COMPENDIUM TST

Team Selection Test

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Toomates Coolección

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Índex.

Fuente.

https://web.evanchen.cc/problems.html

41st IMO Team Selection Test

Lincoln, Nebraska

Day I 1:00 p.m. - 5:30 p.m.

June 10, 2002

1. Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}.$$

2. Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to $\overline{AB}$ and $\overline{CD}$, respectively. Prove that $\overline{EF}$ is perpendicular to the line through the midpoints of $\overline{AD}$ and $\overline{BC}$.

3. Let p be a prime number. For integers r, s such that $rs(r^2 - s^2)$ is not divisible by p, let $f(r, s)$ denote the number of integers $n \in \{1, 2, \ldots, p-1\}$ such that $\{rn/p\}$ and $\{sn/p\}$ are either both less than $1/2$ or both greater than $1/2$. Prove that there exists $N > 0$ such that for $p \geq N$ and all r, s,

$$\left\lceil \frac{p-1}{3} \right\rceil \leq f(r, s) \leq \left\lfloor \frac{2(p-1)}{3} \right\rfloor.$$

41^{st} IMO Team Selection Test

Lincoln, Nebraska

Day II 1:00 p.m. - 5:30 p.m.

June 11, 2002

4. Let n be a positive integer. Prove that

$$\binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1} = \frac{n+1}{2^{n+1}} \left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1} \right).$$

5. Let n be a positive integer. A *corner* is a finite set S of ordered n-tuples of positive integers such that if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are positive integers with $a_k \geq b_k$ for $k = 1, 2, \ldots, n$ and $(a_1, a_2, \ldots, a_n) \in S$, then $(b_1, b_2, \ldots, b_n) \in S$. Prove that among any infinite collection of corners, there exist two corners, one of which is a subset of the other one.

6. Let ABC be a triangle inscribed in a circle of radius R, and let P be a point in the interior of ABC. Prove that

$$\frac{PA}{BC^2} + \frac{PB}{CA^2} + \frac{PC}{AB^2} \geq \frac{1}{R}.$$

42^{nd} IMO Team Selection Test

Washington, D.C.

Day I 1:00 p.m. - 5:30 p.m.

June 9, 2001

1. Let $\{ a_n \}_{n \geq 0}$ be a sequence of real numbers such that $a_{n+1} \geq a_n^2 + \frac{1}{5}$ for all $n \geq 0$. Prove that $\sqrt{a_{n+5}} \geq a_{n-5}^2$ for all $n \geq 5$.

2. Express
$$\sum_{k=0}^{n} (-1)^k (n-k)!(n+k)!$$
in closed form.

3. For a set S, let $|S|$ denote the number of elements in S. Let A be a set of positive integers with $|A| = 2001$. Prove that there exists a set B such that
   (i) $B \subseteq A$;
   (ii) $|B| \geq 668$;
   (iii) for any $u, v \in B$ (not necessarily distinct), $u + v \notin B$.

42^{nd} IMO Team Selection Test

Lincoln, Nebraska

Day II 1:00 p.m. - 5:30 p.m.

June 10, 2001

4. There are 51 senators in a senate. The senate needs to be divided into n committees so that each senator is on one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does *not* necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

5. In triangle ABC, $\angle B = 2\angle C$. Let P and Q be points on the perpendicular bisector of segment BC such that rays AP and AQ trisect $\angle A$. Prove that $PQ < AB$ if and only if $\angle B$ is obtuse.

6. Let a, b, c be positive real numbers such that

$$a + b + c \geq abc.$$

Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

42^{nd} IMO Team Selection Test

Washington, D.C.

Day III 1:00 p.m. - 5:30 p.m.

June 11, 2001

7. Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle ADC = 135^\circ$ and

$$AC^2 \cdot BD^2 = 2AB \cdot BC \cdot CD \cdot DA.$$

Prove that the diagonals of quadrilateral $ABCD$ are perpendicular.

8. Find all pairs of nonnegative integers (m, n) such that

$$(m + n - 5)^2 = 9mn.$$

9. Let A be a finite set of positive integers. Prove that there exists a finite set B of positive integers such that $A \subseteq B$ and

$$\prod_{x \in B} x = \sum_{x \in B} x^2.$$

43^{rd} IMO Team Selection Test

Lincoln, Nebraska

Day I 8:30 a.m. - 1:00 p.m.

June 21, 2002

1. Let ABC be a triangle. Prove that

$$\sin\frac{3A}{2}+\sin\frac{3B}{2}+\sin\frac{3C}{2}\le\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+\cos\frac{C-A}{2}.$$

2. Let p be a prime number greater than 5. For any integer x, define

$$f_p(x)=\sum_{k=1}^{p-1}\frac{1}{(px+k)^2}.$$

Prove that for all positive integers x and y the numerator of $f_p(x)-f_p(y)$, when written in lowest terms, is divisible by p^3.

3. Let n be an integer greater than 2, and $P_1, P_2, \cdots, P_n$ distinct points in the plane. Let $\mathcal{S}$ denote the union of all segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$. Determine if it is always possible to find points A and B in $\mathcal{S}$ such that $P_1P_n \parallel AB$ (segment AB can lie on line P_1P_n) and $P_1P_n = kAB$, where (1) $k = 2.5$; (2) $k = 3$.

43^{rd} IMO Team Selection Test

Lincoln, Nebraska

Day II 8:30 a.m. - 1:00 p.m.

June 22, 2002

4. Let n be a positive integer and let S be a set of $2^n + 1$ elements. Let f be a function from the set of two-element subsets of S to $\{0, \ldots, 2^{n-1} - 1\}$. Assume that for any elements x, y, z of S, one of $f(\{x, y\}), f(\{y, z\}), f(\{z, x\})$ is equal to the sum of the other two. Show that there exist a, b, c in S such that $f(\{a, b\}), f(\{b, c\}), f(\{c, a\})$ are all equal to 0.

5. Consider the family of nonisoceles triangles ABC satisfying the property $AC^2 + BC^2 = 2AB^2$. Points M and D lie on side AB such that $AM = BM$ and $\angle ACD = \angle BCD$. Point E is in the plane such that D is the incenter of triangle CEM. Prove that exactly one of the ratios
$$\frac{CE}{EM}, \quad \frac{EM}{MC}, \quad \frac{MC}{CE}$$
is constant.

6. Find in explicit form all ordered pairs of positive integers (m, n) such that $mn - 1$ divides $m^2 + n^2$.

2 Team Selection Test

44th IMO Team Selection Test

Lincoln, Nebraska

Day I 1:00 PM – 5:30 PM

June 20, 2003

1. For a pair of integers a and b, with $0 < a < b < 1000$, the set $S \subseteq \{1, 2, \ldots, 2003\}$ is called a *skipping set* for (a, b) if for any pair of elements $s_1, s_2 \in S$, $|s_1 - s_2| \notin \{a, b\}$. Let $f(a, b)$ be the maximum size of a skipping set for (a, b). Determine the maximum and minimum values of f.

2. Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, and PC intersect sides BC, CA, and AB at D, E, and F, respectively. Prove that
$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$
if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)

3. Find all ordered triples of primes (p, q, r) such that
$$p \mid q^r + 1, \quad q \mid r^p + 1, \quad r \mid p^q + 1.$$

44th IMO Team Selection Test

Lincoln, Nebraska

Day II 8:30 AM – 1:00 PM

June 21, 2003

4. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(m+n)f(m-n) = f(m^2)$$
for all $m, n \in \mathbb{N}$.

5. Let a, b, c be real numbers in the interval $(0, \frac{\pi}{2})$. Prove that
$$\frac{\sin a \sin(a-b)\sin(a-c)}{\sin(b+c)} + \frac{\sin b \sin(b-c)\sin(b-a)}{\sin(c+a)} + \frac{\sin c \sin(c-a)\sin(c-b)}{\sin(a+b)} \geq 0.$$

6. Let $\overline{AH_1}, \overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute scalene triangle ABC. The incircle of triangle ABC is tangent to $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at T_1, T_2, and T_3, respectively. For $k = 1, 2, 3$, let P_i be the point on line H_iH_{i+1} (where $H_4 = H_1$) such that $H_iT_iP_i$ is an acute isosceles triangle with $H_iT_i = H_iP_i$. Prove that the circumcircles of triangles $T_1P_1T_2$, $T_2P_2T_3$, $T_3P_3T_1$ pass through a common point.

2 Team Selection Test

1. The extremes can be obtained by different approaches. One requires the greedy algorithm, another applies congruence theory.
2. Apply the ingredients that prove **Ceva's Theorem** to convert this into an algebra problem.
3. Prove that one of the primes is 2.
4. Play with the given relation and compute many values of the function.
5. Reduce this to *Schur's Inequality*.
6. The common point is the orthocenter of triangle $T_1T_2T_3$.

2 Team Selection Test

1. For a pair of integers a and b, with $0 < a < b < 1000$, the set $S \subseteq \{1, 2, \ldots, 2003\}$ is called a *skipping set* for (a, b) if for any pair of elements $s_1, s_2 \in S$, $|s_1 - s_2| \notin \{a, b\}$. Let $f(a, b)$ be the maximum size of a skipping set for (a, b). Determine the maximum and minimum values of f.

Note. This problem caused unexpected difficulties for students. It requires two ideas: applying the greedy algorithm to obtain the minimum and applying the Pigeonhole Principle on congruence classes to obtain the maximum. Most students were successful in getting one of the two ideas and obtaining one of the extremal values quickly, but then many of them failed to switch to the other idea. In turn, their solutions for the second extremal value were very lengthy and sometimes unsuccessful.

Solution. The maximum and minimum values of f are 1334 and 338, respectively.

(a) First, we will show that the maximum value of f is 1334. The set $S = \{1, 2, \ldots, 667\} \cup \{1336, 1337, \ldots, 2002\}$ is a skipping set for $(a, b) = (667, 668)$, so $f(667, 668) \geq 1334$.

Now we prove that for any $0 < a < b < 1000$, $f(a, b) \leq 1334$. Because $a \neq b$, we can choose $d \in \{a, b\}$ such that $d \neq 668$. We assume first that $d \geq 669$. Then consider the $2003 - d \leq 1334$ sets $\{1, d+1\}, \{2, d+2\}, \ldots, \{2003-d, 2003\}$. Each can contain at most one element of S, so $|S| \leq 1334$.

We assume second that $d \leq 667$ and that $\lceil \frac{2003}{a} \rceil$ is even, that is, $\lceil \frac{2003}{a} \rceil = 2k$ for some positive integer k. Then each of the congruence classes of $1, 2, \ldots, 2003$ modulo a contains at most $2k$ elements. Therefore at most k members of each of these congruence classes can belong to S. Consequently,

$$\begin{aligned} |S| \leq ka &< \frac{1}{2}\left(\frac{2003}{a} + 1\right) a = \frac{2003 + a}{2} \\ &\leq 1335, \end{aligned}$$

implying that $|S| \leq 1334$.

Finally, we assume that $d \leq 667$ and that $\lceil \frac{2003}{a} \rceil$ is odd, that is, $\lceil \frac{2003}{a} \rceil = 2k + 1$ for some positive integer k. Then, as before, S can contain at most k elements from each congruence class of

$\{1, 2, \ldots, 2ka\}$ modulo a. Then

$$\begin{aligned}|S| &\le ka + (2003 - 2ka) = 2003 - ka \\ &= 2003 - \left(\frac{\lceil \frac{2003}{a} \rceil - 1}{2}\right) a \\ &\le 2003 - \left(\frac{\frac{2003}{a} - 1}{2}\right) a \\ &= \frac{2003 + a}{2} \le 1335.\end{aligned}$$

The last inequality holds if and only if $a = 667$. But if $a = 667$, then $\frac{2003}{a}$ is not an integer, and so the second inequality is strict. Thus, $|S| \le 1334$. Therefore the maximum value of f is 1334.

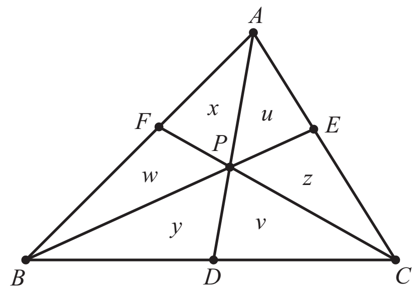
(b) We will now show that the minimum value of f is 668. First, we will show that $f(a, b) \ge 668$ by constructing a skipping set S for any (a, b) with $|S| \ge 668$. Note that if we add x to S, then we are not allowed to add x, $x+a$, or $x+b$ to S at any later time. Then at each step, let us add to S the smallest element of $\{1, 2, \ldots, 2003\}$ that is not already in S and that has not already been disallowed from being in S. Then since adding this element prevents at most three elements from being added at any future time, we can always perform this step $\lceil \frac{2003}{3} \rceil = 668$ times. Thus, $|S| \ge 668$, so $f(a, b) \ge 668$. Now notice that if we let $a = 1, b = 2$, then at most one element from each of the 668 sets $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\ldots$, $\{1999, 2000, 2001\}$, $\{2002, 2003\}$ can belong to S. This implies that $f(1, 2) = 668$, so indeed the minimum value of f is 668.

2. Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, and PC intersect sides BC, CA, and AB at D, E, and F, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$

if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)

Solution. Let $[PAF] = x$, $[PBD] = y$, $[PCE] = z$, $[PAE] = u$, $[PCD] = v$, and $[PBF] = w$.



Note first that

$$\frac{x}{w} = \frac{x+u+z}{w+y+v} = \frac{u+z}{y+v} = \frac{AF}{FB},$$
$$\frac{y}{v} = \frac{x+y+w}{u+v+z} = \frac{x+w}{u+z} = \frac{BD}{DC},$$
$$\frac{z}{u} = \frac{y+z+v}{x+u+w} = \frac{y+v}{x+w} = \frac{CE}{EA}.$$

Point P lies on one of the medians if and only if

$$(x-w)(y-v)(z-u) = 0. \qquad (*)$$

By **Ceva's Theorem**, we have

$$\frac{xyz}{uvw} = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1,$$

or,

$$xyz = uvw. \qquad (1)$$

Multiplying out $\frac{x}{w} = \frac{u+z}{y+v}$ yields $xy+xv = uw+zw$. Likewise, $uy+yz = xv + vw$ and $xz + zw = uy + uv$. Summing up the last three relations, we obtain

$$xy + yz + zx = uv + vw + wu. \qquad (2)$$

Now we are ready to prove the desired result. We first prove the "if" part by assuming that P lies on one of the medians, say AD. Then $y = v$, and so $\frac{y}{v} = \frac{x+w}{u+z}$ and $xyz = uvw$ become $x + w = u + z$ and $xz = uw$, respectively. Then the numbers $x, -z$ and $u, -w$ have the same sum and the same product. It follows that $x = u$ and $z = w$. Therefore $x + y + z = u + v + w$, as desired.

Conversely, we assume that

$$x + y + z = u + v + w. \qquad (3)$$

From (1), (2), and (3) it follows that x, y, z and u, v, w are roots of the same degree three polynomial. Hence $\{x, y, z\} = \{u, v, w\}$. If $x = w$ or $y = v$ or $z = u$, then the conclusion follows by $(*)$. If $x = u$, $y = w$, and $z = v$, then from

$$\frac{x}{w} = \frac{u+z}{y+v} = \frac{u+z-x}{y+v-w} = \frac{z}{v} = 1,$$

we obtain $x = w$. Likewise, we have $y = v$, and so $x = y = z = u = v = w$, that is, P is the centroid of triangle ABC and the conclusion follows. Finally, if $x = v$, $y = u$, $z = w$, then from

$$\frac{x}{w} = \frac{x+u+z}{w+y+v} = \frac{x+y+z}{w+u+v} = 1,$$

we obtain $x = w$. Similarly, $y = v$ and P is again the centroid.

3. Find all ordered triples of primes (p, q, r) such that

$$p \mid q^r + 1, \quad q \mid r^p + 1, \quad r \mid p^q + 1.$$

Solution. Answer: $(2, 5, 3)$ and cyclic permutations.

We check that this is a solution:

$$2 \mid 126 = 5^3 + 1, \quad 5 \mid 10 = 3^2 + 1, \quad 3 \mid 33 = 2^5 + 1.$$

Now let p, q, r be three primes satisfying the given divisibility relations. Since q does not divide $q^r + 1$, $p \neq q$, and similarly $q \neq r$, $r \neq p$, so p, q and r are all distinct. We now prove a lemma.

Lemma. *Let p, q, r be distinct primes with $p \mid q^r + 1$, and $p > 2$. Then either $2r \mid p - 1$ or $p \mid q^2 - 1$.*

Proof. Since $p \mid q^r + 1$, we have

$$q^r \equiv -1 \not\equiv 1 \pmod{p}, \quad \text{because } p > 2,$$

but

$$q^{2r} \equiv (-1)^2 \equiv 1 \pmod{p}.$$

Let d be the order of q mod p; then from the above congruences, d divides $2r$ but not r. Since r is prime, the only possibilities are $d = 2$ or $d = 2r$. If $d = 2r$, then $2r \mid p - 1$ because $d \mid p - 1$. If $d = 2$, then $q^2 \equiv 1 \pmod{p}$ so $p \mid q^2 - 1$. This proves the lemma. ■

Now let's first consider the case where p, q and r are all odd. Since $p \mid q^r + 1$, by the lemma either $2r \mid p - 1$ or $p \mid q^2 - 1$. But $2r \mid p - 1$ is impossible because

$$2r \mid p-1 \Longrightarrow p \equiv 1 \pmod{r} \Longrightarrow 0 \equiv p^q + 1 \equiv 2 \pmod{r}$$

and $r > 2$. So we must have $p \mid q^2 - 1 = (q-1)(q+1)$. Since p is an odd prime and $q-1$, $q+1$ are both even, we must have

$$p \mid \frac{q-1}{2} \quad \text{or} \quad p \mid \frac{q+1}{2};$$

either way,

$$p \leq \frac{q+1}{2} < q.$$

But then by a similar argument we may conclude $q < r$, $r < p$, a contradiction.

Thus, at least one of p, q, r must equal 2. By a cyclic permutation we may assume that $p = 2$. Now $r \mid 2^q + 1$, so by the lemma, either $2q \mid r-1$ or $r \mid 2^2 - 1$. But $2q \mid r-1$ is impossible as before, because q divides $r^2 + 1 = (r^2 - 1) + 2$ and $q > 2$. Hence, we must have $r \mid 2^2 - 1$. We conclude that $r = 3$, and $q \mid r^2 + 1 = 10$. Because $q \neq p$, we must have $q = 5$. Hence (2, 5, 3) and its cyclic permutations are the only solutions.

4. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(m+n)f(m-n) = f(m^2)$$

for all $m, n \in \mathbb{N}$.

Solution. Function $f(n) = 1$, for all $n \in \mathbb{N}$, is the only function satisfying the conditions of the problem.

Note that

$$f(1)f(2n-1) = f(n^2) \quad \text{and} \quad f(3)f(2n-1) = f((n+1)^2)$$

for $n \geq 3$. Thus

$$\frac{f(3)}{f(1)} = \frac{f((n+1)^2)}{f(n^2)}.$$

Setting $\frac{f(3)}{f(1)} = k$ yields $f(n^2) = k^{n-3} f(9)$ for $n \geq 3$. Similarly, for all $h \geq 1$,

$$\frac{f(h+2)}{f(h)} = \frac{f((m+1)^2)}{f(m^2)}$$

for sufficiently large m and is thus also k. Hence $f(2h) = k^{h-1} f(2)$ and $f(2h+1) = k^h f(1)$.

But

$$\frac{f(25)}{f(9)} = \frac{f(25)}{f(23)} \cdots \frac{f(11)}{f(9)} = k^8$$

and

$$\frac{f(25)}{f(9)} = \frac{f(25)}{f(16)} \cdot \frac{f(16)}{f(9)} = k^2,$$

so $k = 1$ and $f(16) = f(9)$. This implies that $f(2h+1) = f(1) = f(2) = f(2j)$ for all j, h, so f is constant. From the original functional equation it is then clear that $f(n) = 1$ for all $n \in \mathbb{N}$.

5. Let a, b, c be real numbers in the interval $(0, \frac{\pi}{2})$. Prove that

$$\frac{\sin a \sin(a-b)\sin(a-c)}{\sin(b+c)} + \frac{\sin b \sin(b-c)\sin(b-a)}{\sin(c+a)} + \frac{\sin c \sin(c-a)\sin(c-b)}{\sin(a+b)} \geq 0.$$

Solution. By the **Product-to-sum formulas** and the **Double-angle formulas**, we have

$$\sin(\alpha-\beta)\sin(\alpha+\beta) = \frac{1}{2}[\cos 2\beta - \cos 2\alpha] = \sin^2\alpha - \sin^2\beta.$$

Hence, we obtain

$$\sin a \sin(a-b)\sin(a-c)\sin(a+b)\sin(a+c) = \sin c(\sin^2 a - \sin^2 b)(\sin^2 a - \sin^2 c)$$

and its analogous forms. Therefore, it suffices to prove that

$$x(x^2-y^2)(x^2-z^2) + y(y^2-z^2)(y^2-x^2) + z(z^2-x^2)(z^2-y^2) \geq 0,$$

where $x = \sin a$, $y = \sin b$, and $z = \sin c$ (hence $x, y, z > 0$). Since the last inequality is symmetric with respect to x, y, z, we may assume that $x \geq y \geq z > 0$. It suffices to prove that

$$x(y^2-x^2)(z^2-x^2) + z(z^2-x^2)(z^2-y^2) \geq y(z^2-y^2)(y^2-x^2),$$

which is evident as

$$x(y^2-x^2)(z^2-x^2) \geq 0$$

and

$$z(z^2 - x^2)(z^2 - y^2) \geq z(y^2 - x^2)(z^2 - y^2) \geq y(z^2 - y^2)(y^2 - x^2).$$

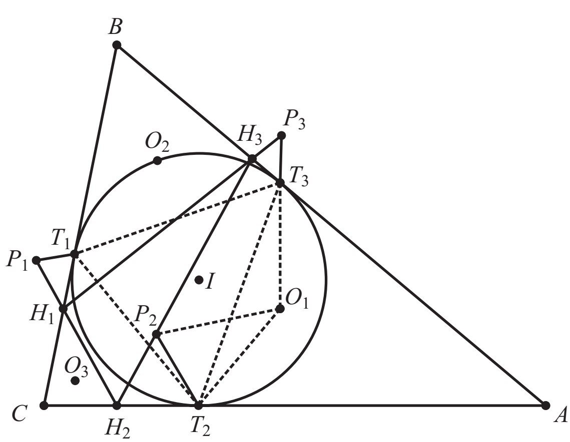
Note. The key step of the proof is an instance of **Schur's Inequality** with $r = \frac{1}{2}$.

6. Let $\overline{AH_1}, \overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute scalene triangle ABC. The incircle of triangle ABC is tangent to $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at T_1, T_2, and T_3, respectively. For $k = 1, 2, 3$, let P_i be the point on line H_iH_{i+1} (where $H_4 = H_1$) such that $H_iT_iP_i$ is an acute isosceles triangle with $H_iT_i = H_iP_i$. Prove that the circumcircles of triangles $T_1P_1T_2$, $T_2P_2T_3$, $T_3P_3T_1$ pass through a common point.

Note. We present three solutions. The first two are synthetic geometry approaches based on the following Lemma. The third solution calculates the exact position of the common point. In these solutions, all angles are directed modulo 180°. If reader is not familiar with the knowledge of directed angles, please refer our proofs with attached Figures. The proofs of the problem for other configurations can be developed in similar fashions.

Lemma. *The circumcenters of triangles $T_2P_2T_3$, $T_3P_3T_1$, and $T_1P_1T_2$ are the incenters of triangles AH_2H_3, BH_3H_1, and CH_1H_2, respectively.*

Proof. We prove that the circumcenter of triangle $T_2P_2T_3$ is the incenter of triangle AH_2H_3; the other two are analogous. It suffices to show that the perpendicular bisectors of T_2T_3 and T_2P_2 are the interior angle bisectors of $\angle H_3AH_2$ and $\angle AH_2H_3$. For the first pair, notice that

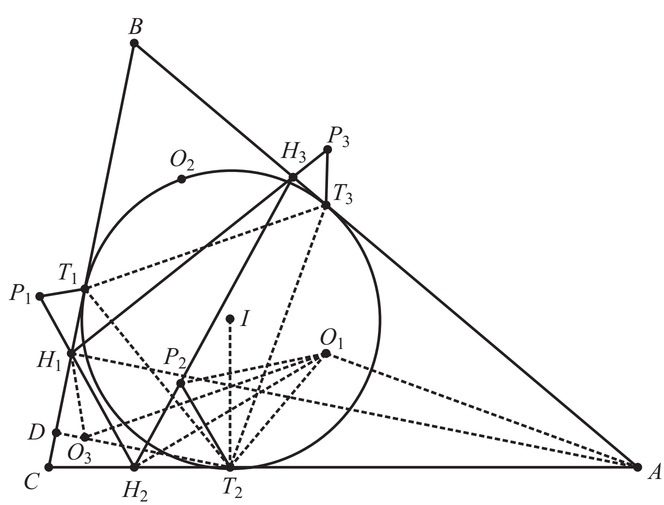


triangle AT_2T_3 is isosceles with $AT_2 = AT_3$ by equal tangents. Also, because triangle ABC is acute, T_2 is on ray AH_2 and T_3 is on ray AH_3. Therefore, the perpendicular bisector of T_2T_3 is the same as the interior angle bisector of $\angle T_3AT_2$, which is the same as the interior angle bisector of $\angle H_3AH_2$.

We prove the second pair similarly. Here, triangle $H_2T_2P_2$ is isosceles with $H_2T_2 = H_2P_2$ by assumption. Also, P_2 is on line H_2H_3 and T_2 is on line H_2A. Because quadrilateral BH_3H_2C is cyclic, $\angle AH_2H_3 = \angle B$ is acute. Now, $\angle T_2H_2P_2$ is also acute by assumption, so P_2 is on ray H_2H_3 if and only if T_2 is on ray H_2A. In other words, $\angle T_2H_2P_2$ either coincides with $\angle AH_2H_3$ or is the vertical angle opposite it. In either case, we see that the perpendicular bisector of T_2P_2 is the same as the interior angle bisector of $\angle T_2H_2P_2$, which is the same as the interior angle bisector of $\angle AH_2H_3$. ■

Let $\omega_1, \omega_2, \omega_3$ denote the circumcircles of triangles $T_2P_2T_3$, $T_3P_3T_1$, $T_1P_1T_2$, respectively. For $i = 1, 2, 3$, let O_i be center of ω_i. By the Lemma, O_1, O_2, O_3 are the incenters of triangles AH_2H_3, BH_3H_1, CH_1H_2, respectively. Let I, ω, and r be the incenter, incircle, and inradius of triangle ABC, respectively.

First Solution. (By Po-Ru Loh) We begin by showing that points O_3, H_2, T_2, and O_3 lie on a cyclic. We will prove this by establishing $\angle O_3O_1H_2 = \angle O_3T_2C = \angle O_3T_2H_2$. To find $\angle O_3O_1H_2$, observe that triangles H_2AH_3 and H_2H_1C are similar. Indeed, quadrilateral BH_3H_2C



is cyclic so $\angle H_2H_3A = \angle C$, and likewise $\angle CH_1H_2 = \angle A$. Now, O_1 and O_3 are corresponding incenters of similar triangles, so it follows that triangles H_2AO_1 and $H_2H_1O_3$ are also similar, and hence are related by a **spiral similarity** about H_2. Thus,

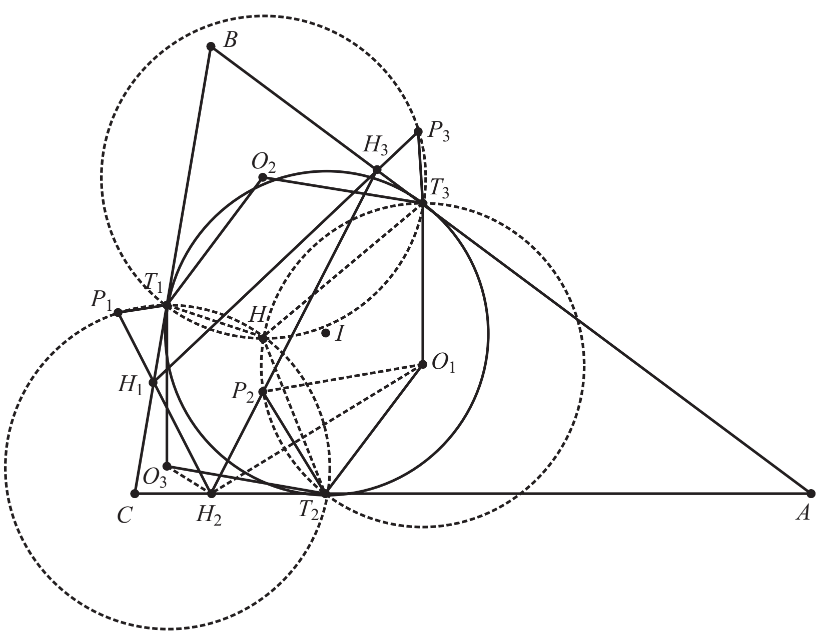
$$\frac{AH_2}{H_1H_2} = \frac{O_1H_2}{O_3H_2}$$

and

$$\begin{aligned}\angle AH_2H_1 &= \angle AH_2O_1 + \angle O_1H_2H_1\\ &= \angle O_1H_2H_1 + \angle H_1H_2O_3 = \angle O_1H_2O_3.\end{aligned}$$

It follows that another spiral similarity about H_2 takes triangle H_2AH_1 to triangle $H_2O_1O_3$. Hence $\angle O_3O_1H_2 = \angle H_1AH_2 = 90^\circ - \angle C$.

We wish to show that $\angle O_3T_2C = 90^\circ - \angle C$ as well, or in other words, $T_2O_3 \perp BC$. To do this, drop the altitude from O_3 to BC and let it intersect BC at D. Triangles ABC and H_1H_2C are similar as before, with corresponding incenters I and O_3. Furthermore, IT_2 and O_3D also correspond. Hence, $CT_2/T_2A = CD/DH_1$, and so $T_2D \parallel AH_1$. Thus, $T_2D \perp BC$, and it follows that $T_2O_3 \perp BC$.



Having shown that $O_1H_2T_2O_3$ is cyclic, we may now write $\angle O_1T_2O_3 = \angle O_1H_2O_3$. Since triangles H_2AO_1 and $H_2H_1O_3$ are related by a spiral similarity about H_2, we have

$$\angle O_1H_2O_3 = \angle AH_2H_1 = 180^\circ - \angle B,$$

by noting that ABH_2H_1 is cyclic. Likewise,

$$\angle O_2T_3O_1 = 180^\circ - \angle C \quad \text{and} \quad \angle O_3T_1O_2 = 180^\circ - \angle A,$$

and so $\angle O_1T_2O_3 + \angle O_2T_3O_1 + \angle O_3T_1O_2 = 360^\circ$. Therefore, $\angle T_3O_1T_2$, $\angle T_1O_2T_3$, and $\angle T_2O_3T_1$ of hexagon $O_1T_2O_3T_1O_2T_3$ also sum to 360°. Now let H be the intersection of circles ω_1 and ω_2. Then $\angle T_2HT_3 = 180^\circ - \frac{1}{2}\angle T_3O_1T_2$ and $\angle T_3HT_1 = 180^\circ - \frac{1}{2}\angle T_1O_2T_3$. Therefore,

$$\begin{aligned}\angle T_1HT_2 &= 360^\circ - \angle T_2HT_3 - \angle T_3HT_1\\ &= \frac{1}{2}\angle T_3O_1T_2 + \frac{1}{2}\angle T_1O_2T_3 = 180^\circ - \frac{1}{2}\angle T_1O_3T_2,\end{aligned}$$

and so H lies on the circle ω_3 as well. Hence, circles ω_1, ω_2, and ω_3 share a common point, as wanted.

Note. Readers might be nervous about the configurations, i.e., what if the hexagon $O_1T_2O_3T_1O_2T_3$ is not convex? Indeed, it is convex. It suffices to show that O_1, O_2, and O_3 are inside triangles AT_2T_3, BT_3T_1, and CT_1T_2, respectively. By symmetry, we only show that O_1 is inside AT_2T_3. Let d denote the distance from A to line T_2T_3. Then

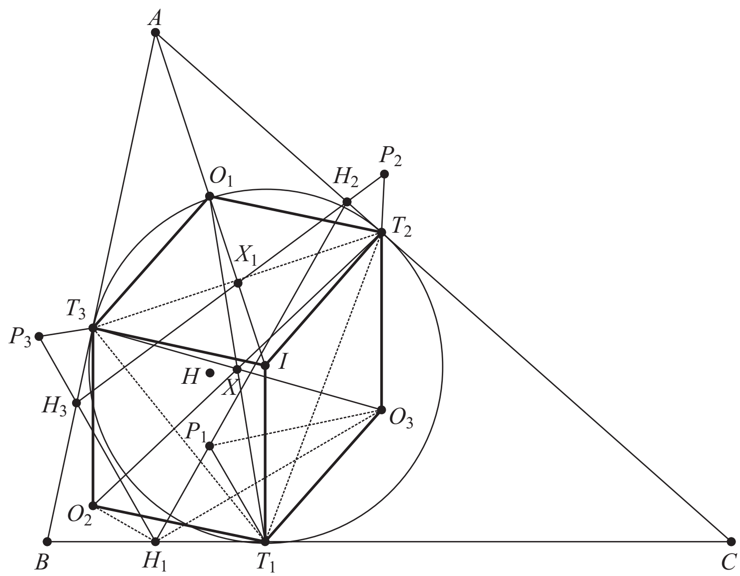
$$\frac{d}{AI} = \frac{d}{AT_2} \cdot \frac{AT_2}{AI} = \cos^2\frac{\angle A}{2}.$$

On the other hand, triangles AH_2H_3 and ABC are similar with ratio $\cos\angle A$. Hence

$$\frac{AO_1}{AI} = \cos\angle A = 2\cos^2\frac{\angle A}{2} - 1 \le \cos^2\frac{\angle A}{2} = \frac{d}{AI},$$

by the **Double-angle formulas**. We conclude that O_1 is inside triangle AT_2T_3. Our second proof is based on above arguments.

Second Solution. (By Anders Kaseorg) Note that $AH_2 = AB\cos\angle A$ and $AH_3 = AC\cos\angle A$, so triangles AH_2H_3 and ABC are similar with ratio $\cos\angle A$. Thus, since O_1 is the incircle of triangle AH_2H_3, $AO_1 = AI\cos\angle A$. If X_1 is the intersection of segments AI and T_2T_3,



we have $\angle IX_1T_2 = \angle AT_2I = 90^\circ$, and so

$$X_1I = T_2I \cos\angle T_2IA = AI\cos^2\angle T_2IA = AI\sin^2\frac{\angle A}{2}$$
$$= AI\cdot\frac{1-\cos\angle A}{2} = \frac{AI - AO_1}{2} = \frac{O_1I}{2}.$$

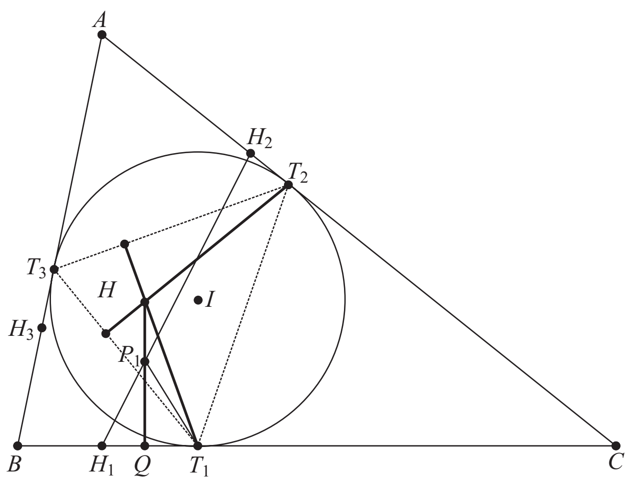
Hence $O_1X_1 = X_1I$, so O_1 is the reflection of I across line T_2T_3, and $O_1T_2 = IT_2 = IT_3 = O_1T_3$. Therefore, $O_1T_2IT_3$, and similarly $O_2T_3IT_1$ and $O_3T_1IT_2$, are rhombi with the same side length r, implying that circles $\omega_1, \omega_2, \omega$ have the same radius r. We also conclude that $O_1T_2 = T_3I = O_2T_1$ and $O_1T_2 \parallel T_3I \parallel O_2T_1$, and so $O_1O_2T_1T_2$ is a parallelogram. Hence the midpoints of O_1T_1 and O_2T_2 (similarly O_3T_3) are the same point P, and $O_1O_2O_3$ is the reflection of $T_1T_2T_3$ across P. If H is the reflection of I across P, we have $O_1H = O_2H = O_3H = r$, that is, H is a common point of the three circumcircles.

Note. Tony Zhang suggested the following finish. Because O_1 is the reflection of I across line T_2T_3 and I is the circumcenter of triangle $T_1T_2T_3$, $\angle T_3O_1T_2 = \angle T_2IT_3 = 2\angle T_2T_1T_3$. If H' is the orthocenter of triangle $T_1T_2T_3$, then

$$\angle T_2H'T_3 = 180^\circ - \angle T_2T_1T_3 = 180^\circ - \frac{\angle T_3O_1T_2}{2},$$

and so H' lies on ω_1. Similarly, H' lies on ω_2 and ω_3.

Third Solution. We use directed lengths (along line BC, with C to B as the positive direction) and directed angles modulo $180°$ in this proof. (For segments not lying on line BC, we assume its direction as the direction of its projection on line BC.) We claim that ω_i, $i = 1, 2, 3$, all pass through H, the orthocenter of triangle $T_1T_2T_3$. Without loss of generality, it suffices to prove that $T_1P_1T_2H$ is cyclic. If $AB = AC$, then $T_1 = H_1 = P_1$ and the case is trivial. Let $AB = c$, $BC = a$, $CA = b$, $\angle BAC = \alpha$, $\angle CBA = \beta$, and $\angle ACB = \gamma$.



Let Q be the intersection of lines HP_1 and BC. Note that

$$\begin{aligned}\angle HT_2T_1 &= 90° - \angle T_2T_1T_3 \\ &= 90° - [180° - \angle T_3T_1B - \angle CT_1T_2] \\ &= 90° - \left[180° - \left(90° - \frac{\beta}{2}\right) - \left(90° - \frac{C}{2}\right)\right] \\ &= \frac{\alpha}{2}.\end{aligned}$$

(Likewise, $\angle T_2T_1H = \beta/2$.) Thus to prove that $T_1P_1T_2H$ is cyclic is equivalent to prove that $\angle QP_1T_1 = \alpha/2$.

Let Q_H and Q_P be the respective feet of perpendiculars from H and P_1 to line BC. Because $\angle AH_1B = \angle AH_2B = 90°$, ABH_1H_2 is cyclic, and so $\angle T_1H_1P_1 = \angle BH_1P_1 = \alpha$. Thus triangles AT_3T_2 and $H_1T_1P_1$

are similar, implying that

$$\angle Q_P P_1 T_1 = 90^\circ - \angle P_1 T_1 H_1 = 90^\circ - \left(90^\circ - \frac{\angle T_1 H_1 P_1}{2}\right) = \frac{\alpha}{2}.$$

Therefore, to prove that $\angle QP_1T_1 = \alpha/2$, we have now reduced to proving that $Q_P = Q_H$, or

$$\frac{T_1Q_P}{T_1H_1} = \frac{T_1Q_H}{T_1H_1}. \qquad (1)$$

Note that

$$T_1H_1 = P_1H_1 \quad \text{and} \quad \frac{T_1Q_P}{T_1H_1} = 1 - \frac{Q_PH_1}{T_1H_1},$$

that is,

$$\frac{T_1Q_P}{T_1H_1} = 1 - \frac{Q_PH_1}{P_1H_1} = 1 - \cos\angle T_1H_1P_1 = 1 - \cos\alpha. \qquad (2)$$

On the other hand, applying the **Law of Cosines** to triangle ABC gives

$$\begin{aligned} T_1H_1 = T_1C - H_1C &= \frac{a+b-c}{2} - b\cos\gamma \\ &= \frac{a+b-c}{2} - \frac{a^2+b^2-c^2}{2a} = \frac{a(b-c)-(b^2-c^2)}{2a}, \end{aligned}$$

or

$$T_1H_1 = \frac{(b-c)(a-b-c)}{2a} = \frac{(c-b)(b+c-a)}{2a}. \qquad (3)$$

Now we calculate T_1Q_H. Because H is the orthocenter of triangle $T_1T_2T_3$,

$$\begin{aligned} \angle T_1HT_2 &= 180^\circ - \angle HT_2T_1 - \angle T_2T_1H \\ &= (90^\circ - \angle HT_2T_1) + (90^\circ - \angle T_2T_1H) \\ &= \angle T_2T_1T_3 + \angle T_3T_2T_1 = 180^\circ - \angle T_1T_3T_2. \end{aligned}$$

Applying the **Law of Sines** to triangle T_1T_2H and applying the **Extended Law of Sines** to triangle $T_1T_2T_3$ gives

$$\frac{T_1H}{\sin\angle HT_2T_1} = \frac{T_1T_2}{\sin\angle T_1HT_2} = \frac{T_1T_2}{\sin\angle T_1T_3T_2} = 2r,$$

and consequently,

$$T_1H = 2r\sin\angle HT_2T_1 = 2r\sin\frac{\alpha}{2}.$$

Because

$$\angle Q_H T_1 H = \angle C T_1 T_2 + \angle T_2 T_1 H = \left(90^\circ - \frac{\gamma}{2}\right) + \frac{\beta}{2}$$
$$= 90^\circ + \frac{\beta - \gamma}{2},$$

we obtain

$$T_1 Q_H = T_1 H \cos \angle H T_1 Q_H = 2r \sin\frac{\alpha}{2} \sin\frac{\gamma - \beta}{2}. \qquad (4)$$

Combining equations (1), (2), (3), and (4), we conclude that it suffices to prove that

$$1 - \cos\alpha = \frac{4ar \sin\frac{\alpha}{2} \sin\frac{\gamma-\beta}{2}}{(c-b)(b+c-a)}. \qquad (5)$$

Applying the fact

$$\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2} = \frac{r}{AT_2} = \frac{2r}{b+c-a},$$

and applying the Law of Sines to triangle ABC, (5) becomes

$$1 - \cos\alpha = \frac{2\sin\alpha \sin^2\frac{\alpha}{2} \sin\frac{\gamma-\beta}{2}}{\cos\frac{\alpha}{2}(\sin\gamma - \sin\beta)}. \qquad (6)$$

By the **Double-angle formulas**, $1 - \cos\alpha = 2\sin^2\frac{\alpha}{2}$ and $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ and so (6) reads

$$\sin\gamma - \sin\beta = 2\sin\frac{\alpha}{2}\sin\frac{\gamma-\beta}{2}.$$

By the **Difference-to-product formulas**, the last equation reduces to

$$2\cos\frac{\beta+\gamma}{2}\sin\frac{\gamma-\beta}{2} = 2\sin\frac{\alpha}{2}\sin\frac{\gamma-\beta}{2},$$

which is evident.

3 IMO

1. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\} \quad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

Note. The size $|S| = 10^6$ is unnecessarily large. See the second solution for a proof of the following stronger statement:

> If A is a k-element subset of $S = \{1, 2, \ldots, n\}$ and m is a positive integer such that $n > (m-1)\left(\binom{k}{2} + 1\right)$, then there exists $t_1, t_2, \ldots, t_m$ in S such that the sets $A_j = \{x + t_j \mid x \in A\}$, $j = 1, 2, \ldots, m$ are pairwise disjoint.

During the jury meeting, people decided to use the easier version as the first problem on the contest.

First Solution. Consider the set $D = \{x - y \mid x, y \in A\}$. There are at most $101 \times 100 + 1 = 10101$ elements in D (where the summand 1 represents the difference $x - y = 0$ for $x = y$). Two sets A_i and A_j have nonempty intersection if and only if $t_i - t_j$ is in D. It suffices to choose 100 numbers $t_1, t_2, \ldots, t_{100}$ in such a way that we do not obtain a difference from D.

We select these elements by induction. Choose one element arbitrarily. Assume that k elements, $k \le 99$, have already been chosen. An element x that is already chosen prevents us from selecting any element from the set $x + D = \{x + d \mid d \in D\}$. Thus, after k elements are chosen, at most $10101k \le 999999$ elements are forbidden. Hence we can select one more element. (Note that the numbers chosen are distinct because 0 is an element in D.)

Second Solution. (By Anders Kaseorg) We construct the set $\{t_j\}$ one element at a time using the following algorithm: Let $t_1 = 1 \in A$. For each j, $1 \le j \le 100$, let t_j be the smallest number in S that has not yet been crossed out, and then cross out t_j and all numbers of the form $t_j + |x - y|$ (with $x, y \in A$, $x \ne y$) that are in S. At each step, we cross out at most $1 + \binom{101}{2} = 5051$ new numbers. After picking t_1 through t_{99}, we have crossed out at most 500049 numbers, so there are always numbers in S that have not been crossed, so there are always candidates for t_j in S. (In fact, we will never need to pick a t_j bigger than 500050.)

2007 USA TST Problems

Contents

Problem 1

Circles ω_1 and ω_2 intersect at P and Q. AC and BD are chords of ω_1 and ω_2, respectively, such that P is on segment AB and on ray CD. Lines AC and BD intersect at X. Let the line through P parallel to AC intersect ω_2 again at Y, and let the line through P parallel to BD intersect ω_1 again at Z. Prove Q, X, Y, Z are collinear.

Solution

Problem 2

Let $a_1 \leq a_2 \leq ... \leq a_n$, $b_1 \leq b_2 \leq ... \leq b_n$ be two nonincreasing sequences of reals such that $a_1 \leq b_1$, $a_1 + a_2 \leq b_1 + b_2, \ldots, a_1 + a_2 + \ldots + a_{n-1} \leq b_1 + b_2 + \ldots + b_{n-1}$ and $a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n$ For any real number m, the number of pairs (i, j) such that $a_i - a_j = m$ is equal to the number of pairs (k, l) such that $b_k - b_l = m$. Prove that $a_i = b_i$ for $i = 1, 2, ..., n$.

Solution

Problem 3

For some $\theta \in (0, \frac{\pi}{2})$, $\cos \theta$ is irrational. If, for some positive integer k, $\cos(k\theta)$ and $\cos([k+1]\theta)$ are both rational, then show $\theta = \frac{\pi}{6}$.

Solution

Problem 4

Are there two positive integers (a, b) such that, for each positive integer n, $b^n - n$ is not divisible by a?

Solution

Problem 5

Let the tangents at B and C to the circumcircle of $\triangle ABC$ meet at T. Let the perpendicular to AT at A meet ray BC at S. Let B_1, C_1 lie on ST such that $B_1T = C_1T = BT$ and so that T lies between S and B_1. Prove that $\triangle AB_1C_1 \sim \triangle ABC$.

Solution

Problem 6

For any polynomial P, let $r(2i-1)$ be the remainder mod 1024 from 0 to 1023, inclusive, of $P(2i-1)$ for $i = 1, 2, ..., 512$. Call the set $\{r(1),\ r(3),\ r(5),\ \ldots,\ r(1023)\}$ the *remainder sequence* of P. Call a remaidner sequence **complete** if it is a permutation of $\{1, 3, 5, ..., 1023\}$. Show that the number of complete remainder sequences is at most 2^{35}.

Solution

See also

- TST

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USA Team Selection Test 2008

www.artofproblemsolving.com/community/c4638

by orl, dule_00, April, rrusczyk

Day 1

1 There is a set of n coins with distinct integer weights $w_1, w_2, \dots, w_n$. It is known that if any coin with weight w_k, where $1 \le k \le n$, is removed from the set, the remaining coins can be split into two groups of the same weight. (The number of coins in the two groups can be different.) Find all n for which such a set of coins exists.

2 Let P, Q, and R be the points on sides BC, CA, and AB of an acute triangle ABC such that triangle PQR is equilateral and has minimal area among all such equilateral triangles. Prove that the perpendiculars from A to line QR, from B to line RP, and from C to line PQ are concurrent.

3 For a pair $A = (x_1, y_1)$ and $B = (x_2, y_2)$ of points on the coordinate plane, let $d(A,B) = |x_1 - x_2| + |y_1 - y_2|$. We call a pair (A,B) of (unordered) points *harmonic* if $1 < d(A,B) \le 2$. Determine the maximum number of harmonic pairs among 100 points in the plane.

Day 2

4 Prove that for no integer n is $n^7 + 7$ a perfect square.

5 Two sequences of integers, $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$, satisfy the equation

$$(a_n - a_{n-1})(a_n - a_{n-2}) + (b_n - b_{n-1})(b_n - b_{n-2}) = 0$$

for each integer n greater than 2. Prove that there is a positive integer k such that $a_k = a_{k+2008}$.

6 Determine the smallest positive real number k with the following property. Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1, and D_1 lie on sides AB, BC, CD, and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1 and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \ge S$.

Author: Zuming Feng and Oleg Golberg, USA

Day 3

7 Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as

P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

8 Mr. Fat and Ms. Taf play a game. Mr. Fat chooses a sequence of positive integers $k_1, k_2, \dots, k_n$. Ms. Taf must guess this sequence of integers. She is allowed to give Mr. Fat a red card and a blue card, each with an integer written on it. Mr. Fat replaces the number on the red card with k_1 times the number on the red card plus the number on the blue card, and replaces the number on the blue card with the number originally on the red card. He repeats this process with number k_2. (That is, he replaces the number on the red card with k_2 times the number now on the red card plus the number now on the blue card, and replaces the number on the blue card with the number that was just placed on the red card.) He then repeats this process with each of the numbers $k_3, \dots k_n$, in this order. After has has gone through the sequence of integers, Mr. Fat then gives the cards back to Ms. Taf. How many times must Ms. Taf submit the red and blue cards in order to be able to determine the sequence of integers $k_1, k_2, \dots k_n$?

9 Let n be a positive integer. Given an integer coefficient polynomial $f(x)$, define its [i]signature modulo n[/i] to be the (ordered) sequence $f(1), \dots, f(n)$ modulo n. Of the n^n such n-term sequences of integers modulo n, how many are the signature of some polynomial $f(x)$ if
a) n is a positive integer not divisible by the square of a prime.
b) n is a positive integer not divisible by the cube of a prime.

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USA Team Selection Test 2009

www.artofproblemsolving.com/community/c4639

by MellowMelon, rrusczyk

Day 1

1 Let m and n be positive integers. Mr. Fat has a set S containing every rectangular tile with integer side lengths and area of a power of 2. Mr. Fat also has a rectangle R with dimensions $2^m \times 2^n$ and a 1×1 square removed from one of the corners. Mr. Fat wants to choose $m+n$ rectangles from S, with respective areas $2^0, 2^1, \ldots, 2^{m+n-1}$, and then tile R with the chosen rectangles. Prove that this can be done in at most $(m+n)!$ ways.

Palmer Mebane.

2 Let ABC be an acute triangle. Point D lies on side BC. Let O_B, O_C be the circumcenters of triangles ABD and ACD, respectively. Suppose that the points B, C, O_B, O_C lies on a circle centered at X. Let H be the orthocenter of triangle ABC. Prove that $\angle DAX = \angle DAH$.

Zuming Feng.

3 For each positive integer n, let $c(n)$ be the largest real number such that

$$c(n) \le \left| \frac{f(a) - f(b)}{a - b} \right|$$

for all triples (f, a, b) such that

–f is a polynomial of degree n taking integers to integers, and
–a, b are integers with $f(a) \neq f(b)$.

Find $c(n)$.

Shaunak Kishore.

Day 2

4 Let ABP, BCQ, CAR be three non-overlapping triangles erected outside of acute triangle ABC. Let M be the midpoint of segment AP. Given that $\angle PAB = \angle CQB = 45^\circ$, $\angle ABP = \angle QBC = 75^\circ$, $\angle RAC = 105^\circ$, and $RQ^2 = 6CM^2$, compute AC^2/AR^2.

Zuming Feng.

5 Find all pairs of positive integers (m, n) such that $mn - 1$ divides $(n^2 - n + 1)^2$.

Aaron Pixton.

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6 Let $N > M > 1$ be fixed integers. There are N people playing in a chess tournament; each pair of players plays each other once, with no draws. It turns out that for each sequence of $M+1$ distinct players $P_0, P_1, \ldots P_M$ such that P_{i-1} beat P_i for each $i = 1, \ldots, M$, player P_0 also beat P_M. Prove that the players can be numbered $1, 2, \ldots, N$ in such a way that, whenever $a \geq b + M - 1$, player a beat player b.

Gabriel Carroll.

Day 3

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7 Find all triples (x, y, z) of real numbers that satisfy the system of equations

$$\begin{cases} x^3 = 3x - 12y + 50, \\ y^3 = 12y + 3z - 2, \\ z^3 = 27z + 27x. \end{cases}$$

Razvan Gelca.

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8 Fix a prime number $p > 5$. Let a, b, c be integers no two of which have their difference divisible by p. Let i, j, k be nonnegative integers such that $i + j + k$ is divisible by $p - 1$. Suppose that for all integers x, the quantity

$$(x - a)(x - b)(x - c)[(x - a)^i (x - b)^j (x - c)^k - 1]$$

is divisible by p. Prove that each of i, j, k must be divisible by $p - 1$.

Kiran Kedlaya and Peter Shor.

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9 Prove that for positive real numbers x, y, z,

$$x^3(y^2 + z^2)^2 + y^3(z^2 + x^2)^2 + z^3(x^2 + y^2)^2 \geq xyz \left[xy(x + y)^2 + yz(y + z)^2 + zx(z + x)^2\right].$$

Zarathustra (Zeb) Brady.

–

USA Team Selection Test 2010

www.artofproblemsolving.com/community/c4640

by MellowMelon, rrusczyk

Day 1

1 Let P be a polynomial with integer coefficients such that $P(0) = 0$ and

$$\gcd(P(0), P(1), P(2), \ldots) = 1.$$

Show there are infinitely many n such that

$$\gcd(P(n) - P(0), P(n+1) - P(1), P(n+2) - P(2), \ldots) = n.$$

2 Let a, b, c be positive reals such that $abc = 1$. Show that

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

3 Let h_a, h_b, h_c be the lengths of the altitudes of a triangle ABC from A, B, C respectively. Let P be any point inside the triangle. Show that

$$\frac{PA}{h_b + h_c} + \frac{PB}{h_a + h_c} + \frac{PC}{h_a + h_b} \geq 1.$$

Day 2

4 Let ABC be a triangle. Point M and N lie on sides AC and BC respectively such that $MN||AB$. Points P and Q lie on sides AB and CB respectively such that $PQ||AC$. The incircle of triangle CMN touches segment AC at E. The incircle of triangle BPQ touches segment AB at F. Line EN and AB meet at R, and lines FQ and AC meet at S. Given that $AE = AF$, prove that the incenter of triangle AEF lies on the incircle of triangle ARS.

5 Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and, for $n > 1$,

$$a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/3 \rfloor} + \ldots + a_{\lfloor n/n \rfloor} + 1.$$

Prove that there are infinitely many n such that $a_n \equiv n \pmod{2^{2010}}$.

6 Let T be a finite set of positive integers greater than 1. A subset S of T is called *good* if for every $t \in T$ there exists some $s \in S$ with $\gcd(s,t) > 1$. Prove that the number of good subsets of T is odd.

Day 3

7 In triangle ABC, let P and Q be two interior points such that $\angle ABP = \angle QBC$ and $\angle ACP = \angle QCB$. Point D lies on segment BC. Prove that $\angle APB + \angle DPC = 180^\circ$ if and only if $\angle AQC + \angle DQB = 180^\circ$.

8 Let m, n be positive integers with $m \geq n$, and let S be the set of all n-term sequences of positive integers $(a_1, a_2, \dots a_n)$ such that $a_1 + a_2 + \cdots + a_n = m$. Show that

$$\sum_S 1^{a_1} 2^{a_2} \cdots n^{a_n} = \binom{n}{n} n^m - \binom{n}{n-1}(n-1)^m + \cdots + (-1)^{n-2}\binom{n}{2} 2^m + (-1)^{n-1}\binom{n}{1}.$$

9 Determine whether or not there exists a positive integer k such that $p = 6k+1$ is a prime and

$$\binom{3k}{k} \equiv 1 \pmod{p}.$$

Team Selection Test for the 54^{th} IMO

December 15, 2011

1. In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.

2. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every pair of real numbers x and y,

$$f(x + y^2) = f(x) + |yf(y)|.$$

3. Determine, with proof, whether or not there exist integers $a, b, c > 2010$ satisfying the equation

$$a^3 + 2b^3 + 4c^3 = 6abc + 1.$$

4. There are 2010 students and 100 classrooms in the Olympiad High School. At the beginning, each of the students is in one of the classrooms. Each minute, as long as not everyone is in the same classroom, somebody walks from one classroom into a different classroom with at least as many students in it (prior to his move). This process will terminate in M minutes. Determine the maximum value of M.

Team Selection Test for the 54^{th} IMO

February 1, 2012

1. Consider (3-variable) polynomials

$$P_n(x,y,z) = (x-y)^{2n}(y-z)^{2n} + (y-z)^{2n}(z-x)^{2n} + (z-x)^{2n}(x-y)^{2n}$$

and

$$Q_n(x,y,z) = [(x-y)^{2n} + (y-z)^{2n} + (z-x)^{2n}]^{2n}.$$

Determine all positive integers n such that the quotient $Q_n(x,y,z)/P_n(x,y,z)$ is a (3-variable) polynomial with rational coefficients.

2. In cyclic quadrilateral $ABCD$, diagonals AC and BD intersect at P. Let E and F be the respective feet of the perpendiculars from P to lines AB and CD. Segments BF and CE meet at Q. Prove that lines PQ and EF are perpendicular to each other.

3. Determine all positive integers n, $n \geq 2$, such that the following statement is true:

If $(a_1, a_2, \ldots, a_n)$ is a sequence of positive integers with $a_1 + a_2 + \cdots + a_n = 2n - 1$, then there is block of (at least two) consecutive terms in the sequence with their (arithmetic) mean being an integer.

4. Find all positive integers $a, n \geq 1$ such that for all primes p dividing $a^n - 1$, there exists a positive integer $m < n$ such that $p \mid a^m - 1$.

Team Selection Test for the 54$^{\text{th}}$ IMO

December 13, 2012

1. A social club has $2k+1$ members, each of whom is fluent in the same k languages. Any pair of members always talk to each other in only one language. Suppose that there were no three members such that they use only one language among them. Let A be the number of three-member subsets such that the three distinct pairs among them use different languages. Find the maximum possible value of A.

2. Find all triples (x, y, z) of positive integers such that $x \le y \le z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

3. Let ABC be a scalene triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. The circumcircle of triangle DKL intersects segment AB at a second point T (other than D). Prove that $\angle ACT = \angle BCT$.

4. Let f be a function from positive integers to positive integers, and let f^m be f applied m times. Suppose that for every positive integer n there exists a positive integer k such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

Team Selection Test for the 54th IMO

January 31, 2013

1. Two incongruent triangles ABC and XYZ are called a pair of *pals* if they satisfy the following conditions:

   (a) the two triangles have the same area;

   (b) let M and W be the respective midpoints of sides BC and YZ. The two sets of lengths $\{AB, AM, AC\}$ and $\{XY, XW, XZ\}$ are identical 3-element sets of pairwise relatively prime integers.

   Determine if there are infinitely many pairs of triangles that are pals of each other.

2. Let ABC be an acute triangle. Circle ω_1, with diameter AC, intersects side BC at F (other than C). Circle ω_2, with diameter BC, intersects side AC at E (other than C). Ray AF intersects ω_2 at K and M with $AK < AM$. Ray BE intersects ω_1 at L and N with $BL < BN$. Prove that lines AB, ML, NK are concurrent.

3. In a table with n rows and $2n$ columns where n is a fixed positive integer, we write either zero or one into each cell so that each row has n zeros and n ones. For $1 \le k \le n$ and $1 \le i \le n$, we define $a_{k,i}$ so that the i^{th} zero in the k^{th} row is the $a_{k,i}^{\text{th}}$ column. Let $\mathcal{F}$ be the set of such tables with $a_{1,i} \ge a_{2,i} \ge \cdots \ge a_{n,i}$ for every i with $1 \le i \le n$. We associate another $n \times 2n$ table $f(C)$ from $C \in \mathcal{F}$ as follows: for the k^{th} row of $f(C)$, we write n ones in the columns $a_{n,k} - k + 1, a_{n-1,k} - k + 2, \ldots, a_{1,k} - k + n$ (and we write zeros in the other cells in the row).

   (a) Show that $f(C) \in \mathcal{F}$.

   (b) Show that $f(f(f(f(f(f(C)))))) = C$ for any $C \in \mathcal{F}$.

4. Determine if there exists a (three-variable) polynomial $P(x, y, z)$ with integer coefficients satisfying the following property: a positive integer n is *not* a perfect square if and only if there is a triple (x, y, z) of positive integers such that $P(x, y, z) = n$.

Team Selection Test for the 55$^{\text{th}}$ International Mathematical Olympiad

United States of America

Day I

Thursday, December 12, 2013

Time limit: 4.5 hours. Each problem is worth 7 points.

IMO TST 1. Let ABC be an acute triangle, and let X be a variable interior point on the minor arc BC of its circumcircle. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point.

IMO TST 2. Let a_1, a_2, a_3, ...be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for all positive integers n and k, the quantity

$$\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}$$

is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to the same perfect square).

IMO TST 3. Let n be an even positive integer, and let G be an n-vertex (simple) graph with exactly $\frac{n^2}{4}$ edges. An unordered pair of distinct vertices $\{x, y\}$ is said to be *amicable* if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ pairs of vertices which are amicable.

Team Selection Test for the 55th International Mathematical Olympiad

United States of America

Day II

Thursday, January 23, 2014

Time limit: 4.5 hours. Each problem is worth 7 points.

IMO TST 4. Let n be a positive even integer, and let c_1, c_2, ..., c_{n-1} be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$

has no real roots.

IMO TST 5. Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, and DA respectively. Let W, X, Y and Z be the orthocenters of triangles AHE, BEF, CFG and DGH, respectively. Prove that the quadrilaterals $ABCD$ and $WXYZ$ have the same area.

IMO TST 6. For a prime p, a subset S of residues modulo p is called a *sum-free multiplicative subgroup* of $\mathbb{F}_p$ if

- there is a nonzero residue α modulo p such that $S = \left\{1, \alpha^1, \alpha^2, \dots\right\}$ (all considered mod p), and
- there are no $a, b, c \in S$ (not necessarily distinct) such that $a + b \equiv c \pmod{p}$.

Prove that for every integer N, there is a prime p and a sum-free multiplicative subgroup S of $\mathbb{F}_p$ such that $|S| \geq N$.

USA TST 2014 Solution Notes

Evan Chen《陳誼廷》

30 September 2023

This is a compilation of solutions for the 2014 USA TST. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let ABC be an acute triangle, and let X be a variable interior point on the minor arc BC of its circumcircle. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point.

2. Let $a_1, a_2, a_3, \dots$ be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for all positive integers n and k, the quantity
$$\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}$$
is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to the same perfect square).

3. Let n be an even positive integer, and let G be an n-vertex (simple) graph with exactly $\frac{n^2}{4}$ edges. An unordered pair of distinct vertices $\{x, y\}$ is said to be *amicable* if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ pairs of vertices which are amicable.

4. Let n be a positive even integer, and let $c_1, c_2, \dots, c_{n-1}$ be real numbers satisfying
$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$
Prove that
$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$
has no real roots.

5. Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, and DA respectively. Let W, X, Y and Z be the orthocenters of triangles AHE, BEF, CFG and DGH, respectively. Prove that the quadrilaterals $ABCD$ and $WXYZ$ have the same area.

6. For a prime p, a subset S of residues modulo p is called a *sum-free multiplicative subgroup* of $\mathbb{F}_p$ if
   - there is a nonzero residue α modulo p such that $S = \{1, \alpha^1, \alpha^2, \dots\}$ (all considered mod p), and
   - there are no $a, b, c \in S$ (not necessarily distinct) such that $a + b \equiv c \pmod p$.

   Prove that for every integer N, there is a prime p and a sum-free multiplicative subgroup S of $\mathbb{F}_p$ such that $|S| \geq N$.

§1 Solutions to Day 1

§1.1 USA TST 2014/1

Available online at https://aops.com/community/p3332310.

Problem statement

Let ABC be an acute triangle, and let X be a variable interior point on the minor arc BC of its circumcircle. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point.

The fixed point is the orthocenter, since ℓ is a Simson line. See Lemma 4.4 of *Euclidean Geometry in Math Olympiads*.

§1.2 USA TST 2014/2, proposed by Victor Wang

Available online at https://aops.com/community/p3332299.

Problem statement

Let $a_1, a_2, a_3, \dots$ be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for all positive integers n and k, the quantity

$$\frac{a_n + a_{n+1} + \dots + a_{n+k-1}}{k}$$

is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to the same perfect square).

Let $\nu_p(n)$ denote the largest exponent of p dividing n. The problem follows from the following proposition.

Proposition

Let (a_n) be a sequence of integers and let p be a prime. Suppose that every consecutive group of a_i's with length at most p averages to a perfect square. Then $\nu_p(a_i)$ is independent of i.

We proceed by induction on the smallest value of $\nu_p(a_i)$ as i ranges (which must be even, as each of the a_i are themselves a square). First we prove two claims.

Claim — If $j \equiv k \pmod p$ then $a_j \equiv a_k \pmod p$.

Proof. Taking groups of length p in our given, we find that $p \mid a_j + \dots + a_{j+p-1}$ and $p \mid a_{j+1} + \dots + a_{j+p}$ for any j. So $a_j \equiv a_{j+p} \pmod p$ and the conclusion follows. □

Claim — If some a_i is divisible by p then all of them are.

Proof. The case $p = 2$ is trivial so assume $p \ge 3$. Without loss of generality (via shifting indices) assume that $a_1 \equiv 0 \pmod p$, and define

$$S_n = a_1 + a_2 + \dots + a_n \equiv a_2 + \dots + a_n \pmod p.$$

Call an integer k with $2 \le k < p$ a **pivot** if $1 - k^{-1}$ is a quadratic nonresidue modulo p.

We claim that for any pivot k, $S_k \equiv 0 \pmod p$. If not, then

$$\frac{a_1 + a_2 + \dots + a_k}{k} \text{ and } \frac{a_2 + \dots + a_k}{k-1}$$

are both qudaratic residues. Division implies that $\frac{k-1}{k} = 1 - k^{-1}$ is a quadratic residue, contradiction.

Next we claim that there is an integer m with $S_m \equiv S_{m+1} \equiv 0 \pmod p$, which implies $p \mid a_{m+1}$. If 2 is a pivot, then we simply take $m = 1$. Otherwise, there are $\frac12(p-1)$ pivots, one for each nonresidue (which includes neither 0 nor 1), and all pivots lie in $[3, p-1]$, so we can find an m such that m and $m+1$ are both pivots.

Repeating this procedure starting with a_{m+1} shows that $a_{2m+1}, a_{3m+1}, \dots$ must all be divisible by p. Combined with the first claim and the fact that $m < p$, we find that all the a_i are divisible by p. $\square$

The second claim establishes the base case of our induction. Now assume all a_i are divisible by p and hence p^2. Then all the averages in our proposition (with length at most p) are divisible by p and hence p^2. Thus the map $a_i \mapsto \frac{1}{p^2} a_i$ gives a new sequence satisfying the proposition, and our inductive hypothesis completes the proof.

Remark. There is a subtle bug that arises if one omits the condition that $k \le p$ in the proposition. When $k = p^2$ the average $\frac{a_1 + \dots + a_{p^2}}{p^2}$ is not necessarily divisible by p even if all the a_i are. Hence it is not valid to divide through by p. This is why the condition $k \le p$ was added.

§1.3 USA TST 2014/3

Available online at https://aops.com/community/p3332307.

Problem statement

Let n be an even positive integer, and let G be an n-vertex (simple) graph with exactly $\frac{n^2}{4}$ edges. An unordered pair of distinct vertices $\{x, y\}$ is said to be *amicable* if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ pairs of vertices which are amicable.

First, we prove the following lemma. (https://en.wikipedia.org/wiki/Friendship_paradox).

Lemma (On average, your friends are more popular than you)

For a vertex v, let $a(v)$ denote the average degree of the neighbors of v (setting $a(v) = 0$ if $\deg v = 0$). Then

$$\sum_v a(v) \ge \sum_v \deg v = 2\#E.$$

Proof. Ignoring isolated vertices, we can write

$$\begin{aligned} \sum_v a(v) &= \sum_v \frac{\sum_{w\sim v} \deg w}{\deg v} \\ &= \sum_v \sum_{w\sim v} \frac{\deg w}{\deg v} \\ &= \sum_{\text{edges } vw} \left(\frac{\deg w}{\deg v} + \frac{\deg v}{\deg w} \right) \\ &\overset{\text{AM-GM}}{\ge} \sum_{\text{edges } vw} 2 = 2\#E = \sum_v \deg v \end{aligned}$$

as desired. □

Corollary (On average, your most popular friend is more popular than you)

For a vertex v, let $m(v)$ denote the maximum degree of the neighbors of v (setting $m(v) = 0$ if $\deg v = 0$). Then

$$\sum_v m(v) \ge \sum_v \deg v = 2\#E.$$

We can use this to count amicable pairs by noting that any particular vertex v is in at least $m(v) - 1$ amicable pairs. So, the number of amicable pairs is at least

$$\frac{1}{2} \sum_v (m(v) - 1) \ge \#E - \frac{1}{2}\#V.$$

Note that up until now we haven't used any information about G. But now if we plug in $\#E = n^2/4$, $\#V = n$, then we get exactly the desired answer. (Equality holds for $G = K_{n/2,n/2}$.)

§2 Solutions to Day 2

§2.1 USA TST 2014/4

Available online at https://aops.com/community/p3476290.

Problem statement

Let n be a positive even integer, and let $c_1, c_2, \dots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \dots - c_1x^1 + 2$$

has no real roots.

We will prove the polynomial is positive for all $x \in \mathbb{R}$. As $c_i > 0$, the result is vacuous for $x \le 0$, so we restrict attention to $x > 0$.

Then letting $c_i = 1 - d_i$ for each i, the inequality we want to prove becomes

$$x^n + 1 + \frac{x^{n+1} + 1}{x + 1} > \sum_{1}^{n-1} d_i x^i \qquad \text{given } \sum |d_i| < 1.$$

But obviously $x^n + 1 > x^i$ for any $1 \le i \le n-1$ and $x > 0$. So in fact $x^n + 1 > \sum_1^{n-1} |d_i| x^i$ holds for $x > 0$, as needed.

§2.2 USA TST 2014/5, proposed by Po-Shen Loh

Available online at https://aops.com/community/p3476291.

Problem statement

Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, and DA respectively. Let W, X, Y and Z be the orthocenters of triangles AHE, BEF, CFG and DGH, respectively. Prove that the quadrilaterals $ABCD$ and $WXYZ$ have the same area.

The following solution is due to Grace Wang.

We begin with:

Claim — Point W has coordinates $\frac{1}{2}(2a+b+d)$.

Proof. The orthocenter of $\triangle DAB$ is $d+a+b$, and $\triangle AHE$ is homothetic to $\triangle DAB$ through A with ratio $1/2$. Hence $w = \frac{1}{2}(a+(d+a+b))$ as needed. □

By symmetry, we have

$$\begin{aligned} w &= \frac{1}{2}(2a+b+d) \\ x &= \frac{1}{2}(2b+c+a) \\ y &= \frac{1}{2}(2c+d+b) \\ z &= \frac{1}{2}(2d+a+c). \end{aligned}$$

We see that $w-y=a-c$, $x-z=b-d$. So the diagonals of $WXYZ$ have the same length as those of $ABCD$ as well as the same directed angle between them. This implies the areas are equal, too.

§2.3 USA TST 2014/6

Available online at https://aops.com/community/p3476292.

Problem statement

For a prime p, a subset S of residues modulo p is called a *sum-free multiplicative subgroup* of $\mathbb{F}_p$ if

- there is a nonzero residue α modulo p such that $S = \{1, \alpha^1, \alpha^2, \dots\}$ (all considered mod p), and
- there are no $a, b, c \in S$ (not necessarily distinct) such that $a + b \equiv c \pmod p$.

Prove that for every integer N, there is a prime p and a sum-free multiplicative subgroup S of $\mathbb{F}_p$ such that $|S| \ge N$.

We first prove the following general lemma.

Lemma

If $f, g \in \mathbb{Z}[X]$ are relatively prime nonconstant polynomials, then for sufficiently large primes p, they have no common root modulo p.

Proof. By Bézout Lemma, there exist polynomials $a(X)$ and $b(X)$ in $\mathbb{Z}[X]$ and a nonzero constant $c \in \mathbb{Z}$ satisfying the identity

$$a(X)f(X) + b(X)g(X) \equiv c.$$

So, plugging in $X = r$ we get $p \mid c$, so the set of permissible primes p is finite. □

With this we can give the construction.

Claim — Suppose that

- n is a positive integer with $n \not\equiv 0 \pmod 3$;
- p is a prime which is 1 mod n; and
- α is a primitive n'th root of unity modulo p.

Then $|S| = n$ and, if p is sufficiently large in n, is also sum-free.

Proof. The assertion $|S| = n$ is immediate from the choice of α. As for sum-free, assume for contradiction that

$$1 + \alpha^k \equiv \alpha^m \pmod p$$

for some integers $k, m \in \mathbb{Z}$. This means $(X+1)^n - 1$ and $X^n - 1$ have common root $X = \alpha^k$.

But

$$\gcd_{\mathbb{Z}[x]}\Big((X+1)^n - 1,\ X^n - 1\Big) = 1 \qquad \forall n \not\equiv 0 \pmod 3$$

because when $3 \nmid n$ the two polynomials have no common complex roots. (Indeed, if $|\omega| = |1 + \omega| = 1$ then $\omega = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.)

Thus p is bounded by the lemma, as desired. $\square$

Team Selection Test for the 56^{th} International Mathematical Olympiad

United States of America

Day I

Thursday, December 11, 2014

Time limit: 4.5 hours. Each problem is worth 7 points.

IMO TST 1. Let ABC be a scalene triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$ and let P be a point in the interior of $\triangle ABC$ so that $MD = MP$ and $\angle PAB = \angle PAC$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

IMO TST 2. Prove that for every positive integer n, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S.

IMO TST 3. A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to isolate two usamons that she is 100% sure are currently in the same state. Is there any series of diode usage that makes this possible?

Team Selection Test for the 56^{th} International Mathematical Olympiad

United States of America

Day II

Thursday, January 22, 2015

Time limit: 4.5 hours. Each problem is worth 7 points.

IMO TST 4. Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be a function such that for any $x, y \in \mathbb{Q}$, the number $f(x+y) - f(x) - f(y)$ is an integer. Decide whether there must exist a constant c such that $f(x) - cx$ is an integer for every rational number x.

IMO TST 5. Fix a positive integer n. A tournament on n vertices has all its edges colored by χ colors, so that any two directed edges $u \to v$ and $v \to w$ have different colors. Over all possible tournaments on n vertices, determine the minimum possible value of χ.

IMO TST 6. Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.

USA TST 2015 Solution Notes

Evan Chen《陳誼廷》

30 September 2023

This is a compilation of solutions for the 2015 USA TST. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let ABC be a scalene triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$ and let P be a point in the interior of $\triangle ABC$ so that $MD = MP$ and $\angle PAB = \angle PAC$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

2. Prove that for every positive integer n, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S.

3. A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to isolate two usamons that she is 100% sure are currently in the same state. Is there any series of diode usage that makes this possible?

4. Let $f \colon \mathbb{Q} \to \mathbb{Q}$ be a function such that for any $x, y \in \mathbb{Q}$, the number $f(x+y) - f(x) - f(y)$ is an integer. Decide whether there must exist a constant c such that $f(x) - cx$ is an integer for every rational number x.

5. Fix a positive integer n. A tournament on n vertices has all its edges colored by χ colors, so that any two directed edges $u \to v$ and $v \to w$ have different colors. Over all possible tournaments on n vertices, determine the minimum possible value of χ.

6. Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.

§1 Solutions to Day 1

§1.1 USA TST 2015/1, proposed by Evan Chen

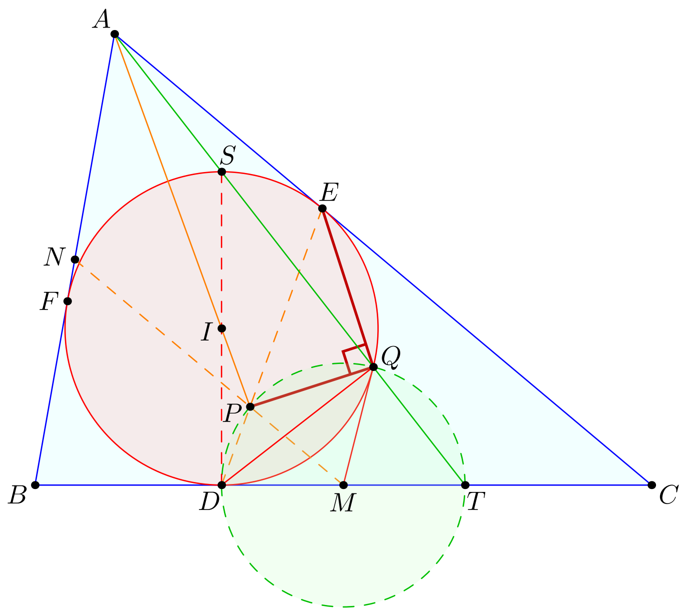
Available online at https://aops.com/community/p3683109.

Problem statement

Let ABC be a scalene triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$ and let P be a point in the interior of $\triangle ABC$ so that $MD = MP$ and $\angle PAB = \angle PAC$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

We present two solutions.

¶ Official solution Assume without loss of generality that $AB < AC$; we show $\angle PQE = 90^\circ$.



First, we claim that D, P, E are collinear. Let N be the midpoint of $\overline{AB}$. It is well-known that the three lines MN, DE, AI are concurrent at a point (see for example problem 6 of USAJMO 2014). Let P' be this intersection point, noting that P' actually lies on segment DE. Then P' lies inside $\triangle ABC$ and moreover

$$\triangle DP'M \sim \triangle DEC$$

so $MP' = MD$. Hence $P' = P$, proving the claim.

Let S be the point diametrically opposite D on the incircle, which is also the second intersection of $\overline{AQ}$ with the incircle. Let $T = \overline{AQ} \cap \overline{BC}$. Then T is the contact point of the A-excircle; consequently,

$$MD = MP = MT$$

and we obtain a circle with diameter $\overline{DT}$. Since $\angle DQT = \angle DQS = 90^\circ$ we have Q on this circle as well.

As $\overline{SD}$ is tangent to the circle with diameter $\overline{DT}$, we obtain

$$\angle PQD = \angle SDP = \angle SDE = \angle SQE.$$

Since $\angle DQS = 90^\circ$, $\angle PQE = 90^\circ$ too.

¶ Solution using spiral similarity We will ignore for now the point P. As before define S, T and note $\overline{AQST}$ collinear, as well as $DPQT$ cyclic on circle ω with diameter $\overline{DT}$.

Let τ be the spiral similarity at Q sending ω to the incircle. We have $\tau(T) = D$, $\tau(D) = S$, $\tau(Q) = Q$. Now

$$I = \overline{DD} \cap \overline{QQ} \implies \tau(I) = \overline{SS} \cap \overline{QQ}$$

and hence we conclude $\tau(I)$ is the pole of $\overline{ASQT}$ with respect to the incircle, which lies on line EF.

Then since $\overline{AI} \perp \overline{EF}$ too, we deduce τ sends line AI to line EF, hence $\tau(P)$ must be either E or F as desired.

¶ Authorship comments Written April 2014. I found this problem while playing with GeoGebra. Specifically, I started by drawing in the points A, B, C, I, D, M, T, common points. I decided to add in the circle with diameter DT, because of the synergy it had with the rest of the picture. After a while of playing around, I intersected ray AI with the circle to get P, and was surprised to find that D, P, E were collinear, which I thought was impossible since the setup should have been symmetric. On further reflection, I realized it was because AI intersected the circle twice, and set about trying to prove this. I noticed the relation $\angle PQE = 90^\circ$ in my attempts to prove the result, even though this ended up being a corollary rather than a useful lemma.

§1.2 USA TST 2015/2, proposed by Iurie Boreico

Available online at https://aops.com/community/p3683110.

Problem statement

Prove that for every positive integer n, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S.

The idea is to look for a sequence $d_1, \dots, d_{n-1}$ of "differences" such that the following two conditions hold. Let $s_i = d_1 + \dots + d_{i-1}$, and $t_{i,j} = d_i + \dots + d_{j-1}$ for $i \le j$.

(i) No two of the $t_{i,j}$ divide each other.

(ii) There exists an integer a satisfying the CRT equivalences

$$a \equiv -s_i \pmod{t_{i,j}} \quad \forall i \le j$$

Then the sequence $a + s_1$, $a + s_2$, ..., $a + s_n$ will work. For example, when $n = 3$ we can take $(d_1, d_2) = (2, 3)$ giving

$$10 \overbrace{\underbrace{\qquad}_{2} 12 \underbrace{\qquad}_{3}}^{5} 15$$

because the only conditions we need satisfy are

$$\begin{aligned} a &\equiv 0 \pmod 2 \\ a &\equiv 0 \pmod 5 \\ a &\equiv -2 \pmod 3. \end{aligned}$$

But with this setup we can just construct the d_i inductively. To go from n to $n+1$, take a $d_1, \dots, d_{n-1}$ and let p be a prime not dividing any of the d_i. Moreover, let M be a multiple of $\prod_{i \le j} t_{i,j}$ coprime to p. Then we claim that $d_1 M, d_2 M, \dots, d_{n-1} M, p$ is such a difference sequence. For example, the previous example extends as follows with $M = 300$ and $p = 7$.

$$\overbrace{a \underbrace{\qquad}_{600} b \overbrace{\underbrace{\qquad}_{900} c \underbrace{\qquad}_{7}}^{907} d}^{1507}$$

The new numbers p, $p + Md_{n-1}$, $p + Md_{n-2}$, …are all relatively prime to everything else. Hence (i) still holds. To see that (ii) still holds, just note that we can still get a family of solutions for the first n terms, and then the last $(n+1)$st term can be made to work by Chinese Remainder Theorem since all the new $p + Md_k$ are coprime to everything.

§1.3 USA TST 2015/3, proposed by Linus Hamilton

Available online at https://aops.com/community/p3683111.

Problem statement

A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to isolate two usamons that she is 100% sure are currently in the same state. Is there any series of diode usage that makes this possible?

The answer is no. Call the usamons $U_1, \dots, U_m$ (here $m = 2015$). Consider models M_k of the following form: $U_1, \dots, U_k$ are all charged for some $0 \le k \le m$ and the other usamons are not charged. Note that for any pair there's a model where they are different states, by construction.

We can consider the physicist as acting on these $m+1$ models simultaneously, and trying to reach a state where there's a pair in all models which are all the same charge. (This is a necessary condition for a winning strategy to exist.)

But we claim that any diode operation $U_i \to U_j$ results in the $m+1$ models being an isomorphic copy of the previous set. If $i < j$ then the diode operation can be interpreted as just swapping U_i with U_j, which doesn't change anything. Moreover if $i > j$ the operation never does anything. The conclusion follows from this.

Remark. This problem is not a "standard" olympiad problem, so I can't say it's trivial. But the idea is pretty natural I think.

You can motivate it as follows: there's a sequence of diode operations you can do which forces the situation to be one of the M_k above: first, use the diode into U_1 for all other U_i's, so that either no electrons exist at all or U_1 has an electron. Repeat with the other U_i. This leaves us at the situation described at the start of the problem. Then you could guess the answer was "no" just based on the fact that it's impossible for $n = 2, 3$ and that there doesn't seem to be a reasonable strategy.

In this way it's possible to give a pretty good description of what it's possible to do. One possible phrasing: "the physicist can arrange the usamons in a line such that all the charged usamons are to the left of the un-charged usamons, but can't determine the *number* of charged usamons".

§2 Solutions to Day 2

§2.1 USA TST 2015/4, proposed by Victor Wang

Available online at https://aops.com/community/p4628083.

Problem statement

Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be a function such that for any $x, y \in \mathbb{Q}$, the number $f(x+y) - f(x) - f(y)$ is an integer. Decide whether there must exist a constant c such that $f(x) - cx$ is an integer for every rational number x.

No, such a constant need not exist.

One possible solution is as follows: define a sequence by $x_0 = 1$ and

$$\begin{aligned} 2x_1 &= x_0 \\ 2x_2 &= x_1 + 1 \\ 2x_3 &= x_2 \\ 2x_4 &= x_3 + 1 \\ 2x_5 &= x_4 \\ 2x_6 &= x_5 + 1 \\ &\vdots \end{aligned}$$

Set $f(2^{-k}) = x_k$ and $f(2^k) = 2^k$ for $k = 0, 1, \dots$. Then, let

$$f\left(a \cdot 2^k + \frac{b}{c}\right) = af\left(2^k\right) + \frac{b}{c}$$

for odd integers a, b, c. One can verify this works.

A second shorter solution (given by the proposer) is to set, whenever $\gcd(p, q) = 1$ and $q > 0$,

$$f\left(\frac{p}{q}\right) = \frac{p}{q}\left(1! + 2! + \dots + q!\right).$$

Remark. Silly note: despite appearances, $f(x) = \lfloor x \rfloor$ is not a counterexample since one can take $c = 0$.

§2.2 USA TST 2015/5, proposed by Po-Shen Loh

Available online at https://aops.com/community/p4628085.

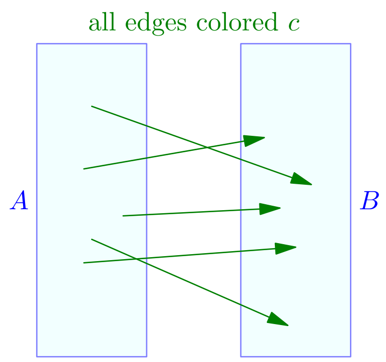
Problem statement

Fix a positive integer n. A tournament on n vertices has all its edges colored by χ colors, so that any two directed edges $u \to v$ and $v \to w$ have different colors. Over all possible tournaments on n vertices, determine the minimum possible value of χ.

The answer is

$$\chi = \lceil \log_2 n \rceil .$$

First, we prove by induction on n that $\chi \ge \log_2 n$ for any coloring and any tournament. The base case $n = 1$ is obvious. Now given any tournament, consider any used color c. Then it should be possible to divide the tournament into two subsets A and B such that all c-colored edges point from A to B (for example by letting A be all vertices which are the starting point of a c-edge).



One of A and B has size at least $n/2$, say A. Since A has no c edges, and uses at least $\log_2 |A|$ colors other than c, we get

$$\chi \ge 1 + \log_2(n/2) = \log_2 n$$

completing the induction.

One can read the construction off from the argument above, but here is a concrete description. For each integer n, consider the tournament whose vertices are the binary representations of $S = \{0, \dots, n-1\}$. Instantiate colors c_1, c_2, ... Then for $v, w \in S$, we look at the smallest order bit for which they differ; say the kth one. If v has a zero in the kth bit, and w has a one in the kth bit, we draw $v \to w$. Moreover we color the edge with color c_k. This works and uses at most $\lceil \log_2 n \rceil$ colors.

Remark (Motivation)**.** The philosophy "combinatorial optimization" applies here. The idea is given any color c, we can find sets A and B such that all c-edges point A to B. Once you realize this, the next insight is to realize that you may as well color *all* the edges from A to B by c; after all, this doesn't hurt the condition and makes your life easier. Hence, if f is the answer, we have already a proof that $f(n) = 1 + \max(f(|A|), f(|B|))$ and we choose $|A| \approx |B|$. This optimization also gives the inductive construction.

§2.3 USA TST 2015/6

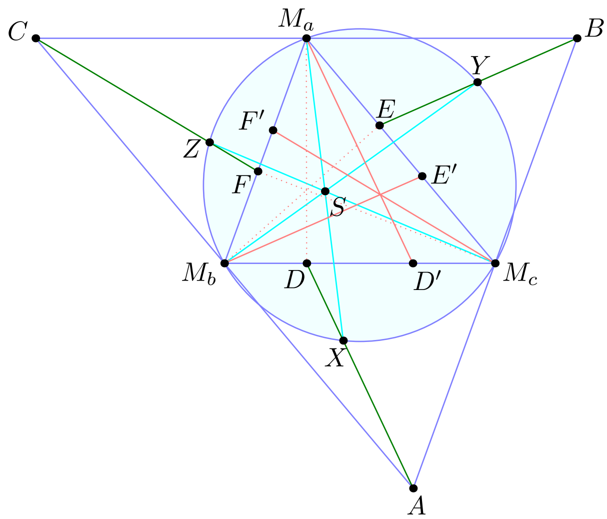
Available online at https://aops.com/community/p4628087.

Problem statement

Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.

We assume now and forever that ABC is scalene since the problem follows by symmetry in the isosceles case. We present four solutions.

¶ First solution by barycentric coordinates (Evan Chen) Let AX meet M_bM_c at D, and let X reflected over M_bM_c's midpoint be X'. Let Y', Z', E, F be similarly defined.



By Cevian Nest Theorem it suffices to prove that M_aD, M_bE, M_cF are concurrent. Taking the isotomic conjugate and recalling that $M_aM_bAM_c$ is a parallelogram, we see that it suffices to prove M_aX', M_bY', M_cZ' are concurrent.

We now use barycentric coordinates on $\triangle M_aM_bM_c$. Let

$$S = \left(a^2S_A + t : b^2S_B + t : c^2S_C + t\right)$$

(possibly $t = \infty$ if S is the centroid). Let $v = b^2S_B + t$, $w = c^2S_C + t$. Hence

$$X = \left(-a^2vw : (b^2w + c^2v)v : (b^2w + c^2v)w\right).$$

Consequently,

$$X' = \left(a^2vw : -a^2vw + (b^2w + c^2v)w : -a^2vw + (b^2w + c^2v)v\right)$$

We can compute

$$b^2w + c^2v = (bc)^2(S_B + S_C) + (b^2 + c^2)t = (abc)^2 + (b^2 + c^2)t.$$

Thus

$$-a^2v + b^2w + c^2v = (b^2 + c^2)t + (abc)^2 - (ab)^2S_B - a^2t = S_A((ab)^2 + t).$$

Finally

$$X' = \left(a^2vw : S_A(c^2S_C + t)\left((ab)^2 + 2t\right) : S_A(b^2S_B + t)\left((ac)^2 + 2t\right)\right)$$

and from this it's evident that AX', BY', CZ' are concurrent.

¶ Second solution by moving points (Anant Mudgal) Let H_a, H_b, H_c be feet of altitudes, and let γ denote the nine-point circle. The main claim is that:

Claim — Lines XH_a, YH_b, ZH_c are concurrent,

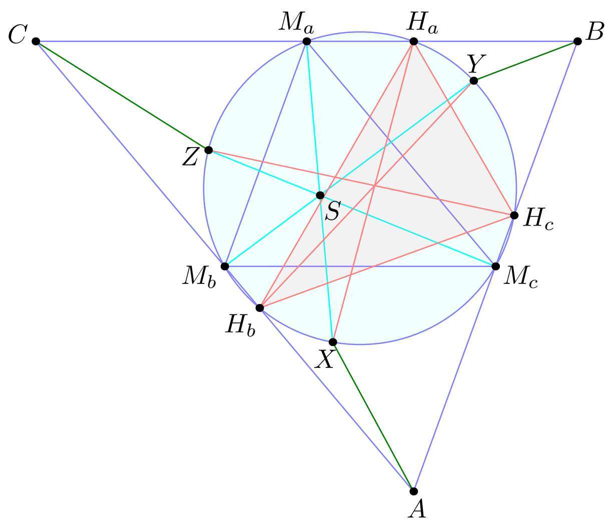
Proof. In fact, we claim that the concurrence point lies on the Euler line ℓ. This gives us a way to apply the moving points method: fix triangle ABC and animate $S \in \ell$; then the map

$$\begin{aligned} \ell &\to \gamma \to \ell \\ S &\mapsto X \mapsto S_a \coloneqq \ell \cap \overline{H_aX} \end{aligned}$$

is projective, because it consists of two perspectivities. So we want the analogous maps $S \mapsto S_b$, $S \mapsto S_c$ to coincide. For this it suffices to check three positions of S; since you're such a good customer here are four.

- If S is the orthocenter of $\triangle M_aM_bM_c$ (equivalently the circumcenter of $\triangle ABC$) then S_a coincides with the circumcenter of $M_aM_bM_c$ (equivalently the nine-point center of $\triangle ABC$). By symmetry S_b and S_c are too.
- If S is the circumcenter of $\triangle M_aM_bM_c$ (equivalently the nine-point center of $\triangle ABC$) then S_a coincides with the de Longchamps point of $\triangle M_aM_bM_c$ (equivalently orthocenter of $\triangle ABC$). By symmetry S_b and S_c are too.
- If S is either of the intersections of the Euler line with γ, then $S = S_a = S_b = S_c$ (as $S = X = Y = Z$).

This concludes the proof. □



We now use Trig Ceva to carry over the concurrence. By sine law,

$$\frac{\sin \angle M_cAX}{\sin \angle AM_cX} = \frac{M_cX}{AX}$$

and a similar relation for M_b gives that

$$\frac{\sin \angle M_cAX}{\sin \angle M_bAX} = \frac{\sin \angle AM_cX}{\sin \angle AM_bX} \cdot \frac{M_cX}{M_bX} = \frac{\sin \angle AM_cX}{\sin \angle AM_bX} \cdot \frac{\sin \angle XM_aM_c}{\sin \angle XM_aM_b}.$$

Thus multiplying cyclically gives

$$\prod_{\text{cyc}} \frac{\sin \angle M_cAX}{\sin \angle M_bAX} = \prod_{\text{cyc}} \frac{\sin \angle AM_cX}{\sin \angle AM_bX} \prod_{\text{cyc}} \frac{\sin \angle XM_aM_c}{\sin \angle XM_aM_b}.$$

The latter product on the right-hand side equals 1 by Trig Ceva on $\triangle M_aM_bM_c$ with cevians $\overline{M_aX}$, $\overline{M_bY}$, $\overline{M_cZ}$. The former product also equals 1 by Trig Ceva for the concurrence in the previous claim (and the fact that $\angle AM_cX = \angle H_cH_aX$). Hence the left-hand side equals 1, implying the result.

¶ Third solution by moving points (Gopal Goel) In this solution, we will instead use barycentric coordinates with resect to $\triangle ABC$ to bound the degrees suitably, and then verify for seven distinct choices of S.

We let R denote the radius of $\triangle ABC$, and N the nine-point center.

First, imagine solving for X in the following way. Suppose $\vec{X} = (1 - t_a)\vec{M}_a + t_a\vec{S}$. Then, using the dot product (with $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$ in general)

$$\begin{aligned}
\frac{1}{4}R^2 &= \left|\vec{X} - \vec{N}\right|^2 \\
&= \left|t_a(\vec{S} - \vec{M}_a) + \vec{M}_a - \vec{N}\right|^2 \\
&= \left|t_a(\vec{S} - \vec{M}_a)\right|^2 + 2t_a\left(\vec{S} - \vec{M}_a\right) \cdot \left(\vec{M}_a - \vec{N}\right) + \left|\vec{M}_a - \vec{N}\right|^2 \\
&= t_a^2\left|(\vec{S} - \vec{M}_a)\right|^2 + 2t_a\left(\vec{S} - \vec{M}_a\right) \cdot \left(\vec{M}_a - \vec{N}\right) + \frac{1}{4}R^2
\end{aligned}$$

Since $t_a \neq 0$ we may solve to obtain

$$t_a = -\frac{2(\vec{M}_a - \vec{N}) \cdot (\vec{S} - \vec{M}_a)}{\left|\vec{S} - \vec{M}_a\right|^2}.$$

Now imagine S varies along the Euler line, meaning there should exist linear functions $\alpha, \beta, \gamma \colon \mathbb{R} \to \mathbb{R}$ such that

$$S = (\alpha(s), \beta(s), \gamma(s)) \qquad s \in \mathbb{R}$$

with $\alpha(s) + \beta(s) + \gamma(s) = 1$. Thus $t_a = \frac{f_a}{g_a} = \frac{f_a(s)}{g_a(s)}$ is the quotient of a linear function $f_a(s)$ and a quadratic function $g_a(s)$.

So we may write:

$$\begin{aligned}
X &= (1 - t_a)\left(0, \frac{1}{2}, \frac{1}{2}\right) + t_a(\alpha, \beta, \gamma) \\
&= \left(t_a\alpha, \frac{1}{2}(1 - t_a) + t_a\beta, \frac{1}{2}(1 - t_a) + t_a\gamma\right)
\end{aligned}$$

$$= (2f_a\alpha : g_a - f_a + 2f_a\beta : g_a - f_a + 2f_a\gamma) .$$

Thus the coordinates of X are quadratic polynomials in s when written in this way.

In a similar way, the coordinates of Y and Z should be quadratic polynomials in s. The Ceva concurrence condition

$$\prod_{\text{cyc}} \frac{g_a - f_a + 2f_a\beta}{g_a - f_a + 2f_a\gamma} = 1$$

is thus a polynomial in s of degree at most six. Our goal is to verify it is identically zero, thus it suffices to check seven positions of S.

- If S is the circumcenter of $\triangle M_aM_bM_c$ (equivalently the nine-point center of $\triangle ABC$) then $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are altitudes of $\triangle ABC$.
- If S is the centroid of $\triangle M_aM_bM_c$ (equivalently the centroid of $\triangle ABC$), then $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are medians of $\triangle ABC$.
- If S is either of the intersections of the Euler line with γ, then $S = X = Y = Z$ and all cevians concur at S.
- If S lies on the $\overline{M_aM_b}$, then $Y = M_a$, $X = M_c$, and thus $\overline{AX} \cap \overline{BY} = C$, which is of course concurrent with $\overline{CZ}$ (regardless of Z). Similarly if S lies on the other sides of $\triangle M_aM_bM_c$.

Thus we are also done.

¶ Fourth solution using Pascal (official one) We give a different proof of the claim that $\overline{XH_a}$, $\overline{YH_b}$, $\overline{ZH_c}$ are concurrent (and then proceed as in the end of the second solution).

Let H denote the orthocenter, N the nine-point center, and moreover let N_a, N_b, N_c denote the midpoints of $\overline{AH}$, $\overline{BH}$, $\overline{CH}$, which also lie on the nine-point circle (and are the antipodes of M_a, M_b, M_c).

- By Pascal's theorem on $M_bN_bH_bM_cN_cH_c$, the point $P = \overline{M_cH_b} \cap \overline{M_bH_c}$ is collinear with $N = \overline{M_bN_b} \cap \overline{M_cN_c}$, and $H = \overline{N_bH_b} \cap \overline{N_cH_c}$. So P lies on the Euler line.
- By Pascal's theorem on $M_bYH_bM_cZH_c$, the point $\overline{YH_b} \cap \overline{ZH_c}$ is collinear with $S = \overline{M_bY} \cap \overline{M_cZ}$ and $P = \overline{M_bH_c} \cap \overline{M_cH_b}$. Hence YH_b and ZH_c meet on the Euler line, as needed.

Team Selection Test for the 57^{th} International Mathematical Olympiad

United States of America

Day I

Thursday, December 10, 2015

Time limit: 4.5 hours. Each problem is worth 7 points.

IMO TST 1. Let $S = \{1, \dots, n\}$. Given a bijection $f : S \to S$ an *orbit* of f is a set of the form $\{x, f(x), f(f(x)), \dots\}$ for some $x \in S$. We denote by $c(f)$ the number of distinct orbits of f. For example, if $n = 3$ and $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, the two orbits are $\{1, 2\}$ and $\{3\}$, hence $c(f) = 2$.

Given k bijections f_1, ..., f_k from S to itself, prove that

$$c(f_1) + \cdots + c(f_k) \leq n(k-1) + c(f)$$

where $f : S \to S$ is the composed function $f_1 \circ \cdots \circ f_k$.

IMO TST 2. Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at K and M. The circumcircle of $\triangle DKM$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq M$. Prove that line AT passes through either S_1 or S_2.

IMO TST 3. Let p be a prime number. Let $\mathbb{F}_p$ denote the integers modulo p, and let $\mathbb{F}_p[x]$ be the set of polynomials with coefficients in $\mathbb{F}_p$. Define $\Psi\colon \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ by

$$\Psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{p^i}.$$

Prove that for nonzero polynomials $F, G \in \mathbb{F}_p[x]$,

$$\Psi(\gcd(F, G)) = \gcd(\Psi(F), \Psi(G)).$$

Team Selection Test for the 57$^{\text{th}}$ International Mathematical Olympiad

United States of America

Day II

Thursday, January 21, 2016

Time limit: 4.5 hours. Each problem is worth 7 points.

IMO TST 4. Let $\sqrt{3} = 1.b_1b_2b_3\ldots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer n, at least one of the digits $b_n, b_{n+1}, \ldots, b_{2n}$ equals 1.

IMO TST 5. Let $n \geq 4$ be an integer. Find all functions $W\colon \{1,\ldots,n\}^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,

$$\sum_{a\in A}\sum_{b\in B}\sum_{c\in C} W(a,b)W(b,c) = |A||B||C|.$$

IMO TST 6. Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90$.

USA IMO TST 2016 Solutions

United States of America — IMO Team Selection Tests

Evan Chen《陳誼廷》

60th IMO 2016 Hong Kong

Contents

§0 Problems

1. Let $S = \{1, \dots, n\}$. Given a bijection $f \colon S \to S$ an *orbit* of f is a set of the form $\{x, f(x), f(f(x)), \dots\}$ for some $x \in S$. We denote by $c(f)$ the number of distinct orbits of f. For example, if $n = 3$ and $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, the two orbits are $\{1, 2\}$ and $\{3\}$, hence $c(f) = 2$.

   Given k bijections $f_1, \dots, f_k$ from S to itself, prove that

   $$c(f_1) + \dots + c(f_k) \le n(k-1) + c(f)$$

   where $f \colon S \to S$ is the composed function $f_1 \circ \dots \circ f_k$.

2. Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at K and M. The circumcircle of $\triangle DKM$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq M$. Prove that line AT passes through either S_1 or S_2.

3. Let p be a prime number. Let $\mathbb{F}_p$ denote the integers modulo p, and let $\mathbb{F}_p[x]$ be the set of polynomials with coefficients in $\mathbb{F}_p$. Define $\Psi \colon \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ by

   $$\Psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{p^i}.$$

   Prove that for nonzero polynomials $F, G \in \mathbb{F}_p[x]$,

   $$\Psi(\gcd(F, G)) = \gcd(\Psi(F), \Psi(G)).$$

4. Let $\sqrt{3} = 1.b_1 b_2 b_3 \dots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer n, at least one of the digits $b_n, b_{n+1}, \dots, b_{2n}$ equals 1.

5. Let $n \ge 4$ be an integer. Find all functions $W \colon \{1, \dots, n\}^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,

   $$\sum_{a \in A} \sum_{b \in B} \sum_{c \in C} W(a,b) W(b,c) = |A||B||C|.$$

6. Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90^\circ$.

§1 Solutions to Day 1

§1.1 USA TST 2016/1, proposed by Maria Monks

Available online at https://aops.com/community/p5679356.

Problem statement

Let $S = \{1, \dots, n\}$. Given a bijection $f \colon S \to S$ an *orbit* of f is a set of the form $\{x, f(x), f(f(x)), \dots\}$ for some $x \in S$. We denote by $c(f)$ the number of distinct orbits of f. For example, if $n = 3$ and $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, the two orbits are $\{1, 2\}$ and $\{3\}$, hence $c(f) = 2$.

Given k bijections $f_1, \dots, f_k$ from S to itself, prove that

$$c(f_1) + \cdots + c(f_k) \le n(k-1) + c(f)$$

where $f \colon S \to S$ is the composed function $f_1 \circ \cdots \circ f_k$.

Most motivated solution is to consider $n - c(f)$ and show this is the transposition distance. Dumb graph theory works as well.

§1.2 USA TST 2016/2, proposed by Evan Chen

Available online at https://aops.com/community/p5679361.

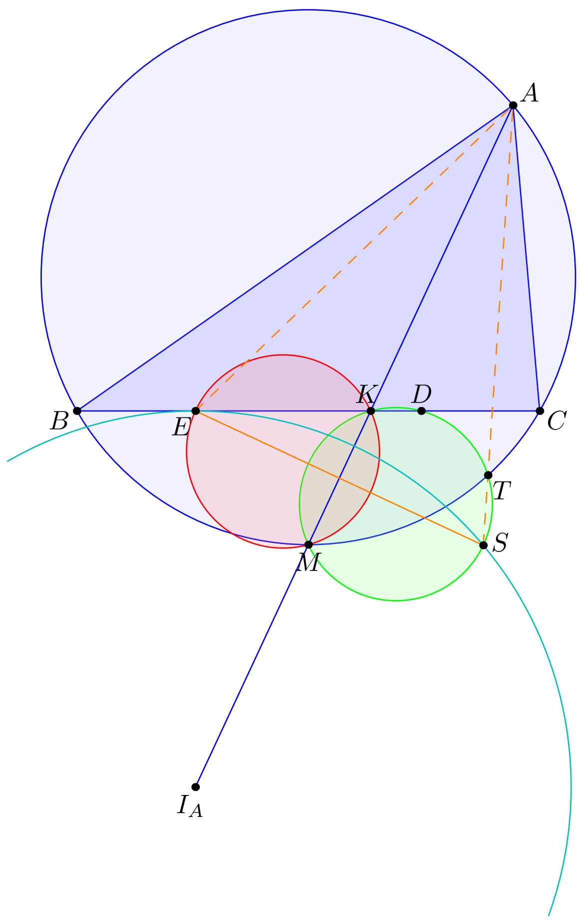
Problem statement

Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at K and M. The circumcircle of $\triangle DKM$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq M$. Prove that line AT passes through either S_1 or S_2.

We present an angle-chasing solution, and then a more advanced alternative finish.

¶ First solution (angle chasing) Assume for simplicity $AB < AC$. Let E be the contact point of the A-excircle on BC; also let ray TD meet Ω again at L. From the fact that $\angle MTL = \angle MTD = 180° - \angle MKD$, we can deduce that $\angle MTL = \angle ACM$, meaning that L is the reflection of A across the perpendicular bisector ℓ of BC. If we reflect T, D, L over ℓ, we deduce A, E and the reflection of T across ℓ are collinear, which implies that $\angle BAT = \angle CAE$.

Now, consider the reflection point E across line AI, say S. Since ray AI passes through the A-excenter, S lies on the A-excircle. Since $\angle BAT = \angle CAE$, S also lies on ray AT. But the circumcircles of triangles DKM and KME are congruent (from $DM = EM$), so S lies on the circumcircle of $\triangle DKM$ too. Hence S is the desired intersection point.



¶ Second solution (advanced) It's known that T is the touch-point of the A-mixtilinear incircle. Let E be contact point of A-excircle on BC. Now the circumcircles of $\triangle DKM$ and $\triangle KME$ are congruent, since $DM = ME$ and the angles at K are supplementary. Let S be the reflection of E across line KM, which by the above the above comment lies on the circumcircle of $\triangle DKM$. Since KM passes through the A-excenter, S also lies on the A-excircle. But S also lies on line AT, since lines AT and AE are isogonal (the mixtilinear cevian is isogonal to the Nagel line). Thus S is the desired intersection point.

¶ Authorship comments This problem comes from an observation of mine: let ABC be a triangle, let the $\angle A$ bisector meet $\overline{BC}$ and (ABC) at E and M. Let W be the tangency point of the A-mixtilinear excircle with the circumcircle of ABC. Then A-Nagel line passed through a common intersection of the circumcircle of $\triangle MEW$ and the A-mixtilinear incircle.

This problem is the inverted version of this observation.

§1.3 USA TST 2016/3, proposed by Mark Sellke

Available online at https://aops.com/community/p5679392.

Problem statement

Let p be a prime number. Let $\mathbb{F}_p$ denote the integers modulo p, and let $\mathbb{F}_p[x]$ be the set of polynomials with coefficients in $\mathbb{F}_p$. Define $\Psi\colon \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ by

$$\Psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{p^i}.$$

Prove that for nonzero polynomials $F, G \in \mathbb{F}_p[x]$,

$$\Psi(\gcd(F,G)) = \gcd(\Psi(F), \Psi(G)).$$

Observe that Ψ is also a linear map of $\mathbb{F}_p$ vector spaces, and that $\Psi(xP) = \Psi(P)^p$ for any $P \in \mathbb{F}_p[x]$. (In particular, $\Psi(1) = x$, not 1, take caution!)

¶ First solution (Ankan Bhattacharya) We start with:

Claim — If $P \mid Q$ then $\Psi(P) \mid \Psi(Q)$.

Proof. Set $Q = PR$, where $R = \sum_{i=0}^{k} r_i x^i$. Then

$$\Psi(Q) = \Psi\left(P \sum_{i=0}^{k} r_i x^i\right) = \sum_{i=0}^{k} \Psi\left(P \cdot r_i x^i\right) = \sum_{i=0}^{k} r_i \Psi(P)^{p^i}$$

which is divisible by $\Psi(P)$. □

This already implies

$$\Psi(\gcd(F,G)) \mid \gcd(\Psi(F), \Psi(G)).$$

For the converse, by Bezout there exists $A, B \in \mathbb{F}_p[x]$ such that $AF + BG = \gcd(F,G)$, so taking Ψ of both sides gives

$$\Psi(AF) + \Psi(BG) = \Psi\left(\gcd(F,G)\right).$$

The left-hand side is divisible by $\gcd(\Psi(F), \Psi(G))$ since the first term is divisible by $\Psi(F)$ and the second term is divisible by $\Psi(G)$. So $\gcd(\Psi(F), \Psi(G)) \mid \Psi(\gcd(F,G))$ and noting both sides are monic we are done.

¶ Second solution Here is an alternative (longer but more conceptual) way to finish without Bezout lemma. Let $\daleth \subseteq \mathbb{F}_p[x]$ denote the set of polynomials in the image of Ψ, thus $\Psi\colon \mathbb{F}_p[x] \to \daleth$ is a bijection on the level of sets.

Claim — If $A, B \in \daleth$ then $\gcd(A,B) \in \daleth$.

Proof. It suffices to show that if A and B are monic, and $\deg A > \deg B$, then the remainder when A is divided by B is in $\daleth$. Suppose $\deg A = p^k$ and $B = x^{p^{k-1}} - c_2 x^{p^{k-2}} - \cdots - c_k$. Then

$$\begin{aligned} x^{p^k} &\equiv \left(c_2 x^{p^{k-2}} + c_3 x^{p^{k-3}} + \cdots + c_k\right)^p \pmod B \\ &\equiv c_2 x^{p^{k-1}} + c_3 x^{p^{k-2}} \cdots + c_k \pmod B \end{aligned}$$

since exponentiation by p commutes with addition in $\mathbb{F}_p$. This is enough to imply the conclusion. The proof if $\deg B$ is smaller less than p^{k-1} is similar. $\square$

Thus, if we view $\mathbb{F}_p[x]$ and $\daleth$ as partially ordered sets under polynomial division, then gcd is the "greatest lower bound" or "meet" in both partially ordered sets. We will now prove that Ψ is an *isomorphism* of the posets. We have already seen that $P \mid Q \implies \Psi(P) \mid \Psi(Q)$ from the first solution. For the converse:

Claim — If $\Psi(P) \mid \Psi(Q)$ then $P \mid Q$.

Proof. Suppose $\Psi(P) \mid \Psi(Q)$, but $Q = PA + B$ where $\deg B < \deg P$. Thus $\Psi(P) \mid \Psi(PA) + \Psi(B)$, hence $\Psi(P) \mid \Psi(B)$, but $\deg \Psi(P) > \deg \Psi(B)$ hence $\Psi(B) = 0 \implies B = 0$. $\square$

This completes the proof.

Remark. In fact $\Psi \colon \mathbb{F}_p[x] \to \daleth$ is a ring isomorphism if we equip $\daleth$ with function composition as the ring multiplication. Indeed in the proof of the first claim (that $P \mid Q \implies \Psi(P) \mid \Psi(Q)$) we saw that

$$\Psi(RP) = \sum_{i=0}^{k} r_i \Psi(P)^{p^i} = \Psi(R) \circ \Psi(P).$$

§2 Solutions to Day 2

§2.1 USA TST 2016/4, proposed by Iurie Boreico

Available online at https://aops.com/community/p6368201.

Problem statement

Let $\sqrt{3} = 1.b_1b_2b_3\dots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer n, at least one of the digits $b_n, b_{n+1}, \dots, b_{2n}$ equals 1.

Assume the contrary, so that for some integer k we have

$$k < 2^{n-1}\sqrt{3} < k + \frac{1}{2^{n+1}}.$$

Squaring gives

$$\begin{aligned} k^2 < 3 \cdot 2^{2n-2} &< k^2 + \frac{k}{2^n} + \frac{1}{2^{2n+2}} \\ &\le k^2 + \frac{2^{n-1}\sqrt{3}}{2^n} + \frac{1}{2^{2n+2}} \\ &= k^2 + \frac{\sqrt{3}}{2} + \frac{1}{2^{2n+2}} \\ &\le k^2 + \frac{\sqrt{3}}{2} + \frac{1}{16} \\ &< k^2 + 1 \end{aligned}$$

and this is a contradiction.

§2.2 USA TST 2016/5, proposed by Zilin Jiang

Available online at https://aops.com/community/p6368185.

Problem statement

Let $n \geq 4$ be an integer. Find all functions $W \colon \{1, \dots, n\}^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,

$$\sum_{a \in A} \sum_{b \in B} \sum_{c \in C} W(a,b)W(b,c) = |A||B||C|.$$

Of course, $W(k,k)$ is arbitrary for $k \in [n]$. We claim that $W(a,b) = \pm 1$ for any $a \neq b$, with the sign fixed. (These evidently work.)

First, let $X_{abc} = W(a,b)W(b,c)$ for all distinct a, b, c, so the given condition is

$$\sum_{a,b,c \in A \times B \times C} X_{abc} = |A||B||C|.$$

Consider the given equation with the particular choices

- $A = \{1\}$, $B = \{2\}$, $C = \{3, 4, \dots, n\}$.
- $A = \{1\}$, $B = \{3\}$, $C = \{2, 4, \dots, n\}$.
- $A = \{1\}$, $B = \{2, 3\}$, $C = \{4, \dots, n\}$.

This gives

$$\begin{aligned} X_{123} + X_{124} + \dots + X_{12n} &= n - 2 \\ X_{132} + X_{134} + \dots + X_{13n} &= n - 2 \\ (X_{124} + \dots + X_{12n}) + (X_{134} + \dots + X_{13n}) &= 2(n-3). \end{aligned}$$

Adding the first two and subtracting the last one gives $X_{123} + X_{132} = 2$. Similarly, $X_{123} + X_{321} = 2$, and in this way we have $X_{321} = X_{132}$. Thus $W(3,2)W(2,1) = W(1,3)W(3,2)$, and since $W(3,2) \neq 0$ (clearly) we get $W(2,1) = W(3,2)$.

Analogously, for any distinct a, b, c we have $W(a,b) = W(b,c)$. For $n \geq 4$ this is enough to imply $W(a,b) = \pm 1$ for $a \neq b$ where the choice of sign is the same for all a and b.

Remark. Surprisingly, the $n = 3$ case has "extra" solutions for $W(1,2) = W(2,3) = W(3,1) = \pm 1$, $W(2,1) = W(3,2) = W(1,3) = \mp 1$.

Remark (Intuition)**.** It should still be possible to solve the problem with X_{abc} in place of $W(a,b)W(b,c)$, because we have about far more equations than variables $X_{a,b,c}$ so linear algebra assures us we almost certainly have a unique solution.

§2.3 USA TST 2016/6, proposed by Ivan Borsenco

Available online at https://aops.com/community/p6368189.

Problem statement

Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90^\circ$.

In complex numbers with ABC the unit circle, it is equivalent to solving the following two cubic equations in p and $q = \overline{p}$:

$$(p-a)(p-b)(p-c) = (abc)^2(q-1/a)(q-1/b)(q-1/c)$$
$$0 = \prod_{\text{cyc}}(p+c-b-bcq) + \prod_{\text{cyc}}(p+b-c-bcq).$$

Viewing this as two cubic curves in $(p,q) \in \mathbb{C}^2$, by *Bézout's Theorem* it follows there are at most nine solutions (unless both curves are not irreducible, but it's easy to check the first one cannot be factored). Moreover it is easy to name nine solutions (for ABC scalene): the three vertices, the three excenters, and I, O, H. Hence the answer is just those three triangle centers I, O and H.

Remark. On the other hand it is not easy to solve the cubics by hand; I tried for an hour without success. So I think this solution is only feasible with knowledge of algebraic geometry.

Remark. These two cubics have names:

- The locus of $\angle PAB + \angle PBC + \angle PCA = 90^\circ$ is the **McCay cubic**, which is the locus of points P for which P, P^*, O are collinear.
- The locus of the pedal condition is the **Darboux cubic**, which is the locus of points P for which P, P^*, L are collinear, L denoting the de Longchamps point.

Assuming $P \neq P^*$, this implies P and P^* both lie on the Euler line of $\triangle ABC$, which is possible only if $P = O$ or $P = H$.

Some other synthetic solutions are posted at https://aops.com/community/c6h1243902p6368189.

Team Selection Test for the 58$^{\text{th}}$ International Mathematical Olympiad

United States of America

Day I

Thursday, December 8, 2016

Time limit: 4.5 hours. Each problem is worth 7 points.

IMO TST 1. In a sports league, each team uses a set of at most t signature colors. A set S of teams is *color-identifiable* if one can assign each team in S one of their signature colors, such that no team in S is assigned *any* signature color of a different team in S. For all positive integers n and t, determine the maximum integer $g(n,t)$ such that: In any sports league with exactly n distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n,t)$.

IMO TST 2. Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.

(a) Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.

(b) Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.

IMO TST 3. Let $P, Q \in \mathbb{R}[x]$ be relatively prime nonconstant polynomials. Show that there can be at most three real numbers λ such that $P + \lambda Q$ is the square of a polynomial.

Team Selection Test for the 58th International Mathematical Olympiad

United States of America

Day II

Thursday, January 19, 2017

Time limit: 4.5 hours. Each problem is worth 7 points.

IMO TST 4. You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestant's guesses before writing your own. For each question, after all guesses are submitted, the emcee announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth -2 points for other contestants, but only -1 point for you, because you hacked the scoring system. After announcing the correct answer, the emcee proceeds to read out the next question. Show that if you are leading by 2^{n-1} points at any time, then you can surely win first place.

IMO TST 5. Let ABC be a triangle with altitude $\overline{AE}$. The A-excircle touches $\overline{BC}$ at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

IMO TST 6. Prove that there are infinitely many triples (a, b, p) of integers, with p prime and $0 < a \le b < p$, for which p^3 divides $(a+b)^p - a^p - b^p$.

Note that in this solutions file, we present slightly stronger versions of problems 4 and 6 on the January TST than actually appeared on the exams.

USA IMO TST 2017 Solutions

United States of America — IMO Team Selection Tests

Evan Chen《陳誼廷》

58th IMO 2017 Brazil

Contents

§0 Problems

1. In a sports league, each team uses a set of at most t signature colors. A set S of teams is *color-identifiable* if one can assign each team in S one of their signature colors, such that no team in S is assigned *any* signature color of a different team in S. For all positive integers n and t, determine the maximum integer $g(n,t)$ such that: In any sports league with exactly n distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n,t)$.

2. Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.
   (a) Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.
   (b) Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.

3. Let $P, Q \in \mathbb{R}[x]$ be relatively prime nonconstant polynomials. Show that there can be at most three real numbers λ such that $P + \lambda Q$ is the square of a polynomial.

4. You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestant's guesses before writing your own. For each question, after all guesses are submitted, the emcee announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth -2 points for other contestants, but only -1 point for you, because you hacked the scoring system. After announcing the correct answer, the emcee proceeds to read out the next question. Show that if you are leading by 2^{n-1} points at any time, then you can surely win first place.

5. Let ABC be a triangle with altitude $\overline{AE}$. The A-excircle touches $\overline{BC}$ at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

6. Prove that there are infinitely many triples (a, b, p) of integers, with p prime and $0 < a \le b < p$, for which p^5 divides $(a+b)^p - a^p - b^p$.

§1 Solutions to Day 1

§1.1 USA TST 2017/1, proposed by Po-Shen Loh

Available online at https://aops.com/community/p7389115.

Problem statement

In a sports league, each team uses a set of at most t signature colors. A set S of teams is *color-identifiable* if one can assign each team in S one of their signature colors, such that no team in S is assigned *any* signature color of a different team in S. For all positive integers n and t, determine the maximum integer $g(n,t)$ such that: In any sports league with exactly n distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n,t)$.

Answer: $\lceil n/t \rceil$.

To see this is an upper bound, note that one can easily construct a sports league with that many teams anyways.

A quick warning:

> **Remark** (Misreading the problem)**.** It is common to misread the problem by ignoring the word "any". Here is an illustration.
>
> Suppose we have two teams, MIT and Harvard; the colors of MIT are red/grey/black, and the colors of Harvard are red/white. (Thus $n = 4$ and $t = 3$.) The assignment of MIT to grey and Harvard to red is *not* acceptable because red is a signature color of MIT, even though not the one assigned.

We present two proofs of the lower bound.

¶ Approach by deleting teams (Gopal Goel) Initially, place all teams in a set S. Then we repeat the following algorithm:

> If there is a team all of whose signature colors are shared by some other team in S already, then we delete that team.

(If there is more than one such team, we pick arbitrarily.)

At the end of the process, all n colors are still present at least once, so at least $\lceil n/t \rceil$ teams remain. Moreover, since the algorithm is no longer possible, the remaining set S is already color-identifiable.

> **Remark** (Gopal Goel)**.** It might seem counter-intuitive that we are *deleting* teams from the full set when the original problem is trying to get a large set S.
>
> This is less strange when one thinks of it instead as "safely deleting useless teams". Basically, if one deletes such a team, the problem statement implies that the task must still be possible, since $g(n,t)$ does not depend on the number of teams: n is the number of colors present, and deleting a useless team does not change this. It turns out that this optimization is already enough to solve the problem.

¶ Approach by adding colors For a constructive algorithmic approach, the idea is to greedy pick by color (rather than by team), taking at each step the least used color. Select the color C_1 with the *fewest* teams using it, and a team T_1 using it. Then delete all colors T_1 uses, and all teams which use C_1. Note that

- By problem condition, this deletes at most t teams total.
- Any remaining color C still has at least one user. Indeed, if not, then C had the same set of teams as C_1 did (by minimality of C), but then it should have deleted as a color of T_1.

Now repeat this algorithm with C_2 and T_2, and so on. This operations uses at most t colors each time, so we select at least $\lceil n/t \rceil$ colors.

Remark. A greedy approach by team *does not work*. For example, suppose we try to "grab teams until no more can be added".

As before, assume our league has teams, MIT and Harvard; the colors of MIT are red/grey/black, and the colors of Harvard are red/white. (Thus $n = 4$ and $t = 3$.) If we start by selecting MIT and red, then it is impossible to select any more teams; but $g(n, t) = 2$.

§1.2 USA TST 2017/2, proposed by Evan Chen

Available online at https://aops.com/community/p7389108.

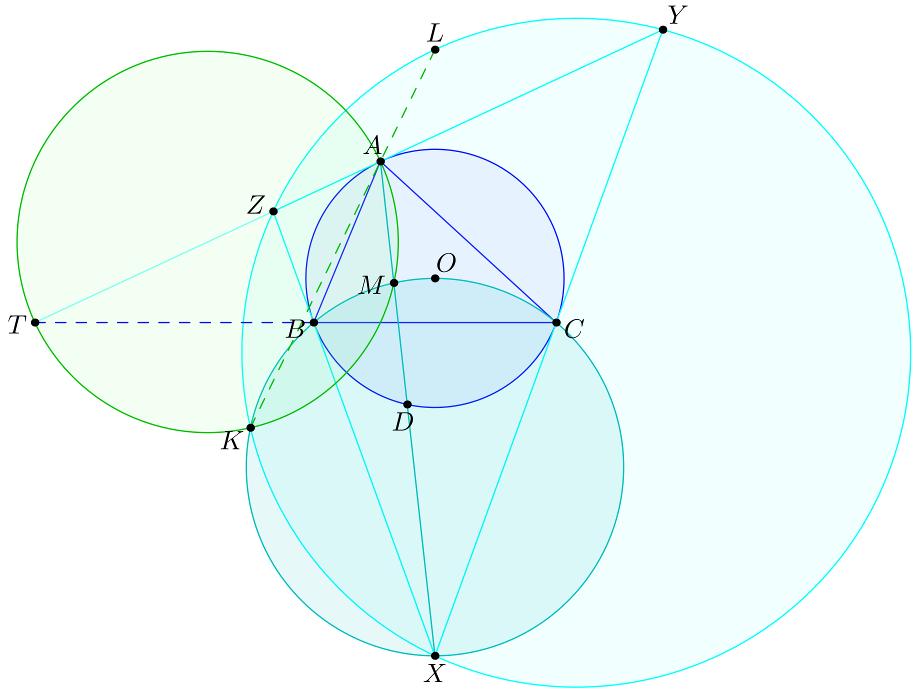
Problem statement

Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.

(a) Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.

(b) Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.

Let triangle ABC have circumcircle Γ. Let $\triangle XYZ$ be the tangential triangle of $\triangle ABC$ (hence Γ is the incircle of $\triangle XYZ$), and denote by Ω its circumcircle. Suppose the symmedian $\overline{AX}$ meets Γ again at D, and let M be the midpoint of $\overline{AD}$. Finally, let K be the Miquel point of quadrilateral $ZBCY$, meaning it is the intersection of Ω and the circumcircle of $\triangle BOC$ (other than X).



We first claim that M and K are A_1 and A_2. In that case $OM < OA < OK$, so $M = A_1$, $K = A_2$.

To see that $M = A_1$, note that $\angle OMX = 90^\circ$, and moreover that $\overline{TA}$, $\overline{TD}$ are tangents to Γ, whence we also have $M = \overline{TO} \cap \overline{AD}$. Thus M lies on both (BOC) and (AT). This solves part (a) of the problem: the concurrency point is the symmedian point of $\triangle ABC$.

Now, note that since K is the Miquel point,

$$\frac{ZK}{YK} = \frac{ZB}{YC} = \frac{ZA}{YA}$$

and hence $\overline{KA}$ is an angle bisector of $\angle ZKY$. Thus from $(TA;YZ) = -1$ we obtain $\angle TKA = 90^\circ$.

It remains to show $\overline{AK}$ passes through a fixed point on the Euler line. We claim it is the exsimilicenter of Γ and Ω. Let L be the midpoint of the arc YZ of $\triangle XYZ$ not containing X. Then we know that K, A, L are collinear. Now the positive homothety sending Γ to Ω maps A to L; this proves the claim. Finally, it is well-known that the line through O and the circumcenter of $\triangle XYZ$ coincides with the Euler line of $\triangle ABC$; hence done.

A second approach to (b) presented by many contestants is to take an inversion around the circumcircle of ABC. In that situation, the part reduces to the following known lemma: if $\overline{AH_a}$, $\overline{BH_b}$, $\overline{CH_c}$ are the altitudes of a triangle, then the circumcircles of triangles OAH_a, BOH_b, COH_c are coaxial, and the radical axis coincides with the Euler line. Indeed one simply observes that the orthocenter has equal power to all three circles.

¶ Authorship comments This problem was inspired by the fact that K, A, L are collinear in the figure, which was produced by one of my students (Ryan Kim) in a solution to a homework problem. I realized for example that this implied that line AK passed through the X_{56} point of $\triangle XYZ$ (which lies on the Euler line of $\triangle ABC$).

This problem was the result of playing around with the resulting very nice picture: all the power comes from the "magic" point K.

§1.3 USA TST 2017/3, proposed by Alison Miller

Available online at https://aops.com/community/p7389123.

Problem statement

Let $P, Q \in \mathbb{R}[x]$ be relatively prime nonconstant polynomials. Show that there can be at most three real numbers λ such that $P + \lambda Q$ is the square of a polynomial.

This is true even with $\mathbb{R}$ replaced by $\mathbb{C}$, and it will be necessary to work in this generality.

¶ First solution using transformations We will prove the claim in the following form:

Claim — Assume $P, Q \in \mathbb{C}[x]$ are relatively prime. If $\alpha P + \beta Q$ is a square for four different choices of the ratio $[\alpha : \beta]$ then P and Q must be constant.

Call pairs (P, Q) as in the claim *bad*; so we wish to show the only bad pairs are pairs of constant polynomials. Assume not, and take a bad pair with $\deg P + \deg Q$ minimal.

By a suitable Möbius transformation, we may transform (P, Q) so that the four ratios are $[1 : 0]$, $[0 : 1]$, $[1 : -1]$ and $[1 : -k]$, so we find there are polynomials A and B such that

$$A^2 - B^2 = C^2$$
$$A^2 - kB^2 = D^2$$

where $A^2 = P + \lambda_1 Q$, $B^2 = P + \lambda_2 Q$, say. Of course $\gcd(A, B) = 1$.

Consequently, we have $C^2 = (A + B)(A - B)$ and $D^2 = (A + \mu B)(A - \mu B)$ where $\mu^2 = k$. Now $\gcd(A, B) = 1$, so $A + B$, $A - B$, $A + \mu B$ and $A - \mu B$ are squares; id est (A, B) is bad. This is a contradiction, since $\deg A + \deg B < \deg P + \deg Q$.

¶ Second solution using derivatives (by Zack Chroman) We will assume without loss of generality that $\deg P \neq \deg Q$; if not, then one can replace (P, Q) with $(P + cQ, Q)$ for a suitable constant c.

Then, there exist $\lambda_i \in \mathbb{C}$ and polynomials R_i for $i = 1, 2, 3, 4$ such that

$$\begin{aligned} P + \lambda_i Q &= R_i^2 \\ \implies P' + \lambda_i Q' &= 2R_i R_i' \\ \implies R_i &\mid Q'(P + \lambda_i Q) - Q(P' + \lambda_i Q') = Q'P - QP'. \end{aligned}$$

On the other hand by Euclidean algorithm it follows that R_i are relatively prime to each other. Therefore

$$R_1 R_2 R_3 R_4 \mid Q'P - QP'.$$

However, we have

$$\sum_1^4 \deg R_i \geq \frac{3 \max(\deg P, \deg Q) + \min(\deg P, \deg Q)}{2} \geq \deg P + \deg Q > \deg(Q'P - QP').$$

This can only occur if $Q'P - QP' = 0$ or $(P/Q)' = 0$ by the quotient rule! But P/Q can't be constant, the end.

Remark. The result is previously known; see e.g. Lemma 1.6 of `http://math.mit.edu/~ebelmont/ec-notes.pdf` or Exercise 6.5.L(a) of Vakil's notes.

§2 Solutions to Day 2

§2.1 USA TST 2017/4, proposed by Linus Hamilton

Available online at https://aops.com/community/p7732191.

Problem statement

You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestant's guesses before writing your own. For each question, after all guesses are submitted, the emcee announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth -2 points for other contestants, but only -1 point for you, because you hacked the scoring system. After announcing the correct answer, the emcee proceeds to read out the next question. Show that if you are leading by 2^{n-1} points at any time, then you can surely win first place.

We will prove the result with 2^{n-1} replaced even by $2^{n-2}+1$.

We first make the following reductions. First, change the weights to be $+1$, -1, 0 respectively (rather than 0, -2, -1); this clearly has no effect. Also, WLOG that all contestants except you initially have score zero (and that your score exceeds 2^{n-2}). WLOG ignore rounds in which all answers are the same. Finally, ignore rounds in which you get the correct answer, since that leaves you at least as well off as before — in other words, we'll assume your score is always fixed, but you can pick any group of people with the same answers and ensure they lose 1 point, while some other group gains 1 point.

The key observation is the following. Consider two rounds R_1 and R_2 such that:

- In round R_1, some set S of contestants gains a point.
- In round R_2, the set S of contestants all have the same answer.

Then, if we copy the answers of contestants in S during R_2, then the sum of the scorings in R_1 and R_2 cancel each other out. In other words we can then ignore R_1 and R_2 forever.

We thus consider the following strategy. We keep a list $\mathcal{L}$ of subsets of $\{1, \dots, n\}$, initially empty. Now do the following strategy:

- On a round, suppose there exists a set S of people with the same answer such that $S \in \mathcal{L}$. (If multiple exist, choose one arbitrarily.) Then, copy the answer of S, causing them to lose a point. Delete S from $\mathcal{L}$. (Importantly, we do not add any new sets to $\mathcal{L}$.)
- Otherwise, copy any set T of contestants, selecting $|T| \ge n/2$ if possible. Let S be the set of contestants who answer correctly (if any), and add S to the list $\mathcal{L}$. Note that $|S| \le n/2$, since S is disjoint from T.

By construction, $\mathcal{L}$ has no duplicate sets. So the score of any contestant c is bounded above by the number of times that c appears among sets in $\mathcal{L}$. The number of such sets is clearly at most $\frac{1}{2} \cdot 2^{n-1}$. So, if you lead by $2^{n-2}+1$ then you ensure victory. This completes the proof!

Remark. Several remarks are in order. First, we comment on the bound $2^{n-2}+1$ itself. The most natural solution using only the list idea gives an upper bound of $(2^n-2)+1$, which is the number of nonempty proper subsets of $\{1, \dots, n\}$. Then, there are two optimizations one can observe:

- In fact we can improve to the number of times any particular contestant c appears in some set, rather than the total number of sets.
- When adding new sets S to $\mathcal{L}$, one can ensure $|S| \le n/2$.

Either observation alone improves the bound from $2^n - 1$ to 2^{n-1}, but both together give the bound $2^{n-2} + 1$. Additionally, when n is odd the calculation of subsets actually gives $2^{n-2} - \frac{1}{2}\binom{n-1}{\frac{n-1}{2}} + 1$. This gives the best possible value at both $n = 2$ and $n = 3$. It seems likely some further improvements are possible, and the true bound is suspected to be polynomial in n.

Secondly, the solution is highly motivated by considering a true/false contest in which only two distinct answers are given per question. However, a natural mistake (which graders assessed as a two-point deduction) is to try and prove that in fact one can "WLOG" we are in the two-question case. The proof of this requires substantially more care than expected. For instance, set $n = 3$. If $\mathcal{L} = \{\{1\}, \{2\}, \{3\}\}$ then it becomes impossible to prevent a duplicate set from appearing in $\mathcal{L}$ if all contestants give distinct answers. One might attempt to fix this by instead adding to $\mathcal{L}$ the *complement* of the set T described above. The example $\mathcal{L} = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$ (followed again by a round with all distinct answers) shows that this proposed fix does not work either. This issue affects all variations of the above approach.

Because the USA TST did not have any solution-writing process at this time, this issue was not noticed until January 15 (less than a week before the exam). When it was brought up by email by Evan, every organizer who had testsolved the problem had apparently made the same error.

Remark. Here are some motivations for the solution:

1. The exponential bound 2^n suggests looking at subsets.
2. The $n = 2$ case suggests the idea of "repeated rounds". (I think this $n = 2$ case is actually really good.)
3. The "two distinct answers" case suggests looking at rounds as partitions (even though the WLOG does not work, at least not without further thought).
4. There's something weird about this problem: it's a finite bound over unbounded time. This is a hint to *not* worry excessively about the actual scores, which turn out to be almost irrelevant.

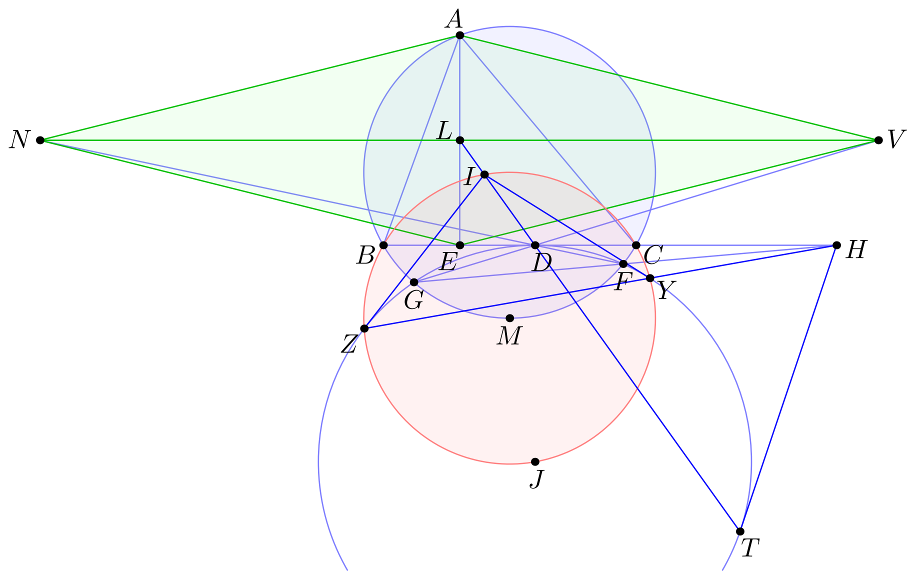
§2.2 USA TST 2017/5, proposed by Danielle Wang, Evan Chen

Available online at https://aops.com/community/p7732197.

Problem statement

Let ABC be a triangle with altitude $\overline{AE}$. The A-excircle touches $\overline{BC}$ at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

Let I denote the incenter, J the A-excenter, and L the midpoint of $\overline{AE}$. Denote by $\overline{IY}$, $\overline{IZ}$ the tangents from I to the A-excircle. Note that lines $\overline{BC}$, $\overline{GF}$, $\overline{YZ}$ then concur at H (unless $AB = AC$, but this case is obvious), as it's the radical center of cyclic hexagon $BICYJZ$, the circumcircle and the A-excircle.



Now let $\overline{HD}$ and $\overline{HT}$ be the tangents from H to the A-excircle. It follows that $\overline{DT}$ is the symmedian of $\triangle DZY$, hence passes through $I = \overline{YY} \cap \overline{ZZ}$. Moreover, it's well known that $\overline{DI}$ passes through L, the midpoint of the A-altitude (for example by homothety). Finally, $(DT; FG) = -1$, hence project through D onto the line through L parallel to $\overline{BC}$ to obtain $(\infty L; VN) = -1$ as desired.

¶ Authorship comments This is a joint proposal with Danielle Wang (mostly by her). The formulation given was that the tangents to the A-excircle at F and G was on line $\overline{DI}$; I solved this formulation using the radical axis argument above. I then got the idea to involve the point L, already knowing it was on $\overline{DI}$. Observing the harmonic quadrilateral, I took perspectivity through M onto the line through L parallel to $\overline{BC}$ (before this I had tried to use the A-altitude with little luck). This yields the rhombus in the problem.

§2.3 USA TST 2017/6, proposed by Noam Elkies

Available online at https://aops.com/community/p7732203.

Problem statement

Prove that there are infinitely many triples (a, b, p) of integers, with p prime and $0 < a \le b < p$, for which p^5 divides $(a+b)^p - a^p - b^p$.

The key claim is that if $p \equiv 1 \pmod 3$, then

$$p(x^2 + xy + y^2)^2 \text{ divides } (x+y)^p - x^p - y^p$$

as polynomials in x and y. Since it's known that one can select a and b such that $p^2 \mid a^2 + ab + b^2$, the conclusion follows. (The theory of quadratic forms tells us we can do it with $p^2 = a^2 + ab + b^2$; Thue's lemma lets us do it by solving $x^2 + x + 1 \equiv 0 \pmod{p^2}$.)

To prove this, it is the same to show that

$$(x^2 + x + 1)^2 \text{ divides } F(x) := (x+1)^p - x^p - 1.$$

since the binomial coefficients $\binom{p}{k}$ are clearly divisible by p. Let ζ be a third root of unity. Then $F(\zeta) = (1+\zeta)^p - \zeta^p - 1 = -\zeta^2 - \zeta - 1 = 0$. Moreover, $F'(x) = p(x+1)^{p-1} - px^{p-1}$, so $F'(\zeta) = p - p = 0$. Hence ζ is a double root of F as needed.

(Incidentally, $p = 2017$ works!)

Remark. One possible motivation for this solution is the case $p = 7$. It is nontrivial even to prove that p^2 can divide the expression if we exclude the situation $a + b = p$ (which provably never achieves p^3). As $p = 3, 5$ fails considering the $p = 7$ polynomial gives

$$(x+1)^7 - x^7 - 1 = 7x(x+1)\left(x^4 + 2x^3 + 3x^2 + 2x + 1\right).$$

The key is now to notice that the last factor is $(x^2 + x + 1)^2$, which suggests the entire solution.

In fact, even if $p \equiv 2 \pmod 3$, the polynomial $x^2 + x + 1$ still divides $(x+1)^p - x^p - 1$. So even the $p = 5$ case can motivate the main idea.

Team Selection Test for the 59^{th} International Mathematical Olympiad

United States of America

Day I

Thursday, December 7, 2017

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, December 11 at noon Eastern time.

IMO TST 1. Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

IMO TST 2. Find all functions $f\colon \mathbb{Z}^2 \to [0,1]$ such that for any integers x and y,

$$f(x,y) = \frac{f(x-1,y)+f(x,y-1)}{2}.$$

IMO TST 3. At a university dinner, there are 2017 mathematicians who each order two distinct entrées, with no two mathematicians ordering the same pair of entrées. The cost of each entrée is equal to the number of mathematicians who ordered it, and the university pays for each mathematician's less expensive entrée (ties broken arbitrarily). Over all possible sets of orders, what is the maximum total amount the university could have paid?

Team Selection Test for the 59$^{\text{th}}$ International Mathematical Olympiad

United States of America

Day II

Thursday, January 18, 2018

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, January 22 at noon Eastern time.

IMO TST 4. Let n be a positive integer and let $S \subseteq \{0,1\}^n$ be a set of binary strings of length n. Given an odd number $x_1, \dots, x_{2k+1} \in S$ of binary strings (not necessarily distinct), their *majority* is defined as the binary string $y \in \{0,1\}^n$ for which the i^{th} bit of y is the most common bit among the i^{th} bits of $x_1, \dots, x_{2k+1}$. (For example, if $n = 4$ the majority of 0000, 0000, 1101, 1100, 0101 is 0100.)

Suppose that for some positive integer k, S has the property P_k that the majority of any $2k+1$ binary strings in S (possibly with repetition) is also in S. Prove that S has the same property P_k for all positive integers k.

IMO TST 5. Let $ABCD$ be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at H. Denote by M and N the midpoints of $\overline{BC}$ and $\overline{CD}$. Rays MH and NH meet $\overline{AD}$ and $\overline{AB}$ at S and T, respectively. Prove there exists a point E, lying outside quadrilateral $ABCD$, such that

- ray EH bisects both angles $\angle BES$, $\angle TED$, and
- $\angle BEN = \angle MED$.

IMO TST 6. Alice and Bob play a game. First, Alice secretly picks a finite set S of lattice points in the Cartesian plane. Then, for every line ℓ in the plane which is horizontal, vertical, or has slope $+1$ or -1, she tells Bob the number of points of S that lie on ℓ. Bob wins if he can then determine the set S.

Prove that if Alice picks S to be of the form

$$S = \left\{(x,y) \in \mathbb{Z}^2 \mid m \le x^2 + y^2 \le n\right\}$$

for some positive integers m and n, then Bob can win. (Bob does not know in advance that S is of this form.)

USA IMO TST 2018 Solutions

United States of America — IMO Team Selection Tests

EVAN CHEN《陳誼廷》

59$^{\text{th}}$ IMO 2018 Romania

Contents

§0 Problems

1. Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

2. Find all functions $f \colon \mathbb{Z}^2 \to [0,1]$ such that for any integers x and y,
$$f(x,y) = \frac{f(x-1,y)+f(x,y-1)}{2}.$$

3. At a university dinner, there are 2017 mathematicians who each order two distinct entrées, with no two mathematicians ordering the same pair of entrées. The cost of each entrée is equal to the number of mathematicians who ordered it, and the university pays for each mathematician's less expensive entrée (ties broken arbitrarily). Over all possible sets of orders, what is the maximum total amount the university could have paid?

4. Let n be a positive integer and let $S \subseteq \{0,1\}^n$ be a set of binary strings of length n. Given an odd number $x_1, \dots, x_{2k+1} \in S$ of binary strings (not necessarily distinct), their *majority* is defined as the binary string $y \in \{0,1\}^n$ for which the i^{th} bit of y is the most common bit among the i^{th} bits of $x_1, \dots, x_{2k+1}$. (For example, if $n=4$ the majority of 0000, 0000, 1101, 1100, 0101 is 0100.)

   Suppose that for some positive integer k, S has the property P_k that the majority of any $2k+1$ binary strings in S (possibly with repetition) is also in S. Prove that S has the same property P_k for all positive integers k.

5. Let $ABCD$ be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at H. Denote by M and N the midpoints of $\overline{BC}$ and $\overline{CD}$. Rays MH and NH meet $\overline{AD}$ and $\overline{AB}$ at S and T, respectively. Prove there exists a point E, lying outside quadrilateral $ABCD$, such that
   - ray EH bisects both angles $\angle BES$, $\angle TED$, and
   - $\angle BEN = \angle MED$.

6. Alice and Bob play a game. First, Alice secretly picks a finite set S of lattice points in the Cartesian plane. Then, for every line ℓ in the plane which is horizontal, vertical, or has slope $+1$ or -1, she tells Bob the number of points of S that lie on ℓ. Bob wins if he can then determine the set S.

   Prove that if Alice picks S to be of the form
$$S = \left\{(x,y) \in \mathbb{Z}^2 \mid m \leq x^2+y^2 \leq n\right\}$$
   for some positive integers m and n, then Bob can win. (Bob does not know in advance that S is of this form.)

§1 Solutions to Day 1

§1.1 USA TST 2018/1, proposed by Ashwin Sah

Available online at https://aops.com/community/p9513094.

Problem statement

Let $n \ge 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

The equality case is $n = p^e$ for p prime and a positive integer e. It is easy to check that this works.

¶ First solution In what follows, by $[a, b]$ we mean $\{a, a+1, \dots, b\}$. First, we make the following easy observation.

Claim — If a and d are positive integers, then precisely $\varphi(d)$ elements of $[a, a+d-1]$ are relatively prime to d.

Let $d_1, d_2, \dots, d_k$ denote denote the divisors of n in some order. Consider the intervals

$$\begin{aligned} I_1 &= [1, d_1], \\ I_2 &= [d_1+1, d_1+d_2] \\ &\vdots \\ I_k &= [d_1 + \dots + d_{k-1} + 1, d_1 + \dots + d_k]. \end{aligned}$$

of length $d_1, \dots, d_k$ respectively. The jth interval will have exactly $\varphi(d_j)$ elements which are relatively prime d_j, hence at most $\varphi(d_j)$ which are relatively prime to n. Consequently, in $I = \bigcup_{j=1}^k I_k$ there are at most

$$\sum_{j=1}^k \varphi(d_j) = \sum_{d \mid n} \varphi(d) = n$$

integers relatively prime to n. On the other hand $I = [1, \sigma(n)]$ so this implies the inequality.

We see that the equality holds for $n = p^e$. Assume now $p < q$ are distinct primes dividing n. Reorder the divisors d_i so that $d_1 = q$. Then $p, q \in I_1$, and so I_1 should contain strictly fewer than $\varphi(d_1) = q - 1$ elements relatively prime to n, hence the inequality is strict.

¶ Second solution (Ivan Borsenco and Evan Chen) Let $n = p_1^{e_1} \dots p_k^{e_k}$, where $p_1 < p_2 < \dots$. We are going to assume $k \ge 2$, since the $k = 1$ case was resolved in the very beginning, and prove the strict inequality.

For a general N, the number of relatively prime integers in $[1, N]$ is given exactly by

$$f(N) = N - \sum_i \left\lfloor \frac{N}{p_i} \right\rfloor + \sum_{i<j} \left\lfloor \frac{N}{p_i p_j} \right\rfloor - \dots$$

according to the inclusion-exclusion principle. So, we wish to show that $f(\sigma(n)) < n$ (as $k \ge 2$). Discarding the error terms from the floors (noting that we get at most 1 from the negative floors) gives

$$\begin{aligned} f(N) &< 2^{k-1} + N - \sum_i \frac{N}{p_i} + \sum_{i<j} \frac{N}{p_i p_j} - \dots \\ &= 2^{k-1} + N \prod_i \left(1 - p_i^{-1}\right) \\ &= 2^{k-1} + \prod_i \left(1 - p_i^{-1}\right)\left(1 + p_i + p_i^2 + \dots + p_i^{e_i}\right) \\ &= 2^{k-1} + \prod_i \left(p_i^{e_i} - p_i^{-1}\right). \end{aligned}$$

The proof is now divided into two cases. If $k = 2$ we have

$$\begin{aligned} f(N) &< 2 + \left(p_1^{e_1} - p_1^{-1}\right)\left(p_2^{e_2} - p_2^{-1}\right) \\ &= 2 + n - \frac{p_2^{e_2}}{p_1} - \frac{p_1^{e_1}}{p_2} + \frac{1}{p_1 p_2} \\ &\le 2 + n - \frac{p_2}{p_1} - \frac{p_1}{p_2} + \frac{1}{p_1 p_2} \\ &= n + \frac{1 - (p_1 - p_2)^2}{p_1 p_2} \le n. \end{aligned}$$

On the other hand if $k \ge 3$ we may now write

$$\begin{aligned} f(N) &< 2^{k-1} + \left[\prod_{i=2}^{k-1} \left(p_i^{e_i}\right)\right]\left(p_1^{e_1} - p_1^{-1}\right) \\ &= 2^{k-1} + n - \frac{p_2^{e_2} \dots p_k^{e_k}}{p_1} \\ &\le 2^{k-1} + n - \frac{p_2 p_3 \dots p_k}{p_1}. \end{aligned}$$

If $p_1 = 2$, then one can show by induction that $p_2 p_3 \dots p_k \ge 2^{k+1} - 1$, which implies the result. If $p_1 > 2$, then one can again show by induction $p_3 \dots p_k \ge 2^k - 1$ (since $p_3 \ge 7$), which also implies the result.

§1.2 USA TST 2018/2, proposed by Michael Kural, Yang Liu

Available online at https://aops.com/community/p9513099.

Problem statement

Find all functions $f\colon \mathbb{Z}^2 \to [0,1]$ such that for any integers x and y,

$$f(x,y) = \frac{f(x-1,y)+f(x,y-1)}{2}.$$

We claim that the only functions f are constant functions. (It is easy to see that they work.)

¶ First solution (hands-on) First, iterating the functional equation relation to the nth level shows that

$$f(x,y) = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}f(x-i,y-(n-i)).$$

In particular,

$$\begin{aligned} |f(x,y)-f(x-1,y+1)| &= \frac{1}{2^n}\left|\sum_{i=0}^{n+1} f(x-i,y-(n-i))\cdot\left(\binom{n}{i}-\binom{n}{i-1}\right)\right| \\ &\le \frac{1}{2^n}\sum_{i=0}^{n+1}\left|\binom{n}{i}-\binom{n}{i-1}\right| \\ &= \frac{1}{2^n}\cdot 2\binom{n}{\lfloor n/2\rfloor} \end{aligned}$$

where we define $\binom{n}{n+1}=\binom{n}{-1}=0$ for convenience. Since

$$\binom{n}{\lfloor n/2\rfloor} = o(2^n)$$

it follows that f must be constant.

> **Remark.** A very similar proof extends to d dimensions.

¶ Second solution (random walks, Mark Sellke) We show that if $x+y=x'+y'$ then $f(x,y)=f(x',y')$. Let Z_n, Z_n' be random walks starting at (x,y) and (x',y') and moving down/left. Then $f(Z_n)$ is a martingale so we have

$$\mathbb{E}[f(Z_n)] = f(x,y), \qquad \mathbb{E}[f(Z_n')] = f(x',y').$$

We'll take Z_n, Z_n' to be independent until they hit each other, after which they will stay together. Then

$$|\mathbb{E}[f(Z_n)-f(Z_n')]| \le \mathbb{E}[|f(Z_n)-f(Z_n')|] \le p_n$$

where p_n is the probability that Z_n, Z_n' never collide. But the distance between Z_n, Z_n' is essentially a 1-dimensional random walk, so they will collide with probability 1, meaning $\lim_{n\to\infty} p_n = 0$. Hence

$$|f(x,y)-f(x',y')| = |\mathbb{E}[f(Z_n)-f(Z_n')]| = o(1)$$

as desired.

> **Remark.** If the problem were in $\mathbb{Z}^d$ for large d, this solution wouldn't work as written because the independent random walks wouldn't hit each other. However, this isn't a serious problem because Z_n, Z'_n don't have to be independent before hitting each other. Indeed, if every time Z_n, Z'_n agree on a new coordinate we force them to agree on that coordinate forever, we can make the two walks individually have the distribution of a coordinate-decreasing random walk but make them intersect eventually with probability 1. The difference in each coordinate will be a 1-dimensional random walk which gets stuck at 0.

¶ Third solution (martingales) Imagine starting at (x, y) and taking a random walk down and to the left. This is a martingale. As f is bounded, this martingale converges with probability 1. Let $X_1, X_2, \dots$ each be random variables that represent either down moves or left moves with equal probability. Note that by the Hewitt-Savage 0-1 law, we have that for any real numbers $a < b$,

$$\Pr\left[\lim_{n\to\infty} f((x,y) + X_1 + X_2 + \cdots + X_n) \in [a,b]\right] \in \{0,1\}.$$

Hence, there exists a single value v such that with probability 1,

$$\Pr\left[\lim_{n\to\infty} f((x,y) + X_1 + X_2 + \cdots + X_n) = v\right] = 1.$$

Obviously, this value v must equal $f(x, y)$. Now, we show this value v is the same for all (x, y). Note that any two starting points have a positive chance of meeting. Therefore, we are done.

§1.3 USA TST 2018/3, proposed by Evan Chen

Available online at https://aops.com/community/p9513105.

Problem statement

At a university dinner, there are 2017 mathematicians who each order two distinct entrées, with no two mathematicians ordering the same pair of entrées. The cost of each entrée is equal to the number of mathematicians who ordered it, and the university pays for each mathematician's less expensive entrée (ties broken arbitrarily). Over all possible sets of orders, what is the maximum total amount the university could have paid?

In graph theoretic terms: we wish to determine the maximum possible value of

$$S(G) := \sum_{e=vw} \min(\deg v, \deg w)$$

across all graphs G with 2017 edges. We claim the answer is $63 \cdot \binom{64}{2} + 1 = 127009$.

¶ First solution (combinatorial, Evan Chen) First define L_k to consist of a clique on k vertices, plus a single vertex connected to exactly one vertex of the clique. Hence L_k has $k+1$ vertices, $\binom{k}{2}+1$ edges, and $S(L_k) = (k-1)\binom{k}{2}+1$. In particular, L_{64} achieves the claimed maximum, so it suffices to prove the upper bound.

Lemma

Let G be a graph such that either

- G has $\binom{k}{2}$ edges for some $k \geq 3$ or
- G has $\binom{k}{2}+1$ edges for some $k \geq 4$.

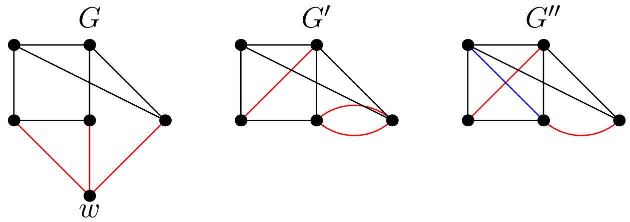
Then there exists a graph G^* with the same number of edges such that $S(G^*) \geq S(G)$, and moreover G^* has a universal vertex (i.e. a vertex adjacent to every other vertex).

Proof. Fix k and the number m of edges. We prove the result by induction on the number n of vertices in G. Since the lemma has two parts, we will need two different base cases:

1. Suppose $n = k$ and $m = \binom{k}{2}$. Then G must be a clique so pick $G^* = G$.

2. Suppose $n = k+1$ and $m = \binom{k}{2}+1$. If G has no universal vertex, we claim we may take $G^* = L_k$. Indeed each vertex of G has degree at most $k-1$, and the average degree is
$$\frac{2m}{n} = \frac{k^2-k+1}{k+1} < k-1$$
using here $k \geq 4$. Thus there exists a vertex w of degree $1 \leq d \leq k-2$. The edges touching w will have label at most d and hence
$$\begin{aligned} S(G) &\leq (k-1)(m-d) + d^2 = (k-1)m - d(k-1-d) \\ &\leq (k-1)m - (k-2) = (k-1)\binom{k}{2} + 1 = S(G^*). \end{aligned}$$

Now we settle the inductive step. Let w be a vertex with minimal degree $0 \le d < k-1$, with neighbors $w_1, \dots, w_d$. By our assumption, for each w_i there exists a vertex v_i for which $v_i w_i \notin E$. Now, we may delete all edges ww_i and in their place put $v_i w_i$, and then delete the vertex w. This gives a graph G', possibly with multiple edges (if $v_i = w_j$ and $w_j = v_i$), and with one fewer vertex.



We then construct a graph G'' by taking any pair of double edges, deleting one of them, and adding any missing edge of G'' in its place. (This is always possible, since when $m = \binom{k}{2}$ we have $n-1 \ge k$ and when $m = \binom{k}{2}+1$ we have $n-1 \ge k+1$.)

Thus we have arrived at a simple graph G'' with one fewer vertex. We also observe that we have $S(G'') \ge S(G)$; after all every vertex in G'' has degree at least as large as it did in G, and the d edges we deleted have been replaced with new edges which will have labels at least d. Hence we may apply the inductive hypothesis to the graph G'' to obtain G^* with $S(G^*) \ge S(G'') \ge S(G)$. □

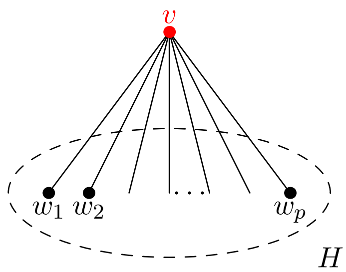
The problem then is completed once we prove the following:

Claim — For any graph G,

- If G has $\binom{k}{2}$ edges for $k \ge 3$, then $S(G) \le \binom{k}{2} \cdot (k-1)$.
- If G has $\binom{k}{2}+1$ edges for $k \ge 4$, then $S(G) \le \binom{k}{2} \cdot (k-1) + 1$.

Proof. We prove both parts at once by induction on k, with the base case $k = 3$ being plain (there is nothing to prove in the second part for $k = 3$). Thus assume $k \ge 4$. By the earlier lemma, we may assume G has a universal vertex v. For notational convenience, we say G has $\binom{k}{2} + \varepsilon$ edges for $\varepsilon \in \{0, 1\}$, and G has $p+1$ vertices, where $p \ge k-1+\varepsilon$.

Let H be the subgraph obtained when v is deleted. Then $m = \binom{k}{2} + \varepsilon - p$ is the number of edges in H; from $p \ge k-1+\varepsilon$ we have $m \le \binom{k-1}{2}$ and so we may apply the inductive hypothesis to H to deduce $S(H) \le \binom{k-1}{2} \cdot (k-2)$.



Now the labels of edges vw_i have sum

$$\sum_{i=1}^{p} \min\left(\deg_G v, \deg_G w_i\right) = \sum_{i=1}^{p} \deg_G w_i = \sum_{i=1}^{p} (\deg_H w_i + 1) = 2m + p.$$

For each of the edges contained in H, the label on that edge has increased by exactly 1, so those edges contribute $S(H) + m$. In total,

$$\begin{aligned} S(G) &= 2m + p + (S(H) + m) = (m + p) + 2m + S(H) \\ &\le \binom{k}{2} + \varepsilon + 2\binom{k-1}{2} + \binom{k-1}{2}(k-2) = \binom{k}{2}(k-1) + \varepsilon. \end{aligned}$$ □

¶ Second solution (algebraic, submitted by contestant James Lin) We give a different proof of $S(G) \le 127009$. The proof proceeds using the following two claims, which will show that $S(G) \le 127010$ for all graphs G. Then a careful analysis of the equality cases will show that this bound is not achieved for any graph G. Since the example L_{64} earlier has $S(L_{64}) = 127009$, this will solve the problem.

Lemma (Combinatorial bound)

Let G be a graph with 2017 edges and let $d_1 \ge d_2 \ge \cdots \ge d_n$ be the degree sequence of the graph (thus $n \ge 65$). Then

$$S(G) \le d_2 + 2d_3 + 3d_4 + \cdots + 63d_{64} + d_{65}.$$

Proof. Let $v_1, \dots, v_n$ be the corresponding vertices. For any edge $e = \{v_i, v_j\}$ with $i < j$, we consider associating each edge e with v_j, and computing the sum $S(G)$ indexing over associated vertices. To be precise, if we let a_i denote the number of edges associated to v_i, we now have $a_i \le i - 1$, $\sum a_i = 2017$, and

$$S(G) = \sum_{i=1}^{n} a_i d_i.$$

The inequality $\sum a_i d_i \le d_2 + 2d_3 + 3d_4 + \cdots + 63d_{64} + d_{65}$ then follows for smoothing reasons (by "smoothing" the a_i), since the d_i are monotone. This proves the given inequality. □

Once we have this property, we handle the bounding completely algebraically.

Lemma (Algebraic bound)

Let $x_1 \ge x_2 \ge \cdots \ge x_{65}$ be any nonnegative integers such that $\sum_{i=1}^{65} x_i \le 4034$. Then

$$x_2 + 2x_3 + \cdots + 63x_{64} + x_{65} \le 127010.$$

Moreover, equality occurs if and only if $x_1 = x_2 = x_3 = \cdots = x_{64} = 63$ and $x_{65} = 2$.

Proof. Let A denote the left-hand side of the inequality. We begin with a smoothing argument.

- Suppose there are indices $1 \le i < j \le 64$ such that $x_i > x_{i+1} \ge x_{j-1} > x_j$. Then replacing (x_i, x_j) by $(x_i - 1, x_j + 1)$ strictly increases A preserving all conditions. Thus we may assume all numbers in $\{x_1, \dots, x_{64}\}$ differ by at most 1.
- Suppose $x_{65} \ge 4$. Then we can replace $(x_1, x_2, x_3, x_4, x_{65})$ by $(x_1 + 1, x_2 + 1, x_3 + 1, x_4 + 1, x_{65} - 4)$ and strictly increase A. Hence we may assume $x_{65} \le 3$.

We will also tacitly assume $\sum x_i = 4034$, since otherwise we can increase x_1. These two properties leave only four sequences to examine:

- $x_1 = x_2 = x_3 = \cdots = x_{63} = 63$, $x_{64} = 62$, and $x_{65} = 3$, which gives $A = 126948$.
- $x_1 = x_2 = x_3 = \cdots = x_{63} = x_{64} = 63$ and $x_{65} = 2$, which gives $A = 127010$.
- $x_1 = 64$, $x_2 = x_3 = \cdots = x_{63} = x_{64} = 63$ and $x_{65} = 1$, which gives $A = 127009$.
- $x_1 = x_2 = 64$, $x_3 = \cdots = x_{63} = x_{64} = 63$ and $x_{65} = 0$, which gives $A = 127009$.

This proves that $A \le 127010$. To see that equality occurs only in the second case above, note that all the smoothing operations other than incrementing x_1 were strict, and that x_1 could not have been incremented in this way as $x_1 = x_2 = 63$. $\square$

This shows that $S(G) \le 127010$ for all graphs G, so it remains to show equality never occurs. Retain the notation d_i and a_i of the combinatorial bound now; we would need to have $d_1 = \cdots = d_{64} = 63$ and $d_{65} = 2$ (in particular, deleting isolated vertices from G, we may assume $n = 65$). In that case, we have $a_i \le i - 1$ but also $a_{65} = 2$ by definition (the last vertex gets all edges associated to it). Finally,

$$\begin{aligned} S(G) &= \sum_{i=1}^{n} a_i d_i = 63(a_1 + \cdots + a_{64}) + a_{65} \\ &= 63(2017 - a_{65}) + a_{65} \le 63 \cdot 2015 + 2 = 126947 \end{aligned}$$

completing the proof.

> **Remark.** Another way to finish once $S(G) \le 127010$ is note there is a unique graph (up to isomorphism and deletion of universal vertices) with degree sequence $(d_1, \dots, d_{65}) = (63, \dots, 63, 2)$. Indeed, the complement of the graph has degree sequence $(1, \dots, 1, 63)$, and so it must be a 63-star plus a single edge. One can then compute $S(G)$ explicitly for this graph.

¶ Some further remarks

> **Remark.** Interestingly, the graph C_4 has $\binom{3}{2} + 1 = 4$ edges and $S(C_4) = 8$, while $S(L_3) = 7$. This boundary case is visible in the combinatorial solution in the base case of the first claim. It also explains why we end up with the bound $S(G) \le 127010$ in the second algebraic solution, and why it is necessary to analyze the equality cases so carefully; observe in $k = 3$ the situation $d_1 = d_2 = d_3 = d_4 = 2$.

> **Remark.** Some comments about further context for this problem:
>
> - The obvious generalization of 2017 to any constant was resolved in September 2018 by Mehtaab Sawhney and Ashwin Sah. The relevant paper is *On the discrepancy between two Zagreb indices*, published in Discrete Mathematics, Volume 341, Issue 9, pages 2575-2589. The arXiv link is https://arxiv.org/pdf/1801.02532.pdf.
> - The quantity
> $$S(G) = \sum_{e=vw} \min(\deg v, \deg w)$$
> in the problem has an interpretation: it can be used to provide a bound on the number of triangles in a graph G. To be precise, $\#E(G) \le \frac{1}{3}S(G)$, since an edge $e = vw$ is part of at most $\min(\deg v, \deg w)$ triangles.
> - For *planar* graphs it is known $S(G) \le 18n - 36$ and it is conjectured that for n large

enough, $S(G) \leq 18n - 72$. See https://mathoverflow.net/a/273694/70654.

¶ Authorship comments I came up with the quantity $S(G)$ in a failed attempt to provide a bound on the number of triangles in a graph, since this is natural to consider when you do a standard double-counting via the edges of the triangle. I think the problem was actually APMO 1989, and I ended up not solving the problem (the solution is much simpler), but the quantity $S(G)$ stuck in my head for a while after that.

Later on that month I was keeping Danielle company while she was working on art project (flower necklace), and with not much to do except doodle on tables I began thinking about $S(G)$ again. I did have the sense that $S(G)$ should be maximized at a graph close to a complete graph. But to my frustration I could not prove it for a long time. Finally after many hours of trying various approaches I was able to at least show that $S(G)$ was maximized for complete graphs if the number of edges was a triangular number.

I had come up with this in March 2016, which would have been perfect since 2016 is a triangular number, but it was too late to submit it to any contest (the USAMO and IMO deadlines were long past). So on December 31, 2016 I finally sat down and solved it for the case 2017, which took another few hours of thought, then submitted it to that year's IMO. To my dismay it was rejected, but I passed it along to the USA TST after that, thus making it just in time for the close of the calendar year.

§2 Solutions to Day 2

§2.1 USA TST 2018/4, proposed by Josh Brakensiek

Available online at https://aops.com/community/p9735607.

Problem statement

Let n be a positive integer and let $S \subseteq \{0,1\}^n$ be a set of binary strings of length n. Given an odd number $x_1, \dots, x_{2k+1} \in S$ of binary strings (not necessarily distinct), their *majority* is defined as the binary string $y \in \{0,1\}^n$ for which the i^{th} bit of y is the most common bit among the i^{th} bits of $x_1, \dots, x_{2k+1}$. (For example, if $n = 4$ the majority of 0000, 0000, 1101, 1100, 0101 is 0100.)

Suppose that for some positive integer k, S has the property P_k that the majority of any $2k+1$ binary strings in S (possibly with repetition) is also in S. Prove that S has the same property P_k for all positive integers k.

Let M denote the majority function (of any length).

¶ First solution (induction) We prove all P_k are equivalent by induction on $n \ge 2$, with the base case $n = 2$ being easy to check by hand. (The case $n = 1$ is also vacuous; however, the inductive step is not able to go from $n = 1$ to $n = 2$.)

For the inductive step, we proceed by contradiction; assume S satisfies P_ℓ, but not P_k, so there exist $x_1, \dots, x_{2k+1} \in S$ whose majority $y = M(x_1, \dots, x_k)$ is not in S. We contend that:

Claim — Let y_i be the string which differs from y only in the i^{th} bit. Then $y_i \in S$.

Proof. For a string $s \in S$ we let $\hat{s}$ denote the string s with the i^{th} bit deleted (hence with $n-1$ bits). Now let

$$T = \{\hat{s} \mid s \in S\}.$$

Since S satisfies P_ℓ, so does T; thus by the induction hypothesis on n, T satisfies P_k.

Consequently, $T \ni M(\hat{x}_1, \dots, \hat{x}_{2k+1}) = \hat{y}$. Thus there exists $s \in S$ such that $\hat{s} = \hat{y}$. This implies $s = y$ or $s = y_i$. But since we assumed $y \notin S$ it follows $y_i \in S$ instead. $\square$

Now take any $2\ell + 1$ copies of the y_i, about equally often (i.e. the number of times any two y_i are taken differs by at most 1). We see the majority of these is y itself, contradiction.

¶ Second solution (circuit construction) Note that $P_k \implies P_1$ for any k, since

$$M(\underbrace{a, \dots, a}_{k}, \underbrace{b, \dots, b}_{k}, c) = M(a, b, c)$$

for any a, b, c.

We will now prove $P_1 + P_k \implies P_{k+1}$ for any k, which will prove the result. Actually, we will show that the majority of any $2k+3$ strings $x_1, \dots, x_{2k+3}$ can be expressed by 3 and $(2k+1)$-majorities. WLOG assume that $M(x_1, \dots, x_{2k+3}) = 0 \dots 0$, and let $\odot$ denote binary AND.

Claim — We have $M(x_1, x_2, M(x_3, \dots, x_{2k+3})) = x_1 \odot x_2$.

Proof. Consider any particular bit. The result is clear if the bits are equal. Otherwise, if they differ, the result follows from the original hypothesis that $M(x_1, \dots, x_{2k+3}) = 0 \dots 0$ (removing two differing bits does not change the majority). □

By analogy we can construct any $x_i \odot x_j$. Finally, note that

$$M(x_1 \odot x_2, x_2 \odot x_3, \dots, x_{2k+1} \odot x_{2k+2}) = 0 \dots 0,$$

as desired. (Indeed, if we look at any index, there were at most $k+1$ 1's in the x_i strings, and hence there will be at most k 1's among $x_i \odot x_{i+1}$ for $i = 1, \dots, 2k+1$.)

Remark. The second solution can be interpreted in circuit language as showing that all "$2k+1$-majority gates" are equivalent. See also `https://cstheory.stackexchange.com/a/21399/48303`, in which Valiant gives a probabilistic construction to prove that one can construct $(2k+1)$-majority gates from a *polynomial* number of 3-majority gates. No explicit construction is known for this.

§2.2 USA TST 2018/5, proposed by Evan Chen

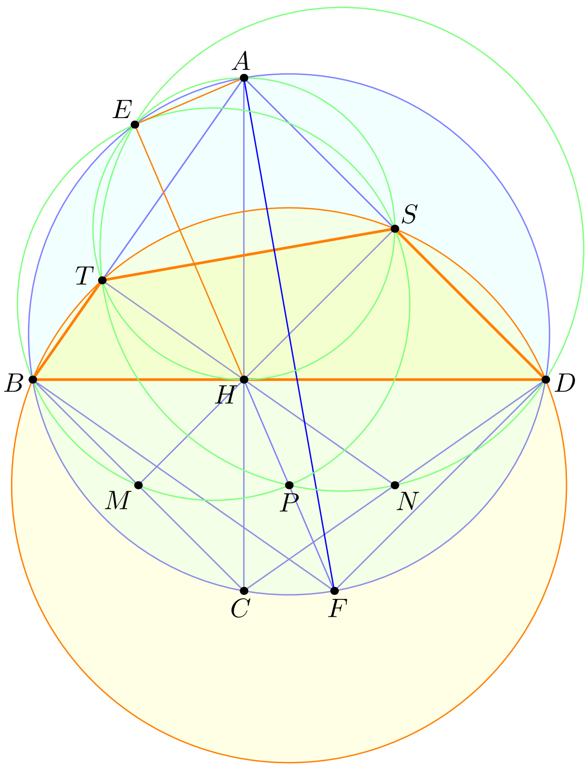
Available online at https://aops.com/community/p9735608.

Problem statement

Let $ABCD$ be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at H. Denote by M and N the midpoints of $\overline{BC}$ and $\overline{CD}$. Rays MH and NH meet $\overline{AD}$ and $\overline{AB}$ at S and T, respectively. Prove there exists a point E, lying outside quadrilateral $ABCD$, such that

- ray EH bisects both angles $\angle BES$, $\angle TED$, and
- $\angle BEN = \angle MED$.

The main claim is that E is the intersection of $(ABCD)$ with the circle with diameter $\overline{AH}$.



The following observation can be quickly made without reference to E.

Lemma

We have $\angle HSA = \angle HTA = 90^\circ$. Consequently, quadrilateral $BTSD$ is cyclic.

Proof. This is direct angle chasing. In fact, $\overline{HM}$ passes through the circumcenter of $\triangle BHC$ and $\triangle HAD \sim \triangle HCB$, so $\overline{HS}$ ought to be the altitude of $\triangle HAD$. □

From here it follows that E is the Miquel point of cyclic quadrilateral $BTSD$. Define F to be the point diametrically opposite A, so that E, H, F are collinear, $\overline{CF} \parallel \overline{BD}$. By now we already have

$$\measuredangle BEH = \measuredangle BEF = \measuredangle BAF = \measuredangle CAD = \measuredangle HAS = \measuredangle HES$$

so $\overline{EH}$ bisects $\angle BES$, and $\angle TED$. Hence it only remains to show $\angle BEM = \angle NED$; we present several proofs below.

¶ First proof (original solution) Let P be the circumcenter of $BTSD$. The properties of the Miquel point imply P lies on the common bisector $\overline{EH}$ already, and it also lies on the perpendicular bisector of $\overline{BD}$, hence it must be the midpoint of $\overline{HF}$.

We now contend quadrilaterals $BMPS$ and $DNPT$ are cyclic. Obviously $\overline{MP}$ is the external angle bisector of $\angle BMS$, and $PB = PS$, so P is the arc midpoint of (BMS). The proof for $DNPT$ is analogous.

It remains to show $\angle BEN = \angle MED$, or equivalently $\angle BEM = \angle NED$. By properties of Miquel point we have $E \in (BMPS) \cap (TPND)$, so

$$\measuredangle BEM = \measuredangle BPM = \measuredangle PBD = \measuredangle BDP = \measuredangle NPD = \measuredangle NED$$

as desired.

¶ Second proof (2011 G4) By 2011 G4, the circumcircle of $\triangle EMN$ is tangent to the circumcircle of $ABCD$. Hence if we extend $\overline{EM}$ and $\overline{EN}$ to meet $(ABCD)$ again at X and Y, we get $\overline{XY} \parallel \overline{MN} \parallel \overline{BD}$. Thus $\measuredangle BEM = \measuredangle BEX = \measuredangle YED = \measuredangle NED$.

¶ Third proof (involutions, submitted by Daniel Liu) Let $G = \overline{BN} \cap \overline{MD}$ denote the centroid of $\triangle BCD$, and note that it lies on $\overline{EHF}$.

Now consider the dual of Desargues involution theorem on complete quadrilateral $BMDNCG$ at point E. We get

$$(EB, ED), \quad (EM, EN), \quad (EC, EG)$$

form an involutive pairing.

However, the bisector of $\angle BED$, say ℓ, is also the angle bisector of $\angle CEF$ (since $\overline{CF} \parallel \overline{BD}$). So the involution we found must coincide with reflection across ℓ. This means $\angle MEN$ is bisected by ℓ as well, as desired.

¶ Authorship comments This diagram actually comes from the inverted picture in IMO 2014/3 (which I attended). I had heard for many years that one could solve this problem quickly by inversion at H afterwards. But when I actually tried to do it during an OTIS class years later, I ended up with the picture in the TST problem, and couldn't see why it was true! In the process of trying to reconstruct this rumored solution, I ended up finding most of the properties that ended up in the January TST problem (but were overkill for the original IMO problem).

Let us make the equivalence explicit by deducing the IMO problem from our work.

Let rays EM and EN meet the circumcircles of $\triangle BHC$ and $\triangle BNC$ again at X and Y, with $EM < EX$ and $EN < EY$. As above we concluded $EM/EX = EN/EY$ and so $\overline{MN} \parallel \overline{XY} \implies \overline{XY} \perp \overline{AHC}$.

Now consider an inversion at H which swaps $B \leftrightarrow D$ and $A \leftrightarrow C$. The point E goes to E^* diametrically opposite A. Points X and Y go to points on $X^* \in \overline{AD}$ and $Y^* \in \overline{AB}$. Since the reflection of E across $\overline{PX}$ is supposed to lie on (BAE), it follows that the circumcenter of $\triangle HX^*E^*$ lies on $\overline{AD}$. Consequently X^* plays the role of point "T" in the IMO problem. Then Y^* plays the role of point "S" in the IMO problem.

Now the fact that (HX^*Y^*) is tangent to $\overline{BD}$ is equivalent to $\overline{XY} \perp \overline{AHC}$ which we already knew.

§2.3 USA TST 2018/6, proposed by Mark Sellke

Available online at https://aops.com/community/p9735613.

Problem statement

Alice and Bob play a game. First, Alice secretly picks a finite set S of lattice points in the Cartesian plane. Then, for every line ℓ in the plane which is horizontal, vertical, or has slope $+1$ or -1, she tells Bob the number of points of S that lie on ℓ. Bob wins if he can then determine the set S.

Prove that if Alice picks S to be of the form

$$S = \left\{(x, y) \in \mathbb{Z}^2 \mid m \le x^2 + y^2 \le n\right\}$$

for some positive integers m and n, then Bob can win. (Bob does not know in advance that S is of this form.)

Clearly Bob can compute the number N of points.

The main claim is that:

Claim — Fix m and n as in the problem statement. Among all sets $T \subseteq \mathbb{Z}^2$ with N points, the set S is the *unique* one which maximizes the value of

$$F(T) := \sum_{(x,y)\in T} (x^2 + y^2)(m + n - (x^2 + y^2)).$$

Proof. Indeed, the different points in T do not interact in this sum, so we simply want the points (x, y) with $x^2 + y^2$ as close as possible to $\frac{m+n}{2}$ which is exactly what S does. □

As a result of this observation, it suffices to show that Bob has enough information to compute $F(S)$ from the data given. (There is no issue with fixing m and n, since Bob can find an upper bound on the magnitude of the points and then check all pairs (m, n) smaller than that.) The idea is that he knows the full distribution of each of X, Y, $X + Y$, $X - Y$ and hence can compute sums over T of any power of a single one of those linear functions. By taking linear combinations we can hence compute $F(S)$.

Let us make the relations explicit. For ease of exposition we take $Z = (X, Y)$ to be a uniformly random point from the set S. The information is precisely the individual distributions of X, Y, $X + Y$, and $X - Y$. Now compute

$$\begin{aligned}\frac{F(S)}{N} &= \mathbb{E}\left[(m+n)(X^2+Y^2) - (X^2+Y^2)^2\right] \\ &= (m+n)\left(\mathbb{E}[X^2] + \mathbb{E}[Y^2]\right) - \mathbb{E}[X^4] - \mathbb{E}[Y^4] - 2\mathbb{E}[X^2Y^2].\end{aligned}$$

On the other hand,

$$\mathbb{E}[X^2Y^2] = \frac{\mathbb{E}[(X+Y)^4] + \mathbb{E}[(X-Y)^4] - 2\mathbb{E}[X^4] - 2\mathbb{E}[Y^4]}{12}.$$

Thus we have written $F(S)$ in terms of the distributions of X, Y, $X - Y$, $X + Y$ which completes the proof.

Remark (Mark Sellke)**.**

- This proof would have worked just as well if we allowed arbitrary $[0, 1]$-valued weights on points with finitely many weights non-zero. There is an obvious continuum generalization one can make concerning the indicator function for an annulus. It's a simpler but fun problem to characterize when just the vertical/horizontal directions determine the distribution.
- An obstruction to purely combinatorial arguments is that if you take an octagon with points $(\pm a, \pm b)$ and $(\pm b, \pm a)$ then the two ways to pick every other point (going around clockwise) are indistinguishable by Bob. This at least shows that Bob's task is far from possible in general, and hints at proving an inequality.
- A related and more standard fact (among a certain type of person) is that given a probability distribution μ on $\mathbb{R}^n$, if I tell you the distribution of *all* 1-dimensional projections of μ, that determines μ uniquely. This works because this information gives me the Fourier transform $\hat{\mu}$, and Fourier transforms are injective.

  For the continuum version of this problem, this connection gives a much larger family of counterexamples to any proposed extension to arbitrary non-annular shapes. Indeed, take a fast-decaying smooth function $f \colon \mathbb{R}^2 \to \mathbb{R}$ which vanishes on the four lines

  $$x = 0, \; y = 0, \; x + y = 0, \; x - y = 0.$$

  Then the Fourier transform $\hat{f}$ will have mean 0 on each line ℓ as in the problem statement. Hence the positive and negative parts of $\hat{f}$ will not be distinguishable by Bob.

USA IMO TST 2018 Statistics

United States of America — IMO Team Selection Tests

Evan Chen

59th IMO 2018 Romania

§1 Summary of scores for TST 2018

N	35	1st Q	7	Max	36
μ	14.06	Median	10	Top 3	29
σ	9.64	3rd Q	22	Top 6	23

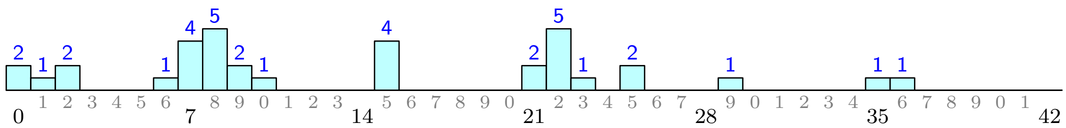
§2 Problem statistics for TST 2018

	P1	**P2**	**P3**	**P4**	**P5**	**P6**
0	15	26	16	12	12	31
1	6	0	16	1	1	2
2	1	0	0	0	4	0
3	0	0	2	0	0	0
4	0	0	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	2	0	1
7	13	9	1	19	18	1
Avg	2.83	1.80	0.83	4.31	3.86	0.43
QM	4.30	3.55	1.54	5.42	5.07	1.58
#5+	**13**	**9**	**1**	**22**	**18**	**2**
%5+	%37.1	%25.7	%2.9	%62.9	%51.4	%5.7

§3 Rankings for TST 2018

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
42	0	0	0.00%	28	0	3	8.57%	14	0	17	48.57%
41	0	0	0.00%	27	0	3	8.57%	13	0	17	48.57%
40	0	0	0.00%	26	0	3	8.57%	12	0	17	48.57%
39	0	0	0.00%	25	2	5	14.29%	11	0	17	48.57%
38	0	0	0.00%	24	0	5	14.29%	10	1	18	51.43%
37	0	0	0.00%	23	1	6	17.14%	9	2	20	57.14%
36	1	1	2.86%	22	5	11	31.43%	8	5	25	71.43%
35	1	2	5.71%	21	2	13	37.14%	7	4	29	82.86%
34	0	2	5.71%	20	0	13	37.14%	6	1	30	85.71%
33	0	2	5.71%	19	0	13	37.14%	5	0	30	85.71%
32	0	2	5.71%	18	0	13	37.14%	4	0	30	85.71%
31	0	2	5.71%	17	0	13	37.14%	3	0	30	85.71%
30	0	2	5.71%	16	0	13	37.14%	2	2	32	91.43%
29	1	3	8.57%	15	4	17	48.57%	1	1	33	94.29%
								0	2	35	100.00%

§4 Histogram for TST 2018



Team Selection Test for the 60^{th} International Mathematical Olympiad

United States of America

Day I

Thursday, December 6, 2018

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, December 10 at noon Eastern time.

IMO TST 1. Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

IMO TST 2. Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x)+x, \quad g(x)+2x, \quad \dots, \quad g(x)+100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

IMO TST 3. A *snake of length k* is an animal which occupies an ordered k-tuple $(s_1, \dots, s_k)$ of cells in an $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \dots, k-1$. If the snake is currently occupying $(s_1, \dots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can *move* to occupy $(s, s_1, \dots, s_{k-1})$ instead. The snake has *turned around* if it occupied $(s_1, s_2, \dots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \dots, s_1)$ instead.

Determine whether there exists an integer $n > 1$ such that one can place some snake of length at least $0.9n^2$ in an $n \times n$ grid which can turn around.

Team Selection Test for the 60$^{\text{th}}$ International Mathematical Olympiad

United States of America

Day II

Thursday, January 17, 2019

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, January 21 at noon Eastern time.

IMO TST 4. We say a function $f\colon \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is *great* if for any nonnegative integers m and n,

$$f(m+1, n+1)f(m, n) - f(m+1, n)f(m, n+1) = 1.$$

If $A = (a_0, a_1, \dots)$ and $B = (b_0, b_1, \dots)$ are two sequences of integers, we write $A \sim B$ if there exists a great function f satisfying $f(n, 0) = a_n$ and $f(0, n) = b_n$ for every nonnegative integer n (in particular, $a_0 = b_0$).

Prove that if A, B, C, and D are four sequences of integers satisfying $A \sim B$, $B \sim C$, and $C \sim D$, then $D \sim A$.

IMO TST 5. Let n be a positive integer. Tasty and Stacy are given a circular necklace with $3n$ sapphire beads and $3n$ turquoise beads, such that no three consecutive beads have the same color. They play a cooperative game where they alternate turns removing three consecutive beads, subject to the following conditions:

- Tasty must remove three consecutive beads which are turquoise, sapphire, and turquoise, in that order, on each of his turns.
- Stacy must remove three consecutive beads which are sapphire, turquoise, and sapphire, in that order, on each of her turns.

They win if all the beads are removed in $2n$ turns. Prove that if they can win with Tasty going first, they can also win with Stacy going first.

IMO TST 6. Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.

USA IMO TST 2019 Solutions

United States of America — IMO Team Selection Tests

Evan Chen《陳誼廷》

60th IMO 2019 United Kingdom

Contents

§0 Problems

1. Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

2. Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x)+x, \quad g(x)+2x, \quad \dots, \quad g(x)+100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

3. A *snake of length k* is an animal which occupies an ordered k-tuple $(s_1, \dots, s_k)$ of cells in an $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \dots, k-1$. If the snake is currently occupying $(s_1, \dots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can *move* to occupy $(s, s_1, \dots, s_{k-1})$ instead. The snake has *turned around* if it occupied $(s_1, s_2, \dots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \dots, s_1)$ instead.

   Determine whether there exists an integer $n > 1$ such that one can place some snake of length at least $0.9n^2$ in an $n \times n$ grid which can turn around.

4. We say a function $f\colon \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0} \to \mathbb{Z}$ is *great* if for any nonnegative integers m and n,

$$f(m+1,n+1)f(m,n) - f(m+1,n)f(m,n+1) = 1.$$

   If $A = (a_0, a_1, \dots)$ and $B = (b_0, b_1, \dots)$ are two sequences of integers, we write $A \sim B$ if there exists a great function f satisfying $f(n,0) = a_n$ and $f(0,n) = b_n$ for every nonnegative integer n (in particular, $a_0 = b_0$).

   Prove that if A, B, C, and D are four sequences of integers satisfying $A \sim B$, $B \sim C$, and $C \sim D$, then $D \sim A$.

5. Let n be a positive integer. Tasty and Stacy are given a circular necklace with $3n$ sapphire beads and $3n$ turquoise beads, such that no three consecutive beads have the same color. They play a cooperative game where they alternate turns removing three consecutive beads, subject to the following conditions:

   - Tasty must remove three consecutive beads which are turquoise, sapphire, and turquoise, in that order, on each of his turns.
   - Stacy must remove three consecutive beads which are sapphire, turquoise, and sapphire, in that order, on each of her turns.

   They win if all the beads are removed in $2n$ turns. Prove that if they can win with Tasty going first, they can also win with Stacy going first.

6. Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

   Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.

§1 Solutions to Day 1

§1.1 USA TST 2019/1, proposed by Merlijn Staps

Available online at https://aops.com/community/p11419585.

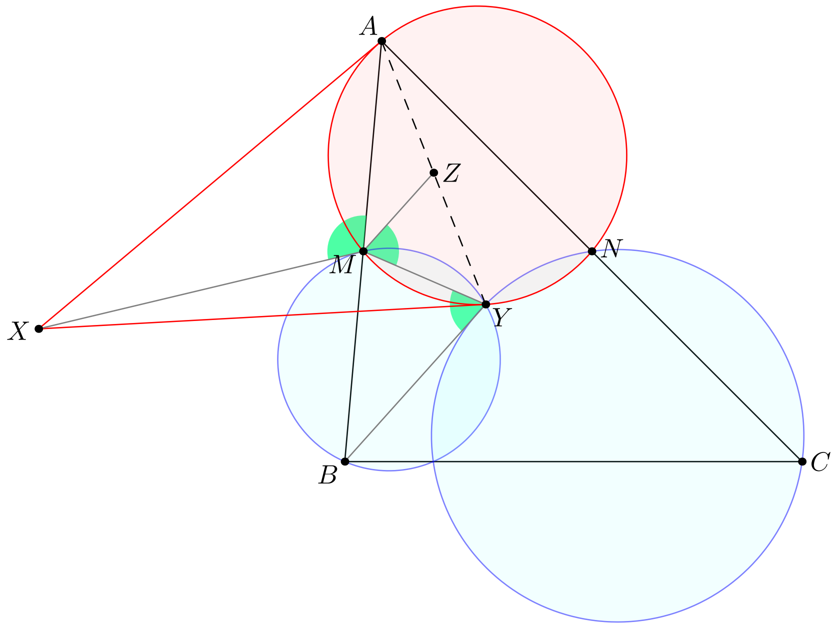
Problem statement

Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

We present four solutions, the second of which shows that M and N can be replaced by any two points on AB and AC satisfying $AM/AB + AN/AC = 1$.

¶ First solution using symmedians (Merlijn Staps) Let $\overline{XY}$ be the other tangent from X to (AMN).

Claim — Line $\overline{XM}$ is tangent to (BMY); hence Y lies on ω_B.

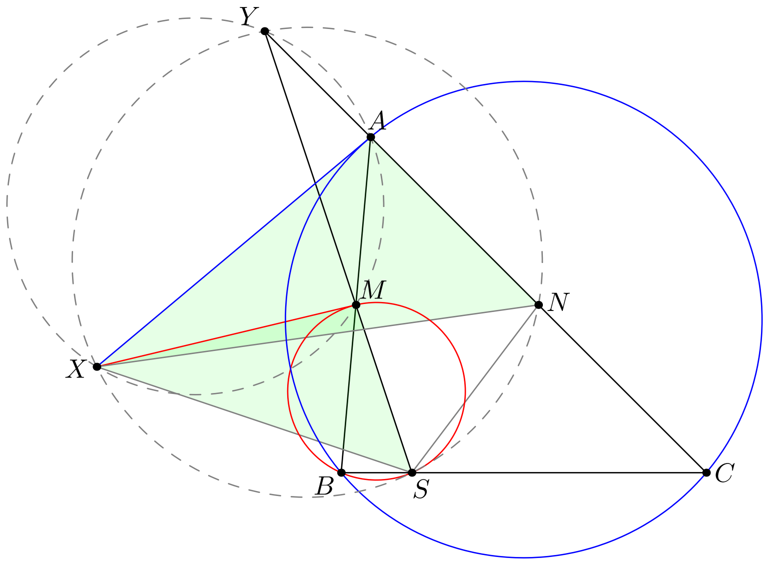


Proof. Let Z be the midpoint of $\overline{AY}$. Then $\overline{MX}$ is the M-symmedian in triangle AMY. Since $\overline{MZ} \parallel \overline{BY}$, it follows that $\measuredangle AMX = \measuredangle ZMY = \measuredangle BYM$. We conclude that $\overline{XM}$ is tangent to the circumcircle of triangle BMY. □

Similarly, ω_C is the circumcircle of triangle CNY. As $AMYN$ is cyclic too, it follows that ω_B and ω_C intersect on $\overline{BC}$, by Miquel's theorem.

Remark. The converse of Miquel's theorem is true, which means the problem is equivalent to showing that the second intersection of the ω_B and ω_C moves along (AMN). Thus the construction of Y above is not so unnatural.

¶ Second solution (Jetze Zoethout) Let ω_B intersect $\overline{BC}$ again at S and let $\overline{MS}$ intersect $\overline{AC}$ again at Y. Angle chasing gives $\measuredangle XMY = \measuredangle XMS = \measuredangle MBS = \measuredangle ABC = \measuredangle XAC = \measuredangle XAY$, so Y is on the circumcircle of triangle AMX. Furthermore, from $\measuredangle XMY = \measuredangle ABC$ and $\measuredangle ACB = \measuredangle XAB = \measuredangle XYM$ it follows that $\triangle ABC \sim \triangle XMY$ and from $\measuredangle XAY = \measuredangle MBS$ and $\measuredangle YXA = \measuredangle YMA = \measuredangle BMS$ it follows that $\triangle AXY \sim \triangle BMS$.



We now find

$$\frac{AN}{AX} = \frac{AN/BM}{AX/BM} = \frac{AC/AB}{MS/XY} = \frac{AB/AB}{MS/XM} = \frac{XM}{MS},$$

which together with $\angle XMS = \angle XAN$ yields $\triangle XMS \sim \triangle XAN$. From $\measuredangle XSY = \measuredangle XSM = \measuredangle XNA = \measuredangle XNY$ we now have that S is on the circumcircle of triangle XNY. Finally, we have $\measuredangle XNS = \measuredangle XYS = \measuredangle XYM = \measuredangle ACB = \measuredangle NCS$ so $\overline{XN}$ is tangent to the circle through C, N, and S, as desired.

¶ Third solution by moving points method Fix triangle ABC and animate X along the tangent at A. We let D denote the second intersection point of ω_C with line $\overline{BC}$.

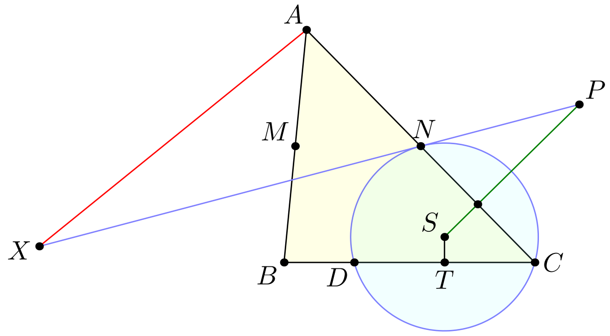
Claim — The composed map $X \mapsto D$ is a fractional linear transformation (i.e. a projective map) in terms of a real coordinate on line $\overline{AA}$, $\overline{BC}$.

Proof. Let ℓ denote the perpendicular bisector of $\overline{CN}$, also equipped with a real coordinate. We let P denote the intersection of $\overline{XM}$ with ℓ, S the circumcenter of $\triangle CMD$. Let T denote the midpoint of $\overline{BD}$.

We claim that the composed map

$$\begin{aligned} &\overline{AA} \to \ell \to \ell \to \overline{BC} \to \overline{BC} \\ \text{by} \quad &X \mapsto P \mapsto S \mapsto T \mapsto D \end{aligned}$$

is projective, by showing each individual map is projective.



- The map $X \mapsto P$ is projective since it is a perspectivity through N from $\overline{AA}$ to ℓ.
- The map $P \mapsto S$ is projective since it is equivalent to a negative inversion on ℓ at the midpoint of $\overline{NC}$ with radius $\frac{1}{2}NC$. (Note $\angle PNS = 90^\circ$ is fixed.)
- The map $S \mapsto T$ is projective since it is a perspectivity $\ell \to \overline{BC}$ through the point at infinity perpendicular to $\overline{BC}$ (in fact, it is linear).
- The map $T \mapsto D$ is projective (in fact, linear) since it is a homothety through C with fixed ratio 2.

Thus the composed map is projective as well. □

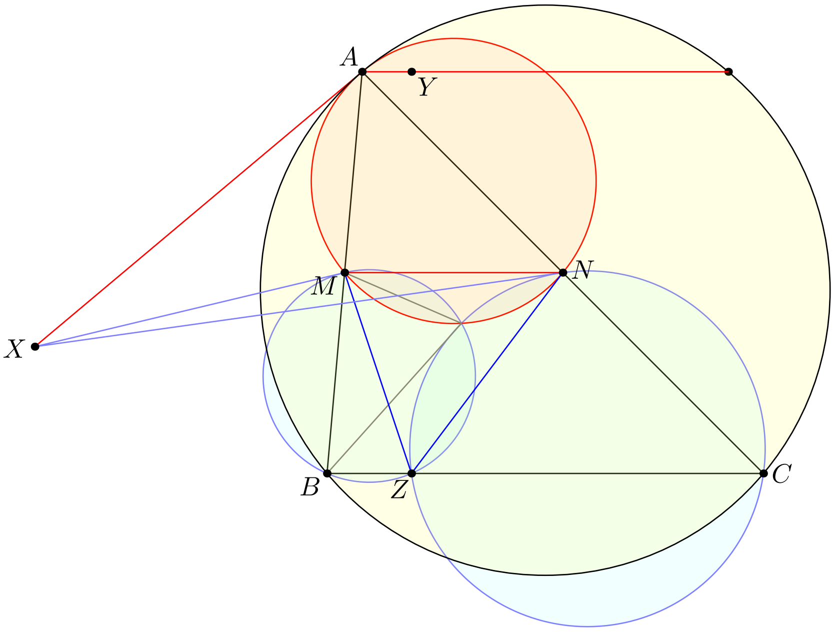
Similarly, if we define D' so that $\overline{XM}$ is tangent to (BMD'), the map $X \mapsto D'$ is projective as well. We aim to show $D = D'$, and since the maps correspond to fractional linear transformations in projective coordinates, it suffices to verify it for three distinct choices of X. We do so:

- If $X = \overline{AA} \cap \overline{MN}$, then D and D' satisfy $MB = MD'$, $NC = ND$. This means they are the feet of the A-altitude on $\overline{BC}$.
- As X approaches A the points D and D' approach the infinity point along $\overline{BC}$.
- If X is a point at infinity along $\overline{AA}$, then D and D' coincide with the midpoint of $\overline{BC}$.

This completes the solution.

> **Remark** (Anant Mudgal)**.** An alternative (shorter) way to show $X \mapsto D$ is projective is to notice $\measuredangle XND$ is a constant angle. I left the longer "original" proof for instructional reasons.

¶ Fourth solution by isogonal conjugates (Anant Mudgal) Let Y be the isogonal conjugate of X in $\triangle AMN$ and Z be the reflection of Y in $\overline{MN}$. As $\overline{AX}$ is tangent to the circumcircle of $\triangle AMN$, it follows that $\overline{AY} \parallel \overline{MN}$. Thus Z lies on $\overline{BC}$ since $\overline{MN}$ bisects the strip made by $\overline{AY}$ and $\overline{BC}$.



Finally,

$$\measuredangle ZMX = \measuredangle ZMN + \measuredangle NMX = \measuredangle NMY + \measuredangle YMA = \measuredangle NMA = \measuredangle ZBM$$

so $\overline{XM}$ is tangent to the circumcircle of $\triangle ZMB$, hence Z lies on ω_B. Similarly, Z lies on ω_C and we're done.

§1.2 USA TST 2019/2, proposed by Ashwin Sah, Yang Liu

Available online at https://aops.com/community/p11419598.

Problem statement

Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x)+x, \quad g(x)+2x, \quad \dots, \quad g(x)+100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

Call a function g *valiant* if it obeys this condition. We claim the answer is all numbers relatively prime to $101!$.

The construction is to just let g be the identity function.

Before proceeding to the converse solution, we make a long motivational remark.

> **Remark** (Motivation for both parts)**.** The following solution is dense, and it is easier to think about some small cases first, to motivate the ideas. We consider the result where 101 is replaced by 2 or 3.
>
> - If we replaced 101 with 2, you can show $2 \nmid n$ easily: write
>
> $$\sum_x x \equiv \sum_x g(x) \equiv \sum_x (g(x)+x) \pmod n$$
>
> which implies
>
> $$0 \equiv \sum_x x = \frac{1}{2}n(n+1) \pmod n$$
>
> which means $\frac{1}{2}n(n+1) \equiv 0 \pmod n$, hence n odd.
>
> - If we replaced 101 with 3, then you can try a similar approach using squares, since
>
> $$\begin{aligned} 0 &\equiv \sum_x \left[(g(x)+2x)^2 - 2\left(g(x)+x\right)^2 + g(x)^2 \right] \pmod n \\ &= \sum_x 2x^2 = 2 \cdot \frac{n(n+1)(2n+1)}{6} \end{aligned}$$
>
> which is enough to force $3 \nmid n$.

We now present several different proofs of the converse, all of which generalize the ideas contained here. In everything that follows we assume $n > 1$ for convenience.

¶ First solution (original one) The proof is split into two essentially orthogonal claims, which we state as lemmas.

Lemma (Lemma I: elimination of g)
Assume valiant $g\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ exists. Then

$$k! \sum_{x \in \mathbb{Z}/n\mathbb{Z}} x^k \equiv 0 \pmod n$$

for $k = 0, 1, \dots, 100$.

Proof. Define $g_x(T) = g(x) + Tx$ for any integer T. If we view $g_x(T)^k$ as a polynomial in $\mathbb{Z}[T]$ of degree k with leading coefficient x^k, then taking the kth finite difference implies that, for any x,

$$k!x^k = \binom{k}{0} g_x(k)^k - \binom{k}{1} g_x(k-1)^k + \binom{k}{2} g_x(k-2)^k - \dots + (-1)^k \binom{k}{k} g_x(0)^k.$$

On the other hand, for any $1 \le k \le 100$ we should have

$$\begin{aligned} \sum_x g_x(0)^k \equiv \sum_x g_x(1)^k \equiv \dots \equiv \sum_x g_x(k)^k & \\ \equiv S_k := 0^k + \dots + (n-1)^k & \pmod n \end{aligned}$$

by the hypothesis. Thus we find

$$k! \sum_x x^k \equiv \left[\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \dots \right] S_k \equiv 0 \pmod n$$

for any $1 \le k \le 100$, but also obviously for $k = 0$. □

We now prove the following self-contained lemma.

Lemma (Lemma II: power sum calculation)
Let p be a prime, and let n, M be positive integers such that

$$M \quad \text{divides} \quad 1^k + 2^k + \dots + n^k$$

for $k = 0, 1, \dots, p-1$. If $p \mid n$ then $\nu_p(M) < \nu_p(n)$.

Proof. The hypothesis means that that any polynomial $f(T) \in \mathbb{Z}[T]$ with $\deg f \le p-1$ will have $\sum_{x=1}^n f(x) \equiv 0 \pmod M$. In particular, we have

$$\begin{aligned} 0 &\equiv \sum_{x=1}^n (x-1)(x-2)\cdots(x-(p-1)) \\ &= (p-1)! \sum_{x=1}^n \binom{x-1}{p-1} = (p-1)! \binom{n}{p} \pmod M. \end{aligned}$$

But now $\nu_p(M) \le \nu_p(\binom{n}{p}) = \nu_p(n) - 1$. □

Now assume for contradiction that valiant $g\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ exists, and $p \le 101$ is the *smallest* prime dividing n. Lemma I implies that $k! \sum_x x^k \equiv 0 \pmod n$ for $k = 1, \dots, p-1$ and hence $\sum_x x^k \equiv 0 \pmod n$ too. Thus $M = n$ holds in the previous lemma, impossible.

¶ A second solution Both lemmas above admit variations where we focus on working modulo p^e rather than working modulo n.

Lemma (Lemma I')

Assume valiant $g\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ exists. Let $p \le 101$ be a prime, and $e = \nu_p(n)$. Then

$$\sum_{x \in \mathbb{Z}/n\mathbb{Z}} x^k \equiv 0 \pmod{p^e}$$

for $k = 0, 1, \dots, p-1$.

Proof. This is weaker than Lemma I, but we give an independent specialized proof. Begin by writing

$$\sum_x (g(x) + Tx)^k \equiv \sum_x x^k \pmod{p^e}.$$

Both sides are integer polynomials in T, which vanish at $T = 0, 1, \dots, p-1$ by hypothesis (since $p - 1 \le 100$).

We now prove the following more general fact: if $f(T) \in \mathbb{Z}[T]$ is an integer polynomial with $\deg f \le p-1$, such that $f(0) \equiv \dots \equiv f(p-1) \equiv 0 \pmod{p^e}$, then all coefficients of f are divisible by p^e. The proof is by induction on $e \ge 1$. When $e = 1$, this is just the assertion that the polynomial has at most $\deg f$ roots modulo p. When $e \ge 2$, we note that the previous result implies all coefficients are divisible by p, and then we divide all coefficients by p.

Applied here, we have that all coefficients of

$$f(T) := \sum_x (g(x) + Tx)^k - \sum_x x^k$$

are divisible by p^e. The leading T^k coefficient is $\sum_k x^k$ as desired. □

Lemma (Lemma II')

If $e \ge 1$ is an integer, and p is a prime, then

$$\nu_p\left(1^{p-1} + 2^{p-1} + \dots + (p^e - 1)^{p-1}\right) = e - 1.$$

Proof. First, note that the cases where $p = 2$ or $e = 1$ are easy; since if $p = 2$ we have $\sum_{x=0}^{2^e-1} x \equiv 2^{e-1}(2^e - 1) \equiv -2^{e-1} \pmod{2^e}$, while if $e = 1$ we have $1^{p-1} + \dots + (p-1)^{p-1} \equiv -1 \pmod{p}$. Henceforth assume that $p > 2$, $e > 1$.

Let g be an integer which is a primitive root modulo p^e. Then, we can sum the terms which are relatively prime to p as

$$S_0 := \sum_{\gcd(x,p)=1} x^{p-1} \equiv \sum_{i=1}^{\varphi(p^e)} g^{(p-1)\cdot i} \equiv \frac{g^{p^{e-1}(p-1)^2} - 1}{g^{p-1} - 1} \pmod{p^e}$$

which implies $\nu_p(S_0) = e - 1$, by lifting the exponent. More generally, for $r \ge 1$ we may set

$$S_r := \sum_{\nu_p(x)=r} x^{p-1} \equiv (p^r)^{p-1} \sum_{i=1}^{\varphi(p^{e-r})} g_r^{(p-1)\cdot i} \pmod{p^e}$$

where g_r is a primitive root modulo p^{e-r}. Repeating the exponent-lifting calculation shows that $\nu_p(S_r) = r(p-1) + ((e-r)-1) > e$, as needed. $\square$

Assume to the contrary that $p \le 101$ is a prime dividing n, and a valiant $g\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ exists. Take $k = p-1$ in Lemma I' to contradict Lemma II'

¶ A third remixed solution We use Lemma I and Lemma II' from before. As before, assume $g\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is valiant, and n has a prime divisor $p \le 101$. Also, let $e = \nu_p(n)$.

Then $(p-1)! \sum_x x^{p-1} \equiv 0 \pmod n$ by Lemma I, and now

$$\begin{aligned} 0 &\equiv \sum_x x^{p-1} \pmod{p^e} \\ &\equiv \frac{n}{p^e} \sum_{x=1}^{p^e-1} x^{p-1} \not\equiv 0 \pmod{p^e} \end{aligned}$$

by Lemma II', contradiction.

¶ A fourth remixed solution We also can combine Lemma I' and Lemma II. As before, assume $g\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is valiant, and let p be the smallest prime divisor of n.

Assume for contradiction $p \le 101$. By Lemma I' we have

$$\sum_x x^k \equiv 0 \pmod{p^e}$$

for $k = 0, \dots, p-1$. This directly contradicts Lemma II with $M = p^e$.

§1.3 USA TST 2019/3, proposed by Nikolai Beluhov

Available online at https://aops.com/community/p11419601.

Problem statement

A *snake of length* k is an animal which occupies an ordered k-tuple $(s_1, \dots, s_k)$ of cells in an $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \dots, k-1$. If the snake is currently occupying $(s_1, \dots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can *move* to occupy $(s, s_1, \dots, s_{k-1})$ instead. The snake has *turned around* if it occupied $(s_1, s_2, \dots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \dots, s_1)$ instead.

Determine whether there exists an integer $n > 1$ such that one can place some snake of length at least $0.9n^2$ in an $n \times n$ grid which can turn around.

The answer is yes (and 0.9 is arbitrary).

¶ First grid-based solution The following solution is due to Brian Lawrence. For illustration reasons, we give below a figure of a snake of length 89 turning around in an 11×11 square (which generalizes readily to odd n). We will see that a snake of length $(n-1)(n-2)-1$ can turn around in an $n \times n$ square, so this certainly implies the problem.

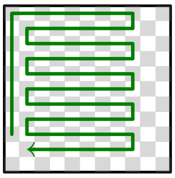
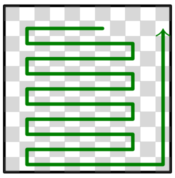


Figure 1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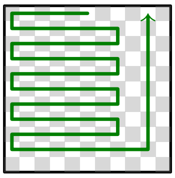
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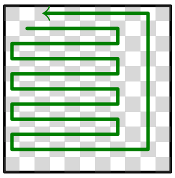


Figure 4

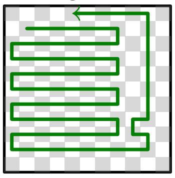


Figure 5

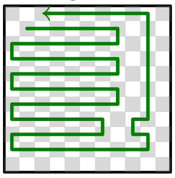


Figure 6

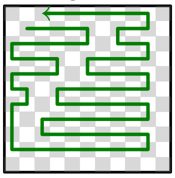


Figure 7

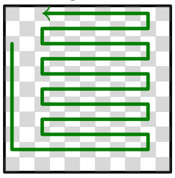


Figure 8

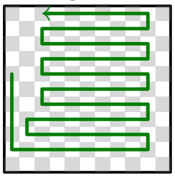


Figure 9

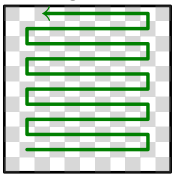


Figure 10

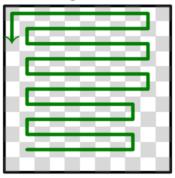


Figure 11

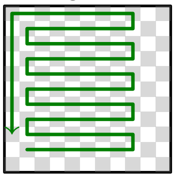


Figure 12

Use the obvious coordinate system with $(1,1)$ in the bottom left. Start with the snake as shown in Figure 1, then have it move to $(2,1)$, $(2,n)$, $(n,n-1)$ as in Figure 2. Then, have the snake shift to the position in Figure 3; this is possible since the snake can just walk to (n,n), then start walking to the left and then follow the route; by the time it reaches the ith row from the top its tail will have vacated by then. Once it achieves Figure 3, move the head of the snake to $(3,n)$ to achieve Figure 4.

In Figure 5 and 6, the snake begins to "deform" its loop continuously. In general, this deformation by two squares is possible in the following way. The snake walks first to $(1,n)$ then retraces the steps left by its tail, except when it reaches $(n-1,3)$ it makes a brief detour to $(n-2,3)$, $(n-2,4)$, $(n-1,4)$ and continues along its way; this gives the position in Figure 5. Then it retraces the entire loop again, except that when it reaches $(n-4,4)$ it turns directly down, and continues retracing its path; thus at the end of this second revolution, we arrive at Figure 6.

By repeatedly doing perturbations of two cells, we can move move all the "bumps" in the path gradually to protrude from the right; Figure 7 shows a partial application of the procedure, with the final state as shown in Figure 8.

In Figure 9, we stretch the bottom-most bump by two more cells; this shortens the "tail" by two units, which is fine. Doing this for all $(n-3)/2$ bumps arrives at the situation in Figure 10, with the snake's head at $(3,n)$. We then begin deforming the turns on the bottom-right by two steps each as in Figure 11, which visually will increase the length of the head. Doing this arrives finally at the situation in Figure 12. Thus the snake has turned around.

¶ Second solution phrased using graph theory (Nikolai Beluhov) Let G be any undirected graph. Consider a snake of length k lying within G, with each segment of the snake occupying one vertex, consecutive segments occupying adjacent vertices, and no two segments occupying the same vertex. One move of the snake consists of the snake's head advancing to an adjacent empty vertex and segment i advancing to the vertex of segment $i-1$ for $i=2, 3, \dots, k$.

The solution proceeds in two stages. First we construct a planar graph G such that it is possible for a snake that occupies nearly all of G to turn around inside G. Then we construct a subgraph H of a grid adjacency graph such that H is isomorphic to G and H occupies nearly all of the grid.

For the first stage of the solution, we construct G as follows.

Let r and ℓ be positive integers. Start with r disjoint *main* paths $p_1, p_2, \dots, p_r$, each of length at least ℓ, with p_i leading from A_i to B_i for $i=1, 2, \dots, r$. Add to those r *linking* paths, one leading from B_i to A_{i+1} for each $i=1, 2, \dots, r-1$, and one leading from B_r to A_1. Finally, add to those two families of *transit* paths, with one family containing one transit path joining A_1 to each of $A_2, A_3, \dots, A_r$ and the other containing one path joining B_r to each of $B_1, B_2, \dots, B_{r-1}$. We require that all paths specified in the construction have no interior vertices in common, with the exception of transit paths in the same family.

We claim that a snake of length $(r-1)\ell$ can turn around inside G.

To this end, let the concatenation $A_1B_1A_2B_2\dots A_rB_r$ of all main and linking paths be the *great cycle*. We refer to $A_1B_1A_2B_2\dots A_rB_r$ as the counterclockwise orientation of the great cycle, and to $B_rA_rB_{r-1}A_{r-1}\dots B_1A_1$ as its clockwise orientation.

Place the snake so that its tail is at A_1 and its body extends counterclockwise along the great cycle. Then let the snake manoeuvre as follows. (We track only the snake's head, as its movement uniquely determines the movement of the complete body of the snake.)

At phase 1, advance counterclockwise along the great cycle to B_{r-1}, take a detour along a transit path to B_r, and advance clockwise along the great cycle to A_r.

For $i=2, 3, \dots, r-1$, at phase i, take a detour along a transit path to A_1, advance counterclockwise along the great cycle to B_{r-i}, take a detour along a transit path to B_r, and advance clockwise along the great cycle to A_{r-i+1}.

At phase r, simply advance clockwise along the great cycle to A_1.

For the second stage of the solution, let n be a sufficiently large positive integer. Consider an $n \times n$ grid S. Number the columns of S from 1 to n from left to right, and its rows from 1 to n from bottom to top.

Let a_1, a_2, …, a_{r+1} be cells of S such that all of a_1, a_2, …, a_{r+1} lie in column 2, a_1 lies in row 2, a_{r+1} lies in row $n-1$, and a_1, a_2, …, a_{r+1} are approximately equally spaced. Let b_1, b_2, …, b_r be cells of S such that all of b_1, b_2, …, b_r lie in column $n-2$ and b_i lies in the row of a_{i+1} for $i = 1, 2, \dots, r$.

Construct H as follows. For $i = 1, 2, \dots, r$, let the main path from a_i to b_i fill up the rectangle bounded by the rows and columns of a_i and b_i nearly completely. Then every main path is of length approximately $\frac{1}{r}n^2$.

For $i = 1, 2, \dots, r-1$, let the linking path that leads from b_i to a_{i+1} lie inside the row of b_i and a_{i+1} and let the linking path that leads from b_r to a_1 lie inside row n, column n, and row 1.

Lastly, let the union of the first family of transit paths be column 1 and let the union of the second family of transit paths be column $n-1$, with the exception of their bottommost and topmost squares.

As in the first stage of the solution, by this construction a snake of length k approximately equal to $\frac{r-1}{r}n^2$ can turn around inside an $n \times n$ grid S. When r is fixed and n tends to infinity, $\frac{k}{n^2}$ tends to $\frac{r-1}{r}$. Furthermore, when r tends to infinity, $\frac{r-1}{r}$ tends to 1. This gives the answer.

§2 Solutions to Day 2

§2.1 USA TST 2019/4, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p11625808.

Problem statement

We say a function $f : \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0} \to \mathbb{Z}$ is *great* if for any nonnegative integers m and n,

$$f(m+1, n+1)f(m, n) - f(m+1, n)f(m, n+1) = 1.$$

If $A = (a_0, a_1, \dots)$ and $B = (b_0, b_1, \dots)$ are two sequences of integers, we write $A \sim B$ if there exists a great function f satisfying $f(n, 0) = a_n$ and $f(0, n) = b_n$ for every nonnegative integer n (in particular, $a_0 = b_0$).

Prove that if A, B, C, and D are four sequences of integers satisfying $A \sim B$, $B \sim C$, and $C \sim D$, then $D \sim A$.

We present two solutions. In what follows, we say (A, B) form a great pair if $A \sim B$.

¶ First solution (Nikolai Beluhov) Let $k = a_0 = b_0 = c_0 = d_0$. We let f, g, h be great functions for (A, B), (B, C), (C, D) and write the following infinite array:

$$\begin{bmatrix} & \vdots & \vdots & b_3 & \vdots & \vdots & \\ \cdots & g(2,2) & g(2,1) & b_2 & f(1,2) & f(2,2) & \cdots \\ \cdots & g(1,2) & g(1,1) & b_1 & f(1,1) & f(2,1) & \cdots \\ c_3 & c_2 & c_1 & k & a_1 & a_2 & a_3 \\ \cdots & h(2,1) & h(1,1) & d_1 & & & \\ \cdots & h(2,2) & h(1,2) & d_2 & & & \\ & \vdots & \vdots & d_3 & & & \ddots \end{bmatrix}$$

The greatness condition is then equivalent to saying that any 2×2 sub-grid has determinant ± 1 (the sign is $+1$ in two quadrants and -1 in the other two), and we wish to fill in the lower-right quadrant. To this end, it suffices to prove the following.

Lemma

Suppose we have a 3×3 sub-grid

$$\begin{bmatrix} a & b & c \\ x & y & z \\ p & q & \end{bmatrix}$$

satisfying the determinant conditions. Then we can fill in the ninth entry in the lower right with an integer while retaining greatness.

Proof. We consider only the case where the 3×3 is completely contained inside the bottom-right quadrant, since the other cases are exactly the same (or even by flipping the signs of the top row or left column appropriately).

If $y = 0$ we have $-1 = bz = bx = xq$, hence $qz = -1$, and we can fill in the entry arbitrarily.

Otherwise, we have $bx \equiv xq \equiv bz \equiv -1 \pmod y$. This is enough to imply $qz \equiv -1 \pmod y$, and so we can fill in the integer $\frac{qz+1}{y}$. □

Remark. In this case (of all +1 determinants), I think it turns out the bottom entry is exactly equal to $qza - cyp - c - p$, which is obviously an integer.

¶ Second solution (Ankan Bhattacharya) We will give an explicit classification of great sequences:

Lemma

The pair (A, B) is great if and only if $a_0 = b_0$, $a_0 \mid a_1 b_1 + 1$, and $a_n \mid a_{n-1} + a_{n+1}$ and $b_n \mid b_{n-1} + b_{n+1}$ for all n.

Proof of necessity. It is clear that $a_0 = b_0$. Then $a_0 f(1,1) - a_1 b_1 = 1$, i.e. $a_0 \mid a_1 b_1 + 1$.

Now, focus on six entries $f(x, y)$ with $x \in \{n-1, n, n+1\}$ and $y \in \{0, 1\}$. Let $f(n-1, 1) = u$, $f(n, 1) = v$, and $f(n+1, 1) = w$, so

$$va_{n-1} - ua_n = 1,$$
$$wa_n - va_{n+1} = 1.$$

Then

$$u + w = \frac{v(a_{n-1} + a_{n+1})}{a_n}$$

and from above $\gcd(v, a_n) = 1$, so $a_n \mid a_{n-1} + a_{n+1}$; similarly for b_n. (If $a_n = 0$, we have $va_{n-1} = 1$ and $va_{n+1} = -1$, so this is OK.) □

Proof of sufficiency. Now consider two sequences $a_0, a_1, \dots$ and $b_0, b_1, \dots$ satisfying our criteria. We build a great function f by induction on (x, y). More strongly, we will assume as part of the inductive hypothesis that any two adjacent entries of f are relatively prime and that for any three consecutive entries horizontally or vertically, the middle one divides the sum of the other two.

First we set $f(1,1)$ so that $a_0 f(1,1) = a_1 b_1 + 1$, which is possible.

Consider an uninitialized $f(s, t)$; without loss of generality suppose $s \ge 2$. Then we know five values of f and wish to set a sixth one z, as in the matrix below:

$$\begin{matrix} u & x \\ v & y \\ w & z \end{matrix}$$

(We imagine a-indices to increase southwards and b-indices to increase eastwards.) If $v \neq 0$, then the choice $y \cdot \frac{u+w}{v} - x$ works as $uy - vx = 1$. If $v = 0$, it easily follows that $\{u, w\} = \{1, -1\}$ and $y = w$ as $yw = 1$. Then we set the uninitialized entry to anything.

Now we verify that this is compatible with the inductive hypothesis. From the determinant 1 condition, it easily follows that $\gcd(w, z) = \gcd(v, z) = 1$. The proof that $y \mid x + z$ is almost identical to a step performed in the "necessary" part of the lemma and we do not repeat it here. By induction, a desired great function f exists. □

We complete the solution. Let A, B, C, and D be integer sequences for which (A, B), (B, C), and (C, D) are great. Then $a_0 = b_0 = c_0 = d_0$, and each term in each sequence (after the zeroth term) divides the sum of its neighbors. Since a_0 divides all three of $a_1b_1 + 1$, $b_1c_1 + 1$, and $c_1d_1 + 1$, it follows a_0 divides $d_1a_1 + 1$, and thus (D, A) is great as desired.

Remark. To simplify the problem, we may restrict the codomain of great functions and elements in great pairs of sequences to $\mathbb{Z}_{>0}$. This allows the parts of the solution dealing with zero entries to be ignored.

Remark. Of course, this solution also shows that any odd path (in the graph induced by the great relation on sequences) completes to an even cycle. If we stipulate that great functions must have $f(0, 0) = \pm 1$, then even paths complete to cycles as well. Alternatively, we could change the great functional equation to

$$f(x + 1, y + 1)f(x, y) - f(x + 1, y)f(x, y + 1) = -1.$$

A quick counterexample to transitivity of $\sim$ as is without the condition $f(0, 0) = 1$, for concreteness: let $a_n = c_n = 3 + n$ and $b_n = 3 + 2n$ for $n \geq 0$.

§2.2 USA TST 2019/5, proposed by Yannick Yao

Available online at https://aops.com/community/p11625809.

Problem statement

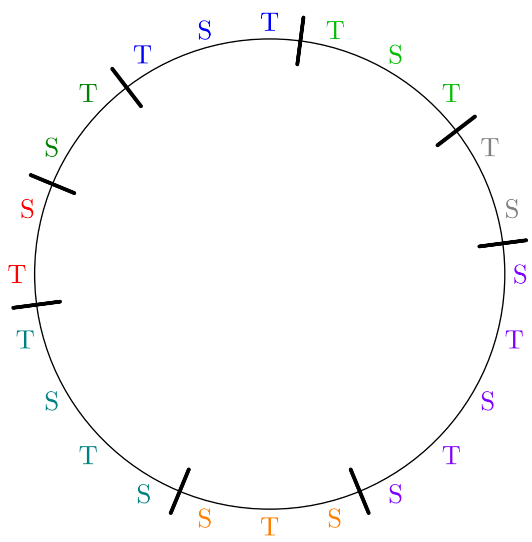
Let n be a positive integer. Tasty and Stacy are given a circular necklace with $3n$ sapphire beads and $3n$ turquoise beads, such that no three consecutive beads have the same color. They play a cooperative game where they alternate turns removing three consecutive beads, subject to the following conditions:

- Tasty must remove three consecutive beads which are turquoise, sapphire, and turquoise, in that order, on each of his turns.
- Stacy must remove three consecutive beads which are sapphire, turquoise, and sapphire, in that order, on each of her turns.

They win if all the beads are removed in $2n$ turns. Prove that if they can win with Tasty going first, they can also win with Stacy going first.

In the necklace, we draw a *divider* between any two beads of the same color. Unless there are no dividers, this divides the necklace into several *zigzags* in which the beads in each zigzag alternate. Each zigzag has two *endpoints* (adjacent to dividers).

Observe that the condition about not having three consecutive matching beads is equivalent to saying there are no zigzags of lengths 1.



The main claim is that the game is winnable (for either player going first) if and only if there are at most $2n$ dividers. We prove this in two parts, the first part not using the hypothesis about three consecutive letters.

Claim — The game cannot be won with Tasty going first if there are more than $2n$ dividers.

Proof. We claim each move removes at most one divider, which proves the result.

Consider removing a TST in some zigzag (necessarily of length at least 3). We illustrate the three possibilities in the following table, with Tasty's move shown in red.

Before	After	Change
... ST | TST | TS ...	... ST | TS ...	One less divider; two zigzags merge
... ST | TSTST ...	... STST ...	One less divider; two zigzags merge
... STSTS ...	... S | S ...	One more divider; a zigzag splits in two

The analysis for Stacy's move is identical. □

Claim — If there are at most $2n$ dividers and there are no zigzags of length 1 then the game can be won (with either player going first).

Proof. By symmetry it is enough to prove Tasty wins going first.

At any point if there are no dividers at all, then the necklace alternates $TSTST\dots$ and the game can be won. So we will prove that on each of Tasty's turns, if there exists at least one divider, then Tasty and Stacy can each make a move at an endpoint of some zigzag (i.e. the first two cases above). As we saw in the previous proof, such moves will (a) decrease the number of dividers by exactly one, (b) not introduce any singleton zigzags (because the old zigzags merge, rather than split). Since there are fewer than $2n$ dividers, our duo can eliminate all dividers and then win.

Note that as the number of S and T's are equal, there must be an equal number of

- zigzags of odd length (≥ 3) with T at the endpoints (i.e. one more T than S), and
- zigzags of odd length (≥ 3) with S at the endpoints (i.e. one more S than T).

Now iff there is at least one of each, then Tasty removes a TST from the end of such a zigzag while Stacy removes an STS from the end of such a zigzag.

Otherwise suppose all zigzags have even size. Then Tasty finds any zigzag of length ≥ 4 (which must exist since the *average* zigzag length is 3) and removes TST from the end containing T. The resulting merged zigzag is odd and hence S endpoints, hence Stacy can move as well. □

Remark. There are many equivalent ways to phrase the solution. For example, the number of dividers is equal to the number of pairs of two consecutive letters (rather than singleton letters). So the win condition can also be phrased in terms of the number of adjacent pairs of letters being at least $2n$, or equivalently the number of differing pairs being at least $4n$.

If one thinks about the game as a process, this is a natural "monovariant" to consider anyways, so the solution is not so unmotivated.

Remark. The constraint of no three consecutive identical beads is actually needed: a counterexample without this constraint is TTSTSTSTTSSS. (They win if Tasty goes first and lose if Stacy goes first.)

Remark (Why induction is unlikely to work)**.** Many contestants attempted induction. However, in doing so they often implicitly proved a different problem: "prove that if they can win with Tasty going first *without ever creating a triplet*, they can also win in such a way with Stacy going first". This essentially means nearly all induction attempts fail.

Amusingly, even the modified problem (which is much more amenable to induction) sill seems difficult without *some* sort of global argument. Consider a position in which Tasty wins going first, with the sequence of winning moves being Tasty's first move in red below and Stacy's second move in blue below:

$$\dots\texttt{TTSSTT}\underbrace{\texttt{S}\overbrace{\texttt{TST}}^{\text{Tasty}}\texttt{TS}}_{\text{Stacy}}\texttt{STTSST}\dots.$$

There is no "nearby" STS that Stacy can remove instead on her first turn, without introducing a triple-T and also preventing Tasty from taking a TST. So it does not seem possible to easily change a Tasty-first winning sequence to a Stacy-first one, even in the modified version.

§2.3 USA TST 2019/6, proposed by Ankan Bhattachrya

Available online at https://aops.com/community/p11625814.

Problem statement

Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.

We present two solutions.

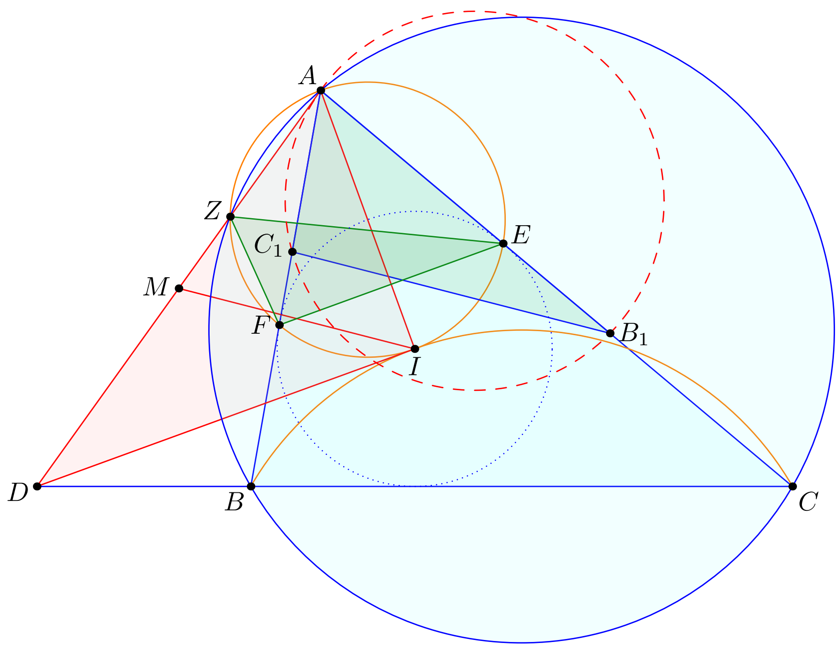
¶ First solution using spiral similarity (Ankan Bhattacharya) First, we prove the part of the problem which does not depend on the condition $AB_1A_1C_1$ is cyclic.

Lemma

Let ABC be a triangle and define I, D, B_1, C_1 as in the problem. Moreover, let M denote the midpoint of $\overline{AD}$. Then $\overline{AD}$ is tangent to (AB_1C_1), and moreover $\overline{B_1C_1} \parallel \overline{IM}$.

Proof. Let E and F be the tangency points of the incircle. Denote by Z the Miquel point of $BFEC$, i.e. the second intersection of the circle with diameter $\overline{AI}$ and the circumcircle.

Note that A, Z, D are collinear, by radical axis on (ABC), $(AFIE)$, (BIC).



Then the spiral similarity gives us

$$\frac{ZF}{ZE} = \frac{BF}{CE} = \frac{AC_1}{AB_1}$$

which together with $\measuredangle FZE = \measuredangle FAE = \measuredangle BAC$ implies that $\triangle ZFE$ and $\triangle AC_1B_1$ are (directly) similar. (See IMO Shortlist 2006 G9 for a similar application of spiral similarity.)

Now the remainder of the proof is just angle chasing. First, since

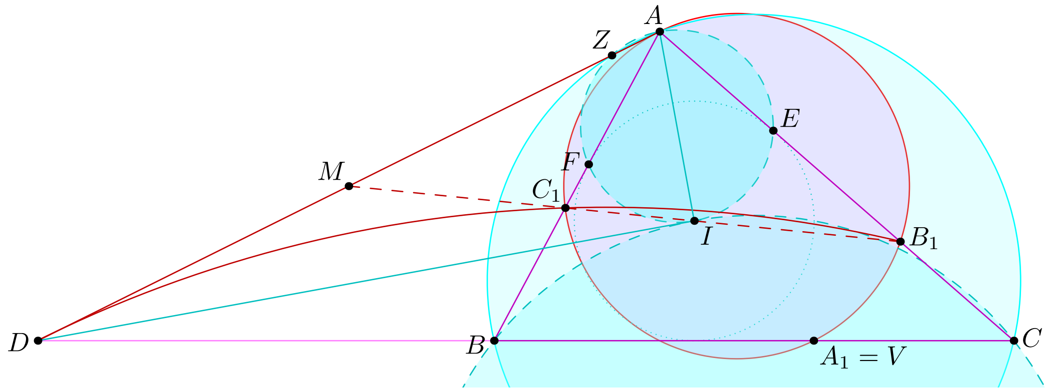
$$\measuredangle DAC_1 = \measuredangle ZAF = \measuredangle ZEF = \measuredangle AB_1C_1$$

we have $\overline{AD}$ is tangent to (AB_1C_1). Moreover, to see that $\overline{IM} \parallel \overline{B_1C_1}$, write

$$\begin{aligned}\measuredangle(\overline{AI}, \overline{B_1C_1}) &= \measuredangle IAC + \measuredangle AB_1C_1 = \measuredangle BAI + \measuredangle ZEF = \measuredangle FAI + \measuredangle ZAF \\ &= \measuredangle ZAI = \measuredangle MAI = \measuredangle AIM\end{aligned}$$

the last step since $\triangle AID$ is right with hypotenuse $\overline{AD}$, and median $\overline{IM}$. □

Now we return to the present problem with the additional condition.



Claim — Given the condition, we actually have $\angle AB_1A_1 = \angle AC_1A_1 = 90°$.

Proof. Let I_A, I_B and I_C be the excenters of $\triangle ABC$. Then the perpendiculars to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ from A_1, B_1, C_1 respectively meet at the so-called *Bevan point* V (which is the circumcenter of $\triangle I_AI_BI_C$).

Now $\triangle AB_1C_1$ has circumdiameter $\overline{AV}$. We are given A_1 lies on this circle, so if $V \neq A_1$ then $\overline{AA_1} \perp \overline{A_1V}$. But $\overline{A_1V} \perp \overline{BC}$ by definition, which would imply $\overline{AA_1} \parallel \overline{BC}$, which is absurd. □

Claim — Given the condition the points B_1, I, C_1 are collinear (hence with M).

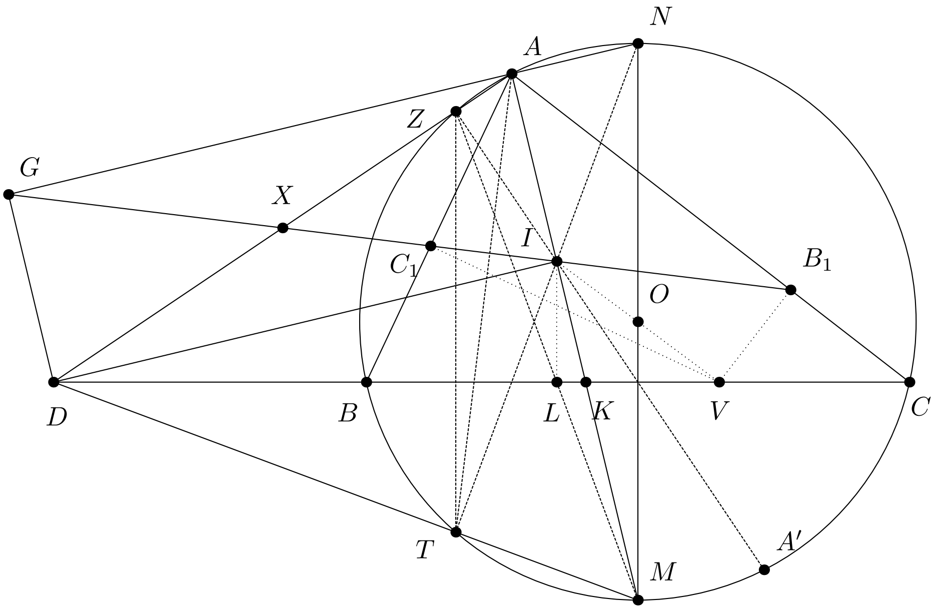
Proof. By Pappus theorem on $\overline{I_BAI_C}$ and $\overline{BA_1C}$ after the previous claim. □

To finish, since $\overline{DMA}$ was tangent to the circumcircle of $\triangle AB_1C_1$, we have $MD^2 = MA^2 = MC_1 \cdot MB_1$, implying the required tangency.

Remark. The triangles satisfying the problem hypothesis are exactly the ones satisfying $r_A = 2R$, where R and r_A denote the circumradius and A-exradius.

Remark. If P is the foot of the A-altitude then this should also imply AB_1PC_1 is harmonic.

¶ Second solution by inversion and mixtilinears (Anant Mudgal) As in the end of the preceding solution, we have $\angle AB_1A_1 = \angle AC_1A_1 = 90^\circ$ and $I \in \overline{B_1C_1}$. Let M be the midpoint of minor arc BC and N be the midpoint of arc $\widehat{BAC}$. Let L be the intouch point on $\overline{BC}$. Let O be the circumcenter of $\triangle ABC$. Let $K = \overline{AI} \cap \overline{BC}$.



Claim — We have $\angle(\overline{AI}, \overline{B_1C_1}) = \angle IAD$.

Proof. Let Z lie on (ABC) with $\angle AZI = 90^\circ$. By radical axis theorem on (AIZ), (BIC), and (ABC), we conclude that D lies on $\overline{AZ}$. Let $\overline{NI}$ meet (ABC) again at $T \neq N$.

Inversion in (BIC) maps $\overline{AI}$ to $\overline{KI}$ and (ABC) to $\overline{BC}$. Thus, Z maps to L, so Z, L, M are collinear. Since $BL = CV$ and $OI = OV$, we see that $MLIN$ is a trapezoid with $\overline{IL} \parallel \overline{MN}$. Thus, $\overline{ZT} \parallel \overline{MN}$.

It is known that $\overline{AT}$ and $\overline{AA_1}$ are isogonal in angle BAC. Since $\overline{AV}$ is a circumdiameter in (AB_1C_1), so $\overline{AT} \perp \overline{B_1C_1}$. So $\measuredangle ZAI = \measuredangle NMT = 90^\circ - \measuredangle TAI = \measuredangle(\overline{AI}, \overline{B_1C_1})$. $\square$

Let X be the midpoint of $\overline{AD}$ and G be the reflection of I in X. Since $AIDG$ is a rectangle, we have $\measuredangle AIG = \measuredangle ZAI = \measuredangle(\overline{AI}, \overline{B_1C_1})$, by the previous claim. So $\overline{IG}$ coincides with $\overline{B_1C_1}$. Now $\overline{AI}$ bisects $\angle B_1AC_1$ and $\angle IAG = 90^\circ$, so $(\overline{IG}; \overline{B_1C_1}) = -1$.

Since $\angle IDG = 90^\circ$, we see that $\overline{DI}$ and $\overline{DG}$ are bisectors of angle B_1DC_1. Now $\angle XDI = \angle XID \implies \angle XDC_1 = \angle XID - \angle IDB_1 = \angle DB_1C_1$, so $\overline{XD}$ is tangent to (DB_1C_1).

USA IMO TST 2019 Statistics

United States of America — IMO Team Selection Tests

EVAN CHEN

60th IMO 2019 United Kingdom

§1 Summary of scores for TST 2019

N	26	1st Q	12	Max	35
μ	15.31	Median	14	Top 3	27
σ	8.59	3rd Q	21	Top 6	21

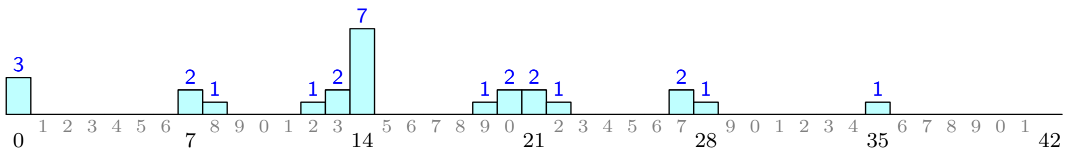
§2 Problem statistics for TST 2019

	P1	P2	P3	P4	P5	P6
0	8	19	25	6	19	16
1	0	1	0	0	1	2
2	0	1	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	1	1	0	0	1	0
6	0	0	0	8	0	0
7	17	4	1	12	5	8
Avg	4.77	1.38	0.27	5.08	1.58	2.23
QM	5.74	2.95	1.37	5.80	3.23	3.89
#5+	**18**	**5**	**1**	**20**	**6**	**8**
%5+	%69.2	%19.2	%3.8	%76.9	%23.1	%30.8

§3 Rankings for TST 2019

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
42	0	0	0.00%	28	1	2	7.69%	14	7	17	65.38%
41	0	0	0.00%	27	2	4	15.38%	13	2	19	73.08%
40	0	0	0.00%	26	0	4	15.38%	12	1	20	76.92%
39	0	0	0.00%	25	0	4	15.38%	11	0	20	76.92%
38	0	0	0.00%	24	0	4	15.38%	10	0	20	76.92%
37	0	0	0.00%	23	0	4	15.38%	9	0	20	76.92%
36	0	0	0.00%	22	1	5	19.23%	8	1	21	80.77%
35	1	1	3.85%	21	2	7	26.92%	7	2	23	88.46%
34	0	1	3.85%	20	2	9	34.62%	6	0	23	88.46%
33	0	1	3.85%	19	1	10	38.46%	5	0	23	88.46%
32	0	1	3.85%	18	0	10	38.46%	4	0	23	88.46%
31	0	1	3.85%	17	0	10	38.46%	3	0	23	88.46%
30	0	1	3.85%	16	0	10	38.46%	2	0	23	88.46%
29	0	1	3.85%	15	0	10	38.46%	1	0	23	88.46%
								0	3	26	100.00%

§4 Histogram for TST 2019



Team Selection Test for the 61st International Mathematical Olympiad

United States of America

Day I

Thursday, December 12, 2019

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, December 16 at noon Eastern time.

IMO TST 1. Choose positive integers b_1, b_2, $\dots$ satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \frac{b_4}{4^2} > \cdots$$

and let r denote the largest real number satisfying $\frac{b_n}{n^2} \geq r$ for all positive integers n. What are the possible values of r across all possible choices of the sequence (b_n)?

IMO TST 2. Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

IMO TST 3. Let $\alpha \geq 1$ be a real number. Hephaestus and Poseidon play a turn-based game on an infinite grid of unit squares. Before the game starts, Poseidon chooses a finite number of cells to be *flooded*. Hephaestus is building a *levee*, which is a subset of unit edges of the grid (called *walls*) forming a connected, non-self-intersecting path or loop*.

The game then begins with Hephaestus moving first. On each of Hephaestus's turns, he adds one or more walls to the levee, as long as the total length of the levee is at most αn after his nth turn. On each of Poseidon's turns, every cell which is adjacent to an already flooded cell and with no wall between them becomes flooded as well.

Hephaestus wins if the levee forms a closed loop such that all flooded cells are contained in the interior of the loop — hence stopping the flood and saving the world. For which α can Hephaestus guarantee victory in a finite number of turns no matter how Poseidon chooses the initial cells to flood?

*More formally, there must exist lattice points A_0, A_1, $\dots$, A_k, pairwise distinct except possibly $A_0 = A_k$, such that the set of walls is exactly $\{A_0A_1, A_1A_2, \dots, A_{k-1}A_k\}$. Once a wall is built it cannot be destroyed; in particular, if the levee is a closed loop (i.e. $A_0 = A_k$) then Hephaestus cannot add more walls. Since each wall has length 1, the length of the levee is k.

Team Selection Test for the 61st International Mathematical Olympiad

United States of America

Day II

Thursday, January 23, 2020

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, January 27 at noon Eastern time.

IMO TST 4. For a finite simple* graph G, we define G' to be the graph on the same vertex set as G, where for any two vertices $u \neq v$, the pair $\{u, v\}$ is an edge of G' if and only if u and v have a common neighbor in G.

Prove that if G is a finite simple graph which is isomorphic to $(G')'$, then G is also isomorphic to G'.

IMO TST 5. Find all integers $n \geq 2$ for which there exists an integer m and a polynomial $P(x)$ with integer coefficients satisfying the following three conditions:

- $m > 1$ and $\gcd(m, n) = 1$;
- the numbers $P(0)$, $P^2(0)$, $\ldots$, $P^{m-1}(0)$ are not divisible by n; and
- $P^m(0)$ is divisible by n.

Here P^k means P applied k times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, etc.

IMO TST 6. Let $P_1P_2 \cdots P_{100}$ be a cyclic 100-gon, and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $\overline{P_{i-2}P_{i+1}}$ and $\overline{P_{i-1}P_{i+2}}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points Q_1, Q_2, $\ldots$, Q_{100} are concyclic.

---

*A *finite simple graph* $G = (V, E)$ is a finite set V of vertices, together with a set E of edges, where each edge in E is a set of two distinct vertices of V. If v is a vertex of G, the *neighbors* of v are the vertices u for which $\{u, v\} \in E$. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection $f\colon V_1 \to V_2$ such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$.

USA IMO TST 2020 Solutions

United States of America — IMO Team Selection Tests

Ankan Bhattacharya and Evan Chen

61$^{\text{th}}$ IMO 2020 Russia

Contents

§0 Problems

1. Choose positive integers b_1, b_2, ... satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \frac{b_4}{4^2} > \cdots$$

and let r denote the largest real number satisfying $\frac{b_n}{n^2} \ge r$ for all positive integers n. What are the possible values of r across all possible choices of the sequence (b_n)?

2. Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

3. Let $\alpha \ge 1$ be a real number. Hephaestus and Poseidon play a turn-based game on an infinite grid of unit squares. Before the game starts, Poseidon chooses a finite number of cells to be *flooded*. Hephaestus is building a *levee*, which is a subset of unit edges of the grid, called *walls*, forming a connected, non-self-intersecting path or loop.

The game then begins with Hephaestus moving first. On each of Hephaestus's turns, he adds one or more walls to the levee, as long as the total length of the levee is at most αn after his nth turn. On each of Poseidon's turns, every cell which is adjacent to an already flooded cell and with no wall between them becomes flooded as well.

Hephaestus wins if the levee forms a closed loop such that all flooded cells are contained in the interior of the loop — hence stopping the flood and saving the world. For which α can Hephaestus guarantee victory in a finite number of turns no matter how Poseidon chooses the initial cells to flood?

4. For a finite simple graph G, we define G' to be the graph on the same vertex set as G, where for any two vertices $u \ne v$, the pair $\{u, v\}$ is an edge of G' if and only if u and v have a common neighbor in G. Prove that if G is a finite simple graph which is isomorphic to $(G')'$, then G is also isomorphic to G'.

5. Find all integers $n \ge 2$ for which there exists an integer m and a polynomial $P(x)$ with integer coefficients satisfying the following three conditions:

- $m > 1$ and $\gcd(m, n) = 1$;
- the numbers $P(0)$, $P^2(0)$, ..., $P^{m-1}(0)$ are not divisible by n; and
- $P^m(0)$ is divisible by n.

Here P^k means P applied k times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, etc.

6. Let $P_1P_2 \cdots P_{100}$ be a cyclic 100-gon, and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $\overline{P_{i-2}P_{i+1}}$ and $\overline{P_{i-1}P_{i+2}}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points Q_1, Q_2, ..., Q_{100} are concyclic.

§1 Solutions to Day 1

§1.1 USA TST 2020/1, proposed by Carl Schildkraut, Milan Haiman

Available online at https://aops.com/community/p13654466.

Problem statement

Choose positive integers $b_1, b_2, \dots$ satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \frac{b_4}{4^2} > \cdots$$

and let r denote the largest real number satisfying $\frac{b_n}{n^2} \ge r$ for all positive integers n. What are the possible values of r across all possible choices of the sequence (b_n)?

The answer is $0 \le r \le 1/2$. Obviously $r \ge 0$.

In one direction, we show that

Claim (Greedy bound) — For all integers n, we have

$$\frac{b_n}{n^2} \le \frac{1}{2} + \frac{1}{2n}.$$

Proof. This is by induction on n. For $n = 1$ it is given. For the inductive step we have

$$\begin{aligned} b_n < n^2 \frac{b_{n-1}}{(n-1)^2} &\le n^2 \left(\frac{1}{2} + \frac{1}{2(n-1)} \right) = \frac{n^3}{2(n-1)} \\ &= \frac{1}{2} \left[n^2 + n + 1 + \frac{1}{n-1} \right] \\ &= \frac{n(n+1)}{2} + \frac{1}{2} \left[1 + \frac{1}{n-1} \right] \\ &\le \frac{n(n+1)}{2} + 1 \end{aligned}$$

So $b_n < \frac{n(n+1)}{2} + 1$ and since b_n is an integer, $b_n \le \frac{n(n+1)}{2}$. This implies the result. □

We now give a construction. For $r = 1/2$ we take $b_n = \frac{1}{2}n(n+1)$ for $r = 0$ we take $b_n = 1$.

Claim (Explicit construction, given by Nikolai Beluhov) — Fix $0 < r < 1/2$. Let N be large enough that $\lceil rn^2 + n \rceil < \frac{1}{2}n(n+1)$ for all $n \ge N$. Then the following sequence works:

$$b_n = \begin{cases} \lceil rn^2 + n \rceil & n \ge N \\ \frac{n^2+n}{2} & n < N. \end{cases}$$

Proof. We certainly have

$$\frac{b_n}{n^2} = \frac{rn^2 + n + O(1)}{n^2} \xrightarrow{n\to\infty} r.$$

Mainly, we contend $b_n n^{-2}$ is strictly decreasing. We need only check this for $n \ge N$; in fact

$$\frac{b_n}{n^2} \ge \frac{rn^2 + n}{n^2} > \frac{[r(n+1)^2 + (n+1)] + 1}{(n+1)^2} > \frac{b_{n+1}}{(n+1)^2}$$

where the middle inequality is true since it rearranges to $\frac{1}{n} > \frac{n+2}{(n+1)^2}$. □

§1.2 USA TST 2020/2, proposed by Merlijn Staps

Available online at https://aops.com/community/p13654481.

Problem statement

Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

We present four solutions.

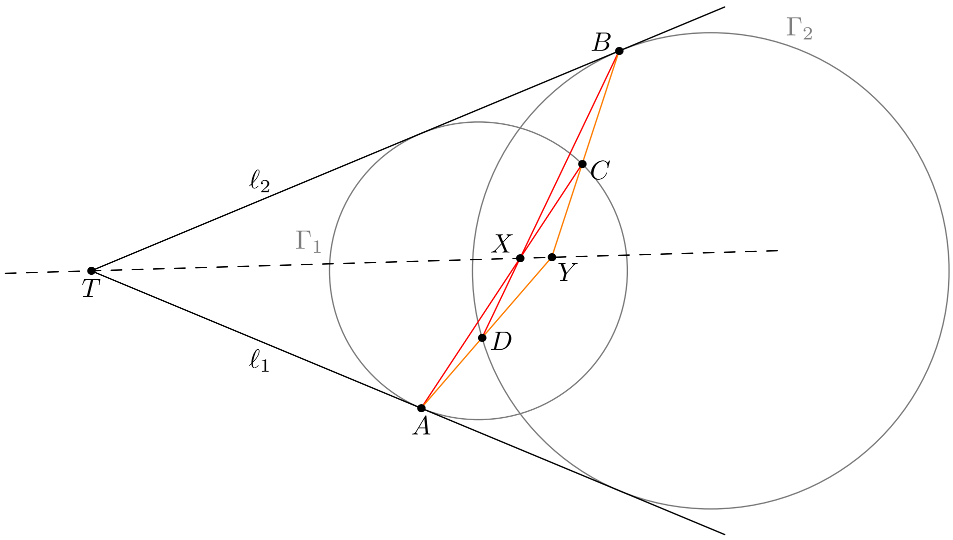
¶ First solution, elementary (original) We have $\triangle YAC \sim \triangle YBD$, from which it follows

$$\frac{d(Y, AC)}{d(Y, BD)} = \frac{AC}{BD}.$$

Moreover, if we denote by r_1 and r_2 the radii of Γ_1 and Γ_2, then

$$\frac{d(T, AC)}{d(T, BD)} = \frac{TA \sin \angle(AC, \ell_1)}{TB \sin \angle(BD, \ell_2)} = \frac{2r_1 \sin \angle(AC, \ell_1)}{2r_2 \sin \angle(BD, \ell_2)} = \frac{AC}{BD}$$

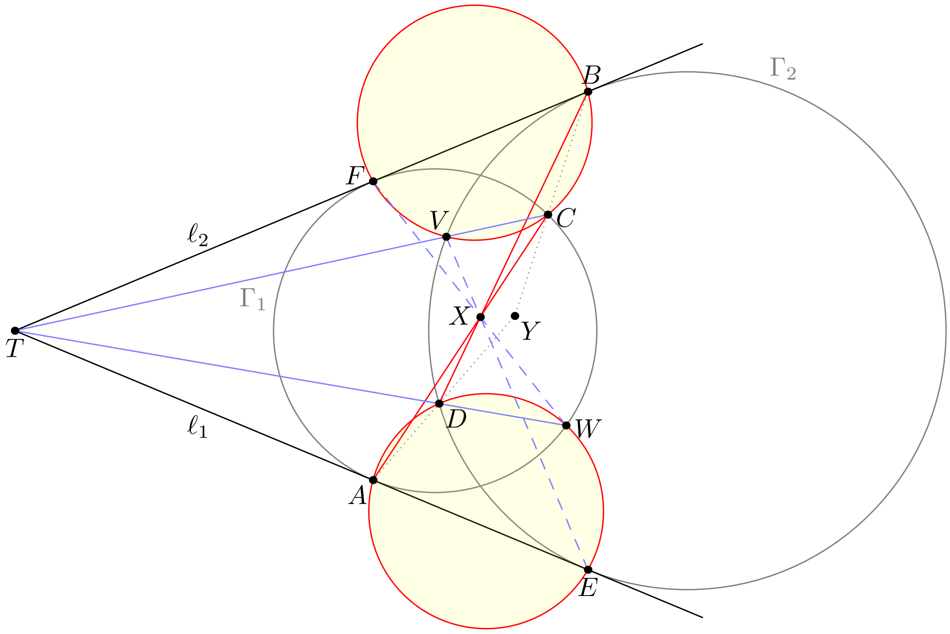
the last step by the law of sines.



This solves the problem up to configuration issues: we claim that Y and T both lie inside $\angle AXB \equiv \angle CXD$. WLOG $TA < TB$.

- The former is since Y lies outside segments BC and AD, since we assumed $ABCD$ was convex.
- For the latter, we note that X lies inside both Γ_1 and Γ_2 in fact on the radical axis of the two circles (since X was an interior point of both chords AC and BD). In particular, X is contained inside $\angle ATB$, and moreover $\angle ATB < 90^\circ$, and this is enough to imply the result.

¶ **Second solution, inversive** This is based on the solution posted by `kapilpavase` on AoPS. Consider the inversion at T swapping Γ_1 and Γ_2; we let it send A to E, B to F, C to V, D to W, as shown. Draw circles $ADWE$ and $BCVF$.



Claim — Points T and Y lie on the radical axis of (ADE) and (BCF).

Proof. Because $TF \cdot TB = TA \cdot TE$ and $YA \cdot YD = YC \cdot YB$. □

Claim — Point X has equal power to (ADE) and (BCF).

Proof. Since $TV \cdot TC = TA \cdot TE$, quadrilateral $VCEA$ is cyclic too, so by radical axis with Γ_1 and Γ_2 we find X lies on VE. Similarly, X lies on FW. Thus, X is the center of negative inversion between (ADE) and (BCF).

But since $AE = BF$ and moreover

$$\begin{aligned} \measuredangle BCF + \measuredangle ADE &= (\measuredangle BCA + \measuredangle ACF) + (\measuredangle ADB + \measuredangle BDE) \\ &= (\measuredangle BCA + \measuredangle ADB) + (\measuredangle ACF + \measuredangle BDE) = 0 + 0 = 0 \end{aligned}$$

we conclude that (ADE) and (BCF) are *congruent*. As X was the center of negative inversion between them, we're done. □

¶ **Third solution, projective (Nikolai Beluhov)** We start with some definitions. Let ℓ_1 touch Γ_2 at E, ℓ_2 touch Γ_1 at F, $K = \ell_1 \cap \overline{BD}$, $L = \ell_2 \cap \overline{AC}$, line FX meet Γ_1 again at M, line EX meet Γ_2 again at N, and lines AB, AD, and BC meet line TX at Z, Y_1, and Y_2. Thus the desired statement is equivalent to $Y_1 = Y_2$.

Claim — $(EB; ND)_{\Gamma_2} = (FA; MC)_{\Gamma_1}$.

Proof. Note that $AX \cdot XC = BX \cdot XD = EX \cdot XN$, so $AECN$ is cyclic. Likewise $BFDM$ is cyclic.

Consider the inversion with center T which swaps Γ_1 and Γ_2; it also swaps the pairs $\{A, E\}$ and $\{B, F\}$. Since $AECN$ is cyclic, C is on Γ_1, and N is on Γ_2, it also swaps $\{C, N\}$; similarly it swaps $\{D, M\}$.

Thus $(EB; ND)_{\Gamma_2} = (AF; CM)_{\Gamma_1} = (FA; MC)_{\Gamma_1}$ as desired. $\square$

With this claim, the remainder of the proof is chasing cross-ratios:

$$(TZ; XY_1) \stackrel{A}{=} (KB; XD) \stackrel{E}{=} (EB; ND)_{\Gamma_2} = (FA; MC)_{\Gamma_1} \stackrel{F}{=} (LA; XC) \stackrel{B}{=} (TZ; XY_2)$$

implies $Y_1 = Y_2$ as desired.

¶ Fourth solution by untethered moving points Fix ℓ_1, ℓ_2, T, Γ_1 and Γ_2, and let Γ_1 and Γ_2 meet at U and V. By the radical axis theorem, X lies on UV.

Thus we instead treat X as a variable point on line UV and let $C = AX \cap \Gamma_1$, $D = BX \cap \Gamma_2$. By definition, X has degree 1 and T has degree 0.

We apply **Zack's lemma** to untethered point Y. Note that C and D move projectively on conics, and therefore have degree 2. Then, lines AD and BC each have degree at most $\deg(A) + \deg(D) = 0 + 2 = 2$, and so their intersection Y has degree at most $2 + 2 = 4$. But when $X \in AB$, the lines AD and BC are the same, so Zack's lemma implies that

$$\deg Y \le 4 - 1 = 3.$$

Thus the assertion that T, X, Y are collinear (which for example can be seen as a certain vanishing determinant) is a statement of degree at most $0 + 1 + 3 = 4$. Thus it suffices to find 5 values of X (other than $X \in \overline{AB}$, which we used already). This is remarkably easy:

1. When $X = U$ or $X = V$, then $X = C = D = Y$ and the statement is obvious
2. When $X \in \ell_1$, say, then $A = C$ and so Y lies on $AC = \ell_1$ as well. The case $X \in \ell_2$ is symmetric.
3. Finally, take X at infinity along UV. Then C and D are the other tangency points of the circles with ℓ_1, ℓ_2, and so $AC = \ell_1, BD = \ell_2$, so $Y = T$.

This finishes the problem.

§1.3 USA TST 2020/3, proposed by Nikolai Beluhov

Available online at https://aops.com/community/p13654498.

Problem statement

Let $\alpha \ge 1$ be a real number. Hephaestus and Poseidon play a turn-based game on an infinite grid of unit squares. Before the game starts, Poseidon chooses a finite number of cells to be *flooded*. Hephaestus is building a *levee*, which is a subset of unit edges of the grid, called *walls*, forming a connected, non-self-intersecting path or loop.

The game then begins with Hephaestus moving first. On each of Hephaestus's turns, he adds one or more walls to the levee, as long as the total length of the levee is at most αn after his nth turn. On each of Poseidon's turns, every cell which is adjacent to an already flooded cell and with no wall between them becomes flooded as well.

Hephaestus wins if the levee forms a closed loop such that all flooded cells are contained in the interior of the loop — hence stopping the flood and saving the world. For which α can Hephaestus guarantee victory in a finite number of turns no matter how Poseidon chooses the initial cells to flood?

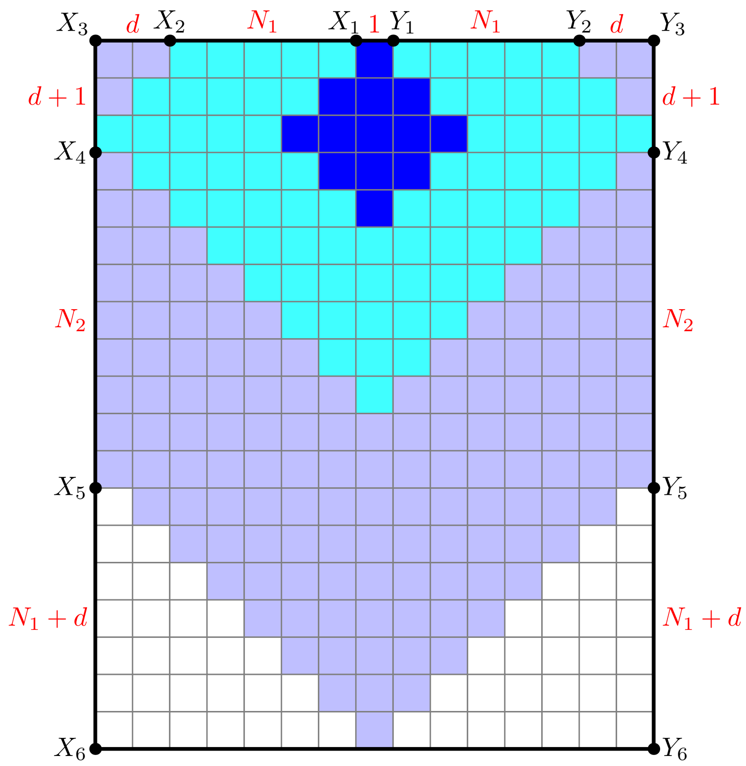
We show that if $\alpha > 2$ then Hephaestus wins, but when $\alpha = 2$ (and hence $\alpha \le 2$) Hephaestus cannot contain even a single-cell flood initially.

Strategy for $\alpha > 2$: Impose $\mathbb{Z}^2$ coordinates on the cells. Adding more flooded cells does not make our task easier, so let us assume that initially the cells (x, y) with $|x| + |y| \le d$ are flooded for some $d \ge 2$; thus on Hephaestus's kth turn, the water is contained in $|x| + |y| \le d + k - 1$. Our goal is to contain the flood with a large rectangle.

We pick large integers N_1 and N_2 such that

$$\alpha N_1 > 2N_1 + (2d + 3)$$
$$\alpha(N_1 + N_2) > 2N_2 + (6N_1 + 8d + 4).$$

Mark the points X_i, Y_i as shown in the figure for $1 \le i \le 6$. The red figures indicate the distance between the marked points on the rectangle.



We follow the following plan.

- Turn 1: place wall X_1Y_1. This cuts off the flood to the north.
- Turns 2 through N_1+1: extend the levee to segment X_2Y_2. This prevents further flooding to the north.
- Turn N_1+2: add in broken lines $X_4X_3X_2$ and $Y_4Y_3Y_2$ all at once. This cuts off the flood west and east.
- Turns N_1+2 to N_1+N_2+1: extend the levee along segments X_4X_5 and Y_4Y_5. This prevents further flooding west and east.
- Turn N_1+N_2+2: add in the broken line $X_5X_6Y_6Y_5$ all at once and win.

Proof for $\alpha = 2$: Suppose Hephaestus contains the flood on his $(n+1)$st turn. We prove that $\alpha > 2$ by showing that in fact at least $2n+4$ walls have been constructed.

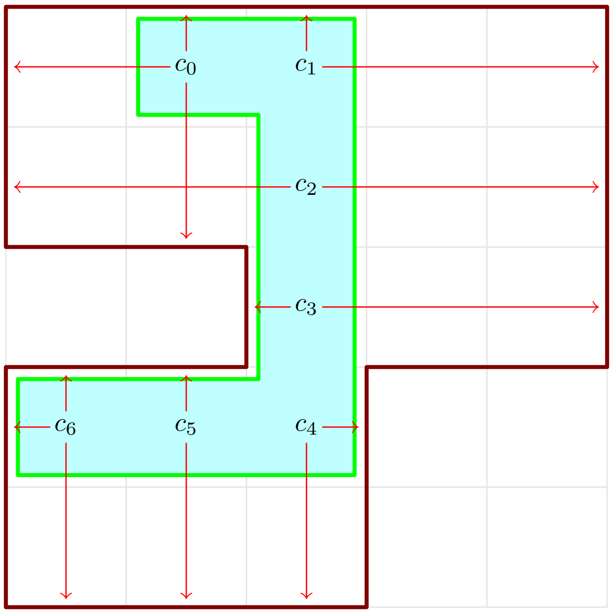
Let c_0, c_1, $\dots$, c_n be a path of cells such that c_0 is the initial cell flooded, and in general c_i is flooded on Poseidon's ith turn from c_{i-1}. The levee now forms a closed loop enclosing all c_i.

Claim — If c_i and c_j are adjacent then $|i-j| = 1$.

Proof. Assume c_i and c_j are adjacent but $|i-j| > 1$. Then the two cells must be separated by a wall. But the levee forms a closed loop, and now c_i and c_j are on opposite sides. □

Thus the c_i actually form a path. We color *green* any edge of the unit grid (wall or not) which is an edge of exactly one c_i (i.e. the boundary of the polyomino). It is easy to see there are exactly $2n+4$ green edges.

Now, from the center of each cell c_i, shine a laser towards each green edge of c_i (hence a total of $2n+4$ lasers are emitted). An example below is shown for $n=6$, with the levee marked in brown.



Claim — No wall is hit by more than one laser.

Proof. Assume for contradiction that a wall w is hit by lasers from c_i and c_j. WLOG that laser is vertical, so c_i and c_j are in the same column (e.g. $(i,j)=(0,5)$ in figure).

We consider two cases on the position of w.

- If w is between c_i and c_j, then we have found a segment intersecting the levee exactly once. But the endpoints of the segment lie inside the levee. This contradicts the assumption that the levee is a closed loop.
- Suppose w lies above both c_i and c_j and assume WLOG $i<j$. Then we have found that there is no levee at all between c_i and c_j.

  Let $\rho \geq 1$ be the distance between the centers of c_i and c_j. Then c_j is flooded in a straight line from c_i within ρ turns, and this is the unique shortest possible path. So this situation can only occur if $j=i+\rho$ and $c_i, \dots, c_j$ form a column. But then no vertical lasers from c_i and c_j may point in the same direction, contradiction.

Since neither case is possible, the proof ends here. □

This implies the levee has at least $2n+4$ walls (the number of lasers) on Hephaestus's $(n+1)$st turn. So $\alpha \geq \frac{2n+4}{n+1} > 2$.

Remark (Author comments)**.** The author provides the following remarks.

- Even though the flood can be stopped when $\alpha = 2+\varepsilon$, it takes a very long time to do that. Starting from a single flooded cell, the strategy I have outlined requires $\Theta(1/\varepsilon^2)$ days. Starting from several flooded cells contained within an area of diameter D, it takes $\Theta(D/\varepsilon^2)$ days. I do not know any strategies that require fewer days than that.
- There is a gaping chasm between $\alpha \leq 2$ and $\alpha > 2$. Since $\alpha \leq 2$ does not suffice even when only one cell is flooded in the beginning, there are in fact no initial

configurations at all for which it is sufficient. On the other hand, $\alpha > 2$ works for all initial configurations.

- The second half of the solution essentially estimates the perimeter of a polyomino in terms of its diameter (where diameter is measured entirely within the polyomino).

  It appears that this has not been done before, or at least I was unable to find any reference for it. I did find tons of references where the perimeter of a polyomino is estimated in terms of its area, but nothing concerning the diameter.

  My argument is a formalisation of the intuition that if P is any shortest path within some weirdly-shaped polyomino, then the boundary of that polyomino must hug P rather closely so that P cannot be shortened.

§2 Solutions to Day 2

§2.1 USA TST 2020/4, proposed by Zack Chroman, Mehtaab Sawhney

Available online at https://aops.com/community/p13913804.

Problem statement

For a finite simple graph G, we define G' to be the graph on the same vertex set as G, where for any two vertices $u \neq v$, the pair $\{u, v\}$ is an edge of G' if and only if u and v have a common neighbor in G. Prove that if G is a finite simple graph which is isomorphic to $(G')'$, then G is also isomorphic to G'.

We say a vertex of a graph is *fatal* if it has degree at least 3, and some two of its neighbors are not adjacent.

Claim — The graph G' has at least as many triangles as G, and has strictly more if G has any fatal vertices.

Proof. Obviously any triangle in G persists in G'. Moreover, suppose v is a fatal vertex of G. Then the neighbors of G will form a clique in G' which was not there already, so there are more triangles. □

Thus we only need to consider graphs G with no fatal vertices. Looking at the connected components, the only possibilities are cliques (including single vertices), cycles, and paths. So in what follows we restrict our attention to graphs G only consisting of such components.

Remark (Warning)**.** Beware: assuming G is connected loses generality. For example, it could be that $G = G_1 \sqcup G_2$, where $G_1' \cong G_2$ and $G_2' \cong G_1$.

First, note that the following are stable under the operation:

- an isolated vertex,
- a cycle of odd length, or
- a clique with at least three vertices.

In particular, $G \cong G''$ holds for such graphs.

On the other hand, cycles of even length or paths of nonzero length will break into more connected components. For this reason, a graph G with any of these components will not satisfy $G \cong G''$ because G' will have strictly more connected components than G, and G'' will have at least as many as G'.

Therefore $G \cong G''$ if and only if G is a disjoint union of the three types of connected components named earlier. Since $G \cong G'$ holds for such graphs as well, the problem statement follows right away.

Remark. Note that the same proof works equally well for an arbitrary number of iterations $G^{''\cdots''} \cong G$, rather than just $G'' \cong G$.

Remark. The proposers included a variant of the problem where given any graph G, the operation stabilized after at most $O(\log n)$ operations, where n was the number of vertices of G.

§2.2 USA TST 2020/5, proposed by Carl Schildkraut

Available online at https://aops.com/community/p13913769.

Problem statement

Find all integers $n \geq 2$ for which there exists an integer m and a polynomial $P(x)$ with integer coefficients satisfying the following three conditions:

- $m > 1$ and $\gcd(m, n) = 1$;
- the numbers $P(0)$, $P^2(0)$, $\dots$, $P^{m-1}(0)$ are not divisible by n; and
- $P^m(0)$ is divisible by n.

Here P^k means P applied k times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, etc.

The answer is that this is possible if and only if there exists primes $p' < p$ such that $p \mid n$ and $p' \nmid n$. (Equivalently, the radical $\operatorname{rad}(n)$ must not be the product of the first several primes.)

For a polynomial P, and an integer N, we introduce the notation

$$\mathbf{zord}(P \bmod N) := \min \{e > 0 \mid P^e(0) \equiv 0 \bmod N\}$$

where we set $\min \varnothing = 0$ by convention. Note that in general we have

$$\mathbf{zord}(P \bmod N) = \operatorname*{lcm}_{q \mid N} (\mathbf{zord}(P \bmod q)) \qquad (\dagger)$$

where the index runs over all prime powers $q \mid N$ (by Chinese remainder theorem). This will be used in both directions.

Construction: First, we begin by giving a construction. The idea is to first use the following prime power case.

Claim — Let p^e be a prime power, and $1 \leq k < p$. Then

$$f(X) = X + 1 - k \cdot \frac{X(X-1)(X-2)\dots(X-(k-2))}{(k-1)!}$$

viewed as a polynomial in $(\mathbb{Z}/p^e)[X]$ satisfies $\mathbf{zord}(f \bmod p^e) = k$.

Proof. Note $f(0) = 1$, $f(1) = 2$, $\dots$, $f(k-2) = k-1$, $f(k-1) = 0$ as needed. □

This gives us a way to do the construction now. For the prime power $p^e \mid n$, we choose $1 \leq p' < p$ and require $\mathbf{zord}(P \bmod p^e) = p'$ and $\mathbf{zord}(P \bmod q) = 1$ for every other prime power q dividing n. Then by $(\dagger)$ we are done.

> **Remark.** The claim can be viewed as a special case of Lagrange interpolation adapted to work over $\mathbb{Z}/p^e$ rather than $\mathbb{Z}/p$. Thus the construction of the polynomial f above is quite natural.

Necessity: by $(\dagger)$ again, it will be sufficient to prove the following claim.

Claim — For any prime power $q = p^e$, and any polynomial $f(x) \in \mathbb{Z}[x]$, if the quantity $\mathbf{zord}(f \bmod q)$ is nonzero then it has all prime factors at most p.

Proof. This is by induction on $e \geq 1$. For $e = 1$, the pigeonhole principle immediately implies that $\mathbf{zord}(P \bmod p) \leq p$.

Now assume $e \geq 2$. Let us define

$$k := \mathbf{zord}(P \bmod p^{e-1}), \qquad Q := P^k.$$

Since being periodic modulo p^e requires periodic modulo p^{e-1}, it follows that

$$\mathbf{zord}(P \bmod p^e) = k \cdot \mathbf{zord}(Q \bmod p^e).$$

However since $Q(0) \equiv 0 \bmod p^{e-1}$, it follows $\{Q(0), Q^2(0), \dots\}$ are actually all multiples of p^{e-1}. There are only p residues modulo p^e which are also multiples of p^{e-1}, so $\mathbf{zord}(Q \bmod p^e) \leq p$, as needed. □

Remark. One reviewer submitted the following test-solving comments:

This is one of these problems where you can make many useful natural observations, and if you make enough of them eventually they cohere into a solution. For example, here are some things I noticed while solving:

- The polynomial $1 - x$ shows that $m = 2$ works for any odd n.
- In general, if ζ is a primitive mth root of unity modulo n, then $\zeta(x + 1) - 1$ has the desired property (assuming $\gcd(m, n) = 1$). We can extend this using the Chinese remainder theorem to find that if $p \mid n$, $m \mid p - 1$, and $\gcd(m, n) = 1$, then n works. So by this point I already have something about the prime factors of n being sort-of closed downwards.
- By iterating P we see it is enough to consider m prime.
- In the case where $n = 2^k$, it is not too difficult to show that no odd prime m works.

§2.3 USA TST 2020/6, proposed by Michael Ren

Available online at https://aops.com/community/p13913742.

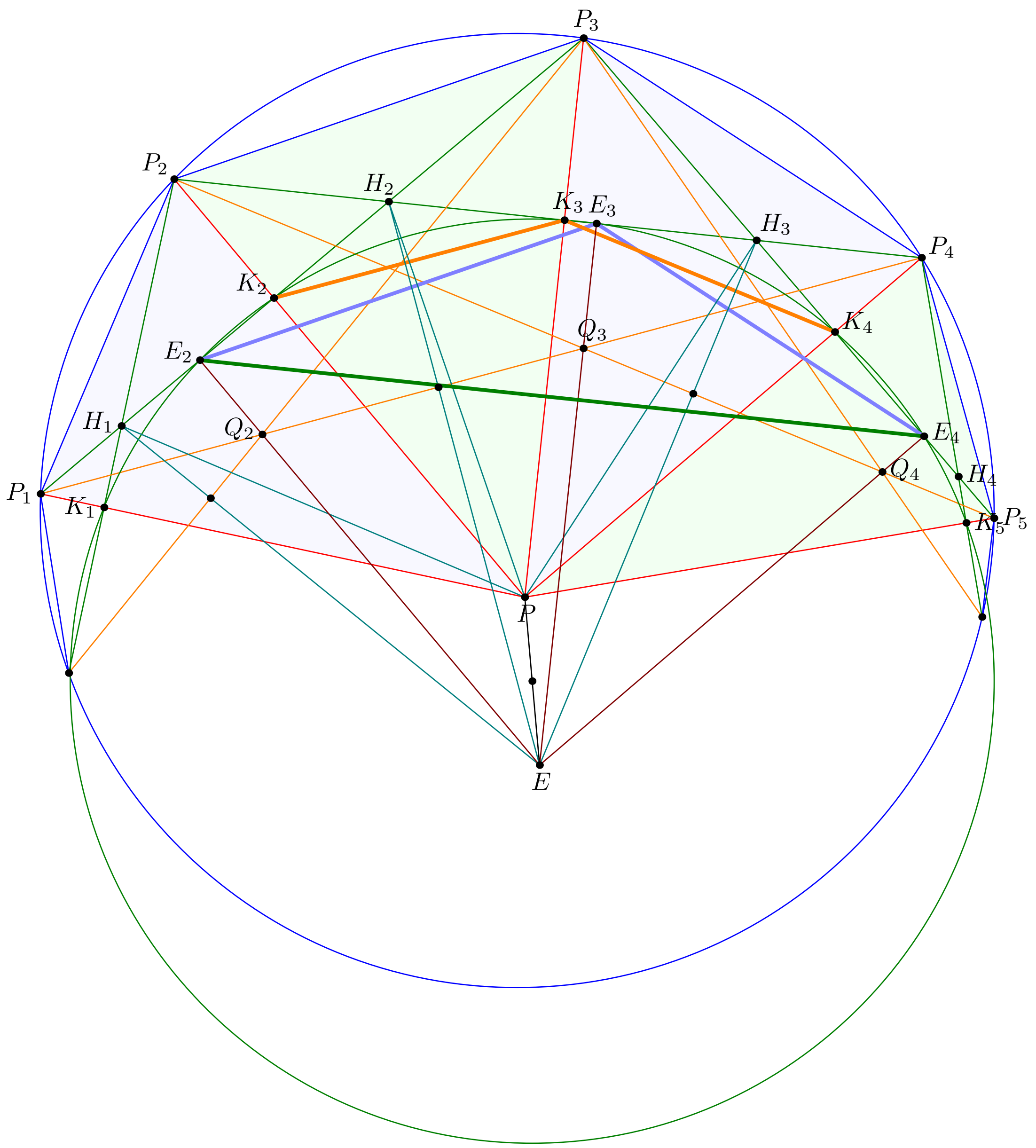
Problem statement

Let $P_1P_2 \cdots P_{100}$ be a cyclic 100-gon, and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $\overline{P_{i-2}P_{i+1}}$ and $\overline{P_{i-1}P_{i+2}}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points Q_1, Q_2, $\dots$, Q_{100} are concyclic.

We show two solutions. In addition, Luke Robitaille has a reasonable complex numbers solution posted at https://aops.com/community/p26795631.

¶ Solution to proposed problem We let $\overline{PP_2}$ and $\overline{P_1P_3}$ intersect (perpendicularly) at point K_2, and define $K_\bullet$ cyclically.



Claim — The points $K_\bullet$ are concyclic say with circumcircle γ.

Proof. Note that $PP_1 \times PK_1 = PP_2 \times PK_2 = \dots$ so the result follows by inversion at P. $\square$

Let E_i be the second intersection of line $\overline{P_{i-1}K_iP_{i+1}}$ with γ; then it follows that the perpendiculars to $\overline{P_{i-1}P_{i+1}}$ at E_i all concur at a point E, which is the reflection of P across the center of γ.

We let $H_2 = \overline{P_1P_3} \cap \overline{P_2P_4}$ denote the orthocenter of $\triangle PP_2P_3$ and define $H_\bullet$ cyclically.

Claim — We have

$$\overline{EH_2} \perp \overline{P_1P_4} \parallel \overline{K_2K_3} \quad \text{and} \quad \overline{PH_2} \perp \overline{E_2E_3} \parallel \overline{P_2P_3}.$$

Proof. Both parallelisms follow by Reim's theorem through $\angle E_2H_2E_3 = \angle K_2H_2K_3$, So we need to show the perpendicularities.

Note that $\overline{H_2P}$ and $\overline{H_2E}$ are respectively circum-diameters of $\triangle H_2K_2K_3$ and $\triangle H_2E_2E_3$. As $\overline{K_2K_3}$ and $\overline{E_2E_3}$ are anti-parallel, it follows $\overline{H_2P}$ and $\overline{H_2E}$ are isogonal and we derive both perpendicularities. □

Claim — The points E, Q_3, E_3 are collinear.

Proof. We use the previous claim. The parallelisms imply that

$$\frac{E_3H_2}{E_3P_2} = \frac{E_2H_2}{E_2P_3} = \frac{E_4H_3}{E_4P_3} = \frac{E_3H_3}{E_3P_4}.$$

Now consider a homothety centered at E_3 sending H_2 to P_2 and H_3 to P_4. Then it should send the orthocenter of $\triangle EH_2H_3$ to Q_3, proving the result. □

From all this it follows that $\triangle EQ_2Q_3 \sim \triangle PK_2K_3$ as the opposite sides are all parallel. Repeating this we actually find a homothety of 100-gons

$$Q_1Q_2Q_3 \cdots \sim K_1K_2K_3 \cdots$$

and that concludes the proof.

Remark. The proposer remarks that in fact, if one lets s be an integer and instead defines $R_i = P_iP_{i+s} \cap P_{i+1}P_{i+s+1}$, then the $R_\bullet$ are concyclic. The present problem is the case $s = 3$. We comment on a few special cases:

- There is nothing to prove for $s = 1$.
- If $s = 0$, this amounts to proving that poles of $\overline{P_iP_{i+1}}$ are concyclic; by inversion this is equivalent to showing the midpoints of the sides are concyclic. This is an interesting problem but not as difficult.
- The problem for $s = 2$ is to show that our $H_\bullet$ are concyclic, which uses the $s = 0$ case as a lemma.

¶ Solution to generalization (Nikolai Beluhov) We are going to need some well-known lemmas.

Lemma

Suppose that $ABCD$ is inscribed in a circle Γ. Let the tangents to Γ at A and B meet at E, let the tangents to Γ at C and D meet at F, and let diagonals AC and BD meet at P. Then points E, F, and P are collinear.

Proof. Let the circle of center E and radius $EA = EB$ meet lines AC and BD for the second time at points U and V. By a simple angle chase, triangles EUV and FCD are homothetic. □

Lemma

Suppose that points X and Y are isogonal conjugates in polygon $\mathcal{A} = A_1A_2\dots A_n$. (This means that lines A_iX and A_iY are symmetric with respect to the interior angle bisector of $\angle A_{i-1}A_iA_{i+1}$ for all i, where $A_{n+j} \equiv A_j$ for all j.) Then the $2n$ projections of X and Y on the sides of $\mathcal{A}$ are concyclic.

Proof. By a simple angle chase, for all i we have that the four projections on sides $A_{i-1}A_i$ and A_iA_{i+1} are concyclic. Say that they lie on circle Γ_i. Consider the midpoint M of segment XY. For every side s of $\mathcal{A}$, we have that M is equidistant from the projections of X and Y on s. Therefore, M is the center of Γ_i for all i, and thus all of the Γ_i coincide. □

Lemma

Let Γ' and Γ'' be two circles and let r be some fixed real number. Then the locus of points X satisfying $\operatorname{Pow}(X, \Gamma') : \operatorname{Pow}(X, \Gamma'') = r$ is a circle.

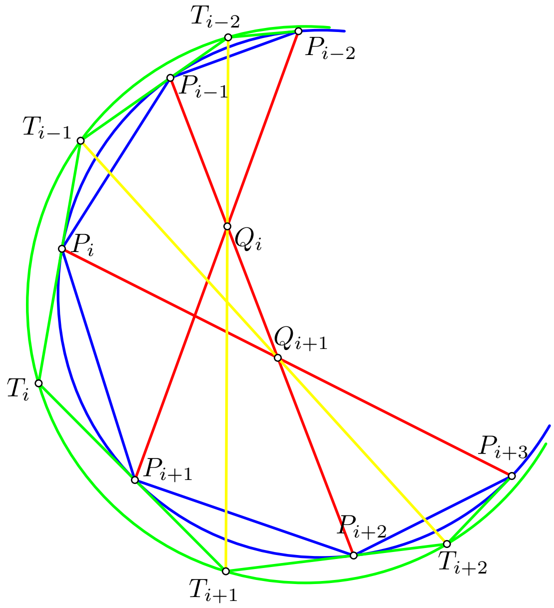
Proof. This is a classical result in circle geometry. A full proof is given, for example, in item 115 of Roger Johnson's *Advanced Euclidean Geometry*. □

We are ready to solve the problem. Let $\mathcal{P}$ be our polygon, let O be its the circumcenter, and let Γ be its circumcircle.

Fix any index i. In triangle $P_{i-1}P_iP_{i+1}$, we have that line P_iP contains the altitude through P_i and line P_iO contains the circumradius through P_i. Therefore, these two lines are symmetric with respect to the interior angle bisector of $\angle P_{i-1}P_iP_{i+1}$.

Thus points P and O are isogonal conjugates in $\mathcal{P}$. By Lemma 2, it follows that the projections of O onto the sides of $\mathcal{P}$ are concyclic. In other words, the midpoints of the sides of $\mathcal{P}$ lie on some circle ω.

Let M_i be the midpoint of segment P_iP_{i+1} and let the tangents to Γ at points P_i and P_{i+1} meet at T_i. Since inversion with respect to Γ swaps M_i and T_i for all i, and also since all of the M_i lie on the same circle ω, we obtain that all of the T_i lie on the same circle Ω.



Again, fix any index i. By Lemma 1 applied to cyclic quadrilateral $P_{i-2}P_{i-1}P_{i+1}P_{i+2}$, we have that point Q_i lies on line $T_{i-2}T_{i+1}$. Similarly, point Q_{i+1} lies on line $T_{i-1}T_{i+2}$.

Define

$$f_i = \frac{\operatorname{Pow}(Q_i, \Gamma)}{\operatorname{Pow}(Q_i, \Omega)}.$$

Claim — We have $f_i = f_{i+1}$ for all i.

Proof. Note that

$$\begin{aligned}
\operatorname{Pow}(Q_i, \Gamma) &= Q_iP_{i-1} \cdot Q_iP_{i+2} \\
\operatorname{Pow}(Q_{i+1}, \Gamma) &= Q_{i+1}P_{i-1} \cdot Q_{i+1}P_{i+2}. \\
\operatorname{Pow}(Q_i, \Omega) &= Q_iT_{i-2} \cdot Q_iT_{i+1} \\
\operatorname{Pow}(Q_{i+1}, \Omega) &= Q_{i+1}T_{i-1} \cdot Q_{i+1}T_{i+2}.
\end{aligned}$$

Consider cyclic quadrilateral $T_{i-2}T_{i-1}T_{i+1}T_{i+2}$. Since Γ touches its opposite sides $T_{i-2}T_{i-1}$ and $T_{i+1}T_{i+2}$ at points P_{i-1} and P_{i+2}, we have that line $P_{i-1}P_{i+2}$ makes equal angles with these opposite sides. From here, a simple angle chase shows that triangles $P_{i-1}Q_iT_{i-2}$ and $P_{i+2}Q_{i+1}T_{i+2}$ are similar. Thus

$$\frac{Q_iP_{i-1}}{Q_iT_{i-2}} = \frac{Q_{i+1}P_{i+2}}{Q_{i+1}T_{i+2}}.$$

Similarly,

$$\frac{Q_iP_{i+2}}{Q_iT_{i+1}} = \frac{Q_{i+1}P_{i-1}}{Q_{i+1}T_{i-1}}.$$

From these, the desired identity $f_i = f_{i+1}$ follows. □

Therefore, the power ratio f_i is the same for all i. By Lemma 3 for circles Γ and Ω, the solution is complete.

Remark. This solution applies to the full generalization (from 3 to s) mentioned in the end of the previous solution, essentially with no change.

Remark. Polygon $T_1T_2 \dots T_{100}$ is both circumscribed about a circle and inscribed inside a circle. Polygons like that are known as *Poncelet polygons*. See, for example, https://en.wikipedia.org/wiki/Poncelet's_closure_theorem. This solution borrows a lot from the discussion of Poncelet's closure theorem in *Advanced Euclidean Geometry*, referenced above for Lemma 3.

USA IMO TST 2020 Statistics

United States of America — IMO Team Selection Tests

Ankan Bhattacharya and Evan Chen

61th IMO 2020 Russia

§1 Summary of scores for TST 2020

N	31	1st Q	17	Max	34
μ	18.87	Median	19	Top 3	29
σ	6.82	3rd Q	22	Top 6	23

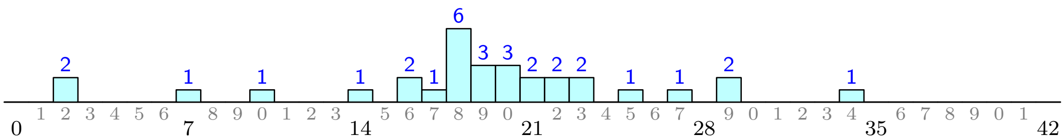
§2 Problem statistics for TST 2020

	P1	P2	P3	P4	P5	P6
0	1	20	19	3	8	31
1	4	1	1	1	0	0
2	0	0	10	0	1	0
3	0	0	0	1	1	0
4	1	1	0	3	2	0
5	1	0	0	1	0	0
6	2	0	0	10	1	0
7	22	9	1	12	18	0
Avg	5.77	2.19	0.90	5.32	4.68	0.00
QM	6.21	3.84	1.70	5.77	5.57	0.00
#5+	**25**	**9**	**1**	**23**	**19**	**0**
%5+	%80.6	%29.0	%3.2	%74.2	%61.3	%0.0

§3 Rankings for TST 2020

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
42	0	0	0.00%	28	0	3	9.68%	14	1	27	87.10%
41	0	0	0.00%	27	1	4	12.90%	13	0	27	87.10%
40	0	0	0.00%	26	0	4	12.90%	12	0	27	87.10%
39	0	0	0.00%	25	1	5	16.13%	11	0	27	87.10%
38	0	0	0.00%	24	0	5	16.13%	10	1	28	90.32%
37	0	0	0.00%	23	2	7	22.58%	9	0	28	90.32%
36	0	0	0.00%	22	2	9	29.03%	8	0	28	90.32%
35	0	0	0.00%	21	2	11	35.48%	7	1	29	93.55%
34	1	1	3.23%	20	3	14	45.16%	6	0	29	93.55%
33	0	1	3.23%	19	3	17	54.84%	5	0	29	93.55%
32	0	1	3.23%	18	6	23	74.19%	4	0	29	93.55%
31	0	1	3.23%	17	1	24	77.42%	3	0	29	93.55%
30	0	1	3.23%	16	2	26	83.87%	2	2	31	100.00%
29	2	3	9.68%	15	0	26	83.87%	1	0	31	100.00%
								0	0	31	100.00%

§4 Histogram for TST 2020



Team Selection Test for the 62$^{\text{nd}}$ International Mathematical Olympiad

United States of America

February 25, 2021

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, March 1 at noon Eastern time.

IMO TST 1. Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

IMO TST 2. Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D. Let O and ρ denote the circumcenter and circumradius of $\triangle XCD$, respectively.

Prove there exists a fixed point K and a real number c, independent of X, for which $OK^2 - \rho^2 = c$ always holds regardless of the choice of X.

IMO TST 3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x}f(x) + \frac{y-x}{z-x}f(z)\right) \leq f\left(\frac{x+z}{2}\right) - \frac{f(x)+f(z)}{2}$$

for all real numbers $x < y < z$.

USA IMO TST 2021 Solutions

United States of America — IMO Team Selection Test

Andrew Gu, Ankan Bhattacharya and Evan Chen

62$^{\text{th}}$ IMO 2021 Russia

Contents

§0 Problems

1. Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

2. Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

   Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

   Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.

3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z)\right) \leq f\left(\frac{x+z}{2}\right) - \frac{f(x)+f(z)}{2}$$

   for all real numbers $x < y < z$.

§1 Solutions to Day 1

§1.1 USA TST 2021/1, proposed by Ankan Bhattacharya, Michael Ren

Available online at https://aops.com/community/p20672573.

Problem statement

Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

The answer is s composite.

¶ Composite construction Write $s = (w + x)(y + z)$, where w, x, y, z are positive integers. Let $a = wy$, $b = wz$, $c = xy$, $d = xz$. Then

$$abc + abd + acd + bcd = wxyz(w + x)(y + z)$$

so this works.

¶ Prime proof Choose suitable a, b, c, d. Then

$$(a + b)(a + c)(a + d) = (abc + abd + acd + bcd) + a^2(a + b + c + d) \equiv 0 \pmod{s}.$$

Hence s divides a product of positive integers less than s, so s is composite.

> **Remark.** Here is another proof that s is composite.
> Suppose that s is prime. Then the polynomial $(x - a)(x - b)(x - c)(x - d) \in \mathbb{F}_s[x]$ is even, so the roots come in two opposite pairs in $\mathbb{F}_s$. Thus the sum of each pair is at least s, so the sum of all four is at least $2s > s$, contradiction.

§1.2 USA TST 2021/2, proposed by Andrew Gu, Frank Han

Available online at https://aops.com/community/p20672623.

Problem statement

Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.

For brevity, we let ℓ_i denote line U_iV_i for $i = 1, 2$.

We first give an explicit description of the fixed point K. Let E and F be points on Γ such that $\overline{AE} \parallel \ell_1$ and $\overline{BF} \parallel \ell_2$. The problem conditions imply that E lies between U_1 and A while F lies between U_2 and B. Then we let

$$K = \overline{AF} \cap \overline{BE}.$$

This point exists because $AEFB$ are the vertices of a convex quadrilateral.

Remark (How to identify the fixed point)**.** If we drop the condition that X lies on the arc, then the choice above is motivated by choosing $X \in \{E, F\}$. Essentially, when one chooses $X \to E$, the point C approaches an infinity point. So in this degenerate case, the only points whose power is finite to (XCD) are bounded are those on line BE. The same logic shows that K must lie on line AF. Therefore, if the problem is going to work, the fixed point must be exactly $\overline{AF} \cap \overline{BE}$.

We give two possible approaches for proving the power of K with respect to (XCD) is fixed.

¶ First approach by Vincent Huang We need the following claim:

Claim — Suppose distinct lines AC and BD meet at X. Then for any point K

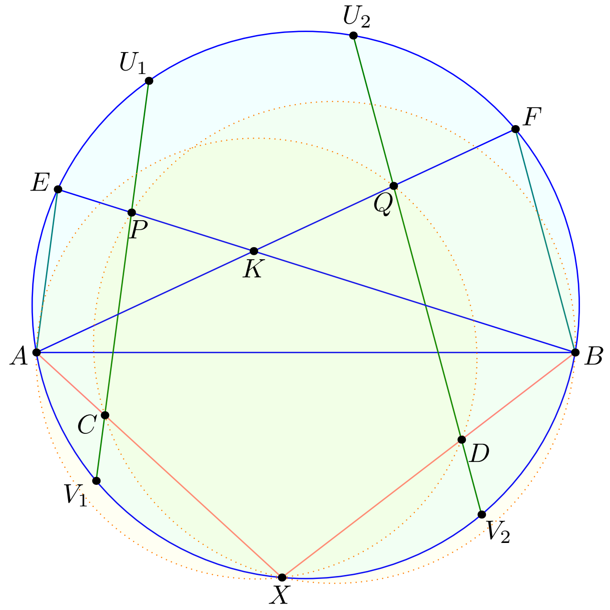
$$\operatorname{pow}(K, XAB) + \operatorname{pow}(K, XCD) = \operatorname{pow}(K, XAD) + \operatorname{pow}(K, XBC).$$

Proof. The difference between the left-hand side and right-hand side is a linear function in K, which vanishes at all of A, B, C, D. □

Construct the points $P = \ell_1 \cap \overline{BE}$ and $Q = \ell_2 \cap \overline{AF}$, which do not depend on X.

Claim — Quadrilaterals $BPCX$ and $AQDX$ are cyclic.

Proof. By Reim's theorem: $\measuredangle CPB = \measuredangle AEB = \measuredangle AXB = \measuredangle CXB$, etc. □



Now, for the particular K we choose, we have

$$\begin{aligned}\operatorname{pow}(K, XCD) &= \operatorname{pow}(K, XAD) + \operatorname{pow}(K, XBC) - \operatorname{pow}(K, XAB) \\ &= KA \cdot KQ + KB \cdot KP - \operatorname{pow}(K, \Gamma).\end{aligned}$$

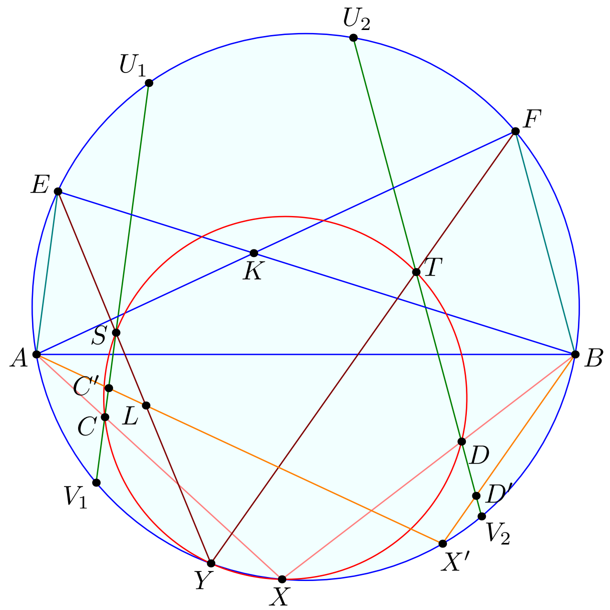
This is fixed, so the proof is completed.

¶ Second approach by authors Let Y be the second intersection of (XCD) with Γ. Let $S = \overline{EY} \cap \ell_1$ and $T = \overline{FY} \cap \ell_2$.

Claim — Points S and T lies on (XCD) as well.

Proof. By Reim's theorem: $\measuredangle CSY = \measuredangle AEY = \measuredangle AXY = \measuredangle CXY$, etc. $\square$

Now let X' be any other choice of X, and define C' and D' in the obvious way. We are going to show that K lies on the radical axis of (XCD) and $(X'C'D')$.



The main idea is as follows:

Claim — The point $L = \overline{EY} \cap \overline{AX'}$ lies on the radical axis. By symmetry, so does the point $M = \overline{FY} \cap \overline{BX'}$ (not pictured).

Proof. Again by Reim's theorem, $SC'YX'$ is cyclic. Hence we have

$$\operatorname{pow}(L, X'C'D') = LC' \cdot LX' = LS \cdot LY = \operatorname{pow}(L, XCD). \qquad \square$$

To conclude, note that by Pascal theorem on

$$EYFAX'B$$

it follows K, L, M are collinear, as needed.

Remark. All the conditions about U_1, V_1, U_2, V_2 at the beginning are there to eliminate configuration issues, making the problem less obnoxious to the contestant.

In particular, without the various assumptions, there exist configurations in which the point K is at infinity. In these cases, the center of XCD moves along a fixed line.

§1.3 USA TST 2021/3, proposed by Gabriel Carroll

Available online at https://aops.com/community/p20672681.

Problem statement

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x}f(x) + \frac{y-x}{z-x}f(z)\right) \le f\left(\frac{x+z}{2}\right) - \frac{f(x)+f(z)}{2}$$

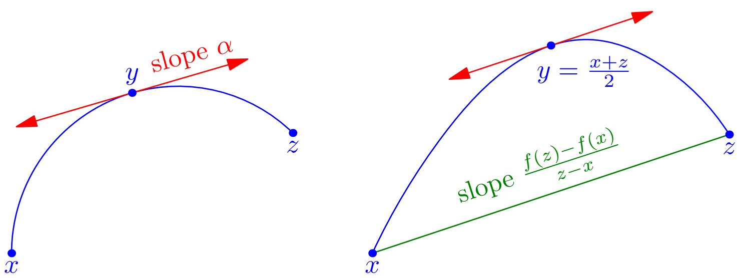
for all real numbers $x < y < z$.

Answer: all functions of the form $f(y) = ay^2 + by + c$, where a, b, c are constants with $a \le 0$.

If $I = (x, z)$ is an interval, we say that a real number α is a *supergradient* of f at $y \in I$ if we always have

$$f(t) \le f(y) + \alpha(t-y)$$

for every $t \in I$. (This inequality may be familiar as the so-called "tangent line trick". A cartoon of this situation is drawn below for intuition.) We will also say α is a supergradient of f at y, without reference to the interval, if α is a supergradient of *some* open interval containing y.



Claim — The problem condition is equivalent to asserting that $\frac{f(z)-f(x)}{z-x}$ is a supergradient of f at $\frac{x+z}{2}$ for the interval (x, z), for every $x < z$.

Proof. This is just manipulation. □

At this point, we may readily verify that all claimed quadratic functions $f(x) = ax^2 + bx + c$ work — these functions are concave, so the graphs lie below the tangent line at any point. Given $x < z$, the tangent at $\frac{x+z}{2}$ has slope given by the derivative $f'(x) = 2ax + b$, that is

$$f'\left(\frac{x+z}{2}\right) = 2a \cdot \frac{x+z}{2} + b = \frac{f(z)-f(x)}{z-x}$$

as claimed. (Of course, it is also easy to verify the condition directly by elementary means, since it is just a polynomial inequality.)

Now suppose f satisfies the required condition; we prove that it has the above form.

Claim — The function f is concave.

Proof. Choose any $\Delta > \max\{z-y, y-x\}$. Since f has a supergradient α at y over the interval $(y-\Delta, y+\Delta)$, and this interval includes x and z, we have

$$\begin{aligned}\frac{z-y}{z-x}f(x)+\frac{y-x}{z-x}f(z) &\le \frac{z-y}{z-x}(f(y)+\alpha(x-y))+\frac{y-x}{z-x}(f(y)+\alpha(z-y)) \\ &= f(y).\end{aligned}$$

That is, f is a concave function. Continuity follows from the fact that any concave function on $\mathbb{R}$ is automatically continuous. □

Lemma (see e.g. https://math.stackexchange.com/a/615161 for picture)
Any concave function f on $\mathbb{R}$ is continuous.

Proof. Suppose we wish to prove continuity at $p \in \mathbb{R}$. Choose any real numbers a and b with $a < p < b$. For any $0 < \varepsilon < \max(b-p, p-a)$ we always have

$$f(p)+\frac{f(b)-f(p)}{b-p}\varepsilon \le f(p+\varepsilon) \le f(p)+\frac{f(p)-f(a)}{p-a}\varepsilon$$

which implies right continuity; the proof for left continuity is the same. □

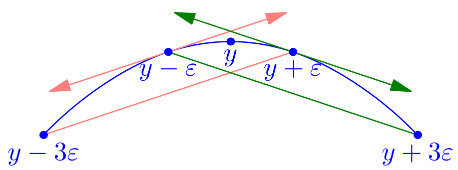
Claim — The function f cannot have more than one supergradient at any given point.

Proof. Fix $y \in \mathbb{R}$. For $t > 0$, let's define the function

$$g(t) = \frac{f(y)-f(y-t)}{t} - \frac{f(y+t)-f(y)}{t}.$$

We contend that $g(\varepsilon) \le \frac{3}{5}g(3\varepsilon)$ for any $\varepsilon > 0$. Indeed by the problem condition,

$$\begin{aligned}f(y) &\le f(y-\varepsilon)+\frac{f(y+\varepsilon)-f(y-3\varepsilon)}{4}\\ f(y) &\le f(y+\varepsilon)-\frac{f(y+3\varepsilon)-f(y-\varepsilon)}{4}.\end{aligned}$$



Summing gives the desired conclusion.

Now suppose that f has two supergradients $\alpha < \alpha'$ at point y. For small enough ε, we should have we have $f(y-\varepsilon) \le f(y)-\alpha'\varepsilon$ and $f(y+\varepsilon) \le f(y)+\alpha\varepsilon$, hence

$$g(\varepsilon) = \frac{f(y)-f(y-\varepsilon)}{\varepsilon} - \frac{f(y+\varepsilon)-f(y)}{\varepsilon} \ge \alpha'-\alpha.$$

This is impossible since $g(\varepsilon)$ may be arbitrarily small. □

Claim — The function f is quadratic on the rational numbers.

Proof. Consider any four-term arithmetic progression $x, x+d, x+2d, x+3d$. Because $(f(x+2d)-f(x+d))/d$ and $(f(x+3d)-f(x))/3d$ are both supergradients of f at the point $x+3d/2$, they must be equal, hence

$$f(x+3d)-3f(x+2d)+3f(x+d)-f(x) = 0. \tag{1}$$

If we fix $d = 1/n$, it follows inductively that f agrees with a quadratic function $\widetilde{f_n}$ on the set $\frac{1}{n}\mathbb{Z}$. On the other hand, for any $m \neq n$, we apparently have $\widetilde{f_n} = \widetilde{f_{mn}} = \widetilde{f_m}$, so the quadratic functions on each "layer" are all equal. $\square$

Since f is continuous, it follows f is quadratic, as needed.

Remark (Alternate finish using differentiability due to Michael Ren)**.** In the proof of the main claim (about uniqueness of supergradients), we can actually notice the two terms $\frac{f(y)-f(y-t)}{t}$ and $\frac{f(y+t)-f(y)}{t}$ in the definition of $g(t)$ are both monotonic (by concavity). Since we supplied a proof that $\lim_{t\to 0} g(t) = 0$, we find f is differentiable.

Now, if the derivative at some point exists, it must coincide with all the supergradients; (informally, this is why "tangent line trick" always has the slope as the derivative, and formally, we use the mean value theorem). In other words, we must have

$$f(x+y) - f(x-y) = 2f'(x) \cdot y$$

holds for all real numbers x and y. By choosing $y = 1$ we obtain that $f'(x) = f(x+1) - f(x-1)$ which means f' is also continuous.

Finally differentiating both sides with respect to y gives

$$f'(x+y) - f'(x-y) = 2f'(x)$$

which means f' obeys Jensen's functional equation. Since f' is continuous, this means f' is linear. Thus f is quadratic, as needed.

USA IMO TST 2021 Solutions

United States of America — IMO Team Selection Test

ANDREW GU, ANKAN BHATTACHARYA AND EVAN CHEN

62[th] IMO 2021 Russia

The Romanian Masters of Mathematics (RMM) was originally scheduled in February and is normally used as a USA team selection test. Since the RMM was postponed due to the COVID-19 pandemic, a single three-problem test was used instead for team selection testing in February. The statistics for that one exam are listed here.

§1 Summary of scores for TST 2021

N	27	1st Q	7	Max	21
μ	10.15	Median	7	Top 3	16
σ	4.30	3rd Q	13	Top 6	14

§2 Problem statistics for TST 2021

	P1	P2	P3
0	0	20	17
1	0	2	1
2	0	0	1
3	0	0	0
4	0	0	2
5	0	0	1
6	1	0	2
7	26	5	3
Avg	6.96	1.37	1.81

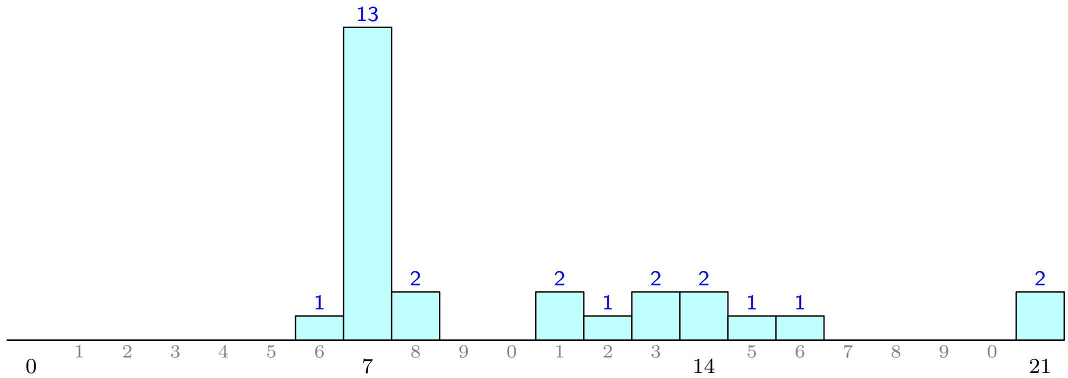
§3 Rankings for TST 2021

Sc	Num	Cu	Per
21	2	2	7.41%
20	0	2	7.41%
19	0	2	7.41%
18	0	2	7.41%
17	0	2	7.41%
16	1	3	11.11%
15	1	4	14.81%

Sc	Num	Cu	Per
14	2	6	22.22%
13	2	8	29.63%
12	1	9	33.33%
11	2	11	40.74%
10	0	11	40.74%
9	0	11	40.74%
8	2	13	48.15%

Sc	Num	Cu	Per
7	13	26	96.30%
6	1	27	100.00%
5	0	27	100.00%
4	0	27	100.00%
3	0	27	100.00%
2	0	27	100.00%
1	0	27	100.00%
0	0	27	100.00%

§4 Histogram for TST 2021



Team Selection Test for the 64^{th} International Mathematical Olympiad

United States of America

Day I

Thursday, December 8, 2022

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, December 12 at noon Eastern time.

IMO TST 1. There are 2022 equally spaced points on a circular track γ of circumference 2022. The points are labeled A_1, A_2, $\ldots$, A_{2022} in some order, each label used once. Initially, Bunbun the Bunny begins at A_1. She hops along γ from A_1 to A_2, then from A_2 to A_3, until she reaches A_{2022}, after which she hops back to A_1. When hopping from P to Q, she always hops along the shorter of the two arcs $\widehat{PQ}$ of γ; if $\overline{PQ}$ is a diameter of γ, she moves along either semicircle.

Determine the maximal possible sum of the lengths of the 2022 arcs which Bunbun traveled, over all possible labellings of the 2022 points.

IMO TST 2. Let ABC be an acute triangle. Let M be the midpoint of side BC, and let E and F be the feet of the altitudes from B and C, respectively. Suppose that the common external tangents to the circumcircles of triangles BME and CMF intersect at a point K, and that K lies on the circumcircle of ABC. Prove that line AK is perpendicular to line BC.

IMO TST 3. Consider pairs (f, g) of functions from the set of nonnegative integers to itself such that

- $f(0) \geq f(1) \geq f(2) \geq \cdots \geq f(300) \geq 0$;
- $f(0) + f(1) + f(2) + \cdots + f(300) \leq 300$;
- for any 20 nonnegative integers n_1, n_2, $\ldots$, n_{20}, not necessarily distinct, we have

$$g(n_1 + n_2 + \cdots + n_{20}) \leq f(n_1) + f(n_2) + \cdots + f(n_{20}).$$

Determine the maximum possible value of $g(0) + g(1) + \cdots + g(6000)$ over all such pairs of functions.

Team Selection Test for the 64$^{\text{th}}$ International Mathematical Olympiad

United States of America

Day II

Thursday, January 12, 2023

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, January 16 at noon Eastern time.

IMO TST 4. Let $\lfloor \bullet \rfloor$ denote the floor function. For nonnegative integers a and b, their *bitwise xor*, denoted $a \oplus b$, is the unique nonnegative integer such that

$$\left\lfloor \frac{a}{2^k} \right\rfloor + \left\lfloor \frac{b}{2^k} \right\rfloor - \left\lfloor \frac{a \oplus b}{2^k} \right\rfloor$$

is even for every integer $k \geq 0$. (For example, $9 \oplus 10 = 1001_2 \oplus 1010_2 = 0011_2 = 3$.)

Find all positive integers a such that for any integers $x > y \geq 0$, we have

$$x \oplus ax \neq y \oplus ay.$$

IMO TST 5. Let m and n be fixed positive integers. Tsvety and Freyja play a game on an infinite grid of unit square cells. Tsvety has secretly written a real number inside of each cell so that the sum of the numbers within every rectangle of size either $m \times n$ or $n \times m$ is zero. Freyja wants to learn all of these numbers.

One by one, Freyja asks Tsvety about some cell in the grid, and Tsvety truthfully reveals what number is written in it. Freyja wins if, at any point, Freyja can simultaneously deduce the number written in every cell of the entire infinite grid. (If this never occurs, Freyja has lost the game and Tsvety wins.)

In terms of m and n, find the smallest number of questions that Freyja must ask to win, or show that no finite number of questions can suffice.

IMO TST 6. Let $\mathbb{N}$ denote the set of positive integers. Fix a function $f \colon \mathbb{N} \to \mathbb{N}$ and for any $m, n \in \mathbb{N}$ define

$$\Delta(m,n) = \underbrace{f(f(\ldots f}_{f(n)\text{ times}}(m)\ldots)) - \underbrace{f(f(\ldots f}_{f(m)\text{ times}}(n)\ldots)).$$

Suppose $\Delta(m,n) \neq 0$ for any distinct $m, n \in \mathbb{N}$. Show that Δ is unbounded, meaning that for any constant C there exist $m, n \in \mathbb{N}$ with $|\Delta(m,n)| > C$.

USA TST 2023 Solutions

United States of America — Team Selection Test

Andrew Gu, Evan Chen, and Gopal Goel

64rd IMO 2022 Japan and 12th EGMO 2023 Slovenia

Contents

§0 Problems

1. There are 2022 equally spaced points on a circular track γ of circumference 2022. The points are labeled A_1, A_2, …, A_{2022} in some order, each label used once. Initially, Bunbun the Bunny begins at A_1. She hops along γ from A_1 to A_2, then from A_2 to A_3, until she reaches A_{2022}, after which she hops back to A_1. When hopping from P to Q, she always hops along the shorter of the two arcs $\widehat{PQ}$ of γ; if $\overline{PQ}$ is a diameter of γ, she moves along either semicircle.

Determine the maximal possible sum of the lengths of the 2022 arcs which Bunbun traveled, over all possible labellings of the 2022 points.

2. Let ABC be an acute triangle. Let M be the midpoint of side BC, and let E and F be the feet of the altitudes from B and C, respectively. Suppose that the common external tangents to the circumcircles of triangles BME and CMF intersect at a point K, and that K lies on the circumcircle of ABC. Prove that line AK is perpendicular to line BC.

3. Consider pairs (f, g) of functions from the set of nonnegative integers to itself such that

- $f(0) \geq f(1) \geq f(2) \geq \cdots \geq f(300) \geq 0$;
- $f(0) + f(1) + f(2) + \cdots + f(300) \leq 300$;
- for any 20 nonnegative integers n_1, n_2, …, n_{20}, not necessarily distinct, we have
$$g(n_1 + n_2 + \cdots + n_{20}) \leq f(n_1) + f(n_2) + \cdots + f(n_{20}).$$

Determine the maximum possible value of $g(0) + g(1) + \cdots + g(6000)$ over all such pairs of functions.

4. For nonnegative integers a and b, denote their *bitwise xor* by $a \oplus b$. (For example, $9 \oplus 10 = 1001_2 \oplus 1010_2 = 0011_2 = 3$.)

Find all positive integers a such that for any integers $x > y \geq 0$, we have
$$x \oplus ax \neq y \oplus ay.$$

5. Let m and n be fixed positive integers. Tsvety and Freyja play a game on an infinite grid of unit square cells. Tsvety has secretly written a real number inside of each cell so that the sum of the numbers within every rectangle of size either $m \times n$ or $n \times m$ is zero. Freyja wants to learn all of these numbers.

One by one, Freyja asks Tsvety about some cell in the grid, and Tsvety truthfully reveals what number is written in it. Freyja wins if, at any point, Freyja can simultaneously deduce the number written in every cell of the entire infinite grid. (If this never occurs, Freyja has lost the game and Tsvety wins.)

In terms of m and n, find the smallest number of questions that Freyja must ask to win, or show that no finite number of questions can suffice.

6. Fix a function $f \colon \mathbb{N} \to \mathbb{N}$ and for any $m, n \in \mathbb{N}$ define
$$\Delta(m, n) = \underbrace{f(f(\dots f}_{f(n)\text{ times}}(m)\dots)) - \underbrace{f(f(\dots f}_{f(m)\text{ times}}(n)\dots)).$$

Suppose $\Delta(m, n) \neq 0$ for any distinct $m, n \in \mathbb{N}$. Show that Δ is unbounded, meaning that for any constant C there exist $m, n \in \mathbb{N}$ with $|\Delta(m, n)| > C$.

§1 Solutions to Day 1

§1.1 USA TST 2023/1, proposed by Kevin Cong

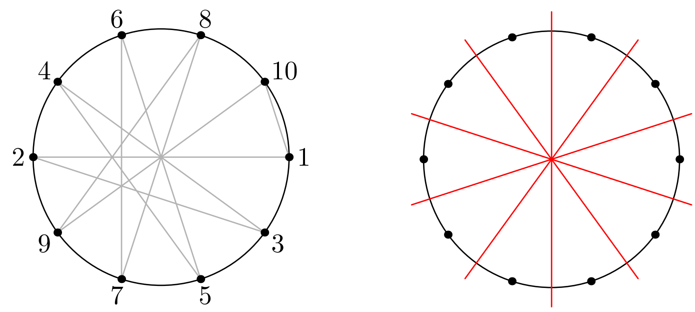
Available online at https://aops.com/community/p26685816.

Problem statement

There are 2022 equally spaced points on a circular track γ of circumference 2022. The points are labeled $A_1, A_2, \ldots, A_{2022}$ in some order, each label used once. Initially, Bunbun the Bunny begins at A_1. She hops along γ from A_1 to A_2, then from A_2 to A_3, until she reaches A_{2022}, after which she hops back to A_1. When hopping from P to Q, she always hops along the shorter of the two arcs $\widehat{PQ}$ of γ; if $\overline{PQ}$ is a diameter of γ, she moves along either semicircle.

Determine the maximal possible sum of the lengths of the 2022 arcs which Bunbun traveled, over all possible labellings of the 2022 points.

Replacing 2022 with $2n$, the answer is $2n^2 - 2n + 2$. (When $n = 1011$, the number is 2042222.)



¶ Construction The construction for $n = 5$ shown on the left half of the figure easily generalizes for all n.

Remark. The validity of this construction can also be seen from the below proof.

¶ First proof of bound Let d_i be the shorter distance from A_{2i-1} to A_{2i+1}.

Claim — The distance of the leg of the journey $A_{2i-1} \to A_{2i} \to A_{2i+1}$ is at most $2n - d_i$.

Proof. Of the two arcs from A_{2i-1} to A_{2i+1}, Bunbun will travel either d_i or $2n-d_i$. One of those arcs contains A_{2i} along the way. So we get a bound of $\max(d_i, 2n-d_i) = 2n-d_i$. □

That means the total distance is at most

$$\sum_{i=1}^{n} (2n - d_i) = 2n^2 - (d_1 + d_2 + \cdots + d_n).$$

Claim — We have
$$d_1 + d_2 + \cdots + d_n \geq 2n - 2.$$

Proof. The left-hand side is the sum of the walk $A_1 \to A_3 \to \cdots \to A_{2n-1} \to A_1$. Among the n points here, two of them must have distance at least $n-1$ apart; the other d_i's contribute at least 1 each. So the bound is $(n-1) + (n-1) \cdot 1 = 2n - 2$. □

¶ Second proof of bound Draw the n diameters through the $2n$ arc midpoints, as shown on the right half of the figure for $n = 5$ in red.

Claim (Interpretation of distances) — The distance between any two points equals the number of diameters crossed to travel between the points.

Proof. Clear. □

With this in mind, call a diameter *critical* if it is crossed by all $2n$ arcs.

Claim — At most one diameter is critical.

Proof. Suppose there were two critical diameters; these divide the circle into four arcs. Then all $2n$ arcs cross both diameters, and so travel between opposite arcs. But this means that points in two of the four arcs are never accessed — contradiction. □

Claim — Every diameter is crossed an even number of times.

Proof. Clear: the diameter needs to be crossed an even number of times for the loop to return to its origin. □

This immediately implies that the maximum possible total distance is achieved when one diameter is crossed all $2n$ times, and every other diameter is crossed $2n - 2$ times, for a total distance of at most

$$n \cdot (2n - 2) + 2 = 2n^2 - 2n + 2.$$

§1.2 USA TST 2023/2, proposed by Kevin Cong

Available online at https://aops.com/community/p26685484.

Problem statement

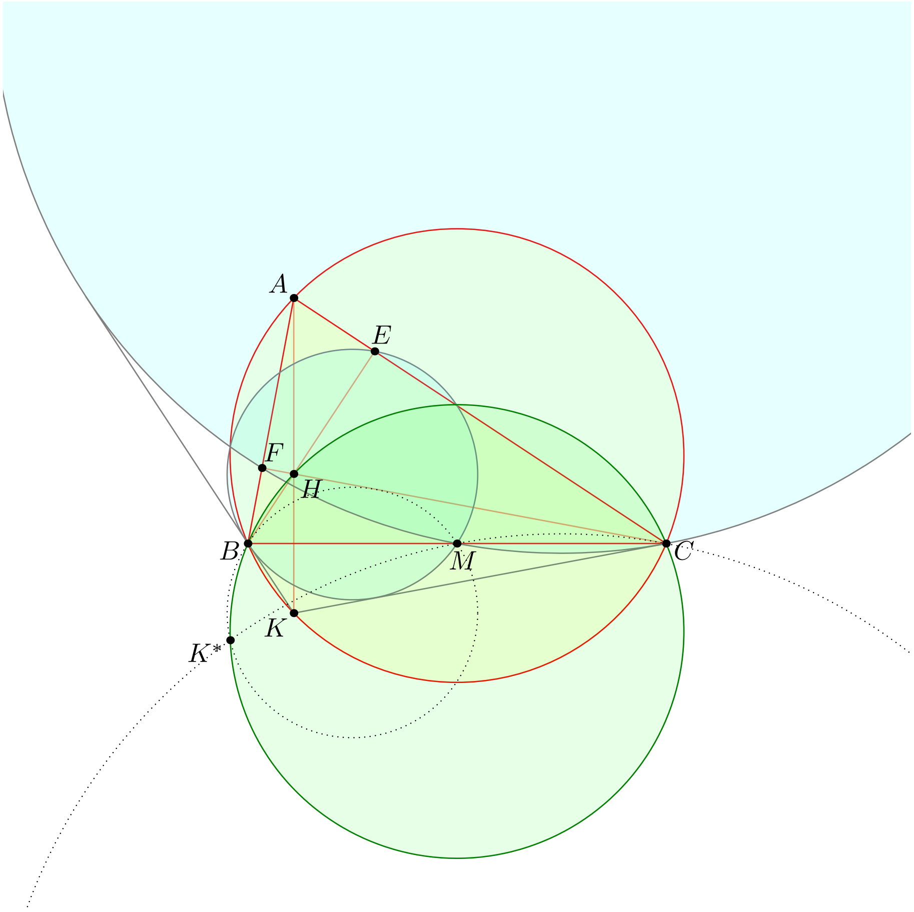
Let ABC be an acute triangle. Let M be the midpoint of side BC, and let E and F be the feet of the altitudes from B and C, respectively. Suppose that the common external tangents to the circumcircles of triangles BME and CMF intersect at a point K, and that K lies on the circumcircle of ABC. Prove that line AK is perpendicular to line BC.

We present several distinct approaches.

¶ Inversion solution submitted by Ankan Bhattacharya and Nikolai Beluhov Let H be the orthocenter of $\triangle ABC$. We use inversion in the circle with diameter $\overline{BC}$. We identify a few images:

- The circumcircles of $\triangle BME$ and $\triangle CMF$ are mapped to lines BE and CF.
- The common external tangents are mapped to the two circles through M which are tangent to lines BE and CF.
- The image of K, denoted K^*, is the second intersection of these circles.
- The assertion that K lies on (ABC) is equivalent to K^* lying on (BHC).

However, now K^* is simple to identify directly: it's just the reflection of M in the bisector of $\angle BHC$.



In particular, $\overline{HK^*}$ is a symmedian of $\triangle BHC$. However, since K^* lies on (BHC), this means $(HK^*; BC) = -1$.

Then, we obtain that $\overline{BC}$ bisects $\angle HMK^* \equiv \angle HMK$. However, K also lies on (ABC), which forces K to be the reflection of H in $\overline{BC}$. Thus $\overline{AK} \perp \overline{BC}$, as wanted.

¶ Solution with coaxial circles (Pitchayut Saengrungkongka) Let H be the orthocenter of $\triangle ABC$. Let Q be the second intersection of $\odot(BME)$ and $\odot(CMF)$. We first prove the following well-known properties of Q.

Claim — Q is the Miquel point of $BCEF$. In particular, Q lies on both $\odot(AEF)$ and $\odot(ABC)$.

Proof. Follows since $BCEF$ is cyclic with M being the circumcenter. □

Claim — $A(Q, H; B, C) = -1$.

Proof. By the radical center theorem on $\odot(AEF)$, $\odot(ABC)$, and $\odot(BCEF)$, we get that AQ, EF, and BC are concurrent. Now, the result follows from a well-known harmonic property. □

Now, we get to the meat of the solution. Let the circumcircle of $\odot(QMK)$ meet BC again at $T \neq M$. The key claim is the following.

Claim — QT is tangent to $\odot(BQC)$.

Proof. We use the "forgotten coaxiality lemma".

$$\begin{aligned}\frac{BT}{TC} &= \frac{TB \cdot TM}{TC \cdot TM} \\ &= \frac{\operatorname{pow}(T, \odot(BME))}{\operatorname{pow}(T, \odot(CMF))} \\ &= \frac{\operatorname{pow}(K, \odot(BME))}{\operatorname{pow}(K, \odot(CMF))} \\ &= \left(\frac{r_{\odot(BME)}}{r_{\odot(CMF)}}\right)^2 \\ &= \left(\frac{BQ/\sin\angle QMB}{CQ/\sin\angle QMC}\right)^2 \\ &= \frac{BQ^2}{CQ^2},\end{aligned}$$

implying the result. □

To finish, let O be the center of $\odot(ABC)$. Then, from the claim, $\angle OQT = 90^\circ = \angle OMT$, so O also lies on $\odot(QMTK)$. Thus, $\angle OKT = 90^\circ$, so KT is also tangent to $\odot(ABC)$ as well. This implies that $QBKC$ is harmonic quadrilateral, and the result follows from the second claim.

¶ Solution by Luke Robitaille Let Q be the second intersection of $\odot(BME)$ and $\odot(CMF)$. We use the first two claims of the previous solution. In particular, $Q \in \odot(ABC)$. We have the following claim.

Claim (Also appeared in ISL 2017 G7) — We have $\measuredangle QKM = \measuredangle QBM + \measuredangle QCM$.

Proof. Let KQ and KM meet $\odot(BME)$ again at Q' and M'. Then, by homothety, $\measuredangle Q'QM' = \measuredangle QCM$, so

$$\begin{aligned}\measuredangle QKM &= \measuredangle Q'QM' + \measuredangle QM'M \\ &= \measuredangle QCM + \measuredangle QBM,\end{aligned}$$

as desired. □

Now, we extend KM to meet $\odot(ABC)$ again at Q_1. We have

$$\begin{aligned}\measuredangle Q_1QB = \measuredangle Q_1KB &= \measuredangle Q_1KQ + \measuredangle QCB \\ &= \measuredangle MKQ + \measuredangle QKB \\ &= (\measuredangle MBQ + \measuredangle MCQ) + \measuredangle QCB \\ &= \measuredangle CBQ,\end{aligned}$$

implying that $QQ_1 \parallel BC$. This implies that $QBKC$ is harmonic quadrilateral, so we are done.

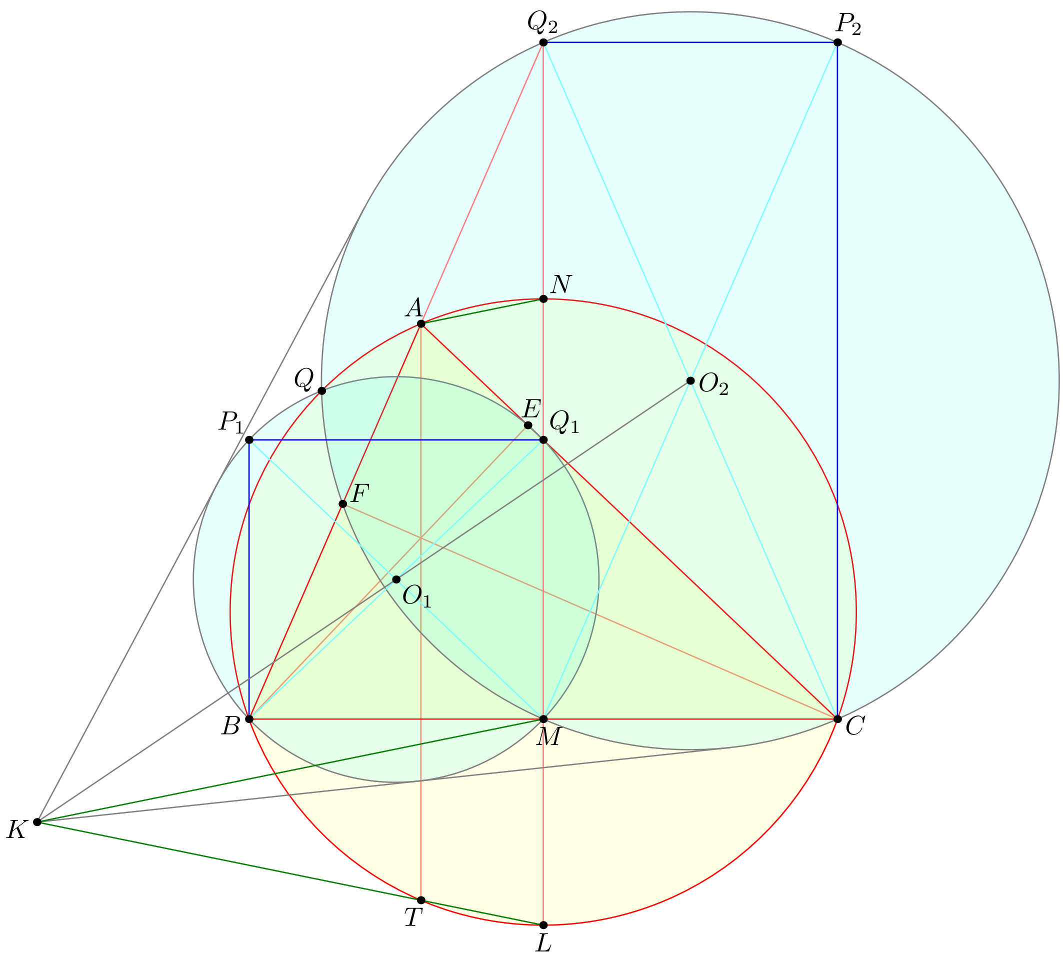
¶ Synthetic solution due to Andrew Gu (Harvard 2026) Define O_1 and O_2 as the circumcenters of (BME) and (CMF). Let T be the point on (ABC) such that $\overline{AT} \perp \overline{BC}$. Denote by L the midpoint of minor arc $\widehat{BC}$.

We are going to ignore the condition that K lies on the circumcircle of ABC, and prove the following unconditional result:

Proposition

The points T, L, K are collinear.

This will solve the problem because if K is on the circumcircle of ABC, it follows $K = T$ or $K = L$; but $K = L$ can never occur since O_1 and O_2 are obviously on different sides of line LM so line LM must meet O_1O_2 inside segment O_1O_2, and K lies outside this segment.



We now turn to the proof of the main lemma. Let P_1 and P_2 be the antipodes of M on these circles.

Claim — Lines AC and LM meet at the antipode Q_1 of B on (BME), so that BP_1Q_1M is a rectangle. Similarly, lines AB and LM meet at the antipode Q_2 of C on (CMF), so that CP_2Q_2M is a rectangle.

Proof. Let $Q_1' = \omega_1 \cap AC \neq E$. Then $\measuredangle BMQ_1' = \measuredangle BEQ_1' = 90^\circ$ hence $Q_1' \in LM$. The other half of the lemma follows similarly. □

From this, it follows that $P_1Q_1 = BM = \frac{1}{2}BC = MC = P_2Q_2$. Letting r_1 denote the radius of ω_1 (and similarly for ω_2), we deduce that $CQ_1 = BQ_1 = 2r_1$.

Claim — $KM = KL$.

Proof. I first claim that $\overline{CL}$ is the external bisector of $\angle Q_1CQ_2$; this follows from

$$\measuredangle Q_1CL = \measuredangle ACL = \measuredangle ABL = \measuredangle Q_2BL = \measuredangle Q_2CL.$$

The external angle bisector theorem then gives an equality of directed ratios

$$\frac{LQ_1}{LQ_2} = \frac{|CQ_1|}{|CQ_2|} = \frac{|BQ_1|}{|CQ_2|} = \frac{2r_1}{2r_2} = \frac{r_1}{r_2}$$

Let the reflection of M over K be P; then P lies on $\overline{P_1P_2}$ and

$$\frac{PP_1}{PP_2} = \frac{2KO_1}{2KO_2} = \frac{KO_1}{KO_2} = \frac{r_1}{r_2} = \frac{LQ_1}{LQ_2}$$

where again the ratios are directed. Projecting everything onto line LM, so that P_1 lands at Q_1 and P_2 lands at Q_2, we find that the projection of P must land exactly at L. □

Claim — Line KM is an external angle bisector of $\angle O_1MO_2$.

Proof. Because $\frac{KO_1}{KO_2} = \frac{r_1}{r_2} = \frac{MO_1}{MO_2}$. □

To finish, note that we know that $\overline{MP_1} \parallel \overline{CQ_1} \equiv \overline{AC}$ and $\overline{MP_2} \parallel \overline{BQ_2} \equiv \overline{AB}$, meaning the angles $\angle O_1MO_2$ and $\angle CAB$ have parallel legs. Hence, if N is the antipode of L, it follows that $\overline{MK} \parallel \overline{AN}$. Now from $MK = KL$ and the fact that $ANLT$ is an isosceles trapezoid, we deduce that $\overline{LT}$ and $\overline{LK}$ are lines in the same direction (namely, the reflection of $MK \parallel AN$ across $\overline{BC}$), as needed.

¶ Complex numbers approach with Apollonian circles, by Carl Schildkraut We use complex numbers. As in the first approach, we will ignore the hypothesis that K lies on (ABC).

Let $Q := (AH) \cap (ABC) \cap (AEF) \neq A$ be the Miquel point of $BFEC$ again. Construct the point T on (ABC) for which $AT \perp BC$; note that $T = -\frac{bc}{a}$. This time the unconditional result is:

Proposition

We have Q, M, T, K are concyclic (or collinear) on an Apollonian circle of $\overline{O_1O_2}$.

This will solve the original problem since once K lies on (ABC) it must be either Q or T. But since K is not on (BME), $K \neq Q$, it will have to be T.

We now prove the proposition. Suppose (ABC) is the unit circle and let $A = a$, $B = b$, $C = c$. Let $H = a + b + c$ be the orthocenter of $\triangle ABC$. By the usual formulas,

$$E := \frac{1}{2}\left(a + b + c - \frac{bc}{a}\right).$$

Let O_1 be the center of (BME) and O_2 be the center of (CMF).

Claim (Calculation of the Miquel point) — We have $Q = \frac{2a+b+c}{a(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})+1}$.

Proof. We now compute that $Q = q$ satisfies $\overline{q} = 1/q$ (since Q is on the unit circle) and $\frac{q-h}{q-a} \in i\mathbb{R}$ (since $AQ \perp QH$), which expands to

$$0 = \frac{q-h}{q-a} + \frac{1/q - \overline{h}}{1/q - 1/a} = \frac{q-h}{q-a} - \frac{a(1-q\overline{h})}{q-a}.$$

This solves to $q = \frac{h+a}{a\overline{h}+1} = \frac{2a+b+c}{a\overline{h}+1}$. □

Claim (Calculation of O_1 and O_2) — We have $O_1 = \frac{b(2a+b+c)}{2(a+b)}$ and $O_2 = \frac{c(2a+b+c)}{2(a+c)}$.

Proof. We now compute O_1 and O_2. For $x, y, z \in \mathbb{C}$, let $\operatorname{Circum}(x, y, z)$ denote the circumcenter of the triangle defined by vertices x, y, and z in $\mathbb{C}$. We have

$$\begin{aligned} O_1 &= \operatorname{Circum}(B, M, E) \\ &= b + \frac{1}{2}\operatorname{Circum}\left(0, c-b, \frac{(a-b)(b-c)}{b}\right) \\ &= b - \frac{b-c}{2b}\operatorname{Circum}(0, b, b-a) \\ &= b - \frac{b-c}{2b}\left(b - \operatorname{Circum}(0, b, a)\right) \\ &= b - \frac{b-c}{2b}\left(b - \frac{ab}{a+b}\right) = b - \frac{b(b-c)}{2(a+b)} = \frac{b(2a+b+c)}{2(a+b)}. \end{aligned}$$

Similarly, $O_2 = \frac{c(2a+b+c)}{2(a+c)}$. □

We are now going to prove the following:

Claim — We have

$$\frac{TO_1}{TO_2} = \frac{MO_1}{MO_2} = \frac{QO_1}{QO_2}.$$

Proof. We now compute

$$MO_1 = BO_1 = \left|b - \frac{b(2a+b+c)}{2(a+b)}\right| = \left|\frac{b(b-c)}{2(a+b)}\right| = \frac{1}{2}\left|\frac{b-c}{a+b}\right|$$

and

$$QO_1 = \left|r - \frac{b(2a+b+c)}{2(a+b)}\right| = \left|1 - \frac{b(a+h)}{2(a+b)r}\right| = \left|1 - \frac{b(a\overline{h}+1)}{2(a+b)}\right| = \left|\frac{a - \frac{ab}{c}}{2(a+b)}\right| = \frac{1}{2}\left|\frac{b-c}{a+b}\right|.$$

This implies both (by symmetry) that $\frac{MO_1}{MO_2} = \frac{QO_1}{QO_2} = \left|\frac{a+c}{a+b}\right|$ and that Q is on (BME) and (CMF). Also,

$$\frac{TO_1}{TO_2} = \frac{\left|\frac{b(2a+b+c)}{2(a+b)} + \frac{bc}{a}\right|}{\left|\frac{c(2a+b+c)}{2(a+c)} + \frac{bc}{a}\right|} = \left|\frac{\frac{b(2a^2+ab+ac+2ac+2bc)}{2a(a+b)}}{\frac{c(2a^2+ab+ac+2ab+2bc)}{2a(a+c)}}\right| = \left|\frac{a+c}{a+b}\right| \cdot \left|\frac{2a^2+2bc+ab+3ac}{2a^2+2bc+3ab+ac}\right|;$$

if $z = 2a^2 + 2bc + ab + 3ac$, then $a^2bc\overline{z} = 2a^2 + 2bc + 3ab + ac$, so the second term has magnitude 1. This means $\frac{TO_1}{TO_2} = \frac{MO_1}{MO_2} = \frac{QO_1}{QO_2}$, as desired. □

To finish, note that this common ratio is the ratio between the radii of these two circles, so it is also $\frac{KO_1}{KO_2}$. By Apollonian circles the points $\{Q, M, T, K\}$ lie on a circle or a line.

§1.3 USA TST 2023/3, proposed by Sean Li

Available online at https://aops.com/community/p26685437.

Problem statement

Consider pairs (f, g) of functions from the set of nonnegative integers to itself such that

- $f(0) \geq f(1) \geq f(2) \geq \cdots \geq f(300) \geq 0$;
- $f(0) + f(1) + f(2) + \cdots + f(300) \leq 300$;
- for any 20 nonnegative integers $n_1, n_2, \ldots, n_{20}$, not necessarily distinct, we have

$$g(n_1 + n_2 + \cdots + n_{20}) \leq f(n_1) + f(n_2) + \cdots + f(n_{20}).$$

Determine the maximum possible value of $g(0) + g(1) + \cdots + g(6000)$ over all such pairs of functions.

Replace $300 = \frac{24 \cdot 25}{2}$ with $\frac{s(s+1)}{2}$ where $s = 24$, and 20 with k. The answer is $115440 = \frac{ks(ks+1)}{2}$. Equality is achieved at $f(n) = \max(s - n, 0)$ and $g(n) = \max(ks - n, 0)$. To prove

$$g(n_1 + \cdots + n_k) \leq f(n_1) + \cdots + f(n_k),$$

write it as

$$\max(x_1 + \cdots + x_k, 0) \leq \max(x_1, 0) + \cdots + \max(x_k, 0)$$

with $x_i = s - n_i$. This can be proven from the $k = 2$ case and induction.

It remains to show the upper bound. For this problem, define a *partition* to be a nonincreasing function $p \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $p(n) = 0$ for some n. The sum of p is defined to be $\sum_{n=0}^{\infty} p(n)$, which is finite under the previous assumption. Let $L = \mathbb{Z}_{\geq 0}^2$. The *Young diagram* of the partition is the set of points

$$\mathcal{P} := \{(x, y) \in L : y < p(x)\}.$$

The number of points in $\mathcal{P}$ is equal to the sum of p. The *conjugate* of a partition defined as

$$p_*(n) = \text{the number of } i \text{ for which } p(i) > n.$$

This is a partition with the same sum as p. Geometrically, the Young diagrams of p and p_* are reflections about $x = y$.

Since each $g(n)$ is independent, we may maximize each one separately for all n and assume that

$$g(n) = \min_{n_1 + \cdots + n_k = n} (f(n_1) + \cdots + f(n_k)). \qquad (*)$$

The conditions of the problem statement imply that $f\left(\frac{s(s+1)}{2}\right) = 0$. Then, for any $n \leq k\frac{s(s+1)}{2}$, there exists an optimal combination $(n_1, \ldots, n_k)$ in (*) where all n_i are at most $\frac{s(s+1)}{2}$, by replacing any term in an optimum greater than $\frac{s(s+1)}{2}$ by $\frac{s(s+1)}{2}$ and shifting the excess to smaller terms (because f is nonincreasing). Therefore we may extend f to a partition by letting $f(n) = 0$ for $n > \frac{s(s+1)}{2}$ without affecting the relevant values of g. Then (*) implies that g is a partition as well.

The problem can be restated as follows: f is a partition with sum $\frac{s(s+1)}{2}$, and g is a partition defined by (*). Find the maximum possible sum of g. The key claim is that the problem is the same under conjugation.

Claim — Under these conditions, we have

$$g_*(n) = \min_{n_1+\cdots+n_k=n} (f_*(n_1) + \cdots + f_*(n_k)).$$

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be the Young diagrams of f and g respectively, and $\overline{\mathcal{F}} = L \setminus \mathcal{F}$ and $\overline{\mathcal{G}} = L \setminus \mathcal{G}$ be their complements. The lower boundary of $\overline{\mathcal{F}}$ is formed by the points $(n, f(n))$ for $i \in \mathbb{Z}_{\geq 0}$. By the definition of g, the lower boundary of $\overline{\mathcal{G}}$ consists of points $(n, g(n))$ which are formed by adding k points of $\overline{\mathcal{F}}$. This means

$$\overline{\mathcal{G}} = \underbrace{\overline{\mathcal{F}} + \cdots + \overline{\mathcal{F}}}_{k\ \overline{\mathcal{F}}\text{'s}}$$

where $+$ denotes set addition. This definition remains invariant under reflection about $x = y$, which swaps f and g with their conjugates. □

Let A be the sum of g. We now derive different bounds on A. First, by Hermite's identity

$$n = \sum_{i=0}^{k-1} \left\lfloor \tfrac{n+i}{k} \right\rfloor$$

we have

$$\begin{aligned} A &= \sum_{n=0}^{\infty} g(n) \\ &\leq \sum_{n=0}^{\infty} \sum_{i=0}^{k-1} f\left(\left\lfloor \tfrac{n+i}{k} \right\rfloor\right) \\ &= k^2 \sum_{n=0}^{\infty} f(n) - \frac{k(k-1)}{2} f(0) \\ &= k^2 \frac{s(s+1)}{2} - \frac{k(k-1)}{2} f(0). \end{aligned}$$

By the claim, we also get the second bound $A \leq k^2 \frac{s(s+1)}{2} - \frac{k(k-1)}{2} f_*(0)$.

For the third bound, note that $f(f_*(0)) = 0$ and thus $g(kf_*(0)) = 0$. Moreover,

$$g(qf_*(0) + r) \leq q \cdot f(f_*(0)) + (k-q-1)f(0) + f(r) = (k-q-1)f(0) + f(r),$$

so we have

$$\begin{aligned} A &= \sum_{\substack{0 \leq q < k \\ 0 \leq r < f_*(0)}} g(qf_*(0) + r) \\ &\leq \frac{k(k-1)}{2} f_*(0) f(0) + k \sum_{0 \leq r < f_*(0)} f(r) \\ &= \frac{k(k-1)}{2} f_*(0) f(0) + k \frac{s(s+1)}{2}. \end{aligned}$$

Now we have three cases:

- If $f(0) \geq s$ then
$$A \leq k^2 \frac{s(s+1)}{2} - \frac{k(k-1)}{2} f(0) \leq \frac{ks(ks+1)}{2}.$$
- If $f_*(0) \geq s$ then
$$A \leq k^2 \frac{s(s+1)}{2} - \frac{k(k-1)}{2} f_*(0) \leq \frac{ks(ks+1)}{2}.$$
- Otherwise, $f(0)f_*(0) \leq s^2$ and
$$A \leq \frac{k(k-1)}{2} f_*(0) f(0) + k \frac{s(s+1)}{2} \leq \frac{ks(ks+1)}{2}.$$

In all cases, $A \leq \frac{ks(ks+1)}{2}$, as desired.

Remark. One can estimate the answer to be around $k^2 \frac{s(s+1)}{2}$ by observing the set addition operation "dilates" $\mathcal{F}$ by a factor of k, but significant care is needed to sharpen the bound.

§2 Solutions to Day 2

§2.1 USA TST 2023/4, proposed by Carl Schildkraut

Available online at https://aops.com/community/p26896062.

Problem statement

For nonnegative integers a and b, denote their *bitwise xor* by $a \oplus b$. (For example, $9 \oplus 10 = 1001_2 \oplus 1010_2 = 0011_2 = 3$.)

Find all positive integers a such that for any integers $x > y \geq 0$, we have

$$x \oplus ax \neq y \oplus ay.$$

Answer: the function $x \mapsto x \oplus ax$ is injective if and only if a is an even integer.

¶ Even case First, assume $\nu_2(a) = k > 0$. We wish to recover x from $c := x \oplus ax$. Notice that:

- The last k bits of c coincide with the last k bits of x.
- Now the last k bits of x give us also the last $2k$ bits of ax, so we may recover the last $2k$ bits of x as well.
- Then the last $2k$ bits of x give us also the last $3k$ bits of ax, so we may recover the last $3k$ bits of x as well.
- ...and so on.

¶ Odd case Conversely, suppose a is odd. To produce the desired collision:

Claim — Let n be any integer such that $2^n > a$, and define

$$x = \underbrace{1\dots1}_{n} = 2^n - 1, \qquad y = 1\underbrace{0\dots0}_{n}1 = 2^n + 1.$$

Then $x \oplus ax = y \oplus ay$.

Proof. Let P be the binary string for a, zero-padded to length n, and let Q be the binary string for $a-1$, zero-padded to length n, Then let R be the bitwise complement of Q. (Hence all three of are binary strings of length n.) Then

$$\begin{aligned} ax = \overline{QR} &\implies x \oplus ax = \overline{QQ} \\ ay = \overline{PP} &\implies y \oplus ay = \overline{QQ}. \end{aligned}$$

We're done. □

§2.2 USA TST 2023/5, proposed by Nikolai Beluhov

Available online at https://aops.com/community/p26896130.

Problem statement

Let m and n be fixed positive integers. Tsvety and Freyja play a game on an infinite grid of unit square cells. Tsvety has secretly written a real number inside of each cell so that the sum of the numbers within every rectangle of size either $m \times n$ or $n \times m$ is zero. Freyja wants to learn all of these numbers.

One by one, Freyja asks Tsvety about some cell in the grid, and Tsvety truthfully reveals what number is written in it. Freyja wins if, at any point, Freyja can simultaneously deduce the number written in every cell of the entire infinite grid. (If this never occurs, Freyja has lost the game and Tsvety wins.)

In terms of m and n, find the smallest number of questions that Freyja must ask to win, or show that no finite number of questions can suffice.

The answer is the following:

- If $\gcd(m, n) > 1$, then Freyja cannot win.
- If $\gcd(m, n) = 1$, then Freyja can win in a minimum of $(m-1)^2 + (n-1)^2$ questions.

First, we dispose of the case where $\gcd(m, n) > 1$. Write $d = \gcd(m, n)$. The idea is that any labeling where each $1 \times d$ rectangle has sum zero is valid. Thus, to learn the labeling, Freyja must ask at least one question in every row, which is clearly not possible in a finite number of questions.

Now suppose $\gcd(m, n) = 1$. We split the proof into two halves.

¶ Lower bound Clearly, any labeling where each $m \times 1$ and $1 \times m$ rectangle has sum zero is valid. These labelings form a vector space with dimension $(m-1)^2$, by inspection. (Set the values in an $(m-1) \times (m-1)$ square arbitrarily and every other value is uniquely determined.)

Similarly, labelings where each $n \times 1$ and $1 \times n$ rectangle have sum zero are also valid, and have dimension $(n-1)^2$.

It is also easy to see that no labeling other than the all-zero labeling belongs to both categories; labelings in the first space are periodic in both directions with period m, while labelings in the second space are periodic in both directions with period n; and hence any labeling in both categories must be constant, ergo all-zero.

Taking sums of these labelings gives a space of valid labelings of dimension $(m-1)^2 + (n-1)^2$. Thus, Freyja needs at least $(m-1)^2 + (n-1)^2$ questions to win.

¶ Proof of upper bound using generating functions, by Ankan Bhattacharya We prove:

Claim (Periodicity) — Any valid labeling is doubly periodic with period mn.

Proof. By Chicken McNugget, there exists N such that N and $N+1$ are both nonnegative integer linear combinations of m and n.

Then both $mn \times N$ and $mn \times (N+1)$ rectangles have zero sum, so $mn \times 1$ rectangles have zero sum. This implies that any two cells with a vertical displacement of mn are equal; similarly for horizontal displacements. □

With that in mind, consider a valid labeling. It naturally corresponds to a generating function

$$f(x,y) = \sum_{a=0}^{mn-1} \sum_{b=0}^{mn-1} c_{a,b} x^a y^b$$

where $c_{a,b}$ is the number in (a,b).

The generating function corresponding to sums over $n \times m$ rectangles is

$$f(x,y)(1+x+\cdots+x^{m-1})(1+y+\cdots+y^{n-1}) = f(x,y) \cdot \frac{x^m-1}{x-1} \cdot \frac{y^n-1}{y-1}.$$

Similarly, the one for $m \times n$ rectangles is

$$f(x,y) \cdot \frac{x^n-1}{x-1} \cdot \frac{y^m-1}{y-1}.$$

Thus, the constraints for f to be valid are equivalent to

$$f(x,y) \cdot \frac{x^m-1}{x-1} \cdot \frac{y^n-1}{y-1} \quad \text{and} \quad f(x,y) \cdot \frac{x^n-1}{x-1} \cdot \frac{y^m-1}{y-1}$$

being zero when reduced modulo $x^{mn}-1$ and $y^{mn}-1$, or, letting $\omega = \exp(2\pi i/mn)$, both terms being zero when powers of ω are plugged in.

To restate the constraints one final time, we need

$$f(\omega^a,\omega^b) \cdot \frac{\omega^{am}-1}{\omega^a-1} \cdot \frac{\omega^{bn}-1}{\omega^b-1} = f(\omega^a,\omega^b) \cdot \frac{\omega^{an}-1}{\omega^a-1} \cdot \frac{\omega^{bm}-1}{\omega^b-1} = 0$$

for all $a, b \in \{0, \dots, mn-1\}$.

Claim — This implies that $f(\omega^a,\omega^b) = 0$ for all but at most $(m-1)^2 + (n-1)^2$ values of $(a,b) \in \{0,\dots,mn-1\}^2$.

Proof. Consider a pair (a,b) such that $f(\omega^a,\omega^b) \neq 0$. Then we need

$$\frac{\omega^{am}-1}{\omega^a-1} \cdot \frac{\omega^{bn}-1}{\omega^b-1} = \frac{\omega^{an}-1}{\omega^a-1} \cdot \frac{\omega^{bm}-1}{\omega^b-1} = 0.$$

This happens when (at least) one fraction in either product is zero.

- If the first fraction is zero, then either $n \mid a$ and $a > 0$, or $m \mid b$ and $b > 0$.
- If the second fraction is zero, then either $m \mid a$ and $a > 0$, or $n \mid b$ and $b > 0$.

If the first condition holds in both cases, then $mn \mid a$, but $0 < a < mn$, a contradiction. Thus if $n \mid a$, then we must have $n \mid b$, and similarly if $m \mid a$ then $m \mid b$.

The former case happens $(m-1)^2$ times, and the latter case happens $(n-1)^2$ times. Thus, at most $(m-1)^2 + (n-1)^2$ values of $f(\omega^a,\omega^b)$ are nonzero. □

Claim — The $(mn)^2$ equations $f(\omega^a, \omega^b) = 0$ are linearly independent when viewed as linear equations in $(mn)^2$ variables $c_{a,b}$. Hence, any subset of these equations is also linearly independent.

Proof. In general, the equation $f(x, y) = 0$ is a polynomial relation with $\deg_x f(x, y) = \deg_y f(x, y) < mn$. However, if we let $S = \{\omega^0, \omega^1, \dots, \omega^{mn-1}\}$, then $|S| = mn$ and we are given $f(s, s') = 0$ for all $s, s' \in S$. This can only happen if f is the zero polynomial, that is, $c_{a,b} = 0$ for all a and b. □

It follows that the dimension of the space of valid labelings is at most $(m-1)^2 + (n-1)^2$, as desired.

¶ Explicit version of winning algorithm by Freyja, from author Suppose that $\gcd(m, n) = 1$ and $m \le n$. Let $[a, b]$ denote the set of integers between a and b inclusive.

Let Freyja ask about all cells (x, y) in the two squares

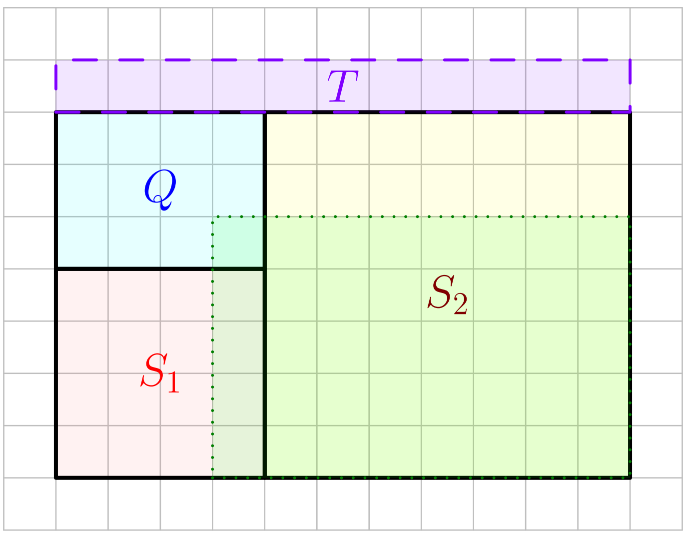
$$S_1 = [1, m-1] \times [1, m-1]$$
$$S_2 = [m, m+n-2] \times [1, n-1].$$

In the beginning, one by one, Freyja determines all values inside of the rectangle $Q := [1, m-1] \times [m, n-1]$. To that end, on each step she considers some rectangle with m rows and n columns such that its top left corner is in Q and all of the other values in it have been determined already. In this way, Freyja uncovers all of Q, starting with its lower right corner and then proceeding upwards and to the left.

Thus Freyja can learn all numbers inside of the rectangle

$$R := [1, m+n-2] \times [1, n-1] = Q \cup S_1 \cup S_2.$$

See the figure below for an illustration for $(m, n) = (5, 8)$. The first cell of Q is uncovered using the dotted green rectangle.



$(m, n) = (5, 8)$

We need one lemma:

Lemma

Let m and n be positive integers with $\gcd(m, n) = 1$. Consider an unknown sequence of real numbers z_1, z_2, …, z_s with $s \geq m+n-2$. Suppose that we know the sums of all contiguous blocks of size either m or n in this sequence. Then we can determine all individual entries in the sequence as well.

Proof. By induction on $m+n$. Suppose, without loss of generality, that $m \leq n$. Our base case is $m = 1$, which is clear. For the induction step, set $\ell = n - m$. Each contiguous block of size ℓ within z_1, z_2, …, z_{s-n} is the difference of two contiguous blocks of sizes m and n within the original sequence. By the induction hypothesis for ℓ and m, it follows that we can determine all of z_1, z_2, …, z_{s-n}. Then we determine the remaining z_i as well, one by one, in order from left to right, by examining on each step an appropriate contiguous block of size m. □

Let T be the rectangle $[1, m+n-2] \times \{n\}$. By looking at appropriate rectangles of sizes $m \times n$ and $n \times m$ such that their top row is contained within T and all of their other rows are contained within R, Freyja can learn the sums of all contiguous blocks of values of sizes m and n within T. By the Lemma, it follows that Freyja can uncover all of T.

In this way, with the help of the Lemma, Freyja can extend her rectangular area of knowledge both upwards and downwards. Once its height reaches $m+n-2$, by the same method she will be able to extend it to the left and right as well. This allows Freyja to determine all values in the grid. Therefore, $(m-1)^2 + (n-1)^2$ questions are indeed sufficient.

Remark. The ideas in the solution also yield a proof of the following result:

> Let m and n be relatively prime positive integers. Consider an infinite grid of unit square cells coloured in such a way that every rectangle of size either $m \times n$ or $n \times m$ contains the same multiset of colours. Then the colouring is either doubly periodic with period length m or doubly periodic with period length n.

(Here, "doubly periodic with period length s" means "both horizontally and vertically periodic with period length s".)

Here is a quick sketch of the proof. Given two positive integers i and j with $1 \leq i, j \leq m-1$, we define

$$f_{ij}(x, y) := \begin{cases} +1 & \text{when } (x, y) \equiv (0, 0) \text{ or } (i, j) \pmod{m}; \\ -1 & \text{when } (x, y) \equiv (0, j) \text{ or } (i, 0) \pmod{m}; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Define $g_{i,j}$ similarly, but with $1 \leq i, j \leq n-1$ and "mod m" everywhere replaced by "mod n". First we show that if a linear combination $h := \sum \alpha_{i,j} f_{i,j} + \sum \beta_{i,j} g_{i,j}$ of the $f_{i,j}$ and $g_{i,j}$ contains only two distinct values, then either all of the $\alpha_{i,j}$ vanish or all of the $\beta_{i,j}$ do. It follows that each colour, considered in isolation, is either doubly periodic with period length m or doubly periodic with period length n. Finally, we check that different period lengths cannot mix.

On the other hand, if m and n are not relatively prime, then there exist infinitely many non-isomorphic valid colourings. Furthermore, when $\gcd(m, n) = 2$, there exist valid colourings which are not horizontally periodic; and, when $\gcd(m, n) \geq 3$, there exist valid colourings which are neither horizontally nor vertically periodic.

§2.3 USA TST 2023/6, proposed by Maxim Li

Available online at https://aops.com/community/p26896222.

Problem statement

Fix a function $f \colon \mathbb{N} \to \mathbb{N}$ and for any $m, n \in \mathbb{N}$ define

$$\Delta(m,n) = \underbrace{f(f(\dots f}_{f(n)\text{ times}}(m)\dots)) - \underbrace{f(f(\dots f}_{f(m)\text{ times}}(n)\dots)).$$

Suppose $\Delta(m,n) \neq 0$ for any distinct $m, n \in \mathbb{N}$. Show that Δ is unbounded, meaning that for any constant C there exist $m, n \in \mathbb{N}$ with $|\Delta(m,n)| > C$.

Suppose for the sake of contradiction that $|\Delta(m,n)| \le N$ for all m, n. Note that f is injective, as

$$f(m) = f(n) \implies \Delta(m,n) = 0 \implies m = n,$$

as desired.

Let G be the "arrow graph" of f, which is the directed graph with vertex set $\mathbb{N}$ and edges $n \to f(n)$. The first step in the solution is to classify the structure of G. Injectivity implies that G is a disjoint collection of chains (infinite and half-infinite) and cycles. We have the following sequence of claims that further refine the structure.

Claim — The graph G has no cycles.

Proof. Suppose for the sake of contradiction that $f^k(n) = n$ for some $k \ge 2$ and $n \in \mathbb{N}$. As m varies over $\mathbb{N}$, we have $|\Delta(m,n)| \le N$, so $f^{f(n)}(m)$ can only take on some finite set of values. In particular, this means that

$$f^{f(n)}(m_1) = f^{f(n)}(m_2)$$

for some $m_1 \neq m_2$, which contradicts injectivity. □

Claim — The graph G has at most $2N+1$ chains.

Proof. Suppose we have numbers $m_1, \dots, m_k$ in distinct chains. Select a positive integer $B > \max\{f(m_1), \dots, f(m_k)\}$. Now,

$$\left|\Delta\left(m_i, f^{B-f(m_i)}(1)\right)\right| \le N \implies \left|f^B(1) - f^{f^{B-f(m_i)+1}(1)}(m_i)\right| \le N.$$

Since the m_is are in different chains, we have that $f^{f^{B-f(m_i)+1}(1)}(m_i)$ are distinct for each i, which implies that $k \le 2N+1$, as desired. □

Claim — The graph G consists of exactly one half-infinite chain.

Proof. Fix some $c \in \mathbb{N}$. Call an element of $\mathbb{N}$ *bad* if it is not of the form $f^k(c)$ for some $k \ge 0$. It suffices to show that there are only finitely many bad numbers.

Since there are only finitely many chains, $f^{f(c)}(n)$ achieves all sufficiently large positive integers, say all positive integers at least M. Fix A and B such that $B > A \ge M$.

If $f^{f(c)}(n) \in [A, B]$, then $f^{f(n)}(c) \in [A-N, B+N]$, and distinct n generate distinct $f^{f(n)}(c)$ due to the structure of G. Therefore, we have at least $B-A+1$ good numbers in $[A-N, B+N]$, so there are at most $2N$ bad numbers in $[A-N, B+N]$.

Varying B, this shows there are at most $2N$ bad numbers at least $A-N$. $\square$

Let c be the starting point of the chain, so every integer is of the form $f^k(c)$, where $k \geq 0$. Define a function $g\colon \mathbb{Z}_{\geq 0} \to \mathbb{N}$ by

$$g(k) \coloneqq f^k(c).$$

Due to the structure of G, g is a bijection. Define

$$\delta(a,b) \coloneqq \Delta(f^a(c), f^b(c)) = g(g(b+1)+a) - g(g(a+1)+b),$$

so the conditions are equivalent to $|\delta(a,b)| \leq N$ for all $a, b \in \mathbb{Z}_{\geq 0}$ and $\delta(a,b) \neq 0$ for $a \neq b$, which is equivalent to $g(a+1) - a \neq g(b+1) - b$ for $a \neq b$. This tells us that $g(x) - x$ is injective for $x \geq 1$.

Lemma

For all M, there exists a nonnegative integer x with $g(x) \leq x - M$.

Proof. Assume for the sake of contradiction that $g(x) - x$ is bounded below. Fix some large positive K. Since $g(x) - x$ is injective, there exists B such that $g(x) - x \geq K$ for all $x \geq B$. Then $\min\{g(B+1), g(B+2), \dots\} \geq B + K$, while $\{g(0), \dots, g(B)\}$ only achieve $B+1$ values. Thus, at least $K-1$ values are not achieved by g, which is a contradiction. $\square$

Now pick B such that $g(B) + N \leq B$ and $g(B) > N$. Note that infinitely many such B exist, since we can take M to be arbitrarily small in the above lemma. Let

$$t = \max\{g^{-1}(g(B)-N), g^{-1}(g(B)-N+1), \dots, g^{-1}(g(B)+N)\}.$$

Note that $g(t) \leq g(B) + N \leq B$, so we have

$$|\delta(t-1, B-g(t))| = |g(B) - g(t-1+g(B+1-g(t)))| \leq N,$$

so

$$t - 1 + g(B+1-g(t)) \in \{g^{-1}(g(B)-N), g^{-1}(g(B)-N+1), \dots, g^{-1}(g(B)+N)\},$$

so by the maximality of t, we must have $g(B+1-g(t)) = 1$, so $B+1-g(t) = g^{-1}(1)$. We have $|g(t) - g(B)| \leq N$, so

$$|(B - g(B)) + 1 - g^{-1}(1)| \leq N.$$

This is true for infinitely many values of B, so infinitely many values of $B - g(B)$ (by injectivity of $g(x) - x$), which is a contradiction. This completes the proof.

USA IMO TST 2023 Statistics

United States of America — IMO Team Selection Tests

Evan Chen, Gopal Goel, and Andrew Gu

64th IMO 2023 Japan and 12th EGMO 2023 Slovenia

§1 Summary of scores for TST 2023

N	31	1st Q	14	Max	32
μ	17.55	Median	18	Top 3	29
σ	7.91	3rd Q	22	Top 12	22

§2 Problem statistics for TST 2023

	P1	**P2**	**P3**	**P4**	**P5**	**P6**
0	1	13	22	4	12	18
1	3	6	7	0	4	13
2	1	3	0	0	4	0
3	0	0	0	0	5	0
4	0	0	0	0	1	0
5	0	0	0	0	0	0
6	0	1	0	5	1	0
7	26	8	2	22	4	0
Avg	6.03	2.39	0.68	5.94	2.10	0.42
QM	6.43	3.79	1.84	6.37	3.18	0.65
#5+	**26**	**9**	**2**	**27**	**5**	**0**
%5+	%83.9	%29.0	%6.5	%87.1	%16.1	%0.0

§3 Rankings for TST 2023

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
42	0	0	0.00%	28	0	3	9.68%	14	3	24	77.42%
41	0	0	0.00%	27	0	3	9.68%	13	1	25	80.65%
40	0	0	0.00%	26	1	4	12.90%	12	0	25	80.65%
39	0	0	0.00%	25	1	5	16.13%	11	0	25	80.65%
38	0	0	0.00%	24	1	6	19.35%	10	0	25	80.65%
37	0	0	0.00%	23	1	7	22.58%	9	1	26	83.87%
36	0	0	0.00%	22	5	12	38.71%	8	0	26	83.87%
35	0	0	0.00%	21	0	12	38.71%	7	2	28	90.32%
34	0	0	0.00%	20	1	13	41.94%	6	0	28	90.32%
33	0	0	0.00%	19	1	14	45.16%	5	0	28	90.32%
32	1	1	3.23%	18	4	18	58.06%	4	0	28	90.32%
31	1	2	6.45%	17	2	20	64.52%	3	0	28	90.32%
30	0	2	6.45%	16	1	21	67.74%	2	2	30	96.77%
29	1	3	9.68%	15	0	21	67.74%	1	1	31	100.00%
								0	0	31	100.00%

§4 Histogram for TST 2023