COMPENDIUM RMM

Romanian Master of Mathematics Competition

Gerard Romo Garrido

Toomates Coolección vol. 103

Toomates Coolección

Los libros de **Toomates** son materiales digitales y gratuitos. Son digitales porque están pensados para ser consultados mediante un ordenador, tablet o móvil. Son gratuitos porque se ofrecen a la comunidad educativa sin coste alguno. Los libros de texto pueden ser digitales o en papel, gratuitos o en venta, y ninguna de estas opciones es necesariamente mejor o peor que las otras. Es más: Suele suceder que los mejores docentes son los que piden a sus alumnos la compra de un libro de texto en papel, esto es un hecho. Lo que no es aceptable, por inmoral y mezquino, es el modelo de las llamadas "licencias digitales", "licencias de uso" y en general cualquier forma de "pago por el acceso a los materiales didácticos", con las que algunas empresas pretenden cobrar a los estudiantes, una y otra vez, por acceder a los mismos contenidos (unos contenidos que, además, son de una bajísima calidad). Este modelo de negocio es miserable, pues impide el compartir un mismo material, incluso entre dos hermanos, pretende convertir a los estudiantes en un mercado cautivo, exige a los estudiantes y a las escuelas costosísimas líneas de Internet, pretende pervertir el conocimiento, que es algo social, público, convirtiéndolo en un producto de propiedad privada, accesible solo a aquellos que se lo puedan permitir, y solo de una manera encapsulada, fragmentada, impidiendo el derecho del alumno de poseer todo el libro, de acceder a todo el libro, de moverse libremente por todo el libro.
Nadie puede pretender ser neutral ante esto: Mirar para otro lado y aceptar el modelo de pago por acceso a los materiales es admitir un mundo más injusto, es participar en la denegación del acceso al conocimiento a aquellos que no disponen de medios económicos, y esto en un mundo en el que las modernas tecnologías actuales permiten, por primera vez en la historia de la Humanidad, poder compartir el conocimiento sin coste alguno, con algo tan simple como es un archivo "pdf". **El conocimiento no es una mercancía**.
El proyecto Toomates tiene como objetivo la promoción y difusión entre el profesorado y el colectivo de estudiantes de unos materiales didácticos libres, gratuitos y de calidad, que fuerce a las empresas comerciales a competir ofreciendo alternativas de pago atractivas aumentando la calidad de los materiales que ofrecen, (que son muy mediocres) y no mediante retorcidas técnicas comerciales.
 Todos los libros se ofrecen en dos versiones: En formato "**pdf**" para una cómoda lectura y en el formato "**doc**" de MSWord para permitir y facilitar su edición y generar versiones parcial o totalmente modificadas. **¡Libérate de la tiranía y mediocridad de los productos comerciales! Crea, utiliza y comparte tus propios materiales didácticos.**

Problem Solving (en español):
Geometría Axiomática Problemas de Geometría Vol. 1 Vol. 2 Vol. 3
Introducción a la Geometría Álgebra Teoría de números Vol. 1 Vol. 2 Combinatoria
Probabilidad Trigonometría Desigualdades Números complejos Calculus & Precalculus

Libros de texto para ESO y bachillerato (en español y en catalán):
Cálculo infinitesimal ESP CAT Álgebra Lineal ESP CAT Geometría Lineal ESP CAT
Números Complejos ESP CAT Combinatoria y probabilidad ESP CAT Estadística ESP CAT
Programación Lineal ESP CAT Álgebra ESP CAT Trigonometria ESP CAT
Geometria analítica ESP CAT Funciones ESP CAT Números (Preálgebra) ESP CAT
Proporcionalidad ESP CAT Medidas geométricas ESP CAT Mates amb Excel

PAU españolas:
Cataluña TEC Cataluña CCSS Valencia Galicia País Vasco Baleares

Reválidas internacionales:
Portugal Italia Francia Rumanía Hungría Polonia Pearson Edexcel Int A Level
China-Gaokao China-Zhongkao Corea-Suneung Israel-Bagrut Cambridge Int A Level
Cambridge IGCSE AQA GCSE International Baccalaureate (IB) Pearson Edexcel IGCSE

Evaluación diagnóstica y pruebas de acceso:
ACM6EP ACM4 CFGS PAP

Competiciones matemáticas:
Canguro: España 1 2 3 Cataluña 1 2 3 Francia Reino Unido Austria 1 2 USA
USA: Mathcounts AMC 8 10 12 AIME USAJMO USAMO TSTST TST ELMO Putnam HMMT
España: OME OMEFL OMEEX OMC OMEA OMEM OMA CDP OFEM
Europa: OMI Arquimede BMO BalkanMO JBMO OPM OMP OMJ ONM Baltic Way RMM
Internacional: IMO IGO SMT INMO CMO EGMO KMO KJMO AMC Australia
AHSME: Book 1 2 3 4 5 6 7 8 9

Otros materiales:
Pizzazz! (Book A B C D E Pre-Algebra Algebra) , REOIM , Llibre3r , Excalibur

¡Genera tus propias versiones de este documento! Siempre que es posible se ofrecen las versiones editables "MS Word" de todos los materiales para facilitar su edición.
¡Ayuda a mejorar! Envía cualquier duda, observación, comentario o sugerencia a **toomates@gmail.com**
¡No utilices una versión anticuada! Todos estos libros se revisan y amplían constantemente. Descarga totalmente gratis la última versión de estos documentos en los correspondientes enlaces superiores, en los que siempre encontrarás la versión más actualizada.
Consulta el **catálogo de libros** completo en http://www.toomates.net
¿Problemas para descargar algún documento? Descarga toda la biblioteca Toomates Aquí MEGA
Visita mi **Canal de Youtube**: https://www.youtube.com/c/GerardRomo
Visita mi **blog:** https://toomatesbloc.blogspot.com/

Versión de este documento: 31/05/2026

Presentación.

El Romanian Master of Mathematics (RMM) es una de las competencias internacionales de matemáticas de nivel preuniversitario más prestigiosas y exigentes del mundo.

Celebrada anualmente en Bucarest (Rumania), reúne a jóvenes talentos matemáticos para resolver problemas de alta dificultad y fomentar el intercambio cultural

El RMM es un escenario de élite diseñado para estudiantes de secundaria excepcionales. Su objetivo principal no es solo poner a prueba el razonamiento lógico-matemático mediante problemas de altísima complejidad, sino también fortalecer las relaciones interculturales entre los futuros líderes científicos del mundo.

Cada país invitado presenta un equipo de hasta cuatro concursantes oficiales, acompañados por un líder y un sublíder.

El certamen se divide en dos días, con tres problemas por jornada.

La puntuación oficial del equipo se calcula sumando las tres notas individuales más altas.

El RMM es un esfuerzo conjunto de instituciones académicas y gubernamentales de primer nivel. Está organizado por el Colegiul Național de Informatică "Tudor Vianu" en cooperación con La Sociedad Matemática Rumana, La Universidad Politécnica de Bucarest, El Ministerio de Educación de Rumania y el Ayuntamiento del Distrito 1 de Bucarest.

Siguiendo los estándares de las olimpiadas internacionales, el comité otorga medallas de oro, plata y bronce, además de menciones honoríficas basadas en los puntajes individuales. El país ganador recibe el trofeo oficial, el cual guarda grabado su nombre hasta la siguiente edición.

Fuentes de los documentos.

https://rmms.lbi.ro/rmm2026/index.php?id=home
https://matematickitalent.mk/

Índice.

Romanian Masters In Mathematics 2008

Bucharest

---

1. Let ABC be an equilateral triangle and P in its interior. The distances from P to the triangle's sides are denoted by a^2, b^2, c^2 respectively, where $a, b, c > 0$. Find the locus of the points P for which a, b, c can be the sides of a non-degenerate triangle.

2. Prove that every bijective function $f : \mathbb{Z} \to \mathbb{Z}$ can be written in the way $f = u + v$ where $u, v : \mathbb{Z} \to \mathbb{Z}$ are bijective functions.

3. Let $a > 1$ be a positive integer. Prove that every non-zero positive integer N has a multiple in the sequence $(a_n)_{n\geq 1}$, $a_n = \left\lfloor \frac{a^n}{n} \right\rfloor$.

4. Consider a square of sidelength n and $(n+1)^2$ interior points. Prove that we can choose 3 of these points so that they determine a triangle (eventually degenerated) of area at most $\frac{1}{2}$.

Romanian Master in Mathematics
First Edition, 2008, Bucharest - SOLUTIONS

Problem 1. Let ABC be an equilateral triangle. P is a variable point internal to the triangle and its perpendicular distances to the sides are denoted by a^2, b^2 and c^2 for positive real numbers a, b and c. Find the locus of points P so that a, b and c can be the sides of a non-degenerate triangle.

[U.K.]

Solution. The required locus is the interior of the inscribed circle of triangle ABC.

To prove this, embed the equilateral triangle in the Cartezian space $Oxyz$, as the set in the plane $x + y + z = 1$ described by $x, y, z \geq 1$. Let the feet of the perpendiculars from P to BC and CA be D and E respectively, and let the feet of the perpendiculars from P to the planes OBC and OCA be Q and R respectively. Then triangles PQD and PRE are similar, so $PQ : PR = PD : PE$; i.e. $x : y = a^2 : b^2$, where (x, y, z) are coordinates of P. In the same way we get $y : z = b^2 : c^2$, so we have $(a^2 : b^2 : c^2) = (x : y : z)$.

Now if a, b and c are the sides of a triangle, the Heron's formula states that the square of the area of that triangle is

$$\frac{1}{16}(a + b + c)(-a + b + c)(a - b + c)(a + b - c).$$

So this quantity is positive. The reverse is also true.

Multiplying the expression out, this means that a, b and c are the sides of a triangle if and only if

$$2\sum b^2c^2 - \sum a^4 > 0.$$

Since a^2, b^2, c^2 are proportional to x, y, z, it follows that a, b and c are the sides of a triangle if and only if

$$2(x^2 + y^2 + z^2) < (x + y + z)^2 = 1.$$

So the required locus of points is the intersection of the solid sphere $x^2 + y^2 + z^2 < 1/2$ with the plane $x + y + z = 1$; that is the interior of the inscribed circle of the equilateral triangle.

Remark. Using a^2, b^2, c^2 as baricentric coordinates for P, in an equilateral triangle of circumradius 1, one can calculate the distance from P to the incenter I, reducing thus the problem to an algebraic one. In fact one can see the similarity to the above solution.

Problem 2. Prove that any bijective function $f : \mathbb{Z} \to \mathbb{Z}$ can be written as $f = u + v$ where $u, v : \mathbb{Z} \to \mathbb{Z}$ are bijective functions.

[Romania]

Solution. To find u, v such that $f = u + v$ it is enough to consider the case $f =$ identity on $\mathbb{Z}$. For that it suffices to write the above relation as $\mathrm{id}_{\mathbb{Z}} = u \circ f^{-1} + v \circ f^{-1}$. Consider the following well-ordering of the nonzero integers: $\mathbb{Z}^* = \{1, -1, 2, -2, \ldots, n, -n, \ldots\}$.

Build the following table

Step	A	#	B
1	1	$+1$	2
2	-1	-2	-3
3	-2	-3	-5
4	3	$+4$	7
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	a_k	$\mathrm{sign}(a_k) \cdot k$	$b_k = a_k + \#(k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The inductive rule in completing the table is as follows: at step 1 write 1, the first in the ordering of $\mathbb{Z}^*$, in column A, in column # put the number of the step, that is 1, with the sign from A, and in column B the sum from A and #. Suppose now that row of step i has been completed. Write on row $i+1$ in column A the first integer in the ordering of $\mathbb{Z}^*$ that has not yet been used in A nor B, in column # the number $i + 1$ with the sign given by that of the number just written in A, and in B the sum of A and #.

It is easy to see that in this manner we get an infinite array where $A \cup B = \mathbb{Z}^*$ and $A \cap B = \emptyset$, while elements in A and B do not repeat.

Define now $u(0) = v(0) = 0$ and for $x \in \mathbb{Z}$

- for $x = a_i \in A$ (meaning that x is in column A and row i), take $u(x) = -\#(i), v(x) = b_i$;
- for $x = b_j \in B$, take $u(x) = \#(j), v(x) = a_j$.

Obviously u and v are both bijections from $\mathbb{Z}$ to $\mathbb{Z}$ and $\mathrm{id}_{\mathbb{Z}} = u + v$. ■

Problem 3. Given positive integer $a > 1$, prove that any positive integer N has a multiple in the sequence

$$(a_n)_{n\geq 1}, \quad a_n = \left\lfloor \frac{a^n}{n} \right\rfloor.$$

[Romania]

Solution. In what follows, all literals will represent non-negative integers. The solution makes use of specific values for n, carefully chosen to facilitate the computation of the *floor* function.

Clearly, there exist $e \geq 0$, $q \geq 1$ and

$$M = a^{a^e - e} q, \quad \gcd(q, a) = 1,$$

such that M is a multiple of N.

Let us consider values $n = a^e p$, with p prime, $p > M$. Then, by Fermat's little Theorem ($p > M \geq a$, so $\gcd(a, p) = 1$)

$$a^{a^e(p-1)} - 1 = (a^{p-1})^{a^e} - 1 \equiv 0 \pmod{p}, \text{ so } a^n = a^{a^e} kp + a^{a^e},$$

therefore, as $n = a^e p > a^e M \geq a^{a^e}$

$$a_n = \left\lfloor \frac{a^n}{n} \right\rfloor = a^{a^e - e} k.$$

On the other hand, $kp = a^{a^e(p-1)} - 1$. Assuming $p - 1 = m\varphi(q)$ we have $a^{\varphi(q)} \equiv 1 \pmod{q}$[1] therefore $kp \equiv 0 \pmod{q}$, so q divides kp. But $p > M > q$, so $\gcd(q, p) = 1$, hence q divides k, so M (and *a fortiori* N) divides a_n.

We are left to prove that we can find such $p - 1 = m\varphi(q)$, that is, $p > M$ must belong to the arithmetic sequence of first-term 1 and ratio $\varphi(q)$. **The existence of such p is guaranteed by Dirichlet's Theorem**[2] and that should suffice in an international math competition. ■

Remarks. We will however, for self-containment, present a proof for this particular case of Dirichlet's Theorem [3]

An arithmetical sequence of first-term 1 and ratio r contains infinitely many primes (assume $r > 2$, as $r = 1$ or $r = 2$ makes it trivially true).

We will denote by d, $1 \leq d < r$, any (proper) divisor of r. Let us consider the polynomial $X^r - 1 \in \mathbb{Z}(X)$, factored in irreducible polynomials. Its roots (the *r-roots of unity*) are

$$\cos\frac{2k\pi}{r} + i \sin\frac{2k\pi}{r}, \text{ with } 1 \leq k \leq r,$$

[1] φ is the Euler *totient* function, and $\gcd(q, a) = 1$.

[2] Dirichlet's Theorem asserts the existence of infinitely many primes in an arithmetic sequence of co-prime first-term and ratio.

[3] This effort is a personal improvement on a proof by A. Rotkiewicz.

and, for $k = 1$, the main *primitive* r-root of unity ζ cannot be the root of any polynomial $X^d - 1$. Therefore ζ must be root of an irreducible factor $f(X)$ for $X^r - 1$, which cannot be a factor for any $X^d - 1$.[4] Now

$$f(X) \text{ divides } \frac{X^r - 1}{X^d - 1} \text{ for all } d, \text{ and } f(X) = \prod_{i=1}^{\deg f} (X - z_i),$$

with z_i among the r-roots of unity, so $|z_i| = 1$. Therefore, for any $n > 2$

$$|f(n)| = \prod_{i=1}^{\deg f} |n - z_i| \geq \prod_{i=1}^{\deg f} |n - |z_i|| = (n-1)^{\deg f} > 1.$$

Assume now there are only finitely many such primes q, and take $n = r \prod q$.[5] As $|f(n)| > 1$, there exists p prime, dividing $f(n)$, and therefore dividing $\frac{n^r-1}{n^d-1}$ for all d. We then cannot have p dividing $n^d - 1$ for any d, because

$$X^{\frac{r}{d}} - 1 = (X-1)P(X), \; P(X) = (X-1)Q(X) + R, \; R = P(1) = \frac{r}{d},$$

so $\frac{n^r-1}{n^d-1} = P(n^d) = (n^d - 1)Q(n^d) + \frac{r}{d}$, while clearly $n^d - 1$ and $\frac{r}{d}$ are co-prime (as r divides n), therefore p cannot divide $\frac{r}{d}$.

This shows that $n^r \equiv 1 \pmod{p}$ and $n^d \not\equiv 1 \pmod{p}$ for any d, so $r = \mathrm{ord}_p(n)$. But $n^{p-1} \equiv 1 \pmod{p}$ (by Fermat's little Theorem), so we must have r dividing $p - 1$, that is, p belongs to the stated arithmetical sequence. However, $p \neq q$ for any q considered in the above, as $\gcd(p, n) = 1$, and thus we have found yet another such prime, contradiction. □

[4]In fact (not needed here), all primitive roots, for $\gcd(k, r) = 1$, are the roots of a **same** irreducible factor $\Phi_r(X)$, of degree $\varphi(r)$, which is the *cyclotomic polynomial* of order r. Then $X^r - 1 = \prod_{d|r} \Phi_d(X)$, the product of the (irreducible) cyclotomic polynomials.

[5]By definition $\prod q := 1$ if no such primes were to be selected.

Problem 4. Prove that from among any $(n+1)^2$ points inside a square of sidelength positive integer n, one can pick three, such that the triangle determined by them has area no more than $\frac{1}{2}$.

[Romania]

Solution. Although the topic of the problem may somehow appear familiar, the solution involves a novel and ingenious mix of ideas, centered around estimating areas of triangles using simple convexity inequalities.

Denote by $A = n^2$ the area of the square, by $P = 4n$ the perimeter of the square, and by $N = (n+1)^2$ the number of points. The convex hull of the set of N points will be a convex k-gon (contained in the given square), $3 \leq k \leq N$, with $N-k$ points in its interior (if any three points are collinear, they will determine a triangle of area 0, thus rendering the result trivially).

We will make use of the following folklore result

Any triangulation of a (convex) k-gon, using $m = N - k$ interior points, is made of $t = (k - 2) + 2m = 2(N - 1) - k$ triangles.[6]

As the area of the convex hull k-gon is at most A, it follows, using an *averaging* argument, that there will exist a triangle Δ_f of area at most

$$\frac{A}{t} = \frac{A}{2(N-1)-k} = f(k).$$

On the other hand, as the perimeter of the convex hull k-gon is at most P, one can find a pair of consecutive sides, be them $\mathbf{a}$, $\mathbf{b}$, of lengths a, b, such that $\frac{a+b}{2} \leq \frac{P}{k}$ (this also is an *averaging* argument). Now, the area of the triangle Δ_g determined by $\mathbf{a}$, $\mathbf{b}$, is

$$\frac{1}{2}ab\sin\angle(\mathbf{a},\mathbf{b}) \leq \frac{1}{2}\left(\frac{a+b}{2}\right)^2 \leq \frac{P^2}{2k^2} = g(k).$$

Clearly, the bounds for the areas of triangles Δ_f, Δ_g depend on k, but $f(k)$ is increasing, while $g(k)$ is decreasing, therefore the worst case occurs for the value calculated in k_0 where the graphs of f and g meet

$$\frac{A}{2(N-1)-k_0} = \frac{P^2}{2k_0^2}, \text{ so } k_0^2 = 16(n+1)^2 - 16 - 8k_0, \text{ hence } k_0 = 4n.$$

Both formulae f and g, calculated in k_0, yield the value $\frac{1}{2}$, as required. ∎

Remarks. One can improve on the bound given by $g(k)$; in fact it may be proven that a triangle Δ_g of area at most $\frac{P^2}{2k^2}\sin\frac{2\pi}{k}$ can be found. However, the minimum value offered by $f(k)$ is greater than $\frac{1}{2}(\frac{n}{n+1})^2$, which converges

[6]The total sum of angles for the t triangles is $t\pi$; but the vertices contribute $(k-2)\pi$, while the interior points contribute $2m\pi$, therefore $t = (k-2) + 2m$.

to $\frac{1}{2}$ when n grows large, thus thwarting any attempt to improve on the $\frac{1}{2}$ bound. The issue is to improve on the bound given by $f(k)$, but it is difficult to find efficient ways to bound from above the size of a least-area triangle for small k.

The author is far from claiming the result is tight (for large n), although better estimates appear elusive; however the naïve attempt to use the pigeonhole principle in its simplest form (partition the side-n square into n^2 unit squares; then for any $2n^2+1$ points inside the square there will exist three within a unit square, thus determining a triangle of area at most $\frac{1}{2}$), necessitates almost twice as many points as those afforded in the problem (except for $n = 2$, when $2 \cdot 2^2 + 1 = (2+1)^2$). On the other hand, for $n = 1$, the result is best possible!

Moreover, using the $\frac{P^2}{2k^2} \sin \frac{2\pi}{k}$ bound for Δ_g, one can prove for $n = 2$ that there exists a triangle of area at most $\frac{4}{9}$ (the critical point k_0 is moving from value 8 to 7, when the correct answer is given by $f(7) = \frac{4}{9}$), a better bound than anything found in the literature!

The 2nd Romanian Master of Mathematics Competition – Solutions
Bucharest, Saturday, February 28, 2009

Problem 1. For any positive integers $a_1, \dots, a_k$, let $n = \sum_{i=1}^{k} a_i$, and let $\binom{n}{a_1, \dots, a_k}$ be the multinomial coefficient $\frac{n!}{\prod_{i=1}^{k}(a_i!)}$. Let $d = \gcd(a_1, \dots, a_k)$ denote the greatest common divisor of $a_1, \dots, a_k$.

Prove that $\frac{d}{n}\binom{n}{a_1, \dots, a_k}$ is an integer.

Romania, Dan Schwarz[1]

Solution. The key idea is the fact that the greatest common divisor is a linear combination with integer coefficients of the numbers involved[2], i.e. there exist $u_i \in \mathbb{Z}$ such that $d = \sum_{i=1}^{k} u_i a_i$. But

$$\binom{n}{a_1, \dots, a_k} = \frac{n}{a_i}\binom{n-1}{a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_k},$$

so

$$\frac{d}{n}\binom{n}{a_1, \dots, a_k} = \sum_{i=1}^{k} u_i \binom{n-1}{a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_k},$$

which clearly is an integer, since multinomial coefficients are known (and easy to prove) to be integer. ■

Problem 2. A set S of points in space satisfies the property that all pairwise distances between points in S are distinct. Given that all points in S have integer coordinates (x, y, z), where $1 \le x, y, z \le n$, show that the number of points in S is less than $\min\left((n+2)\sqrt{n/3},\, n\sqrt{6}\right)$.

Romania, Dan Schwarz[3]

Solution. The critical idea is to estimate the total number possible T of distinct distances realized by pairs of points (x, y, z), of integer coordinates $1 \le x, y, z \le n$. However, any such distance is also realized by a pair anchored at $(1, 1, 1)$, from symmetry considerations.

But the number of distinct distances to points with no coordinates x, y, z equal is at most $\binom{n}{3} = \frac{1}{6}n(n-1)(n-2)$; the number of distinct distances to points with two of the three coordinates x, y, z equal is at most $2\binom{n}{2} = n(n-1)$; while the number of distinct distances to points with all three coordinates x, y, z equal is $n - 1$, hence

$$T \le \frac{1}{6}n(n-1)(n-2) + n(n-1) + (n-1) < \frac{1}{6}(n^3 + 3n^2 + 2n).$$

On the other hand, the total number of distinct distances between the N points in S needs be $\binom{N}{2} = \frac{1}{2}N(N-1) \le T$, yielding

$$(2N-1)^2 < \frac{1}{3}(4n^3 + 12n^2 + 8n) + 1 \le \frac{1}{3}(2n\sqrt{n} + 3\sqrt{n})^2,$$

hence $N < \frac{1}{2}\left((2n+3)\sqrt{n/3} + 1\right) \le (n+2)\sqrt{n/3}$ for $n \ge 3$. One can easily check that the inequality is true for $n = 2$ also, since then[4] $T = 3$.

On the other hand, since the squares of the distances can only take the integer values between 1 and the trivial upper bound $3(n-1)^2$ (for the diagonal of the cube), it follows that $T \le 3(n-1)^2$, yielding $N < n\sqrt{6}$. ■

Problem 3. Given four points A_1, A_2, A_3, A_4 in the plane, no three collinear, such that

$$A_1A_2 \cdot A_3A_4 = A_1A_3 \cdot A_2A_4 = A_1A_4 \cdot A_2A_3,$$

let us denote by O_i the circumcenter of $\Delta A_jA_kA_\ell$, with $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$.

Assuming $A_i \neq O_i$ for all indices i, prove that the four lines A_iO_i are concurrent or parallel.

Bulgaria, Nikolai Ivanov Beluhov

Solution. (D. Schwarz) The given triple equality being invariated by any permutation in $\mathscr{S}_4$, it is enough to prove that the lines A_iO_i for $2 \le i \le 4$ are concurrent or parallel. The relations can then be written

$$\frac{A_1A_2}{A_1A_3} = \frac{A_4A_2}{A_4A_3}, \quad \frac{A_1A_3}{A_1A_4} = \frac{A_2A_3}{A_2A_4}, \quad \frac{A_1A_4}{A_1A_2} = \frac{A_3A_4}{A_3A_2}.$$

Consider the Apollonius circles Γ_k of centers $\omega_k \in A_iA_j$, for $\{i, j, k\} = \{2, 3, 4\}$, determined by the point A_1, which therefore lies on all three, while the points A_k lie on Γ_k. Moreover, the points ω_k are collinear, since the point A'_k which is the other meeting point (than A_1, if any) of Γ_i and Γ_j fulfills

$$\frac{A'_kA_j}{A'_kA_k} = \frac{A_iA_j}{A_iA_k} \text{ and } \frac{A'_kA_i}{A'_kA_k} = \frac{A_jA_i}{A_jA_k}, \text{ thus } \frac{A'_kA_i}{A'_kA_j} = \frac{A_kA_i}{A_kA_j},$$

therefore A'_k also lies on Γ_k, hence all three circles Γ_k share the same meeting point(s), thus their centers are collinear.

Now, the circumcenters O_i and O_j, as well as the point ω_k, lie on the perpendicular bisector of the segment A_1A_k, for $\{i, j, k\} = \{2, 3, 4\}$. It follows that the pairs of lines A_iA_j, O_iO_j meet at the collinear points ω_k. Desargues' theorem for the perspective triangles $\Delta A_iA_jA_k$ and $\Delta O_iO_jO_k$ yields the claim. ■

Alternate Solution. The author's original solution makes use of inversions of poles A_i to reach the same conclusion via Desargues, in a dual-by-inversion to the solution above manner, with a lot more details than concepts. We feel that making use of the well-known properties of the Apollonius circles renders the idea in a more striking way. ■

Remark. There exists a particular (degenerate) case, when the points are the vertices of a kite of $\frac{\pi}{6}$ equal angles, hence one of the associated ratios is 1, so a corresponding Apollonius circle degenerates to the perpendicular bisector. This (together with the use of Desargues) shows the deep projective nature of the problem, better handled through projective methods.

Also, there is no converse implication, since the case of concyclic points trivially warrants the conclusion, without fulfilling the stated condition (as in conflict with Ptolemy's relation).

Problem 4. For a finite set X of positive integers, let

$$\Sigma(X) = \sum_{x\in X} \arctan \frac{1}{x}.$$

Given a finite set S of positive integers for which $\Sigma(S) < \frac{\pi}{2}$, show that there exists at least one finite set T of positive integers for which $S \subset T$ and $\Sigma(T) = \frac{\pi}{2}$.

United Kingdom, Kevin Buzzard

Solution. (D. Schwarz) We will step-by-step augment the set S with positive integers t_n, by taking each time t_n as the least positive integer larger than $\max(S)$, and not already used, such that $\Sigma(S \cup \{t_1, t_2, \dots, t_n\})$ remains at most $\frac{\pi}{2}$ (this is possible since $\arctan \frac{1}{t} \to 0$ when $t \to \infty$). If at some point we get exactly $\frac{\pi}{2}$ we are through, since we have augmented S to a set T as required, so assume the process continues indefinitely. Clearly the sequence $(t_n)_{n\geq 1}$ is built (strictly) increasing, so for all $n \geq 1$ we have $t_{n+1} > t_n > \max(S)$.

We will make some useful notations. Take $S_0 = S$, $S_{n+1} = S_n \cup \{t_{n+1}\}$, for $n \in \mathbb{N}$. Also take $x_n = \tan\left(\frac{\pi}{2} - \Sigma(S_n)\right)$. Using the well-known formula $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ one can easily prove by simple induction that a lesser than $\frac{\pi}{2}$ sum of arcs of rational tangents is as well an arc of rational tangent, therefore $x_n = \frac{p_n}{q_n}$, with $p_n, q_n \in \mathbb{N}^*$, $(p_n, q_n) = 1$. Since arctan is increasing, we need take $t_{n+1} \geq \left\lceil \frac{1}{x_n} \right\rceil$ in order that we may augment S_n with t_{n+1} to obtain S_{n+1}.

Assume that for all $n \geq 1$ we have $\frac{1}{x_n} \leq t_n$. Since we need both $t_{n+1} \geq \left\lceil \frac{1}{x_n} \right\rceil$ and $t_{n+1} > t_n \geq \frac{1}{x_n}$, it follows that $t_{n+1} = t_n + 1$ (the least available value), so $t_{k+1} = t_1 + k$ for all $k \geq 0$. But then $\dfrac{\pi}{2} > \Sigma(\{t_1, t_2, \dots, t_n\}) = \sum_{k=0}^{n-1} \arctan \dfrac{1}{t_1+k} > \dfrac{1}{2}\sum_{k=0}^{n-1} \dfrac{1}{t_1+k} \to \infty$ when $n \to \infty$, absurd (see *Lemma*).

Therefore there exists some $N \geq 1$ for which $\frac{1}{x_N} > t_N$, so $\left\lceil \frac{1}{x_N} \right\rceil$ is available for t_{N+1}. Moreover, for any $n \geq N$ with $t_{n+1} = \left\lceil \frac{1}{x_n} \right\rceil$, we have $x_{n+1} = \dfrac{x_n - \frac{1}{t_{n+1}}}{1 + x_n \frac{1}{t_{n+1}}} = \dfrac{x_n t_{n+1} - 1}{t_{n+1} + x_n} < \dfrac{x_n}{t_{n+1} + x_n} < \dfrac{1}{t_{n+1}}$, since $t_{n+1} = \left\lceil \frac{1}{x_n} \right\rceil$ implies $x_n t_{n+1} - 1 < x_n$; and so we can take $t_{n+1} = \left\lceil \frac{1}{x_n} \right\rceil$ indefinitely for $n \geq N$. Now we use the fact that $x_n = \frac{p_n}{q_n}$.

Then $\dfrac{p_{n+1}}{q_{n+1}} = \dfrac{\frac{p_n}{q_n} - \frac{1}{t_{n+1}}}{1 + \frac{p_n}{q_n}\frac{1}{t_{n+1}}} = \dfrac{p_n t_{n+1} - q_n}{q_n t_{n+1} + p_n}$, hence $p_{n+1} \leq p_n t_{n+1} - q_n < p_n$, since $t_{n+1} = \left\lceil \dfrac{q_n}{p_n} \right\rceil$, and so $t_{n+1} < \dfrac{q_n}{p_n} + 1$.

Therefore the sequence $(p_n)_{n\geq 1}$ of the numerators of x_n eventually becomes (strictly) decreasing, absurd for any sequence of positive integers. ■

Lemma. For $x \in (0, \frac{\pi}{2})$ one has $\arctan x > \frac{x}{2}$.

Proof. We start by proving that under given condition one has $\sin x > \tan \frac{x}{2}$, in turn equivalent to $2\sin\dfrac{x}{2}\cos\dfrac{x}{2} > \dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$, $2\cos^2\frac{x}{2} - 1 > 0$, and finally $\cos x > 0$, patently true.

Now, arctan is increasing, hence applied to the above, together with the well-known inequality $x > \sin x$, true for all $x > 0$, yields $\arctan x > \arctan\sin x > \arctan\tan\frac{x}{2} = \frac{x}{2}$. □

As a corollary, $\arctan\frac{1}{n} > \frac{1}{2n}$, for all positive integers n, inequality used to yield the divergence of the series $\sum_{n\geq 1} \arctan \dfrac{1}{n}$ in the above solution.

Remark. The above solution shows that it is irrelevant that we start with the arc $\dfrac{\pi}{2} - \sum_{s\in S} \arctan \dfrac{1}{s}$; in fact we may state the problem like this

> *Prove that for any arc $\alpha \in (0, \frac{\pi}{2})$ of some rational tangent $\tau = \tan\alpha$, and any finite set S of distinct positive integers, there exists some finite set T of distinct positive integers such that $T \cap S = \emptyset$ and*
>
> $$\sum_{t\in T} \arctan \frac{1}{t} = \alpha.$$

The problem is strongly reminiscent of a strengthened form of the famous *Egyptian fraction*[5] theorem

> *Prove that for any rational number $r \in (0,1)$, and any finite set S of distinct positive integers, there exists a finite set T of distinct positive integers such that $T \cap S = \emptyset$ and*
>
> $$\sum_{t\in T} \frac{1}{t} = r.$$

All the ingredients are there: the greedy algorithm, going beyond the largest element of S, using the divergence of the series $\sum_{n\geq 1} \dfrac{1}{n}$, and the (Fermat) infinite descent method of a (strictly) decreasing sequence of positive integers.

In fact, it is enough to consider a (strictly) increasing function $f : \mathbb{Q}_+ \to \mathbb{R}_+$ with the properties that there exists a function $\varphi : \mathbb{Q}_+ \times \mathbb{Q}_+ \to \mathbb{Q}_+$ such that $f(r) - f(s) = f(\varphi(r,s))$ for any $0 \leq s < r$ in $\mathbb{Q}$, $\lim_{x\to 0} f(x) = 0$, and $\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\dfrac{1}{k}\right) = \infty$.

Moreover, we need that $\varphi(r,s)$ has not larger numerator than $r - s$, and not lesser denominator. Then the Egyptian fraction method extends perfectly. Or $f(x) = \arctan x$ and $\varphi(x,y) = \frac{x-y}{1+xy}$ conform to this model. **END**

---

[1] Based on a property of quasi-Catalan numbers of J. Conway, see [GUY, R.K., *Unsolved Problems in Number Theory*].

[2] Easily proven by induction from the classical Bézout's relation $\gcd(M,N) = uM + vN$ for some integers u, v.

[3] A 3-dimensional extrapolation of a plane lattice points case study of P. Erdös and R.K. Guy.

[4] An example of $N = 3$ points for $n = 2$ is $(1,1,1)$, $(2,2,1)$, $(2,2,2)$; and of $N = 4$ points for $n = 3$ is $(1,1,1)$, $(1,1,2)$, $(2,2,1)$, $(2,3,3)$.

[5] An Egyptian fraction is written as a finite sum of fractions with all unit numerators and all distinct denominators. Such fractions were used by ancient Egyptians, as apparent in the Rhind Papyrus, but their use is discontinued today.

THE 3RD ROMANIAN MASTER OF MATHEMATICS COMPETITION

DAY 1: FRIDAY, FEBRUARY 26, 2010, BUCHAREST

Language: English

Problem 1. For a finite non-empty set of primes P, let $m(P)$ be the largest possible number of consecutive positive integers, each of which is divisible by at least one member of P.

(i) Show that $|P| \leq m(P)$, with equality if and only if $\min(P) > |P|$;

(ii) Show that $m(P) < (|P|+1)(2^{|P|}-1)$.

(The number $|P|$ is the size of the set P.)

Problem 2. For each positive integer n, find the largest real number C_n with the following property. Given any n real-valued functions $f_1(x), f_2(x), \ldots, f_n(x)$ defined on the closed interval $0 \leq x \leq 1$, one can find numbers $x_1, x_2, \ldots, x_n$, such that $0 \leq x_i \leq 1$, satisfying

$$\left| f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) - x_1 x_2 \cdots x_n \right| \geq C_n.$$

Problem 3. Let $A_1A_2A_3A_4$ be a convex quadrilateral with no pair of parallel sides. For each $i = 1, 2, 3, 4$, define ω_i to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i$, A_iA_{i+1} and $A_{i+1}A_{i+2}$ (indices are considered modulo 4, so $A_0 = A_4$, $A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with the side A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.

Each of the three problems is worth 7 points.
Time allowed: $4\frac{1}{2}$ hours.

THE 3RD ROMANIAN MASTER OF MATHEMATICS COMPETITION

DAY 2: SATURDAY, FEBRUARY 27, 2010, BUCHAREST

Language: English

Problem 4. Determine whether there exist a polynomial $f(x_1, x_2)$ in two variables, with integer coefficients, and two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in the plane, satisfying all the following conditions:

(i) A is an integer point (i.e., a_1 and a_2 are integers);
(ii) $|a_1 - b_1| + |a_2 - b_2| = 2010$;
(iii) $f(n_1, n_2) > f(a_1, a_2)$, for all integer points (n_1, n_2) in the plane other than A;
(iv) $f(x_1, x_2) > f(b_1, b_2)$, for all points (x_1, x_2) in the plane other than B.

Problem 5. Let n be a given positive integer. Say that a set K of points with integer coordinates in the plane is *connected* if for every pair of points $R, S \in K$, there exist a positive integer ℓ and a sequence $R = T_0, T_1, \ldots, T_\ell = S$ of points in K, where each T_i is distance 1 away from T_{i+1}. For such a set K, we define the set of vectors

$$\Delta(K) = \{\overrightarrow{RS} \mid R, S \in K\}.$$

What is the maximum value of $|\Delta(K)|$ over all connected sets K of $2n+1$ points with integer coordinates in the plane?

Problem 6. Given a polynomial $f(x)$ with rational coefficients, of degree $d \geq 2$, we define the sequence of sets $f^0(\mathbb{Q}), f^1(\mathbb{Q}), \ldots$ by $f^0(\mathbb{Q}) = \mathbb{Q}$ and $f^{n+1}(\mathbb{Q}) = f(f^n(\mathbb{Q}))$ for $n \geq 0$. (Given a set S, we write $f(S)$ for the set $\{f(x) \mid x \in S\}$.)

Let $f^\omega(\mathbb{Q}) = \bigcap_{n=0}^{\infty} f^n(\mathbb{Q})$ be the set of numbers that are in all of the sets $f^n(\mathbb{Q})$. Prove that $f^\omega(\mathbb{Q})$ is a finite set.

Each of the three problems is worth 7 points.
Time allowed: $4\frac{1}{2}$ hours.

The 3rd Romanian Master of Mathematics Competition – Solutions
Day 1: Friday, February 26, 2010, Bucharest

Problem 1. For a finite non-empty set of primes P, let $m(P)$ be the largest possible number of consecutive positive integers, each of which is divisible by at least one member of P. (In the sequel, the number $|P|$ is the size of the set P.)

(i) Show that $|P| \le m(P)$, with equality if and only if $\min(P) > |P|$;

(ii) Show that $m(P) < (|P|+1)(2^{|P|}-1)$.

Romania, Dan Schwarz

Solution. In the sequel we will consider P being made of the primes $1 < p_1 < p_2 < \cdots < p_k$, with $k = |P| \ge 1$.

(i) By the Chinese Remainder Theorem there will exist some $a \in \mathbb{N}$ such that $a \equiv -i \pmod{p_i}$, hence $p_i \mid a+i$. Then the set $\{a+i\ ;\ i = 1,2,\dots,k\}$ of k consecutive integers has the desired property, hence $m(P) \ge k$. When $\min P > |P|$, within any set of $|P|+1$ consecutive integers at most one is divisible by any $p \in P$, hence by the Pigeonhole Principle there will be one not divisible by any of the primes in P. On the other hand, when $\min P \le |P|$, we will make again use of the Chinese Remainder Theorem, so there will exist some $a \in \mathbb{N}$ such that $a \equiv -r_i \pmod{p_i}$, hence $p_i \mid a + r_i$, where $\{r_i\ ;\ i = 1,2,\dots,k\} = \{1,2,\dots,k\}$ and the extra requirement that $r_1 = k+1-p_1$. It follows that the set $\{a+i\ ;\ i = 1,2,\dots,k,k+1\}$ of $k+1$ consecutive integers has the desired property, hence $m(P) \ge k+1 > |P|$.

(ii) Now, let a set made of m consecutive integers have the desired property. For any $\emptyset \ne I \subseteq \{1,2,\dots,k\}$, the number $N(I)$ of its elements which are divisible by $\prod_{i\in I} p_i$ will satisfy the inequality

$$\frac{m}{\prod_{i\in I} p_i} - 1 < \left\lfloor \frac{m}{\prod_{i\in I} p_i} \right\rfloor \le N(I) \le \left\lceil \frac{m}{\prod_{i\in I} p_i} \right\rceil < \frac{m}{\prod_{i\in I} p_i} + 1.$$

Then, by the Principle of Inclusion/Exclusion, one has

$$m = \sum_{i=1}^{k} (-1)^{i+1} \sum_{|I|=i} N(I) < \sum_{i=1}^{k} \binom{k}{i} + m \sum_{i=1}^{k} (-1)^{i+1} \sum_{|I|=i} \frac{1}{\prod_{i\in I} p_i}.$$

The first term is clearly equal to $2^k - 1$, while the second is equal to

$$m\left(1 - \prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right)\right) \le m\left(1 - \prod_{i=1}^{k}\left(1 - \frac{1}{i+1}\right)\right) = m - \frac{m}{k+1},$$

therefore $m < (k+1)\left(2^k - 1\right)$, and so will be $m(P)$. ■

Remarks. [1] A simpler variant could be
(ii) Prove that $|P| \le m(P) \le \max_{|P'|=|P|} m(P') < \infty$. [2]

Problem 2. For each positive integer n, find the largest real number C_n with the following property. Given any n real-valued functions $f_1(x), f_2(x), \dots, f_n(x)$ defined on the closed interval $0 \le x \le 1$, one can find numbers $x_1, x_2, \dots, x_n$, such that $0 \le x_i \le 1$, satisfying

$$\left|f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) - x_1x_2\cdots x_n\right| \ge C_n.$$

Serbia, Marko Radovanović

Solution. First we will prove that $C_n \ge \frac{n-1}{2n}$, i.e. that for any n functions $f_1, f_2, \dots, f_n : [0,1] \to \mathbb{R}$, there exist numbers $x_1, x_2, \dots, x_n$ in $[0,1]$ such that

$$|f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) - x_1x_2\cdots x_n| \ge \frac{n-1}{2n}.$$

For $n = 1$ this is trivial. For $n \ge 2$ suppose, contrariwise, that for all $x_1, x_2, \dots, x_n$ in $[0,1]$ we have

$$|f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) - x_1x_2\cdots x_n| < \frac{n-1}{2n}.$$

Plugging in $x_i = 1$ for $1 \le i \le n$, we get $\left|\sum_{i=1}^{n} f_i(1) - 1\right| < \frac{n-1}{2n}$.
Plugging in $x_i = 0$ for $1 \le i \le n$, we get $\left|\sum_{i=1}^{n} f_i(0)\right| < \frac{n-1}{2n}$.
Plugging in (for every $1 \le i \le n$) $x_i = 0$ and $x_j = 1$ for all $j \ne i$, we get $\left|f_i(0) + \sum_{j\ne i} f_j(1)\right| < \frac{n-1}{2n}$. Since

$$(n-1)\sum_{i=1}^{n} f_i(1) = \sum_{i=1}^{n}\left(f_i(0) + \sum_{j\ne i} f_j(1)\right) - \sum_{i=1}^{n} f_i(0),$$

by the triangle inequality we have

$$(n-1)\left|\sum_{i=1}^{n} f_i(1)\right| < (n+1)\frac{n-1}{2n}.$$

On the other hand, by again the triangle inequality we have

$$1 \le \left|\sum_{i=1}^{n} f_i(1)\right| + \left|\sum_{i=1}^{n} f_i(1) - 1\right| < \frac{n+1}{2n} + \frac{n-1}{2n} = 1,$$

which is a contradiction.

To prove that $\boxed{C_n = \frac{n-1}{2n}}$ is the largest constant, it will be sufficient to prove that for the n (equal) functions

$$f_i(x) = f(x) := \frac{x^n}{n} - \frac{n-1}{2n^2},\ 1 \le i \le n,$$

and any n numbers $x_1, x_2, \dots, x_n$ in $[0,1]$, we have

$$|f(x_1) + f(x_2) + \cdots + f(x_n) - x_1x_2\cdots x_n| \le \frac{n-1}{2n},$$

equivalent to

$$-\frac{n-1}{2n} \le \frac{1}{n}\sum_{i=1}^{n} x_i^n - \prod_{i=1}^{n} x_i - \frac{n-1}{2n} \le \frac{n-1}{2n}.$$

The LHS inequality follows from the AM-GM inequality. The RHS inequality is equivalent to

$$F(\boldsymbol{x}) = F(x_1, x_2, \dots, x_n) := \frac{1}{n}\sum_{i=1}^{n} x_i^n - \prod_{i=1}^{n} x_i \le \frac{n-1}{n}$$

at all points $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ of the hypercube $[0,1]^n$. Since F is convex in every variable, its maximum is reached at some vertex $\boldsymbol{v}$ of the hypercube (point with $x_i = 0$ or $x_i = 1$, for all $1 \le i \le n$). It is easy to see that for all such points we have $F(v) \le \frac{n-1}{n}$, which completes the proof. ■

Remarks. The choice of the functions

$$f_i(x) := \frac{x^n}{n} - \frac{n-1}{2n^2}, \ 1 \le i \le n$$

could be justified by the fact that if we try all $f_i = f$ and all $x_i = x$, the relation becomes $|nf(x) - x^n| \ge C_n$, as tight as possible for some $x \in [0,1]$. Then $f(x) = \frac{x^n}{n} - \frac{1}{n}C_n$ is a potential candidate.

On a different note, RHS inequality above is equivalent to

$$F(\boldsymbol{x}) = F(x_1, x_2, \dots, x_n) := \frac{1}{n}\sum_{i=1}^{n} x_i^n - \prod_{i=1}^{n} x_i \ge 0$$

at all points $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ of the hypercube $[0,1]^n$, and this again may be justified by F being convex, since it can be easily seen that $F(\boldsymbol{v}) \ge 0$ at any vertex $\boldsymbol{v}$ of the hypercube, so one may use a unifying argument for both sides of that inequality.

Problem 3. Let $A_1A_2A_3A_4$ be a convex quadrilateral with no pair of parallel sides. For each $i = 1,2,3,4$, define ω_i to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i$, A_iA_{i+1} and $A_{i+1}A_{i+2}$ (indices are considered modulo 4, so $A_0 = A_4$, $A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with the side A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.

Russia, Pavel Kozhevnikov

Solution. We start with a reformulation of a well-known statement on harmonic cyclic quadruples (K_1, K_2, K_3, K_4), also provable by polar transformation (projective methods).

Lemma. Being given four pairwise non-parallel lines ℓ_i, $i = 1,2,3,4$, tangent to a circle ω at points K_i, and such that lines ℓ_1, ℓ_3 and K_2K_4 are concurrent, then lines ℓ_2, ℓ_4 and K_1K_3 are also concurrent.

Proof. Let O be the centre of ω, $X = \ell_1 \cap \ell_3 \cap K_2K_4$, $Y = \ell_2 \cap \ell_4$. We have $OX \perp K_1K_3$ and $OY \perp K_2K_4$. Let $Z = OX \cap K_1K_3$, $T = OY \cap K_2K_4$. Notice that triangles $\triangle OK_3X$ and $\triangle OZK_3$ are similar, and also similar are triangles $\triangle OK_4Y$ and $\triangle OTK_4$, hence $OY \cdot OT = OK_4^2 = OK_3^2 = OX \cdot OZ$.

This means that triangles $\triangle OXT$ and $\triangle OYZ$ are similar, hence $YZ \perp OX$, and so $Y \in K_1K_3$. □

Suppose now lines A_2A_3, A_4A_1 and T_1T_3 are concurrent at a point P. Let T_4', T_2' be the tangency points of lines A_4A_1, respectively A_2A_3, to circle ω_1, and let T_3' be the second meeting point of line T_1T_3 and circle ω_1. Let the tangent to ω_1 at T_3' meet the lines A_4A_1, A_2A_3 at points A_4', respectively A_3'. The (direct) homothety of centre P that takes ω_1 to ω_3 maps T_3' to T_3, hence $A_3A_4 \parallel A_3'A_4'$.

Let $Q = A_1A_2 \cap A_3A_4$, $Q' = A_1A_2 \cap A_3'A_4'$. Applying the *Lemma* to circle ω_1 and lines A_2A_3, $A_3'A_4'$, A_4A_1, A_1A_2, yields that points Q', T_2', T_4' are collinear. The (inverse) homothety of centre A_1 that takes $\triangle QA_1A_4$ to $\triangle Q'A_1A_4'$ maps ω_4 to ω_1, so maps T_4 to T_4', hence $Q'T_4' \parallel QT_4$. Similarly, the (inverse) homothety of centre A_2 that takes $\triangle QA_2A_3$ to $\triangle Q'A_2A_3'$ maps ω_2 to ω_1, so maps T_2 to T_2', hence also $Q'T_2' \parallel QT_2$. Since points Q', T_2', T_4' are collinear, it follows points Q, T_2, T_4 are also collinear.

The converse implication is done in a similar way, due to the cyclic nature of the notations used (just increase each index by 1). ■

Alternative Solution. (D. Şerbănescu) Suppose Q, T_2, T_4 are collinear. We will show P, T_1, T_3 are collinear. We will use the notations of the solution above, but also let S_1', S_1'' be the tangency points of line A_1A_2 to circle ω_2, respectively ω_4, and let S_3', S_3'' be the tangency points of line A_3A_4 to circle ω_2, respectively ω_4. Let T_4'' be the (other than T_2) meeting point of line QT_2T_4 and circle ω_2, and let the tangent line to ω_2 at T_4'' (parallel to A_1A_4) meet A_2A_3 at P' (via the (direct) homothety of centre Q that takes ω_4 to ω_2).

Clearly $\triangle PT_2T_4 \sim \triangle P'T_2T_4''$ and $\triangle P'T_2T_4''$ is isosceles, so $PT_2 = PT_4$ (in other words, if Q, T_2, T_4 are collinear then $PT_2 = PT_4$; the other implication trivially also holds, but is irrelevant here).

From $PT_2 = PT_4$ and $PT_2' = PT_4'$ follows $T_2'T_2 = T_4'T_4$. As external tangents, $T_2'T_2 = T_1S_1'$ and $T_4'T_4 = T_1S_1''$, hence T_1 is the midpoint of $S_1'S_1''$. Similarly, T_3 is the midpoint of $S_3'S_3''$. It follows that P, T_1, T_3 lie on the radical axis of the circles ω_2 and ω_4, hence are collinear.

The converse implication is done in a similar way. ■

Remarks.

1. The statement remains true if points $\top_i$ are replaced by their symmetrical $\perp_i$ with respect to the midpoints of the segments A_iA_{i+1}.

2. Via some trigonometrical computations, one obtains that both conditions in the statement are equivalent to the condition $\sin\frac{A_1}{2}\sin\frac{A_3}{2} = \sin\frac{A_2}{2}\sin\frac{A_4}{2}$.

END

---

[1] When the primes in P are the first $|P|$ consecutive primes $q_1, q_2, \dots, q_k$, it is easy to see that the set of integers between 1 and the next prime q_{k+1} has the desired property, so $m(P) \ge q_{k+1} - 2$. It is obvious that one can check just the positive integers less than $\prod_{1\le i\le k} q_i$ in order to verify that this is indeed the value of $m(P)$, and this conjecture seems to hold, since it is true for $k = 1,2,3,4,5$. However, for $k = 6$, there exists a larger set than (prescribed) between 1 and 13, made by the numbers between 113 and 127.

[2] The **proof** goes by simple induction on $k = |P|$. Denote $m(k) = \max_{|P|=k} m(P)$; it is quite clear that $m(1) = 1$. Assume therefore that $m(k) < \infty$ and consider sets P with $|P| = k+1$. If $\max P > 2m(k)+1$, then $m(P) < 2m(k)+2$, since within any set of $2m(k)+2$ consecutive integers at most one is divisible by $\max P$, so there would exist a subset of at least $m(k)+1$ consecutive integers for $P \setminus \{\max P\}$, absurd. If $\max P \le 2m(k)+1$, then there are a finite number of such sets P, and clearly $m(P) < \prod_{p\in P} p$, so all will have a common upper bound. It follows that $m(k+1) < \infty$. □

The 3rd Romanian Master of Mathematics Competition – Solutions
Day 2: Saturday, February 27, 2010, Bucharest

Problem 4. Determine whether there exist a polynomial $f(x_1,x_2)$ in two variables, with integer coefficients, and two points $A=(a_1,a_2)$ and $B=(b_1,b_2)$ in the plane, satisfying all the following conditions:

(i) A is an integer point (i.e., a_1 and a_2 are integers);

(ii) $|a_1-b_1|+|a_2-b_2|=2010$;

(iii) $f(n_1,n_2)>f(a_1,a_2)$, for all integer points (n_1,n_2) in the plane other than A;

(iv) $f(x_1,x_2)>f(b_1,b_2)$, for all points (x_1,x_2) in the plane other than B.

Italy, Massimo Gobbino

Solution. The triple $(f(x_1,x_2),A,B)$ does exist, so $\boxed{\text{YES}}$.

Let $A=O=(0,0)$, $B=(x_0,y_0)=(2009+\frac{2}{3},\frac{1}{3})$. The idea is to search for a polynomial f such that $f(x,y)=0$ is the equation of an ellipse centred at B, passing through O and with tangent line $y=0$ at O. In fact, if f is chosen like this, the ellipse $f(x,y)=0$ is completely contained in the region $\{(x,y)\ ;\ 0\le y\le \frac{2}{3}\}$, with O the only integer point on the ellipse or in its interior; clearly, the absolute minimum of $f(x,y)$ is attained at B and $f(x,y)$ is positive at all integer points other than O. Therefore, we consider polynomials of the type

$$f(X,Y)=9M(X-x_0)^2+9N(X-x_0)(Y-y_0)+9P(Y-y_0)^2-Q$$

where M,N,P,Q are integers with $M,P,Q,4MP-N^2>0$.

The condition that the ellipse $f(x,y)=0$ passes through O, with tangent line $y=0$ at O, is expressed by

$$\begin{cases}6029^2M+6029N+P-Q=0\\ 2\cdot 6029M+N=0.\end{cases}$$

It is then sufficient to choose $M=1$, $N=-2\cdot 6029$, P any integer greater than 6029^2 and $Q=P-6029^2$. ■

Alternative Solution.[1] (D. Schwarz)

Given any integer point $A(a_1,a_2)$, there exist infinitely many points $B(b_1,b_2)$ with $b_1,b_2\in\mathbb{Q}\setminus\mathbb{Z}$, and such that $|a_1-b_1|+|a_2-b_2|=2010$, for example $b_1=a_1+\alpha+r$, $b_2=a_2+\beta+(1-r)$, with $\alpha,\beta\in\mathbb{Z}_+$, $r\in\mathbb{Q}\cap(0,1)$, and $\alpha+\beta=2009$. We now will consider polynomials of the type

$$f(X,Y)=N\left((X-a_1)^2+(Y-a_2)^2+\varepsilon\right)\left((X-b_1)^2+(Y-b_2)^2\right)$$

where $\varepsilon\in\mathbb{Q}_+^*$ and $N\in\mathbb{Z}_+^*$ large enough for $f(X,Y)$ to have integer coefficients.

One then has $f(b_1,b_2)=0$, while $f(x,y)>0$ for all points (x,y) in the plane, other than B.

One also then has $f(a_1,a_2)=N\varepsilon\left((a_1-b_1)^2+(a_2-b_2)^2\right)$, while one has, for all integer points (n_1,n_2) in the plane, other than A, $\min\left((n_1-a_1)^2+(n_2-a_2)^2+\varepsilon\right)=1+\varepsilon$ and $\min\left((n_1-b_1)^2+(n_2-b_2)^2\right)=m$, for some $m\in\mathbb{Q}\cap(0,\frac{1}{2}]$ (for example $m=\frac{1}{2}$ when $r=\frac{1}{2}$), therefore $F(n_1,n_2)>N(1+\varepsilon)m$.

In order to have $f(n_1,n_2)>f(a_1,a_2)$ it is thus enough that $N(1+\varepsilon)m\ge N\varepsilon\left((a_1-b_1)^2+(a_2-b_2)^2\right)$, therefore let us take $\varepsilon=m/\left((a_1-b_1)^2+(a_2-b_2)^2-m\right)\in\mathbb{Q}_+^*$, and then choose some appropriate N. ■

Remarks. It is a case of applying the knowledge that some closed simple curve given by $f(x,y)=0$ separates the plane into two regions, with values of one sign in the interior region, and values of the opposite sign in the exterior region. Once this idea comes to mind, the problem turns into a simple exercise in analytic geometry of the conics.

Notice that the point $C(2x_0,2y_0)$, the symmetrical of A with respect to B, also lies on the ellipse $F(x,y)=0$. What in fact we have done is to take the circle given by equation $\Gamma(x,y)=(x-x_0)^2+(y-y_0)^2-(x_0^2+y_0^2)$ and then stretch it on a direction perpendicular to OB, until a resulting rational ellipse contains no other integer points than A.

Problem 5. Let n be a given positive integer. Say that a set K of points with integer coordinates in the plane is *connected* if for every pair of points $R,S\in K$, there exist a positive integer ℓ and a sequence $R=T_0,T_1,\dots,T_\ell=S$ of points in K, where each T_i is distance 1 away from T_{i+1}. For such a set K, we define the set of vectors

$$\Delta(K)=\{\overrightarrow{RS}\mid R,S\in K\}.$$

What is the maximum value of $|\Delta(K)|$ over all connected sets K of $2n+1$ points with integer coordinates in the plane?

Russia, Grigory Chelnokov

Solution. We claim the answer is $\boxed{2n^2+4n+1}$. A model is $K=\{(0,0)\}\cup\{(i,0)\ ;\ 1\le i\le n\}\cup\{(0,i)\ ;\ 1\le i\le n\}$, when $W=\{(a,-b)\ ;\ 0\le a,b\le n\}\cup\{(-a,b)\ ;\ 0\le a,b\le n\}$. It is left to prove that $|W|\le 2n^2+4n+1$ for any set K.

What the statement of the problem describes is a connected graph $G=(K,E)$ of order $2n+1$, whose vertices are the points in K, while the edges are the horizontal/vertical segments of length 1 that connect (some of) these points. The key to the proof is to sequence the elements of K as $A_0,A_1,\dots,A_{2n}$ such that the graph $G_k:=G[A_0,A_1,\dots,A_k]$ is connected for every $1\le k\le 2n$; this can be done through

Lemma.[2] The vertices of a finite connected graph G can always be enumerated, say as a sequence $v_0,\dots,v_{|G|-1}$, so that $G_k:=G[v_0,\dots,v_k]$ is connected for every $1\le k\le |G|-1$.

Proof. Pick any vertex as v_0, and assume inductively that $v_0,\dots,v_k$ have been chosen for some $0\le k<|G|-1$. Now pick a vertex $v\in G-G_k$. As G is connected, it contains a $v-v_0$ path P. Choose as v_{k+1} the last vertex of P in $G-G_k$; then v_{k+1} has as neighbour in G_k the next vertex of P. The connectedness of every G_k follows by induction on k. □

Moreover, if we just keep the edges through which A_{k+1} has the (selected) neighbour in G_k, then G_k is a tree, and so G_{2n} is a spanning tree of G. Call the vertex *horizontal* (*vertical*) if the edge that connects him is horizontal (vertical). Denote by h, respectively v, the number of horizontal, respectively vertical vertices; since G_{2n} is a tree, it follows $h+v=2n$. The point A_0 contributes $2n+1$ vectors $\overrightarrow{A_0A_i}$. Now, for $0\le k\le 2n$, each point A_{k+1} contributes at most $(2n+1)-x$ new vectors, where $x=h$ if the vertex is horizontal, respectively $x=v$ if the vertex is vertical, since those vectors $\overrightarrow{A_{k+1}A_i}$, with ends at the corresponding edges of same direction, will be duplicated by the vectors determined by the opposite parallel sides of the parallelograms created, which have already been accounted for.

Therefore the total number of distinct vectors will be $|W| \le (2n+1)^2 - h^2 - v^2$. But $h^2 + v^2 \ge \frac{1}{2}(h+v)^2 = 2n^2$, hence $|W| \le (2n+1)^2 - 2n^2 = 2n^2 + 4n + 1$.[3] ■

Problem 6. Given a polynomial $f(x)$ with rational coefficients, of degree $d \ge 2$, we define the sequence of sets $f^0(\mathbb{Q}), f^1(\mathbb{Q}), \ldots$ by $f^0(\mathbb{Q}) = \mathbb{Q}$ and $f^{n+1}(\mathbb{Q}) = f(f^n(\mathbb{Q}))$ for $n \ge 0$. (Given a set S, we write $f(S)$ for the set $\{f(x) \mid x \in S\}$.)

Let $f^\omega(\mathbb{Q}) = \bigcap_{n=0}^\infty f^n(\mathbb{Q})$ be the set of numbers that are in all of the sets $f^n(\mathbb{Q})$. Prove that $f^\omega(\mathbb{Q})$ is a finite set.

Romania, Dan Schwarz

Solution. For any function f, denote its n-th iterate f^n. Take $d = \deg f \ge 2$. One can write $f(x) = \frac{1}{N}(ax^d + g(x))$ for some $N \in \mathbb{Z}_+^*$, $a \in \mathbb{Z}^*$, and some $g \in \mathbb{Z}[x]$, with $\deg g \le d-1$, $g(x) = \sum_{i=0}^{d-1} a_i x^i$, $a_i \in \mathbb{Z}$, for all $0 \le i \le d-1$.

Finally, $f^\omega(\mathbb{Q}) \subset f^n(\mathbb{Q}) \subset f^{n-1}(\mathbb{Q}) \subseteq \mathbb{Q}$, for $n \ge 1$.

For any $x \in \mathbb{Q}$, one can uniquely write $x = \frac{\mu(x)}{\nu(x)}$, with $\mu(x), \nu(x) \in \mathbb{Z}$, and $\nu(x) > 0$, $\gcd(\mu(x), \nu(x)) = 1$. Take now
$$M = \frac{\sum_{i=0}^{d-1} |a_i| + 2N}{|a|} + 1, \text{ and}$$

$$\mathcal{M} := \{x \in \mathbb{Q}\,;\, |x| > M\}, \quad \mathcal{F} := \{x \in \mathbb{Q}\,;\, \nu(x) > a^2\}.$$

Now, $(\mathbb{Q} \setminus \mathcal{F}) \cap (\mathbb{Q} \setminus \mathcal{M})$ is obviously finite; take $\mathfrak{m}$ to be its cardinality.

For $x \in \mathcal{M}$ one has $|f(x)| \ge |x| + 1 > M$ (see *Lemma 1*), hence $f(x) \in \mathcal{M}$. Then $|f^n(x)| \ge |x| + n$. For $x \in \mathcal{F}$ one has $\nu(f(x)) \ge \nu(x) + 1 > a^2$ (see *Lemma 2*), hence $f(x) \in \mathcal{F}$. Then $\nu(f^n(x)) \ge \nu(x) + n$.

Take $x \in f^\omega(\mathbb{Q})$, and take n large enough. Then we will have $x \in f^n(\mathbb{Q})$, hence there will exist $x_n \in \mathbb{Q}$ such that $f^n(x_n) = x$. If $f^k(x_n) \in \mathcal{M}$ for $n-k > |x|$, then $|x| = |f^n(x_n)| = |f^{n-k}(f^k(x_n))| \ge |f^k(x_n)| + (n-k) > |x|$, absurd. If $f^k(x_n) \in \mathcal{F}$ for $n-k > \nu(x)$, then $\nu(x) = \nu(f^n(x_n)) = \nu(f^{n-k}(f^k(x_n))) \ge \nu(f^k(x_n)) + (n-k) > \nu(x)$, absurd.

Take n large enough so that $n > \mathfrak{m} + \max(|x|, \nu(x))$. One then has $f^k(x_n) \in (\mathbb{Q} \setminus \mathcal{F}) \cap (\mathbb{Q} \setminus \mathcal{M})$, for $0 \le k \le \mathfrak{m}$, therefore there will exist $0 \le i < j \le \mathfrak{m}$ such that $f^i(x_n) = f^j(x_n)$, therefore $f^n(x_n) = f^k(x_n)$ for some $i \le k \le j$, hence $x = f^n(x_n) = f^k(x_n) \in (\mathbb{Q} \setminus \mathcal{F}) \cap (\mathbb{Q} \setminus \mathcal{M})$.

This implies $f^\omega(\mathbb{Q}) \subseteq (\mathbb{Q} \setminus \mathcal{F}) \cap (\mathbb{Q} \setminus \mathcal{M})$, thus a finite set. ■

Lemma 1. For $x \in \mathcal{M}$ one has $|f(x)| \ge |x| + 1 > M$.

Proof. Clearly then $|x| > M > 1$. Now $\frac{|a||x|^d}{N} > \frac{\sum_{i=0}^{d-1} |a_i||x|^{d-1}}{N} + |x|^d + |x|^d > \frac{\sum_{i=0}^{d-1} |a_i||x|^i}{N} + |x| + 1 \ge \frac{|g(x)|}{N} + |x| + 1$. It follows that $|f(x)| = \left|\frac{ax^d + g(x)}{N}\right| \ge \left|\frac{|a||x|^d}{N} - \frac{|g(x)|}{N}\right| > |x| + 1 > M$. □

Lemma 2. For $x \in \mathcal{F}$ one has $\nu(f(x)) \ge \nu(x) + 1 > a^2$.

Proof. For $x \in \mathcal{F}$ one can write $f(x) = \frac{1}{N\nu(x)^d}(a\mu(x)^d + \nu(x)z) = \frac{\mu(f(x))}{\nu(f(x))}$, with $z = \nu(x)^{d-1}g(x) \in \mathbb{Z}$. Now, for $e = \gcd(\nu(x), a)$, one has $\nu(x) = er$, $a = eb$, with $\gcd(r, b) = 1$.

Then it follows that $\delta = \gcd(a\mu(x)^d + \nu(x)z, N\nu(x)^d) \le N \cdot \gcd(eb\mu(x)^d + erz, e^d r^d) = Ne \cdot \gcd(b\mu(x)^d + rz, e^{d-1}r^d) = Ne \cdot \gcd(b\mu(x)^d + rz, e^{d-1})$, since from previous relations $\gcd(b\mu(x), r) = 1$. Lastly $\delta \le Ne \cdot \gcd(b\mu(x)^d + rz, e^{d-1}) \le Nee^{d-1} = Ne^d \le N|a|^d$.

Therefore $\nu(f(x)) = \frac{N\nu(x)^d}{\delta} \ge \frac{N\nu(x)^d}{N|a|^d} > \nu(x)$, since $\nu(x) > a^2 \ge |a|^{\frac{d}{d-1}}$. It follows that $\nu(f(x)) \ge \nu(x) + 1 > a^2$. □

While *Lemma 1* is classical, *Lemma 2* is somewhat more computational, even if readily intuitive (it may be shown, with not more trouble, that $x \in f^\omega(\mathbb{Q})$ implies $\nu(x) \mid a$, thus allowing for easier computation of $f^\omega(\mathbb{Q})$).

Remarks.

1. For $\deg f = 1$, one has $f(\mathbb{Q}) = \mathbb{Q}$, therefore $f^\omega(\mathbb{Q}) = \mathbb{Q}$. On the other hand, replacing $\mathbb{Q}$ with $\mathbb{Z}$ allover, results in a much simpler statement (use *Lemma 1* only, or see point 3).

2. If we use the (quite readily established) result that $f(f^\omega(\mathbb{Q})) = f^\omega(\mathbb{Q})$, it follows that the restriction of f to $f^\omega(\mathbb{Q})$ is a permutation (hence a product of cycles) of this finite set. On the other hand, any such cycle clearly belongs to $f^\omega(\mathbb{Q})$; thus the only orbits for f are finitely many, among the elements of $f^\omega(\mathbb{Q})$.

Using a Lagrange interpolation polynomial, one can build as large a finite orbit as wanted. However, all orbits then turn out to be fixed points for some iterate $f^{|f^\omega(\mathbb{Q})|!}$. Conversely, for any nonempty finite set $Q \subset \mathbb{Q}$, we can build the polynomial $f(x) = \prod_{q \in Q}(x-q)^2 + x$, for which $f^\omega(\mathbb{Q}) = Q$, since $Q \subseteq f^\omega(\mathbb{Q})$, while for $x \in f^\omega(\mathbb{Q})$ one has $x \le f(x) \le \cdots \le f^{|f^\omega(\mathbb{Q})|!}(x) = x$, hence $x = f(x)$, whence $x \in Q$.

3. The problem 5 at IMO 2006 (Slovenia) proved that, for $f \in \mathbb{Z}[x]$ with $\deg f > 1$, f^n has at most $\deg f$ fixed points for $n = 1$ or 2 (and no new fixed points for $n > 2$), which under this interpretation translates into $|f^\omega(\mathbb{Z})| \le \deg f$.

According with the above, if f is monic, then $f^\omega(\mathbb{Q}) \subset \mathbb{Z}$, hence the same result holds.[4]

However, polynomial $f(x) = \frac{1}{2}(x-2)(x-3)$ has fixed points 1 and 6, and length-2 orbit $(0, 3)$ (computable to $f^\omega(\mathbb{Q}) = \{0, 1, 3, 6\}$), showing the fact that the above result for $\mathbb{Z}$ stands no more. Also, polynomial $f(x) = \frac{1}{4}(x^2 - 29)$ has a length-3 orbit $(5, -1, -7)$ (example by TIMO ERKAMA).[5]

4. As for the situation at hand, a simple corollary states *A sequence $(x_n)_{n \ge 1}$ of rational numbers, such that $x_n = f(x_{n+1})$, is periodic.*[6] (The proof is that clearly then the terms of the sequence all belong to the finite set $f^\omega(\mathbb{Q})$, whence the claim of its periodicity.)

END

---

[1] Certainly all computations are irrelevant – the idea matters.

[2] To be found in [DIESTEL, R., *Graph Theory*, Springer-Verlag, (2000)], Proposition 1.4.1. We chose to include its proof.

[3] The exactly same argument works in the d-dimensional space, for a set of $dn+1$ latticeal points; then the maximal possible number of vectors will be $(d^2-d)n^2 + 2dn + 1$, with a model made by the origin O, and n points, at distances $1, 2, \ldots, n$ from origin, on each axis of the coordinate system.

[4] This is a corollary to a theorem by NARKIEWICZ (see also the seminal theorems by NORTHCOTT).

[5] Such topics are studied by discrete dynamic systems theory, closely related with the study of MANDELBROT and JULIA sets; also FEIGENBAUM constant, attractors, fractals, chaos theory.

[6] For $f(x) = x^3$ it is said to be an old China TST problem.

THE 4th ROMANIAN MASTER OF MATHEMATICS COMPETITION

DAY 1: FRIDAY, FEBRUARY 25, 2011, BUCHAREST

Language: English

Problem 1. Prove that there exist two functions $f, g\colon \mathbb{R} \to \mathbb{R}$, such that $f \circ g$ is strictly decreasing and $g \circ f$ is strictly increasing.

Problem 2. Determine all positive integers n for which there exists a polynomial $f(x)$ with real coefficients, with the following properties:

(1) for each integer k, the number $f(k)$ is an integer if and only if k is not divisible by n;
(2) the degree of f is less than n.

Problem 3. A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

THE 4th ROMANIAN MASTER OF MATHEMATICS COMPETITION

DAY 2: SATURDAY, FEBRUARY 26, 2011, BUCHAREST

Language: English

Problem 4. Given a positive integer $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, we write $\Omega(n)$ for the total number $\sum_{i=1}^{s} \alpha_i$ of prime factors of n, counted with multiplicity. Let $\lambda(n) = (-1)^{\Omega(n)}$ (so, for example, $\lambda(12) = \lambda(2^2 \cdot 3^1) = (-1)^{2+1} = -1$).

Prove the following two claims:

i) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = +1$;
ii) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = -1$.

Problem 5. For every $n \geq 3$, determine all the configurations of n distinct points $X_1, X_2, \ldots, X_n$ in the plane, with the property that for any pair of distinct points X_i, X_j there exists a permutation σ of the integers $\{1, \ldots, n\}$, such that $\mathrm{d}(X_i, X_k) = \mathrm{d}(X_j, X_{\sigma(k)})$ for all $1 \leq k \leq n$.
(We write $\mathrm{d}(X, Y)$ to denote the distance between points X and Y.)

Problem 6. The cells of a square 2011×2011 array are labelled with the integers $1, 2, \ldots, 2011^2$, in such a way that every label is used exactly once. We then identify the left-hand and right-hand edges, and then the top and bottom, in the normal way to form a torus (the surface of a doughnut).

Determine the largest positive integer M such that, no matter which labelling we choose, there exist two neighbouring cells with the difference of their labels at least M.*

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

*Cells with coordinates (x, y) and (x', y') are considered to be neighbours if $x = x'$ and $y - y' \equiv \pm 1 \pmod{2011}$, or if $y = y'$ and $x - x' \equiv \pm 1 \pmod{2011}$.

The 4th Romanian Master of Mathematics Competition – Solutions
Day 1: Friday, February 25, 2011, Bucharest

Problem 1. Prove that there exist two functions

$$f, g\colon \mathbb{R} \to \mathbb{R},$$

such that $f \circ g$ is strictly decreasing, while $g \circ f$ is strictly increasing.

(POLAND) ANDRZEJ KOMISARSKI & MARCIN KUCZMA

Solution. Let

- $A = \bigcup_{k\in\mathbb{Z}} \left(\left[-2^{2k+1}, -2^{2k}\right) \bigcup \left(2^{2k}, 2^{2k+1}\right]\right)$;
- $B = \bigcup_{k\in\mathbb{Z}} \left(\left[-2^{2k}, -2^{2k-1}\right) \bigcup \left(2^{2k-1}, 2^{2k}\right]\right)$.

Thus $A = 2B$, $B = 2A$, $A = -A$, $B = -B$, $A \cap B = \emptyset$, and finally $A \cup B \cup \{0\} = \mathbb{R}$. Let us take

$$f(x) = \begin{cases} x & \text{for} \quad x \in A; \\ -x & \text{for} \quad x \in B; \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Take $g(x) = 2f(x)$. Thus $f(g(x)) = f(2f(x)) = -2x$ and $g(f(x)) = 2f(f(x)) = 2x$. ■

Problem 2. Determine all positive integers n for which there exists a polynomial $f(x)$ with real coefficients, with the following properties:

(1) for each integer k, the number $f(k)$ is an integer if and only if k is not divisible by n;

(2) the degree of f is less than n.

(HUNGARY) GÉZA KÓS

Solution. We will show that such polynomial exists if and only if $n = 1$ or n is a power of a prime.

We will use two known facts stated in Lemmata 1 and 2.

LEMMA 1. If p^a is a power of a prime and k is an integer, then $\dfrac{(k-1)(k-2)\dots(k-p^a+1)}{(p^a-1)!}$ is divisible by p if and only if k is not divisible by p^a.

Proof. First suppose that $p^a \mid k$ and consider

$$\frac{(k-1)(k-2)\cdots(k-p^a+1)}{(p^a-1)!} = \frac{k-1}{p^a-1}\cdot\frac{k-2}{p^a-2}\cdots\frac{k-p^a+1}{1}.$$

In every fraction on the right-hand side, p has the same maximal exponent in the numerator as in the denominator. Therefore, the product (which is an integer) is not divisible by p.

Now suppose that $p^a \nmid k$. We have

$$\frac{(k-1)(k-2)\cdots(k-p^a+1)}{(p^a-1)!} = \frac{p^a}{k}\cdot\frac{k(k-1)\cdots(k-p^a+1)}{(p^a)!}.$$

The last fraction is an integer. In the fraction $\frac{p^a}{k}$, the denominator k is not divisible by p^a. □

LEMMA 2. If $g(x)$ is a polynomial with degree less than n then

$$\sum_{\ell=0}^{n}(-1)^\ell\binom{n}{\ell}g(x+n-\ell) = 0.$$

Proof. Apply induction on n. For $n = 1$ then $g(x)$ is a constant and

$$\binom{1}{0}g(x+1) - \binom{1}{1}g(x) = g(x+1) - g(x) = 0.$$

Now assume that $n > 1$ and the Lemma holds for $n-1$. Let $h(x) = g(x+1) - g(x)$; the degree of h is less than the degree of g, so the induction hypothesis applies for g and $n-1$:

$$\sum_{\ell=0}^{n-1}(-1)^\ell\binom{n-1}{\ell}h(x+n-1-\ell) = 0$$

$$\sum_{\ell=0}^{n-1}(-1)^\ell\binom{n-1}{\ell}\big(g(x+n-\ell) - g(x+n-1-\ell)\big) = 0$$

$$\binom{n-1}{0}g(x+n) + \sum_{\ell=1}^{n-1}(-1)^\ell\left(\binom{n-1}{\ell-1} + \binom{n-1}{\ell}\right)g(x+n-\ell) - (-1)^{n-1}\binom{n-1}{n-1}g(x) = 0$$

$$\sum_{\ell=0}^{n}(-1)^\ell\binom{n}{\ell}g(x+n-\ell) = 0.$$

□

LEMMA 3. If n has at least two distinct prime divisors then the greatest common divisor of $\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}$ is 1.

Proof. Suppose to the contrary that p is a common prime divisor of $\binom{n}{1}, \dots, \binom{n}{n-1}$. In particular, $p \mid \binom{n}{1} = n$. Let a be the exponent of p in the prime factorization of n. Since n has at least two prime divisors, we have $1 < p^a < n$. Hence, $\binom{n}{p^a-1}$ and $\binom{n}{p^a}$ are listed among $\binom{n}{1}, \dots, \binom{n}{n-1}$ and thus $p \mid \binom{n}{p^a}$ and $p \mid \binom{n}{p^a-1}$. But then p divides $\binom{n}{p^a} - \binom{n}{p^a-1} = \binom{n-1}{p^a-1}$, which contradicts Lemma 1. □

Next we construct the polynomial $f(x)$ when $n=1$ or n is a power of a prime.

For $n=1$, $f(x)=\frac{1}{2}$ is such a polynomial.

If $n=p^a$ where p is a prime and a is a positive integer then let

$$f(x)=\frac{1}{p}\binom{x-1}{p^a-1}=\frac{1}{p}\cdot\frac{(x-1)(x-2)\cdots(x-p^a+1)}{(p^a-1)!}.$$

The degree of this polynomial is $p^a-1=n-1$.

The number $\frac{(k-1)(k-2)\cdots(k-p^a+1)}{(p^a-1)!}$ is an integer for any integer k, and, by Lemma 1, it is divisible by p if and only if k is not divisible by $p^a=n$.

Finally we prove that if n has at least two prime divisors then no polynomial $f(x)$ satisfies (1,2). Suppose that some polynomial $f(x)$ satisfies (1,2), and apply Lemma 2 for $g=f$ and $x=-k$ where $1\le k\le n-1$. We get that

$$\binom{n}{k}f(0)=\sum_{0\le\ell\le n,\,\ell\ne k}(-1)^{k-\ell}\binom{n}{\ell}f(-k+\ell).$$

Since $f(-k),\dots,f(-1)$ and $f(1),\dots,f(n-k)$ are all integers, we conclude that $\binom{n}{k}f(0)$ is an integer for every $1\le k\le n-1$.

By dint of Lemma 3, the greatest common divisor of $\binom{n}{1},\binom{n}{2},\dots,\binom{n}{n-1}$ is 1. Hence, there will exist some integers $u_1,u_2,\dots,u_{n-1}$ for which $u_1\binom{n}{1}+\cdots+u_{n-1}\binom{n}{n-1}=1$. Then

$$f(0)=\left(\sum_{k=1}^{n-1}u_k\binom{n}{k}\right)f(0)=\sum_{k=1}^{n-1}u_k\binom{n}{k}f(0)$$

is a sum of integers. This contradicts the fact that $f(0)$ is not an integer. So such polynomial $f(x)$ does not exist. ■

Alternative Solution. (I. Bogdanov) We claim the answer is $n=p^\alpha$ for some prime p and nonnegative α.

Lemma. For every integers $a_1,\dots,a_n$ there exists an integer-valued polynomial $P(x)$ of degree $<n$ such that $P(k)=a_k$ for all $1\le k\le n$.

Proof. Induction on n. For the base case $n=1$ one may set $P(x)=a_1$. For the induction step, suppose that the polynomial $P_1(x)$ satisfies the desired property for all $1\le k\le n-1$. Then set $P(x)=P_1(x)+(a_n-P_1(n))\binom{x-1}{n-1}$; since $\binom{k-1}{n-1}=0$ for $1\le k\le n-1$ and $\binom{n-1}{n-1}=1$, the polynomial $P(x)$ is a sought one. □

Now, if for some n there exists some polynomial $f(x)$ satisfying the problem conditions, one may choose some integer-valued polynomial $P(x)$ (of degree $<n-1$) coinciding with $f(x)$ at points $1,\dots,n-1$. The difference $f_1(x)=f(x)-P(x)$ also satisfies the problem conditions, therefore we may restrict ourselves to the polynomials vanishing at points $1,\dots,n-1$ — that are, the polynomials of the form $f(x)=c\prod_{i=1}^{n-1}(x-i)$ for some (surely rational) constant c. Let $c=p/q$ be its irreducible form, and $q=\prod_{j=1}^{d}p_j^{\alpha_j}$ be the prime decomposition of the denominator.

1. Assume that a desired polynomial $f(x)$ exists. Since $f(0)$ is not an integer, we have $q\nmid(-1)^{n-1}(n-1)!$ and hence $p_j^{\alpha_j}\nmid(-1)^{n-1}(n-1)!$ for some j. Hence

$$\prod_{i=1}^{n-1}(p_j^{\alpha_j}-i)\equiv(-1)^{n-1}(n-1)!\not\equiv 0 \pmod{p_j^{\alpha_j}},$$

therefore $f(p_i^{\alpha_i})$ is not integer, too. By the condition (i), this means that $n\mid p_i^{\alpha_i}$, and hence n should be a power of a prime.

2. Now let us construct a desired polynomial $f(x)$ for any power of a prime $n=p^\alpha$. We claim that the polynomial

$$f(x)=\frac{1}{p}\binom{x-1}{n-1}=\frac{n}{px}\binom{x}{n}$$

fits. Actually, consider some integer x. From the first representation, the denominator of the irreducible form of $f(x)$ may be 1 or p only. If $p^\alpha\nmid x$, then the prime decomposition of the fraction $n/(px)$ contains p with a nonnegative exponent; hence $f(x)$ is integer. On the other hand, if $n=p^\alpha\mid x$, then the numbers $x-1,x-2,\dots,x-(n-1)$ contain the same exponents of primes as the numbers $n-1,n-2,\dots,1$ respectively; hence the number

$$\binom{x-1}{n-1}=\frac{\prod_{i=1}^{n-1}(x-i)}{\prod_{i=1}^{n-1}(n-i)}$$

is not divisible by p. Thus $f(x)$ is not an integer. ■

Problem 3. A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

(Russia) Vasily Mokin & Fedor Ivlev

Solution. Let P be the meeting point of the common inner tangents to γ_1 and γ_2. Also, let b be the angle bisector of $\angle BAC$. Since $KL\parallel BC$, b is also the angle bisector of $\angle KAL$.

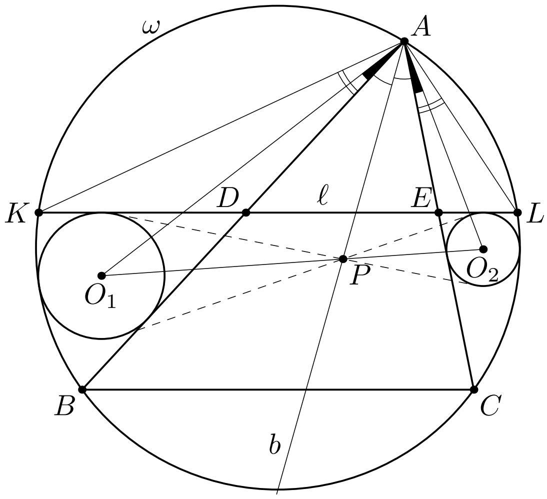
Let $\mathfrak{H}$ be the composition of the symmetry $\mathfrak{S}$ with respect to b and the inversion $\mathfrak{I}$ of centre A and ratio $\sqrt{AK\cdot AL}$ (it is readily seen that $\mathfrak{S}$ and $\mathfrak{I}$ commute, so since $\mathfrak{S}^2=\mathfrak{I}^2=\mathrm{id}$, then also $\mathfrak{H}^2=\mathrm{id}$, the identical transformation). The elements of the configuration interchanged by $\mathfrak{H}$ are summarized in Table I.

Let O_1 and O_2 be the centres of circles γ_1 and γ_2. Since the circles γ_1 and γ_2 are determined by their construction (in a unique way), they are interchanged by $\mathfrak{H}$, therefore the rays AO_1 and AO_2 are symmetrical with respect to b. Denote by ρ_1 and ρ_2 the radii of γ_1 and γ_2. Since $\angle O_1AB = \angle O_2AC$, we have $\rho_1/\rho_2 = AO_1/AO_2$. On the other hand, from the definition of P we have $O_1P/O_2P = \rho_1/\rho_2 = AO_1/AO_2$; this means that AP is the angle bisector of $\angle O_1AO_2$ and therefore of $\angle BAC$.

The limiting, degenerated, cases are when the parallel line passes through A – when P coincides with A; respectively when the parallel line is BC – when P coincides with the foot $A' \in BC$ of the angle bisector of $\angle BAC$ (or any other point on BC). By continuity, any point P on the open segment AA' is obtained for some position of the parallel, therefore the locus is the open segment AA' of the angle bisector b of $\angle BAC$. ■

point K	$\longleftrightarrow$	point L
line KL	$\longleftrightarrow$	circle ω
ray AB	$\longleftrightarrow$	ray AC
point B	$\longleftrightarrow$	point E
point C	$\longleftrightarrow$	point D
segment BD	$\longleftrightarrow$	segment EC
arc BK	$\longleftrightarrow$	segment EL
arc CL	$\longleftrightarrow$	segment DK

TABLE I: Elements interchanged by $\mathfrak{H}$.



The 4$^{\text{th}}$ Romanian Master of Mathematics Competition – Solutions
Day 2: Saturday, February 26, 2011, Bucharest

Problem 4. Given a positive integer $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, we write $\Omega(n)$ for the total number $\sum_{i=1}^{s} \alpha_i$ of prime factors of n, counted with multiplicity. Let $\lambda(n) = (-1)^{\Omega(n)}$ (so, for example, $\lambda(12) = \lambda(2^2 \cdot 3^1) = (-1)^{2+1} = -1$).

Prove the following two claims:

i) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = +1$;

ii) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = -1$.

(ROMANIA) DAN SCHWARZ

Solution. Notice that we have $\Omega(mn) = \Omega(m) + \Omega(n)$ for all positive integers m, n (Ω is a completely additive arithmetic function), translating into $\lambda(mn) = \lambda(m) \cdot \lambda(n)$ (so λ is a completely multiplicative arithmetic function), hence $\lambda(p) = -1$ for any prime p, and $\lambda(k^2) = \lambda(k)^2 = +1$ for all positive integers k.[1]

The start (first 100 terms) of the sequence $\mathfrak{S} = (\lambda(n))_{n\geq 1}$ is

$+1, -1, -1, +1, -1, +1, -1, -1, +1, +1, -1, -1, -1, +1, +1, +1, -1, -1, -1, -1,$
$+1, +1, -1, +1, +1, +1, -1, -1, -1, -1, -1, -1, +1, +1, +1, +1, -1, +1, +1, +1,$
$-1, -1, -1, -1, -1, +1, -1, -1, +1, -1, +1, -1, -1, +1, +1, +1, +1, +1, -1, +1,$
$-1, +1, -1, +1, +1, -1, -1, -1, +1, -1, -1, -1, -1, +1, -1, -1, +1, -1, -1, -1,$
$+1, +1, -1, +1, +1, +1, +1, +1, -1, +1, +1, -1, +1, +1, +1, +1, -1, -1, -1, +1.$

i) The Pell equation $x^2 - 6y^2 = 1$ has infinitely many solutions in positive integers; all solutions are given by (x_n, y_n), where $x_n + y_n\sqrt{6} = (5 + 2\sqrt{6})^n$. Since $\lambda(6y^2) = 1$ and also $\lambda(6y^2 + 1) = \lambda(x^2) = 1$, the thesis is proven.

Alternative Solution. Take any existing pair with $\lambda(n) = \lambda(n+1) = 1$. Then $\lambda((2n+1)^2 - 1) = \lambda(4n^2 + 4n) = \lambda(4) \cdot \lambda(n) \cdot \lambda(n+1) = 1$, and also $\lambda((2n+1)^2) = \lambda(2n+1)^2 = 1$, so we have built a larger $(1,1)$ pair.

ii) The equation $3x^2 - 2y^2 = 1$ (again Pell theory) has also infinitely many solutions in positive integers, given by (x_n, y_n), where $x_n\sqrt{3} + y_n\sqrt{2} = (\sqrt{3} + \sqrt{2})^{2n+1}$. Since $\lambda(2y^2) = -1$ and $\lambda(2y^2 + 1) = \lambda(3x^2) = -1$, the thesis is proven.

Alternative Solution. Assume $(\lambda(n-1), \lambda(n))$ is the largest $(-1,-1)$ pair, therefore $\lambda(n+1) = 1$ and $\lambda(n^2 + n) = \lambda(n) \cdot \lambda(n+1) = -1$, therefore again $\lambda(n^2 + n + 1) = 1$. But then $\lambda(n^3 - 1) = \lambda(n-1) \cdot \lambda(n^2 + n + 1) = -1$, and also $\lambda(n^3) = \lambda(n)^3 = -1$, so we found yet a larger such pair than the one we started with, contradiction.

Alternative Solution. Assume the pairs of consecutive terms $(-1,-1)$ in $\mathfrak{S}$ are finitely many. Then from some rank on we only have subsequences $(1, -1, 1, 1, \ldots, 1, -1, 1)$. By "doubling" such a subsequence (like at point ii)), we will produce

$$(-1, ?, 1, ?, -1, ?, -1, ?, \ldots, ?, -1, ?, 1, ?, -1).$$

According with our assumption, all ?-terms ought to be 1, hence the produced subsequence is

$$(-1, 1, 1, 1, -1, 1, -1, 1, \ldots, 1, -1, 1, 1, 1, -1),$$

and so the "separating packets" of 1's contain either one or three terms. Now assume some far enough $(1,1,1,1)$ or $(-1,1,1,-1)$ subsequence of $\mathfrak{S}$ were to exist. Since it lies within some "doubled" subsequence, it contradicts the structure described above, which thus is the only prevalent from some rank on. But then all the positions of the (-1)-terms will have the same parity. However though, we have $\lambda(p) = \lambda(2p^2) = -1$ for all odd primes p, and these terms have different parity of their positions. A contradiction has been reached.[2]

Alternative Solution for both i) **and** ii). (I. Bogdanov) Take $\varepsilon \in \{-1, 1\}$. There obviously exist infinitely many n such that $\lambda(2n+1) = \varepsilon$ (just take $2n+1$ to be the product of an appropriate number of odd primes). Now, if either $\lambda(2n) = \varepsilon$ or $\lambda(2n+2) = \varepsilon$, we are done; otherwise $\lambda(n) = -\lambda(2n) = -\lambda(2n+2) = \lambda(n+1) = \varepsilon$. Therefore, for such an n, one of the three pairs $(n, n+1)$, $(2n, 2n+1)$ or $(2n+1, 2n+2)$ fits the bill.

We have thus proved the existence in $\mathfrak{S}$ of infinitely many occurrences of all possible subsequences of length 1, viz. $(+1)$ and (-1), and of length 2, viz. $(+1,-1)$, $(-1,+1)$, $(+1,+1)$ and $(-1,-1)$.[3] ■

Problem 5. For every $n \geq 3$, determine all the configurations of n distinct points $X_1, X_2, \ldots, X_n$ in the plane, with the property that for any pair of distinct points X_i, X_j there exists a permutation σ of the integers $\{1, \ldots, n\}$, such that $\mathrm{d}(X_i, X_k) = \mathrm{d}(X_j, X_{\sigma(k)})$ for all $1 \leq k \leq n$.
(We write $\mathrm{d}(X, Y)$ to denote the distance between points X and Y.)

(UNITED KINGDOM) LUKE BETTS

Solution. Let us first prove that the points must be concyclic. Assign to each point X_k the vector x_k in a system of orthogonal coordinates whose origin is the point of mass of the configuration, thus $\frac{1}{n}\sum_{k=1}^{n} x_k = 0$.

Then $\mathrm{d}^2(X_i, X_k) = ||x_i - x_k||^2 = \langle x_i - x_k, x_i - x_k \rangle = ||x_i||^2 - 2\langle x_i, x_k \rangle + ||x_k||^2$, hence $\sum_{k=1}^{n} \mathrm{d}^2(X_i, X_k) = n||x_i||^2 -$

$2\langle x_i, \sum_{k=1}^{n} x_k\rangle + \sum_{k=1}^{n} ||x_k||^2 = n||x_i||^2 + \sum_{k=1}^{n} ||x_k||^2 = n||x_j||^2 + \sum_{k=1}^{n} ||x_{\sigma(k)}||^2 = \sum_{k=1}^{n} \mathrm{d}^2(X_j, X_{\sigma(k)})$, therefore $||x_i|| = ||x_j||$ for all pairs (i,j). The points are thus concyclic (lying on a circle centred at $O(0,0)$).

Let now m be the least angular distance between any two points. Two points situated at angular distance m must be adjacent on the circle. Let us connect each pair of such two points with an edge. The graph G obtained must be regular, of degree $\deg(G) = 1$ or 2. If n is odd, since $\sum_{k=1}^{n} \deg(X_k) = n\deg(G) = 2|E|$, we must have $\deg(G) = 2$, hence the configuration is a regular n-gon.

If n is even, we may have the configuration of a regular n-gon, but we also may have $\deg(G) = 1$. In that case, let M be the next least angular distance between any two points; such points must also be adjacent on the circle. Let us connect each pair of such two points with an edge, in order to get a graph G'. A similar reasoning yields $\deg(G') = 1$, thus the configuration is that of an equiangular n-gon (with alternating equal side-lengths). ■

Problem 6. The cells of a square 2011×2011 array are labelled with the integers $1, 2, \ldots, 2011^2$, in such a way that every label is used exactly once. We then identify the left-hand and right-hand edges, and then the top and bottom, in the normal way to form a torus (the surface of a doughnut).

Determine the largest positive integer M such that, no matter which labelling we choose, there exist two neighbouring cells with the difference of their labels at least M.[4]

(ROMANIA) DAN SCHWARZ

Preamble. For a planar $N \times N$ array, it is folklore that this value is $M = N$, with some easy models shown below. As such, the problem is mentioned in [BÉLA BOLLOBÁS - *The Art of Mathematics*], **21. Neighbours in a Matrix**.

This is not necessarily a flaw on the actual problem, which is presented in a brand novel setting; on the contrary, some general previous knowledge on such type of problems (which we think must be encouraged) is beneficial in searching for the right ideas of a proof.

The idea for a proof goes along the lines of finding a moment in the consecutive filling with numbers of the array, when there are at least N pairs of adjacent filled/yet-unfilled cells (with either distinct filled cells or distinct yet-unfilled cells). Then, when the cell next to that bearing the least label is filled, the difference between its label and the one being filled will be at least N. □

1	2	…	N
N+1	N+2	…	2N
⋮	⋮	⋱	⋮
(N-1)N+1	(N-1)N+2	…	N^2

A planar parallel $N \times N$ model array.

1	2	4	…		N(N-1)/2 + 1
3	5		…	N(N-1)/2 + 2	
6			…		
⋮	⋮	⋮	⋱	⋮	⋮
	N(N+1)/2 - 1		…		N^2 - 2
N(N+1)/2			…	N^2 - 1	N^2

A planar diagonal $N \times N$ model array.

Solution. For the toroidal case, it is clear the statement of the problem is referring to the cells of a $\mathbb{Z}_N \times \mathbb{Z}_N$ lattice on the surface of the torus, labeled with the numbers $1, 2, \ldots, N^2$, where one has to determine the least possible maximal absolute value M of the difference of labels assigned to orthogonally adjacent cells.

The toroidal $N = 2$ case is trivially seen to be $M = 2$ (thus coinciding with the planar case).

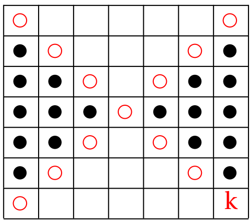
1	2
3	4

The unique 2×2 toroidal array.

For $N \geq 3$ we will prove that value to be at least $M \geq 2N - 1$. Consider such a configuration, and color all cells of the square in white. Go along the cells labeled 1, 2, etc. coloring them in black, stopping just on the cell bearing the least label k which, after assigned and colored in black, makes that all lines of a same orientation (rows, or columns, or both) contain at least two black cells (that is, before coloring in black the cell labeled k, at least one row and at least one column contained at most one black cell). Wlog assume this happens for rows. Then at most one row is all black, since if two were then the stopping condition would have been fulfilled before cell labeled k (if the cell labeled k were to be on one of these rows, then all rows would have contained at least two black cells before, while if not, then all columns would have contained at least two black cells before).

Now color in red all those black cells adjacent to a white cell. Since each row, except the potential all black one, contained at least two black and one white cell, it will now contain at least two red cells. For the potential all black row, any of the neighbouring rows contains at least one white cell, and so the cell adjacent to it has been colored red. In total we have therefore colored red at least $2(N-1) + 1 = 2N - 1$ cells.

The least label of the red cells has therefore at most the value $k + 1 - (2N - 1)$. When the white cell adjacent to it will eventually be labeled, its label will be at least $k + 1$, therefore their difference is at least $(k+1) - (k+1-(2N-1)) = 2N - 1$.



Example of coloring the array.

The models are kind of hard to find, due to the fact that the direct proof offers little as to their structure (it is difficult to determine the equality case during the argument involving the inequality with the bound, and then, even this is not sure to be prone to being prolonged to a full labeling of the array).

The weaker fact the value M is not larger than $2N$ is proved by the general model exhibited below (presented so that partial credits may be awarded).

N+1	N+2	…	2N
3N+1	3N+2	…	4N
⋮	⋮	⋱	⋮
(2ℓ-1)N+1	(2ℓ-1)N+2	…	2ℓN
⋮	⋮	⋱	⋮
2kN+1	2kN+2	…	(2k+1)N
⋮	⋮	⋱	⋮
2N+1	2N+2	…	3N
1	2	…	N

A general model for $M = 2N$ in a $N \times N$ array.

By examining some small $N > 2$ cases, one comes up with the idea of spiral models for the true value $\boxed{M = 2N - 1}$. The models presented are for odd N (since 2011 is odd); similar models exist for even N (but are less symmetric). The color red (preceded by green) marks the moment where the largest difference $M = 2N - 1$ first appears. ■

7	2	6
3	1	5
8	4	9

TABLE I: The spiral 3×3 array.

16	14	7	13	16
12	8	2	6	12
9	3	1	5	9
15	10	4	11	15
16	14	7	13	

TABLE II: The spiral 4×4 array.

23	16	7	15	22
17	8	2	6	14
9	3	1	5	13
18	10	4	12	21
24	19	11	20	25

TABLE III: The spiral 5×5 array.

47	40	29	16	28	39	46
41	30	17	7	15	27	38
31	18	8	2	6	14	26
19	9	3	1	5	13	25
32	20	10	4	12	24	37
42	33	21	11	23	36	45
48	43	34	22	35	44	49

TABLE IV: The spiral 7×7 array.

(2n+1)^2-2	(2n+1)^2-9	…		n(2n-1)+1		…	(2n+1)^2-10	(2n+1)^2-3
(2n+1)^2-8		…	n(2n-1)+2		n(2n-1)	…		(2n+1)^2-11
⋮	⋮	⋱	⋮	⋮	⋮	⋱	2n(n+1)+3	⋮
	2n^2	…	8	2	6	…	2n(n-1)+2	2n(n+1)+2
2n^2 + 1		…	3	1	5	…		2n(n+1)+1
	2n^2+2	…	10	4	12	…	2n(n+1)	
⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
(2n+1)^2-7		…	n(2n+1)		n(2+1)+2	…		(2n+1)^2-4
(2n+1)^2-1	(2n+1)^2-6	…		n(2n+1)+1		…	(2n+1)^2-5	(2n+1)^2

TABLE V: The general spiral $N \times N$ array for $N = 2n + 1 \geq 5$.

---

[1] Also see Sloane's Online Encyclopædia of Integer Sequences (OEIS), sequence A001222 for Ω and sequence A008836 for λ, which is called Liouville's function. Its summatory function $\sum_{d|n} \lambda(d)$ is equal to 1 for a perfect square n, and 0 otherwise. Pólya conjectured that $L(n) := \sum_{k=1}^{n} \lambda(k) \leq 0$ for all n, but this has been proven false by Minoru Tanaka, who in 1980 computed that for $n = 906,151,257$ its value was positive. Turán showed that if $T(n) := \sum_{k=1}^{n} \frac{\lambda(k)}{k} \geq 0$ for all large enough n, that will imply Riemann's Hypothesis; however, Haselgrove proved it is negative infinitely often.

[2] Using the same procedure for point i), we only need notice that $\lambda((2k+1)^2) = \lambda((2k)^2) = 1$, and these terms again are of different parity of their position.

[3] Is this true for subsequences of all lengths $\ell = 3, 4$, etc.? If no, up to which length $\ell \geq 2$?

[4] Cells with coordinates (x, y) and (x', y') are considered to be neighbours if $x = x'$ and $y - y' \equiv \pm 1 \pmod{2011}$, or if $y = y'$ and $x - x' \equiv \pm 1 \pmod{2011}$.

The 5$^{\text{th}}$ Romanian Master of Mathematics Competition

Solutions for the Day 1

Problem 1. Given a finite group of boys and girls, a *covering set of boys* is a set of boys such that every girl knows at least one boy in that set; and a *covering set of girls* is a set of girls such that every boy knows at least one girl in that set. Prove that the number of covering sets of boys and the number of covering sets of girls have the same parity. (Acquaintance is assumed to be mutual.)

Solution 1. A set X of boys is *separated* from a set Y of girls if no boy in X is an acquaintance of a girl in Y. Similarly, a set Y of girls is *separated* from a set X of boys if no girl in Y is an acquaintance of a boy in X. Since acquaintance is assumed mutual, separation is symmetric: X is separated from Y if and only if Y is separated from X.

This enables doubly counting the number n of ordered pairs (X, Y) of separated sets X, of boys, and Y, of girls, and thereby showing that it is congruent modulo 2 to both numbers in question.

Given a set X of boys, let Y_X be the largest set of girls separated from X, to deduce that X is separated from exactly $2^{|Y_X|}$ sets of girls. Consequently, $n = \sum_X 2^{|Y_X|}$ which is clearly congruent modulo 2 to the number of covering sets of boys.

Mutatis mutandis, the argument applies to show n congruent modulo 2 to the number of covering sets of girls.

Remark. The argument in this solution translates verbatim in terms of the adjancency matrix of the associated acquaintance graph.

Solution 2. (Ilya Bogdanov) Let B denote the set of boys, let G denote the set of girls and induct on $|B| + |G|$. The assertion is vacuously true if either set is empty.

Next, fix a boy b, let $B' = B \setminus \{b\}$, and let G' be the set of all girls who do not know b. Notice that:

(1) a covering set of boys in $B' \cup G$ is still one in $B \cup G$; and

(2) a covering set of boys in $B \cup G$ which is no longer one in $B' \cup G$ is precisely the union of a covering set of boys in $B' \cup G'$ and $\{b\}$,

so the number of covering sets of boys in $B \cup G$ is the sum of those in $B' \cup G$ and $B' \cup G'$.

On the other hand,

(1$'$) a covering set of girls in $B \cup G$ is still one in $B' \cup G$; and

(2$'$) a covering set of girls in $B' \cup G$ which is no longer one in $B \cup G$ is precisely a covering set of girls in $B' \cup G'$,

so the number of covering sets of girls in $B \cup G$ is the difference of those in $B' \cup G$ and $B' \cup G'$.

Since the assertion is true for both $B' \cup G$ and $B' \cup G'$ by the induction hypothesis, the conclusion follows.

Solution 3. (Géza Kós) Let B and G denote the sets of boys and girls, respectively. For every pair $(b, g) \in B \times G$, write $f(b, g) = 0$ if they know each other, and $f(b, g) = 1$ otherwise. A set X of boys is covering if and only if

$$\prod_{g \in G} \left(1 - \prod_{b \in X} f(b, g)\right) = 1.$$

Hence the number of covering sets of boys is

$$\begin{aligned} \sum_{X \subseteq B} \prod_{g \in G} \left(1 - \prod_{b \in X} f(b, g)\right) &\equiv \sum_{X \subseteq B} \prod_{g \in G} \left(1 + \prod_{b \in X} f(b, g)\right) \\ &= \sum_{X \subseteq B} \sum_{Y \subseteq G} \prod_{b \in X} \prod_{g \in Y} f(b, g) \pmod{2}. \end{aligned}$$

By symmetry, the same is valid for the number of covering sets of girls.

Problem 2. Given a triangle ABC, let D, E, and F respectively denote the midpoints of the sides BC, CA, and AB. The circle BCF and the line BE meet again at P, and the circle ABE and the line AD meet again at Q. Finally, the lines DP and FQ meet at R. Prove that the centroid G of the triangle ABC lies on the circle PQR.

Solution 1. We will use the following lemma.

Lemma. *Let AD be a median in triangle ABC. Then $\cot\angle BAD = 2\cot A + \cot B$ and $\cot\angle ADC = \frac{1}{2}(\cot B - \cot C)$.*

Proof. Let CC_1 and DD_1 be the perpendiculars from C and D to AB. Using the signed lengths we write

$$\cot BAD = \frac{AD_1}{DD_1} = \frac{(AC_1 + AB)/2}{CC_1/2} = \frac{CC_1\cot A + CC_1(\cot A + \cot B)}{CC_1} = 2\cot A + \cot B.$$

Similarly, denoting by A_1 the projection of A onto BC, we get

$$\cot ADC = \frac{DA_1}{AA_1} = \frac{BC/2 - A_1C}{AA_1} = \frac{(AA_1\cot B + AA_1\cot C)/2 - AA_1\cot C}{AA_1} = \frac{\cot B - \cot C}{2}.$$

The Lemma is proved.

Turning to the solution, by the Lemma we get

$$\cot\angle BPD = 2\cot\angle BPC + \cot\angle PBC = 2\cot\angle BFC + \cot\angle PBC \quad \text{(from circle } BFPC\text{)}$$
$$= 2\cdot\frac{1}{2}(\cot A - \cot B) + 2\cot B + \cot C = \cot A + \cot B + \cot C.$$

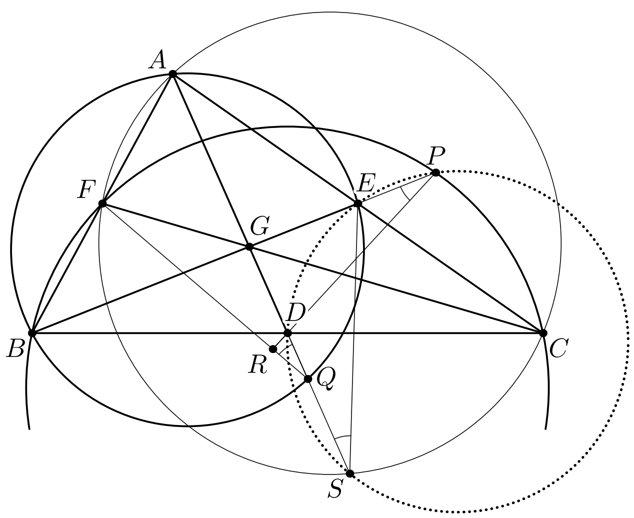
Similarly, $\cot\angle GQF = \cot A + \cot B + \cot C$, so $\angle GPR = \angle GQF$ and $GPRQ$ is cyclic.

Remark. The angle $\angle GPR = \angle GQF$ is the Brocard angle.

Solution 2. (Ilya Bogdanov and Marian Andronache) We also prove that $\angle(RP, PG) = \angle(RQ, QG)$, or $\angle(DP, PG) = \angle(FQ, QG)$.

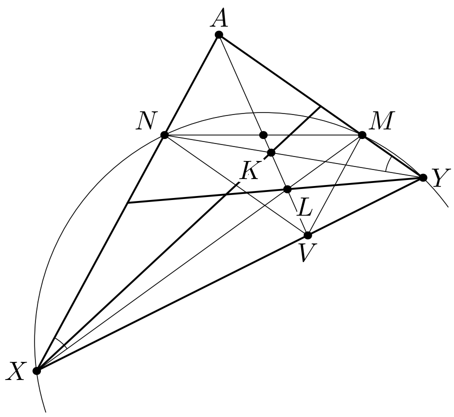
Let S be the point on ray GD such that $AG\cdot GS = CG\cdot GF$ (so the points A, S, C, F are concyclic). Then $GP\cdot GE = GP\cdot\frac{1}{2}GB = \frac{1}{2}CG\cdot GF = \frac{1}{2}AG\cdot GS = GD\cdot GS$, hence the points E, P, D, S are also concyclic, and $\angle(DP, PG) = \angle(GS, SE)$. The problem may therefore be rephrased as follows:

Given a triangle ABC, let D, E and F respectively denote the midpoints of the sides BC, CA and AB. The circle ABE, respectively, ACF, and the line AD meet again at Q, respectively, S. Prove that $\angle AQF = \angle ASE$ (and $ES = FQ$).



Upon inversion of pole A, the problem reads:

Given a triangle $AE'F'$, let the symmedian from A meet the medians from E' and F' at $K = Q'$ and $L = S'$, respectively. Prove that the angles $AE'L$ and $AF'K$ are congruent.

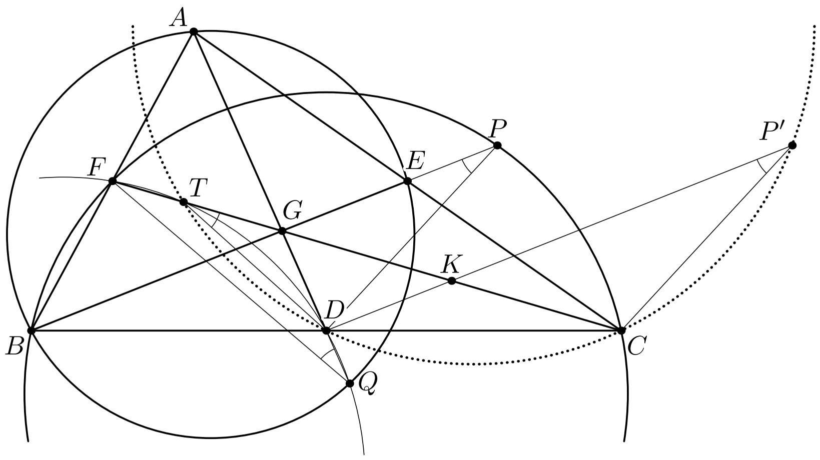


To prove this, denote $E' = X$, $F' = Y$. Let the symmedian from A meet the side XY at V and let the lines XL and YK meet the sides AY and AX at M and N, respectively. Since the points K and L lie on the medians, we have $VM \parallel AX$, $VN \parallel AY$. Hence $AMVN$ is a parallelogram, the symmedian AV of triangle AXY supports the median of triangle AMN, which implies that the triangles AMN and AXY are similar. Hence the points M, N, X, Y are concyclic, and $\angle AXM = \angle AYN$, QED.

Remark 1. We know that the points X, Y, M, N are concyclic. Invert back from A and consider the circles AFQ and AES: the former meets AC again at M' and the latter meets AB again at N'. Then the points E, F, M', N' are concyclic.

Remark 2. The inversion at pole A also allows one to show that $\angle AQF$ is the Brocard angle, thus providing one more solution. In our notation, it is equivalent to the fact that the points Y, K, and Z are collinear, where Z is the Brocard point (so $\angle ZAX = \angle ZYA = \angle ZXY$). This is valid because the lines AV, XK, and YZ are the radical axes of the following circles: (i) passing through X and tangent to AY at A; (ii) passing through Y and tangent to AX at A; and (iii) passing through X and tangent to AY at Y. The point K is the radical center of these three circles.

Solution 3. (Ilya Bogdanov) Again, we will prove that $\angle(DP, PG) = \angle(FQ, QG)$. Mark a point T on the ray GF such that $GF \cdot GT = GQ \cdot GD$; then the points F, Q, D, T are concyclic, and $\angle(FQ, QG) = \angle(TG, TD) = \angle(TC, TD)$.



Shift the point P by the vector $\overrightarrow{BD}$ to obtain point P'. Then $\angle(DP, PG) = \angle(CP', P'D)$, and we need to prove that $\angle(CP', P'D) = \angle(CT, TD)$. This is precisely the condition that the points T, D, C, P' be concyclic.

Denote $GE = x$, $GF = y$. Then $GP \cdot GB = GC \cdot GF$, so $GP = y^2/x$. On the other hand, $GB \cdot GE = GQ \cdot GA = 2GQ \cdot GD = 2GT \cdot GF$, so $GT = x^2/y$. Denote by K the point of intersection of DP' and CT; we need to prove that $TK \cdot KC = DK \cdot KP'$.

Now, $DP' = BP = BG + GP = 2x + y^2/x$, $CT = CG + GT = 2y + x^2/y$, $DK = BG/2 = x$, $CK = CG/2 = y$. Hence the desired equality reads $x(x + y^2/x) = y(y + x^2/y)$ which is obvious.

Remark. The points B, T, E, and C are concyclic, hence the point T is also of the same kind as P and Q.

Problem 3. Each positive integer number is coloured red or blue. A function f from the set of positive integer numbers into itself has the following two properties:

(a) if $x \leq y$, then $f(x) \leq f(y)$; and

(b) if x, y and z are all (not necessarily distinct) positive integer numbers of the same colour and $x + y = z$, then $f(x) + f(y) = f(z)$.

Prove that there exists a positive number a such that $f(x) \leq ax$ for all positive integer numbers x.

Solution. For integer x, y, by a segment $[x, y]$ we always mean the set of all integers t such that $x \leq t \leq y$; the *length* of this segment is $y - x$.

If for every two positive integers x, y sharing the same colour we have $f(x)/x = f(y)/y$, then one can choose $a = \max\{f(r)/r, f(b)/b\}$, where r and b are arbitrary red and blue numbers, respectively. So we can assume that there are two red numbers x, y such that $f(x)/x \neq f(y)/y$.

Set $m = xy$. Then each segment of length m contains a blue number. Indeed, assume that all the numbers on the segment $[k, k+m]$ are red. Then

$$f(k+m) = f(k+xy) = f(k+x(y-1)) + f(x) = \cdots = f(k) + yf(x),$$
$$f(k+m) = f(k+xy) = f(k+(x-1)y) + f(y) = \cdots = f(k) + xf(y),$$

so $yf(x) = xf(y)$ — a contradiction. Now we consider two cases.

Case 1. Assume that there exists a segment $[k, k+m]$ of length m consisting of blue numbers. Define $D = \max\{f(k), \ldots, f(k+m)\}$. We claim that $f(z) - f(z-1) \leq D$, whatever $z > k$, and the conclusion follows. Consider the largest blue number b_1 not exceeding z, so $z - b_1 \leq m$, and some blue number b_2 on the segment $[b_1 + k, b_1 + k + m]$, so $b_2 > z$. Write $f(b_2) = f(b_1) + f(b_2 - b_1) \leq f(b_1) + D$ to deduce that $f(z+1) - f(z) \leq f(b_2) - f(b_1) \leq D$, as claimed.

Case 2. Each segment of length m contains numbers of both colours. Fix any red number $R \geq 2m$ such that $R + 1$ is blue and set $D = \max\{f(R), f(R+1)\}$. Now we claim that $f(z+1) - f(z) \leq D$, whatever $z > 2m$. Consider the largest red number r not exceeding z and the largest blue number b smaller than r; then $0 < z - b = (z - r) + (r - b) \leq 2m$, and $b + 1$ is red. Let $t = b + R + 1$; then $t > z$. If t is blue, then $f(t) = f(b) + f(R+1) \leq f(b) + D$, and $f(z+1) - f(z) \leq f(t) - f(b) \leq D$. Otherwise, $f(t) = f(b+1) + f(R) \leq f(b+1) + D$, hence $f(z+1) - f(z) \leq f(t) - f(b+1) \leq D$, as claimed.

The 5^{th} Romanian Master of Mathematics Competition

Solutions for the Day 2

Problem 4. Prove that there are infinitely many positive integer numbers n such that $2^{2^n+1}+1$ be divisible by n, but 2^n+1 be not.

Solution 1. Throughout the solution n stands for a positive integer. By Euler's theorem, $(2^{3^n}+1)(2^{3^n}-1) = 2^{2\cdot 3^n}-1 \equiv 0 \pmod{3^{n+1}}$. Since $2^{3^n}-1 \equiv 1 \pmod 3$, it follows that $2^{3^n}+1$ is divisible by 3^{n+1}.

The number $(2^{3^{n+1}}+1)/(2^{3^n}+1) = 2^{2\cdot 3^n} - 2^{3^n} + 1$ is greater than 3 and congruent to 3 modulo 9, so it has a prime factor $p_n > 3$ that does not divide $2^{3^n}+1$ (otherwise, $2^{3^n} \equiv -1 \pmod{p_n}$, so $2^{2\cdot 3^n} - 2^{3^n} + 1 \equiv 3 \pmod{p_n}$, contradicting the fact that p_n is a factor greater than 3 of $2^{2\cdot 3^n} - 2^{3^n} + 1$).

We now show that $a_n = 3^n p_n$ satisfies the conditions in the statement. Since $2^{a_n}+1 \equiv 2^{3^n}+1 \not\equiv 0 \pmod{p_n}$, it follows that a_n does not divide $2^{a_n}+1$.

On the other hand, 3^{n+1} divides $2^{3^n}+1$ which in turn divides $2^{a_n}+1$, so $2^{3^{n+1}}+1$ divides $2^{2^{a_n}+1}+1$. Finally, both 3^n and p_n divide $2^{3^{n+1}}+1$, so a_n divides $2^{2^{a_n}+1}+1$.

As n runs through the positive integers, the a_n are clearly pairwise distinct and the conclusion follows.

Solution 2. (Géza Kós) We show that the numbers $a_n = (2^{3^n}+1)/9$, $n \geq 2$, satisfy the conditions in the statement. To this end, recall the following well-known facts:

(1) If N is an odd positive integer, then $\nu_3(2^N+1) = \nu_3(N)+1$, where $\nu_3(a)$ is the exponent of 3 in the decomposition of the integer a into prime factors; and

(2) If M and N are odd positive integers, then $(2^M+1, 2^N+1) = 2^{(M,N)}+1$, where (a,b) is the greatest common divisor of the integers a and b.

By (1), $a_n = 3^{n-1}m$, where m is an odd positive integer not divisible by 3, and by (2),

$$(m, 2^{a_n}+1) \mid (2^{3^n}+1, 2^{a_n}+1) = 2^{(3^n, a_n)}+1 = 2^{3^{n-1}}+1 < \frac{2^{3^n}+1}{3^{n+1}} = m,$$

so m cannot divide $2^{a_n}+1$.

On the other hand, $3^{n-1} \mid 2^{2^{a_n}+1}+1$, for $\nu_3(2^{2^{a_n}+1}+1) > \nu_3(2^{a_n}+1) > \nu_3(a_n) = n-1$, and $m \mid 2^{2^{a_n}+1}+1$, for $3^{n-1} \mid a_n$, so $3^n \mid 2^{a_n}+1$ whence $m \mid 2^{3^n}+1 \mid 2^{2^{a_n}+1}+1$. Since 3^{n-1} and m are coprime, the conclusion follows.

Remarks. There are several variations of these solutions. For instance, let $b_1 = 3$ and $b_{n+1} = 2^{b_n}+1$, $n \geq 1$, and notice that b_n divides b_{n+1}. It can be shown that there are infinitely many indices n such that some prime factor p_n of b_{n+1} does not divide b_n. One checks that for these n's the $a_n = p_n b_{n-1}$ satisfy the required conditions.

Finally, the numbers $3^n \cdot 571$, $n \geq 2$, form yet another infinite set of positive integers fulfilling the conditions in the statement — the details are omitted.

Solution 3. (Dušan Djukić) Assume that n satisfies the conditions of the problem. We claim that the number $N = 2^n+1 > n$ also satisfies these conditions.

Firstly, since $n \nmid N$, the fact (2) from Solution 2 allows to conclude that $2^n+1 \nmid 2^N+1$, or $N \nmid 2^N+1$. Next, since $n \mid 2^{2^n+1}+1 = 2^N+1$, we obtain from the same fact that $N = 2^n+1 \mid 2^{2^N+1}+1$, thus confirming our claim.

Hence, it suffices to provide only one example, hence obtaining an infinite series by the claim. For instance, one may easily check that the number $n = 57$ fits.

Problem 5. Given a positive integer number $n \geq 3$, colour each cell of an $n \times n$ square array one of $[(n+2)^2/3]$ colours, each colour being used at least once. Prove that the cells of some 1×3 or 3×1 rectangular subarray have pairwise distinct colours.

Solution. For more convenience, say that a subarray of the $n \times n$ square array *bears* a colour if at least two of its cells share that colour.

We shall prove that the number of 1×3 and 3×1 rectangular subarrays, which is $2n(n-2)$, exceeds the number of such subarrays, each of which bears some colour. The key ingredient is the estimate in the lemma below.

Lemma. *If a colour is used exactly p times, then the number of 1×3 and 3×1 rectangular subarrays bearing that colour does not exceed $3(p-1)$.*

Assume the lemma for the moment, let $N = [(n+2)^2/3]$ and let n_i be the number of cells coloured the ith colour, $i = 1, \ldots, N$, to deduce that the number of 1×3 and 3×1 rectangular subarrays, each of which bears some colour, is at most

$$\sum_{i=1}^{N} 3(n_i - 1) = 3\sum_{i=1}^{N} n_i - 3N = 3n^2 - 3N < 3n^2 - (n^2 + 4n) = 2n(n-2)$$

and thereby conclude the proof.

Back to the lemma, the assertion is clear if $p = 1$, so let $p > 1$.

We begin by showing that if a row contains exactly q cells coloured C, then the number r of 3×1 rectangular subarrays bearing C does not exceed $3q/2 - 1$; of course, a similar estimate holds for a column. To this end, notice first that the case $q = 1$ is trivial, so we assume that $q > 1$. Consider the incidence of a cell c coloured C and a 3×1 rectangular subarray R bearing C:

$$\langle c, R\rangle = \begin{cases} 1 & \text{if } c \subset R, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, given R, $\sum_c \langle c, R\rangle \geq 2$, and, given c, $\sum_R \langle c, R\rangle \leq 3$; moreover, if c is the leftmost or rightmost cell, then $\sum_R \langle c, R\rangle \leq 2$. Consequently,

$$2r \leq \sum_R \sum_c \langle c, R\rangle = \sum_c \sum_R \langle c, R\rangle \leq 2 + 3(q-2) + 2 = 3q - 2,$$

whence the conclusion.

Finally, let the p cells coloured C lie on k rows and ℓ columns and notice that $k + \ell \geq 3$, for $p > 1$. By the preceding, the total number of 3×1 rectangular subarrays bearing C does not exceed $3p/2 - k$, and the total number of 1×3 rectangular subarrays bearing C does not exceed $3p/2 - \ell$, so the total number of 1×3 and 3×1 rectangular subarrays bearing C does not exceed $(3p/2 - k) + (3p/2 - \ell) = 3p - (k + \ell) \leq 3p - 3 = 3(p-1)$. This completes the proof.

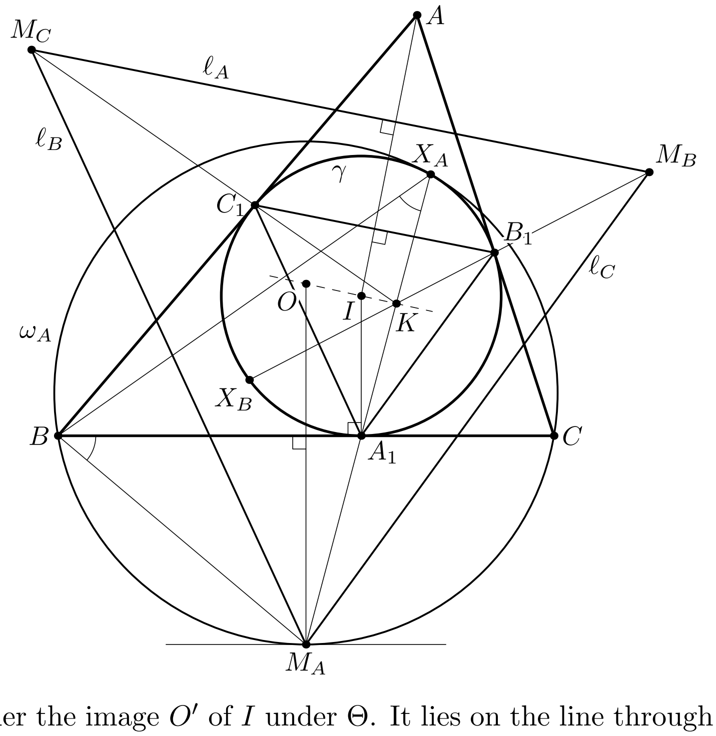
Remarks. In terms of the total number of cells, the number $N = [(n+2)^2/3]$ of colours is asymptotically close to the minimum number of colours required for some 1×3 or 3×1 rectangular subarray to have all cells of pairwise distinct colours, whatever the colouring. To see this, colour the cells with the coordinates (i, j), where $i+j \equiv 0 \pmod 3$ and $i, j \in \{0, 1, \ldots, n-1\}$, one colour each, and use one additional colour C to colour the remaining cells. Then each 1×3 and each 3×1 rectangular subarray has exactly two cells coloured C, and the number of colours is $\lceil n^2/3\rceil + 1$ if $n \equiv 1$ or $2 \pmod 3$, and $\lceil n^2/3\rceil$ if $n \equiv 0 \pmod 3$. Consequently, the minimum number of colours is $n^2/3 + O(n)$.

Problem 6. Let ABC be a triangle and let I and O respectively denote its incentre and circumcentre. Let ω_A be the circle through B and C and tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C through A meet again at A'; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

Solution. Let γ be the incircle of the triangle ABC and let A_1, B_1, C_1 be its contact points with the sides BC, CA, AB, respectively. Let further X_A be the point of contact of the circles γ and ω_A. The latter circle is the image of the former under a homothety centred at X_A. This homothety sends A_1 to a point M_A on ω_A such that the tangent to ω_A at M_A is parallel to BC. Consequently, M_A is the midpoint of the arc BC of ω_A not containing X_A. It follows that the angles M_AX_AB and M_ABC are congruent, so the triangles M_ABA_1 and M_AX_AB are similar: $M_AB/M_AX_A = M_AA_1/M_AB$. Rewrite the latter $M_AB^2 = M_AA_1 \cdot M_AX_A$ to deduce that M_A lies on the radical axis ℓ_B of B and γ. Similarly, M_A lies on the radical axis ℓ_C of C and γ.

Define the points X_B, X_C, M_B, M_C and the line ℓ_A in a similar way and notice that the lines ℓ_A, ℓ_B, ℓ_C support the sides of the triangle $M_AM_BM_C$. The lines ℓ_A and B_1C_1 are both perpendicular to AI, so they are parallel. Similarly, the lines ℓ_B and ℓ_C are parallel to C_1A_1 and A_1B_1, respectively. Consequently, the triangle $M_AM_BM_C$ is the image of the triangle $A_1B_1C_1$ under a homothety Θ. Let K be the centre of Θ and let $k = M_AK/A_1K = M_BK/B_1K = M_CK/C_1K$ be the similitude ratio. Notice that the lines M_AA_1, M_BB_1 and M_CC_1 are concurrent at K.

Since the points A_1, B_1, X_A, X_B are concyclic, $A_1K \cdot KX_A = B_1K \cdot KX_B$. Multiply both sides by k to get $M_AK \cdot KX_A = M_BK \cdot KX_B$ and deduce thereby that K lies on the radical axis CC' of ω_A and ω_B. Similarly, both lines AA' and BB' pass through K.

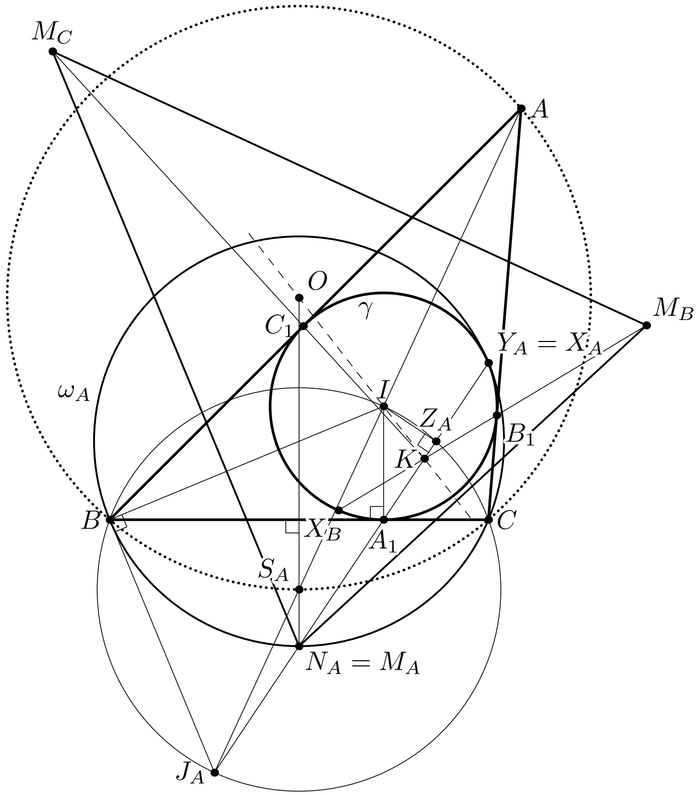


Finally, consider the image O' of I under Θ. It lies on the line through M_A parallel to A_1I (and hence perpendicular to BC); since M_A is the midpoint of the arc BC, this line must be M_AO. Similarly, O' lies on the line M_BO, so $O' = O$. Consequently, the points I, K and O are collinear.

Remark 1. Many steps in this solution allow different reasonings. For instance, one may see that the lines A_1X_A and B_1X_B are concurrent at point K on the radical axis CC' of the circles ω_A and ω_B by applying Newton's theorem to the quadrilateral $X_AX_BA_1B_1$ (since the common tangents at X_A and X_B intersect on CC'). Then one can conclude that $KA_1/KB_1 = KM_A/KM_B$, thus obtaining that the triangles $M_AM_BM_C$ and $A_1B_1C_1$ are homothetical at K (and therefore K is the radical center of ω_A, ω_B, and ω_C). Finally, considering the inversion with the pole K and the power equal to $KX_1 \cdot KM_A$ followed by the reflection at P we see that the circles ω_A, ω_B, and ω_C are invariant under this transform; next, the image of γ is the circumcircle of $M_AM_BM_C$ and it is tangent to all the circles ω_A, ω_B, and ω_C, hence its center is O, and thus O, I, and K are collinear.

Remark 2. Here is an outline of an alternative approach to the first part of the solution. Let J_A be the excentre of the triangle ABC opposite A. The line J_AA_1 meets γ again at Y_A; let Z_A and N_A be the midpoints of the segments A_1Y_A and J_AA_1, respectively. Since the segment IJ_A is a diameter in the circle BCZ_A, it follows that $BA_1 \cdot CA_1 = Z_AA_1 \cdot J_AA_1$, so $BA_1 \cdot CA_1 = N_AA_1 \cdot Y_AA_1$. Consequently, the points B, C, N_A and Y_A lie on some circle ω'_A.

It is well known that N_A lies on the perpendicular bisector of the segment BC, so the tangents to ω'_A and γ at N_A and A_1 are parallel. It follows that the tangents to these circles at Y_A coincide, so ω'_A is in fact ω_A, whence $X_A = Y_A$ and $M_A = N_A$. It is also well known that the midpoint S_A of the segment IJ_A lies both on the circumcircle ABC and on the perpendicular bisector of BC. Since S_AM_A is a midline in the triangle A_1IJ_A, it follows that $S_AM_A = r/2$, where r is the radius of γ (the inradius of the triangle ABC). Consequently, each of the points M_A, M_B and M_C is at distance $R + r/2$ from O (here R is the circumradius). Now proceed as above.



The 6th Romanian Master of Mathematics Competition

Solutions for the Day 1

Problem 1. For a positive integer a, define a sequence of integers $x_1, x_2, \ldots$ by letting $x_1 = a$ and $x_{n+1} = 2x_n + 1$. Let $y_n = 2^{x_n} - 1$. Determine the largest possible k such that, for some positive integer a, the numbers $y_1, \ldots, y_k$ are all prime.

(Russia) Valery Senderov

Solution. The largest such is $k = 2$. Notice first that if y_i is prime, then x_i is prime as well. Actually, if $x_i = 1$ then $y_i = 1$ which is not prime, and if $x_i = mn$ for integer $m, n > 1$ then $2^m - 1 \mid 2^{x_i} - 1 = y_i$, so y_i is composite. In particular, if $y_1, y_2, \ldots, y_k$ are primes for some $k \geq 1$ then $a = x_1$ is also prime.

Now we claim that for every odd prime a at least one of the numbers y_1, y_2, y_3 is composite (and thus $k < 3$). Assume, to the contrary, that y_1, y_2, and y_3 are primes; then x_1, x_2, x_3 are primes as well. Since $x_1 \geq 3$ is odd, we have $x_2 > 3$ and $x_2 \equiv 3 \pmod 4$; consequently, $x_3 \equiv 7 \pmod 8$. This implies that 2 is a quadratic residue modulo $p = x_3$, so $2 \equiv s^2 \pmod p$ for some integer s, and hence $2^{x_2} = 2^{(p-1)/2} \equiv s^{p-1} \equiv 1 \pmod p$. This means that $p \mid y_2$, thus $2^{x_2} - 1 = x_3 = 2x_2 + 1$. But it is easy to show that $2^t - 1 > 2t + 1$ for all integer $t > 3$. A contradiction.

Finally, if $a = 2$, then the numbers $y_1 = 3$ and $y_2 = 31$ are primes, while $y_3 = 2^{11} - 1$ is divisible by 23; in this case we may choose $k = 2$ but not $k = 3$.

Remark. The fact that $23 \mid 2^{11} - 1$ can be shown along the lines in the solution, since 2 is a quadratic residue modulo $x_4 = 23$.

Problem 2. We say a pair (g, h) of functions $g, h\colon \mathbb{R} \to \mathbb{R}$ is a *tester pair* just when the only function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(g(x)) = g(f(x))$ and $f(h(x)) = h(f(x))$ for all $x \in \mathbb{R}$ is the identity function. Does a tester pair exist?

(United Kingdom) Alexander Betts

Solution 1. Such a tester pair exists. We may biject $\mathbb{R}$ with the closed unit interval, so it suffices to find a tester pair for that instead. We give an explicit example: take some positive real numbers α, β (which we will specify further later). Take

$$g(x) = \max(x - \alpha, 0) \qquad \text{and} \qquad h(x) = \min(x + \beta, 1).$$

Say a set $S \subseteq [0, 1]$ is *invariant* if $f(S) \subseteq S$ for all functions f commuting with both g and h. Note that intersections and unions of invariant sets are invariant. Preimages of invariant sets under g and h are also invariant; indeed, if S is invariant and, say, $T = g^{-1}(S)$, then $g(f(T)) = f(g(T)) \subseteq f(S) \subseteq S$, thus $f(T) \subseteq T$.

We claim that (if we choose $\alpha + \beta < 1$) the intervals $[0, n\alpha - m\beta]$ are invariant where n and m are nonnegative integers with $0 \le n\alpha - m\beta \le 1$. We prove this by induction on $m + n$.

The set $\{0\}$ is invariant, as for any f commuting with g we have $g(f(0)) = f(g(0)) = f(0)$, so $f(0)$ is a fixed point of g. This gives that $f(0) = 0$, thus the induction base is established.

Suppose now we have some m, n such that $[0, n'\alpha - m'\beta]$ is invariant whenever $m' + n' < m + n$. At least one of the numbers $(n-1)\alpha - m\beta$ and $n\alpha - (m-1)\beta$ lies in $(0, 1)$. Note however that in the first case $[0, n\alpha - m\beta] = g^{-1}\left([0, (n-1)\alpha - m\beta]\right)$, so $[0, n\alpha - m\beta]$ is invariant. In the second case $[0, n\alpha - m\beta] = h^{-1}\left([0, n\alpha - (m-1)\beta]\right)$, so again $[0, n\alpha - m\beta]$ is invariant. This completes the induction.

We claim that if we choose $\alpha + \beta < 1$, where $0 < \alpha \notin \mathbb{Q}$ and $\beta = 1/k$ for some integer $k > 1$, then all intervals $[0, \delta]$ are invariant for $0 \le \delta < 1$. This occurs, as by the previous claim, for all nonnegative integers n we have $[0, (n\alpha \bmod 1)]$ is invariant. The set of $n\alpha \bmod 1$ is dense in $[0, 1]$, so in particular

$$[0, \delta] = \bigcap_{(n\alpha \bmod 1) > \delta} [0, (n\alpha \bmod 1)]$$

is invariant.

A similar argument establishes that $[\delta, 1]$ is invariant, so by intersecting these $\{\delta\}$ is invariant for $0 < \delta < 1$. Yet we also have $\{0\}, \{1\}$ both invariant, which proves f to be the identity.

Solution 2. Let us agree that a sequence $\mathbf{x} = (x_n)_{n=1,2,\ldots}$ is *cofinally non-constant* if for every index m there exists an index $n > m$ such that $x_m \neq x_n$.

Bijuct $\mathbb{R}$ with the set of cofinally non-constant sequences of 0's and 1's, and define g and h by

$$g(\epsilon, \mathbf{x}) = \begin{cases} \epsilon, \mathbf{x} & \text{if } \epsilon = 0 \\ \mathbf{x} & \text{else} \end{cases} \qquad \text{and} \qquad h(\epsilon, \mathbf{x}) = \begin{cases} \epsilon, \mathbf{x} & \text{if } \epsilon = 1 \\ \mathbf{x} & \text{else} \end{cases}$$

where $\epsilon, \mathbf{x}$ denotes the sequence formed by appending $\mathbf{x}$ to the single-element sequence ϵ. Note that g fixes precisely those sequences beginning with 0, and h fixes precisely those beginning with 1.

Now assume that f commutes with both f and g. To prove that $f(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ we show that $\mathbf{x}$ and $f(\mathbf{x})$ share the same first n terms, by induction on n.

The base case $n = 1$ is simple, as we have noticed above that the set of sequences beginning with a 0 is precisely the set of g-fixed points, so is preserved by f, and similarly for the set of sequences starting with 1.

Suppose that $f(\mathbf{x})$ and $\mathbf{x}$ agree for the first n terms, whatever $\mathbf{x}$. Consider any sequence, and write it as $\mathbf{x} = \epsilon, \mathbf{y}$. Without loss of generality, we may (and will) assume that $\epsilon = 0$, so $f(\mathbf{x}) = 0, \mathbf{y}'$ by the base case. Yet then $f(\mathbf{y}) = f(h(\mathbf{x})) = h(f(\mathbf{x})) = h(0, \mathbf{y}') = \mathbf{y}'$. Consequently, $f(\mathbf{x}) = 0, f(\mathbf{y})$, so $f(\mathbf{x})$ and $\mathbf{x}$ agree for the first $n+1$ terms by the inductive hypothesis.

Thus f fixes all of cofinally non-constant sequences, and the conclusion follows.

Solution 3. (*Ilya Bogdanov*) We will show that there exists a tester pair of *bijective* functions g and h.

First of all, let us find out when a pair of functions is a tester pair. Let $g, h\colon \mathbb{R} \to \mathbb{R}$ be arbitrary functions. We construct a directed graph $G_{g,h}$ with $\mathbb{R}$ as the set of vertices, its edges being painted with two colors: for every vertex $x \in \mathbb{R}$, we introduce a red edge $x \to g(x)$ and a blue edge $x \to h(x)$.

Now, assume that the function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $f(g(x)) = g(f(x))$ and $f(h(x)) = h(f(x))$ for all $x \in \mathbb{R}$. This means exactly that if there exists an edge $x \to y$, then there also exists an edge $f(x) \to f(y)$ of the same color; that is — f is an *endomorphism* of $G_{g,h}$.

Thus, a pair (g, h) is a tester pair if and only if the graph $G_{g,h}$ admits no nontrivial endomorphisms. Notice that each endomorphism maps a component into a component. Thus, to construct a tester pair, it suffices to construct a continuum of components with no nontrivial endomorphisms and no homomorphisms from one to another. It can be done in many ways; below we present one of them.

Let $g(x) = x + 1$; the construction of h is more involved. For every $x \in [0, 1)$ we define the set $S_x = x + \mathbb{Z}$; the sets S_x will be exactly the components of $G_{g,h}$. Now we will construct these components.

Let us fix any $x \in [0, 1)$; let $x = 0.x_1x_2\dots$ be the binary representation of x. Define $h(x-n) = x-n+1$ for every $n > 3$. Next, let $h(x-3) = x$, $h(x) = x-2$, $h(x-2) = x-1$, and $h(x-1) = x+1$ (that would be a "marker" which fixes a point in our component).

Next, for every $i = 1, 2, \dots$, we define

(1) $h(x+3i-2) = x+3i-1$, $h(x+3i-1) = x+3i$, and $h(x+3i) = x+3i+1$, if $x_i = 0$;

(2) $h(x+3i-2) = x+3i$, $h(x+3i) = 3i-1$, and $h(x+3i-1) = x+3i+1$, if $x_i = 1$.

Clearly, h is a bijection mapping each S_x to itself. Now we claim that the graph $G_{g,h}$ satisfies the desired conditions.

Consider any homomorphism $f_x\colon S_x \to S_y$ (x and y may coincide). Since g is a bijection, consideration of the red edges shows that $f_x(x+n) = x+n+k$ for a fixed real k. Next, there exists a blue edge $(x-3) \to x$, and the only blue edge of the form $(y+m-3) \to (y+m)$ is $(y-3) \to y$; thus $f_x(x) = y$, and $k = 0$.

Next, if $x_i = 0$ then there exists a blue edge $(x+3i-2) \to (x+3i-1)$; then the edge $(y+3i-2) \to (y+3i-1)$ also should exist, so $y_i = 0$. Analogously, if $x_i = 1$ then there exists a blue edge $(x+3i-2) \to (x+3i)$; then the edge $(y+3i-2) \to (y+3i)$ also should exist, so $y_i = 1$. We conclude that $x = y$, and f_x is the identity mapping, as required.

Remark. If g and h are injections, then the components of $G_{g,h}$ are at most countable. So the set of possible pairwise non-isomorphic such components is continual; hence there is no bijective tester pair for a hyper-continual set instead of $\mathbb{R}$.

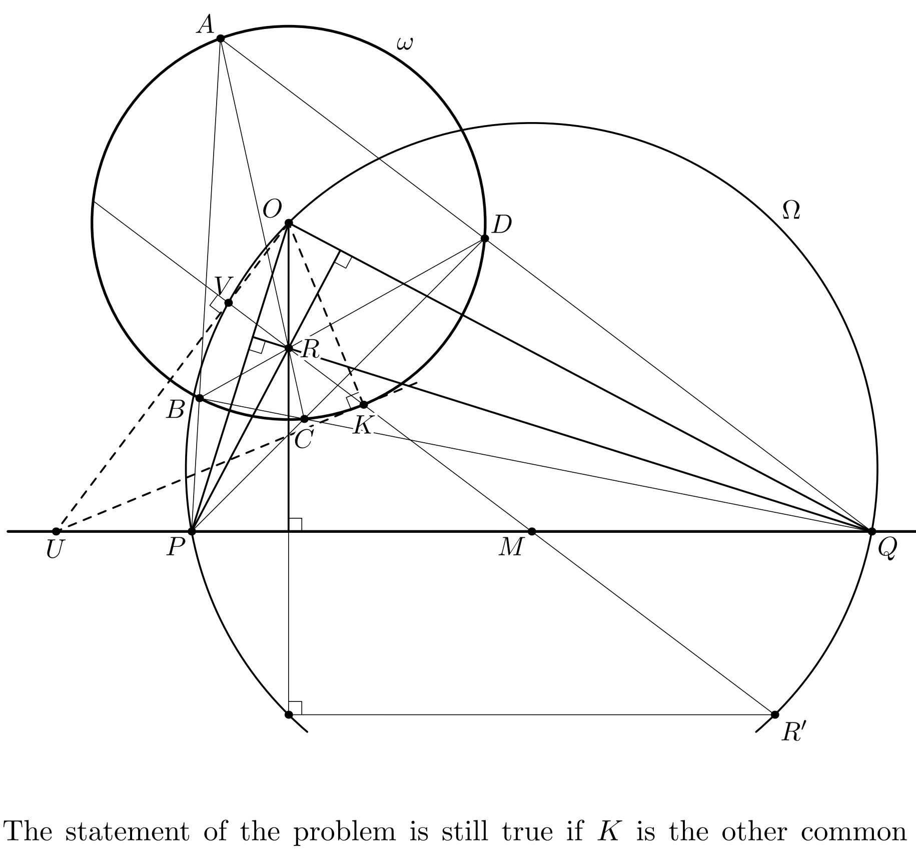
Problem 3. Let $ABCD$ be a quadrangle inscribed in a circle ω. The lines AB and CD meet at P, the lines AD and BC meet at Q, and the diagonals AC and BD meet at R. Let M be the midpoint of the segment PQ, and let K be the common point of the segment MR and the circle ω. Prove that the circles KPQ and ω are tangent to one another.

(Russia) Medeubek Kungozhin

Solution. Let O be the centre of ω. Notice that the points P, Q, and R are the poles (with respect to ω) of the lines QR, RP, and PQ, respectively. Hence we have $OP \perp QR$, $OQ \perp RP$, and $OR \perp PQ$, thus R is the orthocentre of the triangle OPQ. Now, if $MR \perp PQ$, then the points P and Q are the reflections of one another in the line $MR = MO$, and the triangle PQK is symmetrical with respect to this line. In this case the statement of the problem is trivial.

Otherwise, let V be the foot of the perpendicular from O to MR, and let U be the common point of the lines OV and PQ. Since U lies on the polar line of R and $OU \perp MR$, we obtain that U is the pole of MR. Therefore, the line UK is tangent to ω. Hence it is enough to prove that $UK^2 = UP \cdot UQ$, since this relation implies that UK is also tangent to the circle KPQ.

From the rectangular triangle OKU, we get $UK^2 = UV \cdot UO$. Let Ω be the circumcircle of triangle OPQ, and let R' be the reflection of its orthocentre R in the midpoint M of the side PQ. It is well known that R' is the point of Ω opposite to O, hence OR' is the diameter of Ω. Finally, since $\angle OVR' = 90^\circ$, the point V also lies on Ω, hence $UP \cdot UQ = UV \cdot UO = UK^2$, as required.



Remark. The statement of the problem is still true if K is the other common point of the line MR and ω.

The 6$^{\text{th}}$ Romanian Master of Mathematics Competition

Solutions for the Day 2

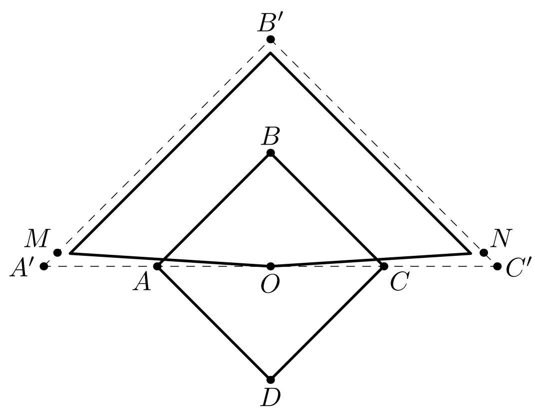
Problem 4. Suppose two convex quadrangles in the plane, P and P', share a point O such that, for every line ℓ through O, the segment along which ℓ and P meet is longer than the segment along which ℓ and P' meet. Is it possible that the ratio of the area of P' to the area of P be greater than 1.9?

(Bulgaria)

Solution. The answer is in the affirmative: Given a positive $\epsilon < 2$, the ratio in question may indeed be greater than $2 - \epsilon$.

To show this, consider a square $ABCD$ centred at O, and let A', B', and C' be the reflections of O in A, B, and C, respectively. Notice that, if ℓ is a line through O, then the segments $\ell \cap ABCD$ and $\ell \cap A'B'C'$ have equal lengths, unless ℓ is the line AC.

Next, consider the points M and N on the segments $B'A'$ and $B'C'$, respectively, such that $B'M/B'A' = B'N/B'C' = (1 - \epsilon/4)^{1/2}$. Finally, let P' be the image of the convex quadrangle $B'MON$ under the homothety of ratio $(1 - \epsilon/4)^{1/4}$ centred at O. Clearly, the quadrangles $P \equiv ABCD$ and P' satisfy the conditions in the statement, and the ratio of the area of P' to the area of P is exactly $2 - \epsilon/2$.



Remarks. **(1)** With some care, one may also construct such example with a point O being interior for both P and P'. In our example, it is enough to replace vertex O of P' by a point on the segment OD close enough to O. The details are left to the reader.

(2) On the other hand, one may show that the ratio of areas of P' and P cannot exceed 2 (even if P and P' are arbitrary convex polygons rather than quadrilaterals). Further on, we denote by $[S]$ the area of S.

In order to see that $[P'] < 2[P]$, let us fix some ray r from O, and let r_α be the ray from O making an (oriented) angle α with r. Denote by X_α and Y_α the points of P and P', respectively, lying on r_α farthest from O, and denote by $f(\alpha)$ and $g(\alpha)$ the lengths of the segments OX_α and OY_α, respectively. Then

$$[P] = \frac{1}{2}\int_0^{2\pi} f^2(\alpha)\, d\alpha = \frac{1}{2}\int_0^{\pi} \left(f^2(\alpha) + f^2(\pi + \alpha)\right) d\alpha,$$

and similarly

$$[P'] = \frac{1}{2}\int_0^{\pi} \left(g^2(\alpha) + g^2(\pi + \alpha)\right) d\alpha.$$

But $X_\alpha X_{\pi+\alpha} > Y_\alpha Y_{\pi+\alpha}$ yields $2 \cdot \frac{1}{2}\left(f^2(\alpha)+f^2(\pi+\alpha)\right) = OX_\alpha^2 + OX_{\pi+\alpha}^2 \geq \frac{1}{2}X_\alpha X_{\pi+\alpha}^2 > \frac{1}{2}Y_\alpha Y_{\pi+\alpha}^2 \geq \frac{1}{2}(OY_\alpha^2 + OY_{\pi+\alpha}^2) = \frac{1}{2}\left(g^2(\alpha)+g^2(\pi+\alpha)\right)$. Integration then gives us $2[P] > [P']$, as needed.

This can also be proved via elementary methods. Actually, we will establish the following more general fact.

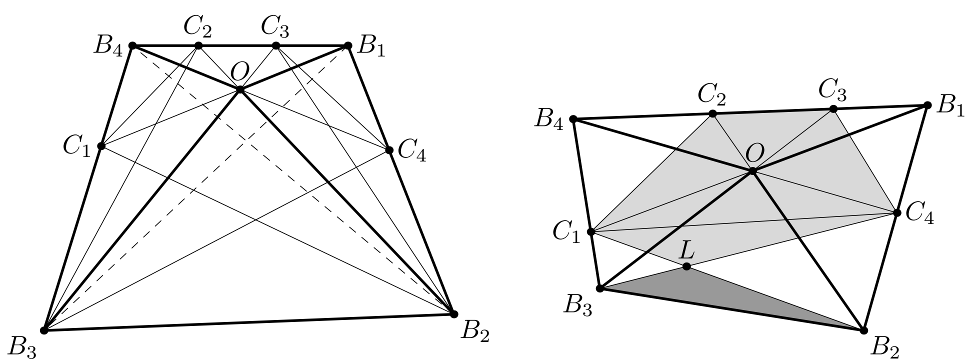
Fact. *Let $P = A_1A_2A_3A_4$ and $P' = B_1B_2B_3B_4$ be two convex quadrangles in the plane, and let O be one of their common points different from the vertices of P'. Denote by ℓ_i the line OB_i, and assume that for every $i = 1,2,3,4$ the length of segment $\ell_i \cap P$ is greater than the length of segment $\ell_i \cap P'$. Then $[P'] < 2[P]$.*

Proof. One of (possibly degenerate) quadrilaterals $OB_1B_2B_3$ and $OB_1B_4B_3$ is convex; the same holds for $OB_2B_3B_4$ and $OB_2B_1B_4$. Without loss of generality, we may (and will) assume that the quadrilaterals $OB_1B_2B_3$ and $OB_2B_3B_4$ are convex.

Denote by C_i such a point that $\ell_i \cap P'$ is the segment B_iC_i; let a_i be the length of $\ell_i \cap P$, and let α_i be the angle between ℓ_i and ℓ_{i+1} (hereafter, we use the cyclic notation, thus $\ell_5 = \ell_1$ and so on). Thus C_2 and C_3 belong to the segment B_1B_4, C_1 lies on B_3B_4, and C_4 lies on B_1B_2. Assume that there exists an index i such that the area of $B_iB_{i+1}C_iC_{i+1}$ is at least $[P']/2$; then we have

$$\frac{[P']}{2} \leq [B_iB_{i+1}C_iC_{i+1}] = \frac{B_iC_i \cdot B_{i+1}C_{i+1} \cdot \sin\alpha_i}{2} < \frac{a_ia_{i+1}\sin\alpha_i}{2} \leq [P],$$

as desired. Assume, to the contrary, that such index does not exist. Two cases are possible.



Case 1. Assume that the rays B_1B_2 and B_4B_3 do not intersect (see the left figure above). This means, in particular, that $d(B_1, B_3B_4) \leq d(B_2, B_3B_4)$.

Since the ray B_3O lies in the angle $B_1B_3B_4$, we obtain $d(B_1, B_3C_3) \leq d(C_4, B_3C_3)$; hence $[B_3B_4B_1] \leq [B_3B_4C_3C_4] < [P']/2$. Similarly, $[B_1B_2B_4] \leq [B_1B_2C_1C_2] < [P']/2$. Thus,

$$[B_2B_3C_2C_3] = [P'] - [B_1B_2C_3] - [B_3B_4C_2] = [P'] - \frac{B_1C_3}{B_1B_4} \cdot [B_1B_2B_4] - \frac{B_4C_2}{B_1B_4} \cdot [B_3B_4B_1]$$
$$> [P']\left(1 - \frac{B_1C_3 + B_4C_2}{2B_1B_4}\right) \geq \frac{[P']}{2}.$$

A contradiction.

Case 2. Assume now that the rays B_1B_2 and B_4B_3 intersect at some point (see the right figure above). Denote by L the common point of B_2C_1 and B_3C_4. We have $[B_2C_4C_1] \geq [B_2C_4B_3]$, hence $[C_1C_4L] \geq [B_2B_3L]$. Thus we have

$$[P'] > [B_1B_2C_1C_2] + [B_3B_4C_3C_4] = [P'] + [LC_1C_2C_3C_4] - [B_2B_3L]$$
$$\geq [P'] + [C_1C_4L] - [B_2B_3L] \geq [P'].$$

A final contradiction.

Problem 5. Given a positive integer $k \geq 2$, set $a_1 = 1$ and, for every integer $n \geq 2$, let a_n be the smallest solution of the equation

$$x = 1 + \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{x}{a_i}} \right\rfloor$$

that exceeds a_{n-1}. Prove that all primes are among the terms of the sequence $a_1, a_2, \ldots$.

(Bulgaria)

Solution 1. We prove that the a_n are precisely the *kth-power-free* positive integers, that is, those divisible by the kth power of no prime. The conclusion then follows.

Let B denote the set of all kth-power-free positive integers. We first show that, given a positive integer c,

$$\sum_{b \in B,\, b \leq c} \left\lfloor \sqrt[k]{\frac{c}{b}} \right\rfloor = c.$$

To this end, notice that every positive integer has a unique representation as a product of an element in B and a kth power. Consequently, the set of all positive integers less than or equal to c splits into

$$C_b = \{x \colon x \in \mathbb{Z}_{>0},\ x \leq c, \text{ and } x/b \text{ is a } k\text{th power}\}, \quad b \in B,\ b \leq c.$$

Clearly, $|C_b| = \lfloor \sqrt[k]{c/b} \rfloor$, whence the desired equality.

Finally, enumerate B according to the natural order: $1 = b_1 < b_2 < \cdots < b_n < \cdots$. We prove by induction on n that $a_n = b_n$. Clearly, $a_1 = b_1 = 1$, so let $n \geq 2$ and assume $a_m = b_m$ for all indices $m < n$. Since $b_n > b_{n-1} = a_{n-1}$ and

$$b_n = \sum_{i=1}^{n} \left\lfloor \sqrt[k]{\frac{b_n}{b_i}} \right\rfloor = \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{b_n}{b_i}} \right\rfloor + 1 = \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{b_n}{a_i}} \right\rfloor + 1,$$

the definition of a_n forces $a_n \leq b_n$. Were $a_n < b_n$, a contradiction would follow:

$$a_n = \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{a_n}{b_i}} \right\rfloor = \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{a_n}{a_i}} \right\rfloor = a_n - 1.$$

Consequently, $a_n = b_n$. This completes the proof.

Solution 2. (*Ilya Bogdanov*) For every $n = 1, 2, 3, \ldots$, introduce the function

$$f_n(x) = x - 1 - \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{x}{a_i}} \right\rfloor.$$

Denote also by $g_n(x)$ the number of the indices $i \leq n$ such that x/a_i is the kth power of an integer. Then $f_n(x+1) - f_n(x) = 1 - g_n(x)$ for every integer $x \geq a_n$; hence $f_n(x)+1 \geq f_n(x+1)$. Moreover, $f_n(a_{n-1}) = -1$ (since $f_{n-1}(a_{n-1}) = 0$). Now a straightforward induction shows that $f_n(x) < 0$ for all integers $x \in [a_{n-1}, a_n)$.

Next, if $g_n(x) > 0$ then $f_n(x) \leq f_n(x-1)$; this means that such an x cannot equal a_n. Thus a_j/a_i is never the kth power of an integer if $j > i$.

Now we are prepared to prove by induction on n that $a_1, a_2, \ldots, a_n$ are exactly all kth-power-free integers in $[1, a_n]$. The base case $n = 1$ is trivial.

Assume that all the kth-power-free integers on $[1, a_n]$ are exactly $a_1, \ldots, a_n$. Let b be the least integer larger than a_n such that $g_n(b) = 0$. We claim that: **(1)** $b = a_{n+1}$; and **(2)** b is the least kth-power-free number greater than a_n.

To prove **(1)**, notice first that all the numbers of the form a_j/a_i with $1 \leq i < j \leq n$ are not kth powers of *rational* numbers since a_i and a_j are kth-power-free. This means that for every integer $x \in (a_n, b)$ there exists exactly one index $i \leq n$ such that x/a_i is the kth power of an integer (certainly, x is not kth-power-free). Hence $f_{n+1}(x) = f_{n+1}(x-1)$ for each such x, so $f_{n+1}(b-1) = f_{n+1}(a_n) = -1$. Next, since b/a_i is not the kth power of an integer, we have $f_{n+1}(b) = f_{n+1}(b-1) + 1 = 0$, thus $b = a_{n+1}$. This establishes **(1)**.

Finally, since all integers in (a_n, b) are not kth-power-free, we are left to prove that b is kth-power-free to establish **(2)**. Otherwise, let $y > 1$ be the greatest integer such that $y^k \mid b$; then b/y^k is kth-power-free and hence $b/y^k = a_i$ for some $i \leq n$. So b/a_i is the kth power of an integer, which contradicts the definition of b.

Thus $a_1, a_2, \ldots$ are exactly all kth-power-free positive integers; consequently all primes are contained in this sequence.

Problem 6. A token is placed at each vertex of a regular $2n$-gon. A *move* consists in choosing an edge of the $2n$-gon and swapping the two tokens placed at the endpoints of that edge. After a finite number of moves have been performed, it turns out that every two tokens have been swapped exactly once. Prove that some edge has never been chosen.

(Russia) Alexander Gribalko

Solution. Step 1. Enumerate all the tokens in the initial arrangement in clockwise circular order; also enumerate the vertices of the $2n$-gon accordingly. Consider any three tokens $i < j < k$. At each moment, their cyclic order may be either i, j, k or i, k, j, counted clockwise. This order changes exactly when two of these three tokens have been switched. Hence the order has been reversed thrice, and in the final arrangement token k stands on the arc passing clockwise from token i to token j. Thus, at the end, token $i + 1$ is a counter-clockwise neighbor of token i for all $i = 1, 2, \ldots, 2n - 1$, so the tokens in the final arrangement are numbered successively in counter-clockwise circular order.

This means that the final arrangement of tokens can be obtained from the initial one by reflection in some line ℓ.

Step 2. Notice that each token was involved into $2n-1$ switchings, so its initial and final vertices have different parity. Hence ℓ passes through the midpoints of two opposite sides of a $2n$-gon; we may assume that these are the sides a and b connecting $2n$ with 1 and n with $n+1$, respectively.

During the process, each token x has crossed ℓ at least once; thus one of its switchings has been made at edge a or at edge b. Assume that some two its switchings were performed at a and at b; we may (and will) assume that the one at a was earlier, and $x \leq n$. Then the total movement of token x consisted at least of: (i) moving from vertex x to a and crossing ℓ along a; (ii) moving from a to b and crossing ℓ along b; (iii) coming to vertex $2n + 1 - x$. This tales at least $x + n + (n - x) = 2n$ switchings, which is impossible.

Thus, each token had a switching at exactly one of the edges a and b.

Step 3. Finally, let us show that either each token has been switched at a, or each token has been switched at b (then the other edge has never been used, as desired). To the contrary, assume that there were switchings at both a and at b. Consider the first such switchings, and let x and y be the tokens which were moved clockwise during these switchings and crossed ℓ at a and b, respectively. By Step 2, $x \neq y$. Then tokens x and y initially were on opposite sides of ℓ.

Now consider the switching of tokens x and y; there was exactly one such switching, and we assume that it has been made on the same side of ℓ as vertex y. Then this switching has been made after token x had traced a. From this point on, token x is on the clockwise arc from token y to b, and it has no way to leave out from this arc. But this is impossible, since token y should trace b after that moment. A contradiction.

Remark. The same statement for $(2n-1)$-gon is also valid. The problem is stated for a polygon with an even number of sides only to avoid case consideration.

Let us outline the solution in the case of a $(2n - 1)$-gon. We prove the existence of line ℓ as in Step 1. This line passes through some vertex x, and through the midpoint of the opposite edge a. Then each token either passes through x, or crosses ℓ along a (but not both; this can be shown as in Step 2). Finally, since a token is involved into an even number of moves, it passes through x but not through a, and a is never used.

The 7^{th} Romanian Master of Mathematics Competition

Day 1: Friday, February 27, 2015, Bucharest

Language: English

Problem 1. Does there exist an infinite sequence of positive integers a_1, a_2, a_3, ... such that a_m and a_n are coprime if and only if $|m - n| = 1$?

Problem 2. For an integer $n \geq 5$, two players play the following game on a regular n-gon. Initially, three consecutive vertices are chosen, and one counter is placed on each. A move consists of one player sliding one counter along any number of edges to another vertex of the n-gon without jumping over another counter. A move is *legal* if the area of the triangle formed by the counters is strictly greater after the move than before. The players take turns to make legal moves, and if a player cannot make a legal move, that player loses. For which values of n does the player making the first move have a winning strategy?

Problem 3. A finite list of rational numbers is written on a blackboard. In an *operation*, we choose any two numbers a, b, erase them, and write down one of the numbers

$$a + b,\ a - b,\ b - a,\ a \times b,\ a/b \text{ (if } b \neq 0),\ b/a \text{ (if } a \neq 0).$$

Prove that, for every integer $n > 100$, there are only finitely many integers $k \geq 0$, such that, starting from the list

$$k + 1,\ k + 2,\ \ldots,\ k + n,$$

it is possible to obtain, after $n - 1$ operations, the value $n!$.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

The 7^{th} Romanian Master of Mathematics Competition

Day 2: Saturday, February 28, 2015, Bucharest

Language: English

Problem 4. Let ABC be a triangle, and let D be the point where the incircle meets side BC. Let J_b and J_c be the incentres of the triangles ABD and ACD, respectively. Prove that the circumcentre of the triangle AJ_bJ_c lies on the angle bisector of $\angle BAC$.

Problem 5. Let $p \geq 5$ be a prime number. For a positive integer k, let $R(k)$ be the remainder when k is divided by p, with $0 \leq R(k) \leq p-1$. Determine all positive integers $a < p$ such that, for every $m = 1, 2, \ldots, p-1$,

$$m + R(ma) > a.$$

Problem 6. Given a positive integer n, determine the largest real number μ satisfying the following condition: for every set C of $4n$ points in the interior of the unit square U, there exists a rectangle T contained in U such that

- the sides of T are parallel to the sides of U;
- the interior of T contains exactly one point of C;
- the area of T is at least μ.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

The 7$^{\text{th}}$ Romanian Master of Mathematics Competition

Solutions for the Day 1

Problem 1. Does there exist an infinite sequence of positive integers a_1, a_2, a_3, $\ldots$ such that a_m and a_n are coprime if and only if $|m-n|=1$?

(Peru) Jorge Tipe

Solution. The answer is in the affirmative.

The idea is to consider a sequence of pairwise distinct primes p_1, p_2, p_3, $\ldots$, cover the positive integers by a sequence of finite non-empty sets I_n such that I_m and I_n are disjoint if and only if m and n are one unit apart, and set $a_n = \prod_{i\in I_n} p_i$, $n = 1, 2, 3, \ldots$.

One possible way of finding such sets is the following. For all positive integers n, let

$$2n \in I_k \qquad \text{for all } k = n, n+3, n+5, n+7, \ldots; \qquad \text{and}$$
$$2n-1 \in I_k \qquad \text{for all } k = n, n+2, n+4, n+6, \ldots.$$

Clearly, each I_k is finite, since it contains none of the numbers greater than $2k$. Next, the number p_{2n} ensures that I_n has a common element with each I_{n+2i}, while the number p_{2n-1} ensures that I_n has a common element with each I_{n+2i+1} for $i = 1, 2, \ldots$. Finally, none of the indices appears in two consecutive sets.

Remark. The sets I_n from the solution above can explicitly be written as

$$I_n = \{2n-4k-1\colon k = 0, 1, \ldots, \lfloor (n-1)/2\rfloor\} \cup \{2n-4k-2\colon k = 1, 2, \ldots, \lfloor n/2\rfloor - 1\} \cup \{2n\},$$

The above construction can alternatively be described as follows: Let p_1, p_1', p_2, p_2', $\ldots$, p_n, p_n', $\ldots$ be a sequence of pairwise distinct primes. With the standard convention that empty products are 1, let

$$P_n = \begin{cases} p_1p_2'p_3p_4'\cdots p_{n-4}p_{n-3}'p_{n-2}, & \text{if } n \text{ is odd,} \\ p_1'p_2p_3'p_4\cdots p_{n-3}'p_{n-2}, & \text{if } n \text{ is even,} \end{cases}$$

and define $a_n = P_np_np_n'$.

Problem 2. For an integer $n \geq 5$, two players play the following game on a regular n-gon. Initially, three consecutive vertices are chosen, and one counter is placed on each. A move consists of one player sliding one counter along any number of edges to another vertex of the n-gon without jumping over another counter. A move is *legal* if the area of the triangle formed by the counters is strictly greater after the move than before. The players take turns to make legal moves, and if a player cannot make a legal move, that player loses. For which values of n does the player making the first move have a winning strategy?

(United Kingdom) Jeremy King

Solution. We shall prove that the first player wins if and only the exponent of 2 in the prime decomposition of $n-3$ is odd.

Since the game is identical for both players, has finitely many possible states and always terminates, we can label the possible states Wins od Losses according as whether a player faced with that position has a winning strategy or not. A state is a Win if and only if there is some legal move taking the state to a Loss, and a state is a Loss if and only if all moves take that state to a Win (including the case where there are no legal moves).

Lemma. *Any configuration in which the triangle formed by the three counters is not isosceles is necessarily a Win.*

Proof. Label the positions of the counters X, Y, Z so that the arc YZ of the circumcircle is shortest and the arc ZX is longest. Begin by moving the counter at Z around the polygon on the arc YZX until it forms an isosceles triangle XYZ' with apex at Y (note that the arc XY is less than half the circle, so that Z does not jump over the counter at X). If this configuration is a Loss, we are done.

If instead this configuration is a Win, then the counters can be moved legally from triangle XYZ' to reach a losing state. This cannot involve the counter at Y, so by symmetry a Loss state can be reached by moving the counter at Z' to a new location Z''. But then the counter at Z could have been moved to Z'' in the first place, so the original configuration was a Win as well. □

For every nonzero integer x, denote by $v_2(x)$ the exponent of 2 in the prime decomposition of x. Now, given a configuration in which the triangle formed by the three counters is isosceles, the arcs between the vertices having lengths a, a, b respectively (in appropriate units so that $2a+b=n$), we show that the configuration is a Win if and only if $a \neq b$ and $v_2(a-b)$ is odd.

Write $b = a \pm |a-b|$ and notice that the only other isosceles triangle that can be reached from the original configuration is one with arc lengths a, $a \pm |a-b|/2$, $a \pm |a-b|/2$. If $|a-b|$ is odd, this is of course impossible, so the configuration is a Loss, since all non-isosceles configurations are Wins, by the lemma.

If instead $|a-b|$ is even, then all states that can be reached from the original configuration are Wins, except possibly the state with arc lengths a, $a \pm |a-b|/2$, $a \pm |a-b|/2$. Consequently, (a,a,b) is a Win if and only if $(a, a \pm |a-b|/2, a \pm |a-b|/2)$ is a Loss. Since the side lengths of this new triangle differ by $|a-b|/2$, the conclusion follows inductively once the exceptional and trivial case $a=b$ is dealt with.

As an immediate corollary, the configuration with arc lengths 1, 1, $n-2$ (the starting configuration of the question) is a Win if and only if $v_2(n-3)$ is odd.

Remark. Relying on the solution presented above, one may also derive an explicit winning strategy. Denote the position in the game by the multiset $\{a,b,c\}$ of thr lengths of the three arcs between the tokens (again in appropriate units so that $a+b+c=n$). A move now consists in choosing two of the three numbers a, b, c, and replacing them by two numbers with the same sum so as to strictly increase the minimum of the pair.

The winning strategy for a player is to obtain at the end of each of his moves the positions of the form $\{a, a, b\}$, where $a = b$ or $v_2(a - b)$ is even; we say that such position is *good*. At the beginning of the game, the position is good exactly if $v_2(n - 3)$ is even.

Now, there is at most one position of the form $\{a', a', b'\}$ which may be obtained by a move from a good position $\{a, a, b\}$ — that is, with $b' = a$. This position is not good, thus it suffices to show that it is possible to obtain a good position from any non-good one by a move.

Let now $\{a, b, c\}$ be a non-good position, with $a \leq b \leq c$. If $a+c = 2b$ then one may get the good position (b, b, b). Assume now that $a + c \neq 2b$. If $v_2(c + a - 2b)$ is even, then it is possible to achieve the good position $\{b, b, c + a - b\}$; otherwise, $c + a$ is necessarily even, and one may get the good position $\{(c + a)/2, (c + a)/2, b\}$.

Problem 3. A finite list of rational numbers is written on a blackboard. In an *operation*, we choose any two numbers a, b, erase them, and write down one of the numbers

$$a + b,\ a - b,\ b - a,\ a \times b,\ a/b \text{ (if } b \neq 0),\ b/a \text{ (if } a \neq 0).$$

Prove that, for every integer $n > 100$, there are only finitely many integers $k \geq 0$, such that, starting from the list

$$k + 1,\ k + 2,\ \dots,\ k + n,$$

it is possible to obtain, after $n - 1$ operations, the value $n!$.

(United Kingdom) Alexander Betts

Solution. We prove the problem statement even for all positive integer n.

There are only finitely many ways of constructing a number from n pairwise distinct numbers $x_1, \dots, x_n$ only using the four elementary arithmetic operations, and each x_k exactly once. Each such formula for $k > 1$ is obtained by an elementary operation from two such formulas on two disjoint sets of the x_i.

A straightforward induction on n shows that the outcome of each such construction is a number of the form

$$\frac{\displaystyle\sum_{\alpha_1,\dots,\alpha_n \in \{0,1\}} a_{\alpha_1,\dots,\alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\displaystyle\sum_{\alpha_1,\dots,\alpha_n \in \{0,1\}} b_{\alpha_1,\dots,\alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}}, \tag{$*$}$$

where the $a_{\alpha_1,\dots,\alpha_n}$ and $b_{\alpha_1,\dots,\alpha_n}$ are all in the set $\{0, \pm 1\}$, not all zero of course, $a_{0,\dots,0} = b_{1,\dots,1} = 0$, and also $a_{\alpha_1,\dots,\alpha_n} \cdot b_{\alpha_1,\dots,\alpha_n} = 0$ for every set of indices.

Since $|a_{\alpha_1,\dots,\alpha_n}| \leq 1$, and $a_{0,0,\dots,0} = 0$, the absolute value of the numerator does not exceed $(1+|x_1|)\cdots(1+|x_n|)-1$; in particular, if c is an integer in the range $-n, \dots, -1$, and $x_k = c+k$, $k = 1, \dots, n$, then the absolute value of the numerator is at most $(-c)!(n+c+1)!-1 \leq n!-1 < n!$.

Consider now the integral polynomials,

$$P = \sum_{\alpha_1,\dots,\alpha_n \in \{0,1\}} a_{\alpha_1,\dots,\alpha_n} (X+1)^{\alpha_1} \cdots (X+n)^{\alpha_n},$$

and

$$Q = \sum_{\alpha_1,\dots,\alpha_n \in \{0,1\}} b_{\alpha_1,\dots,\alpha_n} (X+1)^{\alpha_1} \cdots (X+n)^{\alpha_n},$$

where the $a_{\alpha_1,\dots,\alpha_n}$ and $b_{\alpha_1,\dots,\alpha_n}$ are all in the set $\{0, \pm 1\}$, not all zero, $a_{\alpha_1,\dots,\alpha_n} b_{\alpha_1,\dots,\alpha_n} = 0$ for every set of indices, and $a_{0,\dots,0} = b_{1,\dots,1} = 0$. By the preceding, $|P(c)| < n!$ for every integer c in the range $-n, \dots, -1$; and since $b_{1,\dots,1} = 0$, the degree of Q is less than n.

Since every non-zero polynomial has only finitely many roots, and the number of roots does not exceed the degree, to complete the proof it is sufficient to show that the polynomial $P - n!Q$ does not vanish identically, provided that Q does not (which is the case in the problem).

Suppose, if possible, that $P = n!Q$, where $Q \neq 0$. Since $\deg Q < n$, it follows that $\deg P < n$ as well, and since $P \neq 0$, the number of roots of P does not exceed $\deg P < n$, so $P(c) \neq 0$ for some integer c in the range $-n, \dots, -1$. By the preceding, $|P(c)|$ is consequently a positive integer less than $n!$. On the other hand, $|P(c)| = n!|Q(c)|$ is an integral multiple of $n!$. A contradiction.

Remark. Alternatively, it can be shown by induction on n that

$$\max(|P(c)|,\ 2|Q(c)|) \leq \prod_{k=1}^{n} \max(|c+k|, 2),$$

for all integers c. In case $n > 8$, this provides a solution along the same lines.

The 7^{th} Romanian Master of Mathematics Competition

Solutions for the Day 2

Problem 4. Let ABC be a triangle, let D be the touchpoint of the side BC and the incircle of the triangle ABC, and let J_b and J_c be the incentres of the triangles ABD and ACD, respectively. Prove that the circumcentre of the triangle AJ_bJ_c lies on the bisectrix of the angle BAC.

(Russia) Fedor Ivlev

Solution. Let the incircle of the triangle ABC meet CA and AB at points E and F, respectively. Let the incircles of the triangles ABD and ACD meet AD at points X and Y, respectively. Then $2DX = DA + DB - AB = DA + DB - BF - AF = DA - AF$; similarly, $2DY = DA - AE = 2DX$. Hence the points X and Y coincide, so $J_bJ_c \perp AD$.

Now let O be the circumcentre of the triangle AJ_bJ_c. Then $\angle J_bAO = \pi/2 - \angle AOJ_b/2 = \pi/2 - \angle AJ_cJ_b = \angle XAJ_c = \frac{1}{2}\angle DAC$. Therefore, $\angle BAO = \angle BAJ_b + \angle J_bAO = \frac{1}{2}\angle BAD + \frac{1}{2}\angle DAC = \frac{1}{2}\angle BAC$, and the conclusion follows.

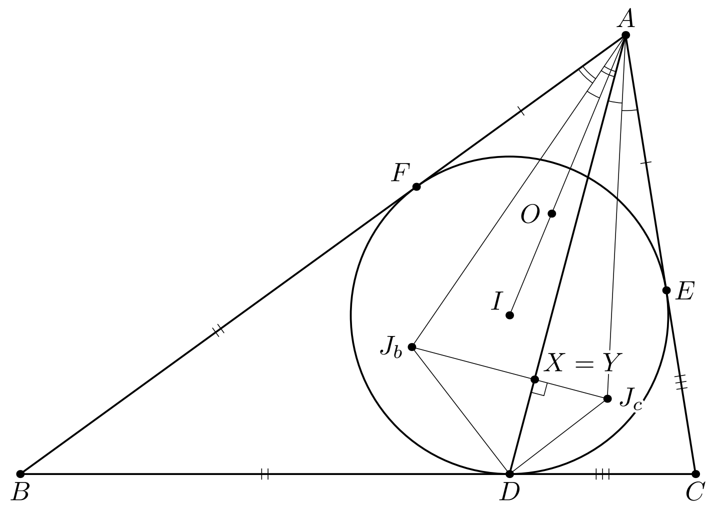


Fig. 1

Problem 5. Let $p \geq 5$ be a prime number. For a positive integer k we denote by $R(k)$ the remainder of k when divided by p. Determine all positive integers $a < p$ such that

$$m + R(ma) > a$$

for every $m = 1, 2, \ldots, p-1$.

(Bulgaria) Alexander Ivanov

Solution. The required integers are $p-1$ along with all the numbers of the form $\lfloor p/q \rfloor$, $q = 2, \ldots, p-1$. In other words, these are $p-1$, along with the numbers $1, 2, \ldots, \lfloor \sqrt{p} \rfloor$, and also the (distinct) numbers $\lfloor p/q \rfloor$, $q = 2, \ldots, \left\lfloor \sqrt{p} - \frac{1}{2} \right\rfloor$.

We begin by showing that these numbers satisfy the conditions in the statement. It is readily checked that $p-1$ satisfies the required inequalities, since $m+R(m(p-1)) = m+(p-m) = p > p-1$ for all $m = 1, \ldots, p-1$.

Now, consider any number a of the form $a = \lfloor p/q \rfloor$, where q is an integer greater than 1 but less than p; then $p = aq + r$ with $0 < r < q$. Choose any integer $m \in (0, p)$ and write $m = xq + y$ with $x, y \in \mathbb{Z}$, $0 < y \leq q$ (notice that x is nonnegative). Then

$$R(ma) = R(ay + xaq) = R(ay + xp - xr) = R(ay - xr).$$

Since $ay - xr \leq ay \leq aq < p$, we obtain $R(ay - xr) \geq ay - xr$ and hence

$$m + R(ma) \geq (xq + y) + (ay - xr) = x(q - r) + y(a + 1) \geq a + 1$$

by $q > r$ and $y \geq 1$. Thus a satisfies the required condition.

Finally, we show that if an integer $a \in (0, p-1)$ satisfies the required condition then a is indeed of the form $a = \lfloor p/q \rfloor$ for some integer $q \in (0, p)$. This is clear for $a = 1$, so we may (and will) assume that $a \geq 2$.

Write $p = aq + r$ with $q, r \in \mathbb{Z}$ and $0 < r < a$; since $a \geq 2$ we have $q < p/2$. Choose $m = q + 1 < p$; we have $R(ma) = R(aq + a) = R(p + (a - r)) = a - r$, so

$$a < m + R(ma) = q + 1 + a - r,$$

which yields $r < q+1$. Moreover, if $r = q$, then $p = q(a+1)$ which is impossible by $1 < a+1 < p$. Thus $r < q$, and we have

$$0 \leq \frac{p}{q} - a = \frac{r}{q} < 1,$$

which proves $a = \lfloor p/q \rfloor$.

Problem 6. Given a positive integer n, determine the largest real number μ satisfying the following condition: for every $4n$-point configuration C in an open unit square U, there exists an open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C, and has an area greater than or equal to μ.

(Bulgaria) Nikolai Beluhov

Solution. The required maximum is $\frac{1}{2n+2}$. To show that the condition in the statement is not met if $\mu > \frac{1}{2n+2}$, let $U = (0,1) \times (0,1)$, choose a small enough positive ϵ, and consider the configuration C consisting of the n four-element clusters of points $\left(\frac{i}{n+1} \pm \epsilon\right) \times \left(\frac{1}{2} \pm \epsilon\right)$, $i = 1, \dots, n$, the four possible sign combinations being considered for each i. Clearly, every open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C, has area at most $\left(\frac{1}{n+1} + \epsilon\right) \cdot \left(\frac{1}{2} + \epsilon\right) < \mu$ if ϵ is small enough.

We now show that, given a finite configuration C of points in an open unit square U, there always exists an open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C, and has an area greater than or equal to $\mu_0 = \dfrac{2}{|C|+4}$.

To prove this, usage will be made of the following two lemmas whose proofs are left at the end of the solution.

Lemma 1. *Let k be a positive integer, and let $\lambda < \dfrac{1}{\lfloor k/2 \rfloor + 1}$ be a positive real number. If $t_1, \dots, t_k$ are pairwise distinct points in the open unit interval $(0,1)$, then some t_i is isolated from the other t_j by an open subinterval of $(0,1)$ whose length is greater than or equal to λ.*

Lemma 2. *Given an integer $k \geq 2$ and positive integers $m_1, \dots, m_k$,*

$$\left\lfloor \frac{m_1}{2} \right\rfloor + \sum_{i=1}^{k} \left\lfloor \frac{m_i}{2} \right\rfloor + \left\lfloor \frac{m_k}{2} \right\rfloor \leq \sum_{i=1}^{k} m_i - k + 2.$$

Back to the problem, let $U = (0,1) \times (0,1)$, project C orthogonally on the x-axis to obtain the points $x_1 < \cdots < x_k$ in the open unit interval $(0,1)$, let ℓ_i be the vertical through x_i, and let $m_i = |C \cap \ell_i|$, $i = 1, \dots, k$.

Setting $x_0 = 0$ and $x_{k+1} = 1$, assume that $x_{i+1} - x_{i-1} > (\lfloor m_i/2 \rfloor + 1)\mu_0$ for some index i, and apply Lemma 1 to isolate one of the points in $C \cap \ell_i$ from the other ones by an open subinterval $x_i \times J$ of $x_i \times (0,1)$ whose length is greater than or equal to $\mu_0/(x_{i+1} - x_{i-1})$. Consequently, $(x_{i-1}, x_{i+1}) \times J$ is an open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C and has an area greater than or equal to μ_0.

Next, we rule out the case $x_{i+1} - x_{i-1} \leq (\lfloor m_i/2 \rfloor + 1)\mu_0$ for all indices i. If this were the case, notice that necessarily $k > 1$; also, $x_1 - x_0 < x_2 - x_0 \leq (\lfloor m_1/2 \rfloor + 1)\mu_0$ and $x_{k+1} - x_k < x_{k+1} - x_{k-1} \leq (\lfloor m_k/2 \rfloor + 1)\mu_0$. With reference to Lemma 2, write

$$\begin{aligned}
2 = 2(x_{k+1} - x_0) &= (x_1 - x_0) + \sum_{i=1}^{k} (x_{i+1} - x_{i-1}) + (x_{k+1} - x_k) \\
&< \left(\left(\left\lfloor \frac{m_1}{2} \right\rfloor + 1 \right) + \sum_{i=1}^{k} \left(\left\lfloor \frac{m_i}{2} \right\rfloor + 1 \right) + \left(\left\lfloor \frac{m_k}{2} \right\rfloor + 1 \right) \right) \cdot \mu_0 \\
&\leq \left(\sum_{i=1}^{k} m_i + 4 \right) \mu_0 = (|C| + 4)\mu_0 = 2,
\end{aligned}$$

and thereby reach a contradiction.

Finally, we prove the two lemmas.

Proof of Lemma 1. Suppose, if possible, that no t_i is isolated from the other t_j by an open subinterval of $(0,1)$ whose length is greater than or equal to λ. Without loss of generality, we may (and will) assume that $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$. Since the open interval (t_{i-1}, t_{i+1}) isolates t_i from the other t_j, its length, $t_{i+1} - t_{i-1}$, is less than λ. Consequently, if k is odd we have $1 = \sum_{i=0}^{(k-1)/2}(t_{2i+2} - t_{2i}) < \lambda\big(1 + \frac{k-1}{2}\big) < 1$; if k is even, we have $1 < 1 + t_k - t_{k-1} = \sum_{i=0}^{k/2-1}(t_{2i+2} - t_{2i}) + (t_{k+1} - t_{k-1}) < \lambda\big(1 + \frac{k}{2}\big) < 1$. A contradiction in either case.

Proof of Lemma 2. Let I_0, respectively I_1, be the set of all indices i in the range 2, …, $k-1$ such that m_i is even, respectively odd. Clearly, I_0 and I_1 form a partition of that range. Since $m_i \geq 2$ if i is in I_0, and $m_i \geq 1$ if i is in I_1 (recall that the m_i are positive integers),

$$\sum_{i=2}^{k-1} m_i = \sum_{i\in I_0} m_i + \sum_{i \in I_1} m_i \geq 2|I_0| + |I_1| = 2(k-2) - |I_1|, \quad \text{or} \quad |I_1| \geq 2(k-2) - \sum_{i=2}^{k-1} m_i.$$

Therefore,

$$\begin{aligned}
\left\lfloor \frac{m_1}{2} \right\rfloor + \sum_{i=1}^{k} \left\lfloor \frac{m_i}{2} \right\rfloor + \left\lfloor \frac{m_k}{2} \right\rfloor &\leq m_1 + \left(\sum_{i=2}^{k-1} \frac{m_i}{2} - \frac{|I_1|}{2} \right) + m_k \\
&\leq m_1 + \left(\frac{1}{2} \sum_{i=2}^{k-1} m_i - (k-2) + \frac{1}{2} \sum_{i=2}^{k-1} m_i \right) + m_k \\
&= \sum_{i=1}^{k} m_i - k + 2.
\end{aligned}$$

□

Remark. In case $4n$ is replaced by a positive integer k not divisible by 4, we do not yet know the maximal μ satisfying the corresponding condition.

The 8th Romanian Master of Mathematics Competition

Day 1: Friday, February 26, 2016, Bucharest

Language: English

Problem 1. Let ABC be a triangle and let D be a point on the segment BC, $D \neq B$ and $D \neq C$. The circle ABD meets the segment AC again at an interior point E. The circle ACD meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).

Problem 2. Given positive integers m and $n \geq m$, determine the largest number of dominoes (1×2 or 2×1 rectangles) that can be placed on a rectangular board with m rows and $2n$ columns consisting of cells (1×1 squares) so that:

(i) each domino covers exactly two adjacent cells of the board;
(ii) no two dominoes overlap;
(iii) no two form a 2×2 square; and
(iv) the bottom row of the board is completely covered by n dominoes.

Problem 3. A *cubic sequence* is a sequence of integers given by $a_n = n^3 + bn^2 + cn + d$, where b, c and d are integer constants and n ranges over all integers, including negative integers.

(a) Show that there exists a cubic sequence such that the only terms of the sequence which are squares of integers are a_{2015} and a_{2016}.

(b) Determine the possible values of $a_{2015} \cdot a_{2016}$ for a cubic sequence satisfying the condition in part **(a)**.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

The 8^{th} Romanian Master of Mathematics Competition

Day 2: Saturday, February 27, 2016, Bucharest

Language: English

Problem 4. Let x and y be positive real numbers such that $x + y^{2016} \geq 1$. Prove that $x^{2016} + y > 1 - 1/100$.

Problem 5. A convex hexagon $A_1B_1A_2B_2A_3B_3$ is inscribed in a circle Ω of radius R. The diagonals A_1B_2, A_2B_3, and A_3B_1 concur at X. For $i = 1, 2, 3$, let ω_i be the circle tangent to the segments XA_i and XB_i, and to the arc A_iB_i of Ω not containing other vertices of the hexagon; let r_i be the radius of ω_i.

(a) Prove that $R \geq r_1 + r_2 + r_3$.

(b) If $R = r_1 + r_2 + r_3$, prove that the six points where the circles ω_i touch the diagonals A_1B_2, A_2B_3, A_3B_1 are concyclic.

Problem 6. A set of n points in Euclidean 3-dimensional space, no four of which are coplanar, is partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$. An $\mathcal{AB}$-*tree* is a configuration of $n - 1$ segments, each of which has an endpoint in $\mathcal{A}$ and the other in $\mathcal{B}$, and such that no segments form a closed polyline. An $\mathcal{AB}$-tree is transformed into another as follows: choose three distinct segments A_1B_1, B_1A_2 and A_2B_2 in the $\mathcal{AB}$-tree such that A_1 is in $\mathcal{A}$ and $A_1B_1 + A_2B_2 > A_1B_2 + A_2B_1$, and remove the segment A_1B_1 to replace it by the segment A_1B_2. Given any $\mathcal{AB}$-tree, prove that every sequence of successive transformations comes to an end (no further transformation is possible) after finitely many steps.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

The 8th Romanian Master of Mathematics Competition

Day 1 — Solutions

Problem 1. Let ABC be a triangle and let D be a point on the segment BC, $D \neq B$ and $D \neq C$. The circle ABD meets the segment AC again at an interior point E. The circle ACD meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).

Solution 1. (*Ilya Bogdanov*) Let σ denote reflection in the line BC. Since $\angle BDF = \angle BAC = \angle CDE$, by concyclicity, the lines DE and DF are images of one another under σ, so the lines AC and DF meet at $P' = \sigma(P)$, and the lines AB and DE meet at $Q' = \sigma(Q)$. Consequently, the lines PQ and $P'Q' = \sigma(PQ)$ meet at some (possibly ideal) point R on the line BC.

Since the pairs of lines (CA, QD), (AB, DP), (BC, PQ) meet at three collinear points, namely P', Q', R respectively, the triangles ABC and DPQ are perspective, i.e., the lines AD, BP, CQ are concurrent, by the Desargues theorem.

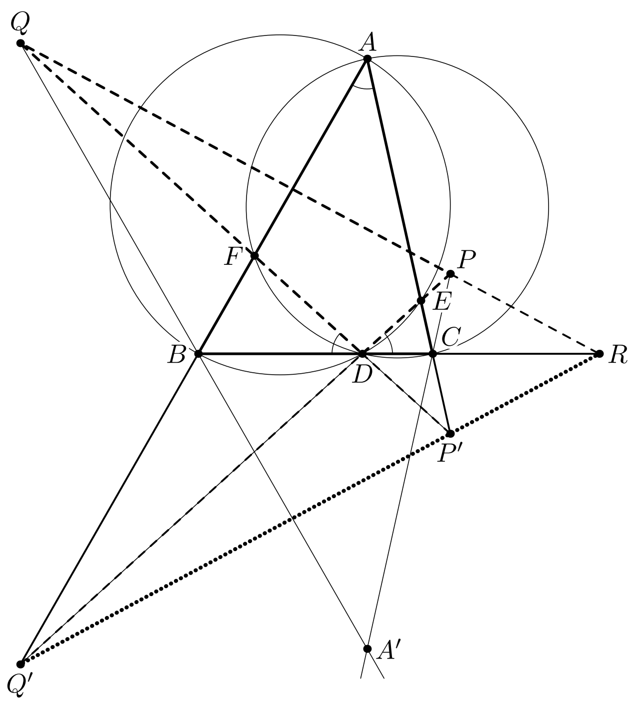


Fig. 1

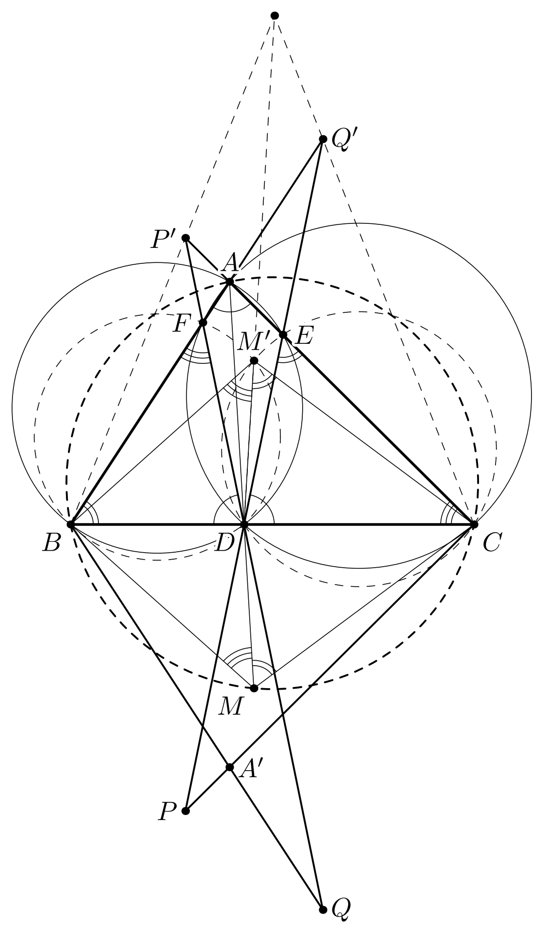


Fig. 2

Solution 2. As in the first solution, σ denotes reflection in the line BC, the lines DE and DF are images of one another under σ, the lines AC and DF meet at $P' = \sigma(P)$, and the lines AB and DE meet at $Q' = \sigma(Q)$.

Let the line AD meet the circle ABC again at M. Letting $M' = \sigma(M)$, it is sufficient to prove that the lines $DM' = \sigma(AD)$, $BP' = \sigma(BP)$ and $CQ' = \sigma(CQ)$ are concurrent.

Begin by noticing that $\angle(BM', M'D) = -\angle(BM, MA) = -\angle(BC, CA) = \angle(BF, FD)$, to infer that M' lies on the circle BDF. Similarly, M' lies on the circle CDE, so the line DM' is the radical axis of the circles BDF and CDE.

Since P' lies on the lines AC and DF, it is the radical centre of the circles ABC, ADC, and BDF; hence the line BP' is the radical axis of the circles BDF and ABC. Similarly, the line CQ' is the radical axis of the circles CDE and ABC. So the conclusion follows: the lines DM', BP' and CQ' are concurrent at the radical centre of the circles ABC, BDF and CDE; thus the lines DM, BP' and CQ' are also concurrent.

Solution 3. (*Ilya Bogdanov*) As in the previous solutions, σ denotes reflection in the line BC. Let the lines BE and CF meet at X. Due to the circles $BDEA$ and $CDFA$, we have $\angle XBD = \angle EAD = \angle XFD$, so the quadrilateral $BFXD$ is cyclic; similarly, the quadrilateral $CEXD$ is cyclic. Hence $\angle XDB = \angle CFA = \angle CDA$, the lines DX and DA are therefore images of one another under σ, and $X' = \sigma(X)$ lies on the line AD. Let $E' = \sigma(E)$ and $F' = \sigma(F)$, and apply the Pappus theorem to the hexagon $BPF'CQE'$ to infer that X', D, and $BP \cap CQ$ are collinear. The conclusion follows.

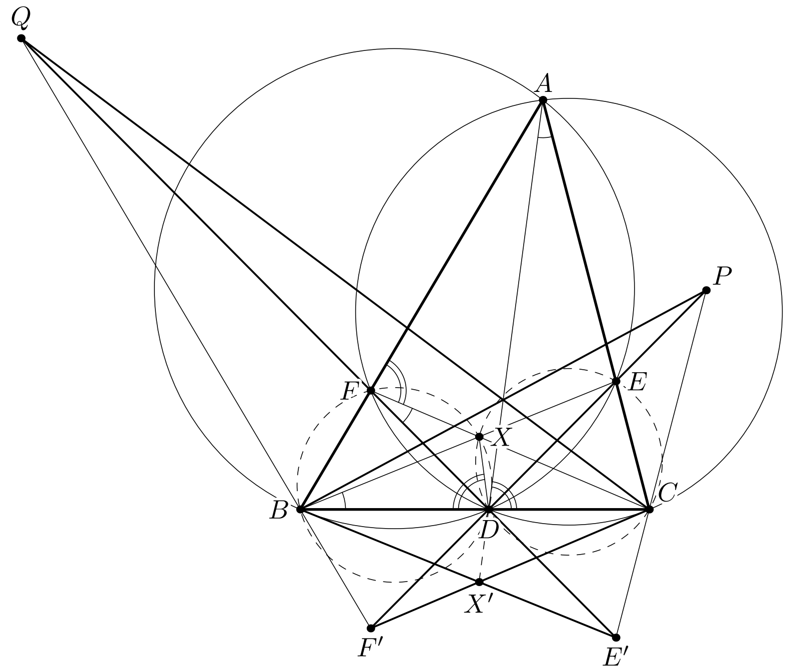


Fig. 3

Remark. In fact, the point X in Solution 3 and the point M in Solution 2 coincide.

Problem 2. Given positive integers m and $n \geq m$, determine the largest number of dominoes (1×2 or 2×1 rectangles) that can be placed on a rectangular board with m rows and $2n$ columns consisting of cells (1×1 squares) so that:

(i) each domino covers exactly two adjacent cells of the board;
(ii) no two dominoes overlap;
(iii) no two form a 2×2 square; and
(iv) the bottom row of the board is completely covered by n dominoes.

Solution 1. The required maximum is $mn - \lfloor m/2 \rfloor$ and is achieved by the brick-like vertically symmetric arrangement of blocks of n and $n-1$ horizontal dominoes placed on alternate rows, so that the bottom row of the board is completely covered by n dominoes.

To show that the number of dominoes in an arrangement satisfying the conditions in the statement does not exceed $mn - \lfloor m/2 \rfloor$, label the rows upwards $0, 1, \ldots, m-1$, and, for each

i in this range, draw a vertically symmetric block of $n-i$ fictitious horizontal dominoes in the i-th row (so the block on the i-th row leaves out i cells on either side) — Figure 4 illustrates the case $m=n=6$. A fictitious domino is *good* if it is completely covered by a domino in the arrangement; otherwise, it is *bad*.

If the fictitious dominoes are all good, then the dominoes in the arrangement that cover no fictitious domino, if any, all lie in two triangular regions of side-length $m-1$ at the upper-left and upper-right corners of the board. Colour the cells of the board chess-like and notice that in each of the two triangular regions the number of black cells and the number of white cells differ by $\lfloor m/2 \rfloor$. Since each domino covers two cells of different colours, at least $\lfloor m/2 \rfloor$ cells are not covered in each of these regions, and the conclusion follows.

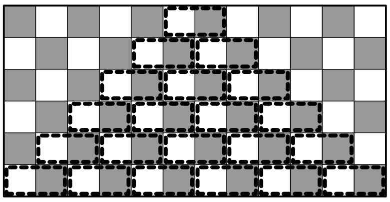


Fig. 4

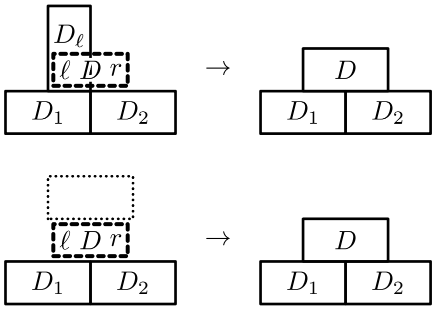


Fig. 5

To deal with the remaining case where bad fictitious dominoes are present, we show that an arrangement satisfying the conditions in the statement can be transformed into another such with at least as many dominoes, but fewer bad fictitious dominoes. A finite number of such transformations eventually leads to an arrangement of at least as many dominoes all of whose fictitious dominoes are good, and the conclusion follows by the preceding.

Consider the row of minimal rank containing bad fictitious dominoes — this is certainly not the bottom row — and let D be one such. Let ℓ, respectively r, be the left, respectively right, cell of D and notice that the cell below ℓ, respectively r, is the right, respectively left, cell of a domino D_1, respectively D_2, in the arrangement.

If ℓ is covered by a domino D_ℓ in the arrangement, since D is bad and no two dominoes in the arrangement form a square, it follows that D_ℓ is vertical. If r were also covered by a domino D_r in the arrangement, then D_r would also be vertical, and would therefore form a square with D_ℓ — a contradiction. Hence r is not covered, and there is room for D_ℓ to be placed so as to cover D, to obtain a new arrangement satisfying the conditions in the statement; the latter has as many dominoes as the former, but fewer bad fictitious dominoes. The case where r is covered is dealt with similarly.

Finally, if neither cell of D is covered, addition of an extra domino to cover D and, if necessary, removal of the domino above D to avoid formation of a square, yields a new arrangement satisfying the conditions in the statement; the latter has at least as many dominoes as the former, but fewer bad fictitious dominoes. (Figure 5 illustrates the two cases.)

Solution 2. (sketch by *Ilya Bogdanov*) We present an alternative proof of the bound.

Label the rows upwards $0,1,\ldots,m-1$, and the columns from the left to the right by $0,1,\ldots,2n-1$; label each cell by the pair of its column's and row's numbers, so that $(1,0)$ is the second left cell in the bottom row. Colour the cells chess-like so that $(0,0)$ is white. For $0 \le i \le n-1$, we say that the *ith white diagonal* is the set of cells of the form $(2i+k,k)$, where k ranges over all appropriate indices. Similarly, the *ith black diagonal* is the set of cells of the form $(2i+1-k,k)$. (Notice that the white cells in the upper-left corner and the black cells in the upper-right corner are not covered by these diagonals.)

Claim. Assume that K lowest cells of some white diagonal are all covered by dominoes. Then all these K dominoes face right or up from the diagonal. (In other words, the black cell of any such domino is to the right or to the top of its white cell.) Similarly, if K lowest cells of some black diagonal are covered by dominoes, then all these dominoes face left or up from the diagonal.

Proof. By symmetry, it suffices to prove the first statement. Assume that K lowest cells of the ith white diagonal is completely covered. We prove by induction on $k < K$ that the required claim holds for the domino covering $(2i + k, k)$. The base case $k = 0$ holds due to the problem condition. To establish the step, one observes that if $(2i + k, k)$ is covered by a domino facing up of right, while $(2i + k + 1, k + 1)$ is covered by a domino facing down or left, then these two dominoes form a square.

We turn to the solution. We will prove that there are at least $d = \lfloor m/2 \rfloor$ empty white cells. Since each domino covers exactly one white cell, the required bound follows.

If each of the first d white diagonals contains an empty cell, the result is clear. Otherwise, let $i < d$ be the least index of a completely covered white diagonal. We say that the dominoes covering our diagonal are *distinguished*. After removing the distinguished dominoes, the board splits into two parts; the left part L contains i empty white cells on the previous diagonals. So, it suffices to prove that the right part R contains at least $d - i$ empty white cells.

Let j be the number of distinguished dominoes facing up. Then at least $j-i$ of these dominoes cover some cells of (distinct) black diagonals (the relation $m \leq n$ is used). Each such domino faces down from the corresponding black diagonal — so, by the Claim, each such black diagonal contains an empty cell in R. Thus, R contains at least $j - i$ empty black cells.

Now, let w be the number of white cells in R. Then the number of black cells in R is $w - d + j$, and at least $i - j$ of those are empty. Thus, the number of dominoes in R is at most $(w - d + j) - (j - i) = w - (d - i)$, so R contains at least $d - i$ empty white cells, as we wanted to show.

Remark. The conclusion still holds if some row, not necessarily the bottom row, is completely covered by n dominoes — apply the result in the problem to the upper and lower parts of the board overlapping along a row completely covered by n dominoes.

However, omission of the condition that the bottom row be covered by n dominoes reduces the minimal number of uncovered cells dramatically. For instance, all but two cells of a $(2k + 1) \times (4k + 2)$ board can be covered by dominoes no two of which form a 2×2 square.

Problem 3. A *cubic sequence* is a sequence of integers given by $a_n = n^3 + bn^2 + cn + d$, where b, c and d are integer constants and n ranges over all integers, including negative integers.

(a) Show that there exists a cubic sequence such that the only terms of the sequence which are squares of integers are a_{2015} and a_{2016}.

(b) Determine the possible values of $a_{2015} \cdot a_{2016}$ for a cubic sequence satisfying the condition in part **(a)**.

Solution. The only possible value of $a_{2015} \cdot a_{2016}$ is 0. For simplicity, by performing a translation of the sequence (which may change the defining constants b, c and d), we may instead concern ourselves with the values a_0 and a_1, rather than a_{2015} and a_{2016}.

Suppose now that we have a cubic sequence a_n with $a_0 = p^2$ and $a_1 = q^2$ square numbers. We will show that $p = 0$ or $q = 0$. Consider the line $y = (q-p)x + p$ passing through $(0, p)$ and $(1, q)$; the latter are two points the line under consideration and the cubic $y^2 = x^3 + bx^2 + cx + d$ share. Hence the two must share a third point whose x-coordinate is the third root of the polynomial $t^3 + (b - (q-p)^2)t^2 + (c - 2(q-p)p)t + (d - p^2)$ (it may well happen that this third point coincide with one of the other two points the line and the cubic share).

Notice that the sum of the three roots is $(q - p)^2 - b$, so the third intersection has integral x-coordinate $X = (q-p)^2 - b - 1$. Its y-coordinate $Y = (q-p)X + p$ is also an integer, and hence $a_X = X^3 + bX^2 + cX + d = Y^2$ is a square. This contradicts our assumption on the sequence unless $X = 0$ or $X = 1$, i.e. unless $(q - p)^2 = b + 1$ or $(q - p)^2 = b + 2$.

Applying the same argument to the line through $(0, -p)$ and $(1, q)$, we find that $(q+p)^2 = b+1$ or $b+2$ also. Since $(q-p)^2$ and $(q+p)^2$ have the same parity, they must be equal, and hence $pq = 0$, as desired.

It remains to show that such sequences exist, say when $p = 0$. Consider the sequence $a_n = n^3 + (q^2-2)n^2 + n$, chosen to satisfy $a_0 = 0$ and $a_1 = q^2$. We will show that when $q = 1$, the only square terms of the sequence are $a_0 = 0$ and $a_1 = 1$. Indeed, suppose that $a_n = n(n^2 - n + 1)$ is square. Since the second factor is positive, and the two factors are coprime, both must be squares; in particular, $n \geq 0$. The case $n = 0$ is clear, so let $n \geq 1$. Finally, if $n > 1$, then $(n-1)^2 < n^2 - n + 1 < n^2$, so $n^2 - n + 1$ is not a square. Consequently, $n = 0$ or $n = 1$, and the conclusion follows.

Remark. The values $q = 3$ and $q = 4$ work as well. In the former case, the only square terms of the sequence $a_n = n(n^2 + 7n + 1)$ are $a_0 = 0$ and $a_1 = 9$. In the other case, the only square terms of the sequence $a_n = n(n^2 + 14n + 1)$ are $a_0 = 0$ and $a_1 = 16$.

The 8^{th} Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Let x and y be positive real numbers such that $x+y^{2016} \geq 1$. Prove that $x^{2016}+y > 1-1/100$.

Solution. If $x \geq 1-1/(100 \cdot 2016)$, then

$$x^{2016} \geq \left(1-\frac{1}{100 \cdot 2016}\right)^{2016} > 1-2016 \cdot \frac{1}{100 \cdot 2016} = 1-\frac{1}{100}$$

by Bernoulli's inequality, whence the conclusion.

If $x < 1-1/(100 \cdot 2016)$, then $y \geq (1-x)^{1/2016} > (100 \cdot 2016)^{-1/2016}$, and it is sufficient to show that the latter is greater than $1-1/100 = 99/100$; alternatively, but equivalently, that

$$\left(1+\frac{1}{99}\right)^{2016} > 100 \cdot 2016.$$

To establish the latter, refer again to Bernoulli's inequality to write

$$\left(1+\frac{1}{99}\right)^{2016} > \left(1+\frac{1}{99}\right)^{99 \cdot 20} > \left(1+99 \cdot \frac{1}{99}\right)^{20} = 2^{20} > 100 \cdot 2016.$$

Remarks. (1) Although the constant $1/100$ is not sharp, it cannot be replaced by the smaller constant $1/400$, as the values $x = 1-1/210$ and $y = 1-1/380$ show.

(2) It is natural to ask whether $x^n+y \geq 1-1/k$, whenever x and y are positive real numbers such that $x+y^n \geq 1$, and k and n are large. Using the inequality $\left(1+\frac{1}{k-1}\right)^k > \mathrm{e}$, it can be shown along the lines in the solution that this is indeed the case if $k \leq \frac{n}{2\log n}(1+o(1))$. It *seems* that this estimate differs from the best one by a constant factor.

Problem 5. A convex hexagon $A_1B_1A_2B_2A_3B_3$ is inscribed in a circle Ω of radius R. The diagonals A_1B_2, A_2B_3, and A_3B_1 concur at X. For $i = 1, 2, 3$, let ω_i be the circle tangent to the segments XA_i and XB_i, and to the arc A_iB_i of Ω not containing other vertices of the hexagon; let r_i be the radius of ω_i.

(a) Prove that $R \geq r_1+r_2+r_3$.

(b) If $R = r_1+r_2+r_3$, prove that the six points where the circles ω_i touch the diagonals A_1B_2, A_2B_3, A_3B_1 are concyclic.

Solution. (a) Let ℓ_1 be the tangent to Ω parallel to A_2B_3, lying on the same side of A_2B_3 as ω_1. The tangents ℓ_2 and ℓ_3 are defined similarly. The lines ℓ_1 and ℓ_2, ℓ_2 and ℓ_3, ℓ_3 and ℓ_1 meet at C_3, C_1, C_2, respectively (see Fig. 1). Finally, the line C_2C_3 meets the rays XA_1 and XB_1 emanating from X at S_1 and T_1, respectively; the points S_2, T_2, and S_3, T_3 are defined similarly.

Each of the triangles $\Delta_1 = \triangle XS_1T_1$, $\Delta_2 = \triangle T_2XS_2$, and $\Delta_3 = \triangle S_3T_3X$ is similar to $\Delta = \triangle C_1C_2C_3$, since their corresponding sides are parallel. Let k_i be the ratio of similitude of Δ_i and Δ (e.g., $k_1 = XS_1/C_1C_2$ and the like). Since $S_1X = C_2T_3$ and $XT_2 = S_3C_1$, it follows that $k_1+k_2+k_3 = 1$, so, if ρ_i is the inradius of Δ_i, then $\rho_1+\rho_2+\rho_3 = R$.

Finally, notice that ω_i is interior to Δ_i, so $r_i \leq \rho_i$, and the conclusion follows by the preceding.

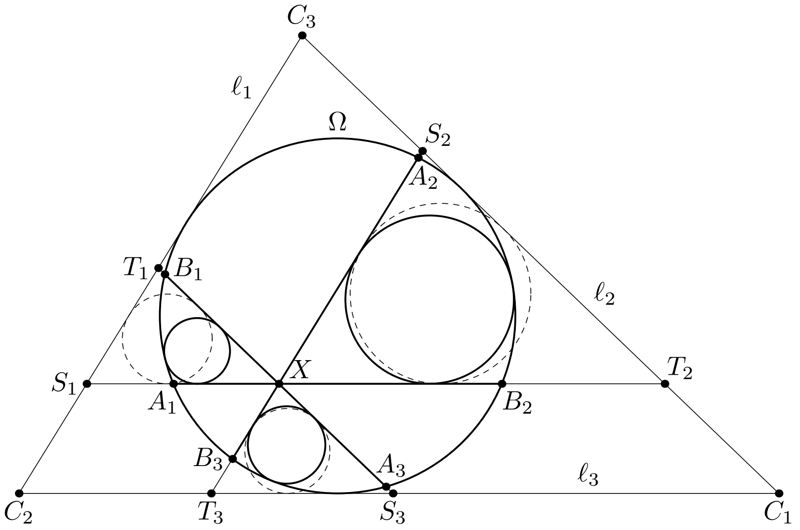


Fig. 1

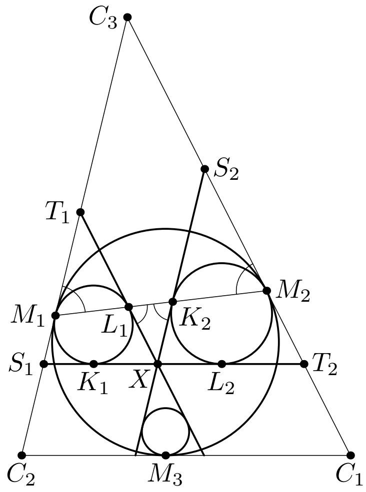


Fig. 2

(b) By part **(a)**, the equality $R = r_1 + r_2 + r_3$ holds if and only if $r_i = \rho_i$ for all i, which implies in turn that ω_i is the incircle of Δ_i. Let K_i, L_i, M_i be the points where ω_i touches the sides XS_i, XT_i, S_iT_i, respectively. We claim that the six points K_i and L_i $(i = 1, 2, 3)$ are equidistant from X.

Clearly, $XK_i = XL_i$, and we are to prove that $XK_2 = XL_1$ and $XK_3 = XL_2$. By similarity, $\angle T_1M_1L_1 = \angle C_3M_1M_2$ and $\angle S_2M_2K_2 = \angle C_3M_2M_1$, so the points M_1, M_2, L_1, K_2 are collinear. Consequently, $\angle XK_2L_1 = \angle C_3M_1M_2 = \angle C_3M_2M_1 = \angle XL_1K_2$, so $XK_2 = XL_1$. Similarly, $XK_3 = XL_2$.

Remark. Under the assumption in part **(b)**, the point M_i is the centre of a homothety mapping ω_i to Ω. Since this homothety maps X to C_i, the points M_i, C_i, X are collinear, so X is the *Gergonne point* of the triangle $C_1C_2C_3$. This condition is in fact equivalent to $R = r_1 + r_2 + r_3$.

Problem 6. A set of n points in Euclidean 3-dimensional space, no four of which are coplanar, is partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$. An $\mathcal{AB}$-*tree* is a configuration of $n-1$ segments, each of which has an endpoint in $\mathcal{A}$ and the other in $\mathcal{B}$, and such that no segments form a closed polyline. An $\mathcal{AB}$-tree is transformed into another as follows: choose three distinct segments A_1B_1, B_1A_2 and A_2B_2 in the $\mathcal{AB}$-tree such that A_1 is in $\mathcal{A}$ and $A_1B_1 + A_2B_2 > A_1B_2 + A_2B_1$, and remove the segment A_1B_1 to replace it by the segment A_1B_2. Given any $\mathcal{AB}$-tree, prove that every sequence of successive transformations comes to an end (no further transformation is possible) after finitely many steps.

Solution. The configurations of segments under consideration are all bipartite geometric trees on the points n whose vertex-parts are $\mathcal{A}$ and $\mathcal{B}$, and transforming one into another preserves the degree of any vertex in $\mathcal{A}$, but not necessarily that of a vertex in $\mathcal{B}$.

The idea is to devise a strict semi-invariant of the process, i.e., assign each $\mathcal{AB}$-tree a real number strictly decreasing under a transformation. Since the number of trees on the n points is finite, the conclusion follows.

To describe the assignment, consider an $\mathcal{AB}$-tree $\mathcal{T} = (\mathcal{A} \sqcup \mathcal{B}, \mathcal{E})$. Removal of an edge e of $\mathcal{T}$ splits the graph into exactly two components. Let $p_{\mathcal{T}}(e)$ be the number of vertices in $\mathcal{A}$ lying in the component of $\mathcal{T} - e$ containing the $\mathcal{A}$-endpoint of e; since $\mathcal{T}$ is a tree, $p_{\mathcal{T}}(e)$ counts the number of paths in $\mathcal{T} - e$ from the $\mathcal{A}$-endpoint of e to vertices in $\mathcal{A}$ (including the one-vertex path). Define $f(\mathcal{T}) = \sum_{e\in\mathcal{E}} p_{\mathcal{T}}(e)|e|$, where $|e|$ is the Euclidean length of e.

We claim that f strictly decreases under a transformation. To prove this, let $\mathcal{T}'$ be obtained from $\mathcal{T}$ by a transformation involving the polyline $A_1B_1A_2B_2$; that is, A_1 and A_2 are in $\mathcal{A}$, B_1

and B_2 are in $\mathcal{B}$, $A_1B_1 + A_2B_2 > A_1B_2 + A_2B_1$, and $\mathcal{T}' = \mathcal{T} - A_1B_1 + A_1B_2$. It is readily checked that $p_{\mathcal{T}'}(e) = p_{\mathcal{T}}(e)$ for every edge e of $\mathcal{T}$ different from A_1B_1, A_2B_1 and A_2B_2, $p_{\mathcal{T}'}(A_1B_2) = p_{\mathcal{T}}(A_1B_1)$, $p_{\mathcal{T}'}(A_2B_1) = p_{\mathcal{T}}(A_2B_1) + p_{\mathcal{T}}(A_1B_1)$, and $p_{\mathcal{T}'}(A_2Bb_2) = p_{\mathcal{T}}(A_2B_2) - p_{\mathcal{T}}(A_1B_1)$. Consequently,

$$\begin{aligned} f(\mathcal{T}') - f(\mathcal{T}) &= p_{\mathcal{T}'}(A_1B_2) \cdot A_1B_2 + (p_{\mathcal{T}'}(A_2B_1) - p_{\mathcal{T}}(A_2B_1)) \cdot A_2B_1 + \\ &\quad (p_{\mathcal{T}'}(A_2B_2) - p_{\mathcal{T}}(A_2B_2)) \cdot A_2B_2 - p_{\mathcal{T}}(A_1B_1) \cdot A_1B_1 \\ &= p_{\mathcal{T}}(A_1B_1)\,(A_1B_2 + A_2B_1 - A_2B_2 - A_1B_1) < 0. \end{aligned}$$

Remarks. (1) The solution above does not involve the geometric structure of the configurations, so the conclusion still holds if the Euclidean length (distance) is replaced by any real-valued function on $\mathcal{A} \times \mathcal{B}$.

(2) There are infinitely many strict semi-invariants that can be used to establish the conclusion, as we are presently going to show. The idea is to devise a non-strict real-valued semi-invariant f_A for each A in $\mathcal{A}$ (i.e., f_A does not increase under a transformation) such that $\sum_{A\in\mathcal{A}} f_A = f$. It then follows that any linear combination of the f_A with positive coefficients is a strict semi-invariant.

To describe f_A, where A is a fixed vertex in $\mathcal{A}$, let $\mathcal{T}$ be an $\mathcal{AB}$-tree. Since $\mathcal{T}$ is a tree, by orienting all paths in $\mathcal{T}$ with an endpoint at A away from A, every edge of $\mathcal{T}$ comes out with a unique orientation so that the in-degree of every vertex of $\mathcal{T}$ other than A is 1. Define $f_A(\mathcal{T})$ to be the sum of the Euclidean lengths of all out-going edges from $\mathcal{A}$. It can be shown that f_A does not increase under a transformation, and it strictly decreases if the paths from A to each of A_1, A_2, B_1, B_2 all pass through A_1 — i.e., of these four vertices, A_1 is combinatorially nearest to A. In particular, this is the case if $A_1 = A$, i.e., the edge-switch in the transformation occurs at A. It is not hard to prove that $\sum_{A\in\mathcal{A}} f_A(\mathcal{T}) = f(\mathcal{T})$.

The conclusion of the problem can also be established by resorting to a single carefully chosen f_A. Suppose, if possible, that the process is infinite, so some tree $\mathcal{T}$ occurs (at least) twice. Let A be the vertex in $\mathcal{A}$ at which the edge-switch occurs in the transformation of the first occurrence of $\mathcal{T}$. By the preceding paragraph, consideration of f_A shows that $\mathcal{T}$ can never occur again.

(3) Recall that the degree of any vertex in $\mathcal{A}$ is invariant under a transformation, so the linear combination $\sum_{A\in\mathcal{A}}(\deg A - 1) f_A$ is a strict semi-invariant for $\mathcal{AB}$-trees $\mathcal{T}$ whose vertices in $\mathcal{A}$ all have degrees exceeding 1. Up to a factor, this semi-invariant can alternatively, but equivalently be described as follows. Fix a vertex $*$ and assign each vertex X a number $g(X)$ so that $g(*) = 0$, and $g(A) - g(B) = AB$ for every A in $\mathcal{A}$ and every B in $\mathcal{B}$ joined by an edge. Next, let $\beta(\mathcal{T}) = \frac{1}{|\mathcal{B}|}\sum_{B\in\mathcal{B}} g(B)$, let $\alpha(\mathcal{T}) = \frac{1}{|\mathcal{E}|-|\mathcal{A}|}\sum_{A\in\mathcal{A}}(\deg A - 1) g(A)$, where $\mathcal{E}$ is the edge-set of $\mathcal{T}$, and set $\mu(\mathcal{T}) = \beta(\mathcal{T}) - \alpha(\mathcal{T})$. It can be shown that μ strictly decreases under a transformation; in fact, μ and $\sum_{A\in\mathcal{A}}(\deg A - 1) f_A$ are proportional to one another.

The 9th Romanian Master of Mathematics Competition

Day 1: Friday, February 24, 2017, Bucharest

Language: English

Problem 1. **(a)** Prove that every positive integer n can be written uniquely in the form

$$n = \sum_{j=1}^{2k+1} (-1)^{j-1} 2^{m_j},$$

where $k \geq 0$ and $0 \leq m_1 < m_2 < \cdots < m_{2k+1}$ are integers.
This number k is called the *weight* of n.

(b) Find (in closed form) the difference between the number of positive integers at most 2^{2017} with even weight and the number of positive integers at most 2^{2017} with odd weight.

Problem 2. Determine all positive integers n satisfying the following condition: for every monic polynomial P of degree at most n with integer coefficients, there exists a positive integer $k \leq n$, and $k+1$ distinct integers $x_1, x_2, \ldots, x_{k+1}$ such that

$$P(x_1) + P(x_2) + \cdots + P(x_k) = P(x_{k+1}).$$

Note. A polynomial is *monic* if the coefficient of the highest power is one.

Problem 3. Let n be an integer greater than 1 and let X be an n-element set. A non-empty collection of subsets $A_1, \ldots, A_k$ of X is *tight* if the union $A_1 \cup \cdots \cup A_k$ is a proper subset of X and no element of X lies in exactly one of the A_is. Find the largest cardinality of a collection of proper non-empty subsets of X, no non-empty subcollection of which is tight.

Note. A subset A of X is *proper* if $A \neq X$. The sets in a collection are assumed to be distinct. The whole collection is assumed to be a subcollection.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

The 9th Romanian Master of Mathematics Competition

Day 2: Saturday, February 25, 2017, Bucharest

Language: English

Problem 4. In the Cartesian plane, let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the graphs of the quadratic functions $f_1(x) = p_1x^2 + q_1x + r_1$ and $f_2(x) = p_2x^2 + q_2x + r_2$, where $p_1 > 0 > p_2$. The graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ cross at distinct points A and B. The four tangents to $\mathcal{G}_1$ and $\mathcal{G}_2$ at A and B form a convex quadrilateral which has an inscribed circle. Prove that the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same axis of symmetry.

Problem 5. Fix an integer $n \geq 2$. An $n \times n$ *sieve* is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A *stick* is a $1 \times k$ or $k \times 1$ array for any positive integer k. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

Problem 6. Let $ABCD$ be any convex quadrilateral and let P, Q, R, S be points on the segments AB, BC, CD, and DA, respectively. It is given that the segments PR and QS dissect $ABCD$ into four quadrilaterals, each of which has perpendicular diagonals. Show that the points P, Q, R, S are concyclic.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

The 9$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 1 — Solutions

Problem 1. **(a)** Prove that every positive integer n can be written uniquely in the form

$$n = \sum_{j=1}^{2k+1} (-1)^{j-1} 2^{m_j},$$

where $k \geq 0$ and $0 \leq m_1 < m_2 < \cdots < m_{2k+1}$ are integers.
This number k is called the *weight* of n.

(b) Find (in closed form) the difference between the number of positive integers at most 2^{2017} with even weight and the number of positive integers at most 2^{2017} with odd weight.

Vjekoslav Kovač, Croatia

Solution. **(a)** We show by induction on the integer $M \geq 0$ that every integer n in the range -2^M+1 through 2^M can uniquely be written in the form $n = \sum_{j=1}^{\ell}(-1)^{j-1}2^{m_j}$ for some integers $\ell \geq 0$ and $0 \leq m_1 < m_2 < \cdots < m_\ell \leq M$ (empty sums are 0); moreover, in this unique representation ℓ is odd if $n > 0$, and even if $n \leq 0$. The integer $w(n) = \lfloor \ell/2 \rfloor$ is called the *weight* of n.

Existence once proved, uniqueness follows from the fact that there are as many such representations as integers in the range $-2^M + 1$ through 2^M, namely, 2^{M+1}.

To prove existence, notice that the base case $M = 0$ is clear, so let $M \geq 1$ and let n be an integer in the range $-2^M + 1$ through 2^M.

If $-2^M + 1 \leq n \leq -2^{M-1}$, then $1 \leq n + 2^M \leq 2^{M-1}$, so $n + 2^M = \sum_{j=1}^{2k+1}(-1)^{j-1}2^{m_j}$ for some integers $k \geq 0$ and $0 \leq m_1 < \cdots < m_{2k+1} \leq M - 1$ by the induction hypothesis, and $n = \sum_{j=1}^{2k+2}(-1)^{j-1}2^{m_j}$, where $m_{2k+2} = M$.

The case $-2^{M-1} + 1 \leq n \leq 2^{M-1}$ is covered by the induction hypothesis.

Finally, if $2^{M-1} + 1 \leq n \leq 2^M$, then $-2^{M-1} + 1 \leq n - 2^M \leq 0$, so $n - 2^M = \sum_{j=1}^{2k}(-1)^{j-1}2^{m_j}$ for some integers $k \geq 0$ and $0 \leq m_1 < \cdots < m_{2k} \leq M - 1$ by the induction hypothesis, and $n = \sum_{j=1}^{2k+1}(-1)^{j-1}2^{m_j}$, where $m_{2k+1} = M$.

(b) First Approach. Let $M \geq 0$ be an integer. The solution for part **(a)** shows that the number of even (respectively, odd) weight integers in the range 1 through 2^M coincides with the number of subsets in $\{0, 1, 2, \ldots, M\}$ whose cardinality has remainder 1 (respectively, 3) modulo 4. Therefore, the difference of these numbers is

$$\sum_{k=0}^{\lfloor M/2 \rfloor} (-1)^k \binom{M+1}{2k+1} = \frac{(1+\mathrm{i})^{M+1} - (1-\mathrm{i})^{M+1}}{2\mathrm{i}} = 2^{(M+1)/2} \sin\frac{(M+1)\pi}{4},$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. Thus, the required difference is 2^{1009}.

Second Approach. For every integer $M \geq 0$, let $A_M = \sum_{n=-2^M+1}^{0}(-1)^{w(n)}$ and let $B_M = \sum_{n=1}^{2^M}(-1)^{w(n)}$; thus, B_M evaluates the difference of the number of even weight integers in the range 1 through 2^M and the number of odd weight integers in that range.

Notice that

$$w(n) = \begin{cases} w(n + 2^M) + 1 & \text{if } -2^M + 1 \leq n \leq -2^{M-1}, \\ w(n - 2^M) & \text{if } 2^{M-1} + 1 \leq n \leq 2^M, \end{cases}$$

to get

$$A_M = -\sum_{n=-2^M+1}^{-2^{M-1}} (-1)^{w(n+2^M)} + \sum_{n=-2^{M-1}+1}^{0} (-1)^{w(n)} = -B_{M-1} + A_{M-1},$$

$$B_M = \sum_{n=1}^{2^{M-1}} (-1)^{w(n)} + \sum_{n=2^{M-1}+1}^{2^M} (-1)^{w(n-2^M)} = B_{M-1} + A_{M-1}.$$

Iteration yields

$$\begin{aligned} B_M &= A_{M-1} + B_{M-1} = (A_{M-2} - B_{M-2}) + (A_{M-2} + B_{M-2}) = 2A_{M-2} \\ &= 2A_{M-3} - 2B_{M-3} = 2(A_{M-4} - B_{M-4}) - 2(A_{M-4} + B_{M-4}) = -4B_{M-4}. \end{aligned}$$

Thus, $B_{2017} = (-4)^{504} B_1 = 2^{1008} B_1$; since $B_1 = (-1)^{w(1)} + (-1)^{w(2)} = 2$, it follows that $B_{2017} = 2^{1009}$.

Problem 2. Determine all positive integers n satisfying the following condition: for every monic polynomial P of degree at most n with integer coefficients, there exists a positive integer $k \le n$, and $k+1$ distinct integers $x_1, x_2, \ldots, x_{k+1}$ such that

$$P(x_1) + P(x_2) + \cdots + P(x_k) = P(x_{k+1}).$$

Semen Petrov, Russia

Note. A polynomial is *monic* if the coefficient of the highest power is one.

Solution. There is only one such integer, namely, $n = 2$. In this case, if P is a constant polynomial, the required condition is clearly satisfied; if $P = X + c$, then $P(c-1) + P(c+1) = P(3c)$; and if $P = X^2 + qX + r$, then $P(X) = P(-X - q)$.

To rule out all other values of n, it is sufficient to exhibit a monic polynomial P of degree at most n with integer coefficients, whose restriction to the integers is injective, and $P(x) \equiv 1 \pmod{n}$ for all integers x. This is easily seen by reading the relation in the statement modulo n, to deduce that $k \equiv 1 \pmod{n}$, so $k = 1$, since $1 \le k \le n$; hence $P(x_1) = P(x_2)$ for some distinct integers x_1 and x_2, which contradicts injectivity.

If $n = 1$, let $P = X$, and if $n = 4$, let $P = X^4 + 7X^2 + 4X + 1$. In the latter case, clearly, $P(x) \equiv 1 \pmod{4}$ for all integers x; and P is injective on the integers, since $P(x) - P(y) = (x-y)\big((x+y)(x^2+y^2+7)+4\big)$, and the absolute value of $(x+y)(x^2+y^2+7)$ is either 0 or at least 7 for integral x and y.

Assume henceforth $n \ge 3$, $n \ne 4$, and let $f_n = (X-1)(X-2)\cdots(X-n)$. Clearly, $f_n(x) \equiv 0 \pmod{n}$ for all integers x. If n is odd, then f_n is non-decreasing on the integers; and if, in addition, $n > 3$, then $f_n(x) \equiv 0 \pmod{n+1}$ for all integers x, since $f_n(0) = -n! = -1 \cdot 2 \cdot \cdots \cdot \frac{n+1}{2} \cdot \cdots \cdot n \equiv 0 \pmod{n+1}$.

Finally, let $P = f_n + nX + 1$ if n is odd, and let $P = f_{n-1} + nX + 1$ if n is even. In either case, P is strictly increasing, hence injective, on the integers, and $P(x) \equiv 1 \pmod{n}$ for all integers x.

Remark. The polynomial $P = f_n + nX + 1$ works equally well for even $n > 2$. To prove injectivity, notice that P is strictly monotone, hence injective, on non-positive (respectively, positive) integers. Suppose, if possible, that $P(a) = P(b)$ for some integers $a \le 0$ and $b > 0$. Notice that $P(a) \ge P(0) = n! + 1 > n^2 + 1 = P(n)$, since $n \ge 4$, to infer that $b \ge n+1$. It is therefore sufficient to show that $P(x) > P(n+1-x) > P(x-1)$ for all integers $x \ge n+1$. The former inequality is trivial, since $f_n(x) = f_n(n+1-x)$ for even n. For the latter, write

$$\begin{aligned} P(n+1-x) - P(x-1) &= (x-1)\cdots(x-n) - (x-2)\cdots(x-n-1) + n(n+2-2x) \\ &= n\big((x-2)\cdots(x-n) + (n-2) - 2(x-2)\big) \ge n(n-2) > 0, \end{aligned}$$

since $(x-3)\cdots(x-n) \ge 2$.

Problem 3. Let n be an integer greater than 1 and let X be an n-element set. A non-empty collection of subsets $A_1, \ldots, A_k$ of X is *tight* if the union $A_1 \cup \cdots \cup A_k$ is a proper subset of X and no element of X lies in exactly one of the A_is. Find the largest cardinality of a collection of proper non-empty subsets of X, no non-empty subcollection of which is tight.

Note. A subset A of X is *proper* if $A \neq X$. The sets in a collection are assumed to be distinct. The whole collection is assumed to be a subcollection.

Alexander Polyansky, Russia

Solution 1. (*Ilya Bogdanov*) The required maximum is $2n-2$. To describe a $(2n-2)$-element collection satisfying the required conditions, write $X = \{1, 2, \ldots, n\}$ and set $B_k = \{1, 2, \ldots, k\}$, $k = 1, 2, \ldots, n-1$, and $B_k = \{k-n+2, k-n+3, \ldots, n\}$, $k = n, n+1, \ldots, 2n-2$. To show that no subcollection of the B_k is tight, consider a subcollection $\mathcal{C}$ whose union U is a proper subset of X, let m be an element in $X \smallsetminus U$, and notice that $\mathcal{C}$ is a subcollection of $\{B_1, \ldots, B_{m-1}, B_{m+n-1}, \ldots, B_{2n-2}\}$, since the other B's are precisely those containing m. If U contains elements less than m, let k be the greatest such and notice that B_k is the only member of $\mathcal{C}$ containing k; and if U contains elements greater than m, let k be the least such and notice that B_{k+n-2} is the only member of $\mathcal{C}$ containing k. Consequently, $\mathcal{C}$ is not tight.

We now proceed to show by induction on $n \geq 2$ that the cardinality of a collection of proper non-empty subsets of X, no subcollection of which is tight, does not exceed $2n-2$. The base case $n = 2$ is clear, so let $n > 2$ and suppose, if possible, that $\mathcal{B}$ is a collection of $2n-1$ proper non-empty subsets of X containing no tight subcollection.

To begin, notice that $\mathcal{B}$ has an empty intersection: if the members of $\mathcal{B}$ shared an element x, then $\mathcal{B}' = \{B \smallsetminus \{x\} : B \in \mathcal{B},\ B \neq \{x\}\}$ would be a collection of at least $2n-2$ proper non-empty subsets of $X \smallsetminus \{x\}$ containing no tight subcollection, and the induction hypothesis would be contradicted.

Now, for every x in X, let $\mathcal{B}_x$ be the (non-empty) collection of all members of $\mathcal{B}$ not containing x. Since no subcollection of $\mathcal{B}$ is tight, $\mathcal{B}_x$ is not tight, and since the union of $\mathcal{B}_x$ does not contain x, some x' in X is covered by a single member of $\mathcal{B}_x$. In other words, *there is a single set in $\mathcal{B}$ covering x' but not x.* In this case, draw an arrow from x to x'. Since there is at least one arrow from each x in X, some of these arrows form a (minimal) cycle $x_1 \to x_2 \to \cdots \to x_k \to x_{k+1} = x_1$ for some suitable integer $k \geq 2$. Let A_i be the unique member of $\mathcal{B}$ containing x_{i+1} but not x_i, and let $X' = \{x_1, x_2, \ldots, x_k\}$.

Remove $A_1, A_2, \ldots, A_k$ from $\mathcal{B}$ to obtain a collection $\mathcal{B}'$ each member of which either contains or is disjoint from X': for if a member B of $\mathcal{B}'$ contained some but not all elements of X', then B should contain x_{i+1} but not x_i for some i, and $B = A_i$, a contradiction. This rules out the case $k = n$, for otherwise $\mathcal{B} = \{A_1, A_2, \ldots, A_n\}$, so $|\mathcal{B}| < 2n-1$.

To rule out the case $k < n$, consider an extra element x^* outside X and let

$$\mathcal{B}^* = \{B : B \in \mathcal{B}',\ B \cap X' = \varnothing\} \cup \{(B \smallsetminus X') \cup \{x^*\} : B \in \mathcal{B}',\ X' \subseteq B\};$$

thus, in each member of $\mathcal{B}'$ containing X', the latter is collapsed to singleton x^*. Notice that $\mathcal{B}^*$ is a collection of proper non-empty subsets of $X^* = (X \smallsetminus X') \cup \{x^*\}$, no subcollection of which is tight. By the induction hypothesis, $|\mathcal{B}'| = |\mathcal{B}^*| \leq 2|X^*| - 2 = 2(n-k)$, so $|\mathcal{B}| \leq 2(n-k) + k = 2n - k < 2n - 1$, a final contradiction.

Solution 2. Proceed again by induction on n to show that the cardinality of a collection of proper non-empty subsets of X, no subcollection of which is tight, does not exceed $2n-2$.

Consider any collection $\mathcal{B}$ of proper non-empty subsets of X with no tight subcollection (we call such collection *good*). Assume that there exist $M, N \in \mathcal{B}$ such that $M \cup N$ is distinct from M, N, and X. In this case, we will show how to modify $\mathcal{B}$ so that it remains good, contains the same number of sets, but the total number of elements in the sets of $\mathcal{B}$ increases.

Consider a maximal (relative to set-theoretic inclusion) subcollection $\mathcal{C} \subseteq \mathcal{B}$ such that the set $C = \bigcup_{A \in \mathcal{C}} A$ is distinct from X and from all members of $\mathcal{C}$. Notice here that the union of *any* subcollection $\mathcal{D} \subset \mathcal{B}$ cannot coincide with any $K \in \mathcal{B} \setminus \mathcal{D}$, otherwise $\{K\} \cup \mathcal{D}$ would be tight. Surely, $\mathcal{C}$ exists (since $\{M, N\}$ is an example of a collection satisfying the requirements on $\mathcal{C}$, except for maximality); moreover, $C \notin \mathcal{B}$ by the above remark.

Since $C \neq X$, there exists an $L \in \mathcal{C}$ and $x \in L$ such that L is the unique set in $\mathcal{C}$ containing x. Now replace in $\mathcal{B}$ the set L by C in order to obtain a new collection $\mathcal{B}'$ (then $|\mathcal{B}'| = |\mathcal{B}|$). We claim that $\mathcal{B}'$ is good.

Assume, to the contrary, that $\mathcal{B}'$ contained a tight subcollection $\mathcal{T}$; clearly, $C \in \mathcal{T}$, otherwise $\mathcal{B}$ is not good. If $\mathcal{T} \subseteq \mathcal{C} \cup \{C\}$, then C is the unique set in $\mathcal{T}$ containing x which is impossible. Therefore, there exists $P \in \mathcal{T} \setminus (\mathcal{C} \cup \{C\})$. By maximality of $\mathcal{C}$, the collection $\mathcal{C} \cup \{P\}$ does not satisfy the requirements imposed on $\mathcal{C}$; since $P \cup C \neq X$, this may happen only if $C \cup P = P$, i.e., if $C \subset P$. But then $\mathcal{G} = (\mathcal{T} \setminus \{C\}) \cup \mathcal{C}$ is a tight subcollection in $\mathcal{B}$: all elements of C are covered by $\mathcal{G}$ at least twice (by P and an element of $\mathcal{C}$), and all the rest elements are covered by $\mathcal{G}$ the same number of times as by $\mathcal{T}$. A contradiction. Thus $\mathcal{B}'$ is good.

Such modifications may be performed finitely many times, since the total number of elements of sets in $\mathcal{B}$ increases. Thus, at some moment we arrive at a good collection $\mathcal{B}$ for which the procedure no longer applies. This means that for every $M, N \in \mathcal{B}$, either $M \cup N = X$ or one of them is contained in the other.

Now let M be a minimal (with respect to inclusion) set in $\mathcal{B}$. Then each set in $\mathcal{B}$ either contains M or forms X in union with M (i.e., contains $X \setminus M$). Now one may easily see that the two collections

$$\mathcal{B}_+ = \{A \setminus M : A \in \mathcal{B},\ M \subset A,\ A \neq M\}, \quad \mathcal{B}_- = \{A \cap M : A \in \mathcal{B},\ X \setminus M \subset A,\ A \neq X \setminus M\}$$

are good as collections of subsets of $X \setminus M$ and M, respectively; thus, by the induction hypothesis, we have $|\mathcal{B}_+| + |\mathcal{B}_-| \leq 2n - 4$.

Finally, each set $A \in \mathcal{B}$ either produces a set in one of the two new collections, or coincides with M or $X \setminus M$. Thus $|\mathcal{B}| \leq |\mathcal{B}_+| + |\mathcal{B}_-| + 2 \leq 2n - 2$, as required.

Solution 3. We provide yet another proof of the estimate $|\mathcal{B}| \leq 2n - 2$, using the notion of a *good* collection from Solution 2. Arguing indirectly, we assume that there exists a good collection $\mathcal{B}$ with $|\mathcal{B}| \geq 2n - 1$, and choose one such for the minimal possible value of n. Clearly, $n > 2$.

Firstly, we perform a different modification of $\mathcal{B}$. Choose any $x \in X$, and consider the subcollection $\mathcal{B}_x = \{B : B \in \mathcal{B},\ x \notin B\}$. By our assumption, $\mathcal{B}_x$ is not tight. As the union of sets in $\mathcal{B}_x$ is distinct from X, either this collection is empty, or there exists an element $y \in X$ contained in a unique member A_x of $\mathcal{B}_x$. In the former case, we add the set $B_x = X \setminus \{x\}$ to $\mathcal{B}$, and in the latter we replace A_x by B_x, to form a new collection $\mathcal{B}'$. (Notice that if $B_x \in \mathcal{B}$, then $B_x \in \mathcal{B}_x$ and $y \in B_x$, so $B_x = A_x$.)

We claim that the collection $\mathcal{B}'$ is also good. Indeed, if $\mathcal{B}'$ has a tight subcollection $\mathcal{T}$, then B_x should lie in $\mathcal{T}$. Then, as the union of the sets in $\mathcal{T}$ is distinct from X, we should have $\mathcal{T} \subseteq \mathcal{B}_x \cup \{B_x\}$. But in this case an element y is contained in a unique member of $\mathcal{T}$, namely B_x, so $\mathcal{T}$ is not tight — a contradiction.

Perform this procedure for every $x \in X$, to get a good collection $\mathcal{B}$ containing the sets $B_x = X \setminus \{x\}$ for all $x \in X$. Consider now an element $x \in X$ such that $|\mathcal{B}_x|$ is maximal. As we have mentioned before, there exists an element $y \in X$ belonging to a unique member (namely, B_x) of $\mathcal{B}_x$. Thus, $\mathcal{B}_x \setminus \{B_x\} \subset \mathcal{B}_y$; also, $B_y \in \mathcal{B}_y \setminus \mathcal{B}_x$. Thus we get $|\mathcal{B}_y| \geq |\mathcal{B}_x|$, which by the maximality assumption yields the equality, which in turn means that $\mathcal{B}_y = (\mathcal{B}_x \setminus \{B_x\}) \cup \{B_y\}$.

Therefore, each set in $\mathcal{B} \setminus \{B_x, B_y\}$ contains either both x and y, or none of them. Collapsing $\{x, y\}$ to singleton x^*, we get a new collection of $|\mathcal{B}| - 2$ subsets of $(X \setminus \{x, y\}) \cup \{x^*\}$ containing no tight subcollection. This contradicts minimality of n.

Remarks. 1. Removal of the condition that subsets be proper would only increase the maximum by 1. The 'non-emptiness' condition could also be omitted, since the empty set forms a tight collection by itself, but the argument is a bit too formal to be considered.

2. There are many different examples of good collections of $2n-2$ sets. E.g., applying the algorithm from the first part of Solution 2 to the example shown in Solution 1, one may get the following example: $B_k = \{1, 2, \ldots, k\}$, $k = 1, 2, \ldots, n-1$, and $B_k = X \setminus \{k-n+1\}$, $k = n, n+1, \ldots, 2n-2$.

The 9th Romanian Master of Mathematics Competition

Day 2 — Solutions

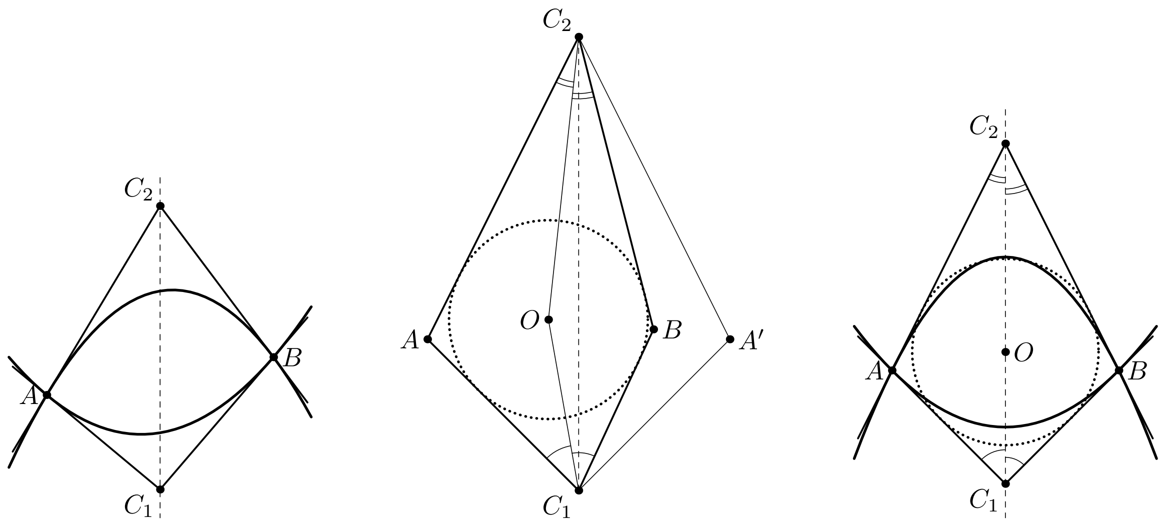
Problem 4. In the Cartesian plane, let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the graphs of the quadratic functions $f_1(x) = p_1x^2 + q_1x + r_1$ and $f_2(x) = p_2x^2 + q_2x + r_2$, where $p_1 > 0 > p_2$. The graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ cross at distinct points A and B. The four tangents to $\mathcal{G}_1$ and $\mathcal{G}_2$ at A and B form a convex quadrilateral which has an inscribed circle. Prove that the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same axis of symmetry.

Alexey Zaslavsky, Russia

Solution 1. Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be the tangents to $\mathcal{G}_i$ at A and B, respectively, and let $C_i = \mathcal{A}_i \cap \mathcal{B}_i$. Since $f_1(x)$ is convex and $f_2(x)$ is concave, the convex quadrangle formed by the four tangents is exactly AC_1BC_2.

Lemma. *If CA and CB are the tangents drawn from a point C to the graph $\mathcal{G}$ of a quadratic trinomial $f(x) = px^2 + qx + r$, $A, B \in \mathcal{G}$, $A \neq B$, then the abscissa of C is the arithmetic mean of the abscissae of A and B.*

Proof. Assume, without loss of generality, that C is at the origin, so the equations of the two tangents have the form $y = k_ax$ and $y = k_bx$. Next, the abscissae x_A and x_B of the tangency points A and B, respectively, are multiple roots of the polynomials $f(x) - k_ax$ and $f(x) - k_bx$, respectively. By the Vieta theorem, $x_A^2 = r/p = x_B^2$, so $x_A = -x_B$, since the case $x_A = x_B$ is ruled out by $A \neq B$.



The Lemma shows that the line C_1C_2 is parallel to the y-axis and the points A and B are equidistant from this line.

Suppose, if possible, that the incentre O of the quadrangle AC_1BC_2 does not lie on the line C_1C_2. Assume, without loss of generality, that O lies inside the triangle AC_1C_2 and let A' be the reflection of A in the line C_1C_2. Then the ray C_iB emanating from C_i lies inside the angle AC_iA', so B lies inside the quadrangle $AC_1A'C_2$, whence A and B are not equidistant from C_1C_2 — a contradiction.

Thus O lies on C_1C_2, so the lines AC_i and BC_i are reflections of one another in the line C_1C_2, and $B = A'$. Hence $y_A = y_B$, and since $f_i(x) = y_A + p_i(x - x_A)(x - x_B)$, the line C_1C_2 is the axis of symmetry of both parabolas, as required.

Solution 2. Use the standard equation of a tangent to a smooth curve in the plane, to deduce that the tangents at two distinct points A and B on the parabola of equation $y = px^2 + qx + r$,

$p \neq 0$, meet at some point C whose coordinates are

$$x_C = \tfrac{1}{2}(x_A + x_B) \quad \text{and} \quad y_C = px_Ax_B + q \cdot \tfrac{1}{2}(x_A + x_B) + r.$$

Usage of the standard formula for Euclidean distance yields

$$CA = \tfrac{1}{2}|x_B - x_A|\sqrt{1 + (2px_A + q)^2} \quad \text{and} \quad CB = \tfrac{1}{2}|x_B - x_A|\sqrt{1 + (2px_B + q)^2},$$

so, after obvious manipulations,

$$CB - CA = \frac{2p(x_B - x_A)|x_B - x_A|(p(x_A + x_B) + q)}{\sqrt{1 + (2px_A + q)^2} + \sqrt{1 + (2px_B + q)^2}}.$$

Now, write the condition in the statement in the form $C_1B - C_1A = C_2B - C_2A$, apply the above formula and clear common factors to get

$$\frac{p_1(p_1(x_A + x_B) + q_1)}{\sqrt{1 + (2p_1x_A + q_1)^2} + \sqrt{1 + (2p_1x_B + q_1)^2}} = \frac{p_2(p_2(x_A + x_B) + q_2)}{\sqrt{1 + (2p_2x_A + q_2)^2} + \sqrt{1 + (2p_2x_B + q_2)^2}}.$$

Next, use the fact that x_A and x_B are the solutions of the quadratic equation $(p_1 - p_2)x^2 + (q_1 - q_2)x + r_1 - r_2 = 0$, so $x_A + x_B = -(q_1 - q_2)/(p_1 - p_2)$, to obtain

$$\frac{p_1(p_1q_2 - p_2q_1)}{\sqrt{1 + (2p_1x_A + q_1)^2} + \sqrt{1 + (2p_1x_B + q_1)^2}} = \frac{p_2(p_1q_2 - p_2q_1)}{\sqrt{1 + (2p_2x_A + q_2)^2} + \sqrt{1 + (2p_2x_B + q_2)^2}}.$$

Finally, since $p_1p_2 < 0$ and the denominators above are both positive, the last equality forces $p_1q_2 - p_2q_1 = 0$; that is, $q_1/p_1 = q_2/p_2$, so the two parabolas have the same axis.

Remarks. The are, of course, several different proofs of the Lemma in Solution 1 — in particular, computational. Another argument relies on the following consequence of focal properties: The tangents to a parabola at two points meet at the circumcentre of the triangle formed by the focus and the orthogonal projections of those points on the directrix. Since the directrix of the parabola in the lemma is parallel to the axis of abscissae, the conclusion follows.

Problem 5. Fix an integer $n \geq 2$. An $n \times n$ *sieve* is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A *stick* is a $1 \times k$ or $k \times 1$ array for any positive integer k. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

Palmer Mebane and Nikolai Beluhov

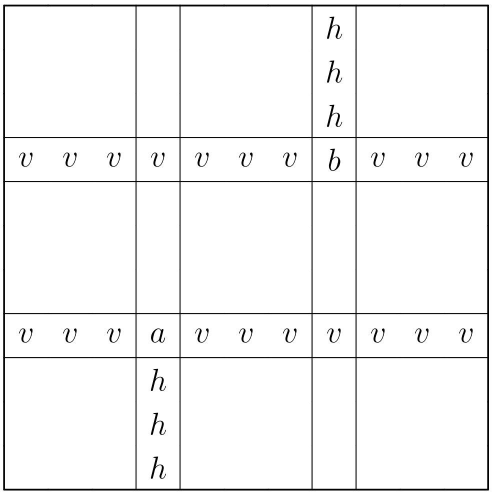
Solution 1. Given A, $m(A) = 2n - 2$, and it is achieved, for instance, by dissecting A along all horizontal (or vertical) grid lines. It remains to prove that $m(A) \geq 2n - 2$ for every A.

By *holes* we mean the cells which are cut out from the board. The *cross* of a hole in A is the union of the row and the column through that hole.

Arguing indirectly, consider a dissection of A into $2n - 3$ or fewer sticks. Horizontal sticks are all labelled h, and vertical sticks are labelled v; 1×1 sticks are both horizontal and vertical, and labelled arbitrarily. Each cell of A inherits the label of the unique containing stick.

Assign each stick in the dissection to the cross of the unique hole on its row, if the stick is horizontal; on its column, if the stick is vertical.

Since there are at most $2n - 3$ sticks and exactly n crosses, there are two crosses each of which is assigned to at most one stick in the dissection. Let the crosses be c and d, centred at $a = (x_a, y_a)$ and $b = (x_b, y_b)$, respectively, and assume, without loss of generality, $x_a < x_b$ and $y_a < y_b$. The sticks covering the cells (x_a, y_b) and (x_b, y_a) have like labels, for otherwise one of the two crosses would be assigned to at least two sticks. Say the common label is v, so each of c and d contains a stick covering one of those two cells. It follows that the lower (respectively, upper) arm of c (respectively, d) is all-h, and the horizontal arms of both crosses are all-v, as illustrated below.



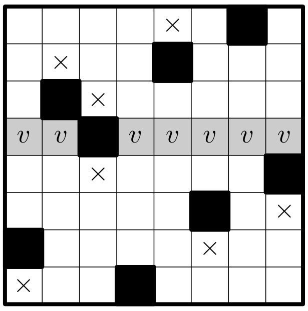
Each of the rows between the rows of a and b, that is, rows $y_a + 1$, $y_a + 2$, $\ldots$, $y_b - 1$, contains a hole. The column of each such hole contains at least two v-sticks. All other columns contain at least one v-stick each. In addition, all rows below a and all rows above b contain at least one h-stick each. This amounts to a total of at least $2(y_b - y_a - 1) + (n - y_b + y_a + 1) + (n - y_b) + (y_a - 1) = 2n - 2$ sticks. A contradiction.

Remark. One may find a different argument finishing the solution. Since c and d are proven to contain one stick each, there is a third cross e centred at (x_*, y_*) also containing at most one stick. It meets the horizontal arms of c and d at two v-cells, so all the cells where two of the three crosses meet are labelled with v. Now, assuming (without loss of generality) that $y_a < y_* < y_b$, we obtain that both vertical arms of e contain v-cells, so e is assigned to two different v-sticks. A contradiction.

Solution 2. (*Ilya Bogdanov*) We provide a different proof that $m(A) \geq 2n - 2$.

Call a stick *vertical* if it is contained in some column, and *horizontal* if it is contained in some row; 1×1 sticks may be called arbitrarily, but any of them is supposed to have only one direction. Assign to each vertical/horizontal stick the column/row it is contained in. If each row and each column is assigned to some stick, then there are at least $2n$ sticks, which is even more than we want. Thus we assume, without loss of generality, that some *exceptional* row R is not assigned to any stick. This means that all $n-1$ existing cells in R belong to $n-1$ distinct vertical sticks; call these sticks *central.*

Now we mark $n-1$ cells on the board in the following manner. ($\downarrow$) For each hole c below R, we mark the cell just under c; ($\uparrow$) for each hole c above R, we mark the cell just above c; and ($\bullet$) for the hole r in R, we mark both the cell just above it and just below it. We have described $n+1$ cells, but exactly two of them are out of the board; so $n-1$ cells are marked within the board. A sample marking is shown in the figure below, where the marked cells are crossed.



Notice that all the marked cells lie in different rows, and all of them are marked in different columns, except for those two marked for ($\bullet$); but the latter two have a hole r between them. So no two marked cells may belong to the same stick. Moreover, none of them lies in a central stick, since the marked cells are separated from R by the holes. Thus the marked cells should be covered by $n-1$ different sticks (call them *border*) which are distinct from the central sticks. This shows that there are at least $(n-1)+(n-1) = 2n-2$ distinct sticks, as desired.

Solution 3. In order to prove $m(A) \geq 2n-2$, it suffices to show that there are $2n-2$ cells in A, no two of which may be contained in the same stick.

To this end, consider the bipartite graph G with parts G_h and G_v, where the vertices in G_h (respectively, G_v) are the $2n-2$ maximal sticks A is dissected into by all horizontal (respectively, vertical) grid lines, two sticks being joined by an edge in G if and only if they share a cell.

We show that G admits a perfect matching by proving that it fulfils the condition in Hall's theorem; the $2n-2$ cells corresponding to the edges of this matching form the desired set. It is sufficient to show that every subset S of G_h has at least $|S|$ neighbours (in G_v, of course).

Let L be the set of all sticks in S that contain a cell in the leftmost column of A, and let R be the set of all sticks in S that contain a cell in the rightmost column of A; let ℓ be the length of the longest stick in L (zero if L is empty), and let r be the length of the longest stick in R (zero if R is empty).

Since every row of A contains exactly one hole, L and R partition S; and since every column of A contains exactly one hole, neither L nor R contains two sticks of the same size, so $\ell \geq |L|$ and $r \geq |R|$, whence $\ell + r \geq |L| + |R| = |S|$.

If $\ell + r \leq n$, we are done, since there are at least $\ell + r \geq |S|$ vertical sticks covering the cells of the longest sticks in L and R. So let $\ell + r > n$, in which case the sticks in S span all n columns, and notice that we are again done if $|S| \leq n$, to assume further $|S| > n$.

Let $S' = G_h \setminus S$, let T be set of all neighbours of S, and let $T' = G_v \setminus T$. Since the sticks in S span all n columns, $|T| \geq n$, so $|T'| \leq n-2$. Transposition of the above argument (replace S by T'), shows that $|T'| \leq |S'|$, so $|S| \leq |T|$.

Remark. Here is an alternative argument for $s = |S| > n$. Add to S two *empty sticks* formally present to the left of the leftmost hole and to the right of the rightmost one. Then there are at least $s-n+2$ rows containing two sticks from S, so two of them are separated by at least $s-n$ other rows. Each hole in those $s-n$ rows separates two vertical sticks from G_v both of which are neighbours of S. Thus the vertices of S have at least $n+(s-n)$ neighbours.

Solution 4. Yet another proof of the estimate $m(A) \geq 2n-2$. We use the induction on n. Now we need the base cases $n = 2, 3$ which can be completed by hands.

Assume now that $n > 3$ and consider any dissection of A into sticks. Define the *cross* of a hole as in Solution 1, and notice that each stick is contained in some cross. Thus, if the dissection contains more than n sticks, then there exists a cross containing at least two sticks. In this case, remove this cross from the sieve to obtain an $(n-1)\times(n-1)$ sieve. The dissection of the original sieve induces a dissection of the new array: even if a stick is partitioned into two by the removed cross, then the remaining two parts form a stick in the new array. After this operation has been performed, the number of sticks decreases by at least 2, and by the induction hypothesis the number of sticks in the new dissection is at least $2n-4$. Hence, the initial dissection contains at least $(2n-4)+2 = 2n-2$ sticks, as required.

It remains to rule out the case when the dissection contains at most n sticks. This can be done in many ways, one of which is removal a cross containing some stick. The resulting dissection of an $(n-1)\times(n-1)$ array contains at most $n-1$ sticks, which is impossible by the induction hypothesis since $n-1 < 2(n-1)-2$.

Remark. The idea of removing a cross containing at least two sticks arises naturally when one follows an inductive approach. But it is much trickier to finish the solution using this approach, **unless** one starts to consider removing **each** cross instead of removing a specific one.

Problem 6. Let $ABCD$ be any convex quadrilateral and let P, Q, R, S be points on the segments AB, BC, CD, and DA, respectively. It is given that the segments PR and QS dissect $ABCD$ into four quadrilaterals, each of which has perpendicular diagonals. Show that the points P, Q, R, S are concyclic.

Nikolai Beluhov

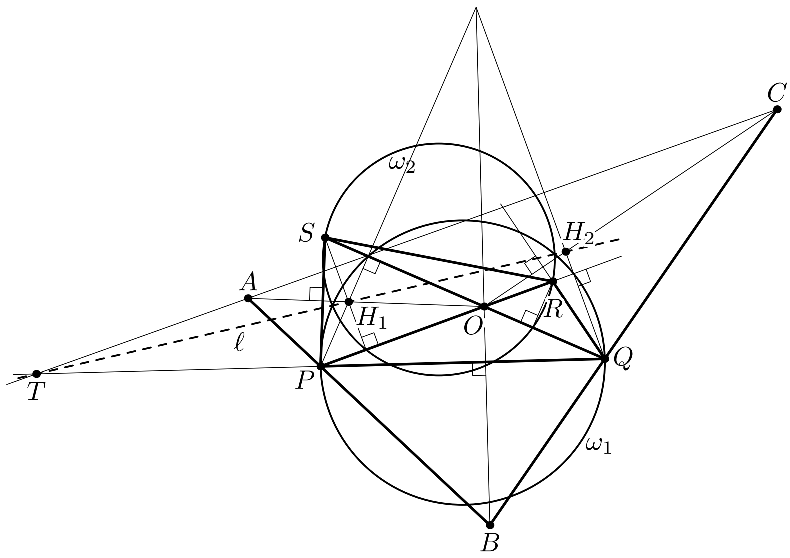
Solution 1. We start with a lemma which holds even in a more general setup.

Lemma 1. *Let $PQRS$ be a convex quadrangle whose diagonals meet at O. Let ω_1 and ω_2 be the circles on diameters PQ and RS, respectively, and let ℓ be their radical axis. Finally, choose the points A, B, and C outside this quadrangle so that: the point P (respectively, Q) lies on the segment AB (respectively, BC); and $AO \perp PS$, $BO \perp PQ$, and $CO \perp QR$. Then the three lines AC, PQ, and ℓ are concurrent or parallel.*

Proof. Assume first that the lines PR and QS are not perpendicular. Let H_1 and H_2 be the orthocentres of the triangles OSP and OQR, respectively; notice that H_1 and H_2 do not coincide.

Since H_1 is the radical centre of the circles on diameters RS, SP, and PQ, it lies on ℓ. Similarly, H_2 lies on ℓ, so the lines H_1H_2 and ℓ coincide.

The corresponding sides of the triangles APH_1 and CQH_2 meet at O, B, and the orthocentre of the triangle OPQ (which lies on OB). By Desargues' theorem, the lines AC, PQ and ℓ are concurrent or parallel.

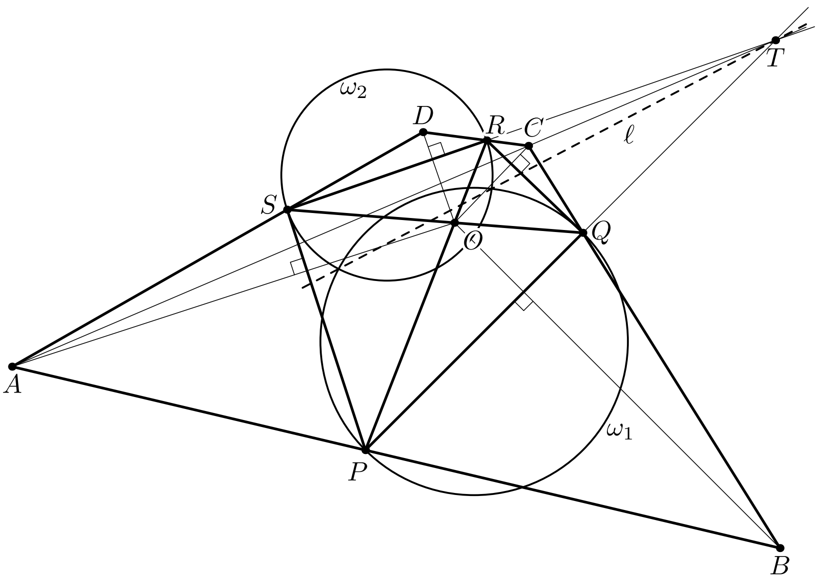


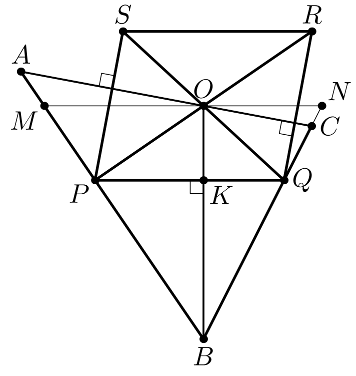
The case when $PR \perp QS$ may be considered as a limit case, since the configuration in the statement of the lemma allows arbitrarily small perturbations. The lemma is proved.

Back to the problem, let the segments PR and QS cross at O, let ω_1 and ω_2 be the circles on diameters PQ and RS, respectively, and let ℓ be their radical axis. By the Lemma, the three lines AC, ℓ, and PQ are concurrent or parallel, and similarly so are the three lines AC, ℓ, and RS. Thus, if the lines AC and ℓ are distinct, all four lines are concurrent or pairwise parallel.

This is clearly the case when the lines PS and QR are not parallel (since ℓ crosses OA and OC at the orthocentres of OSP and OQR, these orthocentres being distinct from A and C). In this case, denote the concurrency point by T. If T is not ideal, then we have $TP \cdot TQ = TR \cdot TS$ (as $T \in \ell$), so $PQRS$ is cyclic. If T is ideal (i.e., all four lines are parallel), then the segments PQ and RS have the same perpendicular bisector (namely, the line of centers of ω_1 and ω_2), and $PQRS$ is cyclic again.

Assume now PS and QR parallel. By symmetry, PQ and RS may also be assumed parallel: otherwise, the preceding argument goes through after relabelling. In this case, we need to prove that the parallelogram $PQRS$ is a rectangle.





Suppose, by way of contradiction, that $OP > OQ$. Let the line through O and parallel to PQ meet AB at M, and CB at N. Since $OP > OQ$, the angle SPQ is acute and the angle PQR is obtuse, so the angle AOB is obtuse, the angle BOC is acute, M lies on the segment AB, and N lies on the extension of the segment BC beyond C. Therefore: $OA > OM$, since the angle OMA is obtuse; $OM > ON$, since $OM : ON = KP : KQ$, where K is the projection of O onto PQ; and $ON > OC$, since the angle OCN is obtuse. Consequently, $OA > OC$.

Similarly, $OR > OS$ yields $OC > OA$: a contradiction. Consequently, $OP = OQ$ and $PQRS$ is a rectangle. This ends the proof.

Solution 2. (*Ilya Bogdanov*) To begin, we establish a useful lemma.

Lemma 2. *If P is a point on the side AB of a triangle OAB, then*

$$\frac{\sin AOP}{OB} + \frac{\sin POB}{OA} = \frac{\sin AOB}{OP}.$$

Proof. Let $[XYZ]$ denote the area of a triangle XYZ, to write

$$0 = 2([AOB] - [POB] - [POC]) = OA \cdot OB \cdot \sin AOB - OB \cdot OP \cdot \sin POB - OP \cdot OA \cdot \sin AOP,$$

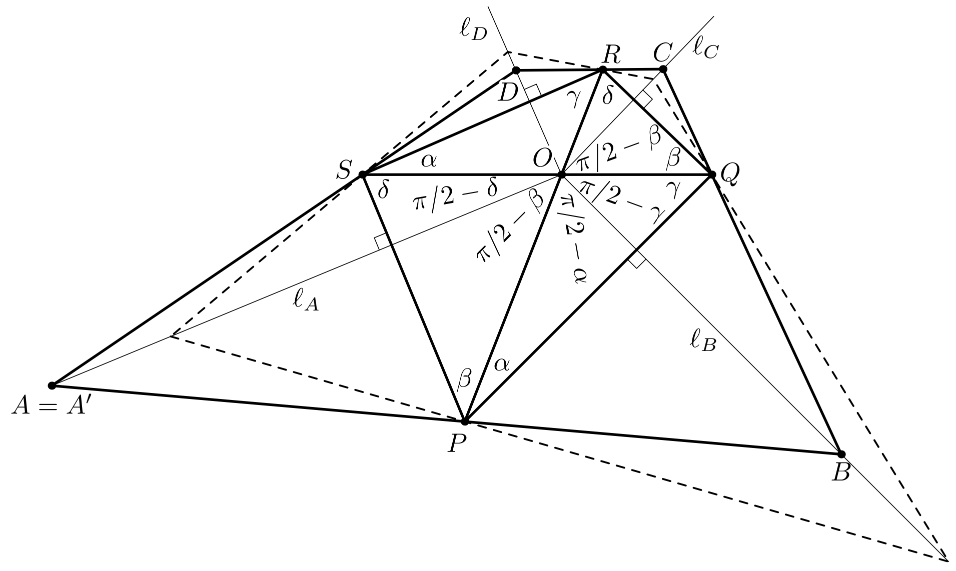
and divide by $OA \cdot OB \cdot OP$ to get the required identity.

A similar statement remains valid if the point C lies on the line AB; the proof is obtained by using signed areas and directed lengths.

We now turn to the solution. We first prove some sort of a converse statement, namely:

Claim. *Let $PQRS$ be a cyclic quadrangle with $O = PR \cap QS$; assume that no its diagonal is perpendicular to a side. Let ℓ_A, ℓ_B, ℓ_C, and ℓ_D be the lines through O perpendicular to SP, PQ, QR, and RS, respectively. Choose any point $A \in \ell_A$ and successively define $B = AP \cap \ell_B$, $C = BQ \cap \ell_C$, $D = CR \cap \ell_D$, and $A' = DS \cap \ell_A$. Then $A' = A$.*

Proof. We restrict ourselves to the case when the points A, B, C, D, and A' lie on ℓ_A, ℓ_B, ℓ_C, ℓ_D, and ℓ_A on the same side of O as their points of intersection with the respective sides of the quadrilateral $PQRS$. Again, a general case is obtained by suitable consideration of directed lengths.



Denote

$$\begin{aligned}
\alpha &= \angle QPR = \angle QSR = \pi/2 - \angle POB = \pi/2 - \angle DOS, \\
\beta &= \angle RPS = \angle RQS = \pi/2 - \angle AOP = \pi/2 - \angle QOC, \\
\gamma &= \angle SQP = \angle SRP = \pi/2 - \angle BOQ = \pi/2 - \angle ROD, \\
\delta &= \angle PRQ = \angle PSQ = \pi/2 - \angle COR = \pi/2 - \angle SOA.
\end{aligned}$$

By Lemma 2 applied to the lines APB, PQC, CRD, and DSA', we get

$$\frac{\sin(\alpha+\beta)}{OP} = \frac{\cos\alpha}{OA} + \frac{\cos\beta}{OB}, \quad \frac{\sin(\beta+\gamma)}{OQ} = \frac{\cos\beta}{OB} + \frac{\cos\gamma}{OC},$$
$$\frac{\sin(\gamma+\delta)}{OR} = \frac{\cos\gamma}{OC} + \frac{\cos\delta}{OD}, \quad \frac{\sin(\delta+\alpha)}{OS} = \frac{\cos\delta}{OD} + \frac{\cos\alpha}{OA'}.$$

Adding the two equalities on the left and subtracting the two on the right, we see that the required equality $A = A'$ (i.e., $\cos\alpha/OA = \cos\alpha/OA'$, in view of $\cos\alpha \neq 0$) is equivalent to the relation

$$\frac{\sin QPS}{OP} + \frac{\sin SRQ}{OR} = \frac{\sin PQR}{OQ} + \frac{\sin RSP}{OS}.$$

Let d denote the circumdiameter of $PQRS$, so $\sin QPS = \sin SRQ = QS/d$ and $\sin RSP = \sin PQR = PR/d$. Thus the required relation reads

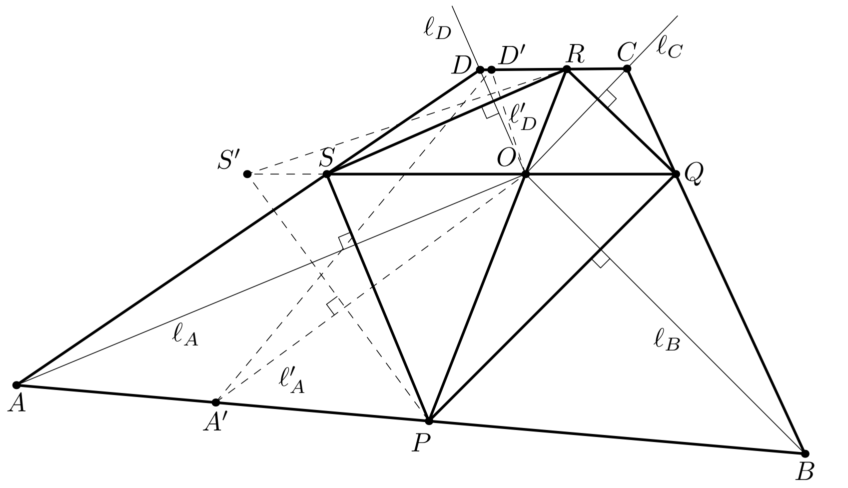
$$\frac{QS}{OP} + \frac{QS}{OR} = \frac{PR}{OS} + \frac{PR}{OQ}, \quad \text{or} \quad \frac{QS \cdot PR}{OP \cdot OR} = \frac{PR \cdot QS}{OS \cdot OQ}.$$

The last relation is trivial, due again to cyclicity.

Finally, it remains to derive the problem statement from our Claim. Assume that $PQRS$ is not cyclic, e.g., that $OP \cdot OR > OQ \cdot OS$, where $O = PR \cap QS$. Mark the point S' on the ray OS so that $OP \cdot OR = OQ \cdot OS'$. Notice that no diagonal of $PQRS$ is perpendicular to a side, so the quadrangle $PQRS'$ satisfies the conditions of the claim.

Let ℓ'_A and ℓ'_D be the lines through O perpendicular to PS' and RS', respectively. Then ℓ'_A and ℓ'_D cross the segments AP and RD, respectively, at some points A' and D'. By the Claim, the line $A'D'$ passes through S'. This is impossible, because the segment $A'D'$ crosses the segment OS at some interior point, while S' lies on the extension of this segment. This contradiction completes the proof.

Remark. According to the author, there is a remarkable corollary that is worth mentioning: Four lines dissect a convex quadrangle into nine smaller quadrangles to make it into a 3×3 array



of quadrangular cells. Label these cells 1 through 9 from left to right and from top to bottom. If the first eight cells are orthodiagonal, then so is the ninth.

The 10^{th} Romanian Master of Mathematics Competition

Day 1 — Solutions

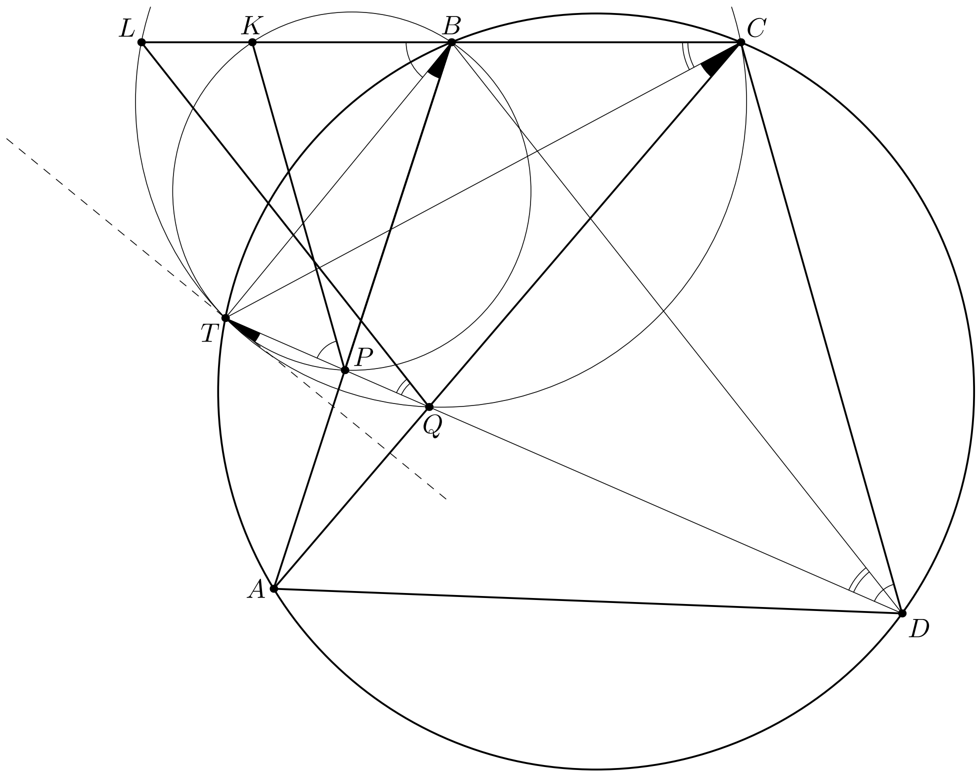
Problem 1. Let $ABCD$ be a cyclic quadrangle and let P be a point on the side AB. The diagonal AC crosses the segment DP at Q. The parallel through P to CD crosses the extension of the side BC beyond B at K, and the parallel through Q to BD crosses the extension of the side BC beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent.

Aleksandr Kuznetsov, Russia

Solution. We show that the circles BKP and CLQ are tangent at the point T where the line DP crosses the circle $ABCD$ again.

Since $BCDT$ is cyclic, we have $\angle KBT = \angle CDT$. Since $KP \parallel CD$, we get $\angle CDT = \angle KPT$. Thus, $\angle KBT = \angle CDT = \angle KPT$, which shows that T lies on the circle BKP. Similarly, the equalities $\angle LCT = \angle BDT = \angle LQT$ show that T also lies on the circle CLQ.

It remains to prove that these circles are indeed tangent at T. This follows from the fact that the chords TP and TQ in the circles $BKTP$ and $CLTQ$, respectively, both lie along the same line and subtend equal angles $\angle TBP = \angle TBA = \angle TCA = \angle TCQ$.



Remarks. The point T may alternatively be defined as the Miquel point of (any four of) the five lines AB, BC, AC, KP, and LQ.

Of course, the result still holds if P is chosen on the line AB, and the other points lie on the corresponding lines rather than segments/rays. The current formulation was chosen in order to avoid case distinction based on the possible configurations of points.

Problem 2. Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}. \tag{$*$}$$

Ilya Bogdanov, Russia

Solution 1. The answer is in the negative. Comparing the degrees of both sides in $(*)$ we get $\deg P = 21n$ and $\deg Q = 10n$ for some positive integer n. Take the derivative of $(*)$ to obtain

$$P' P^8 (10P + 9) = Q' Q^{19} (21Q + 20). \tag{$**$}$$

Since $\gcd(10P + 9, P) = \gcd(10P + 9, P + 1) = 1$, it follows that $\gcd(10P + 9, P^9(P + 1)) = 1$, so $\gcd(10P + 9, Q) = 1$, by $(*)$. Thus $(**)$ yields $10P + 9 \mid Q'(21Q + 20)$, which is impossible since $0 < \deg(Q'(21Q + 20)) = 20n - 1 < 21n = \deg(10P + 9)$. A contradiction.

Remark. A similar argument shows that there are no non-constant solutions of $P^m + P^{m-1} = Q^k + Q^{k-1}$, where k and m are positive integers with $k \geq 2m$. A critical case is $k = 2m$; but in this case there exist more routine ways of solving the problem. Thus, we decided to choose $k = 2m + 1$.

Solution 2. Letting r and s be integers such that $r \geq 2$ and $s \geq 2r$, we show that if $P^r + P^{r-1} = Q^s + Q^{s-1}$, then Q is constant.

Let $m = \deg P$ and $n = \deg Q$. A degree inspection in the given relation shows that $m \geq 2n$.

We will prove that $P(P + 1)$ has at least $m + 1$ distinct complex roots. Assuming this for the moment, notice that Q takes on one of the values 0 or -1 at each of those roots. Since $m + 1 \geq 2n + 1$, it follows that Q takes on one of the values 0 and -1 at more than n distinct points, so Q must be constant.

Finally, we prove that $P(P + 1)$ has at least $m + 1$ distinct complex roots. This can be done either by referring to the Mason–Stothers theorem or directly, in terms of multiplicities of the roots in question.

Since P and $P + 1$ are relatively prime, the Mason–Stothers theorem implies that the number of distinct roots of $P(P + 1)$ is greater than m, hence at least $m + 1$.

For a direct proof, let $z_1, \ldots, z_t$ be the distinct complex roots of $P(P + 1)$, and let z_k have multiplicity α_k, $k = 1, \ldots, t$. Since P and $P + 1$ have no roots in common, and $P' = (P + 1)'$, it follows that P' has a root of multiplicity $\alpha_k - 1$ at z_k. Consequently, $m - 1 = \deg P' \geq \sum_{k=1}^{t} (\alpha_k - 1) = \sum_{k=1}^{t} \alpha_k - t = 2m - t$; that is, $t \geq m + 1$. This completes the prof.

Remark. The Mason–Stothers theorem (in a particular case over the complex field) claims that, given coprime complex polynomials $P(x)$, $Q(x)$, and $R(x)$, not all constant, such that $P(x) + Q(x) = R(x)$, the total number of their complex roots (**not** regarding multiplicities) is at least $\max\{\deg P, \deg Q, \deg R\} + 1$. This theorem was a part of motivation for the famous *abc*-conjecture.

Problem 3. Ann and Bob play a game on an infinite checkered plane making moves in turn; Ann makes the first move. A move consists in orienting any unit grid-segment that has not been oriented before. If at some stage some oriented segments form an oriented cycle, Bob wins. Does Bob have a strategy that guarantees him to win?

Maxim Didin, Russia

Solution. The answer is in the negative: Ann has a strategy allowing her to prevent Bob's victory.

We say that two unit grid-segments form a *low-left corner* (or *LL-corner*) if they share an endpoint which is the lowest point of one and the leftmost point of the other. An *up-right corner* (or *UR-corner*) is defined similarly. The common endpoint of two unit grid-segments at a corner is the *joint* of that corner.

Fix a vertical line on the grid and call it the *midline*; the unit grid-segments along the midline are called *middle segments*. The unit grid-segments lying to the left/right of the midline are called *left/right segments*. Partition all left segments into LL-corners, and all right segments into UR-corners.

We now describe Ann's strategy. Her first move consists in orienting some middle segment arbitrarily. Assume that at some stage, Bob orients some segment s. If s is a middle segment, Ann orients any free middle segment arbitrarily. Otherwise, s forms a corner in the partition with some other segment t. Then Ann orients t so that the joint of the corner is either the source of both arrows, or the target of both. Notice that after any move of Ann's, each corner in the partition is either completely oriented or completely not oriented. This means that Ann can always make a required move.

Assume that Bob wins at some stage, i.e., an oriented cycle C occurs. Let X be the lowest of the leftmost points of C, and let Y be the topmost of the rightmost points of C. If X lies (strictly) to the left of the midline, then X is the joint of some corner whose segments are both oriented. But, according to Ann's strategy, they are oriented so that they cannot occur in a cycle — a contradiction. Otherwise, Y lies to the right of the midline, and a similar argument applies. Thus, Bob will never win, as desired.

Remarks. (1) There are several variations of the argument in the solution above. For instance, instead of the midline, Ann may choose any infinite in both directions down going polyline along the grid (i.e., consisting of steps to the right and steps-down alone). Alternatively, she may split the plane into four quadrants, use their borders as "trash bin" (as the midline was used in the solution above), partition all segments in the upper-right quadrant into UR-corners, all segments in the lower-right quadrant into LR-corners, and so on.

(2) The problem becomes easier if Bob makes the first move. In this case, his opponent just partitions the whole grid into LL-corners. In particular, one may change the problem to say that the first player to achieve an oriented cycle wins (in this case, the result is a draw).

On the other hand, this version is closer to known problems. In particular, the following problem is known:

> *Ann and Bob play the game on an infinite checkered plane making moves in turn (Ann makes the first move). A move consists in painting any unit grid segment that has not been painted before (Ann paints in blue, Bob paints in red). If a player creates a cycle of her/his color, (s)he wins. Does any of the players have a winning strategy?*

Again, the solution is pairing strategy with corners of a fixed orientation (with a little twist for Ann's strategy — in this problem, it is clear that Ann has better chances).

The 10^{th} Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Let a, b, c, d be positive integers such that $ad \neq bc$ and $\gcd(a,b,c,d) = 1$. Prove that, as n runs through the positive integers, the values $\gcd(an+b, cn+d)$ may achieve form the set of all positive divisors of some integer.

Raul Alcantara, Peru

Solution 1. We extend the problem statement by allowing a and c take non-negative integer values, and allowing b and d to take arbitrary integer values. (As usual, the greatest common divisor of two integers is non-negative.) Without loss of generality, we assume $0 \leq a \leq c$. Let $S(a,b,c,d) = \{\gcd(an+b, cn+d) \colon n \in \mathbb{Z}_{>0}\}$.

Now we induct on a. We first deal with the inductive step, leaving the base case $a = 0$ to the end of the solution. So, assume that $a > 0$; we intend to find a 4-tuple (a', b', c', d') satisfying the requirements of the extended problem, such that $S(a',b',c',d') = S(a,b,c,d)$ and $0 \leq a' < a$, which will allow us to apply the induction hypothesis.

The construction of this 4-tuple is provided by the step of the Euclidean algorithm. Write $c = aq + r$, where q and r are both integers and $0 \leq r < a$. Then for every n we have

$$\gcd(an+b, cn+d) = \gcd\big(an+b, q(an+b) + rn + d - qb\big) = \gcd\big(an+b, rn + (d - qb)\big),$$

so a natural intention is to define $a' = r$, $b' = d - qb$, $c' = a$, and $d' = b$ (which are already shown to satisfy $S(a',b',c',d') = S(a,b,c,d)$). The check of the problem requirements is straightforward: indeed,

$$a'd' - b'c' = (c - qa)b - (d - qb)a = -(ad - bc) \neq 0$$

and

$$\gcd(a',b',c',d') = \gcd(c - qa, b - qd, a, b) = \gcd(c,d,a,b) = 1.$$

Thus the step is verified.

It remains to deal with the base case $a = 0$, i.e., to examine the set $S(0,b,c,d)$ with $bc \neq 0$ and $\gcd(b,c,d) = 1$. Let b' be the integer obtained from b by ignoring all primes b and c share (none of them divides $cn + d$ for any integer n, otherwise $\gcd(b,c,d) > 1$). We thus get $\gcd(b',c) = 1$ and $S(0,b',c,d) = S(0,b,c,d)$.

Finally, it is easily seen that $S(0,b',c,d)$ is the set of all positive divisors of b'. Each member of $S(0,b',c,d)$ is clearly a divisor of b'. Conversely, if δ is a positive divisor of b', then $cn + d \equiv \delta \pmod{b'}$ for some n, since b' and c are coprime, so δ is indeed a member of $S(0,b',c,d)$.

Solution 2. (*Alexander Betts*) For positive integers s and t and prime p, we will denote by $\gcd_p(s,t)$ the greatest common p-power divisor of s and t.

Claim 1. For any positive integer n, $\gcd(an+b, cn+d) \mid ad - bc$.

Proof. This is clear from the identity

$$a(cn+d) - c(an+b) = ad - bc. \qquad (\dagger)$$

Claim 2. The set of values taken by $\gcd(an+b, cn+d)$ is exactly the set of values taken by the product

$$\prod_{p \mid ad-bc} \gcd_p(an_p + b, cn_p + d)$$

as the $(n_p)_{p \mid ad-bc}$ each range over positive integers.

Proof. From the identity

$$\gcd(an+b, cn+d) = \prod_{p \mid ad-bc} \gcd_p(an+b, cn+d),$$

it is clear that every value taken by $\gcd(an+b, cn+d)$ is also a value taken by the product (with all $n_p = n$). Conversely, it suffices to show that, given any positive integers $(n_p)_{p \mid ad-bc}$, there is a positive integer n such that $\gcd_p(an+b, cn+d) = \gcd_p(an_p+b, cn_p+d)$ for each $p \mid ad-bc$. This can be achieved by requiring that n be congruent to n_p modulo a sufficiently large[1] power of p (using the Chinese Remainder Theorem).

Using Claim 2, it suffices to determine the sets of values taken by $\gcd_p(an+b, cn+d)$ as n ranges over all positive integers. There are two cases.

Claim 3. If $p \mid a, c$, then $\gcd_p(an+b, cn+d) = 1$ for all n.

Proof. If $p \mid an+b, cn+d$, then we would have $p \mid a, b, c, d$, which is not the case.

Claim 4. If $p \nmid a$ or $p \nmid c$, then the values taken by $\gcd_p(an+b, cn+d)$ are exactly the p-power divisors of $ad-bc$.

Proof. Assume without loss of generality that $p \nmid a$. Then from identity (†) we have $\gcd_p(an+b, cn+d) = \gcd_p(an+b, ad-bc)$. But since $p \nmid a$, the arithmetic progression $an+b$ takes all possible values modulo the highest p-power divisor of $ad-bc$, and in particular the values taken by $\gcd_p(an+b, ad-bc)$ are exactly the p-power divisors of $ad-bc$.

Conclusion. Using claims 2, 3 and 4, we see that the set of values taken by $\gcd(an+b, cn+d)$ is equal to the set of products of p-power divisors of $ad-bc$, where we only consider those primes p not dividing $\gcd(a,c)$. Thus the set of values of $\gcd(an+b, cn+d)$ is equal to the set of divisors of the largest factor of $ad-bc$ coprime to $\gcd(a,c)$.

Remarks. (1) If $S(a,b,c,d)$ is the set of all positive divisors of some integer, then necessarily $ad \neq bc$ and $\gcd(a,b,c,d) = 1$: finiteness of $S(a,b,c,d)$ forces the former, and membership of 1 forces the latter.

(2) One may modify the problem statement according to the first paragraph of the solution. However, it seems that in this case one needs to include a clarification of the agreement on gcd being necessarily non-negative.

[1] For example, $n \equiv n_p$ modulo the largest p-power divisor of $ad-bc$.

Problem 5. Let n be a positive integer and fix $2n$ distinct points on a circumference. Split these points into n pairs and join the points in each pair by an arrow (i.e., an oriented line segment). The resulting configuration is *good* if no two arrows cross, and there are no arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ such that $ABCD$ is a convex quadrangle *oriented clockwise*. Determine the number of good configurations.

Fedor Petrov, Russia

Solution 1. The required number is $\binom{2n}{n}$. To prove this, trace the circumference counterclockwise to label the points $a_1, a_2, \ldots, a_{2n}$.

Let $\mathcal{C}$ be any good configuration and let $O(\mathcal{C})$ be the set of all points *from* which arrows emerge. We claim that every n-element subset S of $\{a_1, \ldots, a_{2n}\}$ is an O-image of a unique good configuration; clearly, this provides the answer.

To prove the claim induct on n. The base case $n = 1$ is clear. For the induction step, consider any n-element subset S of $\{a_1, \ldots, a_{2n}\}$, and assume that $S = O(\mathcal{C})$ for some good configuration $\mathcal{C}$. Take any index k such that $a_k \in S$ and $a_{k+1} \notin S$ (assume throughout that indices are cyclic modulo $2n$, i.e., $a_{2n+1} = a_1$ etc.).

If the arrow from a_k points to some a_ℓ, $k+1 < \ell$ $(< 2n+k)$, then the arrow pointing to a_{k+1} emerges from some a_m, m in the range $k+2$ through $\ell - 1$, since these two arrows do not cross. Then the arrows $a_k \to a_\ell$ and $a_m \to a_{k+1}$ form a prohibited quadrangle. Hence, $\mathcal{C}$ contains an arrow $a_k \to a_{k+1}$.

On the other hand, if any configuration $\mathcal{C}$ contains the arrow $a_k \to a_{k+1}$, then this arrow cannot cross other arrows, neither can it occur in prohibited quadrangles.

Thus, removing the points a_k, a_{k+1} from $\{a_1, \ldots, a_{2n}\}$ and the point a_k from S, we may apply the induction hypothesis to find a unique good configuration $\mathcal{C}'$ on $2n-2$ points compatible with the new set of sources (i.e., points from which arrows emerge). Adjunction of the arrow $a_k \to a_{k+1}$ to $\mathcal{C}'$ yields a unique good configuration on $2n$ points, as required.

Solution 2. Use the counterclockwise labelling $a_1, a_2, \ldots, a_{2n}$ in the solution above.

Letting D_n be the number of good configurations on $2n$ points, we establish a recurrence relation for the D_n. To this end, let $C_n = \frac{(2n)!}{n!(n+1)!}$ the nth Catalan number; it is well-known that C_n is the number of ways to connect $2n$ given points on the circumference by n pairwise disjoint chords.

Since no two arrows cross, in any good configuration the vertex a_1 is connected to some a_{2k}. Fix k in the range 1 through n and count the number of good configurations containing the arrow $a_1 \to a_{2k}$. Let $\mathcal{C}$ be any such configuration.

In $\mathcal{C}$, the vertices $a_2, \ldots, a_{2k-1}$ are paired off with one other, each arrow pointing from the smaller to the larger index, for otherwise it would form a prohibited quadrangle with $a_1 \to a_{2k}$. Consequently, there are C_{k-1} ways of drawing such arrows between $a_2, \ldots, a_{2k-1}$.

On the other hand, the arrows between $a_{2k+1}, \ldots, a_{2n}$ also form a good configuration, which can be chosen in D_{n-k} ways. Finally, it is easily seen that any configuration of the first kind and any configuration of the second kind combine together to yield an overall good configuration.

Thus the number of good configurations containing the arrow $a_1 \to a_{2k}$ is $C_{k-1}D_{n-k}$. Clearly, this is also the number of good configurations containing the arrow $a_{2(n-k+1)} \to a_1$, so

$$D_n = 2\sum_{k=1}^{n} C_{k-1}D_{n-k}. \qquad (*)$$

To find an explicit formula for D_n, let $d(x) = \sum_{n=0}^{\infty} D_n x^n$ and let $c(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$ be the generating functions of the D_n and the C_n, respectively. Since $D_0 = 1$, relation $(*)$

yields $d(x) = 2xc(x)d(x) + 1$, so

$$d(x) = \frac{1}{1 - 2xc(x)} = (1-4x)^{-1/2} = \sum_{n \geq 0} \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)\frac{(-4x)^n}{n!}$$
$$= \sum_{n \geq 0} \frac{2^n(2n-1)!!}{n!}x^n = \sum_{n \geq 0} \binom{2n}{n} x^n.$$

Consequently, $D_n = \binom{2n}{n}$.

Solution 3. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the nth Catalan number and recall that there are exactly C_n ways to join $2n$ distinct points on a circumference by n pairwise disjoint chords. Such a configuration of chords will be referred to as a *Catalan n-configuration*. An orientation of the chords in a Catalan configuration $\mathcal{C}$ making it into a good configuration (in the sense defined in the statement of the problem) will be referred to as a *good* orientation for $\mathcal{C}$.

We show by induction on n that there are exactly $n+1$ good orientations for any Catalan n-configuration, so there are exactly $(n+1)C_n = \binom{2n}{n}$ good configurations on $2n$ points. The base case $n = 1$ is clear.

For the induction step, let $n > 1$, let $\mathcal{C}$ be a Catalan n-configuration, and let ab be a chord of minimal length in $\mathcal{C}$. By minimality, the endpoints of the other chords in $\mathcal{C}$ all lie on the major arc ab of the circumference.

Label the $2n$ endpoints $1, 2, \ldots, 2n$ counterclockwise so that $\{a, b\} = \{1, 2\}$, and notice that the good orientations for $\mathcal{C}$ fall into two disjoint classes: Those containing the arrow $1 \to 2$, and those containing the opposite arrow.

Since the arrow $1 \to 2$ cannot be involved in a prohibited quadrangle, the induction hypothesis applies to the Catalan $(n-1)$-configuration formed by the other chords to show that the first class contains exactly n good orientations.

Finally, the second class consists of a single orientation, namely, $2 \to 1$, every other arrow emerging from the smaller endpoint of the respective chord; a routine verification shows that this is indeed a good orientation. This completes the induction step and ends the proof.

Remark. Combining the arguments from Solutions 1 and 3 one gets a way (though not the easiest) to compute the Catalan number C_n.

Solution 4, sketch. (*Sang-il Oum*) As in the previous solution, we intend to count the number of good orientations of a Catalan n-configuration.

For each such configuration, we consider its *dual graph* T whose vertices are finite regions bounded by chords and the circle, and an edge connects two regions sharing a boundary segment. This graph T is a plane tree with n edges and $n+1$ vertices.

There is a canonical bijection between orientations of chords and orientations of edges of T in such a way that each chord crosses an edge of T from the right to the left of the arrow on that edge. A good orientation of chords corresponds to an orientation of the tree containing no two edges oriented towards each other. Such an orientation is defined uniquely by its *source vertex*, i.e., the unique vertex having no in-arrows.

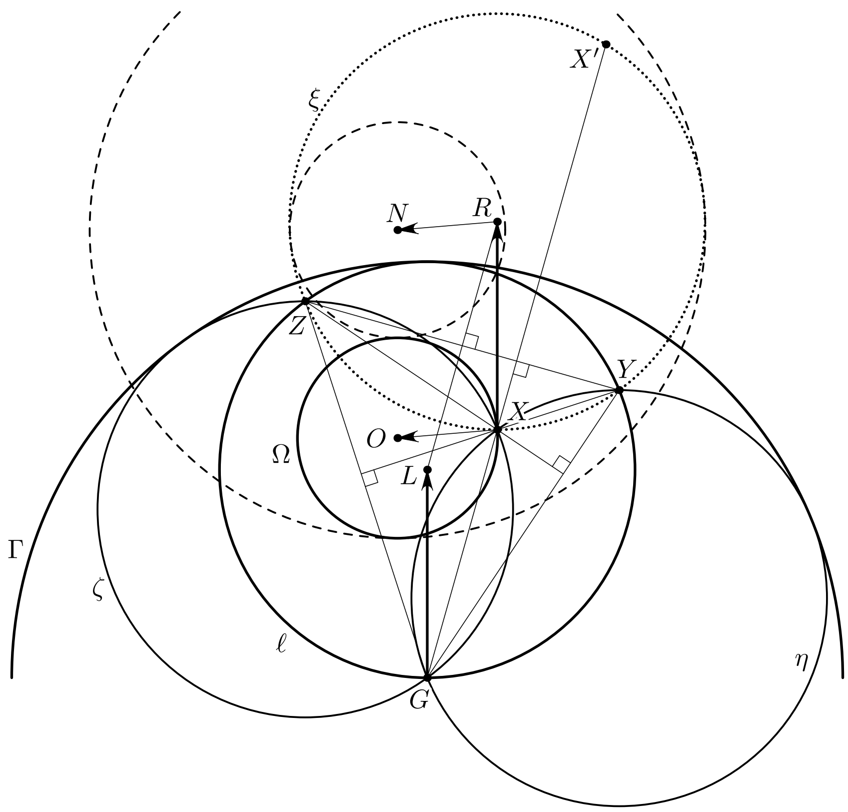
Therefore, for each tree T on $n+1$ vertices, there are exactly $n+1$ ways to orient it so that the source vertex is unique — one for each choice of the source. Thus, the answer is obtained in the same way as above.

Problem 6. Fix a circle Γ, a line ℓ tangent to Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω cross ℓ at Y and Z. Prove that, as X traces Ω, the circle XYZ is tangent to two fixed circles.

Russia, Ivan Frolov

Solution. Assume Γ of unit radius and invert with respect to Γ. No reference will be made to the original configuration, so images will be denoted by the same letters. Letting Γ be centred at G, notice that inversion in Γ maps tangents to Γ to circles of unit diameter through G (hence internally tangent to Γ). Under inversion, the statement reads as follows:

Fix a circle Γ of unit radius centred at G, a circle ℓ of unit diameter through G, and a circle Ω inside ℓ disjoint from ℓ. The circles η and ζ of unit diameter, through G and a variable point X on Ω, cross ℓ again at Y and Z, respectively. Prove that, as X traces Ω, the circle XYZ is tangent to two fixed circles.



Since η and ζ are the reflections of the circumcircle ℓ of the triangle GYZ in its sidelines GY and GZ, respectively, they pass through the orthocentre of this triangle. And since η and ζ cross again at X, the latter is the orthocentre of the triangle GYZ. Hence the circle ξ through X, Y, Z is the reflection of ℓ in the line YZ; in particular, ξ is also of unit diameter.

Let O and L be the centres of Ω and ℓ, respectively, and let R be the (variable) centre of ξ. Let GX cross ξ again at X'; then G and X' are reflections of one another in the line YZ, so $GLRX'$ is an isosceles trapezoid. Then $LR \parallel GX$ and $\angle(LG, GX) = \angle(GX', X'R) = \angle(RX, XG)$, i.e., $LG \parallel RX$; this means that $GLRX$ is a parallelogram, so $\overrightarrow{XR} = \overrightarrow{GL}$ is constant.

Finally, consider the fixed point N defined by $\overrightarrow{ON} = \overrightarrow{GL}$. Then $XRNO$ is a parallelogram, so the distance $RN = OX$ is constant. Consequently, ξ is tangent to the fixed circles centred at N of radii $|1/2 - OX|$ and $1/2 + OX$.

One last check is needed to show that the inverse images of the two obtained circles are indeed circles and not lines. This might happen if one of them contained G; we show that this is impossible. Indeed, since Ω lies inside ℓ, we have $OL < 1/2 - OX$, so

$$NG = \left|\overrightarrow{GL} + \overrightarrow{LO} + \overrightarrow{ON}\right| = \left|2\overrightarrow{GL} + \overrightarrow{LO}\right| \geq 2\left|\overrightarrow{GL}\right| - \left|\overrightarrow{LO}\right| > 1 - (1/2 - OX) = 1/2 + OX;$$

this shows that G is necessarily outside the obtained circles.

Remarks. (1) The last check could be omitted, if we allowed in the problem statement to regard a line as a particular case of a circle. On the other hand, the Problem Selection Committee suggests not to punish students who have not performed this check.

(2) Notice that the required fixed circles are also tangent to Ω.

The 11$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 1 — Solutions

Problem 1. Amy and Bob play the game. At the beginning, Amy writes down a positive integer on the board. Then the players take moves in turn, Bob moves first. On any move of his, Bob replaces the number n on the blackboard with a number of the form $n-a^2$, where a is a positive integer. On any move of hers, Amy replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bob wins if the number on the board becomes zero. Can Amy prevent Bob's win?

Russia, Maxim Didin

Solution. The answer is in the negative. For a positive integer n, we define its *square-free part* $S(n)$ to be the smallest positive integer a such that n/a is a square of an integer. In other words, $S(n)$ is the product of all primes having odd exponents in the prime expansion of n. We also agree that $S(0)=0$.

Now we show that (i) on any move of hers, Amy does not increase the square-free part of the positive integer on the board; and (ii) on any move of his, Bob always can replace a positive integer n with a non-negative integer k with $S(k)<S(n)$. Thus, if the game starts by a positive integer N, Bob can win in at most $S(N)$ moves.

Part (i) is trivial, as the definition of the square-part yields $S(n^k)=S(n)$ whenever k is odd, and $S(n^k)=1\leq S(n)$ whenever k is even, for any positive integer n.

Part (ii) is also easy: if, before Bob's move, the board contains a number $n=S(n)\cdot b^2$, then Bob may replace it with $n'=n-b^2=(S(n)-1)b^2$, whence $S(n')\leq S(n)-1$.

Remarks. **(1)** To make the argument more transparent, Bob may restrict himself to subtract only those numbers which are divisible by the maximal square dividing the current number. This restriction having been put, one may replace any number n appearing on the board by $S(n)$, omitting the square factors.

After this change, Amy's moves do not increase the number, while Bob's moves decrease it. Thus, Bob wins.

(2) In fact, Bob may win even in at most 4 moves of his. For that purpose, use Lagrange's four squares theorem in order to expand $S(n)$ as the sum of at most four squares of positive integers: $S(n)=a_1^2+\cdots+a_s^2$. Then, on every move of his, Bob can replace the number $(a_1^2+\cdots+a_k^2)b^2$ on the board by $(a_1^2+\cdots+a_{k-1}^2)b^2$. The only chance for Amy to interrupt this process is to replace a current number by its even power; but in this case Bob wins immediately.

On the other hand, four is indeed the minimum number of moves in which Bob can guarantee himself to win. To show that, let Amy choose the number 7, and take just the first power on each of her subsequent moves.

Problem 2. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let E be the midpoint of AC. Denote by ω and Ω the circumcircles of the triangles ABE and CDE, respectively. Let P be the crossing poitn of the tangent to ω at A with the tangent to Ω at D. Prove that PE is tangent to Ω.

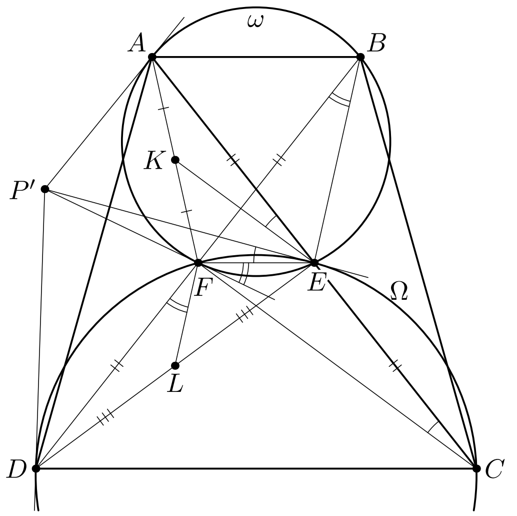
SLOVENIA, JAKOB JURIJ SNOJ

Solution 1. If $ABCD$ is a rectangle, the statement is trivial due to symmetry. Hence, in what follows we assume $AD \nparallel BC$.

Let F be the midpoint of BD; by symmetry, both ω and Ω pass through F. Let P' be the meeting point of tangents to ω at F and to Ω at E. We aim to show that $P' = P$, which yields the required result. For that purpose, we show that $P'A$ and $P'D$ are tangent to ω and Ω, respectively.

Let K be the midpoint of AF. Then EK is a midline in the triangle ACF, so $\angle(AE, EK) = \angle(EC, CF)$. Since $P'E$ is tangent to Ω, we get $\angle(EC, CF) = \angle(P'E, EF)$. Thus, $\angle(AE, EK) = \angle(P'E, EF)$, so EP' is a symmedian in the triangle AEF. Therefore, EP' and the tangents to ω at A and F are concurrent, and the concurrency point is P' itself. Hence $P'A$ is tangent to ω.

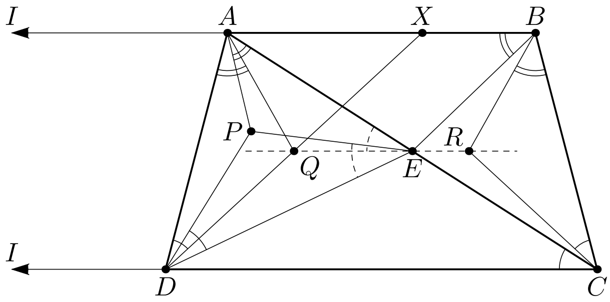
The second claim is similar. Taking L to be the midpoint of DE, we have $\angle(DF, FL) = \angle(FB, BE) = \angle(P'F, FE)$, so $P'F$ is a symmedian in the triangle DEF, and hence P' is the meeting point of the tangents to Ω at D and E.



Remark. The above arguments may come in different orders. E.g., one may define P' to be the point of intersection of the tangents to Ω at D and E — hence obtaining that $P'F$ is a symmedian in $\triangle DEF$, then deduce that $P'F$ is tangent to ω, and then apply a similar argument to show that $P'E$ is a symmedian in $\triangle AEF$, whence $P'A$ is tangent to ω.

Solution 2. Let Q be the isogonal conjugate of P with respect to $\triangle AED$, so $\angle(QA, AD) = \angle(EA, AP) = \angle(EB, BA)$ and $\angle(QD, DA) = \angle(ED, DP) = \angle(EC, CD)$. Now our aim is to prove that $QE \parallel CD$; this will yield that $\angle(EC, CD) = \angle(AE, EQ) = \angle(PE, ED)$, whence PE is tangent to Ω.

Let DQ meet AB at X. Then we have $\angle(XD, DA) = \angle(EC, CD) = \angle(EA, AB)$ and $\angle(DA, AX) = \angle(AB, BC)$, hence the triangles DAX and ABC are similar. Since $\angle(AB, BE) = \angle(DA, AQ)$, the points Q and E correspond to each other in these triangles, hence Q is the midpoint of DX. This yields that the points Q and E lie on the midline of the trapezoid parallel to CD, as desired.



Remark. The last step could be replaced with another application of isogonal conjugacy in the following manner. Reflect Q in the common perpendicular bisector of AB and CD to obtain a point R such that $\angle(CB, BR) = \angle(QA, AD) = \angle(EB, BA)$ and $\angle(BC, CR) = \angle(QD, DA) = \angle(EC, CD)$. These relations yield that the points E and R are isogonally conjugate in a triangle BCI, where I is the (ideal) point of intersection of BA with CD. Since E is equidistant from AB and CD, R is also equidistant from them, which yields what we need. (The last step deserves some explanation, since one vertex of the triangle is ideal. Such explanation may be obtained in many different ways — e.g., by a short computation in sines, or by noticing that, as in the usual case, R is the circumcenter of the triangle formed by the reflections of E in the sidelines AB, BC, and CD.)

Solution 3. *(Dan Carmon)* Let O be the intersection of the diagonals AC and BD. Let F be the midpoint of BD. Let S be the second intersection point of the circumcircles of triangles AOF and DOE. We will prove that SD and SE are tangent to Ω; the symmetric argument would then imply also that SA and SF are tangent to Γ. Thus $S = P$ and the claimed tangency holds.

We first prove that OS is parallel to AB and DC. Compute the powers of the points A, B with respect to the circumcircles of AOF and DOE:

$$d(A, AOF) = 0, \quad d(A, DOE) = AO \cdot AE$$

$$d(B, AOF) = BO \cdot BF, \quad d(B, DOE) = BO \cdot BD = 2BO \cdot BF$$

And therefore

$$d(B, DOE) - d(B, AOF) = BO \cdot BF = AO \cdot AE = d(A, DOE) - d(A, AOF)$$

Thus both A and B belong to a locus of the form

$$d(X, DOE) - d(X, AOF) = \text{const},$$

which is always a lines parallel to the radical axis of the respective circles. Since this radical axis is OS by definition, it follows that AB is parallel to OS, as claimed.

Now by angle chasing in the cyclic quadrilateral $DSOE$, we find

$$\angle(SD, DE) = \angle(SO, OE) = \angle(DC, CE),$$
$$\angle(SE, ED) = \angle(SO, OD) = \angle(DC, DB) = \angle(AC, CD) = \angle(EC, CD),$$

and these angle equalities are exactly the conditions of SD, SE being tanget to Ω, as claimed.

Remarks. **(1)** The solution was motivated by the following observation: Suppose P is the intersection of the tangents to Ω at D and E as claimed. Then by single angle chasing we observe that the isogonal conjugate of P in the triangle DOE is the common ideal point at infinity of DC and EF. This implies that P is on the circumcircle of DOE and that OP is parallel to DC (to be precise, it implies that the reflection of OP in the angle bisector of DOE is parallel to DC and EF – but the angle bisector is also parallel to these lines, so in fact OP is the angle bisector). By symmetry it follows that P is also on the circumcircle of AOF, thus the construction.

(2) The key parts of the proof can be described as (1) Constructing S, (2) Proving that OS is parallel to AB and CD, and (3) Concluding that $S = P$ and finishing the problem. Parts (2) and (3) can be performed in various other ways. For example, part (2) can be proved by showing that the circumcentres of AOF and DOE lie on a line perpendicular to the trapezium's bases; part (3) can be proved considering the spiral map taking the circumcircle of DOC to the circumcircle of DSE. Since O is the second intersction point of these circles, and since OCE are collinear and SO is tangent to the circumcircle of DOC at O (by symmetry), it follows that the spiral map sends C to E and O to S, i.e. the triangle DSE is similar to the isoceles triangle DOC, from which the remainder of the angle chase is trivial.

Problem 3. Given any positive real number ε, prove that, for all but finitely many positive integers v, any graph on v vertices with at least $(1+\varepsilon)v$ edges has two distinct simple cycles of equal lengths.

(Recall that the notion of a *simple cycle* does not allow repetition of vertices in a cycle.)

Russia, Fedor Petrov

Solution. Fix a positive real number ε, and let G be a graph on v vertices with at least $(1+\varepsilon)v$ edges, all of whose simple cycles have pairwise distinct lengths.

Assuming $\varepsilon^2 v \geq 1$, we exhibit an upper bound linear in v and a lower bound quadratic in v for the total number of simple cycles in G, showing thereby that v cannot be arbitrarily large, whence the conclusion.

Since a simple cycle in G has at most v vertices, and each length class contains at most one such, G has at most v pairwise distinct simple cycles. This establishes the desired upper bound.

For the lower bound, consider a spanning tree for each component of G, and collect them all together to form a spanning forest F. Let A be the set of edges of F, and let B be the set of all other edges of G. Clearly, $|A| \leq v-1$, so $|B| \geq (1+\varepsilon)v - |A| \geq (1+\varepsilon)v - (v-1) = \varepsilon v + 1 > \varepsilon v$.

For each edge b in B, adjoining b to F produces a unique simple cycle C_b through b. Let S_b be the set of edges in A along C_b. Since the C_b have pairwise distinct lengths, $\sum_{b\in B} |S_b| \geq 2 + \cdots + (|B|+1) = |B|(|B|+3)/2 > |B|^2/2 > \varepsilon^2 v^2/2$.

Consequently, some edge in A lies in more than $\varepsilon^2 v^2/(2v) = \varepsilon^2 v/2$ of the S_b. Fix such an edge a in A, and let B' be the set of all edges b in B whose corresponding S_b contain a, so $|B'| > \varepsilon^2 v/2$.

For each 2-edge subset $\{b_1, b_2\}$ of B', the union $C_{b_1} \cup C_{b_2}$ of the cycles C_{b_1} and C_{b_2} forms a θ-graph, since their common part is a path in F through a; and since neither of the b_i lies along this path, $C_{b_1} \cup C_{b_2}$ contains a third simple cycle C_{b_1,b_2} through both b_1 and b_2. Finally, since $B' \cap C_{b_1,b_2} = \{b_1, b_2\}$, the assignment $\{b_1, b_2\} \mapsto C_{b_1,b_2}$ is injective, so the total number of simple cycles in G is at least $\binom{|B'|}{2} > \binom{\varepsilon^2 v/2}{2}$. This establishes the desired lower bound and concludes the proof.

Remarks. (1) The problem of finding two cycles of equal lengths in a graph on v vertices with $2v$ edges is known and much easier — simply consider all cycles of the form C_b.

The solution above shows that a graph on v vertices with at least $v + \Theta(v^{3/4})$ edges has two cycles of equal lengths. The constant $3/4$ is not sharp; a harder proof seems to show that $v + \Theta(\sqrt{v \log v})$ edges would suffice. On the other hand, there exist graphs on v vertices with $v + \Theta(\sqrt{v})$ edges having no such cycles.

(2) To avoid graph terminology, the statement of the problem may be rephrased as follows:

> Given any positive real number ε, prove that, for all but finitely many positive integers v, any v-member company, within which there are at least $(1+\varepsilon)v$ friendship relations, satisfies the following condition: For some integer $u \geq 3$, there exist two distinct u-member cyclic arrangements in each of which any two neighbours are friends. (Two arrangements are distinct if they are not obtained from one another through rotation and/or symmetry; a member of the company may be included in neither arrangement, in one of them or in both.)

Sketch of solution 2. *(Po-Shen Loh)* Recall that the *girth* of a graph G is the minimal length of a (simple) cycle in this graph.

Lemma. For any fixed positive δ, a graph on v vertices whose girth is at least δv has at most $v + o(v)$ edges.

Proof. Define $f(v)$ to be the maximal number f such that a graph on v vertices whose girth is at least δv may have $v + f$ edges. We are interested in the recursive estimates for f.

Let G be a graph on v vertices whose gifth is at least δv containing $v + f(v)$ edges. If G contains a leaf (i.e., a vertex of degree 1), then one may remove this vertex along with its edge, obtaining a graph with at most $v - 1 + f(v-1)$ edges. Thus, in this case $f(v) \le f(v-1)$.

Define an *isolated path* of length k to be a sequence of vertices $v_0, v_1, \dots, v_k$, such that v_i is connected to v_{i+1}, and each of the vertices $v_1, \dots, v_{k-1}$ has degree 2 (so, these vertices are connected only to their neighbors in the path). If G contains an isolated path $v_0, \dots, v_k$ of length, say, $k > \sqrt{v}$, then one may remove all its middle vertices $v_1, \dots, v_{k-1}$, along with all their k edges. We obtain a graph on $v - k + 1$ vertices with at most $(v-k+1) + f(v-k+1)$ edges. Thus, in this case $f(v) \le f(v-k+1) + 1$.

Assume now that the lengths of all isolated paths do not exceed $\sqrt{v}$; we show that in this case v is bounded from above. For that purpose, we replace each maximal isolated path by an edge between its endpoints, removing all middle vertices. We obtain a graph H whose girth is at least $\delta v/\sqrt{v} = \delta\sqrt{v}$. Each vertex of H has degree at least 3. By the girth condition, the neighborhood of any vertex x of radius $r = \lfloor(\delta\sqrt{v} - 1)/2\rfloor$ is a tree rooted at x. Any vertex at level $i < r$ has at least two sons; so the tree contains at least $2^{\lfloor(\delta\sqrt{v}-1)/2\rfloor}$ vertices (even at the last level). So, $v \ge 2^{\lfloor(\delta\sqrt{v}-1)/2\rfloor}$ which may happen only for a finite number of values of v.

Thus, for all large enough values of v, we have either $f(v) \le f(v-1)$ or $f(v) \le f(v-k+1)$ for some $k > \sqrt{v}$. This easily yields $f(v) = o(v)$, as desired. □

Now we proceed to the problem. Consider a graph on v vertices containing no two simple cycles of the same length. Take its $\lfloor \varepsilon v/2 \rfloor$ shortest cycles (or all its cycles, if their total number is smaller) and remove an edge from each. We get a graph of girth at least $\varepsilon v/2$. By the lemma, the number of edges in the obtained graph is at most $v + o(v)$, so the number of edges in the initial graph is at most $v + \varepsilon v/2 + o(v)$, which is smaller than $(1+\varepsilon)v$ if v is large enough.

The 11th Romanian Master of Mathematics Competition

Day 2 — Solutions

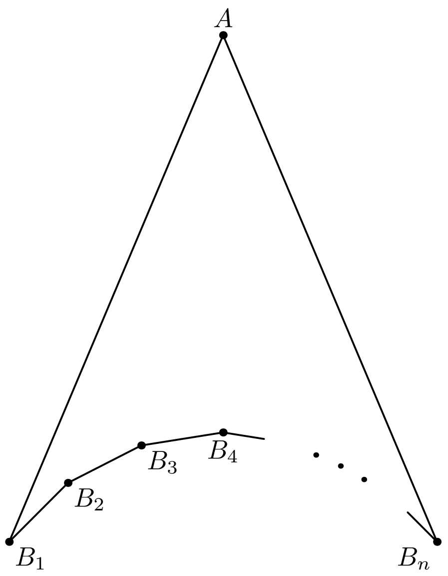
Problem 4. Prove that for every positive integer n there exists a (not necessarily convex) polygon with no three collinear vertices, which admits exactly n different triangulations.

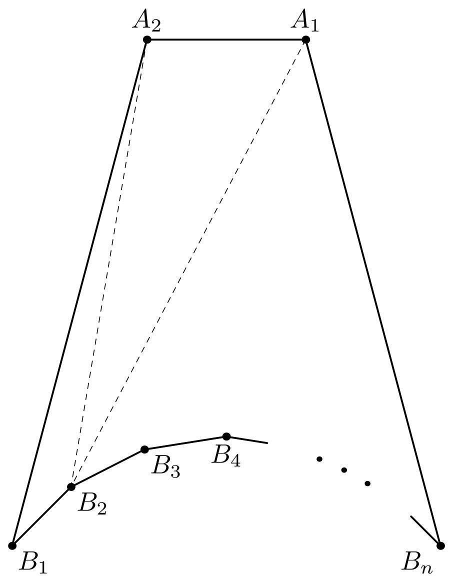
(A *triangulation* is a dissection of the polygon into triangles by interior diagonals which have no common interior points with each other nor with the sides of the polygon.)

Iran, Morteza Saghafian

Solution. The left figure below shows an example of a polygon admitting a unique triangulation: the only its diagonals lying inside the polygon come from A, so all of them must be drawn. (Notice that the "exterior" polygon $B_1B_2\ldots B_n$ is convex.)

Now we prove that the right figure below shows a polygon $A_1A_2B_1B_2\ldots B_n$ with exactly n triangulations. Indeed, any triangulation must contain a triangle with side A_1A_2, and there are n possible such, namely $A_1A_2B_i$ for $i = 1, 2, \ldots, n$. After such triangle has been chosen, the rest part of the polygon splits into two (or one) polygons admitting a unique triangulation. Hence the result.





Problem 5. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + yf(x)) + f(xy) = f(x) + f(2019y),$$

for all real numbers x and y.

Slovenia, Jakob Jurij Snoj

Solution. There are three types of such functions: (i) $f(x) = 2019 - x$; (ii) $f(x) = c$ for an arbitrary constant c; and (iii) $f(x) = 0$ for $x \neq 0$, and $f(0)$ is arbitrary.

A straightforward check shows that all three types satisfy the equation hence we need to show that they are the only ones. Let $N = 2019$.

First of all, setting $x = Nx'$, we arrive at the equation $f(Nx' + yf(Nx')) + f(Nx'y) = f(Nx') + f(Ny)$. After a change $g(x) = f(Nx)/N$ this equation reads

$$g\big(x + yg(x)\big) + g(xy) = g(x) + g(y) \qquad (x, y \in \mathbb{R}), \tag{1}$$

which does not depend on N. Now we investigate the corresponding functions g.

Setting $x = 1$ we get $g(1 + yg(1)) = g(1)$. If $g(1) \neq 0$, then $1 + yg(1)$ attains all real values, so we arrive at the answer (ii). Otherwise, $g(1) = 0$, and by setting $y = 1$ we get $g\big(x + g(x)\big) = 0$. If $a = 1$ is the unique real number with $g(a) = 0$, then we obtain $x + g(x) = 1$, whence $g(x) = 1 - x$, which falls into (i). Hence in the sequel we assume that

$$g(1) = 0, \quad \text{and also } g(a) = 0 \text{ for some } a \neq 1. \tag{$*$}$$

We will make use of the following two arguments.

Claim 1. If b is an arbitrary zero of g, then by substituting $x = z$ we get $g(zy) = g(y)$. Recalling that $g(g(0)) = g(0 + g(0)) = 0$, we obtain also $g(g(0)y) = g(y)$. □

Claim 2. Let a and b are two zeroes of g, and let s be its *non-zero*, i. e. $g(s) \neq 0$. We claim that g is p-periodic, where $p = (a - b)s$. Indeed, substituting $x = as$ and using Claim 1, we get that he expression

$$g(as + yg(s)) = g(as) + g(y) - g(asy) = g(s) + g(y) - g(sy)$$

does not de[pend on a. Hence $g(as + yg(s)) = g(bs + yg(s))$ for all y, which proves the required periodicity, since $yg(s)$ attains all real values. □

Now, if $g(x) = 0$ for all $x \neq 0$, we get the remaining answer (iii). Assume now that there exists $s \neq 0$ with $g(s) \neq 0$, so by Claim 2 g is periodic with some period p. Substituting $x = p$ and using periodicity we get $g(yg(0)) + g(py) = g(0) + g(y)$. Since $g(yg(0)) = g(y)$ by Claim 1, we arrive at $g(py) = g(0)$ which shows g is constant.

Remark. After arriving at $(*)$ and obtaining Claims 1 and 2, alternative approaches are possible.

E. g., denote by $Z = \{x \in \mathbb{R} \colon g(x) = 0\}$ the set of zeros of g. Claim 1 yields that Z is *a-invariant*, i. e., $aZ = Z$. We want to show that $Z - Z = \mathbb{R}$; this will, by means of Claim 2, yield that g is periodic with *every* period, i. e., constant.

For any $\beta \in Z$, we plug in $y = \beta$ and use Claim 1 to obtain $g(x + \beta g(x)) = 0$, so $x + \beta g(x) \in Z$ for all x. Now, setting $\beta = 1$ and $\beta = a$ (from $(*)$) we get $x + g(x), x + ag(x) \in Z$. The first inclusion yields also $a(x + g(x)) \in Z$, and hence $(a - 1)x = a(x + g(x)) - (x + ag(x)) \in Z - Z$. This shows that $Z - Z = \mathbb{R}$.

Problem 6. Find all pairs of integers (c, d), both greater than 1, such that the following holds:

For any monic polynomial Q of degree d with integer coefficients and for any prime $p > c(2c+1)$, there exists a set S of at most $\left(\frac{2c-1}{2c+1}\right)p$ integers, such that

$$\bigcup_{s\in S}\{s,\ Q(s),\ Q(Q(s)),\ Q(Q(Q(s))),\ \dots\}$$

contains a complete residue system modulo p (i.e., intersects with every residue class modulo p).

Croatia, Adrian Beker

Solution. Those pairs are all pairs (c, d) of positive integers greater than 1 such that $d \le c$.

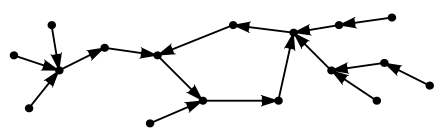
Assume first that $d \ge c+1$. Choose a large prime p (we need $p > 2c^2 + c$) congruent to 1 modulo d (such a prime exists, by a particular case of Dirichlet's theorem; this particular case is easier to prove by using the cyclotomic polynomial Φ_d). Let $Q(X) = X^d$. Since $d \mid p-1$, exactly $1 + (p-1)/d$ residues modulo p are d-th powers; all other $(d-1)(p-1)/d$ residue classes contain no values of P. Hence, if a set S satisfies the requirements, it should contain representatives of all those classes. But this is more than S is allowed to contain, since

$$\frac{d-1}{d}(p-1) > \frac{2c-1}{2c+1}p \quad\Longleftarrow\quad \frac{c}{c+1}(p-1) > \frac{2c-1}{2c+1}p$$
$$\Longleftarrow\quad \frac{p}{(c+1)(2c+1)} > \frac{c}{c+1} \quad\Longleftarrow\quad p > c(2c+1).$$

We now show that such set S exists, whenever $d \le c$. To this end, usage is made of the lemma below.

Lemma. Fix an integer $d \ge 2$. Let $G = (V, E)$ be a directed graph, each vertex of which has exactly one outgoing edge and at most d incoming edges. Assume further that there are at most d loops in this graph. Then there exists a subset V' of V of cardinality $|V'| \le 1 + \frac{d-1}{d}|V|$ such that every vertex in $V \smallsetminus V'$ is the terminus of a directed path emanating from V'.

Proof. Consider any (weak) connected component $G_1 = (V_1, E_1)$ in G — i.e., a component of the corresponding *undirected* graph. Since from each vertex emanates exactly one edge, the component contains a directed cycle (possibly a loop); and since the numbers of vertices and edges in G_1 are equal, even an undirected cycle is unique. Hence, the component is a cycle with some trees rooting out of its vertices. With reference again to uniqueness of outgoing edges, the edges of these trees are all directed towards the cycle.



Now, let V' choose exactly one vertex from each component that is just a cycle; for any other component, let V' choose all its in-degree 0 vertices, i.e., the leaves of all trees rooting out of the vertices of the core-cycle — any vertex of such a tree can be reached from some leaf, and hence so can any vertex of the core-cycle.

To bound $|V'|$ from above, let t be the number of single-vertex components in G, and notice that $t \le d$, since there are at most d loops in the graph. From each other component that is a cycle, V' chooses at most half of its vertices, so at most $\frac{d-1}{d}$-th part of them. Finally, consider a component containing some trees. Since each in-degree is at most d, at least $\frac{1}{d}$-th part of the vertices have incoming edges, hence V' chooses at most $\frac{d-1}{d}$-th part of the vertices. Consequently,

$$|V'| \le t + \frac{d-1}{d}(|V| - t) = \frac{t}{d} + \frac{d-1}{d}|V| \le 1 + \frac{d-1}{d}|V|,$$

as desired. This establishes the lemma. □

Now let p and Q be chosen as in the problem statement. Consider a graph with vertex set $\mathbb{Z}_p$. Regard Q as a polynomial over $\mathbb{Z}_p$, and draw an edge $a \to Q(a)$ for every a in $\mathbb{Z}_p$. Since $\deg Q = d$, each b in $\mathbb{Z}_p$ has at most d preimages, so the in-degree of each vertex is at most d. Since Q is monic and $d > 1$, the equation $Q(x) = x$ has at most d roots in $\mathbb{Z}_p$, hence the graph has at most d loops. Thus, implementation of the lemma provides a set V' which is suitable as the required set S. Indeed, the lemma statement shows that each residue is a repetitive image of some element of S; and the implications below show that the cardinality of V' lies within the required range:

$$|V'| \le \frac{d-1}{d}p + 1 \le \frac{2c-1}{2c+1}p \quad \Longleftarrow \quad \frac{c-1}{c}p + 1 \le \frac{2c-1}{2c+1}p$$
$$\Longleftarrow \quad \frac{p}{c(2c+1)} \ge 1 \quad \Longleftarrow \quad p \ge c(2c+1).$$

The 12th Romanian Master of Mathematics Competition

Day 1 — Solutions

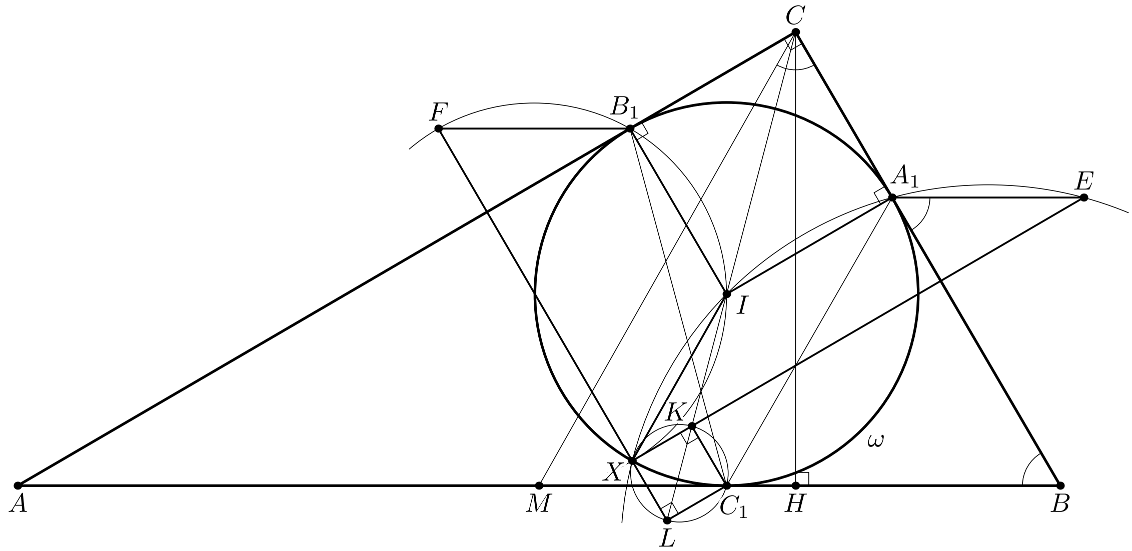
Problem 1. Let ABC be a triangle with a right angle at C, let I be its incentre, and let H be the orthogonal projection of C on AB. The incircle ω of the triangle ABC is tangent to the sides BC, CA, and AB at A_1, B_1, and C_1, respectively. Let E and F be the reflections of C in the lines C_1A_1 and C_1B_1, respectively, and let K and L be the reflections of H in the same lines. Prove that the circles A_1EI, B_1FI, and C_1KL have a common point.

Russia, Dmitry Prokopenko

Solution. The line C_1A_1 is parallel to the external angle bisector of $\angle B$, so the reflection in C_1A_1 maps the segment A_1C to the segment A_1E parallel to AB. Similarly, $B_1F \parallel AB$. Notice also that $A_1E = A_1C = B_1C = B_1F = r$, where r is the inradius of $\triangle ABC$.

Let M be the midpoint of AB. Let X be the point of ω such that $\overrightarrow{IX} \upuparrows \overrightarrow{CM}$. Notice that $\angle EA_1I = 90^\circ + \angle EA_1B = 90^\circ + \angle CBM = 90^\circ + \angle BCM = \angle A_1IX$, and $A_1E = IA_1 = IX$; thus, XIA_1E is an isosceles trapezoid. Hence X lies on the circle IA_1E, and $EX \parallel A_1I$. Similarly, X lies on the circle IB_1F, and $FX \parallel B_1I$. It remains to show that X lies on the circle C_1KL.

Under the symmetry in C_1A_1, the line CH (perpendicular to AB) maps to the line through E perpendicular to BC — i.e., CH maps to EX. Therefore, the projection H of C_1 onto CH maps to the projection K of C_1 onto EX. Similarly, L is the projection of C_1 onto FX. So the quadrilateral C_1KXL is cyclic, due to right angles at K and L.



Comments. (1) In fact, the quadrilateral C_1KXL is a square, since $C_1K = C_1H = C_1L$ and $\angle KC_1L = 2\angle A_1C_1B_1 = 90^\circ$.

(2) One can easily see that the points C_1 and X are symmetric in the angle bisector CI. This yields that K and L both lie on CI. One can show that this conclusion in fact holds in any, not necessarily right-angled, triangle.

Problem 2. Let N be a positive integer, and let $\mathbf{a} = (a(1), \dots, a(N))$ and $\mathbf{b} = (b(1), \dots, b(N))$ be sequences of non-negative integers, each written on a circle (so we assume $a(i \pm N) = a(i)$ and $b(i \pm N) = b(i)$). We say $\mathbf{a}$ is $\mathbf{b}$-*harmonic*, if each $a(i)$ is the arithmetic mean of the counterclockwise nearest $b(i)$ numbers, the clockwise nearest $b(i)$ numbers, and $a(i)$ itself; that is,

$$a(i) = \frac{1}{2b(i)+1} \sum_{s=-b(i)}^{b(i)} a(i+s). \qquad (*)$$

(A term of $\mathbf{a}$ may appear more than once in the above sum.) Suppose that neither $\mathbf{a}$ nor $\mathbf{b}$ is constant, and that both $\mathbf{a}$ is $\mathbf{b}$-harmonic, and $\mathbf{b}$ is $\mathbf{a}$-harmonic. Prove that more than half of the $2N$ terms across both sequences vanish.

United Kingdom, Dominic Yeo

Solution 1. Let $a = \min_i a(i)$ and let $b = \min_i b(i)$. Since $\mathbf{a}$ is not constant, there exists an i such that $a = a(i) < a(i+1)$.

Claim 1. If $a = a(i) < a(i+1)$, then $b(i) = 0$. Similarly, if $a = a(i) < a(i-1)$, then $b(i) = 0$.

Proof. Otherwise the sum in $(*)$ contains a term $a(i+1) > a$ but no terms smaller than a, so the average is greater than a. □

Claim 1 implies $b = 0$; similarly, $a = 0$. With reference again to Claim 1, $a(i) = b(i) = 0$ for some index i.

Say that $[i, j]$ is an $\mathbf{a}$-segment if $a(i) = a(i+1) = \cdots = a(j) = 0$ but $a(i-1) \neq 0 \neq a(j+1)$; define a $\mathbf{b}$-segment similarly. By Claim 1, the endpoints of any such segment satisfy $a(i) = b(i) = a(j) = b(j) = 0$. Since the sequences are non-constant, each i where $a(i) = 0$ is contained in an $\mathbf{a}$-segment.

Claim 2. Let $[i, j]$ be a $\mathbf{b}$-segment, and let $k \in [i, j]$. Then $a(k) \le k - i$ (and, similarly, $a(k) \le j - k$).

Proof. Indeed, since $b(k) = 0$, the elements of $\mathbf{b}$ with indices from $k - a(k)$ to $k + a(k)$ must all be zero as well. □

We now show that every index is contained in either an $\mathbf{a}$- or a $\mathbf{b}$-segment. Since at least one index is contained in both, the conclusion follows.

Assume, to the contrary, that $a(i)$ and $b(i)$ are both positive for some index i; call such indices *bad*. Among all bad indices i, choose one maximising $\max(a(i), b(i))$; by symmetry, we may and will assume that this maximum is $a(i)$. We may and will also assume that either the index $i-1$ is not bad, or $a(i-1) < a(i)$ (otherwise change i to $i-1$, repeat if necessary, recalling that $\mathbf{a}$ is not constant).

Consider the range of indices $\Delta = [i - b(i), i + b(i)]$, and the values $\mathbf{a}$ assumes at those indices. Some indices j in Δ are bad; the corresponding values $a(j)$ do not exceed $a(i)$. Other indices j in Δ are covered by several $\mathbf{a}$- and $\mathbf{b}$-segments. Each $\mathbf{b}$-segment contributes at most $b(i)$ members nearest to one of its endpoints, so the average value of $\mathbf{a}$ over those indices does not exceed $(b(i) - 1)/2 < a(i)$ by Claim 2. The remaining indices j in Δ all lie in $\mathbf{a}$-segments, so the corresponding values $a(j)$ are all zero.

Combining all this, it follows that the average in the right-hand member of $(*)$ does not exceed $a(i)$. Moreover, if some $\mathbf{a}$- or $\mathbf{b}$-segment intersects Δ, then the inequality is strict. Otherwise, $i - 1$ is a bad index contained in Δ, and $a(i-1) < a(i)$, so the inequality is again strict. This contradiction ends the proof and completes the solution.

Solution 2. The solution has a few well-defined steps:

Lemma 1. Assume that $a(i) = M := \max \mathbf{a}$; then $b(i+k) = 0$ for all $k = -M, -M+1, \dots, M$. In particular, $b(i-1) = b(i) = b(i+1) = 0$, as $M \ge 1$.

Proof. Assume that $a(j) = a(j+1) = \cdots = a(j+s) = M > 0$, and $a(j-1), a(j+s+1) < M$, where $i \in [j, j+s]$. Then $b(j) = 0$, as otherwise $a(j)$ is the mean of at least three terms, all $\leq M$, with at least one $< M$. For the same reason, $b(j+s) = 0$ also.

But then $b(j)$ is the mean of $2M+1$ terms of $\mathbf{b}$, which must therefore all also be equal to 0. So $b(j+k) = 0$ for all $k \in [-M, M]$. Iterating this argument gives $b(j+k) = 0$ for all $k \in [-M, M+s]$. which implies the statement of the lemma. □

Corollary. There exist i such that $a(i) = 0$ and j such that $b(j) = 0$. □

Lemma 2. Suppose $\max \mathbf{a} \geq \max \mathbf{b}$. Generate $\mathbf{a}'$ by replacing all copies of $M = \max \mathbf{a}$ with 1 in $\mathbf{a}$. Then $\mathbf{a}'$ is $\mathbf{b}$-harmonic, and $\mathbf{b}$ is $\mathbf{a}'$-harmonic.

Proof. We start with another consequence of Lemma 1. Assume that $a(i) \neq M$; then none of the terms $a(i+k)$ with $k \in [-b(i), b(i)]$ equals M. Indeed, if $a(i+k) = M$ with $|k| \leq b(i) \leq M$, then by Lemma 1 we have $b(i) = b\big((i+k)-k\big) = 0$, which yields $k = 0$ and hence $a(i) = a(i+k) = M$.

We can now check that the harmonic properties are preserved under replacing all copies of M in $\mathbf{a}$ with 1:

If $a(i) \neq M$, then the harmonic property for $b(i)$ is unchanged. If $a(i) = M$, then $a'(i) = 1$ and $b(i-1) = b(i) = b(i+1) = 0$, so $b(i)$ certainly has the $a'(i)$-harmonic property; and

If $a(i) = M$, then $b(i) = 0$, and so $a'(i) = 1$ has the $b(i)$-harmonic property. If $a(i) \neq M$, then we have just shown that none of the terms in the statement of $a(i)$'s harmonic property are changed by this process, so it remains harmonic.

This check completes the proof of the lemma. □

Lemma 3. We have $\min(a(i), b(i)) = 0$ for all i. Moreover, there exists an i with $a(i) = b(i) = 0$.

Proof. Both statements in the lemma are invariant under the procedure in Lemma 2. Apply this procedure repeatedly, to replace all instances of the maximum value in one of the sequences with 1, until both sequences consist of zeroes and ones. It suffices to check the lemma statement for the obtained pair of sequences.

Suppose that $a(i) = b(i) = 1$ for some i. Since the sequences remain non-constant, we may and will assume that $\min(a(i-1), b(i-1)) = 0$, say $a(i-1) = 0$. But then the $b(i)$-harmonic property is violated for $a(i)$, as $a(i+1) \leq 1$.

Suppose now that there is no i with $a(i) = b(i) = 0$. This means that for every index i we have either $a(i) = 1$ and $b(i) = 0$, or $a(i) = 0$ and $b(i) = 1$. There is a pair of adjacent indices having different types, so that $a(i) = b(i+1) = 1$ and $a(i+1) = b(i) = 0$. But then $b(i)$ violates the $a(i)$-harmonic property. □

Lemma 3 readily yields that at least $N+1$ terms across both sequences are zeroes, as required.

Remark. It can be shown that there are at least $N+2$ zeroes across both sequences, a bound achieved if, for instance, $\mathbf{a} = (0, 0, 1, 1, \ldots, 1, 0)$ and $\mathbf{b} = (1, 0, 0, \ldots, 0)$.

Problem 3. In a country there are n airports and n air companies operating return flights. Each company operates an odd number of flights forming a closed route. Prove that a traveller can complete a closed route consisting of an odd number of flights operated by pairwise distinct companies.

Israel, Ron Aharoni

Solution. In graph-theoretic setting, the statement reads:

> Consider a collection of n odd cycles, not necessarily distinct, all on the same vertex set of size n. Prove that at most one edge can be chosen from each of these cycles to form a collection that contains the edges of an odd cycle.

Call a set of edges *rainbow* if it is formed by choosing at most one edge from each cycle. We have to prove that there exists a rainbow cycle of odd length.

Begin by choosing a maximal rainbow forest F, that is, an acyclic rainbow set of edges.

Since F is acyclic, its size is less than n, so there is a cycle C in the collection no edge of which lies in F. The forest F contains every vertex of C, for otherwise an edge of C incident with a vertex outside F could be added to F to form a larger rainbow forest, contradicting maximality. Moreover, no edge of C joins different components of F, for one such could again be added to F to contradict maximality.

Consequently, the vertices of C all lie in some component of F, a tree T. As such, T is bipartite, that is, its vertices split into two disjoint parts, and all edges are between the two. Since C is an odd cycle, it has an edge whose endpoints both lie in the same part. This edge then completes an odd rainbow cycle.

The 12^{th} Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Let $\mathbb{N}$ be the set of all positive integers. A subset A of $\mathbb{N}$ is *sum-free* if, whenever x and y are (not necessarily distinct) members of A, their sum $x+y$ does not belong to A. Determine all *surjective* functions $f\colon \mathbb{N} \to \mathbb{N}$ such that, for each sum-free subset A of $\mathbb{N}$, the image $\{f(a)\colon a \in A\}$ is again sum-free.

India, Sutanay Bhattacharya

Solution. The identity is the only surjection of the positive integers onto themselves sending every sum-free set onto a sum-free set (no verification is needed, of course).

To prove this, fix a function f satisfying the conditions in the statement, and proceed in several steps.

Step 1. Notice that a 2-element set $\{x, y\}$, where $x < y$, is *not* sum-free if and only if $y = 2x$.

Choose any $a \in \mathbb{N}$, and for any $i \geq 0$ choose some x_i such that $f(x_i) = 2^i a$. The set $f(\{x_i, x_{i+1}\})$ is not sum-free, so neither is $\{x_i, x_{i+1}\}$, whence $x_i = 2x_{i+1}$ or $x_{i+1} = 2x_i$. Since the x_i are all distinct, the same option should hold for all i. The former option yields $x_i = x_0 2^{-i}$ which cannot hold for large enough i. So $x_{i+1} = 2x_i$ for all i.

Therefore, $f(2x) = 2f(x)$ for all x, and, moreover, x is the only argument t with $f(t) = f(2x)/2$. Therefore, f is injective (and hence bijective).

Step 2. Say that a 3-element set $\{a, b, c\}$ is *good* if it is not sum-free, but each of its 2-element subsets is (in other words, no element is twice another). It is easily seen that a set $\{a, b, c\}$, where $a < b$, is good only if $c = b \pm a$. Notice that the pre-image of a good set is also a good set, due to Step 1.

Now let $f(1) = a$. We show that $f(n) = an$ by induction on n. The base cases are $n = 1, 2, 3, 4, 5$; for $n = 1, 2, 4$ the result follows from Step 1.

Set $t = f^{-1}(3a)$ and $s = f^{-1}(5a)$. The sets $\{a, 4a, 3a\}$ and $\{a, 4a, 5a\}$ are good, hence so are $\{1, 4, t\}$ and $\{1, 4, s\}$. Therefore, $\{s, t\} = \{3, 5\}$. But the set $\{a, 5a, 6a\}$ is also good, so the pair $\{1, s\}$ is contained in one more good set, which is not the case if $s = 3$, since $\{1, 3\}$ is contained in one single good set, namely, $\{1, 4, 3\}$. Thus $t = 3$ and $s = 5$, which establishes the base.

For the induction step, assume that $f(k) = ak$ for all $k \leq n$, where $n \geq 5$. Choose $t = f^{-1}((n+1)a)$. Then the pair $\{a, na\}$ is contained in two good sets, namely, $\{a, na, (n-1)a\}$ and $\{a, na, (n+1)a\}$. Their pre-images, $\{1, n, n-1\}$ and $\{1, n, t\}$, are also good, and injectivity of f forces $t = n+1$. This completes the induction step.

Finally, since f is surjective, $1 = f(n) = an$ for some positive integer n, so $a = 1 = n$. Consequently, f is the identity, as claimed at the beginning of the solution.

Problem 5. A *lattice point* in the Cartesian plane is a point whose coordinates are both integral. A *lattice polygon* is a polygon whose vertices are lattice points. Let Γ be a convex lattice polygon. Prove that Γ is contained in a convex lattice polygon Δ exactly one vertex of which is not a vertex of Γ, and the vertices of Γ all lie on the boundary of Δ.

Russia, Maxim Didin

Solution 1. Let T be the extra vertex of a desired polygon Δ; then Δ is the convex hull of T and Γ. Thus, a point T fits the bill if and only if this convex hull contains no vertices of Γ in its interior.

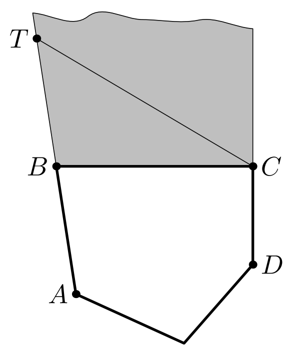
Each segment AB joining two lattice points is partitioned by lattice points into congruent *elementary segments*. Define the *elementary length* $\ell(AB)$ of AB to be the length of any of those elementary segments.

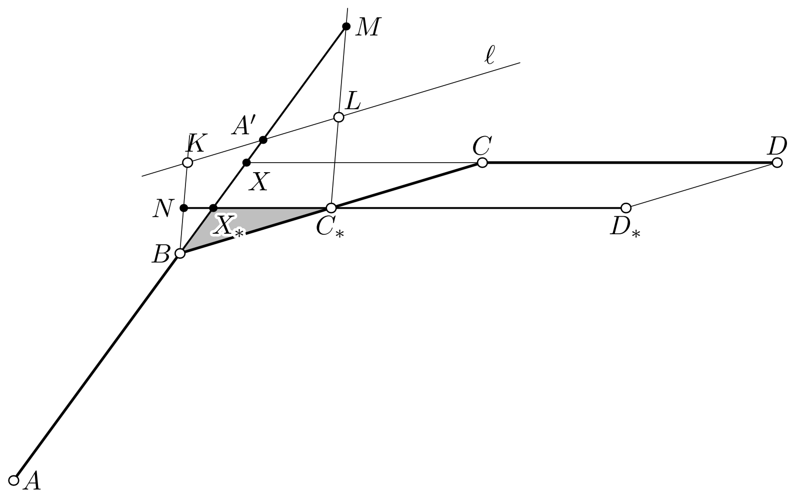
Take any three consecutive sides AB, BC, and CD on the boundary of Γ. Consider the convex region bounded by the segment BC and the rays complementary to the rays BA and CD (including the boundary). If this region is unbounded (see the left figure below), then it contains some lattice point T (e.g., the point with $\overrightarrow{BT} = \overrightarrow{AB}$), and any such point T satisfies the problem requirements. Thus, in what follows we assume that the two rays cross each other at some point X. Assume further that the triangle BCX contains no lattice points outside the segment BC, as any other such point would satisfy the requirements.

Let C_* be a lattice point on the segment BC such that $BC_* = \ell(BC)$, let X_* be the point on BX such that $C_*X_* \parallel CX$, and let D_* be the point such that $\overrightarrow{C_*D_*} = \overrightarrow{CD}$ (see the right figure below). Then the triangle BC_*X_* contains no lattice points apart from B and C_*.

Consider the half-plane determined by the line BC and containing no interior points of Γ. Let ℓ be the line in that half-plane parallel to BC, containing some lattice points, and nearest to BC among such. Let the ray AB meet ℓ at a point A' which belongs to the elementary segment KL on ℓ (we assume that $\overrightarrow{KL} = \overrightarrow{BC_*}$; the point A' may coincide with L but not with K). Then the ray D_*C_* crosses the ray LK (excluding L), otherwise L lies in the triangle BC_*X_*.

The only lattice points contained in the parallelogram $BKLC_*$ are its vertices. This yields that there are no lattice points strictly indise the strip defined by the parallel lines BK and C_*L.





Let M and N be the meeting points of the rays BX_*, C_*L and C_*X_*, BK, respectively. Then the segments BM and C_*N contain no lattice points except their endpoints, so $\ell(AB) \geq BM$ and $\ell(DC) = \ell(D_*C_*) \geq C_*N$. Therefore,

$$\frac{BX_*}{\ell(AB)} + \frac{C_*X_*}{\ell(CD)} \leq \frac{BX_*}{BM} + \frac{C_*X_*}{C_*N} = \frac{BX_*}{BM} + \frac{MX_*}{BM} = 1. \tag{$*$}$$

Choose now BC to be a side of largest elementary length. Then

$$(1 \geq)\frac{BX_*}{\ell(AB)} + \frac{C_*X_*}{\ell(CD)} \geq \frac{BX_* + C_*X_*}{\ell(BC)} = \frac{BX_* + C_*X_*}{BC_*},$$

which contradicts the triangle inequality.

Comments. (1) The usage of elementary length seems to be crucial. In particular, under the assumption that the triangle BCX contains no lattice points outside BC, an inequality

$$\frac{BX}{AB}+\frac{CX}{CD}\leq 1$$

similar to $(*)$ does not necessarily hold.

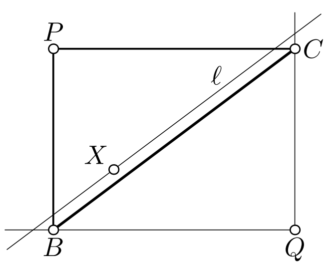
(2) Any lattice parallelogram of unit area may be transformed into a unit square by an affine transform preserving the set of lattice points. If one applies such a transform to the parallelogram $BKLC_*$, inequality $(*)$ may become more transparent.

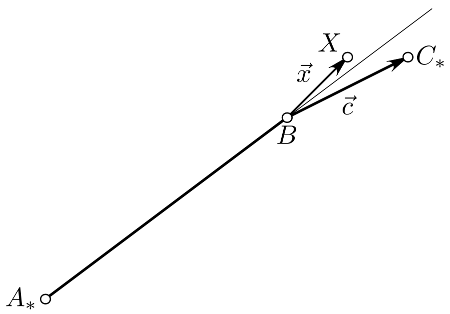
The key inequality $(*)$ is preserved by affine transforms preserving the lattice, so the above solution is in some sense "affine". In contrast, the next solution, although involving similar ideas, is more "Euclidean".

Solution 2. We use the same notions of *elementary segments* and *elementary lengths* as in Solution 1. We say that a segment is *sloped* if it is neither horizontal nor vertical. If Γ has no sloped sides, then Γ is a rectangle, and the problem statement is trivial: one may choose any lattice point on the extension of any side. So, in what follows, we assume that Γ has a sloped side.

Lemma 1. Let BC be a sloped segment of the boundary of Γ with no lattice points in its interior, and let P be a lattice point such that the segments PB and PC are not sloped, and the triangle PBC lies outside Γ. Then there exists a unique lattice point X in the triangle PBC such that the area of the triangle XBC is $1/2$.

Proof. Choose a lattice point Q such that $PBQC$ is a rectangle. As in Solution 1, let ℓ denote the line through some lattice point parallel to BC, lying outside Γ, and nearest to the line BC under these constraints (see the left figure below). The line ℓ crosses the interior of the angle BQC along an interval of length $> BC$, so this interval should contain a lattice point X. The triangle XBC contains no lattice points apart from the vertices, so its area is $1/2$ due to Pick's formula. Moreover, any such point X should lie within the angle BPC, and ℓ crosses this angle along a segment of length $< BC$. Hence X is the required unique lattice point. $\square$





Denote the point X defined in Lemma 1 by $f(AB)$.

Lemma 2. Let AB and BC be two consecutive sides on the boundary of Γ, and let BA_* and BC_* be elementary segments on the sides BA and BC, respectively. Assume that both coordinates of $\overrightarrow{BC}$ are positive, and that the line AB strictly separates the points C and $X=f(BC_*)$. Then both coordinates of the vector $\overrightarrow{A_*B}$ are also positive, and $BA_* > BC_*$.

Proof. Since AB separates X and C, the vector $\overrightarrow{A_*B}$ is a linear combination of $\vec{c}=\overrightarrow{BC_*}$ and $\vec{x}=\overrightarrow{BX}$ with positive coefficients; so the coordinates of $\overrightarrow{A_*B}$ are positive. Since the area of the triangle BXC equals $1/2$, the vectors $\vec{c}$ and $\vec{x}$ span the whole lattice, so the coefficients of the linear combination are integers. Finally, the angle between $\vec{c}$ and $\vec{x}$ is acute, so $BA_*=|\overrightarrow{A_*B}|\geq \sqrt{|\vec{c}|^2+|\vec{x}|^2}>|\vec{c}|=BC_*$, as desired. $\square$

Now, choose a sloped side BC of Γ of a maximal elementary length, and let $ABCD$ be the corresponding part of the boundary of Γ. Let C_* and B_* be the points on the segment BC such that $BC_* = B_*C = \ell(BC)$. Let $X = f(BC_*)$ and $X' = f(B_*C)$. Then, due to Lemma 2, the line AB does not separate X and C, and the line CD does not separate X' and B. Therefore, the segment XX' lies in the same angle of the lines AB and CD as Γ, so X may serve as a suitable vertex T of Δ.

Problem 6. For an integer $n > 1$, let $\operatorname{gpf}(n)$ denote the greatest prime factor of n. A *strange pair* is an unordered pair of distinct primes p and q such that $\{p, q\} = \{\operatorname{gpf}(n), \operatorname{gpf}(n+1)\}$ for no integer $n > 1$. Prove that there exist infinitely many strange pairs.

Russia, Dmitry Krachun

Solution. We show that there are infinitely many strange pairs of the form $\{2, q\}$ where q is an odd prime.

The Lemma below provides a sufficient condition for such a pair to be strange. For an odd prime q, let $\operatorname{ord}_q(2)$ denote the multiplicative order of 2 modulo q, i.e., the least positive integer s satisfying $q \mid 2^s - 1$.

Lemma. If some primes $2 < q_1 < q_2$ satisfy $\operatorname{ord}_{q_1}(2) = \operatorname{ord}_{q_2}(2)$, then $\{2, q_1\}$ is a strange pair.

Proof. Arguing indirectly, suppose first that $2 = \operatorname{gpf}(n)$ and $q_1 = \operatorname{gpf}(n+1)$; in particular, $n = 2^k$ for some positive integer k, and $q_1 \mid 2^k + 1$. This yields $q_1 \mid 2^{2k} - 1$, so $\operatorname{ord}_{q_2}(2) = \operatorname{ord}_{q_1}(2) \mid 2k$. Therefore, $q_2 \mid 2^{2k} - 1 = (2^k - 1)(2^k + 1)$, but $q_2 \nmid 2^k - 1$, hence $q_2 \mid 2^k + 1$. So $\operatorname{gpf}(n+1) \geq q_2$, which is a contradiction.

Similarly, but easier, if $2 = \operatorname{gpf}(n+1)$ and $q_1 = \operatorname{gpf}(n)$, then $n + 1 = 2^k$, so $\operatorname{ord}_{q_2}(2) = \operatorname{ord}_{q_1}(2) \mid k$ and hence $q_2 \mid 2^k - 1$. Therefore, $\operatorname{gpf}(n+1) \geq q_2$, a contradiction. $\square$

It remains to show that there exist infinitely many disjoint pairs of primes $q_1 < q_2$ satisfying the conditions in the Lemma.

Let $p = 2r - 1 > 5$ be a prime, and let $N = 2^{2p} + 1$. We prove that:

(1) N has at least two distinct prime factors greater than 5; and

(2) $\operatorname{ord}_q(2) = 4p$ for every prime factor $q > 5$ of N.

Thus, every prime $p > 5$ provides a pair of odd primes satisfying the conditions in the Lemma. Moreover, **(2)** shows that distinct primes $p > 5$ provide disjoint such pairs, whence the conclusion.

To prove **(1)**, notice that $3 \nmid N$, and write $N = (4+1) \cdot (4^{p-1} - 4^{p-2} + \cdots + 1) \equiv 5p \pmod{25}$, to infer that $25 \nmid N$.

Next, write $N = (2^p + 1)^2 - 2^{p+1} = (2^p - 2^r + 1)(2^p + 2^r + 1)$. The two factors are coprime (since they are odd, and their difference is 2^{r+1}), and each is larger than 5. Hence each has a prime factor greater than 5. This establishes **(1)**.

To prove **(2)**, consider a prime factor $q > 5$ of N, and notice that $\operatorname{ord}_q(2) \mid 4p$, since $q \mid N \mid 2^{4p} - 1$. If $\operatorname{ord}_q(2) < 4p$, then either $\operatorname{ord}_q(2) \mid 2p$ or $\operatorname{ord}_q(2) \mid 4$. The former is impossible due to $2^{2p} - 1 = N - 2 \equiv -2 \pmod{q}$, the latter — due to $q \nmid 15 = 2^4 - 1$. This establishes **(2)** and completes the proof.

The 13$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 1: Tuesday, October 12, 2021, Bucharest

Language: English

Problem 1. Let T_1, T_2, T_3, T_4 be pairwise distinct collinear points such that T_2 lies between T_1 and T_3, and T_3 lies between T_2 and T_4. Let ω_1 be a circle through T_1 and T_4; let ω_2 be the circle through T_2 and internally tangent to ω_1 at T_1; let ω_3 be the circle through T_3 and externally tangent to ω_2 at T_2; and let ω_4 be the circle through T_4 and externally tangent to ω_3 at T_3. A line crosses ω_1 at P and W, ω_2 at Q and R, ω_3 at S and T, and ω_4 at U and V, the order of these points along the line being P, Q, R, S, T, U, V, W. Prove that $PQ + TU = RS + VW$.

Problem 2. Xenia and Sergey play the following game. Xenia thinks of a positive integer N not exceeding 5000. Then she fixes 20 distinct positive integers a_1, a_2, $\ldots$, a_{20} such that, for each $k = 1, 2, \ldots, 20$, the numbers N and a_k are congruent modulo k. By a move, Sergey tells Xenia a set S of positive integers not exceeding 20, and she tells him back the set $\{a_k \colon k \in S\}$ without spelling out which number corresponds to which index. How many moves does Sergey need to determine for sure the number Xenia thought of?

Problem 3. A number of 17 workers stand in a row. Every contiguous group of at least 2 workers is a *brigade*. The chief wants to assign each brigade a leader (which is a member of the brigade) so that each worker's number of assignments is divisible by 4. Prove that the number of such ways to assign the leaders is divisible by 17.

Each of the three problems is worth 7 marks.
Time allowed $4\frac{1}{2}$ hours.

The 13^{th} Romanian Master of Mathematics Competition

Day 2: Wednesday, October 13, 2021, Bucharest

Language: English

Problem 4. Consider an integer $n \geq 2$ and write the numbers $1, 2, \ldots, n$ down on a board. A move consists in erasing any two numbers a and b, then writing down the numbers $a+b$ and $|a-b|$ on the board, and then removing repetitions (e.g., if the board contained the numbers $2, 5, 7, 8$, then one could choose the numbers $a = 5$ and $b = 7$, obtaining the board with numbers $2, 8, 12$). For all integers $n \geq 2$, determine whether it is possible to be left with exactly two numbers on the board after a finite number of moves.

Problem 5. Let n be a positive integer. The kingdom of Zoomtopia is a convex polygon with integer sides, perimeter $6n$, and 60° rotational symmetry (that is, there is a point O such that a 60° rotation about O maps the polygon to itself). In light of the pandemic, the government of Zoomtopia would like to relocate its $3n^2+3n+1$ citizens at $3n^2+3n+1$ points in the kingdom so that every two citizens have a distance of at least 1 for proper social distancing. Prove that this is possible. (The kingdom is assumed to contain its boundary.)

Problem 6. Initially, a non-constant polynomial $S(x)$ with real coefficients is written down on a board. Whenever the board contains a polynomial $P(x)$, not necessarily alone, one can write down on the board any polynomial of the form $P(C+x)$ or $C+P(x)$, where C is a real constant. Moreover, if the board contains two (not necessarily distinct) polynomials $P(x)$ and $Q(x)$, one can write $P(Q(x))$ and $P(x)+Q(x)$ down on the board. No polynomial is ever erased from the board.

Given two sets of real numbers, $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, a polynomial $f(x)$ with real coefficients is (A, B)-*nice* if $f(A) = B$, where $f(A) = \{f(a_i)\colon i = 1, 2, \ldots, n\}$.

Determine all polynomials $S(x)$ that can initially be written down on the board such that, for any two finite sets A and B of real numbers, with $|A| = |B|$, one can produce an (A, B)-nice polynomial in a finite number of steps.

Each of the three problems is worth 7 marks. Time allowed $4\frac{1}{2}$ hours.

The 13th Romanian Master of Mathematics Competition

Day 1 — Solutions

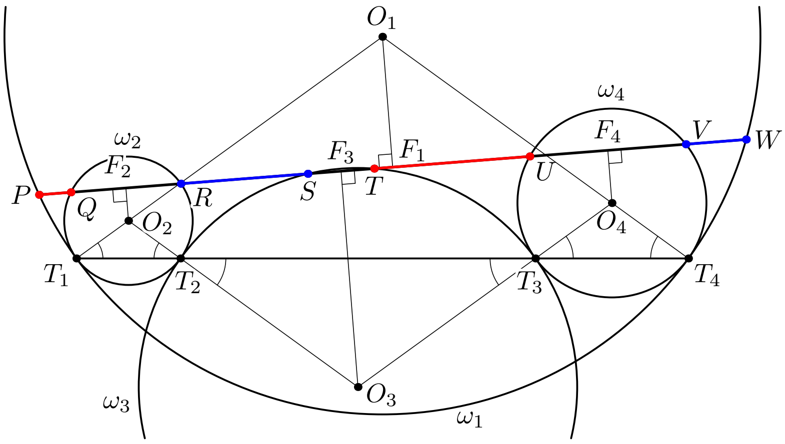
Problem 1. Let T_1, T_2, T_3, T_4 be pairwise distinct collinear points such that T_2 lies between T_1 and T_3, and T_3 lies between T_2 and T_4. Let ω_1 be a circle through T_1 and T_4; let ω_2 be the circle through T_2 and internally tangent to ω_1 at T_1; let ω_3 be the circle through T_3 and externally tangent to ω_2 at T_2; and let ω_4 be the circle through T_4 and externally tangent to ω_3 at T_3. A line crosses ω_1 at P and W, ω_2 at Q and R, ω_3 at S and T, and ω_4 at U and V, the order of these points along the line being P, Q, R, S, T, U, V, W. Prove that $PQ + TU = RS + VW$.

Hungary, Geza Kos

Solution. Let O_i be the centre of ω_i, $i = 1, 2, 3, 4$. Notice that the isosceles triangles $O_iT_iT_{i-1}$ are similar (indices are reduced modulo 4), to infer that ω_4 is internally tangent to ω_1 at T_4, and $O_1O_2O_3O_4$ is a (possibly degenerate) parallelogram.

Let F_i be the foot of the perpendicular from O_i to PW. The F_i clearly bisect the segments PW, QR, ST and UV, respectively.

The proof can now be concluded in two similar ways.



First Approach. Since $O_1O_2O_3O_4$ is a parallelogram, $\overrightarrow{F_1F_2} + \overrightarrow{F_3F_4} = \mathbf{0}$ and $\overrightarrow{F_2F_3} + \overrightarrow{F_4F_1} = \mathbf{0}$; this still holds in the degenerate case, for if the O_i are collinear, then they all lie on the line T_1T_4, and each O_i is the midpoint of the segment T_iT_{i+1}. Consequently,

$$\begin{aligned}
\overrightarrow{PQ} - \overrightarrow{RS} + \overrightarrow{TU} - \overrightarrow{VW} &= \left(\overrightarrow{PF_1} + \overrightarrow{F_1F_2} + \overrightarrow{F_2Q}\right) - \left(\overrightarrow{RF_2} + \overrightarrow{F_2F_3} + \overrightarrow{F_3S}\right) \\
&\quad + \left(\overrightarrow{TF_3} + \overrightarrow{F_3F_4} + \overrightarrow{F_4U}\right) - \left(\overrightarrow{VF_4} + \overrightarrow{F_4F_1} + \overrightarrow{F_1W}\right) \\
&= \left(\overrightarrow{PF_1} - \overrightarrow{F_1W}\right) - \left(\overrightarrow{RF_2} - \overrightarrow{F_2Q}\right) + \left(\overrightarrow{TF_3} - \overrightarrow{F_3S}\right) - \left(\overrightarrow{VF_4} - \overrightarrow{F_4U}\right) \\
&\quad + \left(\overrightarrow{F_1F_2} + \overrightarrow{F_3F_4}\right) - \left(\overrightarrow{F_2F_3} + \overrightarrow{F_4F_1}\right) = \mathbf{0}.
\end{aligned}$$

Alternatively, but equivalently, $\overrightarrow{PQ} + \overrightarrow{TU} = \overrightarrow{RS} + \overrightarrow{VW}$, as required.

Second Approach. This is merely another way of reading the previous argument. Fix an orientation of the line PW, say, from P towards W, and use a lower case letter to denote the coordinate of a point labelled by the corresponding upper case letter.

Since the diagonals of a parallelogram bisect one another, $f_1 + f_3 = f_2 + f_4$, the common value being twice the coordinate of the projection to PW of the point where O_1O_3 and O_2O_4 cross; the relation clearly holds in the degenerate case as well.

Plug $f_1 = \frac{1}{2}(p+w)$, $f_2 = \frac{1}{2}(q+r)$, $f_3 = \frac{1}{2}(s+t)$ and $f_4 = \frac{1}{2}(u+v)$ into the above equality to get $p+w+s+t = q+r+u+v$. Alternatively, but equivalently, $(q-p)+(u-t) = (s-r)+(w-v)$, that is, $PQ + TU = RQ + VW$, as required.

Problem 2. Xenia and Sergey play the following game. Xenia thinks of a positive integer N not exceeding 5000. Then she fixes 20 distinct positive integers $a_1, a_2, \ldots, a_{20}$ such that, for each $k = 1, 2, \ldots, 20$, the numbers N and a_k are congruent modulo k. By a move, Sergey tells Xenia a set S of positive integers not exceeding 20, and she tells him back the set $\{a_k : k \in S\}$ without spelling out which number corresponds to which index. How many moves does Sergey need to determine for sure the number Xenia thought of?

Russia, Sergey Kudrya

Solution. Sergey can determine Xenia's number in 2 but not fewer moves.

We first show that 2 moves are sufficient. Let Sergey provide the set $\{17, 18\}$ on his first move, and the set $\{18, 19\}$ on the second move. In Xenia's two responses, exactly one number occurs twice, namely, a_{18}. Thus, Sergey is able to identify a_{17}, a_{18}, and a_{19}, and thence the residue of N modulo $17 \cdot 18 \cdot 19 = 5814 > 5000$, by the Chinese Remainder Theorem. This means that the given range contains a single number satisfying all congruences, and Sergey achieves his goal.

To show that 1 move is not sufficient, let $M = \operatorname{lcm}(1, 2, \ldots, 10) = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. Notice that M is divisible by the greatest common divisor of every pair of distinct positive integers not exceeding 20. Let Sergey provide the set $S = \{s_1, s_2, \ldots, s_k\}$. We show that there exist pairwise distinct positive integers $b_1, b_2, \ldots, b_k$ such that $1 \equiv b_i \pmod{s_i}$ and $M + 1 \equiv b_{i-1} \pmod{s_i}$ (indices are reduced modulo k). Thus, if in response Xenia provides the set $\{b_1, b_2, \ldots, b_k\}$, then Sergey will be unable to distinguish 1 from $M + 1$, as desired.

To this end, notice that, for each i, the numbers of the form $1 + ms_i$, $m \in \mathbb{Z}$, cover all residues modulo s_{i+1} which are congruent to $1\,(\equiv M + 1)$ modulo $\gcd(s_i, s_{i+1}) \mid M$. Xenia can therefore choose a positive integer b_i such that $b_i \equiv 1 \pmod{s_i}$ and $b_i \equiv M + 1 \pmod{s_{i+1}}$. Clearly, such choices can be performed so as to make the b_i pairwise distinct, as required.

Problem 3. A number of 17 workers stand in a row. Every contiguous group of at least 2 workers is a *brigade*. The chief wants to assign each brigade a leader (which is a member of the brigade) so that each worker's number of assignments is divisible by 4. Prove that the number of such ways to assign the leaders is divisible by 17.

Russia, Mikhail Antipov

Solution. Assume that every single worker also forms a brigade (with a unique possible leader). In this modified setting, we are interested in the number N of ways to assign leadership so that each worker's number of assignments is congruent to 1 modulo 4.

Consider the variables $x_1, x_2, \ldots, x_{17}$ corresponding to the workers. Assign each brigade (from the i-th through the j-th worker) the polynomial $f_{ij} = x_i + x_{i+1} + \cdots + x_j$, and form the product $f = \prod_{1\le i\le j\le 17} f_{ij}$. The number N is the sum $\Sigma(f)$ of the coefficients of all monomials $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_{17}^{\alpha_{17}}$ in the expansion of f, where the α_i are all congruent to 1 modulo 4. For any polynomial P, let $\Sigma(P)$ denote the corresponding sum. From now on, all polynomials are considered with coefficients in the finite field $\mathbb{F}_{17}$.

Recall that for any positive integer n, and any integers $a_1, a_2, \ldots, a_n$, there exist indices $i \le j$ such that $a_i + a_{i+1} + \cdots + a_j$ is divisible by n. Consequently, $f(a_1, a_2, \ldots, a_{17}) = 0$ for all $a_1, a_2, \ldots, a_{17}$ in $\mathbb{F}_{17}$.

Now, if some monomial in the expansion of f is divisible by x_i^{17}, replace that x_i^{17} by x_i; this does not alter the above overall vanishing property (by Fermat's Little Theorem), and preserves $\Sigma(f)$. After several such changes, f transforms into a polynomial g whose degree in each variable does not exceed 16, and $g(a_1, a_2, \ldots, a_{17}) = 0$ for all $a_1, a_2, \ldots, a_{17}$ in $\mathbb{F}_{17}$. For such a polynomial, an easy induction on the number of variables shows that it is identically zero. Consequently, $\Sigma(g) = 0$, so $\Sigma(f) = 0$ as well, as desired.

The 13^{th} Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Consider an integer $n \geq 2$ and write the numbers 1, 2, …, n down on a board. A move consists in erasing any two numbers a and b, and, for each c in $\{a+b, |a-b|\}$, writing c down on the board, unless c is already there; if c is already on the board, do nothing. For all integers $n \geq 2$, determine whether it is possible to be left with exactly two numbers on the board after a finite number of moves.

China

Solution. The answer is in the affirmative for all $n \geq 2$. Induct on n. Leaving aside the trivial case $n = 2$, deal first with particular cases $n = 5$ and $n = 6$.

If $n = 5$, remove first the pair $(2, 5)$, notice that $3 = |2 - 5|$ is already on the board, so $7 = 2 + 5$ alone is written down. Removal of the pair $(3, 4)$ then leaves exactly two numbers on the board, 1 and 7, since $|3 \pm 4|$ are both already there.

If $n = 6$, remove first the pair $(1, 6)$, notice that $5 = |1 - 6|$ is already on the board, so $7 = 1 + 6$ alone is written down. Next, remove the pair $(2, 5)$ and notice that $|2 \pm 5|$ are both already on the board, so no new number is written down. Finally, removal of the pair $(3, 4)$ provides a single number to be written down, $1 = |3 - 4|$, since $7 = 3 + 4$ is already on the board. At this stage, the process comes to an end: 1 and 7 are the two numbers left.

In the remaining cases, the problem for n is brought down to the corresponding problem for $\lceil n/2 \rceil < n$ by a finite number of moves. The conclusion then follows by induction.

Let $n = 4k$ or $4k - 1$, where k is a positive integer. Remove the pairs $(1, 4k-1)$, $(3, 4k-3)$, …, $(2k-1, 2k+1)$ in turn. Each time, two odd numbers are removed, and the corresponding $c = |a \pm b|$ are even numbers in the range 2 through $4k$, of which one is always $4k$. These even numbers are already on the board at each stage, so no c is to be written down, unless $n = 4k - 1$ in which case $4k$ is written down during the first move. The outcome of this k-move round is the string of even numbers 2 through $4k$ written down on the board. At this stage, the problem is clearly brought down to the case where the numbers on the board are 1, 2, …, $2k = \lceil n/2 \rceil$, as desired.

Finally, let $n = 4k + 1$ or $4k + 2$, where $k \geq 2$. Remove first the pair $(4, 2k+1)$ and notice that no new number is to be written down on the board, since $4 + (2k + 1) = 2k + 5 \leq 4k + 1 \leq n$. Next, remove the pairs $(1, 4k + 1)$, $(3, 4k - 1)$, …, $(2k - 1, 2k + 3)$ in turn. As before, at each of these stages, two odd numbers are removed; the corresponding $c = |a \pm b|$ are even numbers, this time in the range 4 through $4k + 2$, of which one is always $4k + 2$; and no new numbers are to be written down on the board, except $4 = |(2k - 1) - (2k + 3)|$ during the last move, and, possibly, $4k + 2 = 1 + (4k + 1)$ during the first move if $n = 4k + 1$. Notice that 2 has not yet been involved in the process, to conclude that the outcome of this $(k + 1)$-move round is the string of even numbers 2 through $4k + 2$ written down on the board. At this stage, the problem is clearly brought down to the case where the numbers on the board are 1, 2, …, $2k + 1 = \lceil n/2 \rceil$, as desired.

Solution 2. We will prove the following, more general statement:

Claim. Write down a finite number (at least two) of pairwise distinct positive integers on a board. A *move* consists in erasing any two numbers a and b, and, for each c in $\{a + b, |a - b|\}$, writing c down on the board, unless c is already there; if c is already on the board, do nothing. Then it is possible to be left with exactly two numbers on the board after a finite number of moves.

Notice that, if we divide all numbers on the board by some common factor, the resulting process goes on equally well. Such a reduction can therefore be performed after any move.

Notice that we cannot be left with less than two numbers. So it suffices to show that, given k positive integers on the board, $k \geq 3$, we can always decrease their number by at least 1. Arguing indirectly, choose a set of $k \geq 3$ positive integers $S = \{a_1, \ldots, a_k\}$ which cannot be reduced in size by a sequence of moves, having a minimal possible sum σ. So, in any sequence of moves applied to S, two numbers are erased and exactly two numbers appear on each move. Moreover, the sum of any resulting set of k numbers is at least σ.

Notice that, given two numbers $a > b$ on the board, we can replace them by $a + b$ and $a - b$, and then, performing a move on the two new numbers, by $(a + b) + (a - b) = 2a$ and $(a + b) - (a - b) = 2b$. So we can double any two numbers on the board.

We now show that, if the board contains two even numbers a and b, we can divide them both by 2, while keeping the other numbers unchanged. If k is even, split the other numbers into pairs to multiply each pair by 2; then clear out the common factor 2. If k is odd, split all numbers but a into pairs to multiply each by 2; then do the same for all numbers but b; finally, clear out the common factor 4.

Back to the problem, if two of the numbers $a_1, \ldots, a_k$ are even, reduce them both by 2 to get a set with a smaller sum, which is impossible. Otherwise, two numbers, say, $a_1 < a_2$, are odd, and we may replace them by the two even numbers $a_1 + a_2$ and $a_2 - a_1$, and then by $\frac{1}{2}(a_1 + a_2)$ and $\frac{1}{2}(a_2 - a_1)$, to get a set with a smaller sum, which is again impossible.

Problem 5. Let n be a positive integer. The kingdom of Zoomtopia is a convex polygon with integer sides, perimeter $6n$, and 60° rotational symmetry (that is, there is a point O such that a 60° rotation about O maps the polygon to itself). In light of the pandemic, the government of Zoomtopia would like to relocate its $3n^2+3n+1$ citizens at $3n^2+3n+1$ points in the kingdom so that every two citizens have a distance of at least 1 for proper social distancing. Prove that this is possible. (The kingdom is assumed to contain its boundary.)

USA, Ankan Bhattacharya

Solution. Let P denote the given polygon, i.e., the kingdom of Zoomtopia. Throughout the solution, we interpret polygons with integer sides and perimeter $6k$ as $6k$-gons with unit sides (some of their angles may equal 180°). The argument hinges on the claim below:

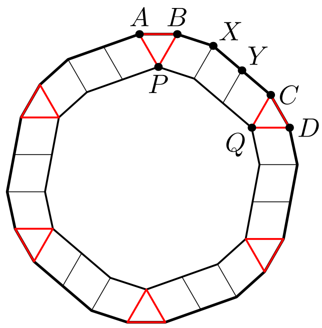
Claim. Let P be a convex polygon satisfying the problem conditions — i.e., it has integer sides, perimeter $6n$, and 60° rotational symmetry. Then P can be tiled with unit equilateral triangles and unit lozenges with angles at least 60°, with tiles meeting completely along edges, so that the tile configuration has a total of exactly $3n^2+3n+1$ distinct vertices.

Proof. Induct on n. The base case, $n=1$, is clear.

Now take a polygon P of perimeter $6n \geq 12$. Place six equilateral triangles inwards on six edges corresponding to each other upon rotation at 60°. It is possible to stick a lozenge to each other edge, as shown in the Figure below.

We show that all angles of the lozenges are at least 60°. Let an edge XY of the polygon bearing some lozenge lie along a boundary segment between edges AB and CD bearing equilateral triangles ABP and CDQ. Then the angle formed by $\overrightarrow{XY}$ and $\overrightarrow{BP}$ is between those formed by $\overrightarrow{AB}$, $\overrightarrow{BP}$ and $\overrightarrow{CD}$, $\overrightarrow{CQ}$, i.e., between 60° and 120°, as desired.

Removing all obtained tiles, we get a 60°-symmetric convex $6(n-1)$-gon with unit sides which can be tiled by the inductive hypothesis. Finally, the number of vertices in the tiling of P is $6n+3(n-1)^2+3(n-1)+1 = 3n^2+3n+1$, as desired.



Using the Claim above, we now show that the citizens may be placed at the $3n^2+3n+1$ tile vertices.

Consider any tile T_1; its vertices are at least 1 apart from each other. Moreover, let BAC be a part of the boundary of some tile T, and let X be any point of the boundary of T, lying outside the half-open intervals $[A, B)$ and $[A, C)$ (in this case, we say that X is not adjacent to A). Then $AX \geq \sqrt{3}/2$.

Now consider any two tile vertices A and B. If they are vertices of the same tile we already know $AB \geq 1$; otherwise, the segment AB crosses the boundaries of some tiles containing A and B at some points X and Y not adjacent to A and B, respectively. Hence $AB \geq AX+YB \geq \sqrt{3} > 1$.

Problem 6. Initially, a non-constant polynomial $S(x)$ with real coefficients is written down on a board. Whenever the board contains a polynomial $P(x)$, not necessarily alone, one can write down on the board any polynomial of the form $P(C+x)$ or $C+P(x)$, where C is a real constant. Moreover, if the board contains two (not necessarily distinct) polynomials $P(x)$ and $Q(x)$, one can write $P(Q(x))$ and $P(x)+Q(x)$ down on the board. No polynomial is ever erased from the board.

Given two sets of real numbers, $A=\{a_1,a_2,\ldots,a_n\}$ and $B=\{b_1,b_2,\ldots,b_n\}$, a polynomial $f(x)$ with real coefficients is (A,B)-*nice* if $f(A)=B$, where $f(A)=\{f(a_i)\colon i=1,2,\ldots,n\}$.

Determine all polynomials $S(x)$ that can initially be written down on the board such that, for any two finite sets A and B of real numbers, with $|A|=|B|$, one can produce an (A,B)-nice polynomial in a finite number of steps.

Iran, Navid Safaei

Solution. The required polynomials are all polynomials of an even degree $d\geq 2$, and all polynomials of odd degree $d\geq 3$ with negative leading coefficient.

Part I. We begin by showing that any (non-constant) polynomial $S(x)$ **not** listed above is not (A,B)-nice for some pair (A,B) with either $|A|=|B|=2$, or $|A|=|B|=3$.

If $S(x)$ is linear, then so are all the polynomials appearing on the board. Therefore, none of them will be (A,B)-nice, say, for $A=\{1,2,3\}$ and $B=\{1,2,4\}$, as desired.

Otherwise, $\deg S=d\geq 3$ is odd, and the leading coefficient is positive. In this case, we make use of the following technical fact, whose proof is presented at the end of the solution.

Claim. There exists a positive constant T such that $S(x)$ satisfies the following condition:

$$S(b)-S(a)\geq b-a \qquad \text{whenever} \quad b-a\geq T. \tag{$*$}$$

Fix a constant T provided by the Claim. Then, an immediate check shows that all newly appearing polynomials on the board also satisfy $(*)$ (with the same value of T). Therefore, none of them will be (A,B)-nice, say, for $A=\{0,T\}$ and $B=\{0,T/2\}$, as desired.

Part II. We show that the polynomials listed in the Answer satisfy the requirements. We will show that for any $a_1<a_2<\cdots<a_n$ and any $b_1\leq b_2\leq\cdots\leq b_n$ there exists a polynomial $f(x)$ satisfying $f(a_i)=b_{\sigma(i)}$ for all $i=1,2,\ldots,n$, where σ is some permutation.

The proof goes by induction on $n\geq 2$. It is based on the following two lemmas, first of which is merely the base case $n=2$; the proofs of the lemmas are also at the end of the solution.

Lemma 1. For any $a_1<a_2$ and any b_1,b_2 one can write down on the board a polynomial $F(x)$ satisfying $F(a_i)=b_i$, $i=1,2$.

Lemma 2. For any distinct numbers $a_1<a_2<\cdots<a_n$ one can produce a polynomial $F(x)$ on the board such that the list $F(a_1)$, $F(a_2)$, …, $F(a_n)$ contains exactly $n-1$ distinct numbers, and $F(a_1)=F(a_2)$.

Now, in order to perform the inductive step, we may replace the polynomial $S(x)$ with its shifted copy $S(C+x)$ so that the values $S(a_i)$ are pairwise distinct. Applying Lemma 2, we get a polynomial $f(x)$ such that only two among the numbers $c_i=f(a_i)$ coincide, namely c_1 and c_2. Now apply Lemma 1 to get a polynomial $g(x)$ such that $g(a_1)=b_1$ and $g(a_2)=b_2$. Apply the inductive hypothesis in order to obtain a polynomial $h(x)$ satisfying $h(c_i)=b_i-g(a_i)$ for all $i=2,3,\ldots,n$. Then the polynomial $h(f(x))+g(x)$ is a desired one; indeed, we have $h(f(a_i))+g(a_i)=h(c_i)+g(a_i)=b_i$ for all $i=2,3,\ldots,n$, and finally $h(f(a_1))+g(a_1)=h(c_1)+g(a_1)=b_2-g(a_2)+g(a_1)=b_1$.

It remains to prove the Claim and the two Lemmas.

Proof of the Claim. There exists some segment $\Delta=[\alpha',\beta']$ such that $S(x)$ is monotone increasing outside that segment. Now one can choose $\alpha\leq\alpha'$ and $\beta\geq\beta'$ such that $S(\alpha)<\min_{x\in\Delta}S(x)$ and

$S(\beta) > \max_{x\in\Delta} S(x)$. Therefore, for any x, y, z with $x \le \alpha \le y \le \beta \le z$ we get $S(x) \le S(\alpha) \le S(y) \le S(\beta) \le S(z)$.

We may decrease α and increase β (preserving the condition above) so that, in addition, $S'(x) > 3$ for all $x \notin [\alpha, \beta]$. Now we claim that the number $T = 3(\beta - \alpha)$ fits the bill.

Indeed, take any a and b with $b - a \ge T$. Even if the segment $[a, b]$ crosses $[\alpha, \beta]$, there still is a segment $[a', b'] \subseteq [a, b] \smallsetminus (\alpha, \beta)$ of length $b' - a' \ge (b - a)/3$. Then

$$S(b) - S(a) \ge S(b') - S(a') = (b' - a') \cdot S'(\xi) \ge 3(b' - a') \ge b - a$$

for some $\xi \in (a', b')$.

Proof of Lemma 1. If $S(x)$ has an even degree, then the polynomial $T(x) = S(x+a_2) - S(x+a_1)$ has an odd degree, hence there exists x_0 with $T(x_0) = S(x_0 + a_2) - S(x_0 + a_1) = b_2 - b_1$. Setting $G(x) = S(x+x_0)$, we see that $G(a_2)-G(a_1) = b_2-b_1$, so a suitable shift $F(x) = G(x)+\big(b_1-G(a_1)\big)$ fits the bill.

Assume now that $S(x)$ has odd degree and a negative leading coefficient. Notice that the polynomial $S^2(x) := S(S(x))$ has an odd degree and a positive leading coefficient. So, the polynomial $S^2(x+a_2) - S^2(x+a_1)$ attains all sufficiently large positive values, while $S(x+a_2) - S(x + a_1)$ attains all sufficiently large negative values. Therefore, the two-variable polynomial $S^2(x + a_2) - S^2(x + a_1) + S(y + a_2) - S(y + a_1)$ attains all real values; in particular, there exist x_0 and y_0 with $S^2(x_0 + a_2) + S(y_0 + a_2) - S^2(x_0 + a_1) - S(y_0 + a_1) = b_2 - b_1$. Setting $G(x) = S^2(x + x_0) + S(x + y_0)$, we see that $G(a_2) - G(a_1) = b_2 - b_1$, so a suitable shift of G fits the bill.

Proof of Lemma 2. Let Δ denote the segment $[a_1; a_n]$. We modify the proof of Lemma 1 in order to obtain a polynomial F convex (or concave) on Δ such that $F(a_1) = F(a_2)$; then F is a desired polynomial. Say that a polynomial $H(x)$ is *good* if H is convex on Δ.

If $\deg S$ is even, and its leading coefficient is positive, then $S(x+c)$ is good for all sufficiently large negative c, and $S(a_2+c) - S(a_1+c)$ attains all sufficiently large negative values for such c. Similarly, $S(x + c)$ is good for all sufficiently large positive c, and $S(a_2 + c) - S(a_1 + c)$ attains all sufficiently large positive values for such c. Therefore, there exist large $c_1 < 0 < c_2$ such that $S(x + c_1) + S(x + c_2)$ is a desired polynomial. If the leading coefficient of H is negative, we similarly find a desired polynomial which is concave on Δ.

If $\deg S \ge 3$ is odd (and the leading coefficient is negative), then $S(x + c)$ is good for all sufficiently large negative c, and $S(a_2+c) - S(a_1+c)$ attains all sufficiently large negative values for such c. Similarly, $S^2(x+c)$ is good for all sufficiently large positive c, and $S^2(a_2+c)-S^2(a_1+c)$ attains all sufficiently large positive values for such c. Therefore, there exist large $c_1 < 0 < c_2$ such that $S(x + c_1) + S^2(x + c_2)$ is a desired polynomial.

Comment. Both parts above allow some variations.

In Part I, the same scheme of the proof works for many conditions similar to $(*)$, e.g.,

$$S(b) - S(a) > T \qquad \text{whenever} \quad b - a > T.$$

Let us sketch an alternative approach for Part II. It suffices to construct, for each i, a polynomial $f_i(x)$ such that $f_i(a_i) = b_i$ and $f_i(a_j) = 0$, $j \ne i$. The construction of such polynomials may be reduced to the construction of those for $n = 3$ similarly to what happens in the proof of Lemma 2. However, this approach (as well as any in this part) needs some care in order to work properly.

The 14^{th} Romanian Master of Mathematics Competition

Day 1: Wednesday, March 1^{st}, 2023, Bucharest

Language: English

Problem 1. Determine all prime numbers p and all positive integers x and y satisfying

$$x^3 + y^3 = p(xy + p).$$

Problem 2. Fix an integer $n \geqslant 3$. Let $\mathcal{S}$ be a set of n points in the plane, no three of which are collinear. Given different points A, B, C in $\mathcal{S}$, the triangle ABC is *nice for AB* if $\text{Area}(ABC) \leqslant \text{Area}(ABX)$ for all X in $\mathcal{S}$ different from A and B. (Note that for a segment AB there could be several nice triangles.) A triangle is *beautiful* if its vertices are all in $\mathcal{S}$ and it is nice for at least two of its sides.

Prove that there are at least $\frac{1}{2}(n-1)$ beautiful triangles.

Problem 3. Let $n \geqslant 2$ be an integer, and let f be a $4n$-variable polynomial with real coefficients. Assume that, for any $2n$ points $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$ in the Cartesian plane, $f(x_1, y_1, \ldots, x_{2n}, y_{2n}) = 0$ if and only if the points form the vertices of a regular $2n$-gon in some order, or are all equal.

Determine the smallest possible degree of f.

(Note, for example, that the degree of the polynomial

$$g(x, y) = 4x^3y^4 + yx + x - 2$$

is 7 because $7 = 3 + 4$.)

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

The 14^{th} Romanian Master of Mathematics Competition

Day 2: Thursday, March 2^{nd}, 2023, Bucharest

Language: English

Problem 4. Given an acute triangle ABC, let H and O be its orthocentre and circumcentre, respectively. Let K be the midpoint of the line segment AH. Also let ℓ be a line through O, and let P and Q be the orthogonal projections of B and C onto ℓ, respectively.

Prove that $KP + KQ \geqslant BC$.

Problem 5. Let $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ be non-constant polynomials with real coefficients such that $P(Q(x)) = R(S(x))$. Suppose that the degree of $P(x)$ is divisible by the degree of $R(x)$.

Prove that there is a polynomial $T(x)$ with real coefficients such that

$$P(x) = R(T(x)).$$

Problem 6. Let r, g, b be non-negative integers. Let Γ be a connected graph on $r + g + b + 1$ vertices. The edges of Γ are each coloured red, green or blue. It turns out that Γ has

- a spanning tree in which exactly r of the edges are red,
- a spanning tree in which exactly g of the edges are green and
- a spanning tree in which exactly b of the edges are blue.

Prove that Γ has a spanning tree in which exactly r of the edges are red, exactly g of the edges are green and exactly b of the edges are blue.

(A *spanning tree* of Γ is a graph which has the same vertices as Γ, with edges which are also edges of Γ, for which there is exactly one path between each pair of different vertices.)

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

Problem 1. Determine all prime numbers p and all positive integers x and y satisfying $x^3+y^3 = p(xy+p)$.

Serbia, Dushan Djukitch

Solution 1. Up to a swap of the first two entries, the only solutions are $(x,y,p) = (1,8,19)$, $(x,y,p) = (2,7,13)$ and $(x,y,p) = (4,5,7)$. The verification is routine.

Set $s = x+y$. Rewrite the equation in the form $s(s^2-3xy) = p(p+xy)$, and express xy:

$$xy = \frac{s^3-p^2}{3s+p}. \qquad (*)$$

In particular,

$$s^2 \geq 4xy = \frac{4(s^3-p^2)}{3s+p},$$

or

$$(s-2p)(s^2+sp+2p^2) \leq p^2-p^3 < 0,$$

so $s < 2p$.

If $p \mid s$, then $s = p$ and $xy = p(p-1)/4$ which is impossible for $x+y = p$ (the equation $t^2-pt+p(p-1)/4 = 0$ has no integer solutions).

If $p \nmid s$, rewrite $(*)$ in the form

$$27xy = (9s^2-3sp+p^2) - \frac{p^2(p+27)}{3s+p}.$$

Since $p \nmid s$, this could be integer only if $3s+p \mid p+27$, and hence $3s+p \mid 27-s$.

If $s \neq 9$, then $|3s-27| \geq 3s+p$, so $27-3s \geq 3s+p$, or $27-p \geq 6s$, whence $s \leq 4$. These cases are ruled out by hand.

If $s = x+y = 9$, then $(*)$ yields $xy = 27-p$. Up to a swap of x and y, all such triples (x,y,p) are $(1,8,19)$, $(2,7,13)$, and $(4,5,7)$.

Solution 2. Set again $s = x+y$. It is readily checked that $s \leq 8$ provides no solutions, so assume $s \geq 9$. Notice that $x^3+y^3 = s(x^2-xy+y^2) \geq \frac{1}{4}s^3$ and $xy \leq \frac{1}{4}s^2$. The condition in the statement then implies $s^2(s-p) \leq 4p^2$, so $s < p+4$.

Notice that p divides one of s and x^2-xy+y^2. The case $p \mid s$ is easily ruled out by the condition $s < p+4$: The latter forces $s = p$, so $x^2-xy+y^2 = xy+p$, i. e., $(x-y)^2 = p$, which is impossible.

Hence $p \mid x^2-xy+y^2$, so $x^2-xy+y^2 = kp$ and $xy+p = ks$ for some positive integer k, implying

$$s^2+3p = k(3s+p). \qquad (**)$$

Recall that $p \nmid s$ to infer that $3k \equiv s \pmod p$. We now present two approaches.

1st Approach. Write $3k = s+mp$ for some integer m and plug $k = \frac{1}{3}(s+mp)$ into $(**)$ to get $s = (9-mp)/(3m+1)$. The condition $s \geq 9$ then forces $m = 0$, so $s = 9$, in which case, up to a swap of the first two entries, the solutions turn out to be $(x,y,p) = (1,8,19)$, $(x,y,p) = (2,7,13)$ and $(x,y,p) = (4,5,7)$.

2nd Approach. Notice that $k = \frac{s^2+3p}{3s+p} = 3 + \frac{s(s-9)}{3s+p} \leq 3+\frac{1}{3}(s-9) = \frac{1}{3}s \leq \frac{1}{3}(p+3)$, since $s < p+4$. Hence $3k \leq p+3$, and the congruence $3k \equiv s \pmod p$ then forces either $3k = s-p$ or $3k = s$.

The case $3k = s-p$ is easily ruled out: Otherwise, $(**)$ boils down to $2s+p+9 = 0$, which is clearly impossible.

Finally, if $3k = s$, then $(**)$ reduces to $s = 9$. In this case, up to a swap of the first two entries, the only solutions are $(x,y,p) = (1,8,19)$, $(x,y,p) = (2,7,13)$ and $(x,y,p) = (4,5,7)$.

Remark. The upper bound for k can equally well be established by considering the variation of $(s^2+3p)/(3s+p)$ for $1 \leq s \leq p+3$. The maximum is achieved at $s = p+3$:

$$\max_{1\leq s\leq p+3} \frac{s^2+3p}{3s+p} = \frac{(p+3)^2+3p}{3(p+3)+p} = \frac{p^2+9p+9}{4p+9} < \frac{p+4}{3},$$

so the integer $3k \leq p+3 < p+9 \leq p+s$ and the remainder of the proof now goes along the few final lines above.

Problem 2. Fix an integer $n \geqslant 3$. Let $\mathcal{S}$ be a set of n points in the plane, no three of which are collinear. Given different points A, B, C in $\mathcal{S}$, the triangle ABC is *nice for AB* if $\text{Area}(ABC) \leqslant \text{Area}(ABX)$ for all X in $\mathcal{S}$ different from A and B. (Note that for a segment AB there could be several nice triangles.) A triangle is *beautiful* if its vertices are all in $\mathcal{S}$ and it is nice for at least two of its sides.

Prove that there are at least $\frac{1}{2}(n-1)$ beautiful triangles.

Bulgaria, Alexander Ivanov

Solution. For convenience, a triangle whose vertices all lie in $\mathcal{S}$ will be referred to as a triangle in $\mathcal{S}$. The argument hinges on the following observation:

Given any partition of $\mathcal{S}$, amongst all triangles in $\mathcal{S}$ with at least one vertex in each part, those of minimal area are all adequate.

Indeed, amongst the triangles under consideration, one of minimal area is suitable for both sides with endpoints in different parts.

We now present two approaches for the lower bound.

1st Approach. By the above observation, the 3-uniform hypergraph of adequate triangles is connected. It is a well-known fact that such a hypergraph has at least $\frac{1}{2}(n-1)$ hyperedges, whence the required lower bound

2nd Approach. For a partition $\mathcal{S} = \mathcal{A} \sqcup \mathcal{B}$, an area minimising triangle as above will be called $(\mathcal{A}, \mathcal{B})$-minimal. Thus, $(\mathcal{A}, \mathcal{B})$-minimal triangles are all adequate.

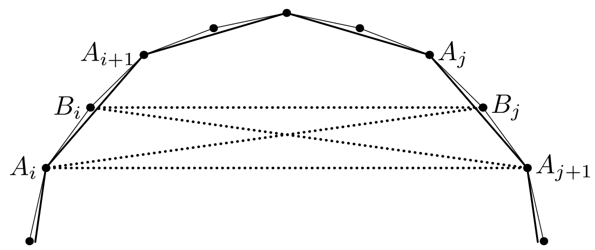
Consider now a partition of $\mathcal{S} = \mathcal{A} \sqcup \mathcal{B}$, where $|\mathcal{A}| = 1$. Choose an $(\mathcal{A}, \mathcal{B})$-minimal triangle and add to $\mathcal{A}$ its vertices from $\mathcal{B}$ to obtain a new partition also written $\mathcal{S} = \mathcal{A} \sqcup \mathcal{B}$. Continuing, choose an $(\mathcal{A}, \mathcal{B})$-minimal triangle and add to $\mathcal{A}$ its vertices/vertex from $\mathcal{B}$ and so on and so forth all the way down for at least another $\frac{1}{2}(n-5)$ steps — this works at least as many times, since at each step, $\mathcal{B}$ loses at most two points. Clearly, each step provides a new adequate triangle, so the overall number of adequate triangles is at least $\frac{1}{2}(n-1)$, as required.

Remark. In fact, $\lfloor n/2 \rfloor$ is the smallest possible number of adequate triangles, as shown by the the configurations described below.

Let first $n = 2k-1$. Consider a regular n-gon $\mathcal{P} = A_1A_2 \dots A_n$. Choose a point B_i on the perpendicular bisector of A_iA_{i+1} outside $\mathcal{P}$ and sufficiently close to the segment A_iA_{i+1}. We claim that there are exactly $k-1 = \lfloor n/2 \rfloor$ adequate triangles in the set

$$\mathcal{S} = \{A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_{k-1}\}.$$

Notice here that the arc $A_1A_2 \dots A_k$ is less than half of the circumcircle of $\mathcal{P}$, so the angles $\angle A_uA_vA_w$, $1 \le u < v < w \le k$, are all obtuse.



To prove the claim, list the suitable triangles for each segment.

For segments A_iA_{i+1}, A_iB_i, and B_iA_{i+1}, it is $A_iB_iA_{i+1}$.

For segment A_iA_{j+1}, $j \ge i+1$, those are $A_iB_iA_{j+1}$ and $A_iB_jA_{j+1}$.

For segment A_iB_j, $j \ge i+1$, it is $A_iB_iB_j$.

For segment B_iA_{j+1}, $j \ge i+1$, it is $B_iB_jA_{j+1}$.

For segment B_iB_j, $i < j$, those are $B_iA_{i+1}B_j$ and $B_iA_jB_j$.

It is easily seen that the only triangles occurring twice are $A_iB_iA_{i+1}$, hence they are the only adequate triangles.

For $n = 2k-2$, just remove A_k from the above example. This removes the adequate triangle $A_{k-1}B_{k-1}A_k$ and provides only one new such instead, namely, $B_{k-2}A_{k-1}B_{k-1}$. Consequently, there are exactly $k-1 = \lfloor n/2 \rfloor$ adequate triangles in the set

$$\mathcal{S} = \{A_1, A_2, \dots, A_{k-1}, B_1, B_2, \dots, B_{k-1}\}.$$

Problem 3. Let $n \geqslant 2$ be an integer, and let f be a $4n$-variable polynomial with real coefficients. Assume that, for any $2n$ points (x_1, y_1), $\dots$, (x_{2n}, y_{2n}) in the plane, $f(x_1, y_1, \dots, x_{2n}, y_{2n}) = 0$ if and only if the points form the vertices of a regular $2n$-gon in some order, or are all equal.

Determine the smallest possible degree of f.

USA

Solution. The smallest possible degree is $2n$. In what follows, we will frequently write $A_i = (x_i, y_i)$, and abbreviate $P(x_1, y_1, \dots, x_{2n}, y_{2n})$ to $P(A_1, \dots, A_{2n})$ or as a function of any $2n$ points.

Suppose that f is valid. First, we note a key property:

Claim (Sign of f). f attains wither only nonnegative values, or only nonpositive values.

Proof. This follows from the fact that the zero-set of f is very sparse: if f takes on a positive and a negative value, we can move A_1, $\dots$, A_{2n} from the negative value to the positive value without ever having them form a regular $2n$-gon — a contradiction. □

The strategy for showing $\deg f \geq 2n$ is the following. We will animate the points A_1, $\dots$, A_{2n} linearly in a variable t; then $g(t) = f(A_1, \dots, A_{2n})$ will have degree at most $\deg f$ (assuming it is not zero). The claim above then establishes that any root of g must be a multiple root, so if we can show that there are at least n roots, we will have shown $\deg g \geq 2n$, and so $\deg f \geq 2n$.

Geometrically, our goal is to exhibit $2n$ linearly moving points so that they form a regular $2n$-gon a total of n times, but not always form one.

We will do this as follows. Draw n *mirrors* through the origin, as lines making angles of $\frac{\pi}{n}$ with each other. Then, any point P has a total of $2n$ reflections in the mirrors, as shown below for $n = 5$. (Some of these reflections may overlap.)

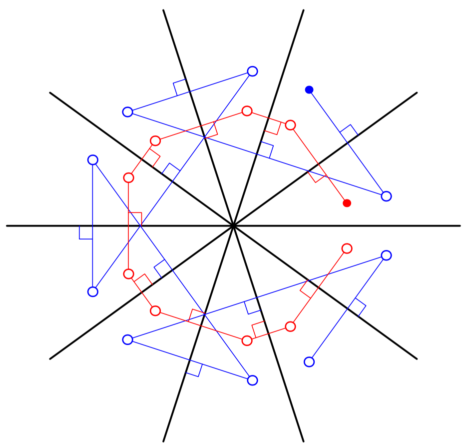
Draw the n angle bisectors of adjacent mirrors. Observe that the reflections of P form a regular $2n$-gon if and only if P lies on one of the bisectors.

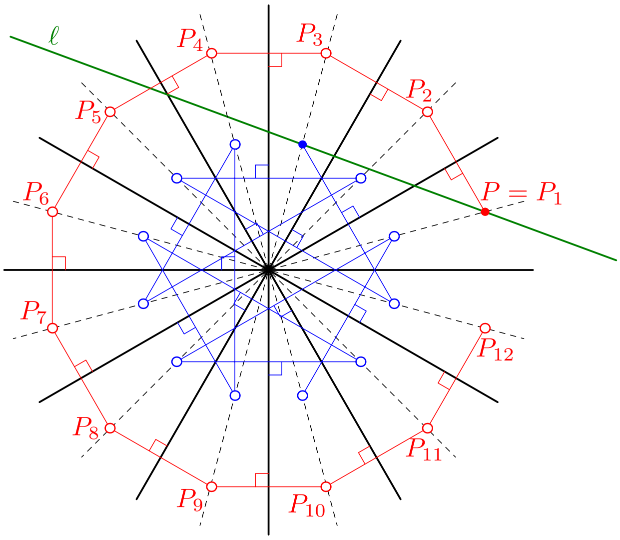
We will animate P on any line ℓ which intersects all n bisectors (but does not pass through the origin), and let P_1, $\dots$, P_{2n} be its reflections. Clearly, these are also all linearly animated, and because of the reasons above, they will form a regular $2n$-gon exactly n times, when ℓ meets each bisector. So this establishes $\deg f \geq 2n$ for the reasons described previously.

Now we pass to constructing a polynomial f of degree $2n$ having the desired property. First of all, we will instead find a polynomial g which has this property, but only when points with sum zero are input. This still solves the problem, because then we can choose

$$f(A_1, A_2, \dots, A_{2n}) = g(A_1 - \bar{A}, \dots, A_{2n} - \bar{A}),$$

where $\bar{A}$ is the centroid of A_1, $\dots$, A_{2n}. This has the upshot that we can now always assume $A_1 + \dots + A_{2n} = 0$, which will simplify the ensuing discussion.





We will now construct a suitable g as a sum of squares. This means that, if we write $g = g_1^2 + g_2^2 + \dots + g_m^2$, then $g = 0$ if and only if $g_1 = \dots = g_m = 0$, and that if their degrees are d_1, $\dots$, d_m, then g has degree at most $2\max(d_1, \dots, d_m)$.

Thus, it is sufficient to exhibit several polynomials, all of degree at most n, such that $2n$ points with zero sum are the vertices of a regular $2n$-gon if and only if the polynomials are all zero at those points.

First, we will impose the constraints that all $|A_i|^2 = x_i^2 + y_i^2$ are equal. This uses multiple degree 2 constraints.

Now, we may assume that the points $A_1, \ldots, A_{2n}$ all lie on a circle with centre 0, and $A_1 + \cdots + A_{2n} = 0$. If this circle has radius 0, then all A_i coincide, and we may ignore this case.

Otherwise, the circle has positive radius. We will use the following lemma.

Lemma. Suppose that $a_1, \ldots, a_{2n}$ are complex numbers of the same non-zero magnitude, and suppose that $a_1^k + \cdots + a_{2n}^k = 0$, $k = 1, \ldots, n$. Then $a_1, \ldots, a_{2n}$ form a regular $2n$-gon centred at the origin. (Conversely, this is easily seen to be sufficient.)

Proof. Since all the hypotheses are homogenous, we may assume (mostly for convenience) that $a_1, \ldots, a_{2n}$ lie on the unit circle. By Newton's sums, the k-th symmetric sums of $a_1, \ldots, a_{2n}$ are all zero for k in the range $1, \ldots, n$.

Taking conjugates yields $a_1^{-k} + \cdots + a_{2n}^{-k} = 0$, $k = 1, \ldots, n$. Thus, we can repeat the above logic to obtain that the k-th symmetric sums of $a_1^{-1}, \ldots, a_{2n}^{-1}$ are also all zero for $k = 1, \ldots, n$. However, these are simply the $(2n - k)$-th symmetric sums of $a_1, \ldots, a_{2n}$ (divided by $a_1 \cdots a_{2n}$), so the first $2n-1$ symmetric sums of $a_1, \ldots, a_{2n}$ are all zero. This implies that $a_1, \ldots, a_{2n}$ form a regular $2n$-gon centred at the origin. □

We will encode all of these constraints into our polynomial. More explicitly, write $a_r = x_r + y_r i$; then the constraint $a_1^k + \cdots + a_{2n}^k = 0$ can be expressed as $p_k + q_k i = 0$, where p_k and q_k are real polynomials in the coordinates. To incorporate this, simply impose the constraints $p_k = 0$ and $q_k = 0$; these are conditions of degree $k \leq n$, so their squares are all of degree at most $2n$.

To recap, taking the sum of squares of all of these constraints gives a polynomial f of degree at most $2n$ which works whenever $A_1 + \cdots + A_{2n} = 0$. Finally, the centroid-shifting trick gives a polynomial which works in general, as wanted.

Remark 1. Here is a more detailed approach of the mirror-reflection argument. Let $re^{i\theta}$ be the polar representation of the point P. The polar representations of its mirrored images are then

$$re^{i\theta}, \quad re^{-i\theta}, \quad re^{i\left(\frac{2\pi}{n}+\theta\right)}, \quad re^{i\left(\frac{2\pi}{n}-\theta\right)},$$
$$\ldots, \quad re^{i\left(\frac{2(n-1)\pi}{n}+\theta\right)}, \quad re^{i\left(\frac{2(n-1)\pi}{n}-\theta\right)}.$$

Clearly, they are all linear with respect to P and lie on the circle of radius r centred at the origin. As listed above, the $2n$ images are not necessarily in circular order around the circle. For convenience, assume $0 \leq \theta \leq \frac{\pi}{n}$, so the list now displays them in circular order. These images form the vertices of a regular $2n$-gon if and only if the angle between every two consecutive terms in the list (read circularly) is $\frac{\pi}{n}$. This is clearly the case if and only if $\theta = \frac{\pi}{2n}$. Consequently, the images are the vertices of a regular $2n$-gon if and only if P lies on the internal bisector of the angle formed by some pair of consecutive mirrors.

Remark 2. We sketch here some versions of the arguments in the solution above.

To show that $\deg f \geq 2n$, we use the same constancy of sign claim and the convention that the polynomial is a function of points (= pairs of coordinates) $A_1, A_2, \ldots, A_{2n}$. Assume that the values of f are all non-negative.

Write $B(\phi) = (\cos\phi, \sin\phi)$. Choose a substitution $A_{2i-1} = B\left((2i-1)\frac{\pi}{n} + \phi\right)$ and $A_{2i} = B\left(2i\frac{\pi}{n} - \phi\right)$, $i = 1, 2, \ldots, n$. Notice that the coordinates of the points $A_1, A_2, \ldots, A_{2n}$ are all linear functions in $c = \cos\phi$ and $s = \sin\phi$, so, substituting these expressions into f, we get a polynomial $g(c, s)$ with $\deg g \leq \deg f$.

Now, the values of g are all non-negative (each being one of f), and, on the circle $c^2 + s^2 = 1$, it vanishes at exactly $2n$ points, namely, $(c, s) = \left(\cos\frac{\pi}{n}k, \sin\frac{\pi}{n}k\right)$, $k = 1, \ldots, 2n$. We show that these properties already yield $\deg g \geq 2n$.

Obviously, if $g(c, s)$ possesses the properties listed above, then so does $g(c, -s)$, and hence so does $\bar{g}(c, s) = g(c, s) + g(c, -s)$.

The polynomial $\bar{g}$ is even in s, so it in fact depends only on s^2, and we may plug $s^2 = 1 - c^2$ into it, to obtain a polynomial $h(c)$ with $\deg h \leq \deg g$ which is non-negative on $[-1, 1]$ and vanishes on this segment exactly at $c = \cos\frac{\pi}{n}k$. These are $n+1$ such points, and, except $c = \pm 1$, they should all be roots of h of even multiplicity, due to sign conservation. All in all, this provides $2n$ roots of h, counted with multiplicity, hence $\deg f \geq \deg g \geq \deg h \geq 2n$, as desired.

For a bit alternative construction of a suitable f, one may notice that the Lemma in the above solution can be changed to impose vanishing of the elementary symmetric polynomials $\sigma_i(a_1, a_2, \ldots, a_{2n})$, $i = 1, 2, \ldots, n$, instead of Newton sums. Indeed, if the σ_i all vanish, then so do the polynomials

$$\sigma_i(\bar{a}_1, a_2, \ldots, \bar{a}_{2n}) = \frac{|a_1|^{2i}\sigma_{2n-i}(a_1, a_2, \ldots, a_{2n})}{\bar{a}_1\bar{a}_2 \ldots \bar{a}_{2n}},$$

so $\sigma_i(a_1, \ldots, a_{2n})$ also vanishes for $i = n+1, \ldots, 2n-1$. Hence $a_1, a_2, \ldots, a_{2n}$ are the roots of $z^{2n} - |a_1|^n$, as desired.

Problem 4. Given a triangle ABC, let H and O be its orthocentre and circumcentre, respectively. Let K be the midpoint of the line segment AH. Let further ℓ be a line through O, and let P and Q be the orthogonal projections of B and C onto ℓ, respectively. Prove that $KP+KQ \geq BC$.

Russia, Vasily Mokin

Solution 1. Fix the origin at O and the real axis along ℓ. A lower case letter denotes the complex coordinate of the corresponding point in the configuration. For convenience, let $|a| = |b| = |c| = 1$.

Clearly, $k = a + \frac{1}{2}(b+c)$, $p = a + \frac{1}{2}\left(b + \frac{1}{b}\right)$ and $q = a + \frac{1}{2}\left(c + \frac{1}{c}\right)$.

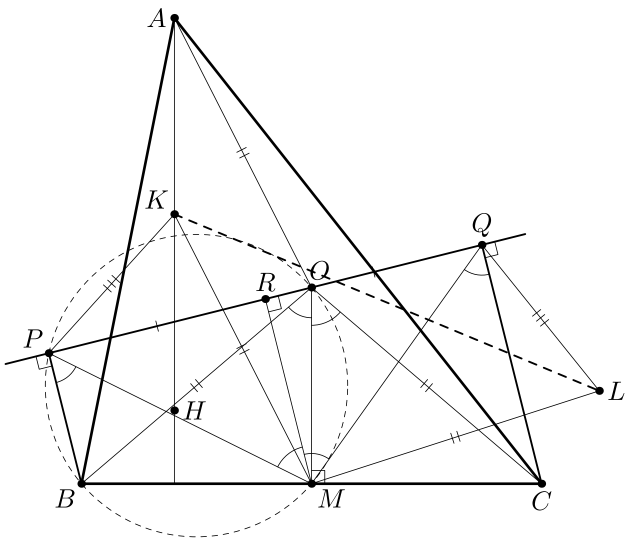
Then $|k-p| = \left|a + \frac{1}{2}\left(c - \frac{1}{b}\right)\right| = \frac{1}{2}|2ab + bc - 1|$, since $|b| = 1$.

Similarly, $|k-q| = \frac{1}{2}|2ac + bc - 1|$, so, since $|a| = 1$,

$$\begin{aligned}|k-p| + |k-q| &= \frac{1}{2}|2ab+bc-1| + \frac{1}{2}|2ac+bc-1| \\ &\geq \frac{1}{2}|2a(b-c)| = |b-c|,\end{aligned}$$

as required.

Solution 2. Let M be the midpoint of BC, and let R be the projection of M onto ℓ. In other words, R is the midpoint of PQ. Since $\angle BPO = \angle BMO = 90^\circ$, the points B, P, O, and M are concyclic, so $\angle(OM, OB) = \angle(PM, PB) = \angle(PM, MR)$, so the right triangles MRP and OMB are similar and have different orientation. Similarly, the triangles MRQ and OMC are similar and have different orientation, hence so are the triangles OBC and MPQ.



Recall that $\overrightarrow{AH} = 2\overrightarrow{OM}$, so $\overrightarrow{OM} = \overrightarrow{AK}$. Hence $AOMK$ is a parallelogram, so $MK = OA = OB = OC$.

Consider the rotation through $\angle(\overrightarrow{OC}, \overrightarrow{OB})$ about M. It maps P to Q; let it map K to some point L. Then $MK = ML = OB = OC$ and $\angle LMK = \angle BOC$, so the triangles OBC and MKL are congruent. Hence $BC = KL \leq KQ + LQ = KQ + KP$, as required.

Solution 3. Let $\alpha = \angle(PB, BC) = \angle(QC, BC)$. Since P lies on the circle of diameter OB, $\angle(OP, OM) = \alpha$. Since also Q lies on the circle of diameter OC, it immediately follows that $MP = MQ = R\sin\alpha$ by sine theorem in triangles $\triangle OPM$ and $\triangle OQM$.

Because PQ is the projection of BC on line ℓ, it follows that $PQ = BC\sin\alpha$. Just like in the first solution, $KM = AO = R$ (the circumradius of triangle $\triangle ABC$).

Now apply Ptolemy's inequality for the quadrilateral $KPMQ$: $KP \cdot MQ + KQ \cdot MP \geq PQ \cdot KM$, and now substitute the relations from above, leading to

$$R\sin\alpha(KP + KQ) \geq R\sin\alpha \cdot BC,$$

which is precisely the conclusion whenever $\sin\alpha \neq 0$. The case when $\sin\alpha = 0$ can be treated either directly, or via a limit argument.

Solution 4. Denote by R and O the circumradius and the circumcentre of triangle ABC, respectively. As in Solution 1, we see that $MK = R$.

Assume now that ℓ is fixed, while A moves along the fixed circle (ABC). Then K will move along a cricle centred at M with radius R. We must show that for each point K on this circle we have $BC \leq KP + KQ$. In doing so, we prove that the afore-mentioned circle contains an ellipse with *foci* at Q and P with distance BC.

Let S be the foot of the perpendicular from M to PQ, it is easy to verify that S is the center of the ellipse. We shall then consider it as the origin. Let $u = \frac{BC}{2}$ and $t = \frac{PQ}{2}$; notice that u is the major semi-axis of the ellipse and $\sqrt{u^2 - t^2}$ is the minor one. Assume $X(x, y)$ is a point on this ellipse. We now need to prove $MX \leq R$.

Since X is on the ellipse, we can write $(x, y) = (u\cos\theta, \sqrt{u^2 - t^2}\sin\theta)$, for some $\theta \in (0, 2\pi)$. Since $MX^2 = x^2 + (y + MS)^2$, we can expand and obtain

$$MX^2 = u^2 + MS^2 - t^2 \cdot \sin^2\theta + 2MS \cdot \sqrt{u^2 - t^2} \cdot \sin\theta.$$

Add and subtract $MS^2(u^2 - t^2)/t^2$ in order to obtain a square on the right hand side: $MX^2 = u^2 + MS^2 + \frac{MS^2(u^2-t^2)}{t^2} - \left(t\sin\theta - \frac{MS\sqrt{u^2-t^2}}{t}\right)^2$. It now suffices to show that $u^2 + MS^2 + \dfrac{MS^2(u^2 - t^2)}{t^2} = R^2$, since then it would immediately follow that $MX^2 \leq R^2$.

Applying *Pythagorean* theorem in triangles OBM and OSM, we obtain $R^2 = u^2 + OM^2$ and $OM^2 = MS^2 + OS^2$, so it remains to prove that $OS^2 = \frac{MS^2(u^2-t^2)}{t^2}$. Let $\alpha = \angle(OP, BM)$, then $OS/MS = \tan\alpha$ and $t/u = \cos\alpha$, so $OS^2 = MS^2\tan^2\alpha = MS^2\left(\frac{1-\cos^2\alpha}{\cos^2\alpha}\right) = MS^2 \cdot \frac{u^2-t^2}{t^2}$, which is the desired result.

Problem 5. Let $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ be non-constant polynomials with real coefficients such that $P(Q(x)) = R(S(x))$. Suppose that the degree of $P(x)$ is divisible by the degree of $R(x)$.

Prove that there is a polynomial $T(x)$ with real coefficients such that $P(x) = R(T(x))$.

Iran, Navid Safaei

Solution 1. Degree comparison of $P(Q(x))$ and $R(S(x))$ implies that $q = \deg Q \mid \deg S = s$. We will show that $S(x) = T(Q(x))$ for some polynomial T. Then $P(Q(x)) = R(S(x)) = R(T(Q(x)))$, so the polynomial $P(t) - R(T(t))$ vanishes upon substitution $t = S(x)$; it therefore vanishes identically, as desired.

Choose the polynomials $T(x)$ and $M(x)$ such that

$$S(x) = T(Q(x)) + M(x), \qquad (*)$$

where $\deg M$ is minimised; if $M = 0$, then we get the desired result. For the sake of contradiction, suppose $M \neq 0$. Then $q \nmid m = \deg M$; otherwise, $M(x) = \beta Q(x)^{m/q} + M_1(x)$, where β is some number and $\deg M_1 < \deg M$, contradicting the choice of M. In particular, $0 < m < s$ and hence $\deg T(Q(x)) = s$.

Substitute now $(*)$ into $R(S(x)) - P(Q(x)) = 0$; let α be the leading coefficient of $R(x)$ and let $r = \deg R(x)$. Expand the brackets to get a sum of powers of $Q(x)$ and other terms including powers of $M(x)$ as well. Amongst the latter, the unique term of highest degree is $\alpha r M(x) T(Q(x))^{r-1}$. So, for some polynomial $N(x)$, $N(Q(x)) = \alpha r M(x) T(Q(x))^{r-1} +$ a polynomial of lower degree.

This is impossible, since q divides the degree of the left-hand member, but not that of the right-hand member.

Solution 2. All polynomials in the solution have real coefficients. As usual, the degree of a polynomial $f(x)$ is denoted $\deg f(x)$.

Of all pairs of polynomials $P(x)$, $R(x)$, satisfying the conditions in the statement, choose one, say, $P_0(x)$, $R_0(x)$, so that $P_0(Q(x)) = R_0(S(x))$ has a minimal (positive) degree. We will show that $\deg R_0(x) = 1$, say, $R_0(x) = \alpha x + \beta$ for some real numbers $\alpha \neq 0$ and β, so $P_0(Q(x)) = \alpha S(x) + \beta$. Hence $S(x) = T(Q(x))$ for some polynomial $T(x)$.

Now, if $P(x)$ and $R(x)$ are polynomials satisfying $P(Q(x)) = R(S(x))$, then $P(Q(x)) = R(T(Q(x)))$. Since $Q(x)$ is not constant, it takes infinitely many values, so $P(x)$ and $R(T(x))$ agree at infinitely many points, implying that $P(x) = R(T(x))$, as required.

It is therefore sufficient to solve the problem in the particular case where $F(x) = P(Q(x)) = R(S(x))$ has a minimal degree. Let $d = \gcd(\deg Q(x), \deg S(x))$ to write $\deg Q(x) = ad$ and $\deg S(x) = bd$, where $\gcd(a, b) = 1$. Then $\deg P(x) = bc$, $\deg R(x) = ac$ and $\deg F(x) = abcd$ for some positive integer c. We will show that minimality of $\deg F(x)$ forces $c = 1$, so $\deg P(x) = b$, $\deg R(x) = a$ and $\deg F(x) = abd$. The conditions $a = \deg R(x) \mid \deg P(x) = b$ and $\gcd(a, b) = 1$ then force $a = 1$, as stated above.

Consequently, the only thing we are left with is the proof of the fact that $c = 1$. For convenience, we may and will assume that $P(x)$, $Q(x)$, $R(x)$, $S(x)$ are all monic; hence so is $F(x)$. The argument hinges on the lemma below.

Lemma. If $f(x)$ is a monic polynomial of degree mn, then there exists a degree n monic polynomial $g(x)$ such that $\deg\big(f(x) - g(x)^m\big) < (m-1)n$. (If $m = 0$ or 1, or $n = 0$, the conclusion is still consistent with the usual convention that the identically zero polynomial has degree $-\infty$.)

Proof. Write $f(x) = \sum_{k=0}^{mn} \alpha_k x^k$, $\alpha_{mn} = 1$, and seek $g(x) = \sum_{k=0}^{n} \beta_k x^k$, $\beta_n = 1$, so as to fit the bill. To this end, notice that, for each positive integer $k \leq n$, the coefficient of x^{mn-k} in the expansion of $g(x)^m$ is of the form $m\beta_{n-k} + \varphi_k(\beta_n, \ldots, \beta_{n-k+1})$, where $\varphi_k(\beta_n, \ldots, \beta_{n-k+1})$ is an algebraic expression in $\beta_n, \ldots, \beta_{n-k+1}$. Recall that $\beta_n = 1$ to determine the β_{n-k} recursively by requiring $\beta_{n-k} = \frac{1}{m}\big(a_{mn-k} - \varphi_k(\beta_n, \ldots, \beta_{n-k+1})\big)$, $k = 1, \ldots, n$.

The outcome is then the desired polynomial $g(x)$.

We are now in a position to prove that $c = 1$. Suppose, if possible, that $c > 1$. By the lemma, there exist monic polynomials $U(x)$ and $V(x)$ of degree b and a, respectively, such that $\deg\big(P(x) - U(x)^c\big) < (c-1)b$ and $\deg\big(R(x) - V(x)^c\big) < (c-1)a$. Then $\deg\big(F(x) - U(Q(x))^c\big) = \deg\big(P(Q(x)) - U(Q(x))^c\big) < (c-1)abd$, $\deg\big(F(x) - V(S(x))^c\big) = \deg\big(R(S(x)) - V(S(x))^c\big) < (c-1)abd$, so $\deg\big(U(Q(x))^c - V(S(x))^c\big) = \deg\Big(\big(F(x) - V(S(x))^c\big) - \big(F(x) - U(Q(x))^c\big)\Big) < (c-1)abd$.

On the other hand, $U(Q(x))^c - V(S(x))^c = \big(U(Q(x)) - V(S(x))\big)\big(U(Q(x))^{c-1} + \cdots + V(S(x))^{c-1}\big)$.

By the preceding, the degree of the left-hand member is (strictly) less than $(c-1)abd$ which is precisely the degree of the second factor in the right-hand member. This forces $U(Q(x)) = V(S(x))$, so $U(Q(x)) = V(S(x))$ has degree $abd < abcd = \deg F(x)$ — a contradiction. Consequently, $c = 1$. This completes the argument and concludes the proof.

Problem 6. Let r, g, b be non-negative integers. Let Γ be a connected graph on $r+g+b+1$ vertices. The edges of Γ are each coloured red, green or blue. It turns out that Γ has

- a spanning tree in which exactly r of the edges are red,
- a spanning tree in which exactly g of the edges are green and
- a spanning tree in which exactly b of the edges are blue.

Prove that Γ has a spanning tree in which exactly r of the edges are red, exactly g of the edges are green and exactly b of the edges are blue.

Russia, Vasily Mokin

Solution 1. Induct on $n = r + g + b$. The base case, $n = 1$, is clear.

Let now $n > 1$. Let V denote the vertex set of Γ, and let T_r, T_g, and T_b be the trees with exactly r red edges, g green edges, and b blue edges, respectively. Consider two cases.

Case 1: There exists a partition $V = A \sqcup B$ of the vertex set into two non-empty parts such that the edges joining the parts all bear the same colour, say, blue.

Since Γ is connected, it has a (necessarily blue) edge connecting A and B. Let e be one such.

Assume that T, one of the three trees, does not contain e. Then the graph $T \cup \{e\}$ has a cycle C through e. The cycle C should contain another edge e' connecting A and B; the edge e' is also blue. Replace e' by e in T to get another tree T' with the same number of edges of each colour as in T, but containing e.

Performing such an operation to all three trees, we arrive at the situation where the three trees T'_r, T'_g, and T'_b all contain e. Now shrink e by identifying its endpoints to obtain a graph Γ^*, and set $r^* = r$, $g^* = g$, and $b^* = b - 1$. The new graph satisfies the conditions in the statement for those new values — indeed, under the shrinking, each of the trees T'_r, T'_g, and T'_b loses a blue edge. So Γ^* has a spanning tree with exactly r red, exactly g green, and exactly $b - 1$ blue edges. Finally, pass back to Γ by restoring e, to obtain the a desired spanning tree in Γ.

Case 2: There is no such a partition.

Consider all possible collections (R, G, B), where R, G and B are acyclic sets consisting of r red edges, g green edges, and b blue edges, respectively. By the problem assumptions, there is at least one such collection. Amongst all such collections, consider one such that the graph on V with edge set $R \cup G \cup B$ has the smallest number k of components. If $k = 1$, then the collection provides the edges of a desired tree (the number of edges is one less than the number of vertices).

Assume now that $k \geq 2$; then in the resulting graph some component K contains a cycle C. Since R, G, and B are acyclic, C contains edges of at least two colours, say, red and green. By assumption, the edges joining $V(K)$ to $V \setminus V(K)$ bear at least two colours; so one of these edges is either red or green. Without loss of generality, consider a red such edge e.

Let e' be a red edge in C and set $R' = R \setminus \{e'\} \cup \{e\}$. Then (R', G, B) is a valid collection providing a smaller number of components. This contradicts minimality of the choice above and concludes the proof.

Problem 6. Let r, g, b be non-negative integers. Let Γ be a connected graph on $r+g+b+1$ vertices. The edges of Γ are each coloured red, green or blue. It turns out that Γ has

- a spanning tree in which exactly r of the edges are red,
- a spanning tree in which exactly g of the edges are green and
- a spanning tree in which exactly b of the edges are blue.

Prove that Γ has a spanning tree in which exactly r of the edges are red, exactly g of the edges are green and exactly b of the edges are blue.

Russia, Vasily Mokin

Solution 2. For a spanning tree T in Γ, denote by $r(T)$, $g(T)$, and $b(T)$ the number of red, green, and blue edges in T, respectively.

Assume that $\mathcal{C}$ is some collection of spanning trees in Γ. Write

$$r(\mathcal{C}) = \min_{T\in\mathcal{C}} r(T), \quad g(\mathcal{C}) = \min_{T\in\mathcal{C}} g(T),$$
$$b(\mathcal{C}) = \min_{T\in\mathcal{C}} b(T), \quad R(\mathcal{C}) = \max_{T\in\mathcal{C}}(T),$$
$$G(\mathcal{C}) = \max_{T\in\mathcal{C}} g(T), \quad B(\mathcal{C}) = \max_{T\in\mathcal{C}} b(T).$$

Say that a collection $\mathcal{C}$ is *good* if $r \in [r(\mathcal{C}, R(\mathcal{C})]$, $g \in [g(\mathcal{C}, G(\mathcal{C})]$, and $b \in [b(\mathcal{C}, B(\mathcal{C})]$. By the problem conditions, the collection of all spanning trees in Γ is good.

For a good collection $\mathcal{C}$, say that an edge e of Γ is *suspicious* if e belongs to some tree in $\mathcal{C}$ but not to all trees in $\mathcal{C}$. Choose now a good collection $\mathcal{C}$ minimizing the number of suspicious edges. If $\mathcal{C}$ contains a desired tree, we are done. Otherwise, without loss of generality, $r(\mathcal{C}) < r$ and $G(\mathcal{C}) > g$.

We now distinguish two cases.

Case 1: $B(\mathcal{C}) = b$.

Let T^0 be a tree in $\mathcal{C}$ with $g(T^0) = g(\mathcal{C}) \le g$. Since $G(\mathcal{C}) > g$, there exists a green edge e contained in some tree in $\mathcal{C}$ but not in T^0; clearly, e is suspicious. Fix one such green edge e.

Now, for every T in $\mathcal{C}$, define a spanning tree T_1 of Γ as follows. If T does not contain e, then $T_1 = T$; in particular, $(T^0)_1 = T^0$. Otherwise, the graph $T \setminus \{e\}$ falls into two components. The tree T_0 contains some edge e' joining those components; this edge is necessarily suspicious. Choose one such edge and define $T_1 = T \setminus \{e\} \cup \{e'\}$.

Let $\mathcal{C}_1 = \{T_1 : T \in \mathcal{C}\}$. All edges suspicious for $\mathcal{C}_1$ are also suspicious for $\mathcal{C}$, but no tree in $\mathcal{C}_1$ contains e. So the number of suspicious edges for $\mathcal{C}_1$ is strictly smaller than that for $\mathcal{C}$.

We now show that $\mathcal{C}_1$ is good, reaching thereby a contradiction with the choice of $\mathcal{C}$. For every T in $\mathcal{C}$, the tree T_1 either coincides with T or is obtained from it by removing a green edge and adding an edge of some colour. This already shows that $g(\mathcal{C}_1) \le g(\mathcal{C}) \le g$, $G(\mathcal{C}_1) \ge G(\mathcal{C}) - 1 \ge g$, $R(\mathcal{C}_1) \ge R(\mathcal{C}) \ge r$, $r(\mathcal{C}_1) \le r(\mathcal{C}) + 1 \le r$, and $B(\mathcal{C}_1) \ge B(\mathcal{C}) \ge b$. Finally, we get $b(T^0) \le B(\mathcal{C}) = b$; since $\mathcal{C}_1$ contains T^0, it follows that $b(\mathcal{C}_1) \le b(T^0) \le b$, which concludes the proof.

Case 2: $B(\mathcal{C}) > b$.

Consider a tree T^0 in $\mathcal{C}$ satisfying $r(T^0) = R(\mathcal{C}) \ge r$. Since $r(\mathcal{C}) < r$, the tree T^0 contains a suspicious red edge. Fix one such edge e.

Now, for every T in $\mathcal{C}$, define a spanning tree T_2 of Γ as follows. If T contains e, then $T_2 = T$; in particular, $(T^0)_2 = T^0$. Otherwise, the graph $T \cup \{e\}$ contains a cycle C through e. This cycle contains an edge e' absent from T^0 (otherwise T^0 would contain the cycle C), so e' is suspicious. Choose one such edge and define $T_2 = T \setminus \{e'\} \cup \{e\}$.

Let $\mathcal{C}_2 = \{T_2 : T \in \mathcal{C}\}$. All edges suspicious for $\mathcal{C}_2$ are also suspicious for $\mathcal{C}$, but all trees in $\mathcal{C}_2$ contain e. So the number of suspicious edges for $\mathcal{C}_2$ is strictly smaller than that for $\mathcal{C}$.

We now show that $\mathcal{C}_2$ is good, reaching again a contradiction. For every T in $\mathcal{C}$, the tree T_2 either coincides with T or is obtained from it by removing some edge and adding a red edge. This shows that $r(\mathcal{C}_2) \le r(\mathcal{C}) + 1 \le r$, $R(\mathcal{C}_2) \ge R(\mathcal{C}) \ge r$, $G(\mathcal{C}_2) \ge G(\mathcal{C}) - 1 \ge g$, $g(\mathcal{C}_2) \le g(\mathcal{C}) \le g$, $b(\mathcal{C}_1) \le b(\mathcal{C}) \le b$ and $B(\mathcal{C}_2) \ge B(\mathcal{C}) - 1 \ge b$. This concludes the proof.

The 15^{th} Romanian Master of Mathematics Competition

Day 1: Wednesday, February 28^{th}, 2024, Bucharest

Language: English

Problem 1. Let n be a positive integer. Initially, a bishop is placed in each square of the top row of a $2^n \times 2^n$ chessboard; those bishops are numbered from 1 to 2^n, from left to right. A *jump* is a simultaneous move made by all bishops such that the following conditions are satisfied:

- each bishop moves diagonally, in a straight line, some number of squares, and
- at the end of the jump, the bishops all stand in different squares of the same row.

Find the total number of permutations σ of the numbers $1, 2, \ldots, 2^n$ with the following property: There exists a sequence of jumps such that all bishops end up on the bottom row arranged in the order $\sigma(1), \sigma(2), \ldots, \sigma(2^n)$, from left to right.

Problem 2. Consider an odd prime p and a positive integer $N < 50p$. Let $a_1, a_2, \ldots, a_N$ be a list of positive integers less than p such that any specific value occurs at most $\frac{51}{100}N$ times and $a_1 + a_2 + \cdots + a_N$ is not divisible by p. Prove that there exists a permutation $b_1, b_2, \ldots, b_N$ of the a_i such that, for all $k = 1, 2, \ldots, N$, the sum $b_1 + b_2 + \cdots + b_k$ is not divisible by p.

Problem 3. Given a positive integer n, a set $\mathcal{S}$ is n*-admissible* if

- each element of $\mathcal{S}$ is an unordered triple of integers in $\{1, 2, \ldots, n\}$,
- $|\mathcal{S}| = n - 2$, and
- for each $1 \leq k \leq n-2$ and each choice of k distinct $A_1, A_2, \ldots, A_k \in \mathcal{S}$,

$$|A_1 \cup A_2 \cup \cdots \cup A_k| \geq k + 2.$$

Is it true that, for all $n > 3$ and for each n-admissible set $\mathcal{S}$, there exist pairwise distinct points $P_1, \ldots, P_n$ in the plane such that the angles of the triangle $P_iP_jP_k$ are all less than 61° for any triple $\{i, j, k\}$ in $\mathcal{S}$?

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

The 15^{th} Romanian Master of Mathematics Competition

Day 2: Thursday, February 29^{th}, 2024, Bucharest

Language: English

Problem 4. Fix integers a and b greater than 1. For any positive integer n, let r_n be the (non-negative) remainder that b^n leaves upon division by a^n. Assume there exists a positive integer N such that $r_n < 2^n/n$ for all integers $n \geq N$. Prove that a divides b.

Problem 5. Let BC be a fixed segment in the plane, and let A be a variable point in the plane not on the line BC. Distinct points X and Y are chosen on the rays $\overrightarrow{CA}$ and $\overrightarrow{BA}$, respectively, such that $\angle CBX = \angle YCB = \angle BAC$. Assume that the tangents to the circumcircle of ABC at B and C meet line XY at P and Q, respectively, such that the points X, P, Y, and Q are pairwise distinct and lie on the same side of BC. Let Ω_1 be the circle through X and P centred on BC. Similarly, let Ω_2 be the circle through Y and Q centred on BC. Prove that Ω_1 and Ω_2 intersect at two fixed points as A varies.

Problem 6. A polynomial P with integer coefficients is *square-free* if it is not expressible in the form $P = Q^2R$, where Q and R are polynomials with integer coefficients and Q is not constant. For a positive integer n, let $\mathcal{P}_n$ be the set of polynomials of the form

$$1 + a_1x + a_2x^2 + \cdots + a_nx^n$$

with $a_1, a_2, \ldots, a_n \in \{0, 1\}$. Prove that there exists an integer N so that, for all integers $n \geq N$, more than 99% of the polynomials in $\mathcal{P}_n$ are square-free.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

The 15^{th} Romanian Master of Mathematics Competition

Day 1 — Solutions

Problem 1. Let n be a positive integer. Initially, in each square of the top row on the $2^n \times 2^n$ chessboard, a bishop is placed; those bishops are numbered from 1 to 2^n, from left to right. A *jump* is a simultaneous move made by all bishops such that the following conditions are satisfied:

Each bishop moves diagonally any number of squares; and

At the end of the jump, the bishops all stand in different squares of the same row.

Find the total number of permutations σ (of numbers $1, 2, \ldots, 2^n$) with the following property: There exists a sequence of jumps such that all bishops end up on the bottom row arranged in the order $\sigma(1)$, $\sigma(2)$, $\ldots$, $\sigma(2^n)$, from left to right.

Israel

Solution 1. The required number is 2^{n-1}. On a jump, every bishop moves the same number of rows up or down; call this number of rows the *length* of the jump.

Step 1. We show that the length of any jump is of the form 2^d for some integer $d \leq n-1$. Assign each bishop the number of the column it is situated on before the jump. Let k be the length of the jump; then each bishop's column number either increases by k, or decreases by k in the jump.

Thus, bishops $1, 2, \ldots, k$ should move to columns $k+1, k+2, \ldots, 2k$, as they cannot move leftwards. On the other hand, after the jump columns $1, 2, \ldots, k$ should be filled by the bishops $k+1, k+2, \ldots, 2k$. So the leftmost $2k$ bishops still fill the columns $1, 2, \ldots, 2k$ after the jump.

Repeating the argument shows that the next k bishops move rightwards, and the next k bishops beyond move leftwards, and so on and so forth. Finally, the bishops all split into contiguous groups of length $2k$, and in each group the leftmost k bishops move rightwards, whereas the rightmost k bishops move leftwards. Hence $2k \mid 2^n$, so k is indeed of the form 2^d with $d \leq n-1$.

Step 2. To make a more explicit description of the column change during the jump, assign each column the n-digit binary expansion of less 1 its number, augmented with zeroes leftwards if necessary. It is then easily seen that a jump of length 2^d just switches the d-th digit from the left, 0 to 1 and vice versa.

Thus, the resulting permutation also has the following form: For every $d = 0, 1, \ldots, n-1$, the d-th digit is either swapped for all bishops, or it is preserved for them all.

Moreover, notice that the total length of all jumps is odd, so there will be an odd number of jumps of length 1. Hence the 0-th (the rightmost) digit will be switched anyway. This leaves the room for 2^{n-1} possible permutations.

Step 3. It remains to show that all 2^{n-1} permutations are indeed possible. Let us show how to reach any of them.

Start by getting to the bottom row by downward jumps of lengths $1, 2, 4, \ldots, 2^{n-1}$ that will switch all n digits.

Now, if we want to switch the i-th digit back, $1 \leq i \leq n-1$, make two upward jumps of length 2^{i-1}, followed by a downward jump of length 2^i. Combine such modifications for all possible digit combinations to get all desired permutations.

Solution 2. Proceed until the end of Step 1 just like in the first solution. Then extend the board to a vertical strip of width 2^n, this will not affect the result, as it will be seen at the end of the proof.

We will show that any two jumps commute. Consider two jumps of length p and q with $p < q$, and call them the p-jump and the q-jump. As described in the first step, the bishops will be split in contiguous groups. For the p-jump, we look at groups of length p, call these p-groups; for the q-jump, we look at groups of length q, call these q-groups.

Since $2p$ divides q, any p-group is fully contained in a single q-group, and a q-group contains an even number of p-groups. First, let's look at the first two q-groups, and denote by $g_1, \ldots, g_{2k}$ the $p-$groups contained in the first q-group, and $g'_1, \ldots, g'_{2k}$ the p-groups contained in the second q-group, where $2k = q/p$. The p-jump will swap any g_{2i-1} with g_{2i}, and same for their g' counterparts, whereas a q-jump will swap g_j with g'_j. When putting these together, it follows that applying both jumps in either order gives the same result: g_{2i-1} is swapped with g'_{2i} and g_{2i} is swapped with g'_{2i-1}.

Repeat now the same argument for the next two q-groups, and so on and so forth, until the entire row will be accounted for.

We will now establish a bijection from the odd numbers between 1 and $2^n - 1$ to the desired permutations. Let $x = \sum 2^k a_k$ be an odd number between 1 and $2^n - 1$, where each a_k is either 0 or 1, in particular $a_0 = 1$. Perform a_k jumps of length 2^k in increasing order of k, then perform $2^n - 1 - x$ additional jumps of length 1 in order to reach the final row. This will result in a permutation σ, and set $f(x) = \sigma$. This provides a well defined function f.

To prove f injective, it suffices to look at bishop numbered 1 and show that it will end up in position $x + 1$. For any of jump of length 2^k, this bishop will move rightwards, as its position just before the jump was $1 + a_0 + 2a_1 + \ldots + 2^{k-1}a_{k-1} \leq 2^k$. Therefore, before the additional jumps of length 1, this bishop will reach position $x + 1$. Any two jumps of same length cancel each other out, and there is an even number of additional jumps of length 1, so the final position will also be $x + 1$. Consequently, f is injective.

To prove f surjective, consider a σ with the desired property, let b_k be the number of jumps of length 2^k, and let a_k be the remainder of b_k modulo 2. Since the total length of all jumps is odd, there has to be an odd number of jumps of length 1, so $a_0 = 1$. Let $x = \sum 2^k a_k$, and note that this is an odd number between 1 and $2^n - 1$. From Step 2 in Solution 1, the order of the jumps does not matter. Since two consecutive jumps of same length cancel each other out, performing a_k jumps of length 2^k is the same as performing b_k jumps of length 2^k. So $f(x) = \sigma$, as the $2^n - 1 - x$ additional jumps of length 1 at the end also cancel each other out.

Solution 3. Run again Step 1 through in Solution 1.

Look at the two *halves* $1, \ldots, 2^{n-1}$ and $2^{n-1} + 1, \ldots, 2^n$ and let $h(i) = i \pm 2^{n-1}$ be the counterpart of i in the other half, where the sign is chosen appropriately. A jump of length 2^{n-1} will swap the halves between themselves, so i will be swapped with $h(i)$. From Step 1, it follows that any shorter jump will only perform swaps inside a single half, and will act the same way on the other half; specifically, if a jump swaps i with j, it will also swap $h(i)$ with $h(j)$.

Furthermore, applying two jumps of 2^{n-1} will just cancel each other out, regardless of any other jumps in between, because we swapped i with $h(i)$ twice, and the inner configuration of each half is changed in the same way.

Induct now on n. There are 2^{n-2} possible permutations for the $2^{n-1} \times 2^{n-1}$ board. Performing jumps of the same length on a $2^n \times 2^n$ board gives the same configuration in each of the two halves. Now we can either apply a jump of length 2^{n-1}, which will swap the halves, or we can apply two jumps of length 2^{n-2}, which will cancel each other. This provides a construction for 2^{n-1} permutations in the $2^n \times 2^n$ board.

To show that these are the only ones, consider now a valid permutation for the $2^n \times 2^n$ board which is obtained from some jumps. First, discard the jumps of length 2^{n-1}, then attempt to apply the rest on the $2^{n-1} \times 2^{n-1}$ board. If a jump would exit the board, then make the jump of the same length in the opposite direction instead, which will stay on the board because its length is at most 2^{n-2}. Since the length is the same, the resulting bishop configuration is also the same. Since the total length of the moves executed so far is odd, we can make an even number of moves of length 1 in order to reach the final row of the $2^{n-1} \times 2^{n-1}$ board, and this will not change the configuration in the end. Therefore, we obtain a corresponding permutation for the $n-1$ case which describes the configuration in each of the halves. From there, the only variations are whether the halves are swapped or not, depending on whether the number of jumps of length 2^{n-1} was odd or even. So this valid permutation corresponds to one constructed in the earlier paragraph, which completes the induction.

Problem 2. Consider an odd prime p and an integer $N < 50p$. Let $a_1, a_2, \ldots, a_N$ be a list of positive integers less than p such that any specific value occurs at most $\frac{51}{100}N$ times and $a_1 + a_2 + \cdots + a_N$ is not divisible by p. Prove that there exists a permutation $b_1, b_2, \ldots, b_N$ of the a_i such that $b_1 + b_2 + \cdots + b_k$ is not divisible by p for all $k = 1, 2, \ldots, N$.

United Kingdom, Will Steinberg

Solution 1. The argument hinges on the lemma below.

Lemma. *Let n be a positive integer and let $c_1, c_2, \ldots, c_n$ be a list of positive integers less than p such that each specific value occurs at most $\frac{1}{2}(n+1)$ times. Fix a residue $r \not\equiv c_1 + c_2 + \cdots + c_n \pmod p$. Then there exists a permutation $d_1, d_2, \ldots, d_n$ of the c_i such that $r \not\equiv d_1 + d_2 + \cdots + d_k \pmod p$ for all $k = 1, 2, \ldots, n$.*

Proof. Induct on n. The base case, $n = 1$, is clear, so let $n \geq 2$. Consider the residue a that occurs the most times amongst the c_i.

If $a \not\equiv r \pmod p$, set $d_1 = a$ and complete the rest of the list using the inductive hypothesis with r replaced by $r - a$, as any residue will occur amongst the remaining c_i at most $\frac{1}{2}\big((n-1)+1\big)$ times. Indeed, if no other residue occurs as many times as a, then the number of occurrences of any residue amongst the remaining c_i is at most $\frac{1}{2}(n+1) - 1 = \frac{1}{2}(n-1) < \frac{1}{2}\big((n-1)+1\big)$. Otherwise, if there is another residue that occurs as many times as a, their number of occurrences has to be at most $\frac{1}{2}n = \frac{1}{2}\big((n-1)+1\big)$.

If $a \equiv r \pmod p$, choose a residue $b \not\equiv a \pmod p$ amongst the c_i; the choice is possible, as $\frac{1}{2}(n+1) < n$. Set $d_1 = b$ and $d_2 = a$, noting that $d_1 + d_2 = b + a \equiv r + b \not\equiv r \pmod p$. If no other residue occurs as many times as a, then each residue occurs amongst the remaining c_i at most $\frac{1}{2}(n+1) - 1 = \frac{1}{2}(n-1) = \frac{1}{2}\big((n-2)+1\big)$ times. If a occurs at most $\frac{1}{2}(n-1)$ times, then clearly the same will hold for any residue in the remaining c_i. The remaining possibility is that a and another residue both occur $\frac{1}{2}n$ times and n is even, meaning that that the other residue has to be b, and there are no other residues; it is clear that the occurrences in the remaining c_i are precisely $\frac{1}{2}(n-2) < \frac{1}{2}\big((n-2)+1\big)$. The inductive hypothesis then applies with r replaced by $-b$ to complete the list. This establishes the lemma.

Back to the problem, if each residue occurs at most $\frac{1}{2}(N+1)$ times, the conclusion follows by the Lemma.

Otherwise, there is exactly one residue a that occurs $M > \frac{1}{2}(N+1)$ times amongst the a_i. Note that $2M - N > (N+1) - N = 1$, to set $b_i = a$, $i = 1, 2, \ldots, 2M - N$. Letting $\alpha = 50$, note also that $2M - N \leq 2 \cdot \frac{\alpha+1}{2\alpha}N - N = \frac{1}{\alpha}N < p$, so none of the first $2M - N$ partial sums is divisible by p, as p is prime.

To complete the proof, apply the Lemma with $r = (N - 2M)a$ to the remaining a_i. There are left $N - (2M - N) = 2(N - M)$ such, a occurs $M - (2M - N) = N - M$ times amongst these and any other residue occurs at most $N - M$ times, both of which do not exceed $\frac{1}{2}\big(2(N-M)+1\big)$.

Solution 2. The permutation will be constructed step by step, first choosing b_1, then b_2, and so on. At every step, sort the remaining values by their number of appearances from most frequent to least frequent, and from largest to smallest for the situations where the number of appearances is the same. After that, attempt to select the first value from the sorted list; if this results in a partial sum that is divisible by p, then attempt to select the second value from the list, which will result in a partial sum that is not divisible by p as the first and second values are different modulo p. Label the step as type (I) if the first value has been selected and as type (II) if the second value has been selected.

Lemma. Any step of type (II) is followed by a step of type (I).

Proof. Suppose that step j is of type (II). Denote the first two values according to the sorting order with a and b, and let s be the partial sum until this step. Since step j is type (II), $s + a \equiv 0$

(mod p). At step $j+1$, the first value according to the sorting order will still be a. Since $s+b+a \not\equiv 0 \pmod{p}$, a can be selected for step $j+1$, so step $j+1$ is of type (I).

Now suppose, for the sake of contradiction, that this procedure fails. Let $k+1 < N$ be the first step when neither (I) nor (II) is possible. In particular, this means that there is a single value a remaining, and it occurs $N-k$ times. Let j be the smallest number with the following property: at *every* step between j and $k+1$, the value a is the first one according to the sorting order; in particular, a is the most frequently occurring one. Such a j exists because $k+1$ satisfies this property.

If $j > 1$, then at step $j-1$ there is another value b that comes before a in the sorted order. Suppose that at step $j-1$ there are q values of a remaining; then there are at least q values of b remaining. According to the lemma, at least half of the steps $j, j+1, \ldots, k$ are of type (I), which means that a is selected in them. Denote $\ell = k-j+1$, then a has been selected at least $\lfloor \ell/2 \rfloor$ steps between j and k. Since at step $k+1$, there are $N-k \geq 2$ values of a remaining, it means that $q \geq \lfloor \ell/2 \rfloor + 2$. At the same time, at step $k+1$ there are no values of b remaining, so between steps j and k all the remaining b values have been selected, and this can happen at most $\ell - \lfloor \ell/2 \rfloor$ times. Combining these two leads to $\ell - \lfloor \ell/2 \rfloor \geq q \geq \lfloor \ell/2 \rfloor + 2$, or $\ell \geq 2\lfloor \ell/2 \rfloor + 2$, a contradiction.

If $j = 1$, then a is always the first in the sorted order. Therefore, the first $p-1$ steps are of type (I), and step p is of type (II). If $k \geq p$, from steps $p+1$ to k, at least every other step is of type (I), meaning that a is selected. Once step k is done, there are $N-k$ remaining values of a. Therefore, in the beginning, there were at least

$$(p-1) + \frac{k-p}{2} + N - k = N - \frac{k}{2} + \frac{p}{2} - 1 > N - \frac{N-2}{2} + \frac{N}{2\alpha} - 1 = \frac{\alpha+1}{2\alpha} N$$

values of a, contradiction. In the situation when $k < p$, there are at least $N-1$ values of a, which is also a contradiction.

Problem 3. Fix an integer $n > 3$ and let $N = \{1, 2, \dots, n\}$. Let $\mathcal{S}$ be a set of $n-2$ pairwise distinct 3-element subsets of N such that $|A_1 \cup A_2 \cup \cdots \cup A_k| \geq k+2$ for any $A_1, A_2, \dots, A_k$ in $\mathcal{S}$ and any $k = 1, 2, \dots, n-2$. Are there n pairwise distinct points $p_1, p_2, \dots, p_n$ in the plane such that the angles of the triangle $p_i p_j p_k$ are all less than 61° for any set $\{i, j, k\}$ in $\mathcal{S}$?

Russia, Ivan Frolov

Solution. The answer is in the affirmative. Note that the condition on $\mathcal{S}$ may be rephrased as follows: For any subset M of N of size $|M| \geq 2$, there are at most $|M|-2$ sets in $\mathcal{S} \cap \mathcal{P}_3(M)$, where $\mathcal{P}_3(M)$ is the set of 3-element subsets of M.

Extend the problem to $n \geq 1$ and proceed by induction on n; for formal correctness, assume that $|\mathcal{S}| \leq \max(0, n-2)$. The cases $n = 1$, 2 are both trivial.

Look upon points in the plane as complex numbers. Begin by considering a collection of points $q_1, q_2, \dots, q_n$, not necessarily distinct, yet not all identical. Fix $q_n = 0$ and associate with each $\{i, j, k\}$ in $\mathcal{S}$ an equation $q_j - q_i = \omega \cdot (q_k - q_i)$, where $\omega = e^{i2\pi/3}$. This equation ensures that q_i,, q_j, q_k either are the vertices of a clockwise oriented non-degenerate equilateral triangle or they all coincide.

We then get a system of $n-2$ linear equations with $n-1$ complex variables $q_1, q_2, \dots, q_{n-1}$. It is well-known that such a system has a non-trivial solution $(q_1, q_2, \dots, q_{n-1}, q_n = 0)$. However, some of the q_i may coincide. So N splits into some classes, $N = N_1 \sqcup \cdots \sqcup N_m$, such that the points q_i and q_j coincide if and only if i and j share the same class and the size of each class is less than n. Note that the elements of any triple in $\mathcal{S}$ either all lie in the same class or no two lie in the same class.

Clearly, each $\mathcal{S}_i = \mathcal{S} \cap \mathcal{P}_3(N_i)$ satisfies the conditions in the statement relative to N_i of size less than n. By the inductive hypothesis, for each $i = 1, 2, \dots, m$, there exists a point-configuration $\{r_{ij} \colon j \in N_i\}$ satisfying the corresponding requirements. For each $j = 1, 2, \dots, n$, choose the index i of the class N_i containing j, and let $p_j = q_j + \varepsilon r_{ij}$, where ε is small enough.

If the elements of a triple $\{i, j, k\}$ in $\mathcal{S}$ lie in three different classes, then the angles of triangle $p_i p_j p_k$ are close to those of triangle $q_i q_j q_k$, provided that ε is small enough, so the triangle satisfies the required angle-condition. Otherwise, the three elements all lie in some class N_ℓ, and the angles of $p_i p_j p_k$ equal those of $r_{\ell i} r_{\ell j} r_{\ell k}$, so they satisfy the requirements by the inductive hypothesis.

The 15$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Fix integers a and b greater than 1. For any positive integer n, let r_n be the (non-negative) remainder b^n leaves upon division by a^n. Assume that there exists a positive integer N such that $r_n < 2^n/n$ for all integers $n \geq N$. Prove that a divides b.

Iran, Pouria Mahmoudkhan Shirazi

Solution 1. Arguing indirectly, assume that $a \nmid b$, so $r_n \neq 0$ for all n. Let $M = \max(b, N)$.

We now prove that $r_{n+1} \geq br_n$ for all $n \geq M$. Indeed, as $r_n < 2^n/b \leq a^n/b$, it follows that $br_n < a^n$ and $b^{n+1} \equiv br_n \pmod{a^n}$. Therefore, br_n is the remainder b^{n+1} leaves upon division by a^n, i.e., br_n is the smallest non-negative integer r such that $a^n \mid b^{n+1} - r$. This implies $br_n \leq r_{n+1}$, as $a^{n+1} \mid b^{n+1} - r_{n+1}$.

To complete the solution, note that $r_M \geq 1$, so $r_{M+k} \geq b^k r_M \geq 2^k$ for all $k \geq 0$. On the other hand, $r_{M+k} < 2^{M+k}/(M+k) < 2^k$ for k sufficiently large. This is a contradiction.

Solution 2. The argument hinges on the lemma below:

Lemma. *Consider two integers $b > a > 1$. If a does not divide b, then $\{b^n/a^n\} > 1/b$ for infinitely many positive integers n; as usual, $\{x\}$ denotes the fractional part of the real number x.*

Proof. For every positive integer n, write $x_n = \lfloor b^n/a^n \rfloor$ and $y_n = \{b^n/a^n\}$ and note that $by_n - ay_{n+1} = ax_{n+1} - bx_n$ is an integer.

Suppose now, if possible, that $y_n \leq 1/b$ for all $n \geq M$. Consider any such n and note that $y_n > 0$, as a does not divide b, so $-1 < -a/b < by_n - b/a \leq by_n - ay_{n+1} \leq 1 - ay_{n+1} < 1$. Hence the integer $by_n - ay_{n+1} = 0$, so $y_{n+1} = (b/a)y_n$.

Consequently, $y_n = (b/a)^{n-M} y_M$ for all $n \geq M$. As $b > a$, it then follows that $y_n \geq 1$ for all large enough n, contradicting the fact that $y_n < 1$ whatever n. This establishes the lemma.

Back to the problem, suppose a does not divide b. Then $b > a$; otherwise $2^n/n > r_n = b^n \geq 2^n$ which is impossible. Note that $a^n\{b^n/a^n\} = r_n < 2^n/n$ for all large enough n. The lemma then implies $1/b < \{b^n/a^n\} < (1/n)(2/a)^n \leq 1/n$ for infinitely many n which is clearly a contradiction.

Remark. The conclusion holds under the more general assumption that $r_n < 2^n/f(n)$, where f is any given function satisfying $\lim_{n\to\infty} f(n) = \infty$.

Problem 5. Let BC be a fixed segment in the plane, and let A be a variable point in that plane outside the line BC. Points X and Y are chosen on the rays CA (emanating from C) and BA (emanating from B), respectively, such that $\angle CBX = \angle YCB = \angle BAC$. Assume that the tangents to the circumcircle of ABC at B and C cross XY at$P$ and Q, respectively. Let Ω_1 be the circle through X and P centred on BC. Similarly, let Ω_2 be the circle through Y and Q centred on BC. Prove that Ω_1 and Ω_2 intersect at two fixed points as A varies.

Denmark, Daniel Pham Nguyen

Solution 1. All angles in the solution are oriented. We will prove that the two intersection points are points D and D' such that the triangles BCD and BCD' are equilateral.

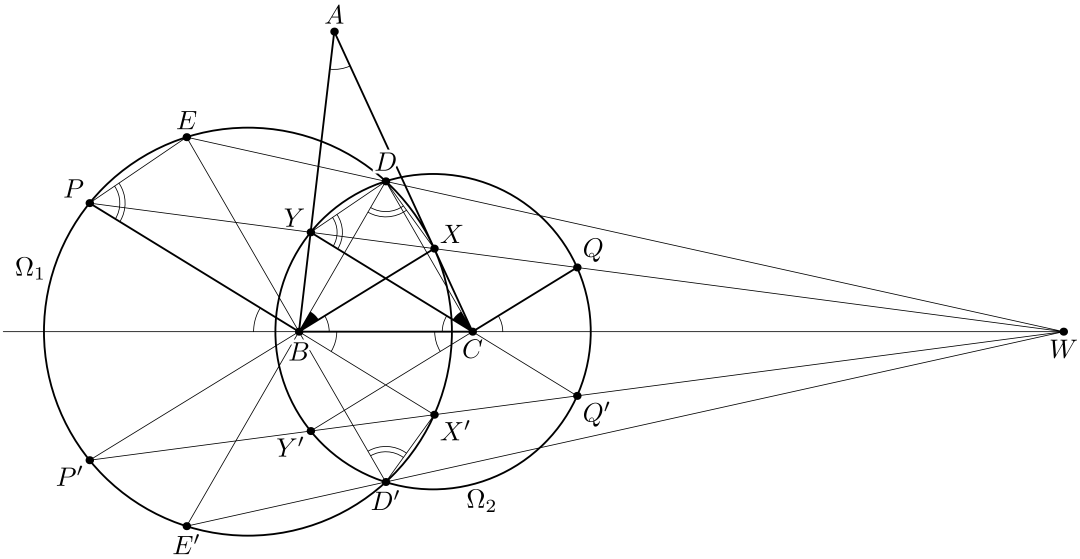
Let X', Y', P', and Q' be the reflections across BC of X, Y, P, and Q, respectively. Then Ω_1 and Ω_2 are just the circles $PXX'P'$ and $QYY'Q'$, respectively.

Denote $\alpha = \angle BAC = \angle CBX = \angle YCB$. Let XY cross BC at W; the case $XY \parallel BC$ may be treated as a limit case. The symmetry yields that W also lies on the line $X'Y'$. The same symmetry, along with tangency of PB and QC to the circle ABC, yields

$$\alpha = \angle X'BC = \angle PBW = \angle WCQ = \angle BCY'. \qquad (*)$$

This yields that each of the triples (P, B, X'), (Q, C, Y'), (P', B, X), and (Q', C, Y) is the collinear, and, moreover, that $PBX' \parallel Q'CY$ and $P'BX \parallel QCY'$. It follows now that quadrilaterals $PXX'P'$ and $YQQ'Y'$ are homothetic at W. Therefore, so are Ω_1 and Ω_2.

Let now Ω_1 nd Ω_2 cross at D and D'. Let WD and WD' meet Ω_1 again at E and E'. Since $W = PX \cap P'X'$ and $B = PX' \cap P'X$, the point B lies on the polar of W with respect to Ω_1. In other words, W and B are inverse with respect to that circle. This yields that the lines DE' and $D'E$ also cross at B.



Now, we have $\angle BDX = \angle E'DX = \angle X'D'E = \angle X'PE = \angle BPE = \angle CYD$ (the last equality holds by means of homothety). Similarly, we have $\angle DXB = \angle DXP' = \angle PX'D' = \angle PED' = \angle PEB = \angle YDC$. Therefore, the triangles BDX and CYD are similar. Firstly, this yields that $\angle DBC = \angle DBX + \angle XBC = \angle YCD + \angle BCY = \angle BCD$, whence $BD = CD$. Secondly, this also implies that $BD/BX = CY/CD$, or $BX \cdot CY = BD \cdot CD = BD^2$. But the triangles BXC and CBY are also similar (as both are similar to ABC), so $BX/BC = BC/CY$, ot $BX \cdot CY = BC^2$. Thus, $BC = BD = CD$, and the triangle BCD is equilateral. This finishes the solution.

Remark. The fact that B and W are inverse to each other with respect to Ω_1 can be obtained (and implemented) in different ways. Two useful conditions equivalent to this fact are: the points P, B, and X are concyclic with the centre O_1 of Ω_1; and Ω_1 is the Apollonius circle with respect to the segment BW; the details follow below.

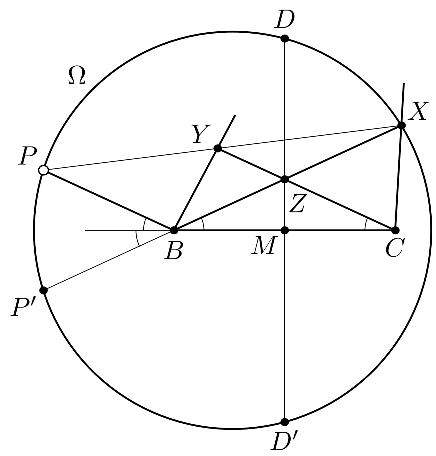
In particular, one may argue as follows. If O_1 is the centre of Ω_1, then O_1 lies on an (external or internal) bisector of $\angle PBX$, as well as on the perpendicular bisector of XP. This yields that O_1 is the midpoint of one of the arcs PX of the circle PBX. That, in turn, implies that the circle Ω_1 contains two of the four points: the incentre, and the three excentres of $\triangle PBX$. It follows that Ω_1 is an Apollonius circle of the segment BW.

So, if D is a point such that the triangle BCD is equilateral, it suffices to show that $BD/DW = BX/XW$ (so D lies on the same Apollonius circle). This can be done by a computation using the cosine law, although not very quickly.

Solution 2. All angles in the solution are directed. All segment lengths on lines BX and CY (and parallel to them) are also oriented; we assume that the directions $\overrightarrow{BX}$ and $\overrightarrow{CY}$ are positive. As in the solution above, we prove that $BP \parallel CY$.

Assume that ABC is oriented anti-clockwise. Let D and D' be the points such that the triangles DBC and $D'CB$ are equilateral, and oriented anti-clockwise. We will show that D and D' lie on the circle Ω_1; similarly, they lie on Ω_2.

Notice that $\alpha = \angle BAC = \angle CBX = \angle YCB = \pi - \angle CBP$; moreover, each of the triangles XBC and BCY is similar to BAC and oriented differently than BAC; hence those two triangles are equi-oriented. Let Ω denote the circle $(DD'X)$; clearly, its center lies on the perpendicular bisector of DD', i.e., on BC. We aim to prove that Ω passes through P; that will yield that $\Omega = \Omega_1$, which establishes what we are aimed to prove.



Denote $Z = XB \cap YC$. Since $\angle CBZ = \angle BAC = \angle ZCB$, we have $ZB = ZC$, and hence Z lies on the perpendicular bisector DD' of BC. By similarity, we get $BX/BC = BC/CY$, or $BC^2 = BX \cdot CY = BX \cdot (ZY + CZ)$. Since $CY \parallel BP$, the triangles XZY and XBP are similar, so $BX \cdot ZY = ZX \cdot BP$. Therefore,

$$BD^2 = BC^2 = BX \cdot ZY + BX \cdot CZ = ZX \cdot BP + (BZ + ZX) \cdot BZ = ZX \cdot (BP + BZ) + BZ^2.$$

On the other hand, let M be the midpoint of BC, and let XB cross Ω again at P'. Write the power of point Z with respect to Ω as

$$XZ \cdot (P'B + BZ) = XZ \cdot P'Z = ZD \cdot ZD' = MZ^2 - DM^2 = BZ^2 - MB^2 - DM^2 = BZ^2 - BD^2.$$

The two obtained relations yield

$$ZX \cdot (BP + BZ) = BD^2 - BZ^2 = ZX \cdot (P'B + BZ),$$

so $BP = P'B$, and so P and P' are reflections of one another in the line BC. Thus, P lies on Ω, as desired.

Remark. It is also possible to solve the problem via the *moving points* method. Introduce the points D and D' as in Solution 2, and introduce the reflections X', Y', P', and Q' of X. Y, P, and Q in the line BC, respectively, as in Solution 1 to read Ω_1 and Ω_2 as the circles $PXX'P'$ and $QYY'Q'$, respectively.

We need to show that D lies on Ω_2 (the other incidences are similar). To this end, it suffices to check that $\angle YDQ = \angle YY'Q = 90° - \angle Y'CB = 90° - \angle BAC$.

Fix B, C, and the circle ABC. As A varies over that circle, the lines BX, CY, BP, and CQ remain constant, and X and Y depend projectively on A. Choosing Q_1 on CQ such that $\angle YDQ_1 = 90° - \angle BAC$, we need to show that $Q_1 = Q$, or that X, Y, and Q_1 are collinear. The point Q_1 also depends projectively on A, so it suffices to check that the points Q_1, X, and Y are collinear for four specific positions of A.

Problem 6. A polynomial P with integer coefficients is *square-free* if it is not expressible in the form $P = Q^2R$, where Q and R are polynomials with integer coefficients and Q is not constant. A polynomial is *suitable* if its constant term and all other non-zero coefficients are equal to 1. Prove that, for all but finitely many integers $n \geq 1$, more than 99% of the suitable polynomials of degree at most n are square-free.

Iran, Navid Safaei

Solution 1. Let $\mathcal{P}_n$ be the set of all suitable polynomials of degree at most n. Clearly, $|\mathcal{P}_n| = 2^n$. Alternatively, but equivalently, we prove that less than $\frac{1}{100} \cdot 2^n$ of the polynomials in $\mathcal{P}_n$ are not square-free for all but finitely many n. Throughout the solution n is always assumed to be sufficiently large to allow room for as large integers $r \leq n$ as the different stages of the argument require. Also, all polynomials have integer coefficients and divisibility is always understood in $\mathbb{Z}[X]$. The proof consists of three parts:

(1) Upper bounding the number of polynomials in $\mathcal{P}_n$ divisible by the square of a non-constant polynomial of degree at most r;

(2) Upper bounding the number of polynomials in $\mathcal{P}_n$ divisible by the square of a polynomial of degree greater than r; and

(3) Choosing a suitable r.

Before dealing with the three parts above, we prove a useful lemma.

Lemma. *The zeroes of any polynomial in $\mathcal{P}_n$ all lie in the open disc $|z| < 2$ and their real parts are all less than $C = \frac{1}{2}(1 + \sqrt{5})$.*

Proof. Let P be a degree m polynomial in $\mathcal{P}_n$. Leaving aside the trivial case $m = 0$, let $m \geq 1$. Write $P = \sum_{k=0}^{m} a_k X^k$ and consider a complex number z of absolute value $|z| \geq 2$. Then

$$|P(z)| \geq |z|^m - \sum_{k=0}^{m-1} |z|^k = \frac{(|z|-2)||z|^m + 1}{|z|-1} \geq \frac{1}{|z|-1} > 0,$$

and the first part follows.

To prove the second part, let z have a real part $\Re z \geq C > 1$. Clearly, $|z| \geq \Re z \geq C > 1$. Leaving aside the trivial cases $m = 0$ and $m = 1$, let $m \geq 2$ to write

$$\begin{aligned}\left|\frac{P(z)}{z^m}\right| &\geq \left|a_m + \frac{a_{m-1}}{z}\right| - \frac{1}{|z|^2} - \cdots - \frac{1}{|z|^m} \geq \left|a_m + \frac{a_{m-1}}{z}\right| - \frac{1}{|z|^2} \\ &> \left|a_m + \frac{a_{m-1}}{z}\right| - \frac{1}{|z|^2 - |z|} \geq \left|a_m + \frac{a_{m-1}}{z}\right| - 1, \qquad \text{as } |z| \geq C, \\ &\geq \Re\left(a_m + \frac{a_{m-1}}{z}\right) - 1 \geq a_m - 1 \geq 0.\end{aligned}$$

This establishes the second part and completes the proof. (The argument in the second part shows in fact that, if $\Re z > 0$ and $|z| \geq C$, then $P(z) \neq 0$. Hence the zeroes of P with a positive real part all lie in the open disc $|z| < C$.)

To deal with **(1)**, consider a polynomial P in $\mathcal{P}_n$ divisible by the square of a non-constant polynomial Q of degree $q \leq r$. We first show that there are at most $(2^{2r+1} - 1)^{r+1}$ such Q's.

Clearly, the leading coefficient of Q is ± 1. The zeroes $z_1, \ldots, z_q$ of Q are amongst those of P, so $|z_k| < 2$, by the lemma. The absolute value of the coefficient of X^{q-k} in Q is then $|\sum z_{i_1} \cdots z_{i_k}| < \binom{q}{k} \cdot 2^k < 2^{q+k} \leq 2^{2q} \leq 2^{2r}$. Hence each coefficient of Q takes on at most $2 \cdot 2^{2r} - 1 = 2^{2r+1} - 1$ values, so there are at most $(2^{2r+1} - 1)^{r+1}$ such Q's, as stated.

Next, upper bound the number of P's in $\mathcal{P}_n$ that are divisible by the same Q^2. Consider such a $P = a_0 + \cdots + a_n X^n$ and let $\mathcal{S}_n(P)$ be the set of all polynomials $R = b_0 + \cdots + b_n X^n$ in $\mathcal{P}_n$ such that $b_k \neq a_k$ for exactly one k; note that $k \geq 1$, as $b_0 = a_0 = 1$. Alternatively, but equivalently, there exists a $k \geq 1$ such that $b_\ell = a_\ell$ for $\ell \neq k$ and $b_k + a_k = 1$. Hence $|\mathcal{S}_n(P)| = n$. Note further that $P - R = \pm X^k$ vanishes at 0, whereas Q does not, as $Q(0)^2$ divides $P(0) \neq 0$. Hence R is not divisible by Q^2, showing that none of the n polynomials in $\mathcal{S}_n(P)$ is.

Consider now distinct P_1 and P_2 in $\mathcal{P}_n$, both divisible by Q^2, to show that $\mathcal{S}_n(P_1)$ and $\mathcal{S}_n(P_2)$ are disjoint: If they shared some R, then $P_1 - R = \pm X^{k_1}$ and $P_2 - R = \pm X^{k_2}$ for some distinct k_1, $k_2 \geq 1$, so $P_1 - P_2 = \pm X^{k_1} \pm X^{k_2}$ would be divisible by Q^2, which is clearly not the case.

By the two paragraphs above, there are then at most $2^n/(n+1)$ polynomials in $\mathcal{P}_n$ that are divisible by the same Q^2.

As there are at most $(2^{2r+1} - 1)^{r+1}$ such Q's of degree at most r, there are at most

$$\frac{(2^{2r+1} - 1)^{r+1}}{n+1} \cdot 2^n$$

polynomials in $\mathcal{P}_n$ divisible by the square of a non-constant polynomial of degree at most r. For a fixed r this upper bound is clearly of order $o(2^n)$.

To deal with **(2)**, consider a polynomial P in $\mathcal{P}_n$ divisible by the square of a non-constant polynomial Q of degree $q > r$.

The zeroes $z_1, \ldots, z_q$ of Q are amongst those of P, so $|z_k| < 2$ and $\Re z_k < C$, by the lemma. As the leading coefficient of Q is clearly ± 1,

$$|Q(3)| = |(3 - z_1) \cdots (3 - z_q)| \geq (3 - \Re z_1) \cdots (3 - \Re z_q) > (3 - C)^q > (3 - C)^r.$$

As $3 - C > 1$, letting r be large enough, $P(3)$ is then divisible by d^2 for some large enough $d > (3 - C)^r$.

Consider the largest integer $s = s_d$ satisfying $3^s < d^2 \leq 3^{s+1}$. Clearly, $s \geq 1$. Write $P = \sum_{k=0}^{n} a_k X^k$. Then $1 = a_0 \leq \sum_{k=0}^{s-1} 3^k a_k \leq \sum_{k=0}^{s-1} 3^k = \frac{1}{2}(3^s - 1) < \frac{1}{2}(d^2 - 1) < d^2$. As for distinct choices of $a_1, \ldots, a_{s-1}$ from $\{0, 1\}$ the sums $\sum_{k=0}^{s-1} 3^k a_k$ are pairwise distinct, they are also pairwise distinct modulo d^2. Noting that these sums are all positive, it follows that there are at most 2^{n-s+1} polynomials P in $\mathcal{P}_n$ such that $P(3)$ is divisible by d^2.

If r is sufficiently large, then so is $d > (3 - C)^r$. Thus, if d is large enough, then $2^s > d^{5/4}$, as $d^2 \leq 3^{s+1}$ and $\log_2 3 < \frac{8}{5}$, so there are at most $2^{n-s+1} \leq 2^{n+1} d^{-5/4}$ polynomials P in $\mathcal{P}_n$ such that $P(3)$ is divisible by d^2. Hence, if d_0 is large enough, then the number of such P's is at most $2^{n+1} \sum_{d > d_0} d^{-5/4}$.

Now, as $\sum_{d \geq 1} d^{-5/4}$ converges, given any $c > 0$, the remainder $\sum_{d > d_0} d^{-5/4} < c$ for some large enough d_0 depending on c, of course. For any such d_0, the number of P's in $\mathcal{P}_n$ such that $P(3)$ is divisible by d^2 is then less than $2c \cdot 2^n$.

Consequently, so is the number of polynomials in $\mathcal{P}_n$ divisible by the square of a polynomial of degree greater than r.

Finally, we deal with **(3)**. Fix any $c > 0$. Then choose r large enough so that the remainder $\sum_{d > (3-C)^r} d^{-5/4} < c$. At this stage $n > r$. Let further $n > \frac{1}{c}(2^{2r+1} - 1)^{r+1}$. By **(1)** and **(2)**, the number of non-square-free polynomials in $\mathcal{P}_n$ is then less than $3c \cdot 2^n$. Setting $c = \frac{1}{300}$ provides the answer to the problem at hand.

Solution 2. We present an alternative approach to parts **(1)** and **(2)**.

To deal with **(1)**, use the first part of the lemma to bound the number of possible polynomials Q by some constant. For every such Q, we then prove that few polynomials in $\mathcal{P}_n$ are divisible by Q. This follows from the clam below:

Claim. *Given a non-constant polynomial Q, the number of polynomials in $\mathcal{P}_n$ that are divisible by Q does not exceed $\binom{n}{\lfloor n/2 \rfloor}$.*

Proof. Let ζ be a non-zero complex root of Q (if there are no such, then no polynomial in $\mathcal{P}_n$ is divisible by Q). Then each polynomial $P = p_n X^n + \cdots + p_1 X + p_0$ in $\mathcal{P}_n$ divisible by Q satisfies

$\sum_{i=0}^{n} p_i \cdot c\zeta^i = 0$, for any complex c. Choose a suitable c so that the $a_i = \operatorname{Re} c\zeta^i$ are all non-zero. Then $\sum_{i=0}^{n} a_i p_i = 0$.

Partially order the (binary) tuples of coefficients by letting $(p_1, p_2, \ldots, p_n) \preceq (p'_1, p'_2, \ldots, p'_n)$ if and only if the non-zero $a_i(p'_i - p_i)$ are all positive. The tuples corresponding to polynomials divisible by Q then form an independent set (anti-chain) in the $\preceq$-partially ordered n-cube.

Assign each tuple $(p_1, p_2, \ldots, p_n)$ the tuple $(\sigma p_1, \sigma p_2, \ldots, \sigma p_n)$, where $\sigma p_i = p_i$, if $a_i > 0$, and $\sigma p_i = 1 - p_i$, otherwise. This assignment shows $\preceq$ isomorphic to the (index) set-inclusion partial order on the binary n-cube, so the length of any $\preceq$-anti-chain is at most $\binom{n}{\lfloor n/2 \rfloor}$, by Sperner's theorem. This proves the claim.

The standard bound provided by Stirling's formula (or any of its elementary relaxations) establishes part **(1)**.

To prove **(2)**, deal more algebraically. Let P be a polynomial in $\mathcal{P}_n$ divisible by some Q^2, where $\deg Q = d > r$. Reduce modulo 2 to get the polynomials $\overline{P}$ and $\overline{Q}$, where $\deg \overline{Q} = d$, as the leading coefficient of Q is ± 1. Then $\overline{P}$ is divisible by $\overline{Q}^2 = \overline{Q}(X^2)$. Write $\overline{P} = \overline{P}_+(X^2) + X\overline{P}_-(X^2)$. Then $\overline{P}_+$ and $\overline{P}_-$ are both divisible by $\overline{Q}$. So, for a fixed $\overline{Q}$, the number of such polynomials does not exceed 2^{n-2d}. Hence for all degree d polynomials $\overline{Q}$, the number of such P's does not exceed $2^{d-1} \cdot 2^{n-2d} = 2^{n-d-1}$, so their fraction in $\mathcal{P}_n$ is at most 2^{-d-1}. Finally, sum over all $d \geq r$, to conclude that the fraction of P's in **(2)** does not exceed 2^{-r}.

The 16^{th} Romanian Master of Mathematics Competition

Day 1: 12 February, 2025, Bucharest

Language: English

Problem 1. Let $n > 10$ be an integer, and let $A_1, A_2, \ldots, A_n$ be distinct points in the plane such that the distances between the points are pairwise different. Define $f_{10}(j, k)$ to be the 10^{th} smallest of the distances from A_j to $A_1, A_2, \ldots, A_k$, excluding A_j if $k \geq j$. Suppose that for all j and k satisfying $11 \leq j \leq k \leq n$, we have $f_{10}(j, j-1) \geq f_{10}(k, j-1)$. Prove that $f_{10}(j, n) \geq \frac{1}{2} f_{10}(n, n)$ for all j in the range $1 \leq j \leq n-1$.

Problem 2. Consider an infinite sequence of positive integers $a_1, a_2, a_3, \ldots$ such that $a_1 > 1$ and $(2^{a_n} - 1)a_{n+1}$ is a square for all positive integers n. Is it possible for two terms of such a sequence to be equal?

Problem 3. Fix an integer $n \geq 3$. Determine the smallest positive integer k satisfying the following condition:

> For any tree T with vertices $v_1, v_2, \ldots, v_n$ and any pairwise distinct complex numbers $z_1, z_2, \ldots, z_n$, there is a polynomial $P(X, Y)$ with complex coefficients of total degree at most k such that for all $i \neq j$ satisfying $1 \leq i, j \leq n$, we have $P(z_i, z_j) = 0$ if and only if there is an edge in T joining v_i to v_j.

Note, for example, that the total degree of the polynomial

$$9X^3Y^4 + XY^5 + X^6 - 2$$

is 7 *because* $7 = 3 + 4$.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

The 16^{th} Romanian Master of Mathematics Competition

Day 2: 13 February, 2025, Bucharest

Language: English

Problem 4. Let $\mathbb{Z}$ denote the set of integers, and let $S \subset \mathbb{Z}$ be the set of integers that are at least 10^{100}. Fix a positive integer c. Determine all functions $f\colon S \to \mathbb{Z}$ satisfying $f(xy + c) = f(x) + f(y)$ for all $x, y \in S$.

Problem 5. Let ABC be an acute triangle with $AB < AC$, and let H and O be its orthocentre and circumcentre, respectively. Let Γ be the circumcircle of triangle BOC. Circle Γ intersects line AO at points O and A', and Γ intersects the circle of radius AO with centre A at points O and F. Prove that the circle which has diameter AA', the circumcircle of triangle AFH, and Γ pass through a common point.

Problem 6. Let k and m be integers greater than 1. Consider k pairwise disjoint sets $S_1, S_2, \ldots, S_k$, each of which has exactly $m + 1$ elements: one red and m blue. Let $\mathcal{F}$ be the family of all subsets T of $S_1 \cup S_2 \cup \cdots \cup S_k$ such that, for every i, the intersection $T \cap S_i$ is monochromatic. Determine the largest possible number of sets in a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that no two sets in $\mathcal{G}$ are disjoint.

A set is monochromatic if all of its elements have the same colour. In particular, the empty set is monochromatic.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

The 16^{th} Romanian Master of Mathematics Competition

Day 1 — Solutions

Problem 1. Let $n > 10$ be an integer, and let $A_1, A_2, \dots A_n$ be distinct points in the plane such that the distances between the points are pairwise different. Define $f_{10}(j,k)$ to be the 10^{th} smallest of the distances from A_j to $A_1, A_2, \dots, A_k$, excluding A_j if $k \geq j$. Suppose that for all j and k satisfying $11 \leq j \leq k \leq n$, we have $f_{10}(j, j-1) \geq f_{10}(k, j-1)$. Prove that $f_{10}(j,n) \geq \frac{1}{2} f_{10}(n,n)$ for all j in the range $1 \leq j \leq n-1$.

Iran, Morteza Saghafian

Solution 1. For every i, denote $a_i = f_{10}(i, i-1)$ and $b_i = f_{10}(i,n)$. So, we need to show that $b_n \leq 2b_i$ for all i. Notice that $a_i \geq b_i$ for all i.

To prove this, choose an arbitrary $i < n$, and let $A_iA_{j_1}, A_iA_{j_2}, \dots, A_iA_{j_{10}}$ be the ten smallest numbers among the A_iA_j with $j \neq i$, ordered so that $j_1 < j_2 < \dots < j_{10}$.

If $j_{10} < i$, then $i > 10$, and the problem condition yields

$$b_i = \max_{1 \leq k \leq 10} A_iA_{j_k} = a_i \geq a_n \geq b_n,$$

which is even stronger than we need.

Otherwise, set $j = j_{10} > i$ (in this case we also have $j_{10} > 10$), and denote $m = b_i = \max\limits_{1 \leq k \leq 10} A_iA_{j_k}$. By the problem condition, we have $a_j \geq a_n = b_n$. On the other hand, we have

$$a_j \leq \max\left(A_jA_i, \max_{1 \leq k \leq 9} A_jA_{j_k}\right) \leq \max\left(A_jA_i, \max_{1 \leq k \leq 9} (A_jA_i + A_iA_{j_k})\right) \leq 2m,$$

as $A_jA_i, A_iA_{j_k} \leq m$. So $b_n \leq a_j \leq 2m = 2b_i$, as desired.

Solution 2. Let $d_j = f_{10}(j,n)$, $j = 1, \dots, n$. To prove that $2d_j \geq d_n$ for every $j = 1, \dots, n-1$, induct on n.

Consider the base case, $n = 11$. Note that each $d_j = \max_{i \neq j} A_iA_j$, as f_{10} is 10-variate. Let $d_{11} = A_{11}A_k$ for some index $k \leq 10$. Clearly, $2d_k \geq 2A_kA_{11} = 2d_{11} \geq d_{11}$ and if $j \neq k$ then $2d_j \geq A_jA_k + A_jA_{11} \geq A_kA_{11} = d_{11}$, by the triangle inequality.

For the induction step, let $n > 11$ and note that

$$\max_{n \geq \ell \geq k} f_{10}(\ell, k-1) = \max_{n-1 \geq \ell \geq k} f_{10}(\ell, k-1),$$

as both maxima are achieved at $\ell = k$, by the condition in the statement. Hence $A_1, A_2, \dots, A_{n-1}$ also satisfy this condition.

Let $d'_j = f_{10}(j, n-1)$, $j = 1, \dots, \leq n-1$. By the induction hypothesis, $2d'_j \geq d'_{n-1}$ for all $j \leq n-2$. Note that $d'_{n-1} = f_{10}(n-1, n-2) \geq f_{10}(n, n-2) \geq f_{10}(n, n-1) = d_n$; the first inequality holds by the condition in the statement for $k = n-1$ and the second because adding more variables to f_{10} does not increase its value.

Let now Δ_i be the closed disc (interior and boundary) of radius d_i, centred at A_i. By the definition of d_i, each Δ_i contains at least 11 points, of which at most 10 (A_i, inclusive) lie strictly inside.

Finally, suppose, if possible, that $2d_j < d_n$ for some index $j < n$. If A_jA_n is not among the first 10 distances from A_j to the other points, then $d_j = d'_j$ and this leads to a contradiction with the induction hypothesis. So A_jA_n has to be among the first 10 distances from A_j to the other points. This means that $d_j \geq A_jA_n$, so $d_n > 2d_j \geq 2A_jA_n$. Hence Δ_j lies strictly inside Δ_n. This is a contradiction, as Δ_j contains at least 11 points, whereas Δ_n contains at most 10 strictly inside. The conclusion follows.

Problem 2. Consider a sequence of integers $a_1, a_2, a_3, \ldots$ such that $a_1 > 1$ and $(2^{a_n} - 1)a_{n+1}$ is a square for all positive integers n. Is it possible that two terms of such a sequence be equal?

Russia, Pavel Kozlov

Solution. The answer is in the negative. Notice first that, if $a_n > 1$, then $2^{a_n} - 1 \equiv 3 \pmod 4$; since $(2^{a_n} - 1)a_{n+1}$ is a perfect square, we should have $a_{n+1} \equiv 0 \pmod 4$ or $a_{n+1} \equiv 3 \pmod 4$, so in particular $a_{n+1} > 1$. As $a_1 > 1$, we conclude that all terms of the sequence are greater than 1.

Denote the largest prime divisor of an integer $k > 1$ by $g(k)$. We will show that $g(a_{n+1}) > g(a_n)$ for all n, which yields the desired result. To this end, usage is made of the lemma below.

Lemma: For any prime p, each prime divisor of $2^p - 1$ is greater than p.

Proof. Let q be a prime factor of $2^p - 1$; then q is odd. The multiplicative order d of 2 modulo q divides p and is larger than 1, so $d = p$. On the other hand, by Fermat's little theorem, $2^{q-1} \equiv 1 \pmod q$, so $p = d \mid q - 1$ and the lemma follow.

Choose now any positive integer n, and denote, for convenience, $k = a_m$ and $\ell = a_{n+1}$. Let $p = g(k)$; then $2^p - 1 \mid 2^k - 1$. Since $2^p - 1 \equiv 3 \pmod 4$, this number is not a square, so there exists a prime q such that $v_q(2^p - 1)$ is odd. By the Lemma, $q > p$, so in particular $q \nmid k$. Therefore, by the Lifting Exponent Lemma,

$$v_q(2^k - 1) = v_q(2^p - 1) + v_q(k/p) = v_q(2^p - 1) + 0,$$

so $v_q(2^k - 1)$ is odd as well. Since $(2^k - 1)\ell$ is a perfect square, we should then have $q \mid \ell$, so $g(\ell) \geq q > p = g(k)$, as desired.

Problem 3. Fix an integer $n \geq 3$. Determine the smallest positive integer k satisfying the following condition:

> For any tree T with vertices $v_1, v_2, \ldots, v_n$ and any pairwise distinct complex numbers $z_1, z_2, \ldots, z_n$, there is a polynomial $P(X, Y)$ with complex coefficients of total degree at most k such that for all $i \neq j$ satisfying $1 \leq i, j \leq n$, we have $P(z_i, z_j) = 0$ if and only if there is an edge in T joining v_i to v_j.

Note, for example, that the total degree of the polynomial

$$9X^3Y^4 + XY^5 + X^6 - 2$$

is 7 *because* $7 = 3 + 4$.

Romania, Andrei Chiriţă

Solution 1. First we provide a proof that $k \geq n-1$. Let T be the path where v_i and v_{i+1} are adjacent for all $1 \leq i \leq n-1$. Let ω be a primitive root of unity of order n and let $a_i = \omega^i$ for all $1 \leq i \leq n$.

If $f(X) = P(X, \omega X)$, then for all $1 \leq i \leq n-1$ we have $f(\omega^i) = P(a_i, a_{i+1}) = 0$. Since $f(1) = P(a_n, a_1) \neq 0$, f is non-zero and has at least $n-1$ roots. This means that $\deg P \geq \deg f \geq n-1$, proving $k \geq n-1$.

It remains to prove that $k = n-1$ is sufficient i.e. for any tree T and any $a_1, a_2, \ldots, a_n$ we can find a polynomial P of degree at most $n-1$. For brevity, we call a two-variable polynomial $A(X, Y)$ *symmetric* if $A(X, Y) = A(Y, X)$.

We begin with the following observation. Suppose that A and B are two variable polynomials of degree at most d. Then we can find $\alpha \in \mathbb{C}$ such that for any $1 \leq i, j \leq n$, $A(a_i, a_j) + \alpha B(a_i, a_j) = 0$ if and only if $A(a_i, a_j) = B(a_i, a_j) = 0$. This means that we can "merge" two conditions of degree at most d into a condition of degree at most d (note that this produces a symmetric polynomial if the initial polynomials are symmetric).

For any integer $t \geq 2$, let a *star* of size t be a collection of t edges for which there is a vertex which belongs to all edges. We will prove the following claims.

Claim 1. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ and E edges. Suppose that we can partition the edges of G into a number of stars. Then for any distinct complex numbers $a_1, a_2, \ldots, a_n$ we can find a symmetric polynomial P of degree at most E such that for all $1 \leq i, j \leq n, i \neq j$, $P(a_i, a_j) = 0$ if and only if there is an edge between v_i and v_j in G.

Proof. We will first prove the claim when G consists of a star of size $E \leq n-1$ and some isolated vertices. Without loss of generality, let $v_1v_2, v_1v_3, \ldots, v_1v_{E+1}$ be the edges of G. Also let $s_1 = a_1 + a_2, s_2 = a_1 + a_3, \ldots, s_E = a_1 + a_{E+1}$.

Consider merging the polynomials $(X - a_1)(Y - a_1)$ and $(X+Y-s_1)(X+Y-s_2)\ldots(X+Y-s_E)$ into a polynomial of degree at most E (which is of course symmetric). They both vanish at a pair a_i, a_j if and only if $1 \in \{i, j\}$ and $a_i + a_j \in \{s_1, s_2, \ldots, s_E\}$. These two happen if and only if v_i and v_j are adjacent, so this produces a valid polynomial.

For the general case, let $S_1 \cup S_2 \cup \cdots \cup S_k$ be the partition of the edges of G. For each $1 \leq i \leq k$, we can find a two variable polynomial P_i of degree at most $|S_i|$ which vanishes only at the edges of S_i. Then we can let $P = P_1P_2 \ldots P_k$, which satisfies the claim as $\deg P \leq |S_1| + |S_2| + \cdots + |S_k| = E$, as desired.

Claim 2. Any tree Γ with odd number of vertices can be partitioned into stars of size 2.

Proof. We prove this by induction on the number of the vertices of Γ. The base case is clear, since Γ is a star of size 2 when Γ has three vertices.

For the inductive step, let Γ be a tree with $2m+1$ vertices, where $m \geq 2$. Let $u_1u_2\ldots u_t$ be a path of maximal length in Γ (of course, $t \geq 3$). Then any neighbour of u_2 except for maybe u_3 must have degree 1, otherwise we can delete u_1 and insert two edges, contradicting the maximality of t. If $\deg u_2 = 2$, we can form the star u_1u_2, u_2u_3 and apply the inductive hypothesis on $\Gamma \setminus \{u_1, u_2\}$. If $\deg u_2 \geq 3$, let $u \neq u_1, u_3$ be a neighbour of u_2. Then create the star u_1u_2, uu_2 and apply the inductive hypothesis on $\Gamma \setminus \{u, u_1\}$. This proves the claim.

The case where n is odd becomes trivial, since T has $n-1$ edges. Assume that n is even. Without loss of generality, let $\deg v_n = 1$ and let v_n be adjacent to v_{n-1}.

Consider the graph T' formed by deleting the edge $v_{n-1}v_n$ from T (but not perturbing the n vertices). Clearly, T' consists of v_n and a tree on $v_1, v_2, \ldots, v_{n-1}$. As $n-1$ is odd, the edges of T' can be partitioned into stars of size 2 from Claim 2. From Claim 1 it follows that there is a symmetric polynomial $Q(X,Y)$ of degree at most $n-2$ such that $Q(a_i, a_j) = 0$ if and only if $i, j \leq n-1$ and v_i, v_j are adjacent in T.

Let $g(X)$ be the polynomial of degree $n-1$ such that $g(a_1) = g(a_2) = \cdots = g(a_{n-1}) = 0$ and $g(a_n) = -Q(a_{n-1}, a_n) = -Q(a_n, a_{n-1})$. Let P be the polynomial of degree at most $n-1$ obtained by merging $F_1(X,Y) = Q(X,Y) + g(X) + g(Y)$ and $F_2(X,Y) = Q(X,Y)(X+Y-s)$, where $s = a_n + a_{n-1}$. It is easy to see that P vanishes at each (a_i, a_j) for which v_i, v_j are adjacent in T.

Suppose that v_i and v_j are not adjacent in T. If $i, j \leq n-1$, then $F_1(a_i, a_j) = Q(a_i, a_j) \neq 0$. If $n \in \{i, j\}$, then $a_i + a_j \neq s$ and $Q(a_i, a_j) \neq 0$, so $F_2(a_i, a_j) \neq 0$. Hence $P(a_i, a_j) \neq 0$. This proves that P satisfies the required conditions, as desired.

Solution 2. Establish the lower bound as in Solution 1. We now address the upper bound differently. Let G be a graph on vertices $v_1, \ldots, v_n$. Say that a polynomial $P(X,Y)$ is G-good if it satisfies the conditions in the statement of the problem. We prove the more general fact below:

Claim. Let d_i be the degree of v_i. Assume that these degrees satisfy $d_i \leq n-i$ for all $i \leq n-1$, and $d_n = 1$. Then there is a G-admissible polynomial of degree at most $n-1$.

Notice here that, if G is a tree with vertices ordered so that $d_1 \geq d_2 \geq \cdots \geq d_n$, then it satisfies the conditions in the Claim. Indeed, we have $d_n = 1$, and if $d_i > n-i$ for some $i \leq n-1$, then we have

$$2n-2\sum_{j=1}^{n} d_j \geq \sum_{j=1}^{i}(n-i+1) + \sum_{j=i+1}^{n} 1 = i(n-i+1) + (n-i) = (i+1)(n-i+1) - 1 \geq 2n-1,$$

which is a contradiction. So, it suffices to prove the Claim.

Proof of the Claim. Let

$$P(X,Y) = \sum_{j=0}^{n-1} R_j(Y)X^j$$

be the sought polynomial; set $R_{n-1}(X) = 1$. Denote

$$Q_i(X) = P(X, a_i) = X^{n-1} + \sum_{j=0}^{n-2} q_{ij}X^j, \qquad \text{where } q_{ij} = R_j(a_i).$$

So, we will seek for the sequences $C_j = (q_{1j}, q_{2j}, \ldots, q_{nj})$ such that there exists a polynomial R_j with $\deg R_j \leq n-j-1$ such that $q_{ij} = R_j(a_i)$. Notice that the first $n-j$ terms of such a sequence determine it uniquely; in particular, there are no restrictions on the sequence C_0.

We know that the polynomial Q_i has d_i prescribed roots. For every $i \leq n-1$, augment this list by some numbers not from the set $A = \{a_1, a_2, \ldots, a_n\}$ to the list $b_{i1}, b_{i2}, \ldots, b_{i,n-i}$. Also, denote by b_{n1} the unique prescribed root of Q_n. Thus, we should have

$$Q_i(X) = S_i(X) \prod_{j=1}^{\max(n-i,1)} (X - b_{ij}), \tag{1}$$

where S_i is a monic polynomial with $\deg S_i = i-1$ for $i \le n-1$ and $\deg S_n = n-2$. The only extra conditions we have are that each S_i should achieve non-zero values at the prescribed finite subset A_i of A (containing no prescribed roots of Q_i).

The polynomial Q_1 is uniquely determined by (1).

Assume that the polynomials $Q_1, \dots, Q_{i-1}$ have already been determined, for some $i \le n-2$. This means that the sequences $S_1, \dots, S_{i-1}$ are also determined. This determines the polynomial S_i up to the constant term. So we may choose this constant term so that S_i has no prohibited roots, thus defining Q_i.

It remains to deal with the indices $i = n-1, n$. At this moment, both polynomials S_{n-1} and S_n are determined up to constant terms. The conditions they must obey are: (i) they should not have roots inside A_{n-1} and A_n, respectively; and (ii) the sequence C_1 should be a sequence of values of a polynomial R_1 with $\deg R_1 \le n-2$. Notice that all such polynomials R_1 are determined up to an additive polynomial of the form $\alpha \prod_{j \le n-2}(X - a_j)$. The coefficients $q_{n-1,1}$ and $q_{n,1}$ depend linearly on α with a non-zero linear term. Hence the constant terms of S_{n-1} and S_n depend linearly on α. Now, again, there are only finitely many restrictions which remove finitely many values of α; any other value fits the bill.

Solution 3. The answer is $n-1$. Use the same argument as the official solution for the lower bound.

We can construct such a polynomial via a method that uses no graph theory other than the fact that T has $n-1$ edges.

Lemma 1. Let k be a positive integer and let $A, B \subseteq \mathbb{C}^2$ be two disjoint finite sets. Suppose $|A| = 2k$ and no line intersects $A \cup B$ in more than $k+1$ points. Then there exists a polynomial of degree k that vanishes on A and is non-zero on B.

Proof. We first argue that we may reduce to the case when $|B| = 1$. This is since if P and P' are polynomials that are zero on A and nonzero on B and B', then a generic linear combination of P and P' is nonzero on A and nonzero on $B \cup B'$. Now suppose $B = \{b\}$.

For $a, a' \in A$, write $a \sim a'$ if a, a', b are collinear. This is an equivalence relation, and each equivalence class has at most k elements. Thus we may pair up the elements of A such that no two paired elements are collinear with b. Now let P be the polynomial vanishing on the union of the k lines determined by the pairs, which is nonzero at b by construction.

Lemma 2. Let $a_1, a_2, \dots, a_n$ be distinct complex numbers. Then a line can intersect at most n points of the form (a_i, a_j).

Proof. If not, then by the pigeonhole principle such a line must contain two points with the same x-coordinate. But then it is vertical and thus can only contain n points.

We are done by applying Lemma 1 with

$$A = \{(a_i, a_j)\colon v_i v_j \in E(T)\} \quad \text{and} \quad B = \{(a_i, a_j)\colon i \ne j, v_i v_j \notin E(T)\}.$$

The 16$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Let $\mathbb{Z}$ denote the set of integers and let $S \subset \mathbb{Z}$ be the set of integers that are at least 10^{100}. Fix a positive integer c. Determine all functions $f\colon S \to \mathbb{Z}$ satisfying

$$f(xy+c) = f(x) + f(y) \quad \text{for all } x, y \in S.$$

United Kingdom

Solution. Observe that if $x_1, y_1, x_2, y_2 \in S$ with $x_1y_1 = x_2y_2$ then

$$f(x_1) + f(y_1) = f(x_2) + f(y_2). \tag{1}$$

This tells us that for $u, v, w \in S$,

$$f(uv) + f(w) = f(u) + f(vw), \quad \text{so} \quad f(uv) - f(u) - f(v) = f(vw) - f(w) - f(v).$$

Notice the RHS is independent of u so the same must be true of the LHS. By replicating the argument with u and v switched, we also see the LHS is independent of v so in fact

$$f(uv) - f(u) - f(v) = k \quad \text{for some constant} \quad k \in \mathbb{Z} \tag{2}$$

Using (1) again we have, for $y, z \in S$

$$\begin{gathered} f(cz) + f(y) = f(z) + f(cy), \\ \text{so} \quad f(cy) - f(y) = l \quad \text{for some constant} \quad \ell \in \mathbb{Z}. \end{gathered} \tag{3}$$

Setting $x = cz$ in the original functional equation for $z \in S$ shows

$$\begin{gathered} f(c(yz+1)) \overset{(3)}{=} f(yz+1) + \ell = f(cz) + f(y) \overset{(3)}{=} f(z) + f(y) + \ell, \\ \text{so} \quad f(yz+1) = f(y) + f(z). \end{gathered}$$

Let $x \in S$ and set $y = x$, $z = x+2$ in the above to get

$$\begin{gathered} f\big((x+1)^2\big) \overset{(2)}{=} 2f(x+1) + k = f(x) + f(x+2) \\ \Rightarrow f(x) + f(x+2) - 2f(x+1) = -k = \text{constant} \end{gathered}$$

which forces f to be a quadratic. By setting $x = y$ in the original functional equation and considering the degree of both sides, we see f must be in fact be constant. The only constant function that satisfies the condition is $f \equiv 0$.

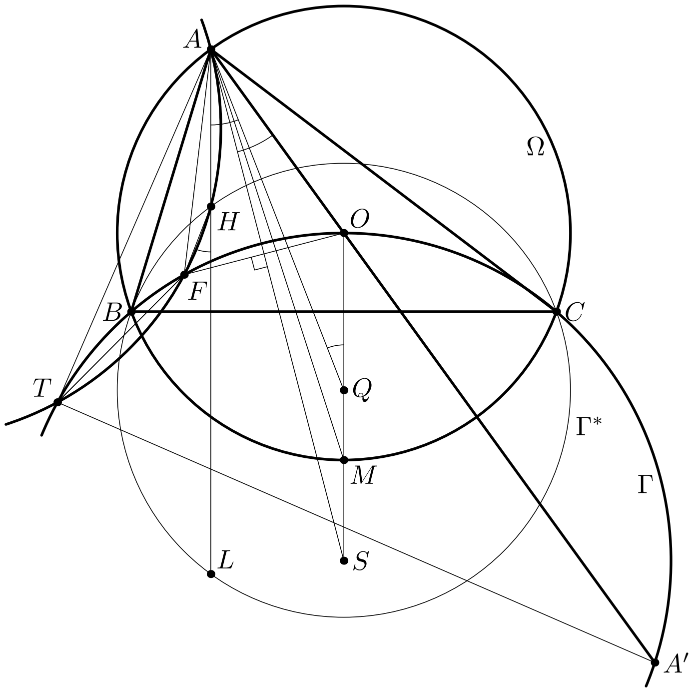
Problem 5. Let ABC be an acute triangle with $AB < AC$ and let H and O be its orthocentre and circumcentre, respectively. Let Γ be the circle BOC. The line AO and the circle of radius AO centred at A cross Γ again at A' and F, respectively. Prove that Γ, the circle on diameter AA' and the circle AFH are concurrent.

Romania, Radu-Andrei Lecoiu

Solution 1. Let Ω denote the circle (ABC) (centred at O), and let M be the midpoint of the minor arc BC of that circle.

Consider the composition ι of an inversion centered at A and the reflection in the bisector AM that swaps B and C. Then ι swaps the circle Ω with the line BC, hence it swaps O with the reflection L of A in BC. Hence $\iota(\Gamma)$ is the circle $\Gamma^* = (BCL)$, i.e., the reflection of Ω in BC which passes through H. Let S and Q be the centers of Γ and Γ^*, respectively; then AM is the angle bisector of $\angle QAS$.

Since ι swaps Γ and Γ^*, they are seen from A at the same angle, so there exists a rotational homothety h centred at A mapping Γ to Γ^*; the angle of h is $\angle SAQ$. Notice that the rays AH and AF are obtained from AO by reflections in AM and AS, respectively, so $\angle HAF = 2\angle MAS = \angle QAS$. This easily yields that $H = h(F)$. Hence the triangles AHF and AQS are similar, and $\angle AHF = \angle AQS$.



Now, let the circle (AHF) meet Γ again at T. Using directed angles, we get $\angle ATF = \angle AHF = \angle AQS$; next, since $FO \perp AS$, we have

$$\angle FTA' = \angle FOA' = \pi/2 - \angle OAS = \pi/2 - \angle QAL = \pi/2 - \angle AQS,$$

(here the equality $\angle OAS = \angle QAL$ holds because these angles are symmetric to each other with respect to AM), so $\angle ATA' = \angle ATF + \angle FTA' = \pi/2$, as desired.

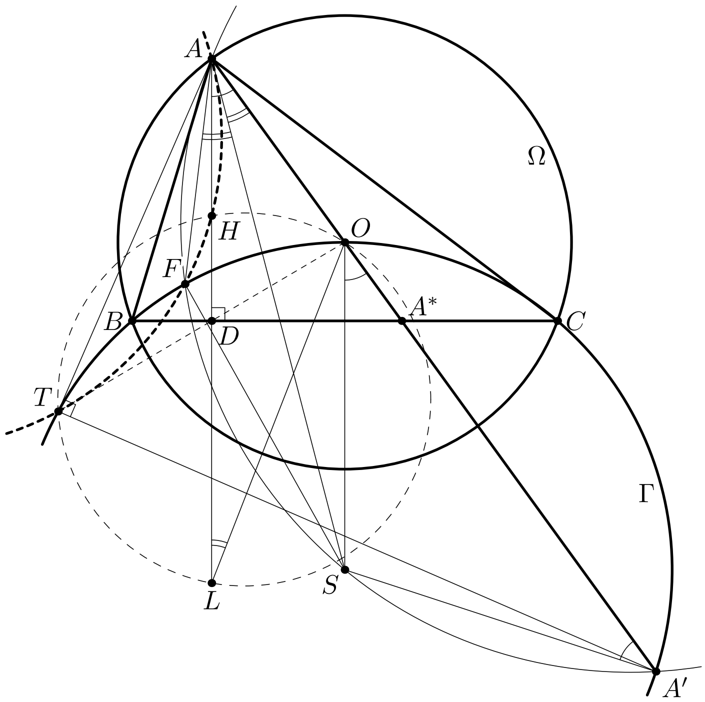
Remark. Existence of the rotational homothety h may be shown in various ways. E.g., one may notice that Ω is an Apollonius circle of the segment QS, so the ratio of the radii of Γ and Γ^* is $BS/BQ = AS/AQ$, which also yields that h exists.

Solution 2. Let S be the centre of Γ and let the circle on diameter AA' cross Γ again at T. It is sufficient to prove that T lies on circle AFH.

The points O and F are reflections of one another in AS; hence S lies on the internal angle bisectrix of $\angle FAA'$. On the other hand, since $SF = SA'$, it lies on the perpendicular bisectrix of FA'; so S is the midpoint of the arc $A'F$ on circle $AA'F$ not containing A. In particular, $AFSA'$ is cyclic.

Let D be the orthogonal projection of A on BC. We will prove that O, D, T are collinear. Invert from O with radius OB. This fixes B and C, so Γ maps to line BC. It follows that A maps to $A^* = AA' \cap BC$. Note that A is fixed under this inversion, as $OA = OB$, so the image of the circle on diameter AA' is a circle δ through A and A^* — and, in fact, δ is the circle of diameter AA^*, as AA' passes through O. Hence T maps to one of the points where line BC crosses δ. As $T \neq A'$, its image is D, so O, D, T are indeed collinear.

Letting L be the reflection of A in BC, we now prove that $HTLO$ is cyclic. As circle HBC is the reflection of Γ in BC, the quadrangle $HLBC$ is cyclic, so $HD \cdot DL = BD \cdot DC = OD \cdot DT$, whence $HTLO$ is indeed cyclic.



Next, we show that triangles ALO and $A'AS$ are similar. Let $\angle BAC = \alpha$ and $\angle CBA = \beta$. As $OS \parallel AL$, it follows that $\angle OAL = \angle A'OS = \angle AA'S$, so by the sine law:

$$\frac{AL}{AA'} = 2 \cdot \frac{AD}{AB} \cdot \frac{AB}{AA'} = 2 \cdot \sin\beta \cdot \frac{\cos\alpha}{\sin\beta} = 2\cos\alpha \quad \text{and} \quad \frac{AO}{A'S} = \frac{BO}{BS} = \frac{\sin 2\alpha}{\sin\alpha} = 2\cos\alpha.$$

Consequently, $AL/AA' = AO/A'S$, so $AL/AO = AA'/A'S$, implying the desired similarity. In particular, $\angle ALO = \angle A'AS$.

Finally, combine the properties established above to chase angles and write successively

$$\begin{aligned}\angle FTH &= \angle FTO - \angle HTO = \frac{1}{2}\angle FSO - \angle HLO = \angle ASO - \angle A'AS \\ &= \angle ASO + \angle A'AS - 2\angle A'AS = \angle SOA' - \angle FAO = \angle HAO - \angle FAO \\ &= \angle HAF\end{aligned}$$

and conclude that T lies on circle AFH, as stated in the first paragraph. This completes the solution.

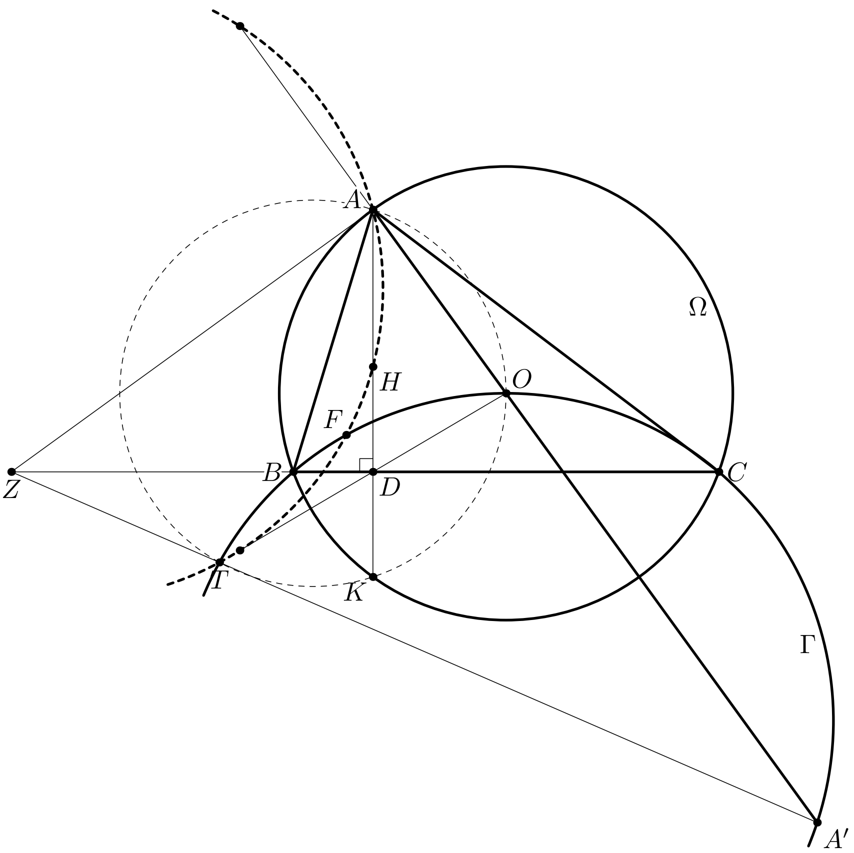
Remark. Let T be the desired intersection point. The property that the points O, D, and T is also useful in other approaches to the problem; in fact, it may be proved for both definitions of point T (as in Solutions 1 and 2), thus providing a different solution.

Here we list several other properties of the figure which appear to be useful in other approaches to the problem.

Let the tangent to the circle (ABC) at A meet BC at Z. Then the desired common intersection point T lies on ZA'.

Let AH meet the circle (ABC) again at K. Then the points A, O, K, and T are concyclic.

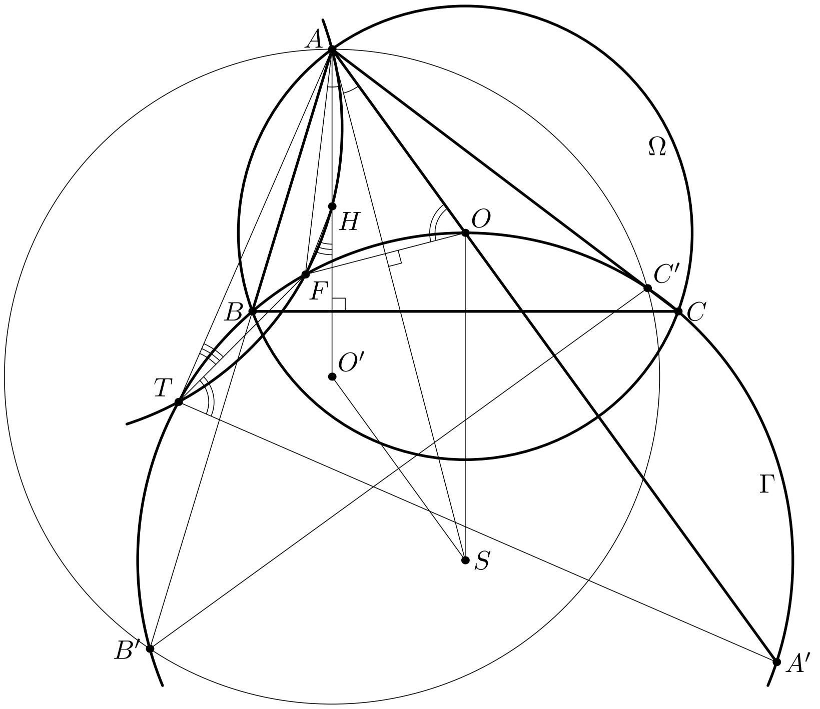
Finally, the circle $(AHFT)$ also passes through K, as well as through the reflection of O in D.



Solution 3. Let S be the centre of Γ and let circle AFH meet Γ again at $T \neq F$. In the sequel, angles are all orientated.

We are to prove that $\angle ATF + \angle FTA' = 90^\circ$. To this end, note the equivalences below:

$$\begin{aligned}
\angle ATF + \angle FTA' = 90^\circ &\Leftrightarrow \angle AHF + \angle FOA = 90^\circ \Leftrightarrow \angle AHF = \angle OAS \quad (\text{as } AS \perp OF) \\
&\Leftrightarrow \angle AHF = \angle SAF \Leftrightarrow AS \text{ is tangent to circle } AFH \\
&\Leftrightarrow \frac{AH}{\sin\angle HAS} = \frac{AF}{\sin\angle FAS} \\
&\Leftrightarrow \frac{AH}{AO} = \frac{\sin\angle HAS}{\sin\angle SAO}. \qquad (*)
\end{aligned}$$



To prove $(*)$, let AB and AC meet Γ again at B' and C', respectively. An easy angle chase shows that O is the orthocentre of triangle $AB'C'$.

As triangles ABC and $AC'B'$ are similar, AH passes through the centre O' of circle $AB'C'$; and as circles $AB'C'$ and $BOCC'B'$ are reflections of one another in $B'C'$ and $AO'SO$ is a parallelogram, it follows that

$$\frac{\sin \angle HAS}{\sin \angle SAO} = \frac{\sin \angle O'AS}{\sin \angle ASO'} = \frac{O'S}{AO'} = \frac{AO}{AO'}. \qquad (**)$$

Further on, as triangles ABC and $AC'B'$ are similar, $(**)$ implies equal corresponding length ratios, so $AO/AO' = AH/AO$. This establishes $(*)$ and concludes the solution.

Remark. Relation $(*)$ is equivalent to AS being the A-symmedian of triangle AOH. This might very well be known and can actually be proved in several different ways.

Problem 6. Let k and m be integers greater than 1. Consider k pairwise disjoint sets $S_1, S_2, \ldots, S_k$; each of these sets has exactly $m+1$ elements, one of which is red and the other m are all blue. Let $\mathcal{F}$ be the family of all subsets F of $S_1 \cup S_2 \cup \ldots \cup S_k$ such that, for every i, the intersection $F \cap S_i$ is monochromatic; the empty set is monochromatic. Determine the largest possible cardinality of a subfamily $\mathcal{G} \subseteq \mathcal{F}$, no two sets of which are disjoint.

Russia, Andrey Kupavskii and Maksim Turevskii

Solution. The required maximum is $2^{m-1}(2^m+1)^{k-1}$ and is achieved if, for instance, $\mathcal{G}$ consists of all sets in $\mathcal{F}$ containing a fixed blue element.

We now prove that $|\mathcal{G}| \leq 2^{m-1}(2^m+1)^{k-1}$ for any $\mathcal{G}$ satisfying the conditions in the statement. For convenience, write $M = 2^m + 1$. Let r_i denote the red element of S_i, and let B_i be the set of blue elements in S_i.

For every subset $X_i \subset B_i$ and every $j \in \mathbb{Z}_M$, define the sets

$$T_{X_i,j} = \begin{cases} \{r_i\}, & \text{if } j = 0; \\ X_i, & \text{if } j \neq 0 \text{ and } j \text{ is even (considered as a number in } [1, M-1]); \\ B_i \setminus X_i, & \text{if } j \neq 0 \text{ and } j \text{ is odd (considered as a number in } [1, M-1]). \end{cases}$$

Note that, for every i and every j, the sets $T_{X_i,j}$ and $T_{X_i,j+1}$ are disjoint. Now, for every sets $X_i \subset B_i$ and every elements $j_i \in \mathbb{Z}_M$, $i = 1, 2, \ldots, k$, denote

$$F(X_1, X_2, \ldots, X_k, j_1, j_2, \ldots, j_k) = \bigcup_{i=1}^{k} T_{X_i,j_i}. \qquad (*)$$

Claim. Every set $F \in \mathcal{F}$ has exactly 2^{mk} representations of the form $(*)$.

Proof. Set $F_i = F \cap S_i$. If $F_i = \{r_i\}$, then there are 2^m possible choices for X_i, and one should necessarily have $j_i = 0$. Otherwise, there are only two possible choices for X_i, namely $X_i = F_i$ and $X_i = B_i \setminus F_i$, and for each of them there are 2^{m-1} possible choices for j_i. So, whatever F, there are 2^m possible choices for each pair (X_i, j_i) all of which can be made independently, whence a total of 2^{mk} possible tuples $(X_1, X_2, \ldots, X_k, j_1, j_2, \ldots, j_k)$. This proves the Claim.

The Claim implies that each $F \in \mathcal{F}$ has the same number of representations of the form $(*)$. Thus, it suffices to show that, among all $N = 2^{km}(2^m+1)^k$ tuples $(X_1, X_2, \ldots, X_k, j_1, j_2, \ldots, j_k)$, at most $\dfrac{2^{m-1}}{2^m+1}N$ satisfy $F(X_1, X_2, \ldots, X_k, j_1, j_2, \ldots, j_k) \in \mathcal{G}$.

To this end, split all these tuples into length M cycles

$$\big(F(X_1, X_2, \ldots, X_k, j_1, j_2, \ldots, j_k), F(X_1, X_2, \ldots, X_k, j_1+1, j_2+1, \ldots, j_k+1), \ \ldots, \\ F(X_1, X_2, \ldots, X_k, j_1+M-1, j_2+M-1, \ldots, j_k+M-1)\big),$$

and note that any two adjacent sets of a cycle are disjoint. Hence each cycle contains at most $\lfloor M/2 \rfloor = 2^{m-1}$ sets from $\mathcal{G}$. This provides the desired upper bound and completes the solution.

The 17$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 1: The 25$^{\text{th}}$ of February, 2026, Bucharest

Language: English

Problem 1. Let n be a positive integer. Alice draws a unit area triangle on the board. Then she draws additional triangles by performing n moves in a row. On each move, she chooses a drawn triangle Δ with no marked points in its interior, marks a point P in its interior, and draws three smaller triangles by joining P to each vertex of Δ with a segment.

Once these n moves have been performed, Bob chooses three distinct drawn triangles Δ_1, Δ_2, and Δ_3 which contain no marked points in their interiors, such that Δ_2 shares one side with Δ_1 and another with Δ_3. In terms of n, determine the largest possible constant c such that Bob can guarantee that the sum of the areas of Δ_1, Δ_2, and Δ_3 is at least c, regardless of Alice's choices.

Problem 2. Let $p \geq 11$ be a prime. Suppose that, if a and b are integers such that $1 \leq a < b \leq p - 3$, then $b! - a!$ is not divisible by p. Prove that $p - 5$ is divisible by 8.

Problem 3. Let $\mathcal{S}$ be a finite subset of $\mathbb{R}^3$. Prove that there exist three polynomials $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ with real coefficients, such that a triple of real numbers (a, b, c) is in $\mathcal{S}$ if and only if the system of equations

$$P(x, y, z) = a,$$

$$Q(x, y, z) = b,$$

$$R(x, y, z) = c,$$

does **not** have a solution in real numbers x, y, and z.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

The 17^{th} Romanian Master of Mathematics Competition

Day 2: The 26^{th} of February, 2026, Bucharest

Language: English

Problem 4. For any positive integer m, let $\varphi(m)$ be the number of positive integers less than or equal to m and coprime to m. Define $\varphi_0(m) = m$ and, for each positive integer k, $\varphi_k(m) = \varphi(\varphi_{k-1}(m))$. For any integer $n \geq 3$, prove that

$$\varphi_0(2^n - 3) \cdot \varphi_1(2^n - 3) \cdot \varphi_2(2^n - 3) \cdot \ldots \cdot \varphi_n(2^n - 3)$$

has at most n distinct prime divisors.

Problem 5. Let ABC be a triangle with $AB < AC$, let O be its circumcentre and let $XYZT$ be a parallelogram inside triangle ABC such that

$$\angle AXB = \angle AZC, \ \angle AZB = \angle AXC,$$

$$\angle AYB = \angle ATC, \ \angle ATB = \angle AYC.$$

Prove that the diagonals XZ and YT of the parallelogram intersect on the circumcircle of BOC.

Problem 6. Let $k > 1$ be an integer, and let S denote the set of all $(k+1)$-tuples of integers $X = (x_1, \ldots, x_{k+1})$ such that $1 \leq x_1 < \cdots < x_{k+1} \leq k^2+1$. If σ is a permutation of the numbers $1, 2, \ldots, k^2+1$, say that an element X of S is σ-*nice* if the sequence $\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_{k+1})$ is monotone. Prove that

$$\min_{1 \leq i \leq k} \left\lfloor \frac{x_i}{i} \right\rfloor + \min_{2 \leq i \leq k+1} \left\lfloor \frac{k^2+2-x_i}{k+2-i} \right\rfloor \geq k+1$$

if and only if there exists a permutation σ such that X is the unique σ-nice tuple in S.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.