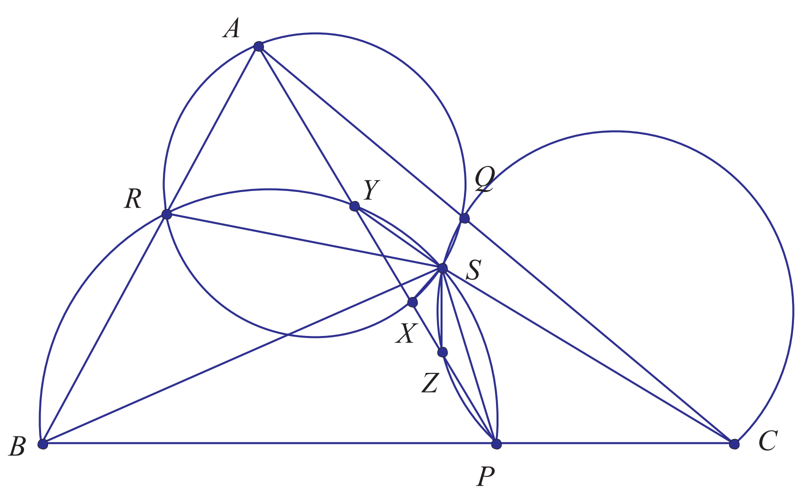
Because $BPSY$ is inscribed in ω_B, $\angle XYS = \angle PYS = \angle PBS$. Because $ARXS$ is inscribed in ω_A, $\angle SXY = \angle SXA = \angle SRA$. Because $BPSR$ is inscribed in ω_B, $\angle SRA = \angle SPB$. Thus, we have $\angle SXY = \angle SRA = \angle SPB$. In triangles SYX and SBP, we have $\angle XYS = \angle PBS$ and $\angle SXY = \angle SPB$. Therefore, triangles SYX and SBP are similar to each other, and, in particular,

$$\frac{YX}{BP} = \frac{SX}{SP}.$$

Similar, we can show that triangles SXZ and SPC are similar to each other and that

$$\frac{SX}{SP} = \frac{XZ}{PC}.$$

Combining the last two equations yields the desired result.



This problem and solution were suggested by Zuming Feng.

Second Solution: Assume that ω_B and ω_C intersect again at another point S (other than P). (The degenerate case of ω_B and ω_C being tangent at P can be dealt with similarly.) Because $BPSR$ and $CPSQ$ are cyclic, we have $\angle RSP = 180^\circ - \angle PBR$ and $\angle PSQ = 180^\circ - \angle QCP$. Hence, we obtain

$$\angle QSR = 360^\circ - \angle RSP - \angle PSQ = \angle PBR + \angle QCP = \angle CBA + \angle ACB = 180^\circ - \angle BAC;$$

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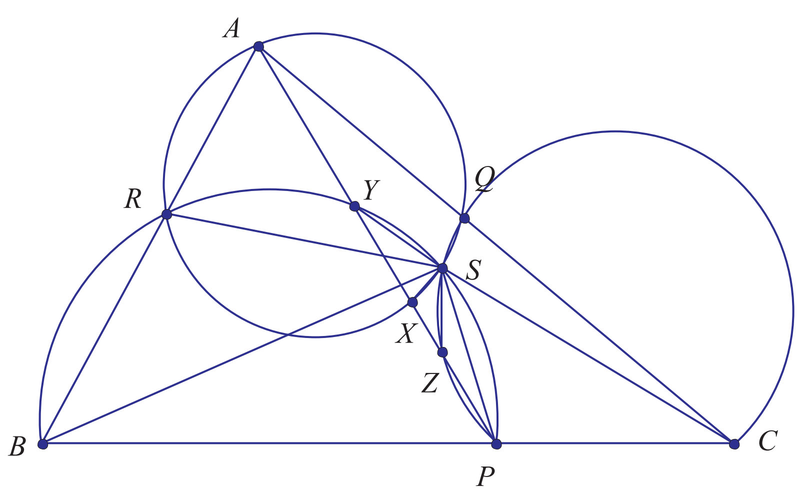
Combining the last two equations yields the desired result.



We consider the configuration shown in the above diagram. (We can adjust the proof below easily for other configurations. In particular, our proof is carried with directed angles modulo 180°.)

Line RY intersects ω_A again at T_Y (other than R). Because $BPYR$ is cyclic, $\angle T_Y YX = \angle T_Y YP = \angle RBP = \angle ABP$. Because $ARXT_Y$ is cyclic, $\angle XT_Y Y = \angle XAR = \angle PAB$. Hence triangles $T_Y YX$ and ABP are similar to each other. In particular,

$$\angle YXT_Y = \angle BPA \quad \text{and} \quad \frac{YX}{BP} = \frac{XT_Y}{PA}. \tag{1}$$

Likewise, if line QZ intersect ω_A again at T_Z (other than R), we can show that triangles $T_Z ZX$ and ACP are similar to each other and that

$$\angle T_Z XZ = \angle APC \quad \text{and} \quad \frac{XT_Z}{PA} = \frac{XZ}{PC}. \tag{2}$$

In the light of the second equations (on lengths proportions) in (1) and (2), it suffices to show that $T_Z = T_Y$. On the other hand, the first equations (on angles) in (1) and (2) imply that X, T_Y, T_Z lie on a line. But this line can only intersect ω_A twice with X being one of them. Hence we must have $T_Y = T_Z$, completing our proof.

Comment: The result remains to be true if segment AP is replaced by line AP. The current statement is given to simplify the configuration issue. Also, a very common mistake in attempts following the second solution is assuming line RY and QZ meet at a point on ω_A.

This solution was suggested by Zuming Feng.

JMO 4. **Solution 1.** The answer is 2047. We shall prove that $f(n)$ is odd iff $n = 2^k - 1$ for $k \geq 1$. It is easy to see that $f(1) = 1$, $f(2) = 2$, and $f(3) = 3$. Assume that the statement holds true for $k \leq m$. We will show that the statement is true for $k = m + 1$.

Let $m \geq 2$ be an integer such that $2^m \leq n \leq 2^{m+1} - 1$.

If $n = 2^m$ we write $n = 2^s + (n - 2^s)$ for $0 \leq s \leq m$. We see that $f(2^m) = f(2^m - 1) + f(2^m - 2) + \ldots + f(2^m - 2^{m-1}) + 1$. By induction hypothesis each of $f(2^m - 2), \ldots, f(2^m - 2^{m-1})$ is even, but $f(2^m - 1)$ is odd, so $f(2^m)$ is even.

If $2^m < n \leq 2^{m+1} - 1$ we have $f(n) = f(n-1) + f(n-2) + \ldots + f(n - 2^m)$.

By induction hypothesis each term on the right hand side is odd iff $n - 2^s = 2^r - 1$ for some positive integer r. For each n of the form $n = 2^s + 2^r - 1$ these odd summands appear in pairs: $n - 2^s$ and $n - 2^r$. Therefore $f(n)$ is odd iff $s = r$, that is iff $n = 2^{s+1} - 1 = 2^{m+1} - 1$.

Solution 2. The answer is 2047. We show that $f(n)$ is odd if and only if n is of the form $2^k - 1$.

We use the method of generating functions. Define the formal power series $b(x) = \sum_{j=0}^{\infty} x^{2^j}$. The desired statement can be interpreted as

$$1/(1 - b(x)) \equiv b(x)/x \pmod{2},$$

where the congruence means that the difference between the two sides has all coefficients divisible by 2. It is equivalent to prove the same thing after clearing denominators, in other words,

$$b(x)^2 - b(x) \equiv x \pmod{2}.$$

But this holds because $b(x)^2 \equiv b(x^2) \pmod{2}$ (all the cross terms in the expansion of $b(x)^2$ being even), so

$$b(x)^2 - b(x) \equiv b(x^2) - b(x) \equiv x \pmod{2}.$$

This problem and solution were suggested by Kiran Kedlaya and David Speyer.

Solution 3. Consider the operation of reversing the order of the sums. Call a sum a palindrome if it is invariant under this symmetry and let $g(n)$ be the number of palindromic decompositions of n. Since non-palindromic sums are paired under reversing order we have

$$f(n) \equiv g(n) \pmod 2.$$

Now suppose $n = 2m + 1$ is odd. By parity a palindromic decomposition of n must have an odd central term (and in particular cannot have even length). Hence the central term must be 1. Thus any palindromic decomposition of $n = 2m + 1$ starts with an arbitrary decomposition of m, followed by a 1 and the reverse of the starting decomposition. Thus

$$g(2m + 1) = f(m).$$

Hence $f(2m + 1) \equiv f(m) \pmod 2$.

Now suppose $n = 2m$ is even and positive. Then there are two kinds of palindromic decompositions of n. The first kind have even length. The second kind have odd length and a central element that is even, hence 2^k for some $k \geq 1$. These two kinds occur equally often since we can add together the two equal terms of a palindrome of equal length into two equal halves to reverse this operation. Thus $f(2m)$ and $g(2m)$ are even.

These two cases easily imply $f(n)$ is odd if and only if n is 1 less than a power of 2. One way to see this is to write n in binary. The first rule $f(2m + 1) \equiv f(m) \pmod 2$ says the parity of $f(n)$ is unchanged if we delete a least significant digit of 1. The second rule says $f(n)$ is even if its least significant digit is zero. Iterating these we see $f(n)$ is odd if and only if its binary representation is all 1s, that is, n is 1 less than a power of 2.

This solution was suggested by Steven Blasberg and Richard Stong.

JMO 5. **First Solution:** Note that $\angle XAY = \angle XBY = \angle XCY = \angle PZX = \angle PZY = 90°$. In right triangles BXY, AXY, AXP, we have

$$BY = XY \cos \angle BYX, \quad AX = XY \cos \angle AXY, \quad XP = \frac{AX}{\cos \angle AXP} = \frac{XY \cos \angle AXY}{\cos \angle AXP},$$

from which it follows that

$$\frac{BY}{XP} = \frac{\cos \angle BYX \cos \angle AXP}{\cos \angle AXY}.$$

Likewise, we have

$$\frac{CY}{XQ} = \frac{\cos \angle CYX \cos \angle AXQ}{\cos \angle AXY}.$$

Adding the last two equations yields

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\cos \angle BYX \cos \angle AXP + \cos \angle CYX \cos \angle AXQ}{\cos \angle AXY}. \tag{3}$$

Because both CY and AZ are perpendicular to XC, $\angle CYX = \angle AZX$. Because $\angle XAP = \angle XZP = 90°$, quadrilateral $AXZP$ is cyclic, from which it follows that $\angle AZX = \angle APX$. Therefore, we have $\angle CYX = \angle AZX = \angle APX = 90° - \angle AXP$ or $\angle CYX + \angle AXP =$

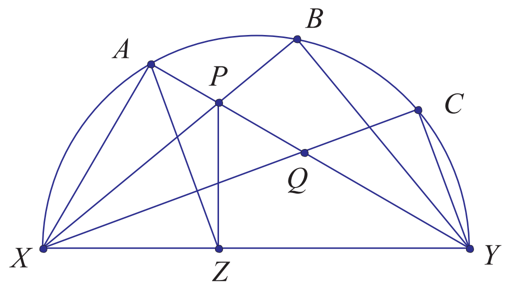
$90°$. Likewise, we can show that $\angle BYX + \angle AXQ = 90°$. Consequently, we conclude that $\cos\angle BYX = \sin\angle AXQ$ and $\sin\angle CYX = \cos\angle AXP$. Thus, by the addition and substraction formula, (4) becomes

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\sin\angle AXQ \sin\angle CYX + \cos\angle CYX \cos\angle AXQ}{\cos\angle AXY} = \frac{\cos(\angle CYX - \angle AXQ)}{\cos\angle AXY}.$$

Because $ACYX$ is cyclic, $\angle AXQ = \angle AXC = \angle CYA$, implying that $\angle CYX - \angle AXQ = \angle CYX - \angle CYA = \angle AYX$. Therefore,

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\cos(\angle CYX - \angle AXQ)}{\cos\angle AXY} = \frac{\cos\angle AYX}{\cos\angle AXY} = \frac{\sin\angle AXY}{\cos\angle AXY} = \tan\angle AXY = \frac{AY}{AX},$$

as desired.



This problem and solution were suggested by Zuming Feng.

Second Solution: Note that $\angle XAY = \angle XBY = \angle XCY = \angle PZX = \angle PZY = 90°$. In right triangles BXY, AXY, AXP, we have

$$BY = XY\cos(\angle BYX), \quad AX = XY\cos(\angle AXY), \quad XP = \frac{AX}{\cos(\angle AXP)} = \frac{XY\cos(\angle AXY)}{\cos(\angle AXP)},$$

from which it follows that

$$\frac{BY}{XP} = \frac{\cos(\angle BYX)\cos(\angle AXP)}{\cos(\angle AXY)}.$$

Likewise, we have

$$\frac{CY}{XQ} = \frac{\cos(\angle CYX)\cos(\angle AXQ)}{\cos(\angle AXY)}.$$

Adding the last two equations yields

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\cos(\angle BYX)\cos(\angle AXP) + \cos(\angle CYX)\cos(\angle AXQ)}{\cos(\angle AXY)}. \tag{4}$$

Because both CY and AZ are perpendicular to XC, $\angle CYX = \angle AZX$. Because $\angle XAP = \angle XZP = 90°$, quadrilateral $AXZP$ is cyclic, from which it follows that $\angle AZX = \angle APX$. Therefore, we have $\angle CYX = \angle AZX = \angle APX = 90° - \angle AXP$ or $\angle CYX + \angle AXP = 90°$. Likewise, we can show that $\angle BYX + \angle AXQ = 90°$. Consequently, we conclude that

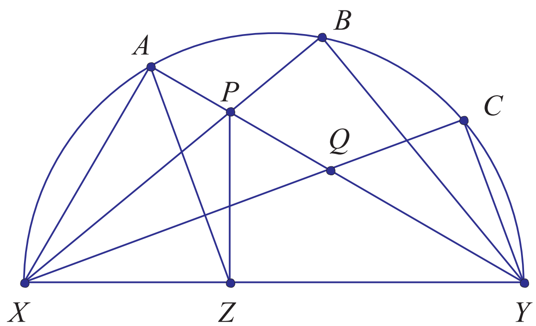
$\cos(\angle BYX) = \sin(\angle AXQ)$ and $\sin(\angle CYX) = \cos(\angle AXP)$. Thus, by the addition and substraction formula, (4) becomes

$$\frac{BY}{XP}+\frac{CY}{XQ} = \frac{\sin(\angle AXQ)\sin(\angle CYX)+\cos(\angle CYX)\cos(\angle AXQ)}{\cos(\angle AXY)} = \frac{\cos(\angle CYX - \angle AXQ)}{\cos(\angle AXY)}.$$

Because $ACYX$ is cyclic, $\angle AXQ = \angle AXC = \angle CYA$, implying that $\angle CYX - \angle AXQ = \angle CYX - \angle CYA = \angle AYX$. Therefore,

$$\frac{BY}{XP}+\frac{CY}{XQ} = \frac{\cos(\angle CYX - \angle AXQ)}{\cos\angle AXY} = \frac{\cos\angle AYX}{\cos\angle AXY} = \frac{\sin\angle AXY}{\cos\angle AXY} = \tan\angle AXY = \frac{AY}{AX},$$

as desired.



Rays YB and YC meet ray XA at B_1 and C_1 respectively. Because $\angle PAB_1 = \angle PBB_1 = 90^\circ$, $APBB_1$ is cyclic, in particular, $\angle XB_1Y = \angle AB_1B = \angle APX$. Because $\angle PAX = \angle PZX = 90^\circ$, $APZX$ is cyclic, in particular, $\angle APX = \angle AZX$. Note that both AC and CY are perpendicular to XC, $AZ \parallel CY$ and so $\angle AZX = \angle CYX = \angle C_1YX$. Therefore, we have $\angle XB_1Y = \angle APX = \angle AZX = \angle C_1YX$. It follows that triangles XYB_1 and XC_1Y are similar to each other, with XB and XC being corresponding altitudes. Hence

$$\frac{BY}{XP} = \frac{CC_1}{XQ} \quad \text{and} \quad \frac{BY}{XP}+\frac{CY}{XQ} = \frac{CC_1}{XQ}+\frac{CY}{XQ} = \frac{C_1Y}{XQ}.$$

It remains to show that

$$\frac{C_1Y}{XQ} = \frac{AY}{AX},$$

which is true because triangles AYC_1 and AXQ are similar to each other ($\angle C_1AY = \angle QAX = 90^\circ$ and $\angle AYC_1 = \angle AYC = \angle AXC = \angle AXQ$.)

This solution was suggested by Zuming Feng.

JMO 6. **First Solution:** Let a, b, c be nonnegative real numbers such that $x = 1 + a^2$, $y = 1 + b^2$ and $z = 1+c^2$. We may assume that $c \le a, b$, so that the condition of the problem becomes

$$(1 + c^2)(1 + (1 + a^2)(1 + b^2)) = (a + b + c)^2.$$

The Cauchy-Schwarz inequality yields

$$(a + b + c)^2 \le (1 + (a + b)^2)(c^2 + 1).$$

Combined with the previous relation, this shows that

$$(1+a^2)(1+b^2) \le (a+b)^2,$$

which can also be written $(ab-1)^2 \le 0$. Hence $ab=1$ and the Cauchy-Schwarz inequality must be an equality, that is, $c(a+b)=1$. Conversely, if $ab=1$ and $c(a+b)=1$, then the relation in the statement of the problem holds, since $c = \frac{1}{a+b} < \frac{1}{b} = a$ and similarly $c < b$.

Thus the solutions of the problem are

$$x = 1+a^2, \quad y = 1+\frac{1}{a^2}, \quad z = 1+\left(\frac{a}{a^2+1}\right)^2$$

for some $a>0$, as well as permutations of this. (Note that we can actually assume $a \ge 1$ by switching x and y if necessary.)

This problem and solution were suggested by Titu Andreescu.

Second Solution: We maintain the notations in the first solution and again consider the equation

$$(a+b+c)^2 = 1+c^2+(1+a^2)(1+b^2)(1+c^2).$$

Expanding both sides of the equation yields

$$a^2+b^2+c^2+2ab+2bc+2ca = 1+c^2+1+a^2+b^2+c^2+a^2b^2+b^2c^2+c^2a^2+a^2b^2c^2$$

or

$$a^2b^2c^2+a^2b^2+b^2c^2+c^2a^2-2ab-2bc-2ca+c^2+2 = 2(ab+bc+ca).$$

Setting $(u,v,w)=(ab,bc,ca)$, we can write the above equation as

$$uvw+u^2+v^2+w^2-2u-2v-2w+\frac{vw}{u}+2 = 2(u+v+w).$$

which is the equality case of the sum of the following three special cases of the AM-GM inequality:

$$uvw+\frac{vw}{u} \ge 2vw, v^2+w^2+2vw+1 = 2(v+w) \ge 0, \quad u^2+1 \ge 2u.$$

Hence we must have the equality cases these AM-GM inequalities; that is, $ab=u=1$ and $a(b+c)=v+w=1$. We can then complete our solution as we did in the first solution.

This solution was suggested by Zuming Feng.

JMO 2013 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2013 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Are there integers a and b such that a^5b+3 and ab^5+3 are both perfect cubes of integers?

2. Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be *adjacent* if their cells share a common side. The filling is called a *garden* if it satisfies the following two conditions:

   (i) The difference between any two adjacent numbers is either 0 or 1.

   (ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0.

   Determine the number of distinct gardens in terms of m and n.

3. In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

4. Let $f(n)$ be the number of ways to write n as a sum of powers of 2, where we keep track of the order of the summation. For example, $f(4) = 6$ because 4 can be written as 4, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Find the smallest n greater than 2013 for which $f(n)$ is odd.

5. Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter $\overline{XY}$. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line $\overline{XY}$. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that
$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$

6. Find all real numbers $x, y, z \geq 1$ satisfying
$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

§1 Solutions to Day 1

§1.1 JMO 2013/1, proposed by Titu Andreescu

Available online at https://aops.com/community/p3041819.

Problem statement

Are there integers a and b such that a^5b+3 and ab^5+3 are both perfect cubes of integers?

No, there do not exist such a and b.

We prove this in two cases.

- Assume $3 \mid ab$. WLOG we have $3 \mid a$, but then $a^5b+3 \equiv 3 \pmod 9$, contradiction.
- Assume $3 \nmid ab$. Then a^5b+3 is a cube not divisible by 3, so it is ± 1 mod 9, and we conclude
$$a^5b \in \{5,7\} \pmod 9.$$
Analogously
$$ab^5 \in \{5,7\} \pmod 9.$$
We claim however these two equations cannot hold simultaneously. Indeed $(ab)^6 \equiv 1 \pmod 9$ by Euler's theorem, despite $5 \cdot 5 \equiv 7$, $5 \cdot 7 \equiv 8$, $7 \cdot 7 \equiv 4$ mod 9.

§1.2 JMO 2013/2, proposed by Sungyoon Kim

Available online at https://aops.com/community/p3041818.

Problem statement

Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be *adjacent* if their cells share a common side. The filling is called a *garden* if it satisfies the following two conditions:

(i) The difference between any two adjacent numbers is either 0 or 1.

(ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0.

Determine the number of distinct gardens in terms of m and n.

The numerical answer is $2^{mn} - 1$. But we claim much more, by giving an explicit description of all gardens:

> Let S be any nonempty subset of the mn cells. Suppose we fill each cell θ with the minimum (taxicab) distance from θ to some cell in S (in particular, we write 0 if $\theta \in S$). Then
>
> - This gives a garden, and
> - All gardens are of this form.

Since there are $2^{mn} - 1$ such nonempty subsets S, this would finish the problem. An example of a garden with $|S| = 3$ is shown below.

$$\begin{bmatrix} 2 & 1 & 2 & 1 & \mathbf{0} & 1 \\ 1 & \mathbf{0} & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 3 & 2 & 3 \\ \mathbf{0} & 1 & 2 & 3 & 3 & 4 \end{bmatrix}$$

It is actually fairly easy to see that this procedure always gives a garden; so we focus our attention on showing that every garden is of this form.

Given a garden, note first that it has at least one cell with a zero in it — by considering the minimum number across the entire garden. Now let S be the (thus nonempty) set of cells with a zero written in them. We contend that this works, i.e. the following sentence holds:

Claim — If a cell θ is labeled d, then the minimum distance from that cell to a cell in S is d.

Proof. The proof is by induction on d, with $d = 0$ being by definition. Now, consider any cell θ labeled $d \geq 1$. Every neighbor of θ has label at least $d - 1$, so any path will necessarily take $d - 1$ steps after leaving θ. Conversely, there is *some* $d - 1$ adjacent to θ by (ii). Stepping on this cell and using the minimal path (by induction hypothesis) gives us a path to a cell in S with length *exactly* d. So the shortest path does indeed have distance d, as desired. □

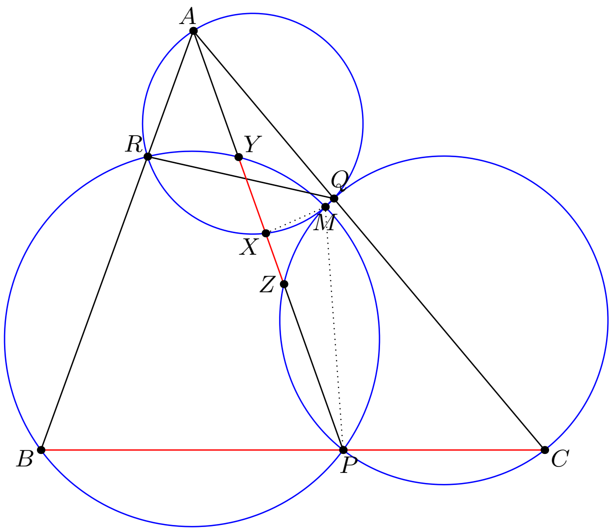
§1.3 JMO 2013/3, proposed by Zuming Feng

Available online at https://aops.com/community/p3041822.

Problem statement

In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

Let M be the concurrence point of ω_A, ω_B, ω_C (by Miquel's theorem).



Then M is the center of a spiral similarity sending $\overline{YZ}$ to $\overline{BC}$. So it suffices to show that this spiral similarity also sends X to P, but

$$\measuredangle MXY = \measuredangle MXA = \measuredangle MRA = \measuredangle MRB = \measuredangle MPB$$

so this follows.

§2 Solutions to Day 2

§2.1 JMO 2013/4, proposed by Kiran Kedlaya

Available online at https://aops.com/community/p3043748.

Problem statement

Let $f(n)$ be the number of ways to write n as a sum of powers of 2, where we keep track of the order of the summation. For example, $f(4) = 6$ because 4 can be written as 4, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Find the smallest n greater than 2013 for which $f(n)$ is odd.

The answer is 2047.

For convenience, we agree that $f(0) = 1$. Then by considering cases on the first number in the representation, we derive the recurrence

$$f(n) = \sum_{k=0}^{\lfloor \log_2 n \rfloor} f(n - 2^k). \qquad (\heartsuit)$$

We wish to understand the parity of f. The first few values are

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 1 \\ f(2) &= 2 \\ f(3) &= 3 \\ f(4) &= 6 \\ f(5) &= 10 \\ f(6) &= 18 \\ f(7) &= 31. \end{aligned}$$

Inspired by the data we make the key claim that

Claim — $f(n)$ is odd if and only if $n+1$ is a power of 2.

Proof. We call a number *repetitive* if it is zero or its binary representation consists entirely of 1's. So we want to prove that $f(n)$ is odd if and only if n is repetitive.

This only takes a few cases:

- If $n = 2^k$, then $(\heartsuit)$ has exactly two repetitive terms on the right-hand side, namely 0 and $2^k - 1$.
- If $n = 2^k + 2^\ell - 1$, then $(\heartsuit)$ has exactly two repetitive terms on the right-hand side, namely $2^{\ell+1} - 1$ and $2^\ell - 1$.
- If $n = 2^k - 1$, then $(\heartsuit)$ has exactly one repetitive terms on the right-hand side, namely $2^{k-1} - 1$.
- For other n, there are no repetitive terms at all on the right-hand side of $(\heartsuit)$.

Thus the induction checks out. $\square$

So the final answer to the problem is 2047.

§2.2 JMO 2013/5, proposed by Zuming Feng

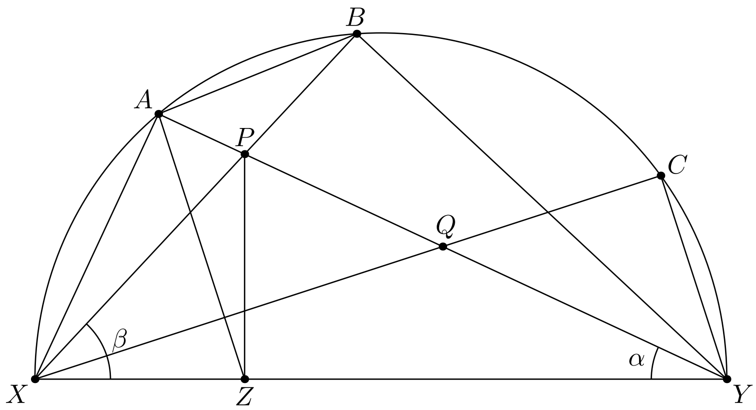
Available online at https://aops.com/community/p3043750.

Problem statement

Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter $\overline{XY}$. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line $\overline{XY}$. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$

Let $\beta = \angle YXP$ and $\alpha = \angle PYX$ and set $XY = 1$. We do not direct angles in the following solution.



Observe that

$$\angle AZX = \angle APX = \alpha + \beta$$

since $APZX$ is cyclic. In particular, $\angle CXY = 90^\circ - (\alpha + \beta)$. It is immediate that

$$BY = \sin\beta, \quad CY = \cos(\alpha+\beta), \quad AY = \cos\alpha, \quad AX = \sin\alpha.$$

The Law of Sines on $\triangle XPY$ gives $XP = XY\frac{\sin\alpha}{\sin(\alpha+\beta)}$, and on $\triangle XQY$ gives $XQ = XY\frac{\sin\alpha}{\sin(90+\beta)} = \frac{\sin\alpha}{\cos\beta}$. So, the given is equivalent to

$$\frac{\sin\beta}{\frac{\sin\alpha}{\sin(\alpha+\beta)}} + \frac{\cos(\alpha+\beta)}{\frac{\sin\alpha}{\cos\beta}} = \frac{\cos\alpha}{\sin\alpha}$$

which is equivalent to $\cos\alpha = \cos\beta\cos(\alpha+\beta) + \sin\beta\sin(\alpha+\beta)$. This is obvious, because the right-hand side is just $\cos((\alpha+\beta)-\beta)$.

§2.3 JMO 2013/6, proposed by Titu Andreescu

Available online at https://aops.com/community/p3043752.

Problem statement

Find all real numbers $x, y, z \geq 1$ satisfying

$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Set $x = 1 + a$, $y = 1 + b$, $z = 1 + c$ which eliminates the $x, y, z \geq 1$ condition. Then the given equation rewrites as

$$\sqrt{(1+a)\left(1+(1+b)(1+c)\right)} = \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

In fact, we are going to prove the left-hand side always exceeds the right-hand side, and then determine the equality cases. We have:

$$\begin{aligned}(1+a)\left(1+(1+b)(1+c)\right) &= (a+1)\left(1+(b+1)(1+c)\right)\\ &\leq (a+1)\left(1+\left(\sqrt{b}+\sqrt{c}\right)^2\right)\\ &\leq \left(\sqrt{a}+\left(\sqrt{b}+\sqrt{c}\right)\right)\end{aligned}$$

by two applications of Cauchy-Schwarz.

Equality holds if $bc = 1$ and $1/a = \sqrt{b} + \sqrt{c}$. Letting $c = t^2$ for $t \geq 1$, we recover $b = t^{-2} \leq t^2$ and $a = \frac{1}{t+1/t} \leq t^2$.

Hence the solution set is

$$(x, y, z) = \left(1 + \left(\frac{t}{t^2+1}\right)^2, 1 + \frac{1}{t^2}, 1 + t^2\right)$$

and permutations, for any $t > 0$.

5^{th} United States of America Junior Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 29, 2014

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

JMO 1. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2-5a+1}{b^2-5b+10}, \frac{10b^2-5b+1}{c^2-5c+10}, \frac{10c^2-5c+1}{a^2-5a+10}\right) \leq abc.$$

JMO 2. Let ΔABC be a non-equilateral, acute triangle with $\angle A = 60^\circ$, and let O and H denote the circumcenter and orthocenter of ΔABC, respectively.

(a) Prove that line OH intersects both segments AB and AC.

(b) Line OH intersects segments AB and AC at P and Q, respectively. Denote by s and t the respective areas of triangle APQ and quadrilateral $BPQC$. Determine the range of possible values for s/t.

JMO 3. Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y)-x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

5^{th} United States of America Junior Mathematical Olympiad

Day II   12:30 PM – 5 PM EDT

April 30, 2014

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

JMO 4. Let $b \geq 2$ be an integer, and let $s_b(n)$ denote the sum of the digits of n when it is written in base b. Show that there are infinitely many positive integers that cannot be represented in the form $n + s_b(n)$, where n is a positive integer.

JMO 5. Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

JMO 6. Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc BAC of Γ.

(a) Prove that I lies on ray CV.

(b) Prove that line XI bisects $\overline{UV}$.

4^{th} United States of America Junior Mathematical Olympiad

Day I, II   12:30 PM – 5 PM EDT

April 29 - April 30, 2014

JMO 1. We start by observing that the denominators of the fractions involved in the statement of the problem are positive. Next, we argue by contradiction and assume that

$$10a^2 - 5a + 1 > abc(b^2 - 5b + 10)$$

and similar inequalities obtained by cyclic permutations. Multiplying these inequalities yields

$$\prod[a^3(a^2 - 5a + 10)] < \prod(10a^2 - 5a + 1).$$

This is impossible, since

$$a^3(a^2 - 5a + 10) - (10a^2 - 5a + 1) = (a - 1)^5 \geq 0$$

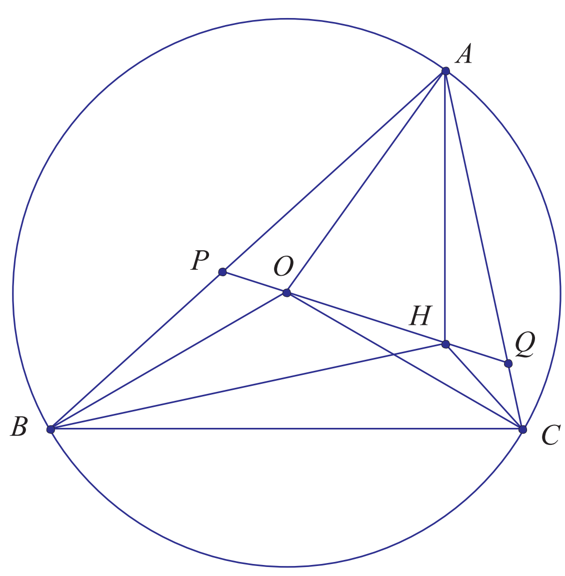
and similarly for b and c.

This problem and solution was suggested by Titu Andreescu.

JMO 2. **(a):** Without loss of generality, we assume that $AB > AC$. Set $\beta = \angle ABC$ and $\gamma = \angle ACB$. We have $\beta < 60^\circ < \gamma$ and $\beta + \gamma = 120^\circ$.

Note that $\angle BAO = 90^\circ - \angle ACB = 90^\circ - \gamma < 90^\circ - \beta = 90^\circ - \angle ABC = \angle BAH$, and so AO lies inside $\angle BAH$. Similarly, $\angle ABO = 90^\circ - \gamma < 30^\circ = \angle ABH$, and so BO lies inside $\angle ABH$. Hence O lies inside $\triangle ABH$, and line OH intersects side AB. In the same way, $\angle CAH = 90^\circ - \gamma < 90^\circ - \beta = \angle CAO$ and $\angle ACH = 30^\circ < 90^\circ - \beta = \angle ACO$; hence H lies inside $\triangle ACO$, and line OH intersects side AC.

(b): The range of s/t is the open interval $(4/5, 1)$.



Based on (a), we may consider the configuration shown above. Note that $\angle BOC = 2\angle BAC = 120^\circ$ and $\angle BHC = 180^\circ - \angle HBC - \angle HCB = 180^\circ - (90^\circ - \gamma) - (90^\circ - \beta) = 120^\circ$, from which it follows that $BOHC$ is cyclic. In particular, $\angle POB = 180^\circ - \angle HOB = \angle HCB = 90^\circ - \beta$, and it follows that

$$\angle APQ = \angle ABO + \angle POB = (90^\circ - \gamma) + (90^\circ - \beta) = 60^\circ.$$

Since $\angle PAQ = 60^\circ$ as well, we see that $\triangle APQ$ is equilateral.

Next note that $\angle POB = 90^\circ - \beta = \angle ACO = \angle QCO$ and $\angle PBO = 90^\circ - \gamma = \angle HBC = \angle HOC = \angle QOC$; since $BO = OC$, we have congruent triangles $\triangle BPO \cong \triangle OQC$. Thus

$$AB + AC = AP + PB + CQ + QA = AP + QO + OP + QA = AP + PQ + QA$$

and so $AP = PQ = QA = \frac{b+c}{3}$, where we write $b = AC$ and $c = AB$. Therefore we have

$$\frac{s}{s+t} = \frac{\text{Area}(\triangle APQ)}{\text{Area}(\triangle ABC)} = \frac{AP}{AB}\frac{AQ}{AC} = \frac{\left(\frac{b+c}{3}\right)^2}{bc} = \frac{2 + m + 1/m}{9},$$

where $m = c/b$.

By our assumptions that $b < c$ and $\triangle ABC$ is acute, it follows that the range of m is $1 \leq m < 2$. (One can see this, for instance, by having A move along the major arc $\overset{\frown}{BC}$ from one extreme, where ABC is equilateral and $c/b = 1$, to the other, where $\angle ACB = 90^\circ$ and $c/b = 2$, and noting that c increases and b decreases during this motion.) For $m \in (1,2)$, the function $f(m) = m + 1/m$ is continuous and increasing: if $1 < m < m' < 2$, then $f(m') - f(m) = \frac{(m'-m)(mm'-1)}{mm'} > 0$. Thus the range of $f(m)$ for $m \in (1,2)$ is $(f(1), f(2)) = (2, \frac{5}{2})$. It follows that the range of $\frac{s}{s+t} = \frac{2+f(m)}{9}$ is $(\frac{4}{9}, \frac{1}{2})$, and the range of $\frac{s}{t}$ is $(\frac{4}{5}, 1)$.

This problem and the first solution was suggested by Zuming Feng.

OR

(b): We use complex numbers. Let $O = 0$, $B = 1$, $C = \omega = e^{2\pi i/3}$, and $A = a$ with $|a| = 1$. Then $H = 1 + \omega + a = a - \omega^2$. Bearing in mind that the equation for the line through complex numbers w_1 and w_2 is $\frac{z-w_1}{w_2-w_1} = \frac{\overline{z}-\overline{w_1}}{\overline{w_2}-\overline{w_1}}$ (i.e., the quotient $\frac{z-w_1}{w_2-w_1}$ is purely real), we see that P, which is the intersection of AB and OH, lies at the point z satisfying

$$\frac{z-1}{a-1} = \frac{\overline{z}-1}{\overline{a}-1} \quad \text{and} \quad \frac{z}{a-\omega^2} = \frac{\overline{z}}{\overline{a}-\omega}.$$

Substituting $\overline{a} = 1/a$, eliminating $\overline{z}$, and solving for z yields $z = \frac{a+1}{1-\omega}$. Thus the vector $\overrightarrow{AP}$ is given by the complex number $\frac{a+1}{1-\omega} - a = \frac{a\omega+1}{1-\omega}$. Similarly Q lies at the point $\frac{a\omega+\omega^2}{\omega-1}$ and the vector $\overrightarrow{AQ}$ is $\frac{a+\omega^2}{\omega-1}$. It follows that $AP = \frac{1}{\sqrt{3}}|\omega a+1| = \frac{1}{\sqrt{3}}|a+\omega^2| = AQ$.

Now $\overrightarrow{AB} = 1 - a$ is collinear with $\overrightarrow{AP} = \frac{a\omega+1}{1-\omega}$, and the ratio of the lengths of these vectors is $\frac{AB}{AP} = (1-a)/\left(\frac{a\omega+1}{1-\omega}\right) = \frac{(1-a)(1-\omega)}{a\omega+1}$; similarly $\overrightarrow{AC} = \omega - a$ is collinear with $\overrightarrow{AQ} = \frac{a+\omega^2}{\omega-1}$, and $\frac{AC}{AQ} = \frac{(\omega-a)(\omega-1)}{a+\omega^2} = \frac{(\omega-a)(\omega^2-\omega)}{a\omega+1}$. Thus

$$\frac{AB+AC}{AP} = \frac{AB}{AP} + \frac{AC}{AQ} = \frac{(1-a)(1-\omega)+(\omega-a)(\omega^2-\omega)}{a\omega+1} = \frac{3a\omega+3}{a\omega+1} = 3,$$

and so

$$\frac{AP}{AB}\frac{AQ}{AC} = \frac{(AB+AC)^2}{9(AB)(AC)}.$$

The second solution was suggested by Razvan Gelca.

JMO 3. Let f be a solution of the problem. Let p be a prime. Since p divides $f(p)^2$, p divides $f(p)$ and so p divides $\frac{f(p)^2}{p}$. Taking $y = 0$ and $x = p$, we deduce that p divides $f(0)$. As p is arbitrary, we must have $f(0) = 0$. Next, take $y = 0$ to obtain $xf(-x) = \frac{f(x)^2}{x}$. Replacing x by $-x$, and combining the two relations yields $f(x) = 0$ or $f(x) = x^2$ for all x.

Suppose now that there exists $x_0 \neq 0$ such that $f(x_0) = 0$. Taking $y = x_0$, we obtain $xf(-x) + x_0^2 f(2x) = \frac{f(x)^2}{x}$, yielding $x_0^2 f(2x) = 0$ for all x and so f vanishes on even numbers. Assume that there exists an odd number y_0 such that $f(y_0) \neq 0$, so $f(y_0) = y_0^2$. Taking $y = y_0$, we obtain

$$xf(2y_0^2 - x) + y_0^2 f(2x - y_0^2) = \frac{f(x)^2}{x} + f(y_0^3).$$

Choosing x even, we deduce that $y_0^2 f(2x - y_0^2) = f(y_0^3)$. This forces $f(y_0^3) = 0$, as otherwise we would have $f(2x - y_0^2) = (2x - y_0^2)^2$ for all even x and so $y_0^2(2x - y_0^2)^2 = f(y_0^3)$ for all such x, obviously impossible. Thus $f(2x - y_0^2) = 0$ for all even numbers x, that is f vanishes on numbers of the form $4k+3$. But since $x^2 f(-x) = f(x)^2$, f also vanishes on all x such that $-x \equiv -1 \pmod 4$, that is on $4\mathbb{Z}+1$. Thus f also vanishes on all odd numbers, contradicting the choice of y_0. Hence, if f is not the zero map, then f does not vanish outside 0 and so $f(x) = x^2$ for all x.

In conclusion, $f(x) = 0$ for all $x \in \mathbb{Z}$ and $f(x) = x^2$ for all $x \in \mathbb{Z}$ are the only possible solutions. The first function clearly satisfies the given relation, while the second also satisfies the Sophie Germaine identity

$$x(2y^2 - x)^2 + y^2(2x - y^2)^2 = x^3 + y^6$$

for all $x, y \in \mathbb{Z}$.

OR

$f(0) = 0$: If $f(0) \neq 0$, set $x = 2f(0)$ to obtain

$$2(f(0))^2 = \frac{(f(2f(0)))^2}{2f(0)} + f(0)$$

that is

$$2(f(0))^2(2f(0)-1) = f(2f(0))^2.$$

But $2(2f(0)-1)$ cannot be a perfect square since it is of the form $4k+2$. So $f(0) = 0$.

This problem and the solutions were suggested by Titu Andreescu and Gabriel Dospinescu.

JMO 4. Let $f(n) = n + s_b(n)$. For a positive integer m, let $k = \lfloor \log_b(m/2) \rfloor$, so that $m \geq 2b^k$. Note that if $b^m - b^k \leq n < b^m$, then the base b expansion of n begins with $m-k$ digits equal to $b-1$, and therefore

$$f(n) > b^m - b^k + (m-k)(b-1) \geq b^m - b^k + (2b^k - k)(b-1) \geq b^m. \qquad (1)$$

Now consider the set $\{f(1), f(2), \ldots, f(b^m)\}$. Any number that is $\leq b^m$ and in the range of f is in this set. However, we see from (1) that $f(n) > b^m$ whenever $b^m - b^k \leq n < b^m$. Therefore, there are at least b^k numbers from 1 to b^m that are not in the range of f. Since k goes to infinity as m goes to infinity, the desired result follows.

This problem and solution was suggested by Palmer Mebane.

OR

We first show that there exist infinitely many pairs $(n_1, m_1), (n_2, m_2), \ldots$ such that $n_i + s_b(n_i) = m_i + s_b(m_i)$ for all i.

- **Case 1** $b = 2$. Let i be a positive integer, and set $j = 2^i + 3$; note $j > i$. Then for $n_i = 2^j - 1$, we have $s_2(n_i) = j$. If we then consider $m_i = 2^j + j - 3$, we have by the definition of j that $m_x = 2^j + 2^i$, so $s_2(m_i) = 2$. It is easy to see that $n_i + s_2(n_i) = m_i + s_2(m_i)$.
- **Case 2** $b > 2$. Let i be a positive integer, and set $j = \frac{b^i+b-2}{b-1} + 1$; note $j > i$. Then for $n_i = b^j - b + 2$, we have $s_b(n_i) = (b-1)(j-1) + 2$. If we then consider $m_i = b^j - b + (b-1)(j-1) + 2$, plugging in our definition for j in the third term gives
$$m_i = b^j - b + (b-1)\left(\frac{b^i + b - 2}{b-1}\right) + 2 = b^j + b^i,$$
so $s_b(m_i) = 2$. We can easily compute that $n_i + s_b(n_i) = m_i + s_b(m_i)$.

In both cases, since j grows exponentially with i, it is easy to check that $n_i < m_i < n_{i+1} < m_{i+1}$, so all of the constructed pairs contain pairwise distinct positive integers.

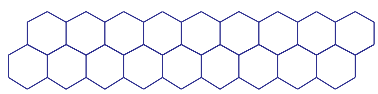
Now we will show at least k positive integers cannot be represented in the form $n + s_b(n)$ for any k. Take $(n_1, m_1), \ldots (n_k, m_k)$ and let A be a number greater than any of the $2k$ numbers in these pairs. For a positive integer x with $x \leq A$, if we have $x = n + s_b(n)$ then we must have $n \leq x \leq A$. So in finding ways to represent the numbers $1, 2, \ldots A$ in the form $n + s_b(n)$, all of them require $n \leq A$. However, among numbers at most A there are at least k pairs n_i, m_i with $n_i + s_b(n_i) = m_i + s_b(m_i)$. Therefore the set

$\{n + s_b(n) \mid n = 1, 2, \ldots A\}$ has at most $A - k$ elements, and so at least k of the numbers $1, 2, \ldots A$ are not members of this set and thus have no representation in the form $n + s_b(n)$. This proves our original claim. Since k is arbitrary there cannot be a finite amount of positive integers with no representation, so there are infinitely many as desired.

The second solution was suggested by Palmer Mebane.

JMO 5. The answer is $k = 6$. First we show that A cannot win for $k \geq 6$. Color the grid in three colors so that no two adjacent spaces have the same color, and arbitrarily pick one color C. B will play by always removing a counter from a space colored C that A just played. If there is no such counter, B plays arbitrarily. Because A cannot cover two spaces colored C simultaneously, it is possible for B to play in this fashion. Now note that any line of six consecutive squares contains two spaces colored C. For A to win he must cover both, but B's strategy ensures at most one space colored C will have a counter at any time.

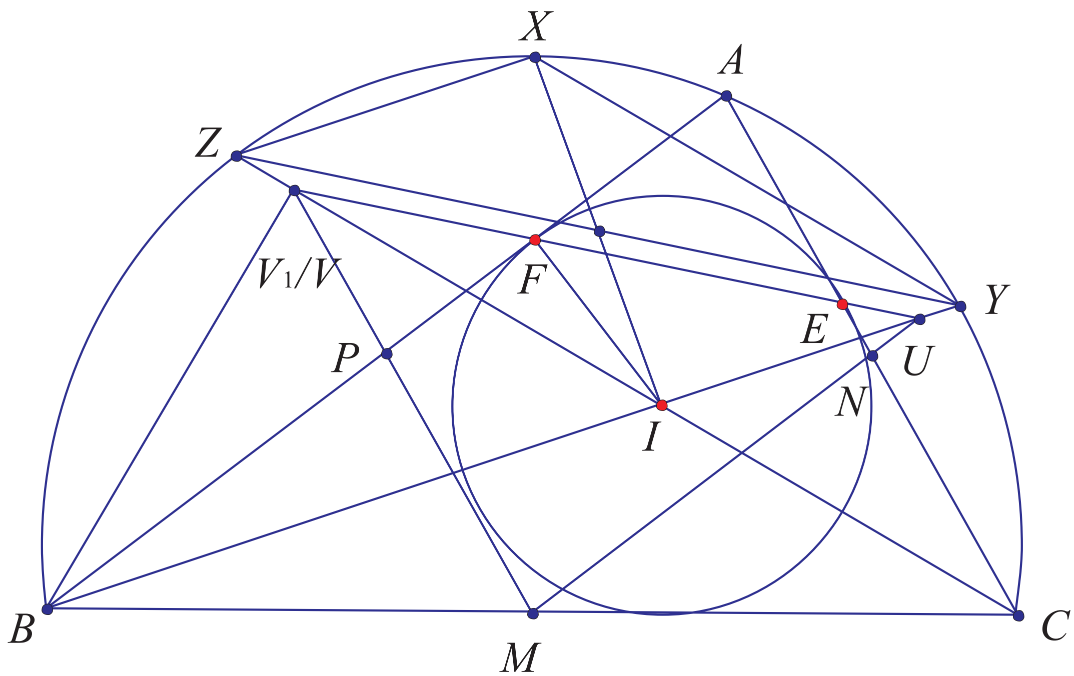
Now we show that A can obtain 5 counters in a row. Take a set of cells in the grid forming the shape shown below. We will have A play counters only in this set of grid cells until this is no longer possible. Since B only removes one counter for every two A places, the number of counters in this set will increase each turn, so at some point it will be impossible for A to play in this set anymore. At that point any two adjacent grid spaces in the set have at least one counter between them.



Consider only the top row of cells in the set, and take the lengths of each consecutive run of cells. If there are two adjacent runs that have a combined length of at least 4, then A gets 5 counters in a row by filling the space in between. Otherwise, a bit of case analysis shows that there exists a run of 1 counter which is neither the first nor last run. This single counter has an empty space on either side of it on the first row. As a result, the four spaces of the second row touching these two empty spaces all must have counters. Then A can play in the 5th cell on either side of these 4 to get 5 counters in a row. So in all cases A can win with $k \leq 5$.

This problem and solution was suggested by Palmer Mebane.

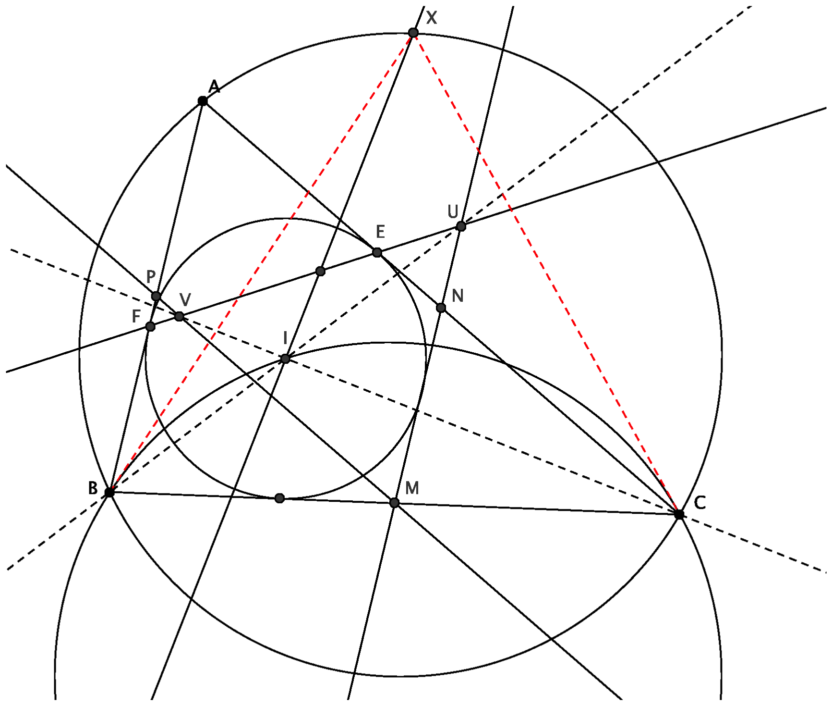
JMO 6. Set $\angle ABC = 2y$ and $\angle BCA = 2z$. First, we start with a known fact that I lies on ray CV. Let V_1 be the foot of the perpendicular from B to ray CI. Then in right triangle BV_1C, $V_1M = MB = MC$ and $\angle MV_1C = \angle MCV_1 = z = \angle V_1CA$, implying that $\overline{MV_1} \| \overline{CA}$; in particular, V_1 lies on line MP. Because $\angle BV_1I = \angle BFI = 90°$, $BIFV_1$ is cyclic, from which it follows that $\angle V_1FB = \angle V_1IB = y + z = \angle AEF = \angle AFE$; in particular, V_1 lies on $\overline{EF}$. Because V_1 lies on both line MP and line EF, $V = V_1$ and V lies on line CI. Likewise we can prove that U lies on line BI.



Rays BI and CI intersect again at Y and Z. Note that $\angle UVC = \angle EVC = \angle AEV - \angle ECV = \angle AEF - \angle ECV = y$. Because $BCYZ$ is cyclic, we have $\angle YZC = \angle YBC = y$. Therefore, $\overline{UV} \| \overline{YZ}$. It suffices to show that IX bisects segment $\overline{YZ}$, which is clearly true because $IYXZ$ is a parallelogram. (Indeed, $\angle YZX = \overset{\frown}{XAY} = \angle XBC - \angle YBC = y + z - y = z = \angle ZYB$, from which it follows that $\overline{ZX} \| \overline{IY}$. Likewise, we can show that $\overline{IZ} \| \overline{XY}$.)

OR

First, note that U and V lie on the bisectors BI and CI, respectively. Indeed, let D be the tangency point of γ with BC and let U' be the intersection of BI with EF. Note that triangles BFU' and BDU' are congruent (by SAS), so $\angle BU'F = \angle BU'D$. In addition, the pencil $(U'F, U'B, U'D, U'C)$ is harmonic; thus, it follows that $U'B \perp U'C$, so, in particular, $U'M = MB$, which gives $\angle MU'B = \angle MBU' = \frac{1}{2}\angle B = \angle ABU'$; thus, $MU' \| AB$; hence $U' = U$, which proves the claim that U lies on BI. Similarly, we get that V is on CI. Also, remember the perpendicularities $IB \perp CU$ and $IC \perp VB$, which we will use soon.



Next, note that the lines XB and XC are tangent to the circumcircle of triangle IBC; indeed, observe that

$$\begin{aligned}
\angle XBI &= \angle ABI - \angle ABX \\
&= \frac{1}{2}\angle B - (\angle BCX - \angle C) \\
&= \frac{1}{2}\angle B - \frac{1}{2}(180^\circ - \angle A) + \angle C \\
&= \frac{1}{2}\angle C \\
&= \angle BCI.
\end{aligned}$$

Similarly, $\angle XCI = \angle IBC$. This means that X is the intersection of the tangents at B and C to the circumcircle of IBC; hence, IX is the $I-$symmedian of triangle IBC.

But we proved before that U and V are on IB and IC, respectively and that $IB \perp CU$ and $IC \perp VB$. In other words, we showed that U and V are the feet of the altitudes from C and B in triangle IBC - so, in particular, we have that $BCUV$ is cyclic and that UV is an antiparallel to BC in triangle IBC. This yields the conclusion, since we know that the $I-$symmedian of IBC is the locus of the midpoints of the antiparallels to BC in triangle IBC; hence we showed that IX bisects UV, as claimed. ■

This problem and and the second solution were suggested by Titu Andreescu and Cosmin Pohoata. The first solution was suggested by Zuming Feng.

JMO 2014 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2014 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the “official” solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered “standard”, then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like “let $\mathbb{R}$ denote the set of real numbers” are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2-5a+1}{b^2-5b+10}, \frac{10b^2-5b+1}{c^2-5c+10}, \frac{10c^2-5c+1}{a^2-5a+10}\right) \le abc.$$

2. Let $\triangle ABC$ be a non-equilateral, acute triangle with $\angle A = 60^\circ$, and let O and H denote the circumcenter and orthocenter of $\triangle ABC$, respectively.

(a) Prove that line OH intersects both segments AB and AC at two points P and Q, respectively.

(b) Denote by s and t the respective areas of triangle APQ and quadrilateral $BPQC$. Determine the range of possible values for s/t.

3. Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y)-x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ such that $x \neq 0$.

4. Let $b \ge 2$ be a fixed integer, and let $s_b(n)$ denote the sum of the base-b digits of n. Show that there are infinitely many positive integers that cannot be represented in the from $n + s_b(n)$ where n is a positive integer.

5. Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In her move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

6. Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc BAC of Γ.

(a) Prove that I lies on ray CV.

(b) Prove that line XI bisects $\overline{UV}$.

§1 Solutions to Day 1

§1.1 JMO 2014/1, proposed by Titu Andreescu

Available online at https://aops.com/community/p3477681.

Problem statement

Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2-5a+1}{b^2-5b+10}, \frac{10b^2-5b+1}{c^2-5c+10}, \frac{10c^2-5c+1}{a^2-5a+10}\right) \le abc.$$

Notice that

$$\frac{10a^2-5a+1}{a^2-5a+10} \le a^3$$

since it rearranges to $(a-1)^5 \ge 0$. Cyclically multiply to get

$$\prod_{\text{cyc}} \left(\frac{10a^2-5a+1}{b^2-5b+10}\right) \le (abc)^3$$

and the minimum is at most the geometric mean.

§1.2 JMO 2014/2, proposed by Zuming Feng

Available online at https://aops.com/community/p3477702.

Problem statement

Let $\triangle ABC$ be a non-equilateral, acute triangle with $\angle A = 60^\circ$, and let O and H denote the circumcenter and orthocenter of $\triangle ABC$, respectively.

(a) Prove that line OH intersects both segments AB and AC at two points P and Q, respectively.

(b) Denote by s and t the respective areas of triangle APQ and quadrilateral $BPQC$. Determine the range of possible values for s/t.

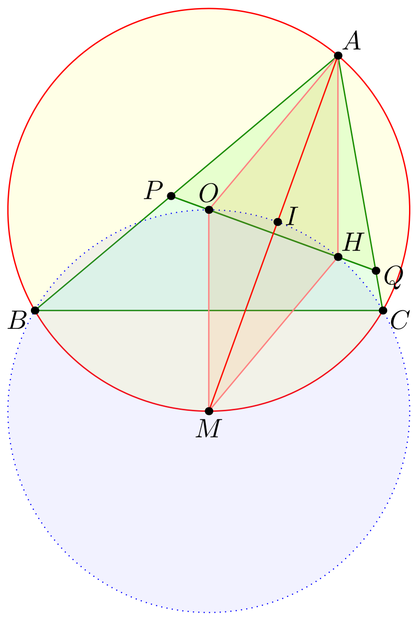
We begin with some synthetic work. Let I denote the incenter, and recall ("fact 5") that the arc midpoint M is the center of (BIC), which we denote by γ.

Now we have that

$$\angle BOC = \angle BIC = \angle BHC = 120^\circ.$$

Since all three centers lie inside ABC (as it was acute), and hence on the opposite side of $\overline{BC}$ as M, it follows that O, I, H lie on minor arc BC of γ.

We note this implies (a) already, as line OH meets line BC outside of segment BC.



Claim — Triangle APQ is equilateral with side length $\frac{b+c}{3}$.

Proof. Let R be the circumradius. We have $R = OM = OA = MH$, and even $AH = 2R\cos A = R$, so $AOMH$ is a rhombus. Thus $\overline{OH} \perp \overline{AM}$ and in this way we derive that $\triangle APQ$ is isosceles, hence equilateral.

Finally, since $\angle PBH = 30^\circ$, and $\angle BPH = 120^\circ$, it follows that $\triangle BPH$ is isosceles and $BP = PH$. Similarly, $CQ = QH$. So $b+c = AP+BP+AQ+QC = AP+AQ+PQ$ as needed. $\square$

Finally, we turn to the boring task of extracting the numerical answer. We have

$$\frac{s}{s+t} = \frac{[APQ]}{[ABC]} = \frac{\frac{\sqrt{3}}{4}\left(\frac{b+c}{3}\right)^2}{\frac{\sqrt{3}}{4}bc} = \frac{b^2+2bc+c^2}{9bc} = \frac{1}{9}\left(2+\frac{b}{c}+\frac{c}{b}\right).$$

So the problem is reduced to analyzing the behavior of b/c. For this, we imagine fixing Γ the circumcircle of ABC, as well as the points B and C. Then as we vary A along the "topmost" arc of measure 120°, we find b/c is monotonic with values $1/2$ and 2 at endpoints, and by continuity all values $b/c \in (1/2, 2)$ can be achieved.

So

$$\frac{1}{2} < \frac{b}{c} < 2 \implies 4/9 < \frac{s}{s+t} < 1/2 \implies 4/5 < \frac{s}{t} < 1$$

as needed.

§1.3 JMO 2014/3, proposed by Titu Andreescu

Available online at https://aops.com/community/p3477690.

Problem statement

Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf\left(2f(y)-x\right)+y^2f\left(2x-f(y)\right)=\frac{f(x)^2}{x}+f\left(yf(y)\right)$$

for all $x, y \in \mathbb{Z}$ such that $x \neq 0$.

The answer is $f(x) \equiv 0$ and $f(x) \equiv x^2$. Check that these work.

Now let's prove these are the only solutions. Put $y = 0$ to obtain

$$xf\left(2f(0)-x\right)=\frac{f(x)^2}{x}+f(0).$$

Now we claim $\boxed{f(0)=0}$. If not, select a prime $p \nmid f(0)$ and put $x = p \neq 0$. In the above, we find that $p \mid f(p)^2$, so $p \mid f(p)$ and hence $p \mid \frac{f(p)^2}{p}$. From here we derive $p \mid f(0)$, contradiction. Hence

$$f(0) = 0.$$

The above then implies that

$$x^2 f(-x) = f(x)^2$$

holds for all nonzero x, but also for $x = 0$. Let us now check that f is an even function. In the above, we may also derive $f(-x)^2 = x^2 f(x)$. If $f(x) \neq f(-x)$ (and hence $x \neq 0$), then subtracting the above and factoring implies that $f(x) + f(-x) = -x^2$; we can then obtain by substituting the relation

$$\left[f(x)+\frac{1}{2}x^2\right]^2=-\frac{3}{4}x^4<0$$

which is impossible. This means $f(x)^2 = x^2 f(x)$, thus

$$f(x) \in \{0, x^2\} \qquad \forall x.$$

Now suppose there exists a nonzero integer t with $f(t) = 0$. We will prove that $f(x) \equiv 0$. Put $y = t$ in the given to obtain that

$$t^2 f(2x) = 0$$

for any integer $x \neq 0$, and hence conclude that $f(2\mathbb{Z}) \equiv 0$. Then selecting $x = 2k \neq 0$ in the given implies that

$$y^2 f(4k - f(y)) = f(yf(y)).$$

Assume for contradiction that $f(m) = m^2$ now for some odd $m \neq 0$. Evidently

$$m^2 f(4k - m^2) = f(m^3).$$

If $f(m^3) \neq 0$ this forces $f(4k - m^2) \neq 0$, and hence $m^2(4k - m^2)^2 = m^6$ for arbitrary $k \neq 0$, which is clearly absurd. That means

$$f(4k - m^2) = f(m^2 - 4k) = f(m^3) = 0$$

for each $k \neq 0$. Since m is odd, $m^2 \equiv 1 \pmod 4$, and so $f(n) = 0$ for all n other than $\pm m^2$ (since we cannot select $k = 0$).

Now $f(m) = m^2$ means that $m = \pm 1$. Hence either $f(x) \equiv 0$ or

$$f(x) = \begin{cases} 1 & x = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

To show that the latter fails, we simply take $x = 5$ and $y = 1$ in the given.

Hence, the only solutions are $f(x) \equiv 0$ and $f(x) \equiv x^2$.

§2 Solutions to Day 2

§2.1 JMO 2014/4, proposed by Palmer Mebane

Available online at https://aops.com/community/p3478579.

Problem statement

Let $b \geq 2$ be a fixed integer, and let $s_b(n)$ denote the sum of the base-b digits of n. Show that there are infinitely many positive integers that cannot be represented in the from $n + s_b(n)$ where n is a positive integer.

For brevity let $f(n) = n + s_b(n)$. Select any integer M. Observe that $f(x) \geq b^{2M}$ for any $x \geq b^{2M}$, but also $f(b^{2M} - k) \geq b^{2M}$ for $k = 1, 2, \ldots, M$, since the base-b expansion of $b^{2M} - k$ will start out with at least M digits $b - 1$.

Thus f omits at least M values in $[1, b^{2M}]$ for any M.

§2.2 JMO 2014/5, proposed by Palmer Mebane

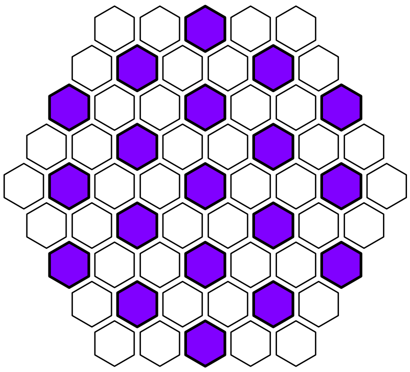
Available online at https://aops.com/community/p3478584.

Problem statement

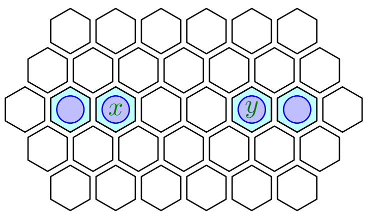
Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In her move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

The answer is $k = 6$.

Proof that A cannot win if $k = 6$. We give a strategy for B to prevent A's victory. Shade in every third cell, as shown in the figure below. Then A can never cover two shaded cells simultaneously on her turn. Now suppose B always removes a counter on a shaded cell (and otherwise does whatever he wants). Then he can prevent A from ever getting six consecutive counters, because any six consecutive cells contain two shaded cells.

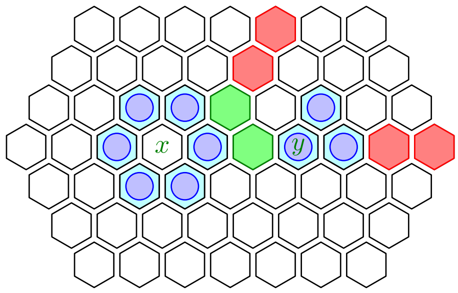


Example of a strategy for A when $k = 5$. We describe a winning strategy for A explicitly. Note that after B's first turn there is one counter, so then A may create an equilateral triangle, and hence after B's second turn there are two consecutive counters. Then, on her third turn, A places a pair of counters two spaces away on the same line. Label the two inner cells x and y as shown below.



Now it is B's turn to move; in order to avoid losing immediately, he must remove either x or y. Then on any subsequent turn, A can replace x or y (whichever was removed) and add one more adjacent counter. This continues until either x or y has all its neighbors filled (we ask A to do so in such a way that she avoids filling in the two central cells between x and y as long as possible).

So, let's say without loss of generality (by symmetry) that x is completely surrounded by tokens. Again, B must choose to remove x (or A wins on her next turn). After x is removed by B, consider the following figure.



We let A play in the two marked green cells. Then, regardless of what move B plays, one of the two choices of moves marked in red lets A win. Thus, we have described a winning strategy when $k = 5$ for A.

§2.3 JMO 2014/6, proposed by Titu Andreescu, Cosmin Pohoata

Available online at https://aops.com/community/p3478583.

Problem statement

Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc BAC of Γ.

(a) Prove that I lies on ray CV.

(b) Prove that line XI bisects $\overline{UV}$.

The fact that $I = \overline{BU} \cap \overline{CV}$ is the so-called *Iran incenter lemma*, and is proved as Lemma 1.45 from my textbook.

As for (b), we note:

Claim — Line IX is a symmedian of $\triangle IBC$.

Proof. Recall that (BIC) has circumcenter coinciding with the antipode of X (by "Fact 5"). So this follows from the fact that $\overline{XB}$ and $\overline{XC}$ are tangent. $\square$

Since $BVUC$ is cyclic with diagonals intersecting at I, and IX is symmedian of $\triangle IBC$, it is median of $\triangle IUV$, as needed.

Remark (Alternate solution to (b) by Gunmay Handa)**.** It's well known that X is the midpoint of $\overline{I_bI_c}$ (by considering the nine-point circle of the excentral triangle). However, $\overline{UV} \parallel \overline{I_bI_c}$ and $I = \overline{I_bU} \cap \overline{I_cV}$, implying the result.

6^{th} United States of America
Junior Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 28, 2015

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in a 1-point automatic deduction.

JMO 1. Given a sequence of real numbers, a move consists of choosing two terms and replacing each by their arithmetic mean. Show that there exists a sequence of 2015 distinct real numbers such that after one initial move is applied to the sequence – no matter what move – there is always a way to continue with a finite sequence of moves so as to obtain in the end a constant sequence.

JMO 2. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

JMO 3. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

6^{th} United States of America
Junior Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 29, 2015

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in a 1-point automatic deduction.

JMO 4. Find all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x) + f(t) = f(y) + f(z)$$

for all rational numbers $x < y < z < t$ that form an arithmetic progression. ($\mathbb{Q}$ is the set of all rational numbers.)

JMO 5. Let $ABCD$ be a cyclic quadrilateral. Prove that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if there exists a point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$ and $\angle CDB = \angle YDA$.

JMO 6. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he is finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i, k), (i, l), (j, k), (j, l) for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k) respectively, or removing one stone from each of (i, l) and (j, k) and moving them to (i, k) and (j, l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

6^{th} United States of America Junior Mathematical Olympiad Solutions

Day I, II 12:30 PM – 5 PM EDT

April 28 - April 29, 2015

JMO 1. Given a sequence of real numbers, a move consists of choosing two terms and replacing each by their arithmetic mean. Show that there exists a sequence of 2015 distinct real numbers such that after one initial move is applied to the sequence – no matter what move – there is always a way to continue with a finite sequence of moves so as to obtain in the end a constant sequence.

Solution: The sequence $(x_1, x_2, \ldots, x_{2015}) = (1, 2, \ldots, 2015)$ satisfies the required property (as does any arithmetic sequence).

Assume that $(x_m, x_n) = (m, n)$ is replaced by $\left(\frac{m+n}{2}, \frac{m+n}{2}\right)$ in the first move. We consider two cases.

In the first case, we assume that none of m and n is equal to 1008. In the second move, we replace $(x_{2016-m}, x_{2016-n}) = (2016 - m, 2016 - n)$ by $\left(2016 - \frac{m+n}{2}, 2016 - \frac{m+n}{2}\right)$. Let all the subsequent moves be applied to the pairs (x_j, x_{2016-j}), $j = 1, 2, \ldots, 1008$. This yields the constant sequence (1008, 1008, ..., 1008).

In the second case, we assume that one of m and n, say, n is equal to 1008. After the first move we have $x_m = x_{1008} = \frac{1008+m}{2}$. Choose k different from 1008, m, and $2016 - m$. We illustrate our next four moves in the following table. (In each move, we operate on the the numbers in bold.)

$$\begin{aligned}
& (x_k, x_m, x_{1008}, x_{2016-m}, x_{2016-k}) \\
= & \left(\boldsymbol{k}, \frac{1008+m}{2}, \frac{1008+m}{2}, 2016-m, \mathbf{2016-\boldsymbol{k}}\right) \\
\to & \left(\mathbf{1008}, \frac{1008+m}{2}, \frac{1008+m}{2}, \mathbf{2016-\boldsymbol{m}}, 1008\right) \\
\to & \left(\frac{\mathbf{3024-\boldsymbol{m}}}{\mathbf{2}}, \frac{\mathbf{1008+\boldsymbol{m}}}{\mathbf{2}}, \frac{1008+m}{2}, \frac{3024-m}{2}, 1008\right) \\
\to & \left(1008, 1008, \frac{\mathbf{1008+\boldsymbol{m}}}{\mathbf{2}}, \frac{\mathbf{3024-\boldsymbol{m}}}{\mathbf{2}}, 1008\right) \\
\to & (1008, 1008, 1008, 1008, 1008)
\end{aligned}$$

Finally apply the move to all the pairs (x_j, x_{2016-j}) (with $j \neq m, k, 2016-m, 2016-k$) to obtain the constant sequence (1008, 1008, ..., 1008).

Query: If the initial sequence is (1, 2, 3, ..., 2013, 2014, 2016), where "2015" is replaced by "2016", is it possible to obtain a constant sequence after a finite sequence of moves?

JMO 2. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

Solution: Let $x + y = 3k$, with $k \in \mathbb{Z}$. Then $x^2 + x(3k - x) + (3k - x)^2 = (k+1)^3$, which reduces to

$$x^2 - (3k)x - (k^3 - 6k^2 + 3k + 1) = 0.$$

Its discriminant Δ is

$$9k^2 + 4(k^3 - 6k^2 + 3k + 1) = 4k^3 - 15k^2 + 12k + 4.$$

We notice the (double) root $k = 2$, so $\Delta = (4k+1)(k-2)^2$. It follows that $4k+1 = (2t+1)^2$ for some nonnegative integer t, hence $k = t^2 + t$ and

$$x = \frac{1}{2}(3(t^2 + t) \pm (2t + 1)(t^2 + t - 2)).$$

We obtain $(x, y) = (t^3 + 3t^2 - 1, -t^3 + 3t + 1)$ and $(x, y) = (-t^3 + 3t + 1, t^3 + 3t^2 - 1)$, $t \in \{0, 1, 2, ...\}$.

OR

One can also try to simplify the original equation as much as possible. First with $k = \frac{x+y}{3} + 1$ we get

$$x^2 - 3xk + 3x = k^3 - 9k^2 + 18k - 9.$$

But then we recognize terms from the expansion of $(k-3)^3$ so we use $s = k - 3$ and obtain

$$x^2 - 3xs - 6x = s^3 - 9s - 9.$$

So again it becomes natural to use $x - 3 = u$. The equation becomes

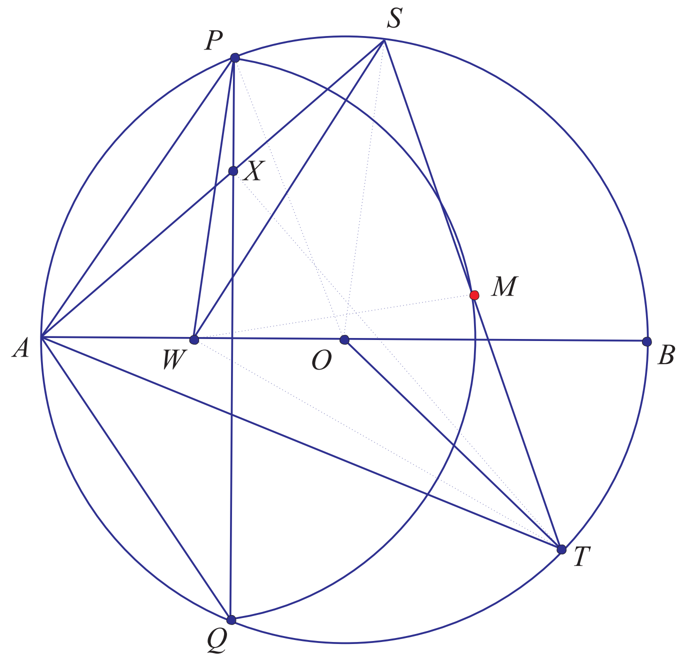
$$u^2 - 3su - s^3 = 0.$$

We view this as a quadratic in u, whose discriminant is $s^2(9 + 4s)$, and so $9 + 4s$ must be a perfect square, and because it is odd, it must be of the form $(2t + 1)^2$. It follows that $s = t^2 + t - 2$, and so $k = t^2 + t + 1$. We obtain the same family of solutions.

JMO 3. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

Solution: Let O denote the center of ω, and let W denote the midpoint of segment $\overline{AO}$. Denote by Ω the circle centered at W with radius WP. We will show that $WM = WP$, which will imply that M always lies on Ω and so solve the problem.

We present two solutions. The first solution is more computational (in particular, with extensive applications of the formula for a median of a triangle); the second is more synthetic.



Set r to be the radius of circle ω. Applying the median formula in triangles APO, SWT, ASO, ATO gives

$$\begin{aligned}
4WP^2 &= 2AP^2 + 2OP^2 - AO^2 = 2AP^2 + r^2,\\
4WM^2 &= 2WS^2 + 2WT^2 - ST^2,\\
2WS^2 &= AS^2 + OS^2 - AO^2/2 = AS^2 + r^2/2,\\
2WT^2 &= AT^2 + OT^2 - AO^2/2 = AT^2 + r^2/2.
\end{aligned}$$

Adding the last three equations yields $4WM^2 = AS^2 + AT^2 - ST^2 + r^2$. It suffices to show that

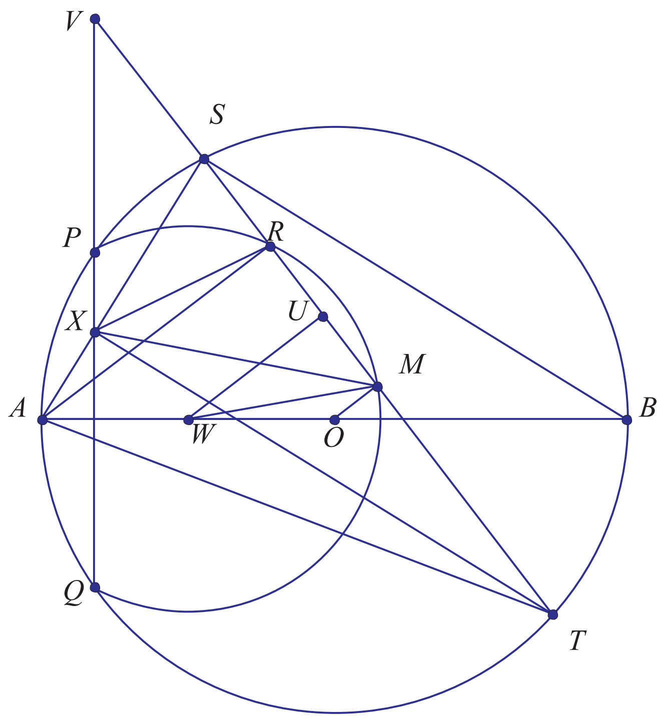
$$4WP^2 = 4WM^2 \quad \text{or} \quad AS^2 + AT^2 - ST^2 = 2AP^2. \tag{1}$$

Because $\overline{XT} \perp \overline{AS}$,

$$\begin{aligned}
AT^2 - ST^2 &= (AX^2 + XT^2) - (SX^2 + XT^2)\\
&= AX^2 - SX^2\\
&= (AX + XS)(AX - XS)\\
&= AS(AX - XS).
\end{aligned}$$

It follows that $AS^2 + AT^2 - ST^2 = AS^2 + AS \cdot (AX - XS) = AS^2 + AS(2AX - AS) = 2AS{\cdot}AX$, and (1) reduces to $AP^2 = AS{\cdot}AX$, which is true because triangle APX is similar to triangle ASP (as $\angle PAX = \angle SAP$ and $\angle APX = \text{arc}(AQ)/2 = \text{arc}(AP)/2 = \angle ASP$).

OR



In the following solution, we use directed distances and directed angles in order to avoid issues with configuration (segments $\overline{ST}$ and $\overline{PQ}$ may intersect, or may not as depicted in the figure.)

Let R be the foot of the perpendicular from A to line ST. Note that $OM \perp ST$, and so $ARMO$ is a right trapezoid. Let U be the midpoint of segment $\overline{RM}$. Then $\overline{WU}$ is the midline of the trapezoid. In particular, $\overline{WU} \perp \overline{RM}$. Hence line WU is the perpendicular bisector of segment $\overline{RM}$. It is also clear that AW is the perpendicular bisector of segment $\overline{PQ}$. Therefore, W is the intersection of the perpendicular bisectors of segments $\overline{RM}$ and $\overline{PQ}$. It suffices to show that quadrilateral $PQMR$ is cyclic, since then W must be its circumcenter, and so $WP = WM$.

(To be precise, this argument fails when ST and PQ are parallel, because then $R = M$ and the perpendicular bisector of $\overline{RM}$ is not defined. However, it is easy to see that this can happen for only one position of X. Because the argument works for all other X, continuity then implies that M lies on Ω for this exceptional case as well.)

Let lines PQ and ST meet in V. By the converse of the power-of-a-point theorem, it suffices to show that $VP \cdot VQ = VR \cdot VM$. On the other hand, because $PQTS$ is cyclic, by the power-of-a-point theorem, we have $VP \cdot VQ = VS \cdot VT$. Therefore, we only need to show that

$$VS \cdot VT = VR \cdot VM. \tag{2}$$

Note that M is the midpoint of segment $\overline{ST}$. Then (2) is equivalent to

$$2VS \cdot VT = VR \cdot (2VM) = VR \cdot (VS + VT)$$

or

$$VS \cdot VT - VS \cdot VR = VT \cdot VR - VT \cdot VS$$

or equivalently

$$VS \cdot RT = VT \cdot SR \quad \text{or} \quad \frac{VS}{SR} = \frac{VT}{RT}. \tag{3}$$

We claim that XS bisects $\angle VXR$. Indeed, because AB is the symmetry line of the kite $APBQ$, $AB \perp PQ$, and so $\angle VXS = \angle QXA = 90° - \angle XAO = 90° - \angle SAO$. Because O is the circumcenter of triangle AST,

$$\angle VXS = 90° - \angle SAO = \angle ATS.$$

On the other hand, because $\angle AXT$ and $\angle ART$ are both right angles, quadrilateral $AXRT$ is cyclic, implying that $\angle SXR = \angle ATR = \angle ATS$. Our claim follows from the last two equations.

Combining our claim and the fact that $XS \perp XT$, we know that XS and XT are the interior and exterior bisectors of $\angle VXR$, from which (3) follows, by the angle-bisector theorem. We saw that (3) was equivalent to (2) and that this was enough to show that $PQMR$ is cyclic, which completes the solution, so we are done.

JMO 4. Find all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x) + f(t) = f(y) + f(z)$$

for all rational numbers $x < y < z < t$ that form an arithmetic progression. ($\mathbb{Q}$ is the set of all rational numbers.)

Solution: Choose any $n \in \mathbb{Z}$, $t \in \mathbb{Q}$. Applying the condition for $nt, (n{+}1)t, (n{+}2)t, (n{+}3)t$ yields

$$f((n+3)t) - f((n+2)t) = f((n+1)t) - f(nt)$$

and similarly

$$f((n+4)t) - f((n+3)t) = f((n+2)t) - f((n+1)t).$$

Adding the two yields

$$f((n+4)t) - f((n+2)t) = f((n+2)t) - f(nt),$$

in particular $f(2kt + 2t) - f(2kt)$ is the same for all $k \in \mathbb{Z}$, which means f is linear on $2t \cdot \mathbb{Z}$. Since $\mathbb{Q}$ is a nested union of such sets, f is linear and all linear functions work.

JMO 5. Let $ABCD$ be a cyclic quadrilateral. Prove that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if there exists a point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$ and $\angle CDB = \angle YDA$.

Solution. By the symmetry, it suffices to show the "only if" part by assuming that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$.

Because $ABCD$ is cyclic, we have $\angle XAD = \angle BAC = \angle BDC = \angle XDC$ and $\angle XDA = \angle BDA = \angle BCA = \angle XCD$. Hence triangles AXD and DXC (and ABC) are similar to each other. In particular,

$$\frac{AX}{DX} = \frac{DX}{XC} \quad \text{or} \quad DX^2 = AX \cdot CX$$

Because $\angle BAC = \angle XAD$, we have $\angle BAX = \angle CAD$. Because $ABCD$ is cyclic, we have $\angle CAD = \angle CBD = \angle CBX$. Consequently, $\angle BAX = \angle CBX$. Note that

$$\angle AXB = \angle XAD + \angle ADX = \angle BAC + \angle ACB = \angle BDC + \angle DCX = \angle CXB.$$

From the above facts, we conclude that triangles ABX and BCX (and ACD) are similar to each other and so we have $BX^2 = AX \cdot CX$. Thus, $BX^2 = AX \cdot CX = DX^2$; that is, X is the midpoint of the segment $\overline{BD}$. Therefore

$$\frac{AB}{BC} = \frac{DX}{XC} = \frac{BX}{XC} = \frac{AD}{DC} \quad \text{or} \quad \frac{BC}{CD} = \frac{BA}{AD}.$$

Construct point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$. From $\angle CBD = \angle YBA$ and $\angle BAY = \angle BAC = \angle BDC$, we conclude that triangles BAY and BDC are similar to each other, from which it follow that

$$\frac{BY}{YA} = \frac{BC}{CD} = \frac{BA}{AD} \quad \text{or} \quad \frac{BY}{BA} = \frac{AY}{AD}.$$

Note also that $\angle YBA = \angle CBD = \angle CAD = \angle YAD$. We conclude that triangles BYA and AYD are similar to each other, implying that $\angle CDB = \angle YAB = \angle YDA$. This is the desired point Y.

OR

By symmetry, it suffices to show that there exists X on the segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if $AB \cdot CD = AD \cdot BC$.

There is a unique point X_1 on segment $\overline{BD}$ such that $\angle X_1AD = \angle BAC$. There is a unique point X_2 on segment $\overline{BD}$ such that $\angle BCA = \angle X_2CD$. Because $ABCD$ is cyclic, $\angle BCA = \angle BDA = \angle X_1DA$. Hence triangles ABC and AX_1D are similar to each other, implying that

$$\frac{AC}{BC} = \frac{AD}{X_1D}.$$

Likewise, we can show that ABC and DX_2C are similar to each other and $\dfrac{AB}{AC} = \dfrac{DX_2}{DC}$. Multiplying the last two equations together gives

$$\frac{AB}{BC} = \frac{AB}{AC} \cdot \frac{AC}{BC} = \frac{DX_2}{DC} \cdot \frac{AD}{X_1D},$$

from which it follows that

$$\frac{AB \cdot CD}{AD \cdot BC} = \frac{DX_2}{DX_1}.$$

Note that point X exists if and only if $X_1 = X_2$, or $DX_2 = DX_1$; that is, $AB \cdot CD = AD \cdot BC$.

JMO 6. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he is finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i,k), (i,l), (j,k), (j,l) for some $1 \leq i,j,k,l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

Solution: We think of the pilings as assigning a positive integer to each square on the grid. Now, we restrict ourselves to the types of moves in which we take a lower left and upper right stone and move them to the upper left and lower right of our chosen rectangle. Call this a Type 1 stone move. We claim that we can perform a sequence of Type 1 stone moves on any piling to obtain an equivalent piling for which we cannot perform any Type 1 move, i.e. in which no square that has stones is above and to the right of any other square that has stones. We call such a piling a "down-right" piling.

To prove that any piling is equivalent to a down-right piling, first consider the squares in the leftmost column and topmost row of the grid. Let a be the entry (number of stones) in the upper left corner, and let b and c be the sum of the remaining entries in the leftmost column and topmost row respectively. If $b < c$, we can perform a sequence of Type 1 stone moves to remove all the stones from the leftmost column except for the top entry, and if $c < b$ we can similarly clear all squares in the top row except for the top left square. In the former case, we can now ignore the leftmost column and repeat the process on the second-to-leftmost column and the top row; similarly, in the latter case, we can ignore the top row and proceed as before. Since the corner square a cannot be part of any Type 1 move at each step in the process, it follows that we end up with a down-right piling.

We next show that down-right pilings in any size grid (not necessarily $n \times n$) are uniquely determined by their row-sums and column-sums, given that the row sums and column sums are nonnegative integers which sum to m both along the rows and the columns. Let the topmost row sum be R_1 and the leftmost column sum be C_1. Then the upper left square must contain $\min(R_1, C_1)$ stones, since otherwise there would be stones both in the first row and first column that are not in the upper left square. Whichever is smaller indicates that either the row or the column respectively is empty save for the upper left square; then we can remove this row or column and are reduced to a smaller grid in which we know all the row and column sums. Since one-row and one-column pilings are clearly uniquely determined by their column and row sums, it follows by induction that down-right pilings are determined uniquely by their row-sums and column sums.

Finally, notice that row sums and column sums are both invariant under stone moves. Therefore every piling is equivalent to a *unique* down-right piling. It therefore suffices to count the number of down-right pilings, which is also equivalent to counting the number of possibilities for the row-sums and column-sums. As stated above, the row sums and column sums can be the sums of any two n-tuples of nonnegative integers that each sum to m. The number of such tuples is $\binom{n+m-1}{m}$, and so the total number of non-equivalent pilings is the number of pairs of these tuples, i.e. $\left(\binom{n+m-1}{m}\right)^2$.

JMO 2015 Solution Notes

EVAN CHEN《陳誼廷》

29 June 2023

This is a compilation of solutions for the 2015 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Given a sequence of real numbers, a move consists of choosing two terms and replacing each with their arithmetic mean. Show that there exists a sequence of 2015 distinct real numbers such that after one initial move is applied to the sequence — no matter what move — there is always a way to continue with a finite sequence of moves so as to obtain in the end a constant sequence.

2. Solve in integers the equation
$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

3. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$.

   As X varies on segment $\overline{PQ}$, show that M moves along a circle.

4. Find all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that
$$f(x) + f(t) = f(y) + f(z)$$
for all rational numbers $x < y < z < t$ that form an arithmetic progression.

5. Let $ABCD$ be a cyclic quadrilateral. Prove that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if there exists a point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$ and $\angle CDB = \angle YDA$.

6. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i,k), (i,l), (j,k), (j,l) for some $1 \leq i,j,k,l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

   Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

§1 Solutions to Day 1

§1.1 JMO 2015/1, proposed by Razvan Gelca

Available online at https://aops.com/community/p4769963.

Problem statement

Given a sequence of real numbers, a move consists of choosing two terms and replacing each with their arithmetic mean. Show that there exists a sequence of 2015 distinct real numbers such that after one initial move is applied to the sequence — no matter what move — there is always a way to continue with a finite sequence of moves so as to obtain in the end a constant sequence.

One valid example of a sequence is 0, 1, ..., 2014. We will show how to achieve the all-1007 sequence based on the first move.

Say two numbers are *opposites* if their average is 1007. We consider 1007 as its own opposite.

We consider two cases:

- First, suppose the first initial move did *not* involve the number 1007. Suppose the two numbers changed were a and b, replaced by $c = \frac{1}{2}(a+b)$ twice.
  - If a and b are opposites, we simply operate on all the other pairs of opposites.
  - Otherwise let a' and b' be the opposites of a and b, so all four of a, b, a', b' are distinct. Then operate on a' and b' to get $c' = 2014 - c$. We work with only these four numbers ande replace them as follows:

$$\begin{array}{cccc} \frac{1}{2}(a+b) & \frac{1}{2}(a+b) & a' & b' \\ \frac{1}{2}(a+b) & \frac{1}{2}(a+b) & \frac{1}{2}(a'+b') & \frac{1}{2}(a'+b') \\ 1007 & \frac{1}{2}(a+b) & 1007 & \frac{1}{2}(a'+b') \\ 1007 & 1007 & 1007 & 1007 \end{array}$$

  Finally, we operate on the remaining 1005 pairs of opposites.

- Now suppose the first initial move involved the number 1007 and some a. Let k be any number other than a or its opposite, and let a', k' be the opposites of a and k. We work with only these five numbers: and replace them in the following way:

$$\begin{array}{ccccc} \frac{1}{2}(a+1007) & \frac{1}{2}(a+1007) & a' & k & k' \\ \frac{1}{2}(a+1007) & \frac{1}{2}(a+1007) & a' & 1007 & 1007 \\ \frac{1}{2}(a+1007) & \frac{1}{2}(a+1007) & \frac{1}{2}(a'+1007) & \frac{1}{2}(a'+1007) & 1007 \\ 1007 & \frac{1}{2}(a+1007) & 1007 & \frac{1}{2}(a'+1007) & 1007 \\ 1007 & 1007 & 1007 & 1007 & 1007 \end{array}$$

  Finally, we operate on the remaining 1005 pairs of opposites.

Remark. In fact, the same proof basically works for any sequence with average m such that m is in the sequence, and every term has an opposite.

However for "most" sequences one expects the result to not be possible. As a simple example, the goal is impossible for $(0, 1, \dots, 2013, 2015)$ since the average of the terms is

$1007 + \frac{1}{2015}$, but in the process the only denominators ever generated are powers of 2. This narrows the search somewhat.

§1.2 JMO 2015/2, proposed by Titu Andreescu

Available online at https://aops.com/community/p4769940.

Problem statement

Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

We do the trick of setting $a = x + y$ and $b = x - y$. This rewrites the equation as

$$\frac{1}{4}\left((a+b)^2 + (a+b)(a-b) + (a-b)^2\right) = \left(\frac{a}{3} + 1\right)^3$$

where $a, b \in \mathbb{Z}$ have the same parity. This becomes

$$3a^2 + b^2 = 4\left(\frac{a}{3} + 1\right)^3$$

which is enough to imply $3 \mid a$, so let $a = 3c$. Miraculously, this becomes

$$b^2 = (c-2)^2(4c+1).$$

So a solution must have $4c + 1 = m^2$, with m odd. This gives

$$x = \frac{1}{8}\left(3(m^2-1) \pm (m^3 - 9m)\right) \quad \text{and} \quad y = \frac{1}{8}\left(3(m^2-1) \mp (m^3 - 9m)\right).$$

For mod 8 reasons, this always generates a valid integer solution, so this is the complete curve of solutions. Actually, putting $m = 2n + 1$ gives the much nicer curve

$$\boxed{x = n^3 + 3n^2 - 1 \quad \text{and} \quad y = -n^3 + 3n + 1}$$

and permutations.

For $n = 0, 1, 2, 3$ this gives the first few solutions are $(-1, 1)$, $(3, 3)$, $(19, -1)$, $(53, -17)$, (and permutations).

§1.3 JMO 2015/3, proposed by Zuming Feng, Jacek Fabrykowski

Available online at https://aops.com/community/p4769957.

Problem statement

Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$.

As X varies on segment $\overline{PQ}$, show that M moves along a circle.

We present three solutions, one by complex numbers, two more synthetic. (A fourth solution using median formulas is also possible.) Most solutions will prove that the center of the fixed circle is the midpoint of $\overline{AO}$ (with O the center of ω); this can be recovered empirically by letting

- X approach P (giving the midpoint of $\overline{BP}$)
- X approach Q (giving the point Q), and
- X at the midpoint of $\overline{PQ}$ (giving the midpoint of $\overline{BQ}$)

which determines the circle; this circle then passes through P by symmetry and we can find the center by taking the intersection of two perpendicular bisectors (which two?).

¶ Complex solution (Evan Chen) Toss on the complex unit circle with $a = -1$, $b = 1$, $z = -\frac{1}{2}$. Let s and t be on the unit circle. We claim Z is the center.

It follows from standard formulas that

$$x = \frac{1}{2}\left(s + t - 1 + s/t\right)$$

thus

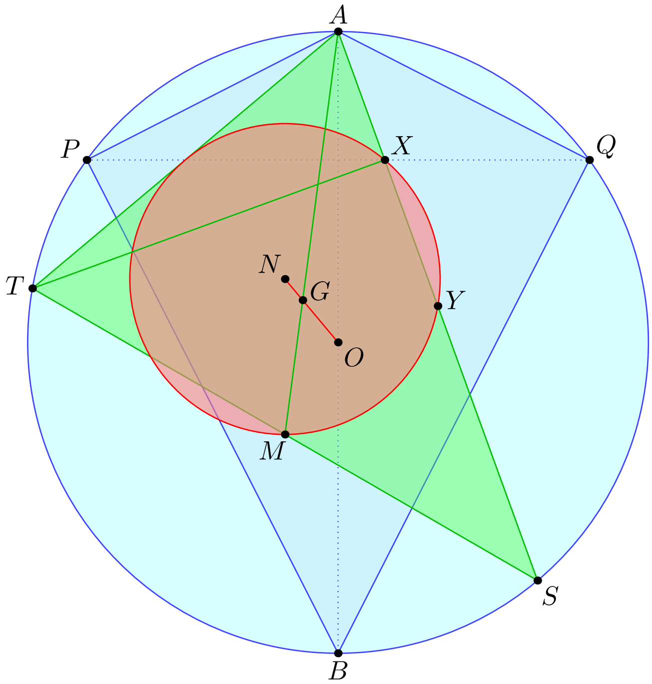
$$4 \operatorname{Re} x + 2 = s + t + \frac{1}{s} + \frac{1}{t} + \frac{s}{t} + \frac{t}{s}$$

which depends only on P and Q, and not on X. Thus

$$4\left|z - \frac{s+t}{2}\right|^2 = |s+t+1|^2 = 3 + (4 \operatorname{Re} x + 2)$$

does not depend on X, done.

¶ Homothety solution (Alex Whatley) Let G, N, O denote the centroid, nine-point center, and circumcenter of triangle AST, respectively. Let Y denote the midpoint of $\overline{AS}$. Then the three points X, Y, M lie on the nine-point circle of triangle AST, which is centered at N and has radius $\frac{1}{2}AO$.



Let R denote the radius of ω. Note that the nine-point circle of $\triangle AST$ has radius equal to $\frac{1}{2}R$, and hence is independent of S and T. Then the power of A with respect to the nine-point circle equals

$$AN^2 - \left(\frac{1}{2}R\right)^2 = AX \cdot AY = \frac{1}{2}AX \cdot AS = \frac{1}{2}AQ^2$$

and hence

$$AN^2 = \left(\frac{1}{2}R\right)^2 + \frac{1}{2}AQ^2$$

which does not depend on the choice of X. So N moves along a circle centered at A.

Since the points O, G, N are collinear on the Euler line of $\triangle AST$ with

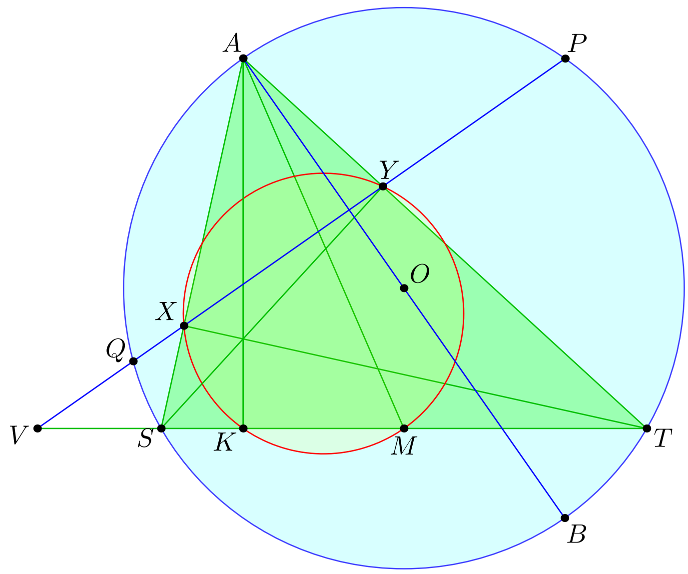
$$GO = \frac{2}{3}NO$$

it follows by homothety that G moves along a circle as well, whose center is situated one-third of the way from A to O. Finally, since A, G, M are collinear with

$$AM = \frac{3}{2}AG$$

it follows that M moves along a circle centered at the midpoint of $\overline{AO}$.

¶ Power of a point solution (Zuming Feng, official solution) We complete the picture by letting $\triangle KYX$ be the orthic triangle of $\triangle AST$; in that case line XY meets the ω again at P and Q.



The main claim is:

Claim — Quadrilateral $PQKM$ is cyclic.

Proof. To see this, we use power of a point: let $V = \overline{QXYP} \cap \overline{SKMT}$. One approach is that since $(VK;ST) = -1$ we have $VQ \cdot VP = VS \cdot VT = VK \cdot VM$. A longer approach is more elementary:

$$VQ \cdot VP = VS \cdot VT = VX \cdot VY = VK \cdot VM$$

using the nine-point circle, and the circle with diameter $\overline{ST}$. $\square$

But the circumcenter of $PQKM$, is the midpoint of $\overline{AO}$, since it lies on the perpendicular bisectors of $\overline{KM}$ and $\overline{PQ}$. So it is fixed, the end.

§2 Solutions to Day 2

§2.1 JMO 2015/4, proposed by Iurie Boreico

Available online at https://aops.com/community/p4774049.

Problem statement

Find all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x) + f(t) = f(y) + f(z)$$

for all rational numbers $x < y < z < t$ that form an arithmetic progression.

Answer: any linear function f. These work.

Here is one approach: for any a and $d > 0$

$$\begin{aligned} f(a) + f(a+3d) &= f(a+d) + f(a+2d) \\ f(a-d) + f(a+2d) &= f(a) + f(a+d) \end{aligned}$$

which imply

$$f(a-d) + f(a+3d) = 2f(a+d).$$

Thus we conclude that for arbitrary x and y we have

$$f(x) + f(y) = 2f\left(\frac{x+y}{2}\right)$$

thus f satisfies Jensen functional equation over $\mathbb{Q}$, so linear.

The solution can be made to avoid appealing to Jensen's functional equation; here is a presentation of such a solution based on the official ones. Let $d > 0$ be a positive integer, and let n be an integer. Consider the two equations

$$\begin{aligned} f\left(\frac{2n-1}{2d}\right) + f\left(\frac{2n+2}{2d}\right) &= f\left(\frac{2n}{2d}\right) + f\left(\frac{2n+1}{2d}\right) \\ f\left(\frac{2n-2}{2d}\right) + f\left(\frac{2n+1}{2d}\right) &= f\left(\frac{2n-1}{2d}\right) + f\left(\frac{2n}{2d}\right) \end{aligned}$$

Summing them and simplifying implies that

$$f\left(\frac{n-1}{d}\right) + f\left(\frac{n+1}{d}\right) = 2f\left(\frac{n}{d}\right)$$

or equivalently $f\left(\frac{n}{d}\right) - f\left(\frac{n-1}{d}\right) = f\left(\frac{n+1}{d}\right) - f\left(\frac{n}{d}\right)$. This implies that on the set of rational numbers with denominator dividing d, the function f is linear.

In particular, we should have $f\left(\frac{n}{d}\right) = f(0) + \frac{n}{d} f(1)$ since $\frac{n}{d}$, 0, 1 have denominators dividing d. This is the same as saying $f(q) = f(0) + q(f(1) - f(0))$ for any $q \in \mathbb{Q}$, which is what we wanted to prove.

§2.2 JMO 2015/5, proposed by Sungyoon Kim

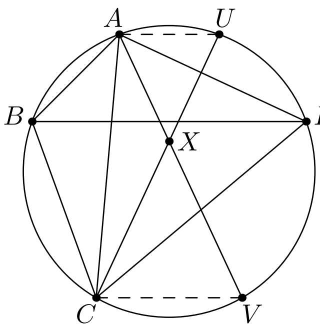
Available online at https://aops.com/community/p4774099.

Problem statement

Let $ABCD$ be a cyclic quadrilateral. Prove that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if there exists a point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$ and $\angle CDB = \angle YDA$.

Both conditions are equivalent to $ABCD$ being harmonic.

Here is a complex solution. Extend U and V and shown. Thus $u = bd/a$ and $v = bd/c$.



Note $\overline{AV} \cap \overline{CU}$ lies on the perpendicular bisector of $\overline{BD}$ unconditionally. Then X exists as described if and only if the midpoint of $\overline{BD}$ lies on $\overline{AV}$. In complex numbers this is $a + v = m + av\overline{m}$, or

$$a + \frac{bd}{c} = \frac{b+d}{2} + \frac{abd}{c} \cdot \frac{b+d}{2bd} \iff 2(ac+bd) = (b+d)(a+c)$$

which is symmetric.

§2.3 JMO 2015/6, proposed by Maria Monks Gillespie

Available online at https://aops.com/community/p4774079.

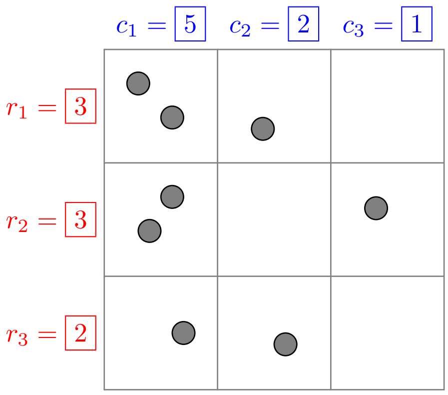
Problem statement

Steve is piling $m \ge 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i,k), (i,l), (j,k), (j,l) for some $1 \le i,j,k,l \le n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

The answer is $\binom{m+n-1}{n-1}^2$. The main observation is that the ordered sequence of column counts (i.e. the number of stones in the first, second, etc. column) is invariant under stone moves, as does the analogous sequence of row counts.

¶ Definitions Call these numbers $(c_1, c_2, \dots, c_m)$ and $(r_1, r_2, \dots, r_m)$ respectively, with $\sum c_i = \sum r_i = n$. We say that the sequence $(c_1, \dots, c_m, r_1, \dots, r_m)$ is the *signature* of the configuration. These are the $2m$ blue and red numbers shown in the example below (in this example we have $m = 8$ and $n = 3$).



Signature: $(5,2,1;3,3,2)$

By stars-and-bars, the number of possible values $(c_1, \dots, c_m)$ is $\binom{m+n-1}{n-1}$. The same is true for $(r_1, \dots, r_m)$. So if we're just counting *signatures*, the total number of possible signatures is $\binom{m+n-1}{n-1}^2$.

¶ Outline and setup We are far from done. To show that the number of non-equivalent ways is also this number, we need to show that signatures correspond to pilings. In other words, we need to prove:

1. Check that signatures are invariant around moves (trivial; we did this already);
2. Check conversely that two configurations are equivalent if they have the same signatures (the hard part of the problem); and

3. Show that each signature is realized by at least one configuration (not immediate, but pretty easy).

Most procedures to the second step are algorithmic in nature, but Ankan Bhattacharya gives the following far cleaner approach. Rather than having a grid of stones, we simply consider the multiset of ordered pairs (x, y) corresponding to the stones. Then:

- a stone move corresponds to switching two y-coordinates in two different pairs.
- we *redefine* the signature to be the multiset (X, Y) of x and y coordinates which appear. Explicitly, X is the multiset that contains c_i copies of the number i for each i.

For example, consider the earlier example which had

- Two stones each at $(1,1)$, $(1,2)$.
- One stone each at $(3,1)$, $(2,1)$, $(2,3)$, $(3,2)$.

Its signature can then be reinterpreted as

$$(5,2,1;3,3,2) \longleftrightarrow \begin{cases} X = \{1,1,1,1,1,2,2,3\} \\ Y = \{1,1,1,2,2,2,3,3\}. \end{cases}$$

In that sense, the entire grid is quite misleading!

¶ Proof that two configurations with the same signature are equivalent The second part is completed just because transpositions generate any permutation. To be explicit, given two sets of stones, we can permute the labels so that the first set is (x_1, y_1), …, (x_m, y_m) and the second set of stones is (x_1, y_1'), …, (x_m, y_m'). Then we just induce the correct permutation on (y_i) to get (y_i').

¶ Proof that any signature has at least one configuration Sort the elements of X and Y arbitrarily (say, in non-decreasing order). Put a stone whose x-coordinate is the ith element of X, and whose y-coordinate is the ith element of Y, for each $i = 1, 2, \dots, m$. Then this gives a stone placement of m stones with signature (X, Y).

For example, if

$$X = \{1,1,1,1,1,2,2,3\}$$
$$Y = \{1,1,1,2,2,2,3,3\}$$

then placing stones at $(1,1)$, $(1,1)$, $(1,1)$, $(1,2)$, $(1,2)$, $(2,2)$, $(2,3)$, $(3,3)$ gives a valid piling with this signature.

45th United States of America Junior Mathematical Olympiad

Day I 12:30PM — 5PM EDT

April 19, 2016

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 1. The isosceles triangle $\triangle ABC$, with $AB = AC$, is inscribed in the circle ω. Let P be a variable point on the arc $\overset{\frown}{BC}$ that does not contain A, and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively.

Prove that as P varies, the circumcircle of triangle $\triangle PI_BI_C$ passes through a fixed point.

USAJMO 2. Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation.

USAJMO 3. Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct non-empty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

45th United States of America Junior Mathematical Olympiad

Day II 12:30PM — 5PM EDT

April 20, 2016

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 4. Find, with proof, the least integer N such that if any 2016 elements are removed from the set $\{1, 2, \ldots, N\}$, one can still find 2016 distinct numbers among the remaining elements with sum N.

USAJMO 5. Let $\triangle ABC$ be an acute triangle, with O as its circumcenter. Point H is the foot of the perpendicular from A to line $\overleftrightarrow{BC}$, and points P and Q are the feet of the perpendiculars from H to the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, respectively.

Given that

$$AH^2 = 2 \cdot AO^2,$$

prove that the points O, P, and Q are collinear.

USAJMO 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2 .$$

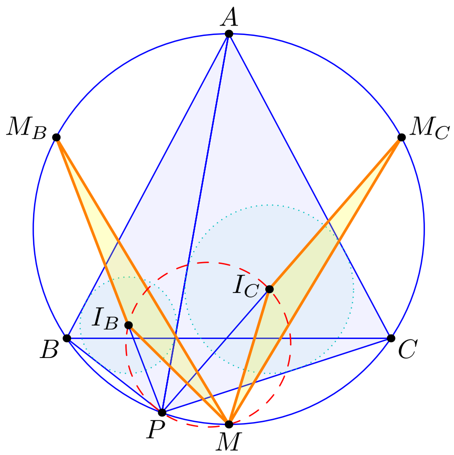
Solutions to USA(J)MO 2016

Evan Chen

57th IMO 2016, Hong Kong

§1 Solution to JMO1

Let M be the midpoint of arc BC not containing A. We claim M is the desired fixed point.



Since $\angle MPA = 90^\circ$ and ray PA bisects $\angle I_B P I_C$, it suffices to show that $MI_B = MI_C$. Let M_B, M_C be the second intersections of PI_B and PI_C with circumcircle. Now $M_B I_B = M_B B = M_C C = M_C I_C$, and moreover $MM_B = MM_C$, and $\angle I_B M_B M = \frac{1}{2}\widehat{PB} = \angle I_C M_C M$, so triangles $\triangle I_B M_B M \cong \triangle I_C M_C M$, done.

§2 Solution to JMO2

One answer is $n = 20 + 2^{19} = 524308$.

First, observe that

$$\begin{aligned} 5^n &\equiv 5^{20} \pmod{5^{20}} \\ 5^n &\equiv 5^{20} \pmod{2^{20}} \end{aligned}$$

the former being immediate and the latter since $\varphi(2^{20}) = 2^{19}$. Hence $5^n \equiv 5^{20} \pmod{10^{20}}$. Moreover, we have

$$5^{20} = \frac{1}{2^{20}} \cdot 10^{20} < \frac{1}{1000^2} \cdot 10^{20} = 10^{-6} \cdot 10^{20}.$$

Thus the last 20 digits of 5^n will begin with six zeros. This completes the proof.

§3 Solution to JMO3 / USAMO1

The answer is that $|S| \geq 8$.

First, we provide a inductive construction for $S = \{1, \dots, 8\}$. Actually, for $n \geq 4$ we will provide a construction for $S = \{1, \dots, n\}$ which has $2^{n-1} + 1$ elements in a line. (This is sufficient, since we then get 129 for $n = 8$.) The idea is to start with the following construction for $|S| = 4$:

$$34 \quad 1 \quad 23 \quad 4 \quad 12 \quad 3 \quad 14 \quad 2 \quad 13\ .$$

Then inductively, we do the following procedure to move from n to $n+1$: take the chain for n elements, delete an element, and make two copies of the chain (which now has even length). Glue the two copies together, joined by $\varnothing$ in between. Then place the element $n+1$ in alternating positions starting with the first (in particular, this hits $n+1$).

Explicitly, when $n = 8$ this construction gives

345678	1	235678	4	125678	3	145678	2	5678
34	15678	23	45678	12	35678	14	678	
345	1678	235	4678	125	3678	145	2678	5
34678	15	23678	45	12678	35	78		
3456	178	2356	478	1256	378	1456	278	56
3478	156	2378	456	1278	356	1478	6	
34578	16	23578	46	12578	36	14578	26	578
346	1578	236	4578	126	8			
34567	18	23567	48	12567	38	14567	28	567
348	1567	238	4567	128	3567	148	67	
3458	167	2358	467	1258	367	1458	267	58
3467	158	2367	458	1267	358	7		
34568	17	23568	47	12568	37	14568	27	568
347	1568	237	4568	127	3568	147	68	
3457	168	2357	468	1257	368	1457	268	57
3468	157	2368	457	1268				

Now let's check $|S| \geq 8$ is sufficient. Consider a chain on a set of size $|S| = 7$. (We need $|S| \geq 7$ else $2^{|S|} < 100$.) Observe that there are sets of size ≥ 4 can only be neighbored by sets of size ≤ 2, of which there are $\binom{7}{1} + \binom{7}{2} = 28$. So there are ≤ 30 sets of size ≥ 4. Also, there are $\binom{7}{3} = 35$ sets of size 3. So the total number of sets in a chain can be at most $30 + 28 + 35 = 93 < 100$.

§4 Solution to USAMO2

We show the exponent of any given prime p is nonnegative in the expression. Recall that the exponent of p in $n!$ is equal to $\sum_{i \geq 1} \lfloor n/p^i \rfloor$. In light of this, it suffices to show that for any prime power P, we have

$$\left\lfloor \frac{k^2}{P} \right\rfloor \geq \sum_{j=0}^{k-1} \left(\left\lfloor \frac{j+k}{P} \right\rfloor - \left\lfloor \frac{j}{P} \right\rfloor \right).$$

Since both sides are integers, we it is equivalent to show:

$$\left\lfloor \frac{k^2}{P} \right\rfloor > -1 + \sum_{j=0}^{k-1} \left(\left\lfloor \frac{j+k}{P} \right\rfloor - \left\lfloor \frac{j}{P} \right\rfloor \right).$$

Suppose we denote by $\{x\}$ the fractional part of x. Since $\lfloor x \rfloor = x - \{x\}$, it suffices to prove that

$$\left\{\frac{k^2}{P}\right\} + \sum_{j=0}^{k-1} \left\{\frac{j}{P}\right\} < 1 + \sum_{j=0}^{k-1} \left\{\frac{j+k}{P}\right\}$$

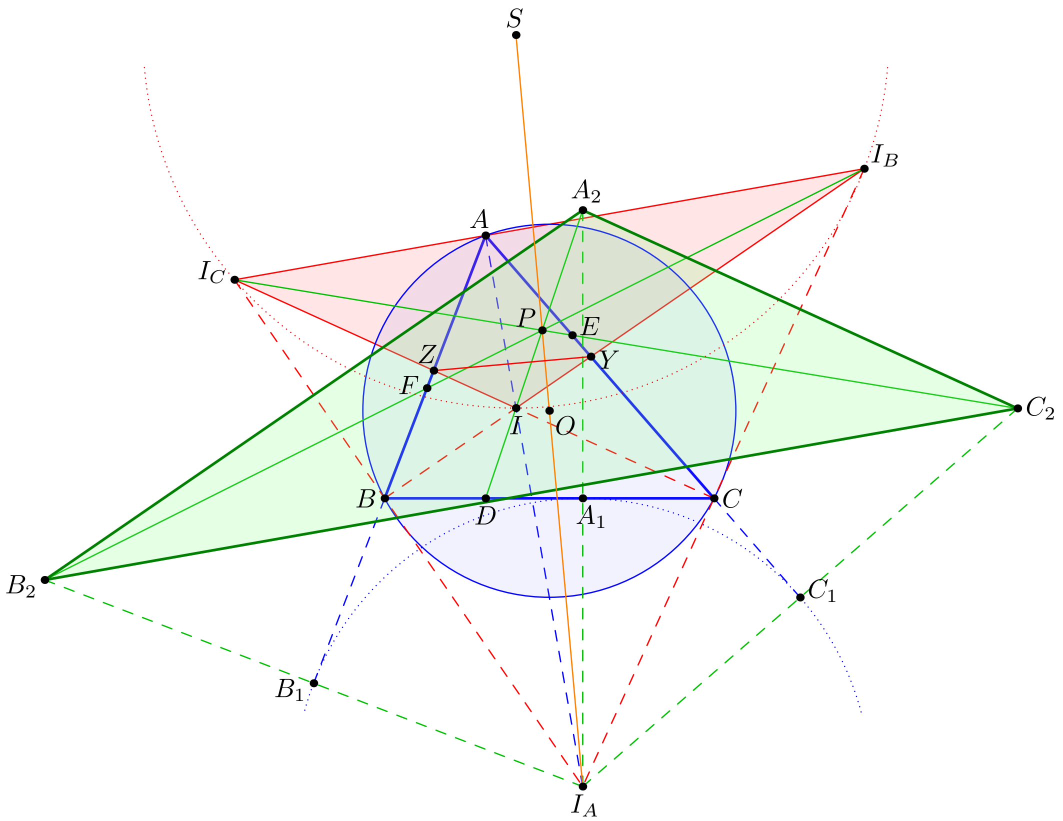
However, the sum of remainders when $(0, 1, \dots, k-1)$ is taken modulo P is easily seen to be less than the sum of remainders when $(k, k+1, \dots, 2k-1)$ is taken modulo P. So

$$\sum_{j=0}^{k-1} \left\{\frac{j}{P}\right\} \le \sum_{j=0}^{k-1} \left\{\frac{j+k}{P}\right\}$$

follows, and we are done upon noting $\left\{k^2/P\right\} < 1$.

§5 Solution to USAMO3

Let I_A denote the A-excenter and I the incenter. Then let D denote the foot of the altitude from A. Suppose the A-excircle is tangent to $\overline{BC}$, $\overline{AB}$, $\overline{AC}$ at A_1, B_1, C_1 and let A_2, B_2, C_2 denote the reflections of I_A across these points. Let S denote the circumcenter of $\triangle II_BI_C$.



We begin with the following observation: points D, I, A_2 are collinear, as are points E, I_C, C_2 are collinear and points F, I_B, B_2 are collinear. This follows from the "midpoints of altitudes" lemma.

Observe that $\overline{B_2C_2} \parallel \overline{B_1C_1} \parallel \overline{I_BI_C}$. Proceeding similarly on the other sides, we discover $\triangle II_BI_C$ and $\triangle A_2B_2C_2$ are homothetic. Hence P is the center of this homothety (in particular, D, I, P, A_2 are collinear). Moreover, P lies on the line joining I_A to S, which is the Euler line of $\triangle II_BI_C$, so it passes through the nine-point center of $\triangle II_BI_C$, which is O. Consequently, P, O, I_A are collinear as well.

To finish, we need only prove that $\overline{OS} \perp \overline{YZ}$. In fact, we claim that $\overline{YZ}$ is the radical axis of the circumcircles of $\triangle ABC$ and $\triangle II_BI_C$. Actually, Y is the radical center of these two circumcircles and the circle with diameter $\overline{II_B}$ (which passes through A and C). Analogously Z is the radical center of the circumcircles and the circle with diameter $\overline{II_C}$, and the proof is complete.

§6 Solution to JMO4

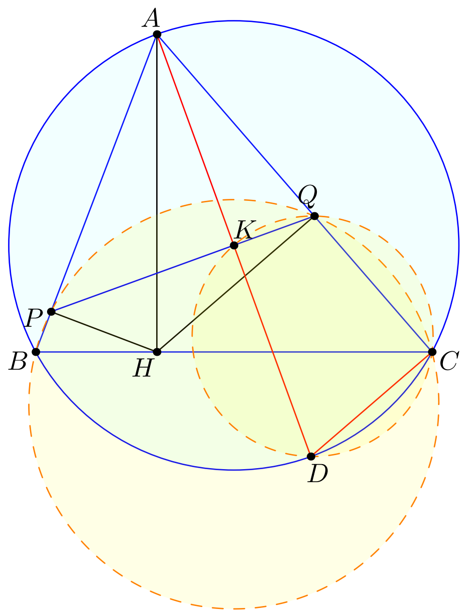
The answer is

$$N = 2017 + 2018 + \cdots + 4032 = 1008 \cdot 6049 = 6097392.$$

To see that N must be at least this large, simply consider the situation when 1, 2, ..., 2016 are removed. Then among the remaining elements, any sum of 2016 elements is certainly at least $2017 + 2018 + \cdots + 6049$.

Now we show this value of N works. Consider the 3024 pairs of numbers $(1, 6048)$, $(2, 6047)$, ..., $(3024, 3025)$. After the elements of $\{1, 2, \ldots, N\}$ are deleted, at least $3024 - 2016 = 1008$ of these pairs have both elements remaining. Since each pair has sum 6049, we can take these pairs to be the desired numbers.

§7 Solution to JMO5



First, since $AP \cdot AB = AH^2 = AQ \cdot AC$, it follows that $PQCB$ is cyclic. Consequently, we have $AO \perp PQ$. Let K be the foot of A onto PQ, and let D be the point diametrically opposite A. Thus A, K, O, D are collinear.

Since quadrilateral $KQCD$ is cyclic ($\angle QKD = \angle QCD = 90^\circ$), we have

$$AK \cdot AD = AQ \cdot QC = AH^2 \implies AK = \frac{AH^2}{AD} = \frac{AH^2}{2AO} = AO$$

so $K = O$.

§8 Solution to JMO6 / USAMO4

First, taking $x = y = 0$ in the given yields $f(0) = 0$, and then taking $x = 0$ gives $f(y)f(-y) = f(y)^2$. So also $f(-y)^2 = f(y)f(-y)$, from which we conclude f is even. Then taking $x = -y$ gives

$$\forall x \in \mathbb{R} : \qquad f(x) = x^2 \qquad \text{or} \qquad f(4x) = 0 \qquad (\star)$$

for all x.

Next, we claim that

$$\forall x \in \mathbb{R} : \qquad f(x) = x^2 \qquad \text{or} \qquad f(x) = 0 \qquad (\heartsuit)$$

To see this assume $f(t) \neq 0$ (hence $t \neq 0$). By $(\star)$ we get $f(t/4) = t^2/16$. Now take $(x, y) = (3t/4, t/4)$ to get

$$\frac{t^2}{4} f(2t) = f(t^2) \implies f(2t) \neq 0.$$

If we apply $(\star)$ again we actually also get $f(t/2) \neq 0$. Together these imply

$$f(t) \neq 0 \iff f(2t) \neq 0 \qquad (\spadesuit).$$

Repeat $(\spadesuit)$ to get $f(4t) \neq 0$, hence $f(t) = t^2$, proving $(\heartsuit)$.

We are now ready to show the claimed solutions are the only ones. Assume there's an $a \neq 0$ for which $f(a) = 0$; we show that $f \equiv 0$. There are two approaches from here, by using inequalities or polynomials.

First approach

Pick $b \in \mathbb{R}$, we show directly $f(b) = 0$.

First, note that $\boxed{f \geq 0}$ always holds by $(\heartsuit)$. By using $(\spadesuit)$ we can generate $c > 100b$ such that $f(c) = 0$ (by taking $c = 2^n a$ for n large). Now, select $x, y > 0$ such that $x - 3y = b$ and $x + y = c$ id est

$$(x, y) = \left(\frac{3c + b}{4}, \frac{c - b}{4} \right).$$

Substitution into the original equation gives

$$0 = (f(x) + xy)\, f(b) + (f(y) + xy)\, f(3x - y).$$

But everything on the right-hand side is nonnegative. Thus it follows that $f(b) = f(3x - y) = 0$ as desired.

Second approach

First, observe that for all $x \in \mathbb{R}$

$$f(4x - a) \neq 0 \implies (f(x) + x(3x - a))\, f(3a - 8x) = f(4x - a)^2 \neq 0$$

by taking $y = 3x - a$ in the original equation. Finally, consider the equations

$$\begin{aligned} 0 &= (4x - a)^4 - (x(3x - a))\,(3a - 8x)^2 \\ 0 &= (4x - a)^4 - \left(x^2 + x(3x - a)\right)(3a - 8x)^2 \end{aligned}$$

Each right-hand side is a nonzero polynomial in x. Thus there are finitely many roots in x, hence there are only finitely many values of x with $f(4x - a) \neq 0$. But $(\spadesuit)$ then implies there cannot be any values of x at all, i.e. we conclude that $f \equiv 0$.

§9 Solution to USAMO5

First solution

In fact, we show that we only need $AM = AQ = NP$ and $MN = QP$.

We use complex numbers with ABC the unit circle, assuming WLOG that A, B, C are labeled counterclockwise. Let x, y, z be the complex numbers corresponding to the arc midpoints of BC, CA, AB, respectively; thus $x + y + z$ is the incenter of $\triangle ABC$. Finally, let $s > 0$ be the side length of $AM = AQ = NP$.

Then, since $MA = s$ and $MA \perp OX$, it follows that

$$m - a = i \cdot sx.$$

Similarly, $n - p = i \cdot sy$ and $a - q = i \cdot sz$, so summing these up gives

$$i \cdot s(x + y + z) = (p - q) + (m - n) = (m - n) - (q - p).$$

Since $MN = PQ$, the argument of $(m - n) - (q - p)$ is along the external angle bisector of the angle formed, which is perpendicular to ℓ. On the other hand, $x + y + z$ is oriented in the same direction as OI, as desired.

Second solution

Let δ and ϵ denote $\angle MNB$ and $\angle CPQ$. Also, assume $AMNPQ$ has side length 1.

In what follows, assume $AB < AC$. First, we note that

$$\begin{aligned} BN &= (c-1)\cos B + \cos\delta \\ CP &= (b-1)\cos C + \cos\epsilon \\ \implies a &= 1 + BN + CP \\ \implies \cos\delta + \cos\epsilon &= \cos B + \cos C - 1. \end{aligned}$$

Also, by Law of Sines, we have $\frac{c-1}{\sin\delta} = \frac{1}{\sin B}$ and similarly on triangle CPQ, and from this we deduce

$$\sin\epsilon - \sin\delta = \sin B - \sin C.$$

Using sum-to-product formulas on our relations implies that

$$\tan\left(\frac{\epsilon - \delta}{2}\right) = \frac{\sin B - \sin C}{\cos B - \cos C + 1}.$$

Now note that ℓ makes an angle of $\frac{1}{2}(\pi + \epsilon - \delta)$ with line BC. Moreover, if line OI intersects line BC with angle φ then

$$\tan\varphi = \frac{r - R\cos A}{\frac{1}{2}(b-c)}.$$

So in order to prove the result, we only need to check that

$$\frac{r - R\cos A}{\frac{1}{2}(b-c)} = \frac{\cos B - \cos C + 1}{\sin B - \sin C}.$$

Using the fact that $b = 2R\sin B$, $c = 2R\sin C$, this just reduces to the fact that $r/R + 1 = \cos A + \cos B + \cos C$, which is the so-called Carnot theorem.

§10 Solution to USAMO6

The game is winnable if and only if $n \neq k$.

First suppose $2 \leq k < n$. Query the cards in positions $\{1, \dots, k\}$, then $\{2, \dots, k+1\}$, and so on, up to $\{2n-k+1, 2n\}$. By taking the diff of any two adjacent queries, we can deduce for certain the values on cards $1, 2, \dots, 2n-k$. If $k \leq n$, this is more than n cards, so we can find a matching pair.

For $k = n$ we remark the following: at each turn after the first, assuming one has not won, there are n cards representing each of the n values exactly once, such that the player has no information about the order of those n cards. We claim that consequently the player cannot guarantee victory. Indeed, let S denote this set of n cards, and $\overline{S}$ the other n cards. The player will never win by picking only cards in S or $\overline{S}$. Also, if the player selects some cards in S and some cards in $\overline{S}$, then it is possible that the choice of cards in S is exactly the complement of those selected from $\overline{S}$; the strategy cannot prevent this since the player has no information on S. This implies the result.

JMO 2016 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2016 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. The isosceles triangle $\triangle ABC$, with $AB = AC$, is inscribed in the circle ω. Let P be a variable point on the arc BC that does not contain A, and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively. Prove that as P varies, the circumcircle of triangle $\triangle PI_BI_C$ passes through a fixed point.

2. Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation.

3. Let $X_1, X_2, \dots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \dots, 99\}$. Find the smallest possible number of elements in S.

4. Find, with proof, the least integer N such that if any 2016 elements are removed from the set $\{1, 2, \dots, N\}$, one can still find 2016 distinct numbers among the remaining elements with sum N.

5. Let $\triangle ABC$ be an acute triangle, with O as its circumcenter. Point H is the foot of the perpendicular from A to line BC, and points P and Q are the feet of the perpendiculars from H to the lines AB and AC, respectively.

   Given that
   $$AH^2 = 2AO^2,$$
   prove that the points O, P, and Q are collinear.

6. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,
   $$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

§1 Solutions to Day 1

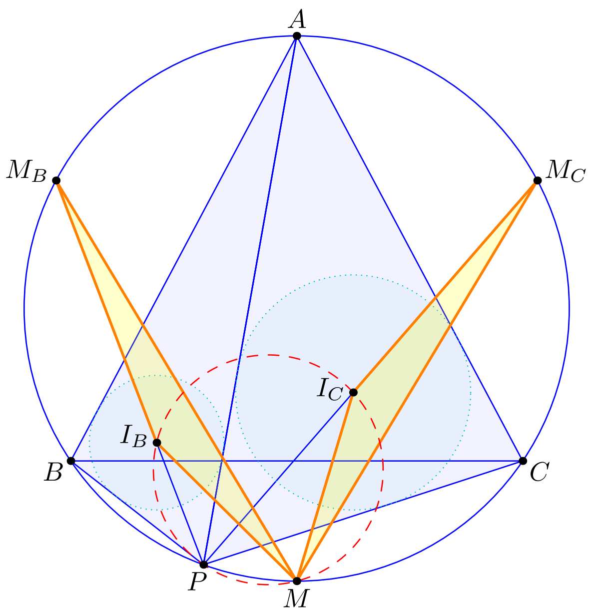
§1.1 JMO 2016/1, proposed by Ivan Borsenco, Zuming Feng

Available online at https://aops.com/community/p6213607.

Problem statement

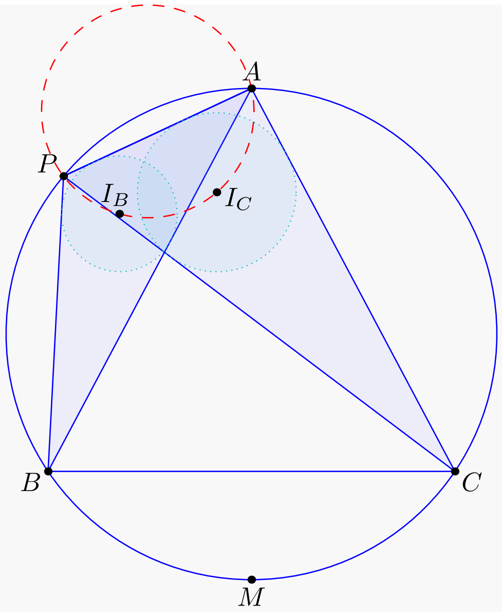
The isosceles triangle $\triangle ABC$, with $AB = AC$, is inscribed in the circle ω. Let P be a variable point on the arc BC that does not contain A, and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively. Prove that as P varies, the circumcircle of triangle $\triangle PI_BI_C$ passes through a fixed point.

Let M be the midpoint of arc BC not containing A. We claim M is the desired fixed point.



Since $\angle MPA = 90^\circ$ and ray PA bisects $\angle I_BPI_C$, it suffices to show that $MI_B = MI_C$. Let M_B, M_C be the second intersections of PI_B and PI_C with circumcircle. Now $M_BI_B = M_BB = M_CC = M_CI_C$, and moreover $MM_B = MM_C$, and $\angle I_BM_BM = \frac{1}{2}\widehat{PM} = \angle I_CM_CM$, so triangles $\triangle I_BM_BM \cong \triangle I_CM_CM$.

Remark 1.1. Complex in the obvious way DOES NOT WORK, because the usual claim ("the fixed point is arc midpoint") is FALSE if the hypothesis that P lies in the interior of the arc is dropped. See figure below.



Fun story, I pointed this out to Zuming during grading; I was the only one that realized the subtlety.

§1.2 JMO 2016/2, proposed by Evan Chen

Available online at https://aops.com/community/p6213569.

Problem statement

Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation.

We will prove that $\boxed{n = 20 + 2^{19} = 524308}$ fits the bill.

First, we claim that

$$5^n \equiv 5^{20} \pmod{5^{20}} \qquad \text{and} \qquad 5^n \equiv 5^{20} \pmod{2^{20}}.$$

Indeed, the first equality holds since both sides are $0 \pmod{5^{20}}$, and the second by $\varphi(2^{20}) = 2^{19}$ and Euler's theorem. Hence

$$5^n \equiv 5^{20} \pmod{10^{20}}.$$

In other words, the last 20 digits of 5^n will match the decimal representation of 5^{20}, with leading zeros. However, we have

$$5^{20} = \frac{1}{2^{20}} \cdot 10^{20} < \frac{1}{1000^2} \cdot 10^{20} = 10^{-6} \cdot 10^{20}$$

and hence those first six of those 20 digits will all be zero. This completes the proof! (To be concrete, it turns out that $5^{20} = 95367431640625$ and so the last 20 digits of 5^n will be 00000095367431640625.)

> **Remark.** Many of the first posts in the JMO 2016 discussion thread (see https://aops.com/community/c5h1230514) claimed that the problem was "super easy". In fact, the problem was solved by only about 10% of contestants.

¶ Authorship comments This problem was inspired by the observation $5^8 \equiv 5^4 \pmod{10^4}$, i.e. that 5^8 ended with 0625.

I noticed this one day back in November, when I was lying on my bed after a long afternoon and was mindlessly computing powers of 5 in my head because I was too tired to do much else. When I reached 5^8 I noticed for the first time that the ending 0625 was actually induced by 5^4. (Given how much MathCounts I did, I really should have known this earlier!)

Thinking about this for a few more seconds, I realized one could obtain arbitrarily long strings of 0's by using a similar trick modulo larger powers of 10. This surprised me, because I would have thought that if this was true, then I would have learned about it back in my contest days. However, I could not find any references, and I thought the result was quite nice, so I submitted it as a proposal for the JMO, where I thought it might be appreciated.

The joke about six consecutive zeros is due to Zuming Feng.

§1.3 JMO 2016/3, proposed by Iurie Boreico

Available online at https://aops.com/community/p6213589.

Problem statement

Let $X_1, X_2, \dots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \dots, 99\}$. Find the smallest possible number of elements in S.

Solution with Danielle Wang: the answer is that $|S| \geq 8$.

Proof of sufficiency Since we must have $2^{|S|} \geq 100$, we must have $|S| \geq 7$.

To see that $|S| = 8$ is the minimum possible size, consider a chain on the set $S = \{1, 2, \dots, 7\}$ satisfying $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$. Because of these requirements any subset of size 4 or more can only be neighbored by sets of size 2 or less, of which there are $\binom{7}{1} + \binom{7}{2} = 28$ available. Thus, the chain can contain no more than 29 sets of size 4 or more and no more than 28 sets of size 2 or less. Finally, since there are only $\binom{7}{3} = 35$ sets of size 3 available, the total number of sets in such a chain can be at most $29 + 28 + 35 = 92 < 100$.

Construction We will provide an inductive construction for a chain of subsets $X_1, X_2, \dots, X_{2^{n-1}+1}$ of $S = \{1, \dots, n\}$ satisfying $X_i \cap X_{i+1} = \varnothing$ and $X_i \cup X_{i+1} \neq S$ for each $n \geq 4$.

For $S = \{1, 2, 3, 4\}$, the following chain of length $2^3 + 1 = 9$ will work:

$$34 \quad 1 \quad 23 \quad 4 \quad 12 \quad 3 \quad 14 \quad 2 \quad 13 \ .$$

Now, given a chain of subsets of $\{1, 2, \dots, n\}$ the following procedure produces a chain of subsets of $\{1, 2, \dots, n+1\}$:

1. take the original chain, delete any element, and make two copies of this chain, which now has even length;
2. glue the two copies together, joined by $\varnothing$ in between; and then
3. insert the element $n + 1$ into the sets in alternating positions of the chain starting with the first.

For example, the first iteration of this construction gives:

$$\begin{array}{ccccccccc} 345 & 1 & 235 & 4 & 125 & 3 & 145 & 2 & 5 \\ 34 & 15 & 23 & 45 & 12 & 35 & 14 & 25 & \end{array}$$

It can be easily checked that if the original chain satisfies the requirements, then so does the new chain, and if the original chain has length $2^{n-1} + 1$, then the new chain has length $2^n + 1$, as desired. This construction yields a chain of length 129 when $S = \{1, 2, \dots, 8\}$.

Remark. Here is the construction for $n = 8$ in its full glory.

345678	1	235678	4	125678	3	145678	2	5678
34	15678	23	45678	12	35678	14	678	
345	1678	235	4678	125	3678	145	2678	5
34678	15	23678	45	12678	35	78		
3456	178	2356	478	1256	378	1456	278	56
3478	156	2378	456	1278	356	1478	6	
34578	16	23578	46	12578	36	14578	26	578
346	1578	236	4578	126	8			
34567	18	23567	48	12567	38	14567	28	567
348	1567	238	4567	128	3567	148	67	
3458	167	2358	467	1258	367	1458	267	58
3467	158	2367	458	1267	358	7		
34568	17	23568	47	12568	37	14568	27	568
347	1568	237	4568	127	3568	147	68	
3457	168	2357	468	1257	368	1457	268	57
3468	157	2368	457	1268				

§2 Solutions to Day 2

§2.1 JMO 2016/4, proposed by Gregory Galperin

Available online at https://aops.com/community/p6220314.

Problem statement

Find, with proof, the least integer N such that if any 2016 elements are removed from the set $\{1, 2, \dots, N\}$, one can still find 2016 distinct numbers among the remaining elements with sum N.

The answer is

$$N = 2017 + 2018 + \cdots + 4032 = 1008 \cdot 6049 = 6097392.$$

To see that N must be at least this large, consider the situation when 1, 2, ..., 2016 are removed. Among the remaining elements, any sum of 2016 elements is certainly at least $2017 + 2018 + \cdots + 4032$.

Now we show this value of N works. Consider the 3024 pairs of numbers $(1, 6048)$, $(2, 6047)$, ..., $(3024, 3025)$. Regardless of which 2016 elements of $\{1, 2, \dots, N\}$ are deleted, at least $3024 - 2016 = 1008$ of these pairs have both elements remaining. Since each pair has sum 6049, we can take these pairs to be the desired numbers.

§2.2 JMO 2016/5, proposed by Zuming Feng, Jacek Fabrykowski

Available online at https://aops.com/community/p6220305.

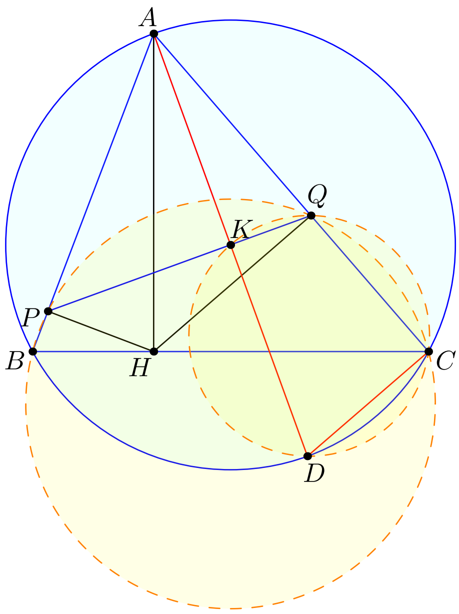
Problem statement

Let $\triangle ABC$ be an acute triangle, with O as its circumcenter. Point H is the foot of the perpendicular from A to line BC, and points P and Q are the feet of the perpendiculars from H to the lines AB and AC, respectively.

Given that
$$AH^2 = 2AO^2,$$
prove that the points O, P, and Q are collinear.

We present two approaches.

¶ First approach (synthetic) First, since $AP \cdot AB = AH^2 = AQ \cdot AC$, it follows that $PQCB$ is cyclic. Consequently, we have $AO \perp PQ$.



Let K be the foot of A onto PQ, and let D be the point diametrically opposite A. Thus A, K, O, D are collinear.

Since quadrilateral $KQCD$ is cyclic ($\angle QKD = \angle QCD = 90^\circ$), we have

$$AK \cdot AD = AQ \cdot AC = AH^2 \implies AK = \frac{AH^2}{AD} = \frac{AH^2}{2AO} = AO$$

so $K = O$.

¶ Second approach (coordinates), with Joshua Hsieh We impose coordinates with H at the origin and $A = (0, a)$, $B = (-b, 0)$, $C = (c, 0)$, for $a, b, c > 0$.

Claim — The circumcenter has coordinates $(\frac{c-b}{2}, \frac{a}{2} - \frac{bc}{2a})$.

Proof. This is a known lemma but but we reproduce its proof for completeness. It uses the following steps:

- By power of a point, the second intersection of line AH with the circumcircle is $(0, -\frac{bc}{a})$.
- Since the orthocenter is the reflection of this point across line BC, the orthocenter is given exactly by $(0, \frac{bc}{a})$.
- The centroid is is $\frac{\vec{A}+\vec{B}+\vec{C}}{3} = (\frac{c-b}{3}, \frac{a}{3})$.
- Since $\vec{H} - \vec{O} = 3(\vec{G} - \vec{O})$ according to the Euler line, we have $\vec{O} = \frac{3}{2}\vec{G} - \frac{1}{2}\vec{H}$. This gives the desired formula. $\square$

Note that $HQ = \frac{HA \cdot HC}{AC} = \frac{ac}{\sqrt{a^2+c^2}}$. If we let T be the foot from Q to BC, then $\triangle HQT \overset{\sim}{+} \triangle AHC$ and so the x-coordinate of Q is given by $HQ \cdot \frac{AH}{AC} = \frac{a^2c}{a^2+c^2}$. Repeating the analogous calculation for Q and P gives

$$\begin{aligned} Q &= \left(\frac{a^2c}{a^2+c^2}, \frac{ac^2}{a^2+c^2} \right) \\ P &= \left(-\frac{a^2b}{a^2+b^2}, \frac{ab^2}{a^2+b^2} \right). \end{aligned}$$

Then, O, P, Q are collinear if and only if the following shoelace determinant vanishes (with denominators cleared out):

$$\begin{aligned} 0 &= \det \begin{bmatrix} -a^2b & ab^2 & a^2+b^2 \\ a^2c & ac^2 & a^2+c^2 \\ a(c-b) & a^2-bc & 2a \end{bmatrix} = a \det \begin{bmatrix} -ab & ab^2 & a^2+b^2 \\ ac & ac^2 & a^2+c^2 \\ c-b & a^2-bc & 2a \end{bmatrix} \\ &= a \det \begin{bmatrix} -a(b+c) & a(b^2-c^2) & b^2-c^2 \\ ac & ac^2 & a^2+c^2 \\ c-b & a^2-bc & 2a \end{bmatrix} = a(b+c) \det \begin{bmatrix} -a & a(b-c) & b-c \\ ac & ac^2 & a^2+c^2 \\ c-b & a^2-bc & 2a \end{bmatrix} \\ &= a(b+c) \cdot \left[-a(a^2c^2 - a^4 + bc(a^2+c^2)) + ac(b-c)\left(-a^2-bc\right) - (b-c)^2 \cdot a^3 \right] \\ &= a^2(b+c)(a^4 - a^2b^2 - b^2c^2 - c^2a^2). \end{aligned}$$

On the other hand,

$$\begin{aligned} AH^2 &= a^2 \\ 2AO^2 &= 2\left[\left(\frac{c-b}{2} \right)^2 + \left(-\frac{a}{2} - \frac{bc}{2a} \right)^2 \right] = \frac{a^2+b^2+c^2+\frac{b^2c^2}{a^2}}{2} \\ \implies AH^2 - 2AO^2 &= \frac{1}{2}\left(a^2 - b^2 - c^2 - \frac{b^2c^2}{a^2} \right). \end{aligned}$$

So the conditions are equivalent.

§2.3 JMO 2016/6, proposed by Titu Andreescu

Available online at https://aops.com/community/p6220308.

Problem statement

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x)+xy)\cdot f(x-3y)+(f(y)+xy)\cdot f(3x-y)=(f(x+y))^2.$$

We claim that the only two functions satisfying the requirements are $f(x) \equiv 0$ and $f(x) \equiv x^2$. These work.

First, taking $x = y = 0$ in the given yields $f(0) = 0$, and then taking $x = 0$ gives $f(y)f(-y) = f(y)^2$. So also $f(-y)^2 = f(y)f(-y)$, from which we conclude f is even. Then taking $x = -y$ gives

$$\forall x \in \mathbb{R}: \qquad f(x) = x^2 \qquad \text{or} \qquad f(4x) = 0 \qquad (\bigstar)$$

for all x.

Remark. Note that an example of a function satisfying $(\bigstar)$ is

$$f(x) = \begin{cases} x^2 & \text{if } |x| < 1 \\ \log(x^{42} + 2016^{\cos(x)}) & \text{if } 1 \le |x| < 4 \\ 0 & \text{if } |x| \ge 4. \end{cases}$$

So, yes, we are currently in a world of trouble, still.

Now we claim

Claim — $f(z) = 0 \iff f(2z) = 0 \qquad (\spadesuit)$.

Proof. Let $(x, y) = (3t, t)$ in the given to get

$$\left(f(t) + 3t^2\right) f(8t) = f(4t)^2.$$

Now if $f(4t) \neq 0$ (in particular, $t \neq 0$), then $f(8t) \neq 0$. Thus we have $(\spadesuit)$ in the reverse direction.

Then $f(4t) \neq 0 \overset{(\bigstar)}{\Longrightarrow} f(t) = t^2 \neq 0 \overset{(\spadesuit)}{\Longrightarrow} f(2t) \neq 0$ implies the forwards direction, the last step being the reverse direction $(\spadesuit)$. $\square$

By putting together $(\bigstar)$ and $(\spadesuit)$ we finally get

$$\forall x \in \mathbb{R}: \qquad f(x) = x^2 \qquad \text{or} \qquad f(x) = 0 \qquad (\heartsuit)$$

We are now ready to approach the main problem. Assume there's an $a \neq 0$ for which $f(a) = 0$; we show that $f \equiv 0$.

Let $b \in \mathbb{R}$ be given. Since f is even, we can assume without loss of generality that $a, b > 0$. Also, note that $f(x) \ge 0$ for all x by $(\heartsuit)$. By using $(\spadesuit)$ we can generate $c > b$ such that $f(c) = 0$ by taking $c = 2^n a$ for a large enough integer n. Now, select $x, y > 0$ such that $x - 3y = b$ and $x + y = c$. That is,

$$(x, y) = \left(\frac{3c+b}{4}, \frac{c-b}{4}\right).$$

Substitution into the original equation gives

$$0 = (f(x) + xy)\, f(b) + (f(y) + xy)\, f(3x - y) \geq (f(x) + xy)\, f(b).$$

But since $f(b) \geq 0$, it follows $f(b) = 0$, as desired.

8^{th} United States of America Junior Mathematical Olympiad
Day 1. 12:30 PM – 5:00 PM EDT
April 19, 2017

Note: For any geometry problem whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 1. Prove that there are infinitely many distinct pairs (a, b) of relatively prime integers $a > 1$ and $b > 1$ such that $a^b + b^a$ is divisible by $a + b$.

USAJMO 2. Consider the equation

$$(3x^3 + xy^2)(x^2y + 3y^3) = (x - y)^7.$$

(a) Prove that there are infinitely many pairs (x, y) of positive integers satisfying the equation.

(b) Describe all pairs (x, y) of positive integers satisfying the equation.

USAJMO 3. ($*$) Let ABC be an equilateral triangle and let P be a point on its circumcircle. Let lines PA and BC intersect at D; let lines PB and CA intersect at E; and let lines PC and AB intersect at F. Prove that the area of triangle DEF is twice the area of triangle ABC.

8^{th} United States of America Junior Mathematical Olympiad
Day 2. 12:30 PM – 5:00 PM EDT
April 20, 2017

Note: For any geometry problem whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 4. Are there any triples (a, b, c) of positive integers such that $(a-2)(b-2)(c-2)+12$ is a prime that properly divides the positive number $a^2+b^2+c^2+abc-2017$?

USAJMO 5. ($*$) Let O and H be the circumcenter and the orthocenter of an acute triangle ABC. Points M and D lie on side BC such that $BM = CM$ and $\angle BAD = \angle CAD$. Ray MO intersects the circumcircle of triangle BHC in point N. Prove that $\angle ADO = \angle HAN$.

USAJMO 6. Let $P_1, \ldots, P_{2n}$ be $2n$ distinct points on the unit circle $x^2+y^2=1$ other than $(1, 0)$. Each point is colored either red or blue, with exactly n of them red and n of them blue. Let $R_1, \ldots, R_n$ be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 traveling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

8^{th} United States of America Junior Mathematical Olympiad Solutions

USAJMO 1. (Proposed by Gregory Galperin)

Let n be an odd positive integer, and take $a = 2n - 1$, $b = 2n + 1$. Then $a^b + b^a \equiv 1 + 3 \equiv 0 \pmod 4$, and $a^b + b^a \equiv -1 + 1 \equiv 0 \pmod n$. Therefore $a + b = 4n$ divides $a^b + b^a$.

Alternate solution: Let $p > 5$ be a prime and let $p \not\equiv 1 \pmod 5$. For each such prime p we construct a pair of relatively prime numbers (a, b) that satisfy the conclusion of the problem. Thus, we will get infinitely many distinct pairs (a, b) as required.

Let $a = 3p + 2$, $b = 7p - 2$. Then $a + b = 10p$. We have $\varphi(10p) = 4(p - 1) = b - a$, where φ is Euler's function.

Obviously, a and b are odd and not divisible by p. They are not divisible by 5 because $p \not\equiv 1 \pmod 5$. Thus, a and b are relatively prime to $10p = a + b$, and therefore relatively prime to each other.

Therefore, using Euler's theorem,

$$a^b = a^{a+\varphi(10p)} = a^a \cdot a^{\varphi(10p)} \equiv a^a \pmod{10p} ,$$

and since $10p = a + b$,

$$a^b + b^a \equiv a^a + b^a \pmod{a + b} .$$

However, since a is odd, $a^a + b^a$ is divisible by $a + b$. Hence, $a^b + b^a$ is divisible by $a + b$.

USAJMO 2. (Proposed by Titu Andreescu)

For $x > 0$ and $y > 0$, the left-hand side of the equation is positive, implying that $x > y$.

(a) Set $\dfrac{x}{y} = k + 1$, for some positive rational number k. Then the equation is equivalent to

$$(k + 1)(3k^2 + 6k + 4)(k^2 + 2k + 4) = (k^7)y.$$

Take any positive integer n. Letting $k = \frac{1}{n}$ yields an infinite family of solutions

$$(x, y) = (n(n + 1)^2(4n^2 + 6n + 3)(4n^2 + 2n + 1), n^2(n + 1)(4n^2 + 6n + 3)(4n^2 + 2n + 1))$$

to the given equation.

(b) Write the equation as

$$x(3x^2 + y^2)y(x^2 + 3y^2) = (x - y)^7,$$

which is equivalent to

$$(x^3 + 3xy^2)(3x^2y + y^3) = (x - y)^7.$$

Let $x^3+3xy^2=a$ and $3x^2y+y^3=b$. Then $a+b=(x+y)^3$, $a-b=(x-y)^3$ and the equation becomes

$$(ab)^3=(a-b)^7.$$

Let $d=\gcd(a,b)$. Then $a=du$ and $b=dv$ for some relatively prime positive integers u and v. Hence

$$(uv)^3=d(u-v)^7.$$

Because $\gcd(u,v)=1$, we have $\gcd(u-v,u)=1$, $\gcd(u-v,v)=1$, hence $\gcd(u-v,uv)=1$. It follows that $u-v=1$ and $d=(uv)^3$. Hence $u=k+1$ and $v=k$, where k is a positive integer, and so $a=(k+1)^4k^3$ and $b=k^4(k+1)^3$. Then

$$(x-y)^3=a-b=[k(k+1)]^3$$

and

$$(x+y)^3=a+b=[k(k+1)]^3(2k+1).$$

It follows that $2k+1=n^3$ for some odd integer $n>1$ and that $x+y=nk(k+1)$ and $x-y=k(k+1)$. Hence

$$(x,y)=\left(\frac{(n+1)k(k+1)}{2},\ \frac{(n-1)k(k+1)}{2}\right)$$

where $k=\frac{n^3-1}{2}$. Thus

$$(x,y)=\left(\frac{(n+1)(n^6-1)}{8},\ \frac{(n-1)(n^6-1)}{8}\right)$$

where n is an odd integer greater than 1, and it is easy to check that these are solutions to the given equation. Hence these pairs describe all the solutions to the equation.

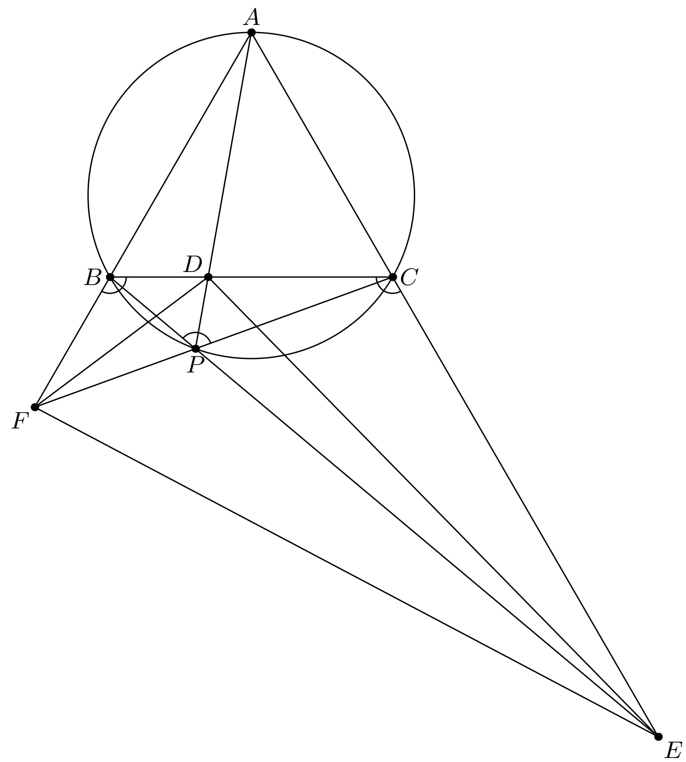
USAJMO 3. (Proposed by Titu Andreescu, Luis Gonzalez, and Cosmin Pohoata)

We offer several solutions. Throughout, we use bracket notation for areas: for example, $[ABC]$ means the area of triangle ABC.

We first present three down-to-earth approaches. One of them is a coordinate geometry approach. The other two approaches utilize the fact of many pairs of similar triangles in this configuration:

- BPC, FPA, FBC, APE, and BCE;
- FBP and FCA;
- ECP and EBA.

In these solutions, we assume the points are configured so that P is on minor arc $\widehat{BC}$ of the circle, as shown in the figure.



Solution 1. (By USA(J)MO packet reviewers.) We may assume that $AB = 1$. Then $[ABC] = \sqrt{3}/4$. Set $b = PB$, $c = PC$, $e = PE$, and $f = PF$. Note that $\angle FBD = \angle ECD = \angle BPC = 120^\circ$. Hence

$$[DEF] = [BCEF] - [FBD] - [ECD] = \frac{1}{2}\sin 120^\circ (BE \cdot CF - BF \cdot BD - CE \cdot CD).$$

It suffices to show that $[DEF] = \sqrt{3}/2$ or

$$2 = (BE \cdot CF - BF \cdot BD - CE \cdot CD) = (b+e)(c+f) - BF \cdot BD - CE \cdot CD.$$

Because $\angle FBC = \angle BPC$ and $\angle FCB = \angle PCB$, triangles FCB and BCP are similar to each other, implying that

$$\frac{FC}{BC} = \frac{CB}{CP} = \frac{BF}{PB} \quad \text{or} \quad \frac{c+f}{1} = \frac{1}{c} = \frac{BF}{b}.$$

Thus, $c + f = 1/c$ and $BF = b/c$. Analogously, $b + e = 1/b$ and $CE = c/b$. It remains to show that

$$2 = (b+e)(c+f) - BF \cdot BD - CE \cdot CD = \frac{1}{bc} - \frac{b}{c} \cdot BD - \frac{c}{b} \cdot CD.$$

Note that $\angle BPD = \angle CPD = 60^\circ$, so we have $BD/CD = BP/CP$ by the Angle-Bisector theorem. Consequently, we have $BD = b/(b+c)$ and $CD = c/(b+c)$. Thus, we want to show that

$$2 \quad = \quad \frac{1}{bc} - \frac{b}{c} \cdot BD - \frac{c}{b} \cdot CD = \frac{1}{bc} - \frac{b^2}{c(b+c)} - \frac{c^2}{b(b+c)}$$

$$= \quad \frac{1}{bc} - \frac{b^3+c^3}{bc(b+c)} = \frac{1-b^2-c^2+bc}{bc},$$

or $b^2+c^2+bc=1$, which is true by applying the Law of Cosines in triangle BPC.

Solution 2. (By USA(J)MO packet reviewers.) Note that $\angle DPF = \angle DPE = \angle EPF = 120^\circ$. We have

$$[DEF] = \frac{1}{2} \cdot \sin 120^\circ \, (PD \cdot PE + PE \cdot PF + PF \cdot PD).$$

To show that $[DEF] = 2[ABC]$, it suffices to show that

$$PD \cdot PE + PE \cdot PF + PF \cdot PD = 2BC^2.$$

Set $b = PB$ and $c = PC$. We will express the lengths of BC, PD, PE, and PF in terms of b and c. Note that $\angle BPC = 120^\circ$. Applying the Law of Cosines in triangle BPC gives $BC^2 = b^2+bc+c^2$. Applying Ptolemy's theorem to cyclic quadrilateral $ABCP$ yields $AP \cdot BC = BP \cdot AC + CP \cdot AB$ or $AP = b+c$. Because $\angle ACB = \angle ABC = \angle APC = 60^\circ$, triangles ACD and APC are similar, and so

$$\frac{AC}{AP} = \frac{CD}{PC} = \frac{DA}{CA},$$

or $b^2+bc+c^2 = AC^2 = AP \cdot AD = (b+c) \cdot AD$. We conclude that

$$AD = \frac{b^2+bc+c^2}{b+c} \quad \text{and} \quad PD = AP - AD = b+c - \frac{b^2+bc+c^2}{b+c} = \frac{bc}{b+c}.$$

Finally, because $\angle FBP = 180^\circ - \angle ABP = \angle ACP$ and $\angle BPF = \angle APC = 60^\circ$, triangles FBP and ACP are similar. Hence

$$\frac{FB}{AC} = \frac{BP}{CP} = \frac{PF}{PA},$$

from which it follows that $PF = AP \cdot BP/CP = b(b+c)/c$. In exactly the same way, we get $PE = c(b+c)/b$. It follows that

$$\begin{aligned} PD \cdot PE + PE \cdot PF + PF \cdot PD &= \frac{bc}{b+c}\left(\frac{c(b+c)}{b} + \frac{b(b+c)}{c}\right) + \frac{c(b+c)}{b} \cdot \frac{b(b+c)}{c} \\ &= 2(b^2+bc+c^2), \end{aligned}$$

as desired.

Solution 3. (By USA(J)MO packet reviewers.) Without loss of generality, we may assume that $A = (0,2)$, $B = (-\sqrt{3},-1)$, and $C = (\sqrt{3},-1)$. Set $P = (a,b)$ with $a^2+b^2=4$.

Solving for line equations $y=-1$ and $y = \dfrac{(b-2)}{a} \cdot x + 2$ gives $D = \left(-\dfrac{3a}{b-2}, -1\right)$.

Solving for line equations $y = \sqrt{3}x + 2$ and $y = \dfrac{(b+1)}{a-\sqrt{3}} \cdot (x - \sqrt{3}) - 1$ gives

$$F = \left(\frac{3a+\sqrt{3}b-2\sqrt{3}}{b+4-\sqrt{3}a}, \ \frac{\sqrt{3}a+5b+2}{b+4-\sqrt{3}a}\right).$$

Solving for line equations $y = -\sqrt{3}x + 2$ and $y = \dfrac{(b+1)}{a+\sqrt{3}} \cdot (x+\sqrt{3}) - 1$ gives

$$E = \left(\frac{3a - \sqrt{3}b + 2\sqrt{3}}{b + 4 + \sqrt{3}a},\ \frac{-\sqrt{3}a + 5b + 2}{b + 4 + \sqrt{3}a} \right).$$

Hence

$$\overrightarrow{DF} = \left[\frac{3a + \sqrt{3}b - 2\sqrt{3}}{b + 4 - \sqrt{3}a} + \frac{3a}{b-2},\ \frac{6(b+1)}{b + 4 - \sqrt{3}a} \right]$$

and

$$\overrightarrow{DE} = \left[\frac{3a - \sqrt{3}b + 2\sqrt{3}}{b + 4 + \sqrt{3}a} + \frac{3a}{b-2},\ \frac{6(b+1)}{b + 4 + \sqrt{3}a} \right].$$

Therefore,

$$\begin{aligned}
2[DEF] &= \frac{6(b+1)}{b+4+\sqrt{3}a} \cdot \left(\frac{3a + \sqrt{3}b - 2\sqrt{3}}{b+4-\sqrt{3}a} + \frac{3a}{b-2} \right) - \frac{6(b+1)}{b+4-\sqrt{3}a} \cdot \left(\frac{3a - \sqrt{3}b + 2\sqrt{3}}{b+4+\sqrt{3}a} + \frac{3a}{b-2} \right) \\
&= \frac{12\sqrt{3}(b+1)(b-2)}{(b+4)^2 - 3a^2} + \frac{18a(b+1)}{b-2} \cdot \left(\frac{1}{b+4+\sqrt{3}a} - \frac{1}{b+4-\sqrt{3}a} \right) \\
&= \frac{12\sqrt{3}(b+1)(b-2)}{(b+4)^2 - 3a^2} - \frac{36\sqrt{3}a^2(b+1)}{(b-2)((b+4)^2 - 3a^2)} \\
&= \frac{12\sqrt{3}(b+1)(b-2)}{(b+4)^2 - 3(4-b^2)} - \frac{36\sqrt{3}(4-b^2)(b+1)}{(b-2)((b+4)^2 - 3(4-b^2))} \\
&= \frac{12\sqrt{3}(b+1)(b-2)}{4b^2 + 8b + 4} - \frac{36\sqrt{3}(2-b)(2+b)(b+1)}{(b-2)(4b^2 + 8b + 4)} \\
&= \frac{3\sqrt{3}(b-2)}{b+1} + \frac{9\sqrt{3}(2+b)}{b+1} = \frac{3\sqrt{3}(b - 2 + 6 + 3b)}{b+1} = 12\sqrt{3},
\end{aligned}$$

implying that $[DEF] = 6\sqrt{3} = 2[ABC]$, as desired.

The next solution is by the problem authors. It uses more advanced tools that USAJMO participants are not expected to know, but offers some additional insight into the origins of the problem.

Solution 4. (By the posers.) Without loss of generality, let us assume that P lies on the arc AC, which does not contain vertex B. Because P is on the circumcircle, its isogonal conjugate, say Q, is a point at infinity. Furthermore, the intersections D', E', F' of lines QA, QB, QC with lines BC, CA, AB, respectively, are the reflections of D, E, F across the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. This essentially follows from the fact that $\triangle ABC$ is equilateral: isogonal conjugates with respect to it are also isotomic conjugates. We are thus led to the following lemma.

Lemma 1. *Let ABC be a triangle with D, E, F points lying on the lines BC, CA, AB, respectively. Let D', E', F' be the reflections of D, E, F with respect to the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Then, triangles DEF and $D'E'F'$ have the same area.*

Proof. The statement holds regardless of the position of points D, E, F on lines BC, CA, AB, so, for convenience, in the computations below we shall assume that these all lie close enough to the midpoints of the sides so that all points D, E, F, D', E', F' lie on the sides of $\triangle ABC$. The proof for the other scenarios is similar.

We begin by writing

$$[CD'E'] = [AD'E] = [AD'C] - [CD'E].$$

Analogously, $[AE'F'] = [BE'A] - [AE'F]$ and $[BF'D'] = [CF'B] - [BF'D]$. Adding these three together, we get

$$\begin{aligned} & [CD'E'] + [AE'F'] + [BF'D'] \\ = \ & [AD'C] + [BE'A] + [CF'B] - [CD'E] - [AE'F] - [BF'D]. \end{aligned}$$

Furthermore,

$$[CDE] = [BD'E] = [BEC] - [CD'E],$$

and similarly $[AEF] = [CFA] - [AE'F]$ and $[BFD] = [ADB] - [BF'D]$. Therefore,

$$\begin{aligned} & [CDE] + [AEF] + [BFD] \\ = \ & [BEC] + [CFA] + [ADB] - [CD'E] - [AE'F] - [BF'D]. \end{aligned}$$

But $D'C = DB$, $E'A = EC$, $F'B = FA$, so $[AD'C] = [ADB]$, $[BE'A] = [BEC]$, $[CF'B] = [CFA]$. Using all of the above, we get

$$[CD'E'] + [AE'F'] + [BF'D'] = [CDE] + [AEF] + [BFD],$$

and so $[ABC] - [D'E'F'] = [ABC] - [DEF]$, i.e., $[DEF] = [D'E'F']$, establishing the lemma. $\square$

Assuming Lemma 1, we just have to check that $[D'E'F'] = 2[ABC]$. Because P lies on the small arc AC, points D and F lie on the extensions of segments BC and AB, respectively, and so D' and F' do too. Furthermore, B lies in the interior of triangle $D'E'F'$, therefore

$$[D'E'F'] = [D'BF'] + [F'BE'] + [E'BD'].$$

On the other hand, $AD' \| CF'$ implies $[D'CF'] = [ACF']$, which, after subtracting $[BCF']$ from both sides, gives $[D'BF'] = [ABC]$. Likewise, $BE' \| CF'$ gives $[F'BE'] = [CBE']$ and $AD' \| BE'$ gives $[E'BD'] = [E'BA]$. Hence, it follows that

$$[D'E'F'] = [ABC] + [CBE'] + [E'BA] = 2[ABC],$$

as claimed.

Note: One can also establish the lemma using barycentric coordinates. Suppose points D, E, F are dividing the sides BC, CA, AB in the ratios

$$BD : DC = x : 1-x, \quad CE : EA = y : 1-y, \quad AF : FB = z : 1-z.$$

In terms of barycentric coordinates with respect to triangle ABC, we have

$$D = (1-x)B + xC, \quad E = (1-y)C + yA, \quad F = (1-z)A + zB.$$

Consequently, by definition, points D', E', F' satisfy

$$D' = xB + (1-x)C, \quad E' = yC + (1-y)A, \quad F' = zA + (1-z)B.$$

Now, without loss of generality, rescale so that $[ABC] = 1$. It can then be easily checked that

$$\begin{aligned}
[DEF] &= [ABC] - ([AEF] + [BFD] + [CDE]) \\
&= (1 - ((1-y)z + (1-z)x + (1-x)y)) \\
&= (1 - (x+y+z) + (xy + yz + zx)) \\
&= (1 - (y(1-z) + z(1-x) + x(1-y))) \\
&= [ABC] - ([AE'F'] + [BF'D'] + [CD'E']) \\
&= [D'E'F'].
\end{aligned}$$

This proves Lemma 1. The rest of the solution is as before.

USAJMO 4. (Proposed by Titu Andreescu)

Suppose (a, b, c) is such a triple. The prime $(a-2)(b-2)(c-2) + 12$ also divides

$$\begin{aligned}
&a^2 + b^2 + c^2 + abc - 2017 - (a-2)(b-2)(c-2) - 12 \\
&= (a+b+c)^2 - 4(a+b+c) + 4 - 2025 \\
&= (a+b+c-2)^2 - 45^2 \\
&= (a+b+c-47)(a+b+c+43).
\end{aligned}$$

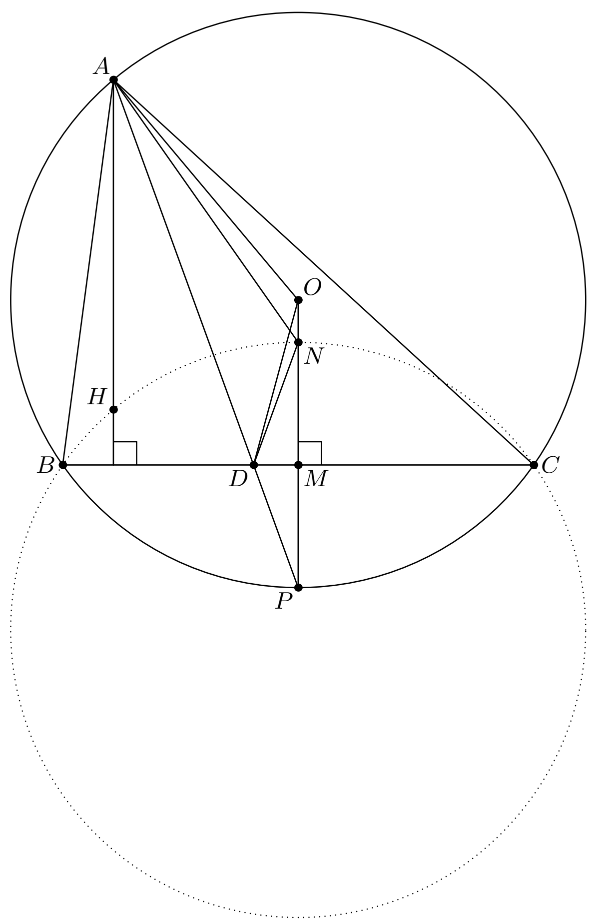
We may assume without loss of generality that $a \le b \le c$. If $a = b = 1$, $c + 10$ must be a prime that properly divides $c^2 + c - 2015$, implying $c + 10$ divides $1925 = 5^2 \cdot 7 \cdot 11$. So $c + 10 = 11$, and we obtain the triple $(1, 1, 1)$. However, this does not make $a^2 + b^2 + c^2 + abc - 2017$ positive.

If $a = 1$ and $b = 2$, then $(a-2)(b-2)(c-2) + 12 = 12$ is not prime. If $a = 1$ and $b = 3$, $14 - c$ must be a prime. The allowable choices for c are $3, 7, 9, 11$ and 12, but none of these work. If $a = 1$ and $b = 4$, the prime is even, so must be 2 and hence $c = 7$, but this doesn't work either. If $a = 1$ and $b \ge 5$ then $c \ge 5$ also, so $(a-2)(b-2)(c-2) + 12 \le 12 - 9 = 3$, and the only possibility is $b = c = 5$, but this also doesn't work. This rules out the cases with $a = 1$. Also $a = 2$ is impossible, again because 12 is not prime.

Now let $x = a - 2$, $y = b - 2$, $z = c - 2$. We now know that $1 \le x \le y \le z$ and $(x+2) + (y+2) + (z+2) > 47$. So $x + y + z \ge 41$, and therefore $z \ge 14$. The prime $xyz + 12$ cannot divide $(x+2) + (y+2) + (z+2) - 47$ since $xyz - 4 > x + y + z - 41$. Indeed, this latter inequality reduces to $x(yz - 1) > y + z - 37$, which will follow if we can prove that $yz - 1 > y + z - 37$ (since $x \ge 1$). The last statement is equivalent to $(y-1)(z-1) > -36$, which is evidently true.

Hence $xyz+12$ divides $(x+2)+(y+2)+(z+2)+43$. They cannot be equal: x, y, z must all be odd, otherwise $xyz+12$ is not prime, but then $(x+2)+(y+2)+(z+2)+43$ is even and so not equal to $xyz+12$. Thus $2(xyz+12) \le x+y+z+49$, implying $2yz-1 \le x(2yz-1) \le y+z+25$. It follows that $(2y-1)(2z-1) \le 53$. Earlier we proved that $z \ge 14$; since z is odd, we must in fact have $z \ge 15$. Moreover, $2y-1 \le 53/(2z-1) \le 53/29 < 2$. Therefore $x = y = 1$. It follows that $z+12$ is prime and $15 \le z \le 27$; therefore $z = 17, 19$, or 25. Also, $z+12$ divides $(x+2)+(y+2)+(z+2)+43 = z+51$. However, this is false for $z = 17, 19$, or 25. Consequently, the answer is negative; i.e., the requested triples (a, b, c) do not exist.

USAJMO 5. (Proposed by Ivan Borsenco)



Set $\angle CAB = A$, $\angle ABC = B$, and $\angle BCA = C$. Because H is the orthocenter, we have $\angle HBC = 90^\circ - C$ and $\angle HCB = 90^\circ - B$. In triangle BHC, we have $\angle BHC = 180^\circ - \angle HBC - \angle HCB = B + C$. Because $BHNC$ is cyclic, we have $\angle BNC = \angle BHC = B + C$. Extend segment AD through D to meet the circumcircle (denoted by ω) of triangle ABC at P. It is clear that P is the midpoint of minor arc $\widehat{BC}$ (of ω) and O, M, P all lie on the perpendicular bisector of segment

BC. In particular, $BPCN$ is a kite with symmetry axis PN. Because $ABPC$ is cyclic, we have $\angle BPC = 180^\circ - \angle BAC = B + C = \angle BNC$. We can further conclude that $BPCN$ is a rhombus, implying that line BC is the perpendicular bisector of segment NP, and so $DN = NP$ and $\angle DPN = \angle DNP$.

Set $x = \angle HAP$. Because $AH \parallel OP$, we have $\angle DNP = \angle DPN = \angle HAP = x$. Because O is the circumcenter of triangle ABC, we have $\angle AOC = 2B$ and $\angle CAO = \angle ACO = 90^\circ - B$. Because H is the orthocenter of triangle ABC, we have $\angle BAH = 90^\circ - B$. Because $\angle BAH = 90^\circ - B = \angle CAO$, $\angle BAC$ and $\angle HAO$ share common angle bisector AD; that is,

$$\angle DNP = \angle DPN = \angle HAP = \angle OAP = \angle OAD = x.$$

Consequently, we have

$$\angle ADO = \angle ADN - \angle ODN = \angle DNP + \angle DPN - \angle ODN = 2x - \angle ODN$$

and

$$\angle HAN = \angle HAO - \angle OAN = \angle HAP + \angle OAP - \angle OAN = 2x - \angle OAN.$$

It suffices to show that $\angle ODN = \angle OAN$, which is clearly true because $ADNO$ is cyclic as $\angle DNP = \angle OAD = x$.

Alternate solution (by Titu Andreescu and Cosmin Pohoata). The key idea is to prove that $ADNO$ is cyclic. Once this is proven, the problem follows by noticing that $\angle ADO = \angle ANO = \angle HAN$, where the latter holds due to the fact that $ON \| AH$.

To prove the concyclicity, one can simply use Power of a Point. First, one has to construct P as in the first solution, and notice that M is the midpoint of segment $\overline{PN}$. This follows from the fact that the reflection of H across line BC lies on the circumcircle Ω of $\triangle ABC$. This implies that the circumcircle of $\triangle BHC$ is the reflection of Ω across line BC, so line BC must indeed bisect $\overline{PN}$ by symmetry. Next, let O' denote the orthogonal projection of O on AD. Clearly $OO'DM$ is cyclic, so Power of a Point yields $PM \cdot PO = PD \cdot PO'$. But O' is the midpoint of PA, so $PO' = PA/2$. Since $PM = PN/2$, this yields

$$PN \cdot PO = PD \cdot PA,$$

which by Power of a Point gives the concyclity of $ADNO$. This completes the proof.

USAJMO 6. (Proposed by Maria Monks Gillespie)

We may assume the points have been labeled as $P_1, P_2, \ldots, P_{2n}$ in order, going counterclockwise from $(1, 0)$. Now, write out the color of each point in order, and replace each R with a $+1$ and each B with a -1, to get a list $p_1, \ldots, p_{2n}$ of $+1$'s and -1's. Consider the partial sums $p_1 + \cdots + p_k$ of this sequence, and choose the index k such that the kth partial sum has as small a value as possible; if several partial sums are tied for smallest, let k be the lowest index among them. Now, rotate the circle clockwise so that points $P_1, \ldots, P_k$ are moved past $(1, 0)$; the resulting sequence of $+1$'s and -1's from the new orientation now has all nonnegative partial sums, and the total sum is 0.

Consider any red point in the rotated diagram and label it R_1. The arc $R_1 \to B_1$ does not cross $(1, 0)$, for otherwise the sequence ends with a string of $+1$'s and the partial sums before those $+1$'s would be negative. Furthermore, the sequence of entries from R_1 to B_1 looks like $+1, +1, +1, \ldots, +1, -1$, and so removing R_1 and B_1 is equivalent to removing a consecutive pair of a $+1$ and -1, so the partial sums remain all nonnegative. It follows that the next pairing also doesn't cross $(1, 0)$, and so on, so no matter which way we pick the ordering of the red points in the rotated circle, there are no counterclockwise arcs $R_i \to B_i$ containing $(1, 0)$.

Finally, note that in any ordering of the red points, the blue points among $P_1, \ldots, P_k$ are all paired with red points, and those red points among $P_1, \ldots, P_k$ are paired with blue points in this same subsequence since there are no crossings in the rotated picture. Let m be the difference between the number of blue and red points among $P_1, \ldots, P_k$. Then it follows that exactly m blue points in $P_1, \ldots, P_k$ were matched with red points from $P_{k+1}, \ldots, P_{2n}$. Therefore, when we rotate the circle back to its original position, there are exactly m crossings, no matter which ordering we pick for the red points. Since m is independent of the ordering, the proof is complete.

JMO 2017 Solution Notes

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2 June 2023

This is a compilation of solutions for the 2017 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Prove that there exist infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b$ divides $a^b + b^a$.

2. Show that the Diophantine equation
$$\left(3x^3 + xy^2\right)\left(x^2y + 3y^3\right) = (x - y)^7$$
has infinitely many solutions in positive integers, and characterize all the solutions.

3. Let ABC be an equilateral triangle and P a point on its circumcircle. Set $D = \overline{PA} \cap \overline{BC}$, $E = \overline{PB} \cap \overline{CA}$, $F = \overline{PC} \cap \overline{AB}$. Prove that the area of triangle DEF is twice the area of triangle ABC.

4. Are there any triples (a, b, c) of positive integers such that $(a-2)(b-2)(c-2)+12$ is a prime number that properly divides the positive number $a^2+b^2+c^2+abc-2017$?

5. Let O and H be the circumcenter and the orthocenter of an acute triangle ABC. Points M and D lie on side BC such that $BM = CM$ and $\angle BAD = \angle CAD$. Ray MO intersects the circumcircle of triangle BHC in point N. Prove that $\angle ADO = \angle HAN$.

6. Let P_1, P_2, $\dots$, P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, $\dots$, R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points B_1, $\dots$, B_n. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering R_1, $\dots$, R_n of the red points.

§1 Solutions to Day 1

§1.1 JMO 2017/1, proposed by Gregory Galperin

Available online at https://aops.com/community/p8108366.

Problem statement

Prove that there exist infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b$ divides $a^b + b^a$.

One construction: let $d \equiv 1 \pmod 4$, $d > 1$. Let $x = \frac{d^d+2^d}{d+2}$. Then set

$$a = \frac{x+d}{2}, \qquad b = \frac{x-d}{2}.$$

To see this works, first check that b is odd and a is even. Let $d = a - b$ be odd. Then:

$$\begin{aligned} a + b \mid a^b + b^a &\iff (-b)^b + b^a \equiv 0 \pmod{a+b} \\ &\iff b^{a-b} \equiv 1 \pmod{a+b} \\ &\iff b^d \equiv 1 \pmod{d+2b} \\ &\iff (-2)^d \equiv d^d \pmod{d+2b} \\ &\iff d + 2b \mid d^d + 2^d. \end{aligned}$$

So it would be enough that

$$d + 2b = \frac{d^d + 2^d}{d+2} \implies b = \frac{1}{2}\left(\frac{d^d+2^d}{d+2} - d\right)$$

which is what we constructed. Also, since $\gcd(x, d) = 1$ it follows $\gcd(a, b) = \gcd(d, b) = 1$.

Remark. Ryan Kim points out that in fact, $(a, b) = (2n - 1, 2n + 1)$ is always a solution.

§1.2 JMO 2017/2, proposed by Titu Andreescu

Available online at https://aops.com/community/p8108503.

Problem statement

Show that the Diophantine equation

$$\left(3x^3 + xy^2\right)\left(x^2y + 3y^3\right) = (x - y)^7$$

has infinitely many solutions in positive integers, and characterize all the solutions.

Let $x = da$, $y = db$, where $\gcd(a, b) = 1$ and $a > b$. The equation is equivalent to

$$(a - b)^7 \mid ab\left(a^2 + 3b^2\right)\left(3a^2 + b^2\right) \qquad (\star)$$

with the ratio of the two becoming d. Note that

- If a and b are both odd, then $a^2 + 3b^2 \equiv 4 \pmod 8$. Similarly $3a^2 + b^2 \equiv 4 \pmod 8$. Hence 2^4 exactly divides right-hand side, contradiction.
- Now suppose $a - b$ is odd. We have $\gcd(a - b, a) = \gcd(a - b, b) = 1$ by Euclid, but also
$$\gcd(a - b, a^2 + 3b^2) = \gcd(a - b, 4b^2) = 1$$
and similarly $\gcd(a - b, 3a^2 + b^2) = 1$. Thus $a - b$ is coprime to each of a, b, $a^2 + 3b^2$, $3a^2 + b^2$ and this forces $a - b = 1$.

Of course $(\star)$ holds whenever $a - b = 1$ as well, and thus $(\star) \iff a - b = 1$. This describes all solutions.

Remark. For cosmetic reasons, one can reconstruct the curve explicitly by selecting $b = \frac{1}{2}(n - 1)$, $a = \frac{1}{2}(n + 1)$ with $n > 1$ an odd integer. Then $d = ab(a^2 + 3b^2)(3a^2 + b^2) = \frac{(n-1)(n+1)(n^2+n+1)(n^2-n+1)}{4} = \frac{n^6-1}{4}$, and hence the solution is

$$(x, y) = (da, db) = \left(\frac{(n + 1)(n^6 - 1)}{8}, \frac{(n - 1)(n^6 - 1)}{8}\right).$$

The smallest solutions are $(364, 182)$, $(11718, 7812)$, $\dots$.

§1.3 JMO 2017/3, proposed by Titu Andreescu, Luis Gonzalez, Cosmin Pohoata

Available online at https://aops.com/community/p8108450.

Problem statement

Let ABC be an equilateral triangle and P a point on its circumcircle. Set $D = \overline{PA} \cap \overline{BC}$, $E = \overline{PB} \cap \overline{CA}$, $F = \overline{PC} \cap \overline{AB}$. Prove that the area of triangle DEF is twice the area of triangle ABC.

¶ First solution (barycentric) We invoke barycentric coordinates on ABC. Let $P = (u : v : w)$, with $uv + vw + wu = 0$ (circumcircle equation with $a = b = c$). Then $D = (0 : v : w)$, $E = (u : 0 : w)$, $F = (u : v : 0)$. Hence

$$\begin{aligned}
\frac{[DEF]}{[ABC]} &= \frac{1}{(u+v)(v+w)(w+u)} \det \begin{bmatrix} 0 & v & w \\ u & 0 & w \\ u & v & 0 \end{bmatrix} \\
&= \frac{2uvw}{(u+v)(v+w)(w+u)} \\
&= \frac{2uvw}{(u+v+w)(uv+vw+wu) - uvw} \\
&= \frac{2uvw}{-uvw} = -2
\end{aligned}$$

as desired (areas signed).

¶ Second solution ("nice" lengths) WLOG $ABPC$ is convex. Let $x = AB = BC = CA$. By Ptolemy's theorem and strong Ptolemy,

$$\begin{aligned}
PA &= PB + PC \\
PA^2 &= PB \cdot PC + AB \cdot AC = PB \cdot PC + x^2 \\
\implies x^2 &+ PB^2 + PB \cdot PC + PC^2.
\end{aligned}$$

Also, $PD \cdot PA = PB \cdot PC$ and similarly since $\overline{PA}$ bisects $\angle BPC$ (causing $\triangle BPD \sim \triangle APC$).

Now P is the Fermat point of $\triangle DEF$, since $\angle DPF = \angle FPE = \angle EPD = 120^\circ$. Thus

$$\begin{aligned}
[DEF] &= \frac{\sqrt{3}}{4} \sum_{\text{cyc}} PE \cdot PF \\
&= \frac{\sqrt{3}}{4} \sum_{\text{cyc}} \left(\frac{PA \cdot PC}{PB}\right)\left(\frac{PA \cdot PB}{PC}\right) \\
&= \frac{\sqrt{3}}{4} \sum_{\text{cyc}} PA^2 \\
&= \frac{\sqrt{3}}{4} \left((PB+PC)^2 + PB^2 + PC^2\right) \\
&= \frac{\sqrt{3}}{4} \cdot 2x^2 = 2[ABC].
\end{aligned}$$

§2 Solutions to Day 2

§2.1 JMO 2017/4, proposed by Titu Andreescu

Available online at https://aops.com/community/p8117256.

Problem statement

Are there any triples (a, b, c) of positive integers such that $(a-2)(b-2)(c-2)+12$ is a prime number that properly divides the positive number $a^2+b^2+c^2+abc-2017$?

No such (a, b, c).

Assume not. Let $x = a-2$, $y = b-2$, $z = c-2$, hence $x, y, z \ge -1$.

$$\begin{aligned} a^2+b^2+c^2+abc-2017 &= (x+2)^2+(y+2)^2+(z+2)^2 \\ &\quad + (x+2)(y+2)(z+2) - 2017 \\ &= (x+y+z+4)^2 + (xyz+12) - 45^2. \end{aligned}$$

Thus the divisibility relation becomes

$$p = xyz + 12 \mid (x+y+z+4)^2 - 45^2 > 0$$

so either

$$\begin{aligned} p = xyz+12 &\mid x+y+z-41 \\ p = xyz+12 &\mid x+y+z+49 \end{aligned}$$

Assume $x \ge y \ge z$, hence $x \ge 14$ (since $x+y+z \ge 41$). We now eliminate several edge cases to get $x, y, z \ne -1$ and a little more:

Claim — We have $x \ge 17$, $y \ge 5$, $z \ge 1$, and $\gcd(xyz, 6) = 1$.

Proof. First, we check that neither y nor z is negative.

- If $x > 0$ and $y = z = -1$, then we want $p = x+12$ to divide either $x-43$ or $x+47$. We would have $0 \equiv x-43 \equiv -55 \pmod p$ or $0 \equiv x+47 \equiv 35 \pmod p$, but $p > 11$ contradiction.
- If $x, y > 0$, and $z = -1$, then $p = 12 - xy > 0$. However, this is clearly incompatible with $x \ge 14$.

Finally, obviously $xyz \ne 0$ (else $p = 12$). So $p = xyz + 12 \ge 14 \cdot 1^2 + 12 = 26$ or $p \ge 29$. Thus $\gcd(6, p) = 1$ hence $\gcd(6, xyz) = 1$.

We finally check that $y = 1$ is impossible, which forces $y \ge 5$. If $y = 1$ and hence $z = 1$ then $p = x + 12$ should divide either $x + 51$ or $x - 39$. These give $39 \equiv 0 \pmod p$ or $25 \equiv 0 \pmod p$, but we are supposed to have $p \ge 29$. □

In that situation $x+y+z-41$ and $x+y+z+49$ are both even, so whichever one is divisible by p is actually divisible by $2p$. Now we deduce that:

$$x+y+z+49 \ge 2p = 2xyz + 24 \implies 25 \ge 2xyz - x - y - z.$$

But $x \geq 17$ and $y \geq 5$ thus

$$\begin{aligned} 2xyz - x - y - z &= z(2xy - 1) - x - y \\ &\geq 2xy - 1 - x - y \\ &> (x-1)(y-1) > 60 \end{aligned}$$

which is a contradiction. Having exhausted all the cases we conclude no solutions exist.

The condition that $x+y+z-41 > 0$ (which comes from "properly divides") cannot be dropped. Examples of solutions in which $x+y+z-41 = 0$ include $(x,y,z) = (5,5,31)$ and $(x,y,z) = (1,11,29)$.

§2.2 JMO 2017/5, proposed by Ivan Borsenco

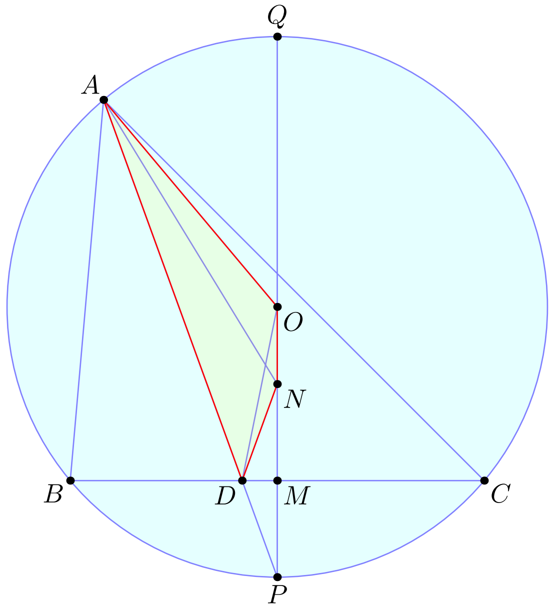
Available online at https://aops.com/community/p8117237.

Problem statement

Let O and H be the circumcenter and the orthocenter of an acute triangle ABC. Points M and D lie on side BC such that $BM = CM$ and $\angle BAD = \angle CAD$. Ray MO intersects the circumcircle of triangle BHC in point N. Prove that $\angle ADO = \angle HAN$.

It's known that N is the reflection of the arc midpoint P across M.

The main claim is that $ADNO$ is cyclic. To see this let P and Q be the arc midpoints of $\widehat{BC}$, so that $ADMQ$ is cyclic (as $\measuredangle QAD = \measuredangle QMD = 90^\circ$) . Then $PN \cdot PO = PM \cdot PQ = PD \cdot PA$ as advertised.



To finish, note that $\measuredangle HAN = \measuredangle ONA = \measuredangle ODA$.

Remark. The orthocenter H is superficial and can be deleted basically immediately. One can reverse-engineer the fact that $ADNO$ is cyclic from the truth of the problem statement.

Remark. One can also show $ADNO$ concyclic by just computing $\measuredangle DAO = \measuredangle PAO$ and $\measuredangle DNO = \measuredangle DPN = \measuredangle APQ$ in terms of the angles of the triangle, or even more directly just because

$$\measuredangle DNO = \measuredangle DNP = \measuredangle NPD = \measuredangle OPD = \measuredangle ONA = \measuredangle HAN.$$

§2.3 JMO 2017/6, proposed by Maria Monks

Available online at https://aops.com/community/p8117190.

Problem statement

Let P_1, P_2, …, P_{2n} be $2n$ distinct points on the unit circle $x^2+y^2=1$, other than $(1,0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, …, R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points B_1, …, B_n. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1,0)$ is independent of the way we chose the ordering R_1, …, R_n of the red points.

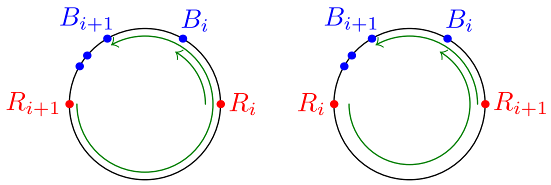
We present two solutions, one based on swapping and one based on an invariant.

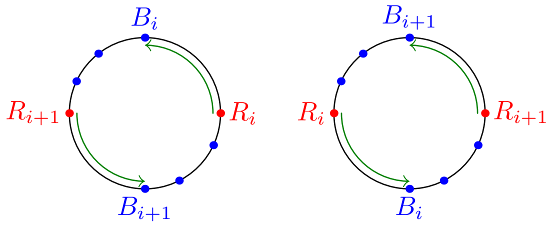
¶ First "local" solution by swapping two points Let $1 \le i < n$ be any index and consider the two red points R_i and R_{i+1}. There are two blue points B_i and B_{i+1} associated with them.

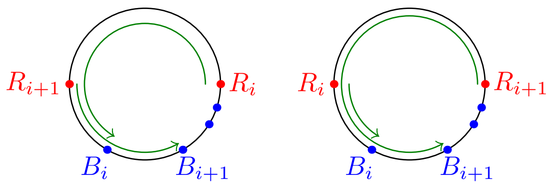
Claim — If we swap the locations of points R_i and R_{i+1} then the new arcs $R_i \to B_i$ and $R_{i+1} \to B_{i+1}$ will cover the same points.

Proof. Delete all the points R_1, …, R_{i-1} and B_1, …, B_{i-1}; instead focus on the positions of R_i and R_{i+1}.

The two blue points can then be located in three possible ways: either 0, 1, or 2 of them lie on the arc $R_i \to R_{i+1}$. For each of the cases below, we illustrate on the left the locations of B_i and B_{i+1} and the corresponding arcs in green; then on the right we show the modified picture where R_i and R_{i+1} have swapped. (Note that by hypothesis there are no other blue points in the green arcs).

Case 1


Case 2


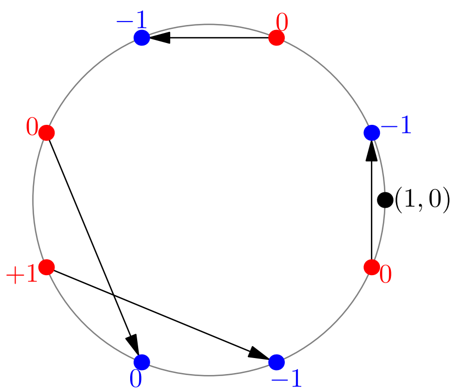
Case 3


Observe that in all cases, the number of arcs covering any given point on the circumference is not changed. Consequently, this proves the claim. □

Finally, it is enough to recall that any permutation of the red points can be achieved by swapping consecutive points (put another way: $(i\ i+1)$ generates the permutation group S_n). This solves the problem.

> **Remark.** This proof does *not* work if one tries to swap R_i and R_j if $|i-j| \neq 1$. For example if we swapped R_i and R_{i+2} then there are some issues caused by the possible presence of the blue point B_{i+1} in the green arc $R_{i+2} \to B_{i+2}$.

¶ Second longer solution using an invariant Visually, if we draw all the segments $R_i \to B_i$ then we obtain a set of n chords. Say a chord is *inverted* if satisfies the problem condition, and *stable* otherwise. The problem contends that the number of stable/inverted chords depends only on the layout of the points and not on the choice of chords.



In fact we'll describe the number of inverted chords explicitly. Starting from $(1,0)$ we keep a running tally of $R-B$; in other words we start the counter at 0 and decrement by 1 at each blue point and increment by 1 at each red point. Let $x \le 0$ be the lowest number ever recorded. Then:

Claim — The number of inverted chords is $-x$ (and hence independent of the choice of chords).

This is by induction on n. I think the easiest thing is to delete chord R_1B_1; note that the arc cut out by this chord contains no blue points. So if the chord was stable certainly no change to x. On the other hand, if the chord is inverted, then in particular the last point before $(1,0)$ was red, and so $x < 0$. In this situation one sees that deleting the chord changes x to $x+1$, as desired.

9^{th} United States of America Junior Mathematical Olympiad
Day 1. 12:30 PM – 5:00 PM EDT
April 18, 2018

Note: For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 1. For each positive integer n, find the number of n-digit positive integers that satisfy both of the following conditions:

- no two consecutive digits are equal, and
- the last digit is a prime.

USAJMO 2. Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

USAJMO 3. $(*)$ Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ = FR$.

9^{th} United States of America Junior Mathematical Olympiad
Day 2. 12:30 PM – 5:00 PM EDT
April 19, 2018

Note: For any geometry problem whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 4. Triangle ABC is inscribed in a circle of radius 2 with $\angle ABC \geq 90°$, and x is a real number satisfying the equation $x^4 + ax^3 + bx^2 + cx + 1 = 0$, where $a = BC, b = CA, c = AB$. Find all possible values of x.

USAJMO 5. Let p be a prime, and let $a_1, a_2, \ldots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, a_2 + 2k, \ldots, a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

USAJMO 6. Karl starts with n cards labeled $1, 2, 3, \ldots, n$ lined up in a random order on his desk. He calls a pair (a, b) of these cards *swapped* if $a > b$ and the card labeled a is to the left of the card labeled b. For instance, in the sequence of cards $3, 1, 4, 2$, there are three swapped pairs of cards, $(3, 1)$, $(3, 2)$, and $(4, 2)$.

He picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i cards to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on until he has picked up and put back each of the cards $1, 2, \ldots, n$ exactly once in that order. (For example, the process starting at $3, 1, 4, 2$ would be $3, 1, 4, 2 \to 3, 4, 1, 2 \to 2, 3, 4, 1 \to 2, 4, 3, 1 \to 2, 3, 4, 1$.)

Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.

2018 U.S.A. Junior Mathematical Olympiad Solutions

USAJMO 1.

First solution. Let us call a positive integer *great* if it has no consecutive digits equal and its last digit is prime. Let $p(n)$ denote the number of great n-digit numbers, so the problem is asking us to compute $p(n)$. We claim that $p(n) = 2 \cdot \frac{9^n-(-1)^n}{5}$.

For $n \geq 2$, we say an n-digit number is *good* if it ends in a prime digit and has no two consecutive digits equal among its first $n-1$ digits. Since the first $n-1$ digits and the last digit may be treated independently, the number of good n-digit numbers is $4 \cdot 9^{n-1}$.

Clearly, any great number is good. On the other hand, a good n-digit number fails to be great if its last two digits are equal. By disregarding the last digit, such good-but-not-great numbers are in bijection with great $(n-1)$-digit numbers. Thus, for $n \geq 2$, we have the equation $p(n) = 4 \cdot 9^{n-1} - p(n-1)$. (If $n = 1$, we have $p(1) = 4 \cdot 9^0 = 4$.) Applying this recursively, we find that

$$p(n) = 4 \cdot (9^{n-1} - 9^{n-2} + 9^{n-3} - \cdots + (-1)^{n-2} \cdot 9 + (-1)^{n-1}) = 4 \cdot \frac{9^n - (-1)^n}{10},$$

as claimed.

Second solution. Define great numbers and $p(n)$ as above. For $n \geq 3$, we will count the number of great n-digit numbers by considering two cases:

- If the second digit is 0, then note that the third digit must be non-zero, so the last $n-2$ digits form a great number. Meanwhile, the first digit can be any non-zero digit. Thus, there are $9 \cdot p(n-2)$ great n-digit numbers of this form.

- If the second digit is not 0, then the last $n-1$ digits form a great number, while there are 8 possibilities for the first digit (it can be any non-zero digit not equal to the second digit). This gives $8 \cdot p(n-1)$ great n-digit numbers of this form.

We conclude that $p(n) = 8p(n-1) + 9p(n-2)$ for all $n \geq 3$. This is a second order recurrence, which we may solve by factoring its characteristic polynomial $t^2 - 8t - 9 = (t-9)(t+1)$. The factorization implies that $p(n)$ takes the form $p(n) = A \cdot 9^n + B \cdot (-1)^n$ for some constants A and B. We can solve the system

$$\begin{aligned} 9A - B &= p(1) = 4 \\ 81A + B &= p(2) = 32, \end{aligned}$$

which yields $A = \frac{2}{5}$ and $B = -\frac{2}{5}$, so that

$$p(n) = \frac{2\,(9^n - (-1)^n)}{5}.$$

USAJMO 2.

First solution. Assume without loss of generality that $c = \min(a, b, c)$. By the AM-GM inequality and the given condition, we have

$$\begin{aligned} 4c(a+b+c) + 4ab &\geq 2\sqrt{16 \cdot abc(a+b+c)} \\ &= 2\sqrt{16\left(\frac{a+b+c}{4}\right)^3 (a+b+c)} \\ &= (a+b+c)^2. \end{aligned}$$

Subtracting $2(ab + bc + ca)$ from both sides, this gives

$$2(ab + bc + ca) + 4c^2 \geq a^2 + b^2 + c^2,$$

as desired.

Remark. The equality in the AM-GM step occurs if and only if $c(a + b + c) = ab$. Solving for $a+b+c$ and substituting into the condition $a+b+c = 4\sqrt[3]{abc}$, this implies $8c^2 = ab$. Substituting this back into the equation $c(a + b + c) = ab$, we conclude that

$$c(a + b + c) = 8c^2 \implies a + b = 7c.$$

We then have

$$a - b = \pm\sqrt{(a + b)^2 - 4ab} = \pm\sqrt{49c^2 - 32c^2} = \pm\sqrt{17}c.$$

It follows that $\{2a, 2b\} = \{(7 - \sqrt{17})c, (7 + \sqrt{17})c\}$. Hence, equality holds if and only if (a, b, c) is a permutation of

$$\left((7 - \sqrt{17})r, (7 + \sqrt{17})r, 2r\right)$$

for some positive real number r.

Second solution. Suppose, as above, that $c = \min(a, b, c)$, and write $A = a/c$, $B = b/c$, and $D = A+B$. The given condition becomes $A+B+1 = 4\sqrt[3]{AB}$, or equivalently, $AB = (D+1)^3/64$. In terms of A and B, the problem asks us to prove that

$$2(AB + A + B) + 4 \geq A^2 + B^2 + 1,$$

which can be rearranged as

$$2(A + B) + 3 - (A + B)^2 + 4AB \geq 0.$$

After substituting in D, this inequality becomes

$$2D + 3 - D^2 + (D + 1)^3/16 \geq 0.$$

Since the left-hand side factors as $(D + 1)(D - 7)^2/16$, the inequality always holds.

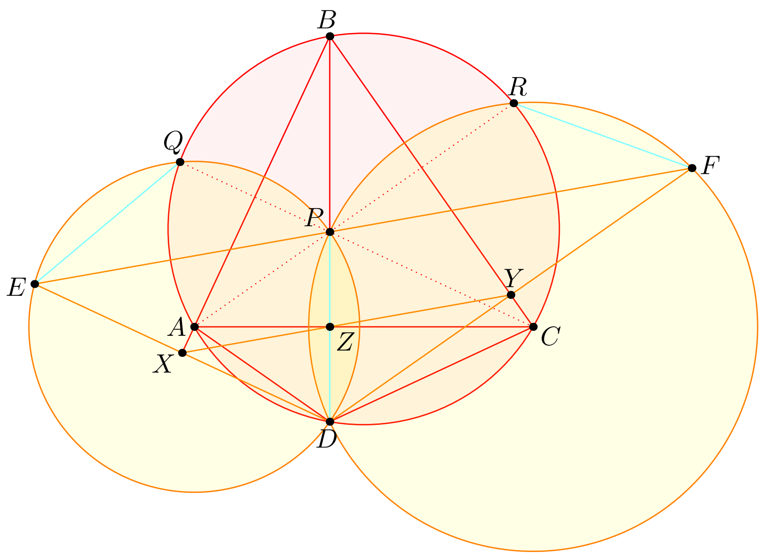
Third solution: Assuming that $c = \min(a, b, c)$ and by adding $2(ab + bc + ca)$ to both sides, our inequality becomes

$$4c(a + b + c) + 4ab \geq (a + b + c)^2.$$

Since both the given condition and the desired claim are homogeneous, we may assume without loss of generality that $a+b+c=8$, so our task is to prove that if $ab = 8/c$, then $32c + 4ab \geq 64$. This clearly holds, since for any positive real number c we have $32\left(c+\frac{1}{c}\right) \geq 64$.

USAJMO 3.

First solution. Let X and Y be the feet of the perpendiculars from D to lines BA and BC, respectively, and let Z be the intersection of lines BD and AC. By Simson's theorem, the points X, Y, Z are collinear. A homothety with ratio 2 about D maps X, Y, Z to E, F, P', respectively, where P' is the orthocenter of $\triangle ABC$. Hence, P' lies on line EF as well as line BD, so $P' = P$.



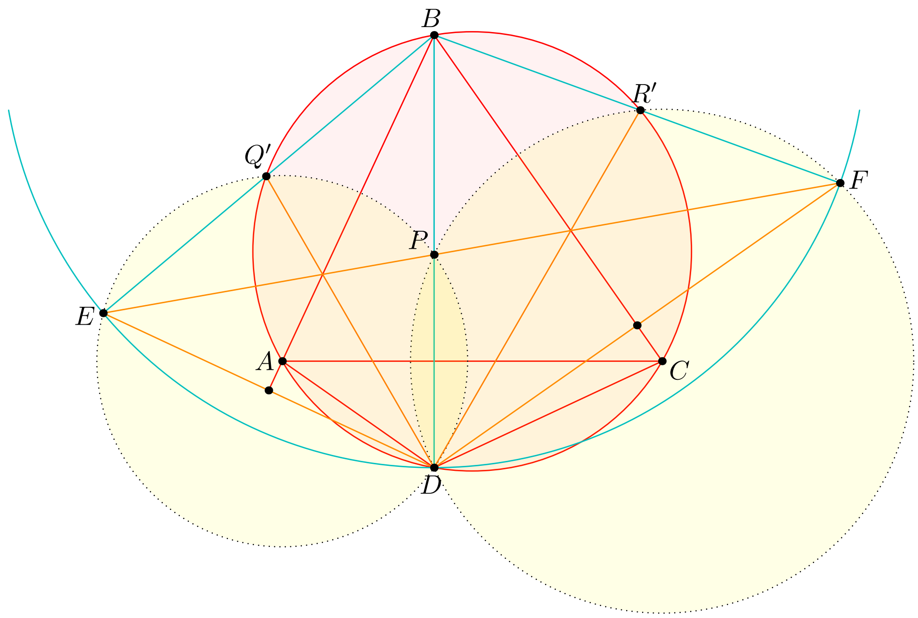
Suppose now we extend ray $\overrightarrow{CP}$ to meet ω again at Q'. Then line BA is the perpendicular bisector of both $\overline{PQ'}$ and $\overline{DE}$; consequently, $PQ'ED$ is an isosceles trapezoid. In particular, it is cyclic, and so $Q' = Q$. In the same way, R is the second intersection of ray $\overrightarrow{AP}$ with ω.

Now, because of the two isosceles trapezoids we have found, we conclude

$$EQ = PD = FR,$$

as desired.

Second solution. Here is a solution which does not identify the point P at all. We know that $BE = BD = BF$, by construction.



Claim 1. *The points B, Q, E are collinear. Similarly the points B, R, F are collinear.*

Proof. Work with directed angles modulo 180°. Let Q' be the intersection of line BE with circle ω (distinct from B). Let $\alpha = \angle DEB = \angle BDE$ and $\beta = \angle BFD = \angle FDB$.

We know that $BE = BD = BF$, so B is the circumcenter of $\triangle DEF$. Thus, $\angle DEP = \angle DEF = 90^\circ - \beta$. Then

$$\begin{aligned}
\angle DPE &= \angle DEP + \angle PDE = (90^\circ - \beta) + \alpha \\
&= \alpha - \beta + 90^\circ; \\
\angle DQ'B &= \angle DCB = \angle DCA + \angle ACB \\
&= \angle DBA - (90^\circ - \angle DBC) = -(90^\circ - \alpha) - (90^\circ - (90^\circ - \beta)) \\
&= \alpha - \beta + 90^\circ.
\end{aligned}$$

Thus Q' lies on the circumcircle of $\triangle DPE$, so $Q' = Q$. Similarly for R. □

Now, by power of a point we have $BQ \cdot BE = BP \cdot BD = BR \cdot BF$, so $BQ = BP = BR$. Hence $EQ = DP = FR$.

USAJMO 4.

The given equation can be rewritten as

$$\left(x^2 + \frac{ax}{2}\right)^2 + \left(b - \frac{a^2 + c^2}{4}\right)x^2 + \left(\frac{cx}{2} + 1\right)^2 = 0.$$

Noting that we must have $x \neq 0$, the equation holds if and only if

$$b = \frac{a^2 + c^2}{4} \quad \text{and} \quad x = -\frac{a}{2} = -\frac{2}{c}.$$

The assumption $\angle ABC \geq 90°$ and the fact that the circle's diameter is 4 imply $a^2 + c^2 \leq b^2 \leq 4b$; but since we saw that $b = (a^2 + c^2)/4$, both of these inequalities are equalities. We conclude that $\angle ABC = 90°$, $b = 4$, $a^2 + c^2 = 16$, and $ac = 4$. These last two equations imply $(a + c)^2 = 16 + 2 \cdot 4 = 24$ and $(a - c)^2 = 16 - 2 \cdot 4 = 8$. Since $a, c > 0$, we have $a + c = 2\sqrt{6}$ and $a - c = \pm 2\sqrt{2}$. Hence the only possible values of $x = -a/2$ are $-\frac{1}{2}(\sqrt{6} + \sqrt{2})$ or $-\frac{1}{2}(\sqrt{6} - \sqrt{2})$. Conversely, these are indeed possible, by having a right triangle with sides $a = \sqrt{6} + \sqrt{2}, b = 4, c = \sqrt{6} - \sqrt{2}$ or $a = \sqrt{6} - \sqrt{2}, b = 4, c = \sqrt{6} + \sqrt{2}$, respectively.

Remark. One can also show that the acute angles of the triangle are 15 degrees and 75 degrees.

USAJMO 5.

The statement is trivial for $p = 2$, so assume $p = 2q + 1$ is odd. Create a $p \times p$ table of numbers, as follows:

$$\begin{array}{cccc}
a_1 + 1 \cdot 0 & a_2 + 2 \cdot 0 & \cdots & a_p + p \cdot 0 \\
a_1 + 1 \cdot 1 & a_2 + 2 \cdot 1 & \cdots & a_p + p \cdot 1 \\
\vdots & \vdots & \ddots & \vdots \\
a_1 + 1 \cdot (p-1) & a_2 + 2 \cdot (p-1) & \cdots & a_p + p \cdot (p-1)
\end{array}$$

Interpret all the numbers above modulo p. Examine two different columns, say columns i and j. We claim they agree (modulo p) in exactly one row. Indeed, $a_i + ik \equiv a_j + jk \pmod{p}$ holds if and only if $(i - j)k \equiv a_j - a_i \pmod{p}$. Since p is prime and $i \not\equiv j \pmod{p}$, this condition holds for a unique value of k (namely, $k \equiv (a_j - a_i)(i - j)^{-1} \pmod{p}$).

Thus, there are $\binom{p}{2} = \frac{p(p-1)}{2} = pq$ pairs of integers that are congruent modulo p and lie in the same row of the table. Since there are only p rows, some row, say $\{a_n + nk\}_n$, must contain at most q such pairs.

We claim that this k satisfies our requirement. Indeed, if we read the p entries in this row one by one, each entry either is distinct from all the previous ones, or is congruent to at least one previous entry and thereby completes a pair. Since the latter case happens at most q times, there must be at least $p - q = (p + 1)/2$ distinct entries (modulo p), completing the proof.

USAJMO 6.

First solution. Consider the following alternative procedure: When Karl removes the card labeled 1, before he inserts it, he adds n to its label to make it a card labeled $n + 1$. Then he reinserts the card as in the original procedure. Now, the new arrangement of cards has the same number of swapped pairs as before, since the 1 used to be part of i swapped pairs using the cards to its left, and now the $n + 1$ is part of i swapped pairs using the cards to its right.

By the same argument, if he next removes the card labeled 2 and adds n to its label before reinserting it in its new position, and so on, he ends up with a permutation of $n + 1, n + 2, \ldots, 2n$

that has the same number of swapped pairs as the one he started with. But this permutation clearly corresponds to the ending permutation from Karl's original procedure upon subtracting n from all the labels, and this subtraction doesn't change the number of swapped pairs. This completes the proof.

Second solution. At each moment during the procedure, define the "charge" on a card to be the net (positive or negative) number of steps it would take to the left if it were to be moved next. The charge depends only on the card's location. For example, if there are 4 cards, their charges from left to right are $-3, -1, +1, +3$.

At each stage, let X be the number of swapped pairs, and let Y be the sum of the charges on all of the cards that have not yet moved. We claim that each move leaves $X + Y$ unchanged. To see this, suppose that card i is being moved c steps to the left. (We take c to be positive; the case of c negative is similar.) When card i passes a lower-numbered card, this creates a swapped pair, increasing X by $+1$. When card i passes a higher-numbered card, it removes a swapped pair, thus changing X by -1; but it also moves the higher-numbered card one step to the right, increasing its charge (which is included in Y) by $+2$. Thus the net increase in $X + Y$ is again $+1$. So the total effect of passing c cards is to increase $X + Y$ by $+c$. But also, after we move card i, its own charge (which was $+c$) is no longer included in Y. So on balance, $X + Y$ is unchanged.

So $X + Y$ is unchanged by the entire process. But Y is zero at the beginning of the process (all the charges sum to zero, by symmetry), and also at the end (when Y is just the empty sum). So X, the number of swapped pairs, is also the same at the beginning as at the end. This is what we needed to prove.

JMO 2018 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

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Corrections and comments are welcome!

Contents

§0 Problems

1. For each positive integer n, find the number of n-digit positive integers for which no two consecutive digits are equal, and the last digit is a prime.

2. Let a, b, c be positive real numbers such that $a+b+c = 4\sqrt[3]{abc}$. Prove that
$$2(ab+bc+ca) + 4\min(a^2,b^2,c^2) \geq a^2+b^2+c^2.$$

3. Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over $\overline{BA}$ and $\overline{BC}$, respectively, and let P be the intersection of $\overline{BD}$ and $\overline{EF}$. Suppose that the circumcircles of EPD and FPD meet ω at Q and R different from D. Show that $EQ = FR$.

4. Find all real numbers x for which there exists a triangle ABC with circumradius 2, such that $\angle ABC \geq 90^\circ$, and
$$x^4 + ax^3 + bx^2 + cx + 1 = 0$$
where $a = BC$, $b = CA$, $c = AB$.

5. Let p be a prime, and let $a_1, \dots, a_p$ be integers. Show that there exists an integer k such that the numbers
$$a_1 + k, \; a_2 + 2k, \; \dots, \; a_p + pk$$
produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

6. Karl starts with n cards labeled $1, 2, \dots n$ lined up in random order on his desk. He calls a pair (a, b) of cards *swapped* if $a > b$ and the card labeled a is to the left of the card labeled b.

   Karl picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i cards to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on, until he has picked up and put back each of the cards $1, \dots, n$ exactly once in that order.

   For example, if $n = 4$, then one example of a process is
$$3142 \longrightarrow 3412 \longrightarrow 2341 \longrightarrow 2431 \longrightarrow 2341$$
   which has three swapped pairs both before and after.

   Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.

§1 Solutions to Day 1

§1.1 JMO 2018/1, proposed by Zachary Franco, Zuming Feng

Available online at https://aops.com/community/p10226138.

Problem statement

For each positive integer n, find the number of n-digit positive integers for which no two consecutive digits are equal, and the last digit is a prime.

Almost trivial. Let a_n be the desired answer. We have

$$a_n + a_{n-1} = 4 \cdot 9^{n-1}$$

for all n, by padding the $(n-1)$ digit numbers with a leading zero.

Since $a_0 = 0$, $a_1 = 4$, solving the recursion gives

$$a_n = \frac{2}{5}\left(9^n - (-1)^n\right).$$

The end.

> **Remark.** For concreteness, the first few terms are 0, 4, 32, 292, ...

§1.2 JMO 2018/2, proposed by Titu Andreescu

Available online at https://aops.com/community/p10226140.

Problem statement

Let a, b, c be positive real numbers such that $a+b+c = 4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca) + 4\min(a^2, b^2, c^2) \ge a^2+b^2+c^2.$$

WLOG let $c = \min(a,b,c) = 1$ by scaling. The given inequality becomes equivalent to

$$4ab + 2a + 2b + 3 \ge (a+b)^2 \qquad \forall a+b = 4(ab)^{1/3} - 1.$$

Now, let $t = (ab)^{1/3}$ and eliminate $a+b$ using the condition, to get

$$4t^3 + 2(4t-1) + 3 \ge (4t-1)^2 \iff 0 \le 4t^3 - 16t^2 + 16t = 4t(t-2)^2$$

which solves the problem.

Equality occurs only if $t = 2$, meaning $ab = 8$ and $a+b = 7$, which gives

$$\{a,b\} = \left\{ \frac{7 \pm \sqrt{17}}{2} \right\}$$

with the assumption $c = 1$. Scaling gives the curve of equality cases.

§1.3 JMO 2018/3, proposed by Ray Li

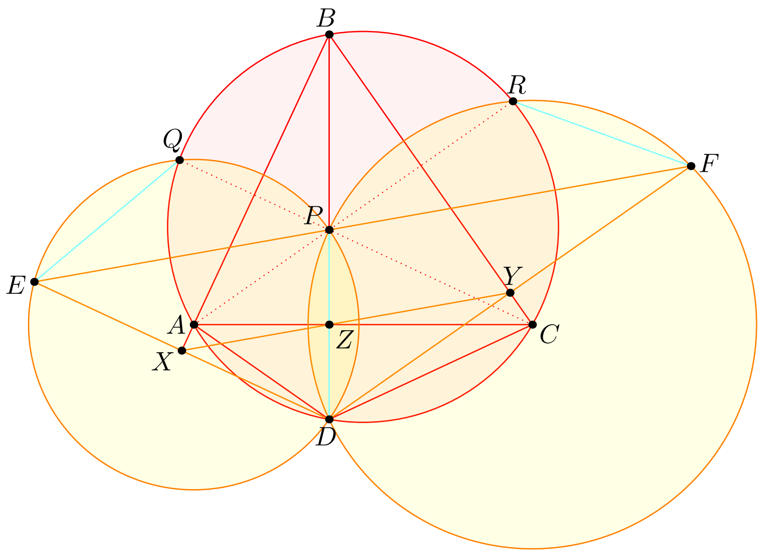
Available online at https://aops.com/community/p10226149.

Problem statement

Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over $\overline{BA}$ and $\overline{BC}$, respectively, and let P be the intersection of $\overline{BD}$ and $\overline{EF}$. Suppose that the circumcircles of EPD and FPD meet ω at Q and R different from D. Show that $EQ = FR$.

Most of this problem is about realizing where the points P, Q, R are.

¶ First solution (Evan Chen) Let X, Y, be the feet from D to $\overline{BA}$, $\overline{BC}$, and let $Z = \overline{BD} \cap \overline{AC}$. By Simson theorem, the points X, Y, Z are collinear. Consequently, the point P is the reflection of D over Z, and so we conclude P is the orthocenter of $\triangle ABC$.



Suppose now we extend ray CP to meet ω again at Q'. Then $\overline{BA}$ is the perpendicular bisector of both $\overline{PQ'}$ and $\overline{DE}$; consequently, $PQ'ED$ is an isosceles trapezoid. In particular, it is cyclic, and so $Q' = Q$. In the same way R is the second intersection of ray $\overline{AP}$ with ω.

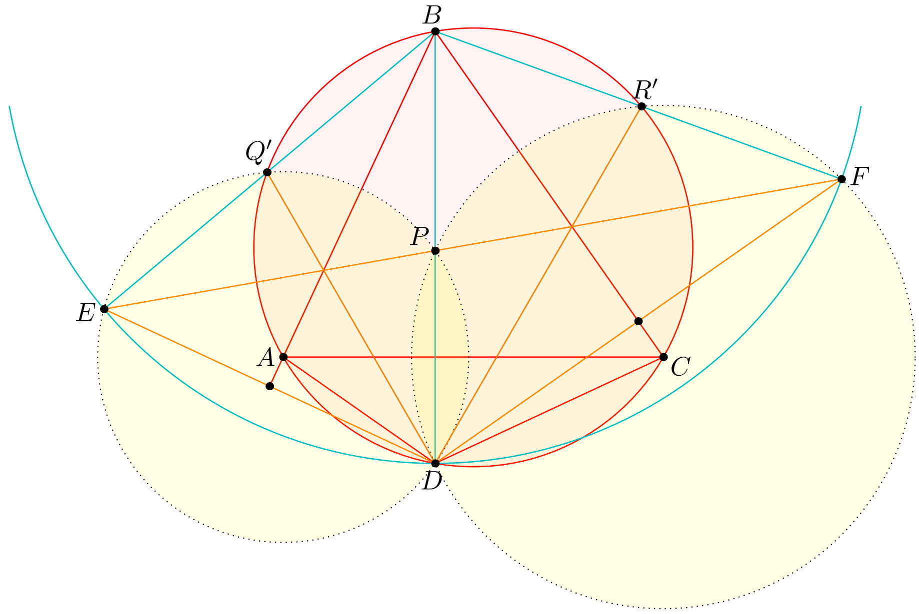
Now, because of the two isosceles trapezoids we have found, we conclude

$$EQ = PD = FR$$

as desired.

> **Remark.** Alternatively, after identifying P, one can note $\overline{BQE}$ and $\overline{BRF}$ are collinear. Since $BE = BD = BF$, upon noticing $BQ = BP = BR$ we are also done.

¶ Second solution (Danielle Wang) Here is a solution which does not identify the point P at all. We know that $BE = BD = BF$, by construction.



Claim — The points B, Q, E are collinear. Similarly the points B, R, F are collinear.

Proof. Work directed modulo 180°. Let Q' be the intersection of $\overline{BE}$ with $(ABCD)$. Let $\alpha = \measuredangle DEB = \measuredangle BDE$ and $\beta = \measuredangle BFD = \measuredangle FDB$.

Observe that $BE = BD = BF$, so B is the circumcenter of $\triangle DEF$. Thus, $\measuredangle DEP = \measuredangle DEF = 90^\circ - \beta$. Then

$$\begin{aligned} \measuredangle DPE &= \measuredangle DEP + \measuredangle PDE = (90^\circ - \beta) + \alpha \\ &= \alpha - \beta + 90^\circ \\ \measuredangle DQ'B &= \measuredangle DCB = \measuredangle DCA + \measuredangle ACB \\ &= \measuredangle DBA - (90^\circ - \measuredangle DBC) = -(90^\circ - \alpha) - (90^\circ - (90^\circ - \beta)) \\ &= \alpha - \beta + 90^\circ. \end{aligned}$$

Thus Q' lies on the desired circle, so $Q' = Q$. □

Now, by power of a point we have $BQ \cdot BE = BP \cdot BD = BR \cdot BF$, so $BQ = BP = BR$. Hence $EQ = PD = FR$.

§2 Solutions to Day 2

§2.1 JMO 2018/4, proposed by Titu Andreescu

Available online at https://aops.com/community/p10232384.

Problem statement

Find all real numbers x for which there exists a triangle ABC with circumradius 2, such that $\angle ABC \geq 90^\circ$, and

$$x^4 + ax^3 + bx^2 + cx + 1 = 0$$

where $a = BC$, $b = CA$, $c = AB$.

The answer is $x = -\frac{1}{2}(\sqrt{6} \pm \sqrt{2})$.

We prove this the only possible answer. Evidently $x < 0$. Now, note that

$$a^2 + c^2 \leq b^2 \leq 4b$$

since $b \leq 4$ (the diameter of its circumcircle). Then,

$$\begin{aligned} 0 &= x^4 + ax^3 + bx^2 + cx + 1 \\ &= x^2 \left[\left(x + \frac{1}{2}a \right)^2 + \left(\frac{1}{x} + \frac{1}{2}c \right)^2 + \left(b - \frac{a^2 + c^2}{4} \right) \right] \\ &\geq 0 + 0 + 0 = 0. \end{aligned}$$

In order for equality to hold, we must have $x = -\frac{1}{2}a$, $1/x = -\frac{1}{2}c$, and $a^2 + c^2 = b^2 = 4b$. This gives us $b = 4$, $ac = 4$, $a^2 + c^2 = 16$. Solving for $a, c > 0$ implies

$$\{a, c\} = \left\{ \sqrt{6} \pm \sqrt{2} \right\}.$$

This gives the x values claimed above; by taking a, b, c as deduced here, we find they work too.

> **Remark.** Note that by perturbing $\triangle ABC$ slightly, we see *a priori* that the set of possible x should consist of unions of intervals (possibly trivial). So it makes sense to try inequalities no matter what.

§2.2 JMO 2018/5, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p10232389.

Problem statement

Let p be a prime, and let $a_1, \dots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k,\ a_2 + 2k,\ \dots,\ a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

For each $k = 0, \dots, p-1$ let G_k be the graph on $\{1, \dots, p\}$ where we join $\{i, j\}$ if and only if

$$a_i + ik \equiv a_j + jk \pmod p \iff k \equiv -\frac{a_i - a_j}{i - j} \pmod p.$$

So we want a graph G_k with at least $\frac{1}{2}p$ connected components.

However, each $\{i, j\}$ appears in exactly one graph G_k, so some graph has at most $\frac{1}{p}\binom{p}{2} = \frac{1}{2}(p-1)$ edges (by "pigeonhole"). This graph has at least $\frac{1}{2}(p+1)$ connected components, as desired.

Remark. Here is an example for $p = 5$ showing equality can occur:

$$\begin{bmatrix} 0 & 0 & 3 & 4 & 3 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 & 1 \\ 0 & 3 & 4 & 3 & 0 \\ 0 & 4 & 1 & 1 & 4 \end{bmatrix}.$$

Ankan Bhattacharya points out more generally that $a_i = i^2$ is sharp in general.

§2.3 JMO 2018/6, proposed by Maria Monks Gillespie

Available online at https://aops.com/community/p10232393.

Problem statement

Karl starts with n cards labeled $1, 2, \dots n$ lined up in random order on his desk. He calls a pair (a, b) of cards *swapped* if $a > b$ and the card labeled a is to the left of the card labeled b.

Karl picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i cards to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on, until he has picked up and put back each of the cards $1, \dots, n$ exactly once in that order.

For example, if $n = 4$, then one example of a process is

$$3142 \longrightarrow 3412 \longrightarrow 2341 \longrightarrow 2431 \longrightarrow 2341$$

which has three swapped pairs both before and after.

Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.

The official solution is really tricky. Call the process P.

We define a new process P' where, when re-inserting card i, we additionally change its label from i to $n + i$. An example of P' also starting with 3142 is:

$$3142 \longrightarrow 3452 \longrightarrow 6345 \longrightarrow 6475 \longrightarrow 6785.$$

Note that now, each step of P' preserves the number of inversions. Moreover, the final configuration of P' is the same as the final configuration of P with all cards incremented by n, and of course thus has the same number of inversions. Boom.

2019 USAJMO Problems

Contents

Day 1

Note: For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

Problem 1

There are $a+b$ bowls arranged in a row, numbered 1 through $a+b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear.

A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$, provided that the difference $i-j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

Solution

Problem 2

Let $\mathbb{Z}$ be the set of all integers. Find all pairs of integers (a,b) for which there exist functions $f:\mathbb{Z}\rightarrow\mathbb{Z}$ and $g:\mathbb{Z}\rightarrow\mathbb{Z}$ satisfying

$$f(g(x))=x+a \quad\text{and}\quad g(f(x))=x+b$$

for all integers x.

Solution

Problem 3

$(*)$ Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2+BC^2=AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD=\angle BPC$. Show that line PE bisects $\overline{CD}$.

Solution

Day 2

Problem 4

$(*)$ Let ABC be a triangle with $\angle ABC$ obtuse. The A-*excircle* is a circle in the exterior of $\triangle ABC$ that is tangent to side $\overline{BC}$ of the triangle and tangent to the extensions of the other two sides. Let E, F be the feet of the altitudes from B and C to lines AC and AB, respectively. Can line EF be tangent to the A-excircle?

Solution

Problem 5

Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, ..., 2n\}$, for integers i, j with $0 \leq i, j \leq n$ such that:

- for all $0 \leq i, j \leq n$, the set $S_{i,j}$ has $i+j$ elements; and
- $S_{i,j} \subseteq S_{k,l}$ whenever $0 \leq i \leq k \leq n$ and $0 \leq j \leq l \leq n$

Solution

Problem 6

Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

Solution

2019 USAJMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=176&year=2019))	
Preceded by **2018 USAJMO**	Followed by **2020 USAJMO**
1 • 2 • 3 • 4 • 5 • 6	
All USAJMO Problems and Solutions	

JMO 2019 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

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Corrections and comments are welcome!

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§0 Problems

1. There are $a+b$ bowls arranged in a row, numbered 1 through $a+b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear. A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$, provided that the difference $i-j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

2. For which pairs of integers (a,b) do there exist functions $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ obeying
$$f(g(x)) = x+a \quad \text{and} \quad g(f(x)) = x+b$$
for all integers x?

3. Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

4. Let ABC be a triangle with $\angle B > 90^\circ$ and let E and F be the feet of the altitudes from B and C. Can line EF be tangent to the A-excircle?

5. Let n be a nonnegative integer. Determine the number of ways to choose sets $S_{ij} \subseteq \{1, 2, \dots, 2n\}$, for all $0 \le i \le n$ and $0 \le j \le n$ (not necessarily distinct), such that
   - $|S_{ij}| = i + j$, and
   - $S_{ij} \subseteq S_{kl}$ if $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

6. Let m and n be relatively prime positive integers. The numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{1}{2}(x+y)$ or their harmonic mean $\frac{2xy}{x+y}$. For which (m,n) can Evan write 1 on the board in finitely many steps?

§1 Solutions to Day 1

§1.1 JMO 2019/1, proposed by Jim Propp

Available online at https://aops.com/community/p12189456.

Problem statement

There are $a+b$ bowls arranged in a row, numbered 1 through $a+b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear. A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$, provided that the difference $i-j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

First we show that if ab is even then the goal is possible. We prove the result by induction on $a+b$.

- If $\min(a,b)=0$ there is nothing to check.
- If $\min(a,b)=1$, say $a=1$, then b is even, and we can swap the (only) leftmost apple with the rightmost pear by working only with those fruits.
- Now assume $\min(a,b)\ge 2$ and $a+b$ is odd. Then we can swap the leftmost apple with rightmost pear by working only with those fruits, reducing to the situation of $(a-1,b-1)$ which is possible by induction (at least one of them is even).
- Finally assume $\min(a,b)\ge 2$ and $a+b$ is even (i.e. a and b are both even). Then we can swap the apple in position 1 with the pear in position $a+b-1$, and the apple in position 2 with the pear in position $a+b$. This reduces to the situation of $(a-2,b-2)$ which is also possible by induction.

Now we show that the result is impossible if ab is odd. Define

$$\begin{aligned} X &= \text{number apples in odd-numbered bowls} \\ Y &= \text{number pears in odd-numbered bowls.} \end{aligned}$$

Note that $X-Y$ does not change under this operation. However, if a and b are odd, then we initially have $X=\frac{1}{2}(a+1)$ and $Y=\frac{1}{2}(b-1)$, while the target position has $X=\frac{1}{2}(a-1)$ and $Y=\frac{1}{2}(b+1)$. So when ab is odd this is not possible.

Remark. Another proof that ab must be even is as follows.

First, note that apples only move right and pears only move left, a successful operation must take exactly ab moves. So it is enough to prove that the *number of moves* made must be even.

However, the number of fruits in odd-numbered bowls either increases by $+2$ or -2 in each move (according to whether i and j are both even or both odd), and since it ends up being the same at the end, the number of moves must be even.

Alternatively, as pointed out in the official solutions, one can consider the sums of squares of positions of fruits. The quantity changes by

$$\left[(i+1)^2+(j-1)^2\right]-(i^2+j^2)=2(i-j)+2\equiv 2 \pmod 4$$

at each step, and eventually the sums of squares returns to zero, as needed.

§1.2 JMO 2019/2, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p12189493.

Problem statement

For which pairs of integers (a, b) do there exist functions $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ obeying

$$f(g(x)) = x + a \quad \text{and} \quad g(f(x)) = x + b$$

for all integers x?

The answer is if $a = b$ or $a = -b$. In the former case, one can take $f(x) \equiv x + a$ and $g(x) \equiv x$. In the latter case, one can take $f(x) \equiv -x + a$ and $g(x) = -x$.

Now we prove these are the only possibilities. First:

Claim — The functions f and g are bijections.

Proof. Surjectivity is obvious. To see injective, note that if $f(u) = f(v)$ then $g(f(u)) = g(f(v)) \implies u + b = v + b \implies u = v$, and similarly for g. □

Note also that for any x, we have

$$\begin{aligned} f(x + b) &= f(g(f(x))) = f(x) + a \\ g(x + a) &= g(f(g(x))) = g(x) + b. \end{aligned}$$

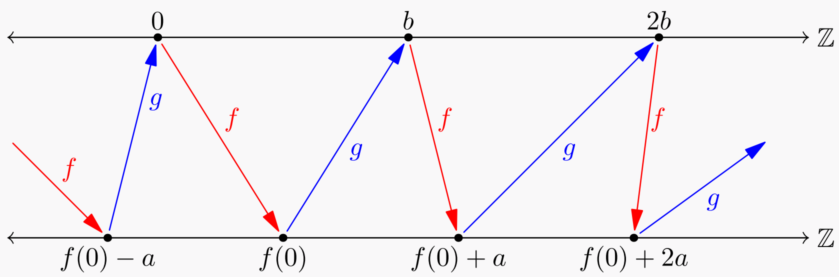
If either a is zero or b is zero, we immediately get the other is zero, and hence done. So assume $ab \neq 0$.

If $|b| > |a|$, then two of

$$\{f(0), f(1), \dots, f(b-1)\} \pmod{|a|}$$

coincide, which together with repeatedly applying the first equation above will then give a contradiction to injectivity of f. Similarly, if $|a| > |b|$ swapping the roles of f and g (and a and b) will give a contradiction to injectivity of g. This completes the proof.

Remark. Here is a way to visualize the argument, so one can see pictorially what is going on. We draw two parallel number lines indexed by $\mathbb{Z}$. Starting from 0, we draw red arrow from 0 to $f(0)$, and then a blue arrow from $f(0)$ to $g(f(0)) = b$, and then a red arrow from b to $g(b) = f(0) + a$, and so on. These arrows can be extended both directions, leading to an infinite "squaretooth" wave. The following is a picture of an example with $a, b > 0$.



The problem is essentially trying to decompose our two copies of $\mathbb{Z}$ into multiple squaretooth waves. We expect for this to be possible, the "width" of the waves on the top and bottom must be the same — i.e., that $|a| = |b|$.

Remark. This also suggests how to classify all functions f and g satisfying the condition. If $a = b = 0$ then any pair of functions f and g which are inverses to each other is okay. There are thus uncountably many pairs of functions (f, g) here.

If $a = b > 0$, then one sets $f(0)$, $f(1)$, $\dots$, $f(a-1)$ as any values which are distinct modulo b, at which point f and g are uniquely determined. An example for $a = b = 3$ is

$$f(x) = \begin{cases} x + 42 & x \equiv 0 \pmod 3 \\ x + 13 & x \equiv 1 \pmod 3 \\ x - 37 & x \equiv 2 \pmod 3, \end{cases} \qquad g(x) = \begin{cases} x - 39 & x \equiv 0 \pmod 3 \\ x + 40 & x \equiv 1 \pmod 3 \\ x - 10 & x \equiv 2 \pmod 3. \end{cases}$$

The analysis for $a = b < 0$ and $a = -b$ are similar, but we don't include the details here.

§1.3 JMO 2019/3, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p12189455.

Problem statement

Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

Here are three solutions. The first two are similar although the first one makes use of symmedians. The last solution by inversion is more advanced.

¶ First solution using symmedians We define point P to obey

$$\frac{AP}{BP} = \frac{AD^2}{BC^2} = \frac{AE^2}{BE^2}$$

so that $\overline{PE}$ is the E-symmedian of $\triangle EAB$, therefore the E-median of $\triangle ECD$.

Now, note that

$$AD^2 = AP \cdot AB \quad \text{and} \quad BC^2 = BP \cdot BA.$$

This implies $\triangle APD \sim \triangle ADB$ and $\triangle BPC \sim \triangle BCA$. Thus

$$\measuredangle DPA = \measuredangle ADB = \measuredangle ACB = \measuredangle BCP$$

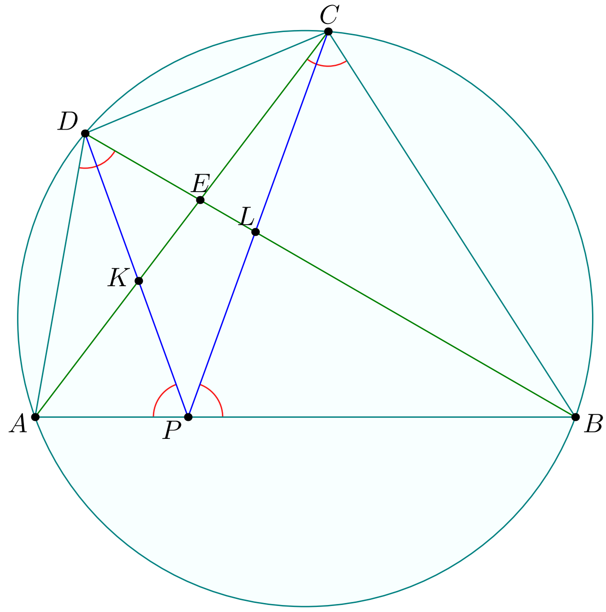
and so P satisfies the condition as in the statement (and is the unique point to do so), as needed.

¶ Second solution using only angle chasing (by proposer) We again re-define P to obey $AD^2 = AP \cdot AB$ and $BC^2 = BP \cdot BA$. As before, this gives $\triangle APD \sim \triangle ABD$ and $\triangle BPC \sim \triangle BDP$ and so we let

$$\theta := \measuredangle DPA = \measuredangle ADB = \measuredangle ACB = \measuredangle BCP.$$

Our goal is to now show $\overline{PE}$ bisects $\overline{CD}$.

Let $K = \overline{AC} \cap \overline{PD}$ and $L = \overline{AD} \cap \overline{PC}$. Since $\measuredangle KPA = \theta = \measuredangle ACB$, quadrilateral $BPKC$ is cyclic. Similarly, so is $APLD$.



Finally $AKLB$ is cyclic since

$$\measuredangle BKA = \measuredangle BKC = \measuredangle BPC = \theta = \measuredangle DPA = \measuredangle DLA = \measuredangle BLA.$$

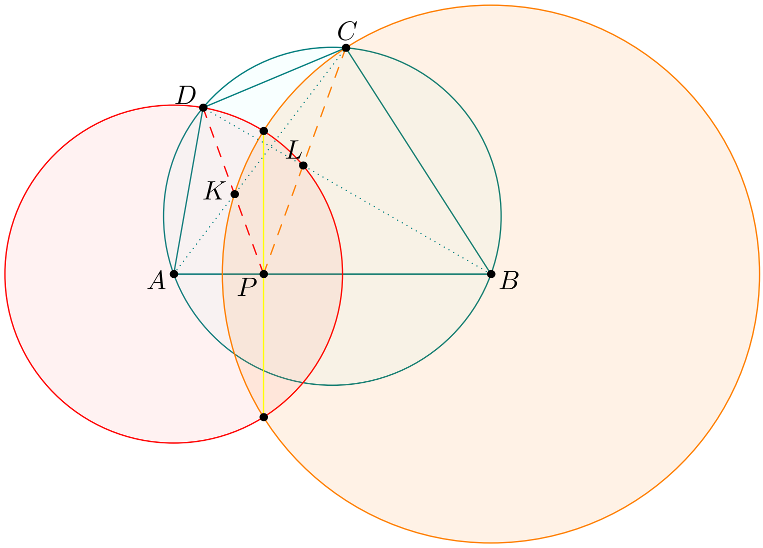
This implies $\measuredangle CKL = \measuredangle LBA = \measuredangle DCK$, so $\overline{KL} \parallel \overline{BC}$. Then PE bisects $\overline{BC}$ by Ceva's theorem on $\triangle PCD$.

¶ Third solution (using inversion) By hypothesis, the circle ω_a centered at A with radius AD is orthogonal to the circle ω_b centered at B with radius BC. For brevity, we let $\mathbf{I}_a$ and $\mathbf{I}_b$ denote inversion with respect to ω_a and ω_b.

We let P denote the intersection of $\overline{AB}$ with the radical axis of ω_a and ω_b; hence $P = \mathbf{I}_a(B) = \mathbf{I}_b(A)$. This already implies that

$$\measuredangle DPA \overset{\mathbf{I}_a}{=} \measuredangle ADB = \measuredangle ACB \overset{\mathbf{I}_b}{=} \measuredangle BPC$$

so P satisfies the angle condition.



Claim — The point $K = \mathbf{I}_a(C)$ lies on ω_b and $\overline{DP}$. Similarly $L = \mathbf{I}_b(D)$ lies on ω_a and $\overline{CP}$.

Proof. The first assertion follows from the fact that ω_b is orthogonal to ω_a. For the other, since (BCD) passes through A, it follows $P = \mathbf{I}_a(B)$, $K = \mathbf{I}_a(C)$, and $D = \mathbf{I}_a(D)$ are collinear. $\square$

Finally, since C, L, P are collinear, we get A is concyclic with $K = \mathbf{I}_a(C)$, $L = \mathbf{I}_a(L)$, $B = \mathbf{I}_a(B)$, i.e. that $AKLB$ is cyclic. So $\overline{KL} \parallel \overline{CD}$ by Reim's theorem, and hence $\overline{PE}$ bisects $\overline{CD}$ by Ceva's theorem.

§2 Solutions to Day 2

§2.1 JMO 2019/4, proposed by Ankan Bhattacharya, Zack Chroman, Anant Mudgal

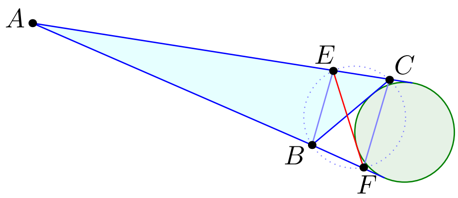
Available online at https://aops.com/community/p12195848.

Problem statement

Let ABC be a triangle with $\angle B > 90^\circ$ and let E and F be the feet of the altitudes from B and C. Can line EF be tangent to the A-excircle?

We show it is not possible, by contradiction (assuming EF is indeed tangent). Thus $BECF$ is a convex cyclic quadrilateral inscribed in a circle with diameter $\overline{BC}$. Note also that the A-excircle lies on the opposite side from A as line EF, since A, E, C are collinear in that order.

¶ First solution by similarity Note that $\triangle AEF$ is similar to $\triangle ABC$ (and oppositely oriented). However, since they have the same A-exradius, it follows they are congruent.



Consequently we get $EF = BC$. But this implies $BFCE$ is a rectangle, contradiction.

¶ Second length solution by tangent lengths By $t(\bullet)$ we mean the length of the tangent from P to the A-excircle. It is a classical fact for example that $t(A) = s$. The main idea is to use the fact that

$$a \cos A = EF = t(E) + t(F).$$

Here $EF = a \cos A$ follows from the extended law of sines applied to the circle with diameter $\overline{BC}$, since there we have $EF = BC \sin \angle ECF = a \sin \angle ACF = a \cos A$. We may now compute

$$\begin{aligned} t(E) &= t(A) - AE = s - c \cos A \\ t(F) &= t(A) - AF = s - b \cos A. \end{aligned}$$

Therefore,

$$\begin{aligned} a \cos A = 2s - (b+c) \cos A &\implies (a+b+c) \cos A = 2s \\ &\implies \cos A = 1. \end{aligned}$$

This is an obvious contradiction.

> **Remark.** On the other hand, there really is an *equality case* with A being some point at infinity (meaning $\cos A = 1$). So, this problem is "sharper" than one might expect; the answer is not "obviously no".

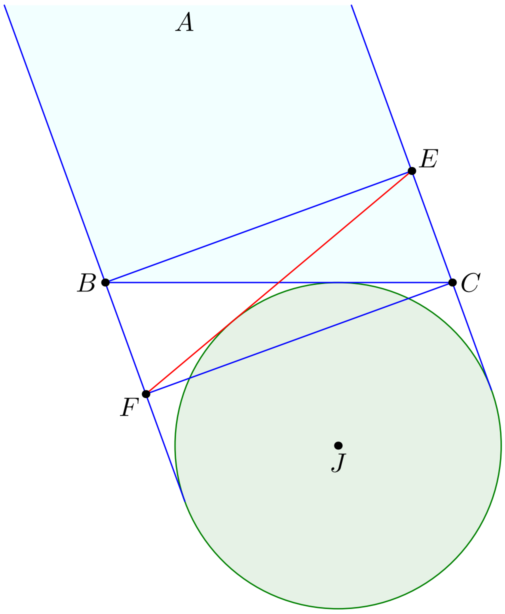
¶ Third solution by Pitot and trigonometry In fact, the $t(\bullet)$ notation from the previous solution gives us a classical theorem once we note the A-excircle is tangent to all four lines EF, BC, BF and CE:

Claim (Pitot theorem) — We have $BF + EF = BC + CE$.

Proof. Here is a proof for completeness. By $t(B)$ we mean the length of the tangent from B to the A-excircle, and define $t(C)$, $t(E)$, $t(F)$ similarly. Then

$$\begin{aligned} BF &= t(B) - t(F) & \qquad EF &= t(E) + t(F) \\ BC &= t(B) + t(C) & \qquad CE &= t(E) - t(C) \end{aligned}$$

and summing gives the result. □



We now calculate all the lengths using trigonometry:

$$\begin{aligned} BC &= a \\ BF &= a\cos(180^\circ - B) = a\cos(A + C) \\ CE &= a\cos C \\ EF &= BC\sin\angle ECF = a\sin\angle ACF = a\cos A. \end{aligned}$$

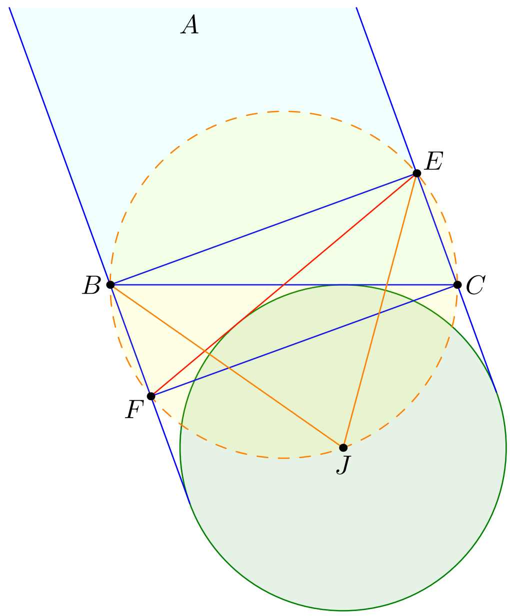
Thus, we apparently have

$$\cos(A + C) + \cos A = 1 + \cos C$$

but this is impossible since $\cos(A + C) < \cos C$ (since $A + C = 180 - B < 90^\circ$) and $\cos A < 1$.

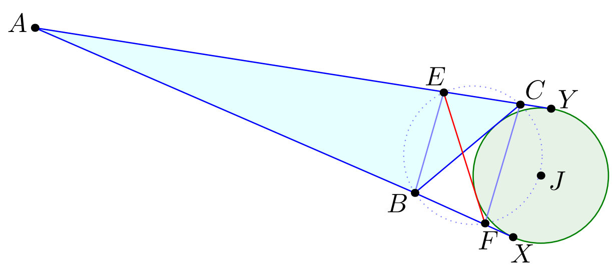
¶ Fourth solution by Pitot and Ptolemy (Evan Chen) We give a trig-free way to finish from Pitot's theorem
$$BF + EF = BC + CE.$$
Assume that $x = BF$, $y = CE$, and $BC = 1$; then the above relation becomes
$$1 + y - x = BC + CE - BF = EF = EF \cdot 1 = xy + \sqrt{(1-x^2)(1-y^2)}$$
with the last step by Ptolemy's theorem. This rearranges to give
$$(1+y)(1-x) = \sqrt{(1-x^2)(1-y^2)} \implies \frac{1+y}{1-y} = \frac{1+x}{1-x} \implies x = y$$
but that means $BECF$ is a rectangle: contradicting the fact that lines BE and CF meet at a point A.

¶ Fifth solution, by angle chasing only! Let J denote the A-excenter. Then J should be the intersection of the internal bisectors of $\angle FEC$ and $\angle FBC$, so it is the midpoint of arc $\widehat{FC}$ on the circle with diameter $\overline{BC}$.



But now we get $\angle BJC = 90^\circ$ from J lying on this circle. Yet $\angle BJC = 90^\circ - \frac{1}{2}\angle A$ in general, so $\angle A = 0^\circ$ which is impossible.

¶ Sixth solution (Zuming Feng) This is similar to the preceding solution, but phrased using contradiction and inequalities. We let X and Y denote the tangency points of the A-excircle on lines AB and AC. Moreover, let J denote the A-excenter.



Note that $AB > AE$ and $AX = AY$, therefore $BX < EY$. By considering the right triangles XBJ and YEJ (which both have $JX = JY$), we conclude $\tan\angle XBJ > \tan\angle YEJ$, thus

$$\angle XBJ > \angle YEJ.$$

However, if line EF was actually tangent to the A-excircle, we would have

$$2\angle XBJ = \angle XBC = \angle FBC = \angle FEC = \angle FEY = 2\angle JEY$$

which is a contradiction.

¶ Seventh solution, by complex numbers, for comedic effect (Evan Chen) Let us denote the tangency points of the A-excircle with sides BC, CA, AB as x, y, z. Assume moreover that line EF is tangent to the A-excircle at a point P.

Also, for brevity let $s = xy + yz + zx$. Then, we have

$$\begin{aligned} E = \frac{2py}{p+y} &= \frac{1}{2}(b + y + y - y^2\overline{b}) = \frac{zx}{z+x} + y - \frac{y^2}{z+x} \\ \implies \frac{2}{\frac{1}{p}+\frac{1}{y}} &= \frac{xy+xz+zx-y^2}{z+x} \implies \frac{\frac{1}{p}+\frac{1}{y}}{2} = \frac{x+z}{s-y^2}. \end{aligned}$$

Similarly by considering the point F,

$$\frac{\frac{1}{p}+\frac{1}{z}}{2} = \frac{x+y}{s-z^2}.$$

Thus we can eliminate P and obtain

$$\begin{aligned} \implies \frac{\frac{1}{y}-\frac{1}{z}}{2} &= \frac{x+z}{s-y^2} - \frac{x+y}{s-z^2} = \frac{-s(y-z)+x(y^2-z^2)+(y^3-z^3)}{(s-y^2)(s-z^2)} \\ \iff \frac{1}{2yz} &= \frac{s - x(y+z) - (y^2+yz+z^2)}{(s-y^2)(s-z^2)} = \frac{-(y^2+z^2)}{(s-y^2)(s-z^2)} \\ \iff 0 &= (s-y^2)(s-z^2) + 2yz(y^2+z^2) \\ &= [x(y+z)+y(z-y)]\,[x(y+z)+z(y-z)] + 2yz(y^2+z^2) \\ &= x^2(y+z)^2 - (y-z)^2 \cdot x(y+z) + yz(2y^2+2z^2-(y-z)^2) \\ &= x^2(y+z)^2 - (y-z)^2 \cdot x(y+z) + yz(y+z)^2 \\ &= xyz(y+z)\left[\frac{x}{y} + \frac{x}{z} - \frac{y}{z} - \frac{z}{y} + 2 + \frac{y}{x} + \frac{z}{x}\right]. \end{aligned}$$

However, $\triangle XYZ$ is obtuse with $\angle X > 90^\circ$, we have $y + z \neq 0$. Note that

$$\begin{aligned} \tfrac{x}{y} + \tfrac{y}{x} &= 2\operatorname{Re}\tfrac{x}{y} = 2\cos(2\angle XZY) \\ \tfrac{x}{z} + \tfrac{z}{x} &= 2\operatorname{Re}\tfrac{x}{z} = 2\cos(2\angle XYZ) \\ \tfrac{y}{z} + \tfrac{z}{y} &= 2\operatorname{Re}\tfrac{y}{z} < 2 \end{aligned}$$

and since $\cos(2\angle XZY) + \cos(2\angle XYZ) > 0$ (say by sum-to-product), we are done.

§2.2 JMO 2019/5, proposed by Ricky Liu

Available online at https://aops.com/community/p12195861.

Problem statement

Let n be a nonnegative integer. Determine the number of ways to choose sets $S_{ij} \subseteq \{1, 2, \dots, 2n\}$, for all $0 \le i \le n$ and $0 \le j \le n$ (not necessarily distinct), such that

- $|S_{ij}| = i + j$, and
- $S_{ij} \subseteq S_{kl}$ if $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

The answer is $(2n)! \cdot 2^{n^2}$. First, we note that $\varnothing = S_{00} \subsetneq S_{01} \subsetneq \dots \subsetneq S_{nn} = \{1, \dots, 2n\}$ and thus multiplying by $(2n)!$ we may as well assume $S_{0i} = \{1, \dots, i\}$ and $S_{in} = \{1, \dots, n+i\}$. We illustrate this situation by placing the sets in a grid, as below for $n = 4$; our goal is to fill in the rest of the grid.

$$\begin{bmatrix} 1234 & 12345 & 123456 & 1234567 & 12345678 \\ 123 & & & & \\ 12 & & & & \\ 1 & & & & \\ \varnothing & & & & \end{bmatrix}$$

We claim the number of ways to do so is 2^{n^2}. In fact, more strongly even the partial fillings are given exactly by powers of 2.

Claim — Fix a choice T of cells we wish to fill in, such that whenever a cell is in T, so are all the cells above and left of it. (In other words, T is a Young tableau.) The number of ways to fill in these cells with sets satisfying the inclusion conditions is $2^{|T|}$.

An example is shown below, with an indeterminate set marked in red (and the rest of T marked in blue).

$$\begin{bmatrix} 1234 & 12345 & 123456 & 1234567 & 12345678 \\ 123 & 1234 & 12346 & 123467 & \\ 12 & 124 & 1234 \text{ or } 1246 & & \\ 1 & 12 & & & \\ \varnothing & 2 & & & \end{bmatrix}$$

Proof. The proof is by induction on $|T|$, with $|T| = 0$ being vacuous.

Now suppose we have a corner $\begin{bmatrix} B & C \\ A & S \end{bmatrix}$ where A, B, C are fixed and S is to be chosen. Then we may write $B = A \cup \{x\}$ and $C = A \cup \{x, y\}$ for $x, y \notin A$. Then the two choices of S are $A \cup \{x\}$ (i.e. B) and $A \cup \{y\}$, and both of them are seen to be valid.

In this way, we gain a factor of 2 any time we add one cell as above to T. Since we can achieve any Young tableau in this way, the induction is complete. $\square$

§2.3 JMO 2019/6, proposed by Yannick Yao

Available online at https://aops.com/community/p12195834.

Problem statement

Let m and n be relatively prime positive integers. The numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{1}{2}(x+y)$ or their harmonic mean $\frac{2xy}{x+y}$. For which (m,n) can Evan write 1 on the board in finitely many steps?

We claim this is possible if and only $m+n$ is a power of 2. Let $q = m/n$, so the numbers on the board are q and $1/q$.

Impossibility: The main idea is the following.

Claim — Suppose p is an odd prime. Then if the initial numbers on the board are $-1 \pmod p$, then all numbers on the board are $-1 \pmod p$.

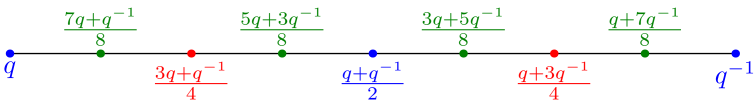
Proof. Let $a \equiv b \equiv -1 \pmod p$. Note that $2 \not\equiv 0 \pmod p$ and $a+b \equiv -2 \not\equiv 0 \pmod p$. Thus $\frac{a+b}{2}$ and $\frac{2ab}{a+b}$ both make sense modulo p and are equal to $-1 \pmod p$. $\square$

Thus if there exists *any* odd prime divisor p of $m+n$ (implying $p \nmid mn$), then

$$q \equiv \frac{1}{q} \equiv -1 \pmod p.$$

and hence all numbers will be $-1 \pmod p$ forever. This implies that it's impossible to write 1, whenever $m+n$ is divisible by some odd prime.

Construction: Conversely, suppose $m+n$ is a power of 2. We will actually construct 1 without even using the harmonic mean.



Note that

$$\frac{n}{m+n} \cdot q + \frac{m}{m+n} \cdot \frac{1}{q} = 1$$

and obviously by taking appropriate midpoints (in a binary fashion) we can achieve this using arithmetic mean alone.

2020 USOJMO Problems

Contents

Day 1

Note: For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

Problem 1

Let $n \geq 2$ be an integer. Carl has n books arranged on a bookshelf. Each book has a height and a width. No two books have the same height, and no two books have the same width. Initially, the books are arranged in increasing order of height from left to right. In a move, Carl picks any two adjacent books where the left book is wider and shorter than the right book, and swaps their locations. Carl does this repeatedly until no further moves are possible. Prove that regardless of how Carl makes his moves, he must stop after a finite number of moves, and when he does stop, the books are sorted in increasing order of width from left to right.

Solution

Problem 2

Let ω be the incircle of a fixed equilateral triangle ABC. Let ℓ be a variable line that is tangent to ω and meets the interior of segments BC and CA at points P and Q, respectively. A point R is chosen such that $PR = PA$ and $QR = QB$. Find all possible locations of the point R, over all choices of ℓ.

Solution

Problem 3

An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A beam is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions:

- The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.)
- No two beams have intersecting interiors.
- The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam.

What is the smallest positive number of beams that can be placed to satisfy these conditions?

Solution

Day 2

Problem 4

Let $ABCD$ be a convex quadrilateral inscribed in a circle and satisfying $DA < AB = BC < CD$. Points E and F are chosen on sides CD and AB such that $BE \perp AC$ and $EF \parallel BC$. Prove that $FB = FD$.

Solution

Problem 5

Suppose that $(a_1, b_1), (a_2, b_2), \ldots, (a_{100}, b_{100})$ are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \leq i < j \leq 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

Solution

Problem 6

Let $n \geq 2$ be an integer. Let $P(x_1, x_2, \ldots, x_n)$ be a nonconstant n-variable polynomial with real coefficients. Assume that whenever $r_1, r_2, \ldots, r_n$ are real numbers, at least two of which are equal, we have $P(r_1, r_2, \ldots, r_n) = 0$. Prove that $P(x_1, x_2, \ldots, x_n)$ cannot be written as the sum of fewer than $n!$ monomials. (A monomial is a polynomial of the form $cx_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}$, where c is a nonzero real number and $d_1, d_2, \ldots, d_n$ are nonnegative integers.)

Solution

2020 USAJMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=176&year={{{year}}}))	
Preceded by **2019 USAJMO**	Followed by **2021 USAJMO**
1 • 2 • 3 • 4 • 5 • 6	
All USAJMO Problems and Solutions	

JMO 2020 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2020 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let $n \geq 2$ be an integer. Carl has n books arranged on a bookshelf. Each book has a height and a width. No two books have the same height, and no two books have the same width.

   Initially, the books are arranged in increasing order of height from left to right. In a *move*, Carl picks any two adjacent books where the left book is wider and shorter than the right book, and swaps their locations. Carl does this repeatedly until no further moves are possible.

   Prove that regardless of how Carl makes his moves, he must stop after a finite number of moves, and when he does stop, the books are sorted in increasing order of width from left to right.

2. Let ω be the incircle of a fixed equilateral triangle ABC. Let ℓ be a variable line that is tangent to ω and meets the interior of segments BC and CA at points P and Q, respectively. A point R is chosen such that $PR = PA$ and $QR = QB$. Find all possible locations of the point R, over all choices of ℓ.

3. An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A *beam* is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions:

   - The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.)
   - No two beams have intersecting interiors.
   - The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam.

   What is the smallest positive number of beams that can be placed to satisfy these conditions?

4. Let $ABCD$ be a convex quadrilateral inscribed in a circle and satisfying

   $$DA < AB = BC < CD.$$

   Points E and F are chosen on sides CD and AB such that $\overline{BE} \perp \overline{AC}$ and $\overline{EF} \parallel \overline{BC}$. Prove that $FB = FD$.

5. Suppose that (a_1, b_1), (a_2, b_2), $\dots$, (a_{100}, b_{100}) are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \leq i < j \leq 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

6. Let $n \geq 2$ be an integer. Let $P(x_1, x_2, \dots, x_n)$ be a nonconstant n-variable polynomial with real coefficients. Assuming that P vanishes whenever two of its arguments are equal, prove that P has at least $n!$ terms.

§1 Solutions to Day 1

§1.1 JMO 2020/1, proposed by Milan Haiman

Available online at https://aops.com/community/p15952780.

Problem statement

Let $n \geq 2$ be an integer. Carl has n books arranged on a bookshelf. Each book has a height and a width. No two books have the same height, and no two books have the same width.

Initially, the books are arranged in increasing order of height from left to right. In a *move*, Carl picks any two adjacent books where the left book is wider and shorter than the right book, and swaps their locations. Carl does this repeatedly until no further moves are possible.

Prove that regardless of how Carl makes his moves, he must stop after a finite number of moves, and when he does stop, the books are sorted in increasing order of width from left to right.

We say that a pair of books (A, B) is *height-inverted* if A is to the left of B and taller than A. Similarly define *width-inverted* pairs.

Note that every operation decreases the number of width-inverted pairs. This proves the procedure terminates, since the number of width-inverted pairs starts at $\binom{n}{2}$ and cannot increase indefinitely.

Now consider a situation where no more moves are possible. Assume for contradiction two consecutive books (A, B) are still width-inverted. Since the operation isn't possible anymore, they are also height-inverted. In particular, the operation could never have swapped A and B. But this contradicts the assumption there were no height-inverted pairs initially.

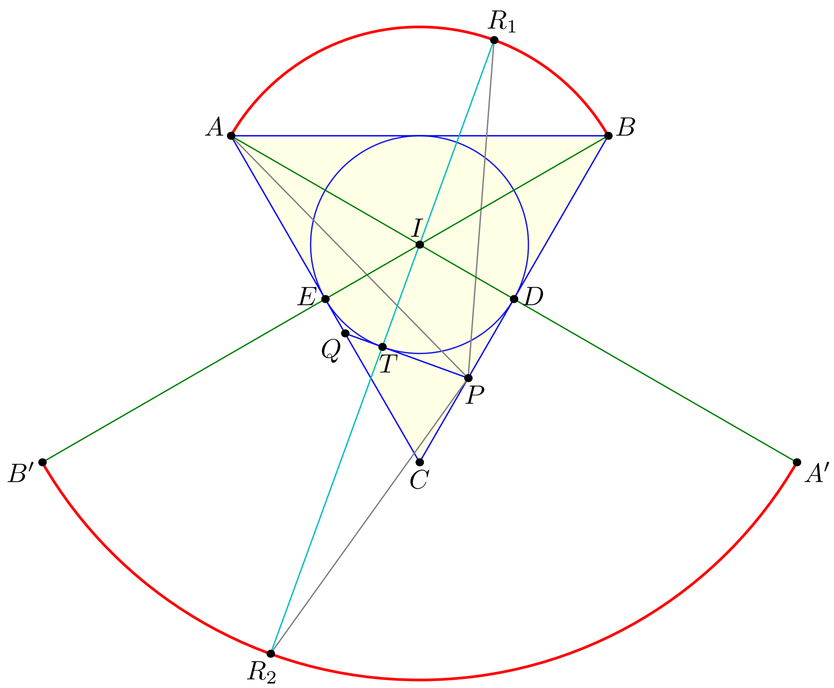
§1.2 JMO 2020/2, proposed by Titu Andreescu, Waldemar Pompe

Available online at https://aops.com/community/p15952801.

Problem statement

Let ω be the incircle of a fixed equilateral triangle ABC. Let ℓ be a variable line that is tangent to ω and meets the interior of segments BC and CA at points P and Q, respectively. A point R is chosen such that $PR = PA$ and $QR = QB$. Find all possible locations of the point R, over all choices of ℓ.

Let r be the inradius. Let T be the tangency point of $\overline{PQ}$ on arc $\widehat{DE}$ of the incircle, which we consider varying. We define R_1 and R_2 to be the two intersections of the circle centered at P with radius PA, and the circle centered at Q with radius QB. We choose R_1 to lie on the opposite side of C as line PQ.



Claim — The point R_1 is the unique point on ray TI with $R_1I = 2r$.

Proof. Define S to be the point on ray TI with $SI = 2r$. Note that there is a homothety at I which maps $\triangle DTE$ to $\triangle ASB$, for some point S.

Note that since $TASD$ is an isosceles trapezoid, it follows $PA = PS$. Similarly, $QB = QS$. So it follows that $S = R_1$. $\square$

Since T can be any point on the open arc $\widehat{DE}$, it follows that the locus of R_1 is exactly the open 120° arc of $\widehat{AB}$ of the circle centered at I with radius $2r$ (i.e. the circumcircle of ABC).

It remains to characterize R_2. Since $TI = r$, $IR_1 = 2r$, it follows $TR_2 = 3r$ and $IR_2 = 4r$. Define A' on ray DI such that $A'I = 4r$, and B' on ray IE such that $B'I = 4r$. Then it follows, again by homothety, that the locus of R_2 is the 120° arc $\widehat{A'B'}$ of the circle centered at I with radius $4r$.

In conclusion, the locus of R is the two open 120° arcs we identified.

§1.3 JMO 2020/3, proposed by Alex Zhai

Available online at https://aops.com/community/p15952773.

Problem statement

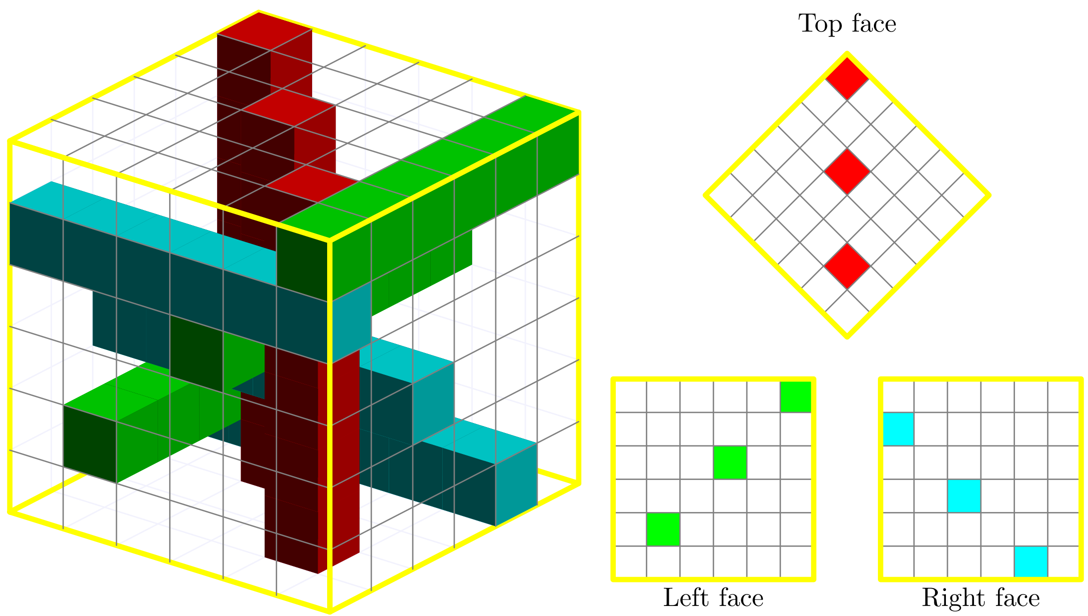
An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A *beam* is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions:

- The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.)
- No two beams have intersecting interiors.
- The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam.

What is the smallest positive number of beams that can be placed to satisfy these conditions?

Answer: 3030 beams.

Construction: We first give a construction with $3n/2$ beams for any $n \times n \times n$ box, where n is an even integer. Shown below is the construction for $n = 6$, which generalizes. (The left figure shows the cube in 3d; the right figure shows a direct view of the three visible faces.)



To be explicit, impose coordinate axes such that one corner of the cube is the origin. We specify a beam by two opposite corners. The $3n/2$ beams come in three directions, $n/2$ in each direction:

- $(0, 0, 0) \to (1, 1, n)$, $(2, 2, 0) \to (3, 3, n)$, $(4, 4, 0) \to (5, 5, n)$, and so on;
- $(1, 0, 0) \to (2, n, 1)$, $(3, 0, 2) \to (4, n, 3)$, $(5, 0, 4) \to (6, n, 5)$, and so on;

- $(0, 1, 1) \to (n, 2, 2)$, $(0, 3, 3) \to (n, 4, 4)$, $(0, 5, 5) \to (n, 6, 6)$, and so on.

This gives the figure we drew earlier and shows 3030 beams is possible.

Necessity: We now show at least $3n/2$ beams are necessary. Maintain coordinates, and call the beams x-beams, y-beams, z-beams according to which plane their long edges are perpendicular too. Let N_x, N_y, N_z be the number of these.

Claim — If $\min(N_x, N_y, N_z) = 0$, then at least n^2 beams are needed.

Proof. Assume WLOG that $N_z = 0$. Orient the cube so the z-plane touches the ground. Then each of the n layers of the cube (from top to bottom) must be completely filled, and so at least n^2 beams are necessary, □

We henceforth assume $\min(N_x, N_y, N_z) > 0$.

Claim — If $N_z > 0$, then we have $N_x + N_y \geq n$.

Proof. Again orient the cube so the z-plane touches the ground. We see that for each of the n layers of the cube (from top to bottom), there is at least one x-beam or y-beam. (Pictorially, some of the x and y beams form a "staircase".) This completes the proof. □

Proceeding in a similar fashion, we arrive at the three relations

$$\begin{aligned} N_x + N_y &\geq n \\ N_y + N_z &\geq n \\ N_z + N_x &\geq n. \end{aligned}$$

Summing gives $N_x + N_y + N_z \geq 3n/2$ too.

Remark. The problem condition has the following "physics" interpretation. Imagine the cube is a metal box which is sturdy enough that all beams must remain orthogonal to the faces of the box (i.e. the beams cannot spin). Then the condition of the problem is exactly what is needed so that, if the box is shaken or rotated, the beams will not move.

Remark. Walter Stromquist points out that the number of constructions with 3030 beams is actually enormous: not dividing out by isometries, the number is $(2 \cdot 1010!)^3$.

§2 Solutions to Day 2

§2.1 JMO 2020/4, proposed by Milan Haiman

Available online at https://aops.com/community/p15952890.

Problem statement

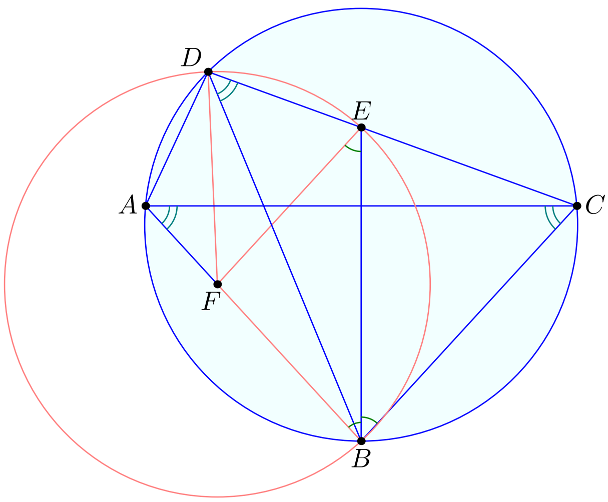
Let $ABCD$ be a convex quadrilateral inscribed in a circle and satisfying

$$DA < AB = BC < CD.$$

Points E and F are chosen on sides CD and AB such that $\overline{BE} \perp \overline{AC}$ and $\overline{EF} \parallel \overline{BC}$. Prove that $FB = FD$.

We present three approaches. We note that in the second two approaches, the result remains valid even if $AB \neq BC$, as long E is replaced by the point on $\overline{AC}$ satisfying $EA = EC$. So the result is actually somewhat more general.

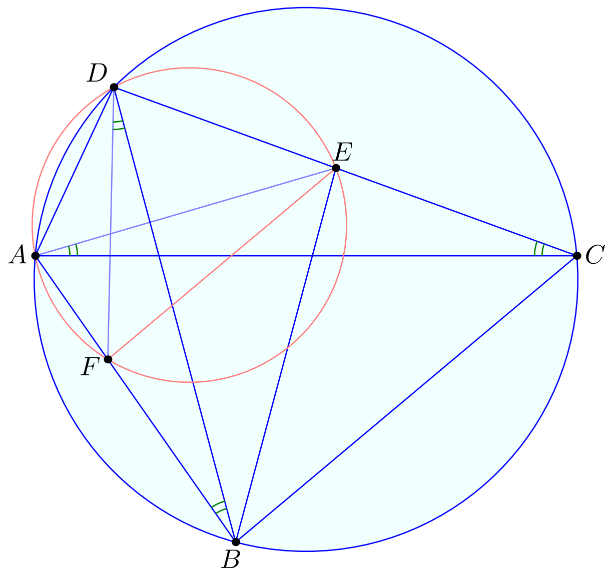
¶ First solution by inscribed angle theorem Since $\overline{EF} \parallel \overline{BC}$ we may set $\theta = \angle FEB = \angle CBE = \angle EBF$. This already implies $FE = FB$, so we will in fact prove that F is the circumcenter of $\triangle BED$.



Note that $\angle BDC = \angle BAC = 90^\circ - \theta$. However, $\angle BFE = 180^\circ - 2\theta$. So by the inscribed angle theorem, D lies on the circle centered at F with radius $FE = FB$, as desired.

> **Remark.** Another approach to the given problem is to show that B is the D-excenter of $\triangle DAE$, and F is the arc midpoint of $\widehat{DAE}$ of the circumcircle of $\triangle DAE$. In my opinion, this approach is much clumsier.

¶ Second general solution by angle chasing By Reim's theorem, $AFED$ is cyclic.

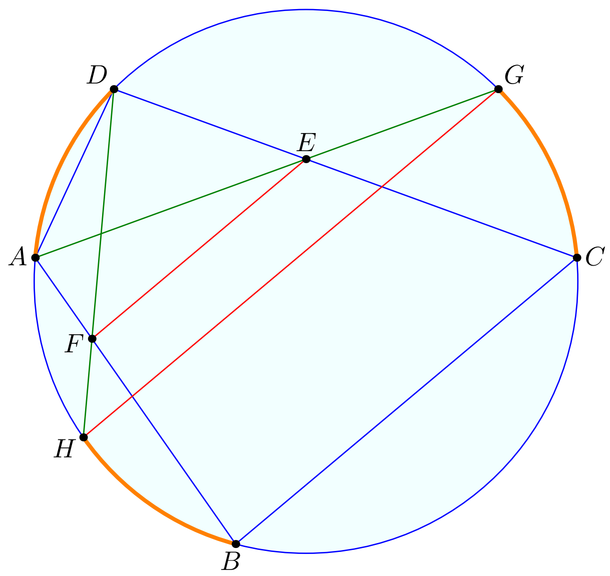


Hence

$$\begin{aligned}\measuredangle FDB &= \measuredangle FDC - \measuredangle BDC = \measuredangle FAE - \measuredangle FAC \\ &= \measuredangle CAE = \measuredangle ECA = \measuredangle DCA = \measuredangle DBA = \measuredangle DBF\end{aligned}$$

as desired.

¶ Third general solution by Pascal Extend rays AE and DF to meet the circumcircle again at G and H. By Pascal's theorem on $HDCBAG$, it follows that E, F, and $GH \cap BC$ are collinear, which means that $\overline{EF} \parallel \overline{GH} \parallel \overline{BC}$.



Since $EA = EC$, it follows $DAGC$ in isosceles trapezoid. But also $GHBC$ is an isosceles trapezoid. Thus $\text{m}\widehat{DA} = \text{m}\widehat{GC} = \text{m}\widehat{BH}$, so $DAHB$ is an isosceles trapezoid. Thus $FD = FB$.

> **Remark.** Addicts of projective geometry can use Pascal on $DBCAHG$ to finish rather than noting the equal arcs.

§2.2 JMO 2020/5, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p15952792.

Problem statement

Suppose that (a_1, b_1), (a_2, b_2), ..., (a_{100}, b_{100}) are distinct ordered pairs of non-negative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \le i < j \le 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

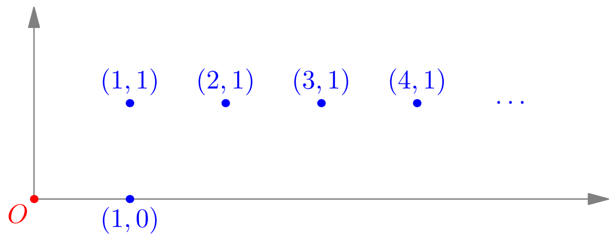
The answer is 197. In general, if 100 is replaced by $n \ge 2$ the answer is $2n - 3$.

The idea is that if we let $P_i = (a_i, b_i)$ be a point in the coordinate plane, and let $O = (0, 0)$ then we wish to maximize the number of triangles $\triangle OP_iP_j$ which have area $1/2$. Call such a triangle *good*.

Construction of 197 points: It suffices to use the points $(1, 0)$, $(1, 1)$, $(2, 1)$, $(3, 1)$, ..., $(99, 1)$ as shown. Notice that:

- There are 98 good triangles with vertices $(0, 0)$, $(k, 1)$ and $(k+1, 1)$ for $k = 1, \dots, 98$.
- There are 99 good triangles with vertices $(0, 0)$, $(1, 0)$ and $(k, 1)$ for $k = 1, \dots, 99$.

This is a total of $98 + 99 = 197$ triangles.

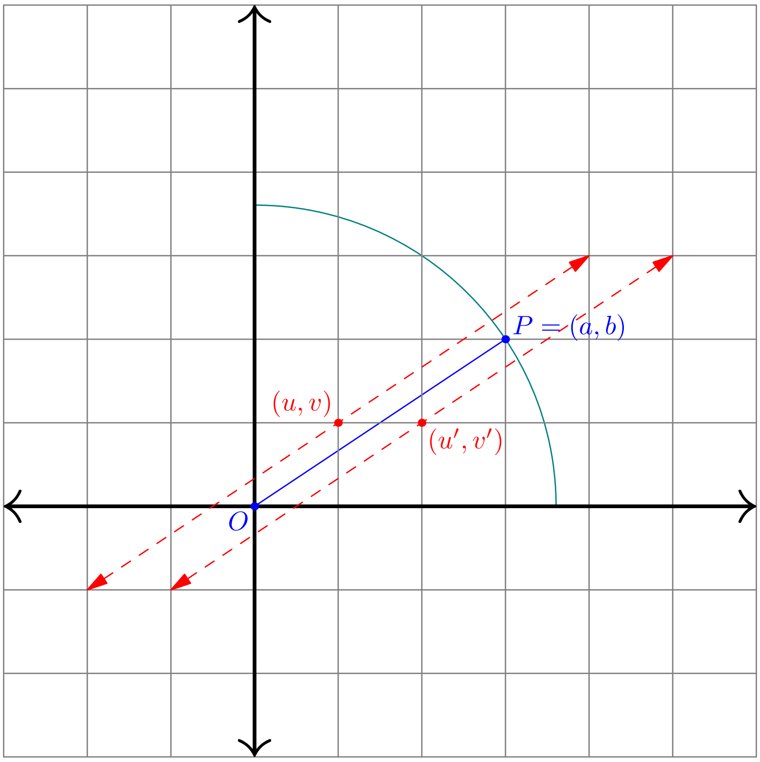


Proof that 197 points is optimal: We proceed by induction on n to show the bound of $2n - 3$. The base case $n = 2$ is evident.

For the inductive step, suppose (without loss of generality) that the point $P = P_n = (a, b)$ is the farthest away from the point O among all points.

Claim — This farthest point $P = P_n$ is part of at most two good triangles.

Proof. We must have $\gcd(a, b) = 1$ for P to be in any good triangles at all, since otherwise any divisor of $\gcd(a, b)$ also divides $2[OPQ]$. Now, we consider the locus of all points Q for which $[OPQ] = 1/2$. It consists of two parallel lines passing with slope OP, as shown.



Since $\gcd(a, b) = 1$, see that only two lattice points on this locus actually lie inside the quarter-circle centered at O with radius OP. Indeed if one of the points is (u, v) then the others on the line are $(u \pm a, v \pm b)$ where the signs match. This proves the claim. $\square$

This claim allows us to complete the induction by simply deleting P_n.

§2.3 JMO 2020/6, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p15952921.

Problem statement

Let $n \geq 2$ be an integer. Let $P(x_1, x_2, \dots, x_n)$ be a nonconstant n-variable polynomial with real coefficients. Assuming that P vanishes whenever two of its arguments are equal, prove that P has at least $n!$ terms.

We present two solutions.

¶ First solution using induction (by Ankan) Begin with the following observation:

Claim — Let $1 \leq i < j \leq n$. There is no term of P which omits both x_i and x_j.

Proof. Note that P ought to become identically zero if we set $x_i = x_j = 0$, since it is zero for any choice of the remaining $n-2$ variables, and the base field $\mathbb{R}$ is infinite. $\square$

Remark (Technical warning for experts)**.** The fact we used is not true if $\mathbb{R}$ is replaced by a field with finitely many elements, such as $\mathbb{F}_p$, even with one variable. For example the one-variable polynomial $X^p - X$ vanishes on every element of $\mathbb{F}_p$, by Fermat's little theorem.

We proceed by induction on $n \geq 2$ with the base case $n = 2$ being clear. Assume WLOG P is not divisible by any of x_1, ..., x_n, since otherwise we may simply divide out this factor. Now for the inductive step, note that

- The polynomial $P(0, x_2, x_3, \dots, x_n)$ obviously satisfies the inductive hypothesis and is not identically zero since $x_1 \nmid P$, so it has at least $(n-1)!$ terms.
- Similarly, $P(x_1, 0, x_3, \dots, x_n)$ also has at least $(n-1)!$ terms.
- Similarly, $P(x_1, x_2, 0, \dots, x_n)$ also has at least $(n-1)!$ terms.
- ...and so on.

By the claim, all the terms obtained in this way came from different terms of the original polynomial P. Therefore, P itself has at least $n \cdot (n-1)! = n!$ terms.

Remark. Equality is achieved by the Vandermonde polynomial $P = \prod_{1 \leq i < j \leq n}(x_i - x_j)$.

¶ Second solution using Vandermonde polynomial (by Yang Liu) Since $x_i - x_j$ divides P for any $i \neq j$, it follows that P should be divisible by the Vandermonde polynomial

$$V = \prod_{i<j}(x_j - x_i) = \sum_{\sigma} \operatorname{sgn}(\sigma) x_1^{\sigma(0)} x_2^{\sigma(1)} \dots x_n^{\sigma(n-1)}$$

where the sum runs over all permutations σ on $\{0, \dots, n-1\}$.

Consequently, we may write

$$P = \sum_\sigma \operatorname{sgn}(\sigma) x_1^{\sigma(0)} x_2^{\sigma(1)} \dots x_n^{\sigma(n-1)} Q$$

The main idea is that each of the $n!$ terms of the above sum has a monomial not appearing in any of the other terms.

As an example, consider $x_1^{n-1} x_2^{n-2} \dots x_{n-1}^1 x_n^0$. Among all monomial in Q, consider the monomial $x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$ with the largest e_1, then largest e_2, ... (In other words, take the lexicographically largest $(e_1, \dots, e_n)$.) This term

$$x_1^{e_1+(n-1)} x_2^{e_2+(n-2)} \dots x_n^{e_n}$$

can't appear anywhere else because it is strictly lexicographically larger than any other term appearing in any other expansion.

Repeating this argument with every σ gives the conclusion.

2021 USAJMO Problems

Contents

Day 1

$\textbf{Note:}$ For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

Problem 1

Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for positive integers a and b,

$$f(a^2 + b^2) = f(a)f(b) \text{ and } f(a^2) = f(a)^2.$$

Solution

Problem 2

Rectangles $BCC_1B_2,$ $CAA_1C_2,$ and ABB_1A_2 are erected outside an acute triangle $ABC.$ Suppose that

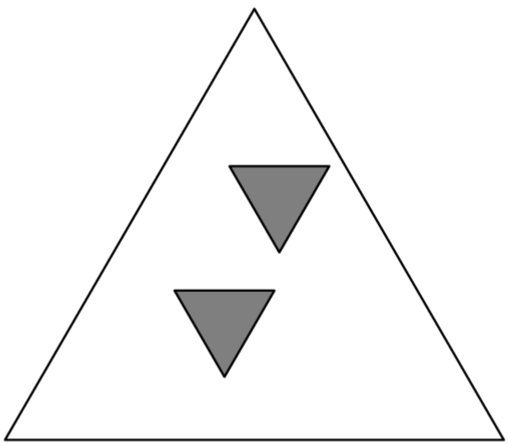
$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^{\circ}.$$

Prove that lines $B_1C_2,$ $C_1A_2,$ and A_1B_2 are concurrent.

Solution

Problem 3

An equilateral triangle Δ of side length $L > 0$ is given. Suppose that n equilateral triangles with side length 1 and with non-overlapping interiors are drawn inside Δ, such that each unit equilateral triangle has sides parallel to Δ, but with opposite orientation. (An example with $n = 2$ is drawn below.)



Prove that

$$n \leq \frac{2}{3}L^2.$$

Solution

Day 2

Problem 4

Carina has three pins, labeled A, B, and C, respectively, located at the origin of the coordinate plane. In a move, Carina may move a pin to an adjacent lattice point at distance 1 away. What is the least number of moves that Carina can make in order for triangle ABC to have area 2021?

(A lattice point is a point (x, y) in the coordinate plane where x and y are both integers, not necessarily positive.)

Solution

Problem 5

A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s, t) = d$. (The elements s and t could be equal.) Given this information, find all possible values for the number of elements of S.

Solution

Problem 6

Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\
a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\
&\vdots & & \\
a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1
\end{aligned}$$

Solution

2021 USAJMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=176&year={{{year}}}))	
Preceded by **2020 USOJMO**	Followed by **2022 USAJMO**
1 • 2 • 3 • 4 • 5 • 6	
All USAJMO Problems and Solutions	

JMO 2021 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2021 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfy $f(a^2+b^2) = f(a)f(b)$ and $f(a^2) = f(a)^2$ for all positive integers a and b.

2. Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that
$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$
Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

3. An equilateral triangle Δ of side length $L > 0$ is given. Suppose that n equilateral triangles with side length 1 and with non-overlapping interiors are drawn inside Δ, such that each unit equilateral triangle has sides parallel to Δ, but with opposite orientation. Prove that
$$n \le \frac{2}{3}L^2.$$

4. Carina has three pins, labeled A, B, and C, respectively, located at the origin of the coordinate plane. In a *move*, Carina may move a pin to an adjacent lattice point at distance 1 away. What is the least number of moves that Carina can make in order for triangle ABC to have area 2021?

5. A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

   Given this information, find all possible values for the number of elements of S.

6. Let $n \ge 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:
$$\begin{aligned} a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\ a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\ a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\ &\vdots & &\vdots \\ a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1. \end{aligned}$$

§1 Solutions to Day 1

§1.1 JMO 2021/1, proposed by Vincent Huang

Available online at https://aops.com/community/p21498724.

Problem statement

Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfy $f(a^2+b^2) = f(a)f(b)$ and $f(a^2) = f(a)^2$ for all positive integers a and b.

The answer is $f \equiv 1$ only, which works. We prove it's the only one.

The bulk of the problem is:

Claim — If $f(a) = f(b) = 1$ and $a > b$, then $f(a^2-b^2) = f(2ab) = 1$.

Proof. Write

$$\begin{aligned} 1 = f(a)f(b) = f(a^2+b^2) &= \sqrt{f\left((a^2+b^2)^2\right)} \\ &= \sqrt{f\left((a^2-b^2)^2+(2ab)^2\right)} \\ &= \sqrt{f(a^2-b^2)f(2ab)}. \end{aligned}$$ □

By setting $a = b = 1$ in the given statement we get $f(1) = f(2) = 1$. Now a simple induction on n shows $f(n) = 1$:

- If $n = 2k$ take $(u,v) = (k,1)$ hence $2uv = n$.
- If $n = 2k+1$ take $(u,v) = (k+1,k)$ hence $u^2 - v^2 = n$.

§1.2 JMO 2021/2, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p21498558.

Problem statement

Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

The angle condition implies the circumcircles of the three rectangles concur at a single point P. Then $\measuredangle CPB_2 = \measuredangle CPA_1 = 90^\circ$, hence P lies on A_1B_2 etc., so we're done.

> **Remark.** As one might guess from the two-sentence solution, the entire difficulty of the problem is getting the characterization of the concurrence point.

§1.3 JMO 2021/3, proposed by Alex Zhai

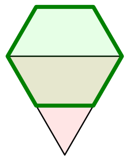
Available online at https://aops.com/community/p21499596.

Problem statement

An equilateral triangle Δ of side length $L > 0$ is given. Suppose that n equilateral triangles with side length 1 and with non-overlapping interiors are drawn inside Δ, such that each unit equilateral triangle has sides parallel to Δ, but with opposite orientation. Prove that

$$n \le \frac{2}{3}L^2.$$

We present the approach of Andrew Gu. For each triangle, we draw a green regular hexagon of side length 1/2 as shown below.



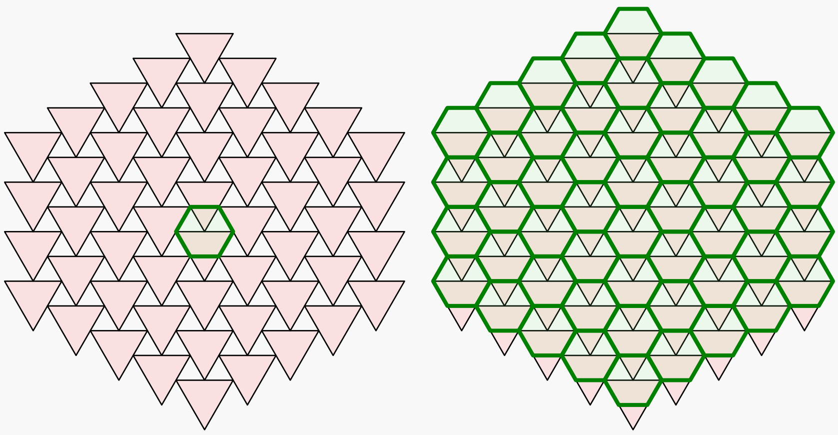
Claim — All the hexagons are disjoint and lie inside Δ.

Proof. Annoying casework. $\square$

Since each hexagon has area $\frac{3\sqrt{3}}{8}$ and lies inside Δ, we conclude

$$\frac{3\sqrt{3}}{8} \cdot n \le \frac{\sqrt{3}}{4}L^2 \implies n \le \frac{2}{3}L^2.$$

Remark. The constant $\frac{2}{3}$ is sharp and cannot be improved. The following tessellation shows how to achieve the $\frac{2}{3}$ density. In the figure on the left, one of the green hexagons is drawn in for illustration. The version on the right has all the hexagons.



§2 Solutions to Day 2

§2.1 JMO 2021/4, proposed by Brandon Wang

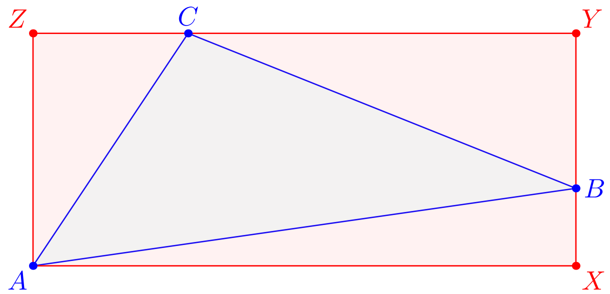
Available online at https://aops.com/community/p21498566.

Problem statement

Carina has three pins, labeled A, B, and C, respectively, located at the origin of the coordinate plane. In a *move*, Carina may move a pin to an adjacent lattice point at distance 1 away. What is the least number of moves that Carina can make in order for triangle ABC to have area 2021?

The answer is 128.

Define the **bounding box** of triangle ABC to be the smallest axis-parallel rectangle which contains all three of the vertices A, B, C.



Lemma

The area of a triangle ABC is at most half the area of the bounding box.

Proof. This can be proven by explicit calculation in coordinates. Nonetheless, we outline a geometric approach. By considering the smallest/largest x coordinate and the smallest/largest y coordinate, one can check that some vertex of the triangle must coincide with a corner of the bounding box (there are four "extreme" coordinates across the $3 \cdot 2 = 6$ coordinates of our three points).

So, suppose the bounding box is $AXYZ$. Imagine fixing C and varying B along the perimeter entire rectangle. The area is a linear function of B, so the maximal area should be achieved when B coincides with one of the vertices $\{A, X, Y, Z\}$. But obviously the area of $\triangle ABC$ is

- exactly 0 if $B = A$,
- at most half the bounding box if $B \in \{X, Z\}$ by one-half-base-height,
- at most half the bounding box if $B = Y$, since $\triangle ABC$ is contained inside either $\triangle AYZ$ or $\triangle AXZ$. $\square$

We now proceed to the main part of the proof.

Claim — If n moves are made, the bounding box has area at most $(n/2)^2$. (In other words, a bounding box of area A requires at least $\left\lceil 2\sqrt{A} \right\rceil$ moves.)

Proof. The sum of the width and height of the bounding box increases by at most 1 each move, hence the width and height have sum at most n. So, by AM-GM, their product is at most $(n/2)^2$. $\square$

This immediately implies $n \geq 128$, since the bounding box needs to have area at least $4042 > 63.5^2$.

On the other hand, if we start all the pins at the point $(3, 18)$ then we can reach the following three points in 128 moves:

$$\begin{aligned} A &= (0, 0) \\ B &= (64, 18) \\ C &= (3, 64) \end{aligned}$$

and indeed triangle ABC has area exactly 2021.

§2.2 JMO 2021/5, proposed by Carl Schildkraut

Available online at https://aops.com/community/p21498580.

Problem statement

A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

The answer is that $|S|$ must be a power of 2 (including 1), or $|S| = 0$ (a trivial case we do not discuss further).

Construction: For any nonnegative integer k, a construction for $|S| = 2^k$ is given by

$$S = \{(p_1 \text{ or } q_1) \times (p_2 \text{ or } q_2) \times \cdots \times (p_k \text{ or } q_k)\}$$

for $2k$ distinct primes $p_1, \dots, p_k, q_1, \dots, q_k$.

Converse: the main claim is as follows.

Claim — In any valid set S, for any prime p and $x \in S$, $\nu_p(x) \le 1$.

Proof. Assume for contradiction $e = \nu_p(x) \ge 2$.

- On the one hand, by taking x in the statement, we see $\frac{e}{e+1}$ of the elements of S are divisible by p.
- On the other hand, consider a $y \in S$ such that $\nu_p(y) = 1$ which must exist (say if $\gcd(x,y) = p$). Taking y in the statement, we see $\frac{1}{2}$ of the elements of S are divisible by p.

So $e = 1$, contradiction. $\square$

Now since $|S|$ equals the number of divisors of any element of S, we are done.

§2.3 JMO 2021/6, proposed by Mohsen Jamaali

Available online at https://aops.com/community/p21498967.

Problem statement

Let $n \ge 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned} a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\ a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\ a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\ &\vdots & &\vdots \\ a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1. \end{aligned}$$

The answer is that the only solution is $(1, 2, 1, 2, \dots, 1, 2)$ which works.

We will prove a_{2k} is a constant sequence, at which point the result is obvious.

¶ First approach (Andrew Gu) Apparently, with indices modulo $2n$, we should have

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}}$$

for every index k (this eliminates all a_{odd}'s). Define

$$m = \min_k a_{2k} \qquad \text{and} \qquad M = \max_k a_{2k}.$$

Look at the indices i and j achieving m and M to respectively get

$$m = \frac{2}{m} + \frac{1}{a_{2i-2}} + \frac{1}{a_{2i+2}} \ge \frac{2}{m} + \frac{1}{M} + \frac{1}{M} = \frac{2}{m} + \frac{2}{M}$$
$$M = \frac{2}{M} + \frac{1}{a_{2j-2}} + \frac{1}{a_{2j+2}} \le \frac{2}{M} + \frac{1}{m} + \frac{1}{m} = \frac{2}{m} + \frac{2}{M}.$$

Together this gives $m \ge M$, so $m = M$. That means a_{2i} is constant as i varies, solving the problem.

¶ Second approach (author's solution) As before, we have

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}}$$

The proof proceeds in three steps.

- Define
$$S = \sum_k a_{2k}, \quad \text{and} \quad T = \sum_k \frac{1}{a_{2k}}.$$
Summing gives $S = 4T$. On the other hand, Cauchy-Schwarz says $S \cdot T \ge n^2$, so $T \ge \frac{1}{2}n$.

- On the other hand,
$$1 = \frac{1}{a_{2k-2}a_{2k}} + \frac{2}{a_{2k}^2} + \frac{1}{a_{2k}a_{2k+2}}$$
Sum this modified statement to obtain
$$n = \sum_k \left(\frac{1}{a_{2k}} + \frac{1}{a_{2k+2}} \right)^2 \overset{\text{QM-AM}}{\geq} \frac{1}{n} \left(\sum_k \frac{1}{a_{2k}} + \frac{1}{a_{2k+2}} \right)^2 = \frac{1}{n} (2T)^2$$
So $T \leq \frac{1}{2}n$.
- Since $T \leq \frac{1}{2}n$ and $T \geq \frac{1}{2}n$, we must have equality everywhere above. This means a_{2k} is a constant sequence.

Remark. The problem is likely intractable over $\mathbb{C}$, in the sense that one gets a high-degree polynomial which almost certainly has many complex roots. So it seems likely that most solutions must involve some sort of inequality, using the fact we are over $\mathbb{R}_{>0}$ instead.

2022 USAJMO Problems

Contents

Day 1

$\textbf{Note:}$ For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

Problem 1

For which positive integers m does there exist an infinite arithmetic sequence of integers $a_1, a_2, \cdots$ and an infinite geometric sequence of integers $g_1, g_2, \cdots$ satisfying the following properties?

• $a_n - g_n$ is divisible by m for all integers $n > 1$;

• $a_2 - a_1$ is not divisible by m.

Solution

Problem 2

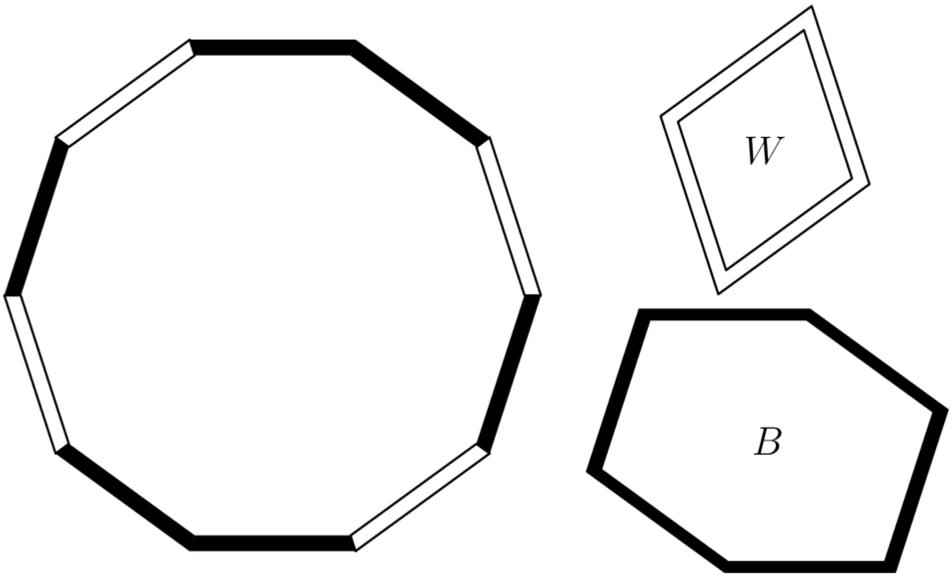
Let a and b be positive integers. The cells of an $(a+b+1) \times (a+b+1)$ grid are colored amber and bronze such that there are at least $a^2 + ab - b$ amber cells and at least $b^2 + ab - a$ bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a + b$ chosen cells lie in the same row or column.

Solution

Problem 3

Let $b \geq 2$ and $w \geq 2$ be fixed integers, and $n = b + w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

We assemble a regular $2n$-gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b = 3$ and $w = 2$ is shown below, as well as the resulting polygons B and W.



Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

Solution

Day 2

Problem 4

$(*)$ Let $ABCD$ be a rhombus, and let K and L be points such that K lies inside the rhombus, L lies outside the rhombus, and $KA = KB = LC = LD$. Prove that there exist points X and Y on lines AC and BD such that $KXLY$ is also a rhombus.

Solution

Problem 5

Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.

Solution

Problem 6

Let a_0, b_0, c_0 be complex numbers, and define

$$a_{n+1} = a_n^2 + 2b_nc_n$$

$$b_{n+1} = b_n^2 + 2c_na_n$$

$$c_{n+1} = c_n^2 + 2a_nb_n$$

for all nonnegative integers n.

Suppose that $\max(|a_n|, |b_n|, |c_n|) \leq 2022$ for all n. Prove that

$$|a_0|^2 + |b_0|^2 + |c_0|^2 \leq 1.$$

Solution

2022 USAJMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=176&year={{{year}}}))	
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JMO 2022 Solution Notes

EVAN CHEN《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2022 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

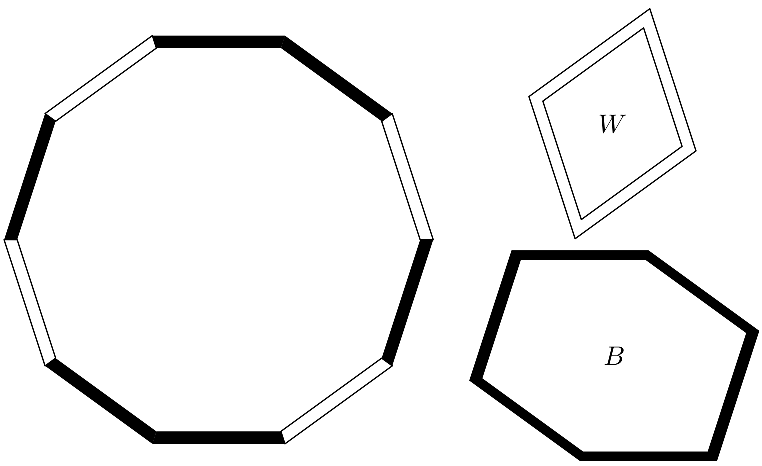
§0 Problems

1. For which positive integers m does there exist an infinite sequence in $\mathbb{Z}/m\mathbb{Z}$ which is both an arithmetic progression and a geometric progression, but is nonconstant?

2. Let a and b be positive integers. Every cell of an $(a+b+1) \times (a+b+1)$ grid is colored either amber or bronze such that there are at least a^2+ab-b amber cells and at least b^2+ab-a bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a+b$ chosen cells lie in the same row or column.

3. Let $b \geq 2$ and $w \geq 2$ be fixed integers, and $n = b + w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

   We assemble a regular $2n$-gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b=3$ and $w=2$ is shown below, as well as the resulting polygons B and W.



   Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

4. Let $ABCD$ be a rhombus, and let K and L be points such that K lies inside the rhombus, L lies outside the rhombus, and $KA = KB = LC = LD$. Prove that there exist points X and Y on lines AC and BD such that $KXLY$ is also a rhombus.

5. Find all pairs of primes (p, q) for which $p-q$ and $pq-q$ are both perfect squares.

6. Let a_0, b_0, c_0 be complex numbers, and define

$$\begin{aligned} a_{n+1} &= a_n^2 + 2b_nc_n \\ b_{n+1} &= b_n^2 + 2c_na_n \\ c_{n+1} &= c_n^2 + 2a_nb_n \end{aligned}$$

   for all nonnegative integers n. Suppose that $\max\{|a_n|, |b_n|, |c_n|\} \leq 2022$ for all $n \geq 0$. Prove that

$$|a_0|^2 + |b_0|^2 + |c_0|^2 \leq 1.$$

§1 Solutions to Day 1

§1.1 JMO 2022/1, proposed by Holden Mui

Available online at https://aops.com/community/p24774800.

Problem statement

For which positive integers m does there exist an infinite sequence in $\mathbb{Z}/m\mathbb{Z}$ which is both an arithmetic progression and a geometric progression, but is nonconstant?

Answer: m must *not* be squarefree.

The problem is essentially asking when there exists a nonconstant arithmetic progression in $\mathbb{Z}/m\mathbb{Z}$ which is also a geometric progression. Now,

- If m is squarefree, then consider three $(s-d, d, s+d)$ in arithmetic progression. It's geometric if and only if $d^2 = (s-d)(s+d) \pmod m$, meaning $d^2 \equiv 0 \pmod m$. Then $d \equiv 0 \pmod m$. So any arithmetic progression which is also geometric is constant in this case.
- Conversely if $p^2 \mid m$ for some prime p, then any arithmetic progression with common difference m/p is geometric by the same calculation.

§1.2 JMO 2022/2, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p24774812.

Problem statement

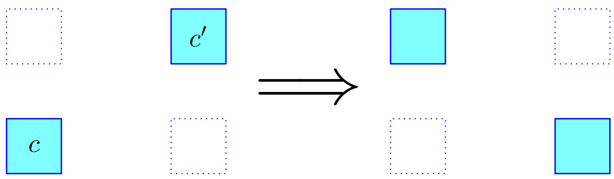
Let a and b be positive integers. Every cell of an $(a+b+1)\times(a+b+1)$ grid is colored either amber or bronze such that there are at least a^2+ab-b amber cells and at least b^2+ab-a bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a+b$ chosen cells lie in the same row or column.

Claim — There exists a transversal T_a with at least a amber cells. Analogously, there exists a transversal T_b with at least b bronze cells.

Proof. If one picks a random transversal, the expected value of the number of amber cells is at least

$$\frac{a^2+ab-b^2}{a+b+1} = (a-1) + \frac{1}{a+b+1} > a-1. \qquad \square$$

Now imagine we transform T_a to T_b in some number of steps, by repeatedly choosing cells c and c' and swapping them with the two other corners of the rectangle formed by their row/column, as shown in the figure.



By "discrete intermediate value theorem", the number of amber cells will be either a or $a+1$ at some point during this transformation. This completes the proof.

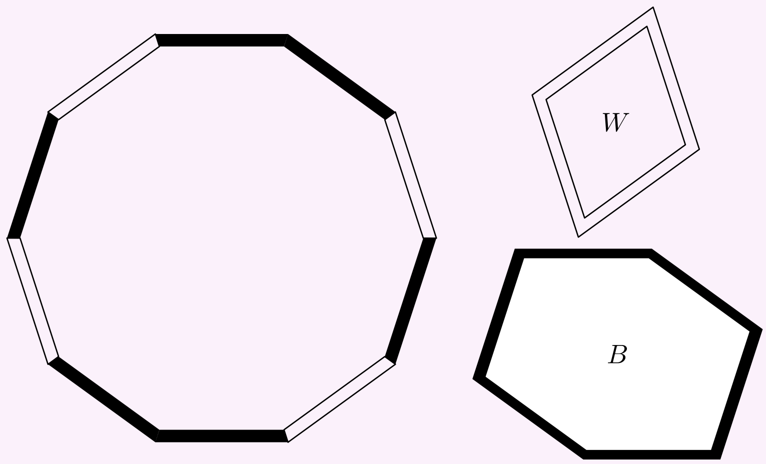
§1.3 JMO 2022/3, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p24775345.

Problem statement

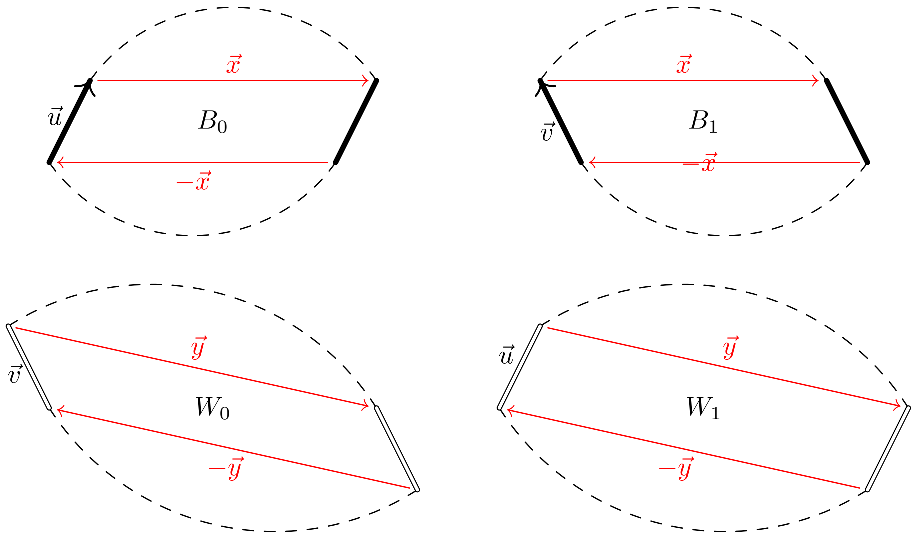
Let $b \geq 2$ and $w \geq 2$ be fixed integers, and $n = b + w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

We assemble a regular $2n$-gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b = 3$ and $w = 2$ is shown below, as well as the resulting polygons B and W.



Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

We are going to prove that one may swap a black rod with an adjacent white rod (as well as the rods parallel to them) without affecting the difference in the areas of $B - W$. Let $\vec{u}$ and $\vec{v}$ denote the originally black and white vectors that were adjacent on the $2n$-gon and are now going to be swapped. Let $\vec{x}$ denote the sum of all the other black vectors between $\vec{u}$ and $-\vec{u}$, and define $\vec{y}$ similarly. See the diagram below, where B_0 and W_0 are the polygons before the swap, and B_1 and W_1 are the resulting changed polygons.



Observe that the only change in B and W is in the parallelograms shown above in each diagram. Letting $\wedge$ denote the wedge product, we need to show that

$$\vec{u} \wedge \vec{x} - \vec{v} \wedge \vec{y} = \vec{v} \wedge \vec{x} - \vec{u} \wedge \vec{y}$$

which can be rewritten as

$$(\vec{u} - \vec{v}) \wedge (\vec{x} + \vec{y}) = 0.$$

In other words, it would suffice to show $\vec{u} - \vec{v}$ and $\vec{x} + \vec{y}$ are parallel. (Students not familiar with wedge products can replace every $\wedge$ with the cross product $\times$ instead.)

Claim — Both $\vec{u} - \vec{v}$ and $\vec{x} + \vec{y}$ are perpendicular to vector $\vec{u} + \vec{v}$.

Proof. We have $(\vec{u} - \vec{v}) \perp (\vec{u} + \vec{v})$ because $\vec{u}$ and $\vec{v}$ are the same length.

For the other perpendicularity, note that $\vec{u} + \vec{v} + \vec{x} + \vec{y}$ traces out a diameter of the circumcircle of the original $2n$-gon; call this diameter AB, so

$$A + \vec{u} + \vec{v} + \vec{x} + \vec{y} = B.$$

Now point $A + \vec{u} + \vec{v}$ is a point on this semicircle, which means (by the inscribed angle theorem) the angle between $\vec{u} + \vec{v}$ and $\vec{x} + \vec{y}$ is $90°$. $\square$

§2 Solutions to Day 2

§2.1 JMO 2022/4, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p24774800.

Problem statement

Let $ABCD$ be a rhombus, and let K and L be points such that K lies inside the rhombus, L lies outside the rhombus, and $KA = KB = LC = LD$. Prove that there exist points X and Y on lines AC and BD such that $KXLY$ is also a rhombus.

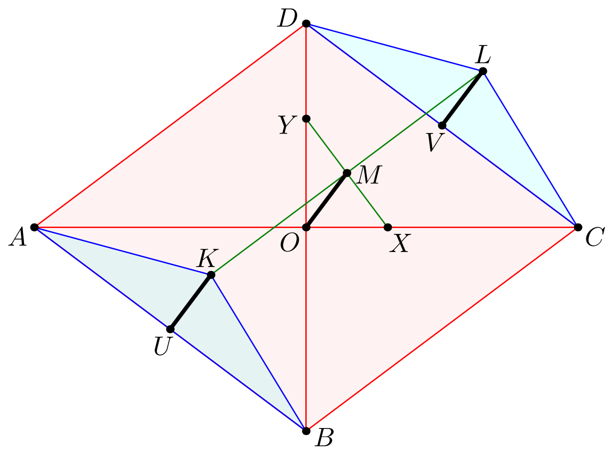
To start, notice that $\triangle AKB \cong \triangle DLC$ by SSS. Then by the condition K lies inside the rhombus while L lies outside it, we find that the two congruent triangles are just translations of each other (i.e. they have the same orientation).

¶ First solution Let M be the midpoint of $\overline{KL}$ and is O the center of the rhombus.

Claim — $\overline{MO} \perp \overline{AB}$.

Proof. Let U and V denote the midpoint of $\overline{AB}$ and $\overline{CD}$ respectively. Then $\overline{KU}$ and $\overline{LV}$ are obviously translates, and perpendicular to $\overline{AB} \parallel \overline{CD}$. Since M is the midpoint of $\overline{KL}$ and O is the midpoint of $\overline{UV}$, the result follows. □

We choose X and Y to be the intersections of the perpendicular bisector of $\overline{KL}$ with $\overline{AC}$ and $\overline{BD}$.



Claim — The midpoint of $\overline{XY}$ coincides with the midpoint of $\overline{KL}$.

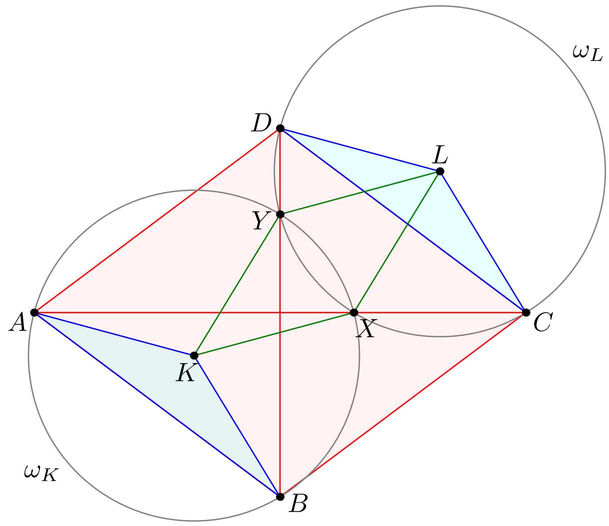
Proof. Because

$$\overline{XY} \perp \overline{KL} \parallel \overline{BC}$$
$$\overline{MO} \perp \overline{AB}$$
$$\overline{BD} \perp \overline{AC}$$

it follows that $\triangle MOY$, which was determined by the three lines $\overline{XY}$, $\overline{MO}$, $\overline{BD}$, is similar to $\triangle ABC$. In particular, it is isosceles with $MY = MO$. Analogously, $MX = MO$. $\square$

> **Remark.** It is also possible to simply use coordinates to prove both claims.

¶ Second solution (author's solution) In this solution, we instead define X and Y as the intersections of the circles centered at K and L of equal radii KA, which will be denoted ω_K and ω_L. It is clear that $KXLY$ is a rhombus under this construction, so it suffices to show that X and Y lie on AC and BD (in some order).



To see this, let $\overline{AC}$ meet ω_K again at X'. We have

$$\measuredangle CXD = \measuredangle BXC = \measuredangle AXB = \frac{1}{2}\,\mathrm{m}\,\widehat{AB} = \mathrm{m}\,\widehat{CD}$$

where the arcs are directed modulo 360°; here $\widehat{AB}$ is the arc of ω_K cut out by $\measuredangle AXB$, and $\widehat{DC}$ is the analogous arc of ω_L. This implies X' lies on ω_L by the inscribed angle theorem. Hence $X = X'$, and it follows X lies on $\overline{AC}$.

Analogously Y lies on BD.

> **Remark.** The angle calculation above can also be replaced with a length calculation, as follows.
>
> Let M and N be the projections of K and L onto $\overline{AC}$, respectively. Then X' is the reflection of A across M; analogously, the second intersection X'' with $\overline{AC}$ should be the reflection of C across N. So to get $X = X' = X''$, we would need to show $AC = 2MN$.
>
> However, note that $AKLD$ is a parallelogram. As MN was the projection of $\overline{KL}$ onto $\overline{AC}$, its length should be the same as the projection of $\overline{AD}$ onto $\overline{AC}$, which is obviously $\frac{1}{2}AC$ because the projection of D onto $\overline{AC}$ is exactly the midpoint of $\overline{AC}$ (i.e. the center of the rhombus).

§2.2 JMO 2022/5, proposed by Holden Mui

Available online at https://aops.com/community/p24774670.

Problem statement

Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.

The answer is $(3, 2)$ only.

Set

$$\begin{aligned} a^2 &= p - q \\ b^2 &= pq - q. \end{aligned}$$

Note that $0 < a < p$, and $0 < b < p$ (because $q \le p$). Now subtracting gives

$$\underbrace{(b-a)}_{<p}\underbrace{(b+a)}_{<2p} = b^2 - a^2 = p(q-1)$$

The inequalities above now force $b + a = p$. Hence $q - 1 = b - a$.

This means p and $q - 1$ have the same parity, which can only occur if $q = 2$. Finally, taking mod 3 shows $p \equiv 0 \pmod 3$. So $(3, 2)$ is the only possibility (and it does work).

§2.3 JMO 2022/6, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p24775314.

Problem statement

Let a_0, b_0, c_0 be complex numbers, and define

$$\begin{aligned} a_{n+1} &= a_n^2 + 2b_n c_n \\ b_{n+1} &= b_n^2 + 2c_n a_n \\ c_{n+1} &= c_n^2 + 2a_n b_n \end{aligned}$$

for all nonnegative integers n. Suppose that $\max\{|a_n|, |b_n|, |c_n|\} \le 2022$ for all $n \ge 0$. Prove that

$$|a_0|^2 + |b_0|^2 + |c_0|^2 \le 1.$$

For brevity, set $s_n := |a_n|^2 + |b_n|^2 + |c_n|^2$. Note that the s_n are real numbers.

Claim (Key miraculous identity) — We have

$$s_{n+1} - s_n^2 = 2|a_n\overline{b_n} + b_n\overline{c_n} + c_n\overline{a_n}|^2.$$

Proof. We prove this by mechanical calculation. First,

$$\begin{aligned} s_{n+1} &= \left|a_n^2 + 2b_nc_n\right|^2 + \left|b_n^2 + 2c_na_n\right|^2 + \left|c_n^2 + 2a_nb_n\right|^2 \\ &= \sum_{\text{cyc}} \left|a_n^2 + 2b_nc_n\right|^2 \\ &= \sum_{\text{cyc}} (a_n^2 + 2b_nc_n)(\overline{a_n}^2 + 2\overline{b_n}\,\overline{c_n}) \\ &= \sum_{\text{cyc}} \left(|a_n|^4 + 2\overline{a_n}^2 b_nc_n + 2a_n^2\overline{b_n}\,\overline{c_n} + 4|b_n|^2|c_n|^2\right) \\ &= s_n^2 + 2\sum_{\text{cyc}} \left(\overline{a_n}^2 b_nc_n + a_n^2\overline{b_n}\,\overline{c_n} + |b_n|^2|c_n|^2\right). \end{aligned}$$

Meanwhile,

$$\begin{aligned} |a_n\overline{b_n} + b_n\overline{c_n} + c_n\overline{a_n}|^2 &= \left(a_n\overline{b_n} + b_n\overline{c_n} + c_n\overline{a_n}\right)\left(\overline{a_n}b_n + \overline{b_n}c_n + \overline{c_n}a_n\right) \\ &= |a_n|^2|b_n^2| + a_n\overline{b_n}^2 c_n + a_n^2\overline{b_n}\,\overline{c_n} \\ &\quad + \overline{a_n}b_n^2\overline{c_n} + |b_n|^2|c_n|^2 + a_nb_n\overline{c_n}^2 \\ &\quad + \overline{a_n}^2 b_nc_n + \overline{a_n}\,\overline{b_n}c_n^2 + |a_n|^2|c_n|^2 \end{aligned}$$

which exactly matches the earlier sum, term for term. □

In particular, $s_{n+1} \ge s_n^2$, so applying repeatedly,

$$s_n \ge s_0^{2^n}.$$

Hence if $s_0 > 1$, it follows s_n is unbounded, contradicting $\max\{|a_n|, |b_n|, |c_n|\} \le 2022$.

Remark. The originally intended solution was to capture all three recursions in the following way. First, change the recursion to

$$\begin{aligned} a_{n+1} &= a_n^2 + 2b_nc_n \\ c_{n+1} &= b_n^2 + 2c_na_n \\ b_{n+1} &= c_n^2 + 2a_nb_n \end{aligned}$$

which is OK because we are just rearranging the terms in each triple. Then if ω is any complex number with $\omega^3 = 1$, and we define

$$z_n := a_n + b_n\omega + c_n\omega^2,$$

the recursion amounts to saying that $z_{n+1} = z_n^2$. This allows us to analyze $|z_n|$ in a similar way as above, as now $|z_n| = |z_0|^{2^n}$.

2023 USAJMO Problems

Contents

Day 1

Problem 1

Find all triples of positive integers (x, y, z) that satisfy the equation

$$2(x+y+z+2xyz)^2 = (2xy+2yz+2zx+1)^2 + 2023$$

Problem 2

In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

Problem 3

Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2-1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes. Find the maximum value of $k(C)$ as a function of n.

Day 2

Problem 4

Two players, B and R, play the following game on an infinite grid of unit squares, all initially colored white. The players take turns starting with B. On B's turn, B selects one white unit square and colors it blue. On R's turn, R selects two white unit squares and colors them red. The players alternate until B decides to end the game. At this point, B gets a score, given by the number of unit squares in the largest (in terms of area) simple polygon containing only blue unit squares. What is the largest score B can guarantee?

(A simple polygon is a polygon (not necessarily convex) that does not intersect itself and has no holes.

Problem 5

A positive integer a is selected, and some positive integers are written on a board. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n+a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.

After analyzing the integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

Problem 6

Isosceles triangle ABC, with $AB = AC$, is inscribed in circle ω. Let D be an arbitrary point inside BC such that $BD \neq DC$. Ray AD intersects ω again at E (other than A). Point F (other than E) is chosen on ω such that $\angle DFE = 90^\circ$. Line FE intersects rays AB and AC at points X and Y, respectively. Prove that $\angle XDE = \angle EDY$.

2023 USAJMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=176&year=2023))	
Preceded by **2022 USAJMO**	Followed by **2024 USAJMO**
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All USAJMO Problems and Solutions	

JMO 2023 Solution Notes

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This is a compilation of solutions for the 2023 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Find all triples of positive integers (x, y, z) satisfying

$$2(x + y + z + 2xyz)^2 = (2xy + 2yz + 2zx + 1)^2 + 2023.$$

2. In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

3. Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2 - 1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes.

   Find the maximum possible value of $k(C)$ as a function of n.

4. Two players, Blake and Ruby, play the following game on an infinite grid of unit squares, all initially colored white. The players take turns starting with Blake. On Blake's turn, Blake selects one white unit square and colors it blue. On Ruby's turn, Ruby selects two white unit squares and colors them red. The players alternate until Blake decides to end the game. At this point, Blake gets a score, given by the number of unit squares in the largest (in terms of area) simple polygon containing only blue unit squares.

   What is the largest score Blake can guarantee?

5. Positive integers a and N are fixed, and N positive integers are written on a blackboard. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n + a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.

   After analyzing the N integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these N integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

6. Isosceles triangle ABC, with $AB = AC$, is inscribed in circle ω. Let D be an arbitrary point inside BC such that $BD \neq DC$. Ray AD intersects ω again at E (other than A). Point F (other than E) is chosen on ω such that $\angle DFE = 90°$. Line FE intersects rays AB and AC at points X and Y, respectively. Prove that $\angle XDE = \angle EDY$.

§1 Solutions to Day 1

§1.1 JMO 2023/1, proposed by Titu Andreescu

Available online at https://aops.com/community/p27349258.

Problem statement

Find all triples of positive integers (x, y, z) satisfying

$$2(x+y+z+2xyz)^2 = (2xy+2yz+2zx+1)^2 + 2023.$$

Answer: $(3, 3, 2)$ and permutations.

The solution hinges upon the following claim:

Claim — The identity

$$2(x+y+z+2xyz)^2 - (2xy+2yz+2zx+1)^2 = (2x^2-1)(2y^2-1)(2z^2-1)$$

is true.

Proof. This can be proved by manually expanding; we show where it "came from". In algebraic number theory, there is a *norm function* $\operatorname{Norm}\colon \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}$ defined by

$$\operatorname{Norm}(a+b\sqrt{2}) = a^2 - 2b^2$$

which is *multiplicative*, meaning

$$\operatorname{Norm}(u \cdot v) = \operatorname{Norm}(u) \cdot \operatorname{Norm}(v).$$

This means that for any rational numbers x, y, z, we should have

$$\begin{aligned} &\operatorname{Norm}\left((1+\sqrt{2}x)(1+\sqrt{2}y)(1+\sqrt{2}z)\right) \\ &= \operatorname{Norm}(1+\sqrt{2}x) \cdot \operatorname{Norm}(1+\sqrt{2}y) \cdot \operatorname{Norm}(1+\sqrt{2}z). \end{aligned}$$

But $(1+\sqrt{2}x)(1+\sqrt{2}y)(1+\sqrt{2}z) = (2xy+2yz+2zx+1) + (x+y+z+2xyz)\sqrt{2}$ so the above equation is the negative of the desired identity. $\square$

We are thus reduced to find positive integers x, y, z satisfying

$$(2x^2-1)(2y^2-1)(2z^2-1) = 2023 = 7 \cdot 17^2.$$

Each of the factors is a positive integer greater than 1. The only divisors of 2023 of the form $2t^2 - 1$ are 1, 7, 17. This gives the answers claimed.

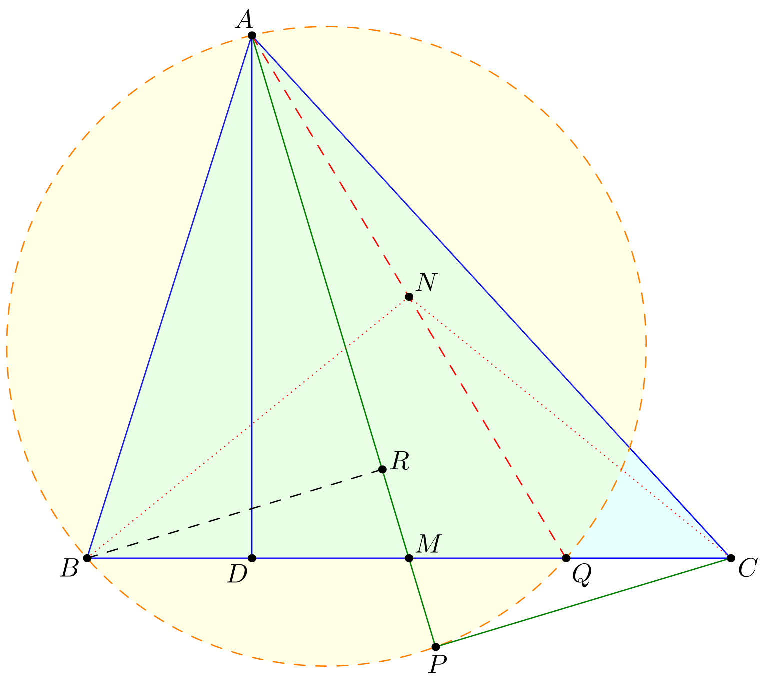
§1.2 JMO 2023/2, proposed by Holden Mui

Available online at https://aops.com/community/p27349297.

Problem statement

In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

We show several different approaches. In all solutions, let D denote the foot of the altitude from A.



¶ Most common synthetic approach The solution hinges on the following claim:

Claim — Q coincides with the reflection of D across M.

Proof. Note that $\measuredangle ADC = \measuredangle APC = 90°$, so $ADPC$ is cyclic. Then by power of a point (with the lengths directed),

$$MB \cdot MQ = MA \cdot MP = MC \cdot MD.$$

Since $MB = MC$, the claim follows. □

It follows that $\overline{MN} \parallel \overline{AD}$, as M and N are respectively the midpoints of $\overline{AQ}$ and $\overline{DQ}$. Thus $\overline{MN} \perp \overline{BC}$, and so N lies on the perpendicular bisector of $\overline{BC}$, as needed.

Remark (David Lin)**.** One can prove the main claim without power of a point as well, as follows: Let R be the foot from B to $\overline{AM}$, so $BRCP$ is a parallelogram. Note that $ABDR$

is cyclic, and hence

$$\measuredangle DRM = \measuredangle DBA = QBA = \measuredangle QPA = \measuredangle QPM.$$

Thus, $\overline{DR} \parallel \overline{PQ}$, so DRQ is also a parallelogram.

¶ Synthetic approach with no additional points at all

Claim — $\triangle BPC \sim \triangle ANM$ (oppositely oriented).

Proof. We have $\triangle BMP \sim \triangle AMQ$ from the given concyclicity of $ABPQ$. Then

$$\frac{BM}{BP} = \frac{AM}{AQ} \implies \frac{2BM}{BP} = \frac{AM}{AQ/2} \implies \frac{BC}{BP} = \frac{AM}{AN}$$

implying the similarity (since $\measuredangle MAQ = \measuredangle BPM$). $\square$

This similarity gives us the equality of directed angles

$$\measuredangle(BC, MN) = -\measuredangle(PC, AM) = 90^\circ$$

as desired.

¶ Synthetic approach using only the point R Again let R be the foot from B to $\overline{AM}$, so $BRCP$ is a parallelogram.

Claim — $ARQC$ is cyclic; equivalently, $\triangle MAQ \sim \triangle MCR$.

Proof. $MR \cdot MA = MP \cdot MA = MB \cdot MQ = MC \cdot MQ$. $\square$

Note that in $\triangle MCR$, the M-median is parallel to $\overline{CP}$ and hence perpendicular to $\overline{RM}$. The same should be true in $\triangle MAQ$ by the similarity, so $\overline{MN} \perp \overline{MQ}$ as needed.

¶ Cartesian coordinates approach with power of a point Suppose we set $B = (-1, 0)$, $M = (0, 0)$, $C = (1, 0)$, and $A = (a, b)$. One may compute:

$$\begin{aligned} \overleftrightarrow{AM} &: 0 = bx - ay \iff y = \frac{b}{a}x \\ \overleftrightarrow{CP} &: 0 = a(x-1) + by \iff y = -\frac{a}{b}(x-1) = -\frac{a}{b}x + \frac{a}{b}. \\ P &= \left(\frac{a^2}{a^2+b^2}, \frac{ab}{a^2+b^2} \right) \end{aligned}$$

Now note that

$$AM = \sqrt{a^2+b^2}, \qquad PM = \frac{a}{\sqrt{a^2+b^2}}$$

together with power of a point

$$AM \cdot PM = BM \cdot QM$$

to immediately deduce that $Q = (a, 0)$. Hence $N = (0, b/2)$ and we're done.

¶ Cartesian coordinates approach without power of a point (outline) After computing A and P as above, one could also directly calculate

$$\begin{aligned}
\text{Perpendicular bisector of } \overline{AB} &: y = -\frac{a+1}{b}x + \frac{a^2+b^2-1}{2b} \\
\text{Perpendicular bisector of } \overline{PB} &: y = -\left(\frac{2a}{b}+\frac{b}{a}\right)x - \frac{b}{2a} \\
\text{Perpendicular bisector of } \overline{PA} &: y = -\frac{a}{b}x + \frac{a+a^2+b^2}{2b}. \\
\text{Circumcenter of } \triangle PAB &= \left(-\frac{a+1}{2}, \frac{2a^2+2a+b^2}{2b}\right).
\end{aligned}$$

This is enough to extract the coordinates of $Q = (\bullet, 0)$, because $B = (-1, 0)$ is given, and the x-coordinate of the circumcenter should be the average of the x-coordinates of B and Q. In other words, $Q = (-a, 0)$. Hence, $N = \left(0, \frac{b}{2}\right)$, as needed.

¶ Ill-advised barycentric approach (outline) Use reference triangle ABC. The A-median is parametrized by $(t : 1 : 1)$ for $t \in \mathbb{R}$. So because of $\overline{CP} \perp \overline{AM}$, we are looking for t such that

$$\left(\frac{t\vec{A}+\vec{B}+\vec{C}}{t+2} - \vec{C}\right) \perp \left(A - \frac{\vec{B}+\vec{C}}{2}\right).$$

This is equivalent to

$$\left(t\vec{A}+\vec{B}-(t+1)\vec{C}\right) \perp \left(2\vec{A}-\vec{B}-\vec{C}\right).$$

By the perpendicularity formula for barycentric coordinates (EGMO 7.16), this is equivalent to

$$\begin{aligned}
0 &= a^2t - b^2\cdot(3t+2) + c^2\cdot(2-t) \\
&= \left(a^2-3b^2-c^2\right)t - 2(b^2-c^2) \\
\implies t &= \frac{2(b^2-c^2)}{a^2-3b^2-c^2}.
\end{aligned}$$

In other words,

$$P = \left(2(b^2-c^2) : a^2-3b^2-c^2 : a^2-3b^2-c^2\right).$$

A long calculation gives $a^2y_Pz_P + b^2z_Px_P + c^2x_Py_P = (a^2-3b^2-c^2)(a^2-b^2+c^2)(a^2-2b^2-2c^2)$. Together with $x_P+y_P+z_P = 2a^2-4b^2-4c^2$, this makes the equation of (ABP) as

$$0 = -a^2yz - b^2zx - c^2xy + \frac{a^2-b^2+c^2}{2}z(x+y+z).$$

To solve for Q, set $x = 0$ to get to get

$$a^2yz = \frac{a^2-b^2+c^2}{2}z(y+z) \implies \frac{y}{z} = \frac{a^2-b^2+c^2}{a^2+b^2-c^2}.$$

In other words,

$$Q = \left(0 : a^2-b^2+c^2 : a^2+b^2-c^2\right).$$

Taking the average with $A = (1, 0, 0)$ then gives

$$N = \left(2a^2 : a^2-b^2+c^2 : a^2+b^2-c^2\right).$$

The equation for the perpendicular bisector of $\overline{BC}$ is given by (see EGMO 7.19)

$$0 = a^2(z-y) + x(c^2-b^2)$$

which contains N, as needed.

¶ Extremely ill-advised complex numbers approaches (outline) Suppose we pick a, b, c as the unit circle, and let $m = (b+c)/2$. Using the fully general "foot" formula, one can get

$$p = \frac{(a-m)\overline{c} + (\overline{a}-\overline{m})c + \overline{a}m - a\overline{m}}{2(\overline{a}-\overline{m})} = \frac{a^2b - a^2c - ab^2 - 2abc - ac^2 + b^2c + 3bc^2}{4bc - 2a(b+c)}$$

Meanwhile, an extremely ugly calculation will eventually yield

$$q = \frac{\frac{bc}{a} + b + c - a}{2}$$

so

$$n = \frac{a+q}{2} = \frac{a+b+c+\frac{bc}{a}}{4} = \frac{(a+b)(a+c)}{2a}.$$

There are a few ways to then verify $NB = NC$. The simplest seems to be to verify that

$$\frac{n - \frac{b+c}{2}}{b-c} = \frac{a-b-c+\frac{bc}{a}}{4(b-c)} = \frac{(a-b)(a-c)}{2a(b-c)}$$

is pure imaginary, which is clear.

§1.3 JMO 2023/3, proposed by Holden Mui

Available online at https://aops.com/community/p27349423.

Problem statement

Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2-1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes.

Find the maximum possible value of $k(C)$ as a function of n.

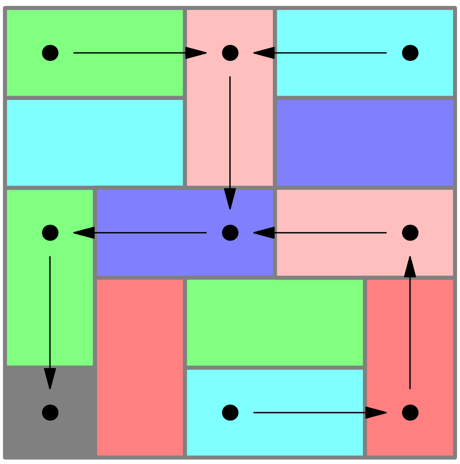
The answer is that

$$k(C) \le \left(\frac{n+1}{2}\right)^2.$$

Remark (Comparison with USAMO version)**.** In the USAMO version of the problem, students instead are asked to find all possible values of $k(C)$. The answer is $k(C) \in \left\{1, 2, \dots, \left(\frac{n-1}{2}\right)^2\right\} \cup \left\{\left(\frac{n+1}{2}\right)^2\right\}$.

Index the squares by coordinates $(x, y) \in \{1, 2, \dots, n\}^2$. We say a square is *special* if it is empty or it has the same parity in both coordinates as the empty square.

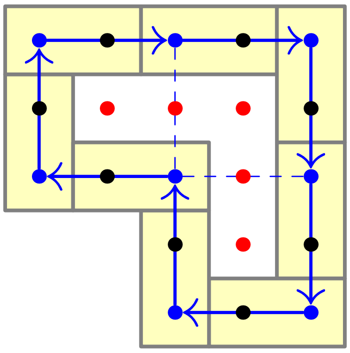
Construct a directed graph $G = G(C)$ whose vertices are special squares as follows: for each domino on a special square s, we draw a directed edge from s to the special square that domino points to, if any. (If the special square has both odd coordinates, all special squares have an outgoing edge except the empty cell. In the even-even case, some arrows may point "off the board" and not be drawn.)



Claim — Any undirected connected component of G is acyclic unless the cycle contains the empty square inside it.

Proof. Consider a cycle of G; we are going to prove that the number of chessboard cells enclosed is always odd.

This can be proven directly by induction, but for theatrical effect, we use Pick's theorem. Mark the center of every chessboard cell on or inside the cycle to get a lattice. The dominoes of the cycle then enclose a polyominoe which actually consists of 2×2 squares, meaning its area is a multiple of 4.



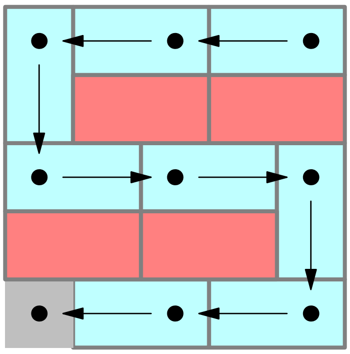
Hence $B/2 + I - 1$ is a multiple of 4, in the notation of Pick's theorem. As B is twice the number of dominoes, and a parity argument on the special squares shows that number is even, it follows that B is also a multiple of 4 (these correspond to blue and black in the figure above). This means I is odd (the red dots in the figure above), as desired. □

Consider the connected component T of the graph containing the empty square; it's acyclic, so it's a tree. Notice that all the arrows along T point towards the empty cell, and moving a domino corresponds to flipping an arrow. Therefore:

Claim — $k(C)$ is exactly the number of vertices of T.

Proof. Starting with the underlying tree, the set of possible graphs is described by picking one vertex to be the sink (the empty cell) and then directing all arrows towards it. □

This implies that $k(C) \le \left(\frac{n+1}{2}\right)^2$, the total number of vertices of G (this could only occur if the special squares are odd-odd, not even-even). Equality is achieved as long as T is a spanning tree; one example of a way to achieve this is using the snake configuration below.



Remark. In Russia 1997/11.8 it's shown that as long as the missing square is a corner, we have $G = T$. The proof is given implicitly from our work here: when the empty cell is in a corner, it cannot be surrounded, ergo G has no cycles at all. Since it has one fewer edge than vertex, it's a tree.

§2 Solutions to Day 2

§2.1 JMO 2023/4, proposed by David Torres

Available online at https://aops.com/community/p27349414.

Problem statement

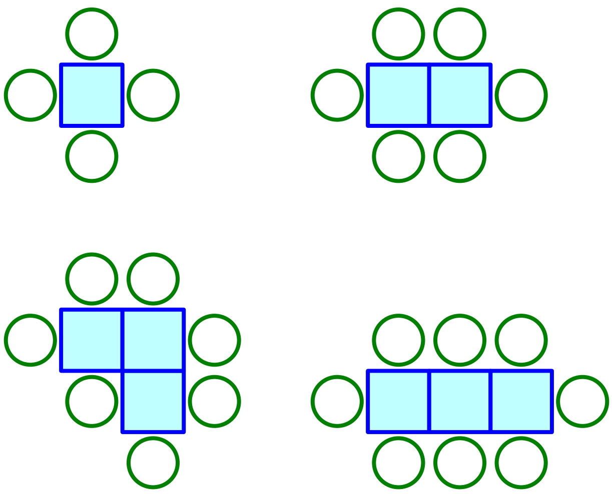
Two players, Blake and Ruby, play the following game on an infinite grid of unit squares, all initially colored white. The players take turns starting with Blake. On Blake's turn, Blake selects one white unit square and colors it blue. On Ruby's turn, Ruby selects two white unit squares and colors them red. The players alternate until Blake decides to end the game. At this point, Blake gets a score, given by the number of unit squares in the largest (in terms of area) simple polygon containing only blue unit squares.

What is the largest score Blake can guarantee?

The answer is 4 squares.

¶ Algorithm for Blake to obtain at least 4 squares We simply let Blake start with any cell blue, then always draw adjacent to a previously drawn blue cell until this is no longer possible.

Note that for $n \leq 3$, any connected region of n blue cells has more than $2n$ liberties (non-blue cells adjacent to a blue cell); up to translation, rotation, and reflection, all the cases are shown in the figure below with liberties being denoted by circles.



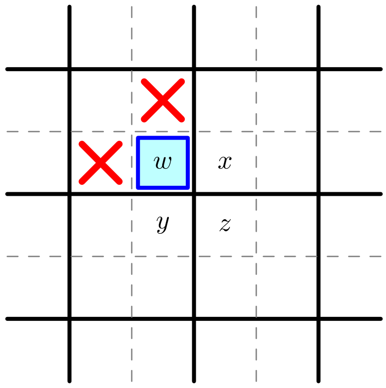
So as long as $n \leq 3$, it's impossible that Ruby has blocked every liberty, since Ruby has colored exactly $2n$ cells red. Therefore, this algorithm could only terminate once $n \geq 4$.

¶ Algorithm for Ruby to prevent more than 4 squares Divide the entire grid into 2×2 squares, which we call *windows*. Any time Blake makes a move in a cell c, let Ruby mark any orthogonal neighbors of c in its window; then place any leftover red cells arbitrarily.

Claim — It's impossible for any window to contain two orthogonally adjacent blue cells.

Proof. By construction: if there were somehow two adjacent blue cells in the same window, whichever one was played first should have caused red cells to be added. □

We show this gives the upper bound of 4 squares. Consider a blue cell w, and assume WLOG it is in the southeast corner of a window. Label squares x, y, z as shown below.



Note that by construction, the blue polygon cannot leave the square $\{w, x, y, z\}$, since whenever one of these four cells is blue, its neighbours outside that square are guaranteed to be red. This implies the bound.

Remark (For Tetris fans)**.** Here is a comedic alternative finish after proving the claim. Consider the possible tetrominoes (using the notation of https://en.wikipedia.org/wiki/Tetromino#One-sided_tetrominoes). We claim that only the square (`O`) is obtainable; as

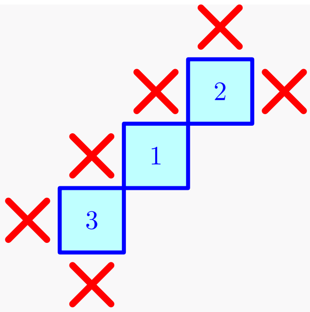
- `T`, `J`/`L`, and `I` all have three cells in a row, so they can't occur;
- `S` and `Z` can't occur either; if the bottom row of an `S` crossed a window boundary, then the top row doesn't for example.

Moreover, the only way a blue `O` could be obtained is if each of it cells is in a different window. In that case, no additional blue cells can be added: it's fully surrounded by red.

Finally, for any k-omino with $k > 4$, one can find a tetromino as a subset. (Proof: take the orthogonal adjacency graph of the k-omino, choose a spanning tree, and delete leaves from the tree until there are only four vertices left.)

Remark (Common wrong approach)**.** Suppose Ruby employs the following algorithm whenever Blake places a square x. If either the north and west neighbors of x are unoccupied, place red squares on both of them. With any leftover red squares, place them at other neighbors of x if possible. Finally, place any other red squares arbitrarily. (Another variant, the one Evan originally came up with, is to place east if possible when west is occupied, place south if possible when north is occupied, and then place any remaining red squares arbitrarily.)

As written, this strategy *does not work*. The reason is that one can end up in the following situation (imagine the blue square in the center is played first; moves for Ruby are drawn as red X's):



In order to prevent Blake from winning, Ruby would need to begin playing moves not adjacent to Blake's most recent move.

Thus in order for this solution to be made correct, one needs a careful algorithm for how Ruby should play when the north and west neighbors are not available. As far as I am aware, there are some specifications that work (and some that don't), but every working algorithm I have seen seems to involve some amount of casework.

It is even more difficult to come up with a solution involving playing on just "some" two neighbors of recently added blue squares without the "prefer north and west" idea.

§2.2 JMO 2023/5, proposed by Carl Schildkraut

Available online at https://aops.com/community/p27349336.

Problem statement

Positive integers a and N are fixed, and N positive integers are written on a blackboard. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n+a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.

After analyzing the N integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these N integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

For $N=1$, there is nothing to prove. We address $N \geq 2$ only henceforth. Let S denote the numbers on the board.

Claim — When $N \geq 2$, if $\nu_2(x) < \nu_2(a)$ for all $x \in S$, the game must terminate no matter what either player does.

Proof. The ν_2 of a number is unchanged by Alice's move and decreases by one on Bob's move. The game ends when every ν_2 is zero.

Hence, in fact the game will always terminate in exactly $\sum_{x \in S} \nu_2(x)$ moves in this case, regardless of what either player does. □

Claim — When $N \geq 2$, if there exists a number x on the board such that $\nu_2(x) \geq \nu_2(a)$, then Alice can cause the game to go on forever.

Proof. Denote by x the first entry of the board (its value changes over time). Then Alice's strategy is to:

- Operate on the first entry if $\nu_2(x) = \nu_2(a)$ (the new entry thus has $\nu_2(x+a) > \nu_2(a)$);
- Operate on any other entry besides the first one, otherwise.

A double induction then shows that

- Just before each of Bob's turns, $\nu_2(x) > \nu_2(a)$ always holds; and
- After each of Bob's turns, $\nu_2(x) \geq \nu_2(a)$ always holds.

In particular Bob will never run out of legal moves, since halving x is always legal. □

§2.3 JMO 2023/6, proposed by Anton Trygub

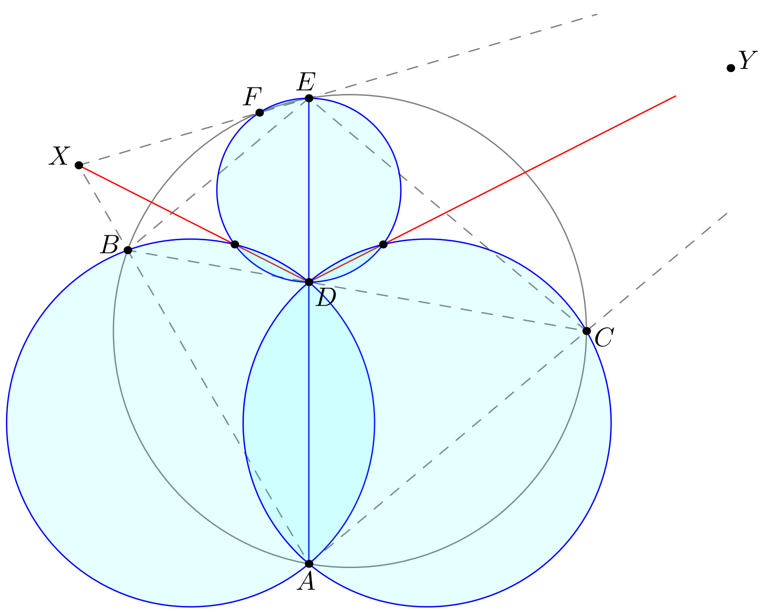
Available online at https://aops.com/community/p27349508.

Problem statement

Isosceles triangle ABC, with $AB = AC$, is inscribed in circle ω. Let D be an arbitrary point inside BC such that $BD \neq DC$. Ray AD intersects ω again at E (other than A). Point F (other than E) is chosen on ω such that $\angle DFE = 90^\circ$. Line FE intersects rays AB and AC at points X and Y, respectively. Prove that $\angle XDE = \angle EDY$.

We present three solutions.

¶ Angle chasing solution Note that (BDA) and (CDA) are congruent, since $BA = CA$ and $\angle BDA + \angle CDA = 180^\circ$. So these two circles are reflections around line ED. Moreover, (DEF) is obviously also symmetric around line ED.



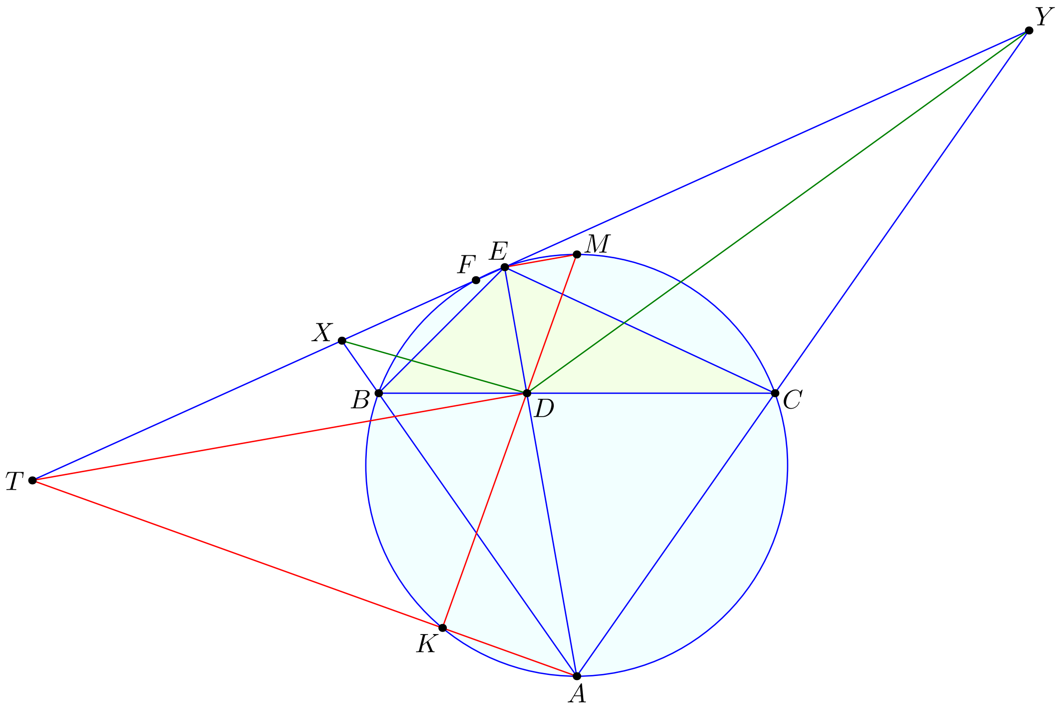
Hence, the radical axis of (BDA) and (DEF), and the radical axis of (CDA) and (DEF), should be symmetric about line DE. But these radical axii are exactly lines XD and YD, so we're done.

> **Remark** (Motivation)**.** The main idea is that you can replace DX and DY with the radical axii, letting X' and Y' be the second intersections of the blue circles. Then for the problem to be true, you'd need X' and Y' to be reflections. That's equivalent to (BDA) and (CDA) being congruent; you check it and it's indeed true.

¶ Harmonic solution (mine) Let T be the point on line $\overline{XFEY}$ such that $\angle EDT = 90^\circ$, and let $\overline{AT}$ meet ω again at K. Then

$$TD^2 = TF \cdot TE = TK \cdot TA \implies \angle DKT = 90^\circ$$

so line DK passes through the antipode M of A.



Thus,

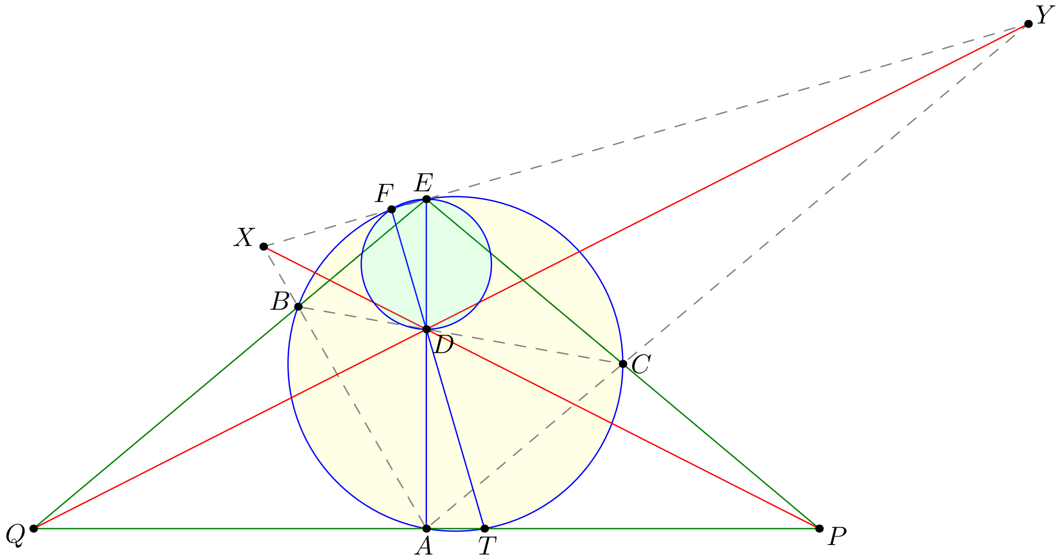
$$-1 = (AM;CB)_\omega \overset{D}{=} (EK;BC)_\omega \overset{A}{=} (TE;XY)$$

and since $\angle EDT = 90^\circ$ we're done.

> **Remark** (Motivation)**.** The idea is to kill the points X and Y by reinterpreting the desired condition as $(TD;XY) = -1$ and then projecting through A onto ω. This eliminates points X and Y altogether and reduces the problem to showing that $\overline{TA}$ passes through the harmonic conjugate of E with respect to BC on ω.
>
> The labels on the diagram are slightly misleading in that $\triangle EBC$ should probably be thought of as the "reference" triangle.

¶ Pascal solution (Zuming Feng) Extend ray FD to the antipode T of E on ω. Then,

- By Pascal's theorem on $EFTABC$, the points X, D, and $P := \overline{EC} \cap \overline{AT}$ are collinear.
- Similarly by Pascal's theorem on $EFTACB$, the points the points Y, D, and $Q := \overline{EB} \cap \overline{AT}$ are collinear.



Now it suffices to prove $\overline{ED}$ bisects $\angle QDP$. However, $\overline{ED}$ is the angle bisector of $\angle QEP = \angle BEC$, but also $\overline{EA} \perp \overline{QP}$. Thus triangle QEP is isosceles with $QE = PE$, and $\overline{EA}$ cuts it in half. Since D is on $\overline{EA}$, the result follows now.