COMPENDIUM TSTST

Team Selection Test Selection Test

Gerard Romo Garrido

Toomates Coolección vol. 70

Toomates Coolección

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Índex.

Fuente.

https://web.evanchen.cc/problems.html

53^{rd} IMO Team Selection Test

Lincoln, Nebraska

Day I 1:30 PM - 6:00 PM

June 26, 2011

1. Find all real-valued functions f defined on pairs of real numbers, having the following property: for all real numbers a, b, c, the median of $f(a, b)$, $f(b, c)$, $f(c, a)$ equals the median of a, b, c.
   (The *median* of three real numbers, not necessarily distinct, is the number that is in the middle when the three numbers are arrange in nondecreasing order.)

2. Two circles ω_1 and ω_2 intersect at points A and B. Line ℓ is tangent to ω_1 at P and to ω_2 at Q so that A is closer to ℓ than B. Let X and Y be points on major arcs PA (on ω_1) and AQ (on ω_2), respectively, such that $AX/PX = AY/QY = c$. Extend segments PA and QA through A to R and S, respectively, such that $AR = AS = c \cdot PQ$. Given that the circumcenter of triangle ARS lies on line XY, prove that $\angle XPA = \angle AQY$.

3. Prove that there exists a real constant c such that for any pair (x, y) of real numbers, there exist relatively prime integers m and n satisfying the relation

$$\sqrt{(x-m)^2 + (y-n)^2} < c \log(x^2 + y^2 + 2).$$

53^{rd} IMO Team Selection Test

Lincoln, Nebraska

Day II 1:30 PM - 6:00 PM

June 27, 2011

4. Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet at ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.

5. At a certain orphanage, every pair of orphans are either friends or enemies. For every three of an orphan's friends, an even number of pairs of them are enemies. Prove that it's possible to assign each orphan two parents such that every pair of friends shares exactly one parent, but no pair of enemies does, and no three parents are in a love triangle (where each pair of them has a child).

6. Let a,b,c be positive real numbers in the interval $[0, 1]$ with $a + b$, $b + c$, $c + a \geq 1$, prove that

$$1 \leq (1-a)^2 + (1-b)^2 + (1-c)^2 + \frac{2\sqrt{2}abc}{\sqrt{a^2+b^2+c^2}}.$$

53^{rd} IMO Team Selection Test

Lincoln, Nebraska

Day III 1:30 PM - 6:00 PM

June 29, 2011

7. Let ABC be a triangle. Its excircles touch sides BC, CA, AB at D, E, F, respectively. Prove that the perimeter of triangle ABC is at most twice that of triangle DEF.

8. Let $x_0, x_1, \cdots, x_{n_0-1}$ be integers, and let $d_1, d_2, \cdots, d_k$ be positive integers with $n_0 = d_1 > d_2 > \cdots > d_k$ and $\gcd(d_1, d_2, \cdots d_k) = 1$. For every integer $n \geq n_0$, define

$$x_n = \left\lfloor \frac{x_{n-d_1} + x_{n-d_2} + \cdots + x_{n-d_k}}{k} \right\rfloor.$$

Show that the sequence $\{x_n\}$ is eventually constant.

9. Let n be a positive integer. Suppose we are given $2^n + 1$ distinct sets, each containing finitely many objects. Place each set into one of two categories, the red sets and the blue sets, so that there is at least one set in each category. We define the *symmetric difference* of two sets as the set of objects belonging to exactly one of the two sets. Prove that there are at least 2^n different sets which can be obtained as the symmetric difference of a red set and a blue set.

TSTST 2011 Solution Notes

Lincoln, Nebraska

Evan Chen《陳誼廷》

28 October 2023

This is a compilation of solutions for the 2011 TSTST. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Find all real-valued functions f defined on pairs of real numbers, having the following property: for all real numbers a, b, c, the median of $f(a,b), f(b,c), f(c,a)$ equals the median of a, b, c.

   (The *median* of three real numbers, not necessarily distinct, is the number that is in the middle when the three numbers are arranged in nondecreasing order.)

2. Two circles ω_1 and ω_2 intersect at points A and B. Line ℓ is tangent to ω_1 at P and to ω_2 at Q so that A is closer to ℓ than B. Let X and Y be points on major arcs $\widehat{PA}$ (on ω_1) and AQ (on ω_2), respectively, such that $AX/PX = AY/QY = c$. Extend segments PA and QA through A to R and S, respectively, such that $AR = AS = c \cdot PQ$. Given that the circumcenter of triangle ARS lies on line XY, prove that $\angle XPA = \angle AQY$.

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$$\sqrt{(x-m)^2 + (y-n)^2} < c \log(x^2 + y^2 + 2).$$

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5. At a certain orphanage, every pair of orphans are either friends or enemies. For every three of an orphan's friends, an even number of pairs of them are enemies. Prove that it's possible to assign each orphan two parents such that every pair of friends shares exactly one parent, but no pair of enemies does, and no three parents are in a love triangle (where each pair of them has a child).

6. Let a, b, c be real numbers in the interval $[0, 1]$ with $a + b, b + c, c + a \geq 1$. Prove that

$$1 \leq (1-a)^2 + (1-b)^2 + (1-c)^2 + \frac{2\sqrt{2}abc}{\sqrt{a^2+b^2+c^2}}.$$

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8. Let $x_0, x_1, \dots, x_{n_0-1}$ be integers, and let $d_1, d_2, \dots, d_k$ be positive integers with $n_0 = d_1 > d_2 > \dots > d_k$ and $\gcd(d_1, d_2, \dots, d_k) = 1$. For every integer $n \geq n_0$, define

$$x_n = \left\lfloor \frac{x_{n-d_1} + x_{n-d_2} + \dots + x_{n-d_k}}{k} \right\rfloor.$$

   Show that the sequence (x_n) is eventually constant.

9. Let n be a positive integer. Suppose we are given $2^n + 1$ distinct sets, each containing finitely many objects. Place each set into one of two categories, the red sets and the blue sets, so that there is at least one set in each category. We define the *symmetric difference* of two sets as the set of objects belonging to exactly one of the two sets. Prove that there are at least 2^n different sets which can be obtained as the symmetric difference of a red set and a blue set.

§1 Solutions to Day 1

§1.1 TSTST 2011/1

Available online at https://aops.com/community/p2374841.

Problem statement

Find all real-valued functions f defined on pairs of real numbers, having the following property: for all real numbers a, b, c, the median of $f(a, b), f(b, c), f(c, a)$ equals the median of a, b, c.

(The *median* of three real numbers, not necessarily distinct, is the number that is in the middle when the three numbers are arranged in nondecreasing order.)

The following solution is joint with Andrew He.

We prove the following main claim, from which repeated applications can deduce the problem.

Claim — Let $a < b < c$ be arbitrary. On $\{a, b, c\}^2$, f takes one of the following two forms, where the column indicates the x-value and the row indicates the y-value.

f	a	b	c
a	a	b	$\geq c$
b	$\leq a$	b	$\geq c$
c	$\leq a$	b	c

or

f	a	b	c
a	a	$\leq a$	$\leq a$
b	b	b	b
c	$\geq c$	$\geq c$	c

Proof. First, we of course have $f(x, x) = x$ for all x. Now:

- By considering the assertion for (a, a, c) and (a, c, c) we see that one of $f(a, c)$ and $f(c, a)$ is $\geq c$ and the other is $\leq a$.
- Hence, by considering (a, b, c) we find that one of $f(a, b)$ and $f(b, c)$ must be b, and similarly for $f(b, a)$ and $f(c, b)$.
- Now, WLOG $f(b, a) = b$; we prove we get the first case.
- By considering (a, a, b) we deduce $f(a, b) \leq a$, so $f(b, c) = b$ and then $f(c, b) \geq c$.
- Finally, considering (c, b, a) once again in conjunction with the first bullet, we arrive at the conclusion that $f(a, c) \leq a$; similarly $f(c, a) \geq c$. □

From this it's easy to obtain that $f(x, y) \equiv x$ or $f(x, y) \equiv y$ are the only solutions.

§1.2 TSTST 2011/2

Available online at https://aops.com/community/p2374843.

Problem statement

Two circles ω_1 and ω_2 intersect at points A and B. Line ℓ is tangent to ω_1 at P and to ω_2 at Q so that A is closer to ℓ than B. Let X and Y be points on major arcs $\widehat{PA}$ (on ω_1) and AQ (on ω_2), respectively, such that $AX/PX = AY/QY = c$. Extend segments PA and QA through A to R and S, respectively, such that $AR = AS = c \cdot PQ$. Given that the circumcenter of triangle ARS lies on line XY, prove that $\angle XPA = \angle AQY$.

We begin as follows:

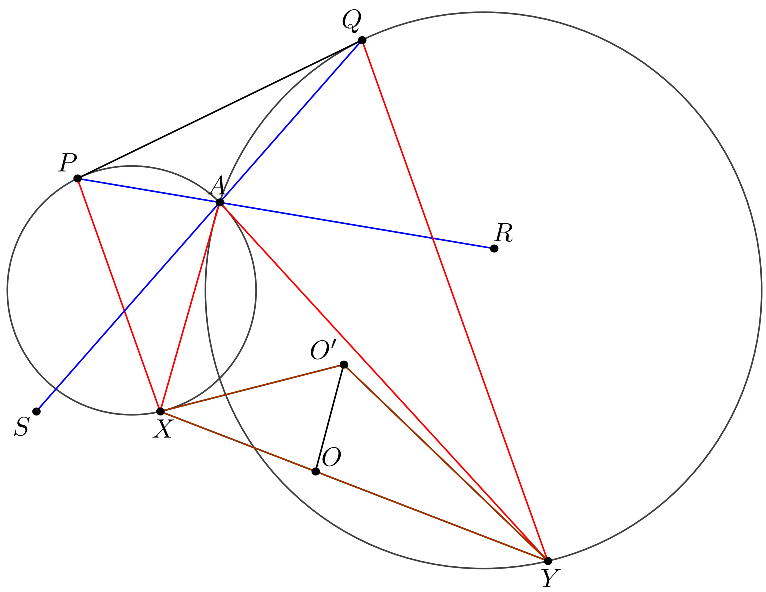
Claim — There is a spiral similarity centered at X mapping AR to PQ. Similarly there is a spiral similarity centered at Y mapping SA to PQ.

Proof. Since $\measuredangle XAR = \measuredangle XAP = \measuredangle XPQ$, and $AR/AX = PQ/PX$ is given. □

Now the composition of the two spiral similarities

$$AR \overset{X}{\longmapsto} PQ \overset{Y}{\longmapsto} SA$$

must be a rotation, since $AR = AS$. The center of this rotation must coincide with the circumcenter O of $\triangle ARS$, which is known to lie on line XY.



As O is a fixed-point of the composed map above, we may let O' be the image of O under the rotation at X, so that

$$\triangle XPA \overset{+}{\sim} \triangle XO'O, \qquad \triangle YQA \overset{+}{\sim} \triangle YO'O.$$

Because

$$\frac{XO}{XO'} = \frac{XA}{XP} = c\frac{YQ}{YA} = \frac{YO}{YO'}$$

it follows $\overline{O'O}$ bisects $\angle XO'Y$. Finally, we have

$$\measuredangle XPA = \measuredangle XO'O = \measuredangle OO'Y = \measuredangle AQY.$$

Remark. Indeed, this also shows $\overline{XP} \parallel \overline{YQ}$; so the positive homothety from ω_1 to ω_2 maps P to Q and X to Y.

§1.3 TSTST 2011/3

Available online at https://aops.com/community/p2374845.

Problem statement

Prove that there exists a real constant c such that for any pair (x, y) of real numbers, there exist relatively prime integers m and n satisfying the relation

$$\sqrt{(x-m)^2 + (y-n)^2} < c \log(x^2 + y^2 + 2).$$

This is actually the same problem as USAMO 2014/6. Surprise!

§2 Solutions to Day 2

§2.1 TSTST 2011/4

Available online at https://aops.com/community/p2374848.

Problem statement

Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$.

Let MH and NH meet the nine-point circle again at P' and Q', respectively. Recall that H is the center of the homothety between the circumcircle and the nine-point circle. From this we can see that P and Q are the images of this homothety, meaning that

$$HQ = 2HQ' \quad \text{and} \quad HP = 2HP'.$$

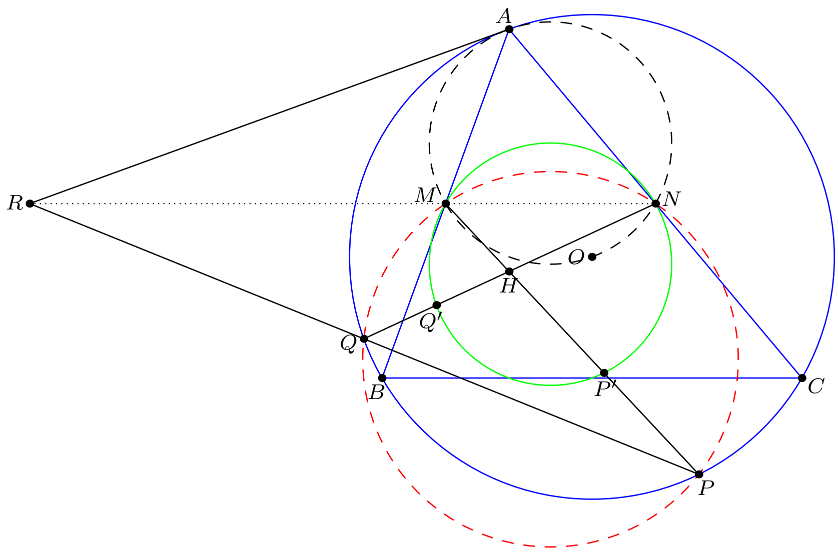
Since M, P', Q', N are cyclic, Power of a Point gives us

$$MH \cdot HP' = HN \cdot HQ'.$$

Multiplying both sides by two, we thus derive

$$HM \cdot HP = HN \cdot HQ.$$

It follows that the points M, N, P, Q are concyclic.



Let ω_1, ω_2, ω_3 denote the circumcircles of $MNPQ$, AMN, and ABC, respectively. The radical axis of ω_1 and ω_2 is line MN, while the radical axis of ω_1 and ω_3 is line PQ. Hence the line R lies on the radical axis of ω_2 and ω_3.

But we claim that ω_2 and ω_3 are internally tangent at A. This follows by noting the homothety at A with ratio 2 sends M to B and N to C. Hence the radical axis of ω_2 and ω_3 is a line tangent to both circles at A.

Hence $\overline{RA}$ is tangent to ω_3. Therefore, $\overline{RA} \perp \overline{OA}$.

§2.2 TSTST 2011/5

Available online at https://aops.com/community/p2374849.

Problem statement

At a certain orphanage, every pair of orphans are either friends or enemies. For every three of an orphan's friends, an even number of pairs of them are enemies. Prove that it's possible to assign each orphan two parents such that every pair of friends shares exactly one parent, but no pair of enemies does, and no three parents are in a love triangle (where each pair of them has a child).

Of course, we consider the graph with vertices as children and edges as friendships. Consider all the maximal cliques in the graph (i.e. repeatedly remove maximal cliques until no edges remain; thus all edges are in some clique).

Claim — Every vertex is in at most two maximal cliques.

Proof. Indeed, consider a vertex v adjacent to w_1 and w_2, but with w_1 not adjacent to w_2. Then by condition, any third vertex u must be adjacent to exactly one of w_1 and w_2. Moreover, given vertices u and u' adjacent to w_1, vertices u and u' are adjacent too. This proves the claim. □

Now, for every maximal clique we assign a particular parent to all vertices in that clique, adding in additional distinct parents if there are any deficient children. This satisfies the friendship/enemy condition. Moreover, one can readily check that there are no love triangles: given children a, b, c such that a and b share a parent while a and c share another parent, according to the claim b and c can't share a third parent. This completes the problem.

Remark. This solution is highly motivated for the following reason: by experimenting with small cases, one quickly finds that given some vertices which form a clique, one *must* assign some particular parent to all vertices in that clique. That is, the requirements of the problem are sufficiently rigid that there is no room for freedom on our part, so we know *a priori* that an assignment based on cliques (as above) must work. From there we know exactly what to prove, and everything else follows through.

Ironically, the condition that there be no love triangle actually makes the problem easier, because it tells us exactly what to do!

§2.3 TSTST 2011/6

Available online at https://aops.com/community/p2374852.

Problem statement

Let a, b, c be real numbers in the interval $[0, 1]$ with $a+b, b+c, c+a \geq 1$. Prove that

$$1 \leq (1-a)^2 + (1-b)^2 + (1-c)^2 + \frac{2\sqrt{2}abc}{\sqrt{a^2+b^2+c^2}}.$$

The following approach is due to Ashwin Sah.

We will prove the inequality for any a, b, c the sides of a possibly degenerate triangle (which is implied by the condition), ignoring the particular constant 1. Homogenizing, we instead prove the problem in the following form:

Claim — We have

$$k^2 \leq (k-a)^2 + (k-b)^2 + (k-c)^2 + \frac{2\sqrt{2}abc}{\sqrt{a^2+b^2+c^2}}$$

for any a, b, c, k with (a, b, c) the sides of a possibly degenerate triangle.

Proof. For any particular (a, b, c) this is a quadratic in k of the form $2k^2 - 2(a+b+c)k + C \geq 0$; thus we will verify it holds for $k = \frac{1}{2}(a+b+c)$.

Letting $x = \frac{1}{2}(b+c-a)$ as is usual, the problem rearranges to In that case, the problem amounts to

$$(x+y+z)^2 \leq x^2 + y^2 + z^2 + \frac{2(x+y)(y+z)(z+x)}{\sqrt{x^2+y^2+z^2+xy+yz+zx}}$$

or equivalently

$$x^2 + y^2 + z^2 + xy + yz + zx \leq \left(\frac{(x+y)(y+z)(z+x)}{xy+yz+zx} \right)^2.$$

To show this, one may let $t = xy + yz + zx$, then using $(x+y)(x+z) = x^2 + B$ this becomes

$$t^2(x^2+y^2+z^2+t) \leq (x^2+t)(y^2+t)(z^2+t)$$

which is obvious upon expansion. □

Remark. The inequality holds actually for all real numbers a, b, c, with very disgusting proofs.

§3 Solutions to Day 3

§3.1 TSTST 2011/7

Available online at https://aops.com/community/p2374855.

Problem statement

Let ABC be a triangle. Its excircles touch sides BC, CA, AB at D, E, F. Prove that the perimeter of triangle ABC is at most twice that of triangle DEF.

Solution by August Chen: It turns out that it is enough to take the orthogonal projection of EF onto side BC (which has length $a-(s-a)(\cos B+\cos C)$) and sum cyclically:

$$\begin{aligned} -s+\sum_{\text{cyc}} EF &\ge -s+\sum_{\text{cyc}}\left[a-(s-a)\left(\cos B+\cos C\right)\right] \\ &= s-\sum_{\text{cyc}} a\cos A=\sum_{\text{cyc}} a\left(\frac{1}{2}-\cos A\right) \\ &= R\sum_{\text{cyc}}\sin A(1-2\cos A) \\ &= R\sum_{\text{cyc}}\left(\sin A-\sin 2A\right). \end{aligned}$$

Thus we're done upon noting that

$$\frac{\sin 2B+\sin 2C}{2}=\sin(B+C)\cos(B-C)=\sin A\cos(B-C)\le \sin A.$$

(Alternatively, one can avoid trigonometry by substituting $\cos A=\frac{b^2+c^2-a^2}{2bc}$ and doing some routine but long calculation.)

§3.2 TSTST 2011/8

Available online at https://aops.com/community/p2374856.

Problem statement

Let x_0, x_1, $\dots$, x_{n_0-1} be integers, and let d_1, d_2, $\dots$, d_k be positive integers with $n_0 = d_1 > d_2 > \dots > d_k$ and $\gcd(d_1, d_2, \dots, d_k) = 1$. For every integer $n \ge n_0$, define

$$x_n = \left\lfloor \frac{x_{n-d_1} + x_{n-d_2} + \dots + x_{n-d_k}}{k} \right\rfloor .$$

Show that the sequence (x_n) is eventually constant.

Note that if the initial terms are contained in some interval $[A, B]$ then they will remain in that interval. Thus the sequence is eventually periodic. Discard initial terms and let the period be T; we will consider all indices modulo T from now on.

Let M be the maximal term in the sequence (which makes sense since the sequence is periodic). Note that if $x_n = M$, we must have $x_{n-d_i} = M$ for all i as well. By taking a linear combination $\sum c_i d_i \equiv 1 \pmod T$ (possibly be Bezout's theorem, since $\gcd_i(d_i) = 1$), we conclude $x_{n-1} = M$, as desired.

§3.3 TSTST 2011/9

Available online at https://aops.com/community/p2374857.

Problem statement

Let n be a positive integer. Suppose we are given 2^n+1 distinct sets, each containing finitely many objects. Place each set into one of two categories, the red sets and the blue sets, so that there is at least one set in each category. We define the *symmetric difference* of two sets as the set of objects belonging to exactly one of the two sets. Prove that there are at least 2^n different sets which can be obtained as the symmetric difference of a red set and a blue set.

We can interpret the problem as working with binary strings of length $\ell \ge n+1$, with ℓ the number of elements across all sets.

Let F be a field of cardinality 2^ℓ, hence $F \cong \mathbb{F}_2^{\oplus \ell}$.

Then, we can think of red/blue as elements of F, so we have some $B \subseteq F$, and an $R \subseteq F$. We wish to prove that $|B+R| \ge 2^n$. Want $|B+R| \ge 2^n$.

Equivalently, any element of a set X with $|X| = 2^n - 1$ should omit some element of $|B+R|$. To prove this: we know $|B|+|R| = 2^n+1$, and define

$$P(b,r) = \prod_{x \in X} (b+r-x).$$

Consider $b^{|B|-1}r^{|R|-1}$. The coefficient of is $\binom{2^n-1}{|B|-1}$, which is odd (say by Lucas theorem), so the nullstellensatz applies.

Team Selection Test for the Selection Team of 54th IMO

Lincoln, Nebraska

Day I   1:30 PM - 6:00 PM

June 22, 2012

1. Find all infinite sequences $a_1, a_2, \ldots$ of positive integers satisfying the following properties:

   (a) $a_1 < a_2 < a_3 < \cdots$,

   (b) there are no positive integers i, j, k, not necessarily distinct, such that $a_i + a_j = a_k$,

   (c) there are infinitely many k such that $a_k = 2k - 1$.

2. Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and the circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $MN \parallel O_1O_2$.

3. Let $\mathbb{N}$ be the set of positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following two conditions:

   (a) $f(m)$ and $f(n)$ are relatively prime whenever m and n are relatively prime.

   (b) $n \leq f(n) \leq n + 2012$ for all n.

   Prove that for any natural number n and any prime p, if p divides $f(n)$ then p divides n.

Team Selection Test for the Selection Team of 54^{th} IMO

Lincoln, Nebraska

Day II 1:30 PM - 6:00 PM

June 24, 2012

4. In scalene triangle ABC, let the feet of the perpendiculars from A to BC, B to CA, C to AB be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides BC, CA, AB. Show that the perpendiculars from D to AA_2, E to BB_2 and F to CC_2 are concurrent.

5. A rational number x is given. Prove that there exists a sequence $x_0, x_1, x_2, \ldots$ of rational numbers with the following properties:

   (a) $x_0 = x$;

   (b) for every $n \geq 1$, either $x_n = 2x_{n-1}$ or $x_n = 2x_{n-1} + \frac{1}{n}$;

   (c) x_n is an integer for some n.

6. Positive real numbers x, y, z satisfy $xyz + xy + yz + zx = x + y + z + 1$. Prove that

$$\frac{1}{3}\left(\sqrt{\frac{1+x^2}{1+x}} + \sqrt{\frac{1+y^2}{1+y}} + \sqrt{\frac{1+z^2}{1+z}}\right) \leq \left(\frac{x+y+z}{3}\right)^{5/8}.$$

Team Selection Test for the Selection Team of 54$^{\text{th}}$ IMO

Lincoln, Nebraska

Day III 1:30 PM - 6:00 PM

June 26, 2012

7. Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

8. Let n be a positive integer. Consider a triangular array of nonnegative integers as follows:

$$\begin{array}{rccccc}
\text{Row } 1: & & & a_{0,1} & & \\
\text{Row } 2: & & a_{0,2} & & a_{1,2} & \\
 & & \vdots & \vdots & \vdots & \\
\text{Row } n-1: & a_{0,n-1} & a_{1,n-1} & \cdots & a_{n-2,n-1} & \\
\text{Row } n: & a_{0,n} \quad a_{1,n} & a_{2,n} & \cdots & & a_{n-1,n}.
\end{array}$$

Call such a triangular array *stable* if for every $0 \le i < j < k \le n$ we have

$$a_{i,j} + a_{j,k} \le a_{i,k} \le a_{i,j} + a_{j,k} + 1.$$

For $s_1, \dots s_n$ any nondecreasing sequence of nonnegative integers, prove that there exists a unique stable triangular array such that the sum of all of the entries in row k is equal to s_k.

9. Given a set S of n variables, a binary operation $\times$ on S is called *simple* if it satisfies $(x \times y) \times z = x \times (y \times z)$ for all $x, y, z \in S$ and $x \times y \in \{x, y\}$ for all $x, y \in S$. Given a simple operation $\times$ on S, any string of elements in S can be reduced to a single element, such as $xyz \to x \times (y \times z)$. A string of variables in S is called *full* if it contains each variable in S at least once, and two strings are *equivalent* if they evaluate to the same variable regardless of which simple $\times$ is chosen. For example xxx, xx, and x are equivalent, but these are only full if $n = 1$. Suppose T is a set of strings such that any full string is equivalent to exactly one element of T. Determine the number of elements of T.

TSTST 2012 Solution Notes

Lincoln, Nebraska

Evan Chen《陳誼廷》

30 September 2023

This is a compilation of solutions for the 2012 TSTST. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Determine all infinite strings of letters with the following properties:
   (a) Each letter is either T or S,
   (b) If position i and j both have the letter T, then position $i+j$ has the letter S,
   (c) There are infinitely many integers k such that position $2k-1$ has the kth T.

2. Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $\overline{MN} \parallel \overline{O_1O_2}$.

3. Let $\mathbb{N}$ be the set of positive integers. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function satisfying the following two conditions:
   (a) $f(m)$ and $f(n)$ are relatively prime whenever m and n are relatively prime.
   (b) $n \le f(n) \le n + 2012$ for all n.

   Prove that for any natural number n and any prime p, if p divides $f(n)$ then p divides n.

4. In scalene triangle ABC, let the feet of the perpendiculars from A to $\overline{BC}$, B to $\overline{CA}$, C to $\overline{AB}$ be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Show that the perpendiculars from D to $\overline{AA_2}$, E to $\overline{BB_2}$ and F to $\overline{CC_2}$ are concurrent.

5. A rational number x is given. Prove that there exists a sequence $x_0, x_1, x_2, \dots$ of rational numbers with the following properties:
   (a) $x_0 = x$;
   (b) for every $n \ge 1$, either $x_n = 2x_{n-1}$ or $x_n = 2x_{n-1} + \frac{1}{n}$;
   (c) x_n is an integer for some n.

6. Positive real numbers x, y, z satisfy $xyz + xy + yz + zx = x + y + z + 1$. Prove that

$$\frac{1}{3}\left(\sqrt{\frac{1+x^2}{1+x}} + \sqrt{\frac{1+y^2}{1+y}} + \sqrt{\frac{1+z^2}{1+z}}\right) \le \left(\frac{x+y+z}{3}\right)^{5/8}.$$

7. Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

8. Let n be a positive integer. Consider a triangular array of nonnegative integers as follows:

$$
\begin{array}{lccccccccc}
\text{Row 1:} & & & & & a_{0,1} & & & & \\
\text{Row 2:} & & & & a_{0,2} & & a_{1,2} & & & \\
\vdots & & & \iddots & & \vdots & & \ddots & & \\
\text{Row } n-1\text{:} & & a_{0,n-1} & & a_{1,n-1} & & \cdots & & a_{n-2,n-1} & \\
\text{Row } n\text{:} & a_{0,n} & & a_{1,n} & & a_{2,n} & & \cdots & & a_{n-1,n}
\end{array}
$$

Call such a triangular array *stable* if for every $0 \le i < j < k \le n$ we have

$$a_{i,j} + a_{j,k} \le a_{i,k} \le a_{i,j} + a_{j,k} + 1.$$

For $s_1, \dots, s_n$ any nondecreasing sequence of nonnegative integers, prove that there exists a unique stable triangular array such that the sum of all of the entries in row k is equal to s_k.

9. Given a set S of n variables, a binary operation $\times$ on S is called *simple* if it satisfies $(x \times y) \times z = x \times (y \times z)$ for all $x, y, z \in S$ and $x \times y \in \{x, y\}$ for all $x, y \in S$. Given a simple operation $\times$ on S, any string of elements in S can be reduced to a single element, such as $xyz \to x \times (y \times z)$. A string of variables in S is called *full* if it contains each variable in S at least once, and two strings are *equivalent* if they evaluate to the same variable regardless of which simple $\times$ is chosen. For example xxx, xx, and x are equivalent, but these are only full if $n = 1$. Suppose T is a set of full strings such that any full string is equivalent to exactly one element of T. Determine the number of elements of T.

§1 Solutions to Day 1

§1.1 TSTST 2012/1, proposed by Palmer Mebane

Available online at https://aops.com/community/p2745864.

Problem statement

Determine all infinite strings of letters with the following properties:

(a) Each letter is either T or S,

(b) If position i and j both have the letter T, then position $i+j$ has the letter S,

(c) There are infinitely many integers k such that position $2k-1$ has the kth T.

We wish to find all infinite sequences $a_1, a_2, \dots$ of positive integers satisfying the following properties:

(a) $a_1 < a_2 < a_3 < \cdots$,

(b) there are no positive integers i, j, k, not necessarily distinct, such that $a_i + a_j = a_k$,

(c) there are infinitely many k such that $a_k = 2k-1$.

If $a_k = 2k-1$ for some $k > 1$, let $A_k = \{a_1, a_2, \dots, a_k\}$. By (b) and symmetry, we have

$$2k-1 \ge \frac{|A_k - A_k| - 1}{2} + |A_k| \ge \frac{2|A_k| - 2}{2} + |A_k| = 2k-1.$$

But in order for $|A_k - A_k| = 2|A_k| - 1$, we must have A_k an arithmetic progression, whence $a_n = 2n-1$ for all n by taking k arbitrarily large.

§1.2 TSTST 2012/2

Available online at https://aops.com/community/p2745851.

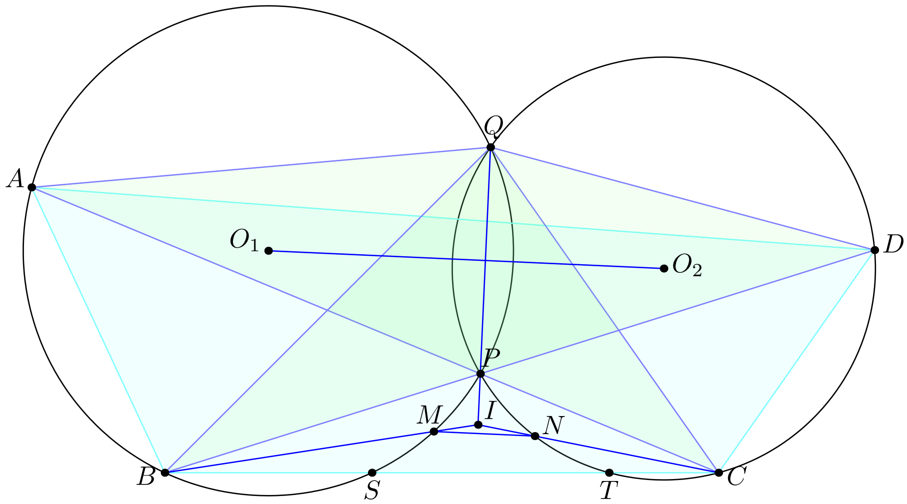
Problem statement

Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $\overline{MN} \parallel \overline{O_1O_2}$.

Let Q be the second intersection point of ω_1, ω_2. Suffice to show $\overline{QP} \perp \overline{MN}$. Now Q is the center of a spiral *congruence* which sends $\overline{AC} \mapsto \overline{BD}$. So $\triangle QAB$ and $\triangle QCD$ are similar isosceles. Now,

$$\measuredangle QPA = \measuredangle QBA = \measuredangle DCQ = \measuredangle DPQ$$

and so $\overline{QP}$ is bisects $\angle BPC$.



Now, let $I = \overline{BM} \cap \overline{CN} \cap \overline{PQ}$ be the incenter of $\triangle PBC$. Then $IM \cdot IB = IP \cdot IQ = IN \cdot IC$, so $BMNC$ is cyclic, meaning $\overline{MN}$ is antiparallel to $\overline{BC}$ through $\angle BIC$. Since $\overline{QPI}$ passes through the circumcenter of $\triangle BIC$, it follows now $\overline{QPI} \perp \overline{MN}$ as desired.

§1.3 TSTST 2012/3

Available online at https://aops.com/community/p2745877.

Problem statement

Let $\mathbb{N}$ be the set of positive integers. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function satisfying the following two conditions:

(a) $f(m)$ and $f(n)$ are relatively prime whenever m and n are relatively prime.

(b) $n \le f(n) \le n + 2012$ for all n.

Prove that for any natural number n and any prime p, if p divides $f(n)$ then p divides n.

¶ First short solution, by Jeffrey Kwan Let p_0, p_1, p_2, ... denote the sequence of all prime numbers, in any order. Pick *any* primes q_i such that

$$q_0 \mid f(p_0), \quad q_1 \mid f(p_1), \quad q_2 \mid f(p_2), \quad \text{etc.}$$

This is possible since each f value above exceeds 1. Also, since by hypothesis the $f(p_i)$ are pairwise coprime, the primes q_i are all pairwise distinct.

Claim — We must have $q_i = p_i$ for each i. (Therefore, $f(p_i)$ is a power of p_i for each i.)

Proof. Assume to the contrary that $q_0 \ne p_0$. By changing labels if necessary, assume $\min(p_1, p_2, \dots, p_{2012}) > 2012$. Then by Chinese remainder theorem we can choose an integer m such that

$$\begin{aligned} m + i &\equiv 0 \pmod{q_i} \\ m &\not\equiv 0 \pmod{p_i} \end{aligned}$$

for $0 \le i \le 2012$. But now $f(m)$ should be coprime to all $f(p_i)$, ergo coprime to $q_0 q_1 \dots q_{2012}$, violating $m \le f(m) \le m + 2012$. □

All that is left to do is note that whenever $p \nmid n$, we have $\gcd(f(p), f(n)) = 1$, hence $p \nmid f(n)$. This is the contrapositive of the problem statement.

¶ Second solution with a grid Fix n and p, and assume for contradiction $p \nmid n$.

Claim — There exists a large integer N with $f(N) = N$, that also satisfies $N \equiv 1 \pmod{n}$ and $N \equiv 0 \pmod{p}$.

Proof. We'll need to pick both N and an ancillary integer M. Here is how: pick $2012 \cdot 2013$ distinct primes $q_{i,j} > n + p + 2013$ for every $i = 1, \dots, 2012$ and $j = 0, \dots, 2012$, and use it to fill in the following table:

$$\begin{array}{c|cccc} & N+1 & N+2 & \dots & N+2012 \\ \hline M & q_{0,1} & q_{0,2} & \cdots & q_{0,2012} \\ M+1 & q_{1,1} & q_{1,2} & \cdots & q_{1,2012} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M+2012 & q_{2012,1} & q_{2012,2} & \cdots & q_{2012,2012} \end{array}.$$

By the Chinese Remainder Theorem, we can construct N such that $N+1 \equiv 0 \pmod{q_{i,1}}$ for every i, and similarly for $N+2$, and so on. Moreover, we can also tack on the extra conditions $N \equiv 0 \pmod p$ and $N \equiv 1 \pmod n$ we wanted.

Notice that N cannot be divisible by any of the $q_{i,j}$'s, since the $q_{i,j}$'s are greater than 2012.

After we've chosen N, we can pick M such that $M \equiv 0 \pmod{q_{0,j}}$ for every j, and similarly $M+1 \equiv 0 \pmod{q_{1,j}}$, et cetera. Moreover, we can tack on the condition $M \equiv 1 \pmod N$, which ensures $\gcd(M, N) = 1$.

What does this do? We claim that $f(N) = N$ now. Indeed $f(M)$ and $f(N)$ are relatively prime; but look at the table! The table tells us that $f(M)$ must have a common factor with each of $N+1, \dots, N+2012$. So the only possibility is that $f(N) = N$. $\square$

Now we're basically done. Since $N \equiv 1 \pmod n$, we have $\gcd(N, n) = 1$ and hence $1 = \gcd(f(N), f(n)) = \gcd(N, f(n))$. But $p \mid N$ and $p \mid f(n)$, contradiction.

§2 Solutions to Day 2

§2.1 TSTST 2012/4

Available online at https://aops.com/community/p2745854.

Problem statement

In scalene triangle ABC, let the feet of the perpendiculars from A to $\overline{BC}$, B to $\overline{CA}$, C to $\overline{AB}$ be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Show that the perpendiculars from D to $\overline{AA_2}$, E to $\overline{BB_2}$ and F to $\overline{CC_2}$ are concurrent.

We claim that they pass through the orthocenter H. Indeed, consider the circle with diameter $\overline{BC}$, which circumscribes quadrilateral BCB_1C_1 and has center D. Then by Brokard theorem, $\overline{AA_2}$ is the polar of line H. Thus $\overline{DH} \perp \overline{AA_2}$.

§2.2 TSTST 2012/5

Available online at https://aops.com/community/p2745867.

Problem statement

A rational number x is given. Prove that there exists a sequence $x_0, x_1, x_2, \dots$ of rational numbers with the following properties:

(a) $x_0 = x$;

(b) for every $n \ge 1$, either $x_n = 2x_{n-1}$ or $x_n = 2x_{n-1} + \frac{1}{n}$;

(c) x_n is an integer for some n.

Think of the sequence as a process over time. We'll show that:

Claim — At any given time t, if the denominator of x_t has some odd prime power $q = p^e$, then we can delete a factor of p from the denominator, while only adding powers of two to the denominator.

(Thus we can just delete off all the odd primes one by one and then double appropriately many times.)

Proof. The idea is to add only fractions of the form $(2^k q)^{-1}$.

Indeed, let n be large, and suppose $t < 2^{r+1}q < 2^{r+2}q < \dots < 2^{r+m}q < n$. For some binary variables $\varepsilon_i \in \{0,1\}$ we can have

$$x_n = 2^{n-t}x_t + c_1 \cdot \frac{\varepsilon_1}{q} + c_2 \cdot \frac{\varepsilon_2}{q} \dots + c_s \cdot \frac{\varepsilon_m}{q}$$

where c_i is some power of 2 (to be exact, $c_i = \frac{2^{n-2^{r+i}q}}{2^{r+1}}$, but the exact value doesn't matter).

If m is large enough the set $\{0, c_1\} + \{0, c_2\} + \dots + \{0, c_m\}$ spans everything modulo p. (Actually, Cauchy-Davenport implies $m = p$ is enough, but one can also just use Pigeonhole to notice some residue appears more than p times, for $m = O(p^2)$.) Thus we can eliminate one factor of p from the denominator, as desired. □

§2.3 TSTST 2012/6, proposed by Sung-Yoon Kim

Available online at https://aops.com/community/p2745861.

Problem statement

Positive real numbers x, y, z satisfy $xyz + xy + yz + zx = x + y + z + 1$. Prove that

$$\frac{1}{3}\left(\sqrt{\frac{1+x^2}{1+x}} + \sqrt{\frac{1+y^2}{1+y}} + \sqrt{\frac{1+z^2}{1+z}}\right) \le \left(\frac{x+y+z}{3}\right)^{5/8}.$$

The key is the identity

$$\begin{aligned}\frac{x^2+1}{x+1} &= \frac{(x^2+1)(y+1)(z+1)}{(x+1)(y+1)(z+1)} \\ &= \frac{x(xyz+xy+xz)+x^2+yz+y+z+1}{2(1+x+y+z)} \\ &= \frac{x(x+y+z+1-yz)+x^2+yz+y+z+1}{2(1+x+y+z)} \\ &= \frac{(x+y)(x+z)+x^2+(x-xyz+y+z+1)}{2(1+x+y+z)} \\ &= \frac{2(x+y)(x+z)}{2(1+x+y+z)} \\ &= \frac{(x+y)(x+z)}{1+x+y+z}.\end{aligned}$$

Remark. The "trick" can be rephrased as $(x^2+1)(y+1)(z+1) = 2(x+y)(x+z)$.

After this, straight Cauchy in the obvious way will do it (reducing everything to an inequality in $s = x+y+z$). One writes

$$\begin{aligned}\left(\sum_{\text{cyc}} \frac{\sqrt{(x+y)(x+z)}}{\sqrt{1+s}}\right)^2 &\le \frac{\left(\sum_{\text{cyc}} x+y\right)\left(\sum_{\text{cyc}} x+z\right)}{1+s} \\ &= \frac{4s^2}{1+s}\end{aligned}$$

and so it suffices to check that $\frac{4s^2}{1+s} \le 9(s/3)^{5/4}$, which is true because

$$(s/3)^5 \cdot 9^4 \cdot (1+s)^4 - (4s^2)^4 = s^5(s-3)^2(27s^2+14s+3) \ge 0.$$

§3 Solutions to Day 3

§3.1 TSTST 2012/7

Available online at https://aops.com/community/p2745857.

Problem statement

Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

By angle chasing, equivalent to show $\overline{MN} \parallel \overline{AD}$, so discard the point H. We now present a three solutions.

¶ First solution using vectors We first contend that:

Claim — We have $QB = PC$.

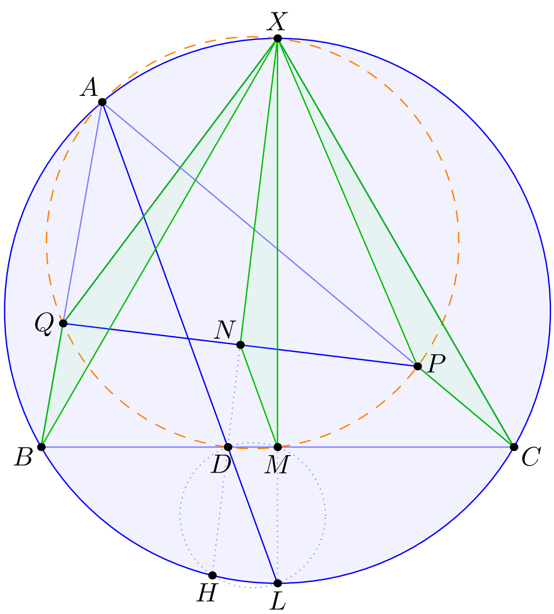
Proof. Power of a Point gives $BM \cdot BD = AB \cdot QB$. Then use the angle bisector theorem. □

Now notice that the vector

$$\overrightarrow{MN} = \frac{1}{2}\left(\overrightarrow{BQ} + \overrightarrow{CP}\right)$$

which must be parallel to the angle bisector since $\overrightarrow{BQ}$ and $\overrightarrow{CP}$ have the same magnitude.

¶ Second solution using spiral similarity let X be the arc midpoint of BAC. Then $ADMX$ is cyclic with diameter $\overline{AM}$, and hence X is the Miquel point X of $QBPC$ is the midpoint of arc BAC. Moreover $\overline{XND}$ collinear (as $XP = XQ$, $DP = DQ$) on (APQ).



Then $\triangle XNM \sim \triangle XPC$ spirally, and

$$\measuredangle XMN = \measuredangle XCP = \measuredangle XCA = \measuredangle XLA$$

thus done.

¶ Third solution using barycentrics (mine) Once reduced to $\overline{MN} \parallel \overline{AB}$, straight bary will also work. By power of a point one obtains

$$\begin{aligned} P &= \left(a^2 : 0 : 2b(b+c) - a^2\right) \\ Q &= \left(a^2 : 2c(b+c) - a^2 : 0\right) \\ \implies N &= \left(a^2(b+c) : 2bc(b+c) - ba^2 : 2bc(b+c) - ca^2\right). \end{aligned}$$

Now the point at infinity along $\overline{AD}$ is $(-(b+c) : b : c)$ and so we need only verify

$$\det \begin{bmatrix} a^2(b+c) & 2bc(b+c) - ba^2 & 2bc(b+c) - ca^2 \\ 0 & 1 & 1 \\ -(b+c) & b & c \end{bmatrix} = 0$$

which follows since the first row is $-a^2$ times the third row plus $2bc(b+c)$ times the second row.

§3.2 TSTST 2012/8, proposed by Palmer Mebane

Available online at https://aops.com/community/p2745872.

Problem statement

Let n be a positive integer. Consider a triangular array of nonnegative integers as follows:

$$\begin{array}{lccccccccc}
\text{Row 1:} & & & & & a_{0,1} & & & & \\
\text{Row 2:} & & & & a_{0,2} & & a_{1,2} & & & \\
\vdots & & & \iddots & & \vdots & & \ddots & & \\
\text{Row } n-1\text{:} & & a_{0,n-1} & & a_{1,n-1} & & \dots & & a_{n-2,n-1} & \\
\text{Row } n\text{:} & a_{0,n} & & a_{1,n} & & a_{2,n} & & \dots & & a_{n-1,n}
\end{array}$$

Call such a triangular array *stable* if for every $0 \le i < j < k \le n$ we have

$$a_{i,j} + a_{j,k} \le a_{i,k} \le a_{i,j} + a_{j,k} + 1.$$

For $s_1, \dots, s_n$ any nondecreasing sequence of nonnegative integers, prove that there exists a unique stable triangular array such that the sum of all of the entries in row k is equal to s_k.

Firstly, here are illustrative examples showing the arrays for $(s_1, s_2, s_3, s_4) = (2, 5, 9, x)$ where $9 \le x \le 14$. (The array has been left justified.)

$$\begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ 5 & 3 & 1 & 0 \end{bmatrix}
\begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ {\color{red}\mathbf{6}} & 3 & 1 & 0 \end{bmatrix}
\begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ 6 & 3 & {\color{red}\mathbf{2}} & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ 6 & {\color{red}\mathbf{4}} & 2 & 0 \end{bmatrix}
\begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ 6 & 4 & 2 & {\color{red}\mathbf{1}} \end{bmatrix}
\begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ {\color{red}\mathbf{7}} & 4 & 2 & 1 \end{bmatrix}$$

Now we outline the proof. By induction on n, we may assume the first $n-1$ rows are fixed. Now, let $N = s_n$ vary. Now, we prove our result by (another) induction on $N \ge s_{n-1}$.

The base case $N = s_{n-1}$ is done by copying the $n-1$st row and adding a zero at the end. This is also unique, since $a_{i,n} \ge a_{i-1,n} + a_{n-1,n}$ for all $i = 0, \dots, n-2$, whence $\sum a_{i,n} \ge s_{n-1}$ follows.

Now the inductive step is based on the following lemma, which illustrates the idea of a "unique increasable entry".

Lemma

Fix a stable array Construct a tournament on the n entries of the last row as follows: for $i < j$,

- $a_{i,n} \to a_{j,n}$ if $a_{i,n} = a_{i,j} + a_{j,n}$, and
- $a_{j,n} \to a_{i,n}$ if $a_{i,n} = a_{i,j} + a_{j,n} + 1$.

Then this tournament is transitive. Also, except for $N = s_{n-1}$, a 0 entry is never a source.

Intuitively, $a_{i,n} \to a_{j,n}$ if $a_{i,n}$ blocks $a_{j,n}$ from increasing. For instance, in the example

$$\begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ \mathbf{6} & 3 & 1 & 0 \end{bmatrix}$$

the tournament is $1 \to 3 \to 0 \to 6$.

Proof of lemma. Let $0 \le i < j < k < n$ be indices. Let $x = a_{i,n}$, $y = a_{j,n}$, $z = a_{k,n}$, $p = a_{i,j}$, $s = a_{i,k}$, $q = a_{j,k}$. Picture:

$$\begin{bmatrix} p & \swarrow & \\ s & q & \swarrow \\ x & y & z \end{bmatrix}$$

If $x \to y \to z \to x$ happens, that means $x = y + p$, $y = q + z$, $x = s + z + 1$, which gives $s = p + q - 1$, contradiction. Similarly if $x \leftarrow y \leftarrow z \leftarrow x$ then $x = y + p + 1$, $y = q + z + 1$, $x = s + z$, which gives $s = p + q + 2$, also contradiction. $\square$

Now this allows us to perform our induction. Indeed, to show existence from N to $N+1$ we take a source of the tournament above and increase it. Conversely, to show uniqueness for N, note that we can take the (nonzero) sink of the tournament and decrement it, which gives $N - 1$; our uniqueness inductive hypothesis now finishes.

Remark. Colin Tang found a nice proof of uniqueness:

$$s_k + \sum_{i=1}^{k-1} a_{0,i} \le k a_{0,k} \le s_k + \sum_{i=1} (a_{0,i} + 1)$$

and similarly for other entries.

§3.3 TSTST 2012/9, proposed by John Berman

Available online at https://aops.com/community/p2745874.

Problem statement

Given a set S of n variables, a binary operation $\times$ on S is called *simple* if it satisfies $(x \times y) \times z = x \times (y \times z)$ for all $x, y, z \in S$ and $x \times y \in \{x, y\}$ for all $x, y \in S$. Given a simple operation $\times$ on S, any string of elements in S can be reduced to a single element, such as $xyz \to x \times (y \times z)$. A string of variables in S is called *full* if it contains each variable in S at least once, and two strings are *equivalent* if they evaluate to the same variable regardless of which simple $\times$ is chosen. For example xxx, xx, and x are equivalent, but these are only full if $n = 1$. Suppose T is a set of full strings such that any full string is equivalent to exactly one element of T. Determine the number of elements of T.

The answer is $(n!)^2$. In fact it is possible to essentially find all $\times$: one assigns a real number to each variable in S. Then $x \times y$ takes the larger of $\{x, y\}$, and in the event of a tie picks either "left" or "right", where the choice of side is fixed among elements of each size.

¶ First solution (Steven Hao) The main trick is the two lemmas, which are not hard to show (and are motivated by our conjecture).

$$xx = x$$
$$xyxzx = xyzx.$$

Consequently, define a **double rainbow** to be the concatenation of two full strings of length n, of which there are $(n!)^2$. We claim that these form equivalence classes for T.

To see that any string s is equivalent to a double rainbow, note that $s = ss$, and hence using the second identity above repeatedly lets us reduce ss to a double rainbow.

To see two distinct double rainbows R_1 and R_2 aren't equivalent, one can use the construction mentioned in the beginning. Specifically, take two variables a and b which do not appear in the same order in R_1 and R_2. Then it's not hard to see that $abab$, $abba$, $baab$, $baba$ are pairwise non-equivalent by choosing "left" or "right" appropriately. Now construct $\times$ on the whole set by having a and b be the largest variables, so the rest of the variables don't matter in the evaluation of the string.

¶ Second solution outline (Ankan Bhattacharya) We outline a proof of the characterization claimed earlier, which will also give the answer $(n!)^2$. We say $a \sim b$ if $ab \neq ba$. Also, say $a > b$ if $ab = ba = a$. The following are proved by finite casework, using the fact that $\{ab, bc, ca\}$ always has exactly two distinct elements for any different a, b, c.

- If $a > b$ and $b > c$ then $a > c$.
- If $a \sim b$ and $b \sim c$ then $ab = a$ if and only if $bc = b$.
- If $a \sim b$ and $b \sim c$ then $a \sim c$.
- If $a \sim b$ and $a > c$ then $b > c$.
- If $a \sim b$ and $c > a$ then $c > b$.

This gives us the total ordering on the elements and the equivalence classes by $\sim$. In this we way can check the claimed operations are the only ones.

We can then (as in the first solution) verify that every full string is equivalent to a unique double rainbow — but this time we prove it by simply considering all possible $\times$, because we have classified them all.

Team Selection Test for the Selection Team of 55$^{\text{th}}$ IMO

Lincoln, Nebraska

Day I 1:00 PM - 5:30 PM

June 21, 2013

1. Let ABC be a triangle and D, E, F be the midpoints of arcs BC, CA, AB on the circumcircle. Line ℓ_a passes through the feet of the perpendiculars from A to DB and DC. Line m_a passes through the feet of the perpendiculars from D to AB and AC. Let A_1 denote the intersection of lines ℓ_a and m_a. Define points B_1 and C_1 similarly. Prove that triangles DEF and $A_1B_1C_1$ are similar to each other.

2. A finite sequence of integers $a_1, a_2, \ldots, a_n$ is called *regular* if there exists a real number x satisfying
$$\lfloor kx \rfloor = a_k \quad \text{for } 1 \le k \le n.$$
Given a regular sequence $a_1, a_2, \ldots, a_n$, for $1 \le k \le n$ we say that the term a_k is *forced* if the following condition is satisfied: the sequence
$$a_1, a_2, \ldots, a_{k-1}, b$$
is regular if and only if $b = a_k$. Find the maximum possible number of forced terms in a regular sequence with 1000 terms.

3. Divide the plane into an infinite square grid by drawing all the lines $x = m$ and $y = n$ for $m, n \in \mathbb{Z}$. Next, if a square's upper-right corner has both coordinates even, color it black; otherwise, color it white (in this way, exactly $1/4$ of the squares are black and no two black squares are adjacent). Let r and s be odd integers, and let (x, y) be a point in the interior of any white square such that $rx - sy$ is irrational. Shoot a laser out of this point with slope r/s; lasers pass through white squares and reflect off black squares. Prove that the path of this laser will from a closed loop.

Committee on the American Mathematics Competitions,
Mathematical Association of America

Team Selection Test for the Selection Team of 55$^{\text{th}}$ IMO

Lincoln, Nebraska

Day II 1:00 PM - 5:30 PM

June 23, 2013

4. Circle ω, centered at X, is internally tangent to circle Ω, centered at Y, at T. Let P and S be variable points on Ω and ω, respectively, such that line PS is tangent to ω (at S). Determine the locus of O – the circumcenter of triangle PST.

5. Let p be a prime. Prove that any complete graph with $1000p$ vertices, whose edges are labelled with integers, has a cycle whose sum of labels is divisible by p.

6. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$

for all $a, b, c \geq 2$.
(Here $f^1(n) = f(n)$ and $f^k(n) = f(f^{k-1}(n))$ for every integer k greater than 1.)

Committee on the American Mathematics Competitions,
Mathematical Association of America

Team Selection Test for the Selection Team of 55^{th} IMO

Lincoln, Nebraska

Day III 1:00 PM - 5:30 PM

June 25, 2013

7. A country has n cities, labelled 1, 2, 3, ..., n. It wants to build exactly $n-1$ roads between certain pairs of cities so that every city is reachable from every other city via some sequence of roads. However, it is not permitted to put roads between pairs of cities that have labels differing by exactly 1, and it is also not permitted to put a road between cities 1 and n. Let T_n be the total number of possible ways to build these roads.

   (a) For all odd n, prove that T_n is divisible by n.

   (b) For all even n, prove that T_n is divisible by $n/2$.

8. Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$, $f(n+1) = f(n) + 2^{f(n)}$ for every positive integer n. Prove that $f(1), f(2), \ldots, f(3^{2013})$ leave distinct remainders when divided by 3^{2013}.

9. Let r be a rational number in the interval $[-1, 1]$ and let $\theta = \cos^{-1} r$. Call a subset S of the plane *good* if S is unchanged upon rotation by θ around any point of S (in both clockwise and counterclockwise directions). Determine all values of r satisfying the following property: The midpoint of any two points in a good set also lies in the set.

Committee on the American Mathematics Competitions,
Mathematical Association of America

TSTST 2013 Solution Notes

Lincoln, Nebraska

EVAN CHEN《陳誼廷》

8 November 2023

This is a compilation of solutions for the 2013 TSTST. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let ABC be a triangle and D, E, F be the midpoints of arcs BC, CA, AB on the circumcircle. Line ℓ_a passes through the feet of the perpendiculars from A to $\overline{DB}$ and $\overline{DC}$. Line m_a passes through the feet of the perpendiculars from D to $\overline{AB}$ and $\overline{AC}$. Let A_1 denote the intersection of lines ℓ_a and m_a. Define points B_1 and C_1 similarly. Prove that triangles DEF and $A_1B_1C_1$ are similar to each other.

2. A finite sequence of integers $a_1, a_2, \dots, a_n$ is called *regular* if there exists a real number x satisfying
$$\lfloor kx \rfloor = a_k \quad \text{for } 1 \le k \le n.$$
Given a regular sequence $a_1, a_2, \dots, a_n$, for $1 \le k \le n$ we say that the term a_k is *forced* if the following condition is satisfied: the sequence
$$a_1,\ a_2,\ \dots,\ a_{k-1},\ b$$
is regular if and only if $b = a_k$.
Find the maximum possible number of forced terms in a regular sequence with 1000 terms.

3. Divide the plane into an infinite square grid by drawing all the lines $x = m$ and $y = n$ for $m, n \in \mathbb{Z}$. Next, if a square's upper-right corner has both coordinates even, color it black; otherwise, color it white (in this way, exactly $1/4$ of the squares are black and no two black squares are adjacent). Let r and s be odd integers, and let (x, y) be a point in the interior of any white square such that $rx - sy$ is irrational. Shoot a laser out of this point with slope r/s; lasers pass through white squares and reflect off black squares. Prove that the path of this laser will from a closed loop.

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5. Let p be a prime. Prove that in a complete graph with $1000p$ vertices whose edges are labelled with integers, one can find a cycle whose sum of labels is divisible by p.

6. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f \colon \mathbb{N} \to \mathbb{N}$ that satisfy the equation
$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$
for all $a, b, c \ge 2$. (Here f^k means f applied k times.)

7. A country has n cities, labelled $1, 2, 3, \dots, n$. It wants to build exactly $n - 1$ roads between certain pairs of cities so that every city is reachable from every other city via some sequence of roads. However, it is not permitted to put roads between pairs of cities that have labels differing by exactly 1, and it is also not permitted to put a road between cities 1 and n. Let T_n be the total number of possible ways to build these roads.
   (a) For all odd n, prove that T_n is divisible by n.
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8. Define a function $f\colon \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$, $f(n+1) = f(n) + 2^{f(n)}$ for every positive integer n. Prove that $f(1)$, $f(2)$, ..., $f(3^{2013})$ leave distinct remainders when divided by 3^{2013}.

9. Let r be a rational number in the interval $[-1, 1]$ and let $\theta = \cos^{-1} r$. Call a subset S of the plane good if S is unchanged upon rotation by θ around any point of S (in both clockwise and counterclockwise directions). Determine all values of r satisfying the following property: The midpoint of any two points in a good set also lies in the set.

§1 Solutions to Day 1

§1.1 TSTST 2013/1

Available online at https://aops.com/community/p3181479.

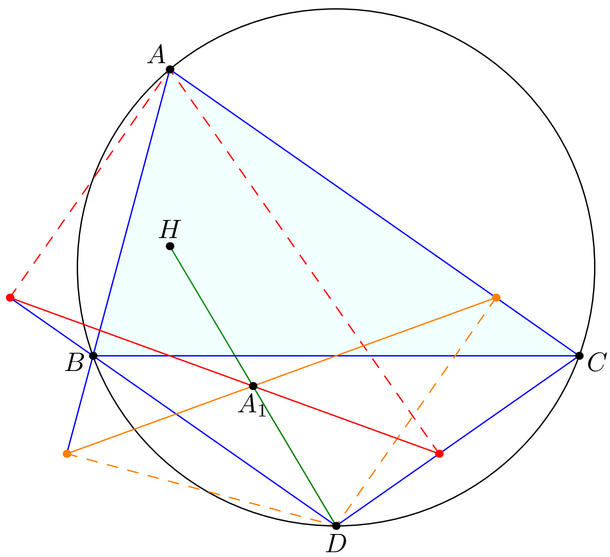
Problem statement

Let ABC be a triangle and D, E, F be the midpoints of arcs BC, CA, AB on the circumcircle. Line ℓ_a passes through the feet of the perpendiculars from A to $\overline{DB}$ and $\overline{DC}$. Line m_a passes through the feet of the perpendiculars from D to $\overline{AB}$ and $\overline{AC}$. Let A_1 denote the intersection of lines ℓ_a and m_a. Define points B_1 and C_1 similarly. Prove that triangles DEF and $A_1B_1C_1$ are similar to each other.

In fact, it is true for any points D, E, F on the circumcircle. More strongly we contend:

Claim — Point A_1 is the midpoint of $\overline{HD}$.

Proof. Lines m_a and ℓ_a are Simson lines, so they both pass through the point $(a+b+c+d)/2$ in complex coordinates. □



Hence $A_1B_1C_1$ is similar to DEF through a homothety at H with ratio $\frac{1}{2}$.

§1.2 TSTST 2013/2

Available online at https://aops.com/community/p3181480.

Problem statement

A finite sequence of integers $a_1, a_2, \dots, a_n$ is called *regular* if there exists a real number x satisfying

$$\lfloor kx \rfloor = a_k \quad \text{for } 1 \le k \le n.$$

Given a regular sequence $a_1, a_2, \dots, a_n$, for $1 \le k \le n$ we say that the term a_k is *forced* if the following condition is satisfied: the sequence

$$a_1,\ a_2,\ \dots,\ a_{k-1},\ b$$

is regular if and only if $b = a_k$.

Find the maximum possible number of forced terms in a regular sequence with 1000 terms.

The answer is 985. WLOG, by shifting $a_1 = 0$ (clearly a_1 isn't forced). Now, we construct regular sequences inductively using the following procedure. Start with the inequality

$$\frac{0}{1} \le x < \frac{1}{1}.$$

Then for each $k = 2, 3, \dots, 1000$ we perform the following procedure. If there is no fraction of the form $F = \frac{m}{k}$ in the interval $A \le x < B$, then a_k is forced, and the interval of possible x values does not change. Otherwise, a_k is not forced, and we pick a value of a_k and update the interval accordingly.

The theory of Farey sequences tells us that when we have a stage $\frac{a}{b} \le x < \frac{c}{d}$ then the next time we will find a fraction in that interval is exactly $\frac{a+c}{b+d}$ (at time $k = b + d$), and it will be the only such fraction.

So essentially, starting with $\frac{0}{1} \le x < \frac{1}{1}$ we repeatedly replace one of the endpoints of the intervals with the mediant, until one of the denominators exceeds 1000; we are trying to minimize the number of non-forced terms, which is the number of denominators that appear in this process. It is not hard to see that this optimum occurs by always replacing the smaller of the denominators, so that the sequence is

$$\begin{aligned} \tfrac{0}{1} &\le x < \tfrac{1}{1} \\ \tfrac{0}{1} &\le x < \tfrac{1}{2} \\ \tfrac{1}{3} &\le x < \tfrac{1}{2} \\ \tfrac{1}{3} &\le x < \tfrac{2}{5} \\ \tfrac{3}{8} &\le x < \tfrac{2}{5} \\ \tfrac{3}{8} &\le x < \tfrac{5}{13} \end{aligned}$$

and so on; we see that the non-forced terms in this optimal configuration are exactly the Fibonacci numbers. There are 15 Fibonacci numbers less than 1000, hence the answer $1000 - 15 = 985$.

§1.3 TSTST 2013/3

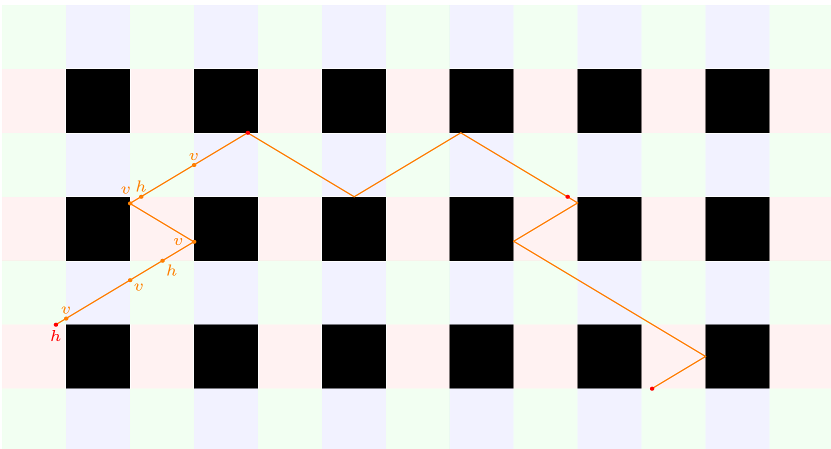
Available online at https://aops.com/community/p3181481.

Problem statement

Divide the plane into an infinite square grid by drawing all the lines $x = m$ and $y = n$ for $m, n \in \mathbb{Z}$. Next, if a square's upper-right corner has both coordinates even, color it black; otherwise, color it white (in this way, exactly $1/4$ of the squares are black and no two black squares are adjacent). Let r and s be odd integers, and let (x, y) be a point in the interior of any white square such that $rx - sy$ is irrational. Shoot a laser out of this point with slope r/s; lasers pass through white squares and reflect off black squares. Prove that the path of this laser will from a closed loop.

Here is Sammy Luo's solution. Fix the speed of light at $\sqrt{r^2 + s^2}$ units per second. We prove periodicity every six seconds.

We re-color the white squares as red, blue, or green according as to whether they have a black square directly to the left/right, above/below, or neither, as shown below. Finally, we fix time zero to be a moment just before the laser passes a horizontal (WLOG) lattice line (not necessarily a wall). Shown below is an example for $(r, s) = (3, 5)$.



The main idea is to keep track of every time the laser passes a lattice line (again, not necessarily a wall). There are four possible types of events:

- A horizontal h event where the laser switches from red to green (or vice-versa);
- A horizontal h event where the laser rebounds off a wall, remaining in a blue square, but flips the x-component of its velocity;
- A vertical v event where the laser switches from blue to green (or vice-versa)
- A vertical v event where the laser rebounds off a wall, remaining in a red square, but flips the y-component of its velocity.

The first key observation is that:

Claim — In the first second, the laser will encounter exactly r horizontal events and s vertical events. In every second after that, the same sequence of $r+s$ events occurs.

Proof. Bouncing off a wall doesn't change this as opposed to if the laser had passed through the wall. □

We let the *key-word* be the sequence w of $r+s$ letters corresponding to the sequence. For example, the picture above denotes an example with keyword $w = hvvhvvhv$; so no matter what, every second, the laser will encounter eight lattice lines, which are horizontal and vertical in that order.

Claim — Color is periodic every 3 seconds.

Proof. The free group generated by h and v acts on the set $\{R, G, B\}$ of colors in an obvious way; consider this right action. First we consider the color of the square after each second. Note that with respect to color, each letter is an involution; so as far as color changes are concerned, it's enough to work with the reduced word w' obtained by modding out by $h^2 = 1$ and $v^2 = 1$. (For example, $w' = hv$ in our example.) In general, $w' = (hv)^k$ or $w' = (vh)^k$, for some odd integer k (since $k \equiv r \equiv s \equiv 1 \pmod 2$). Now we see that the action of hv on the set of colors is red $\mapsto$ blue $\mapsto$ green $\mapsto$ red, and similarly for vh (being the inverse). This implies that the color is periodic every three seconds. □

Now in a 3-second period, consider the $3r$ horizontal events and $3s$ vertical events (both are odd). In order for the color to remain the same (as the only color changes are $R \leftrightarrow G$ for h and $B \leftrightarrow G$ for v) there must have been an even number of color swaps for each orientation. Therefore there was an odd number of wall collisions of each orientation. So, the laser is pointing in the opposite direction at the end of 3 seconds.

Finally, let x_t be the fractional part of the x coordinate after t seconds (the y-coordinate is always zero by our setup at these moments). Note that

$$x_{t+1} = \begin{cases} x_t & \text{even number of vertical wall collisions} \\ 1 - x_t & \text{odd numbers of vertical wall collisions} \end{cases}$$

Since over the there seconds there were an odd number of vertical collisions; it follows $x_3 = 1 - x_0$. Thus at the end of three seconds, the laser is in a symmetric position from the start; and in 6 seconds it will form a closed loop.

§2 Solutions to Day 2

§2.1 TSTST 2013/4

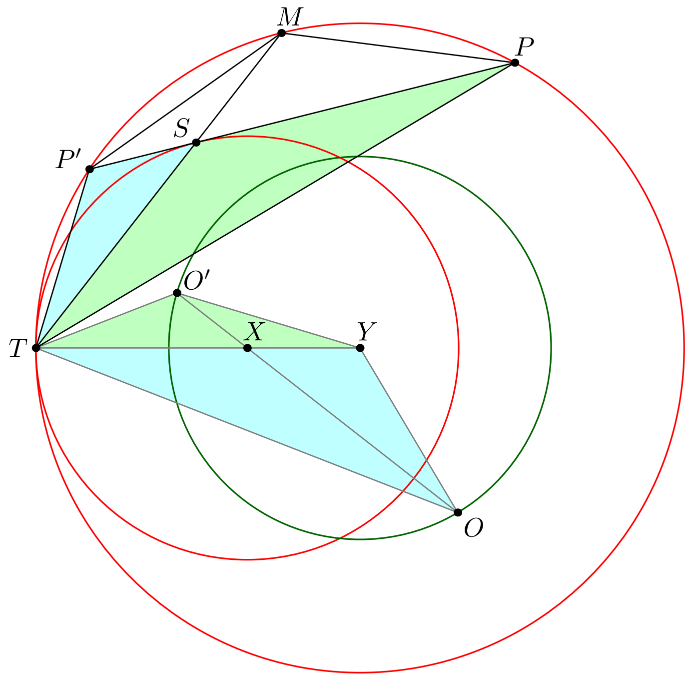
Available online at https://aops.com/community/p3181482.

Problem statement

Circle ω, centered at X, is internally tangent to circle Ω, centered at Y, at T. Let P and S be variable points on Ω and ω, respectively, such that line PS is tangent to ω (at S). Determine the locus of O – the circumcenter of triangle PST.

The answer is a circle centered at Y with radius $\sqrt{YX \cdot YT}$, minus the two points on line XY itself.

We let PS meet Ω again at P', and let O' be the circumcenter of $\triangle TPS'$. Note that O', X, O are collinear on the perpendicular bisector of line $\overline{TS}$ Finally, we let M denote the arc midpoint of PP' which lies on line TS (by homothety).



By three applications of Salmon theorem, we have the following spiral similarities all centered at T:

$$\triangle TSP \overset{+}{\sim} \triangle TO'Y$$
$$\triangle TP'S \overset{+}{\sim} \triangle TYO$$
$$\triangle TP'P \overset{+}{\sim} \triangle TO'O.$$

However, the shooting lemma also gives us two similarities:

$$\triangle TP'M \overset{+}{\sim} \triangle TSP$$
$$\triangle TMP \overset{+}{\sim} \triangle TP'S.$$

Putting everything together, we find that

$$TP'MP \overset{+}{\sim} TO'YO.$$

Then by shooting lemma, $YO'^2 = YX \cdot YT$, so O indeed lies on the claimed circle.

As the line $\overline{O'O}$ may be any line through X other than line XY (one takes S to be the reflection of T across this line) one concludes the only two non-achievable points are the diametrically opposite ones on line XY of this circle (because this leads to the only degenerate situation where $S = T$).

§2.2 TSTST 2013/5

Available online at https://aops.com/community/p3181483.

Problem statement

Let p be a prime. Prove that in a complete graph with $1000p$ vertices whose edges are labelled with integers, one can find a cycle whose sum of labels is divisible by p.

Select $p-1$ disjoint triangles arbitrarily. If any of these triangles have 0 sum modulo p we are done. Otherwise, we may label the vertices u_i, x_i, and v_i (where $1 \le i \le p-1$) in such a way that $u_i x_i + x_i v_i \neq u_i v_i$.

Let $A_i = \{u_i x_i + x_i v_i, u_i v_i\}$. We can show that $|A_1 + A_2 + \dots + A_t| \ge \min\{p, t+1\}$ for each $1 \le t \le p-1$, by using induction on t alongside Cauchy-Davenport. So, $A_1 + A_2 + \dots + A_{p-1}$ spans all of $\mathbb{Z}_p$. All that's left to do is join the triangles together to form a cycle, and then delete either $u_i x_i, x_i v_i$ or $u_i v_i$ from each triangle in such a way that the final sum is 0 mod p.

§2.3 TSTST 2013/6

Available online at https://aops.com/community/p3181484.

Problem statement

Let $\mathbb{N}$ be the set of positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ that satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$

for all $a, b, c \ge 2$. (Here f^k means f applied k times.)

The answer is $f(n) = n - 1$ for $n \ge 3$ with $f(1)$ and $f(2)$ arbitrary; check these work.

Lemma

We have $f^{t^2-t}(t^2) = t$ for all t.

Proof. We say $1 \le k \le 8$ is good if $f^{t^9-t^k}(t^9) = t^k$ for all t. First, we observe that

$$f^{t^9-t^3}(t^9) = t^3 \quad \text{and} \quad f^{t^3-t}(t^3) = t \implies f^{t^9-t}(t^9) = t.$$

so $k = 1$ and $k = 3$ are good. Then taking $(a, b, c) = (t, t^4, t^4)$, $(a, b, c) = (t^2, t^3, t^4)$ gives that $k = 4$ and $k = 2$ are good, respectively. The lemma follows from this $k = 1$ and $k = 2$ being good. □

Now, letting $t = abc$ we combine

$$\begin{aligned} f^{t-a}(a) + f^{t-b}(b) + f^{t-c}(c) &= a + b + c \\ f^{t^2-ab}(t^2) + f^{t^2-t}(t^2) + f^{t^2-c}(t^2) &= ab + t + c \\ \implies \left[f^{t-a}(t) - a\right] + \left[f^{t-b}(t) - b\right] &= \left[f^{t-ab}(t) - ab\right] \end{aligned}$$

by subtracting and applying the lemma repeatedly. In other words, we have proven the second lemma:

Lemma

Let t be fixed, and define $g_t(n) = f^{t-n}(t) - n$ for $n < t$. If $a, b \ge 2$ and $ab \mid t$, $ab < t$, then $g_t(a) + g_t(b) = g_t(ab)$.

Now let $a, b \ge 2$ be arbitrary, and let $p > q > \max\{a, b\}$ be primes. Suppose $s = a^p b^q$ and $t = s^2$; then

$$pg_t(a) + qg_t(b) = g_t\left(a^p b^q\right) = g_t(s) = f^{s^2-s}(s) - s = 0.$$

Now

$$q \mid g_t(a) > -a \quad \text{and} \quad p \mid g_t(b) > -b \implies g_t(a) = g_t(b) = 0.$$

and so we conclude $f^{t-a}(t) = a$ and $f^{t-b}(t) = b$ for $a, b \ge 2$.

In particular, if $a = n$ and $b = n + 1$ then we deduce $f(n + 1) = n$ for all $n \ge 2$, as desired.

Remark. If you let $c = (ab)^2$ after the first lemma, you recover the 2-variable version!

§3 Solutions to Day 3

§3.1 TSTST 2013/7

Available online at https://aops.com/community/p3181485.

Problem statement

A country has n cities, labelled $1, 2, 3, \dots, n$. It wants to build exactly $n-1$ roads between certain pairs of cities so that every city is reachable from every other city via some sequence of roads. However, it is not permitted to put roads between pairs of cities that have labels differing by exactly 1, and it is also not permitted to put a road between cities 1 and n. Let T_n be the total number of possible ways to build these roads.

(a) For all odd n, prove that T_n is divisible by n.

(b) For all even n, prove that T_n is divisible by $n/2$.

You can just spin the tree!

Fixing n, the group $G = \mathbb{Z}/n\mathbb{Z}$ acts on the set of trees by rotation (where we imagine placing $1, 2, \dots, n$ along a circle).

Claim — For odd n, all trees have trivial stabilizer.

Proof. One way to see this is to look at the degree sequence. Suppose g^e fixes a tree T. Then so does g^k, for $k = \gcd(e, n)$. Then it follows that n/k divides $\sum_v \deg v = 2n - 2$. Since $\gcd(2n-2, n) = 1$ we must then have $k = n$. □

The proof for even n is identical except that $\gcd(2n-2, n) = 2$ and hence each tree either has stabilizer with size ≤ 2.

There is also a proof using linear algebra, using Kirchoff's tree formula. (Overkill.)

§3.2 TSTST 2013/8

Available online at https://aops.com/community/p3181486.

Problem statement

Define a function $f \colon \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$, $f(n+1) = f(n) + 2^{f(n)}$ for every positive integer n. Prove that $f(1)$, $f(2)$, $\dots$, $f(3^{2013})$ leave distinct remainders when divided by 3^{2013}.

I'll prove by induction on $k \ge 1$ that any 3^k consecutive values of f produce distinct residues modulo 3^k. The base case $k = 1$ is easily checked (f is always odd, hence f cycles 1, 0, 2 mod 3).

For the inductive step, assume it's true up to k. Since $2^\bullet \pmod{3^{k+1}}$ cycles every $2 \cdot 3^k$, and f is always odd, it follows that

$$\begin{aligned} f(n+3^k) - f(n) &= 2^{f(n)} + 2^{f(n+1)} + \dots + 2^{f(n+3^k-1)} \pmod{3^{k+1}} \\ &\equiv 2^1 + 2^3 + \dots + 2^{2 \cdot 3^k - 1} \pmod{3^{k+1}} \\ &= 2 \cdot \frac{4^{3^k} - 1}{4 - 1}. \end{aligned}$$

Hence

$$f(n+3^k) - f(n) \equiv C \pmod{3^{k+1}} \qquad \text{where} \qquad C = 2 \cdot \frac{4^{3^k} - 1}{4 - 1}$$

noting that C does not depend on n. Exponent lifting gives $\nu_3(C) = k$ hence $f(n)$, $f(n+3^k)$, $f(n+2 \cdot 3^k)$ differ mod 3^{k+1} for all n, and the inductive hypothesis now solves the problem.

§3.3 TSTST 2013/9

Available online at https://aops.com/community/p3181487.

Problem statement

Let r be a rational number in the interval $[-1, 1]$ and let $\theta = \cos^{-1} r$. Call a subset S of the plane good if S is unchanged upon rotation by θ around any point of S (in both clockwise and counterclockwise directions). Determine all values of r satisfying the following property: The midpoint of any two points in a good set also lies in the set.

The answer is that r has this property if and only if $r = \frac{4n-1}{4n}$ for some integer n.

Throughout the solution, we will let $r = \frac{a}{b}$ with $b > 0$ and $\gcd(a, b) = 1$. We also let

$$\omega = e^{i\theta} = \frac{a}{b} \pm \frac{\sqrt{b^2 - a^2}}{b} i.$$

This means we may work with complex multiplication in the usual way; the rotation of z through center c is given by $z \mapsto \omega(z - c) + c$.

For most of our proof, we start by constructing a good set as follows.

- Start by letting $S_0 = \{0, 1\}$.
- Let S_i consist of S_{i-1} plus all points that can be obtained by rotating a point of S_{i-1} through a different point of S_{i-1} (with scale factor ω).
- Let $S_\infty = \bigcup_{i \ge 0} S_i$.

The set S_∞ is the (minimal, by inclusion) good set containing 0 and 1. We are going to show that for most values of r, we have $\frac{1}{2} \notin S_\infty$.

Claim — If b is odd, then $\frac{1}{2} \notin S_\infty$.

Proof. Idea: denominators that appear are always odd.

Consider the ring

$$A = \mathbb{Z}_{\{b\}} = \left\{ \frac{s}{t} \mid s, t \in \mathbb{Z}, t \mid b^\infty \right\}$$

which consists of all rational numbers whose denominators divide b^∞. Then, $0, 1, \omega \in A[\sqrt{b^2 - a^2}]$ and hence $S_\infty \subseteq A[\sqrt{b^2 - a^2}]$ too. (This works even if $\sqrt{b^2 - a^2} \in \mathbb{Z}$, in which case $S_\infty \subseteq A = A[\sqrt{b^2 - a^2}]$.)

But $\frac{1}{2} \notin A[\sqrt{b^2 - a^2}]$. □

Claim — If b is even and $|b - a| \neq 1$, then $\frac{1}{2} \notin S_\infty$.

Proof. Idea: take modulo a prime dividing $b - a$.

Let $D = b^2 - a^2 \equiv 3 \pmod 4$. Let p be a prime divisor of $b - a$ with odd multiplicity. Because $\gcd(a, b) = 1$, we have $p \neq 2$ and $p \nmid b$.

Consider the ring

$$A = \mathbb{Z}_{(p)} = \left\{ \frac{s}{t} \mid s, t \in \mathbb{Z}, p \perp t \right\}$$

which consists of all rational numbers whose denominators are coprime to p. Then, $0, 1, \omega \in A[\sqrt{-D}]$ and hence $S_\infty \subseteq A[\sqrt{-D}]$ too.

Now, there is a well-defined "mod-p" ring homomorphism

$$\Psi\colon A[\sqrt{-D}] \to \mathbb{F}_p \quad \text{by} \quad x + y\sqrt{-D} \mapsto x \bmod p$$

which commutes with addition and multiplication (as $p \mid D$). Under this map,

$$\omega \mapsto \frac{a}{b} \bmod p = 1.$$

Consequently, the rotation $z \mapsto \omega(z-c)+c$ is just the identity map modulo p. In other words, the pre-image of any point in S_∞ under Ψ must be either $\Psi(0)=0$ or $\Psi(1)=1$.

However, $\Psi(1/2) = 1/2$ is neither of these. So this point cannot be achieved. $\square$

Claim — Suppose $a = 2n-1$ and $b = 2n$ for n an odd integer. Then $\frac{1}{2} \notin S_\infty$

Proof. Idea: ω is "algebraic integer" sans odd denominators.

This time, we define the ring

$$B = \mathbb{Z}_{(2)} = \left\{ \frac{s}{t} \mid s, t \in \mathbb{Z}, t \text{ odd} \right\}$$

of rational numbers with odd denominator. We carefully consider the ring $B[\omega]$ where

$$\omega = \frac{2n-1 \pm \sqrt{1-4n}}{2n}.$$

So $S_\infty \subseteq B[\omega]$ as $0, 1, \omega \in B[\omega]$.

I claim that $B[\omega]$ is an integral extension of B; equivalently that ω is integral over B. Indeed, ω is the root of the monic polynomial

$$(T-1)^2 + \frac{1}{n}(T-1) - \frac{1}{n} = 0$$

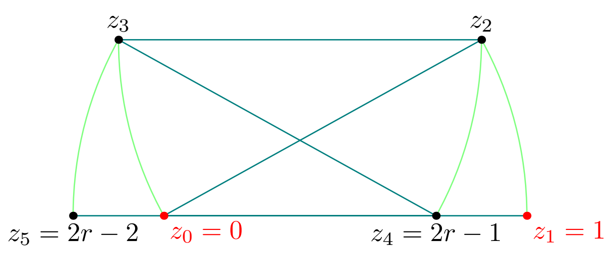
where $\frac{1}{n} \in B$ makes sense as n is odd.

On the other hand, $\frac{1}{2}$ is not integral over B so it is not an element of $B[\omega]$. $\square$

It remains to show that if $r = \frac{4n-1}{4n}$, then goods sets satisfy the midpoint property. Again starting from the points $z_0 = 0$, $z_1 = 1$ construct the sequence

$$\begin{aligned} z_2 &= \omega(z_1 - z_0) + z_0 \\ z_3 &= \omega^{-1}(z_0 - z_2) + z_2 \\ z_4 &= \omega^{-1}(z_2 - z_3) + z_3 \\ z_5 &= \omega(z_3 - z_4) + z_4 \end{aligned}$$

as shown in the diagram below.



This construction shows that if we have the length-one segment $\{0, 1\}$ then we can construct the length-one segment $\{2r-2, 2r-1\}$. In other words, we can shift the segment to the left by

$$1-(2r-1)=2(1-r)=\frac{1}{2n}.$$

Repeating this construction n times gives the desired midpoint $\frac{1}{2}$.

Team Selection Test Selection Test 1
June 23, 2014
1:15 – 5:45pm

Problems:

1. Let $\leftarrow$ denote the left arrow[1] key on a standard keyboard. If one opens a text editor and types the keys "ab$\leftarrow$cd$\leftarrow\leftarrow$e$\leftarrow\leftarrow$f", the result is "faecdb". We say that a string B is *reachable* from a string A if it is possible to insert some amount of $\leftarrow$'s into A, such that typing the resulting characters produces B. So, our example shows that "faecdb" is reachable from "abcdef".

   Prove that for any two strings A and B, A is reachable from B if and only if B is reachable from A.

2. Consider a convex pentagon circumscribed about a circle. We name the lines that connect vertices of the pentagon with the opposite points of tangency with the circle *gergonnians*.

   (a) Prove that if four gergonnians are concurrent, then all five of them are concurrent.

   (b) Prove that if there is a triple of gergonnians that are concurrent, then you can find another triple of gergonnians that are concurrent.

3. Find all polynomial functions $P(x)$ with real coefficients that satisfy

$$P(x\sqrt{2}) = P(x + \sqrt{1 - x^2})$$

   for all real x with $|x| \leq 1$.

---

[1] Here is a short explanation of how the $\leftarrow$ key works. A computer's text editor always starts with an empty screen, and a cursor which we denote by "|". When you type a letter x, the cursor | is replaced by x|. So if the screen shows "m|th", and you press the "o" key, the result is "mo|th".

The $\leftarrow$ key moves the cursor one space backwards. That is, "mo|th" becomes "m|oth", and finally "|moth". If the cursor is already at the beginning of the string, the $\leftarrow$ key has no effect.

Note that the cursor is not considered to be a part of the final string. In the example above, after typing "ab$\leftarrow$cd$\leftarrow\leftarrow$e$\leftarrow\leftarrow$f", the screen displays "f|aecdb", so we take the result to be "faecdb".

Team Selection Test Selection Test 2
June 25, 2014
1:15 – 5:45pm

Problems:

4. Let $P(x)$ and $Q(x)$ be arbitrary polynomials with real coefficients, and let d be the degree of $P(x)$. Assume that $P(x)$ is not the zero polynomial. Prove that there exist polynomials $A(x)$ and $B(x)$ with real coefficients, such that:

   (i) both A and B have degree at most $d/2$, and

   (ii) at most one of A and B is the zero polynomial, and

   (iii) $\frac{A(x)+Q(x)B(x)}{P(x)}$ is a polynomial with real coefficients. That is, there is some polynomial $C(x)$ with real coefficients such that $A(x) + Q(x)B(x) = P(x)C(x)$.

5. Find the maximum number E such that the following holds: there is an edge-colored graph with 60 vertices and E edges, with each edge colored either red or blue, such that in that coloring, there are no monochromatic cycles of length 3 and no monochromatic cycles of length 5.

6. Suppose we have distinct positive integers a, b, c, d, and an odd prime p not dividing any of them, and an integer M such that if one considers the infinite sequence

$$\begin{aligned}&ca - db\\&ca^2 - db^2\\&ca^3 - db^3\\&ca^4 - db^4\\&\dots\end{aligned}$$

   and looks at the highest power of p that divides each of them, these powers are not all zero, and are all at most M. Prove that there exists some T (which may depend on a, b, c, d, p, M) such that whenever p divides an element of this sequence, the maximum power of p that divides that element is exactly p^T.

TSTST 2014 Solution Notes

Lincoln, Nebraska

Evan Chen《陳誼廷》

30 September 2023

This is a compilation of solutions for the 2014 TSTST. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

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   Prove that for any two strings A and B, A is reachable from B if and only if B is reachable from A.

2. Consider a convex pentagon circumscribed about a circle. We name the lines that connect vertices of the pentagon with the opposite points of tangency with the circle *gergonnians*.

   (a) Prove that if four gergonnians are concurrent, then all five of them are concurrent.

   (b) Prove that if there is a triple of gergonnians that are concurrent, then there is another triple of gergonnians that are concurrent.

3. Find all polynomials $P(x)$ with real coefficients that satisfy

$$P(x\sqrt{2}) = P(x + \sqrt{1 - x^2})$$

   for all real numbers x with $|x| \le 1$.

4. Let $P(x)$ and $Q(x)$ be arbitrary polynomials with real coefficients, with $P \neq 0$, and let $d = \deg P$. Prove that there exist polynomials $A(x)$ and $B(x)$, not both zero, such that $\max\{\deg A, \deg B\} \le d/2$ and $P(x) \mid A(x) + Q(x) \cdot B(x)$.

5. Find the maximum number E such that the following holds: there is an edge-colored graph with 60 vertices and E edges, with each edge colored either red or blue, such that in that coloring, there is no monochromatic cycles of length 3 and no monochromatic cycles of length 5.

6. Suppose we have distinct positive integers a, b, c, d and an odd prime p not dividing any of them, and an integer M such that if one considers the infinite sequence

$$\begin{gathered} ca - db \\ ca^2 - db^2 \\ ca^3 - db^3 \\ ca^4 - db^4 \\ \vdots \end{gathered}$$

   and looks at the highest power of p that divides each of them, these powers are not all zero, and are all at most M. Prove that there exists some T (which may depend on a, b, c, d, p, M) such that whenever p divides an element of this sequence, the maximum power of p that divides that element is exactly p^T.

§1 Solutions to Day 1

§1.1 TSTST 2014/1

Available online at https://aops.com/community/p3549404.

Problem statement

Let $\leftarrow$ denote the left arrow key on a standard keyboard. If one opens a text editor and types the keys "ab$\leftarrow$ cd $\leftarrow\leftarrow$ e $\leftarrow\leftarrow$ f", the result is "faecdb". We say that a string B is *reachable* from a string A if it is possible to insert some amount of $\leftarrow$'s in A, such that typing the resulting characters produces B. So, our example shows that "faecdb" is reachable from "abcdef".

Prove that for any two strings A and B, A is reachable from B if and only if B is reachable from A.

Obviously A and B should have the same multiset of characters, and we focus only on that situation.

Claim — If $A = 123\dots n$ and $B = \sigma(1)\sigma(2)\dots\sigma(n)$ is a permutation of A, then B is reachable if and only if it is **213-avoiding**, i.e. there are no indices $i < j < k$ such that $\sigma(j) < \sigma(i) < \sigma(k)$.

Proof. This is clearly necessary. To see its sufficient, one can just type B inductively: after typing k, the only way to get stuck is if $k+1$ is to the right of k and there is some character in the way; this gives a 213 pattern. □

Claim — A permutation σ on $\{1, \dots, n\}$ is 213-avoiding if and only if the inverse σ^{-1} is.

Proof. Suppose $i < j < k$ and $\sigma(j) < \sigma(i) < \sigma(k)$. Let $i' = \sigma(j)$, $j' = \sigma(i)$, $k' = \sigma(k)$; then $i' < j' < k'$ and $\sigma^{-1}(j') < \sigma^{-1}(i') < \sigma^{-1}(k')$. □

This essentially finishes the problem. Suppose B is reachable from A. By using the typing pattern, we get some permutation $\sigma \colon \{1, \dots, n\}$ such that the ith character of A is the $\sigma(i)$th character of B, and which is 213-avoiding by the claim. (The permutation is unique if A has all distinct characters, but there could be multiple if A has repeated ones.) Then σ^{-1} is 213-avoiding too and gives us a way to change B into A.

§1.2 TSTST 2014/2

Available online at https://aops.com/community/p3549405.

Problem statement

Consider a convex pentagon circumscribed about a circle. We name the lines that connect vertices of the pentagon with the opposite points of tangency with the circle *gergonnians*.

(a) Prove that if four gergonnians are concurrent, then all five of them are concurrent.

(b) Prove that if there is a triple of gergonnians that are concurrent, then there is another triple of gergonnians that are concurrent.

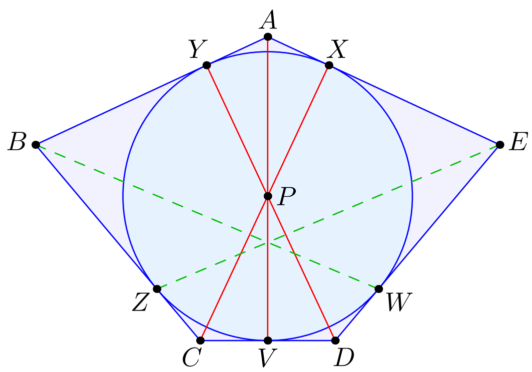
This problem is insta-killed by taking a homography sending the concurrency point (in either part) to the center of the circle while fixing the incircle. Alternatively, one may send any four of the tangency points to a rectangle.

Here are the details. Let $ABCDE$ be a pentagon with gergonnians $\overline{AV}$, $\overline{BW}$, $\overline{CX}$, $\overline{DY}$, $\overline{EZ}$. We prove the following lemma, which (up to a suitable permutation of point names) solves both parts (a) and (b).

Lemma

The gergonnians $\overline{AV}$, $\overline{CX}$, $\overline{DY}$ are concurrent if and only if the gergonnians $\overline{AV}$, $\overline{BW}$, $\overline{EZ}$ are concurrent.

Proof. We prove the first set implies the second (the converse direction being identical). Suppose $\overline{AV}$, $\overline{CX}$, $\overline{DY}$ intersect at P and take a homography fixing the circle and moving P to its center.



Then X and Y are symmetric around $\overline{APV}$ by hypothesis. Since $D = \overline{VV} \cap \overline{PY}$, $C = \overline{VV} \cap \overline{PX}$, it follows that C and D, and hence Z and W, are also symmetric around $\overline{APV}$. Consequently B and E are symmetric too. So $\overline{BW}$ and $\overline{EZ}$ meet on $\overline{AV}$. □

§1.3 TSTST 2014/3

Available online at https://aops.com/community/p3549407.

Problem statement

Find all polynomials $P(x)$ with real coefficients that satisfy

$$P(x\sqrt{2}) = P(x + \sqrt{1-x^2})$$

for all real numbers x with $|x| \le 1$.

The answer is any polynomial of the form $P(x) = f(U(x/\sqrt{2}))$, where $f \in \mathbb{R}[x]$ and U is the unique polynomial satisfying $U(\cos\theta) = \cos(8\theta)$.

Let $Q(x) = P(x\sqrt{2})$, then the condition reads

$$Q(\cos\theta) = Q\left(\frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)\right) = Q(\cos(\theta - 45^\circ)) \qquad \forall\, 0 \le \theta \le 180^\circ.$$

We call a polynomial *good* if it satisfies this functional equation.

Lemma

The minimal (by degree) good nonconstant polynomial is U.

Proof. Since U works, it suffices to show that $\deg Q \ge 8$. Note that:

$$\begin{aligned} Q(\cos 136^\circ) &= Q(\cos 91^\circ) = Q(\cos 46^\circ) = Q(\cos 1^\circ) = Q(\cos -44^\circ) \\ &= Q(\cos 44^\circ) = Q(\cos 89^\circ) = Q(\cos 134^\circ) = Q(\cos 179^\circ). \end{aligned}$$

Hence Q is equal at eight distinct values (not nine since $\cos -44^\circ = \cos 44^\circ$ is repeated), so $\deg Q \ge 8$ (unless Q is constant). □

Now, we claim $Q(x) \equiv f(U(x))$ for some $f \in \mathbb{R}[x]$. Indeed, if Q is good, then by minimality the quotient $Q \bmod U$ must be constant, so $Q(x) = \widetilde{Q}(x) \cdot U(x) + c$ for some constant c, but then $\widetilde{Q}(x)$ is good too and we finish iteratively.

§2 Solutions to Day 2

§2.1 TSTST 2014/4

Available online at https://aops.com/community/p3549409.

Problem statement

Let $P(x)$ and $Q(x)$ be arbitrary polynomials with real coefficients, with $P \neq 0$, and let $d = \deg P$. Prove that there exist polynomials $A(x)$ and $B(x)$, not both zero, such that $\max\{\deg A, \deg B\} \le d/2$ and $P(x) \mid A(x) + Q(x) \cdot B(x)$.

Let V be the vector space of real polynomials with degree at most $d/2$. Consider maps of linear spaces

$$V^{\oplus 2} \to \mathbb{R}[x]/(P(x))$$
$$\text{by} \qquad (A, B) \mapsto A + QB \pmod{P}.$$

The domain has dimension

$$2\left(\lfloor d/2 \rfloor + 1\right)$$

while the codomain has dimension d. For dimension reasons it has nontrivial kernel.

§2.2 TSTST 2014/5

Available online at https://aops.com/community/p3549412.

Problem statement

Find the maximum number E such that the following holds: there is an edge-colored graph with 60 vertices and E edges, with each edge colored either red or blue, such that in that coloring, there is no monochromatic cycles of length 3 and no monochromatic cycles of length 5.

The answer is $E = 30^2 + 2 \cdot 15^2 = 6 \cdot 15^2 = 1350$.

First, we prove $E \le 1350$. Observe that:

Claim — G contains no K_5.

Proof. It's a standard fact that the only triangle-free two-coloring of the edges of K_5 is the union of two monochromatic C_5's. □

Hence by Turán theorem we have $E \le \binom{4}{2} \cdot 15^2 = 1350$.

To show this is achievable, take a red $K_{30,30}$, and on each side draw a blue $K_{15,15}$. This graph has no monochromatic odd cycles at all as desired.

§2.3 TSTST 2014/6

Available online at https://aops.com/community/p3549417.

Problem statement

Suppose we have distinct positive integers a, b, c, d and an odd prime p not dividing any of them, and an integer M such that if one considers the infinite sequence

$$\begin{gathered} ca - db \\ ca^2 - db^2 \\ ca^3 - db^3 \\ ca^4 - db^4 \\ \vdots \end{gathered}$$

and looks at the highest power of p that divides each of them, these powers are not all zero, and are all at most M. Prove that there exists some T (which may depend on a, b, c, d, p, M) such that whenever p divides an element of this sequence, the maximum power of p that divides that element is exactly p^T.

By orders, the indices of terms divisible by p is an arithmetic subsequence of $\mathbb{N}$: say they are κ, $\kappa + \lambda$, $\kappa + 2\lambda$, $\dots$, where λ is the order of a/b. That means we want

$$\nu_p \left(ca^{\kappa+n\lambda} - db^{\kappa+n\lambda} \right) = \nu_p \left(\left(\frac{a^\lambda}{b^\lambda} \right)^n - \frac{da^\kappa}{cb^\kappa} \right)$$

to be constant. Thus, we have reduced the problem to the following proposition:

Proposition

Let p be an odd prime. Let $x, y \in \mathbb{Q}_{>0}$ such that $x \equiv y \equiv 1 \pmod p$. If the sequence $\nu_p(x^n - y)$ of positive integers is nonconstant, then it is unbounded.

For this it would be sufficient to prove the following claim.

Claim — Let p be an odd prime. Let $x, y \in \mathbb{Q}_{>0}$ such that $x \equiv y \equiv 1 \pmod p$. Suppose m and n are positive integers such that

$$d = \nu_p(x^n - y) < \nu_p(x^m - y) = e.$$

Then there exists ℓ such that $\nu_p(x^\ell - y) \ge e + 1$.

Proof. First, note that $\nu_p(x^m - x^n) = \nu_p\left((x^m - y) - (x^n - y)\right) = d$ and so by exponent lifting we can find *some* k such that

$$\nu_p(x^k - 1) = e$$

namely $k = p^{e-d}|m - n|$. (In fact, one could also choose more carefully $k = p^{e-d} \cdot \gcd(m - n, p^\infty)$, so that k is a power of p.)

Suppose we set $x^k = p^e u + 1$ and $x^m = p^e v + y$ where $u, v \in \mathbb{Q}$ aren't divisible by p. Now for any integer $1 \le r \le p-1$ we consider

$$\begin{aligned} x^{kr+m} - y &= (p^e u + 1)^r \cdot (p^e v + y) - y \\ &= p^e (v + yu \cdot r) + p^{2e} (\dots). \end{aligned}$$

By selecting r with $r \equiv -v/u \pmod p$, we ensure $p^{e+1} \mid x^{kr+m} - y$, hence $\ell = kr + m$ is as desired. □

Remark. One way to motivate the proof of the claim is as follows. Suppose we are given $\nu_p(x^m - y) = e$, and we wish to find ℓ such that $\nu_p(x^\ell - y) > e$. Then, it is necessary (albeit insufficient) that

$$x^{\ell - m} \equiv 1 \pmod{p^e} \text{ but } x^{\ell-m} \not\equiv 1 \pmod{p^{e+1}}.$$

In particular, we need $\nu_p(x^{\ell-m} - 1) = e$ exactly. So the k in the claim must exist if we are going to succeed.

On the other hand, if k is *some* integer for which $\nu_p(x^k - 1) = e$, then by choosing $\ell - m$ to be some multiple of k with no extra factors of p, we hope that we can get $\nu_p(x^\ell - y) = e + 1$. That's why we write $\ell = kr + m$ and see what happens when we expand.

2015 USA Team Selection Test Selection Test Day 1
Carnegie Mellon University
June 23, 2015
1:15 – 5:45pm

1. Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers, and let m be a fixed positive integer less than n. We say an index k with $1 \leq k \leq n$ is *good* if there exists some ℓ with $1 \leq \ell \leq m$ such that
$$a_k + a_{k+1} + \cdots + a_{k+\ell-1} \geq 0,$$
where the indices are taken modulo n. Let T be the set of all good indices. Prove that
$$\sum_{k \in T} a_k \geq 0.$$

2. Let ABC be a scalene triangle. Let K_a, L_a, and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at a point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

   (The Euler line of ABC is the line passing through the circumcenter, centroid, and orthocenter of ABC.)

3. Let P be the set of all primes, and let M be a non-empty subset of P. Suppose that for any non-empty subset $\{p_1, p_2, \ldots, p_k\}$ of M, all prime factors of $p_1p_2\cdots p_k + 1$ are also in M. Prove that $M = P$.

2015 USA Team Selection Test Selection Test Day 2
Carnegie Mellon University
June 25, 2015
1:15 – 5:45pm

4. Let x, y, and z be real numbers (not necessarily positive) such that $x^4 + y^4 + z^4 + xyz = 4$. Show that

$$x \leq 2 \quad \text{and} \quad \sqrt{2-x} \geq \frac{y+z}{2}.$$

5. Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

6. A *Nim-style game* is defined as follows. Two positive integers k and n are specified, along with a finite set S of k-tuples of integers (not necessarily positive). At the start of the game, the k-tuple $(n, 0, 0, \ldots, 0)$ is written on the blackboard.

   A legal move consists of erasing the tuple $(a_1, a_2, \ldots, a_k)$ which is written on the blackboard and replacing it with $(a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k)$, where $(b_1, b_2, \ldots, b_k)$ is an element of the set S. Two players take turns making legal moves, and the first to write a negative integer loses. In the event that neither player is ever forced to write a negative integer, the game is a draw.

   Prove that there is a choice of k and S with the following property: the first player has a winning strategy if n is a power of 2, and otherwise the second player has a winning strategy.

TSTST 2015 Solution Notes

Pittsburgh, PA

Evan Chen《陳誼廷》

30 September 2023

This is a compilation of solutions for the 2015 TSTST. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let a_1, a_2, …, a_n be a sequence of real numbers, and let m be a fixed positive integer less than n. We say an index k with $1 \le k \le n$ is *good* if there exists some ℓ with $1 \le \ell \le m$ such that

$$a_k + a_{k+1} + \cdots + a_{k+\ell-1} \ge 0,$$

where the indices are taken modulo n. Let T be the set of all good indices. Prove that

$$\sum_{k \in T} a_k \ge 0.$$

2. Let ABC be a scalene triangle. Let K_a, L_a, and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at a point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

3. Let P be the set of all primes, and let M be a non-empty subset of P. Suppose that for any non-empty subset $\{p_1, p_2, \dots, p_k\}$ of M, all prime factors of $p_1p_2 \cdots p_k + 1$ are also in M. Prove that $M = P$.

4. Let x, y, z be real numbers (not necessarily positive) such that $x^4+y^4+z^4+xyz = 4$. Prove that $x \le 2$ and

$$\sqrt{2-x} \ge \frac{y+z}{2}.$$

5. Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

6. A *Nim-style game* is defined as follows. Two positive integers k and n are specified, along with a finite set S of k-tuples of integers (not necessarily positive). At the start of the game, the k-tuple $(n, 0, 0, \dots, 0)$ is written on the blackboard.

   A legal move consists of erasing the tuple $(a_1, a_2, \dots, a_k)$ which is written on the blackboard and replacing it with $(a_1+b_1, a_2+b_2, \dots, a_k+b_k)$, where $(b_1, b_2, \dots, b_k)$ is an element of the set S. Two players take turns making legal moves, and the first to write a negative integer loses. In the event that neither player is ever forced to write a negative integer, the game is a draw.

   Prove that there is a choice of k and S with the following property: the first player has a winning strategy if n is a power of 2, and otherwise the second player has a winning strategy.

§1 Solutions to Day 1

§1.1 TSTST 2015/1, proposed by Mark Sellke

Available online at https://aops.com/community/p5017901.

Problem statement

Let $a_1, a_2, \dots, a_n$ be a sequence of real numbers, and let m be a fixed positive integer less than n. We say an index k with $1 \le k \le n$ is *good* if there exists some ℓ with $1 \le \ell \le m$ such that

$$a_k + a_{k+1} + \dots + a_{k+\ell-1} \ge 0,$$

where the indices are taken modulo n. Let T be the set of all good indices. Prove that

$$\sum_{k \in T} a_k \ge 0.$$

First we prove the result if the indices are not taken modulo n. Call a number ℓ-good if ℓ is the *smallest* number such that $a_k + a_{k+1} + \dots + a_{k+\ell-1} \ge 0$, and $\ell \le m$. Then if a_k is ℓ-good, the numbers $a_{k+1}, \dots, a_{k+\ell-1}$ are good as well.

Then by greedy from left to right, we can group all the good numbers into blocks with nonnegative sums. Repeatedly take the first good number, if ℓ-good, group it with the next ℓ numbers. An example for $m = 3$:

$$\langle 4 \rangle \quad \langle -1 \quad -2 \quad 6 \rangle \quad -9 \quad -7 \quad \langle 3 \rangle \quad \langle -2 \quad 4 \rangle \quad \langle -1.$$

We can now return to the original problem. Let N be a large integer; applying the algorithm to N copies of the sequence, we deduce that

$$N \sum_{k \in T} a_k + c_N \ge 0$$

where c_N represents some "error" from left-over terms. As $|c_N| \le \sum |a_i|$, by taking N large enough we deduce the problem.

> **Remark.** This solution was motivated by looking at the case $m = 1$, realizing how dumb it was, then looking at $m = 2$, and realizing it was equally dumb.

§1.2 TSTST 2015/2, proposed by Ivan Borsenco

Available online at https://aops.com/community/p5017915.

Problem statement

Let ABC be a scalene triangle. Let K_a, L_a, and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at a point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

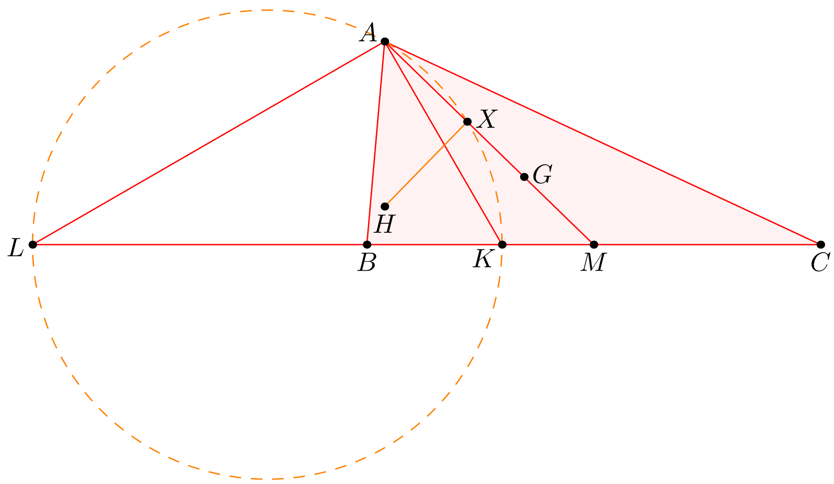
The main content of the problem:

Claim — $\angle HX_aG = 90^\circ$.

This implies the result, since then the desired circumcenter is the midpoint of $\overline{GH}$. (This is the main difficulty; the Euler line is a red herring.)

In what follows, we abbreviate K_a L_a, M_a, X_a to K, L, M, X.

First proof by Brokard. To do this, it suffices to show that M has the same power with respect to the circle with diameter $\overline{AH}$ and the circle with diameter $\overline{KL}$. In fact I claim both circles are orthogonal to the circle with diameter $\overline{BC}$! The former follows from Brokard's theorem, noting that A is on the polar of H, and the latter follows from the harmonic bundle.



Then $\overline{AM}$ is the radical axis, so X lies on both circles. □

Second proof by orthocenter reflection, Bendit Chan. As before, we know $MX \cdot MA = MK \cdot ML = MB \cdot MC$, but X lies inside segment AM. Construct parallelogram $ABA'C$. Then $MX \cdot MA' = MB \cdot MC$, so $XBA'C$ is concyclic.

However, it is well-known the circumcircle of $\triangle BA'C$ (which is the reflection of (ABC) across $\overline{BC}$) passes through H and in fact has diameter $\overline{A'H}$. So this gives $\angle HXA' = 90^\circ$ as needed. □

Third proof by barycentric coordinates. Alternatively we may just compute $X = (a^2 : 2S_A : 2S_A)$. Let $F = (0 : S_C : S_B)$ be the foot from H. Then we check that $XHFM$ is cyclic, which is power of a point from A. □

§1.3 TSTST 2015/3, proposed by Alex Zhai

Available online at https://aops.com/community/p5017928.

Problem statement

Let P be the set of all primes, and let M be a non-empty subset of P. Suppose that for any non-empty subset $\{p_1, p_2, \dots, p_k\}$ of M, all prime factors of $p_1 p_2 \cdots p_k + 1$ are also in M. Prove that $M = P$.

The following solution was found by user `Aiscrim` on AOPS.

Obviously $|M| = \infty$. Assume for contradiction $p \notin M$. We say a prime $q \in M$ is *sparse* if there are only finitely many elements of M which are $q \pmod p$ (in particular there are finitely many sparse primes).

Now let C be the product of all sparse primes (note $p \nmid C$). First, set $a_0 = 1$. For $k \ge 0$, consider then the prime factorization of the number

$$Ca_k + 1.$$

No prime in its factorization is sparse, so consider the number a_{k+1} obtained by **replacing each prime in its factorization with some arbitrary representative of that prime's residue class**. In this way we select a number a_{k+1} such that

- $a_{k+1} \equiv Ca_k + 1 \pmod p$, and
- a_{k+1} is a product of distinct primes in M.

In particular, $a_k \equiv C^k + C^{k-1} + \cdots + 1 \pmod p$

But since $C \not\equiv 0 \pmod p$, we can find a k such that $a_k \equiv 0 \pmod p$ (namely, $k = p-1$ if $C \equiv 1$ and $k = p-2$ else) which is clearly impossible since a_k is a product of primes in M!

§2 Solutions to Day 2

§2.1 TSTST 2015/4, proposed by Alyazeed Basyoni

Available online at https://aops.com/community/p5017801.

Problem statement

Let x, y, z be real numbers (not necessarily positive) such that $x^4+y^4+z^4+xyz=4$. Prove that $x \le 2$ and

$$\sqrt{2-x} \ge \frac{y+z}{2}.$$

We prove that the condition $x^4+y^4+z^4+xyz=4$ implies

$$\sqrt{2-x} \ge \frac{y+z}{2}.$$

We first prove the easy part.

Claim — We have $x \le 2$.

Proof. Indeed, AM-GM gives that

$$\begin{aligned} 5 = x^4+y^4+(z^4+1)+xyz &= \frac{3x^4}{4} + \left(\frac{x^4}{4}+y^4\right) + (z^4+1) + xyz \\ &\ge \frac{3x^4}{4} + x^2y^2 + 2z^2 + xyz. \end{aligned}$$

We evidently have that $x^2y^2+2z^2+xyz \ge 0$ because the quadratic form $a^2+ab+2b^2$ is positive definite, so $x^4 \le \frac{20}{3} \implies x \le 2$. □

Now, the desired statement is implied by its square, so it suffices to show that

$$2-x \ge \left(\frac{y+z}{2}\right)^2$$

We are going to proceed by contradiction (it seems that many solutions do this) and assume that

$$2-x < \left(\frac{y+z}{2}\right)^2 \iff 4x+y^2+2yz+z^2 > 8.$$

By AM-GM,

$$\begin{aligned} x^4+3 &\ge 4x \\ \tfrac{y^4+1}{2} &\ge y^2 \\ \tfrac{z^4+1}{2} &\ge z^2 \end{aligned}$$

which yields that

$$x^4 + \frac{y^4+z^4}{2} + 2yz + 4 > 8.$$

If we replace $x^4 = 4-(y^4+z^4+xyz)$ now, this gives

$$-\frac{y^4+z^4}{2} + (2-x)yz > 0 \implies (2-x)yz > \frac{y^4+z^4}{2}.$$

Since $2-x$ and the right-hand side are positive, we have $yz \geq 0$. Now

$$\frac{y^4+z^4}{2yz} < 2-x < \left(\frac{y+z}{2}\right)^2 \implies 2y^4+2z^4 < yz(y+z)^2 = y^3z+2y^2z^2+yz^3.$$

This is clearly false by AM-GM, so we have a contradiction.

§2.2 TSTST 2015/5

Available online at https://aops.com/community/p5017821.

Problem statement

Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

Here are two explicit solutions.

¶ First solution with ad-hoc subsets, by Evan Chen I consider the following eleven prime numbers:

$$S = \{11, 13, 17, 19, 29, 31, 37, 41, 43, 61, 71\}.$$

This has the property that for any $p \in S$, all prime factors of $p - 1$ are one digit.

Let $N = (210)^{\text{billion}}$, and consider $M = \varphi(N)$. For any subset $T \subset S$, we have

$$M = \varphi\left(\frac{N}{\prod_{p\in T}(p-1)} \prod_{p\in T} p\right).$$

Since $2^{|S|} > 2015$ we're done.

> **Remark.** This solution is motivated by the deep fact that $\varphi(11 \cdot 1000) = \varphi(10 \cdot 1000)$, for example.

¶ Second solution with smallest primes, by Yang Liu Let $2 = p_1 < p_2 < \dots < p_{2015}$ be the *smallest* 2015 primes. Then the 2015 numbers

$$\begin{aligned} n_1 &= (p_1 - 1)p_2 \dots p_{2015} \\ n_2 &= p_1(p_2 - 1) \dots p_{2015} \\ &\vdots \\ n_{2015} &= p_1 p_2 \dots (p_{2015} - 1) \end{aligned}$$

all have the same phi value, namely

$$\varphi(p_1 p_2 \dots p_{2015}) = \prod_{i=1}^{2015} (p_i - 1).$$

§2.3 TSTST 2015/6, proposed by Linus Hamilton

Available online at https://aops.com/community/p5017871.

Problem statement

A *Nim-style game* is defined as follows. Two positive integers k and n are specified, along with a finite set S of k-tuples of integers (not necessarily positive). At the start of the game, the k-tuple $(n, 0, 0, \dots, 0)$ is written on the blackboard.

A legal move consists of erasing the tuple $(a_1, a_2, \dots, a_k)$ which is written on the blackboard and replacing it with $(a_1 + b_1, a_2 + b_2, \dots, a_k + b_k)$, where $(b_1, b_2, \dots, b_k)$ is an element of the set S. Two players take turns making legal moves, and the first to write a negative integer loses. In the event that neither player is ever forced to write a negative integer, the game is a draw.

Prove that there is a choice of k and S with the following property: the first player has a winning strategy if n is a power of 2, and otherwise the second player has a winning strategy.

Here we present a solution with 14 registers and 22 moves. Initially $X = n$ and all other variables are zero.

	X	Y	Go	S_X^0	S_X	S'_X	S_Y^0	S_Y	S'_Y	Cl	A	B	Die	Die'
Init	-1		1									1	1	1
Begin	1		-1	1							-1	1		
Sleep											1	-1		
StartX				-1	1						-1	1		
WorkX	-1				-1	1					-1	1		
WorkX'	-1	1			1	-1					-1	1		
DoneX					-1		1				-1	1		
WrongX	-1						-1					-1		
StartY							-1	1			-1	1		
WorkY		-1						-1	1		-1	1		
WorkY'	1	-1						1	-1		-1	1		
DoneY				1				-1			-1	1		
WrongY		-1		-1								-1		
ClaimX	-1			-1						1	-1	1		
ClaimY		-1					-1			1	-1	1		
FakeX	-1									-1		-1		
FakeY		-1								-1		-1		
Win										-1	-1			
PunA												-2		
PunB											-1	-1		
Kill												-1	-2	1
Kill'												-1	1	-2

Now, the "game" is played as follows. The mechanics are controlled by the *turn counters* A and B.

Observe the game starts with Alice playing Init. Thereafter, we say that the game is

- In the *main part* if $A + B = 1$, and no one has played Init a second time.
- In the *death part* otherwise.

Observe that in the main state, on Alice's turn we always have $(A, B) = (1, 0)$ and on Bob's turn we always have $(A, B) = (0, 1)$.

Claim — A player who plays Init a second time must lose. In particular, a player who makes a move when $A = B = 0$ must lose.

Proof. Situations with $A + B \geq 2$ cannot occur during main part, so there are only a few possibilities.

- Suppose the offending player is in a situation where $A = B = 0$. Then he/she must play Init. At this point, the opposing player can respond by playing Kill. Then the offending player must play Init again. The opposing player now responds with Kill'. This iteration continues until X reaches a negative number and the offending player loses.
- Suppose Alice has $(A, B) = (1, 0)$ but plays Init again anyways. Then Bob responds with PunB to punish her; he then wins as in the first case.
- Suppose Bob has $(A, B) = (0, 1)$ but plays Init again anyways. Alice responds with PunA in the same way. □

Thus we may assume that players avoid the death part at all costs. Hence the second moves consist of Bob playing Sleep, and then Alice playing Begin (thus restoring the value of n in X), then Bob playing Sleep.

Now we return to analysis of the main part. We say the game is in *state* S for $S \in \{S_X^0, S_X, S'_X, S_Y^0, S_Y, S'_Y, \mathrm{Cl}\}$ if $S = 1$ and all other variables are zero. By construction, this is always the case. From then on the main part is divided into several phases:

- An X*-phase*: this begins with Alice at S_X^0, and ends when the game is in a state other than S_X and S'_X. (She can never return to S_X^0 during an X-phase.)
- A Y*-phase*: this begins with Alice at S_Y^0, and ends when the game is in a state other than S_Y and S'_Y. (She can never return to S_Y^0 during a Y-phase.)

Claim — Consider an X-phase in which $(X, Y) = (x, 0)$, $x > 1$. Then Alice can complete the phase without losing if and only if x is even; if so she begins a Y-phase with $(X, Y) = (0, x/2)$.

Proof. As $x > 1$, Alice cannot play ClaimX since Bob will respond with FakeX and win. Now by alternating between WorkX and WorkX', Alice can repeatedly deduct 2 from X and add 1 to Y, leading to $(x - 2, y + 1)$, $(x - 4, y + 2)$, and so on. (During this time, Bob can only play Sleep.) Eventually, she must stop this process by playing DoneX, which begins a Y-phase.

Now note that unless $X = 0$, Bob now has a winning move WrongX. Conversely he may only play Sleep if $X = 0$. □

We have an analogous claim for Y-phases. Thus if n is not a power of 2, we see that Alice eventually loses.

Now suppose $n = 2^k$; then Alice reaches $(X, Y) = (0, 2^{k-1})$, $(2^{k-2}, 0)$, ... until either reaching $(1, 0)$ or $(0, 1)$. At this point she can play ClaimX or ClaimY, respectively; the game is now in state Cl. Bob cannot play either FakeX or FakeY, so he must play Sleep, and then Alice wins by playing Win. Thus Alice has a winning strategy when $n = 2^k$.

58$^{\text{th}}$ IMO TST Selection Test

Pittsburgh, PA

Day I 1:15pm – 5:45pm

June 25, 2016

1. Let $A = A(x, y)$ and $B = B(x, y)$ be two-variable polynomials with real coefficients. Suppose that $A(x, y)/B(x, y)$ is a polynomial in x for infinitely many values of y, and a polynomial in y for infinitely many values of x. Prove that B divides A, meaning there exists a third polynomial C with real coefficients such that $A = B \cdot C$.

2. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.

3. Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \ldots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n.

58$^{\text{th}}$ IMO TST Selection Test

Pittsburgh, PA

Day II 1:15pm – 5:45pm

June 27, 2016

4. Suppose that n and k are positive integers such that
$$1 = \underbrace{\varphi(\varphi(\dots\varphi(}_{k \text{ times}} n)\dots)).$$
Prove that $n \le 3^k$.

   Here $\varphi(n)$ denotes Euler's totient function, i.e. $\varphi(n)$ denotes the number of elements of $\{1, \dots, n\}$ which are relatively prime to n. In particular, $\varphi(1) = 1$.

5. In the coordinate plane are finitely many *walls*, which are disjoint line segments, none of which are parallel to either axis. A bulldozer starts at an arbitrary point and moves in the $+x$ direction. Every time it hits a wall, it turns at a right angle to its path, away from the wall, and continues moving. (Thus the bulldozer always moves parallel to the axes.)

   Prove that it is impossible for the bulldozer to hit both sides of every wall.

6. Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

USA TSTST 2016 Solutions

United States of America — TST Selection Test

Evan Chen《陳誼廷》

58th IMO 2017 Brazil and 6th EGMO 2017 Switzerland

Contents

§0 Problems

1. Let $A = A(x, y)$ and $B = B(x, y)$ be two-variable polynomials with real coefficients. Suppose that $A(x, y)/B(x, y)$ is a polynomial in x for infinitely many values of y, and a polynomial in y for infinitely many values of x. Prove that B divides A, meaning there exists a third polynomial C with real coefficients such that $A = B \cdot C$.

2. Let ABC be a scalene triangle with orthocenter H and circumcenter O and denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line $\overline{AN}$ at $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect on $\overline{PN}$.

3. Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers
$$Q(0),\ Q(1), Q(2),\ \dots,\ Q(n-1)$$
produce at most $0.499n$ distinct residues when taken modulo n.

4. Prove that if n and k are positive integers satisfying $\varphi^k(n) = 1$, then $n \le 3^k$. (Here φ^k denotes k applications of the Euler phi function.)

5. In the coordinate plane are finitely many *walls*, which are disjoint line segments, none of which are parallel to either axis. A bulldozer starts at an arbitrary point and moves in the $+x$ direction. Every time it hits a wall, it turns at a right angle to its path, away from the wall, and continues moving. (Thus the bulldozer always moves parallel to the axes.)

   Prove that it is impossible for the bulldozer to hit both sides of every wall.

6. Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

§1 Solutions to Day 1

§1.1 TSTST 2016/1, proposed by Victor Wang

Available online at https://aops.com/community/p6575197.

Problem statement

Let $A = A(x, y)$ and $B = B(x, y)$ be two-variable polynomials with real coefficients. Suppose that $A(x, y)/B(x, y)$ is a polynomial in x for infinitely many values of y, and a polynomial in y for infinitely many values of x. Prove that B divides A, meaning there exists a third polynomial C with real coefficients such that $A = B \cdot C$.

This is essentially an application of the division algorithm, but the details require significant care.

First, we claim that A/B can be written as a polynomial in x whose coefficients are rational functions in y. To see this, use the division algorithm to get

$$A = Q \cdot B + R \qquad Q, R \in (\mathbb{R}(y))[x]$$

where Q and R are polynomials in x whose coefficients are rational functions in y, and moreover $\deg_x B > \deg_x R$.

Now, we claim that $R \equiv 0$. Indeed, we have by hypothesis that for infinitely many values of y_0 that $B(x, y_0)$ divides $A(x, y_0)$, which means $B(x, y_0) \mid R(x, y_0)$ as polynomials in $\mathbb{R}[x]$. Now, we have $\deg_x B(x, y_0) > \deg_x R(x, y_0)$ outside of finitely many values of y_0 (but not all of them!); this means for infinitely many y_0 we have $R(x, y_0) \equiv 0$. So each coefficient of x^i (in $\mathbb{R}(y)$) has infinitely many roots, hence is a zero polynomial.

Consequently, we are able to write $A/B = F(x, y)/M(y)$ where $F \in \mathbb{R}[x, y]$ and $M \in \mathbb{R}[y]$ are each polynomials. Repeating the same argument now gives

$$\frac{A}{B} = \frac{F(x, y)}{M(y)} = \frac{G(x, y)}{N(x)}.$$

Now, by unique factorization of polynomials in $\mathbb{R}[x, y]$, we can discuss GCD's. So, we tacitly assume $\gcd(F, M) = \gcd(G, N) = (1)$. Also, we obviously have $\gcd(M, N) = (1)$. But $F \cdot N = G \cdot M$, so $M \mid F \cdot N$, thus we conclude M is the constant polynomial. This implies the result.

> **Remark.** This fact does not generalize to arbitrary functions that are separately polynomial: see e.g. http://aops.com/community/c6h523650p2978180.

§1.2 TSTST 2016/2, proposed by Evan Chen

Available online at https://aops.com/community/p6575204.

Problem statement

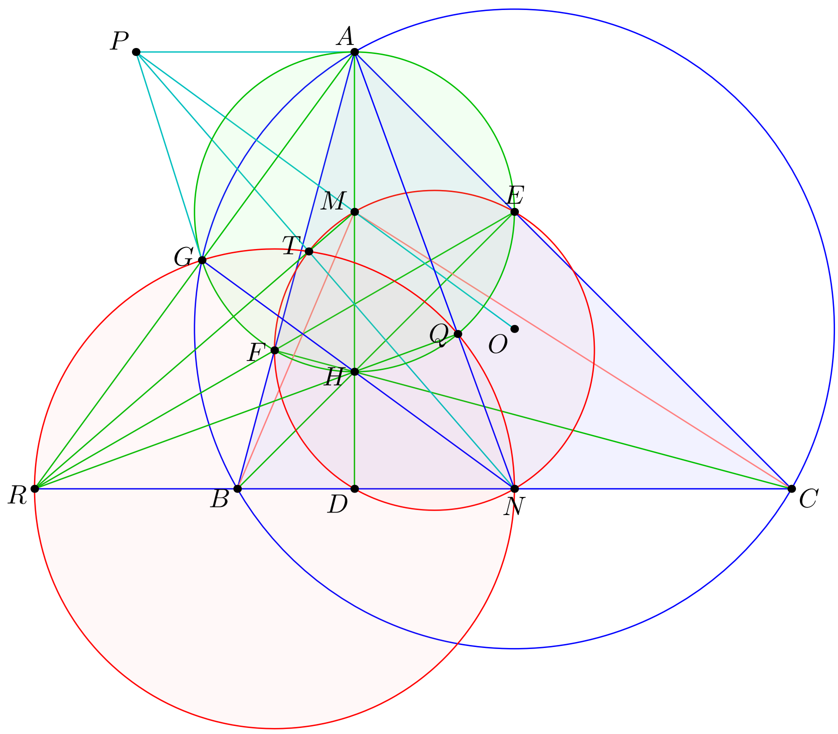
Let ABC be a scalene triangle with orthocenter H and circumcenter O and denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line $\overline{AN}$ at $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect on $\overline{PN}$.

We present two solutions, one using essentially only power of a point, and the other more involved.

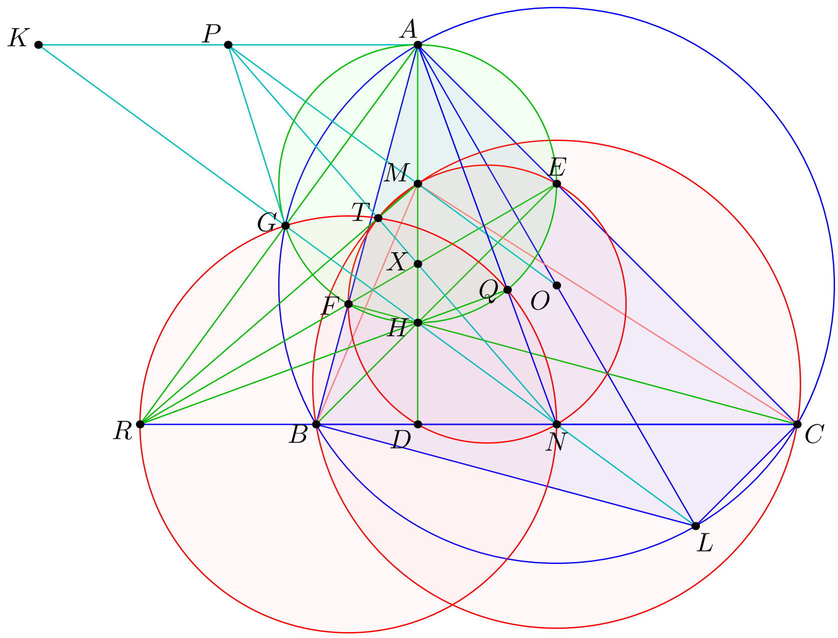
¶ First solution (found by contestants) Denote by $\triangle DEF$ the orthic triangle. Observe $\overline{PA}$ and $\overline{PG}$ are tangents to γ, since $\overline{OM}$ is the perpendicular bisector of $\overline{AG}$. Also note that $\overline{AG}$, $\overline{EF}$, $\overline{BC}$ are concurrent at some point R by radical axis on (ABC), γ, (BC).

Now, consider circles $(PAGM)$, $(MFDNE)$, and (MBC). They intersect at M but have radical center R, so are coaxial; assume they meet again at $T \in \overline{RM}$, say. Then $\angle PTM$ and $\angle MTN$ are both right angles, hence T lies on $\overline{PN}$.

Finally H is the orthocenter of $\triangle ARN$, and thus the circle with diameter $\overline{RN}$ passes through G, Q, N.



¶ Alternate solution (by proposer) Let L be diametrically opposite A on the circumcircle. Denote by $\triangle DEF$ the orthic triangle. Let $X = \overline{AH} \cap \overline{EF}$. Finally, let T be the second intersection of $(MFDNE)$ and (MBC).



We begin with a few easy observations. First, points H, G, N, L are collinear and $\angle AGL = 90^\circ$. Also, Q is the foot from H to $\overline{AN}$. Consequently, lines AG, EF, HQ, BC, TM concur at a point R (radical axis). Moreover, we already know $\angle MTN = 90^\circ$. This implies T lies on the circle with diameter $\overline{RN}$, which is exactly the circumcircle of $\triangle GQN$.

Note by Brokard's Theorem on $AFHE$, the point X is the orthocenter of $\triangle MBC$. But $\angle MTN = 90^\circ$ already, and N is the midpoint of $\overline{BC}$. Consequently, points T, X, N are collinear.

Finally, we claim P, X, N are collinear, which solves the problem. Note $P = \overline{GG} \cap \overline{AA}$. Set $K = \overline{HNL} \cap \overline{AP}$. Then by noting

$$-1 = (D, X; A, H) \overset{N}{=} (\infty, \overline{NX} \cap \overline{AK}; A, K)$$

we see that $\overline{NX}$ bisects segment $\overline{AK}$, as desired. (A more projective finish is to show that $\overline{PXN}$ is the polar of R to γ).

> **Remark.** The original problem proposal reads as follows:
>
> > Let ABC be a triangle with orthocenter H and circumcenter O and denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose ray OM meets the line parallel to $\overline{BC}$ through A at P. Prove that the line through the circumcenter of $\triangle MBC$ and the midpoint of $\overline{OH}$ is parallel to $\overline{NP}$.
>
> The points G and Q were added to the picture later to prevent the problem from being immediate by coordinates.

§1.3 TSTST 2016/3, proposed by Yang Liu

Available online at https://aops.com/community/p6575217.

Problem statement

Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \dots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n.

We claim that

$$Q(x) = 420(x^2-1)^2$$

works. Clearly, it suffices to prove the result when $n = 4$ and when n is an odd prime p. The case $n = 4$ is trivial, so assume now $n = p$ is an odd prime.

First, we prove the following easy claim.

Claim — For any odd prime p, there are at least $\frac{1}{2}(p-3)$ values of a for which $\left(\frac{1-a^2}{p}\right) = +1$.

Proof. Note that if $k \neq 0$, $k \neq \pm 1$, $k^2 \neq -1$, then $a = 2(k + k^{-1})$ works. Also $a = 0$ works. $\square$

Let $F(x) = (x^2-1)^2$. The range of F modulo p is contained within the $\frac{1}{2}(p+1)$ quadratic residues modulo p. On the other hand, if for some t neither of $1 \pm t$ is a quadratic residue, then t^2 is omitted from the range of F as well. Call such a value of t *useful*, and let N be the number of useful residues. We aim to show $N \geq \frac{1}{4}p - 2$.

We compute a lower bound on the number N of useful t by writing

$$\begin{aligned}
N &= \frac{1}{4}\left(\sum_t \left[\left(1-\left(\frac{1-t}{p}\right)\right)\left(1-\left(\frac{1+t}{p}\right)\right)\right] - \left(1-\left(\frac{2}{p}\right)\right) - \left(1-\left(\frac{-2}{p}\right)\right)\right) \\
&\geq \frac{1}{4}\sum_t \left[\left(1-\left(\frac{1-t}{p}\right)\right)\left(1-\left(\frac{1+t}{p}\right)\right)\right] - 1 \\
&= \frac{1}{4}\left(p + \sum_t \left(\frac{1-t^2}{p}\right)\right) - 1 \\
&\geq \frac{1}{4}\left(p + (+1)\cdot \tfrac{1}{2}(p-3) + 0\cdot 2 + (-1)\cdot\left((p-2) - \tfrac{1}{2}(p-3)\right)\right) - 1 \\
&\geq \frac{1}{4}\left(p-5\right).
\end{aligned}$$

Thus, the range of F has size at most

$$\frac{1}{2}(p+1) - \frac{1}{2}N \leq \frac{3}{8}(p+3).$$

This is less than $0.499p$ for any $p \geq 11$.

Remark. In fact, the computation above is essentially an equality. There are only two points where terms are dropped: one, when $p \equiv 3 \pmod 4$ there are no $k^2 = -1$ in the lemma, and secondly, the terms $1 - (2/p)$ and $1 - (-2/p)$ are dropped in the initial estimate for N. With suitable modifications, one can show that in fact, the range of F is exactly equal to

$$\frac{1}{2}(p+1) - \frac{1}{2}N = \begin{cases} \frac{1}{8}(3p+5) & p \equiv 1 \pmod 8 \\ \frac{1}{8}(3p+7) & p \equiv 3 \pmod 8 \\ \frac{1}{8}(3p+9) & p \equiv 5 \pmod 8 \\ \frac{1}{8}(3p+3) & p \equiv 7 \pmod 8. \end{cases}$$

§2 Solutions to Day 2

§2.1 TSTST 2016/4, proposed by Linus Hamilton

Available online at https://aops.com/community/p6580534.

Problem statement

Prove that if n and k are positive integers satisfying $\varphi^k(n) = 1$, then $n \le 3^k$. (Here φ^k denotes k applications of the Euler phi function.)

The main observation is that the exponent of 2 decreases by at most 1 with each application of φ. This will give us the desired estimate.

Define the *weight* function w on positive integers as follows: it satisfies

$$\begin{aligned} w(ab) &= w(a) + w(b); \\ w(2) &= 1; \quad \text{and} \\ w(p) &= w(p-1) \quad \text{for any prime } p > 2. \end{aligned}$$

By induction, we see that $w(n)$ counts the powers of 2 that are produced as φ is repeatedly applied to n. In particular, $k \ge w(n)$.

From $w(2) = 1$, it suffices to prove that $w(p) \ge \log_3 p$ for every $p > 2$. We use strong induction and note that

$$w(p) = w(2) + w\left(\frac{p-1}{2}\right) \ge 1 + \log_3(p-1) - \log_3 2 \ge \log_3 p$$

for any $p > 2$. This solves the problem.

Remark. One can motivate this solution through small cases 2^x3^y like 2^x17^w, $2^x3^y7^z$, 2^x11^t.

Moreover, the stronger bound

$$n \le 2 \cdot 3^{k-1}$$

is true and best possible.

§2.2 TSTST 2016/5, proposed by Linus Hamilton, Cynthia Stoner

Available online at https://aops.com/community/p6580545.

Problem statement

In the coordinate plane are finitely many *walls*, which are disjoint line segments, none of which are parallel to either axis. A bulldozer starts at an arbitrary point and moves in the $+x$ direction. Every time it hits a wall, it turns at a right angle to its path, away from the wall, and continues moving. (Thus the bulldozer always moves parallel to the axes.)

Prove that it is impossible for the bulldozer to hit both sides of every wall.

We say a wall v is *above* another wall w if some point on v is directly above a point on w. (This relation is anti-symmetric, as walls do not intersect).

The critical claim is as follows:

Claim — There exists a lowest wall, i.e. a wall not above any other walls.

Proof. Assume not. Then we get a directed cycle of some length $n \ge 3$: it's possible to construct a series of points P_i, Q_i, for $i = 1, \dots, n$ (indices modulo n), such that the point Q_i is directly above P_{i+1} for each i, the segment $\overline{Q_i P_{i+1}}$ does not intersect any wall in its interior, and finally each segment $\overline{P_i Q_i}$ is contained inside a wall. This gives us a broken line on $2n$ vertices which is not self-intersecting.

Now consider the leftmost vertical segment $\overline{Q_i P_{i+1}}$ and the rightmost vertical segment $\overline{Q_j P_{j+1}}$. The broken line gives a path from P_{i+1} to Q_j, as well as a path from P_{j+1} to Q_i. These clearly must intersect, contradiction. □

Remark. This claim is Iran TST 2010.

Thus if the bulldozer eventually moves upwards indefinitely, it may never hit the bottom side of the lowest wall. Similarly, if the bulldozer eventually moves downwards indefinitely, it may never hit the upper side of the highest wall.

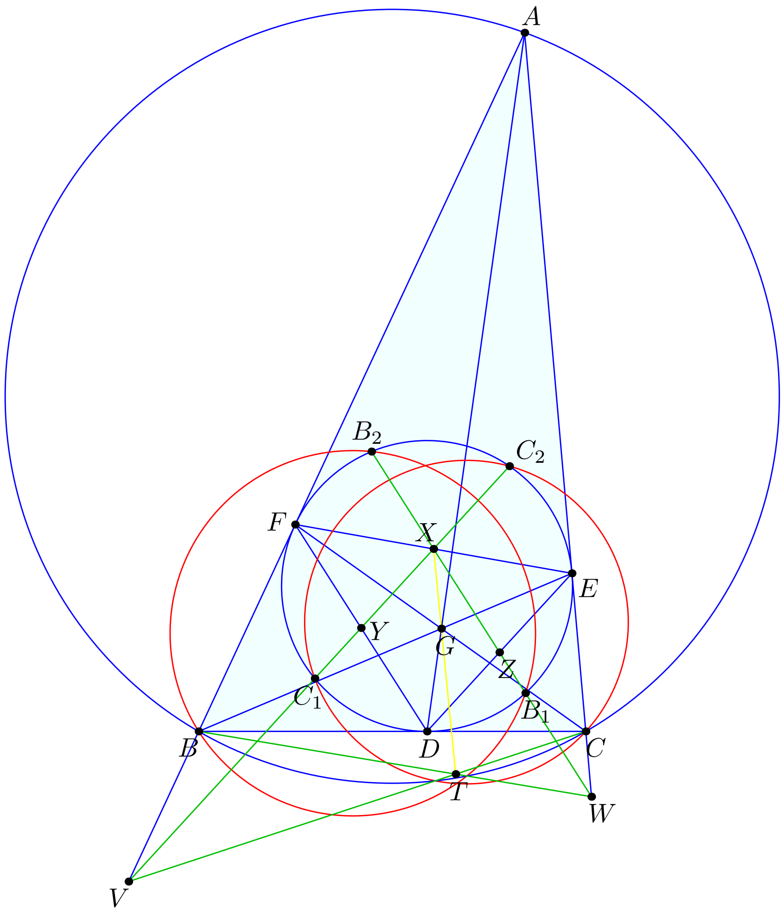
§2.3 TSTST 2016/6, proposed by Danielle Wang

Available online at https://aops.com/community/p6580553.

Problem statement

Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

¶ First solution (Allen Liu) Let X, Y, Z be midpoints of EF, FD, DE, and let G be the Gergonne point. By radical axis on $(AEIF)$, (DEF), (AIC) we see that B_1, X, B_2 are collinear. Likewise, B_1, Z, B_2 are collinear, so lines B_1B_2 and XZ coincide. Similarly, lines C_1C_2 and XY coincide. In particular lines B_1B_2 and C_1C_2 meet at X.



Note G is the symmedian point of DEF, so it is well-known that XG passes through the midpoint of DK. So we just have to prove G lies on the radical axis.

First, note that $\triangle DEF$ is the cevian triangle of the Gergonne point G. Set $V = \overline{XY} \cap \overline{AB}$, $W = \overline{XZ} \cap \overline{AC}$, and $T = \overline{BW} \cap \overline{CV}$.

We begin with the following completely projective claim.

Claim — The points X, G, T are collinear.

Proof. It suffices to view $\triangle XYZ$ as any cevian triangle of $\triangle DEF$ (which is likewise any cevian triangle of $\triangle ABC$). Then

- By Cevian Nest on $\triangle ABC$, it follows that $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are concurrent.
- Hence $\triangle BYV$ and $\triangle CZW$ are perspective.
- Hence $\triangle BZW$ and $\triangle CYV$ are perspective too.
- Hence we deduce by Desargues theorem that T, X, and $\overline{BZ} \cap \overline{CY}$ are collinear.
- Finally, the Cevian Nest theorem applied on $\triangle GBC$ (which has cevian triangles $\triangle DFE$, $\triangle XZY$) we deduce G, X, and $\overline{BZ} \cap \overline{CY}$, proving the claim.

One could also proceed by using barycentric coordinates on $\triangle DEF$. □

Remark (Eric Shen)**.** The first four bullets can be replaced by non-projective means: one can check that $\overline{BZ} \cap \overline{CY}$ is the radical center of (BIC), (BB_1B_2), (CC_1C_2) and therefore it lies on line $\overline{XT}$.

Now, we contend point V is the radical center (CC_1C_2), (ABC) and (DEF). To see this, let $V' = \overline{ED} \cap \overline{AB}$; then $(FV'; AB)$ is harmonic, and V is the midpoint of $\overline{FV'}$, and thus $VA \cdot VB = VF^2 = VC_1 \cdot VC_2$.

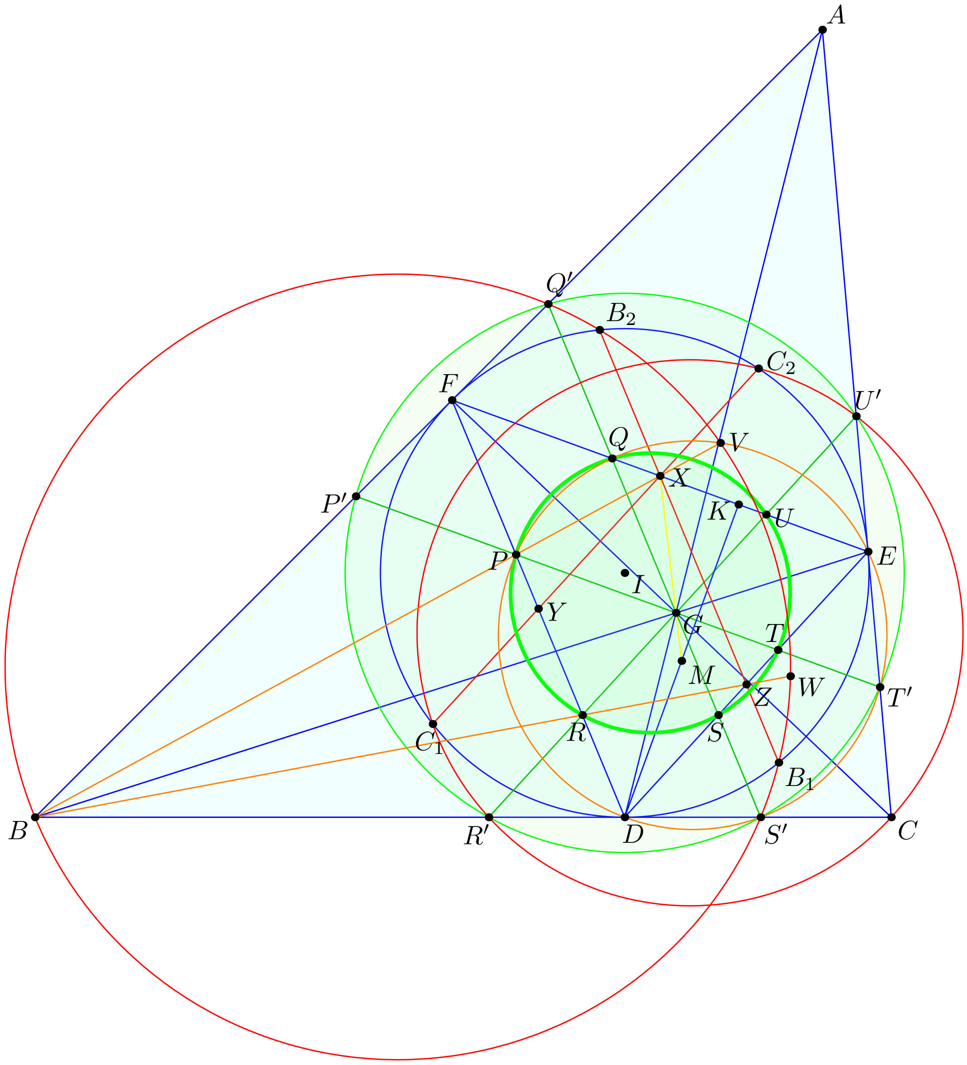
So in fact $\overline{CV}$ is the radical axis of (ABC) and (CC_1C_2).

Similarly, $\overline{BW}$ is the radical axis of (ABC) and (BB_1B_2). Thus T is the radical center of (ABC), (BB_1B_2), (CC_1C_2).

This completes the proof, as now $\overline{XT}$ is the desired radical axis.

¶ Second solution (Evan Chen) Let X, Y, Z be midpoints of EF, FD, DE, and let G be the Gergonne point. By radical axis on $(AEIF)$, (DEF), (AIC) we see that B_1, X, B_2 are collinear. Likewise, B_1, Z, B_2 are collinear, so lines B_1B_2 and XZ coincide. Similarly, lines C_1C_2 and XY coincide. In particular lines B_1B_2 and C_1C_2 meet at X.

Note G is the symmedian point of DEF, so it is well-known that XG passes through the midpoint of DK. So we just have to prove G lies on the radical axis.



Construct parallelograms $GPFQ$, $GRDS$, $GTUE$ such that $P, R \in DF$, $S, T \in DE$, $Q, U \in EF$. As FG bisects PQ and is isogonal to FZ, we find $PQED$, hence $PQRU$, is cyclic. Repeating the same logic and noticing PR, ST, QU not concurrent, all six points $PQRSTU$ are cyclic. Moreover, since PQ bisects GF, we see that a dilation with factor 2 at G sends PQ to $P', Q' \in AB$, say, with F the midpoint of $P'Q'$. Define $R', S' \in BC$ similarly now and $T', U' \in CA$.

Note that $EQPDS'$ is in cyclic too, as $\measuredangle DS'Q = \measuredangle DRS = \measuredangle DEF$. By homothety through B, points B, P, X are collinear; assume they meet $(EQPDS')$ again at V. Thus $EVQPDS'$ is cyclic, and now

$$\measuredangle BVS' = \measuredangle PVS' = \measuredangle PQS = \measuredangle PTS = \measuredangle FED = \measuredangle XEZ = \measuredangle XVZ$$

hence V lies on $(BQ'S')$.

Since $FB \parallel QP$, we get $EVFB$ is cyclic too, so $XV \cdot XB = XE \cdot XF$ now; thus X lies on the radical axis of $(BS'Q')$ and (DEF). By the same argument with $W \in BZ$, we get Z lies on the radical axis too. Thus the radical axis of $(BS'Q')$ and (DEF) must be line XZ, which coincides with B_1B_2; so $(BB_1B_2) = (BS'Q')$.

Analogously, $(CC_1C_2) = (CR'U')$. Since $G = Q'S' \cap R'U'$, we need only prove that $Q'R'S'U'$ is cyclic. But $QRSU$ is cyclic, so we are done.

The circle $(PQRSTU)$ is called the *Lemoine circle* of ABC.

USA TST Selection Test for 59^{th} IMO and 7^{th} EGMO

Pittsburgh, PA

Day I   1:15pm – 5:45pm

Saturday, June 24, 2017

Problem 1. Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and $\angle A \neq 90^\circ$. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection point of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection point of lines AQ and EF. Prove that $\overline{PR} \perp \overline{OH}$.

Problem 2. Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

Problem 3. Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)}$$

where f and g are nonzero polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.

USA TST Selection Test for 59$^{\text{th}}$ IMO and 7$^{\text{th}}$ EGMO

Pittsburgh, PA

Day II 1:15pm – 5:45pm

Monday, June 26, 2017

Problem 4. Find all nonnegative integer solutions to $2^a + 3^b + 5^c = n!$.

Problem 5. Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

Problem 6. A sequence of positive integers $(a_n)_{n\geq 1}$ is of *Fibonacci type* if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

USA TSTST 2017 Solutions

United States of America — TST Selection Test

EVAN CHEN《陳誼廷》

59th IMO 2018 Romania and 7th EGMO 2018 Italy

Contents

§0 Problems

1. Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and $\angle A \neq 90^\circ$. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection point of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection point of lines AQ and EF. Prove that $\overline{PR} \perp \overline{OH}$.

2. Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

3. Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)}$$

where f and g are nonzero polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.

4. Find all nonnegative integer solutions to

$$2^a + 3^b + 5^c = n!.$$

5. Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

6. A sequence of positive integers $(a_n)_{n \geq 1}$ is of *Fibonacci type* if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

§1 Solutions to Day 1

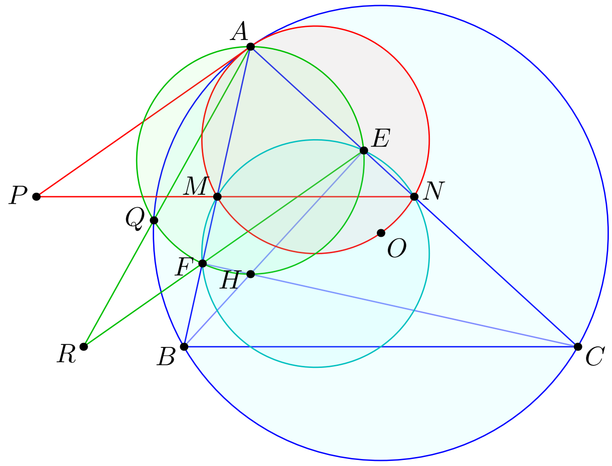
§1.1 TSTST 2017/1, proposed by Ray Li

Available online at `https://aops.com/community/p8526098`.

Problem statement

Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and $\angle A \neq 90^\circ$. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection point of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection point of lines AQ and EF. Prove that $\overline{PR} \perp \overline{OH}$.

¶ First solution (power of a point) Let γ denote the nine-point circle of ABC.



Note that

- $PA^2 = PM \cdot PN$, so P lies on the radical axis of Γ and γ.
- $RA \cdot RQ = RE \cdot RF$, so R lies on the radical axis of Γ and γ.

Thus $\overline{PR}$ is the radical axis of Γ and γ, which is evidently perpendicular to $\overline{OH}$.

> **Remark.** In fact, by power of a point one may also observe that R lies on $\overline{BC}$, since it is on the radical axis of $(AQFHE)$, $(BFEC)$, (ABC). Ironically, this fact is not used in the solution.

¶ Second solution (barycentric coordinates) Again note first $R \in \overline{BC}$ (although this can be avoided too). We compute the points in much the same way as before. Since $\overline{AP} \cap \overline{BC} = (0 : b^2 : -c^2)$ we have

$$P = \left(b^2 - c^2 : b^2 : -c^2\right)$$

(since $x = y + z$ is the equation of line $\overline{MN}$). Now in Conway notation we have

$$R = \overline{EF} \cap \overline{BC} = (0 : S_C : -S_B) = \left(0 : a^2 + b^2 - c^2 : -a^2 + b^2 - c^2\right).$$

Hence

$$\overrightarrow{PR} = \frac{1}{2(b^2 - c^2)} \left(b^2 - c^2, c^2 - a^2, a^2 - b^2\right).$$

On the other hand, we have $\overrightarrow{OH} = \vec{A} + \vec{B} + \vec{C}$. So it suffices to check that

$$\sum_{\text{cyc}} a^2 \left((a^2 - b^2) + (c^2 - a^2)\right) = 0$$

which is immediate.

¶ Third solution (complex numbers) Let ABC be the unit circle. We first compute P as the midpoint of A and $\overline{AA} \cap \overline{BC}$:

$$\begin{aligned} p &= \frac{1}{2}\left(a + \frac{a^2(b+c) - bc \cdot 2a}{a^2 - bc}\right) \\ &= \frac{a(a^2 - bc) + a^2(b+c) - 2abc}{2(a^2 - bc)}. \end{aligned}$$

Using the remark above, R is the inverse of D with respect to the circle with diameter $\overline{BC}$, which has radius $\left|\frac{1}{2}(b-c)\right|$. Thus

$$\begin{aligned} r - \frac{b+c}{2} &= \frac{\frac{1}{4}(b-c)\left(\frac{1}{b} - \frac{1}{c}\right)}{\overline{\frac{1}{2}\left(a - \frac{bc}{a}\right)}} \\ r &= \frac{b+c}{2} + \frac{-\frac{1}{2}\frac{(b-c)^2}{bc}}{\frac{1}{a} - \frac{a}{bc}} \\ &= \frac{b+c}{2} + \frac{a(b-c)^2}{2(a^2 - bc)} \\ &= \frac{a(b-c)^2 + (b+c)(a^2 - bc)}{2(a^2 - bc)}. \end{aligned}$$

Expanding and subtracting gives

$$p - r = \frac{a^3 - abc - ab^2 - ac^2 + b^2c + bc^2}{2(a^2 - bc)} = \frac{(a+b+c)(a-b)(a-c)}{2(a^2 - bc)}$$

which is visibly equal to the negation of its conjugate once the factor of $a + b + c$ is deleted.

(Actually, one can guess this factorization ahead of time by noting that if $A = B$, then $P = B = R$, so $a - b$ must be a factor; analogously $a - c$ must be as well.)

§1.2 TSTST 2017/2, proposed by Kevin Sun

Available online at https://aops.com/community/p8526115.

Problem statement

Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

First we introduce some notation. Define a *block* of letters to be a maximal contiguous subsequence of consecutive letters. Throughout the solution, we fix the word A that Ana picks, and introduce the following notation for its m blocks:

$$A = A_1 A_2 \dots A_m = \underbrace{a_1 \dots a_1}_{x_1} \underbrace{a_2 \dots a_2}_{x_2} \dots \underbrace{a_m \dots a_m}_{x_m}.$$

A *rainbow* will be a subsequence equal to Ana's initial word A (meaning Ana seeks words with exactly k rainbows). Finally, for brevity, let $A_i = \underbrace{a_i \dots a_i}_{x_i}$, so $A = A_1 \dots A_m$.

We prove two claims that resolve the problem.

Claim — If $x_i = 1$ for some i, then for any $k \ge 1$, the word

$$W = A_1 \dots A_{i-1} \underbrace{a_i \dots a_i}_{k} A_{i+1} \dots A_m$$

obtained by repeating the ith letter k times has exactly k rainbows.

Proof. Obviously there are at least $\binom{k}{k-1} = k$ rainbows, obtained by deleting $k-1$ choices of the letter a_i in the repeated block. We show they are the only ones.

Given a rainbow, consider the location of this singleton block in W. It cannot occur within the first $|A_1|+\dots+|A_{i-1}|$ letters, nor can it occur within the final $|A_{i+1}|+\dots+|A_m|$ letters. So it must appear in the ith block of W. That implies that all the other a_i's in the ith block of W must be deleted, as desired. (This last argument is actually nontrivial, and has some substance; many students failed to realize that the upper bound requires care.) □

Claim — If $x_i \ge 2$ for all i, then no word W has exactly two rainbows.

Proof. We prove if there are two rainbows of W, then we can construct at least three rainbows.

Let $W = w_1 \dots w_n$ and consider the two rainbows of W. Since they are not the same, there must be a block A_p of the rainbow, of length $\ell \ge 2$, which do not occupy the same locations in W.

Assume the first rainbow uses $w_{i_1}, \ldots, w_{i_\ell}$ for this block and the second rainbow uses $w_{j_1}, \ldots, w_{j_\ell}$ for this block. Then among the letters w_q for $\min(i_1, j_1) \le q \le \max(i_\ell, j_\ell)$, there must be at least $\ell + 1$ copies of the letter a_p. Moreover, given a choice of ℓ copies of the letter a_p in this range, one can complete the subsequence to a rainbow. So the number of rainbows is at least $\binom{\ell+1}{\ell} \ge \ell + 1$.

Since $\ell \ge 2$, this proves W has at least three rainbows. □

In summary, Ana wins if and only if $x_i = 1$ for some i, since she can duplicate the isolated letter k times; but if $x_i \ge 2$ for all i then Banana only needs to supply $k = 2$.

§1.3 TSTST 2017/3, proposed by Calvin Deng, Linus Hamilton

Available online at https://aops.com/community/p8526130.

Problem statement

Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)}$$

where f and g are nonzero polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.

First, if $c \ge 2$ then we claim no such f and g exist. Indeed, one simply takes $x = 1$ to get $f(1)/g(1) \le 0$, impossible.

For $c < 2$, let $c = 2\cos\theta$, where $0 < \theta < \pi$. We claim that f exists and has minimum degree equal to n, where n is defined as the smallest integer satisfying $\sin n\theta \le 0$. In other words

$$n = \left\lceil \frac{\pi}{\arccos(c/2)} \right\rceil .$$

First we show that this is necessary. To see it, write explicitly

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-2} x^{n-2}$$

with each $a_i \ge 0$, and $a_{n-2} \ne 0$. Assume that n is such that $\sin(k\theta) \ge 0$ for $k = 1, \dots, n-1$. Then, we have the following system of inequalities:

$$\begin{aligned} a_1 &\ge 2\cos\theta \cdot a_0 \\ a_0 + a_2 &\ge 2\cos\theta \cdot a_1 \\ a_1 + a_3 &\ge 2\cos\theta \cdot a_2 \\ &\vdots \\ a_{n-5} + a_{n-3} &\ge 2\cos\theta \cdot a_{n-4} \\ a_{n-4} + a_{n-2} &\ge 2\cos\theta \cdot a_{n-3} \\ a_{n-3} &\ge 2\cos\theta \cdot a_{n-2}. \end{aligned}$$

Now, multiply the first equation by $\sin\theta$, the second equation by $\sin 2\theta$, et cetera, up to $\sin((n-1)\theta)$. This choice of weights is selected since we have

$$\sin(k\theta) + \sin((k+2)\theta) = 2\sin((k+1)\theta)\cos\theta$$

so that summing the entire expression cancels nearly all terms and leaves only

$$\sin((n-2)\theta)\, a_{n-2} \ge \sin((n-1)\theta) \cdot 2\cos\theta \cdot a_{n-2}$$

and so by dividing by a_{n-2} and using the same identity gives us $\sin(n\theta) \le 0$, as claimed.

This bound is best possible, because the example

$$a_k = \sin((k+1)\theta) \ge 0$$

makes all inequalities above sharp, hence giving a working pair (f, g).

Remark. Calvin Deng points out that a cleaner proof of the lower bound is to take $\alpha = \cos\theta + i\sin\theta$. Then $f(\alpha) = 0$, but by condition the imaginary part of $f(\alpha)$ is apparently strictly positive, contradiction.

Remark. Guessing that $c < 2$ works at all (and realizing $c \ge 2$ fails) is the first part of the problem.

The introduction of trigonometry into the solution may seem magical, but is motivated in one of two ways:

- Calvin Deng points out that it's possible to guess the answer from small cases: For $c \le 1$ we have $n = 3$, tight at $\frac{x^3+1}{x+1} = x^2 - x + 1$, and essentially the "sharpest $n = 3$ example". A similar example exists at $n = 4$ with $\frac{x^4+1}{x^2+\sqrt{2}x+1} = x^2 - \sqrt{2}x + 1$ by the Sophie-Germain identity. In general, one can do long division to extract an optimal value of c for any given n, although c will be the root of some polynomial.

  The thresholds $c \le 1$ for $n = 3$, $c \le \sqrt{2}$ for $n = 4$, $c \le \frac{1+\sqrt{5}}{2}$ for $n = 5$, and $c \le 2$ for $n < \infty$ suggest the unusual form of the answer via trigonometry.

- One may imagine trying to construct a polynomial recursively / greedily by making all inequalities above hold (again the "sharpest situation" in which f has few coefficients). If one sets $c = 2t$, then we have

  $$a_0 = 1, \quad a_1 = 2t, \quad a_2 = 4t^2 - 1, \quad a_3 = 8t^3 - 4t, \quad \dots$$

  which are the Chebyshev polynomials of the second type. This means that trigonometry is essentially mandatory. (One may also run into this when by using standard linear recursion techniques, and noting that the characteristic polynomial has two conjugate complex roots.)

Remark. Mitchell Lee notes that an IMO longlist problem from 1997 shows that if $P(x)$ is any polynomial satisfying $P(x) > 0$ for $x > 0$, then $(x+1)^n P(x)$ has nonnegative coefficients for large enough n. This show that f and g at least exist for $c \le 2$, but provides no way of finding the best possible $\deg f$.

Meghal Gupta also points out that showing f and g exist is possible in the following way:

$$\left(x^2 - 1.99x + 1\right)\left(x^2 + 1.99x + 1\right) = \left(x^4 - 1.9601x^2 + 1\right)$$

and so on, repeatedly multiplying by the "conjugate" until all coefficients become positive. To my best knowledge, this also does not give any way of actually minimizing $\deg f$, although Ankan Bhattacharya points out that this construction is actually optimal in the case where n is a power of 2.

Remark. It's pointed out that Matematicheskoe Prosveshchenie, issue 1, 1997, page 194 contains a nearly analogous result, available at https://mccme.ru/free-books/matpros/pdf/mp-01.pdf with solutions presented in https://mccme.ru/free-books/matpros/pdf/mp-05.pdf, pages 221–223; and https://mccme.ru/free-books/matpros/pdf/mp-10.pdf, page 274.

§2 Solutions to Day 2

§2.1 TSTST 2017/4, proposed by Mark Sellke

Available online at https://aops.com/community/p8526131.

Problem statement

Find all nonnegative integer solutions to

$$2^a + 3^b + 5^c = n!.$$

For $n \le 4$, one can check the only solutions are:

$$\begin{aligned} 2^2 + 3^0 + 5^0 &= 3! \\ 2^1 + 3^1 + 5^0 &= 3! \\ 2^4 + 3^1 + 5^1 &= 4!. \end{aligned}$$

Now we prove there are no solutions for $n \ge 5$.

A tricky way to do this is to take modulo 120, since

$$\begin{aligned} 2^a &\pmod{120} \in \{1, 2, 4, 8, 16, 32, 64\} \\ 3^b &\pmod{120} \in \{1, 3, 9, 27, 81\} \\ 5^c &\pmod{120} \in \{1, 5, 25\} \end{aligned}$$

and by inspection one notes that no three elements have vanishing sum modulo 120.

I expect most solutions to instead use casework. Here is one possible approach with cases (with $n \ge 5$). First, we analyze the cases where $a < 3$:

- $a = 0$: No solutions for parity reasons.
- $a = 1$: since $3^b + 5^c \equiv 6 \pmod 8$, we find b even and c odd (hence $c \neq 0$). Now looking modulo 5 gives that $3^b + 5^c \equiv 3 \pmod 5$,
- $a = 2$: From $3^b + 5^c \equiv 4 \pmod 8$, we find b is odd and c is even. Now looking modulo 5 gives a contradiction, even if $c = 0$, since $3^b \in \{2, 3 \pmod 5\}$ but $3^b + 5^c \equiv 1 \pmod 5$.

Henceforth assume $a \ge 3$. Next, by taking modulo 8 we have $3^b + 5^c \equiv 0 \pmod 8$, which forces both b and c to be odd (in particular, $b, c > 0$). We now have

$$\begin{aligned} 2^a + 5^c &\equiv 0 \pmod 3 \\ 2^a + 3^b &\equiv 0 \pmod 5. \end{aligned}$$

The first equation implies a is even, but the second equation requires a to be odd, contradiction. Hence no solutions with $n \ge 5$.

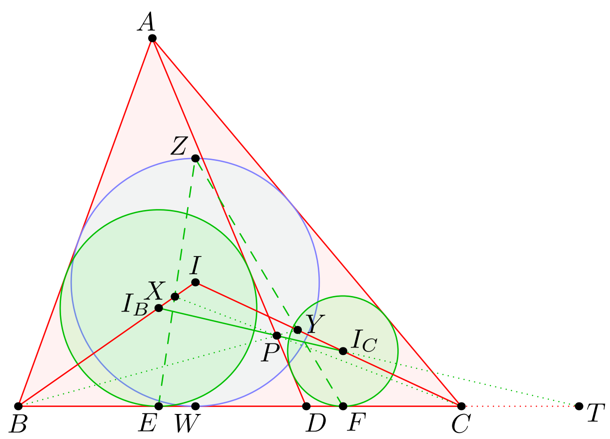
§2.2 TSTST 2017/5, proposed by Ray Li

Available online at https://aops.com/community/p8526136.

Problem statement

Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

¶ First solution (homothety) Let Z be the diametrically opposite point on the incircle. We claim this is the desired intersection.



Note that:

- P is the insimilicenter of ω_B and ω_C
- C is the exsimilicenter of ω and ω_C.

Thus by Monge theorem, the insimilicenter of ω_B and ω lies on line CP.

This insimilicenter should also lie on the line joining the centers of ω and ω_B, which is $\overline{BI}$, hence it coincides with the point X. So $X \in \overline{EZ}$ as desired.

¶ Second solution (harmonic) Let $T = \overline{I_B I_C} \cap \overline{BC}$, and W the foot from I to $\overline{BC}$. Define $Z = \overline{FY} \cap \overline{IW}$. Because $\angle I_B D I_C = 90^\circ$, we have

$$-1 = (I_B I_C; PT) \overset{B}{=} (II_C; YC) \overset{F}{=} (I\infty; ZW)$$

So I is the midpoint of $\overline{ZW}$ as desired.

¶ Third solution (outline, barycentric, Andrew Gu) Let $AD = t$, $BD = x$, $CD = y$, so $a = x + y$ and by Stewart's theorem we have

$$(x+y)(xy+t^2) = b^2x + c^2y. \quad (1)$$

We then have $D = (0 : y : x)$ and so

$$\overline{AI_B} \cap \overline{BC} = \left(0 : y + \frac{tx}{c+t} : \frac{cx}{c+t}\right)$$

hence intersection with BI gives

$$I_B = (ax : cy + at : cx).$$

Similarly,

$$I_C = (ay : by : bx + at).$$

Then, we can compute

$$P = (2axy : y(at + bx + cy) : x(at + bx + cy))$$

since $P \in \overline{I_B I_C}$, and clearly $P \in \overline{AD}$. Intersection now gives

$$\begin{aligned} X &= (2ax : at + bx + cy : 2cx) \\ Y &= (2ay : 2by : at + bx + cy) \,. \end{aligned}$$

Finally, we have $BE = \frac{1}{2}(c + x - t)$, and similarly for CF. Now if we reflect $D = (0, \frac{s-c}{a}, \frac{s-b}{a})$ over $I = (\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s})$, we get the antipode

$$Q := \left(4a^2 : -a^2 + 2ab - b^2 + c^2 : -a^2 + 2ac - c^2 + b^2\right).$$

We may then check Q lies on each of lines EX and FY (by checking $\det(Q, E, X) = 0$ using the equation (1)).

§2.3 TSTST 2017/6, proposed by Ivan Borsenco

Available online at https://aops.com/community/p8526142.

Problem statement

A sequence of positive integers $(a_n)_{n\ge1}$ is of *Fibonacci type* if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \ge 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

Yes, it is possible. The following solutions were written for me by Kevin Sun and Mark Sellke. We let $F_1 = F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, . . .denote the Fibonacci numbers.

¶ First solution (Kevin Sun) We are going to appeal to the so-called Zeckendorf theorem:

Theorem (Zeckendorf)

Every positive integer can be uniquely expressed as the sum of nonconsecutive Fibonacci numbers.

This means every positive integer has a Zeckendorf ("Fibonacci-binary") representation where we put 1 in the ith digit from the right if F_{i+1} is used. The idea is then to take the following so-called *Wythoff array*:

- **Row 1**: $1, 2, 3, 5, \dots$
- **Row 101**: $1+3, 2+5, 3+8, \dots$
- **Row 1001**: $1+5, 2+8, 3+13, \dots$
- **Row 10001**: $1+8, 2+13, 3+21, \dots$
- **Row 10101**: $1+3+8, 2+5+13, 3+8+21, \dots$
- . . .et cetera.

More concretely, the array has the following rows to start:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots \\ 4 & 7 & 11 & 18 & 29 & 47 & 76 & \cdots \\ 6 & 10 & 16 & 26 & 42 & 68 & 110 & \cdots \\ 9 & 15 & 24 & 39 & 63 & 102 & 165 & \cdots \\ 12 & 20 & 32 & 52 & 84 & 136 & 220 & \cdots \\ 14 & 23 & 37 & 60 & 97 & 157 & 254 & \cdots \\ 17 & 28 & 45 & 73 & 118 & 191 & 309 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Here are the full details.

We begin by outlining a proof of Zeckendorf's theorem, which implies the representation above is unique. Note that if F_k is the greatest Fibonacci number at most n, then

$$n - F_k < F_{k+1} - F_k = F_{k-1}.$$

In particular, repeatedly subtracting off the largest F_k from n will produce one such representation with no two consecutive Fibonacci numbers. On the other hand, this F_k must be used, as

$$n \ge F_k > F_{k-1} + F_{k-3} + F_{k-5} + \cdots$$

This shows, by a simple inductive argument, that such a representation exists and unique.

We write $n = \overline{a_k \cdots a_1}_{\text{Fib}}$ for the Zeckendorf representation as we described (where $a_i = 1$ if F_{i+1} is used). Now for each $\overline{a_k \cdots a_1}_{\text{Fib}}$ with $a_1 = 1$, consider the sequence

$$\overline{a_k \cdots a_1}_{\text{Fib}}, \ \overline{a_k \cdots a_1 0}_{\text{Fib}}, \ \overline{a_k \cdots a_1 00}_{\text{Fib}}, \ \dots$$

These sequences are Fibonacci-type by definition, and partition the positive integers since each positive integer has exactly one Fibonacci base representation.

¶ Second solution Call an infinite set of integers S *sandwiched* if there exist increasing sequences $\{a_i\}_{i=0}^{\infty}, \{b_i\}_{i=0}^{\infty}$ such that the following are true:

- $a_i + a_{i+1} = a_{i+2}$ and $b_i + b_{i+1} = b_{i+2}$.
- The intervals $[a_i + 1, b_i - 1]$ are disjoint and are nondecreasing in length.
- $S = \bigcup_{i=0}^{\infty} [a_i + 1, b_i - 1]$.

We claim that if S is any nonempty sandwiched set, then S can be partitioned into a Fibonacci-type sequence (involving the smallest element of S) and two smaller sandwiched sets. If this claim is proven, then we can start with $\mathbb{N} \setminus \{1, 2, 3, 5, \dots\}$, which is a sandwiched set, and repeatedly perform this partition, which will eventually sort each natural number into a Fibonacci-type sequence.

Let S be a sandwiched set given by $\{a_i\}_{i=0}^{\infty}, \{b_i\}_{i=0}^{\infty}$, so the smallest element in S is $x = a_0 + 1$. Note that $y = a_1 + 1$ is also in S and $x < y$. Then consider the Fibonacci-type sequence given by $f_0 = x, f_1 = y$, and $f_{k+2} = f_{k+1} + f_k$. We can then see that $f_i \in [a_i + 1, b_i - 1]$, as the sum of numbers in the intervals $[a_k + 1, b_k - 1]$, $[a_{k+1} + 1, b_{k+1} - 1]$ lies in the interval

$$[a_k + a_{k+1} + 2, b_k + b_{k+1} - 2] = [a_{k+2} + 2, b_{k+2} - 2] \subset [a_{k+2} + 1, b_{k+2} - 1].$$

Therefore, this gives a natural partition of S into this sequence and two sets:

$$S_1 = \bigcup_{i=0}^{\infty} [a_i + 1, f_i - 1]$$
$$\text{and } S_2 = \bigcup_{i=0}^{\infty} [f_i + 1, b_i - 1].$$

(For convenience, $[x, x - 1]$ will be treated as the empty set.)

We now show that S_1 and S_2 are sandwiched. Since $\{a_i\}$, $\{f_i\}$, and $\{b_i\}$ satisfy the Fibonacci recurrence, it is enough to check that the intervals have nondecreasing lengths. For S_1, that is equivalent to $f_{k+1} - a_{k+1} \ge f_k - a_k$ for each k. Fortunately, for $k \ge 1$, the difference is $f_{k-1} - a_{k-1} \ge 0$, and for $k = 0$, $f_1 - a_1 = 1 = f_0 - a_0$. Similarly for S_2, checking $b_{k+1} - f_{k+1} \ge b_k - f_k$ is easy for $k \ge 1$ as $b_{k-1} - f_{k-1} \ge 0$, and

$$(b_1 - f_1) - (b_0 - f_0) = (b_1 - a_1) - (b_0 - a_0),$$

which is nonnegative since the lengths of intervals in S are nondecreasing.

Therefore we have shown that S_1 and S_2 are sandwiched. (Note that some of the $[a_i+1, f_i-1]$ may be empty, which would shift some indices back.) Since this gives us a procedure to take a set S and produce a Fibonacci-type sequence with its smallest element, along which two other sandwiched types, we can partition $\mathbb{N}$ into an infinite number of Fibonacci-type sequences.

¶ Third solution We add Fibonacci-type sequences one-by-one. At each step, let x be the smallest number that has not been used in any previous sequence. We generate a new Fibonacci-type sequence as follows. Set $a_0 = x$ and for $i \ge 1$, set

$$a_i = \left\lfloor \varphi a_{i-1} + \frac{1}{2} \right\rfloor .$$

Equivalently, a_i is the closest integer to φa_{i-1}.
It suffices to show that this sequence is Fibonacci-type and that no two sequences generated in this way overlap. We first show that for a positive integer n,

$$\left\lfloor \varphi \left\lfloor \varphi n + \frac{1}{2} \right\rfloor + \frac{1}{2} \right\rfloor = n + \left\lfloor \varphi n + \frac{1}{2} \right\rfloor .$$

Indeed,

$$\begin{aligned} \left\lfloor \varphi \left\lfloor \varphi n + \frac{1}{2} \right\rfloor + \frac{1}{2} \right\rfloor &= \left\lfloor (1+\varphi^{-1}) \left\lfloor \varphi n + \frac{1}{2} \right\rfloor + \frac{1}{2} \right\rfloor \\ &= \left\lfloor \varphi n + \frac{1}{2} \right\rfloor + \left\lfloor \varphi^{-1} \left\lfloor \varphi n + \frac{1}{2} \right\rfloor + \frac{1}{2} \right\rfloor . \end{aligned}$$

Note that $\left\lfloor \varphi n + \frac{1}{2} \right\rfloor = \varphi n + c$ for some $|c| \le \frac{1}{2}$; this implies that $\varphi^{-1} \left\lfloor \varphi n + \frac{1}{2} \right\rfloor$ is within $\varphi^{-1} \cdot \frac{1}{2} < \frac{1}{2}$ of n, so its closest integer is n, proving the claim.

Therefore these sequences are Fibonacci-type. Additionally, if $a \neq b$, then $|\varphi a - \varphi b| \ge \varphi > 1$. Then

$$a \neq b \implies \left\lfloor \varphi a + \frac{1}{2} \right\rfloor \neq \left\lfloor \varphi b + \frac{1}{2} \right\rfloor ,$$

and since the first term of each sequence is chosen to not overlap with any previous sequences, these sequences are disjoint.

Remark. Ankan Bhattacharya points out that the same sequence essentially appears in IMO 1993, Problem 5 — in other words, a strictly increasing function $f \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with $f(1) = 2$, and $f(f(n)) = f(n) + n$.

Nikolai Beluhov sent us an older reference from March 1977, where Martin Gardner wrote in his column about Wythoff's Nim. The relevant excerpt goes:

> "Imagine that we go through the infinite sequence of safe pairs (in the manner of Eratosthenes' sieve for sifting out primes) and cross out the infinite set of all safe pairs that are pairs in the Fibonacci sequence. The smallest pair that is not crossed out is 4/7. We can now cross out a second infinite set of safe pairs, starting with 4/7, that are pairs in the Lucas sequence. An infinite number of safe pairs, of which the lowest is now 6/10, remain. This pair too begins another infinite Fibonacci sequence, all of whose pairs are safe. The process continues forever. Robert Silber, a mathematician at North Carolina State University, calls a safe pair "primitive" if it is the first safe pair that generates a Fibonacci sequence."

> The relevant article by Robert Silber is *A Fibonacci Property of Wythoff Pairs*, from The Fibonacci Quarterly 11/1976.

¶ Fourth solution (Mark Sellke) For later reference let

$$f_1 = 0, f_2 = 1, f_3 = 1, \dots$$

denote the ordinary Fibonacci numbers. We will denote the Fibonacci-like sequences by F^i and the elements with subscripts; hence F_1^2 is the first element of the second sequence. Our construction amounts to just iteratively add new sequences; hence the following claim is the whole problem.

Lemma

For any disjoint collection of Fibonacci-like sequences $F^1, \dots, F^k$ and any integer m contained in none of them, there is a new Fibonacci-like sequence F^{k+1} beginning with $F_1^{k+1} = m$ which is disjoint from the previous sequences.

Observe first that for each sequence F^j there is $c^j \in \mathbb{R}^n$ such that

$$F_n^j = c^j \phi^n + o(1)$$

where

$$\phi = \frac{1+\sqrt{5}}{2}.$$

Collapse the group $(\mathbb{R}^+, \times)$ into the half-open interval $J = \{x \mid 1 \le x < \phi\}$ by defining $T(x) = y$ for the unique $y \in J$ with $x = y\phi^n$ for some integer n.

Fix an interval $I = [a, b] \subseteq [1.2, 1.3]$ (the last condition is to avoid wrap-around issues) which contains none of the c^j, and take $\varepsilon < 0.001$ to be small enough that in fact each c^j has distance at least 10ε from I; this means any c_j and element of I differ by at least a $(1 + 10\varepsilon)$ factor. The idea will be to take $F_1^{k+1} = m$ and F_2^{k+1} to be a large such that the induced values of F_j^{k+1} grow like $k\phi^j$ for $j \in T^{-1}(I)$, so that F_n^{k+1} is separated from the c^j after applying T. What's left to check is the convergence.

Now let

$$c = \lim_{n \to \infty} \frac{f_n}{\phi^n}$$

and take M large enough that for $n > M$ we have

$$\left| \frac{f_n}{c\phi^n} - 1 \right| < \varepsilon.$$

Now $\frac{T^{-1}(I)}{c}$ contains arbitrarily large integers, so there are infinitely many N with $cN \in T^{-1}(I)$ with $N > \frac{10m}{\varepsilon}$. We claim that for any such N, the sequence $F^{(N)}$ defined by

$$F_1^{(N)} = m, F_2^{(N)} = N$$

will be very multiplicatively similar to the normal Fibonacci numbers up to rescaling; indeed for $j = 2, j = 3$ we have $\frac{F_2^{(N)}}{f_2} = N$, $\frac{F_3^{(N)}}{f_3} = N + m$ and so by induction we will have

$$\frac{F_j^{(N)}}{f_j} \in [N, N + m] \subseteq [N, N(1+\varepsilon)]$$

for $j \geq 2$. Therefore, up to small multiplicative errors, we have

$$F_j^{(N)} \approx N f_j \approx cN\phi^j.$$

From this we see that for $j > M$ we have

$$T(F_j^{(N)}) \in T(cN) \cdot [1 - 2\varepsilon, 1 + 2\varepsilon].$$

In particular, since $T(cN) \in I$ and I is separated from each c_j by a factor of $(1 + 10\varepsilon)$, we get that $F_j^{(N)}$ is not in any of $F^1, F^2, \dots, F^k$.

Finishing is easy, since we now have a uniform estimate on how many terms we need to check for a new element before the exponential growth takes over. We will just use pigeonhole to argue that there are few possible collisions among those early terms, so we can easily pick a value of N which avoids them all. We write it out below.

For large L, the set

$$S_L = (I \cdot \phi^L) \cap \mathbb{Z}$$

contains at least $k_I \phi^L$ elements. As N ranges over S_L, for each fixed j, the value of $F_j^{(N)}$ varies by at most a factor of 1.1 because we imposed $I \subseteq [1.2, 1.3]$ and so this is true for the first two terms, hence for all subsequent terms by induction. Now suppose L is very large, and consider a fixed pair (i, j) with $i \leq k$ and $j \leq M$. We claim there is at most 1 possible value k such that the term F_k^i could equal $F_j^{(N)}$ for some $N \in S_L$; indeed, the terms of F^i are growing at exponential rate with factor $\phi > 1.1$, so at most one will be in a given interval of multiplicative width at most 1.1.

Hence, of these $k_I \phi^L$ values of N, at most kM could cause problems, one for each pair (i, j). However by monotonicity of $F_j^{(N)}$ in N, at most 1 value of N causes a collision for each pair (i, j). Hence for large L so that $k_I \phi^L > 10kM$ we can find a suitable $N \in S_L$ by pigeonhole and the sequence $F^{(N)}$ defined by $(m, N, N + m, \dots)$ works.

USA TST Selection Test for 60th IMO and 8th EGMO

Pittsburgh, PA

Day I   1:15pm – 5:45pm

Tuesday, June 19, 2018

Problem 1. As usual, let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta\colon \mathbb{Z}[x] \to \mathbb{Z}$ such that for any polynomials $p, q \in \mathbb{Z}[x]$,

- $\theta(p+1) = \theta(p) + 1$, and
- if $\theta(p) \neq 0$ then $\theta(p)$ divides $\theta(p \cdot q)$.

Problem 2. In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

We wish to close half the roads of Onewaynia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1 (i.e. of the form 2^n for some integer $n \geq 1$).

Problem 3. Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of $\overline{AB}$. Ray AI meets $\overline{BC}$ at D. Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\angle BAC \neq 60°$, prove that these four lines are concurrent on Γ.

USA TST Selection Test for 60$^{\text{th}}$ IMO and 8$^{\text{th}}$ EGMO

Pittsburgh, PA

Day II   1:15pm – 5:45pm

Thursday, June 21, 2018

Problem 4. For an integer $n > 0$, denote by $\mathcal{F}(n)$ the set of integers $m > 0$ for which the polynomial $p(x) = x^2 + mx + n$ has an integer root.

(a) Let S denote the set of integers $n > 0$ for which $\mathcal{F}(n)$ contains two consecutive integers. Show that S is infinite but

$$\sum_{n\in S} \frac{1}{n} \le 1.$$

(b) Prove that there are infinitely many positive integers n such that $\mathcal{F}(n)$ contains three consecutive integers.

Problem 5. Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of $\triangle AE_1F_1$ and $\triangle AE_2F_2$ are congruent, and the line through their centers is parallel to the tangent to ω at A.

Problem 6. Let $S = \{1, \dots, 100\}$, and for every positive integer n define

$$T_n = \{(a_1, \dots, a_n) \in S^n \mid a_1 + \cdots + a_n \equiv 0 \pmod{100}\}.$$

Determine which n have the following property: if we color any 75 elements of S red, then at least half of the n-tuples in T_n have an even number of coordinates with red elements.

USA TST Selection Test for 60^{th} IMO and 8^{th} EGMO

Pittsburgh, PA

Day III 1:15pm – 5:45pm

Saturday, June 23, 2018

Problem 7. Let n be a positive integer. A frog starts on the number line at 0. Suppose it makes a finite sequence of hops, subject to two conditions:

- The frog visits only points in $\{1, 2, \dots, 2^n - 1\}$, each at most once.
- The length of each hop is in $\{2^0, 2^1, 2^2, \dots\}$. (The hops may be either direction, left or right.)

Let S be the sum of the (positive) lengths of all hops in the sequence. What is the maximum possible value of S?

Problem 8. For which positive integers $b > 2$ do there exist infinitely many positive integers n such that n^2 divides $b^n + 1$?

Problem 9. Show that there is an absolute constant $c < 1$ with the following property: whenever $\mathcal{P}$ is a polygon with area 1 in the plane, one can translate it by a distance of $\frac{1}{100}$ in some direction to obtain a polygon $\mathcal{Q}$, for which the intersection of the interiors of $\mathcal{P}$ and $\mathcal{Q}$ has total area at most c.

USA TSTST 2018 Solutions

United States of America — TST Selection Test

Evan Chen《陳誼廷》

60th IMO 2019 United Kingdom and 8th EGMO 2019 Ukraine

Contents

§0 Problems

1. As usual, let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta\colon \mathbb{Z}[x] \to \mathbb{Z}$ such that for any polynomials $p, q \in \mathbb{Z}[x]$,

- $\theta(p+1) = \theta(p) + 1$, and
- if $\theta(p) \neq 0$ then $\theta(p)$ divides $\theta(p \cdot q)$.

2. In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

We wish to close half the roads of Onewaynia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1 (i.e. of the form 2^n for some integer $n \geq 1$).

3. Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of $\overline{AB}$. Ray AI meets $\overline{BC}$ at D. Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ.

4. For an integer $n > 0$, denote by $\mathcal{F}(n)$ the set of integers $m > 0$ for which the polynomial $p(x) = x^2 + mx + n$ has an integer root.

(a) Let S denote the set of integers $n > 0$ for which $\mathcal{F}(n)$ contains two consecutive integers. Show that S is infinite but

$$\sum_{n \in S} \frac{1}{n} \leq 1.$$

(b) Prove that there are infinitely many positive integers n such that $\mathcal{F}(n)$ contains three consecutive integers.

5. Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of $\triangle AE_1F_1$ and $\triangle AE_2F_2$ are congruent, and the line through their centers is parallel to the tangent to ω at A.

6. Let $S = \{1, \dots, 100\}$, and for every positive integer n define

$$T_n = \{(a_1, \dots, a_n) \in S^n \mid a_1 + \dots + a_n \equiv 0 \pmod{100}\}.$$

Determine which n have the following property: if we color any 75 elements of S red, then at least half of the n-tuples in T_n have an even number of coordinates with red elements.

7. Let n be a positive integer. A frog starts on the number line at 0. Suppose it makes a finite sequence of hops, subject to two conditions:

- The frog visits only points in $\{1, 2, \dots, 2^n - 1\}$, each at most once.
- The length of each hop is in $\{2^0, 2^1, 2^2, \dots\}$. (The hops may be either direction, left or right.)

Let S be the sum of the (positive) lengths of all hops in the sequence. What is the maximum possible value of S?

8. For which positive integers $b > 2$ do there exist infinitely many positive integers n such that n^2 divides $b^n + 1$?

9. Show that there is an absolute constant $c < 1$ with the following property: whenever $\mathcal{P}$ is a polygon with area 1 in the plane, one can translate it by a distance of $\frac{1}{100}$ in some direction to obtain a polygon $\mathcal{Q}$, for which the intersection of the interiors of $\mathcal{P}$ and $\mathcal{Q}$ has total area at most c.

§1 Solutions to Day 1

§1.1 TSTST 2018/1, proposed by Evan Chen, Yang Liu

Available online at https://aops.com/community/p10570981.

Problem statement

As usual, let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta \colon \mathbb{Z}[x] \to \mathbb{Z}$ such that for any polynomials $p, q \in \mathbb{Z}[x]$,

- $\theta(p+1) = \theta(p) + 1$, and
- if $\theta(p) \neq 0$ then $\theta(p)$ divides $\theta(p \cdot q)$.

The answer is $\theta : p \mapsto p(c)$, for each choice of $c \in \mathbb{Z}$. Obviously these work, so we prove these are the only ones. In what follows, $x \in \mathbb{Z}[x]$ is the identity polynomial, and $c = \theta(x)$.

¶ First solution (Merlijn Staps) Consider an integer $n \neq c$. Because $x - n \mid p(x) - p(n)$, we have

$$\theta(x-n) \mid \theta(p(x) - p(n)) \implies c - n \mid \theta(p(x)) - p(n).$$

On the other hand, $c - n \mid p(c) - p(n)$. Combining the previous two gives $c - n \mid \theta(p(x)) - p(c)$, and by letting n large we conclude $\theta(p(x)) - p(c) = 0$, so $\theta(p(x)) = p(c)$.

¶ Second solution First, we settle the case $\deg p = 0$. In that case, from the second property, $\theta(m) = m + \theta(0)$ for every integer $m \in \mathbb{Z}$ (viewed as a constant polynomial). Thus $m + \theta(0) \mid 2m + \theta(0)$, hence $m + \theta(0) \mid -\theta(0)$, so $\theta(0) = 0$ by taking m large. Thus $\theta(m) = m$ for $m \in \mathbb{Z}$.

Next, we address the case of $\deg p = 1$. We know $\theta(x+b) = c + b$ for $b \in \mathbb{Z}$. Now for each particular $a \in \mathbb{Z}$, we have

$$c + k \mid \theta(x+k) \mid \theta(ax + ak) = \theta(ax) + ak \implies c + k \mid \theta(ax) - ac.$$

for any $k \neq -c$. Since this is true for large enough k, we conclude $\theta(ax) = ac$. Thus $\theta(ax + b) = ac + b$.

We now proceed by induction on $\deg p$. Fix a polynomial p and assume it's true for all p of smaller degree. Choose a large integer n (to be determined later) for which $p(n) \neq p(c)$. We then have

$$\frac{p(c) - p(n)}{c - n} = \theta\left(\frac{p - p(n)}{x - n}\right) \mid \theta\left(p - p(n)\right) = \theta(p) - p(n).$$

Subtracting off $c - n$ times the left-hand side gives

$$\frac{p(c) - p(n)}{c - n} \mid \theta(p) - p(c).$$

The left-hand side can be made arbitrarily large by letting $n \to \infty$, since $\deg p \geq 2$. Thus $\theta(p) = p(c)$, concluding the proof.

¶ **Authorship comments** I will tell you a story about the creation of this problem. Yang Liu and I were looking over the drafts of December and January TST in October 2017, and both of us had the impression that the test was too difficult. This sparked a non-serious suggestion that we should try to come up with a problem *now* that would be easy enough to use. While we ended up just joking about changing the TST, we did get this problem out of it.

Our idea was to come up with a functional equation that was different from the usual fare: at first we tried $\mathbb{Z}[x] \to \mathbb{Z}[x]$, but then I suggested the idea of using $\mathbb{Z}[x] \to \mathbb{Z}$, with the answer being the "evaluation" map. Well, what properties does that satisfy? One answer was $a - b \mid p(a) - p(b)$; this didn't immediately lead to anything, but eventually we hit on the form of the problem above off this idea. At first we didn't require $\theta(p) \neq 0$ in the bullet, but without the condition the problem was too easy, since 0 divides only itself; and so the condition was added and we got the functional equation.

I proposed the problem to USAMO 2018, but it was rejected (unsurprisingly; I think the problem may be too abstract for novice contestants). Instead it was used for TSTST, which I thought fit better.

§1.2 TSTST 2018/2, proposed by Victor Wang

Available online at https://aops.com/community/p10570985.

Problem statement

In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

We wish to close half the roads of Onewaynia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1 (i.e. of the form 2^n for some integer $n \ge 1$).

In the language of graph theory, we have a simple digraph G which is 2-regular and we seek the number of sub-digraphs which are 1-regular. We now present two solution paths.

¶ First solution, combinatorial We construct a simple undirected bipartite graph Γ as follows:

- the vertex set consists of two copies of $V(G)$, say V_{out} and V_{in}; and
- for $v \in V_{\text{out}}$ and $w \in V_{\text{in}}$ we have an undirected edge $vw \in E(\Gamma)$ if and only if the directed edge $v \to w$ is in G.

Moreover, the desired sub-digraphs of H correspond exactly to perfect matchings of Γ.

However the graph Γ is 2-regular and hence consists of several disjoint (simple) cycles of even length. If there are n such cycles, the number of perfect matchings is 2^n, as desired.

> **Remark.** The construction of Γ is not as magical as it may first seem.
> Suppose we pick a road $v_1 \to v_2$ to use. Then, the other road $v_3 \to v_2$ is certainly *not* used; hence some other road $v_3 \to v_4$ must be used, etc. We thus get a cycle of forced decisions until we eventually return to the vertex v_1.
> These cycles in the original graph G (where the arrows alternate directions) correspond to the cycles we found in Γ. It's merely that phrasing the solution in terms of Γ makes it cleaner in a linguistic sense, but not really in a mathematical sense.

¶ Second solution by linear algebra over $\mathbb{F}_2$ (Brian Lawrence) This is actually not that different from the first solution. For each edge e, we create an indicator variable x_e. We then require for each vertex v that:

- If e_1 and e_2 are the two edges leaving v, then we require $x_{e_1} + x_{e_2} \equiv 1 \pmod 2$.
- If e_3 and e_4 are the two edges entering v, then we require $x_{e_3} + x_{e_4} \equiv 1 \pmod 2$.

We thus get a large system of equations. Moreover, the solutions come in natural pairs $\vec{x}$ and $\vec{x} + \vec{1}$ and therefore the number of solutions is either zero, or a power of two. So we just have to prove there is at least one solution.

For linear algebra reasons, there can only be zero solutions if some nontrivial linear combination of the equations gives the sum $0 \equiv 1$. So suppose we added up some subset S

of the equations for which every variable appeared on the left-hand side an even number of times. Then every variable that did appear appeared exactly twice; and accordingly we see that the edges corresponding to these variables form one or more even cycles as in the previous solution. Of course, this means $|S|$ is even, so we really have $0 \equiv 0 \pmod 2$ as needed.

> **Remark.** The author's original proposal contained a second part asking to show that it was not always possible for the resulting H to be connected, even if G was strongly connected. This problem is related to IMO Shortlist 2002 C6, which gives an example of a strongly connected graph which does have a full directed Hamiltonian cycle.

§1.3 TSTST 2018/3, proposed by Yannick Yao, Evan Chen

Available online at https://aops.com/community/p10570988.

Problem statement

Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of $\overline{AB}$. Ray AI meets $\overline{BC}$ at D. Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ.

Henceforth assume $\angle A \neq 60^\circ$; we prove the concurrence. Let L denote the center of ω, which is the midpoint of minor arc BC.

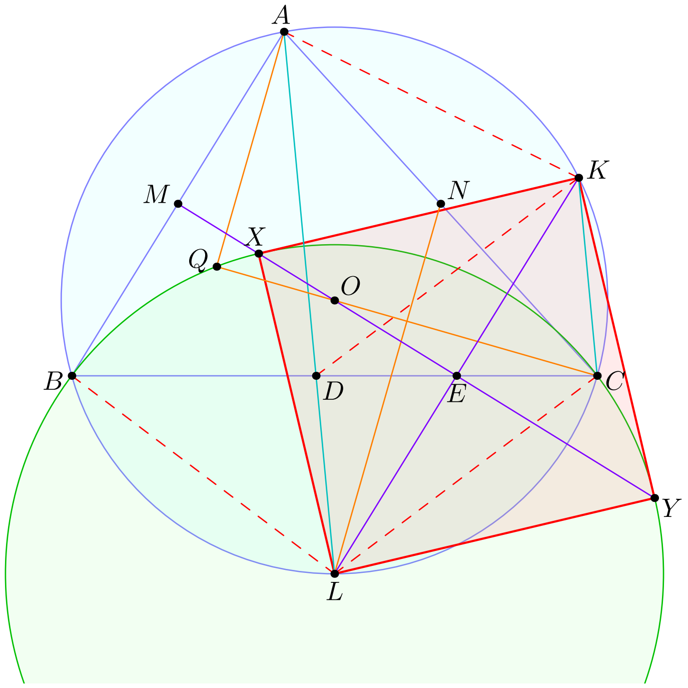
Claim — Let K be the point on ω such that $\overline{KL} \parallel \overline{AB}$ and $\overline{KC} \parallel \overline{AL}$. Then $\overline{KA}$ is tangent to γ, and we may put

$$x = KA = LB = LC = LX = LY = KX = KY.$$

Proof. By construction, $KA = LB = LC$. Also, $\overline{MO}$ is the perpendicular bisector of $\overline{KL}$ (since the chords $\overline{KL}$, $\overline{AB}$ of ω are parallel) and so $KXLY$ is a rhombus as well.

Moreover, $\overline{KA}$ is tangent to γ as well since

$$\measuredangle KAD = \measuredangle KAL = \measuredangle KAC + \measuredangle CAL = \measuredangle KBC + \measuredangle ABK = \measuredangle ABC. \qquad \square$$



Up to now we have not used the existence of Q; we henceforth do so.

Note that $Q \neq O$, since $\angle A \neq 60^\circ \implies O \notin \omega$. Moreover, we have $\angle AOM = \angle ACB$ too. Since O and Q both lie inside $\triangle ABC$, this implies that A, M, O, Q are concyclic. As $Q \neq O$ we conclude $\angle CQA = 90^\circ$.

The main claim is now:

Claim — Assuming Q exists, the rhombus $LXKY$ is a square. In particular, $\overline{KX}$ and $\overline{KY}$ are tangent to ω.

First proof of Claim, communicated by Milan Haiman. Observe that $\triangle QLC \sim \triangle LOC$ since both triangles are isosceles and share a base angle. Hence, $CL^2 = CO \cdot CQ$.

Let N be the midpoint of $\overline{AC}$, which lies on $(AMOQ)$. Then,

$$x^2 = CL^2 = CO \cdot CQ = CN \cdot CA = \frac{1}{2}CA^2 = \frac{1}{2}LK^2$$

where we have also used the fact $AQON$ is cyclic. Thus $LK = \sqrt{2}x$ and so the rhombus $LXKY$ is actually a square. □

Second proof of Claim, Evan Chen. Observe that Q lies on the circle with diameter $\overline{AC}$, centered at N, say. This means that O lies on the radical axis of ω and (N), hence $\overline{NL} \perp \overline{CO}$ implying

$$\begin{aligned} NO^2 + CL^2 &= NC^2 + LO^2 = NC^2 + OC^2 = NC^2 + NO^2 + NC^2 \\ \implies x^2 &= 2NC^2 \\ \implies x &= \sqrt{2}NC = \frac{1}{\sqrt{2}}AC = \frac{1}{\sqrt{2}}LK. \end{aligned}$$

So $LXKY$ is a rhombus with $LK = \sqrt{2}x$. Hence it is a square. □

Third proof of Claim. A solution by trig is also possible. As in the previous claims, it suffices to show that $AC = \sqrt{2}x$.

First, we compute the length CQ in two ways; by angle chasing one can show $\angle CBQ = 180^\circ - (\angle BQC + \angle QCB) = \frac{1}{2}\angle A$, and so

$$\begin{aligned} AC \sin B = CQ &= \frac{BC}{\sin(90^\circ + \frac{1}{2}\angle A)} \cdot \sin\frac{1}{2}\angle A \\ \iff \sin^2 B &= \frac{\sin A \cdot \sin\frac{1}{2}\angle A}{\cos\frac{1}{2}\angle A} \\ \iff \sin^2 B &= 2\sin^2\frac{1}{2}\angle A \\ \iff \sin B &= \sqrt{2}\sin\frac{1}{2}\angle A \\ \iff 2R\sin B &= \sqrt{2}\left(2R\sin\frac{1}{2}\angle A\right) \\ \iff AC &= \sqrt{2}x \end{aligned}$$

as desired (we have here used the fact $\triangle ABC$ is acute to take square roots).

It is interesting to note that $\sin^2 B = 2\sin^2\frac{1}{2}\angle A$ can be rewritten as

$$\cos A = \cos^2 B$$

since $\cos^2 B = 1 - \sin^2 B = 1 - 2\sin^2\frac{1}{2}\angle A = \cos A$; this is the condition for the existence of the point Q. □

We finish by proving that

$$KD = KA$$

and hence line $\overline{KD}$ is tangent to γ. Let $E = \overline{BC} \cap \overline{KL}$. Then

$$LE \cdot LK = LC^2 = LX^2 = \frac{1}{2}LK^2$$

and so E is the midpoint of $\overline{LK}$. Thus $\overline{MXOY}$, $\overline{BC}$, $\overline{KL}$ are concurrent at E. As $\overline{DL} \parallel \overline{KC}$, we find that $DLCK$ is a parallelogram, so $KD = CL = KA$ as well. Thus $\overline{KD}$ and $\overline{KA}$ are tangent to γ.

Remark. The condition $\angle A \neq 60^\circ$ cannot be dropped, since if $Q = O$ the problem is not true.

On the other hand, nearly all solutions begin by observing $Q \neq O$ and then obtaining $\angle AQO = 90^\circ$. This gives a way to construct the diagram by hand with ruler and compass. One draws an arbitrary chord $\overline{BC}$ of a circle ω centered at L, and constructs O as the circumcenter of $\triangle BLC$ (hence obtaining Γ). Then Q is defined as the intersection of ray CO with ω, and A is defined by taking the perpendicular line through Q on the circle Γ. In this way we can draw a triangle ABC satisfying the problem conditions.

¶ Authorship comments In the notation of the present points, the question originally sent to me by Yannick Yao read:

> Circles (L) and (O) are drawn, meeting at B and C, with L on (O). Ray CO meets (L) at Q, and A is on (O) such that $\angle CQA = 90^\circ$. The angle bisector of $\angle AOB$ meets (L) at X and Y. Show that $\angle XLY = 90^\circ$.

Notice the points M and K are absent from the problem. I am told this was found as part of the computer game "Euclidea". Using this as the starting point, I constructed the TSTST problem by recognizing the significance of that special point K, which became the center of attention.

§2 Solutions to Day 2

§2.1 TSTST 2018/4, proposed by Ivan Borsenco

Available online at https://aops.com/community/p10570991.

Problem statement

For an integer $n > 0$, denote by $\mathcal{F}(n)$ the set of integers $m > 0$ for which the polynomial $p(x) = x^2 + mx + n$ has an integer root.

(a) Let S denote the set of integers $n > 0$ for which $\mathcal{F}(n)$ contains two consecutive integers. Show that S is infinite but

$$\sum_{n \in S} \frac{1}{n} \le 1.$$

(b) Prove that there are infinitely many positive integers n such that $\mathcal{F}(n)$ contains three consecutive integers.

We prove the following.

Claim — The set S is given explicitly by $S = \{x(x+1)y(y+1) \mid x, y > 0\}$.

Proof. Note that $m, m+1 \in \mathcal{F}(n)$ if and only if there exist integers $q > p \ge 0$ such that

$$m^2 - 4n = p^2$$
$$(m+1)^2 - 4n = q^2.$$

Subtraction gives $2m + 1 = q^2 - p^2$, so p and q are different parities. We can thus let $q - p = 2x + 1$, $q + p = 2y + 1$, where $y \ge x \ge 0$ are integers. It follows that

$$\begin{aligned}
4n &= m^2 - p^2 \\
&= \left(\frac{q^2 - p^2 - 1}{2}\right)^2 - p^2 = \left(\frac{q^2 - p^2 - 1}{2} - p\right)\left(\frac{q^2 - p^2 - 1}{2} + p\right) \\
&= \frac{q^2 - (p^2 + 2p + 1)}{2} \cdot \frac{q^2 - (p^2 - 2p + 1)}{2} \\
&= \frac{1}{4}(q - p - 1)(q - p + 1)(q + p - 1)(q + p + 1) = \frac{1}{4}(2x)(2x + 2)(2y)(2y + 2) \\
\implies n &= x(x + 1)y(y + 1).
\end{aligned}$$

Since $n > 0$ we require $x, y > 0$. Conversely, if $n = x(x+1)y(y+1)$ for positive x and y then $m = \sqrt{p^2 + 4n} = \sqrt{(y - x)^2 + 4n} = 2xy + x + y = x(y+1) + (x+1)y$ and $m + 1 = 2xy + x + y + 1 = xy + (x+1)(y+1)$. Thus we conclude the main claim. $\square$

From this, part (a) follows as

$$\sum_{n \in S} n^{-1} \le \left(\sum_{x \ge 1} \frac{1}{x(x+1)}\right)\left(\sum_{y \ge 1} \frac{1}{y(y+1)}\right) = 1 \cdot 1 = 1.$$

As for (b), retain the notation in the proof of the claim. Now $m+2 \in S$ if and only if $(m+2)^2 - 4n$ is a square, say r^2. Writing in terms of p and q as parameters we find

$$\begin{aligned} r^2 &= (m+2)^2 - 4n = m^2 - 4n + 4m + 4 = p^2 + 2 + 2(2m+1) \\ &= p^2 + 2(q^2 - p^2) + 2 = 2q^2 - p^2 + 2 \\ \iff 2q^2 + 2 &= p^2 + r^2 \qquad (\dagger) \end{aligned}$$

with $q > p$ of different parity and $n = \frac{1}{16}(q-p-1)(q-p+1)(q+p-1)(q+p+1)$.

Note that (by taking modulo 8) we have $q \not\equiv p \equiv r \pmod 2$, and so there are no parity issues and we will always assume $p < q < r$ in ($\dagger$). Now, for every q, the equation ($\dagger$) has a canonical solution $(p, r) = (q-1, q+1)$, but this leaves $n = 0$. Thus we want to show for infinitely many q there is a third way to write $2q^2 + 2$ as a sum of squares, which will give the desired p.

To do this, choose large integers q such that $q^2 + 1$ is divisible by at least three distinct 1 mod 4 primes. Since each such prime can be written as a sum of two squares, using Lagrange identity, we can deduce that $2q^2 + 2$ can be written as a sum of two squares in at least three different ways, as desired.

Remark. We can see that $n = 144$ is the smallest integer such that $\mathcal{F}(n)$ contains three consecutive integers and $n = 15120$ is the smallest integer such that $\mathcal{F}(n)$ contains four consecutive integers. It would be interesting to determine whether the number of consecutive elements in $\mathcal{F}(n)$ can be arbitrarily large or is bounded.

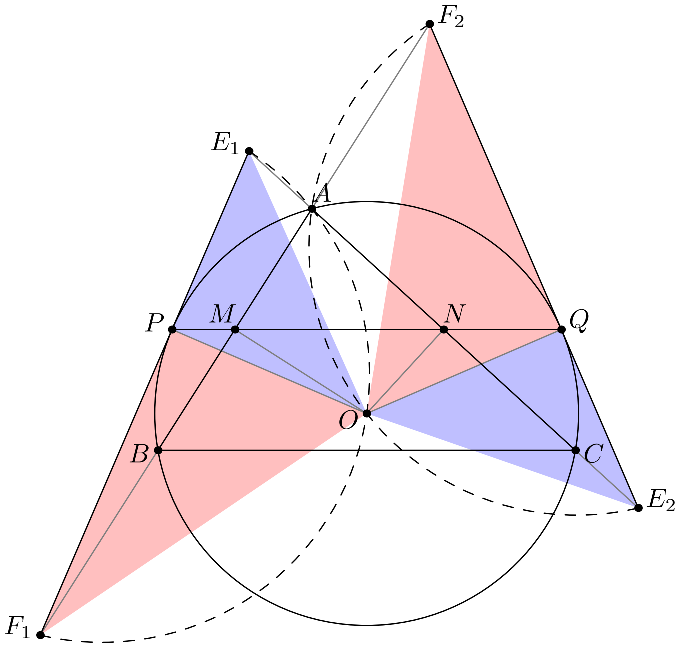
§2.2 TSTST 2018/5, proposed by Ankan Bhattacharya, Evan Chen

Available online at https://aops.com/community/p10571000.

Problem statement

Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of $\triangle AE_1F_1$ and $\triangle AE_2F_2$ are congruent, and the line through their centers is parallel to the tangent to ω at A.

Let O be the center of ω, and let $M = \overline{PQ} \cap \overline{AB}$ and $N = \overline{PQ} \cap \overline{AC}$ be the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively. Refer to the diagram below.



The main idea is to prove two key claims involving O, which imply the result:

(i) quadrilaterals AOE_1F_1 and AOE_2F_2 are cyclic (giving the radical axis is $\overline{AO}$),

(ii) $\triangle OE_1F_1 \cong \triangle OE_2F_2$ (giving the congruence of the circles).

We first note that (i) and (ii) are equivalent. Indeed, because $OP = OQ$, (ii) is equivalent to just the similarity $\triangle OE_1F_1 \sim \triangle OE_2F_2$, and then by the spiral similarity lemma (or even just angle chasing) we have (i) $\iff$ (ii).

We now present five proofs, two of (i) and three of (ii). Thus, we are essentially presenting five different solutions.

¶ Proof of (i) by angle chasing Note that

$$\measuredangle F_2E_2O = \measuredangle QE_2O = \measuredangle QNO = \measuredangle MNO = \measuredangle MAO = \measuredangle F_2AO$$

and hence E_2OAF_2 is cyclic. Similarly, E_1OAF_1 is cyclic.

¶ Proof of (i) by Simson lines Since P, M, N are collinear, we see that $\overline{PMN}$ is the Simson line of O with respect to $\triangle AE_1F_1$.

¶ Proof of (ii) by butterfly theorem By BUTTERFLY THEOREM on the three chords $\overline{AC}$, $\overline{PQ}$, $\overline{PQ}$, it follows that $E_1N = NE_2$. Thus

$$E_1P = \sqrt{E_1A \cdot E_1C} = \sqrt{E_2A \cdot E_2C} = E_2P.$$

But also $OP = OQ$ and hence $\triangle OPE_1 \cong \triangle OQE_2$. Similarly for the other pair.

¶ Proof of (ii) by projective geometry Let $T = \overline{PP} \cap \overline{QQ}$. Let S be on $\overline{PQ}$ with $\overline{ST} \parallel \overline{AC}$; then $\overline{TS} \perp \overline{ON}$, and it follows $\overline{ST}$ is the polar of N (it passes through T by La Hire).

Now,

$$-1 = (PQ; NS) \overset{T}{=} (E_1E_2; N\infty)$$

with $\infty = \overline{AC} \cap \overline{ST}$ the point at infinity. Hence $E_1N = NE_2$ and we can proceed as in the previous solution.

> **Remark.** The assumption that $\triangle ABC$ is acute is not necessary; it is only present to ensure that P lies on segment E_1F_1 and Q lies on segment E_2F_2, which may be helpful for contestants. The argument presented above is valid in all configurations. When one of $\angle B$ and $\angle C$ is a right angle, some of the points E_1, F_1, E_2, F_2 lie at infinity; when one of them is obtuse, both P and Q lie outside segments E_1F_1 and E_2F_2 respectively.

¶ Proof of (ii) by complex numbers We will give using complex numbers on $\triangle ABC$ a proof that $|E_1P| = |E_2Q|$.

We place $APBCQ$ on the unit circle. Since $\overline{PQ} \parallel \overline{BC}$, we have $pq = bc$. Also, the midpoint of $\overline{AB}$ lies on $\overline{PQ}$, so

$$\begin{aligned} p + q &= \frac{a+b}{2} + \overline{\left(\frac{a+b}{2}\right)} \cdot pq \\ &= \frac{a+b}{2} + \frac{a+b}{2ab} \cdot bc \\ &= \frac{a(a+b)}{2a} + \frac{c(a+b)}{2a} \\ &= \frac{(a+b)(a+c)}{2a}. \end{aligned}$$

Now,

$$\begin{aligned} p - e_1 &= p - \frac{pp(a+c) - ac(p+p)}{pp - ac} \\ &= \frac{p(p^2 - p(a+c) + ac)}{pp - ac} = \frac{(p-a)(p-c)}{p^2 - ac}. \\ |PE_1|^2 &= (p - e_1) \cdot \overline{p - e_1} = \frac{(p-a)(p-c)}{p^2 - ac} \cdot \frac{(\frac{1}{p} - \frac{1}{a})(\frac{1}{p} - \frac{1}{c})}{\frac{1}{p^2} - \frac{1}{ac}} \\ &= -\frac{(p-a)^2(p-c)^2}{(p^2 - ac)^2}. \end{aligned}$$

Similarly,

$$|QE_2|^2 = -\frac{(q-a)^2(q-c)^2}{(q^2-ac)^2}.$$

But actually, we claim that

$$\frac{(p-a)(p-c)}{p^2-ac} = \frac{(q-a)(q-c)}{q^2-ac}.$$

One calculates

$$(p-a)(p-c)(q^2-ac) = p^2q^2 - pq^2a - pq^2c + q^2ac - p^2ac + pa^2c + pac^2 - (ac)^2$$

Thus $(p-a)(p-c)(q^2-ac) - (q-a)(q-c)(p^2-ac)$ is equal to

$$\begin{aligned} -(a+c)(pq)(q-p) &+ (q^2-p^2)ac - (p^2-q^2)ac + ac(a+c)(p-q) \\ &= (p-q)\left[(a+c)pq - 2(p+q)ac + ac(a+c)\right] \\ &= (p-q)\left[(a+c)bc - 2\cdot\frac{(a+b)(a+c)}{2a}\cdot ac + ac(a+c)\right] \\ &= (p-q)(a+c)\left[bc - c(a+b) + ac\right] = 0. \end{aligned}$$

This proves $|E_1P| = |E_2Q|$. Together with the similar $|F_1P| = |F_2Q|$, we have proved (ii).

¶ **Authorship comments** Ankan provides an extensive dialogue at https://aops.com/community/c6h1664170p10571644 of how he came up with this problem, which at first was intended just to be an AMC-level question about an equilateral triangle. Here, we provide just the change-log of the versions of this problem.

0. (*Original version*) Let ABC be an equilateral triangle with side 2 inscribed in circle ω, and let P be a point on small arc AB of its circumcircle. The tangent line to ω at P intersects lines AC and AB at E and F. If $PE = PF$, find EF. (Answer: 4.)

1. (*Generalize to isosceles triangle*) Let ABC be an isosceles triangle with $AB = AC$, and let M be the midpoint of $\overline{BC}$. Let P be a point on the circumcircle with $PA = PM$. The tangent to the circumcircle at P intersects lines AC and AB at E and F, respectively. Show that $PE = PF$.

2. (*Block coordinate bashes*) Let ABC be an isosceles triangle with $AB = AC$ and circumcircle ω, and let M be the midpoint of $\overline{BC}$. Let P be a point on ω with $PA = PM$. The tangent to ω at P intersects lines AC and AB at E and F, respectively. Show that the circumcircle of $\triangle AEF$ passes through the center of ω.

3. (*Delete isosceles condition*) Let ABC be a triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P be a point on ω with $PA = PH$. The tangent to ω at P intersects lines AC and AB at E and F, respectively. Show that the circumcircle of $\triangle AEF$ passes through the center of ω.

4. (*Add in both tangents*) Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that $E_1F_1 = E_2F_2$.

5. (*Merge v3 and v4*) Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of $\triangle AE_1F_1$ and $\triangle AE_2F_2$ are congruent, and the line through their centers is parallel to the tangent to ω at A.

The problem bears Evan's name only because he suggested the changes v2 and v5.

§2.3 TSTST 2018/6, proposed by Ray Li

Available online at https://aops.com/community/p10570994.

Problem statement

Let $S = \{1, \dots, 100\}$, and for every positive integer n define

$$T_n = \{(a_1, \dots, a_n) \in S^n \mid a_1 + \dots + a_n \equiv 0 \pmod{100}\}.$$

Determine which n have the following property: if we color any 75 elements of S red, then at least half of the n-tuples in T_n have an even number of coordinates with red elements.

We claim this holds exactly for n even.

¶ First solution by generating functions Define

$$R(x) = \sum_{s \text{ red}} x^s, \qquad B(x) = \sum_{s \text{ blue}} x^s.$$

(Here "blue" means "not-red", as always.) Then, the number of tuples in T_n with exactly k red coordinates is exactly equal to

$$\binom{n}{k} \cdot \frac{1}{100} \sum_{\omega} R(\omega)^k B(\omega)^{n-k}$$

where the sum is over all 100th roots of unity. So, we conclude the number of tuples in T_n with an even (resp odd) number of red elements is exactly

$$\begin{aligned}
X &= \frac{1}{100} \sum_{\omega} \sum_{k \text{ even}} \binom{n}{k} R(\omega)^k B(\omega)^{n-k} \\
Y &= \frac{1}{100} \sum_{\omega} \sum_{k \text{ odd}} \binom{n}{k} R(\omega)^k B(\omega)^{n-k} \\
\implies X - Y &= \frac{1}{100} \sum_{\omega} (B(\omega) - R(\omega))^n \\
&= \frac{1}{100} \left[(B(1) - R(1))^n + \sum_{\omega \neq 1} (2B(\omega))^n \right] \\
&= \frac{1}{100} \left[(B(1) - R(1))^n - (2B(1))^n + 2^n \sum_{\omega} B(\omega)^n \right] \\
&= \frac{1}{100} \left[(B(1) - R(1))^n - (2B(1))^n \right] + 2^n Z \\
&= \frac{1}{100} \left[(-50)^n - 50^n \right] + 2^n Z.
\end{aligned}$$

where

$$Z := \frac{1}{100} \sum_{\omega} B(\omega)^n \geq 0$$

counts the number of tuples in T_n which are all blue. Here we have used the fact that $B(\omega) + R(\omega) = 0$ for $\omega \neq 1$.

We wish to show $X - Y \ge 0$ holds for n even, but may fail when n is odd. This follows from two remarks:

- If n is even, then $X - Y = 2^n Z \ge 0$.
- If n is odd, then if we choose the coloring for which s is red if and only if $s \not\equiv 2 \pmod 4$; we thus get $Z = 0$. Then $X - Y = -\frac{2}{100} \cdot 50^n < 0$.

¶ Second solution by strengthened induction and random coloring We again prove that n even work. Let us define

$$T_n(a) = \{(a_1, \dots, a_n) \in S^n \mid a_1 + \dots + a_n \equiv a \pmod{100}\}.$$

Also, call an n-tuple good if it has an even number of red elements. We claim that $T_n(a)$ also has at least 50% good tuples, by induction.

This follows by induction on $n \ge 2$. Indeed, the base case $n = 2$ can be checked by hand, since $T_2(a) = \{(x, a - x) \mid x \in S\}$. With the stronger claim, one can check the case $n = 2$ manually and proceed by induction to go from $n - 2$ to n, noting that

$$T_n(a) = \bigsqcup_{b+c=a} T_{n-2}(b) \oplus T_2(c)$$

where $\oplus$ denotes concatenation of tuples, applied set-wise. The concatenation of an $(n-2)$-tuple and 2-tuple is good if and only if both or neither are good. Thus for each b and c, if the proportion of $T_{n-2}(b)$ which is good is $p \ge \frac{1}{2}$ and the proportion of $T_2(c)$ which is good is $q \ge \frac{1}{2}$, then the proportion of $T_{n-2}(b) \oplus T_2(c)$ which is good is $pq + (1-p)(1-q) \ge \frac{1}{2}$, as desired. Since each term in the union has at least half the tuples good, all of $T_n(a)$ has at least half the tuples good, as desired.

It remains to fail all odd n. We proceed by a suggestion of Yang Liu and Ankan Bhattacharya by showing that if we pick the 75 elements *randomly*, then any particular tuple in S^n has strictly less than 50% chance of being good. This will imply (by linearity of expectation) that T_n (or indeed any subset of S^n) will, for some coloring, have less than half good tuples.

Let $(a_1, \dots, a_n)$ be such an n-tuple. If any element appears in the tuple more than once, keep *discarding pairs* of that element until there are zero or one; this has no effect on the good-ness of the tuple. If we do this, we obtain an m-tuple $(b_1, \dots, b_m)$ with no duplicated elements where $m \equiv n \equiv 1 \pmod 2$. Now, the probability that any element is red is $\frac{3}{4}$, so the probability of being good is

$$\begin{aligned}\sum_{k \text{ even}}^{m} \binom{m}{k} \left(\frac{3}{4}\right)^k \left(-\frac{1}{4}\right)^{m-k} &= \frac{1}{2}\left[\left(\frac{3}{4}+\frac{1}{4}\right)^m - \left(\frac{3}{4}-\frac{1}{4}\right)^m\right] \\ &= \frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^m\right] < \frac{1}{2}.\end{aligned}$$

Remark (Adam Hesterberg)**.** Here is yet another proof that n even works. Group elements of T_n into equivalence classes according to the $n/2$ sums of pairs of consecutive elements (first and second, third and fourth, ...). For each such pair sum, there are at least as many monochrome pairs with that sum as nonmonochrome ones, since every nonmonochrome pair uses one of the 25 non-reds. The monochromaticity of the pairs is independent.

If $p_i \le \frac{1}{2}$ is the probability that the ith pair is nonmonochrome, then the probability that k pairs are nonmonochrome is the coefficient of x^k in $f(x) = \prod_i (xp_i + (1 - p_i))$. Then the probability that evenly many pairs are nonmonochrome (and hence that evenly many coordinates are red) is the sum of the coefficients of even powers of x in f, which is

$(f(1) + f(-1))/2 = (1 + \prod_i (1 - 2p_i))/2 \geq \frac{1}{2}$, as desired.

§3 Solutions to Day 3

§3.1 TSTST 2018/7, proposed by Ashwin Sah

Available online at https://aops.com/community/p10570996.

Problem statement

Let n be a positive integer. A frog starts on the number line at 0. Suppose it makes a finite sequence of hops, subject to two conditions:

- The frog visits only points in $\{1, 2, \dots, 2^n - 1\}$, each at most once.
- The length of each hop is in $\{2^0, 2^1, 2^2, \dots\}$. (The hops may be either direction, left or right.)

Let S be the sum of the (positive) lengths of all hops in the sequence. What is the maximum possible value of S?

We claim the answer is $\frac{4^n-1}{3}$.

We first prove the bound. First notice that the hop sizes are in $\{2^0, 2^1, \dots, 2^{n-1}\}$, since the frog must stay within bounds the whole time. Let a_i be the number of hops of size 2^i the frog makes, for $0 \le i \le n-1$.

Claim — For any $k = 1, \dots, n$ we have

$$a_{n-1} + \dots + a_{n-k} \le 2^n - 2^{n-k}.$$

Proof. Let $m = n - k$ and look modulo 2^m. Call a jump *small* if its length is at most 2^{m-1}, and *large* if it is at least 2^m; the former changes the residue class of the frog modulo 2^m while the latter does not.

Within each fixed residue modulo 2^m, the frog can make at most $\frac{2^n}{2^m} - 1$ large jumps. So the total number of large jumps is at most $2^m\left(\frac{2^n}{2^m} - 1\right) = 2^n - 2^m$. □

(As an example, when $n = 3$ this means there are at most four hops of length 4, at most six hops of length 2 or 4, and at most seven hops total. Of course, if we want to max the length of the hops, we see that we want $a_2 = 4$, $a_1 = 2$, $a_0 = 1$, and in general equality is achieved when $a_m = 2^m$ for any m.)

Now, the total distance the frog travels is

$$S = a_0 + 2a_1 + 4a_2 + \dots + 2^{n-1}a_{n-1}.$$

We rewrite using the so-called "summation by parts":

$$\begin{aligned} S = a_0 + a_1 + {} & a_2 + a_3 + \ldots + a_{n-1} \\ + \quad & a_1 + a_2 + a_3 + \ldots + a_{n-1} \\ + \quad & 2a_2 + 2a_3 + \ldots + 2a_{n-1} \\ + \quad & 4a_3 + \ldots + 4a_{n-1} \\ \vdots \quad & \ddots \quad \vdots \\ + \quad & 2^{n-2}a_{n-1}. \end{aligned}$$

Hence

$$\begin{aligned} S &\le (2^n - 2^0) + (2^n - 2^1) + 2(2^n - 2^2) + \cdots + 2^{n-2}(2^n - 2^{n-1}) \\ &= \frac{4^n - 1}{3}. \end{aligned}$$

It remains to show that equality can hold. There are many such constructions but most are inductive. Here is one approach. We will construct two family of paths such that there are 2^k hops of size 2^k, for every $0 \le k \le n-1$, and we visit each of $\{0, \dots, 2^n - 1\}$ once, starting on 0 and ending on x, for the two values $x \in \{1, 2^n - 1\}$.

The base case $n = 1$ is clear. To take a path from 0 to $2^{n+1} - 1$.

- Take a path on $\{0, 2, 4, \dots, 2^{n+1} - 2\}$ starting from 0 and ending on 2 (by inductive hypothesis).
- Take a path on $\{1, 3, 5, \dots, 2^{n+1} - 1\}$ starting from 1 and ending on $2^{n+1} - 1$ (by inductive hypothesis).
- Link them together by adding a single jump $2 \to 1$.

The other case is similar, but we route $0 \to (2^{n+1} - 2) \to (2^{n+1} - 1) \to 1$ instead. (This can also be visualized as hopping along a hypercube of binary strings; each inductive step takes two copies of the hypercube and links them together by a single edge.)

Remark (Ashwin Sah)**.** The problem can also be altered to ask for the minimum value of the sum of the reciprocals of the hop sizes, where further we stipulate that the frog must hit every point precisely once (to avoid triviality). With a nearly identical proof that also exploits the added condition $a_0 + \cdots + a_{n-1} = 2^n - 1$, the answer is n. This yields a nicer form for the generalization. The natural generalization changes the above problem by replacing 2^k with a_k where $a_k \mid a_{k+1}$, so that the interval covered by hops is of size a_n and the hop sizes are restricted to the a_i, where $a_0 = 1$. In this case, similar bounding yields

$$\sum_{i=1}^{2^n-1} \frac{1}{b_k} \ge \sum_{i=0}^{n-1} \left(\frac{a_{k+1}}{a_k} - 1 \right).$$

Bounds for the total distance traveled happen in the same way as the solution above, and equality for both can be constructed in an analogous fashion.

§3.2 TSTST 2018/8, proposed by Ankan Bhattacharya, Evan Chen

Available online at https://aops.com/community/p10570998.

Problem statement

For which positive integers $b > 2$ do there exist infinitely many positive integers n such that n^2 divides $b^n + 1$?

This problem is sort of the union of IMO 1990/3 and IMO 2000/5.

The answer is any b such that $b + 1$ is not a power of 2. In the forwards direction, we first prove more carefully the following claim.

Claim — If $b + 1$ is a power of 2, then the only n which is valid is $n = 1$.

Proof. Assume $n > 1$ and let p be the smallest prime dividing n. We cannot have $p = 2$, since then $4 \mid b^n + 1 \equiv 2 \pmod 4$. Thus,

$$b^{2n} \equiv 1 \pmod p$$

so the order of $b \pmod p$ divides $\gcd(2n, p - 1) = 2$. Hence $p \mid b^2 - 1 = (b - 1)(b + 1)$.

But since $b + 1$ was a power of 2, this forces $p \mid b - 1$. Then $0 \equiv b^n + 1 \equiv 2 \pmod p$, contradiction. □

On the other hand, suppose that $b + 1$ is not a power of 2 (and that $b > 2$). We will inductively construct an infinite sequence of distinct primes $p_0, p_1, \dots,$ such that the following two properties hold for each $k \geq 0$:

- $p_0^2 \dots p_{k-1}^2 p_k \mid b^{p_0 \cdots p_{k-1}} + 1$,
- and hence $p_0^2 \dots p_{k-1}^2 p_k^2 \mid b^{p_0 \cdots p_{k-1} p_k} + 1$ by exponent lifting lemma.

This will solve the problem.

Initially, let p_0 be any odd prime dividing $b + 1$. For the inductive step, we contend there exists an *odd* prime $q \notin \{p_0, \dots, p_k\}$ such that $q \mid b^{p_0 \cdots p_k} + 1$. Indeed, this follows immediately by Zsigmondy theorem since $p_0 \dots p_k$ divides $b^{p_0 \cdots p_{k-1}} + 1$. Since $(b^{p_0 \cdots p_k})^q \equiv b^{p_0 \cdots p_k} \pmod q$, it follows we can then take $p_{k+1} = q$. This finishes the induction.

To avoid the use of Zsigmondy, one can instead argue as follows: let $p = p_k$ for brevity, and let $c = b^{p_0 \cdots p_{k-1}}$. Then $\frac{c^p+1}{c+1} = c^{p-1} - c^{p-2} + \dots + 1$ has GCD exactly p with $c + 1$. Moreover, this quotient is always odd. Thus as long as $c^p + 1 > p \cdot (c + 1)$, there will be some new prime dividing $c^p + 1$ but not $c + 1$. This is true unless $p = 3$ and $c = 2$, but we assumed $b > 2$ so this case does not appear.

Remark (On new primes)**.** In going from $n^2 \mid b^n + 1$ to $(nq)^2 \mid b^{nq} + 1$, one does not necessarily need to pick a q such that $q \nmid n$, as long as $\nu_q(n^2) < \nu_q(b^n + 1)$. In other words it suffices to just check that $\frac{b^n+1}{n^2}$ is not a power of 2 in this process.

However, this calculation is a little more involved with this approach. One proceeds by noting that n is odd, hence $\nu_2(b^n + 1) = \nu_2(b + 1)$, and thus $\frac{b^n+1}{n^2} = 2^{\nu_2(b+1)} \leq b + 1$, which is a little harder to bound than the analogous $c^p + 1 > p \cdot (c + 1)$ from the previous solution.

¶ Authorship comments I came up with this problem by simply mixing together the main ideas of IMO 1990/3 and IMO 2000/5, late one night after a class. On the other hand, I do not consider it very original; it is an extremely "routine" number theory problem for experienced contestants, using highly standard methods. Thus it may not be that interesting, but is a good discriminator of understanding of fundamentals.

IMO 1990/3 shows that if $b = 2$, then the only n which work are $n = 1$ and $n = 3$. Thus $b = 2$ is a special case and for this reason the problem explicitly requires $b > 2$.

An alternate formulation of the problem is worth mentioning. Originally, the problem statement asked whether there existed n with at least 3 (or 2018, etc.) prime divisors, thus preventing the approach in which one takes a prime q dividing $\frac{b^n+1}{n^2}$. Ankan Bhattacharya suggested changing it to "infinitely many n", which is more natural.

These formulations are actually not so different though. Explicitly, suppose $k^2 \mid b^k + 1$ and $p \mid b^k + 1$. Consider any $k \mid n$ with $n^2 \mid b^n + 1$, and let p be an odd prime dividing $b^k + 1$. Then $2\nu_p(n) \le \nu_p(b^n + 1) = \nu_p(n/k) + \nu_p(b^k + 1)$ and thus

$$\nu_p(n/k) \le \nu_p\left(\frac{b^k + 1}{k^2}\right).$$

Effectively, this means we can only add each prime a certain number of times.

§3.3 TSTST 2018/9, proposed by Linus Hamilton

Available online at https://aops.com/community/p10571003.

Problem statement

Show that there is an absolute constant $c < 1$ with the following property: whenever $\mathcal{P}$ is a polygon with area 1 in the plane, one can translate it by a distance of $\frac{1}{100}$ in some direction to obtain a polygon $\mathcal{Q}$, for which the intersection of the interiors of $\mathcal{P}$ and $\mathcal{Q}$ has total area at most c.

The following solution is due to Brian Lawrence. We will prove the result with the generality of any measurable set $\mathcal{P}$ (rather than a polygon). For a vector v in the plane, write $\mathcal{P} + v$ for the translate of $\mathcal{P}$ by v.

Suppose $\mathcal{P}$ is a polygon of area 1, and $\varepsilon > 0$ is a constant, such that for any translate $\mathcal{Q} = \mathcal{P} + v$, where v has length exactly $\frac{1}{100}$, the intersection of $\mathcal{P}$ and $\mathcal{Q}$ has area at least $1 - \varepsilon$. The problem asks us to prove a lower bound on ε.

Lemma

Fix a sequence of n vectors $v_1, v_2, \dots, v_n$, each of length $\frac{1}{100}$. A grasshopper starts at a random point x of $\mathcal{P}$, and makes n jumps to $x + v_1 + \dots + v_n$. Then it remains in $\mathcal{P}$ with probability at least $1 - n\varepsilon$.

Proof. In order for the grasshopper to leave $\mathcal{P}$ at step i, the grasshopper's position before step i must be inside the difference set $\mathcal{P} \backslash (\mathcal{P} - v_i)$. Since this difference set has area at most ε, the probability the grasshopper leaves $\mathcal{P}$ at step i is at most ε. Summing over the n steps, the probability that the grasshopper ever manages to leave $\mathcal{P}$ is at most $n\varepsilon$. □

Corollary

Fix a vector w of length at most 8. A grasshopper starts at a random point x of $\mathcal{P}$, and jumps to $x + w$. Then it remains in $\mathcal{P}$ with probability at least $1 - 800\varepsilon$.

Proof. Apply the previous lemma with 800 jumps. Any vector w of length at most 8 can be written as $w = v_1 + v_2 + \dots + v_{800}$, where each v_i has length exactly $\frac{1}{100}$. □

Now consider the process where we select a random starting point $x \in \mathcal{P}$ for our grasshopper, *and* a random vector w of length at most 8 (sampled uniformly from the closed disk of radius 8). Let q denote the probability of staying inside $\mathcal{P}$ we will bound q from above and below.

- On the one hand, suppose we pick w first. By the previous corollary, $q \ge 1 - 800\varepsilon$ (irrespective of the chosen w).
- On the other hand, suppose we pick x first. Then the possible landing points $x + w$ are uniformly distributed over a closed disk of radius 8, which has area 64π. The probability of landing in $\mathcal{P}$ is certainly at most $\frac{[\mathcal{P}]}{64\pi}$.

Consequently, we deduce

$$1 - 800\varepsilon \le q \le \frac{[\mathcal{P}]}{64\pi} \implies \varepsilon > \frac{1 - \frac{[\mathcal{P}]}{64\pi}}{800} > 0.001$$

as desired.

> **Remark.** The choice of 800 jumps is only for concreteness; any constant n for which $\pi(n/100)^2 > 1$ works. I think $n = 98$ gives the best bound following this approach.

TSTST 2018 Statistics

Mathematical Olympiad Summer Program

Evan Chen《陳誼廷》

June 25, 2018

§1 Summary of scores for TSTST 2018

N	90	1st Q	16	Max	62
μ	28.24	Median	25	Top 3	57
σ	16.08	3rd Q	38	Top 12	54

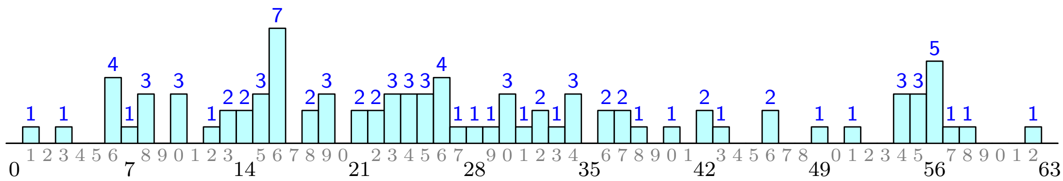
§2 Problem statistics for TSTST 2018

	P1	P2	P3	P4	P5	P6	P7	P8	P9
0	22	48	58	4	38	59	3	51	80
1	6	2	3	0	2	7	9	3	0
2	4	2	0	0	0	0	38	2	1
3	0	0	3	1	1	5	0	4	1
4	0	0	0	14	0	0	2	0	0
5	1	0	0	3	0	0	9	1	0
6	0	3	15	1	1	1	1	1	1
7	57	35	11	67	48	18	28	28	7
Avg	4.64	2.99	1.99	6.10	3.86	1.71	3.78	2.51	0.67
QM	5.62	4.51	3.51	6.35	5.16	3.28	4.50	4.06	2.09
#5+	**58**	**38**	**26**	**71**	**49**	**19**	**38**	**30**	**8**
%5+	%64.4	%42.2	%28.9	%78.9	%54.4	%21.1	%42.2	%33.3	%8.9

§3 Rankings for TSTST 2018

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
63	0	0	0.00%	42	2	21	23.33%	21	2	57	63.33%
62	1	1	1.11%	41	0	21	23.33%	20	0	57	63.33%
61	0	1	1.11%	40	1	22	24.44%	19	3	60	66.67%
60	0	1	1.11%	39	0	22	24.44%	18	2	62	68.89%
59	0	1	1.11%	38	1	23	25.56%	17	0	62	68.89%
58	1	2	2.22%	37	2	25	27.78%	16	7	69	76.67%
57	1	3	3.33%	36	2	27	30.00%	15	3	72	80.00%
56	5	8	8.89%	35	0	27	30.00%	14	2	74	82.22%
55	3	11	12.22%	34	3	30	33.33%	13	2	76	84.44%
54	3	14	15.56%	33	1	31	34.44%	12	1	77	85.56%
53	0	14	15.56%	32	2	33	36.67%	11	0	77	85.56%
52	0	14	15.56%	31	1	34	37.78%	10	3	80	88.89%
51	1	15	16.67%	30	3	37	41.11%	9	0	80	88.89%
50	0	15	16.67%	29	1	38	42.22%	8	3	83	92.22%
49	1	16	17.78%	28	1	39	43.33%	7	1	84	93.33%
48	0	16	17.78%	27	1	40	44.44%	6	4	88	97.78%
47	0	16	17.78%	26	4	44	48.89%	5	0	88	97.78%
46	2	18	20.00%	25	3	47	52.22%	4	0	88	97.78%
45	0	18	20.00%	24	3	50	55.56%	3	1	89	98.89%
44	0	18	20.00%	23	3	53	58.89%	2	0	89	98.89%
43	1	19	21.11%	22	2	55	61.11%	1	1	90	100.00%
								0	0	90	100.00%

§4 Histogram for TSTST 2018



USA TST Selection Test for 61st IMO and 9th EGMO

Pittsburgh, PA

Day I 1:15pm – 5:45pm

Tuesday, June 18, 2019

Problem 1. Find all binary operations $\diamondsuit\colon \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning $\diamondsuit$ takes pairs of positive real numbers to positive real numbers) such that for any real numbers $a, b, c > 0$,

- the equation $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ holds; and
- if $a \geq 1$ then $a \diamondsuit a \geq 1$.

Problem 2. Let ABC be an acute triangle with circumcircle Ω and orthocenter H. Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q, respectively. Denote by ω the circumcircle of $\triangle ADE$.

(a) Show that lines PE and QD meet on ω.

(b) Prove that if ω passes through H, then lines PD and QE meet on ω as well.

Problem 3. On an infinite square grid we place finitely many *cars*, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that the cell immediately in front of each car is empty, and moreover no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.). In a *move*, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times.

USA TST Selection Test for 61st IMO and 9th EGMO

Pittsburgh, PA

Day II   1:15pm – 5:45pm

Thursday, June 20, 2019

Problem 4. Consider coins with positive real denominations not exceeding 1. Find the smallest $C > 0$ such that the following holds: if we are given any 100 such coins with total value 50, then we can always split them into two stacks of 50 coins each such that the absolute difference between the total values of the two stacks is at most C.

Problem 5. Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.

Problem 6. Suppose P is a polynomial with integer coefficients such that for every positive integer n, the sum of the decimal digits of $|P(n)|$ is not a Fibonacci number. Must P be constant?

(A *Fibonacci number* is an element of the sequence F_0, F_1, $\ldots$ defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$.)

USA TST Selection Test for 61$^{\text{st}}$ IMO and 9$^{\text{th}}$ EGMO

Pittsburgh, PA

Day III 1:15pm – 5:45pm

Saturday, June 22, 2019

Problem 7. Let $f\colon \mathbb{Z} \to \{1, 2, \dots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x - y)$$

for all integers x and y. Show that there exist positive integers m and n such that $f(x) = \gcd(m + x, n)$ for all integers x.

Problem 8. Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(\mathcal{S})$ denote the number of ways to draw 8 line segments with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

Problem 9. Let ABC be a triangle with incenter I. Points K and L are chosen on segment BC such that the incircles of $\triangle ABK$ and $\triangle ABL$ are tangent at P, and the incircles of $\triangle ACK$ and $\triangle ACL$ are tangent at Q. Prove that $IP = IQ$.

USA TSTST 2019 Solutions

United States of America — TST Selection Test

Ankan Bhattacharya and Evan Chen

61^{st} IMO 2020 Russia and 9^{th} EGMO 2020 Netherlands

Contents

§0 Problems

1. Find all binary operations $\diamondsuit\colon \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning $\diamondsuit$ takes pairs of positive real numbers to positive real numbers) such that for any real numbers $a, b, c > 0$,

- the equation $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ holds; and
- if $a \ge 1$ then $a \diamondsuit a \ge 1$.

2. Let ABC be an acute triangle with circumcircle Ω and orthocenter H. Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q, respectively. Denote by ω the circumcircle of $\triangle ADE$.

(a) Show that lines PE and QD meet on ω.

(b) Prove that if ω passes through H, then lines PD and QE meet on ω as well.

3. On an infinite square grid we place finitely many *cars*, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that the cell immediately in front of each car is empty, and moreover no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.). In a *move*, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times.

4. Consider coins with positive real denominations not exceeding 1. Find the smallest $C > 0$ such that the following holds: if we are given any 100 such coins with total value 50, then we can always split them into two stacks of 50 coins each such that the absolute difference between the total values of the two stacks is at most C.

5. Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.

6. Suppose P is a polynomial with integer coefficients such that for every positive integer n, the sum of the decimal digits of $|P(n)|$ is not a Fibonacci number. Must P be constant?

7. Let $f\colon \mathbb{Z} \to \{1, 2, \dots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x - y)$$

for all integers x and y. Show that there exist positive integers m and n such that $f(x) = \gcd(m + x, n)$ for all integers x.

8. Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(\mathcal{S})$ denote the number of ways to draw 8 line segments with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

9. Let ABC be a triangle with incenter I. Points K and L are chosen on segment BC such that the incircles of $\triangle ABK$ and $\triangle ABL$ are tangent at P, and the incircles of $\triangle ACK$ and $\triangle ACL$ are tangent at Q. Prove that $IP = IQ$.

§1 Solutions to Day 1

§1.1 TSTST 2019/1, proposed by Evan Chen

Available online at https://aops.com/community/p12608849.

Problem statement

Find all binary operations $\diamondsuit \colon \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning $\diamondsuit$ takes pairs of positive real numbers to positive real numbers) such that for any real numbers $a, b, c > 0$,

- the equation $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ holds; and
- if $a \ge 1$ then $a \diamondsuit a \ge 1$.

The answer is only multiplication and division, which both obviously work.

We present two approaches, one appealing to theorems on Cauchy's functional equation, and one which avoids it.

¶ First solution using Cauchy FE We prove:

Claim — We have $a \diamondsuit b = af(b)$ where f is some involutive and totally multiplicative function. (In fact, this classifies all functions satisfying the first condition completely.)

Proof. Let $P(a, b, c)$ denote the assertion $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$.

- Note that for any x, the function $y \mapsto x \diamondsuit y$ is injective, because if $x \diamondsuit y_1 = x \diamondsuit y_2$ then take $P(1, x, y_i)$ to get $y_1 = y_2$.
- Take $P(1, x, 1)$ and injectivity to get $x \diamondsuit 1 = x$.
- Take $P(1, 1, y)$ to get $1 \diamondsuit (1 \diamondsuit y) = y$.
- Take $P(x, 1, 1 \diamondsuit y)$ to get
$$x \diamondsuit y = x \cdot (1 \diamondsuit y).$$

Henceforth let us define $f(y) = 1 \diamondsuit y$, so $f(1) = 1$, f is involutive and

$$x \diamondsuit y = xf(y).$$

Plugging this into the original condition now gives $f(bf(c)) = f(b)c$, which (since f is an involution) gives f completely multiplicative. □

In particular, $f(1) = 1$. We are now interested only in the second condition, which reads $f(x) \ge 1/x$ for $x \ge 1$.

Define the function

$$g(t) = \log f(e^t)$$

so that g is additive, and also $g(t) \ge -t$ for all $t \ge 0$. We appeal to the following theorem:

Lemma

If $h\colon \mathbb{R} \to \mathbb{R}$ is an additive function which is not linear, then it is *dense* in the plane: for any point (x_0, y_0) and $\varepsilon > 0$ there exists (x, y) such that $h(x) = y$ and $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \varepsilon$.

Applying this lemma with the fact that $g(t) \geq -t$ implies readily that g is linear. In other words, f is of the form $f(x) = x^r$ for some fixed real number r. It is easy to check $r = \pm 1$ which finishes.

¶ Second solution manually As before we arrive at $a \diamondsuit b = af(b)$, with f an involutive and totally multiplicative function.

We prove that:

Claim — For any $a > 0$, we have $f(a) \in \{1/a, a\}$.

Proof. WLOG $b > 1$, and suppose $f(b) = a \geq 1/b$ hence $f(a) = b$.

Assume that $ab > 1$; we show $a = b$. Note that for integers m and n with $a^n b^m \geq 1$, we must have

$$a^m b^n = f(b)^m f(a)^n = f(a^n b^m) \geq \frac{1}{a^n b^m} \implies (ab)^{m+n} \geq 1$$

and thus we have arrived at the proposition

$$m + n < 0 \implies n \log_b a + m < 0$$

for all integers m and n. Due to the density of $\mathbb{Q}$ in the real numbers, this can only happen if $\log_b a = 1$ or $a = b$. $\square$

Claim — The function f is continuous.

Proof. Indeed, it's equivalent to show $g(t) = \log f(e^t)$ is continuous, and we have that

$$|g(t) - g(s)| = \left|\log f(e^{t-s})\right| = |t - s|$$

since $f(e^{t-s}) = e^{\pm|t-s|}$. Therefore g is Lipschitz. Hence g continuous, and f is too. $\square$

Finally, we have from f multiplicative that

$$f(2^q) = f(2)^q$$

for every rational number q, say. As f is continuous this implies $f(x) \equiv x$ or $f(x) \equiv 1/x$ identically (depending on whether $f(2) = 2$ or $f(2) = 1/2$, respectively).

Therefore, $a \diamondsuit b = ab$ or $a \diamondsuit b = a \div b$, as needed.

> **Remark.** The Lipschitz condition is one of several other ways to proceed. The point is that if $f(2) = 2$ (say), and $x/2^q$ is close to 1, then $f(x)/2^q = f(x/2^q)$ is close to 1, which is enough to force $f(x) = x$ rather than $f(x) = 1/x$.

Remark. Compare to AMC 10A 2016 #23, where the second condition is $a \diamondsuit a = 1$.

§1.2 TSTST 2019/2, proposed by Merlijn Staps

Available online at https://aops.com/community/p12608478.

Problem statement

Let ABC be an acute triangle with circumcircle Ω and orthocenter H. Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q, respectively. Denote by ω the circumcircle of $\triangle ADE$.

(a) Show that lines PE and QD meet on ω.

(b) Prove that if ω passes through H, then lines PD and QE meet on ω as well.

We will give one solution to (a), then several solutions to (b).

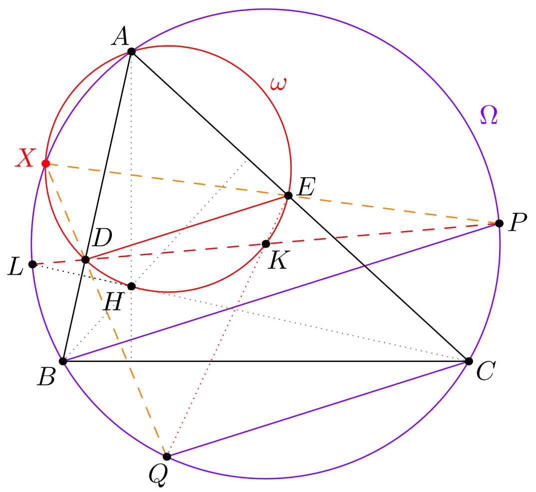
¶ Solution to (a) Note that $\measuredangle AQP = \measuredangle ABP = \measuredangle ADE$ and $\measuredangle APQ = \measuredangle ACQ = \measuredangle AED$, so we have a spiral similarity $\triangle ADE \sim \triangle AQP$. Therefore, lines PE and QD meet at the second intersection of ω and Ω other than A. Call this point X.

¶ Solution to (b) using angle chasing Let L be the reflection of H across $\overline{AB}$, which lies on Ω.

Claim — Points L, D, P are collinear.

Proof. This is just angle chasing:

$$\begin{aligned}\measuredangle CLD &= \measuredangle DHL = \measuredangle DHA + \measuredangle AHL = \measuredangle DEA + \measuredangle AHC \\ &= \measuredangle ADE + \measuredangle CBA = \measuredangle ABP + \measuredangle CBA = \measuredangle CBP = \measuredangle CLP. \qquad \square\end{aligned}$$



Now let $K \in \omega$ such that $DHKE$ is an isosceles trapezoid, i.e. $\measuredangle BAH = \measuredangle KAE$.

Claim — Points D, K, P are collinear.

Proof. Using the previous claim,

$$\measuredangle KDE = \measuredangle KAE = \measuredangle BAH = \measuredangle LAB = \measuredangle LPB = \measuredangle DPB = \measuredangle PDE. \qquad \square$$

By symmetry, $\overline{QE}$ will then pass through the same K, as needed.

> **Remark.** These two claims imply each other, so guessing one of them allows one to realize the other. It is likely the latter is easiest to guess from the diagram, since it does not need any additional points.

¶ Solution to (b) by orthogonal circles (found by contestants) We define K as in the previous solution, but do not claim that K is the desired intersection. Instead, we note that:

Claim — Point K is the orthocenter of isosceles triangle APQ.

Proof. Notice that $AH = AK$ and $BC = PQ$. Moreover from $\overline{AH} \perp \overline{BC}$ we deduce $\overline{AK} \perp \overline{PQ}$ by reflection across the angle bisector.

In light of the formula "$AH^2 = 4R^2 - a^2$", this implies the conclusion. $\square$

Let M be the midpoint of $\overline{PQ}$. Since $\triangle APQ$ is isosceles,

$$\overline{AKM} \perp \overline{PQ} \implies MK \cdot MA = MP^2$$

by orthocenter properties.

So to summarize

- The circle with diameter $\overline{PQ}$ is orthogonal to ω.
- The point $X = \overline{QD} \cap \overline{PE}$ is on ω.

Combined with (a), this implies the result by Brokard theorem.

¶ Solution to (b) by complex numbers (Yang Liu and Michael Ma) Let M be the arc midpoint of $\widehat{BC}$. We use the standard arc midpoint configuration. We have that

$$A = a^2,\ B = b^2,\ C = c^2,\ M = -bc,\ H = a^2 + b^2 + c^2,\ P = \frac{a^2c}{b},\ Q = \frac{a^2b}{c},$$

where M is the arc midpoint of $\widehat{BC}$. By direct angle chasing we can verify that $\overline{MB} \parallel \overline{DH}$. Also, $D \in \overline{AB}$. Therefore, we can compute D as follows.

$$d + a^2b^2\bar{d} = a^2 + b^2 \text{ and } \frac{d-h}{\bar{d}-\bar{h}} = -mb^2 = b^3c \implies d = \frac{a^2(a^2c + b^2c + c^3 - b^3)}{c(bc + a^2)}.$$

By symmetry, we have that

$$e = \frac{a^2(a^2b + bc^2 + b^3 - c^3)}{b(bc + a^2)}.$$

To finish, we want to show that the angle between $\overline{DP}$ and $\overline{EQ}$ is angle A. To show this, we compute $\frac{d-p}{e-q} \Big/ \overline{\frac{d-p}{e-q}}$. First, we compute

$$d - p = \frac{a^2(a^2c + b^2c + c^3 - b^3)}{c(bc + a^2)} - \frac{a^2c}{b}$$

$$= a^2 \left(\frac{a^2c + b^2c + c^3 - b^3}{c(bc + a^2)} - \frac{c}{b} \right) = \frac{a^2(a^2c - b^3)(b - c)}{bc(bc + a^2)}.$$

By symmetry,

$$\frac{d-p}{e-q} = -\frac{a^2c - b^3}{a^2b - c^3} \implies \frac{d-p}{e-q} \Big/ \overline{\frac{d-p}{e-q}} = \frac{a^2b^3c}{a^2bc^3} = \frac{b^2}{c^2}$$

as desired.

¶ Solution to (b) using untethered moving points (Zack Chroman) We work in the real projective plane $\mathbb{RP}^2$, and animate C linearly on a fixed line through A.

Recall:

Lemma (Zack's lemma)

Suppose points A, B have degree d_1, d_2, and there are k values of t for which $A = B$. Then line AB has degree at most $d_1 + d_2 - k$. Similarly, if lines ℓ_1, ℓ_2 have degrees d_1, d_2, and there are k values of t for which $\ell_1 = \ell_2$, then the intersection $\ell_1 \cap \ell_2$ has degree at most $d_1 + d_2 - k$.

Now, note that H moves linearly in C on line BH. Furthermore, angles $\angle AHE$, $\angle AHF$ are fixed, we get that D and E have degree 2. One way to see this is using the lemma; D lies on line AB, which is fixed, and line HD passes through a point at infinity which is a constant rotation of the point at infinity on line AH, and therefore has degree 1. Then D, E have degree at most $1 + 1 - 0 = 2$.

Now, note that P, Q move linearly in C. Both of these are because the circumcenter O moves linearly in C, and P, Q are reflections of B, C in a line through O with fixed direction, which also moves linearly.

So by the lemma, the lines PD, QE have degree at most 3. I claim they actually have degree 2; to show this it suffices to give an example of a choice of C for which $P = D$ and one for which $Q = E$. But an easy angle chase shows that in the unique case when $P = B$, we get $D = B$ as well and thus $P = D$. Similarly when $Q = C$, $E = C$. It follows from the lemma that lines PD, QE have degree at most 2.

Let ℓ_∞ denote the line at infinity. I claim that the points $P_1 = PD \cap \ell_\infty$, $P_2 = QE \cap \ell_\infty$ are projective in C. Since ℓ_∞ is fixed, it suffices to show by the lemma that there exists some value of C for which $QE = \ell_\infty$ and $PD = \ell_\infty$. But note that as $C \to \infty$, all four points P, D, Q, E go to infinity. It follows that P_1, P_2 are projective in C.

Then to finish, recall that we want to show that $\angle(PD, QE)$ is constant. It suffices then to show that there's a constant rotation sending P_1 to P_2. Since P_1, P_2 are projective, it suffices to verify this for 3 values of C.

We can take C such that $\angle ABC = 90$, $\angle ACB = 90$, or $AB = AC$, and all three cases are easy to check.

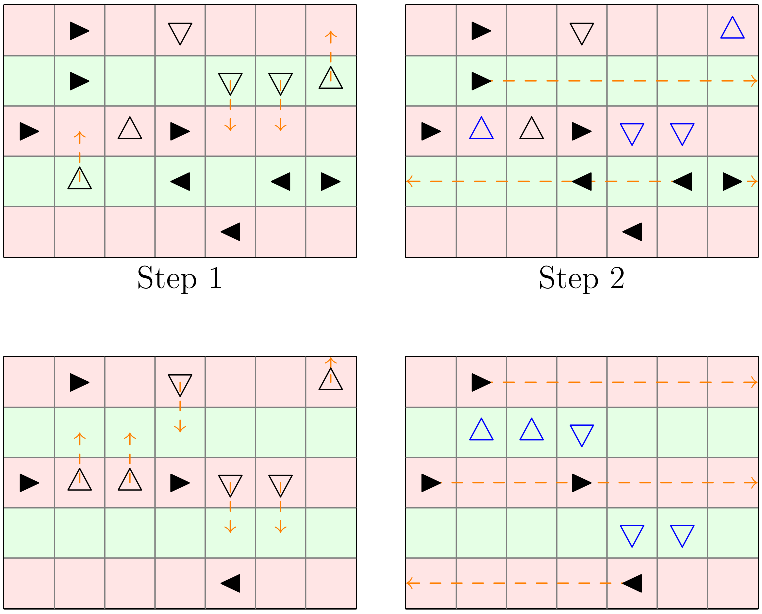
§1.3 TSTST 2019/3, proposed by Nikolai Beluhov

Available online at https://aops.com/community/p12608769.

Problem statement

On an infinite square grid we place finitely many *cars*, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that the cell immediately in front of each car is empty, and moreover no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.). In a *move*, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times.

Let S be any rectangle containing all the cars. Partition S into horizontal strips of height 1, and color them red and green in an alternating fashion. It is enough to prove all the cars may exit S.



Step 1 Step 2

Step 3 Step 4

To do so, we outline a five-stage plan for the cars.

1. All vertical cars in a green cell may advance one cell into a red cell (or exit S altogether), by the given condition. (This is the only place where the hypothesis about empty space is used!)
2. All horizontal cars on green cells may exit S, as no vertical cars occupy green cells.
3. All vertical cars in a red cell may advance one cell into a green cell (or exit S altogether), as all green cells are empty.
4. All horizontal cars within red cells may exit S, as no vertical car occupy red cells.
5. The remaining cars exit S, as they are all vertical. The solution is complete.

Remark (Author's comments)**.** The solution I've given for this problem is so short and simple that it might appear at first to be about IMO 1 difficulty. I don't believe that's true! There are very many approaches that look perfectly plausible at first, and then fall apart in this or that twisted special case.

Remark (Higher-dimensional generalization by author)**.** The natural higher-dimensional generalization is true, and can be proved in largely the same fashion. For example, in three dimensions, one may let S be a rectangular prism and partition S into horizontal slabs and color them red and green in an alternating fashion. Stages 1, 3, and 5 generalize immediately, and stages 2 and 4 reduce to an application of the two-dimensional problem. In the same way, the general problem is handled by induction on the dimension.

Remark (Historical comments)**.** For $k > 1$, we could consider a variant of the problem where cars are $1 \times k$ rectangles (moving parallel to the longer edge) instead of occupying single cells. In that case, if there are $2k - 1$ empty spaces in front of each car, the above proof works (with the red and green strips having height k instead). On the other hand, at least k empty spaces are necessary. We don't know the best constant in this case.

§2 Solutions to Day 2

§2.1 TSTST 2019/4, proposed by Merlijn Staps

Available online at https://aops.com/community/p12608513.

Problem statement

Consider coins with positive real denominations not exceeding 1. Find the smallest $C > 0$ such that the following holds: if we are given any 100 such coins with total value 50, then we can always split them into two stacks of 50 coins each such that the absolute difference between the total values of the two stacks is at most C.

The answer is $C = \frac{50}{51}$. The lower bound is obtained if we have 51 coins of value $\frac{1}{51}$ and 49 coins of value 1. (Alternatively, 51 coins of value $1 - \frac{\varepsilon}{51}$ and 49 coins of value $\frac{\varepsilon}{49}$ works fine for $\varepsilon > 0$.) We now present two (similar) proofs that this $C = \frac{50}{51}$ suffices.

¶ First proof (original) Let $a_1 \le \dots \le a_{100}$ denote the values of the coins in ascending order. Since the 51 coins $a_{50}, \dots, a_{100}$ are worth at least $51a_{50}$, it follows that $a_{50} \le \frac{50}{51}$; likewise $a_{51} \ge \frac{1}{51}$.

We claim that choosing the stacks with coin values

$$a_1, a_3, \dots, a_{49}, \quad a_{52}, a_{54}, \dots, a_{100}$$

and

$$a_2, a_4, \dots, a_{50}, \quad a_{51}, a_{53}, \dots, a_{99}$$

works. Let D denote the (possibly negative) difference between the two total values. Then

$$\begin{aligned} D &= (a_1 - a_2) + \dots + (a_{49} - a_{50}) - a_{51} + (a_{52} - a_{53}) + \dots + (a_{98} - a_{99}) + a_{100} \\ &\le 25 \cdot 0 - \frac{1}{51} + 24 \cdot 0 + 1 = \frac{50}{51}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} D &= a_1 + (a_3 - a_2) + \dots + (a_{49} - a_{48}) - a_{50} + (a_{52} - a_{51}) + \dots + (a_{100} - a_{99}) \\ &\ge 0 + 24 \cdot 0 - \frac{50}{51} + 25 \cdot 0 = -\frac{50}{51}. \end{aligned}$$

It follows that $|D| \le \frac{50}{51}$, as required.

¶ Second proof (Evan Chen) Again we sort the coins in increasing order $0 < a_1 \le a_2 \le \dots \le a_{100} \le 1$. A *large gap* is an index $i \ge 2$ such that $a_i > a_{i-1} + \frac{50}{51}$; obviously there is at most one such large gap.

Claim — If there is a large gap, it must be $a_{51} > a_{50} + \frac{50}{51}$.

Proof. If $i < 50$ then we get $a_{50}, \dots, a_{100} > \frac{50}{51}$ and the sum $\sum_1^{100} a_i > 50$ is too large. Conversely if $i > 50$ then we get $a_1, \dots, a_{i-1} < \frac{1}{51}$ and the sum $\sum_1^{100} a_i < 1/51 \cdot 51 + 49$ is too small. □

Now imagine starting with the coins a_1, a_3, $\dots$, a_{99}, which have total value $S \le 25$. We replace a_1 by a_2, then a_3 by a_4, and so on, until we replace a_{99} by a_{100}. At the end of the process we have $S \ge 25$. Moreover, since we did not cross a large gap at any point, the quantity S changed by at most $C = \frac{50}{51}$ at each step. So at some point in the process we need to have $25 - C/2 \le S \le 25 + C/2$, which proves C works.

§2.2 TSTST 2019/5, proposed by Gunmay Handa

Available online at https://aops.com/community/p12608496.

Problem statement

Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.

We present several solutions. (There are more in the official packet; some are omitted here, which explains the numbering.)

¶ First solution (Andrew Gu) We begin with the following two observations.

Claim — Point K lies on the radical axis of (BEH) and (CFH).

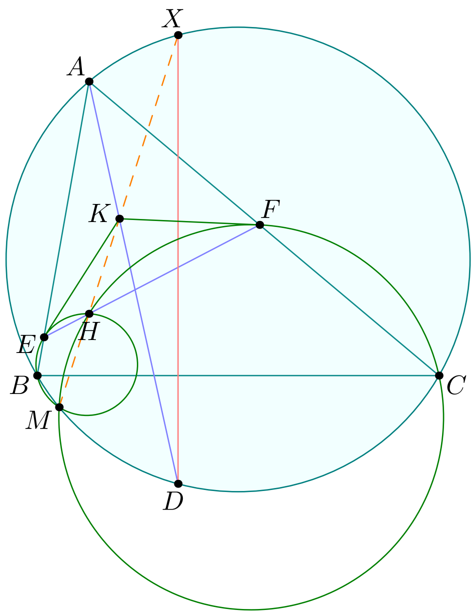
Proof. Actually we claim $\overline{KE}$ and $\overline{KF}$ are tangents. Indeed,

$$\measuredangle HEK = 90^\circ - \measuredangle EAF = 90^\circ - \measuredangle BAC = \measuredangle HBE$$

implying the result. Since $KE = KF$, this implies the result. □

Claim — The second intersection M of (BEH) and (CFH) lies on Γ.

Proof. By Miquel's theorem on $\triangle AEF$ with $H \in \overline{EF}$, $B \in \overline{AE}$, $C \in \overline{AF}$. □



In particular, M, H, K are collinear. Let X be on Γ with $\overline{DX} \perp \overline{BC}$; we then wish to show X lies on the line MHK we found. This is angle chasing: compute

$$\measuredangle XMB = \measuredangle XDB = 90^\circ - \measuredangle DBC = 90^\circ - \measuredangle DAC$$

$$= 90^\circ - \measuredangle KAF = \measuredangle FEA = \measuredangle HEB = \measuredangle HMB$$

as needed.

¶ Second solution (Ankan Bhattacharya) We let D' be the second intersection of $\overline{EF}$ with (BHC) and redefine D as the reflection of D' across $\overline{BC}$. We will first prove that this point D coincides with the point D given in the problem statement. The idea is that:

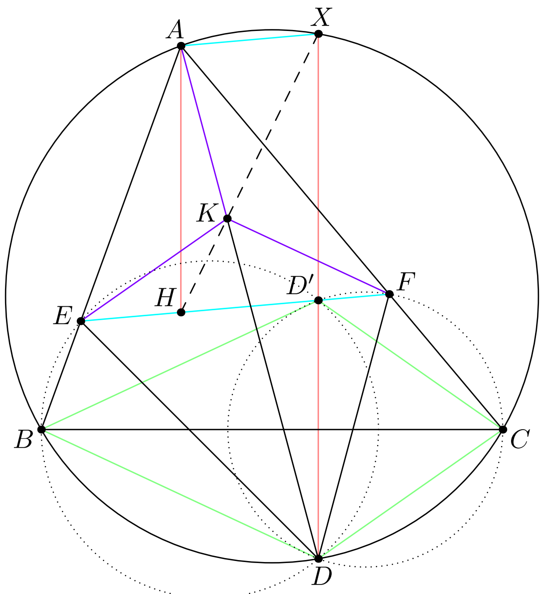
Claim — A is the D-excenter of $\triangle DEF$.

Proof. We contend $BED'D$ is cyclic. This follows by angle chasing:

$$\begin{aligned}\measuredangle D'DB = \measuredangle BD'D = \measuredangle D'BC + 90^\circ = \measuredangle D'HC + 90^\circ \\ = \measuredangle D'HC + \measuredangle(HC, AB) = \measuredangle(D'H, AB) = \measuredangle D'EB.\end{aligned}$$

Now as $BD = BD'$, we obtain $\overline{BEA}$ externally bisects $\angle DED' \cong \angle DEF$. Likewise $\overline{FA}$ externally bisects $\angle DFE$, so A is the D-excenter of $\triangle DEF$. $\square$

Hence, by the so-called "Fact 5", point K lies on $\overline{DA}$, so this point D is the one given in the problem statement.



Now choose point X on (ABC) satisfying $\overline{DX} \perp \overline{BC}$.

Claim — Point K lies on line HX.

Proof. Clearly $AHD'X$ is a parallelogram. By Ptolemy on $DEKF$,

$$\frac{KD}{KA} = \frac{KD}{KE} = \frac{DE + DF}{EF}.$$

On the other hand, if we let r_D denote the D-exradius of $\triangle DEF$ then

$$\frac{XD}{XD'} = \frac{[DEX] + [DFX]}{[XEF]} = \frac{[DEX] + [DFX]}{[AEF]} = \frac{DE \cdot r_D + DF \cdot r_D}{EF \cdot r_D} = \frac{DE + DF}{EF}.$$

Thus

$$[AKX] = \frac{KA}{KD} \cdot [DKX] = \frac{KA}{KD} \cdot \frac{XD}{XD'} \cdot [KD'X] = [D'KX].$$

This is sufficient to prove K lies on $\overline{HX}$. □

The solution is complete: X is the desired concurrency point.

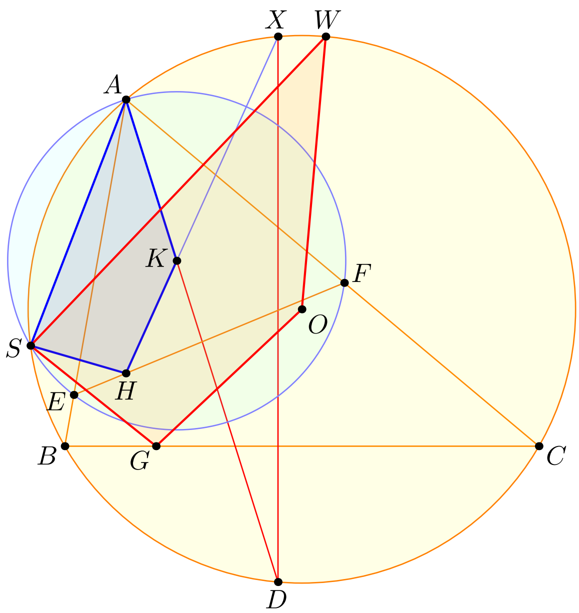
¶ Fourth solution, complex numbers with spiral similarity (Evan Chen) First if $\overline{AD} \perp \overline{BC}$ there is nothing to prove, so we assume this is not the case. Let W be the antipode of D. Let S denote the second intersection of (AEF) and (ABC). Consider the spiral similarity sending $\triangle SEF$ to $\triangle SBC$:

- It maps H to a point G on line BC,
- It maps K to O.
- It maps the A-antipode of $\triangle AEF$ to D.
- Hence (by previous two observations) it maps A to W.
- Also, the image of line AD is line WO, which does not coincide with line BC (as O does not lie on line BC).

Therefore, K is the *unique* point on line $\overline{AD}$ for one can get a direct similarity

$$\triangle AKH \sim \triangle WOG \qquad (\heartsuit)$$

for some point G lying on line $\overline{BC}$.



On the other hand, let us re-define K as $\overline{XH} \cap \overline{AD}$. We will show that the corresponding G making $(\heartsuit)$ true lies on line BC.

We apply complex numbers with Γ the unit circle, with a, b, c, d taking their usual meanings, $H = a + b + c$, $X = -bc/d$, and $W = -d$. Then point K is supposed to satisfy

$$k + ad\overline{k} = a + d$$

$$\begin{aligned}
\frac{k+\frac{bc}{d}}{a+b+c+\frac{bc}{d}} &= \frac{\overline{k}+\frac{d}{bc}}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}} \\
\iff \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}}{a+b+c+\frac{bc}{d}}\left(k+\frac{bc}{d}\right) &= \overline{k}+\frac{d}{bc}
\end{aligned}$$

Adding ad times the last line to the first line and cancelling $ad\overline{k}$ now gives

$$\left(ad\cdot\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}}{a+b+c+\frac{bc}{d}}+1\right)k = a+d+\frac{ad^2}{bc}-abc\cdot\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}}{a+b+c+\frac{bc}{d}}$$

or

$$\begin{aligned}
\left(ad\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}\right)+a+b+c+\frac{bc}{d}\right)k &= \left(a+b+c+\frac{bc}{d}\right)\left(a+d+\frac{ad^2}{bc}\right) \\
&\qquad -abc\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}\right).
\end{aligned}$$

We begin by simplifying the coefficient of k:

$$\begin{aligned}
ad\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}\right)+a+b+c+\frac{bc}{d} &= a+b+c+d+\frac{bc}{d}+\frac{ad}{b}+\frac{ad}{c}+\frac{ad^2}{bc} \\
&= a+\frac{bc}{d}+\left(1+\frac{ad}{bc}\right)(b+c+d) \\
&= \frac{ad+bc}{bcd}\left[bc+d(b+c+d)\right] \\
&= \frac{(ad+bc)(d+b)(d+c)}{bcd}.
\end{aligned}$$

Meanwhile, the right-hand side expands to

$$\begin{aligned}
\text{RHS} &= \left(a+b+c+\frac{bc}{d}\right)\left(a+d+\frac{ad^2}{bc}\right)-abc\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{d}{bc}\right) \\
&= \left(a^2+ab+ac+\frac{abc}{d}\right)+(da+db+dc+bc) \\
&\quad+\left(\frac{a^2d^2}{bc}+\frac{ad^2}{c}+\frac{ad^2}{b}+ad\right)-(ab+bc+ca+ad) \\
&= a^2+d(a+b+c)+\frac{abc}{d}+\frac{a^2d^2}{bc}+\frac{ad^2}{b}+\frac{ad^2}{c} \\
&= a^2+\frac{abc}{d}+d(a+b+c)\cdot\frac{ad+bc}{bc} \\
&= \frac{ad+bc}{bcd}\left[abc+d^2(a+b+c)\right].
\end{aligned}$$

Therefore, we get

$$k=\frac{abc+d^2(a+b+c)}{(d+b)(d+c)}.$$

In particular,

$$\begin{aligned}
k-a &= \frac{abc+d^2(a+b+c)-a(d+b)(d+c)}{(d+b)(d+c)} \\
&= \frac{d^2(b+c)-da(b+c)}{(d+b)(d+c)} = \frac{d(b+c)(d-a)}{(d+b)(d+c)}.
\end{aligned}$$

Now the corresponding point G obeying ($\heartsuit$) satisfies

$$\begin{aligned} \frac{g-(-d)}{0-(-d)} &= \frac{(a+b+c)-a}{k-a} \\ \implies g &= -d + \frac{d(b+c)}{k-a} \\ &= -d + \frac{(d+b)(d+c)}{d-a} = \frac{db+dc+bc+ad}{d-a}. \\ \implies bc\overline{g} &= \frac{bc \cdot \frac{ac+ab+ad+bc}{abcd}}{\frac{a-d}{ad}} = -\frac{ab+ac+ad+bc}{d-a}. \\ \implies g + bc\overline{g} &= \frac{(d-a)(b+c)}{d-a} = b+c. \end{aligned}$$

Hence G lies on BC and this completes the proof.

¶ Seventh solution using moving points (Zack Chroman) We state the converse of the problem as follows:

> Take a point D on Γ, and let $G \in \Gamma$ such that $\overline{DG} \perp \overline{BC}$. Then define K to lie on $\overline{GH}, \overline{AD}$, and take $L \in \overline{AD}$ such that K is the midpoint of $\overline{AL}$. Then if we define E and F as the projections of L onto $\overline{AB}$ and $\overline{AC}$ we want to show that E, H, F are collinear.

It's clear that solving this problem will solve the original. In fact we will show later that each line EF through H corresponds bijectively to the point D.

We animate D projectively on Γ (hence $\deg D = 2$). Since $D \mapsto G$ is a projective map $\Gamma \to \Gamma$, it follows $\deg G = 2$. By Zack's lemma, $\deg(\overline{AD}) \le 0 + 2 - 1 = 1$ (since D can coincide with A), and $\deg(\overline{HG}) \le 0 + 2 - 0 = 2$. So again by Zack's lemma, $\deg K \le 1 + 2 - 1 = 2$, since lines AD and GH can coincide once if D is the reflection of H over $\overline{BC}$. It follows $\deg L = 2$, since it is obtained by dilating K by a factor of 2 across the fixed point A.

Let ∞_C be the point at infinity on the line perpendicular to AC, and similarly ∞_B. Then

$$F = \overline{AC} \cap \overline{\infty_C L}, \quad E = \overline{AB} \cap \overline{\infty_B L}.$$

We want to use Zack's lemma again on line $\overline{\infty_B L}$. Consider the case $G = B$; we get $\overline{HG} \parallel \overline{AD}$, so $ADGH$ is a parallelogram, and then $K = L = \infty_B$. Thus there is at least one t where $L = \infty_B$ and by Zack's lemma we get $\deg\left(\overline{\infty_B L}\right) \le 0 + 2 - 1 = 1$. Again by Zack's lemma, we conclude $\deg E \le 0 + 1 - 0 = 1$. Similarly, $\deg F \le 1$.

We were aiming to show E, F, H collinear which is a condition of degree at most $1 + 1 + 0 = 2$. So it suffices to verify the problem for three distinct choices of D.

- If $D = A$, then line GH is line AH, and $L = \overline{AD} \cap \overline{AH} = A$. So $E = F = A$ and the statement is true.
- If $D = B$, G is the antipode of C on Γ. Then $K = \overline{HG} \cap \overline{AD}$ is the midpoint of $\overline{AB}$, so $L = B$. Then $E = B$ and F is the projection of B onto AC, so E, H, F collinear.
- We finish similarly when $D = C$.

This completes the proof.

Remark. Less careful approaches are possible which give a worse bound on the degrees, requiring to check (say) five choices of D instead. We present the most careful one showing $\deg D = 2$ for instructional reasons, but the others may be easier to find.

§2.3 TSTST 2019/6, proposed by Nikolai Beluhov

Available online at https://aops.com/community/p12608536.

Problem statement

Suppose P is a polynomial with integer coefficients such that for every positive integer n, the sum of the decimal digits of $|P(n)|$ is not a Fibonacci number. Must P be constant?

The answer is yes, P must be constant. By $S(n)$ we mean the sum of the decimal digits of $|n|$.

We need two claims.

Claim — If $P(x) \in \mathbb{Z}[x]$ is nonconstant with positive leading coefficient, then there exists an integer polynomial $F(x)$ such that all coefficients of $P \circ F$ are positive except for the second one, which is negative.

Proof. We will actually construct a cubic F. We call a polynomial *good* if it has the property.

First, consider $T_0(x) = x^3 + x + 1$. Observe that in $T_0^{\deg P}$, every coefficient is strictly positive, except for the second one, which is zero.

Then, let $T_1(x) = x^3 - \frac{1}{D}x^2 + x + 1$. Using continuity as $D \to \infty$, it follows that if D is large enough (in terms of $\deg P$), then $T_1^{\deg P}$ is good, with $-\frac{3}{D}x^{3\deg P - 1}$ being the only negative coefficient.

Finally, we can let $F(x) = CT_1(x)$ where C is a sufficiently large multiple of D (in terms of the coefficients of P); thus the coefficients of $(CT_1(x))^{\deg P}$ dominate (and are integers), as needed. $\square$

Claim — There are infinitely many Fibonacci numbers in each residue class modulo 9.

Proof. Note the Fibonacci sequence is periodic modulo 9 (indeed it is periodic modulo any integer). Moreover (allowing negative indices),

$$\begin{aligned} F_0 &= 0 \equiv 0 \pmod 9 \\ F_1 &= 1 \equiv 1 \pmod 9 \\ F_3 &= 2 \equiv 2 \pmod 9 \\ F_4 &= 3 \equiv 3 \pmod 9 \\ F_7 &= 13 \equiv 4 \pmod 9 \\ F_5 &= 5 \equiv 5 \pmod 9 \\ F_{-4} &= -3 \equiv 6 \pmod 9 \\ F_9 &= 34 \equiv 7 \pmod 9 \\ F_6 &= 8 \equiv 8 \pmod 9. \end{aligned}$$

$\square$

We now show how to solve the problem with the two claims. WLOG P satisfies the conditions of the first claim, and choose F as above. Let

$$P(F(x)) = c_N x^N - c_{N-1}x^{N-1} + c_{N-2}x^{N-2} + \cdots + c_0$$

where $c_i > 0$ (and $N = 3 \deg P$). Then if we select $x = 10^e$ for e large enough (say $x > 10 \max_i c_i$), the decimal representation $P(F(10^e))$ consists of the concatenation of

- the decimal representation of $c_N - 1$,
- the decimal representation of $10^e - c_{N-1}$
- the decimal representation of c_{N-2}, with several leading zeros,
- the decimal representation of c_{N-3}, with several leading zeros,
- ...
- the decimal representation of c_0, with several leading zeros.

(For example, if $P(F(x)) = 15x^3 - 7x^2 + 4x + 19$, then $P(F(1000)) = 14{,}993{,}004{,}019$.) Thus, the sum of the digits of this expression is equal to

$$S(P(F(10^e))) = 9e + k$$

for some constant k depending only on P and F, independent of e. But this will eventually hit a Fibonacci number by the second claim, contradiction.

> **Remark.** It is important to control the number of negative coefficients in the created polynomial. If one tries to use this approach on a polynomial P with $m > 0$ negative coefficients, then one would require that the Fibonacci sequence is surjective modulo $9m$ for any $m > 1$, which is not true: for example the Fibonacci sequence avoids all numbers congruent to $4 \mod 11$ (and thus $4 \mod 99$).
>
> In bases b for which surjectivity modulo $b - 1$ fails, the problem is false. For example, $P(x) = 11x + 4$ will avoid all Fibonacci numbers if we take sum of digits in base 12, since that base-12 sum is necessarily $4 \pmod{11}$, hence not a Fibonacci number.

§3 Solutions to Day 3

§3.1 TSTST 2019/7, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p12608512.

Problem statement

Let $f\colon \mathbb{Z} \to \{1, 2, \dots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x - y)$$

for all integers x and y. Show that there exist positive integers m and n such that $f(x) = \gcd(m + x, n)$ for all integers x.

Let $\mathcal{P}$ be the set of primes not exceeding 10^{100}. For each $p \in \mathcal{P}$, let $e_p = \max_x \nu_p(f(x))$ and let $c_p \in \operatorname{argmax}_x \nu_p(f(x))$.

We show that this is good enough to compute all values of x, by looking at the exponent at each individual prime.

Claim — For any $p \in \mathcal{P}$, we have

$$\nu_p(f(x)) = \min(\nu_p(x - c_p), e_p).$$

Proof. Note that for any x, we have

$$\gcd(f(c_p), f(x)) = \gcd(f(c_p), x - c_p).$$

We then take ν_p of both sides and recall $\nu_p(f(x)) \le \nu_p(f(c_p)) = e_p$; this implies the result. □

This essentially determines f, and so now we just follow through. Choose n and m such that

$$n = \prod_{p \in \mathcal{P}} p^{e_p}$$
$$m \equiv -c_p \pmod{p^{e_p}} \quad \forall p \in \mathcal{P}$$

the latter being possible by Chinese remainder theorem. Then, from the claim we have

$$f(x) = \prod_{p \in \mathcal{P}} p^{\nu_p(f(x))} = \prod_{p \mid n} p^{\min(\nu_p(x - c_p), e_p)}$$
$$= \prod_{p \mid n} p^{\min(\nu_p(x+m), \nu_p(n))} = \gcd(x + m, n)$$

for every $x \in \mathbb{Z}$, as desired.

Remark. The functions $f(x) = x$ and $f(x) = |2x - 1|$ are examples satisfying the gcd equation (the latter always being strictly positive). Hence the hypothesis f bounded cannot be dropped.

Remark. The pair (m, n) is essentially unique: every other pair is obtained by shifting m by a multiple of n. Hence there is not really any choice in choosing m and n.

§3.2 TSTST 2019/8, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p12608780.

Problem statement

Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(\mathcal{S})$ denote the number of ways to draw 8 line segments with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

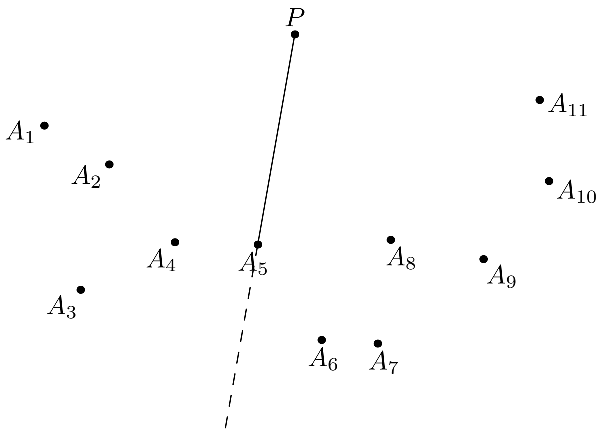
The answer is 1430. In general, we prove that with $2n$ points the answer is the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

First of all, it is well-known that if $\mathcal{S}$ is a convex $2n$-gon, then $\chi(\mathcal{S}) = C_n$.

It remains to prove the lower bound. We proceed by (strong) induction on n, with the base case $n = 0$ and $n = 1$ clear. Suppose the statement is proven for $0, 1, \dots, n$ and consider a set $\mathcal{S}$ with $2(n+1)$ points.

Let P be a point on the convex hull of $\mathcal{S}$, and label the other $2n+1$ points $A_1, \dots, A_{2n+1}$ in order of angle from P.

Consider drawing a segment $\overline{PA_{2k+1}}$. This splits the $2n$ remaining points into two halves $\mathcal{U}$ and $\mathcal{V}$, with $2k$ and $2(n-k)$ points respectively.



Note that by choice of P, no segment in $\mathcal{U}$ can intersect a segment in $\mathcal{V}$. By the inductive hypothesis,

$$\chi(\mathcal{U}) \geq C_k \quad \text{and} \quad \chi(\mathcal{V}) \geq C_{n-k}.$$

Thus, drawing $\overline{PA_{2k+1}}$, we have at least $C_k C_{n-k}$ ways to complete the drawing. Over all choices of k, we obtain

$$\chi(\mathcal{S}) \geq C_0 C_n + \dots + C_n C_0 = C_{n+1}$$

as desired.

Remark. It is possible to show directly from the lower bound proof that convex $2n$-gons achieve the minimum: indeed, every inequality is sharp, and no segment $\overline{PA_{2k}}$ can be drawn (since this splits the rest of the points into two halves with an odd number of points, and no crossing segment can be drawn).

Bobby Shen points out that in the case of 6 points, a regular pentagon with its center also achieves equality, so this is not the only equality case.

Remark. The result that $\chi(S) \geq 1$ for all S is known (consider the choice of 8 segments with smallest sum), and appeared on Putnam 1979. However, it does not seem that knowing this gives an advantage for this problem, since the answer is much larger than 1.

§3.3 TSTST 2019/9, proposed by Ankan Bhattacharya

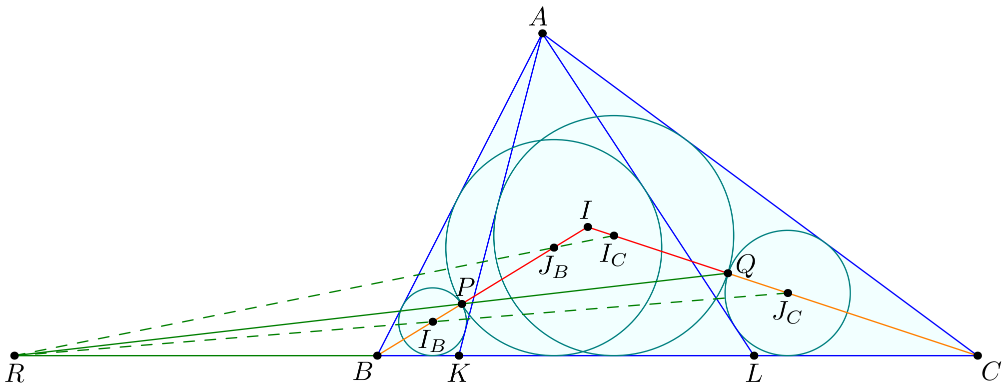
Available online at https://aops.com/community/p12608472.

Problem statement

Let ABC be a triangle with incenter I. Points K and L are chosen on segment BC such that the incircles of $\triangle ABK$ and $\triangle ABL$ are tangent at P, and the incircles of $\triangle ACK$ and $\triangle ACL$ are tangent at Q. Prove that $IP = IQ$.

We present two solutions.

¶ First solution, mostly elementary (original) Let I_B, J_B, I_C, J_C be the incenters of $\triangle ABK$, $\triangle ABL$, $\triangle ACK$, $\triangle ACL$ respectively.



We begin with the following claim which does not depend on the existence of tangency points P and Q.

Claim — Lines BC, I_BJ_C, J_BI_C meet at a point R (possibly at infinity).

Proof. By rotating by $\frac{1}{2}\angle A$ we have the equality

$$A(BI; I_BJ_B) = A(IC; I_CJ_C).$$

It follows $(BI; I_BJ_B) = (IC; I_CJ_C) = (CI; J_CI_C)$. (One could also check directly that both cross ratios equal $\frac{\sin \angle BAK/2}{\sin \angle CAK/2} \div \frac{\sin \angle BAL/2}{\sin \angle CAL/2}$, rather than using rotation.)

Therefore, the concurrence follows from the so-called *prism lemma* on $\overline{IBI_BJ_B}$ and $\overline{ICJ_CI_C}$. □

> **Remark** (Nikolai Beluhov). This result is known; it appears as 4.5.32 in Akopyan's *Geometry in Figures*. The cross ratio is not necessary to prove this claim: it can be proven by length chasing with circumscribed quadrilaterals. (The generalization mentioned later also admits a trig-free proof for the analogous step.)

We now bring P and Q into the problem.

Claim — Line PQ also passes through R.

Proof. Note $(BP; I_BJ_B) = -1 = (CQ; J_CI_C)$, so the conclusion again follows by prism lemma. □

We are now ready to complete the proof. Point R is the exsimilicenter of the incircles of $\triangle ABK$ and $\triangle ACL$, so $\frac{PI_B}{RI_B} = \frac{QJ_C}{RJ_C}$. Now by Menelaus,

$$\frac{I_BP}{PI} \cdot \frac{IQ}{QJ_C} \cdot \frac{J_CR}{RI_B} = -1 \implies IP = IQ.$$

> **Remark** (Author's comments on drawing the diagram)**.** Drawing the diagram directly is quite difficult. If one draws $\triangle ABC$ first, they must locate both K and L, which likely involves some trial and error due to the complex interplay between the two points.
>
> There are alternative simpler ways. For example, one may draw $\triangle AKL$ first; then the remaining points B and C are not related and the task is much simpler (though some trial and error is still required).
>
> In fact, by breaking symmetry, we may only require one application of guesswork. Start by drawing $\triangle ABK$ and its incircle; then the incircle of $\triangle ABL$ may be constructed, and so point L may be drawn. Thus only the location of point C needs to be guessed. I would be interested in a method to create a general diagram without any trial and error.

¶ **Second solution, inversion (Nikolai Beluhov)** As above, the lines BC, I_BJ_C, J_BI_C meet at some point R (possibly at infinity). Let ω_1, ω_2, ω_3, ω_4 be the incircles of $\triangle ABK$, $\triangle ACL$, $\triangle ABL$, and $\triangle ACK$.

> **Claim** — There exists an inversion ι at R swapping $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$.

Proof. Consider the inversion at R swapping ω_1 and ω_2. Since ω_1 and ω_3 are tangent, the image of ω_3 is tangent to ω_2 and is also tangent to BC. The circle ω_4 is on the correct side of ω_3 to be this image. □

> **Claim** — Circles ω_1, ω_2, ω_3, ω_4 share a common radical center.

Proof. Let Ω be the circle with center R fixed under ι, and let k be the circle through P centered at the radical center of Ω, ω_1, ω_3.

Then k is actually orthogonal to Ω, ω_1, ω_3, so k is fixed under ι and k is also orthogonal to ω_2 and ω_4. Thus the center of k is the desired radical center. □

The desired statement immediately follows. Indeed, letting S be the radical center, it follows that $\overline{SP}$ and $\overline{SQ}$ are the common internal tangents to $\{\omega_1, \omega_3\}$ and $\{\omega_2, \omega_4\}$.

Since S is the radical center, $SP = SQ$. In light of $\angle SPI = \angle SQI = 90^\circ$, it follows that $IP = IQ$, as desired.

> **Remark** (Nikolai Beluhov)**.** There exists a circle tangent to all four incircles, because circle k is orthogonal to all four, and line BC is tangent to all four; thus the inverse of line BC in k is a circle tangent to all four incircles.
>
> The amusing thing here is that Casey's theorem is completely unhelpful for proving this fact: all it can tell us is that there is a line or circle tangent to these incircles, and line BC already satisfies this property.

> **Remark** (Generalization by Nikolai Beluhov)**.** The following generalization holds:
>
> > Let $ABCD$ be a quadrilateral circumscribed about a circle with center I. A line through A meets $\overrightarrow{BC}$ and $\overrightarrow{DC}$ at K and L; another line through A meets $\overrightarrow{BC}$ and $\overrightarrow{DC}$ at M and N. Suppose that the incircles of $\triangle ABK$ and $\triangle ABM$ are tangent at P, and the incircles of $\triangle ACL$ and $\triangle ACN$ are tangent at Q. Prove that $IP = IQ$.
>
> The first approach can be modified to the generalization. There is an extra initial step required: by Monge, the exsimilicenter of the incircles of $\triangle ABK$ and $\triangle ADN$ lies on line BD; likewise for the incircles of $\triangle ABL$ and $\triangle ADM$. Now one may prove using the same trig approach that these pairs of incircles have a common exsimilicenter, and the rest of the solution plays out similarly. The second approach can also be modified in the same way, once we obtain that a common exsimilicenter exists. (Thus in the generalization, it seems we also get there exists a circle tangent to all four incircles.)

TSTST 2019 Statistics

Mathematical Olympiad Summer Program

Evan Chen《陳誼廷》

June 23, 2019

§1 Summary of scores for TSTST 2019

N	75	1st Q	16	Max	56
μ	25.79	Median	26	Top 3	50
σ	14.81	3rd Q	36	Top 12	42

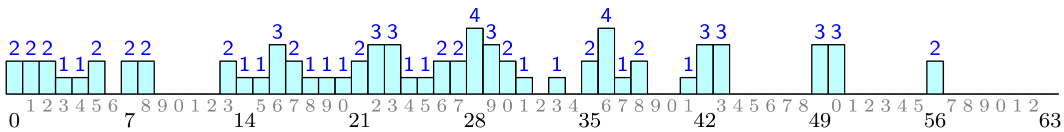
§2 Problem statistics for TSTST 2019

	P1	P2	P3	P4	P5	P6	P7	P8	P9
0	11	14	68	17	41	67	20	24	61
1	10	29	1	12	5	6	1	0	3
2	1	5	0	0	0	1	0	0	0
3	7	0	0	1	0	0	0	1	0
4	8	0	0	0	1	0	2	2	0
5	1	1	0	2	1	0	1	0	0
6	4	1	0	1	0	0	0	0	0
7	33	25	6	42	27	1	51	48	11
Avg	4.33	3.00	0.57	4.33	2.71	0.20	4.95	4.63	1.07
QM	5.15	4.22	1.98	5.37	4.27	0.89	5.84	5.65	2.69
#5+	**38**	**27**	**6**	**45**	**28**	**1**	**52**	**48**	**11**
%5+	%50.7	%36.0	%8.0	%60.0	%37.3	%1.3	%69.3	%64.0	%14.7

§3 Rankings for TSTST 2019

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
63	0	0	0.00%	42	3	14	18.67%	21	2	49	65.33%
62	0	0	0.00%	41	1	15	20.00%	20	1	50	66.67%
61	0	0	0.00%	40	0	15	20.00%	19	1	51	68.00%
60	0	0	0.00%	39	0	15	20.00%	18	1	52	69.33%
59	0	0	0.00%	38	2	17	22.67%	17	2	54	72.00%
58	0	0	0.00%	37	1	18	24.00%	16	3	57	76.00%
57	0	0	0.00%	36	4	22	29.33%	15	1	58	77.33%
56	2	2	2.67%	35	2	24	32.00%	14	1	59	78.67%
55	0	2	2.67%	34	0	24	32.00%	13	2	61	81.33%
54	0	2	2.67%	33	1	25	33.33%	12	0	61	81.33%
53	0	2	2.67%	32	0	25	33.33%	11	0	61	81.33%
52	0	2	2.67%	31	1	26	34.67%	10	0	61	81.33%
51	0	2	2.67%	30	2	28	37.33%	9	0	61	81.33%
50	3	5	6.67%	29	3	31	41.33%	8	2	63	84.00%
49	3	8	10.67%	28	4	35	46.67%	7	2	65	86.67%
48	0	8	10.67%	27	2	37	49.33%	6	0	65	86.67%
47	0	8	10.67%	26	2	39	52.00%	5	2	67	89.33%
46	0	8	10.67%	25	1	40	53.33%	4	1	68	90.67%
45	0	8	10.67%	24	1	41	54.67%	3	1	69	92.00%
44	0	8	10.67%	23	3	44	58.67%	2	2	71	94.67%
43	3	11	14.67%	22	3	47	62.67%	1	2	73	97.33%
								0	2	75	100.00%

§4 Histogram for TSTST 2019



USA Team Selection Test for 62$^{\text{nd}}$ IMO and 10$^{\text{th}}$ EGMO

United States of America

Day I

November 12, 2020

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 1. Let a, b, c be fixed positive integers. There are $a+b+c$ ducks sitting in a circle, one behind the other. Each duck picks either *rock*, *paper*, or *scissors*, with a ducks picking rock, b ducks picking paper, and c ducks picking scissors.

A *move* consists of an operation of one of the following three forms:

- If a duck picking rock sits behind a duck picking scissors, they switch places.
- If a duck picking paper sits behind a duck picking rock, they switch places.
- If a duck picking scissors sits behind a duck picking paper, they switch places.

Determine, in terms of a, b, and c, the maximum number of moves which could take place, over all possible initial configurations.

Problem 2. Let ABC be a scalene triangle with incenter I. The incircle of ABC touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at points D, E, F, respectively. Let P be the foot of the altitude from D to $\overline{EF}$, and let M be the midpoint of $\overline{BC}$. The rays AP and IP intersect the circumcircle of triangle ABC again at points G and Q, respectively. Show that the incenter of triangle GQM coincides with D.

Problem 3. We say a nondegenerate triangle whose angles have measures θ_1, θ_2, θ_3 is *quirky* if there exist integers r_1, r_2, r_3, not all zero, such that

$$r_1\theta_1 + r_2\theta_2 + r_3\theta_3 = 0.$$

Find all integers $n \geq 3$ for which a triangle with side lengths $n-1$, n, $n+1$ is quirky.

USA Team Selection Test for 62nd IMO and 10th EGMO

United States of America

Day II

December 10, 2020

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 4. Find all pairs of positive integers (a, b) satisfying the following conditions:

(i) a divides $b^4 + 1$,

(ii) b divides $a^4 + 1$,

(iii) $\lfloor\sqrt{a}\rfloor = \lfloor\sqrt{b}\rfloor$.

Problem 5. Let $\mathbb{N}^2$ denote the set of ordered pairs of positive integers. A finite subset S of $\mathbb{N}^2$ is *stable* if whenever (x, y) is in S, then so are all points (x', y') of $\mathbb{N}^2$ with both $x' \leq x$ and $y' \leq y$.

Prove that if S is a stable set, then among all stable subsets of S (including the empty set and S itself), at least half of them have an even number of elements.

Problem 6. Let A, B, C, D be four points such that no three are collinear and D is not the orthocenter of triangle ABC. Let P, Q, R be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that lines AP, BQ, CR are pairwise distinct and are concurrent. Show that the four points A, B, C, D lie on a circle.

USA Team Selection Test for 62$^{\text{nd}}$ IMO and 10$^{\text{th}}$ EGMO

United States of America

Day III

January 21, 2021

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 7. Find all nonconstant polynomials $P(z)$ with complex coefficients for which all complex roots of the polynomials $P(z)$ and $P(z)-1$ have absolute value 1.

Problem 8. For every positive integer N, let $\sigma(N)$ denote the sum of the positive integer divisors of N. Find all integers $m \geq n \geq 2$ satisfying

$$\frac{\sigma(m)-1}{m-1} = \frac{\sigma(n)-1}{n-1} = \frac{\sigma(mn)-1}{mn-1}.$$

Problem 9. Ten million fireflies are glowing in $\mathbb{R}^3$ at midnight. Some of the fireflies are friends, and friendship is always mutual. Every second, one firefly moves to a new position so that its distance from each one of its friends is the same as it was before moving. This is the only way that the fireflies ever change their positions. No two fireflies may ever occupy the same point.

Initially, no two fireflies, friends or not, are more than a meter away. Following some finite number of seconds, all fireflies find themselves at least ten million meters away from their original positions. Given this information, find the greatest possible number of friendships between the fireflies.

USA TSTST 2020 Solutions

United States of America — TST Selection Test

Ankan Bhattacharya and Evan Chen

62nd IMO 2021 Russia and 10th EGMO 2021 Georgia

Contents

§0 Problems

1. Let a, b, c be fixed positive integers. There are $a+b+c$ ducks sitting in a circle, one behind the other. Each duck picks either *rock*, *paper*, or *scissors*, with a ducks picking rock, b ducks picking paper, and c ducks picking scissors.

   A *move* consists of an operation of one of the following three forms:

   - If a duck picking rock sits behind a duck picking scissors, they switch places.
   - If a duck picking paper sits behind a duck picking rock, they switch places.
   - If a duck picking scissors sits behind a duck picking paper, they switch places.

   Determine, in terms of a, b, and c, the maximum number of moves which could take place, over all possible initial configurations.

2. Let ABC be a scalene triangle with incenter I. The incircle of ABC touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at points D, E, F, respectively. Let P be the foot of the altitude from D to $\overline{EF}$, and let M be the midpoint of $\overline{BC}$. The rays AP and IP intersect the circumcircle of triangle ABC again at points G and Q, respectively. Show that the incenter of triangle GQM coincides with D.

3. We say a nondegenerate triangle whose angles have measures θ_1, θ_2, θ_3 is *quirky* if there exists integers r_1, r_2, r_3, not all zero, such that

$$r_1\theta_1 + r_2\theta_2 + r_3\theta_3 = 0.$$

   Find all integers $n \ge 3$ for which a triangle with side lengths $n-1$, n, $n+1$ is quirky.

4. Find all pairs of positive integers (a,b) satisfying the following conditions:

   (i) a divides b^4+1,

   (ii) b divides a^4+1,

   (iii) $\lfloor \sqrt{a} \rfloor = \lfloor \sqrt{b} \rfloor$.

5. Let $\mathbb{N}^2$ denote the set of ordered pairs of positive integers. A finite subset S of $\mathbb{N}^2$ is *stable* if whenever (x,y) is in S, then so are all points (x',y') of $\mathbb{N}^2$ with both $x' \le x$ and $y' \le y$.

   Prove that if S is a stable set, then among all stable subsets of S (including the empty set and S itself), at least half of them have an even number of elements.

6. Let A, B, C, D be four points such that no three are collinear and D is not the orthocenter of triangle ABC. Let P, Q, R be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that lines AP, BQ, CR are pairwise distinct and are concurrent. Show that the four points A, B, C, D lie on a circle.

7. Find all nonconstant polynomials $P(z)$ with complex coefficients for which all complex roots of the polynomials $P(z)$ and $P(z)-1$ have absolute value 1.

8. For every positive integer N, let $\sigma(N)$ denote the sum of the positive integer divisors of N. Find all integers $m \ge n \ge 2$ satisfying

$$\frac{\sigma(m)-1}{m-1} = \frac{\sigma(n)-1}{n-1} = \frac{\sigma(mn)-1}{mn-1}.$$

9. Ten million fireflies are glowing in $\mathbb{R}^3$ at midnight. Some of the fireflies are friends, and friendship is always mutual. Every second, one firefly moves to a new position so that its distance from each one of its friends is the same as it was before moving. This is the only way that the fireflies ever change their positions. No two fireflies may ever occupy the same point.

Initially, no two fireflies, friends or not, are more than a meter away. Following some finite number of seconds, all fireflies find themselves at least ten million meters away from their original positions. Given this information, find the greatest possible number of friendships between the fireflies.

§1 Solutions to Day 1

§1.1 TSTST 2020/1, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p18933796.

Problem statement

Let a, b, c be fixed positive integers. There are $a+b+c$ ducks sitting in a circle, one behind the other. Each duck picks either *rock*, *paper*, or *scissors*, with a ducks picking rock, b ducks picking paper, and c ducks picking scissors.

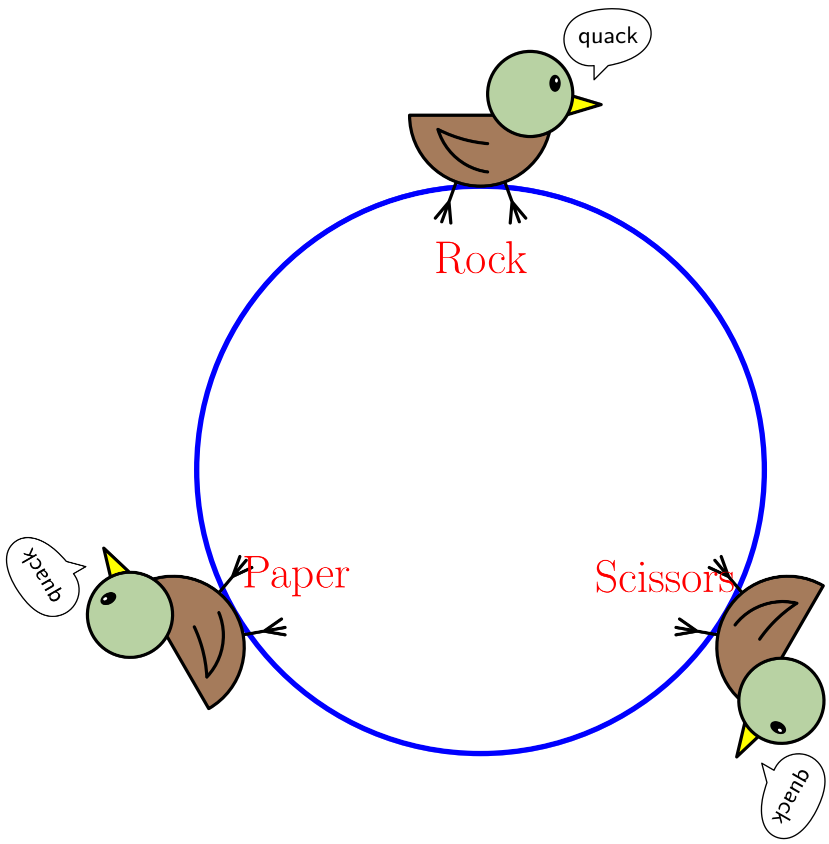
A *move* consists of an operation of one of the following three forms:

- If a duck picking rock sits behind a duck picking scissors, they switch places.
- If a duck picking paper sits behind a duck picking rock, they switch places.
- If a duck picking scissors sits behind a duck picking paper, they switch places.

Determine, in terms of a, b, and c, the maximum number of moves which could take place, over all possible initial configurations.

The maximum possible number of moves is $\max(ab, ac, bc)$.

First, we prove this is best possible. We define a *feisty triplet* to be an unordered triple of ducks, one of each of rock, paper, scissors, such that the paper duck is between the rock and scissors duck and facing the rock duck, as shown. (There may be other ducks not pictured, but the orders are irrelevant.)

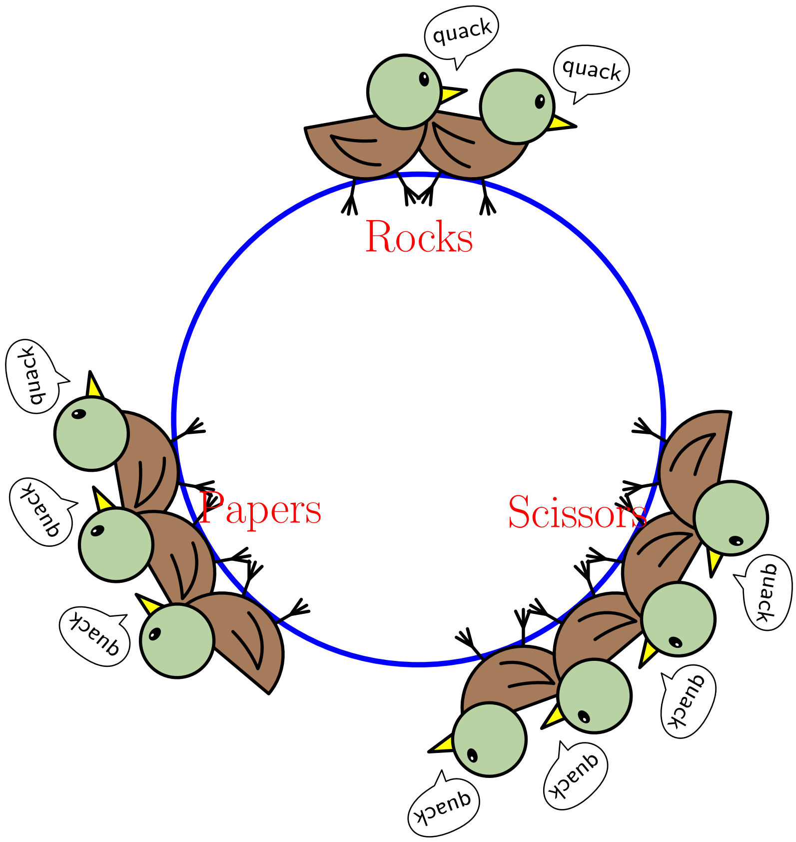


Claim — The number of feisty triplets decreases by c if a paper duck swaps places with a rock duck, and so on.

Proof. Clear. □

Obviously the number of feisty triples is at most abc to start. Thus at most $\max(ab, bc, ca)$ moves may occur, since the number of feisty triplets should always be nonnegative, at which point no moves are possible at all.

To see that this many moves is possible, assume WLOG $a = \min(a, b, c)$ and suppose we have a rocks, b papers, and c scissors in that clockwise order.



Then, allow the scissors to filter through the papers while the rocks stay put. Each of the b papers swaps with c scissors, for a total of $bc = \max(ab, ac, bc)$ swaps.

Remark (Common errors)**.** One small possible mistake: it is not quite kösher to say that "WLOG $a \le b \le c$" because the condition is not symmetric, only cyclic. Therefore in this solution we only assume $a = \min(a, b, c)$.

It is true here that every pair of ducks swaps at most once, and some solutions make use of this fact. However, this fact implicitly uses the fact that $a, b, c > 0$ and is false without this hypothesis.

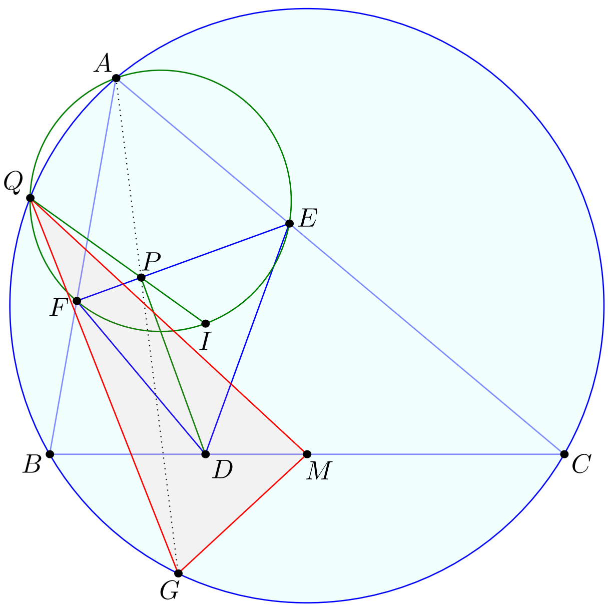
§1.2 TSTST 2020/2, proposed by Zack Chroman, Daniel Liu

Available online at https://aops.com/community/p18933557.

Problem statement

Let ABC be a scalene triangle with incenter I. The incircle of ABC touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at points D, E, F, respectively. Let P be the foot of the altitude from D to $\overline{EF}$, and let M be the midpoint of $\overline{BC}$. The rays AP and IP intersect the circumcircle of triangle ABC again at points G and Q, respectively. Show that the incenter of triangle GQM coincides with D.

Refer to the figure below.



Claim — The point Q is the Miquel point of $BFEC$. Also, $\overline{QD}$ bisects $\angle BQC$.

Proof. Inversion around the incircle maps line EF to $(AIEF)$ and the nine-point circle of $\triangle DEF$ to the circumcircle of $\triangle ABC$ (as the midpoint of $\overline{EF}$ maps to A, etc.). This implies P maps to Q; that is, Q coincides with the second intersection of $(AFIE)$ with (ABC). This is the claimed Miquel point.

The spiral similarity mentioned then gives $\frac{QB}{BF} = \frac{QC}{CE}$, so $\overline{QD}$ bisects $\angle BQC$. □

Remark. The point Q and its properties mentioned in the first claim have appeared in other references. See for example Canada 2007/5, ELMO 2010/6, HMMT 2016 T-10, USA TST 2017/2, USA TST 2019/6 for a few examples.

Claim — We have $(QG;BC) = -1$, so in particular $\overline{GD}$ bisects $\angle BGC$.

Proof. Note that

$$-1 = (AI;EF) \overset{Q}{=} (\overline{AQ} \cap \overline{EF}, P; E, F) \overset{A}{=} (QG;BC).$$

The last statement follows from Apollonian circle, or more bluntly $\frac{GB}{GC} = \frac{QB}{QC} = \frac{BD}{DC}$. □

Hence $\overline{QD}$ and $\overline{GD}$ are angle bisectors of $\angle BQC$ and $\angle BGC$. However, $\overline{QM}$ and $\overline{QG}$ are isogonal in $\angle BQC$ (as median and symmedian), and similarly for $\angle BGC$, as desired.

§1.3 TSTST 2020/3, proposed by Evan Chen, Danielle Wang

Available online at https://aops.com/community/p18933954.

Problem statement

We say a nondegenerate triangle whose angles have measures θ_1, θ_2, θ_3 is *quirky* if there exists integers r_1, r_2, r_3, not all zero, such that

$$r_1\theta_1 + r_2\theta_2 + r_3\theta_3 = 0.$$

Find all integers $n \ge 3$ for which a triangle with side lengths $n-1$, n, $n+1$ is quirky.

The answer is $n = 3,\ 4,\ 5,\ 7$.

We first introduce a variant of the kth Chebyshev polynomials in the following lemma (which is standard, and easily shown by induction).

Lemma

For each $k \ge 0$ there exists $P_k(X) \in \mathbb{Z}[X]$, monic for $k \ge 1$ and with degree k, such that

$$P_k(X + X^{-1}) \equiv X^k + X^{-k}.$$

The first few are $P_0(X) \equiv 2$, $P_1(X) \equiv X$, $P_2(X) \equiv X^2 - 2$, $P_3(X) \equiv X^3 - 3X$.

Suppose the angles of the triangle are $\alpha < \beta < \gamma$, so the law of cosines implies that

$$2\cos\alpha = \frac{n+4}{n+1} \qquad \text{and} \qquad 2\cos\gamma = \frac{n-4}{n-1}.$$

Claim — The triangle is quirky iff there exists $r, s \in \mathbb{Z}_{\ge 0}$ not both zero such that

$$\cos(r\alpha) = \pm\cos(s\gamma) \qquad \text{or equivalently} \qquad P_r\left(\frac{n+4}{n+1}\right) = \pm P_s\left(\frac{n-4}{n-1}\right).$$

Proof. If there are integers x, y, z for which $x\alpha + y\beta + z\gamma = 0$, then we have that $(x-y)\alpha = (y-z)\gamma - \pi y$, whence it follows that we may take $r = |x-y|$ and $s = |y-z|$ (noting $r = s = 0$ implies the absurd $x = y = z$). Conversely, given such r and s with $\cos(r\alpha) = \pm\cos(s\gamma)$, then it follows that $r\alpha \pm s\gamma = k\pi = k(\alpha+\beta+\gamma)$ for some k, so the triangle is quirky. □

If $r = 0$, then by rational root theorem on $P_s(X) \pm 2$ it follows $\frac{n-4}{n-1}$ must be an integer which occurs only when $n = 4$ (recall $n \ge 3$). Similarly we may discard the case $s = 0$.

Thus in what follows assume $n \ne 4$ and $r, s > 0$. Then, from the fact that P_r and P_s are nonconstant monic polynomials, we find

Corollary

If $n \ne 4$ works, then when $\frac{n+4}{n+1}$ and $\frac{n-4}{n-1}$ are written as fractions in lowest terms, the denominators have the same set of prime factors.

But $\gcd(n+1, n-1)$ divides 2, and $\gcd(n+4, n+1)$, $\gcd(n-4, n-1)$ divide 3. So we only have three possibilities:

- $n+1 = 2^u$ and $n-1 = 2^v$ for some $u, v \ge 0$. This is only possible if $n = 3$. Here $2\cos\alpha = \frac{7}{4}$ and $2\cos\gamma = -\frac{1}{2}$, and indeed $P_2(-1/2) = -7/4$.
- $n+1 = 3 \cdot 2^u$ and $n-1 = 2^v$ for some $u, v \ge 0$, which implies $n = 5$. Here $2\cos\alpha = \frac{3}{2}$ and $2\cos\gamma = \frac{1}{4}$, and indeed $P_2(3/2) = 1/4$.
- $n+1 = 2^u$ and $n-1 = 3 \cdot 2^v$ for some $u, v \ge 0$, which implies $n = 7$. Here $2\cos\alpha = \frac{11}{8}$ and $2\cos\gamma = \frac{1}{2}$, and indeed $P_3(1/2) = -11/8$.

Finally, $n = 4$ works because the triangle is right, completing the solution.

Remark (Major generalization due to Luke Robitaille)**.** In fact one may find all quirky triangles whose sides are integers in arithmetic progression.

Indeed, if the side lengths of the triangle are $x-y$, x, $x+y$ with $\gcd(x, y) = 1$ then the problem becomes

$$P_r\left(\frac{x+4y}{x+y}\right) = \pm P_s\left(\frac{x-4y}{x-y}\right)$$

and so in the same way as before, we ought to have $x+y$ and $x-y$ are both of the form $3 \cdot 2^*$ unless $rs = 0$. This time, when $rs = 0$, we get the extra solutions $(1, 0)$ and $(5, 2)$.

For $rs \neq 0$, by triangle inequality, we have $x - y \le x + y < 3(x - y)$, and $\min(\nu_2(x - y), \nu_2(x + y)) \le 1$, so it follows one of $x - y$ or $x + y$ must be in $\{1, 2, 3, 6\}$. An exhaustive check then leads to

$$(x, y) \in \{(3, 1), (5, 1), (7, 1), (11, 5)\} \cup \{(1, 0), (5, 2), (4, 1)\}$$

as the solution set. And in fact they all work.

In conclusion the equilateral triangle, $3-5-7$ triangle (which has a 120° angle) and $6-11-16$ triangle (which satisfies $B = 3A + 4C$) are exactly the new quirky triangles (up to similarity) whose sides are integers in arithmetic progression.

§2 Solutions to Day 2

§2.1 TSTST 2020/4, proposed by Yang Liu

Available online at https://aops.com/community/p19444614.

Problem statement

Find all pairs of positive integers (a, b) satisfying the following conditions:

(i) a divides $b^4 + 1$,

(ii) b divides $a^4 + 1$,

(iii) $\lfloor \sqrt{a} \rfloor = \lfloor \sqrt{b} \rfloor$.

The only solutions are $(1,1)$, $(1,2)$, and $(2,1)$, which clearly work. Now we show there are no others.

Obviously, $\gcd(a, b) = 1$, so the problem conditions imply

$$ab \mid (a-b)^4 + 1$$

since each of a and b divide the right-hand side. We define

$$k \overset{\text{def}}{=} \frac{(b-a)^4 + 1}{ab}.$$

Claim (Size estimate) — We must have $k \le 16$.

Proof. Let $n = \lfloor \sqrt{a} \rfloor = \lfloor \sqrt{b} \rfloor$, so that $a, b \in [n^2, n^2 + 2n]$. We have that

$$ab \ge n^2(n^2+1) \ge n^4 + 1$$
$$(b-a)^4 + 1 \le (2n)^4 + 1 = 16n^4 + 1$$

which shows $k \le 16$. □

Claim (Orders argument) — In fact, $k = 1$.

Proof. First of all, note that k cannot be even: if it was, then a, b have opposite parity, but then $4 \mid (b-a)^4 + 1$, contradiction.

Thus k is odd. However, every odd prime divisor of $(b-a)^4 + 1$ is congruent to 1 (mod 8) and is thus at least 17, so $k = 1$ or $k \ge 17$. It follows that $k = 1$. □

At this point, we have reduced to solving

$$ab = (b-a)^4 + 1$$

and we need to prove the claimed solutions are the only ones. Write $b = a + d$, and assume WLOG that $d \ge 0$: then we have $a(a+d) = d^4 + 1$, or

$$a^2 - da - (d^4 + 1) = 0.$$

The discriminant $d^2 + 4(d^4 + 1) = 4d^4 + d^2 + 4$ must be a perfect square.

- The cases $d = 0$ and $d = 1$ lead to pairs $(1, 1)$ and $(1, 2)$.
- If $d \geq 2$, then we can sandwich

  $$(2d^2)^2 < 4d^4 + d^2 + 4 < 4d^4 + 4d^2 + 1 = (2d^2 + 1)^2,$$

  so the discriminant is not a square.

The solution is complete.

Remark (Author remarks on origin)**.** This comes from the problem of the existence of a pair of elliptic curves over $\mathbb{F}_a$, $\mathbb{F}_b$ respectively, such that the number of points on one is the field size of the other. The bound $n^2 \leq a, b < (n+1)^2$ is the Hasse bound. The divisibility conditions correspond to asserting that the embedding degree of each curve is 8, so that they are *pairing friendly*. In this way, the problem is essentially the key result of `https://arxiv.org/pdf/1803.02067.pdf`, shown in Proposition 3.

§2.2 TSTST 2020/5, proposed by Ashwin Sah, Mehtaab Sawhney

Available online at https://aops.com/community/p19444403.

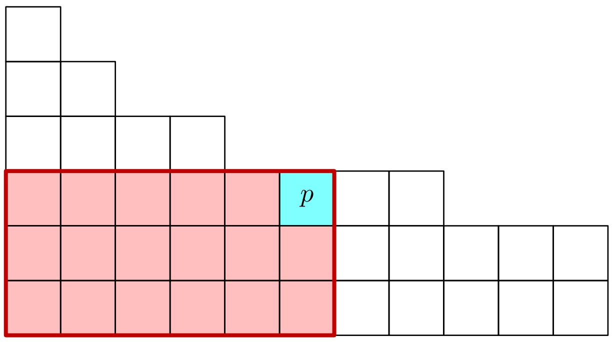
Problem statement

Let $\mathbb{N}^2$ denote the set of ordered pairs of positive integers. A finite subset S of $\mathbb{N}^2$ is *stable* if whenever (x, y) is in S, then so are all points (x', y') of $\mathbb{N}^2$ with both $x' \le x$ and $y' \le y$.

Prove that if S is a stable set, then among all stable subsets of S (including the empty set and S itself), at least half of them have an even number of elements.

The following inductive solution was given by Nikolai Beluhov. We proceed by induction on $|S|$, with $|S| \le 1$ clear.

Suppose $|S| \ge 2$. For any $p \in S$, let $R(p)$ denote the stable rectangle with upper-right corner p. We say such p is *pivotal* if $p + (1, 1) \notin S$ and $|R(p)|$ is even.



Claim — If $|S| \ge 2$, then a pivotal p always exists.

Proof. Consider the top row of S.

- If it has length at least 2, one of the two rightmost points in it is pivotal.
- Otherwise, the top row has length 1. Now either the top point or the point below it (which exists as $|S| \ge 2$) is pivotal. □

We describe how to complete the induction, given some pivotal $p \in S$. There is a partition

$$S = R(p) \sqcup S_1 \sqcup S_2$$

where S_1 and S_2 are the sets of points in S above and to the right of p (possibly empty).

Claim — The desired inequality holds for stable subsets containing p.

Proof. Let E_1 denote the number of even stable subsets of S_1; denote E_2, O_1, O_2 analogously. The stable subsets containing p are exactly $R(p) \sqcup T_1 \sqcup T_2$, where $T_1 \subseteq S_1$ and $T_2 \subseteq S_2$ are stable.

Since $|R(p)|$ is even, exactly $E_1E_2 + O_1O_2$ stable subsets containing p are even, and exactly $E_1O_2 + E_2O_1$ are odd. As $E_1 \ge O_1$ and $E_2 \ge O_2$ by inductive hypothesis, we obtain $E_1E_2 + O_1O_2 \ge E_1O_2 + E_2O_1$ as desired. □

By the inductive hypothesis, the desired inequality also holds for stable subsets not containing p, so we are done.

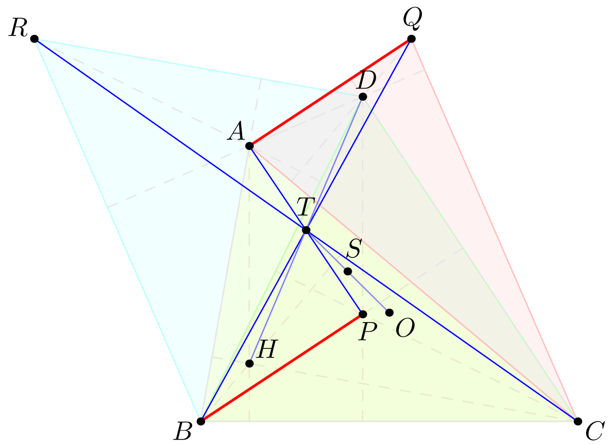
§2.3 TSTST 2020/6, proposed by Andrew Gu

Available online at https://aops.com/community/p19444197.

Problem statement

Let A, B, C, D be four points such that no three are collinear and D is not the orthocenter of triangle ABC. Let P, Q, R be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that lines AP, BQ, CR are pairwise distinct and are concurrent. Show that the four points A, B, C, D lie on a circle.

Let T be the concurrency point, and let H be the orthocenter of $\triangle ABC$.



Claim (Key claim) — T is the midpoint of $\overline{AP}$, $\overline{BQ}$, $\overline{CR}$, $\overline{DH}$, and D is the orthocenter of $\triangle PQR$.

Proof. Note that $\overline{AQ} \parallel \overline{BP}$, as both are perpendicular to $\overline{CD}$. Since lines AP and BQ are distinct, lines AQ and BP are distinct.

By symmetric reasoning, we get that $AQCPBR$ is a hexagon with *opposite sides parallel* and *concurrent diagonals* as $\overline{AP}$, $\overline{BQ}$, $\overline{CR}$ meet at T. This implies that the *hexagon is centrally symmetric* about T; indeed

$$\frac{AT}{TP} = \frac{TQ}{BT} = \frac{CT}{TR} = \frac{TP}{AT}$$

so all the ratios are equal to $+1$.

Next, $\overline{PD} \perp \overline{BC} \parallel \overline{QR}$, so by symmetry we get D is the orthocenter of $\triangle PQR$. This means that T is the midpoint of $\overline{DH}$ as well. □

Corollary

The configuration is now symmetric: we have four points A, B, C, D, and their reflections in T are four orthocenters P, Q, R, H.

Let S be the centroid of $\{A, B, C, D\}$, and let O be the reflection of T in S. We are ready to conclude:

Claim — A, B, C, D are equidistant from O.

Proof. Let A', O', S', T', D' be the projections of A, O, S, T, D onto line BC.

Then T' is the midpoint of $\overline{A'D'}$, so $S' = \frac{1}{4}(A' + D' + B + C)$ gives that O' is the midpoint of $\overline{BC}$.

Thus $OB = OC$ and we're done. □

§3 Solutions to Day 3

§3.1 TSTST 2020/7, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p20020202.

Problem statement

Find all nonconstant polynomials $P(z)$ with complex coefficients for which all complex roots of the polynomials $P(z)$ and $P(z) - 1$ have absolute value 1.

The answer is $P(x)$ should be a polynomial of the form $P(x) = \lambda x^n - \mu$ where $|\lambda| = |\mu|$ and $\operatorname{Re}\mu = -\frac{1}{2}$. One may check these all work; let's prove they are the only solutions.

¶ First approach (Evan Chen) We introduce the following notations:

$$\begin{aligned} P(x) &= c_n x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0 \\ &= c_n(x+\alpha_1)\dots(x+\alpha_n) \\ P(x) - 1 &= c_n(x+\beta_1)\dots(x+\beta_n) \end{aligned}$$

By taking conjugates,

$$\begin{aligned} &(x+\alpha_1)\cdots(x+\alpha_n) = (x+\beta_1)\cdots(x+\beta_n) + c_n^{-1} \\ \implies &\left(x+\frac{1}{\alpha_1}\right)\cdots\left(x+\frac{1}{\alpha_n}\right) = \left(x+\frac{1}{\beta_1}\right)\cdots\left(x+\frac{1}{\beta_n}\right) + (\overline{c_n})^{-1} \qquad (\spadesuit) \end{aligned}$$

The equation $(\spadesuit)$ is the main player:

Claim — We have $c_k = 0$ for all $k = 1, \dots, n-1$.

Proof. By comparing coefficients of x^k in $(\spadesuit)$ we obtain

$$\frac{c_{n-k}}{\prod_i \alpha_i} = \frac{c_{n-k}}{\prod_i \beta_i}$$

but $\prod_i \alpha_i - \prod_i \beta_i = \frac{1}{c_n} \neq 0$. Hence $c_k = 0$. □

It follows that $P(x)$ must be of the form $P(x) = \lambda x^n - \mu$, so that $P(x) = \lambda x^n - (\mu + 1)$. This requires $|\mu| = |\mu + 1| = |\lambda|$ which is equivalent to the stated part.

¶ Second approach (from the author) We let $A = P$ and $B = P - 1$ to make the notation more symmetric. We will as before show that A and B have all coefficients equal to zero other than the leading and constant coefficient; the finish is the same.

First, we rule out double roots.

Claim — Neither A nor B have double roots.

Proof. Suppose that b is a double root of B. By differentiating, we obtain $A' = B'$, so $A'(b) = 0$. However, by Gauss-Lucas, this forces $A(b) = 0$, contradiction. □

Let $\omega = e^{2\pi i/n}$, let $a_1, \dots, a_n$ be the roots of A, and let $b_1, \dots, b_n$ be the roots of B. For each k, let A_k and B_k be the points in the complex plane corresponding to a_k and b_k.

Claim (Main claim) — For any i and j, $\frac{a_i}{a_j}$ is a power of ω.

Proof. Note that

$$\frac{a_i - b_1}{a_j - b_1} \cdots \frac{a_i - b_n}{a_j - b_n} = \frac{B(a_i)}{B(a_j)} = \frac{A(a_i) - 1}{A(a_j) - 1} = \frac{0 - 1}{0 - 1} = 1.$$

Since the points A_i, A_j, B_k all lie on the unit circle, interpreting the left-hand side geometrically gives

$$\measuredangle A_i B_1 A_j + \cdots + \measuredangle A_i B_n A_j = 0 \implies n\widehat{A_i A_j} = 0,$$

where angles are directed modulo 180° and arcs are directed modulo 360°. This implies that $\frac{a_i}{a_j}$ is a power of ω. □

Now the finish is easy: since $a_1, \ldots, a_n$ are all different, they must be $a_1\omega^0, \ldots, a_1\omega^{n-1}$ in some order; this shows that A is a multiple of $x^n - a_1^n$, as needed.

§3.2 TSTST 2020/8, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p20020195.

Problem statement

For every positive integer N, let $\sigma(N)$ denote the sum of the positive integer divisors of N. Find all integers $m \ge n \ge 2$ satisfying

$$\frac{\sigma(m)-1}{m-1} = \frac{\sigma(n)-1}{n-1} = \frac{\sigma(mn)-1}{mn-1}.$$

The answer is that m and n should be powers of the same prime number. These all work because for a prime power we have

$$\frac{\sigma(p^e)-1}{p^e-1} = \frac{(1+p+\cdots+p^e)-1}{p^e-1} = \frac{p(1+\cdots+p^{e-1})}{p^e-1} = \frac{p}{p-1}.$$

So we now prove these are the only ones. Let λ be the common value of the three fractions.

Claim — Any solution (m,n) should satisfy $d(mn) = d(m)+d(n)-1$.

Proof. The divisors of mn include the divisors of m, plus m times the divisors of n (counting m only once). Let λ be the common value; then this gives

$$\begin{aligned}\sigma(mn) &\ge \sigma(m) + m\sigma(n) - m \\ &= (\lambda m - \lambda + 1) + m(\lambda n - \lambda + 1) - m \\ &= \lambda mn - \lambda + 1\end{aligned}$$

and so equality holds. Thus these are all the divisors of mn, for a count of $d(m)+d(n)-1$. □

Claim — If $d(mn) = d(m)+d(n)-1$ and $\min(m,n) \ge 2$, then m and n are powers of the same prime.

Proof. Let A denote the set of divisors of m and B denote the set of divisors of n. Then $|A \cdot B| = |A|+|B|-1$ and $\min(|A|,|B|) > 1$, so $|A|$ and $|B|$ are geometric progressions with the same ratio. It follows that m and n are powers of the same prime. □

Remark (Nikolai Beluhov)**.** Here is a completion not relying on $|A \cdot B| = |A|+|B|-1$. By the above arguments, we see that every divisor of mn is either a divisor of n, or n times a divisor of m.

Now suppose that some prime $p \mid m$ but $p \nmid n$. Then $p \mid mn$ but p does not appear in the above classification, a contradiction. By symmetry, it follows that m and n have the same prime divisors.

Now suppose we have different primes $p \mid m$ and $q \mid n$. Write $\nu_p(m) = \alpha$ and $\nu_p(n) = \beta$. Then $p^{\alpha+\beta} \mid mn$, but it does not appear in the above characterization, a contradiction. Thus, m and n are powers of the same prime.

Remark (Comments on the function in the problem)**.** Let $f(n) = \frac{\sigma(n)-1}{n-1}$. Then f is not really injective even outside the above solution; for example, we have $f(6 \cdot 11^k) = \frac{11}{5}$ for all k, plus sporadic equivalences like $f(14) = f(404)$, as pointed out by one reviewer during test-solving. This means that both relations should be used at once, not independently.

Remark (Authorship remarks)**.** Ankan gave the following story for how he came up with the problem while thinking about so-called *almost perfect* numbers.

> I was in some boring talk when I recalled a conjecture that if $\sigma(n) = 2n - 1$, then n is a power of 2. For some reason (divine intervention, maybe) I had the double idea of (1) seeing whether m, n, mn all almost perfect implies m, n powers of 2, and (2) trying the naive divisor bound to resolve this. Through sheer dumb luck this happened to work out perfectly. I thought this was kinda cool but I felt that I hadn't really unlocked a lot of the potential this idea had: then I basically tried to find the "general situation" which allows for this manipulation, and was amazed that it led to such a striking statement.

§3.3 TSTST 2020/9, proposed by Nikolai Beluhov

Available online at https://aops.com/community/p20020206.

Problem statement

Ten million fireflies are glowing in $\mathbb{R}^3$ at midnight. Some of the fireflies are friends, and friendship is always mutual. Every second, one firefly moves to a new position so that its distance from each one of its friends is the same as it was before moving. This is the only way that the fireflies ever change their positions. No two fireflies may ever occupy the same point.

Initially, no two fireflies, friends or not, are more than a meter away. Following some finite number of seconds, all fireflies find themselves at least ten million meters away from their original positions. Given this information, find the greatest possible number of friendships between the fireflies.

In general, we show that when $n \geq 70$, the answer is $f(n) = \lfloor \frac{n^2}{3} \rfloor$.

Construction: Choose three pairwise parallel lines ℓ_A, ℓ_B, ℓ_C forming an infinite equilateral triangle prism (with side larger than 1). Split the n fireflies among the lines as equally as possible, and say that two fireflies are friends iff they lie on different lines.

To see this works:

1. Reflect ℓ_A and all fireflies on ℓ_A in the plane containing ℓ_B and ℓ_C.
2. Reflect ℓ_B and all fireflies on ℓ_B in the plane containing ℓ_C and ℓ_A.
3. Reflect ℓ_C and all fireflies on ℓ_C in the plane containing ℓ_A and ℓ_B.

$\vdots$

Proof: Consider a valid configuration of fireflies. If there is no 4-clique of friends, then by Turán's theorem, there are at most $f(n)$ pairs of friends.

Let $g(n)$ be the answer, given that there exist four pairwise friends (say a, b, c, d). Note that for a firefly to move, all its friends must be coplanar.

Claim (No coplanar K_4) — We can't have four coplanar fireflies which are pairwise friends.

Proof. If we did, none of them could move (unless three are collinear, in which case they can't move). □

Claim (Key claim — tetrahedrons don't share faces often) — There are at most 12 fireflies e which are friends with at least three of a, b, c, d.

Proof. First denote by A, B, C, D the locations of fireflies a, b, c, d. These four positions change over time as fireflies move, but the tetrahedron $ABCD$ always has a fixed shape, and we will take this tetrahedron as our reference frame for the remainder of the proof.

WLOG, will assume that e is friends with a, b, c. Then e will always be located at one of two points E_1 and E_2 relative to ABC, such that E_1ABC and E_2ABC are two congruent tetrahedrons with fixed shape. We note that points D, E_1, and E_2 are all different: clearly $D \neq E_1$ and $E_1 \neq E_2$. (If $D = E_2$, then some fireflies won't be able to move.)

Consider the moment where firefly a moves. Its friends must be coplanar at that time, so one of E_1, E_2 lies in plane BCD. Similar reasoning holds for planes ACD and ABD.

So, WLOG E_1 lies on both planes BCD and ACD. Then E_1 lies on line CD, and E_2 lies in plane ABD. This uniquely determines (E_1, E_2) relative to $ABCD$:

- E_1 is the intersection of line CD with the reflection of plane ABD in plane ABC.
- E_2 is the intersection of plane ABD with the reflection of line CD in plane ABC.

Accounting for WLOGs, there are at most 12 possibilities for the set $\{E_1, E_2\}$, and thus at most 12 possibilities for E. (It's not possible for both elements of one pair $\{E_1, E_2\}$ to be occupied, because then they couldn't move.) $\square$

Thus, the number of friendships involving exactly one of a, b, c, d is at most $(n-16) \cdot 2 + 12 \cdot 3 = 2n + 4$, so removing these four fireflies gives

$$g(n) \leq 6 + (2n+4) + \max\{f(n-4), g(n-4)\}.$$

The rest of the solution is bounding. When $n \geq 24$, we have $(2n+10) + f(n-4) \leq f(n)$, so

$$g(n) \leq \max\{f(n), (2n+10) + g(n-4)\} \quad \forall n \geq 24.$$

By iterating the above inequality, we get

$$g(n) \leq \max\Big\{f(n),\ (2n+10) + (2(n-4)+10) + \cdots + (2(n-4r)+10) + g(n-4r-4)\Big\},$$

where r satisfies $n - 4r - 4 < 24 \leq n - 4r$.

Now

$$\begin{aligned} &(2n+10) + (2(n-4)+10) + \cdots + (2(n-4r)+10) + g(n-4r-4) \\ &= (r+1)(2n-4r+10) + g(n-4r-4) \\ &\leq \left(\frac{n}{4} - 5\right)(n+37) + \binom{24}{2}. \end{aligned}$$

This is less than $f(n)$ for $n \geq 70$, which concludes the solution.

Remark. There are positive integers n such that it is possible to do better than $f(n)$ friendships. For instance, $f(5) = 8$, whereas five fireflies a, b, c, d, and e as in the proof of the Lemma (E_1 being the intersection point of line CD with the reflection of plane (ABD) in plane (ABC), E_2 being the intersection point of plane (ABD) with the reflection of line CD in plane (ABC), and tetrahedron $ABCD$ being sufficiently arbitrary that points E_1 and E_2 exist and points D, E_1, and E_2 are pairwise distinct) give a total of nine friendships.

Remark (Author comments)**.** It is natural to approach the problem by looking at the two-dimensional version first. In two dimensions, the following arrangement suggests itself almost immediately: We distribute all fireflies as equally as possible among two parallel lines, and two fireflies are friends if and only if they are on different lines.

Similarly to the three-dimensional version, this attains the greatest possible number of friendships for all sufficiently large n, though not for all n. For instance, at least one friendlier arrangements exists for $n = 4$, similarly to the above friendlier arrangement for $n = 5$ in three dimensions.

> This observation strongly suggests that in three dimensions we should distribute the fireflies as equally as possible among two parallel planes, and that two fireflies should be friends if and only if they are on different planes. It was a great surprise for me to discover that this arrangement does not in fact give the correct answer!

> **Remark.** On the other hand, Ankan Bhattacharya gives the following reasoning as to why the answer should not be that surprising:
>
> > I think the answer $(10^{14}-1)/3$ is quite natural if you realize that $(n/2)^2$ is probably optimal in 2D and $\binom{n}{2}$ is optimal in super high dimensions (i.e. around n). So going from dimension 2 to 3 should increase the answer (and indeed it does).

TSTST 2020 Statistics

Mathematical Olympiad Summer Program

Evan Chen《陳誼廷》

February 11, 2021

§1 Summary of scores for TSTST 2020

N	57	1st Q	17	Max	63
μ	29.67	Median	29	Top 3	53
σ	15.24	3rd Q	42	Top 12	43

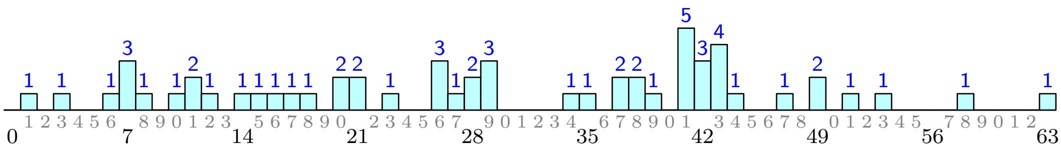
§2 Problem statistics for TSTST 2020

	P1	P2	P3	P4	P5	P6	P7	P8	P9
0	1	23	42	8	27	34	15	35	41
1	3	1	2	0	0	2	7	2	1
2	1	2	1	1	0	0	3	0	11
3	2	0	0	2	0	0	7	0	0
4	4	0	0	0	0	0	1	0	1
5	6	0	2	2	0	0	1	0	1
6	3	0	0	16	3	3	3	1	0
7	37	31	10	28	27	18	20	19	2
Avg	5.86	3.89	1.47	5.44	3.63	2.56	3.53	2.47	0.81
QM	6.16	5.18	3.09	5.95	5.01	4.17	4.61	4.12	1.80
#5+	**46**	**31**	**12**	**46**	**30**	**21**	**24**	**20**	**3**
%5+	%80.7	%54.4	%21.1	%80.7	%52.6	%36.8	%42.1	%35.1	%5.3

§3 Rankings for TSTST 2020

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
63	1	1	1.75%	42	3	15	26.32%	21	2	39	68.42%
62	0	1	1.75%	41	5	20	35.09%	20	2	41	71.93%
61	0	1	1.75%	40	0	20	35.09%	19	0	41	71.93%
60	0	1	1.75%	39	1	21	36.84%	18	1	42	73.68%
59	0	1	1.75%	38	2	23	40.35%	17	1	43	75.44%
58	1	2	3.51%	37	2	25	43.86%	16	1	44	77.19%
57	0	2	3.51%	36	0	25	43.86%	15	1	45	78.95%
56	0	2	3.51%	35	1	26	45.61%	14	1	46	80.70%
55	0	2	3.51%	34	1	27	47.37%	13	0	46	80.70%
54	0	2	3.51%	33	0	27	47.37%	12	1	47	82.46%
53	1	3	5.26%	32	0	27	47.37%	11	2	49	85.96%
52	0	3	5.26%	31	0	27	47.37%	10	1	50	87.72%
51	1	4	7.02%	30	0	27	47.37%	9	0	50	87.72%
50	0	4	7.02%	29	3	30	52.63%	8	1	51	89.47%
49	2	6	10.53%	28	2	32	56.14%	7	3	54	94.74%
48	0	6	10.53%	27	1	33	57.89%	6	1	55	96.49%
47	1	7	12.28%	26	3	36	63.16%	5	0	55	96.49%
46	0	7	12.28%	25	0	36	63.16%	4	0	55	96.49%
45	0	7	12.28%	24	0	36	63.16%	3	1	56	98.25%
44	1	8	14.04%	23	1	37	64.91%	2	0	56	98.25%
43	4	12	21.05%	22	0	37	64.91%	1	1	57	100.00%
								0	0	57	100.00%

§4 Histogram for TSTST 2020



USA Team Selection Test for 63$^{\text{rd}}$ IMO and 11$^{\text{th}}$ EGMO

United States of America

Day I

Thursday, November 4, 2021

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 1. Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Points X and Y lie on sides AB and CD, respectively. Suppose the circumcircles of ADX and BCY meet line XY again at P and Q, respectively. Show that $OP = OQ$.

Problem 2. Let $a_1 < a_2 < a_3 < a_4 < \cdots$ be an infinite sequence of real numbers in the interval $(0, 1)$. Show that there exists a number that occurs exactly once in the sequence

$$\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots.$$

Problem 3. Find all positive integers $k > 1$ for which there exists a positive integer n such that $\binom{n}{k}$ is divisible by n, and $\binom{n}{m}$ is not divisible by n for $2 \leq m < k$.

USA Team Selection Test for 63rd IMO and 11th EGMO

United States of America

Day II

Thursday, December 9, 2021

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 4. Let a and b be positive integers. Suppose that there are infinitely many pairs of positive integers (m, n) for which $m^2 + an + b$ and $n^2 + am + b$ are both perfect squares. Prove that a divides $2b$.

Problem 5. Let T be a tree on n vertices with exactly k leaves. Suppose that there exists a subset of at least $\frac{n+k-1}{2}$ vertices of T, no two of which are adjacent. Show that the longest path in T contains an even number of edges.*

Problem 6. Triangles ABC and DEF share circumcircle Ω and incircle ω so that points A, F, B, D, C, and E occur in this order along Ω. Let Δ_A be the triangle formed by lines AB, AC, and EF, and define triangles Δ_B, $\Delta_C, \dots, \Delta_F$ similarly. Furthermore, let Ω_A and ω_A be the circumcircle and incircle of triangle Δ_A, respectively, and define circles Ω_B, $\omega_B, \dots, \Omega_F$, ω_F similarly.

(a) Prove that the two common external tangents to circles Ω_A and Ω_D and the two common external tangents to circles ω_A and ω_D are either concurrent or pairwise parallel.

(b) Suppose that these four lines meet at point T_A, and define points T_B and T_C similarly. Prove that points T_A, T_B, and T_C are collinear.

*A tree is a connected graph with no cycles. A leaf is a vertex of degree 1.

USA Team Selection Test for 63$^{\text{rd}}$ IMO and 11$^{\text{th}}$ EGMO

United States of America

Day III

Thursday, January 13, 2022

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 7. Let M be a finite set of lattice points and n be a positive integer. A *mine-avoiding path* is a path of lattice points with length n, beginning at $(0, 0)$ and ending at a point on the line $x + y = n$, that does not contain any point in M. Prove that if there exists a mine-avoiding path, then there exist at least $2^{n-|M|}$ mine-avoiding paths.*

Problem 8. Let ABC be a scalene triangle. Points A_1, B_1 and C_1 are chosen on segments BC, CA, and AB, respectively, such that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar. Let A_2 be the unique point on line B_1C_1 such that $AA_2 = A_1A_2$. Points B_2 and C_2 are defined similarly. Prove that $\triangle A_2B_2C_2$ and $\triangle ABC$ are similar.

Problem 9. Let $q = p^r$ for a prime number p and positive integer r. Let $\zeta = e^{\frac{2\pi i}{q}}$. Find the least positive integer n such that

$$\sum_{\substack{1 \le k \le q \\ \gcd(k,p)=1}} \frac{1}{(1-\zeta^k)^n}$$

is not an integer. (The sum is over all $1 \le k \le q$ with p not dividing k.)

*A lattice point is a point (x, y) where x and y are integers. A path of lattice points with length n is a sequence of lattice points $P_0, P_1, \ldots, P_n$ in which any two adjacent points in the sequence have distance 1 from each other.

USA TSTST 2021 Solutions

United States of America — TST Selection Test

ANDREW GU AND EVAN CHEN

63rd IMO 2022 Norway and 11th EGMO 2022 Hungary

Contents

§0 Problems

1. Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Points X and Y lie on sides AB and CD, respectively. Suppose the circumcircles of ADX and BCY meet line XY again at P and Q, respectively. Show that $OP = OQ$.

2. Let $a_1 < a_2 < a_3 < a_4 < \cdots$ be an infinite sequence of real numbers in the interval $(0, 1)$. Show that there exists a number that occurs exactly once in the sequence

$$\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots.$$

3. Find all positive integers $k > 1$ for which there exists a positive integer n such that $\binom{n}{k}$ is divisible by n, and $\binom{n}{m}$ is not divisible by n for $2 \le m < k$.

4. Let a and b be positive integers. Suppose that there are infinitely many pairs of positive integers (m, n) for which $m^2 + an + b$ and $n^2 + am + b$ are both perfect squares. Prove that a divides $2b$.

5. Let T be a tree on n vertices with exactly k leaves. Suppose that there exists a subset of at least $\frac{n+k-1}{2}$ vertices of T, no two of which are adjacent. Show that the longest path in T contains an even number of edges.

6. Triangles ABC and DEF share circumcircle Ω and incircle ω so that points A, F, B, D, C, and E occur in this order along Ω. Let Δ_A be the triangle formed by lines AB, AC, and EF, and define triangles $\Delta_B, \Delta_C, \ldots, \Delta_F$ similarly. Furthermore, let Ω_A and ω_A be the circumcircle and incircle of triangle Δ_A, respectively, and define circles $\Omega_B, \omega_B, \ldots, \Omega_F, \omega_F$ similarly.

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7. Let M be a finite set of lattice points and n be a positive integer. A *mine-avoiding path* is a path of lattice points with length n, beginning at $(0, 0)$ and ending at a point on the line $x + y = n$, that does not contain any point in M. Prove that if there exists a mine-avoiding path, then there exist at least $2^{n-|M|}$ mine-avoiding paths.

8. Let ABC be a scalene triangle. Points A_1, B_1 and C_1 are chosen on segments BC, CA, and AB, respectively, such that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar. Let A_2 be the unique point on line B_1C_1 such that $AA_2 = A_1A_2$. Points B_2 and C_2 are defined similarly. Prove that $\triangle A_2B_2C_2$ and $\triangle ABC$ are similar.

9. Let $q = p^r$ for a prime number p and positive integer r. Let $\zeta = e^{\frac{2\pi i}{q}}$. Find the least positive integer n such that

$$\sum_{\substack{1 \le k \le q \\ \gcd(k,p)=1}} \frac{1}{(1-\zeta^k)^n}$$

is not an integer. (The sum is over all $1 \le k \le q$ with p not dividing k.)

§1 Solutions to Day 1

§1.1 TSTST 2021/1, proposed by Holden Mui

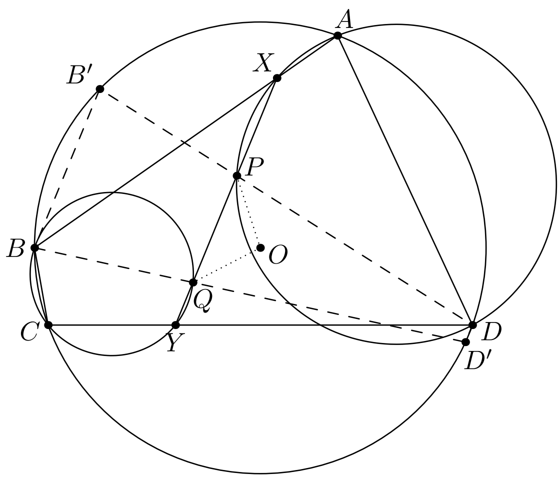
Available online at https://aops.com/community/p23586650.

Problem statement

Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Points X and Y lie on sides AB and CD, respectively. Suppose the circumcircles of ADX and BCY meet line XY again at P and Q, respectively. Show that $OP = OQ$.

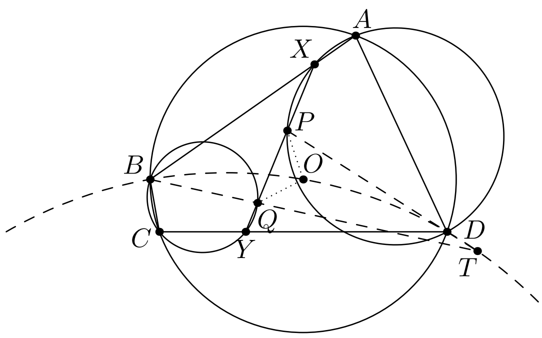
We present many solutions.

¶ First solution, angle chasing only (Ankit Bisain) Let lines BQ and DP meet $(ABCD)$ again at D' and B', respectively.



Then $BB' \parallel PX$ and $DD' \parallel QY$ by Reim's theorem. Segments BB', DD', and PQ share a perpendicular bisector which passes through O, so $OP = OQ$.

¶ Second solution via isosceles triangles (from contestants) Let $T = \overline{BQ} \cap \overline{DP}$.



Note that PQT is isosceles because

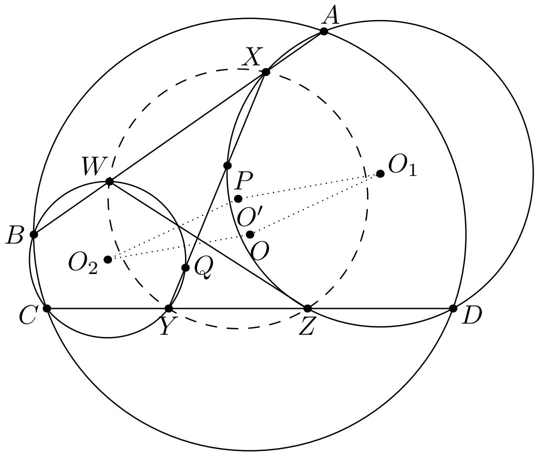
$$\measuredangle PQT = \measuredangle YQB = \measuredangle BCD = \measuredangle BAD = \measuredangle XPD = \measuredangle TPQ.$$

Then $(BODT)$ is cyclic because

$$\measuredangle BOD = 2\measuredangle BCD = \measuredangle PQT + \measuredangle TPQ = \measuredangle BTD.$$

Since $BO = OD$, $\overline{TO}$ is an angle bisector of $\measuredangle BTD$. Since $\triangle PQT$ is isosceles, $\overline{TO} \perp \overline{PQ}$, so $OP = OQ$.

¶ Third solution using a parallelogram (from contestants) Let (BCY) meet $\overline{AB}$ again at W and let (ADX) meet $\overline{CD}$ again at Z. Additionally, let O_1 be the center of (ADX) and O_2 be the center of (BCY).



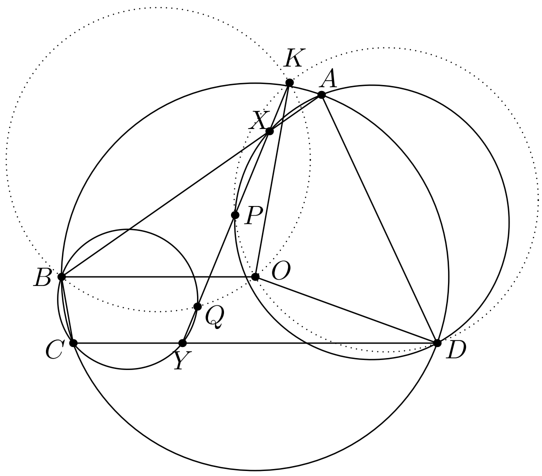
Note that $(WXYZ)$ is cyclic since

$$\measuredangle XWY + \measuredangle YZX = \measuredangle YWB + \measuredangle XZD = \measuredangle YCB + \measuredangle XAD = 0^\circ,$$

so let O' be the center of $(WXYZ)$. Since $\overline{AD} \parallel \overline{WY}$ and $\overline{BC} \parallel \overline{XZ}$ by Reim's theorem, $OO_1O'O_2$ is a parallelogram.

To finish the problem, note that projecting O_1, O_2, and O' onto $\overline{XY}$ gives the midpoints of $\overline{PX}$, $\overline{QY}$, and $\overline{XY}$. Since $OO_1O'O_2$ is a parallelogram, projecting O onto $\overline{XY}$ must give the midpoint of $\overline{PQ}$, so $OP = OQ$.

¶ Fourth solution using congruent circles (from contestants) Let the angle bisector of $\measuredangle BOD$ meet $\overline{XY}$ at K.

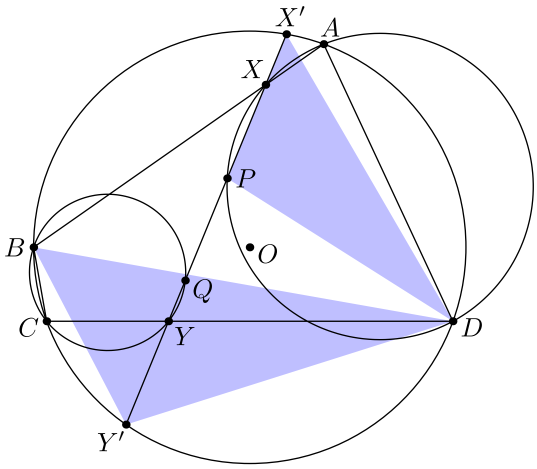


Then $(BQOK)$ is cyclic because $\measuredangle KOD = \measuredangle BAD = \measuredangle KPD$, and $(DOPK)$ is cyclic similarly. By symmetry over KO, these circles have the same radius r, so

$$OP = 2r \sin \angle OKP = 2r \sin \angle OKQ = OQ$$

by the Law of Sines.

¶ Fifth solution by ratio calculation (from contestants) Let $\overline{XY}$ meet $(ABCD)$ at X' and Y'.



Since $\measuredangle Y'BD = \measuredangle PX'D$ and $\measuredangle BY'D = \measuredangle BAD = \measuredangle X'PD$,

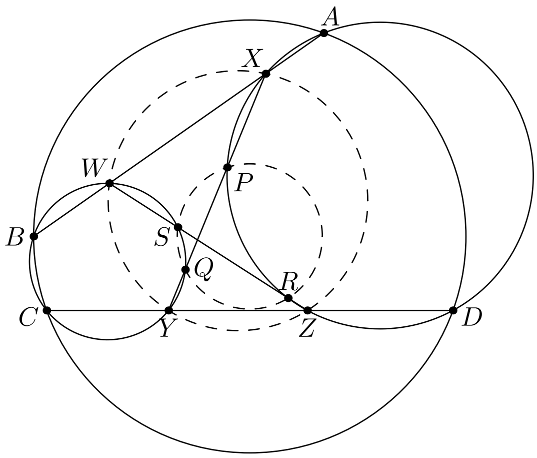
$$\triangle BY'D \sim \triangle XP'D \implies PX' = BY' \cdot \frac{DX'}{BD}.$$

Similarly,

$$\triangle BX'D \sim \triangle BQY' \implies QY' = DX' \cdot \frac{BY'}{BD}.$$

Thus $PX' = QY'$, which gives $OP = OQ$.

¶ Sixth solution using radical axis (from author) Without loss of generality, assume $\overline{AD} \nparallel \overline{BC}$, as this case holds by continuity. Let (BCY) meet $\overline{AB}$ again at W, let (ADX) meet $\overline{CD}$ again at Z, and let $\overline{WZ}$ meet (ADX) and (BCY) again at R and S.



Note that $(WXYZ)$ is cyclic since

$$\measuredangle XWY + \measuredangle YZX = \measuredangle YWB + \measuredangle XZD = \measuredangle YCB + \measuredangle XAD = 0^\circ$$

and $(PQRS)$ is cyclic since

$$\measuredangle PQS = \measuredangle YQS = \measuredangle YWS = \measuredangle PXZ = \measuredangle PRZ = \measuredangle SRP.$$

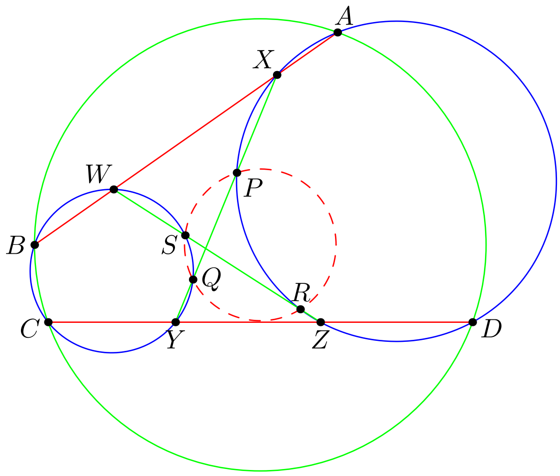
Additionally, $\overline{AD} \parallel \overline{PR}$ since

$$\measuredangle DAX + \measuredangle AXP + \measuredangle XPR = \measuredangle YWX + \measuredangle WXY + \measuredangle XYW = 0^\circ,$$

and $\overline{BC} \parallel \overline{SQ}$ similarly.

Lastly, $(ABCD)$ and $(PQRS)$ are concentric; if not, using the radical axis theorem twice shows that their radical axis must be parallel to both $\overline{AD}$ and $\overline{BC}$, contradiction.

¶ Seventh solution using Cayley-Bacharach (author) Define points W, Z, R, S as in the previous solution.



The quartics $(ADXZ) \cup (BCWY)$ and $\overline{XY} \cup \overline{WZ} \cup (ABCD)$ meet at the 16 points

$$A, B, C, D, W, X, Y, Z, P, Q, R, S, I, I, J, J,$$

where I and J are the circular points at infinity. Since $\overline{AB} \cup \overline{CD} \cup (PQR)$ contains the 13 points

$$A, B, C, D, P, Q, R, W, X, Y, Z, I, J,$$

it must contain S, I, and J as well, by quartic Cayley-Bacharach. Thus, $(PQRS)$ is cyclic and intersects $(ABCD)$ at I, I, J, and J, implying that the two circles are concentric, as desired.

> **Remark** (Author comments)**.** Holden says he came up with this problem via the Cayley-Bacharach solution, by trying to get two quartics to intersect.

§1.2 TSTST 2021/2, proposed by Merlijn Staps

Available online at https://aops.com/community/p23586635.

Problem statement

Let $a_1 < a_2 < a_3 < a_4 < \cdots$ be an infinite sequence of real numbers in the interval $(0, 1)$. Show that there exists a number that occurs exactly once in the sequence

$$\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \dots.$$

We present three solutions.

¶ Solution 1 (Merlijn Staps) We argue by contradiction, so suppose that for each λ for which the set $S_\lambda = \{k : a_k/k = \lambda\}$ is non-empty, it contains at least two elements. Note that S_λ is always a finite set because $a_k = k\lambda$ implies $k < 1/\lambda$.

Write m_λ and M_λ for the smallest and largest element of S_λ, respectively, and define $T_\lambda = \{m_\lambda, m_\lambda + 1, \dots, M_\lambda\}$ as the smallest set of consecutive positive integers that contains S_λ. Then all T_λ are sets of at least two consecutive positive integers, and moreover the T_λ cover $\mathbb{N}$. Additionally, each positive integer is covered finitely many times because there are only finitely many possible values of m_λ smaller than any fixed integer.

Recall that if three intervals have a point in common then one of them is contained in the union of the other two. Thus, if any positive integer is covered more than twice by the sets T_λ, we may throw out one set while maintaining the property that the T_λ cover $\mathbb{N}$. By using the fact that each positive integer is covered finitely many times, we can apply this process so that each positive integer is eventually covered at most twice.

Let Λ denote the set of the λ-values for which T_λ remains in our collection of sets; then $\bigcup_{\lambda \in \Lambda} T_\lambda = \mathbb{N}$ and each positive integer is contained in at most two sets T_λ.

We now obtain

$$\sum_{\lambda \in \Lambda} \sum_{k \in T_\lambda} (a_{k+1} - a_k) \le 2 \sum_{k \ge 1} (a_{k+1} - a_k) \le 2.$$

On the other hand, because $a_{m_\lambda} = \lambda m_\lambda$ and $a_{M_\lambda} = \lambda M_\lambda$, we have

$$\begin{aligned} 2 \sum_{k \in T_\lambda} (a_{k+1} - a_k) \ge 2 \sum_{m_\lambda \le k < M_\lambda} (a_{k+1} - a_k) = 2(a_{M_\lambda} - a_{m_\lambda}) = 2(M_\lambda - m_\lambda)\lambda \\ = 2(M_\lambda - m_\lambda) \cdot \frac{a_{m_\lambda}}{m_\lambda} \ge (M_\lambda - m_\lambda + 1) \cdot \frac{a_1}{m_\lambda} \ge a_1 \cdot \sum_{k \in T_\lambda} \frac{1}{k}. \end{aligned}$$

Combining this with our first estimate, and using the fact that the T_λ cover $\mathbb{N}$, we obtain

$$4 \ge 2 \sum_{\lambda \in \Lambda} \sum_{k \in T_\lambda} (a_{k+1} - a_k) \ge a_1 \sum_{\lambda \in \Lambda} \sum_{k \in T_\lambda} \frac{1}{k} \ge a_1 \sum_{k \ge 1} \frac{1}{k},$$

contradicting the fact that the harmonic series diverges.

¶ Solution 2 (Sanjana Das) Assume for the sake of contradiction that no number appears exactly once in the sequence. For every $i < j$ with $a_i/i = a_j/j$, draw an edge between i and j, so every i has an edge (and being connected by an edge is a transitive property). Call i *good* if it has an edge with some $j > i$.

First, each i has finite degree – otherwise

$$\frac{a_{x_1}}{x_1} = \frac{a_{x_2}}{x_2} = \cdots$$

for an infinite increasing sequence of positive integers x_i, but then the a_{x_i} are unbounded.

Now we use the following process to build a sequence of indices whose a_i we can lower-bound:

- Start at $x_1 = 1$, which is good.
- If we're currently at good index x_i, then let s_i be the largest positive integer such that x_i has an edge to $x_i + s_i$. (This exists because the degrees are finite.)
- Let t_i be the smallest positive integer for which $x_i + s_i + t_i$ is good, and let this be x_{i+1}. This exists because if all numbers $k \le x \le 2k$ are bad, they must each connect to some number less than k (if two connect to each other, the smaller one is good), but then two connect to the same number, and therefore to each other – this is the idea we will use later to bound the t_i as well.

Then $x_i = 1 + s_1 + t_1 + \cdots + s_{i-1} + t_{i-1}$, and we have

$$a_{x_{i+1}} > a_{x_i+s_i} = \frac{x_i + s_i}{x_i} a_{x_i} = \frac{1 + (s_1 + \cdots + s_{i-1} + s_i) + (t_1 + \cdots + t_{i-1})}{1 + (s_1 + \cdots + s_{i-1}) + (t_1 + \cdots + t_{i-1})} a_{x_i}.$$

This means

$$c_n := \frac{a_{x_n}}{a_1} > \prod_{i=1}^{n-1} \frac{1 + (s_1 + \cdots + s_{i-1} + s_i) + (t_1 + \cdots + t_{i-1})}{1 + (s_1 + \cdots + s_{i-1}) + (t_1 + \cdots + t_{i-1})}.$$

Lemma

$t_1 + \cdots + t_n \le s_1 + \cdots + s_n$ for each n.

Proof. Consider $1 \le i \le n$. Note that for every i, the $t_i - 1$ integers strictly between $x_i + s_i$ and $x_i + s_i + t_i$ are all bad, so each such index x must have an edge to some $y < x$.

First we claim that if $x \in (x_i + s_i, x_i + s_i + t_i)$, then x cannot have an edge to x_j for any $j \le i$. This is because $x > x_i + s_i \ge x_j + s_j$, contradicting the fact that $x_j + s_j$ is the largest neighbor of x_j.

This also means x doesn't have an edge to $x_j + s_j$ for any $j \le i$, since if it did, it would have an edge to x_j.

Second, no two bad values of x can have an edge, since then the smaller one is good. This also means no two bad x can have an edge to the same y.

Then each of the $\sum (t_i - 1)$ values in the intervals $(x_i + s_i, x_i + s_i + t_i)$ for $1 \le i \le n$ must have an edge to an unique y in one of the intervals $(x_i, x_i + s_i)$ (not necessarily with the same i). Therefore

$$\sum (t_i - 1) \le \sum (s_i - 1) \implies \sum t_i \le \sum s_i. \qquad \square$$

Now note that if $a > b$, then $\frac{a+x}{b+x} = 1 + \frac{a-b}{b+x}$ is decreasing in x. This means

$$c_n > \prod_{i=1}^{n-1} \frac{1 + 2s_1 + \cdots + 2s_{i-1} + s_i}{1 + 2s_1 + \cdots + 2s_{i-1}} > \prod_{i=1}^{n-1} \frac{1 + 2s_1 + \cdots + 2s_{i-1} + 2s_i}{1 + 2s_1 + \cdots + 2s_{i-1} + s_i},$$

By multiplying both products, we have a telescoping product, which results in

$$c_n^2 \geq 1 + 2s_1 + \cdots + 2s_n + 2s_{n+1}.$$

The right hand side is unbounded since the s_i are positive integers, while $c_n = a_{x_n}/a_1 < 1/a_1$ is bounded, contradiction.

¶ Solution 3 (Gopal Goel) Suppose for sake of contradiction that the problem is false. Call an index i a *pin* if

$$\frac{a_j}{j} = \frac{a_i}{i} \implies j \geq i.$$

Lemma

There exists k such that if we have $\frac{a_i}{i} = \frac{a_j}{j}$ with $j > i \geq k$, then $j \leq 1.1i$.

Proof. Note that for any i, there are only finitely many j with $\frac{a_j}{j} = \frac{a_i}{i}$, otherwise $a_j = \frac{ja_i}{i}$ is unbounded. Thus it suffices to find k for which $j \leq 1.1i$ when $j > i \geq k$.

Suppose no such k exists. Then, take a pair $j_1 > i_1$ such that $\frac{a_{j_1}}{j_1} = \frac{a_{i_1}}{i_1}$ and $j_1 > 1.1i_1$, or $a_{j_1} > 1.1a_{i_1}$. Now, since $k = j_1$ can't work, there exists a pair $j_2 > i_2 \geq i_1$ such that $\frac{a_{j_2}}{j_2} = \frac{a_{i_2}}{i_2}$ and $j_2 > 1.1i_2$, or $a_{j_2} > 1.1a_{i_2}$. Continuing in this fashion, we see that

$$a_{j_\ell} > 1.1a_{i_\ell} > 1.1a_{j_{\ell-1}},$$

so we have that $a_{j_\ell} > 1.1^\ell a_{i_1}$. Taking $\ell > \log_{1.1}(1/a_1)$ gives the desired contradiction. □

Lemma

For $N > k^2$, there are at most $0.8N$ pins in $[\sqrt{N}, N)$.

Proof. By the first lemma, we see that the number of pins in $[\sqrt{N}, \frac{N}{1.1})$ is at most the number of non-pins in $[\sqrt{N}, N)$. Therefore, if the number of pins in $[\sqrt{N}, N)$ is p, then we have

$$p - N\left(1 - \frac{1}{1.1}\right) \leq N - p,$$

so $p \leq 0.8N$, as desired. □

We say that i is the pin of j if it is the smallest index such that $\frac{a_i}{i} = \frac{a_j}{j}$. The pin of j is always a pin.

Given an index i, let $f(i)$ denote the largest index less than i that is not a pin (we leave the function undefined when no such index exists, as we are only interested in the behavior for large i). Then f is weakly increasing and unbounded by the first lemma. Let N_0 be a positive integer such that $f(\sqrt{N_0}) > k$.

Take any $N > N_0$ such that N is not a pin. Let $b_0 = N$, and b_1 be the pin of b_0. Recursively define $b_{2i} = f(b_{2i-1})$, and b_{2i+1} to be the pin of b_{2i}.

Let ℓ be the largest odd index such that $b_\ell \geq \sqrt{N}$. We first show that $b_\ell \leq 100\sqrt{N}$. Since $N > N_0$, we have $b_{\ell+1} > k$. By the choice of ℓ we have $b_{\ell+2} < \sqrt{N}$, so

$$b_{\ell+1} < 1.1 b_{\ell+2} < 1.1\sqrt{N}$$

by the first lemma. We see that all the indices from $b_{\ell+1}+1$ to b_ℓ must be pins, so we have at least $b_\ell - 1.1\sqrt{N}$ pins in $[\sqrt{N}, b_\ell)$. Combined with the second lemma, this shows that $b_\ell \leq 100\sqrt{N}$.

Now, we have that $a_{b_{2i}} = \frac{b_{2i}}{b_{2i+1}} a_{b_{2i+1}}$ and $a_{b_{2i+1}} > a_{b_{2i+2}}$, so combining gives us

$$\frac{a_{b_0}}{a_{b_\ell}} > \frac{b_0}{b_1}\frac{b_2}{b_3}\cdots\frac{b_{\ell-1}}{b_\ell}.$$

Note that there are at least

$$(b_1 - b_2) + (b_3 - b_4) + \cdots + (b_{\ell-2} - b_{\ell-1})$$

pins in $[\sqrt{N}, N)$, so by the second lemma, that sum is at most $0.8N$. Thus,

$$\begin{aligned}(b_0 - b_1) + (b_2 - b_3) + \cdots + (b_{\ell-1} - b_\ell) &= b_0 - [(b_1 - b_2) + \cdots + (b_{\ell-2} - b_{\ell-1})] - b_\ell \\ &\geq 0.2N - 100\sqrt{N}.\end{aligned}$$

Then

$$\begin{aligned}\frac{b_0}{b_1}\frac{b_2}{b_3}\cdots\frac{b_{\ell-1}}{b_\ell} &\geq 1 + \frac{b_0 - b_1}{b_1} + \cdots + \frac{b_{\ell-1} - b_\ell}{b_\ell} \\ &> 1 + \frac{b_0 - b_1}{b_0} + \cdots + \frac{b_{\ell-1} - b_\ell}{b_0} \\ &\geq 1 + \frac{0.2N - 100\sqrt{N}}{N},\end{aligned}$$

which is at least 1.01 if N_0 is large enough. Thus, we see that

$$a_N > 1.01 a_{b_\ell} \geq 1.01 a_{\lfloor\sqrt{N}\rfloor}$$

if $N > N_0$ is not a pin. Since there are arbitrarily large non-pins, this implies that the sequence (a_n) is unbounded, which is the desired contradiction.

§1.3 TSTST 2021/3, proposed by Merlijn Staps

Available online at https://aops.com/community/p23586679.

Problem statement

Find all positive integers $k > 1$ for which there exists a positive integer n such that $\binom{n}{k}$ is divisible by n, and $\binom{n}{m}$ is not divisible by n for $2 \le m < k$.

Such an n exists for any k.

First, suppose k is prime. We choose $n = (k-1)!$. For $m < k$, it follows from $m! \mid n$ that

$$\begin{aligned}(n-1)(n-2)\cdots(n-m+1) &\equiv (-1)(-2)\cdots(-m+1) \\ &\equiv (-1)^{m-1}(m-1)! \\ &\not\equiv 0 \mod m!.\end{aligned}$$

We see that $\binom{n}{m}$ is not a multiple of m. For $m = k$, note that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$. Because $k \nmid n$, we must have $k \mid \binom{n-1}{k-1}$, and it follows that $n \mid \binom{n}{k}$.

Now suppose k is composite. We will choose n to satisfy a number of congruence relations. For each prime $p \le k$, let

$$t_p = \nu_p(\operatorname{lcm}(1, 2, \dots, k-1)) = \max(\nu_p(1), \nu_p(2), \dots, \nu_p(k-1))$$

and choose $k_p \in \{1, 2, \dots, k-1\}$ as large as possible such that $\nu_p(k_p) = t_p$. We now require

$$\begin{aligned} n &\equiv 0 \mod p^{t_p+1} && \text{if } p \nmid k; && (1)\\ \nu_p(n-k_p) &= t_p + \nu_p(k) && \text{if } p \mid k. && (2)\end{aligned}$$

for all $p \le k$. From the Chinese Remainder Theorem, we know that an n exists that satisfies (1) and (2) (indeed, a sufficient condition for (2) is the congruence $n \equiv k_p + p^{t_p+\nu_p(k)} \mod p^{t_p+\nu_p(k)+1}$). We show that this n has the required property.

We first will compute $\nu_p(n-i)$ for primes $p < k$ and $1 \le i < k$.

- If $p \nmid k$, then we have $\nu_p(i), \nu_p(n-i) \le t_p$ and $\nu_p(n) > t_p$, so $\nu_p(n-i) = \nu_p(i)$;
- If $p \mid k$ and $i \ne k_p$, then we have $\nu_p(i), \nu_p(n-i) \le t_p$ and $\nu_p(n) \ge t_p$, so again $\nu_p(n-i) = \nu_p(i)$;
- If $p \mid k$ and $i = k_p$, then we have $\nu_p(n-i) = \nu_p(i) + \nu_p(k)$ by (2).

We conclude that $\nu_p(n-i) = \nu_p(i)$ always holds, except when $i = k_p$, when we have $\nu_p(n-i) = \nu_p(i) + \nu_p(k)$ (this formula holds irrespective of whether $p \mid k$ or $p \nmid k$).

We can now show that $\binom{n}{k}$ is divisible by n, which amounts to showing that $k!$ divides $(n-1)(n-2)\cdots(n-k+1)$. Indeed, for each prime $p \le k$ we have

$$\begin{aligned}\nu_p\left((n-1)(n-2)\dots(n-k+1)\right) &= \nu_p(n-k_p) + \sum_{i<k, i\ne k_p} \nu_p(n-i) \\ &= \nu_p(k_p) + \nu_p(k) + \sum_{i<k, i\ne k_p} \nu_p(i)\end{aligned}$$

$$= \sum_{i=1}^{k} \nu_p(i) = \nu_p(k!),$$

so it follows that $(n-1)(n-2)\cdots(n-k+1)$ is a multiple of $k!$.

Finally, let $1 < m < k$. We will show that n does *not* divide $\binom{n}{m}$, which amounts to showing that $m!$ does not divide $(n-1)(n-2)\cdots(n-m+1)$. First, suppose that m has a prime divisor q that does not divide k. Then we have

$$\begin{aligned} \nu_q\left((n-1)(n-2)\dots(n-m+1)\right) &= \sum_{i=1}^{m-1} \nu_q(n-i) \\ &= \sum_{i=1}^{m-1} \nu_q(i) \\ &= \nu_q((m-1)!) < \nu_q(m!), \end{aligned}$$

as desired. Therefore, suppose that m is only divisible by primes that divide k. If there is such a prime p with $\nu_p(m) > \nu_p(k)$, then it follows that

$$\begin{aligned} \nu_p\left((n-1)(n-2)\dots(n-m+1)\right) &= \nu_p(k) + \sum_{i=1}^{m-1} \nu_p(i) \\ &< \nu_p(m) + \sum_{i=1}^{m-1} \nu_p(i) \\ &= \nu_p(m!), \end{aligned}$$

so $m!$ cannot divide $(n-1)(n-2)\dots(n-m+1)$. On the other hand, suppose that $\nu_p(m) \le \nu_p(k)$ for all $p \mid k$, which would mean that $m \mid k$ and hence $m \le \frac{k}{2}$. Consider a prime p dividing m. We have $k_p \ge \frac{k}{2}$, because otherwise $2k_p$ could have been used instead of k_p. It follows that $m \le \frac{k}{2} \le k_p$. Therefore, we obtain

$$\begin{aligned} \nu_p\left((n-1)(n-2)\dots(n-m+1)\right) &= \sum_{i=1}^{m-1} \nu_p(n-i) \\ &= \sum_{i=1}^{m-1} \nu_p(i) \\ &= \nu_p((m-1)!) < \nu_p(m!), \end{aligned}$$

showing that $(n-1)(n-2)\cdots(n-m+1)$ is not divisible by $m!$. This shows that $\binom{n}{m}$ is not divisible by n for $m < k$, and hence n does have the required property.

§2 Solutions to Day 2

§2.1 TSTST 2021/4, proposed by Holden Mui

Available online at https://aops.com/community/p23864177.

Problem statement

Let a and b be positive integers. Suppose that there are infinitely many pairs of positive integers (m, n) for which $m^2 + an + b$ and $n^2 + am + b$ are both perfect squares. Prove that a divides $2b$.

Treating a and b as fixed, we are given that there are infinitely many quadrpules (m, n, r, s) which satisfy the system

$$m^2 + an + b = (m + r)^2$$
$$n^2 + am + b = (n + s)^2$$

We say that (r, s) is *exceptional* if there exists infinitely many (m, n) that satisfy.

Claim — If (r, s) is exceptional, then either

- $0 < r < a/2$, and $0 < s < \frac{1}{4}a^2$; or
- $0 < s < a/2$, and $0 < r < \frac{1}{4}a^2$; or
- $r^2 + s^2 \le 2b$.

In particular, finitely many pairs (r, s) can be exceptional.

Proof. Sum the two equations to get:

$$r^2 + s^2 - 2b = (a - 2r)m + (a - 2s)n. \qquad (\dagger)$$

If $0 < r < a/2$, then the idea is to use the bound $an + b \ge 2m + 1$ to get $m \le \frac{an+b-1}{2}$. Consequently,

$$(n + s)^2 = n^2 + am + b \le n^2 + a \cdot \frac{an + b - 1}{2} + b$$

For this to hold for infinitely many integers n, we need $2s \le \frac{a^2}{2}$, by comparing coefficients.

A similar case occurs when $0 < s < a/2$.

If $\min(r, s) > a/2$, then $(\dagger)$ forces $r^2 + s^2 \le 2b$, giving the last case. □

Hence, there exists some particular pair (r, s) for which there are infinitely many solutions (m, n). Simplifying the system gives

$$an = 2rm + r^2 - b$$
$$2sn = am + b - s^2$$

Since the system is linear, for there to be infinitely many solutions (m, n) the system must be dependent. This gives

$$\frac{a}{2s} = \frac{2r}{a} = \frac{r^2 - b}{b - s^2}$$

so $a = 2\sqrt{rs}$ and $b = \frac{s^2\sqrt{r}+r^2\sqrt{s}}{\sqrt{r}+\sqrt{s}}$. Since rs must be square, we can reparametrize as $r = kx^2$, $s = ky^2$, and $\gcd(x, y) = 1$. This gives

$$\begin{aligned} a &= 2kxy \\ b &= k^2xy(x^2 - xy + y^2). \end{aligned}$$

Thus, $a \mid 2b$, as desired.

§2.2 TSTST 2021/5, proposed by Vincent Huang

Available online at https://aops.com/community/p23864182.

Problem statement

Let T be a tree on n vertices with exactly k leaves. Suppose that there exists a subset of at least $\frac{n+k-1}{2}$ vertices of T, no two of which are adjacent. Show that the longest path in T contains an even number of edges.

The longest path in T must go between two leaves. The solutions presented here will solve the problem by showing that in the unique 2-coloring of T, all leaves are the same color.

¶ Solution 1 (Ankan Bhattacharya, Jeffery Li)

Lemma

If S is an independent set of T, then

$$\sum_{v \in S} \deg(v) \le n-1.$$

Equality holds if and only if S is one of the two components of the unique 2-coloring of T.

Proof. Each edge of T is incident to at most one vertex of S, so we obtain the inequality by counting how many vertices of S each edge is incident to. For equality to hold, each edge is incident to exactly one vertex of S, which implies the 2-coloring. □

We are given that there exists an independent set of at least $\frac{n+k-1}{2}$ vertices. By greedily choosing vertices of smallest degree, the sum of the degrees of these vertices is at least

$$k + 2 \cdot \frac{n-k-1}{2} = n-1.$$

Thus equality holds everywhere, which implies that the independent set contains every leaf and is one of the components of the 2-coloring.

¶ Solution 2 (Andrew Gu)

Lemma

The vertices of T can be partitioned into $k-1$ paths (i.e. the induced subgraph on each set of vertices is a path) such that all edges of T which are not part of a path are incident to an endpoint of a path.

Proof. Repeatedly trim the tree by taking a leaf and removing the longest path containing that leaf such that the remaining graph is still a tree. □

Now given a path of a vertices, at most $\frac{a+1}{2}$ of those vertices can be in an independent set of T. By the lemma, T can be partitioned into $k-1$ paths of $a_1, \dots, a_{k-1}$ vertices, so the maximum size of an independent set of T is

$$\sum \frac{a_i+1}{2} = \frac{n+k-1}{2}.$$

For equality to hold, each path in the partition must have an odd number of vertices, and has a unique 2-coloring in red and blue where the endpoints are red. The unique independent set of T of size $\frac{n+k-1}{2}$ is then the set of red vertices. By the lemma, the edges of T which are not part of a path connect an endpoint of a path (which is colored red) to another vertex (which must be blue, because the red vertices are independent). Thus the coloring of the paths extends to the unique 2-coloring of T. The leaves of T are endpoints of paths, so they are all red.

§2.3 TSTST 2021/6, proposed by Nikolai Beluhov

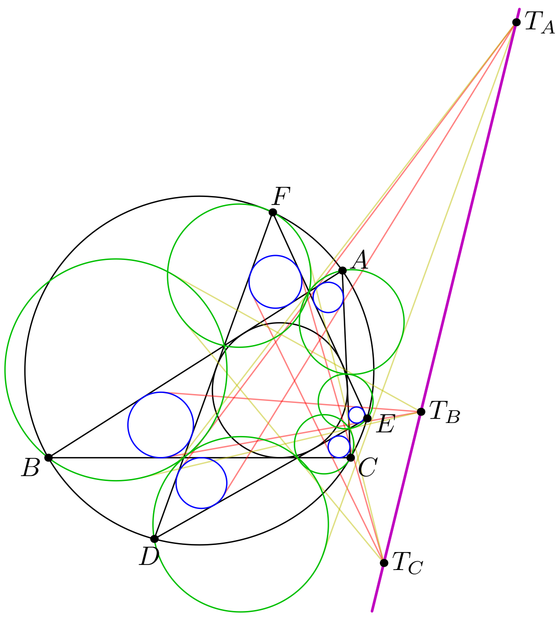
Available online at https://aops.com/community/p23864189.

Problem statement

Triangles ABC and DEF share circumcircle Ω and incircle ω so that points A, F, B, D, C, and E occur in this order along Ω. Let Δ_A be the triangle formed by lines AB, AC, and EF, and define triangles $\Delta_B, \Delta_C, \dots, \Delta_F$ similarly. Furthermore, let Ω_A and ω_A be the circumcircle and incircle of triangle Δ_A, respectively, and define circles $\Omega_B, \omega_B, \dots, \Omega_F, \omega_F$ similarly.

(a) Prove that the two common external tangents to circles Ω_A and Ω_D and the two common external tangents to circles ω_A and ω_D are either concurrent or pairwise parallel.

(b) Suppose that these four lines meet at point T_A, and define points T_B and T_C similarly. Prove that points T_A, T_B, and T_C are collinear.



Let I and r be the center and radius of ω, and let O and R be the center and radius of Ω. Let O_A and I_A be the circumcenter and incenter of triangle Δ_A, and define O_B, $I_B, \dots, I_F$ similarly. Let ω touch EF at A_1, and define B_1, $C_1, \dots, F_1$ similarly.

¶ Part (a) All solutions to part (a) will prove the stronger claim that

$$(\Omega_A \cup \omega_A) \sim (\Omega_D \cup \omega_D).$$

The four lines will concur at the homothetic center of these figures (possibly at infinity).

Solution 1 (author) Let the second tangent to ω parallel to EF meet lines AB and AC at P and Q, respectively, and let the second tangent to ω parallel to BC meet lines DE and DF at R and S, respectively. Furthermore, let ω touch PQ and RS at U and V, respectively.

Let h be inversion with respect to ω. Then h maps A, B, and C onto the midpoints of the sides of triangle $D_1E_1F_1$. So h maps k onto the Euler circle of triangle $D_1E_1F_1$.

Similarly, h maps k onto the Euler circle of triangle $A_1B_1C_1$. Therefore, triangles $A_1B_1C_1$ and $D_1E_1F_1$ share a common nine-point circle γ. Let K be its center; its radius equals $\frac{1}{2}r$.

Let H be the reflection of I in K. Then H is the common orthocenter of triangles $A_1B_1C_1$ and $D_1E_1F_1$.

Let γ_U of center K_U and radius $\frac{1}{2}r$ be the Euler circle of triangle UE_1F_1, and let γ_V of center K_V and radius $\frac{1}{2}r$ be the Euler circle of triangle VB_1C_1.

Let H_U be the orthocenter of triangle UE_1F_1. Since quadrilateral $D_1E_1F_1U$ is cyclic, vectors $\overrightarrow{HH_U}$ and $\overrightarrow{D_1U}$ are equal. Consequently, $\overrightarrow{KK_U} = \frac{1}{2}\overrightarrow{D_1U}$. Similarly, $\overrightarrow{KK_V} = \frac{1}{2}\overrightarrow{A_1V}$.

Since both of A_1U and D_1V are diameters in ω, vectors $\overrightarrow{D_1U}$ and $\overrightarrow{A_1V}$ are equal. Therefore, K_U and K_V coincide, and so do γ_U and γ_V.

As above, h maps γ_U onto the circumcircle of triangle APQ and γ_V onto the circumcircle of triangle DRS. Therefore, triangles APQ and DRS are inscribed inside the same circle Ω_{AD}.

Since EF and PQ are parallel, triangles Δ_A and APQ are homothetic, and so are figures $\Omega_A \cup \omega_A$ and $\Omega_{AD} \cup \omega$. Consequently, we have

$$(\Omega_A \cup \omega_A) \sim (\Omega_{AD} \cup \omega) \sim (\Omega_D \cup \omega_D),$$

which solves part (a).

Solution 2 (Michael Ren) As in the previous solution, let the second tangent to ω parallel to EF meet AB and AC at P and Q, respectively. Let (APQ) meet Ω again at D', so that D' is the Miquel point of $\{AB, AC, BC, PQ\}$. Since the quadrilateral formed by these lines has incircle ω, it is classical that $D'I$ bisects $\angle PD'C$ and $BD'Q$ (e.g. by DDIT).

Let ℓ be the tangent to Ω at D' and $D'I$ meet Ω again at M. We have

$$\measuredangle(\ell, D'B) = \measuredangle D'CB = \measuredangle D'QP = \measuredangle(D'Q, EF).$$

Therefore $D'I$ also bisects the angle between ℓ and the line parallel to EF through D'. This means that M is one of the arc midpoints of EF. Additionally, D' lies on arc BC not containing A, so $D' = D$.

Similarly, letting the second tangent to ω parallel to BC meet DE and DF again at R and S, we have $ADRS$ cyclic.

Lemma

There exists a circle Ω_{AD} tangent to Ω_A and Ω_D at A and D, respectively.

Proof. (This step is due to Ankan Bhattacharya.) It is equivalent to have $\measuredangle OAO_A = \measuredangle O_DDO$. Taking isogonals with respect to the shared angle of $\triangle ABC$ and Δ_A, we see that

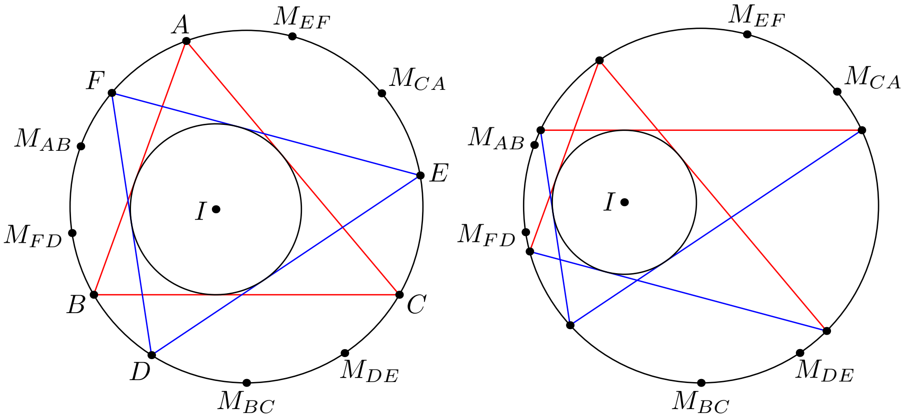
$$\measuredangle OAO_A = \measuredangle(\perp EF, \perp BC) = \measuredangle(EF, BC).$$

(Here, $\perp EF$ means the direction perpendicular to EF.) By symmetry, this is equal to $\measuredangle O_DDO$. □

The circle $(ADPQ)$ passes through A and D, and is tangent to Ω_A by homothety. Therefore it coincides with Ω_{AD}, as does $(ADRS)$. Like the previous solution, we finish with

$$(\Omega_A \cup \omega_A) \sim (\Omega_{AD} \cup \omega) \sim (\Omega_D \cup \omega_D).$$

Solution 3 (Andrew Gu) Construct triangles homothetic to Δ_A and Δ_D (with positive ratio) which have the same circumcircle; it suffices to show that these copies have the same incircle as well. Let O be the center of this common circumcircle, which we take to be the origin, and M_{XY} denote the point on the circle such that the tangent at that point is parallel to line XY (from the two possible choices, we make the choice that corresponds to the arc midpoint on Ω, e.g. M_{AB} should correspond to the arc midpoint on the internal angle bisector of ACB). The left diagram below shows the original triangles ABC and DEF, while the right diagram shows the triangles homothetic to Δ_A and Δ_D.



Using the fact that the incenter is the orthocenter of the arc midpoints, the incenter of Δ_A in this reference frame is $M_{AB} + M_{AC} - M_{EF}$ and the incenter of Δ_D in this reference frame is $M_{DE} + M_{DF} - M_{BC}$. Since ABC and DEF share a common incenter, we have

$$M_{AB} + M_{BC} + M_{CA} = M_{DE} + M_{EF} + M_{FD}.$$

Thus the copies of Δ_A and Δ_D have the same incenter, and therefore the same incircle as well (Euler's theorem determines the inradius).

¶ Part (b) We present several solutions for this part of the problem. Solutions 3 and 4 require solving part (a) first, while the others do not. Solutions 1, 4, and 5 define T_A solely as the exsimilicenter of ω_A and ω_D, whereas solutions 2 and 3 define T_A solely as the exsimilicenter of Ω_A and Ω_D.

Solution 1 (author) By Monge's theorem applied to ω, ω_A, and ω_D, points A, D, and T_A are collinear. Therefore, $T_A = AD \cap I_A I_D$.

Let p be pole-and-polar correspondence with respect to ω. Then p maps A onto line E_1F_1 and D onto line B_1C_1. Consequently, p maps line AD onto $X_A = B_1C_1 \cap E_1F_1$.

Furthermore, p maps the line that bisects the angle formed by lines AB and EF and does not contain I onto the midpoint of segment A_1F_1. Similarly, p maps the line that bisects the angle formed by lines AC and EF and does not contain I onto the midpoint of segment A_1E_1. So p maps I_A onto the midline of triangle $A_1E_1F_1$ opposite A_1. Similarly, p maps I_D onto the midline of triangle $D_1B_1C_1$ opposite D_1. Consequently, p maps line

I_AI_D onto the intersection point Y_A of this pair of midlines, and p maps T_A onto line X_AY_A.

As in the solution to part (a), let H be the common orthocenter of triangles $A_1B_1C_1$ and $D_1E_1F_1$. Let H_A be the foot of the altitude from A_1 in triangle $A_1B_1C_1$ and let H_D be the foot of the altitude from D_1 in triangle $D_1E_1F_1$. Furthermore, let $L_A = HA_1 \cap E_1F_1$ and $L_D = HD_1 \cap B_1C_1$.

Since the reflection of H in line B_1C_1 lies on ω, $A_1H \cdot HH_A$ equals half the power of H with respect to ω. Similarly, $D_1H \cdot HH_D$ equals half the power of H with respect to ω.

Then $A_1H \cdot HH_A = D_1H \cdot HH_D$ and $A_1HH_D \sim D_1HH_A$. Since $\angle HH_DL_A = 90^\circ = \angle HH_AL_D$, figures $A_1HH_DL_A$ and $D_1HH_AL_D$ are similar as well. Consequently,

$$\frac{HL_A}{L_AA_1} = \frac{HL_D}{L_DD_1} = s$$

as a signed ratio.

Let the line through A_1 parallel to E_1F_1 and the line through D_1 parallel to B_1C_1 meet at Z_A. Then $HX_A/X_AZ_A = s$ and Y_A is the midpoint of segment X_AZ_A. Therefore, H lies on line X_AY_A. This means that T_A lies on the polar of H with respect to ω, and by symmetry so do T_B and T_C.

Solution 2 (author) As in the first solution to part (a), let h be inversion with respect to ω, let γ of center K be the common Euler circle of triangles $A_1B_1C_1$ and $D_1E_1F_1$, and let H be their common orthocenter.

Again as in the solution to part (a), h maps Ω_A onto the nine-point circle γ_A of triangle $A_1E_1F_1$ and Ω_D onto the nine-point circle γ_D of triangle $D_1B_1C_1$.

Let K_A and K_D be the centers of γ_A and γ_D, respectively, and let ℓ_A be the perpendicular bisector of segment K_AK_D. Since γ_A and γ_D are congruent (both of them are of radius $\frac{1}{2}r$), they are reflections of each other across ℓ_A.

Inversion h maps the two common external tangents of Ω_A and Ω_D onto the two circles α and β through I that are tangent to both of γ_A and γ_D in the same way. (That is, either internally to both or externally to both.) Consequently, α and β are reflections of each other in ℓ_A and so their second point of intersection S_A, which h maps T_A onto, is the reflection of I in ℓ_A.

Define ℓ_B, ℓ_C, S_B, and S_C similarly. Then S_B is the reflection of I in ℓ_B and S_C is the reflection of I in ℓ_C.

As in the solution to part (a), $\overrightarrow{KK_A} = \frac{1}{2}\overrightarrow{D_1A_1}$ and $\overrightarrow{KK_D} = \frac{1}{2}\overrightarrow{A_1D_1}$. Consequently, K is the midpoint of segment K_AK_D and so K lies on ℓ_A. Similarly, K lies on ℓ_B and ℓ_C.

Therefore, all four points I, S_A, S_B, and S_C lie on the circle of center K that contains I. (This is also the circle on diameter IH.) Since points S_A, S_B, and S_C are concyclic with I, their images T_A, T_B, and T_C under h are collinear, and the solution is complete.

Solution 3 (Ankan Bhattacharya) From either of the first two solutions to part (a), we know that there is a circle Ω_{AD} passing through A and D which is (internally) tangent to Ω_A and Ω_D. By Monge's theorem applied to Ω_A, Ω_D, and Ω_{AD}, it follows that A, D, and T_A are collinear.

The inversion at T_A swapping Ω_A with Ω_D also swaps A with D because T_A lies on AD and A is not homologous to D. Let Ω_A and Ω_D meet Ω again at L_A and L_D. Since ADL_AL_D is cyclic, the same inversion at T_A also swaps L_A and L_D, so $T_A = AD \cap L_AL_D$.

By Taiwan TST 2014, L_A and L_D are the tangency points of the A-mixtilinear and D-mixtilinear incircles, respectively, with Ω. Therefore $AL_A \cap DL_D$ is the exsimilicenter X of Ω and ω. Then T_A, T_B, and T_C all lie on the polar of X with respect to Ω.

Solution 4 (Andrew Gu) We show that T_A lies on the radical axis of the point circle at I and Ω, which will solve the problem. Let I_A and I_D be the centers of ω_A and ω_D respectively. By the Monge's theorem applied to ω, ω_A, and ω_D, points A, D, and T_A are collinear. Additionally, these other triples are collinear: $(A, I_A, I), (D, I_D, I), (I_A, I_D, T)$. By Menelaus's theorem, we have

$$\frac{T_A D}{T_A A} = \frac{I_A I}{I_A A} \cdot \frac{I_D D}{I_D I}.$$

If s is the length of the side opposite A in Δ_A, then we compute

$$\begin{aligned}\frac{I_A I}{I_A A} &= \frac{s/\cos(A/2)}{r_A/\sin(A/2)} \\ &= \frac{2R_A \sin(A)\sin(A/2)}{\cos(A/2)} \\ &= \frac{4R_A \sin^2(A/2)}{r_A} \\ &= \frac{4R_A r^2}{r_A AI^2}.\end{aligned}$$

From part (a), we know that $\frac{R_A}{r_A} = \frac{R_D}{r_D}$. Therefore, doing a similar calculation for $\frac{I_D D}{I_D I}$, we get

$$\begin{aligned}\frac{T_A D}{T_A A} &= \frac{I_A I}{I_A A} \cdot \frac{I_D D}{I_D I} \\ &= \frac{4R_A r^2}{r_A AI^2} \cdot \frac{r_D DI^2}{4R_D r^2} \\ &= \frac{DI^2}{AI^2}.\end{aligned}$$

Thus T_A is the point where the tangent to (AID) at I meets AD and $T_A I^2 = T_A A \cdot T_A D$. This shows what we claimed at the start.

Solution 5 (Ankit Bisain) As in the previous solution, it suffices to show that $\frac{I_A I}{AI_A} \cdot \frac{DI_D}{I_D I} = \frac{DI^2}{AI^2}$. Let AI and DI meet Ω again at M and N, respectively. Let ℓ be the line parallel to BC and tangent to ω but different from BC. Then

$$\frac{DI_D}{I_D I} = \frac{d(D, BC)}{d(BC, \ell)} = \frac{DB \cdot DC/2R}{2r} = \frac{MI^2 - MD^2}{4Rr}.$$

Since $IDM \sim IAN$, we have

$$\frac{DI_D}{I_D I} \cdot \frac{I_A I}{AI_A} = \frac{MI^2 - MD^2}{NI^2 - NA^2} = \frac{DI^2}{AI^2},$$

as desired.

> **Remark** (Author comments on generalization of part (b) with a circumscribed hexagram). Let triangles ABC and DEF be circumscribed about the same circle ω so that they form a hexagram. However, we do not require anymore that they are inscribed in the same circle.
>
> Define circles Ω_A, $\omega_A, \dots, \omega_F$ as in the problem. Let T_A^{Circ} be the intersection point of the two common external tangents to circles Ω_A and Ω_D, and define points T_B^{Circ} and T_C^{Circ} similarly. Also let T_A^{In} be the intersection point of the two common external tangents to circles ω_A and ω_D, and define points T_B^{In} and T_C^{In} similarly.
>
> Then points T_A^{Circ}, T_B^{Circ}, and T_C^{Circ} are collinear and points T_A^{In}, T_B^{In}, and T_C^{In} are also collinear.

The second solution to part (b) of the problem works also for the circumcircles part of the generalisation. To see that segments K_AK_D, K_BK_E, and K_CK_F still have a common midpoint, let M be the centroid of points A, B, C, D, E, and F. Then the midpoint of segment K_AK_D divides segment OM externally in ratio $3:1$, and so do the other two midpoints as well.

For the incircles part of the generalisation, we start out as in the first solution to part (b) of the problem, and eventually we reduce everything to the following:

Let points A_1, B_1, C_1, D_1, E_1, and F_1 lie on circle ω. Let lines B_1C_1 and E_1F_1 meet at point X_A, let the line through A_1 parallel to B_1C_1 and the line through D_1 parallel to E_1F_1 meet at point Z_A, and define points X_B, Z_B, X_C, and Z_C similarly. Then lines X_AZ_A, X_BZ_B, and X_CZ_C are concurrent.

Take ω as the unit circle and assign complex numbers u, v, w, x, y, and z to points A_1, F_1, B_1, D_1, C_1, and E_1, respectively, so that when we permute u, v, w, x, y, and z cyclically the configuration remains unchanged. Then by standard complex bash formulas we obtain that each two out of our three lines meet at φ/ψ, where

$$\varphi = \sum_{\text{Cyc}} u^2vw(wx - wy + xy)(y - z)$$

and

$$\psi = -u^2w^2y^2 - v^2x^2z^2 - 4uvwxyz + \sum_{\text{Cyc}} u^2(vwxy - vwxz + vwyz - vxyz + wxyz).$$

(But the calculations were too difficult for me to do by hand, so I used SymPy.)

Remark (Author comments on generalization of part (b) with an inscribed hexagram)**.** Let triangles ABC and DEF be inscribed inside the same circle Ω so that they form a hexagram. However, we do not require anymore that they are circumscribed about the same circle.

Define points T_A^{Circ}, T_B^{Circ}, ..., T_C^{In} as in the previous remark. It looks like once again points T_A^{Circ}, T_B^{Circ}, and T_C^{Circ} are collinear and points T_A^{In}, T_B^{In}, and T_C^{In} are also collinear. However, I do not have proofs of these claims.

Remark (Further generalization from Andrew Gu)**.** Let ABC and DEF be triangles which share an inconic, or equivalently share a circumconic. Define points $T_A^{\text{Circ}}, T_B^{\text{Circ}}, \dots, T_C^{\text{In}}$ as in the previous remarks. Then it is conjectured that points T_A^{Circ}, T_B^{Circ}, and T_C^{Circ} are collinear and points T_A^{In}, T_B^{In}, and T_C^{In} are also collinear. (Note that extraversion may be required depending on the configuration of points, e.g. excircles instead of incircles.) Additionally, it appears that the insimilicenters of the circumcircles lie on a line perpendicular to the line through T_A^{Circ}, T_B^{Circ}, and T_C^{Circ}.

§3 Solutions to Day 3

§3.1 TSTST 2021/7, proposed by Ankit Bisain, Holden Mui

Available online at https://aops.com/community/p24130213.

Problem statement

Let M be a finite set of lattice points and n be a positive integer. A *mine-avoiding path* is a path of lattice points with length n, beginning at $(0,0)$ and ending at a point on the line $x+y=n$, that does not contain any point in M. Prove that if there exists a mine-avoiding path, then there exist at least $2^{n-|M|}$ mine-avoiding paths.

We present two approaches.

¶ Solution 1 We prove the statement by induction on n. We use $n=0$ as a base case, where the statement follows from $1 \geq 2^{-|M|}$. For the inductive step, let $n>0$. There exists at least one mine-avoiding path, which must pass through either $(0,1)$ or $(1,0)$. We consider two cases:

Case 1: there exist mine-avoiding paths starting at both $(1,0)$ and $(0,1)$.

By the inductive hypothesis, there are at least $2^{n-1-|M|}$ mine-avoiding paths starting from each of $(1,0)$ and $(0,1)$. Then the total number of mine-avoiding paths is at least $2^{n-1-|M|}+2^{n-1-|M|}=2^{n-|M|}$.

Case 2: only one of $(1,0)$ and $(0,1)$ is on a mine-avoiding path.

Without loss of generality, suppose no mine-avoiding path starts at $(0,1)$. Then some element of M must be of the form $(0,k)$ for $1 \leq k \leq n$ in order to block the path along the y-axis. This element can be ignored for any mine-avoiding path starting at $(1,0)$. By the inductive hypothesis, there are at least $2^{n-1-(|M|-1)}=2^{n-|M|}$ mine-avoiding paths.

This completes the induction step, which solves the problem.

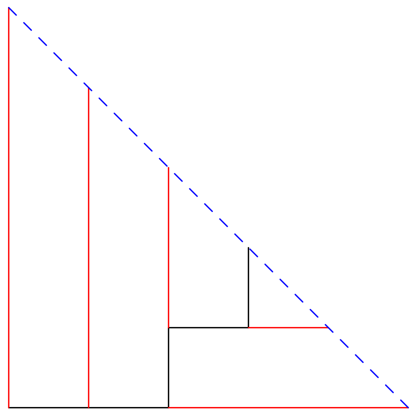
¶ Solution 2

Lemma

If $|M|<n$, there is more than one mine-avoiding path.

Proof. Let $P_0, P_1, \ldots, P_n$ be a mine-avoiding path. Set $P_i=(x_i, y_i)$. For $0 \leq i<n$, define a path Q_i as follows:

- Make the first $i+1$ points $P_0, P_1, \ldots, P_i$.
- If $P_i \to P_{i+1}$ is one unit up, go right until $(n-y_i, y_i)$.
- If $P_i \to P_{i+1}$ is one unit right, go up until $(x_i, n-x_i)$.

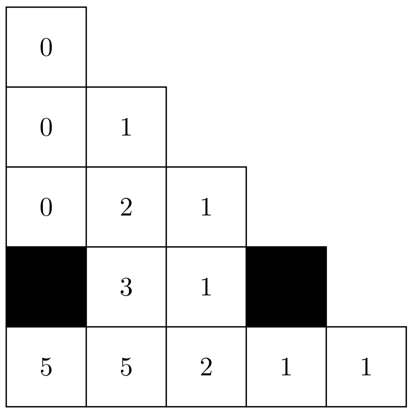


The diagram above is an example for $n = 5$ with the new segments formed by the Q_i in red, and the line $x + y = n$ in blue.

By definition, M has less than n points, none of which are in the original path. Since all Q_i only intersect in the original path, each mine is in at most one of $Q_0, Q_1, \ldots, Q_{n-1}$. By the Pigeonhole Principle, one of the Q_i is mine-avoiding. $\square$

Now, consider the following process:

- Start at $(0, 0)$.
- If there is only one choice of next step that is part of a mine-avoiding path, make that choice.
- Repeat the above until at a point with two possible steps that are part of mine-avoiding paths.
- Add a mine to the choice of next step with more mine-avoiding paths through it. If both have the same number of mine-avoiding paths through them, add a mine arbitrarily.



For instance, consider the above diagram for $n = 4$. Lattice points are replaced with squares. Mines are black squares and each non-mine is labelled with the number of mine-avoiding paths passing through it. This process would start at $(0, 0)$, go to $(1, 0)$, then place a mine at $(1, 1)$.

This path increases the size of M by one, and reduces the number of mine-avoiding paths to a nonzero number at most half of the original. Repeat this process until there is only one path left. By our lemma, the number of mines must be at least n by the end of the process, so the process was iterated at least $n - |M|$ times. By the halving property, there must have been at least $2^{n-|M|}$ mine-avoiding paths before the process, as desired.

§3.2 TSTST 2021/8, proposed by Fedir Yudin

Available online at https://aops.com/community/p24130228.

Problem statement

Let ABC be a scalene triangle. Points A_1, B_1 and C_1 are chosen on segments BC, CA, and AB, respectively, such that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar. Let A_2 be the unique point on line B_1C_1 such that $AA_2 = A_1A_2$. Points B_2 and C_2 are defined similarly. Prove that $\triangle A_2B_2C_2$ and $\triangle ABC$ are similar.

We give three solutions.

¶ Solution 1 (author) We'll use the following lemma.

Lemma

Suppose that $PQRS$ is a convex quadrilateral with $\angle P = \angle R$. Then there is a point T on QS such that $\angle QPT = \angle SRP$, $\angle TRQ = \angle RPS$, and $PT = RT$.

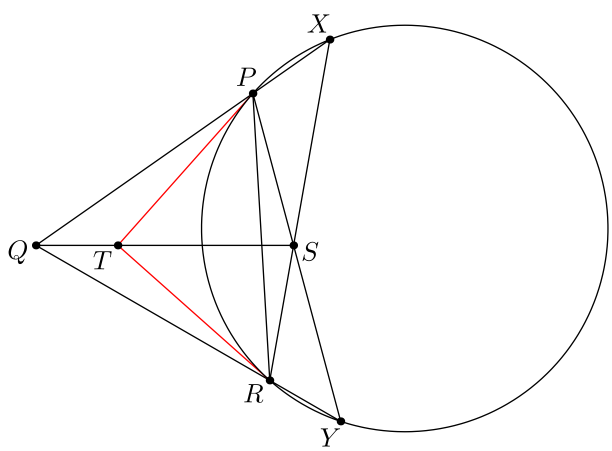
Before proving the lemma, we will show how it solves the problem. The lemma applied for the quadrilateral $AB_1A_1C_1$ with $\angle A = \angle A_1$ shows that $\angle B_1A_1A_2 = \angle C_1AA_1$. This implies that the point A_2 in $\triangle A_1B_1C_1$ corresponds to the point A_1 in $\triangle ABC$. Then $\triangle A_2B_2C_2 \sim \triangle A_1B_1C_1 \sim \triangle ABC$, as desired.

Additionally, $PT = RT$ is a corollary of the angle conditions because

$$\measuredangle PRT = \measuredangle SRQ - \measuredangle TRQ - \measuredangle SRP = \measuredangle QPS - \measuredangle RPS - \measuredangle QPT = \measuredangle TPR.$$

Therefore we only need to prove the angle conditions.

Proof 1 of lemma Denote $X = PQ \cap RS$ and $Y = PS \cap RQ$. Note that $\angle XPY = \angle XRY$, so $PRXY$ is cyclic. Let T be the point of intersection of tangents to this circle at P and R. By Pascal's theorem for the degenerate hexagon $PPXRRY$, we have $T \in QS$ (alternatively, Q, S, and T are collinear on the pole of $PR \cap XY$ with respect to the circle). Also, $\measuredangle QPT = \measuredangle XRP = \measuredangle SRP$ and similarly $\measuredangle TRQ = \measuredangle RPY = \measuredangle RPS$, so we're done.



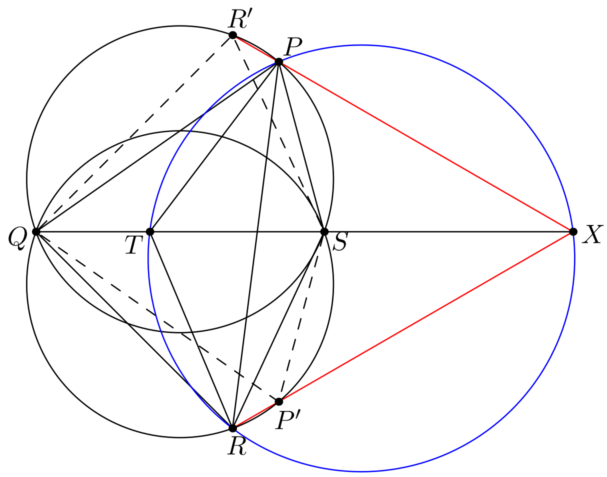
Proof 2 of lemma Let P' and R' be the reflections of P and R in QS. Note that PR' and RP' intersect at a point X on QS. Let T be the second intersection of the circumcircle of $\triangle PRX$ with QS. Note that

$$\begin{aligned}\measuredangle PXT &= \measuredangle R'PQ + \measuredangle PQS \\ &= \measuredangle R'SQ + \measuredangle PQS \\ &= \measuredangle QSR + \measuredangle PQS \\ &= \measuredangle (PQ, SR) \\ &= \measuredangle QPR + \measuredangle PRS.\end{aligned}$$

This means that

$$\begin{aligned}\measuredangle QPT &= \measuredangle QPR - \measuredangle TPR \\ &= \measuredangle QPR - \measuredangle TXR \\ &= \measuredangle QPR - \measuredangle PXT \\ &= \measuredangle QPR - \measuredangle QPR - \measuredangle PRS \\ &= \measuredangle SRP.\end{aligned}$$

Similarly, $\measuredangle QRT = \measuredangle SPR$, so we're done.



Proof 3 of lemma Let T be the point on QS such that $\angle QPT = \angle SRP$. Then we have

$$\frac{QT}{TS} = \frac{\sin QPT \cdot PT / \sin PQT}{\sin TPS \cdot PT / \sin TSP} = \frac{PQ / \sin PRQ}{PS / \sin SRP} = \frac{R(\triangle PQR)}{R(\triangle PRS)},$$

which is symmetric in P and R, so we're done.

¶ Solution 2 (Ankan Bhattacharya) We prove the main claim $\frac{B_1A_2}{A_2C_1} = \frac{BA_1}{A_1C}$.

Let $\triangle A_0B_0C_0$ be the medial triangle of $\triangle ABC$. In addition, let A_1' be the reflection of A_1 over $\overline{B_1C_1}$, and let X be the point satisfying $\triangle XBC \overset{-}{\sim} \triangle AB_1C_1$, so that we have a compound similarity

$$\triangle ABC \sqcup X \overset{-}{\sim} \triangle A_1'B_1C_1 \sqcup A.$$

Finally, let O_A be the circumcenter of $\triangle A_1'B_1C_1$, and let A_2^* be the point on $\overline{B_1C_1}$ satisfying $\frac{B_1A_2^*}{A_2^*C_1} = \frac{BA_1}{A_1C}$.

Recall that O is the Miquel point of $\triangle A_1B_1C_1$, as well as its orthocenter.

Claim — $\overline{AA_1'} \parallel \overline{BC}$.

Proof. We need to verify that the foot from A_1 to $\overline{B_1C_1}$ lies on the A-midline. This follows from the fact that O is both the Miquel point and the orthocenter. □

Claim — $\overline{AX} \parallel \overline{B_1C_1}$.

Proof. From the compound similarity,

$$\measuredangle(\overline{BC}, \overline{AX}) = \measuredangle(\overline{AA_1'}, \overline{B_1C_1}).$$

As $\overline{AA_1'} \parallel \overline{BC}$, we obtain $\overline{AX} \parallel \overline{B_1C_1}$. □

Claim — $\overline{AX} \perp \overline{A_1O}$.

Proof. Because O is the orthocenter of $\triangle A_1B_1C_1$. □

Claim — $\overline{AA_1'} \perp \overline{A_2^*O_A}$.

Proof. Follows from $\overline{AX} \perp \overline{A_1O}$ after the similarity

$$\triangle ABC \sqcup X \mathrel{\overset{-}{\sim}} \triangle A_1'B_1C_1 \sqcup A.$$

□

Claim — $AA_2^* = A_1'A_2$.

Proof. Since $\measuredangle C_1AB_1 = \measuredangle C_1A_1'B_1$, $AO_A = A_1'O_A$, so $\overline{AA_1'} \perp \overline{A_2^*O_A}$ implies $AA_2^* = A_1'A_2^*$. □

Finally, $A_1'A_2^* = A_1A_2^*$ by reflections, so $AA_2^* = A_1A_2^*$, and $A_2^* = A_2$.

§3.3 TSTST 2021/9, proposed by Victor Wang

Available online at https://aops.com/community/p24130243.

Problem statement

Let $q = p^r$ for a prime number p and positive integer r. Let $\zeta = e^{\frac{2\pi i}{q}}$. Find the least positive integer n such that

$$\sum_{\substack{1 \le k \le q \\ \gcd(k,p)=1}} \frac{1}{(1-\zeta^k)^n}$$

is not an integer. (The sum is over all $1 \le k \le q$ with p not dividing k.)

Let S_q denote the set of primitive qth roots of unity (thus, the sum in question is a sum over S_q).

¶ Solution 1 (author) Let $\zeta_p = e^{2\pi i/p}$ be a fixed primitive pth root of unity. Observe that the given sum is an integer for all $n \le 0$ (e.g. because the sum is an integer symmetric polynomial in the primitive qth roots of unity). By expanding polynomials in the basis $(1-x)^k$, it follows that if the sum in the problem statement is an integer for all $n < n_0$, then

$$\sum_{\omega \in S_q} \frac{f(\omega)}{(1-\omega)^n} \in \mathbb{Z}$$

for all $n < n_0$ and $f \in \mathbb{Z}[x]$, whereas for $n = n_0$ there is some $f \in \mathbb{Z}[x]$ for which the sum is not an integer (e.g. $f = 1$).

Let $z_q = r\phi(q) - q/p = p^{r-1}[r(p-1)-1]$. We claim that the answer is $n = z_q + 1$. We prove this by induction on r. First is the base case $r = 1$.

Lemma

There exist polynomials $u, v \in \mathbb{Z}[x]$ such that $(1-\omega)^{p-1}/p = u(\omega)$ and $p/(1-\omega)^{p-1} = v(\omega)$ for all $\omega \in S_p$.

(What we are saying is that p is $(1-\omega)^{p-1}$ times a *unit* (invertible algebraic integer), namely $v(\omega)$.)

Proof. Note that $p = (1-\omega)\cdots(1-\omega^{p-1})$. Thus we can write

$$\frac{p}{(1-\omega)^{p-1}} = \frac{1-\omega}{1-\omega} \cdot \frac{1-\omega^2}{1-\omega} \cdots \frac{1-\omega^{p-1}}{1-\omega}$$

and take

$$v(x) = \prod_{k=1}^{p-1} \frac{1-x^k}{1-x}.$$

Similarly, the polynomial u is

$$u(x) = \prod_{k=1}^{p-1} \frac{1-x^{k\ell_k}}{1-x^k}$$

where ℓ_k is a multiplicative inverse of k modulo p. $\square$

Now, the main idea: given $g \in \mathbb{Z}[x]$, observe that

$$S = \sum_{\omega \in S_p} (1-\omega) g(\omega)$$

is divisible by $1-\zeta_p^k$ (i.e. it is $1-\zeta_p^k$ times an algebraic integer) for every k coprime to p. By symmetric sums, S is an integer; since S^{p-1} is divisible by $(1-\zeta_p)\cdots(1-\zeta_p^{p-1}) = p$, the integer S must itself be divisible by p. (Alternatively, since $h(x) := (1-x)g(x)$ vanishes at $x = 1$, one can interpret S using a roots of unity filter: $S = p \cdot h([x^0] + [x^p] + \cdots) \equiv 0 \pmod p$.) Now write

$$\mathbb{Z} \ni \frac{S}{p} = \sum_{\omega \in S_p} \frac{(1-\omega)^{p-1}}{p} \frac{g(\omega)}{(1-\omega)^{p-2}} = \sum_{\omega \in S_p} u(\omega) \frac{g(\omega)}{(1-\omega)^{p-2}}.$$

Taking $g = v \cdot (1-x)^k$ for $k \ge 0$, we see that the sum in the problem statement is an integer for any $n \le p-2$.

Finally, we have

$$\sum_{\omega \in S_p} \frac{u(\omega)}{(1-\omega)^{p-1}} = \sum_{\omega \in S_p} \frac{1}{p} = \frac{p-1}{p} \notin \mathbb{Z},$$

so the sum is not an integer for $n = p-1$.

Now let $r \ge 2$ and assume the induction hypothesis for $r-1$.

Lemma

There exist polynomials $U, V \in \mathbb{Z}[x]$ such that $(1-\omega)^p/(1-\omega^p) = U(\omega)$ and $(1-\omega^p)/(1-\omega)^p = V(\omega)$ for all $\omega \in S_q$. (Again, these are units.)

Proof. Similarly to the previous lemma, we write $1-\omega^p = (1-\omega\zeta_p^0)\cdots(1-\omega\zeta_p^{p-1})$. The polynomials U and V are

$$U(x) = \prod_{k=1}^{p-1} \frac{1-x^{(kq/p+1)\ell_k}}{1-x^{kq/p+1}}$$
$$V(x) = \prod_{k=1}^{p-1} \frac{1-x^{kq/p+1}}{1-x}$$

where ℓ_k is a multiplicative inverse of $kq/p+1$ modulo q. $\square$

Corollary

If $\omega \in S_q$, then $(1-\omega)^{\phi(q)}/p$ is a unit.

Proof. Induct on r. For $r = 1$, this is the first lemma. For the inductive step, we are given that $(1-\omega^p)^{\phi(q/p)}/p$ is a unit. By the second lemma, $(1-\omega)^{\phi(q)}/(1-\omega^p)^{\phi(q/p)}$ is also a unit. Multiplying these together yields another unit. $\square$

Thus we have polynomials $A, B \in \mathbb{Z}[x]$ such that

$$A(\omega) = \frac{p}{(1-\omega)^{\phi(q)}} V(\omega)^{z_{q/p}}$$
$$B(\omega) = \frac{(1-\omega)^{\phi(q)}}{p} U(\omega)^{z_{q/p}}$$

for all $\omega \in S_q$.

Given $g \in \mathbb{Z}[x]$, consider the pth roots of unity filter

$$S(x) := \sum_{k=0}^{p-1} g(\zeta_p^k x) = p \cdot h(x^p),$$

where $h \in \mathbb{Z}[x]$. Then

$$ph(\eta) = S(\omega) = \sum_{\omega^p = \eta} g(\omega)$$

for all $\eta \in S_{q/p}$, so

$$\begin{aligned} \frac{h(\eta)}{(1-\eta)^{z_{q/p}}} = \frac{S(\omega)}{p(1-\eta)^{z_{q/p}}} &= \sum_{\omega^p=\eta} \frac{(1-\omega)^{pz_{q/p}}}{(1-\omega^p)^{z_{q/p}}} \frac{g(\omega)}{p(1-\omega)^{pz_{q/p}}} \\ &= \sum_{\omega^p=\eta} U(\omega)^{z_{q/p}} \frac{(1-\omega)^{\phi(q)}}{p} \frac{g(\omega)}{(1-\omega)^{z_q}}. \end{aligned}$$

(Implicit in the last line is $z_q = \phi(q) + pz_{q/p}$.) Since $U(\omega)$ and $(1-\omega)^{\phi(q)}/p$ are units, we can let $g = A \cdot f$ for arbitrary $f \in \mathbb{Z}[x]$, so that the expression in the summation simplifies to $f(\omega)/(1-\omega)^{z_q}$. From this we conclude that for any $f \in \mathbb{Z}[x]$, there exists $h \in \mathbb{Z}[x]$ such that

$$\begin{aligned} \sum_{\omega \in S_q} \frac{f(\omega)}{(1-\omega)^{z_q}} &= \sum_{\eta \in S_{q/p}} \sum_{\omega^p=\eta} \frac{f(\omega)}{(1-\omega)^{z_q}} \\ &= \sum_{\eta \in S_{q/p}} \frac{h(\eta)}{(1-\eta)^{z_{q/p}}}. \end{aligned}$$

By the inductive hypothesis, this is always an integer.

In the other direction, for $\eta \in S_{q/p}$ we have

$$\begin{aligned} \sum_{\omega^p=\eta} \frac{B(\omega)}{(1-\omega)^{1+z_q}} &= \sum_{\omega^p=\eta} \frac{1}{p(1-\omega^p)^{z_{q/p}}(1-\omega)} \\ &= \frac{1}{p(1-\eta)^{z_{q/p}}} \sum_{\omega^p=\eta} \frac{1}{1-\omega} \\ &= \frac{1}{p(1-\eta)^{z_{q/p}}} \left[\frac{px^{p-1}}{x^p-\eta} \right]_{x=1} \\ &= \frac{1}{(1-\eta)^{1+z_{q/p}}}. \end{aligned}$$

Summing over all $\eta \in S_{q/p}$, we conclude by the inductive hypothesis that

$$\sum_{\omega \in S_q} \frac{B(\omega)}{(1-\omega)^{1+z_q}} = \sum_{\eta \in S_{q/p}} \frac{1}{(1-\eta)^{1+z_{q/p}}}$$

is not an integer. This completes the solution.

¶ Solution 2 (Nikolai Beluhov) Suppose that the complex numbers $\frac{1}{1-\omega}$ for $\omega \in S_q$ are the roots of

$$P(x) = x^d - c_1 x^{d-1} + c_2 x^{d-2} - \cdots \pm c_d,$$

so that c_k is their k-th elementary symmetric polynomial and $d = \phi(q) = (p-1)p^{r-1}$. Additionally denote

$$S_n = \sum_{\omega \in S_q} \frac{1}{(1-\omega)^n}.$$

Then, by Newton's identities,

$$\begin{aligned} S_1 &= c_1, \\ S_2 &= c_1 S_1 - 2c_2, \\ S_3 &= c_1 S_2 - c_2 S_1 + 3c_3, \end{aligned}$$

and so on. The general pattern when $n \le d$ is

$$S_n = \left[\sum_{j=1}^{n-1} (-1)^{j+1} c_j S_{n-j}\right] + (-1)^{n+1} n c_n.$$

After that, when $n > d$, the pattern changes to

$$S_n = \sum_{j=1}^{d} (-1)^{j+1} c_j S_{n-j}.$$

Lemma

All of the c_i are integers except for c_d. Furthermore, c_d is $1/p$ times an integer.

Proof. The qth cyclotomic polynomial is

$$\Phi_q(x) = 1 + x^{p^{r-1}} + x^{2p^{r-1}} + \cdots + x^{(p-1)p^{r-1}}.$$

The polynomial

$$Q(x) = 1 + (1+x)^{p^{r-1}} + (1+x)^{2p^{r-1}} + \cdots + (1+x)^{(p-1)p^{r-1}}$$

has roots $\omega - 1$ for $\omega \in S_q$, so it is equal to $p(-x)^d P(-1/x)$ by comparing constant coefficients. Comparing the remaining coefficients, we find that c_n is $1/p$ times the x^n coefficient of Q.

Since $(x+y)^p \equiv x^p + y^p \pmod p$, we conclude that, modulo p,

$$\begin{aligned} Q(x) &\equiv 1 + \left(1 + x^{p^{r-1}}\right) + \left(1 + x^{p^{r-1}}\right)^2 + \cdots + \left(1 + x^{p^{r-1}}\right)^{p-1} \\ &\equiv \left[\left(1 + x^{p^{r-1}}\right)^p - 1\right] / x^{p^{r-1}}. \end{aligned}$$

Since $\binom{p}{j}$ is a multiple of p when $0 < j < p$, it follows that all coefficients of $Q(x)$ are multiples of p save for the leading one. Therefore, c_n is an integer when $n < d$, while c_d is $1/p$ times an integer. □

By the recurrences above, S_n is an integer for $n < d$. When $r = 1$, we get that dc_d is not an integer, so S_d is not an integer, either. Thus the answer for $r = 1$ is $n = p - 1$.

Suppose now that $r \ge 2$. Then dc_d does become an integer, so S_d is an integer as well.

Lemma

For all n with $1 \le n \le d$, we have $\nu_p(nc_n) \ge r-2$. Furthermore, the smallest n such that $\nu_p(nc_n) = r-2$ is $d - p^{r-1} + 1$.

Proof. The value of nc_n is $1/p$ times the coefficient of x^{n-1} in the derivative $Q'(x)$. This derivative is

$$p^{r-1}(1+x)^{p^{r-1}-1}\left[\sum_{k=1}^{p-1} k(1+x)^{(k-1)p^{r-1}}\right].$$

What we want to prove reduces to showing that all coefficients of the polynomial in the square brackets are multiples of p except for the leading one.

Using the same trick $(x+y)^p \equiv x^p + y^p \pmod p$ as before and also writing w for $x^{p^{r-1}}$, modulo p the polynomial in the square brackets becomes

$$1 + 2(1+w) + 3(1+w)^2 + \cdots + (p-1)(1+w)^{p-2}.$$

This is the derivative of

$$1 + (1+w) + (1+w)^2 + \cdots + (1+w)^{p-1} = [(1+w)^p - 1]/w$$

and so, since $\binom{p}{j}$ is a multiple of p when $0 < j < p$, we are done. □

Finally, we finish the problem with the following claim.

Claim — Let $m = d - p^{r-1}$. Then for all $k \ge 0$ and $1 \le j \le d$, we have

$$\nu_p(S_{kd+m+1}) = r-2-k$$
$$\nu_p(S_{kd+m+j}) \ge r-2-k.$$

Proof. First, $S_1, S_2, \dots, S_m$ are all divisible by p^{r-1} by Newton's identities and the second lemma. Then $\nu_p(S_{m+1}) = r-2$ because

$$\nu_p((m+1)c_{m+1}) = r-2,$$

and all other terms in the recurrence relation are divisible by p^{r-1}. We can similarly check that $\nu_p(S_n) \ge r-2$ for $m+1 \le n \le d$. Newton's identities combined with the first lemma now imply the following for $n > d$:

- If $\nu_p(S_{n-j}) \ge \ell$ for all $1 \le j \le d$ and $\nu_p(S_{n-d}) \ge \ell+1$, then $\nu_p(S_n) \ge \ell$.
- If $\nu_p(S_{n-j}) \ge \ell$ for all $1 \le j \le d$ and $\nu_p(S_{n-d}) = \ell$, then $\nu_p(S_n) = \ell - 1$.

Together, these prove the claim by induction. □

By the claim, the smallest n for which $\nu_p(S_n) < 0$ (equivalent to S_n not being an integer, by the recurrences) is

$$n = (r-1)d + m + 1 = ((p-1)r - 1)p^{r-1} + 1.$$

Remark. The original proposal was the following more general version:

> Let n be an integer with prime power factorization $q_1 \cdots q_m$. Let S_n denote the set of primitive nth roots of unity. Find all tuples of nonnegative integers $(z_1, \dots, z_m)$ such that
>
> $$\sum_{\omega \in S_n} \frac{f(\omega)}{(1-\omega^{n/q_1})^{z_1} \cdots (1-\omega^{n/q_m})^{z_m}} \in \mathbb{Z}$$
>
> for all polynomials $f \in \mathbb{Z}[x]$.

The maximal z_i are exponents in the prime ideal factorization of the different ideal of the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$.

Remark. Let $F = (x^p - 1)/(x - 1)$ be the minimal polynomial of $\zeta_p = e^{2\pi i/p}$ over $\mathbb{Q}$. A calculation of Euler shows that

$$(\mathbb{Z}[\zeta_p])^* := \{\alpha = g(\zeta_p) \in \mathbb{Q}[\zeta_p] : \sum_{\omega \in S_p} f(\omega)g(\omega) \in \mathbb{Z} \ \forall f \in \mathbb{Z}[x]\} = \frac{1}{F'(\zeta_p)} \cdot \mathbb{Z}[\zeta_p],$$

where

$$F'(\zeta_p) = \frac{p\zeta_p^{p-1} - [1 + \zeta_p + \cdots + \zeta_p^{p-1}]}{1 - \zeta_p} = p(1-\zeta_p)^{-1}\zeta_p^{p-1}$$

is $(1-\zeta_p)^{[p-1]-1} = (1-\zeta_p)^{p-2}$ times a unit of $\mathbb{Z}[\zeta_p]$. Here, $(\mathbb{Z}[\zeta_p])^*$ is the dual lattice of $\mathbb{Z}[\zeta_p]$.

Remark. Let $K = \mathbb{Q}(\omega)$, so (p) factors as $(1-\omega)^{p-1}$ in the ring of integers $\mathcal{O}_K$ (which, for cyclotomic fields, can be shown to be $\mathbb{Z}[\omega]$). In particular, the *ramification index* e of $(1-\omega)$ over p is the exponent, $p-1$. Since $e = p-1$ is not divisible by p, we have so-called *tame ramification*. Now by the ramification theory of Dedekind's different ideal, the exponent z_1 that works when $n = p$ is $e - 1 = p - 2$.

Higher prime powers are more interesting because of wild ramification: p divides $\phi(p^r) = p^{r-1}(p-1)$ if and only if $r > 1$. (This is a similar phenomena to how Hensel's lemma for $x^2 - c$ is more interesting mod powers of 2 than mod odd prime powers.)

Remark. Let $F = (x^q - 1)/(x^{q/p} - 1)$ be the minimal polynomial of $\zeta_q = e^{2\pi i/q}$ over $\mathbb{Q}$. The aforementioned calculation of Euler shows that

$$(\mathbb{Z}[\zeta_q])^* := \{\alpha = g(\zeta_q) \in \mathbb{Q}[\zeta_q] : \sum_{\omega \in S_q} f(\omega)g(\omega) \in \mathbb{Z} \ \forall f \in \mathbb{Z}[x]\} = \frac{1}{F'(\zeta_q)} \cdot \mathbb{Z}[\zeta_q],$$

where the chain rule implies (using the computation from the prime case)

$$F'(\zeta_q) = [p(1-\zeta_p)^{-1}\zeta_p^{p-1}] \cdot \frac{q}{p}\zeta_q^{(q/p)-1} = q(1-\zeta_p)^{-1}\zeta_q^{-1}.$$

is $(1-\zeta_q)^{r\phi(q) - q/p} = (1-\zeta_q)^{z_q}$ times a unit of $\mathbb{Z}[\zeta_q]$.

TSTST 2021 Statistics

Mathematical Olympiad Summer Program

Evan Chen《陳誼廷》

January 28, 2022

§1 Summary of scores for TSTST 2021

N	60	1st Q	14	Max	50
μ	24.80	Median	28	Top 3	49
σ	13.79	3rd Q	36	Top 12	38

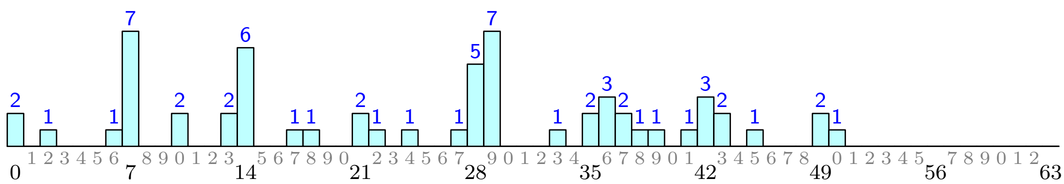
§2 Problem statistics for TSTST 2021

	P1	P2	P3	P4	P5	P6	P7	P8	P9
0	15	37	54	23	16	48	20	23	49
1	4	3	5	1	1	11	0	6	5
2	1	0	0	3	0	0	0	4	1
3	0	0	0	1	1	0	0	0	1
4	0	1	0	0	0	0	0	0	0
5	0	0	0	0	1	0	1	0	1
6	1	4	1	8	1	0	0	0	0
7	39	15	0	24	40	1	39	27	3
Avg	4.75	2.27	0.18	3.77	4.92	0.30	4.63	3.38	0.60
QM	5.71	3.87	0.83	4.98	5.82	1.00	5.68	4.73	1.78
#5+	**40**	**19**	**1**	**32**	**42**	**1**	**40**	**27**	**4**
%5+	%66.7	%31.7	%1.7	%53.3	%70.0	%1.7	%66.7	%45.0	%6.7

§3 Rankings for TSTST 2021

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
63	0	0	0.00%	42	3	9	15.00%	21	2	37	61.67%
62	0	0	0.00%	41	1	10	16.67%	20	0	37	61.67%
61	0	0	0.00%	40	0	10	16.67%	19	0	37	61.67%
60	0	0	0.00%	39	1	11	18.33%	18	1	38	63.33%
59	0	0	0.00%	38	1	12	20.00%	17	1	39	65.00%
58	0	0	0.00%	37	2	14	23.33%	16	0	39	65.00%
57	0	0	0.00%	36	3	17	28.33%	15	0	39	65.00%
56	0	0	0.00%	35	2	19	31.67%	14	6	45	75.00%
55	0	0	0.00%	34	0	19	31.67%	13	2	47	78.33%
54	0	0	0.00%	33	1	20	33.33%	12	0	47	78.33%
53	0	0	0.00%	32	0	20	33.33%	11	0	47	78.33%
52	0	0	0.00%	31	0	20	33.33%	10	2	49	81.67%
51	0	0	0.00%	30	0	20	33.33%	9	0	49	81.67%
50	1	1	1.67%	29	7	27	45.00%	8	0	49	81.67%
49	2	3	5.00%	28	5	32	53.33%	7	7	56	93.33%
48	0	3	5.00%	27	1	33	55.00%	6	1	57	95.00%
47	0	3	5.00%	26	0	33	55.00%	5	0	57	95.00%
46	0	3	5.00%	25	0	33	55.00%	4	0	57	95.00%
45	1	4	6.67%	24	1	34	56.67%	3	0	57	95.00%
44	0	4	6.67%	23	0	34	56.67%	2	1	58	96.67%
43	2	6	10.00%	22	1	35	58.33%	1	0	58	96.67%
								0	2	60	100.00%

§4 Histogram for TSTST 2021



USA TST Selection Test for 64^{th} IMO and 12^{th} EGMO

Pittsburgh, PA

Day I 1:15pm – 5:45pm

Tuesday, June 21, 2022

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 1. Let n be a positive integer. Find the smallest positive integer k such that for any set S of n points in the interior of the unit square, there exists a set of k rectangles such that the following hold:

- The sides of each rectangle are parallel to the sides of the unit square.
- Each point in S is *not* in the interior of any rectangle.
- Each point in the interior of the unit square but *not* in S is in the interior of at least one of the k rectangles.

(The interior of a polygon does not contain its boundary.)

Problem 2. Let ABC be a triangle. Let θ be a fixed angle for which

$$\theta < \frac{1}{2}\min(\angle A, \angle B, \angle C).$$

Points S_A and T_A lie on segment BC such that $\angle BAS_A = \angle T_AAC = \theta$. Let P_A and Q_A be the feet from B and C to $\overline{AS_A}$ and $\overline{AT_A}$ respectively. Then ℓ_A is defined as the perpendicular bisector of $\overline{P_AQ_A}$.

Define ℓ_B and ℓ_C analogously by repeating this construction two more times (using the same value of θ). Prove that ℓ_A, ℓ_B, and ℓ_C are concurrent or all parallel.

Problem 3. Determine all positive integers N for which there exists a strictly increasing sequence of positive integers $s_0 < s_1 < s_2 < \cdots$ satisfying the following properties:

- the sequence $s_1 - s_0,\ s_2 - s_1,\ s_3 - s_2, \ldots$ is periodic; and
- $s_{s_n} - s_{s_{n-1}} \leq N < s_{1+s_n} - s_{s_{n-1}}$ for all positive integers n.

USA TST Selection Test for 64^{th} IMO and 12^{th} EGMO

Pittsburgh, PA

Day II 1:15pm – 5:45pm

Thursday, June 23, 2022

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 4. Let $\mathbb{N}$ denote the set of positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ has the property that for all positive integers m and n, exactly one of the $f(n)$ numbers

$$f(m+1), f(m+2), \ldots, f(m+f(n))$$

is divisible by n. Prove that $f(n) = n$ for infinitely many positive integers n.

Problem 5. Let $A_1, \ldots, A_{2022}$ be the vertices of a regular 2022-gon in the plane. Alice and Bob play a game. Alice secretly chooses a line and colors all points in the plane on one side of the line blue, and all points on the other side of the line red. Points on the line are colored blue, so every point in the plane is either red or blue. (Bob cannot see the colors of the points.)

In each round, Bob chooses a point in the plane (not necessarily among $A_1, \ldots, A_{2022}$) and Alice responds truthfully with the color of that point. What is the smallest number Q for which Bob has a strategy to always determine the colors of points $A_1, \ldots, A_{2022}$ in Q rounds?

Problem 6. Let O and H be the circumcenter and orthocenter, respectively, of an acute scalene triangle ABC. The perpendicular bisector of $\overline{AH}$ intersects $\overline{AB}$ and $\overline{AC}$ at X_A and Y_A respectively. Let K_A denote the intersection of the circumcircles of triangles OX_AY_A and BOC other than O.

Define K_B and K_C analogously by repeating this construction two more times. Prove that K_A, K_B, K_C, and O are concyclic.

USA TST Selection Test for 64^{th} IMO and 12^{th} EGMO

Pittsburgh, PA

Day III 1:15pm – 5:45pm

Saturday, June 25, 2022

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 7. Let $ABCD$ be a parallelogram. Point E lies on segment CD such that

$$2\angle AEB = \angle ADB + \angle ACB,$$

and point F lies on segment BC such that

$$2\angle DFA = \angle DCA + \angle DBA.$$

Let K be the circumcenter of triangle ABD. Prove that $KE = KF$.

Problem 8. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{Z}$ such that

$$\left\lfloor \frac{f(mn)}{n} \right\rfloor = f(m)$$

for all positive integers m, n.

Problem 9. Let $k > 1$ be a fixed positive integer. Prove that if n is a sufficiently large positive integer, there exists a sequence of integers with the following properties:

- Each element of the sequence is between 1 and n, inclusive.
- For any two different contiguous subsequences of the sequence with length between 2 and k inclusive, the multisets of values in those two subsequences is not the same.
- The sequence has length at least $0.499n^2$.

USA TSTST 2022 Solutions

United States of America — TST Selection Test

Andrew Gu and Evan Chen

64th IMO 2023 Japan and 12th EGMO 2023 Slovenia

Contents

§0 Problems

1. Let n be a positive integer. Find the smallest positive integer k such that for any set S of n points in the interior of the unit square, there exists a set of k rectangles such that the following hold:
   - The sides of each rectangle are parallel to the sides of the unit square.
   - Each point in S is *not* in the interior of any rectangle.
   - Each point in the interior of the unit square but *not* in S is in the interior of at least one of the k rectangles.

   (The interior of a polygon does not contain its boundary.)

2. Let ABC be a triangle. Let θ be a fixed angle for which
$$\theta < \frac{1}{2}\min(\angle A, \angle B, \angle C).$$
   Points S_A and T_A lie on segment BC such that $\angle BAS_A = \angle T_AAC = \theta$. Let P_A and Q_A be the feet from B and C to $\overline{AS_A}$ and $\overline{AT_A}$ respectively. Then ℓ_A is defined as the perpendicular bisector of $\overline{P_AQ_A}$.

   Define ℓ_B and ℓ_C analogously by repeating this construction two more times (using the same value of θ). Prove that ℓ_A, ℓ_B, and ℓ_C are concurrent or all parallel.

3. Determine all positive integers N for which there exists a strictly increasing sequence of positive integers $s_0 < s_1 < s_2 < \cdots$ satisfying the following properties:
   - the sequence $s_1 - s_0$, $s_2 - s_1$, $s_3 - s_2$, ... is periodic; and
   - $s_{s_n} - s_{s_{n-1}} \le N < s_{1+s_n} - s_{s_{n-1}}$ for all positive integers n.

4. A function $f\colon \mathbb{N} \to \mathbb{N}$ has the property that for all positive integers m and n, exactly one of the $f(n)$ numbers
$$f(m+1), f(m+2), \dots, f(m+f(n))$$
   is divisible by n. Prove that $f(n) = n$ for infinitely many positive integers n.

5. Let A_1, ..., A_{2022} be the vertices of a regular 2022-gon in the plane. Alice and Bob play a game. Alice secretly chooses a line and colors all points in the plane on one side of the line blue, and all points on the other side of the line red. Points on the line are colored blue, so every point in the plane is either red or blue. (Bob cannot see the colors of the points.)

   In each round, Bob chooses a point in the plane (not necessarily among A_1, ..., A_{2022}) and Alice responds truthfully with the color of that point. What is the smallest number Q for which Bob has a strategy to always determine the colors of points A_1, ..., A_{2022} in Q rounds?

6. Let O and H be the circumcenter and orthocenter, respectively, of an acute scalene triangle ABC. The perpendicular bisector of $\overline{AH}$ intersects $\overline{AB}$ and $\overline{AC}$ at X_A and Y_A respectively. Let K_A denote the intersection of the circumcircles of triangles OX_AY_A and BOC other than O.

   Define K_B and K_C analogously by repeating this construction two more times. Prove that K_A, K_B, K_C, and O are concyclic.

7. Let $ABCD$ be a parallelogram. Point E lies on segment CD such that

$$2\angle AEB = \angle ADB + \angle ACB,$$

and point F lies on segment BC such that

$$2\angle DFA = \angle DCA + \angle DBA.$$

Let K be the circumcenter of triangle ABD. Prove that $KE = KF$.

8. Find all functions $f\colon \mathbb{N} \to \mathbb{Z}$ such that

$$\left\lfloor \frac{f(mn)}{n} \right\rfloor = f(m)$$

for all positive integers m, n.

9. Let $k > 1$ be a fixed positive integer. Prove that if n is a sufficiently large positive integer, there exists a sequence of integers with the following properties:

- Each element of the sequence is between 1 and n, inclusive.
- For any two different contiguous subsequences of the sequence with length between 2 and k inclusive, the multisets of values in those two subsequences is not the same.
- The sequence has length at least $0.499n^2$.

§1 Solutions to Day 1

§1.1 TSTST 2022/1, proposed by Holden Mui

Available online at https://aops.com/community/p25516960.

Problem statement

Let n be a positive integer. Find the smallest positive integer k such that for any set S of n points in the interior of the unit square, there exists a set of k rectangles such that the following hold:

- The sides of each rectangle are parallel to the sides of the unit square.
- Each point in S is *not* in the interior of any rectangle.
- Each point in the interior of the unit square but *not* in S is in the interior of at least one of the k rectangles.

(The interior of a polygon does not contain its boundary.)

We give the author's solution. In terms of n, we wish find the smallest integer k for which $(0,1)^2 \setminus S$ is always a union of k open rectangles for every set $S \subset (0,1)^2$ of size n.

We claim the answer is $k = \boxed{2n+2}$.

The lower bound is given by picking

$$S = \{(s_1, s_1), (s_2, s_2), \dots, (s_n, s_n)\}$$

for some real numbers $0 < s_1 < s_2 < \dots < s_n < 1$. Consider the $4n$ points

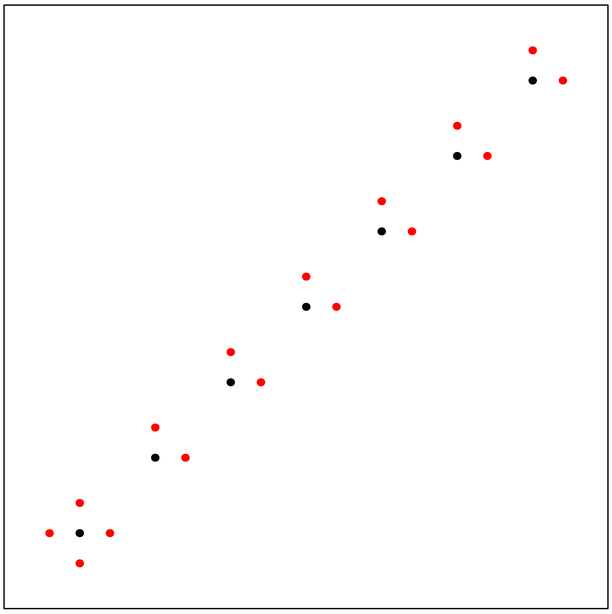
$$S' = S + \{(\varepsilon, 0), (0, \varepsilon), (-\varepsilon, 0), (0, -\varepsilon)\} \subset (0,1)^2$$

for some sufficiently small $\varepsilon > 0$. The four rectangles covering each of

$$(s_1 - \varepsilon, s_1), (s_1, s_1 - \varepsilon), (s_n + \varepsilon, s_n), (s_n, s_n + \varepsilon)$$

cannot cover any other points in S'; all other rectangles can only cover at most 2 points in S', giving a bound of

$$k \geq 4 + \frac{|S'| - 4}{2} = 2n + 2.$$



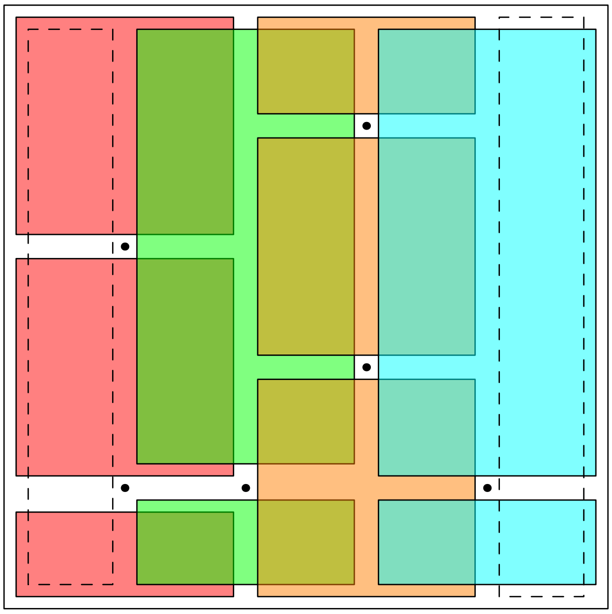
To prove that $2n+2$ rectangles are sufficient, assume that the number of distinct y-coordinates is at least the number of distinct x-coordinates. Let

$$0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1,$$

where $x_1, \ldots, x_m$ are the distinct x-coordinates of points in S, and let Y_i be the set of y-coordinates of points with x-coordinate x_i. For each $1 \le i \le m$, include the $|Y_i|+1$ rectangles

$$(x_{i-1}, x_{i+1}) \times ((0,1) \setminus Y_i)$$

in the union, and also include $(0, x_1) \times (0,1)$ and $(x_m, 1) \times (0,1)$; this uses $m+n+2$ rectangles.



All remaining uncovered points are between pairs of points with the same y-coordinate and adjacent x-coordinates $\{x_i, x_{i+1}\}$. There are at most $n-m$ such pairs by the initial assumption, so covering the points between each pair with

$$(x_i, x_{i+1}) \times (y-\varepsilon, y+\varepsilon)$$

for some sufficiently small $\varepsilon > 0$ gives a total of

$$(m+n+2)+(n-m) = 2n+2$$

rectangles.

§1.2 TSTST 2022/2, proposed by Hongzhou Lin

Available online at https://aops.com/community/p25516988.

Problem statement

Let ABC be a triangle. Let θ be a fixed angle for which

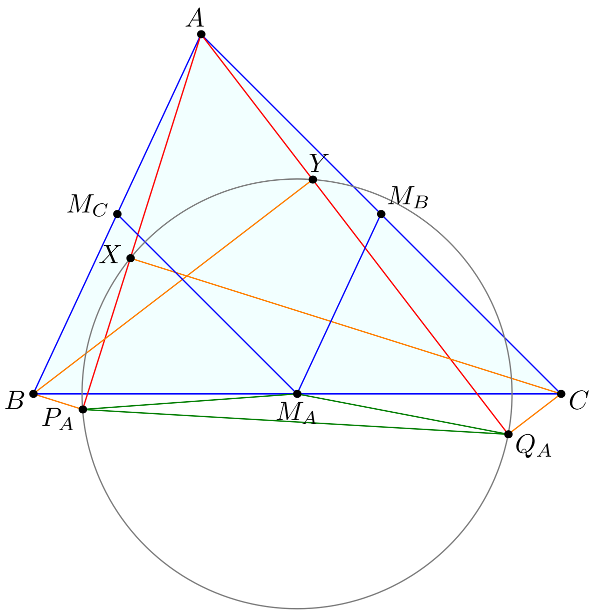
$$\theta < \frac{1}{2}\min(\angle A, \angle B, \angle C).$$

Points S_A and T_A lie on segment BC such that $\angle BAS_A = \angle T_AAC = \theta$. Let P_A and Q_A be the feet from B and C to $\overline{AS_A}$ and $\overline{AT_A}$ respectively. Then ℓ_A is defined as the perpendicular bisector of $\overline{P_AQ_A}$.

Define ℓ_B and ℓ_C analogously by repeating this construction two more times (using the same value of θ). Prove that ℓ_A, ℓ_B, and ℓ_C are concurrent or all parallel.

We discard the points S_A and T_A since they are only there to direct the angles correctly in the problem statement.

¶ First solution, by author Let X be the projection from C to AP_A, Y be the projection from B to AQ_A.



Claim — Line ℓ_A passes through M_A, the midpoint of BC. Also, quadrilateral P_AQ_AYX is cyclic with circumcenter M_A.

Proof. Since

$$AP_A \cdot AX = AB \cdot AC \cdot \cos\theta \cos(\angle A - \theta) = AQ_A \cdot AY,$$

it follows that P_A, Q_A, Y, X are concyclic by power of a point. Moreover, by projection, the perpendicular bisector of P_AX passes through M_A, similar for Q_AY, implying that M_A is the center of P_AQ_AYX. Hence ℓ_A passes through M_A. □

Claim — $\measuredangle(M_AM_C, \ell_A) = \measuredangle YP_AQ_A$.

Proof. Indeed, $\ell_A \perp P_AQ_A$, and $M_AM_C \perp P_AY$ (since $M_AP_A = M_AY$ from $(P_AQ_AY_AX)$ and $M_CP_A = M_CM_A = M_CY$ from the circle with diameter AB). Hence $\measuredangle(M_AM_C, \ell_A) = \measuredangle(P_AY, P_AQ_A) = \measuredangle YP_AQ_A$. □

Therefore,

$$\frac{\sin\angle(M_AM_C, \ell_A)}{\sin\angle(\ell_A, M_AM_B)} = \frac{\sin\angle YP_AQ_A}{\sin\angle P_AQ_AX} = \frac{YQ_A}{XP_A} = \frac{BC\sin(\angle C+\theta)}{BC\sin(\angle B+\theta)} = \frac{\sin(\angle C+\theta)}{\sin(\angle B+\theta)},$$

and we conclude by trig Ceva theorem.

¶ Second solution via Jacobi, by Maxim Li Let D be the foot of the A-altitude. Note that line BC is the external angle bisector of $\angle P_ADQ_A$.

Claim — (DP_AQ_A) passes through the midpoint M_A of BC.

Proof. Perform $\sqrt{bc}$ inversion. Then the intersection of BC and (DP_AQ_A) maps to the second intersection of (ABC) and $(A'P_AQ_A)$, where A' is the antipode to A on (ABC), i.e. the center of spiral similarity from BC to P_AQ_A. Since $BP_A : CQ_A = AB : AC$, we see the center of spiral similarity is the intersection of the A-symmedian with (ABC), which is the image of M_A in the inversion. □

It follows that M_A lies on ℓ_A; we need to identify a second point. We'll use the circumcenter O_A of (DP_AQ_A). The perpendicular bisector of DP_A passes through M_C; indeed, we can easily show the angle it makes with M_CM_A is $90^\circ - \theta - C$, so $\angle O_AM_CM_A = 90 - \theta - C$, and then by analogous angle-chasing we can finish with Jacobi's theorem on $\triangle M_AM_BM_C$.

§1.3 TSTST 2022/3

Available online at https://aops.com/community/p25517008.

Problem statement

Determine all positive integers N for which there exists a strictly increasing sequence of positive integers $s_0 < s_1 < s_2 < \cdots$ satisfying the following properties:

- the sequence $s_1 - s_0$, $s_2 - s_1$, $s_3 - s_2$, ... is periodic; and
- $s_{s_n} - s_{s_{n-1}} \le N < s_{1+s_n} - s_{s_{n-1}}$ for all positive integers n.

¶ Answer All N such that $t^2 \le N < t^2 + t$ for some positive integer t.

¶ Solution 1 (local) If $t^2 \le N < t^2 + t$ then the sequence $s_n = tn + 1$ satisfies both conditions. It remains to show that no other values of N work.

Define $a_n := s_n - s_{n-1}$, and let p be the minimal period of $\{a_n\}$. For each $k \in \mathbb{Z}_{\ge 0}$, let $f(k)$ be the integer such that

$$s_{f(k)} - s_k \le N < s_{f(k)+1} - s_k.$$

Note that $f(s_{n-1}) = s_n$ for all n. Since $\{a_n\}$ is periodic with period p, $f(k+p) = f(k)+p$ for all k, so $k \mapsto f(k) - k$ is periodic with period p. We also note that f is nondecreasing: if $k < k'$ but $f(k') < f(k)$ then

$$N < s_{f(k')+1} - s_{k'} < s_{f(k)} - s_k \le N,$$

which is absurd. We now claim that

$$\max_k (f(k) - k) < p + \min_k (f(k) - k).$$

Indeed, if $f(k') - k' \ge p + f(k) - k$ then we can shift k and k' so that $0 \le k - k' < p$, and it follows that $k \le k' \le f(k') < f(k)$, violating the fact that f is nondecreasing. Therefore $\max_k(f(k) - k) < p + \min_k(f(k) - k)$, so $f(k) - k$ is uniquely determined by its value modulo p. In particular, since $a_n = f(s_{n-1}) - s_{n-1}$, a_n is also uniquely determined by its value modulo p, so $\{a_n \bmod p\}$ also has minimal period p.

Now work in $\mathbb{Z}/p\mathbb{Z}$ and consider the sequence $s_0, f(s_0), f(f(s_0)), \dots$. This sequence must be eventually periodic; suppose it has minimal period p', which must be at most p. Then, since

$$f^n(s_0) - f^{n-1}(s_0) = s_n - s_{n-1} = a_n,$$

and $\{a_n \bmod p\}$ has minimal period p, we must have $p' = p$. Therefore the directed graph G on $\mathbb{Z}/p\mathbb{Z}$ given by the edges $k \to f(k)$ is simply a p-cycle, which implies that the map $k \mapsto f(k)$ is a bijection on $\mathbb{Z}/p\mathbb{Z}$. Therefore, $f(k+1) \ne f(k)$ for all k (unless $p = 1$, but in this case the following holds anyways), hence

$$f(k) < f(k+1) < \cdots < f(k+p) = f(k) + p.$$

This implies $f(k+1) = f(k) + 1$ for all k, so $f(k) - k$ is constant, therefore $a_n = f(s_{n-1}) - s_{n-1}$ is also constant. Let $a_n \equiv t$. It follows that $t^2 \le N < t^2 + t$ as we wanted.

¶ Solution 2 (global). Define $\{a_n\}$ and f as in the previous solution. We first show that $s_i \not\equiv s_j \pmod p$ for all $i < j < i + p$. Suppose the contrary, i.e. that $s_i \equiv s_j \pmod p$ for some i, j with $i < j < i + p$. Then $a_{s_i+k} = a_{s_j+k}$ for all $k \geq 0$, therefore $s_{s_i+k} - s_{s_i} = s_{s_j+k} - s_{s_j}$ for all $k \geq 0$, therefore

$$a_{i+1} = f(s_i) - s_i = f(s_j) - s_j = a_{j+1} \quad \text{and} \quad s_{i+1} = f(s_i) \equiv f(s_j) = s_{j+1} \pmod p.$$

Continuing this inductively, we obtain $a_{i+k} = a_{j+k}$ for all k, so $\{a_n\}$ has period $j - i < p$, which is a contradiction. Therefore $s_i \not\equiv s_j \pmod p$ for all $i < j < i + p$, and this implies that $\{s_i, \dots, s_{i+p-1}\}$ forms a complete residue system modulo p for all i. Consequently we must have $s_{i+p} \equiv s_i \pmod p$ for all i.

Let $T = s_p - s_0 = a_1 + \cdots + a_p$. Since $\{a_n\}$ is periodic with period p, and $\{i+1, \dots, i+kp\}$ contains exactly k values of each residue class modulo p,

$$s_{i+kp} - s_i = a_{i+1} + \cdots + a_{i+kp} = kT$$

for all i, k. Since $p \mid T$, it follows that $s_{s_p} - s_{s_0} = \frac{T}{p} \cdot T = \frac{T^2}{p}$. Summing up the inequalities

$$s_{s_n} - s_{s_{n-1}} \leq N < s_{s_n+1} - s_{s_{n-1}} = s_{s_n} - s_{s_{n-1}} + a_{s_n+1}$$

for $n \in \{1, \dots, p\}$ then implies

$$\frac{T^2}{p} = s_{s_p} - s_{s_0} \leq Np < \frac{T^2}{p} + a_{s_1+1} + a_{s_2+1} + \cdots + a_{s_p+1} = \frac{T^2}{p} + T,$$

where the last equality holds because $\{s_1 + 1, \dots, s_p + 1\}$ is a complete residue system modulo p. Dividing this by p yields $t^2 \leq N < t^2 + t$ for $t := \frac{T}{p} \in \mathbb{Z}^+$.

> **Remark** (Author comments)**.** There are some similarities between this problem and IMO 2009/3, mainly that they both involve terms of the form s_{s_n} and s_{s_n+1} and the sequence $s_0, s_1, \dots$ turns out to be an arithmetic progression. Other than this, I don't think knowing about IMO 2009/3 will be that useful on this problem, since in this problem the fact that $\{s_{n+1} - s_n\}$ is periodic is fundamentally important.
>
> The motivation for this problem comes from the following scenario: assume we have boxes that can hold some things of total size $\leq N$, and a sequence of things of size $a_1, a_2, \dots$ (where $a_i := s_{i+1} - s_i$). We then greedily pack the things in a sequence of boxes, 'closing' each box when it cannot fit the next thing. The number of things we put in each box gives a sequence $b_1, b_2, \dots$. This problem asks when we can have $\{a_n\} = \{b_n\}$, assuming that we start with a sequence $\{a_n\}$ that is periodic.
>
> (Extra motivation: I first thought about this scenario when I was pasting some text repeatedly into the Notes app and noticed that the word at the end of lines are also (eventually) periodic.)

§2 Solutions to Day 2

§2.1 TSTST 2022/4, proposed by Merlijn Staps

Available online at https://aops.com/community/p25517031.

Problem statement

A function $f\colon \mathbb{N} \to \mathbb{N}$ has the property that for all positive integers m and n, exactly one of the $f(n)$ numbers

$$f(m+1), f(m+2), \dots, f(m+f(n))$$

is divisible by n. Prove that $f(n) = n$ for infinitely many positive integers n.

We start with the following claim:

Claim — If $a \mid b$ then $f(a) \mid f(b)$.

Proof. From applying the condition with $n = a$, we find that the set $S_a = \{n \ge 2 : a \mid f(n)\}$ is an arithmetic progression with common difference $f(a)$. Similarly, the set $S_b = \{n \ge 2 : b \mid f(n)\}$ is an arithmetic progression with common difference $f(b)$. From $a \mid b$ it follows that $S_b \subseteq S_a$. Because an arithmetic progression with common difference x can only be contained in an arithmetic progression with common difference y if $y \mid x$, we conclude $f(a) \mid f(b)$. □

In what follows, let $a \ge 2$ be any positive integer. Because $f(a)$ and $f(2a)$ are both divisible by $f(a)$, there are $a+1$ consecutive values of f of which at least two divisible by $f(a)$. It follows that $f(f(a)) \le a$.

However, we also know that exactly one of $f(2)$, $f(3)$, …, $f(1+f(a))$ is divisible by a; let this be $f(t)$. Then we have $S_a = \{t, t+f(a), t+2f(a), \dots\}$. Because $a \mid f(t) \mid f(2t)$, we know that $2t \in S_a$, so t is a multiple of $f(a)$. Because $2 \le t \le 1 + f(a)$, and $f(a) \ge 2$ for $a \ge 2$, we conclude that we must have $t = f(a)$, so $f(f(a))$ is a multiple of a. Together with $f(f(a)) \le a$, this yields $f(f(a)) = a$. Because $f(f(a)) = a$ also holds for $a = 1$ (from the given condition for $n = 1$ it immediately follows that $f(1) = 1$), we conclude that $f(f(a)) = a$ for all a, and hence f is a bijection.

Moreover, we now have that $f(a) \mid f(b)$ implies $f(f(a)) \mid f(f(b))$, i.e. $a \mid b$, so $a \mid b$ if and only if $f(a) \mid f(b)$. Together with the fact that f is a bijection, this implies that $f(n)$ has the same number of divisors of n. Let p be a prime. Then $f(p) = q$ must be a prime as well. If $q \ne p$, then from $f(p) \mid f(pq)$ and $f(q) \mid f(pq)$ it follows that $pq \mid f(pq)$, so $f(pq) = pq$ because $f(pq)$ and pq must have the same number of divisors. Therefore, for every prime number p we either have that $f(p) = p$ or $f(pf(p)) = pf(p)$. From here, it is easy to see that $f(n) = n$ for infinitely many n.

§2.2 TSTST 2022/5, proposed by Ray Li

Available online at https://aops.com/community/p25517063.

Problem statement

Let $A_1, \dots, A_{2022}$ be the vertices of a regular 2022-gon in the plane. Alice and Bob play a game. Alice secretly chooses a line and colors all points in the plane on one side of the line blue, and all points on the other side of the line red. Points on the line are colored blue, so every point in the plane is either red or blue. (Bob cannot see the colors of the points.)

In each round, Bob chooses a point in the plane (not necessarily among $A_1, \dots, A_{2022}$) and Alice responds truthfully with the color of that point. What is the smallest number Q for which Bob has a strategy to always determine the colors of points $A_1, \dots, A_{2022}$ in Q rounds?

The answer is 22. To prove the lower bound, note that there are $2022 \cdot 2021 + 2 > 2^{21}$ possible colorings. If Bob makes less than 22 queries, then he can only output 2^{21} possible colorings, which means he is wrong on some coloring.

Now we show Bob can always win in 22 queries. A key observation is that the set of red points is convex, as is the set of blue points, so if a set of points is all the same color, then their convex hull is all the same color.

Lemma

Let $B_0, \dots, B_{k+1}$ be equally spaced points on a circular arc such that colors of B_0 and B_{k+1} differ and are known. Then it is possible to determine the colors of $B_1, \dots, B_k$ in $\lceil \log_2 k \rceil$ queries.

Proof. There exists some $0 \le i \le k$ such that $B_0, \dots, B_i$ are the same color and $B_{i+1}, \dots, B_{k+1}$ are the same color. (If $i < j$ and B_0 and B_j were red and B_i and B_{k+1} were blue, then segment B_0B_j is red and segment B_iB_{k+1} is blue, but they intersect). Therefore we can binary search. □

Lemma

Let $B_0, \dots, B_{k+1}$ be equally spaced points on a circular arc such that colors of B_0, $B_{\lceil k/2 \rceil}$, B_{k+1} are both red and are known. Then at least one of the following holds: all of $B_1, \dots, B_{\lceil k/2 \rceil}$ are red or all of $B_{\lceil k/2 \rceil}, \dots, B_k$ are red. Furthermore, in one query we can determine which one of the cases holds.

Proof. The existence part holds for similar reason to previous lemma. To figure out which case, choose a point P such that all of $B_0, \dots, B_{k+1}$ lie between rays PB_0 and $PB_{\lceil k/2 \rceil}$, and such that $B_1, \dots, B_{\lceil k/2 \rceil -1}$ lie inside triangle $PB_0B_{\lceil k/2 \rceil}$ and such that $B_{\lceil k/2 \rceil +1}, \dots, B_{k+1}$ lie outside (this point should always exist by looking around the intersections of lines B_0B_1 and $B_{\lceil k/2 \rceil -1}B_{\lceil k/2 \rceil}$). Then if P is red, all the inside points are red because they lie in the convex hull of red points P, B_0, $B_{\lceil k/2 \rceil}$. If P is blue, then all the outside points are red: if B_i were blue for $i > \lceil k/2 \rceil$, then the segment PB_i is blue and intersect the segment $B_0B_{\lceil k/2 \rceil}$, which is red, contradiction. □

Now the strategy is: Bob picks A_1. WLOG it is red. Now suppose Bob does not know the colors of $\le 2^k - 1$ points $A_i, \dots, A_j$ with $j - i + 1 \le 2^k - 1$ and knows the rest are red. I claim Bob can win in $2k - 1$ queries. First, if $k = 1$, there is one point and he wins by querying the point, so the base case holds, so assume $k > 1$. Bob queries $A_{i+\lceil (j-i+1)/2 \rceil}$. If it is blue, he finishes in $2 \log_2 \lceil (j - i + 1)/2 \rceil \le 2(k - 1)$ queries by the first lemma, for a total of $2k - 1$ queries. If it is red, he can query one more point and learn some half of $A_i, \dots, A_j$ that are red by the second lemma, and then he has reduced it to the case with $\le 2^{k-1} - 1$ points in two queries, at which point we induct.

§2.3 TSTST 2022/6, proposed by Hongzhou Lin

Available online at https://aops.com/community/p25516957.

Problem statement

Let O and H be the circumcenter and orthocenter, respectively, of an acute scalene triangle ABC. The perpendicular bisector of $\overline{AH}$ intersects $\overline{AB}$ and $\overline{AC}$ at X_A and Y_A respectively. Let K_A denote the intersection of the circumcircles of triangles OX_AY_A and BOC other than O.

Define K_B and K_C analogously by repeating this construction two more times. Prove that K_A, K_B, K_C, and O are concyclic.

We present several approaches.

¶ First solution, by author Let $\odot OX_AY_A$ intersects AB, AC again at U, V. Then by Reim's theorem $UVCB$ are concyclic. Hence the radical axis of $\odot OX_AY_A$, $\odot OBC$ and $\odot(UVCB)$ are concurrent, i.e. OK_A, BC, UV are concurrent, Denote the intersection as K_A^*, which is indeed the inversion of K_A with respect to $\odot O$. (The inversion sends $\odot OBC$ to the line BC).

Let P_A, P_B, P_C be the circumcenters of $\triangle OBC$, $\triangle OCA$, $\triangle OAB$ respectively.

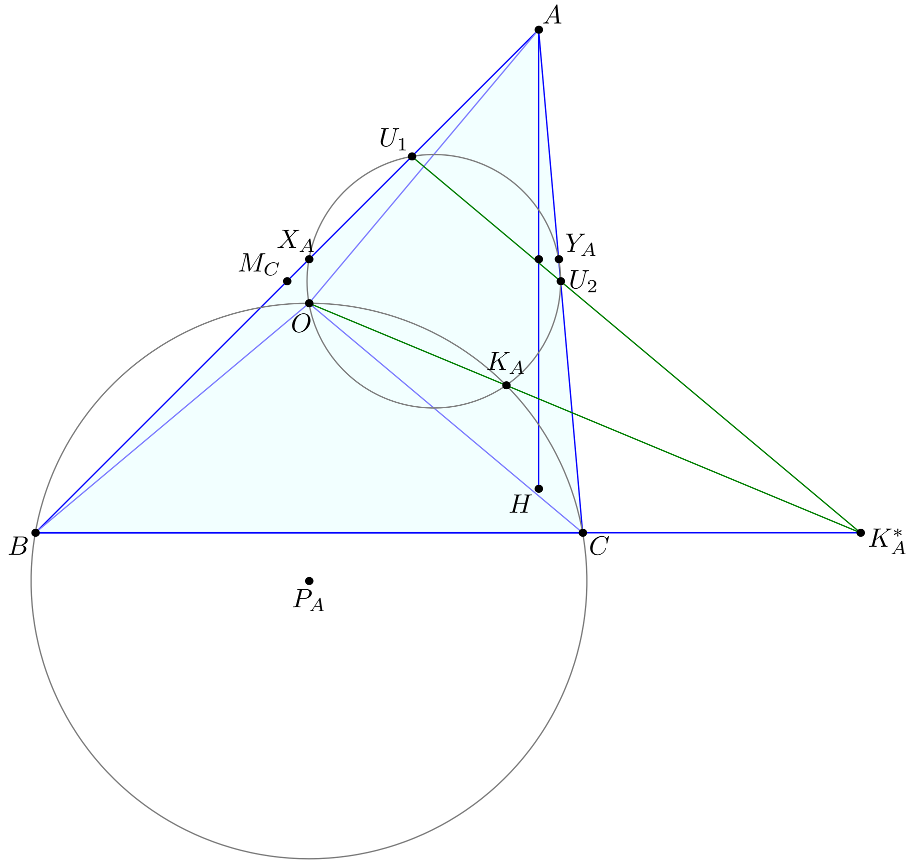
Claim — K_A^* coincides with the intersection of P_BP_C and BC.

Proof. Note that $d(O, BC) = 1/2AH = d(A, X_AY_A)$. This means the midpoint M_C of AB is equal distance to X_AY_A and the line through O parallel to BC. Together with $OM_C \perp AB$ implies that $\angle M_CX_AO = \angle B$. Hence $\angle UVO = \angle B = \angle AVU$. Similarly $\angle VUO = \angle AUV$, hence $\triangle AUV \simeq \triangle OUV$. In other words, UV is the perpendicular bisector of AO, which pass through P_B, P_C. Hence K_A^* is indeed $P_BP_C \cap BC$. □

Finally by Desargue's theorem, it suffices to show that AP_A, BP_B, CP_C are concurrent. Note that

$$d(P_A, AB) = P_AB \sin(90^\circ + \angle C - \angle A),$$
$$d(P_A, AC) = P_AC \sin(90^\circ + \angle B - \angle A).$$

Hence the symmetric product and trig Ceva finishes the proof.



¶ Second solution, from Jeffrey Kwan Let O_A be the circumcenter of $\triangle AX_AY_A$. The key claim is that:

> **Claim** — $O_AX_AY_AO$ is cyclic.

Proof. Let DEF be the orthic triangle; we will show that $\triangle OX_AY_A \sim \triangle DEF$. Indeed, since AO and AD are isogonal, it suffices to note that

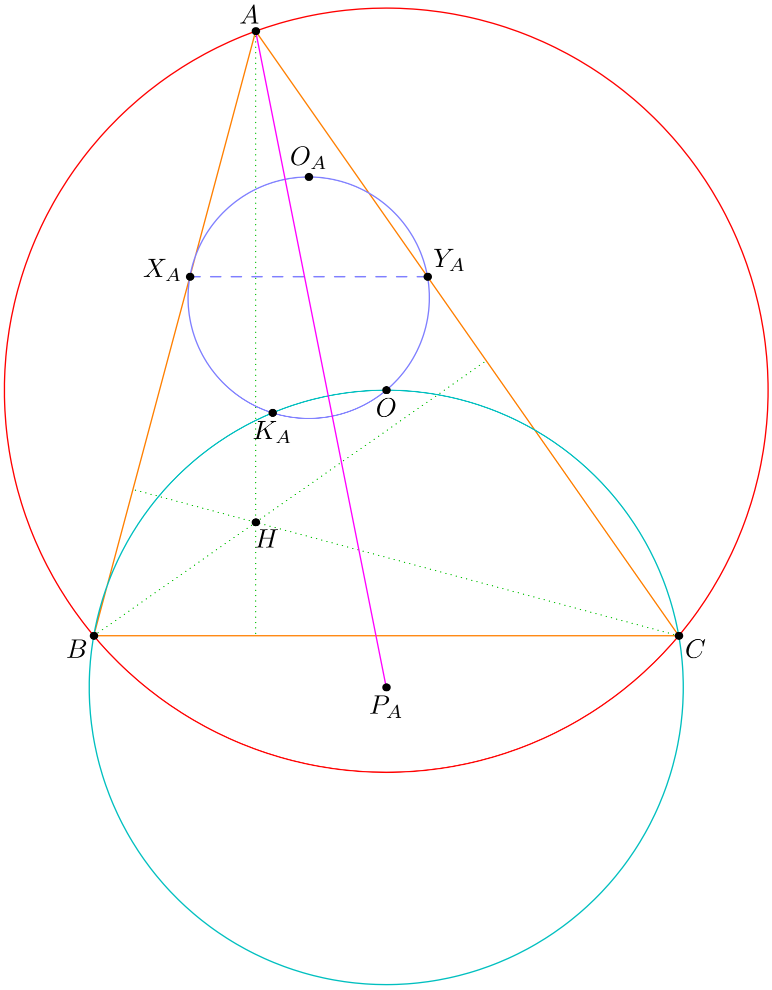
$$\frac{AX_A}{AB} = \frac{AH/2}{AD} = \frac{R\cos A}{AD},$$

and so

$$\frac{AO}{AD} = R \cdot \frac{AX_A}{AB \cdot R\cos A} = \frac{AX_A}{AE} = \frac{AY_A}{AF}.$$

Hence $\angle X_AOY_A = 180^\circ - 2\angle A = 180^\circ - \angle X_AO_AY_A$, which proves the claim. □

Let P_A be the circumcenter of $\triangle OBC$, and define P_B, P_C similarly. By the claim, A is the exsimilicenter of (OX_AY_A) and (OBC), so AP_A is the line between their two centers. In particular, AP_A is the perpendicular bisector of OK_A.



Claim — AP_A, BP_B, CP_C concur at T.

Proof. The key observation is that O is the incenter of $\triangle P_AP_BP_C$, and that A, B, C are the reflections of O across the sides of $\triangle P_AP_BP_C$. Hence P_AA, P_BB, P_CC concur by Jacobi. □

Now T lies on the perpendicular bisectors of OK_A, OK_B, and OK_C. Hence $OK_AK_BK_C$ is cyclic with center T, as desired.

§3 Solutions to Day 3

§3.1 TSTST 2022/7, proposed by Merlijn Staps

Available online at https://aops.com/community/p25516961.

Problem statement

Let $ABCD$ be a parallelogram. Point E lies on segment CD such that

$$2\angle AEB = \angle ADB + \angle ACB,$$

and point F lies on segment BC such that

$$2\angle DFA = \angle DCA + \angle DBA.$$

Let K be the circumcenter of triangle ABD. Prove that $KE = KF$.

Let the circle through A, B, and E intersect CD again at E', and let the circle through D, A, and F intersect BC again at F'. Now $ABEE'$ and $DAF'F$ are cyclic quadrilaterals with two parallel sides, so they are isosceles trapezoids. From $KA = KB$, it now follows that $KE = KE'$, whereas from $KA = KD$ it follows that $KF = KF'$.

Next, let the circle through A, B, and E intersect AC again at S. Then

$$\angle ASB = \angle AEB = \frac{1}{2}(\angle ADB + \angle ACB) = \frac{1}{2}(\angle ADB + \angle DAC) = \frac{1}{2}\angle AMB,$$

where M is the intersection of AC and BD. From $\angle ASB = \frac{1}{2}\angle AMB$, it follows that $MS = MB$, so S is the point on MC such that $MS = MB = MD$. By symmetry, the circle through A, D, and F also passes through S, and it follows that the line AS is the radical axis of the circles (ABE) and (ADF).

By power of a point, we now obtain

$$CE \cdot CE' = CS \cdot CA = CF \cdot CF',$$

from which it follows that E, F, E', and F' are concyclic. The segments EE' and FF' are not parallel, so their perpendicular bisectors only meet at one point, which is K. Hence $KE = KF$.

§3.2 TSTST 2022/8, proposed by Merlijn Staps

Available online at https://aops.com/community/p25516968.

Problem statement

Find all functions $f\colon \mathbb{N} \to \mathbb{Z}$ such that

$$\left\lfloor \frac{f(mn)}{n} \right\rfloor = f(m)$$

for all positive integers m, n.

There are two families of functions that work: for each $\alpha \in \mathbb{R}$ the function $f(n) = \lfloor \alpha n \rfloor$, and for each $\alpha \in \mathbb{R}$ the function $f(n) = \lceil \alpha n \rceil - 1$. (For irrational α these two functions coincide.) It is straightforward to check that these functions indeed work; essentially, this follows from the identity

$$\left\lfloor \frac{\lfloor xn \rfloor}{n} \right\rfloor = \lfloor x \rfloor$$

which holds for all positive integers n and real numbers x.

We now show that every function that works must be of one of the above forms. Let f be a function that works, and define the sequence a_1, a_2, ... by $a_n = f(n!)/n!$. Applying the give condition with $(n!, n+1)$ yields $a_{n+1} \in [a_n, a_n + \frac{1}{n!})$. It follows that the sequence a_1, a_2, ... is non-decreasing and bounded from above by $a_1 + e$, so this sequence must converge to some limit α.

If there exists a k such that $a_k = \alpha$, then we have $a_\ell = \alpha$ for all $\ell > k$. For each positive integer m, there exists $\ell > k$ such that $m \mid \ell!$. Plugging in $mn = \ell!$, it then follows that

$$f(m) = \left\lfloor \frac{f(\ell!)}{\ell!/m} \right\rfloor = \lfloor \alpha m \rfloor$$

for all m, so f is of the desired form.

If there does not exist a k such that $a_k = \alpha$, we must have $a_k < \alpha$ for all k. For each positive integer m, we can now pick an ℓ such that $m \mid \ell!$ and $a_\ell = \alpha - x$ with x arbitrarily small. It then follows from plugging in $mn = \ell!$ that

$$f(m) = \left\lfloor \frac{f(\ell!)}{\ell!/m} \right\rfloor = \left\lfloor \frac{\ell!(\alpha - x)}{\ell!/m} \right\rfloor = \lfloor \alpha m - mx \rfloor .$$

If αm is an integer we can choose ℓ such that $mx < 1$, and it follows that $f(m) = \lceil \alpha m \rceil - 1$. If αm is not an integer we can choose ℓ such that $mx < \{\alpha m\}$, and it also follows that $f(m) = \lceil \alpha m \rceil - 1$. We conclude that in this case f is again of the desired form.

§3.3 TSTST 2022/9, proposed by Vincent Huang

Available online at `https://aops.com/community/p25517112`.

Problem statement

Let $k > 1$ be a fixed positive integer. Prove that if n is a sufficiently large positive integer, there exists a sequence of integers with the following properties:

- Each element of the sequence is between 1 and n, inclusive.
- For any two different contiguous subsequences of the sequence with length between 2 and k inclusive, the multisets of values in those two subsequences is not the same.
- The sequence has length at least $0.499n^2$.

For any positive integer n, define an (n, k)-good sequence to be a finite sequence of integers each between 1 and n inclusive satisfying the second property in the problem statement. The problems asks to show that, for all sufficiently large integers n, there is an (n, k)-good sequence of length at least $0.499n^2$.

Fix $k \ge 2$ and consider some prime power $n = p^m$ with $p > k + 1$. Consider some $0 < g < \frac{n}{k} - 1$ with $\gcd(g, n) = 1$ and let a be the smallest positive integer with $g^a \equiv \pm 1 \pmod{n}$.

Claim (Main claim) — For k, n, g, a defined as above, there is an (n, k)-good sequence of length $a(n + 2) + 2$.

To prove the main claim, we need some results about the structure of $\mathbb{Z}/n\mathbb{Z}$. Specifically, we'll first show that any nontrivial arithmetic sequence is uniquely recoverable.

Lemma

Consider any arithmetic progression of length $i \le k$ whose common difference is relatively prime to n, and let S be the set of residues it takes modulo n. Then there exists a unique integer $0 < d \le \frac{n}{2}$ and a unique integer $0 \le a < n$ such that

$$S = \{a, a + d, \dots, a + (i - 1)d\}.$$

Proof of lemma. We'll split into cases, based on if i is odd or not.

- *Case 1:* i is odd, so $i = 2j + 1$ for some j. Then the middle term of the arithmetic progression is the average of all residues in S, which we can uniquely identify as some u (and we know n is coprime to i, so it is possible to average the residues). We need to show that there is only one choice of d, up to $\pm$, so that $S = \{u - jd, u - (j - 1)d, \dots, u + jd\}$.

  Let X be the sum of squares of the residues in S, so we have

  $$X \equiv (u - jd)^2 + (u - (j - 1)d)^2 + \dots + (u + jd)^2 = (2j + 1)u^2 + d^2\frac{j(j + 1)(2j + 1)}{3},$$

  which therefore implies

  $$3(X - (2j + 1)u^2)(j(j + 1)(2j + 1))^{-1} \equiv d^2,$$

thus identifying d uniquely up to sign as desired.

- *Case 2:* i is even, so $i = 2j$ for some j. Once again we can compute the average u of the residues in S, and we need to show that there is only one choice of d, up to $\pm$, so that $S = \{u-(2j-1)d, u-(2j-3)d, \dots, u+(2j-1)d\}$. Once again we compute the sum of squares X of the residues in S, so that

$$X \equiv (u-(2j-1)d)^2+(u-(2j-3)d)^2+\dots+(u+(2j-1)d)^2 = 2ju^2+\frac{(2j-1)2j(2j+1)}{3}$$

which therefore implies

$$3(X-2ju^2)((2j-1)2j(2j+1))^{-1} \equiv d^2,$$

again identifying d uniquely up to sign as desired.

Thus we have shown that given the set of residues an arithmetic progression takes on modulo n, we can recover that progression up to sign. Here we have used the fact that given $d^2 \pmod n$, it is possible to recover d up to sign provided that n is of the form p^m with $p \neq 2$ and $\gcd(d, n) = 1$. □

Now, we will proceed by chaining many arithmetic sequences together.

Definition. For any integer l between 0 and $a-1$, inclusive, define C_l to be the sequence $0, g^l, g^l, 2g^l, 3g^l, \dots, (n-1)g^l, (n-1)g^l$ taken $\pmod n$. (This is just a sequence where the ith term is $(i-1)g^l$, except the terms $g^l, (n-1)g^l$ is repeated once.)

Definition. Consider the sequence S_n of residues mod n defined as follows:

- The first term of S_n is 0.
- For each $0 \le l < a$, the next $n+2$ terms of S_n are the terms of C_l in order.
- The next and final term of S_n is 0.

We claim that S_n constitutes a k-good string with respect to the alphabet of residues modulo n. We first make some initial observations about S_n.

Lemma

S_n has the following properties:

- S_n has length $a(n+2)+2$.
- If a contiguous subsequence of S_n of length $\le k$ contains two of the same residue $\pmod n$, those two residues occur consecutively in the subsequence.

Proof of lemma. The first property is clear since each C_l has length $n+2$, and there are a of them, along with the 0s at beginning and end.

To prove the second property, consider any contiguous subsequence $S_n[i : i+k-1]$ of length k which contains two of the same residue modulo n. If $S_n[i : i+k-1]$ is wholly contained within some C_l, it's clear that the only way $S_n[i : i+k-1]$ could repeat residues if it repeats one of the two consecutive values g^l, g^l or $(n-1)g^l, (n-1)g^l$, so assume that is not the case.

Now, it must be true that $S_n[i : i+k-1]$ consists of one contiguous subsequence of the form

$$(n-k_1)g^{l-1}, (n-(k_1-1))g^{l-1}, \dots, (n-1)g^{l-1}, (n-1)g^{l-1},$$

which are the portions of $S_n[i : i+k-1]$ contained in C_{l-1}, and then a second contiguous subsequence of the form

$$0, g^l, g^l, 2g^l, \dots, k_2 g^l,$$

which are the portions of $S_n[i : i+k-1]$ contained in C_l, and we obviously have $k_2 + k_1 = k-3$. For $S_n[i : i+k-1]$ to contain two of the same residue in non-consecutive positions, there would have to exist some $0 < u \le k_1, 0 < v \le k_2$ with $(n-u)g^{l-1} \equiv vg^l \pmod n$, meaning that $u + gv \equiv 0 \pmod n$. But we know since $k_1 + k_2 < k$ that $0 < u + gv < k + kg < n$, so this is impossible, as desired. □

Now we can prove the main claim.

Proof of main claim. Consider any multiset M of $2 \le i \le k$ residues $\pmod n$ which corresponds to some unknown contiguous subsequence of S_n. We will show that it is possible to uniquely identify which contiguous subsequence M corresponds to, thereby showing that S_n has no twins of length i for each $2 \le i \le k$, and then the result will follow.

First suppose M contains some residue twice. By the last lemma there are only a few possible cases:

- M contains multiple copies of the residue 0. In this case we know M contains the beginning of S_n, so the corresponding contiguous subsequence is just the first i terms of S_n.

- M contains multiple copies of multiple residues. By the last lemma and the structure of S_n, we can easily see that M must contain two copies of $-g^{i-1}$ and two copies of g^i for some $0 \le i < a$ that can be identified uniquely, and M must contain portions of both C_{i-1}, C_i. It follows M's terms can be partitioned into two portions, the first one being
$$-i_1 g^{i-1}, -(i_1-1)g^{i-1}, \dots, -g^{i-1}, -g^{i-1},$$
and the second one being
$$0, g^i, g^i, 2g^i, \dots, i_2 g^i$$
for some i_1, i_2 with $i_1 + i_2 = i-3$, and we just need to uniquely identify i_1, i_2. Luckily, by dividing the residues in M by g^{i-1}, we know we can partition M's terms into
$$-i_1, -(i_1-1), \dots, -1, -1$$
as well as
$$0, g, g, 2g, \dots, i_2 g.$$
Now since $i_2 g \le kg < n-k$ and $-i_1 \equiv n - i_1 \ge n-k$ it is easy to see that i_1, i_2 can be identified uniquely, as desired.

- M contains multiple copies of only one residue g^i, for some $0 \le i < a$ that can be identified uniquely. Then by the last lemma M must be located at the beginning of C_i and possibly contain the last few terms of C_{i-1}, so M must be of the form $g^i, g^i, 2g^i, \dots, i_1 g^i$, along with possibly the term 0 or the terms $0, -g^{i-1}$. So when we divide M by g^{i-1} we should be left with terms of the form $g, g, 2g, \dots, i_1 g$ along with possibly 0 or $0, -1$. Since $i_1 g \le kg < n-k$, we can easily disambiguate these cases and uniquely identify the contiguous subsequence corresponding to M.

- M contains multiple copies of only one residue $-g^i$, for some $0 \le i < a$ that can be identified uniquely. Then by the last lemma M must be located at the end of C_i and possibly the first terms of C_{i+1}, so M must be of the form $-g^i, -g^i, -2g^i, \dots, -i_1 g^i$, along with possibly the term 0 or the terms $0, g^{i+1}$. So when we divide M by g^{i-1} we should be left with terms of the form $-1, -1, -2, \dots, -i_1$, along with possibly 0 or $0, g$. Since $-i_1 \equiv n - i_1 \ge \frac{n}{2}$ and $g < \frac{n}{2}$, we can disambiguate these cases and uniquely identify the contiguous subsequence corresponding to M.

Thus in all cases where M contains a repeated residue, we can identify the unique contiguous subsequence of S_n corresponding to M.

When M does not contain a repeated residue, it follows that M cannot contain both of the g^i terms or $(n-1)g^i$ terms at the beginning or end of each C_i. It follows that M is either entirely contained in some C_i or contained in the union of the end of some C_i with the beginning of some C_{i+1}, meaning M corresponds to a contiguous subsequence of $(-g^i, 0, g^{i+1})$. In the first case, since each C_i is an arithmetic progression when the repeated terms are ignored, Lemma 1 implies that we can uniquely determine the location of M, and in the second case, it is easy to tell which contiguous subsequence of $(-g^i, 0, g^{i+1})$ corresponds to M.

Therefore, in all cases, for any multiset M corresponding to some contiguous subsequence of S_n of length $i \le k$, we can uniquely identify the contiguous subsequence, meaning S_n is k-good with respect to the alphabet of residues modulo n, as desired. □

Now we will finish the problem. We observe the following.

Claim — Fix k and let $p > k+1$ be a prime. Then for $n = p^2$ we can find a (n,k)-good sequence of length $\frac{p(p-1)(p^2+2)}{2}$.

Proof of last claim. Let g be the smallest primitive root modulo $n = p^2$, so that $a = \frac{p(p-1)}{2}$. As long as we can show that $g < \frac{n}{k} - 1$, we can apply the previous claim to get the desired bound.

We will prove a stronger statement that $g < p$. Indeed, consider any primitive root $g_0 \pmod p$. Then $g_0 + ap$ has order $p-1$ modulo p, so its order modulo p^2 is divisible by $p-1$, hence $g_0 + ap$ is a primitive root modulo p^2 as long as $(g_0+ap)^{p-1} \not\equiv 1 \pmod{p^2}$. Now

$$(g_0 + ap)^{p-1} = \sum_i g_0^{p-1-i}(ap)^i \binom{p-1}{i} \equiv g_0^{p-1} + g_0^{p-2}(ap) \pmod{p^2}.$$

In particular, of the values $g_0, g_0+p, \dots, g_0+p(p-1)$, only one has order $p-1$ and the rest are primitive roots.

So for each $0 < g_0 < p$ which is a primitive root modulo p, either g_0 is a primitive root modulo p^2 or g_0 has order $p-1$ but $g_0+p, g_0+2p, \dots, g_0+p(p-1)$ are all primitive roots. By considering all choices of g_0, we either find a primitive root $\pmod{p^2}$ which is between 0 and p, or we find that all residues $\pmod{p^2}$ of order $p-1$ are between 0 and p. But if $\operatorname{ord}_{p^2}(a) = p-1$ then $\operatorname{ord}_{p^2}(a^{-1}) = p-1$, and two residues between $0, p$ cannot be inverses modulo p^2 (because with the exception of 1, they cannot multiply to something $\ge p^2+1$), so there is always a primitive root between $0, p$ as desired. □

Now for arbitrarily large n we can choose $p < \sqrt{n}$ with $\frac{p}{\sqrt{n}}$ arbitrarily close to 1; by the previous claim, we can get an (n,k)-good sequence of length at least $\frac{p-1}{p} \cdot \frac{p^4}{2}$ for any constant, so for sufficiently large n, p we get (n,k)-good sequences of length $0.499n^2$.

TSTST 2022 Statistics

Mathematical Olympiad Summer Program

Evan Chen《陳誼廷》

June 27, 2022

§1 Summary of scores for TSTST 2022

N	59	1st Q	9	Max	50
μ	16.86	Median	16	Top 3	38
σ	10.94	3rd Q	22	Top 12	25

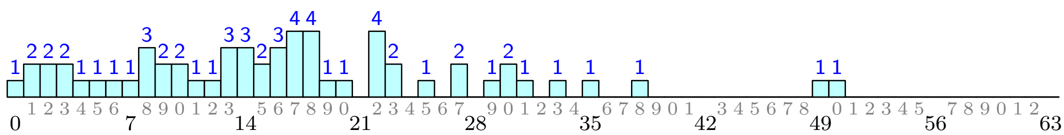
§2 Problem statistics for TSTST 2022

	P1	**P2**	**P3**	**P4**	**P5**	**P6**	**P7**	**P8**	**P9**
0	8	47	39	6	28	49	36	39	55
1	7	0	19	2	12	0	0	5	2
2	23	4	0	7	3	1	0	3	1
3	8	0	0	1	1	1	1	0	0
4	4	0	1	0	0	1	0	0	0
5	0	0	0	0	1	0	1	0	0
6	1	1	0	2	2	2	1	2	0
7	8	7	0	41	12	5	20	10	1
Avg	2.63	1.07	0.39	5.39	2.07	0.95	2.61	1.58	0.19
QM	3.35	2.59	0.77	5.99	3.49	2.42	4.22	3.13	0.97
#5+	**9**	**8**	**0**	**43**	**15**	**7**	**22**	**12**	**1**
%5+	%15.3	%13.6	%0.0	%72.9	%25.4	%11.9	%37.3	%20.3	%1.7

§3 Rankings for TSTST 2022

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
63	0	0	0.00%	42	0	2	3.39%	21	0	18	30.51%
62	0	0	0.00%	41	0	2	3.39%	20	1	19	32.20%
61	0	0	0.00%	40	0	2	3.39%	19	1	20	33.90%
60	0	0	0.00%	39	0	2	3.39%	18	4	24	40.68%
59	0	0	0.00%	38	1	3	5.08%	17	4	28	47.46%
58	0	0	0.00%	37	0	3	5.08%	16	3	31	52.54%
57	0	0	0.00%	36	0	3	5.08%	15	2	33	55.93%
56	0	0	0.00%	35	1	4	6.78%	14	3	36	61.02%
55	0	0	0.00%	34	0	4	6.78%	13	3	39	66.10%
54	0	0	0.00%	33	1	5	8.47%	12	1	40	67.80%
53	0	0	0.00%	32	0	5	8.47%	11	1	41	69.49%
52	0	0	0.00%	31	1	6	10.17%	10	2	43	72.88%
51	0	0	0.00%	30	2	8	13.56%	9	2	45	76.27%
50	1	1	1.69%	29	1	9	15.25%	8	3	48	81.36%
49	1	2	3.39%	28	0	9	15.25%	7	1	49	83.05%
48	0	2	3.39%	27	2	11	18.64%	6	1	50	84.75%
47	0	2	3.39%	26	0	11	18.64%	5	1	51	86.44%
46	0	2	3.39%	25	1	12	20.34%	4	1	52	88.14%
45	0	2	3.39%	24	0	12	20.34%	3	2	54	91.53%
44	0	2	3.39%	23	2	14	23.73%	2	2	56	94.92%
43	0	2	3.39%	22	4	18	30.51%	1	2	58	98.31%
								0	1	59	100.00%

§4 Histogram for TSTST 2022



USA TST Selection Test for 65th IMO and 13th EGMO

Pittsburgh, PA

Day I 1:15pm – 5:45pm

Tuesday, June 20, 2023

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 1. Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that $\angle RAS + \angle BAC = \angle BGC$.

Problem 2. Let $n \geq m \geq 1$ be integers. Prove that

$$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3} \right) \geq m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2 .$$

Problem 3. Find all positive integers n for which it is possible to color some cells of an infinite grid of unit squares red, such that each rectangle consisting of exactly n cells (and whose edges lie along the lines of the grid) contains an odd number of red cells.

USA TST Selection Test for 65$^{\text{th}}$ IMO and 13$^{\text{th}}$ EGMO

Pittsburgh, PA

Day II 1:15pm – 5:45pm

Thursday, June 22, 2023

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 4. Let $n \geq 3$ be an integer and let K_n be the complete graph on n vertices. Each edge of K_n is colored either red, green, or blue. Let A denote the number of triangles in K_n with all edges of the same color, and let B denote the number of triangles in K_n with all edges of different colors. Prove that

$$B \leq 2A + \frac{n(n-1)}{3}.$$

(The *complete graph* on n vertices is the graph on n vertices with $\binom{n}{2}$ edges, with exactly one edge joining every pair of vertices. A *triangle* consists of the set of $\binom{3}{2} = 3$ edges between 3 of these n vertices.)

Problem 5. Suppose a, b, and c are three complex numbers with product 1. Assume that none of a, b, and c are real or have absolute value 1. Define

$$p = (a+b+c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \qquad \text{and} \qquad q = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

Given that both p and q are real numbers, find all possible values of the ordered pair (p, q).

Problem 6. Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of $\overline{PQ}$ lies on the Euler line of ABC.

(The Euler line is the line through the circumcenter and orthocenter of a triangle.)

USA TST Selection Test for 65^{th} IMO and 13^{th} EGMO

Pittsburgh, PA

Day III 1:15pm – 5:45pm

Saturday, June 24, 2023

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 7. The Bank of Pittsburgh issues coins that have a heads side and a tails side. Vera has a row of 2023 such coins alternately tails-up and heads-up, with the leftmost coin tails-up.

In a *move*, Vera may flip over one of the coins in the row, subject to the following rules:

- On the first move, Vera may flip over any of the 2023 coins.
- On all subsequent moves, Vera may only flip over a coin adjacent to the coin she flipped on the previous move. (We do not consider a coin to be adjacent to itself.)

Determine the smallest possible number of moves Vera can make to reach a state in which every coin is heads-up.

Problem 8. Let ABC be an equilateral triangle with side length 1. Points A_1 and A_2 are chosen on side BC, points B_1 and B_2 are chosen on side CA, and points C_1 and C_2 are chosen on side AB such that $BA_1 < BA_2$, $CB_1 < CB_2$, and $AC_1 < AC_2$.

Suppose that the three line segments B_1C_2, C_1A_2, and A_1B_2 are concurrent, and the perimeters of triangles AB_2C_1, BC_2A_1, and CA_2B_1 are all equal. Find all possible values of this common perimeter.

Problem 9. For every integer $m \geq 1$, let $\mathbb{Z}/m\mathbb{Z}$ denote the set of integers modulo m.

Let p be a fixed prime and let $a \geq 2$ and $e \geq 1$ be fixed integers. Given a function $f\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ and an integer $k \geq 0$, the *kth finite difference*, denoted $\Delta^k f$, is the function from $\mathbb{Z}/a\mathbb{Z}$ to $\mathbb{Z}/p^e\mathbb{Z}$ defined recursively by

$$\begin{aligned}
\Delta^0 f(n) &= f(n) \\
\Delta^k f(n) &= \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n) \qquad \text{for } k = 1, 2, \ldots.
\end{aligned}$$

Determine the number of functions f such that there exists some $k \geq 1$ for which $\Delta^k f = f$.

USA TSTST 2023 Solutions

United States of America — TST Selection Test

Andrew Gu, Evan Chen, Gopal Goel

65th IMO 2024 United Kingdom and 13th EGMO 2024 Georgia

Contents

§0 Problems

1. Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that $\angle RAS + \angle BAC = \angle BGC$.

2. Let $n \geq m \geq 1$ be integers. Prove that

$$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3} \right) \geq m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2 .$$

3. Find all positive integers n for which it is possible to color some cells of an infinite grid of unit squares red, such that each rectangle consisting of exactly n cells (and whose edges lie along the lines of the grid) contains an odd number of red cells.

4. Let $n \geq 3$ be an integer and let K_n be the complete graph on n vertices. Each edge of K_n is colored either red, green, or blue. Let A denote the number of triangles in K_n with all edges of the same color, and let B denote the number of triangles in K_n with all edges of different colors. Prove that

$$B \leq 2A + \frac{n(n-1)}{3}.$$

5. Suppose a, b, and c are three complex numbers with product 1. Assume that none of a, b, and c are real or have absolute value 1. Define

$$p = (a+b+c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \qquad \text{and} \qquad q = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

Given that both p and q are real numbers, find all possible values of the ordered pair (p, q).

6. Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of $\overline{PQ}$ lies on the Euler line of ABC.

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Suppose that the three line segments B_1C_2, C_1A_2, and A_1B_2 are concurrent, and the perimeters of triangles AB_2C_1, BC_2A_1, and CA_2B_1 are all equal. Find all possible values of this common perimeter.

9. Let p be a fixed prime and let $a \geq 2$ and $e \geq 1$ be fixed integers. Given a function $f\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ and an integer $k \geq 0$, the *kth finite difference*, denoted $\Delta^k f$, is the function from $\mathbb{Z}/a\mathbb{Z}$ to $\mathbb{Z}/p^e\mathbb{Z}$ defined recursively by

$$\begin{aligned} \Delta^0 f(n) &= f(n) \\ \Delta^k f(n) &= \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n) \qquad \text{for } k = 1, 2, \dots. \end{aligned}$$

Determine the number of functions f such that there exists some $k \geq 1$ for which $\Delta^k f = f$.

§1 Solutions to Day 1

§1.1 TSTST 2023/1, proposed by Merlijn Staps

Available online at https://aops.com/community/p28015679.

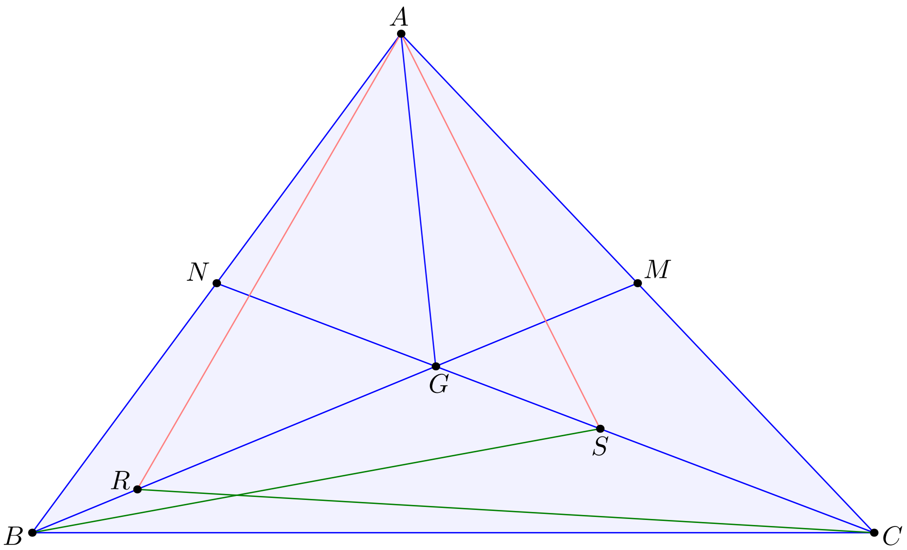
Problem statement

Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that $\angle RAS + \angle BAC = \angle BGC$.

In all the following solutions, let M and N denote the midpoints of $\overline{AC}$ and $\overline{AB}$, respectively.



¶ Solution 1 using power of a point From the given condition that $\measuredangle ACR = \measuredangle CGM$, we get that

$$MA^2 = MC^2 = MG \cdot MR \implies \measuredangle RAC = \measuredangle MGA.$$

Analogously,

$$\measuredangle BAS = \measuredangle AGN.$$

Hence,

$$\measuredangle RAS + \measuredangle BAC = \measuredangle RAC + \measuredangle BAS = \measuredangle MGA + \measuredangle AGN = \measuredangle MGN = \measuredangle BGC.$$

¶ Solution 2 using similar triangles As before, $\triangle MGC \sim \triangle MCR$ and $\triangle NGB \sim \triangle NBS$. We obtain

$$\frac{|AC|}{|CR|} = \frac{2|MC|}{|CR|} = \frac{2|MG|}{|GC|} = \frac{|GB|}{2|NG|} = \frac{|BS|}{2|BN|} = \frac{|BS|}{|AB|}$$

which together with $\angle ACR = \angle ABS$ yields

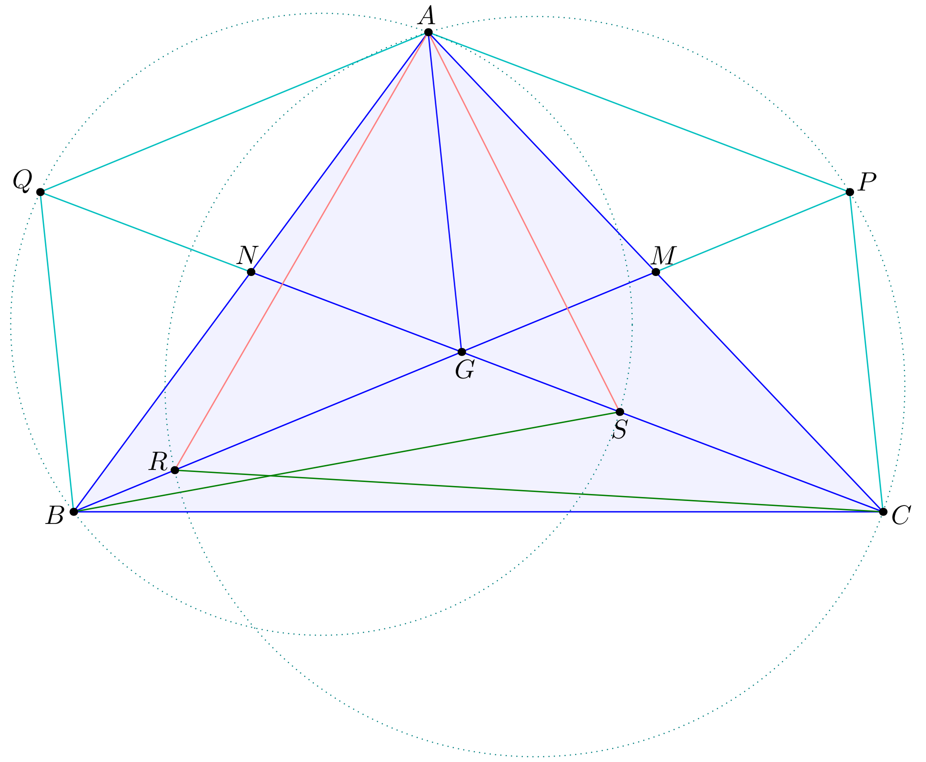
$$\triangle ACR \sim \triangle SBA \implies \measuredangle BAS = \measuredangle CRA.$$

Hence

$$\measuredangle RAS + \measuredangle BAC = \measuredangle RAC + \measuredangle BAS = \measuredangle RAC + \measuredangle CRA = -\measuredangle ACR = \measuredangle BGC,$$

which proves the statement.

¶ Solution 3 using parallelograms Let M and N be defined as above. Let P be the reflection of G in M and let Q the reflection of G in N. Then $AGCP$ and $AGBQ$ are parallelograms.



Claim — Quadrilaterals $APCR$ and $AQBS$ are concyclic.

Proof. Because $\measuredangle APR = \measuredangle APG = \measuredangle CGP = -\measuredangle BGC = \measuredangle ACR$. □

Thus from $\overline{PC} \parallel \overline{GA}$ we get

$$\measuredangle RAC = \measuredangle RPC = \measuredangle GPC = \measuredangle PGA$$

and similarly

$$\measuredangle BAS = \measuredangle BQS = \measuredangle BQG = \measuredangle AGQ.$$

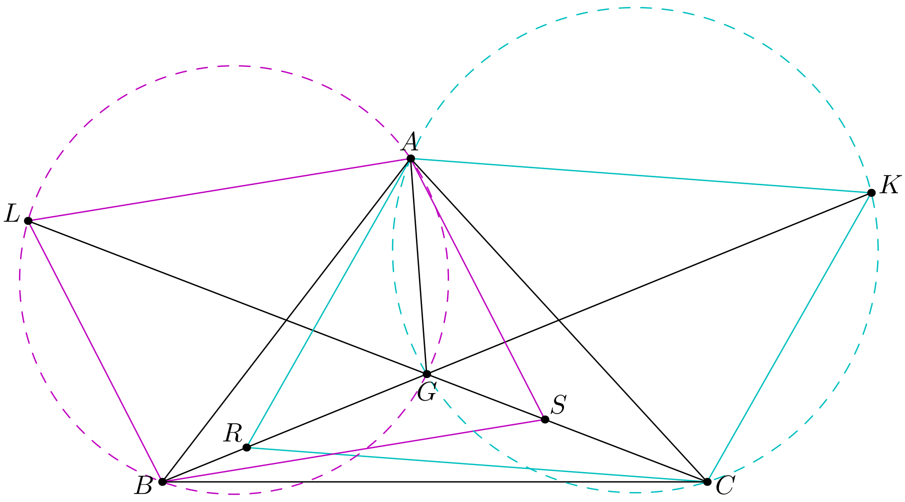
We conclude that

$$\measuredangle RAS + \measuredangle BAC = \measuredangle RAC + \measuredangle BAS = \measuredangle PGA + \measuredangle AGQ = \measuredangle PGQ = \measuredangle BGC.$$

¶ Solution 4 also using parallelograms, by Ankan Bhattacharya Construct parallelograms $ARCK$ and $ASBL$. Since

$$\measuredangle CAK = \measuredangle ACR = \measuredangle CGB = \measuredangle CGK,$$

it follows that $AGCK$ is cyclic. Similarly, $AGBL$ is also cyclic.



Finally, observe that

$$\begin{aligned} \angle RAS + \angle BAC &= \measuredangle BAS + \measuredangle RAC \\ &= \measuredangle ABL + \measuredangle KCA \\ &= \measuredangle AGL + \measuredangle KGA \\ &= \measuredangle KGL \\ &= \angle BGC \end{aligned}$$

as requested.

¶ Solution 5 using complex numbers, by Milan Haiman Note that $\angle RAS + \angle BAC = \angle BAS + \angle RAC$. We compute $\angle BAS$ in complex numbers; then $\angle RAC$ will then be known by symmetry.

Let a, b, c be points on the unit circle representing A, B, C respectively. Let $g = \frac{1}{3}(a+b+c)$ represent the centroid G, and let s represent S.

Claim — We have
$$\frac{s-a}{b-a} = \frac{ab-2bc+ca}{2ab-bc-ca}.$$

Proof. Since S is on line CG, which passes through the midpoint of segment AB, we have that
$$s = \frac{a+b}{2} + t(c-g)$$
for some $t \in \mathbb{R}$.

By the given angle condition, we have that
$$\frac{(s-b)/(b-a)}{(c-g)/(g-b)} \in \mathbb{R}.$$

Note that
$$\frac{s-b}{b-a} = t\frac{c-g}{b-a} - \frac{1}{2}.$$

So,
$$t\frac{g-b}{b-a} - \frac{g-b}{2(c-g)} \in \mathbb{R}.$$

Thus

$$t = \frac{\operatorname{Im}\left(\frac{g-b}{2(c-g)}\right)}{\operatorname{Im}\left(\frac{g-b}{b-a}\right)} = \frac{1}{2} \cdot \frac{\left(\frac{g-b}{c-g}\right) - \overline{\left(\frac{g-b}{c-g}\right)}}{\left(\frac{g-b}{b-a}\right) - \overline{\left(\frac{g-b}{b-a}\right)}}.$$

Let N and D be the numerator and denominator of the second factor above.

We want to compute

$$\frac{s-a}{b-a} = \frac{1}{2} + t\frac{c-g}{b-a} = \frac{(b-a)+2t(c-g)}{2(b-a)} = \frac{(b-a)D + (c-g)N}{2(b-a)D}.$$

We have

$$\begin{aligned} (c-g)N &= g - b - (c-g)\overline{\left(\frac{g-b}{c-g}\right)} \\ &= \frac{a+b+c}{3} - b - \left(c - \frac{a+b+c}{3}\right)\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{b}}{\frac{3}{c}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}} \\ &= \frac{(a+c-2b)(2ab-bc-ca)-(2c-a-b)(ab+bc-2ca)}{3(2ab-bc-ca)} \\ &= \frac{3(a^2b+b^2c+c^2a-ab^2-bc^2-ca^2)}{3(2ab-bc-ca)} \\ &= \frac{(a-b)(b-c)(a-c)}{2ab-bc-ca} \end{aligned}$$

We also compute

$$\begin{aligned} (b-a)D &= g - b - (b-a)\overline{\left(\frac{g-b}{b-a}\right)} \\ &= \frac{a+b+c}{3} - b - (b-a)\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{b}}{\frac{3}{b}-\frac{3}{a}} \\ &= \frac{(a+c-2b)c + (ab+bc-2ca)}{3c} \\ &= \frac{ab-bc-ca+c^2}{3c} \\ &= \frac{(a-c)(b-c)}{3c} \end{aligned}$$

So, we obtain

$$\frac{s-a}{b-a} = \frac{\frac{1}{3c}+\frac{a-b}{2ab-bc-ca}}{\frac{2}{3c}} = \frac{2ab-bc-ca+3c(a-b)}{2(2ab-bc-ca)} = \frac{ab-2bc+ca}{2ab-bc-ca}.$$

□

By symmetry,

$$\frac{r-a}{c-a} = \frac{ab-2bc+ca}{2ca-ab-bc}.$$

Hence their ratio

$$\frac{s-a}{b-a} \div \frac{r-a}{c-a} = \frac{2ab-bc-ca}{2ca-ab-bc}$$

has argument $\angle RAC + \angle BAS$.

We also have that $\angle BGC$ is the argument of

$$\frac{b-g}{c-g} = \frac{2b-a-c}{2c-a-b}.$$

Note that these two complex numbers are inverse-conjugates, and thus have the same argument. So we're done.

§1.2 TSTST 2023/2, proposed by Raymond Feng, Luke Robitaille

Available online at https://aops.com/community/p28015692.

Problem statement

Let $n \ge m \ge 1$ be integers. Prove that

$$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3}\right) \ge m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2}\right)^2 .$$

We show several approaches.

¶ First solution (authors) By Cauchy-Schwarz, we have

$$\begin{aligned} \sum_{k=m}^{n} \frac{k+1}{k^3} &= \sum_{k=m}^{n} \frac{\left(\frac{1}{k^2}\right)^2}{\frac{1}{k(k+1)}} \\ &\ge \frac{\left(\frac{1}{m^2} + \frac{1}{(m+1)^2} + \dots + \frac{1}{n^2}\right)^2}{\frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)} + \dots + \frac{1}{n(n+1)}} \\ &= \frac{\left(\frac{1}{m^2} + \frac{1}{(m+1)^2} + \dots + \frac{1}{n^2}\right)^2}{\frac{1}{m} - \frac{1}{n+1}} \\ &> \frac{\left(\sum_{k=m}^{n} \frac{1}{k^2}\right)^2}{\frac{1}{m}} \end{aligned}$$

as desired.

Remark (Bound on error)**.** Let $A = \sum_{k=m}^{n} k^{-2}$ and $B = \sum_{k=m}^{n} k^{-3}$. The inequality above becomes tighter for large m and $n \gg m$. If we use Lagrange's identity in place of Cauchy-Schwarz, we get

$$A + B - mA^2 = m \cdot \sum_{m \le a < b} \frac{(a-b)^2}{a^3 b^3 (a+1)(b+1)}.$$

We can upper bound this error by

$$\le m \cdot \sum_{m \le a < b} \frac{1}{a^3(a+1)b(b+1)} = m \cdot \sum_{m \le a} \frac{1}{a^3(a+1)^2} \approx m \cdot \frac{1}{m^4} = \frac{1}{m^3},$$

which is still generous as $(a-b)^2 \ll b^2$ for b not much larger than a, so the real error is probably around $\frac{1}{10m^3}$. This exhibits the tightness of the inequality since it implies

$$mA^2 + O(B/m) > A + B.$$

Remark (Construction commentary, from author)**.** My motivation was to write an inequality where Titu could be applied creatively to yield a telescoping sum. This can be difficult because most of the time, such a reverse-engineered inequality will be so loose it's trivial anyways. My first attempt was the not-so-amazing inequality

$$\frac{n^2+3n}{2} = \sum_1^n i+1 = \sum_1^n \frac{\frac{1}{i}}{\frac{1}{i(i+1)}} > \left(\sum_1^n \frac{1}{\sqrt{i}}\right)^2,$$

which is really not surprising given that $\sum \frac{1}{\sqrt{i}} \ll \frac{n}{\sqrt{2}}$. The key here is that we need "near-equality" as dictated by the Cauchy-Schwarz equality case, i.e. the square root of the numerators should be approximately proportional to the denominators.

This motivates using $\frac{1}{i^4}$ as the numerator, which works like a charm. After working out the resulting statement, the LHS and RHS even share a sum, which adds to the simplicity of the problem.

The final touch was to unrestrict the starting value of the sum, since this allows the strength of the estimate $\frac{1}{i^2} \approx \frac{1}{i(i+1)}$ to be fully exploited.

¶ Second approach by inducting down, Luke Robitaille and Carl Schildkraut Fix n; we'll induct downwards on m. For the base case of $n = m$ the result is easy, since the left side is $\frac{m+1}{m^3}$ and the right side is $\frac{m}{m^4} = \frac{1}{m^3}$.

For the inductive step, suppose we have shown the result for $m+1$. Let

$$A = \sum_{k=m+1}^{n} \frac{1}{k^2} \qquad \text{and} \qquad B = \sum_{k=m+1}^{n} \frac{1}{k^3}.$$

We know $A + B \ge (m+1)A^2$, and we want to show

$$\left(A + \frac{1}{m^2}\right) + \left(B + \frac{1}{m^3}\right) \ge m\left(A + \frac{1}{m^2}\right)^2.$$

Indeed,

$$\begin{aligned}\left(A + \frac{1}{m^2}\right) + \left(B + \frac{1}{m^3}\right) - m\left(A + \frac{1}{m^2}\right)^2 &= A + B + \frac{m+1}{m^3} - mA^2 - \frac{2A}{m} - \frac{1}{m^3} \\ &= \left(A + B - (m+1)A^2\right) + \left(A - \frac{1}{m}\right)^2 \ge 0,\end{aligned}$$

and we are done.

¶ Third approach by reducing $n \to \infty$, Michael Ren and Carl Schildkraut First, we give:

Claim (Reduction to $n \to \infty$) — If the problem is true when $n \to \infty$, it is true for all n.

Proof. Let $A = \sum_{k=m}^{n} k^{-2}$ and $B = \sum_{k=m}^{n} k^{-3}$. Consider the region of the xy-plane defined by $y > mx^2 - x$. We are interested in whether (A, B) lies in this region.

However, the region is bounded by a convex curve, and the sequence of points $(0,0)$, $\left(\frac{1}{m^2}, \frac{1}{m^3}\right)$, $\left(\frac{1}{m^2} + \frac{1}{(m+1)^2}, \frac{1}{m^3} + \frac{1}{(m+1)^3}\right)$, ... has successively decreasing slopes between consecutive points. Thus it suffices to check that the inequality is true when $n \to \infty$. □

Set $n = \infty$ henceforth. Let

$$A = \sum_{k=m}^{\infty} \frac{1}{k^2} \text{ and } B = \sum_{k=m}^{\infty} \frac{1}{k^3};$$

we want to show $B \geq mA^2 - A$, which rearranges to

$$1 + 4mB \geq (2mA - 1)^2.$$

Write

$$C = \sum_{k=m}^{\infty} \frac{1}{k^2(2k-1)(2k+1)} \text{ and } D = \sum_{k=m}^{\infty} \frac{8k^2-1}{k^3(2k-1)^2(2k+1)^2}.$$

Then

$$\frac{2}{2k-1} - \frac{2}{2k+1} = \frac{1}{k^2} + \frac{1}{k^2(2k-1)(2k+1)},$$

and

$$\frac{2}{(2k-1)^2} - \frac{2}{(2k+1)^2} = \frac{1}{k^3} + \frac{8k^2-1}{k^3(2k-1)^2(2k+1)^2},$$

so that

$$A = \frac{2}{2m-1} - C \text{ and } B = \frac{2}{(2m-1)^2} - D.$$

Our inequality we wish to show becomes

$$\frac{2m+1}{2m-1} C \geq D + mC^2.$$

We in fact show two claims:

Claim — We have

$$\frac{2m+1/2}{2m-1} C \geq D.$$

Proof. We compare termwise; we need

$$\frac{2m+1/2}{2m-1} \cdot \frac{1}{k^2(2k-1)(2k+1)} \geq \frac{8k^2-1}{k^3(2k-1)^2(2k+1)^2}$$

for $k \geq m$. It suffices to show

$$\frac{2k+1/2}{2k-1} \cdot \frac{1}{k^2(2k-1)(2k+1)} \geq \frac{8k^2-1}{k^3(2k-1)^2(2k+1)^2},$$

which is equivalent to $k(2k+1/2)(2k+1) \geq 8k^2 - 1$. This holds for all $k \geq 1$. □

Claim — We have

$$\frac{1/2}{2m-1} C \geq mC^2.$$

Proof. We need $C \leq 1/(2m(2m-1))$; indeed,

$$\frac{1}{2m(2m-1)} = \sum_{k=m}^{\infty} \left(\frac{1}{2k(2k-1)} - \frac{1}{2(k+1)(2k+1)} \right) = \sum_{k=m}^{\infty} \frac{4k+1}{2k(2k-1)(k+1)(2k+1)};$$

comparing term-wise with the definition of C and using the inequality $k(4k+1) \geq 2(k+1)$ for $k \geq 1$ gives the desired result. □

Combining the two claims finishes the solution.

¶ Fourth approach by bashing, Carl Schildkraut With a bit more work, the third approach can be adapted to avoid the $n \to \infty$ reduction. Similarly to before, define

$$A = \sum_{k=m}^{n} \frac{1}{k^2} \text{ and } B = \sum_{k=m}^{n} \frac{1}{k^3};$$

we want to show $1 + 4mB \geq (2mA - 1)^2$. Writing

$$C = \sum_{k=m}^{n} \frac{1}{k^2(2k-1)(2k+1)} \text{ and } D = \sum_{k=m}^{n} \frac{8k^2-1}{k^3(2k-1)^2(2k+1)^2}.$$

We compute

$$A = \frac{2}{2m-1} - \frac{2}{2n+1} - C \text{ and } B = \frac{2}{(2m-1)^2} - \frac{2}{(2n+1)^2} - D.$$

Then, the inequality we wish to show reduces (as in the previous solution) to

$$\frac{2m+1}{2m-1}C + \frac{2(2m+1)}{(2m-1)(2n+1)} \geq D + mC^2 + \frac{2(2m+1)}{(2n+1)^2} + \frac{4m}{2n+1}C.$$

We deal first with the terms not containing the variable n, i.e. we show that

$$\frac{2m+1}{2m-1}C \geq D + mC^2.$$

For this part, the two claims from the previous solution go through exactly as written above, and we have $C \leq 1/(2m(2m-1))$. We now need to show

$$\frac{2(2m+1)}{(2m-1)(2n+1)} \geq \frac{2(2m+1)}{(2n+1)^2} + \frac{4m}{2n+1}C$$

(this is just the inequality between the remaining terms); our bound on C reduces this to proving

$$\frac{4(2m+1)(n-m+1)}{(2m-1)(2n+1)^2} \geq \frac{2}{(2m-1)(2n+1)}.$$

Expanding and writing in terms of n, this is equivalent to

$$n \geq \frac{1+2(m-1)(2m+1)}{4m} = m - \frac{2m+1}{4m},$$

which holds for all $n \geq m$.

§1.3 TSTST 2023/3, proposed by Merlijn Staps

Available online at https://aops.com/community/p28015682.

Problem statement

Find all positive integers n for which it is possible to color some cells of an infinite grid of unit squares red, such that each rectangle consisting of exactly n cells (and whose edges lie along the lines of the grid) contains an odd number of red cells.

We claim that this is possible for all positive integers n. Call a positive integer for which such a coloring is possible *good*. To show that all positive integers n are good we prove the following:

(i) If n is good and p is an odd prime, then pn is good;

(ii) For every $k \geq 0$, the number $n = 2^k$ is good.

Together, (i) and (ii) imply that all positive integers are good.

¶ Proof of (i) We simply observe that if every rectangle consisting of n cells contains an odd number of red cells, then so must every rectangle consisting of pn cells. Indeed, because p is prime, a rectangle consisting of pn cells must have a dimension (length or width) divisible by p and can thus be subdivided into p rectangles consisting of n cells.

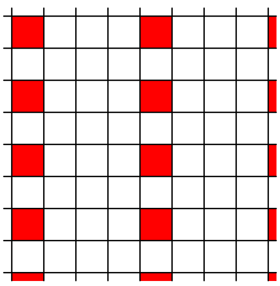
Thus every coloring that works for n automatically also works for pn.

¶ Proof of (ii) Observe that rectangles with $n = 2^k$ cells have $k + 1$ possible shapes: $2^m \times 2^{k-m}$ for $0 \leq m \leq k$.

Claim — For each of these $k + 1$ shapes, there exists a coloring with two properties:

- Every rectangle with n cells and shape $2^m \times 2^{k-m}$ contains an odd number of red cells.
- Every rectangle with n cells and a different shape contains an even number of red cells.

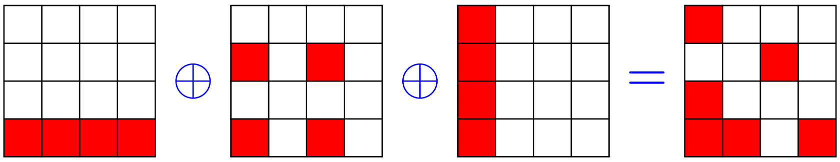
Proof. This can be achieved as follows: assuming the cells are labeled with $(x, y) \in \mathbb{Z}^2$, color a cell red if $x \equiv 0 \pmod{2^m}$ and $y \equiv 0 \pmod{2^{k-m}}$. For example, a 4×2 rectangle gets the following coloring:



A $2^m \times 2^{k-m}$ rectangle contains every possible pair $(x \mod 2^m, y \mod 2^{k-m})$ exactly once, so such a rectangle will contain one red cell (an odd number).

On the other hand, consider a $2^\ell \times 2^{k-\ell}$ rectangle with $\ell > m$. The set of cells this covers is (x, y) where x covers a range of size 2^ℓ and y covers a range of size $2^{k-\ell}$. The number of red cells is the count of x with $x \equiv 0 \mod 2^m$ multiplied by the count of y with $y \equiv 0 \mod 2^{k-m}$. The former number is exactly $2^{\ell-k}$ because 2^k divides 2^ℓ (while the latter is 0 or 1) so the number of red cells is even. The $\ell < m$ case is similar. □

Finally, given these $k+1$ colorings, we can add them up modulo 2, i.e. a cell will be colored red if it is red in an odd number of these $k+1$ colorings. We illustrate $n = 4$ as an example; the coloring is 4-periodic in both axes so we only show one 4×4 cell.



This solves the problem.

Remark. The final coloring can be described as follows: color (x, y) red if

$$\max(0, \min(\nu_2(x), k) + \min(\nu_2(y), k) - k + 1)$$

is odd.

Remark (Luke Robitaille)**.** Alternatively for (i), if $n = 2^e k$ for odd k then one may dissect an $a \times b$ rectangle with area n into k rectangles of area 2^e, each $2^{\nu_2(a)} \times 2^{\nu_2(b)}$. This gives a way to deduce the problem from (ii) without having to consider odd prime numbers.

¶ Alternate proof of (ii) using generating functions We will commit to constructing a coloring which is n-periodic in both directions. (This is actually forced, so it's natural to do so.) With that in mind, let

$$f(x, y) = \sum_{i=0}^{2^k-1} \sum_{j=0}^{2^k-1} \lambda_{i,j} x^i y^j$$

denote its generating function, where $f \in \mathbb{F}_2[x, y]$.

For this to be valid, we need that for any $2^p \times 2^q$ rectangle with area n, the sum of the coefficients of f over it should be one, modulo $x^{2^k} = y^{2^k} = 1$. In other words, whenever $p + q = k$, we must have

$$f(x, y)(1 + \cdots + x^{2^p-1})(1 + \cdots + y^{2^q-1}) = (1 + \cdots + x^{2^k-1})(1 + \cdots + y^{2^k-1}),$$

taken modulo $x^{2^k} = y^{2^k} = 1$. The idea is to rewrite these expressions: because we're in characteristic 2, the given assertion is $(x+1)^{2^k} = (y+1)^{2^k} = 0$, and the requested property is

$$f(x, y)(x+1)^{2^p-1}(y+1)^{2^q-1} = (x+1)^{2^k-1}(y+1)^{2^k-1}.$$

This suggests the substitution $g(x, y) = f(x+1, y+1)$: then we can replace $(x+1, y+1) \mapsto (x, y)$ to simplify the requested property significantly:

Whenever $p + q = k$, we must have

$$g(x, y)x^{2^p-1}y^{2^q-1} = x^{2^k-1}y^{2^k-1},$$

modulo x^{2^k} and y^{2^k}.

However, now the construction of g is very simple: for example, the choice

$$g(x,y) = \sum_{p+q=k} x^{2^k-2^p} y^{2^k-2^q}$$

works. The end.

> **Remark.** Unraveling the substitutions seen here, it's possible to show that this is actually the same construction provided in the first solution.

§2 Solutions to Day 2

§2.1 TSTST 2023/4, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p28015691.

Problem statement

Let $n \geq 3$ be an integer and let K_n be the complete graph on n vertices. Each edge of K_n is colored either red, green, or blue. Let A denote the number of triangles in K_n with all edges of the same color, and let B denote the number of triangles in K_n with all edges of different colors. Prove that

$$B \leq 2A + \frac{n(n-1)}{3}.$$

Consider all unordered pairs of different edges which share exactly one vertex (call these *vees* for convenience). Assign each vee a *charge* of $+2$ if its edge colors are the same, and a charge of -1 otherwise.

We compute the total charge in two ways.

¶ Total charge by summing over triangles Note that

- each monochromatic triangle has a charge of $+6$,
- each bichromatic triangle has a charge of 0, and
- each trichromatic triangle has a charge of -3.

Since each vee contributes to exactly one triangle, we obtain that the total charge is $6A - 3B$.

¶ Total charge by summing over vertices We can also calculate the total charge by examining the centers of the vees. If a vertex has a red edges, b green edges, and c blue edges, the vees centered at that vertex contribute a total charge of

$$\begin{aligned} & 2\left[\binom{a}{2} + \binom{b}{2} + \binom{c}{2}\right] - (ab + ac + bc) \\ &= (a^2 - a + b^2 - b + c^2 - c) - (ab + ac + bc) \\ &= (a^2 + b^2 + c^2 - ab - ac - bc) - (a + b + c) \\ &= (a^2 + b^2 + c^2 - ab - ac - bc) - (n-1) \\ &\geq -(n-1). \end{aligned}$$

In particular, the total charge is at least $-n(n-1)$.

¶ Conclusion Thus, we obtain

$$6A - 3B \geq -n(n-1) \iff B \leq 2A + \frac{n(n-1)}{3}$$

as desired.

§2.2 TSTST 2023/5, proposed by David Altizio

Available online at https://aops.com/community/p28015713.

Problem statement

Suppose a, b, and c are three complex numbers with product 1. Assume that none of a, b, and c are real or have absolute value 1. Define

$$p = (a+b+c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \qquad \text{and} \qquad q = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

Given that both p and q are real numbers, find all possible values of the ordered pair (p, q).

We show $(p, q) = (-3, 3)$ is the only possible ordered pair.

¶ First solution

Setup for proof Let us denote $a = y/x$, $b = z/y$, $c = x/z$, where x, y, z are nonzero complex numbers. Then

$$\begin{aligned} p + 3 &= 3 + \sum_{\text{cyc}} \left(\frac{x}{y} + \frac{y}{x}\right) = 3 + \frac{x^2(y+z) + y^2(z+x) + z^2(x+y)}{xyz} \\ &= \frac{(x+y+z)(xy+yz+zx)}{xyz}. \\ q - 3 &= -3 + \sum_{\text{cyc}} \frac{y^2}{zx} = \frac{x^3+y^3+z^3-3xyz}{xyz} \\ &= \frac{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}{xyz}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{R} &\ni 3(p+3) + (q-3) \\ &= \frac{(x+y+z)(x^2+y^2+z^2+2(xy+yz+zx))}{xyz} \\ &= \frac{(x+y+z)^3}{xyz}. \end{aligned}$$

Now, note that if $x + y + z = 0$, then $p = -3$, $q = 3$ so we are done.

Main proof We will prove that if $x+y+z \neq 0$ then we contradict either the hypothesis that $a, b, c \notin \mathbb{R}$ or that a, b, c do not have absolute value 1.

Scale x, y, z in such a way that $x + y + z$ is nonzero and real; hence so is xyz. Thus, as $p + 3 \in \mathbb{R}$, we conclude $xy + yz + zx \in \mathbb{R}$ as well. Hence, x, y, z are the roots of a cubic with real coefficients. Thus,

- either all three of $\{x, y, z\}$ are real (which implies $a, b, c \in \mathbb{R}$),
- or two of $\{x, y, z\}$ are a complex conjugate pair (which implies one of a, b, c has absolute value 1).

Both of these were forbidden by hypothesis.

Construction As we saw in the setup, $(p, q) = (-3, 3)$ will occur as long as $x+y+z = 0$, and no two of x, y, z to share the same magnitude or are collinear with the origin. This is easy to do; for example, we could choose $(x, y, z) = (3, 4i, -(3+4i))$. Hence $a = \frac{3}{4i}$, $b = -\frac{4i}{3+4i}$, $c = -\frac{3+4i}{3}$ satisfies the hypotheses of the problem statement.

¶ Second solution, found by contestants The main idea is to make the substitution

$$x = a + \frac{1}{c}, \qquad y = b + \frac{1}{a}, \qquad z = c + \frac{1}{b}.$$

Then we can check that

$$\begin{aligned} x + y + z &= p \\ xy + yz + zx &= p + q + 3 \\ xyz &= p + 2. \end{aligned}$$

Therefore x, y, z are the roots of a cubic with real coefficients. As in the previous solution, we note that this cubic must either have all real roots, or a complex conjugate pair of roots. We also have the relation $a(y+1) = ab + a + 1 = x + 1$, and likewise $b(z+1) = y+1$, $c(x+1) = z+1$. This means that if any of x, y, z are equal to -1, then all are equal to -1.

Assume for the sake of contradiction that none are equal to -1. In the case where the cubic has three real roots, $a = \frac{x+1}{y+1}$ would be real. On the other hand, if there is a complex conjugate pair (without loss of generality, x and y) then a has magnitude 1. Therefore this cannot occur.

We conclude that $x = y = z = -1$, so $p = -3$ and $q = 3$. The solutions (a, b, c) can then be parameterized as $(a, -1 - \frac{1}{a}, -\frac{1}{1+a})$. To construct a solution, we need to choose a specific value of a such that none of the wrong conditions hold; when $a = 2i$, say, we obtain the solution $(2i, -1 + \frac{i}{2}, \frac{-1+2i}{5})$.

¶ Third solution by Luke Robitaille and Daniel Zhu The answer is $p = -3$ and $q = 3$. Let's first prove that no other (p, q) work.

Let $e_1 = a + b + c$ and $e_2 = a^{-1} + b^{-1} + c^{-1} = ab + ac + bc$. Also, let $f = e_1 e_2$. Note that $p = e_1 + e_2$.

Our main insight is to consider the quantity $q' = \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$. Note that $f = q + q' + 3$. Also,

$$\begin{aligned} qq' &= 3 + \frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} + \frac{bc}{a^2} + \frac{ac}{b^2} + \frac{ab}{c^2} \\ &= 3 + a^3 + b^3 + c^3 + a^{-3} + b^{-3} + c^{-3} \\ &= 9 + a^3 + b^3 + c^3 - 3abc + a^{-3} + b^{-3} + c^{-3} - 3a^{-1}b^{-1}c^{-1} \\ &= 9 + e_1(e_1^2 - 3e_2) + e_2(e_2^2 - 3e_1) \\ &= 9 + e_1^3 + e_2^3 - 6e_1e_2 \\ &= 9 + p(p^2 - 3f) - 6f \\ &= p^3 - (3p+6)f + 9. \end{aligned}$$

As a result, the quadratic with roots q and q' is $x^2 - (f-3)x + (p^3 - (3p+6)f + 9)$, which implies that

$$q^2 - qf + 3q + p^3 - (3p+6)f + 9 = 0 \iff (3p + q + 6)f = p^3 + q^2 + 3q + 9.$$

At this point, two miracles occur. The first is the following claim:

Claim — f is not real.

Proof. Suppose f is real. Since $(x-e_1)(x-e_2) = x^2 - px + f$, there are two cases:

- e_1 and e_2 are real. Then, a, b, and c are the roots of $x^3 - e_1x^2 + e_2x - 1$, and since every cubic with real coefficients has at least one real root, at least one of a, b, and c is real, contradiction.

- e_1 and e_2 are conjugates. Then, the polynomial $x^3 - \bar{e}_2x^2 + \bar{e}_1x - 1$, which has roots $\bar{a}^{-1}$, $\bar{b}^{-1}$, and $\bar{c}^{-1}$, is the same as the polynomial with a, b, c as roots. We conclude that the multiset $\{a, b, c\}$ is invariant under inversion about the unit circle, so one of a, b, and c must lie on the unit circle. This is yet another contradiction. □

As a result, we know that $3p + q + 6 = p^3 + q^2 + 3q + 9 = 0$. The second miracle is that substituting $q = -3p - 6$ into $q^2 + 3q + p^3 + 9 = 0$, we get

$$0 = p^3 + 9p^2 + 27p + 27 = (p+3)^3,$$

so $p = -3$. Thus $q = 3$.

It remains to construct valid a, b, and c. To do this, let's pick some e_1, let $e_2 = -3 - e_1$, and let a, b, and c be the roots of $x^3 - e_1x^2 + e_2x - 1$. It is clear that this guarantees $p = -3$. By our above calculations, q and q' are the roots of the quadratic $x^2 - (f-3)x + (3f - 18)$, so one of q and q' must be 3; by changing the order of a, b, and c if needed, we can guarantee this to be q. It suffices to show that for some choice of e_1, none of a, b, or c are real or lie on the unit circle.

To do this, note that we can rewrite $x^3 - e_1x^2 + (-3 - e_1)x - 1 = 0$ as

$$e_1 = \frac{x^3 - 3x - 1}{x^2 + x},$$

so all we need is a value of e_1 that is not $\frac{x^3-3x-1}{x^2+x}$ for any real x or x on the unit circle. One way to do this is to choose any nonreal e_1 with $|e_1| < 1/2$. This clearly rules out any real x. Also, if $|x| = 1$, by the triangle inequality $\left|x^3 - 3x - 1\right| \ge |3x| - \left|x^3\right| - |1| = 1$ and $\left|x^2 + x\right| \le 2$, so $\left|\frac{x^3-3x-1}{x^2+x}\right| \ge 1/2$.

§2.3 TSTST 2023/6, proposed by Holden Mui

Available online at https://aops.com/community/p28015708.

Problem statement

Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of $\overline{PQ}$ lies on the Euler line of ABC.

We present three approaches.

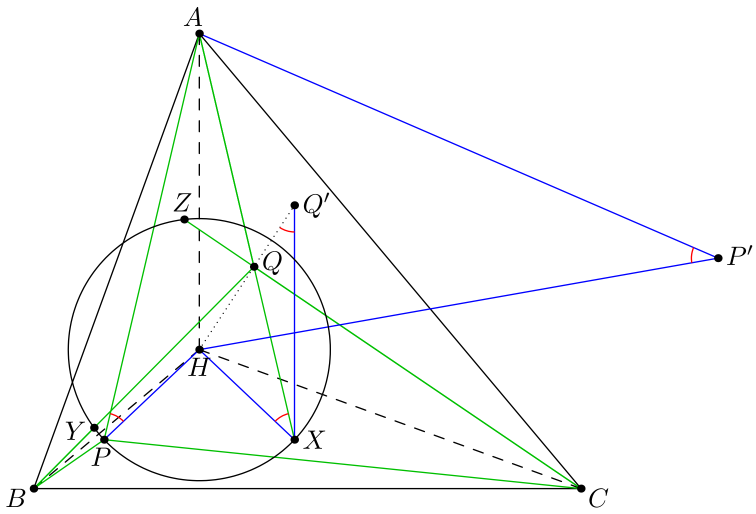
¶ Solution 1 (Ankit Bisain) Let H be the orthocenter of ABC, and construct P' using the following claim.

Claim — There is a point P' for which

$$\measuredangle APH + \measuredangle AP'H = \measuredangle BPH + \measuredangle BP'H = \measuredangle CPH + \measuredangle CP'H = 0.$$

Proof. After inversion at H, this is equivalent to the fact that P's image has an isogonal conjugate in ABC's image. □

Now, let X, Y, and Z be the reflections of P over $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$ respectively. Additionally, let Q' be the image of Q under inversion about $(PXYZ)$.



Claim — $ABCP' \overset{-}{\sim} XYZQ'$.

Proof. Since

$$\measuredangle YXZ = \measuredangle YPZ = \measuredangle(\overline{BH}, \overline{CH}) = -\measuredangle BAC$$

and cyclic variants, triangles ABC and XYZ are similar. Additionally,

$$\measuredangle HQ'X = -\measuredangle HXQ = -\measuredangle HXA = \measuredangle HPA = -\measuredangle HP'A$$

and cyclic variants, so summing in pairs gives $\measuredangle YQ'Z = -\measuredangle BP'C$ and cyclic variants; this implies the similarity. □

Claim — Q' lies on the Euler line of triangle XYZ.

Proof. Let O be the circumcenter of ABC so that $ABCOP' \overset{-}{\sim} XYZHQ'$. Then $\measuredangle HP'A = -\measuredangle HQ'X = \measuredangle OP'A$, so P' lies on $\overline{OH}$. By the similarity, Q' must lie on the Euler line of XYZ. □

To finish the problem, let G_1 be the centroid of ABC and G_2 be the centroid of XYZ. Then with signed areas,

$$\begin{aligned}
[G_1HP] + [G_1HQ] &= \frac{[AHP] + [BHP] + [CHP]}{3} + \frac{[AHQ] + [BHQ] + [CHQ]}{3} \\
&= \frac{[AHQ] - [AHX] + [BHQ] - [BHY] + [CHQ] - [CHZ]}{3} \\
&= \frac{[HQX] + [HQY] + [HQZ]}{3} \\
&= [QG_2H] \\
&= 0
\end{aligned}$$

where the last line follows from the last claim. Therefore $\overline{G_1H}$ bisects $\overline{PQ}$, as desired.

Remark. This solution characterizes the set of all points P for which such a point Q exists. It is the image of the Euler line under the mapping described in the first claim.

¶ Solution 2 using complex numbers (Carl Schildkraut and Milan Haiman) Let (ABC) be the unit circle in the complex plane, and let $A = a$, $B = b$, $C = c$ such that $|a| = |b| = |c| = 1$. Let $P = p$ and $Q = q$, and $O = 0$ and $H = h = a + b + c$ be the circumcenter and orthocenter of ABC respectively.

The first step is to translate the given geometric conditions into a single usable equation:

Claim — We have the equation

$$(p + q) \sum_{\text{cyc}} a^3(b^2 - c^2) = (\overline{p} + \overline{q})abc \sum_{\text{cyc}} (bc(b^2 - c^2)). \tag{1}$$

Proof. The condition that the altitude $\overline{AH}$ bisects $\angle PAQ$ is equivalent to

$$\begin{aligned}
&\frac{(p-a)(q-a)}{(h-a)^2} = \frac{(p-a)(q-a)}{(b+c)^2} \in \mathbb{R} \\
\implies &\frac{(p-a)(q-a)}{(b+c)^2} = \overline{\left(\frac{(p-a)(q-a)}{(b+c)^2}\right)} = \frac{(a\overline{p}-1)(a\overline{q}-1)b^2c^2}{(b+c)^2a^2} \\
\implies &a^2(p-a)(q-a) = b^2c^2(a\overline{p}-1)(a\overline{q}-1) \\
\implies &a^2pq - a^2b^2c^2\overline{pq} + (a^4 - b^2c^2) = a^3(p+q) - ab^2c^2(\overline{p}+\overline{q}).
\end{aligned}$$

Writing the symmetric conditions that $\overline{BH}$ and $\overline{CH}$ bisect $\angle PBQ$ and $\angle PCQ$ gives three equations:

$$\begin{aligned}
a^2pq - a^2b^2c^2\overline{pq} + (a^4 - b^2c^2) &= a^3(p+q) - ab^2c^2(\overline{p}+\overline{q}) \\
b^2pq - a^2b^2c^2\overline{pq} + (b^4 - c^2a^2) &= b^3(p+q) - bc^2a^2(\overline{p}+\overline{q})
\end{aligned}$$

$$c^2pq - a^2b^2c^2\overline{pq} + (c^4 - a^2b^2) = c^3(p+q) - ca^2b^2(\overline{p} + \overline{q}).$$

Now, sum $(b^2 - c^2)$ times the first equation, $(c^2 - a^2)$ times the second equation, and $(a^2 - b^2)$ times the third equation. On the left side, the coefficients of pq and $\overline{pq}$ are 0. Additionally, the coefficient of 1 (the parenthesized terms on the left sides of each equation) sum to 0, since

$$\sum_{\text{cyc}} (a^4 - b^2c^2)(b^2 - c^2) = \sum_{\text{cyc}} (a^4b^2 - b^4c^2 - a^4c^2 + c^4b^2).$$

This gives (1) as desired. □

We can then factor (1):

Claim — The left-hand side of (1) factors as

$$-(p+q)(a-b)(b-c)(c-a)(ab+bc+ca)$$

while the right-hand side factors as

$$-(\overline{p} + \overline{q})(a-b)(b-c)(c-a)(a+b+c).$$

Proof. This can of course be verified by direct expansion, but here is a slightly more economic indirect proof.

Consider the cyclic sum on the left as a polynomial in a, b, and c. If $a = b$, then it simplifies as $a^3(a^2 - c^2) + a^3(c^2 - a^2) + c^3(a^2 - a^2) = 0$, so $a - b$ divides this polynomial. Similarly, $a - c$ and $b - c$ divide it, so it can be written as $f(a,b,c)(a-b)(b-c)(c-a)$ for some symmetric quadratic polynomial f, and thus it is some linear combination of $a^2 + b^2 + c^2$ and $ab + bc + ca$. When $a = 0$, the whole expression is $b^2c^2(b - c)$, so $f(0,b,c) = -bc$, which implies that $f(a,b,c) = -(ab + bc + ca)$.

Similarly, consider the cyclic sum on the right as a polynomial in a, b, and c. If $a = b$, then it simplifies as $ac(a^2 - c^2) + ca(c^2 - a^2) + a^2(a^2 - a^2) = 0$, so $a - b$ divides this polynomial. Similarly, $a - c$ and $b - c$ divide it, so it can be written as $g(a,b,c)(a-b)(b-c)(c-a)$ where g is a symmetric linear polynomial; hence, it is a scalar multiple of $a+b+c$. When $a = 0$, the whole expression is $bc(b^2 - c^2)$, so $g(0,b,c) = -b-c$, which implies that $g(a,b,c) = -(a+b+c)$. □

Since A, B, and C are distinct, we may divide by $(a-b)(b-c)(c-a)$ to obtain

$$(p+q)(ab+bc+ca) = (\overline{p} + \overline{q})abc(a+b+c) \implies (p+q)\overline{h} = (\overline{p} + \overline{q})h.$$

This implies that $\frac{\frac{p+q}{2} - 0}{h - 0}$ is real, so the midpoint of $\overline{PQ}$ lies on line $\overline{OH}$.

¶ Solution 3 also using complex numbers (Michael Ren) We use complex numbers as in the previous solution. The angle conditions imply that $\frac{(a-p)(a-q)}{(b-c)^2}$, $\frac{(b-p)(b-q)}{(c-a)^2}$, and $\frac{(c-p)(c-q)}{(a-b)^2}$ are real numbers. Take a linear combination of these with real coefficients X, Y, and Z to be determined; after expansion, we obtain

$$\left[\frac{X}{(b-c)^2} + \frac{Y}{(c-a)^2} + \frac{Z}{(a-b)^2}\right] pq$$
$$-\left[\frac{aX}{(b-c)^2} + \frac{bY}{(c-a)^2} + \frac{cZ}{(a-b)^2}\right] (p+q)$$

$$+\left[\frac{a^2X}{(b-c)^2}+\frac{b^2Y}{(c-a)^2}+\frac{c^2Z}{(a-b)^2}\right]$$

which is a real number. To get something about the midpoint of PQ, the pq coefficient should be zero, which motivates the following lemma.

Lemma

There exist real X, Y, Z for which

$$\frac{X}{(b-c)^2}+\frac{Y}{(c-a)^2}+\frac{Z}{(a-b)^2}=0 \text{ and}$$
$$\frac{aX}{(b-c)^2}+\frac{bY}{(c-a)^2}+\frac{cZ}{(a-b)^2}\neq 0.$$

Proof. Since $\mathbb{C}$ is a 2-dimensional vector space over $\mathbb{R}$, there exist real X, Y, Z such that $(X, Y, Z) \neq (0, 0, 0)$ and the first condition holds. Suppose for the sake of contradiction that $\frac{aX}{(b-c)^2}+\frac{bY}{(c-a)^2}+\frac{cZ}{(a-b)^2}=0$. Then

$$\begin{aligned}&\frac{(b-a)Y}{(c-a)^2}+\frac{(c-a)Z}{(a-b)^2}\\ =&\frac{aX}{(b-c)^2}+\frac{bY}{(c-a)^2}+\frac{cZ}{(a-b)^2}-a\left(\frac{X}{(b-c)^2}+\frac{Y}{(c-a)^2}+\frac{Z}{(a-b)^2}\right)\\ =&0.\end{aligned}$$

We can easily check that $(Y, Z) = (0, 0)$ is impossible, therefore $\frac{(b-a)^3}{(c-a)^3}=-\frac{Z}{Y}$ is real. This means $\angle BAC = 60^\circ$ or 120°. By symmetry, the same is true of $\angle CBA$ and $\angle ACB$. This is impossible because ABC is scalene. $\square$

With the choice of X, Y, Z as in the lemma, there exist complex numbers α and β, depending only on a, b, and c, such that $\alpha \neq 0$ and $\alpha(p+q)+\beta$ is real. Therefore the midpoint of PQ, which corresponds to $\frac{p+q}{2}$, lies on a fixed line. It remains to show that this line is the Euler line. First, choose $P = Q$ to be the orthocenter to show that the orthocenter lies on the line. Secondly, choose P and Q to be the foci of the Steiner circumellipse to show that the centroid lies on the line. (By some ellipse properties, the external angle bisector of $\angle PAQ$ is the tangent to the circumellipse at A, which is the line through A parallel to BC. Therefore these points are valid.) Therefore the fixed line of the midpoint is the Euler line.

Remark. This solution does not require fixing the origin of the complex plane or setting (ABC) to be the unit circle.

§3 Solutions to Day 3

§3.1 TSTST 2023/7, proposed by Luke Robitaille

Available online at https://aops.com/community/p28015706.

Problem statement

The Bank of Pittsburgh issues coins that have a heads side and a tails side. Vera has a row of 2023 such coins alternately tails-up and heads-up, with the leftmost coin tails-up.

In a *move*, Vera may flip over one of the coins in the row, subject to the following rules:

- On the first move, Vera may flip over any of the 2023 coins.
- On all subsequent moves, Vera may only flip over a coin adjacent to the coin she flipped on the previous move. (We do not consider a coin to be adjacent to itself.)

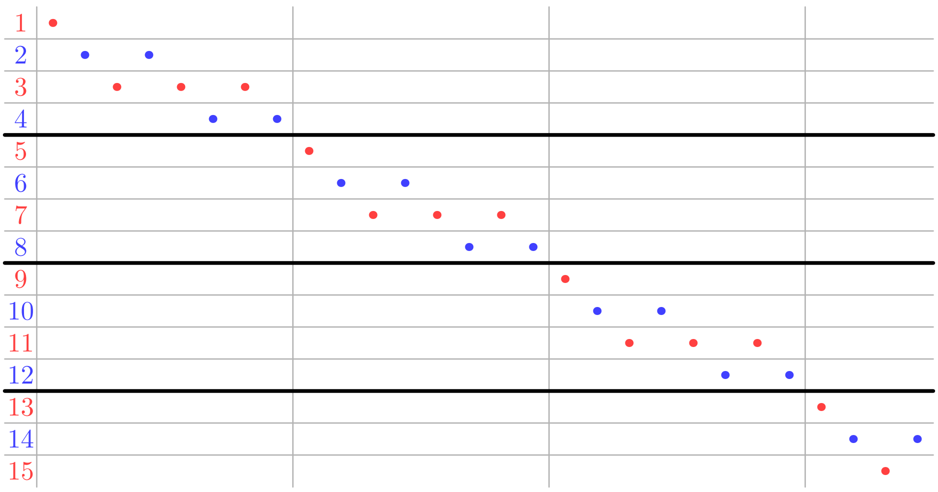
Determine the smallest possible number of moves Vera can make to reach a state in which every coin is heads-up.

The answer is $\boxed{4044}$. In general, replacing 2023 with $4n+3$, the answer is $8n+4$.

¶ Bound Observe that the first and last coins must be flipped, and so every coin is flipped at least once. Then, the $2n+1$ even-indexed coins must be flipped at least twice, so they are flipped at least $4n+2$ times.

The $2n+2$ odd-indexed coins must then be flipped at least $4n+1$ times. Since there are an even number of these coins, the total flip count must be even, so they are actually flipped a total of at least $4n+2$ times, for a total of at least $8n+4$ flips in all.

¶ Construction For $k = 0, 1, \dots, n-1$, flip $(4k+1, 4k+2, 4k+3, 4k+2, 4k+3, 4k+4, 4k+3, 4k+4)$ in that order; then at the end, flip $4n+1$, $4n+2$, $4n+3$, $4n+2$. This is illustrated below for $4n+3 = 15$.



It is easy to check this works, and there are 4044 flips, as desired.

§3.2 TSTST 2023/8, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p28015680.

Problem statement

Let ABC be an equilateral triangle with side length 1. Points A_1 and A_2 are chosen on side BC, points B_1 and B_2 are chosen on side CA, and points C_1 and C_2 are chosen on side AB such that $BA_1 < BA_2$, $CB_1 < CB_2$, and $AC_1 < AC_2$.

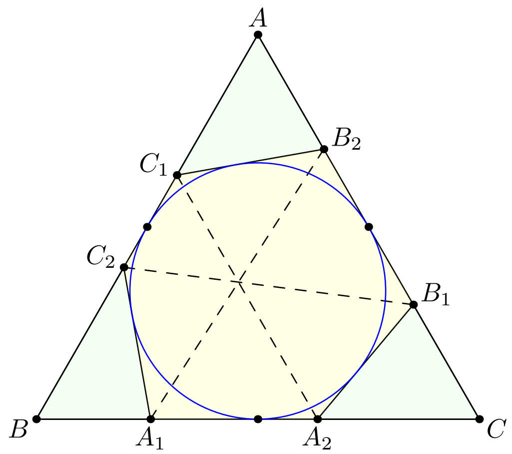
Suppose that the three line segments B_1C_2, C_1A_2, and A_1B_2 are concurrent, and the perimeters of triangles AB_2C_1, BC_2A_1, and CA_2B_1 are all equal. Find all possible values of this common perimeter.

The only possible value of the common perimeter, denoted p, is 1.

¶ Synthetic approach (from author) We prove the converse of the problem first:

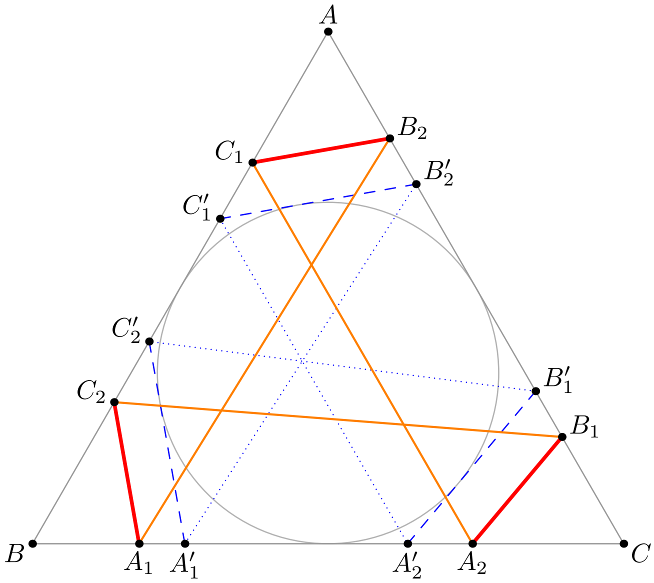
Claim ($p = 1$ implies concurrence) — Suppose the six points are chosen so that triangles AB_2C_1, BC_2A_1, CA_2B_1 all have perimeter 1. Then lines $\overline{B_1C_2}$, $\overline{C_1A_2}$, and $\overline{A_1B_2}$ are concurrent.

Proof. The perimeter conditions mean that $\overline{B_2C_1}$, $\overline{C_2A_1}$, and $\overline{A_2B_1}$ are tangent to the incircle of $\triangle ABC$.



Hence the result follows by *Brianchon's theorem*. □

Now suppose $p \neq 1$. Let $\overline{B_2'C_1'}$ be the dilation of $\overline{B_2C_1}$ with ratio $\frac{1}{p}$ at center A, and define C_2', A_1', A_2', B_1' similarly. The following diagram showcases the situation $p < 1$.



By the reasoning in the $p = 1$ case, note that $\overline{B_1'C_2'}$, $\overline{C_1'A_2'}$, and $\overline{A_1'B_2'}$ are concurrent. However, $\overline{B_1C_2}$, $\overline{C_1A_2}$, $\overline{A_1B_2}$ lie in the interior of quadrilaterals $BCB_1'C_2'$, $CAC_1'A_2'$, and $ABA_1'B_2'$, and these quadrilaterals do not share an interior point, a contradiction.

Thus $p \geq 1$. Similarly, we can show $p \leq 1$, and so $p = 1$ is forced (and achieved if, for example, the three triangles are equilateral with side length $1/3$).

¶ Barycentric solution (by Carl, Krit, Milan) We show that, if the common perimeter is 1, then the lines concur. To do this, we use barycentric coordinates. Let $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, and $C = (0 : 0 : 1)$. Let $A_1 = (0 : 1 - a_1 : a_1)$, $A_2 = (0 : a_2 : 1 - a_2)$, $B_1 = (b_1 : 0 : 1 - b_1)$, $B_2 = (1 - b_2 : 0 : b_2)$, $C_1 = (1 - c_1 : c_1 : 0)$, and $C_2 = (c_2 : 1 - c_2 : 0)$. The line B_1C_2 is defined by the equation

$$\det \begin{bmatrix} x & y & z \\ b_1 & 0 & 1 - b_1 \\ c_2 & 1 - c_2 & 0 \end{bmatrix} = 0;$$

i.e.

$$x\big(-(1 - b_1)(1 - c_2)\big) + y\big((1 - b_1)c_2\big) + z\big(b_1(1 - c_2)\big) = 0.$$

Computing the equations for the other lines cyclically, we get that the lines B_1C_2, C_1A_2, and A_1B_2 concur if and only if

$$\det \begin{bmatrix} -(1 - b_1)(1 - c_2) & (1 - b_1)c_2 & b_1(1 - c_2) \\ c_1(1 - a_2) & -(1 - c_1)(1 - a_2) & (1 - c_1)a_2 \\ (1 - a_1)b_2 & a_1(1 - b_2) & -(1 - a_1)(1 - b_2) \end{bmatrix} = 0.$$

Let this matrix be M. We also define the similar matrix

$$N = \begin{bmatrix} -(1 - b_2)(1 - c_1) & (1 - b_2)c_1 & b_2(1 - c_1) \\ c_2(1 - a_1) & -(1 - c_2)(1 - a_1) & (1 - c_2)a_1 \\ (1 - a_2)b_1 & a_2(1 - b_1) & -(1 - a_2)(1 - b_1) \end{bmatrix}.$$

Geometrically, $\det N = 0$ if and only if $B_2'C_1'$, $C_2'A_1'$, and $A_2'B_1'$ concur, where for a point P on a side of triangle ABC, P' denotes its reflection over that side's midpoint.

Claim — We have $\det M = \det N$.

Proof. To show $\det M = \det N$, it suffices to demonstrate that the determinant above is invariant under swapping subscripts of "1" and "2," an operation we call Ψ.

We use the definition of the determinant as a sum over permutations. The even permutations give us the following three terms:

$$\begin{aligned} -(1-b_1)(1-c_2)(1-c_1)(1-a_2)(1-a_1)(1-b_2) &= -\prod_{i=1}^{2}\left((1-a_i)(1-b_i)(1-c_i)\right) \\ (1-a_1)b_2(1-b_1)c_2(1-c_1)a_2 &= \left((1-a_1)(1-b_1)(1-c_1)\right)\left(a_2b_2c_2\right) \\ c_1(1-a_2)a_1(1-b_2)b_1(1-c_2) &= \left((1-a_2)(1-b_2)(1-c_2)\right)\left(a_1b_1c_1\right). \end{aligned}$$

The first term is invariant under Ψ, while the second and third terms are swapped under Ψ. For the odd permutations, we have a contribution to the determinant of

$$\sum_{\text{cyc}}(1-b_1)(1-c_2)(1-c_1)a_2a_1(1-b_2);$$

each summand is invariant under Ψ. This finishes the proof of our claim. $\square$

Now, it suffices to show that, if AB_2C_1, BC_2A_1, and CA_2B_1 each have perimeter 1, then

$$\det\begin{bmatrix} -(1-b_2)(1-c_1) & (1-b_2)c_1 & b_2(1-c_1) \\ c_2(1-a_1) & -(1-c_2)(1-a_1) & (1-c_2)a_1 \\ (1-a_2)b_1 & a_2(1-b_1) & -(1-a_2)(1-b_1). \end{bmatrix} = 0.$$

Indeed, we have $AB_2 = b_2$ and $AC_1 = c_1$, so by the law of cosines,

$$1 - b_2 - c_1 = 1 - AB_2 - AC_1 = B_2C_1 = \sqrt{b_2^2 + c_1^2 - b_2c_1}.$$

This gives

$$(1-b_2-c_1)^2 = b_2^2 + c_1^2 - b_2c_1 \implies 1 - 2b_2 - 2c_1 + 3b_2c_1 = 0.$$

Similarly, $1 - 2c_2 - 2a_1 + 3c_2a_1 = 0$ and $1 - 2a_2 - 2b_1 + 3a_2b_1 = 0$.

Now,

$$\begin{aligned} N\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} -(1-b_2)(1-c_1) + (1-b_2)c_1 + b_2(1-c_1) \\ -(1-c_2)(1-a_1) + (1-c_2)a_1 + c_2(1-a_1) \\ -(1-a_2)(1-b_1) + (1-a_2)b_1 + a_2(1-b_1) \end{bmatrix} \\ &= \begin{bmatrix} -1 + 2b_2 + 2c_1 - 3b_2c_1 \\ -1 + 2c_2 + 2a_1 - 3c_2a_1 \\ -1 + 2a_2 + 2b_1 - 2a_2b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

So it follows $\det N = 0$, as desired.

§3.3 TSTST 2023/9, proposed by Holden Mui

Available online at https://aops.com/community/p28015688.

Problem statement

Let p be a fixed prime and let $a \geq 2$ and $e \geq 1$ be fixed integers. Given a function $f \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ and an integer $k \geq 0$, the *kth finite difference*, denoted $\Delta^k f$, is the function from $\mathbb{Z}/a\mathbb{Z}$ to $\mathbb{Z}/p^e\mathbb{Z}$ defined recursively by

$$\begin{aligned} \Delta^0 f(n) &= f(n) \\ \Delta^k f(n) &= \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n) \qquad \text{for } k = 1, 2, \dots. \end{aligned}$$

Determine the number of functions f such that there exists some $k \geq 1$ for which $\Delta^k f = f$.

The answer is

$$(p^e)^a \cdot p^{-ep^{\nu_p(a)}} = p^{e(a - p^{\nu_p(a)})}.$$

¶ First solution by author For convenience in what follows, set $d = \nu_p(a)$, let $a = p^d \cdot b$, and let a function $f \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be *essential* if it equals one of its iterated finite differences.

The key claim is the following.

Claim (Characterization of essential functions) — A function f is essential if and only if

$$f(x) + f(x + p^d) + \cdots + f(x + (b-1)p^d) = 0 \qquad (2)$$

for all x.

As usual, we split the proof into two halves.

Proof that essential implies the equation First, suppose that f is essential, with $\Delta^N f = f$. Observe that f is in the image of Δ^k for any k, because $\Delta^{mN} f = f$ for any m. The following lemma will be useful.

Lemma

Let $g \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be any function, and let $h = \Delta^{p^d} g$. Then

$$h(x) + h(x + p^d) + \cdots + h(x + (b-1)p^d) \equiv 0 \pmod p$$

for all x.

Proof. By definition,

$$h(x) = \Delta^{p^d} g(x) = \sum_{k=0}^{p^d} (-1)^k \binom{p^d}{k} g(x + p^d - k).$$

However, it is known that $\binom{p^d}{k}$ is a multiple of p if $1 \leq k \leq p^d - 1$, so

$$h(x) \equiv g(x + p^d) + (-1)^{p^d} g(x) \pmod p.$$

Using this, we easily obtain

$$\begin{aligned} &h(x) + h(x+p^d) + \cdots + h(x+(b-1)p^d) \\ &\equiv \begin{cases} 0 & p > 2 \\ 2(g(x) + g(x+p^d) + \cdots + g(x+(b-1)p^d)) & p = 2 \end{cases} \\ &\equiv 0 \pmod p, \end{aligned}$$

as desired. □

Corollary

Let $g\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be any function, and let $h = \Delta^{ep^d} g$. Then

$$h(x) + h(x+p^d) + \cdots + h(x+(b-1)p^d) = 0$$

for all x.

Proof. Starting with the lemma, define

$$h_1(x) = \frac{h(x) + h(x+p^d) + \cdots + h(x+(b-1)p^d)}{p}.$$

Applying the lemma to h_1 shows the corollary for $e = 2$, since $h_1(x)$ is divisible by p, hence the numerator is divisible by p^2. Continue in this manner to get the result for general $e > 2$. □

This immediately settles this direction, since f is in the image of Δ^{ep^d}.

Proof the equation implies essential Let $\mathcal{S}$ be the set of all functions satisfying 2; then it's easy to see that Δ is a function on $\mathcal{S}$. To show that all functions in $\mathcal{S}$ are essential, it's equivalent to show that Δ is a permutation on $\mathcal{S}$.

We will show that Δ is injective on $\mathcal{S}$. Suppose otherwise, and consider two functions f, g in $\mathcal{S}$ with $\Delta f = \Delta g$. Then, we obtain that f and g differ by a constant; say $g = f + \lambda$. However, then

$$\begin{aligned} &g(0) + g(p^e) + \cdots + g((b-1)p^e) \\ &= (f(0) + \lambda) + (f(p^e) + \lambda) + \cdots + (f((b-1)p^e) + \lambda) \\ &= b\lambda. \end{aligned}$$

This should also be zero. Since $p \nmid b$, we obtain $\lambda = 0$, as desired.

Counting Finally, we can count the essential functions: all but the last p^d entries can be chosen arbitrarily, and then each remaining entry has exactly one possible choice. This leads to a count of

$$(p^e)^{a-p^d} = p^{e(a-p^{\nu_p(a)})},$$

as promised.

¶ Second solution by Daniel Zhu There are two parts to the proof: solving the $e = 1$ case, and using the $e = 1$ result to solve the general problem by induction on e. These parts are independent of each other.

The case $e = 1$ Represent functions f as elements

$$\alpha_f := \sum_{k \in \mathbb{Z}/a\mathbb{Z}} f(-k)x^k \in \mathbb{F}_p[x]/(x^a - 1)$$

Then, since $\alpha_{\Delta f} = (x-1)\alpha_f$, we wish to find the number of $\alpha \in \mathbb{F}_p[x]/(x^a - 1)$ such that $(x-1)^m \alpha = \alpha$ for some m.

Now, make the substitution $y = x - 1$ and let $P(y) = (y+1)^a - 1$; we want to find $\alpha \in \mathbb{F}_p[y]/(P(y))$ such that $y^m \alpha = \alpha$ for some m.

If we write $P(y) = y^d Q(y)$ with $Q(0) \neq 0$, then by the Chinese Remainder Theorem we have the ring isomorphism

$$\mathbb{F}_p[y]/(P(y)) \cong \mathbb{F}_p[y]/(y^d) \times \mathbb{F}_p[y]/(Q(y)).$$

Note that y is nilpotent in the first factor, while it is a unit in the second factor. So the α that work are exactly those that are zero in the first factor; thus there are p^{a-d} such α. We can calculate $d = p^{v_p(a)}$ (via, say, Lucas's Theorem), so we are done.

The general problem The general idea is as follows: call a $f\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ *e-good* if $\Delta^m f = f$ for some m. Our result above allows us to count the 1-good functions. Then, if $e \geq 1$, every $(e+1)$-good function, when reduced mod p^e, yields an e-good function, so we count $(e+1)$-good functions by counting how many reduce to any given e-good function.

Formally, we use induction on e, with the $e = 1$ case being treated above. Suppose now we have solved the problem for a given $e \geq 1$, and we now wish to solve it for $e+1$. For any function $g\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^{e+1}\mathbb{Z}$, let $\bar{g}\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be its reduction mod p^e. For a given e-good f, let $n(f)$ be the number of $(e+1)$-good g with $\bar{g} = f$. The following two claims now finish the problem:

Claim — If f is e-good, then $n(f) > 0$.

Proof. Suppose m is such that $\Delta^m f = f$. Pick any g with $\bar{g} = f$, and consider the sequence of functions

$$g, \Delta^m g, \Delta^{2m} g, \dots.$$

Since there are finitely many functions $\mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^{e+1}\mathbb{Z}$, there must exist $a < b$ such that $\Delta^{am} g = \Delta^{bm} g$. We claim $\Delta^{am} g$ is the desired $(e+1)$-good function. To see this, first note that since $\overline{\Delta^k g} = \Delta^k \bar{g}$, we must have $\overline{\Delta^{am} g} = \Delta^{am} f = f$. Moreover,

$$\Delta^{(b-a)m}(\Delta^{am} g) = \Delta^{bm} g = \Delta^{am} g,$$

so $\Delta^{am} g$ is $(e+1)$-good. □

Claim — If f is e-good, and $n(f) > 0$, then $n(f)$ is exactly the number of 1-good functions, i.e. $p^{a-p^{v_p(a)}}$.

Proof. Let g be any $(e+1)$-good function with $\bar{g} = f$. We claim that the $(e+1)$-good g_1 with $\bar{g}_1 = f$ are exactly the functions of the form $g + p^e h$ for any 1-good h. Since these functions are clearly distinct, this characterization will prove the claim.

To show that this condition is sufficient, note that $\overline{g + p^e h} = \bar{g} = f$. Moreover, if $\Delta^m g = g$ and $\Delta^{m'} h = h$, then

$$\Delta^{mm'}(g + p^e h) = \Delta^{mm'} g + p^e \Delta^{mm'} h = g + p^e h.$$

To show that this condition is necessary, let g_1 be any $(e+1)$-good function such that $\bar{g}_1 = f$. Then $g_1 - g$ is also $(e+1)$-good, since if $\Delta^m g = g$, $\Delta^{m'} g_1 = g_1$, we have

$$\Delta^{mm'}(g_1 - g) = \Delta^{mm'} g_1 - \Delta^{mm'} g = g_1 - g.$$

On the other hand, we also know that $g_1 - g$ is divisible by p^e. This means that it must be $p^e h$ for some function $f \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$, and it is not hard to show that $g_1 - g$ being $(e+1)$-good means that h is 1-good. $\square$

TSTST 2023 Statistics

Mathematical Olympiad Summer Program

EVAN CHEN《陳誼廷》

June 25, 2023

The cutoff for TST group is **24 points** (this was 38 students of 65 eligible students).

§1 Summary of scores for TSTST 2023

N	65	1st Q	20	Max	58
μ	26.92	Median	26	Top 3	53
σ	12.78	3rd Q	32	Top 12	35

§2 Problem statistics for TSTST 2023

	P1	P2	P3	P4	P5	P6	P7	P8	P9
0	10	36	35	21	32	52	2	35	47
1	5	6	11	2	11	1	2	4	7
2	0	0	3	0	4	0	3	4	1
3	0	0	2	0	4	0	0	1	3
4	0	0	0	0	0	0	0	3	1
5	0	0	0	1	4	1	1	1	1
6	0	1	2	0	3	1	6	3	0
7	50	22	12	41	7	10	51	14	5
Avg	5.46	2.55	1.83	4.52	1.82	1.26	6.25	2.28	0.95
QM	6.15	4.15	3.28	5.60	3.07	2.91	6.51	3.71	2.23
#5+	**50**	**23**	**14**	**42**	**14**	**12**	**58**	**18**	**6**
%5+	%76.9	%35.4	%21.5	%64.6	%21.5	%18.5	%89.2	%27.7	%9.2

§3 Rankings for TSTST 2023

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
63	0	0	0.00%	42	1	9	13.85%	21	6	48	73.85%
62	0	0	0.00%	41	0	9	13.85%	20	3	51	78.46%
61	0	0	0.00%	40	1	10	15.38%	19	0	51	78.46%
60	0	0	0.00%	39	0	10	15.38%	18	0	51	78.46%
59	0	0	0.00%	38	1	11	16.92%	17	1	52	80.00%
58	1	1	1.54%	37	0	11	16.92%	16	0	52	80.00%
57	0	1	1.54%	36	0	11	16.92%	15	2	54	83.08%
56	1	2	3.08%	35	5	16	24.62%	14	1	55	84.62%
55	0	2	3.08%	34	0	16	24.62%	13	1	56	86.15%
54	0	2	3.08%	33	0	16	24.62%	12	1	57	87.69%
53	1	3	4.62%	32	4	20	30.77%	11	0	57	87.69%
52	0	3	4.62%	31	2	22	33.85%	10	2	59	90.77%
51	1	4	6.15%	30	5	27	41.54%	9	3	62	95.38%
50	1	5	7.69%	29	2	29	44.62%	8	0	62	95.38%
49	2	7	10.77%	28	1	30	46.15%	7	1	63	96.92%
48	0	7	10.77%	27	0	30	46.15%	6	0	63	96.92%
47	0	7	10.77%	26	3	33	50.77%	5	0	63	96.92%
46	0	7	10.77%	25	3	36	55.38%	4	0	63	96.92%
45	0	7	10.77%	24	2	38	58.46%	3	0	63	96.92%
44	1	8	12.31%	23	2	40	61.54%	2	1	64	98.46%
43	0	8	12.31%	22	2	42	64.62%	1	0	64	98.46%
								0	1	65	100.00%

§4 Histogram for TSTST 2023