COMPENDIUM INMO

Indian National Math Olympiad 1986-2021

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Índex.

INDIAN NATIONAL MATH OLYMPIAD 1986

Time : 3 hours] [Max Marks 100

Attempt all questions.

Q.1 A person who left home between 4 p.m. and 5 p.m. returned between 5 p.m. and 6 p.m. and found that the hands of his watch had exactly exchanged place, when did he go out ?

Q.2 Solve.

$$\mathrm{Log}_2 x + \log_4 y + \log_4 z = 2$$

$$\mathrm{Log}_3 y + \log_9 z + \log_9 x = 2$$

$$\log_4 z + \log_{16} x + \log_{16} y = 2$$

Q.3 Two circles with radii a and b respectively touch each other externally. Let c be the radius of a circle that touches these two circles as well as a common tangent to the two circles. Prove that

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

Q.4 Find the least natural number whose last digit is 7 such that it becomes 5 times larger when this last digit is carried to the beginning of the number.

Q.5 If P (x) is a polynomial with integer coefficients and a, b, c, three distinct integers, then show that it is impossible to have P (a) = b, P (b) = c, P (c) = a.

Q.6 Construct a quadrilateral which is not a parallelogram, in which a pair of opposite angles and a pair of opposite sides are equal.

Q.7 If a, b, x, y are integers greater than 1 such that a and b have no common factor except 1 and $x^a = y^b$ show that $x = n^b$,$y = n^a$ for some integer n greater than 1.

Q.8 Suppose A_1, A_6 are six sets each with four elements and B_1 …… Bn are n sets each with two elements, Let S = A_1UA_2 U…UA_6 = B_1U…UB_n. Given that each elements of S belogs to exactly four of the A's and to exactly three of the B's, find n.

Q.9 Show that among all quadrilaterals of a given perimeter the square has the largest area.

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INDIAN NATIONAL MATHEMATICS OLYMPIAD 1987

SECOND PHASE

Time 3 hours] **INMO 1987** [Max Marks 100

Attempt all questions.

Q.1 Given m and n as relatively prime positive integers greater than one, show that $\log_{10}m/\log_{10}n$ is not a rational number.

Q.2 Determine the largest number in the infinite sequence

$1, \sqrt[2]{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots \sqrt[n]{n} \ldots$

Q.3 Let T be the set of all triplets (a, b, c) of integers such that $1 \leq a \angle b \angle c \leq 6$ For each triplet (a,b,c) in T, take number axbxc Add all these numbers corresponding to all the triplets in T. Prove that the answer is divisible by 7.

Q.4 If x, y, z, and n are natural numbers, and $n \geq z$ then prove that the relation $x^n + y^n = z^n$ does not hold.

Q.5 Find a finite sequence of 16 numbers such that:

[a] it reads same from left to right as from right to left.

[b] the sum of any 7 consecutive terms is – 1,

[c] the sum of any 11 consecutive terms is + 1.

Q.6 Prove that if coefficients of the quadratic equation $ax^2+bx+c=0$ are odd integers, then the roots of the equation cannot be rational numbers.

Q.7 Construct the Δ ABC, given h_a, h_b (the altitudes from A and B) and m_a, the median from the vertex A.

Q. 8 Three congruent circles have a common point 0 and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the in-centre and the circum-centre of the triangle and the common point 0 are collinear.

Q. 9 Prove that any triangle having two equal internal angle bisectors (each measured from a vertex to the opposite side) is isosceles.

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NBHM(DAE)

INDIAN NATIONAL MATH OLYMPIAD 1988

Time 3 hours] [Max Marks 100

Attempt all questions.

Q. 1 Let m_1, $m_{2,}$ m_3 …….,m_n be a rearrangement of the numbers 1,2 ……., n. Suppose that n is odd. Prove that the product $(m_1 - 1)(m_2 - 2)$ …… $(m_n - n)$ is an even integer.

Q. 2 Prove that the product of 4 consecutive natural numbers cannot be a perfect cube.

Q. 3 Five men, A, B, C, D, D, E are wearing caps of block or white colour without each knowing the colour of his cap. It is known that a man wearing black cap always speaks the truth while the ones wearing white always tell lies. If they make the following statements, find the colour worn by each of them:

A : I see three black caps and one white cap.

B : I see four white caps

C : I see one black cap and three white caps

D : I see your four black caps.

Q. 4 If a and b are positive and a + b = 1, prove that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq 12\frac{1}{2}$$

Q.5 Show that there do not exist any distinct natural numbers a, b, c, d such that

$a^3 + b^3 = c^3 + d^3$ and

$a + b = c + d$

Q. 6 If a_0, a_1, ………… a_{50} are confidents of the polynomial $(1+x+x^2)^{25}$ show that $a_0+a_2+a_4+\ldots\ldots+a_{50}$ is even.

Q. 7 Given an angle Q B P and a point L outside the angle OBP. Draw a straight line through L meeting BQ in A and BP in C such that the triangle ABC has a given perimeter.

Q.8 A river flows between two houses A and B, the houses standing some distances away from the banks. Where should a bridge be built on the river so that a person going from A to B, using the bridge to cross the river may do so by the shortest path? Assume that the banks of the river are straight and parallel, and the bridge must be perpendicular to the banks.

Q. 9 Show that for a triangle with radii of circum-circle and in-circle equal to R, r respectively, the inequality $R \geq 2r$ holds

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4-th Indian Mathematical Olympiad 1989

1. Prove that the polynomial
$$f(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$$
is irreducible over $\mathbb{Z}[x]$.

2. Let a,b,c,d be real numbers, not all zero. Prove that the roots of the polynomial $x^6 + ax^3 + bx^2 + cx + d$ cannot all be real.

3. Let A be a subset of the set $\{1, 11, 21, 31, \dots, 551\}$ whose no two elements add up to 552. Show that A has not more than 28 elements.

4. Find all natural numbers n such that

   (i) n is not a square, and

   (ii) $[\sqrt{n}]^3$ divides n^2.

5. Let a,b,c be the sides of a triangle. Show that the quantity
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$
must lie between $\frac{3}{2}$ and 2. Can the equality hold at either limit?

6. In a scalene triangle ABC the angle at A is obtuse. Determine the set of points on the extended side BC such that $AD = \sqrt{BD \cdot CD}$.

7. A triangle ABC is acute-angled. For any point P within the triangle, D,E, and F denote the projections of P onto BC, CA, AB respectively. Find the locus of P for which triangle DEF is isosceles. When is $\triangle DEF$ equilateral?

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5-th Indian Mathematical Olympiad 1990

1. If the equation $x^4 + px^3 + qx^2 + rx + s = 0$ has four positive real roots, prove that

   (a) $pr - 16r \geq 0$

   (b) $q^2 - 36s \geq 0$

   with equality in each case if and only if the four roots are equal.

2. Find all pairs of nonnegative integers (x, y) satisfying $(xy - 7)^2 = x^2 + y^2$.

3. Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be a function that satisfies

   (i) $x - f(x) = 19[\frac{x}{19}] - 90[\frac{f(x)}{90}]$ for all nonnegative integers x;

   (ii) $1900 < f(1990) < 2000$.

   Find all possible values that $f(1990)$ can take.

4. Determine the number of three-element subsets of $\{1, 2, 3, \dots, 300\}$ for which the sum of the elements is a multiple of 3.

5. Let a, b, c be the sides of a triangle. Show that the quantity

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

   must lie between $\frac{3}{2}$ and 2. Can equality hold at either limit?

6. In a scalene triangle ABC the angle at A is obtuse. Determine the set of points D lying on the extended line BC for which $AD = \sqrt{BD \cdot CD}$.

7. For any point P lying within a given acute-angled triangle ABC, let D, E, F denote the feet of the perpendiculars from P onto AB, BC, CA respectively. Find the set of all positions of P for which the triangle DEF is isosceles. For which position of P is the triangle DEF equilateral?

6-th Indian Mathematical Olympiad 1991

1. Find the number of positive integers n such that

   (i) $n \leq 1991$,

   (ii) $n^2 + 3n + 2$ is a multiple of 6.

2. In an acute-angled triangle ABC, the altitude from A meets the semicircle with diameter BC constructed outwards at point A'. Points B' and C' are defined analogously. Prove that
$$S^2_{BCA'} + S^2_{CAB'} + S^2_{ABC'} = S^2_{ABC},$$
where S_{XYZ} denotes the area of triangle XYZ.

3. Given a triangle ABC, denote
$$\begin{aligned} x &= \tan\frac{B-C}{2}\tan\frac{A}{2}, \\ y &= \tan\frac{C-A}{2}\tan\frac{B}{2}, \\ z &= \tan\frac{A-B}{2}\tan\frac{C}{2}. \end{aligned}$$
Prove that $x + y + z + xyz = 0$.

4. Let a, b, c be real numbers in the interval $(0,1)$ with $a + b + c = 2$. Prove that
$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8.$$

5. In a triangle ABC with incenter I, points X, Y are taken on the segments AB, AC respectively such that $BX \cdot AB = IB^2$ and $CY \cdot AC = IC^2$. Given that the points X, I, Y are collinear, find the possible values of $\angle A$.

6. (a) Find all positive integers n for which 3^{n+1} divides $2^{3^n} + 1$.

   (b) Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

7. Determine all real solutions x, y, z of the system
$$\begin{cases} x + y - z = 4, \\ x^2 - y^2 + z^2 = -4, \\ xyz = 6. \end{cases}$$

8. We are given 10 objects of integer weights with the total weight 20. Prove that if none of the weights exceds 10, then the objects can be divided into two groups of equal weights.

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9. The incircle l of a triangle ABC is centered at I and touches the side BC at T. The line through T parallel to IA meets the incircle again at S and the tangent to the incircle at S meets AB, AC at points C', B', respectively. Prove that the triangle $AB'C'$ is similar to the triangle ABC.

10. For any positive integer n, let $s(n)$ denote the number of ordered pairs (x,y) of positive integers for which
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n}\,.$$
Determine all those n for which $s(n)=5$.

7-th Indian Mathematical Olympiad 1992

1. In a triangle ABC, $\angle A = 2\angle B$. Prove that $a^2 = b(b+c)$.

2. If real numbers x,y,z satisfy $x+y+z=4$ and $x^2+y^2+z^2=6$, show that each of x,y,z lies in the segment $\left[\frac{2}{3},2\right]$. Can x attain either of the endpoints of the segment?

3. Determine the remainder of 19^{92} upon division by 92.

4. Find the number of permutations $(p_1,\dots,p_6)$ of $1,2,\dots,6$ such that for any k, $1 \le k \le 5$, $(p_1,\dots,p_k)$ does not form a permutation of $1,2,\dots,k$.

5. Two circles C_1 and C_2 in the plane meet at points P and $Q \neq P$. A line through P meets C_1 at A and C_2 at B. Let Y be the midpoint of AB and let QY meet the cirlces C_1 and C_2 again at X and Z respectively. Show that Y is the midpoint of XZ.

6. Let $f(x)$ be a polynomial with integer coefficients such that there exist distinct integers $a_1,\dots,a_5$ at which f takes the value 2. Show that there does not exist an integer b with $f(b)=9$.

7. For each integer $n \ge 3$, find the number of ways in which one can place the numbers $1,2,\dots,n^2$ in the squares of an $n \times n$ chessboard (one on each) such that the numbers in each row and in each column form an arithmetic progression.

8. Find all pairs (m,n) of positive integers for which $2^m + 3^n$ is a perfect square.

9. Find n such that in a regular n-gon $A_1A_2\dots A_n$ we have
$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}.$$

10. Determine all functions $f : \mathbb{R} \setminus [0,1] \to \mathbb{R}$ such that for all x,
$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}.$$

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8-th Indian Mathematical Olympiad 1993

1. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at point P. Let O be the circumcenter of triangle APB and H be the orthocenter of triangle CPD. Show that the points H,P, and O lie on a line.

2. Consider a quadratic polynomial $P(x) = x^2 + ax + b$ with $a,b \in \mathbb{Z}$. Show that for any integer n there is an integer m such that $P(n)P(n+1) = P(m)$.

3. If a,b,c,d are positive numbers with $a+b+c+d = 1$, prove that
$$ab+bc+cd \leq \frac{1}{4}\,.$$
Does the analogous inequality hold for n variables?

4. Find the set of all points P in the set of a triangle ABC such that $P \neq A,B,C$ and the triangles ABP, BCP, and CAP have the same circumradii.

5. Show that there exists a natural number n such that $n!$ in decimal system ends in exactly 1993 zeros.

6. Let $\mathscr{S}$ be the circumcircle of a right triangle ABC with $\angle A = 90^\circ$. Circle $\mathscr{S}_1$ is tangent to the lines AB and AC and internally to $\mathscr{S}$. Circle $\mathscr{S}_2$ is tangent to AB and AC and externally to $\mathscr{S}$. If r_1 and r_2 are the radii of S_1 and S_2 prove that $r_1 \cdot r_2$ equals four times the area of $\triangle ABC$.

7. Let B be a 53-element subset of $A = \{1,2,3,\dots,100\}$. Prove that there are two distinct elements $x,y \in B$ whose sum is divisible by 11.

8. Let f be a bijective function from $A = \{1,2,\dots,n\}$ to itself. Prove that there is a positive integer M such that $f^M(i) = i$ for each $i \in A$, where $f^M = f \circ f \circ \cdots \circ f$ (M times).

9. Prove that there exists a convex hexagon in the plane whose all interior angles are equal and whose side lengths are $1,2,3,4,5,6$ in some order.

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9-th Indian Mathematical Olympiad 1994

1. In a triangle ABC with an obtuse angle at C, AD and CF are the medians and G the centroid.

   (a) If points B, D, G, F lie on a circle, show that $\frac{AC}{BC} \geq \sqrt{2}$.

   (b) Moreover, if P is the fourth vertex of the parallelogram $AGCP$, prove that triangle GAP is similar to $\triangle ABC$.

2. Prove that if x is a real root of $x^5 - x^3 + x = a$, then $x^6 \geq 2a - 1$.

3. Prove that among any 181 perfect squares there exist 19 whose sum is divisible by 19.

4. Find the number of (nondegenerate) triangles whose vertices lie in the set of points (s, t) in the plane with $s, t \in \{0, 1, 2, 3, 4\}$.

5. A circle through vertex C of a rectangle $ABCD$ is tangent to sides AB and AD at M and N. Given that the distance from C to the line MN equals 5, compute the area of rectangle $ABCD$.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(-x) = f(x) \quad \text{and} \quad f(x+1) = f(x) + 1 \quad \text{for all } x \text{, and}$$
$$f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2} \quad \text{for all } x \neq 0.$$

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10-th Indian Mathematical Olympiad 1995

1. In an acute-angled triangle ABC with $\angle A = 30^\circ$, H is the orthocenter and M the midpoint of BC. Point T is symmetric to H with respect to M. Show that $AT = 2BC$.

2. Show that there are infinitely many pairs (a,b) of coprime integers such that both the quadratic equations $x^2+ax+b=0$ and $x^2+2ax+b=0$ have integer roots.

3. Show that the number of three-element subsets $\{a,b,c\}$ of $\{1,2,\dots,63\}$ with $a+b+c<95$ is less than the number of those with $a+b+c>95$.

4. Let Γ' be the circle lying inside a triangle ABC and touching the sides AB and AC and the incircle Γ of the triangle externally. Show that the ratio of the radii of the circles Γ' and Γ equals $\tan^2 \frac{\pi-\alpha}{4}$.

5. The real numbers $a_1, a_2, \dots, a_n$ are all greater than 1 and satisfy $|a_k - a_{k+1}| < 1$ for $1 \le k \le n-1$. Prove that
$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1} < 2n-1.$$

6. Find all primes p for which $\dfrac{2^{p-1}-1}{p}$ is a perfect square.

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11-th Indian Mathematical Olympiad 1996

1. (a) Show that, for any positive integer n, there exist distinct positive integers x and y such that $x+j$ divides $y+j$ for $j=1,2,\ldots,n$.

   (b) If for some positive integers x and y, $x+j$ divides $y+j$ for all positive integers j, show that $x=y$.

2. Let C_1 and C_2 be two concentric circles in the plane with radii R and $3R$ respectively. Show that the orthocenter of any triangle inscribed in C_1 lies in the interior of C_2. Conversely, show that every point in the interior of C_2 is the orthocenter of some triangle inscribed in C_1.

3. Solve in real numbers a,b,c,d,e the following system of equations:

$$3a=(b+c+d)^3, \quad 3b=(c+d+e)^3, \quad 3c=(d+e+a)^3,$$
$$3d=(e+a+b)^3, \quad 3e=(a+b+c)^3.$$

4. Find the number of ordered triples (A,B,C) of subsets of a given n-element set X such that $A\subset B\subsetneq C$.

5. The sequence $(a_n)_{n\in\mathbb{N}}$ is defined by $a_1=1$, $a_2=2$, and

$$a_{n+2}=2a_{n+1}-a_n+2 \quad \text{for } n\geq 1.$$

   Prove that for any m, a_ma_{m+1} is also a term of the sequence.

6. Given a $2n\times 2n$ array of 0's and 1's containing exactly $3n$ zeros, show that it is possible to remove all the zeros by deleting some n rows and n columns.

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12-th Indian Mathematical Olympiad 1997

1. A line through the vertex C of a parallelogram $ABCD$ meets the extensions of sides AB and AD at E and F respectively. Prove that

$$AC^2 + CE \cdot CF = AB \cdot AE + AD \cdot AF.$$

2. Show that there do not exist positive integers m and n such that

$$\frac{m}{n} + \frac{n+1}{m} = 4.$$

3. Suppose that a, b, c are distinct real numbers and t a real number such that $a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a} = t$. Show that $abc + t = 0$.

4. One hundred rays emanating from the center of a square divide the square into 100 parts, all having equal parameter p. Show that $1.4 < p < 1.5$.

5. Find the number of 4×4 arrays tables whose entries are from the set $\{0, 1, 2, 3\}$ such that the sum of the numbers in each of the four rows and in each of the four columns is divisible by 4.

6. Let a and b be positive numbers for which the cubic equation $x^3 - ax + b = 0$ has three (not necessarily distinct) real roots. If α is the one with minimal absolute value, prove that

$$\frac{b}{a} < \alpha < \frac{3b}{2a}.$$

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13-th Indian Mathematical Olympiad 1998

1. Let AB be a chord of a circle $\mathscr{C}_1$ that is not a diameter, and M be the midpoint of AB. Let T be a point on the circle $\mathscr{C}_2$ with OM as diameter. The tangent to $\mathscr{C}_2$ at T meets $\mathscr{C}_1$ at P. Show that
$$PA^2 + PB^2 = 4PT^2.$$

2. Let a and b be positive rational numbers. Prove that if $\sqrt[3]{a} + \sqrt[3]{b}$ is a rational number, then so are $\sqrt[3]{a}$ and $\sqrt[3]{b}$.

3. Let p,q,r,s be four integers with $5 \nmid s$. If there is an integer a for which $pa^3 + qa^2 + ra + s$ is divisible by 5, prove that there is an integer b such that $sb^3 + rb^2 + qb + p$ is also divisible by 5.

4. A convex quadrilateral $ABCD$ is inscribed in a circle of unit radius. Show that if $AB \cdot BC \cdot CD \cdot DA \geq 4$, then $ABCD$ is a square.

5. Suppose a,b,c are real numbers such that the quadratic equation
$$x^2 - (a+b+c)x + (ab+bc+ca) = 0$$
has roots of the form $\alpha \pm i\beta$, where $\alpha > 0$ and $\beta \neq 0$ are real numbers. Show that:
   (a) The numbers a,b,c are all positive.
   (b) The numbers $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are the sides of a triangle.

6. We want to choose n of the $2n$ integers $0,0,1,1,2,2,\dots,n-1,n-1$ such that the average of the n chosen integers is an integer and as small as possible. Show that this can be done for each positive integer n and find this smallest value.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
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14-th Indian Mathematical Olympiad 1999

1. Points D,E,F are taken on the sides BC,CA,AB of an acute-angled triangle ABC such that $AD \perp BC$, $AE = BC$, and CF bisects $\angle C$. Suppose that CF meets AD at M and DE at N such that $FM = 2$, $MN = 1$, and $NC = 3$. Find the perimeter of $\triangle ABC$.

2. In a village 1998 persons volunteered to clean up, for a fair, a rectangular field with integer sides and perimeter equal to 3996 feet. For this purpose, the field was divided into 1998 equal parts. If each part had an integer area, find the length and width of the field.

3. Prove that there are no nonconstant polynomials $p(x)$ and $q(x)$ with integer coefficients such that $p(x)q(x) = x^5 + 2x + 1$.

4. The equilateral triangles ABC and $A_1B_1C_1$ are inscribed in concentric circles Γ and Γ' respectively. If P and P' are arbitrary points on Γ and Γ' respectively, prove that
$$P'A^2 + P'B^2 + P'C^2 = PA'^2 + PB'^2 + PC'^2.$$

5. Show that among any four distinct positive numbers one can choose three, say A,B,C, such that the three quadratic equations
$$\begin{aligned} Bx^2 + x + C &= 0 \\ Cx^2 + x + A &= 0 \\ Ax^2 + x + B &= 0 \end{aligned}$$
either all have real roots or all have non-real roots.

6. For which positive integers n can the set $\{1,2,3,\dots,4n\}$ be split into n disjoint four-element subsets $\{a,b,c,d\}$ such that in each of them $a = \frac{b+c+d}{3}$?

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The IMO Compendium Group,
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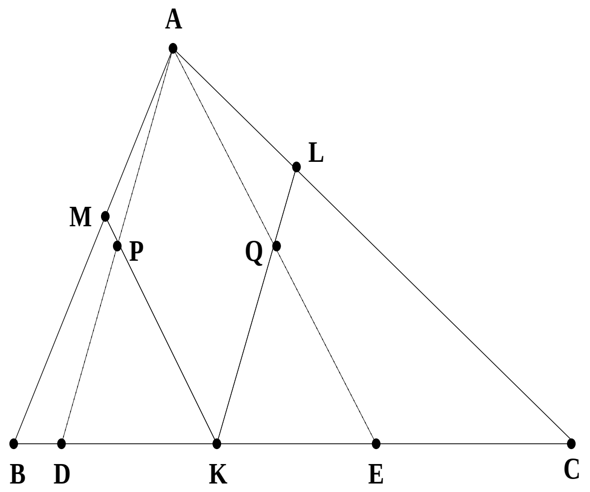
INNO-2000
Problems and Solutions

1. The in-circle of triangle ABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of triangle ABC.

   Solution. : Let AP, AQ produced meet BC in D, E respectively.



   Since MK is parallel to AE, we have $\angle AEK = \angle MKB$. Since $BK = BM$, both being tangents to the circle from B, $\angle MKB = \angle BMK$. This with the fact that MK is parallel to AE gives us $\angle AEK = \angle MAE$. This shows that $MAEK$ is an isosceles trapezoid. We conclude that $MA = KE$. Similarly, we can prove that $AL = DK$. But $AM = AL$. We get that $DK = KE$. Since KP is parallel to AE, we get $DP = PA$ and similarly $EQ = QA$. This implies that PQ is parallel to DE and hence bisects AB, AC when produced.

   [The same argument holds even if one or both of P and Q lie outside triangle ABC.]

2. Solve for integers x, y, z:

$$x + y = 1 - z, \qquad x^3 + y^3 = 1 - z^2.$$

   Sol. : Eliminating z from the given set of equations, we get

$$x^3 + y^3 + \{1 - (x + y)\}^2 = 1.$$

This factors to

$$(x+y)(x^2-xy+y^2+x+y-2)=0.$$

Case 1. Suppose $x+y=0$. Then $z=1$ and $(x,y,z)=(m,-m,1)$, where m is an integer give one family of solutions.

Case 2. Suppose $x+y\neq 0$. Then we must have

$$x^2-xy+y^2+x+y-2=0.$$

This can be written in the form

$$(2x-y+1)^2+3(y+1)^2=12.$$

Here there are two possibilities:

$$2x-y+1=0, y+1=\pm 2; \qquad 2x-y+1=\pm 3, y+1=\pm 1.$$

Analysing all these cases we get

$$(x,y,z)=(0,1,0),(-2,-3,6),(1,0,0),(0,-2,3),(-2,0,3),(-3,-2,6).$$

3. If a,b,c,x are real numbers such that $abc\neq 0$ and

$$\frac{xb+(1-x)c}{a}=\frac{xc+(1-x)a}{b}=\frac{xa+(1-x)b}{c},$$

then prove that either $a+b+c=0$ or $a=b=c$.

Sol. : Suppose $a+b+c\neq 0$ and let the common value be λ. Then

$$\lambda=\frac{xb+(1-x)c+xc+(1-x)a+xa+(1-x)b}{a+b+c}=1.$$

We get two equations:

$$-a+xb+(1-x)c=0, \qquad (1-x)a-b+xc=0.$$

(The other equation is a linear combination of these two.) Using these two equations, we get the relations

$$\frac{a}{1-x+x^2}=\frac{b}{x^2-x+1}=\frac{c}{(1-x)^2+x}.$$

Since $1-x+x^2\neq 0$, we get $a=b=c$.

4. In a convex quadrilateral $PQRS$, $PQ = RS$, $(\sqrt{3}+1)QR = SP$ and $\angle RSP - \angle SPQ = 30^\circ$. Prove that

$$\angle PQR - \angle QRS = 90^\circ.$$

Sol. : Let [Fig] denote the area of Fig. We have

$$[PQRS] = [PQR] + [RSP] = [QRS] + [SPQ].$$

Let us write $PQ = p, QR = q, RS = r, SP = s$. The above relations reduce to

$$pq \sin\angle PQR + rs \sin\angle RSP = qr \sin\angle QRS + sp \sin\angle SPQ.$$

Using $p = r$ and $(\sqrt{3}+1)q = s$ and dividing by pq, we get

$$\sin\angle PQR + (\sqrt{3}+1)\sin\angle RSP = \sin\angle QRS + (\sqrt{3}+1)\sin\angle SPQ.$$

Therefore, $\sin\angle PQR - \sin\angle QRS = (\sqrt{3}+1)(\sin\angle SPQ - \sin\angle RSP)$.

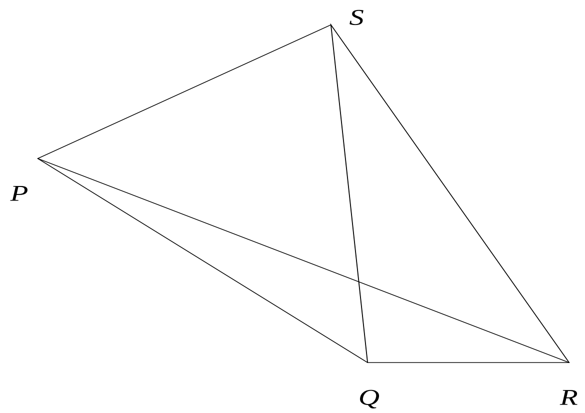


Fig. 2.

This can be written in the form

$$2\sin\frac{\angle PQR - \angle QRS}{2}\cos\frac{\angle PQR + \angle QRS}{2}$$
$$= (\sqrt{3}+1)2\sin\frac{\angle SPQ - \angle RSP}{2}\cos\frac{\angle SPQ + \angle RSP}{2}.$$

Using the relations

$$\cos\frac{\angle PQR + \angle QRS}{2} = -\cos\frac{\angle SPQ + \angle RSP}{2}$$

and

$$\sin\frac{\angle SPQ - \angle RSP}{2} = -\sin 15^\circ = -\frac{(\sqrt{3}-1)}{2\sqrt{2}},$$

we obtain

$$\sin\frac{\angle PQR-\angle QRS}{2}=(\sqrt{3}+1)[-\frac{(\sqrt{3}-1)}{2\sqrt{2}}]=\frac{1}{\sqrt{2}}.$$

This shows that

$$\frac{\angle PQR-\angle QRS}{2}=\frac{\pi}{4}\quad\text{or}\quad\frac{3\pi}{4}.$$

Using the convexity of $PQRS$, we can rule out the latter alternative. We obtain

$$\angle PQR-\angle QRS=\frac{\pi}{2}.$$

5. Let a,b,c be three real numbers such that $1\geq a\geq b\geq c\geq 0$. Prove that if λ is a root of the cubic equation $x^3+ax^2+bx+c=0$ (real or complex),then $|\lambda|\leq 1$.

Sol. : Since λ is a root of the equation $x^3+ax^2+bx+c=0$, we have

$$\lambda^3=-a\lambda^2-b\lambda-c.$$

This implies that

$$\begin{aligned}\lambda^4 &= -a\lambda^3-b\lambda^2-c\lambda\\ &= (1-a)\lambda^3+(a-b)\lambda^2+(b-c)\lambda+c\end{aligned}$$

where we have used again

$$-\lambda^3-a\lambda^2-b\lambda-c=0.$$

Suppose $|\lambda|\geq 1$. Then we obtain

$$\begin{aligned}|\lambda|^4 &\leq (1-a)|\lambda|^3+(a-b)|\lambda|^2+(b-c)|\lambda|+c\\ &\leq (1-a)|\lambda|^3+(a-b)|\lambda|^3+(b-c)|\lambda|^3+c|\lambda|^3\\ &\leq |\lambda|^3.\end{aligned}$$

This shows that $|\lambda|\leq 1$. Hence the only possibility in this case is $|\lambda|=1$. We conclude that $|\lambda|\leq 1$ is always true.

6. For any natural number n, $(n\geq 3)$, let $f(n)$ denote the number of non-congruent integer-sided triangles with perimeter n (e.g., $f(3)=1, f(4)=0, f(7)=2$). Show that

$$\begin{aligned}&\text{(a)}\quad f(1999)>f(1996);\\ &\text{(b)}\quad f(2000)=f(1997).\end{aligned}$$

Sol. :

(a) Let a,b,c be the sides of a triangle with $a+b+c=1996$, and each being a positive integer. Then $a+1,b+1,c+1$ are also sides of a triangle with perimeter 1999 because

$$a<b+c\quad\Longrightarrow\quad a+1<(b+1)+(c+1),$$

and so on. Moreover $(999, 999, 1)$ form the sides of a triangle with perimeter 1999, which is not obtainable in the form $(a+1, b+1, c+1)$ where a, b, c are the integers and the sides of a triangle with $a + b + c = 1996$. We conclude that $f(1999) > f(1996)$.

(b) As in the case (a) we conclude that $f(2000) \geq f(1997)$. On the other hand, if x, y, z are the integer sides of a triangle with $x+y+z = 2000$, and say $x \geq y \geq z \geq 1$, then we cannot have $z = 1$; for otherwise we would get $x + y = 1999$ forcing x, y to have opposite parity so that $x - y \geq 1 = z$ violating triangle inequality for x, y, z. Hence $x \geq y \geq z > 1$. This implies that $x - 1 \geq y - 1 \geq z - 1 > 0$. We already have $x < y + z$. If $x \geq y + z - 1$, then we see that $y + z - 1 \leq x < y + z$, showing that $y + z - 1 = x$. Hence we obtain $2000 = x + y + z = 2x + 1$ which is impossible. We conclude that $x < y + z - 1$. This shows that $x - 1 < (y - 1) + (z - 1)$ and hence $x - 1, y - 1, z - 1$ are the sides of a triangle with perimeter 1997. This gives $f(2000) \leq f(1997)$. Thus we obtain the desired result.

INNMO-2000
Problems and Solutions

1. The in-circle of triangle ABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of triangle ABC.

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   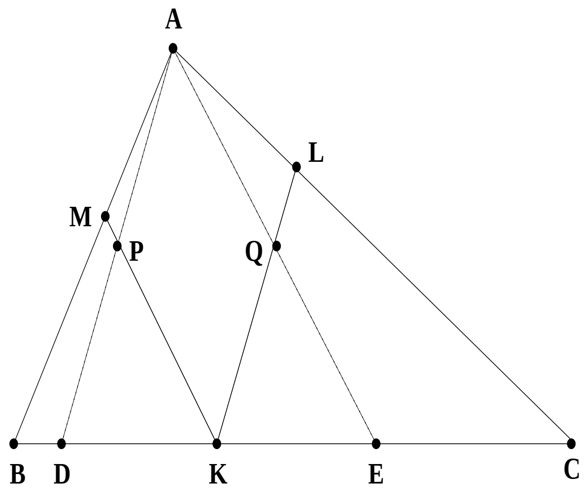

   Since MK is parallel to AE, we have $\angle AEK = \angle MKB$. Since $BK = BM$, both being tangents to the circle from B, $\angle MKB = \angle BMK$. This with the fact that MK is parallel to AE gives us $\angle AEK = \angle MAE$. This shows that $MAEK$ is an isosceles trapezoid. We conclude that $MA = KE$. Similarly, we can prove that $AL = DK$. But $AM = AL$. We get that $DK = KE$. Since KP is parallel to AE, we get $DP = PA$ and similarly $EQ = QA$. This implies that PQ is parallel to DE and hence bisects AB, AC when produced.

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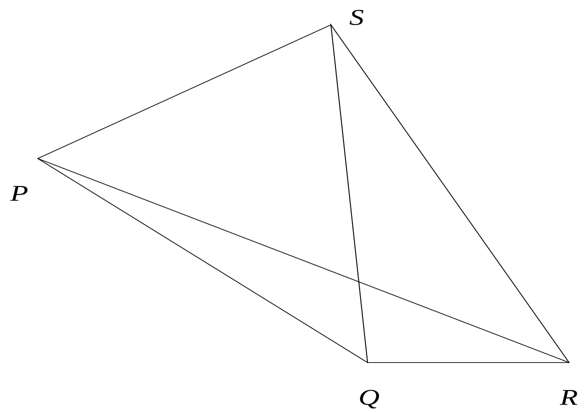

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where we have used again

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Suppose $|\lambda|\geq 1$. Then we obtain

$$\begin{aligned}|\lambda|^4 &\leq (1-a)|\lambda|^3+(a-b)|\lambda|^2+(b-c)|\lambda|+c\\ &\leq (1-a)|\lambda|^3+(a-b)|\lambda|^3+(b-c)|\lambda|^3+c|\lambda|^3\\ &\leq |\lambda|^3.\end{aligned}$$

This shows that $|\lambda|\leq 1$. Hence the only possibility in this case is $|\lambda|=1$. We conclude that $|\lambda|\leq 1$ is always true.

6. For any natural number n, $(n\geq 3)$, let $f(n)$ denote the number of non-congruent integer-sided triangles with perimeter n (e.g., $f(3)=1, f(4)=0, f(7)=2$). Show that

$$\begin{aligned}&\text{(a)}\quad f(1999)>f(1996);\\ &\text{(b)}\quad f(2000)=f(1997).\end{aligned}$$

Sol. :

(a) Let a, b, c be the sides of a triangle with $a+b+c=1996$, and each being a positive integer. Then $a+1, b+1, c+1$ are also sides of a triangle with perimeter 1999 because

$$a<b+c\quad\Longrightarrow\quad a+1<(b+1)+(c+1),$$

and so on. Moreover $(999, 999, 1)$ form the sides of a triangle with perimeter 1999, which is not obtainable in the form $(a+1, b+1, c+1)$ where a, b, c are the integers and the sides of a triangle with $a + b + c = 1996$. We conclude that $f(1999) > f(1996)$.

(b) As in the case (a) we conclude that $f(2000) \geq f(1997)$. On the other hand, if x, y, z are the integer sides of a triangle with $x+y+z = 2000$, and say $x \geq y \geq z \geq 1$, then we cannot have $z = 1$; for otherwise we would get $x + y = 1999$ forcing x, y to have opposite parity so that $x - y \geq 1 = z$ violating triangle inequality for x, y, z. Hence $x \geq y \geq z > 1$. This implies that $x - 1 \geq y - 1 \geq z - 1 > 0$. We already have $x < y + z$. If $x \geq y + z - 1$, then we see that $y + z - 1 \leq x < y + z$, showing that $y + z - 1 = x$. Hence we obtain $2000 = x + y + z = 2x + 1$ which is impossible. We conclude that $x < y + z - 1$. This shows that $x - 1 < (y - 1) + (z - 1)$ and hence $x - 1, y - 1, z - 1$ are the sides of a triangle with perimeter 1997. This gives $f(2000) \leq f(1997)$. Thus we obtain the desired result.

INMO-2001
Problems and Solutions

1. Let ABC be a triangle in which *no* angle is $90°$. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove the following statements:

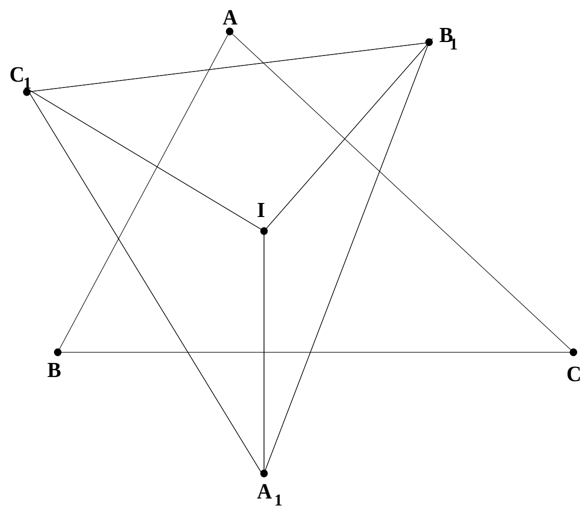
   (a) If P is the incentre or an excentre of ABC, then P is the circumcentre of $A_1B_1C_1$;

   (b) If P is the circumcentre of ABC, then P is the orthocentre of $A_1B_1C_1$;

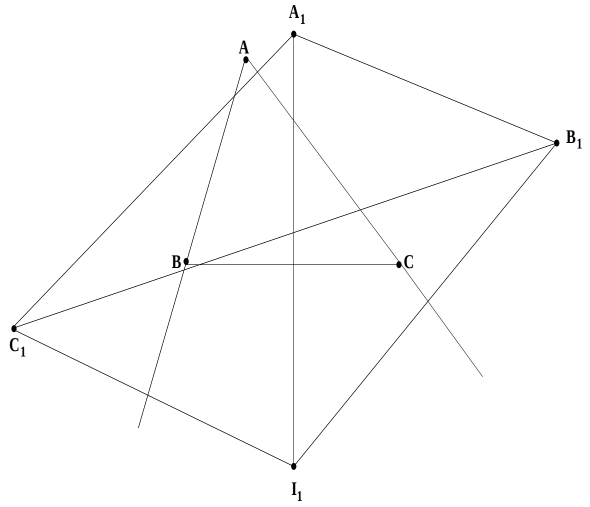
   (c) If P is the orthocentre of ABC, then P is either the incentre or an excentre of $A_1B_1C_1$.

Solution:

(a)



If $P = I$ is the incentre of triangle ABC, and r its inradius, then it is clear that $A_1I = B_1I = C_1I = 2r$. It follows that I is the circumcentre of $A_1B_1C_1$. On the otherhand if $P = I_1$ is the excentre of ABC opposite A and r_1 the corresponding ex-radius, then again we see that $A_1I_1 = B_1I_1 = C_1I_1 = 2r_1$. Thus I_1 is the circumcentre of $A_1B_1C_1$.



(b)

Let $P = O$ be the circumcentre of ABC. By definition, it follows that OA_1 bisects and is bisected by BC and so on. Let D, E, F be the mid-points of BC, CA, AB respectively. Then FE is parallel to BC. But E, F are also mid-points of OB_1, OC_1 and hence FE is parallel to B_1C_1 as well. We conclude that BC is parallel to B_1C_1. Since OA_1 is perpendicular to BC, it follows that OA_1 is perpendicular to B_1C_1. Similarly OB_1 is perpendicular to C_1A_1 and OC_1 is perpendicular to A_1B_1. These imply that O is the orthocentre of $A_1B_1C_1$. (This applies whether O is inside or outside ABC.)

(c)

let $P = H$, the orthocentre of ABC. We consider two possibilities; H falls inside ABC and H falls outside ABC.

Suppose H is inside ABC; this happens if ABC is an acute triangle. It is known that A_1, B_1, C_1 lie on the circumcircle of ABC. Thus $\angle C_1A_1A = \angle C_1CA = 90° - A$. Similarly $\angle B_1A_1A = \angle B_1BA = 90° - A$. These show that $\angle C_1A_1A = \angle B_1A_1A$. Thus A_1A is an internal bisector of $\angle C_1A_1B_1$. Similarly we can show that B_1 bisects $\angle A_1B_1C_1$ and C_1C bisects $\angle B_1C_1A_1$. Since A_1A, B_1B, C_1C concur at H, we conclude that H is the incentre of $A_1B_1C_1$.

OR If D, E, F are the feet of perpendiculars of A, B, C to the sides BC, CA, AB respectively, then we see that EF, FD, DE are respectively parallel to B_1C_1, C_1A_1, A_1B_1. This implies that $\angle C_1A_1H = \angle FDH = \angle ABE = 90° - A$, as $BDHF$ is a cyclic quadrilateral. Similarly, we can show that $\angle B_1A_1H = 90° - A$. It follows that A_1H is the internal bisector of $\angle C_1A_1B_1$. We can proceed as in the earlier case.

If H is outside ABC, the same proofs go through again, except that two of A_1H, B_1H, C_1H are external angle bisectors and one of these is an internal angle bisector. Thus H becomes an excentre of triangle $A_1B_1C_1$.

2. Show that the equation

$$x^2 + y^2 + z^2 = (x - y)(y - z)(z - x)$$

has infinitely many solutions in integers x, y, z.

Solution: We seek solutions (x, y, z) which are in arithmetic progression. Let us put $y - x = z - y = d > 0$ so that the equation reduces to the form

$$3y^2 + 2d^2 = 2d^3.$$

Thus we get $3y^2 = 2(d - 1)d^2$. We conclude that $2(d - 1)$ is 3 times a square. This is satisfied if $d - 1 = 6n^2$ for some n. Thus $d = 6n^2 + 1$ and $3y^2 = d^2 \cdot 2(6n^2)$ giving us $y^2 = 4d^2n^2$. Thus we can take $y = 2dn = 2n(6n^2 + 1)$. From this we obtain $x = y - d = (2n - 1)(6n^2 + 1)$, $z = y + d = (2n + 1)(6n^2 + 1)$. It is easily verified that

$$(x, y, z) = ((2n - 1)(6n^2 + 1), 2n(6n^2 + 1), (2n + 1)(6n^2 + 1)),$$

is indeed a solution for a fixed n and this gives an infinite set of solutions as n varies over natural numbers.

3. If a, b, c are positive real numbers such that $abc = 1$, prove that

$$a^{b+c}\, b^{c+a}\, c^{a+b} \leq 1.$$

Solution: Note that the inequality is symmetric in a, b, c so that we may assume that $a \geq b \geq c$. Since $abc = 1$, it follows that $a \geq 1$ and $c \leq 1$. Using $b = 1/ac$, we get

$$a^{b+c}\, b^{c+a}\, c^{a+b} = \frac{a^{b+c}c^{a+b}}{a^{c+a}c^{c+a}} = \frac{c^{b-c}}{a^{a-b}} \leq 1,$$

because $c \leq 1$, $b \geq c$, $a \geq 1$ and $a \geq b$.

4. Given any nine integers show that it is possible to choose, from among them, four integers a, b, c, d such that $a + b - c - d$ is divisible by 20. Further show that such a selection is not possible if we start with eight integers instead of nine.

Solution:

Suppose there are four numbers a, b, c, d among the given nine numbers which leave the same remainder modulo 20. Then $a + b \equiv c + d \pmod{20}$ and we are done.

If not, there are two possibilities:

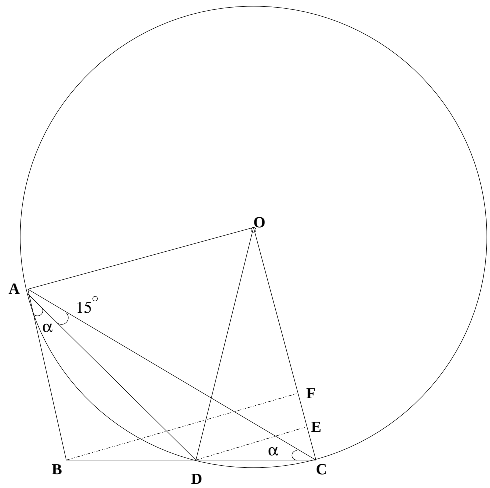
(1) We may have two disjoint pairs $\{a, c\}$ and $\{b, d\}$ obtained from the given nine numbers such that $a \equiv c \pmod{20}$ and $b \equiv d \pmod{20}$. In this case we get $a+b \equiv c+d \pmod{20}$.

(2) Or else there are at most three numbers having the same remainder modulo 20 and the remaining six numbers leave distinct remainders which are also different from the first remainder (i.e., the remainder of the three numbers). Thus there are at least 7 disinct remainders modulo 20 that can be obtained from the given set of nine numbers. These 7 remainders give rise to $\binom{7}{2} = 21$ pairs of numbers. By pigeonhole principle, there must be two pairs $(r_1, r_2), (r_3, r_4)$ such that $r_1 + r_2 \equiv r_3 + r_4 \pmod{20}$. Going back we get four numbers a, b, c, d such that $a + b \equiv c + d \pmod{20}$.

If we take the numbers $0, 0, 0, 1, 2, 4, 7, 12$, we check that the result is not true for these eight numbers.

5. Let ABC be a triangle and D be the mid-point of side BC. Suppose $\angle DAB = \angle BCA$ and $\angle DAC = 15^\circ$. Show that $\angle ADC$ is obtuse. Further, if O is the circumcentre of ADC, prove that triangle AOD is equilateral.

Solution:



Let α denote the equal angles $\angle BAD = \angle DCA$. Using sine rule in triangles DAB and DAC, we get

$$\frac{AD}{\sin B} = \frac{BD}{\sin \alpha}, \quad \frac{CD}{\sin 15^\circ} = \frac{AD}{\sin \alpha}.$$

Eliminating α (using $BD = DC$ and $2\alpha + B + 15^\circ = \pi$), we obtain $1 + \cos(B + 15^\circ) = 2 \sin B \sin 15^\circ$. But we know that $2 \sin B \sin 15^\circ = \cos(B - 15^\circ) - \cos(B + 15^\circ)$. Putting $\beta = B - 15^\circ$, we get a relation $1 + 2\cos(\beta + 30) = \cos \beta$. We write this in the form

$$(1 - \sqrt{3}) \cos \beta + \sin \beta = 1.$$

Since $\sin \beta \leq 1$, it follows that $(1 - \sqrt{3}) \cos \beta \geq 0$. We conclude that $\cos \beta \leq 0$ and hence that β is obtuse. So is angle B and hence $\angle ADC$.

We have the relation $(1-\sqrt{3})\cos\beta+\sin\beta=1$. If we set $x=\tan(\beta/2)$, then we get, using $\cos\beta=(1-x^2)/(1+x^2), \sin\beta=2x/(1+x^2)$,

$$(\sqrt{3}-2)x^2+2x-\sqrt{3}=0.$$

Solving for x, we obtain $x=1$ or $x=\sqrt{3}(2+\sqrt{3})$. If $x=\sqrt{3}(2+\sqrt{3})$, then $\tan(\beta/2)>2+\sqrt{3}=\tan 75°$ giving us $\beta>150°$. This forces that $B>165°$ and hence $B+A>165°+15°=180°$, a contradiction. thus $x=1$ giving us $\beta=\pi/2$. This gives $B=105°$ and hence $\alpha=30°$. Thus $\angle DAO=60°$. Since $OA=OD$, the result follows.

OR

Let m_a denote the median AD. Then we can compute

$$\cos\alpha=\frac{c^2+m_a^2-(a^2/4)}{2cm_a}, \quad \sin\alpha=\frac{2\Delta}{cm_a},$$

where Δ denotes the area of triangle ABC. These two expressions give

$$\cot\alpha=\frac{c^2+m_a^2-(a^2/4)}{4\Delta}.$$

Similarly, we obtain

$$\cot\angle CAD=\frac{b^2+m_a^2-(a^2/4)}{4\Delta}.$$

Thus we get

$$\cot\alpha-\cot 15°=\frac{c^2-a^2}{4\Delta}.$$

Similarly we can also obtain

$$\cot B-\cot\alpha=\frac{c^2-a^2}{4\Delta},$$

giving us the relation

$$\cot B=2\cot\alpha-\cot 15°.$$

If B is acute then $2\cot\alpha>\cot 15°=2+\sqrt{3}>2\sqrt{3}$. It follows that $\cot\alpha>\sqrt{3}$. This implies that $\alpha<30°$ and hence

$$B=180°-2\alpha-15°>105°.$$

This contradiction forces that angle B is obtuse and consequently $\angle ADC$ is obtuse.

Since $\angle BAD=\alpha=\angle ACD$, the line AB is tangent to the circumcircle Γ of ADC at A. Hence OA is perpendicular to AB. Draw DE and BF perpendicular to AC, and join OD. Since $\angle DAC=15°$, we see that $\angle DOC=30°$ and hence $DE=OD/2$. But DE is parallel to BF and $BD=DC$ shows that $BF=2DE$. We conclude that

$BF = DO$. But $DO = AO$, both being radii of Γ. Thus $BF = AO$. Using right triangles BFO and BAO, we infer that $AB = OF$. We conclude that $ABFO$ is a rectangle. In particular $\angle AOF = 90°$. It follows that

$$\angle AOD = 90° - \angle DOC = 90° - 30° = 60°.$$

Since $OA = OD$, we conclude that AOD is equilateral.

OR

Note that triangles ABD and CBA are similar. Thus we have the ratios

$$\frac{AB}{BD} = \frac{CB}{BA}.$$

This reduces to $a^2 = 2c^2$ giving us $a = \sqrt{2}c$. This is equivalent to $\sin^2(\alpha + 15°) = 2\sin^2\alpha$. We write this in the form

$$\cos 15° + \cot\alpha \sin 15° = \sqrt{2}.$$

Solving for $\cot\alpha$, we get $\cot\alpha = \sqrt{3}$. We conclude that $\alpha = 30°$, and the result follows.

6. Let $\mathbf{R}$ denote the set of all real numbers. Find all functions $f : \mathbf{R} \to \mathbf{R}$ satisfying the condition

$$f(x + y) = f(x)f(y)f(xy)$$

for all x, y in $\mathbf{R}$.

Solution: Putting $x = 0, y = 0$, we get $f(0) = f(0)^3$ so that $f(0) = 0, 1$ or -1. If $f(0) = 0$, then taking $y = 0$ in the given equation, we obtain $f(x) = f(x)f(0)^2 = 0$ for all x.

Suppose $f(0) = 1$. Taking $y = -x$, we obtain

$$1 = f(0) = f(x - x) = f(x)f(-x)f(-x^2).$$

This shows that $f(x) \neq 0$ for any $x \in \mathbf{R}$. Taking $x = 1, y = x - 1$, we obtain

$$f(x) = f(1)f(x-1)^2 = f(1)\left[f(x)f(-x)f(-x)\right]^2.$$

Using $f(x) \neq 0$, we conclude that $1 = kf(x)(f(-x))^2$, where $k = f(1)(f(-1))^2$. Changing x to $-x$ here, we also infer that $1 = kf(-x)(f(x))^2$. Comparing these expressions we see that $f(-x) = f(x)$. It follows that $1 = kf(x)^3$. Thus $f(x)$ is constant for all x. Since $f(0) = 1$, we conclude that $f(x) = 1$ for all real x.

If $f(0) = -1$, a similar analysis shows that $f(x) = -1$ for all $x \in \mathbf{R}$. We can verify that each of these functions satisfies the given functional equation. Thus there are three solutions, all of them being constant functions.

INMO-2001
Problems and Solutions

1. Let ABC be a triangle in which *no* angle is $90°$. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove the following statements:

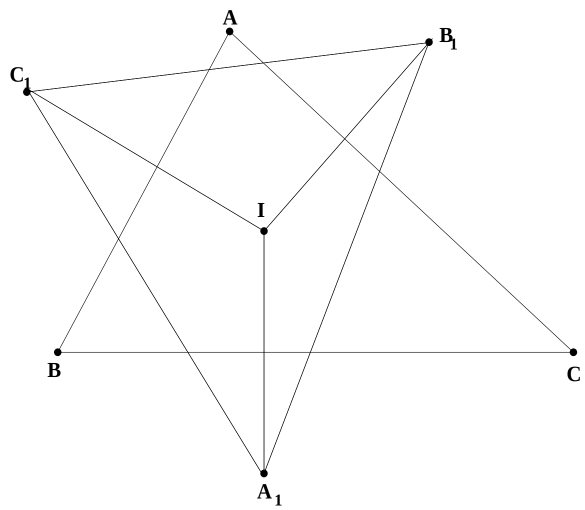
   (a) If P is the incentre or an excentre of ABC, then P is the circumcentre of $A_1B_1C_1$;

   (b) If P is the circumcentre of ABC, then P is the orthocentre of $A_1B_1C_1$;

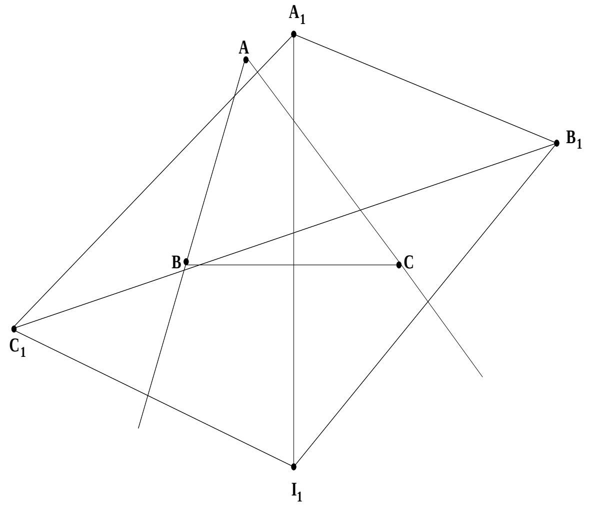
   (c) If P is the orthocentre of ABC, then P is either the incentre or an excentre of $A_1B_1C_1$.

   Solution:

   (a)



If $P = I$ is the incentre of triangle ABC, and r its inradius, then it is clear that $A_1I = B_1I = C_1I = 2r$. It follows that I is the circumcentre of $A_1B_1C_1$. On the otherhand if $P = I_1$ is the excentre of ABC opposite A and r_1 the corresponding ex-radius, then again we see that $A_1I_1 = B_1I_1 = C_1I_1 = 2r_1$. Thus I_1 is the circumcentre of $A_1B_1C_1$.



(b)

Let $P = O$ be the circumcentre of ABC. By definition, it follows that OA_1 bisects and is bisected by BC and so on. Let D, E, F be the mid-points of BC, CA, AB respectively. Then FE is parallel to BC. But E, F are also mid-points of OB_1, OC_1 and hence FE is parallel to B_1C_1 as well. We conclude that BC is parallel to B_1C_1. Since OA_1 is perpendicular to BC, it follows that OA_1 is perpendicular to B_1C_1. Similarly OB_1 is perpendicular to C_1A_1 and OC_1 is perpendicular to A_1B_1. These imply that O is the orthocentre of $A_1B_1C_1$. (This applies whether O is inside or outside ABC.)

(c)

let $P = H$, the orthocentre of ABC. We consider two possibilities; H falls inside ABC and H falls outside ABC.

Suppose H is inside ABC; this happens if ABC is an acute triangle. It is known that A_1, B_1, C_1 lie on the circumcircle of ABC. Thus $\angle C_1A_1A = \angle C_1CA = 90° - A$. Similarly $\angle B_1A_1A = \angle B_1BA = 90° - A$. These show that $\angle C_1A_1A = \angle B_1A_1A$. Thus A_1A is an internal bisector of $\angle C_1A_1B_1$. Similarly we can show that B_1 bisects $\angle A_1B_1C_1$ and C_1C bisects $\angle B_1C_1A_1$. Since A_1A, B_1B, C_1C concur at H, we conclude that H is the incentre of $A_1B_1C_1$.

OR If D, E, F are the feet of perpendiculars of A, B, C to the sides BC, CA, AB respectively, then we see that EF, FD, DE are respectively parallel to B_1C_1, C_1A_1, A_1B_1. This implies that $\angle C_1A_1H = \angle FDH = \angle ABE = 90° - A$, as $BDHF$ is a cyclic quadrilateral. Similarly, we can show that $\angle B_1A_1H = 90° - A$. It follows that A_1H is the internal bisector of $\angle C_1A_1B_1$. We can proceed as in the earlier case.

If H is outside ABC, the same proofs go through again, except that two of A_1H, B_1H, C_1H are external angle bisectors and one of these is an internal angle bisector. Thus H becomes an excentre of triangle $A_1B_1C_1$.

2. Show that the equation

$$x^2 + y^2 + z^2 = (x - y)(y - z)(z - x)$$

has infinitely many solutions in integers x, y, z.

Solution: We seek solutions (x, y, z) which are in arithmetic progression. Let us put $y - x = z - y = d > 0$ so that the equation reduces to the form

$$3y^2 + 2d^2 = 2d^3.$$

Thus we get $3y^2 = 2(d - 1)d^2$. We conclude that $2(d - 1)$ is 3 times a square. This is satisfied if $d - 1 = 6n^2$ for some n. Thus $d = 6n^2 + 1$ and $3y^2 = d^2 \cdot 2(6n^2)$ giving us $y^2 = 4d^2n^2$. Thus we can take $y = 2dn = 2n(6n^2 + 1)$. From this we obtain $x = y - d = (2n - 1)(6n^2 + 1)$, $z = y + d = (2n + 1)(6n^2 + 1)$. It is easily verified that

$$(x, y, z) = ((2n - 1)(6n^2 + 1), 2n(6n^2 + 1), (2n + 1)(6n^2 + 1)),$$

is indeed a solution for a fixed n and this gives an infinite set of solutions as n varies over natural numbers.

3. If a, b, c are positive real numbers such that $abc = 1$, prove that

$$a^{b+c}\, b^{c+a}\, c^{a+b} \leq 1.$$

Solution: Note that the inequality is symmetric in a, b, c so that we may assume that $a \geq b \geq c$. Since $abc = 1$, it follows that $a \geq 1$ and $c \leq 1$. Using $b = 1/ac$, we get

$$a^{b+c}\, b^{c+a}\, c^{a+b} = \frac{a^{b+c}c^{a+b}}{a^{c+a}c^{c+a}} = \frac{c^{b-c}}{a^{a-b}} \leq 1,$$

because $c \leq 1$, $b \geq c$, $a \geq 1$ and $a \geq b$.

4. Given any nine integers show that it is possible to choose, from among them, four integers a, b, c, d such that $a + b - c - d$ is divisible by 20. Further show that such a selection is not possible if we start with eight integers instead of nine.

Solution:

Suppose there are four numbers a, b, c, d among the given nine numbers which leave the same remainder modulo 20. Then $a + b \equiv c + d \pmod{20}$ and we are done.

If not, there are two possibilities:

(1) We may have two disjoint pairs $\{a, c\}$ and $\{b, d\}$ obtained from the given nine numbers such that $a \equiv c \pmod{20}$ and $b \equiv d \pmod{20}$. In this case we get $a+b \equiv c+d \pmod{20}$.

(2) Or else there are at most three numbers having the same remainder modulo 20 and the remaining six numbers leave distinct remainders which are also different from the first remainder (i.e., the remainder of the three numbers). Thus there are at least 7 disinct remainders modulo 20 that can be obtained from the given set of nine numbers. These 7 remainders give rise to $\binom{7}{2} = 21$ pairs of numbers. By pigeonhole principle, there must be two pairs $(r_1, r_2), (r_3, r_4)$ such that $r_1 + r_2 \equiv r_3 + r_4 \pmod{20}$. Going back we get four numbers a, b, c, d such that $a + b \equiv c + d \pmod{20}$.

If we take the numbers $0, 0, 0, 1, 2, 4, 7, 12$, we check that the result is not true for these eight numbers.

5. Let ABC be a triangle and D be the mid-point of side BC. Suppose $\angle DAB = \angle BCA$ and $\angle DAC = 15°$. Show that $\angle ADC$ is obtuse. Further, if O is the circumcentre of ADC, prove that triangle AOD is equilateral.

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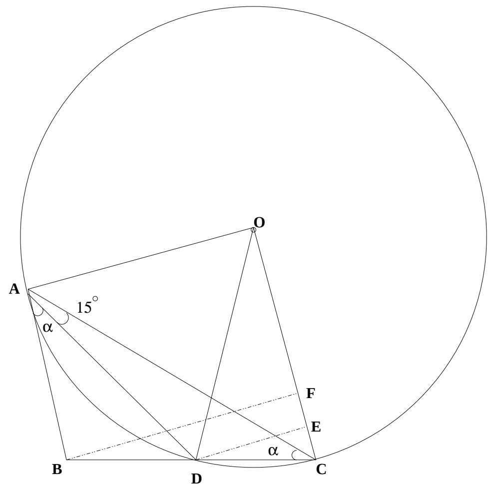

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Eliminating α (using $BD = DC$ and $2\alpha + B + 15° = \pi$), we obtain $1 + \cos(B + 15°) = 2 \sin B \sin 15°$. But we know that $2 \sin B \sin 15° = \cos(B - 15°) - \cos(B + 15°)$. Putting $\beta = B - 15°$, we get a relation $1 + 2\cos(\beta + 30) = \cos\beta$. We write this in the form

$$(1 - \sqrt{3})\cos\beta + \sin\beta = 1.$$

Since $\sin\beta \leq 1$, it follows that $(1 - \sqrt{3})\cos\beta \geq 0$. We conclude that $\cos\beta \leq 0$ and hence that β is obtuse. So is angle B and hence $\angle ADC$.

We have the relation $(1-\sqrt{3})\cos\beta + \sin\beta = 1$. If we set $x = \tan(\beta/2)$, then we get, using $\cos\beta = (1-x^2)/(1+x^2), \sin\beta = 2x/(1+x^2)$,

$$(\sqrt{3}-2)x^2 + 2x - \sqrt{3} = 0.$$

Solving for x, we obtain $x = 1$ or $x = \sqrt{3}(2+\sqrt{3})$. If $x = \sqrt{3}(2+\sqrt{3})$, then $\tan(\beta/2) > 2+\sqrt{3} = \tan 75°$ giving us $\beta > 150°$. This forces that $B > 165°$ and hence $B + A > 165° + 15° = 180°$, a contradiction. thus $x = 1$ giving us $\beta = \pi/2$. This gives $B = 105°$ and hence $\alpha = 30°$. Thus $\angle DAO = 60°$. Since $OA = OD$, the result follows.

OR

Let m_a denote the median AD. Then we can compute

$$\cos\alpha = \frac{c^2 + m_a^2 - (a^2/4)}{2cm_a}, \quad \sin\alpha = \frac{2\Delta}{cm_a},$$

where Δ denotes the area of triangle ABC. These two expressions give

$$\cot\alpha = \frac{c^2 + m_a^2 - (a^2/4)}{4\Delta}.$$

Similarly, we obtain

$$\cot\angle CAD = \frac{b^2 + m_a^2 - (a^2/4)}{4\Delta}.$$

Thus we get

$$\cot\alpha - \cot 15° = \frac{c^2 - a^2}{4\Delta}.$$

Similarly we can also obtain

$$\cot B - \cot\alpha = \frac{c^2 - a^2}{4\Delta},$$

giving us the relation

$$\cot B = 2\cot\alpha - \cot 15°.$$

If B is acute then $2\cot\alpha > \cot 15° = 2+\sqrt{3} > 2\sqrt{3}$. It follows that $\cot\alpha > \sqrt{3}$. This implies that $\alpha < 30°$ and hence

$$B = 180° - 2\alpha - 15° > 105°.$$

This contradiction forces that angle B is obtuse and consequently $\angle ADC$ is obtuse.

Since $\angle BAD = \alpha = \angle ACD$, the line AB is tangent to the circumcircle Γ of ADC at A. Hence OA is perpendicular to AB. Draw DE and BF perpendicular to AC, and join OD. Since $\angle DAC = 15°$, we see that $\angle DOC = 30°$ and hence $DE = OD/2$. But DE is parallel to BF and $BD = DC$ shows that $BF = 2DE$. We conclude that

$BF = DO$. But $DO = AO$, both being radii of Γ. Thus $BF = AO$. Using right triangles BFO and BAO, we infer that $AB = OF$. We conclude that $ABFO$ is a rectangle. In particular $\angle AOF = 90°$. It follows that

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Note that triangles ABD and CBA are similar. Thus we have the ratios

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This reduces to $a^2 = 2c^2$ giving us $a = \sqrt{2}c$. This is equivalent to $\sin^2(\alpha + 15°) = 2\sin^2\alpha$. We write this in the form

$$\cos 15° + \cot\alpha \sin 15° = \sqrt{2}.$$

Solving for $\cot\alpha$, we get $\cot\alpha = \sqrt{3}$. We conclude that $\alpha = 30°$, and the result follows.

6. Let $\mathbf{R}$ denote the set of all real numbers. Find all functions $f : \mathbf{R} \to \mathbf{R}$ satisfying the condition

$$f(x + y) = f(x)f(y)f(xy)$$

for all x, y in $\mathbf{R}$.

Solution: Putting $x = 0, y = 0$, we get $f(0) = f(0)^3$ so that $f(0) = 0, 1$ or -1. If $f(0) = 0$, then taking $y = 0$ in the given equation, we obtain $f(x) = f(x)f(0)^2 = 0$ for all x.

Suppose $f(0) = 1$. Taking $y = -x$, we obtain

$$1 = f(0) = f(x - x) = f(x)f(-x)f(-x^2).$$

This shows that $f(x) \neq 0$ for any $x \in \mathbf{R}$. Taking $x = 1, y = x - 1$, we obtain

$$f(x) = f(1)f(x-1)^2 = f(1)\left[f(x)f(-x)f(-x)\right]^2.$$

Using $f(x) \neq 0$, we conclude that $1 = kf(x)(f(-x))^2$, where $k = f(1)(f(-1))^2$. Changing x to $-x$ here, we also infer that $1 = kf(-x)(f(x))^2$. Comparing these expressions we see that $f(-x) = f(x)$. It follows that $1 = kf(x)^3$. Thus $f(x)$ is constant for all x. Since $f(0) = 1$, we conclude that $f(x) = 1$ for all real x.

If $f(0) = -1$, a similar analysis shows that $f(x) = -1$ for all $x \in \mathbf{R}$. We can verify that each of these functions satisfies the given functional equation. Thus there are three solutions, all of them being constant functions.

INMO–2002

February 3, 2002

1. For a convex hexagon $ABCDEF$, consider the following six statements :

$$\begin{array}{llll} (a_1) & AB is parallel to DE : & (a_2) & AE = BD; \\ (b_1) & BC is parallel to EF : & (b_2) & BF = CE; \\ (c_1) & CD is parallel to FA : & (c_2) & CA = DF. \end{array}$$

   (a) Show that if all the six statements are true , then the hexagon is cyclic (i. e. , it can be inscribed in a circle) .

   (b) Prove that , in fact , any five of these six statements also imply that the hexagon is cyclic .

2. Determine the least positive value taken by the expression $a^3 + b^3 + c^3 = 3abc$ as a, b, c vary over all positive integers . Find also all triples (a, b, c) for which this least value is attained .

3. Let x, y be positive reals such that $x + y = 2$. Prove that

$$x^3y^3(x^3 + y^3) \leq 2.$$

4. Do there exist 100 lines in the plane , no three of them concurrent , such that they intersect exactly in 2002 points ?

5. Do there exist three distinct positive real numbers a, b, c such that the numbers $a, b, c, b + c - a, c + a - b, a + b - c$ and $a + b + c$ form a 7-term arithmetic progression in some order ?

6. Suppose the n^2 numbers $1, 2, 3, \cdots, n^2$ are arranged to form an n by n array consisting of n rows and n columns such that the numbers in each row (from left to right) and each column (from top to bottom) are in increasing order . Denote by a_{jk} the number in the $j - th$ row and $k - th$ column . Suppose b_j is the maximum possible number of entries that can occur as $a_{jj}, 1 \leq j \leq n$. Prove that

$$b_1 + b_2 + b_3 + \cdots b_n \leq \frac{n}{3}(n^2 - 3n + 5).$$

   (Example : In the case $n = 3$, the only numbers which can occur as a_{22} are $4, 5$ or 6 so that $b_2 = 3$.)

Solution to INMO-2002 Problems

1. For a convex hexagon $ABCDEF$ in which each pair of opposite sides is unequal, consider the following six statements:

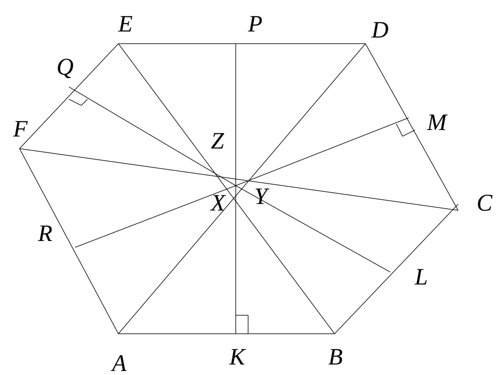
(a_1) AB is parallel to DE;	(a_2) $AE = BD$;
(b_1) BC is parallel to EF;	(b_2) $BF = CE$;
(c_1) CD is parallel to FA;	(c_2) $CA = DF$.

(a) Show that if all the six statements are true, then the hexagon is cyclic(i.e., it can be inscribed in a circle).

(b) Prove that, in fact, any five of these six statements also imply that the hexagon is cyclic.

Solution:

(a) Suppose all the six statements are true. Then $ABDE$, $BCEF$, $CDFA$ are isosceles trapeziums; if K, L, M, P, Q, R are the mid-points of AB, BC, CD, DE, EF, FA respectively, then we see that $KP \perp AB, ED$; $LQ \perp BC, EF$ and $MR \perp CD, FA$.



If AD, BE, CF themselves concur at a point O, then $OA = OB = OC = OD = OE = OF$. ($O$ is on the perpendicular bisector of each of the sides.) Hence A, B, C, D, E, F are concyclic and lie on a circle with centre O. Otherwise these lines AD, BE, CF form a triangle, say XYZ. (See Fig.) Then KX, MY, QZ, when extended, become the internal angle bisectors of the triangle XYZ and hence concur at the incentre O' of XYZ. As earlier O' lies on the perpendicular bisector of each of the sides. Hence $O'A = O'B = O'C = O'D = O'E = O'F$, giving the concyclicity of A, B, C, D, E, F.

(b) Suppose (a_1), (a_2), (b_1), (b_2) are true. Then we see that $AD = BE = CF$. Assume that (c_1) is true. Then CD is parallel to AF. It follows that triangles YCD and YFA are similar. This gives

$$\frac{FY}{AY} = \frac{YC}{YD} = \frac{FY+YC}{AY+YD} = \frac{FC}{AD} = 1.$$

We obtain $FY = AY$ and $YC = YD$. This forces that triangles CYA and DYF are congruent. In particular $AC = DF$ so that (c_2) is true. The conclusion follows from (a). Now assume that (c_2) is true; i.e., $AC = FD$. We have seen that $AD = BE = CF$. It follows that triangles FDC and ACD are congruent. In particular $\angle ADC = \angle FCD$. Similarly, we can show that $\angle CFA = \angle DAF$. We conclude that CD is parallel to AF giving (c_1).

2. Determine the least positive value taken by the expression $a^3+b^3+c^3-3abc$ as a, b, c vary over all positive integers. Find also all triples (a, b, c) for which this least value is attained.

Solution: We observe that

$$Q = a^3 + b^3 + c^3 - 3abc = \frac{1}{2}\Big(a+b+c\Big)\Big((a-b)^2 + (b-c)^2 + (c-a)^2\Big).$$

Since we are looking for the least positive value taken by Q, it follows that a, b, c are not all equal. Thus $a+b+c \geq 1+1+2 = 4$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 1+1+0 = 2$. Thus we see that $Q \geq 4$. Taking $a = 1$, $b = 1$ and $c = 2$, we get $Q = 4$. Therefore the least value of Q is 4 and this is achieved only by $a+b+c = 4$ and $(a-b)^2+(b-c)^2+(c-a)^2 = 2$. The triples for which $Q = 4$ are therefore given by

$$(a, b, c) = (1, 1, 2),\ (1, 2, 1),\ (2, 1, 1).$$

3. Let x, y be positive reals such that $x + y = 2$. Prove that

$$x^3y^3(x^3+y^3) \leq 2.$$

Solution: We have from the AM-GM inequality, that

$$xy \leq \left(\frac{x+y}{2}\right)^2 = 1.$$

Thus we obtain $0 < xy \leq 1$. We write

$$\begin{aligned} x^3y^3\left(x^3+y^3\right) &= \left(xy\right)^3\left(x+y\right)\left(x^2-xy+y^2\right) \\ &= 2\left(xy\right)^3\left(\left(x+y\right)^2 - 3xy\right) \\ &= 2\left(xy\right)^3\left(4-3xy\right). \end{aligned}$$

Thus we need to prove that

$$(xy)^3(4 - 3xy) \leq 1.$$

Putting $z = xy$, this inequality reduces to

$$z^3(4 - 3z) \leq 1,$$

for $0 < z \leq 1$. We can prove this in different ways. We can put the inequality in the form

$$3z^4 - 4z^3 + 1 \geq 0.$$

Here the expression in the **LHS** factors to $(z-1)^2(3z^2 + 2z + 1)$ and $(3z^2 + 2z + 1)$ is positive since its discriminant $D = -8 < 0$. Or applying the AM-GM inequality to the positive reals $4 - 3z, z, z, z$, we obtain

$$z^3(4 - 3z) \leq \left(\frac{4 - 3z + 3z}{4}\right)^4 \leq 1.$$

4. Do there exist 100 lines in the plane, no three of them concurrent, such that they intersect exactly in 2002 points?

Solution: Any set of 100 lines in the plane can be partitioned into a finite number of disjoint sets, say $A_1, A_2, A_3, \ldots, A_k$, such that

(i) Any two lines in each A_j are parallel to each other, for $1 \leq j \leq k$ (provided, of course, $|A_j| \geq 2$);

(ii) for $j \neq l$, the lines in A_j and A_l are not parallel.

If $|A_j| = m_j$, $1 \leq j \leq k$, then the total number of points of intersection is given by $\sum_{1 \leq j \leq l \leq k} m_j m_l$, as no three lines are concurrent. Thus we have to find positive integers $m_1, m_2, \ldots, m_k$ such that

$$\sum_{j=1}^{k} m_j = 100, \quad \sum m_j m_l = 2002,$$

for an affirmative answer to the given question.

We observe that

$$\begin{aligned} \sum_{j=1}^{k} m_j^2 &= \left(\sum_{j=1}^{k} m_j\right)^2 - 2\left(\sum m_j m_l\right) \\ &= 100^2 - 2(2002) = 5996. \end{aligned}$$

Thus we have to choose $m_1, m_2, \ldots, m_k$ such that

$$\sum_{j=1}^{k} m_j = 100, \quad \sum_{j=1}^{k} m_j^2 = 5996.$$

We observe that $\left[\sqrt{5996}\ \right] = 77$. So we may take $m_1 = 77$, so that

$$\sum_{j=2}^{k} m_j = 23, \quad \sum j = 2^k m_j^2 = 67.$$

Now we may choose $m_2 = 5$, $m_3 = m_4 = 4$, $m_5 = m_6 = \cdots = m_{14} = 1$. Finally, we can take

$$k = 14, \quad (m_1, m_2, \ldots, m_{14}) = (77, 5, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

proving the existence of 100 lines with exactly 2002 points of intersection.

5. Do there exist three distinct positive real numbers a, b, c such that the numbers a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order?

Solution: We show that the answer is **NO**. Suppose, if possible, let a, b, c be three distinct positive real numbers such that a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order. We may assume that $a < b < c$. Then there are only two cases we need to check: (I) $a+b-c < a < c+a-b < b < c < b+c-a < a+b+c$ and (II) $a+b-c < a < b < c+a-b < c < b+c-a < a+b+c$.

Case I. Suppose the chain of inequalities $a+b-c < a < c+a-b < b < c < b+c-a < a+b+c$ holds good. let d be the common difference. Thus we see that

$$c = a+b+c-2d, \ b = a+b+c-3d, \ a = a+b+c-5d.$$

Adding these, we see that $a+b+c = 5d$. But then $a = 0$ contradicting the positivity of a.

Case II. Suppose the inequalities $a+b-c < a < b < c+a-b < c < b+c-a < a+b+c$ are true. Again we see that

$$c = a+b+c-2d, \ b = a+b+c-4d, \ a = a+b+c-5d.$$

We thus obtain $a+b+c = (11/2)d$. This gives

$$a = \frac{1}{2}d, \ b = \frac{3}{2}d, \ c = \frac{7}{2}d.$$

Note that $a+b-c = a+b+c-6d = -(1/2)d$. However we also get $a+b-c = \big[(1/2)+(3/2)-(7/2)\big]d = -(3/2)d$. It follows that $3e = e$ giving $d = 0$. But this is impossible.

Thus there are no three distinct positive real numbers a, b, c such that a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order.

6. Suppose the n^2 numbers $1, 2, 3, \ldots, n^2$ are arranged to form an n by n array consisting of n rows and n columns such that the numbers in each row(from left to right) and each column(from top to bottom) are in increasing order. Denote by a_{jk} the number in j-th row and k-th column. Suppose b_j is the maximum possible number of entries that can occur as a_{jj}, $1 \le j \le n$. Prove that

$$b_1 + b_2 + b_3 + \cdots b_n \le \frac{n}{3}\big(n^2 - 3n + 5\big).$$

(Example: In the case $n = 3$, the only numbers which can occur as a_{22} are 4, 5 or 6 so that $b_2 = 3$.)

Solution: Since a_{jj} has to exceed all the numbers in the top left $j \times j$ submatrix (excluding itself), and since there are $j^2 - 1$ entries, we must have $a_{jj} \ge j^2$. Similarly, a_{jj} must not exceed eac of the numbers in the bottom right $(n-j+1) \times (n-j+1)$ submatrix (other than itself) and there are $(n-j+1)^2 - 1$ such entries giving $a_{jj} \le n^2 - (n-j+1)^2 + 1$. Thus we see that

$$a_{jj} \in \Big\{j^2, j^2+1, j^2+2, \ldots, n^2 - (n-j+1)^2 + 1\Big\}.$$

The number of elements in this set is $n^2 - (n-j+1)^2 - j^2 + 2$. This implies that

$$b_j \le n^2 - (n-j+1)^2 - j^2 + 2 = (2n+2)j - 2j^2 - (2n-1).$$

It follows that

$$\begin{aligned}
\sum_{j=1}^{n} b_j &\le \Big(2n+2\Big)\sum_{j=1}^{n} j - 2\sum_{j=1}^{n} j^2 - n(2n-1) \\
&= \Big(2n+2\Big)\left(\frac{n(n+1)}{2}\right) - 2\left(\frac{n(n+1)(2n+1)}{6}\right) - n(2n-1) \\
&= \frac{n}{3}\Big(n^2 - 3n + 5\Big),
\end{aligned}$$

which is the required bound.

INMO–2003

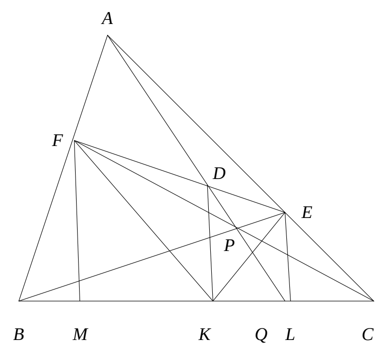
February 2, 2003

1. Consider an acute triangle ABC and let P be an interior point of ABC . Suppose the lines BP and CP , when produced , meet AC and AB in E and F respectively . Let D be the point where AP intersects the line segment EF and K be the foot of perpendicular from D on to BC . Show that DK bisects $\angle EKF$.
2. Find all primes p and q ,and even numbers $n > 2$, satisfying the equation
$$p^n + p^{n-1} + \cdots + p + 1 = q^2 + q + 1.$$
3. Show that for every real number a the equation
$$8x^4 - 16x^3 + 16x^2 - 8x + a = 0$$
has at least one non-real root and find the sum of all the non-real roots of the equation .
4. Find all 7-digit numbers formed by using only the digits 5 and 7 , and divisible by both 5 and 7 .
5. Let ABC be a triangle with sides a, b, c . Consider a triangle $A_1B_1C_1$ with sides equal to $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$. Show that
$$[A_1B_1C_1] \geq \frac{9}{4}[ABC],$$
where $[XYZ]$ denotes the area of the triangle XYZ .
6. In a lottery tickets are given nine-digit numbers using only the digits $1, 2, 3$. THey are also coloured red , blue or green in such a way that two tickets whose numbers differ in all the nine places get different colours . Suppose the ticket bearing the number 122222222 is red and that bearing the number 222222222 is green . Determine , with proof , the colour of the ticket bearing the number 123123123 .

Solutions to INMO-2003 problems

1. Consider an acute triangle ABC and let P be an interior point of ABC. Suppose the lines BP and CP, when produced, meet AC and AB in E and F respectively. Let D be the point where AP intersects the line segment EF and K be the foot of perpendicular from D on to BC. Show that DK bisects $\angle EKF$.

   Solution: Produce AP to meet BC in Q. Join KE and KF. Draw perpendiculars from F and E on to BC to meet it in M and L respectively. Let us denote $\angle BKF$ by α and $\angle CKE$ by β. We show that $\alpha = \beta$ by proving $\tan\alpha = \tan\beta$. This implies that $\angle DKF = \angle DKE$.(See Figure below.)



   Since the cevians AQ, BE and CF concur, we may write

$$\frac{BQ}{QC} = \frac{z}{y}, \frac{CE}{EA} = \frac{x}{z}, \frac{AF}{FB} = \frac{y}{x}.$$

   We observe that

$$\frac{FD}{DE} = \frac{[AFD]}{[AED]} = \frac{[PFD]}{[PED]} = \frac{[AFP]}{[AEP]}.$$

   However standard computations involving bases give

$$[AFP] = \frac{y}{y+x}[ABP], \quad [AEP] = \frac{z}{z+x}[ACP],$$

   and

$$[ABP] = \frac{z}{x+y+z}[ABC], \quad [ACP] = \frac{y}{x+y+z}[ABC].$$

   Thus we obtain

$$\frac{FD}{DE} = \frac{x+z}{x+y}.$$

On the other hand

$$\tan\alpha = \frac{FM}{KM} = \frac{FB\sin B}{KM}, \tan\beta = \frac{EL}{KL} = \frac{EC\sin C}{KL}.$$

Using $FB = \left(\frac{x}{x+y}\right)AB$, $EC = \left(\frac{x}{x+z}\right)AC$ and $AB\sin B = AC\sin C$, we obtain

$$\begin{aligned}\frac{\tan\alpha}{\tan\beta} &= \left(\frac{x+z}{x+y}\right)\left(\frac{KL}{KM}\right)\\ &= \left(\frac{x+z}{x+y}\right)\left(\frac{DE}{FD}\right)\\ &= \left(\frac{x+z}{x+y}\right)\left(\frac{x+y}{x+z}\right) = 1.\end{aligned}$$

We conclude that $\alpha = \beta$.

2. Find all primes p and q, and even numbers $n > 2$, satisfying the equation

$$p^n + p^{n-1} + \cdots + p + 1 = q^2 + q + 1.$$

Solution: Obviously $p \neq q$. We write this in the form

$$p\left(p^{n-1} + p^{n-2} + \cdots + 1\right) = q(q+1).$$

If $q \leq p^{n/2} - 1$, then $q < p^{n/2}$ and hence we see that $q^2 < p^n$. Thus we obtain

$$q^2 + q < p^n + p^{n/2} < p^n + p^{n-1} + \cdots + p,$$

since $n > 2$. It follows that $q \geq p^{n/2}$. Since $n > 2$ and is an even number, $n/2$ is a natural number larger than 1. This implies that $q \neq p^{n/2}$ by the given condition that q is a prime. We conclude that $q \geq p^{n/2} + 1$. We may also write the above relation in the form

$$p\left(p^{n/2} - 1\right)\left(p^{n/2} + 1\right) = (p-1)q(q+1).$$

This shows that q divides $\left(p^{n/2} - 1\right)\left(p^{n/2} + 1\right)$. But $q \geq p^{n/2} + 1$ and q is a prime. Hence the only possibility is $q = p^{n/2} + 1$. This gives

$$p\left(p^{n/2} - 1\right) = (p-1)(q+1) = (p-1)\left(p^{n/2} + 2\right).$$

Simplification leads to $3p = p^{n/2} + 2$. This shows that p divide 2. Thus $p = 2$ and hence $q = 5$, $n = 4$. It is easy to verify that these indeed satisfy the given equation.

3. Show that for every real number a the equation

$$8x^4 - 16x^3 + 16x^2 - 8x + a = 0 \tag{1}$$

has at least one non-real root and find the sum of all the non-real roots of the equation.

Solution: Substituting $x = y + (1/2)$ in the equation, we obtain the equation in y:

$$8y^4 + 4y^2 + a - \frac{3}{2} = 0. \tag{2}$$

Using the transformation $z = y^2$, we get a quadratic equation in z:

$$8z^2 + 4z + a - \frac{3}{2} = 0. \tag{3}$$

The discriminant of this equation is $32(2 - a)$ which is nonnegative if and only if $a \le 2$. For $a \le 2$, we obtain the roots

$$z_1 = \frac{-1 + \sqrt{2(2-a)}}{4}, \quad z_2 = \frac{-1 - \sqrt{2(2-a)}}{4}.$$

For getting real y we need $z \ge 0$. Obviously $z_2 < 0$ and hence it gives only non-real values of y. But $z_1 \ge 0$ if and only if $a \le \frac{3}{2}$. In this case we obtain two real values for y and hence two real roots for the original equation (1). Thus we conclude that there are two real roots and two non-real roots for $a \le \frac{3}{2}$ and four non-real roots for $a > \frac{3}{2}$. Obviously the sum of all the roots of the equation is 2. For $a \le \frac{3}{2}$, two real roots of (2) are given by $y_1 = +\sqrt{z_1}$ and $y_2 = -\sqrt{z_1}$. Hence the sum of real roots of (1) is given by $y_1 + \frac{1}{2} + y_2 + \frac{1}{2}$ which reduces to 1. It follows the sum of the non-real roots of (1) for $a \le \frac{3}{2}$ is also 1. Thus

$$\text{The sum of nonreal roots} = \begin{cases} 1 & \text{for } a \le \frac{3}{2} \\ 2 & \text{for } a > \frac{3}{2} \end{cases}$$

4. Find all 7-digit numbers formed by using only the digits 5 and 7, and divisible by both 5 and 7.

Solution: Clearly, the last digit must be 5 and we have to determine the remaining 6 digits. For divisibility by 7, it is sufficient to consider the number obtained by replacing 7 by 0; for example 5775755 is divisible by 7 if and only 5005055 is divisible by 7. Each such number is obtained by adding some of the numbers from the set $\{50, 500, 5000, 50000, 500000, 5000000\}$ along with 5. We look at the remainders of these when divided by 7; they are $\{1, 3, 2, 6, 4, 5\}$. Thus it is sufficient to check for those combinations of remainders which add up to a number of the from $2+7k$, since the last digit is already 5. These are $\{2\}$, $\{3,6\}$, $\{4,5\}$, $\{2,3,4\}$, $\{1,3,5\}$, $\{1,2,6\}$, $\{2,3,5,6\}$, $\{1,4,5,6\}$ and $\{1,2,3,4,6\}$. These correspond to the numbers 7775775, 7757575, 5577775, 7575575, 5777555, 7755755, 5755575, 5557755, 755555.

5. Let ABC be a triangle with sides a, b, c. Consider a triangle $A_1B_1C_1$ with sides equal to $a+\frac{b}{2}$, $b+\frac{c}{2}$, $c+\frac{a}{2}$. Show that

$$[A_1B_1C_1] \geq \frac{9}{4}[ABC],$$

where $[XYZ]$ denotes the area of the triangle XYZ.

Solution: It is easy to observe that there is a triangle with sides $a+\frac{b}{2}$, $b+\frac{c}{2}$, $c+\frac{a}{2}$. Using Heron's formula, we get

$$16[ABC]^2 = (a+b+c)(a+b-c)(b+c-a)(c+a-b),$$

and

$$16\left[A_1B_1C_1\right]^2 = \frac{3}{16}(a+b+c)(-a+b+3c)(-b+c+3a)(-c+a+3b).$$

Since a,b,c are the sides of a triangle, there are positive real numbers p,q,r such that $a=q+r, b=r+p, c=p+q$. Using these relations we obtain

$$\frac{[ABC]^2}{\left[A_1B_1C_1\right]^2} = \frac{16pqr}{3(2p+q)(2q+r)(2r+p)}.$$

Thus it is sufficient to prove that

$$(2p+q)(2q+r)(2r+p) \geq 27pqr,$$

for positive real numbers p,q,r. Using AM-GM inequality, we get

$$2p+q \geq 3(p^2q)^{1/3}, 2q+r \geq 3(q^2r)^{1/3}, 2r+p \geq 3(r^2p)^{1/3}.$$

Multiplying these, we obtain the desired result. We also observe that equality holds if and only if $p=q=r$. This is equivalent to the statement that ABC is equilateral.

6. In a lottery, tickets are given nine-digit numbers using only the digits $1,2,3$. They are also coloured red, blue or green in such a way that two tickets whose numbers differ in all the nine places get different colours. Suppose

the ticket bearing the number 122222222 is red and that bearing the number 222222222 is green. Determine, with proof, the colour of the ticket bearing the number 123123123.

Solution: The following sequence of moves lead to the colour of the ticket bearing the number 123123123:

Line Number	Ticket Number	Colour	Reason
1	122222222	red	Given
2	222222222	green	Given
3	313113113	blue	Lines 1 & 2
4	231331331	green	Lines 1 & 3
5	331331331	blue	Lines 1 & 2
6	123123123	red	Lines 4 & 5

If 123123123 is reached by some other root, red colour must be obtained even along that root. For if for example 123123123 gets blue from some other root, then the following sequence leads to a contradiction:

Line Number	Ticket Number	Colour	Reason
1	122222222	red	Given
2	123123123	blue	Given
3	231311311	green	Lines 1 & 2
4	211331311	green	Lines 1 & 2
5	332212212	red	Lines 4 & 2
6	113133133	blue	Lines 3 & 5
7	331331331	green	Lines 1 & 2
8	222222222	red	Line 6 & 7

Thus the colour of 22222222 is red contradicting that it is grren.

INMO–2004

February 1, 2004

1. Consider a convex quadrilateral $ABCD$, in which K, L, M, N are the midpoints of the sides AB, BC, CD, DA respectively . Suppose

   (a) BD bisects KM at Q ;

   (b) $QA = QB = QC = QD$; and

   (c) $LK/LM = CD/CB$.

   Prove that $ABCD$ is a square .

2. Suppose p is a prime greater than 3. Find all pairs of integers (a, b) satisfying the equation

$$a^2 + 3ab + 2p(a + b) + p^2 = 0.$$

3. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, prove that $[\alpha^6] = 3$. (For any real number a , we denote by $[a]$ the greatest integer not exceeding a .)

4. Let R denote the circumradius of a triangle ABC; a, b, c its sides BC, CA, AB ; and r_a, r_b, r_c its exradii opposite A, B, C. If $2R \leq r_a$, prove that

   (i) $a > b$ and $a > c$;

   (ii) $2R > r_b$ and $2R > r_c$.

5. Let S denote the set of all 6-tuples (a, b, c, d, e, f) of positive integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Consider the set

$$T = \{abcdef : (a, b, c, d, e, f) \in S\}.$$

   Find the greatest common divisor of all the members of T.

6. Prove that the number of 5-tuples of positive integers (a,b,c,d,e) satisfying the equation

$$abcde = 5(bcde + acde + abde + abce + abcd)$$

   is an odd integer .

INMO 2004 - Solutions

1. Consider a convex quadrilateral $ABCD$, in which K, L, M, N are the midpoints of the sides AB, BC, CD, DA respectively. Suppose

   (a) BD bisects KM at Q;

   (b) $QA = QB = QC = QD$; and

   (c) $LK/LM = CD/CB$.

   Prove that $ABCD$ is a **square**.

   Solution:

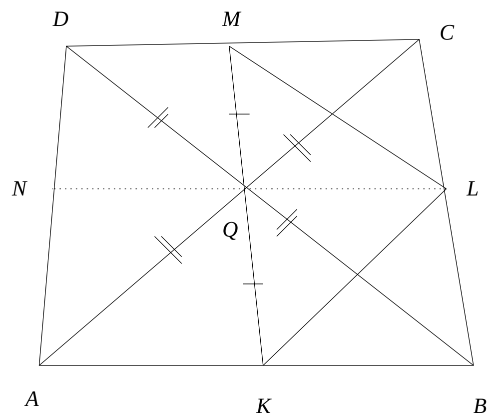


Fig. 1.

Observe that $KLMN$ is a paralellogram, Q is the midpoint of MK and hence NL also passes through Q. Let T be the point of intersection of AC and BD; and let S be the point of intersection of BD and MN.

Consider the triangle MNK. Note that SQ is parallel to NK and Q is the midpoint of MK. Hence S is the mid-point of MN. Since MN is parallel to AC, it follows that T is the mid-point of AC. Now Q is the circumcentre of $\triangle ABC$ and the median BT passes through Q. Here there are two possibilities:

(i) ABC is a right triangle with $\angle ABC = 90°$ and $T = Q$; and

(ii) $T \neq Q$ in which case BT is perpendicular to AC.

Suppose $\angle ABC = 90°$ and $T = Q$. Observe that Q is the circumcentre of the triangle DCB and hence $\angle DCB = 90°$. Similarly $\angle DAB = 90°$. It follows that $\angle ADC = 90°$. and $ABCD$ is a rectangle. This implies that $KLMN$ is a rhombus. Hence $LK/LM = 1$ and this gives $CD = CB$. Thus $ABCD$ is a square.

In the second case, observe that BD is perpendicular to AC, KL is parallel to AC and LM is parallel to BD. Hence it follows that ML is perpendicular to LK. Similar reasoning shows that $KLMN$ is a rectangle.

Using $LK/LM = CD/CB$, we get that CBD is similar to LMK. In particular, $\angle LMK = \angle CBD = \alpha$ say. Since LM is parallel to DB, we also get $\angle BQK = \alpha$. Since $KLMN$ is a cyclic quadrilateral we also get $\angle LNK = \angle LMK = \alpha$. Using the fact that BD is parallel to NK, we get $\angle LQB = \angle LNK = \alpha$. Since BD bisects $\angle CBA$, we also have $\angle KBQ = \alpha$. Thus

$$QK = KB = BL = LQ$$

and BL is parallel to QK. This gives QM is parallel to LC and

$$QM = QL = BL = LC$$

It follows that $QLCM$ is a parallelogram. But $\angle LCM = 90°$. Hence $\angle MQL = 90°$. This implies that $KLMN$ is a square. Also observe that $\angle LQK = 90°$ and hence $\angle CBA = \angle LQK = 90°$. This gives $\angle CDA = 90°$ and hence $ABCD$ is a rectangle. Since $BA = BC$, it follows that $ABCD$ is a square.

2. Suppose p is a prime greater than 3. Find all pairs of integers (a, b) satisfying the equation

$$a^2 + 3ab + 2p(a + b) + p^2 = 0.$$

<u>Solution:</u> We write the equation in the form

$$a^2 + 2ap + p^2 + b(3a + 2p) = 0$$

Hence

$$b = \frac{-(a + p)^2}{3a + 2p}$$

is an integer. This shows that $3a + 2p$ divides $(a + p)^2$ and hence also divides $(3a + 3p)^2$. But, we have

$$(3a + 3p)^2 = (3a + 2p + p)^2 = (3a + 2p)^2 + 2p(3a + 2p) + p^2.$$

It follows that $3a + 2p$ divides p^2. Since p is a prime, the only divisors of p^2 are $\pm 1, \pm p$ and $\pm p^2$. Since $p > 3$, we also have $p = 3k + 1$ or $3k + 2$.

<u>Case 1:</u> Suppose $p = 3k + 1$. Obviously $3a + 2p = 1$ is not possible. Infact, we get $1 = 3a + 2p = 3a + 2(3k + 1) \Rightarrow 3a + 6k = -1$ which is impossible. On the other hand $3a + 2p = -1$ gives $3a = -2p - 1 = -6k - 3 \Rightarrow a = -2k - 1$ and $a + p = -2k - 1 + 3k + 1 = k$.

Thus $b = \dfrac{-(a + p)^2}{(3k + 2p)} = k^2$. Thus $(a, b) = (-2k - 1, k^2)$ when $p = 3k + 1$. Similarly, $3a + 2p = p \Rightarrow 3a = -p$ which is not possible. Considering $3a + 2p = -p$, we get $3a = -3p$ or $a = -p \Rightarrow b = 0$. Hence $(a, b) = (-3k - 1, 0)$ where $p = 3k + 1$.

Let us consider $3a + 2p = p^2$. Hence $3a = p^2 - 2p = p(p - 2)$ and neither p nor $p - 2$ is divisible by 3. If $3a + 2p = -p^2$, then $3a = -p(p + 2) \Rightarrow a = -(3k + 1)(k + 1)$.

Hence $a + p = (3k + 1)(-k - 1 + 1) = -(3k + 1)k$. This gives $b = k^2$. Again $(a, b) = \Big(-(k + 1)(3k + 1), k^2\Big)$ when $p = 3k + 1$.

<u>Case 2:</u> Suppose $p = 3k - 1$. If $3a + 2p = 1$, then $3a = -6k + 3$ or $a = -2k + 1$. We also get

$$b = \frac{-(a + p)^2}{1} = \frac{-(-2k + 1 + 3k - 1)^2}{1} = -k^2$$

and we get the solution $(a, b) = (-2k + 1, k^2)$. On the other hand $3a + 2p = -1$ does not have any solution integral solution for a. Similarly, there is no solution in the case $3a + 2p = p$. Taking $3a + 2p = -p$, we get $a = -p$ and hence $b = 0$. We get the solution $(a, b) = (-3k + 1, 0)$. If $3a + 2p = p^2$, then $3a = p(p - 2) = (3k - 1)(3k - 3)$ giving $a = (3k - 1)(k - 1)$ and hence $a + p = (3k - 1)(1 + k - 1) = k(3k - 1)$. This gives $b = -k^2$ and hence $(a, b) = (3k - 1, -k^2)$. Finally $3a + 2p = -p^2$ does not have any solution.

3. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, prove that $[\alpha^6] = 3$. (For any real number a, we denote by $[a]$ the greatest integer not exceeding a.)

Solution: Suppose α is a real root of the given equation. Then

$$\alpha^5 - \alpha^3 + \alpha - 2 = 0. \qquad \cdots(\mathbf{1})$$

This gives $\alpha^5 - \alpha^3 + \alpha - 1 = 1$ and hence $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) = 1$. Observe that $\alpha^4 + \alpha^3 + 1 \geq 2\alpha^2 + \alpha^3 = \alpha^2(\alpha + 2)$. If $-1 \leq \alpha < 0$, then $\alpha + 2 > 0$, giving $\alpha^2(\alpha + 2) > 0$ and hence $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$. If $\alpha < -1$, then $\alpha^4 + \alpha^3 = \alpha^3(\alpha + 1) > 0$ and hence $\alpha^4 + \alpha^3 + 1 > 0$. This again gives $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$.

The above resoning shows that for $\alpha < 0$, we have $\alpha^5 - \alpha^3 + \alpha - 1 < 0$ and hence cannot be equal to 1. We conclude that a real root α of $x^5 - x^3 + x - 2 = 0$ is positive (obviously $\alpha \neq 0$).

Now using $\alpha^5 - \alpha^3 + \alpha - 2 = 0$, we get

$$\alpha^6 = \alpha^4 - \alpha^2 + 2\alpha$$

The statement $[\alpha^6] = 3$ is equivalent to $3 \leq \alpha^6 < 4$.

Consider $\alpha^4 - \alpha^2 + 2\alpha < 4$. Since $\alpha > 0$, this is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 < 4\alpha$. Using the relation (1), we can write $2\alpha^2 - \alpha + 2 < 4\alpha$ or $2\alpha^2 - 5\alpha + 2 < 0$. Treating this as a quadratic, we get this is equivalent to $\frac{1}{2} < \alpha < 2$. Now observe that if $\alpha \geq 2$ then $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) \geq 25$ which is impossible. If $0 < \alpha \leq \frac{1}{2}$, then $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$ which again is impossible. We conlude that $\frac{1}{2} < \alpha < 2$. Similarly $\alpha^4 - \alpha^2 + 2\alpha \geq 3$ is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \geq 0$ which is equivalent to $2\alpha^2 - 4\alpha + 2 \geq 0$. But this is $2(\alpha - 1)^2 \geq 0$ which is valid. Hence $3 \leq \alpha^6 < 4$ and we get $[\alpha^6] = 3$.

4. Let R denote the circumradius of a triangle ABC; a, b, c its sides BC, CA, AB; and r_a, r_b, r_c its exradii opposite A, B, C. If $2R \leq r_a$, prove that

   (i) $a > b$ and $a > c$;

   (ii) $2R > r_b$ and $2R > r_c$.

Solution: We know that $2R = \dfrac{abc}{2\triangle}$ and $r_a = \dfrac{\triangle}{s - a}$, where a, b, c are the sides of the triangle ABC, $s = \dfrac{a + b + c}{2}$ and $\triangle$ is the area of ABC. Thus the given condition $2R \leq r_a$ translates to

$$abc \leq \frac{2\triangle^2}{s - a}$$

Putting $s-a=p, s-b=q, s-c=r$, we get $a=q+r, b=r+p, c=p+q$ and the condition now is

$$p(p+q)(q+r)(r+p) \leq 2\triangle^2$$

But Heron's formula gives, $\triangle^2 = s(s-a)(s-b)(s-c) = pqr(p+q+r)$. We obtain $(p+q)(q+r)(r+p) \leq 2qr(p+q+r)$. Expanding and effecting some cancellations, we get

$$p^2(q+r) + p(q^2+r^2) \leq qr(q+r). \qquad (\star)$$

Suppose $a \leq b$. This implies that $q+r \leq r+p$ and hence $q \leq p$. This implies that $q^2r \leq p^2r$ and $qr^2 \leq pr^2$ giving $qr(q+r) \leq p^2r + pr^2 < p^2r + pr^2 + p^2q + pq^2 = p^2(q+r) + p(q^2+r^2)$ which contradicts $(\star)$. Similarly, $a \leq c$ is also not possible. This proves (i).

Suppose $2R \leq r_b$. As above this takes the form

$$q^2(r+p) + q(r^2+p^2) \leq pr(p+r). \qquad (\star\star)$$

Since $a > b$ and $a > c$, we have $q > p, r > p$. Thus $q^2r > p^2r$ and $qr^2 > pr^2$. Hence

$$q^2(r+p) + q(r^2+p^2) > q^2r + qr^2 > p^2r + pr^2 = pr(p+r)$$

which contradicts $(\star\star)$. Hence $2R > r_b$. Similarly, we can prove that $2R > r_c$. This proves (ii)

5. Let S denote the set of all 6-tuples (a,b,c,d,e,f) of positive integers such that $a^2+b^2+c^2+d^2+e^2 = f^2$. Consider the set

$$T = \Big\{abcdef \ : \ (a,b,c,d,e,f) \in S\Big\}.$$

Find the greatest common divisor of all the members of T.

Solution: We show that the required gcd is 24. Consider an element $(a,d,c,d,e,f) \in S$. We have

$$a^2 + b^2 + c^2 + d^2 + e^2 = f^2.$$

We first observe that not all a,b,c,d,e can be odd. Otherwise, we have $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod 8$ and hence $f^2 \equiv 5 \pmod 8$, which is impossible because no square can be congruent to 5 modulo 8. Thus at least one of a,b,c,d,e is even.

Similarly if none of a,b,c,d,e is divisible by 3, then $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod 3$ and hence $f^2 \equiv 2 \pmod 3$ which again is impossible because no square is congruent to 2 modulo 3. Thus 3 divides $abcdef$.

There are several possibilities for a,b,c,d,e.

Case 1: Suppose one of them is even and the other four are odd; say a is even, b,c,d,e are odd. Then $b^2+c^2+d^2+e^2 \equiv 4 \pmod 8$. If $a^2 \equiv 4 \pmod 8$, then $f^2 \equiv 0 \pmod 8$ and hence $2|a, 4|f$ giving $8|af$. If $a^2 \equiv 0 \pmod 8$, then $f^2 \equiv 4 \pmod 8$ which again gives that $4|a$ and $2|f$ so that $8|af$. It follows that $8|abcdef$ and hence $24|abcdef$.

Case 2: Suppose a,b are even and c,d,e are odd. Then $c^2+d^2+e^2 \equiv 3 \pmod 8$. Since $a^2+b^2 \equiv 0$ or 4 modulo 8, it follows that $f^2 \equiv 3$ or $7 \pmod 8$ which is impossible. Hence this case does not arise.

Case 3: If three of a,b,c,d,e are even and two odd, then $8|abcdef$ and hence $24|abcdef$.

Case 4: If four of a,b,c,d,e are even, then again $8|abcdef$ and $24|abcdef$. Here again for any six tuple (a,b,c,d,e,f) in S, we observe that $24|abcdef$. Since

$$1^2 + 1^2 + 1^2 + 2^2 + 3^2 = 4^2.$$

We see that (1,1,1,2,3,4)$\in S$ and hence $24 \in T$. Thus 24 is the gcd of T.

6. Prove that the number of 5-tuples of positive integers (a, b, c, d, e) satisfying the equation

$$abcde = 5\big(bcde + acde + abde + abce + abcd\big)$$

is an **odd** integer.

Solution: We write the equation in the form:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = \frac{1}{5}.$$

The number of five tuple (a, b, c, d, e) which satisfy the given relation and for which $a \neq b$ is even, because for if (a, b, c, d, e) is a solution, then so is (b, a, c, d, e)which is distinct from (a, b, c, d, e). Similarly the number of five tuples which satisfy the equation and for which $c \neq d$ is also even. Hence it suffices to count only those five tuples (a, b, c, d, e) for which $a = b, c = d$. Thus the equation reduces to

$$\frac{2}{a} + \frac{2}{c} + \frac{1}{e} = \frac{1}{5}.$$

Here again the tuple (a, a, c, c, e) for which $a \neq c$ is even because we can associate different solution (c, c, a, a, e) to this five tuple. Thus it suffices to consider the equation

$$\frac{4}{a} + \frac{1}{e} = \frac{1}{5},$$

and show that the number of pairs (a, e) satisfying this equation is odd.

This reduces to

$$ae = 20e + 5a$$

or

$$(a - 20)(e - 5) = 100.$$

But observe that

$$100 = 1 \times 100 = 2 \times 50 = 4 \times 25 = 5 \times 20$$
$$= 10 \times 10 = 20 \times 5 = 25 \times 4 = 50 \times 2 = 100 \times 1.$$

Note that no factorisation of 100 as product of two negative numbers yield a positive tuple (a, e). Hence we get these 9 solutions. This proves that the total number of five tuples (a, b, c, d, e) satisfying the given equation is odd.

INMO–2005

February 6, 2005

1. Let M be the midpoint of side BC of a triangle ABC. Let the median AM intersect the incircle of ABC at K and L, K being nearer to A than L. If $AK = KL = LM$, prove that the sides of triangle ABC are in the ratio $5 : 10 : 13$ in some order.

2. Let α and β be positive integers such that
$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}$$
.
Find the minimum possible value of β.

3. Let p, q, r be positive real numbers, not all equal, such that some two of the equations
$$px^2 + 2qx + r = 0, qx^2 + 2rx + p = 0, rx^2 + 2px + q = 0,$$
have a common root, say α. Prove that

   (a) α is real and negative; and

   (b) the remaining third equation has non-real roots.

4. All possible 6-digit numbers, in each of which the digits occur in non-increasing order (from left to right, e.g.,877550) are written as a sequence in increasing order. Find the 2005-th number in this sequence.

5. Let x_1 be a given positive integer. A sequence$(x_n)_{n=1}^{\infty} = (x_1, x_2, x_3, \cdots)$ of positive integers is such that x_n, for $n \geq 2$, is obtained from x_{n-1} by adding some nonzero digit of x_{n-1}. Prove that

   (a) the sequence has an even number;

   (b) the sequence has infinitely many even numbers.

6. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that
$$f(x^2 + yf(z)) = xf(x) + zf(y),$$
for all x, y, z in $\mathbf{R}$. (Here $\mathbf{R}$ denotes the set of all real numbers.)

INMO 2005: Problems and Solutions

1. Let M be the midpoint of side BC of a triangle ABC. Let the median AM intersect the incircle of ABC at K and L, K being nearer to A than L. If $AK{=}KL{=}LM$, prove that the sides of triangle ABC are in the ratio $5 : 10 : 13$ in some order.

Solution:
Let I be the incentre of triangle ABC and D be its projection on BC. Observe that $AB \neq AC$ as $AB = AC$ implies that $D = L = M$. So assume that $AC > AB$. Let N be the projection of I on KL. Then the perpendicular IN from I to KL is a bisector of KL and as $AK = LM$, it is a bisector of AM also. Hence $AI = IM$.

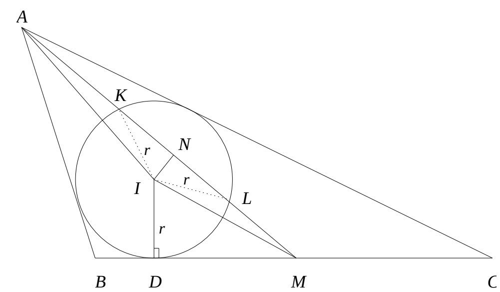


Fig. 1.

But $AI = \dfrac{r}{\sin\left(A/2\right)} = r\ \text{cosec}\left(A/2\right)$ and

$$\begin{aligned} IM^2 &= ID^2 + DM^2 = r^2 + (BM - BD)^2 \\ &= r^2 + \left(\frac{a}{2} - (s-b)\right)^2. \end{aligned}$$

Hence $r^2\ \text{cosec}^2\left(A/2\right) = r^2 + \left((a/2) - (s-b)^2\right)^2$ giving $r^2\ \cot^2\left(A/2\right) = \left((b-c)/2\right)^2$. Since $b > c$, we obtain $r\ \cot\ \left(A/2\right) = \left((b-c)/2\right)$. So $s - a = \left((b-c)/2\right)$. This gives $a = 2c$.

As $KN = NL$ and $AK = KL = LM$, we have $NL = AM/6$. We also have $AN = NM$. Now

$$\begin{aligned} r^2 = IL^2 = IN^2 + NL^2 &= AI^2 - AN^2 + NL^2 \\ &= AI^2 - \frac{1}{4}m_a^2 + \frac{1}{36}m_a^2 \\ &= r^2\ \text{cosec}^2\left(A/2\right) - \frac{2}{9}m_a^2. \end{aligned}$$

Hence $r^2\ \cot^2\left(A/2\right) = \frac{2}{9}m_a^2$. From the above,we get

$$\left(\frac{b-c}{2}\right)^2 = \frac{2}{9}\cdot\frac{1}{4}\left(2b^2 + 2c^2 - a^2\right).$$

Simplification gives $5b^2 + 13c^2 - 18bc = 0$. This can be written as $(b-c)(5b - 13c) = 0$. As $b \neq c$, we get $5b - 13c = 0$. To conclude, $a = 2c, 5b = 13c$ yield

$$\frac{a}{10} = \frac{b}{13} = \frac{c}{5}.$$

2. Let α and β be positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}.$$

Find the minimum possible value of β.

Solution:
We have

$$\frac{77}{17} < \frac{\beta}{\alpha} < \frac{197}{43}.$$

That is,

$$4 + \frac{9}{17} < \frac{\beta}{\alpha} < 4 + \frac{25}{43}.$$

Thus $4 < \frac{\beta}{\alpha} < 5$. Since α and β are positive integers, we may write $\beta = 4\alpha + x$, where $0 < x < \alpha$. Now we get

$$4 + \frac{9}{17} < 4 + \frac{x}{\alpha} < 4 + \frac{25}{43}.$$

So $\frac{9}{17} < \frac{x}{\alpha} < \frac{25}{43}$; that is, $\frac{43x}{25} < \alpha < \frac{17x}{9}$.

We find the smallest value of x for which α becomes a well-defined integer. For $x = 1, 2, 3$ the bounds of α are respectively $\left(1\frac{18}{25}, 1\frac{8}{9}\right)$, $\left(3\frac{11}{25}, 3\frac{7}{9}\right)$, $\left(5\frac{4}{9}, 5\frac{2}{3}\right)$. None of these pairs contain an integer between them.

For $x = 4$, we have $\frac{43x}{25} = 6\frac{12}{25}$ and $\frac{17x}{9} = 7\frac{5}{9}$. Hence, in this case $\alpha = 7$, and $\beta = 4\alpha + x = 28 + 4 = 32$.

This is also the least possible value, because, if $x \geq 5$, then $\alpha > \frac{43x}{25} \geq \frac{43}{5} > 8$, and so $\beta > 37$. Hence the minimum possible value of β is 32.

3. Let p, q, r be positive real numbers, not all equal, such that some two of the equations

$$px^2 + 2qx + r = 0, \quad qx^2 + 2rx + p = 0, \quad rx^2 + 2px + q = 0,$$

have a common root, say α. Prove that

(a) α is real and negative; and

(b) the third equation has non-real roots.

Solution:
Consider the discriminants of the three equations

$$\begin{aligned} px^2 + qr + r &= 0 & (1)\\ qx^2 + rx + p &= 0 & (2)\\ rx^2 + px + q &= 0. & (3) \end{aligned}$$

Let us denote them by D_1, D_2, D_3 respectively. Then we have

$$D_1 = 4(q^2 - rp), D_2 = 4(r^2 - pq), D_3 = 4(p^2 - qr).$$

We observe that

$$\begin{aligned} D_1 + D_2 + D_3 &= 4(p^2 + q^2 + r^2 - pq - qr - rp)\\ &= 2\left\{(p-q)^2 + (q-r)^2 + (r-p)^2\right\} > 0 \end{aligned}$$

since p, q, r are not all equal. Hence at least one of D_1, D_2, D_3 must be positive. We may assume $D_1 > 0$.

Suppose $D_2 < 0$ and $D_3 < 0$. In this case both the equations (2) and (3) have only non-real roots and equation (1) has only real roots. Hence the common root α must be between (2) and (3). But then $\bar{\alpha}$ is the other root of both (2) and (3). Hence it follows that (2) and (3) have same set of roots. This implies that

$$\frac{q}{r} = \frac{r}{p} = \frac{p}{q}.$$

Thus $p = q = r$ contradicting the given condition. Hence both D_2 and D_3 cannot be negative. We may assume $D_2 \geq 0$. Thus we have

$$q^2 - rp > 0, \; r^2 - pq \geq 0.$$

These two give

$$q^2r^2 > p^2qr$$

since p, q, r are all positive. Hence we obtain $qr > p^2$ or $D_3 < 0$. We conclude that the common root must be between equations (1) and (2).
Thus

$$\begin{aligned} p\alpha^2 + q\alpha + r &= 0 \\ q\alpha^2 + r\alpha + p &= 0 \end{aligned}$$

Eliminating α^2, we obtain

$$2(q^2 - pr)\alpha = p^2 - qr.$$

Since $q^2 - pr > 0$ and $p^2 - qr < 0$, we conclude that $\alpha < 0$.

The condition $p^2 - qr < 0$ implies that the equation (3) has only non-real roots.

Alternately one can argue as follows. Suppose α is a common root of two equations, say, (1) and (2). If α is non-real, then $\bar{\alpha}$ is also a root of both (1) and (2). Hence The coefficients of (1) and (2) are proportional. This forces $p = q = r$, a contradiction. Hence the common root between any two equations cannot be non-real. Looking at the coefficients, we conclude that the common root α must be negative. If (1) and (2) have common root α, then $q^2 \geq rp$ and $r^2 \geq pq$. Here at least one inequality is strict for $q^2 = pr$ and $r^2 = pq$ forces $p = q = r$. Hence $q^2r^2 > p^2qr$. This gives $p^2 < qr$ and hence (3) has nonreal roots.

4. All possible 6-digit numbers, in each of which the digits occur in **non-increasing** order (from left to right, e.g., 877550) are written as a sequence in **increasing** order. Find the 2005-th number in this sequence.

Solution I:
Consider a 6-digit number whose digits from left to right are in non increasing order. If 1 is the first digit of such a number, then the subsequent digits cannot exceed 1. The set of all such numbers with initial digit equal to 1 is

$$\{100000, 110000, 111000, 111100, 111110, 1111111\}.$$

There are elements in this set.

Let us consider 6-digit numbers with initial digit 2. Starting form 200000, we can go up to 222222. We count these numbers as follows:

200000	-	211111	:	6
220000	-	221111	:	5
222000	-	222111	:	4
222200	-	222211	:	3
222220	-	222221	:	2
222222	-	222222	:	1

The number of such numbers is 21. Similarly we count numbers with initial digit 3; the sequence starts from 300000 and ends with 333333. We have

300000	-	322222	:	21
330000	-	332222	:	15
333000	-	333222	:	10
333300	-	333322	:	6
333330	-	333332	:	3
333333	-	333333	:	1

We obtain the total number of numbers starting from 3 equal to 56. Similarly,

400000	-	433333	:	56
440000	-	443333	:	35
444000	-	444333	:	20
444400	-	444433	:	10
444440	-	444443	:	4
444444	-	444444	:	1
				126

500000	-	544444	:	126
550000	-	554444	:	70
555000	-	555444	:	35
555500	-	555544	:	15
555550	-	555554	:	5
555555	-	555555	:	1
				252

600000	-	655555	:	252
660000	-	665555	:	126
666000	-	666555	:	56
666600	-	666655	:	21
666660	-	666665	:	6
666666	-	666666	:	1
				462

700000	-	766666	:	462
770000	-	776666	:	210
777000	-	777666	:	84
777700	-	777766	:	28
777770	-	777776	:	7
777777	-	777777	:	1
				792

Thus the number of 6-digit numbers where digits are non-increasing starting from 100000 and ending with 777777 is

$$792 + 462 + 252 + 126 + 56 + 21 + 6 = 1715.$$

Since 2005-1715=290, we have to consider only 290 numbers in the sequence with initial digit 8. We have

800000	-	855555	:	252
860000	-	863333	:	35
864000	-	864110	:	3

Thus the required number is 864110.

Solution: II

It is known that the number of ways of choosing r objects from n different types of objects (with repetitions allowed) is $\binom{n+r-1}{r}$. In particular, if we want to write r-digit numbers using n digits allowing for repetitions with the additional condition that the digits appear in non-increasing order, we see that this can be done in $\binom{n+r-1}{r}$ ways.

Now we group the given numbers into different classes and write the number of ways in which each class can be obtained. To keep track we also write the cumulative sums of the number of numbers so obtained. Observe that the numbers themselves are written in ascending order. So we exhaust numbers beginning with 1, then beginning with 2 and so on.

Numbers	Digits used other than the fixed part	n	r	$\binom{n+r-1}{r}$	Cumulative sum
beginning with 1	1,0	2	5	$\binom{6}{5} = 6$	6
2	2,1,0	3	5	$\binom{7}{5} = 21$	27
3	3,2,1,0	4	5	$\binom{8}{5} = 56$	83
4	4,3,2,1,0	5	5	$\binom{9}{5} = 126$	209
5	5,4,3,2,1,0	6	5	$\binom{10}{5} = 252$	461
6	6,5,4,3,2,1,0	7	5	$\binom{11}{5} = 462$	923
7	7,6,5,4,3,2,1,0	8	5	$\binom{12}{5} = 792$	1715
from 800000 to 855555	5,4,3,2,1,0	6	5	$\binom{10}{5} = 252$	1967
from 860000 to 863333	3,2,1,0	4	4	$\binom{7}{4} = 35$	2002

The next three 6-digit numbers are 864000, 864100, 864110.

Hence the 2005th number in the sequence is 864110.

5. Let x_1 be a given positive integer. A sequence $\langle x_n\rangle_{n=1}^{\infty} = \langle x_1, x_2, x_3, \cdots\rangle$ of positive integers is such that x_n, for $n \geq 2$, is obtained from x_{n-1} by adding some nonzero digit of x_{n-1}. Prove that

   (a) the sequence has an **even** number;

   (b) the sequence has infinitely many even numbers.

Solution:

(a) Let us assume that there are no even numbers in the sequence. This means that x_{n+1} is obtained from x_n, by adding a nonzero even digit of x_n to x_n, for each $n \geq 1$.
Let E be the left most even digit in x_1 which may be taken in the form

$$x_1 = O_1 O_2 \cdots O_k E D_1 D_2 \cdots D_l$$

where $O_1, O_2, \ldots, O_k$ are odd digits ($k \geq 0$); $D_1, D_2, \ldots, D_{l-1}$ are even or odd; and D_l odd, $l \geq 1$.
Since each time we are adding at least 2 to a term of the sequence to get the next term, at some stage, we will have a term of the form

$$x_r = O_1 O_2 \cdots O_k E 999 \cdots 9F$$

where $F = 3, 5, 7$ or 9. Now we are forced to add E to x_r to get x_{r+1}, as it is the only even digit available. After at most four steps of addition, we see that some next term is of the form

$$x_s = O_1 O_2 \cdots O_k G 000 \cdots M$$

where G replaces E of x_r, $G = E + 1$, $M = 1, 3, 5,$ or 7. But x_s has no nonzero even digit contradicting our assumption. Hence the sequence has some even number as its term.

(b) If there are only finitely many even terms and x_t is the last term, then the sequence $\langle x_n \rangle_{n=t+1}^{\infty} = \langle x_{t+1}, x_{t+2}, \ldots \rangle$ is obtained in a similar manner and hence must have an even term by (a), a contradiction. Thus $\langle x_n \rangle_{n=1}^{\infty}$, has infinitely many even terms.

6. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f\left(x^2 + yf(z)\right) = xf(x) + zf(y) \tag{1}$$

for all x, y, z in $\mathbf{R}$. (Here $\mathbf{R}$ denotes the set of all real numbers.)

Solution: Taking $x = y = 0$ in (1), we get $zf(0) = f(0)$ for all $z \in \mathbf{R}$. Hence we obtain $f(0) = 0$. Taking $y = 0$ in (1), we get

$$f(x^2) = xf(x) \tag{2}$$

Similarly $x = 0$ in (1) gives

$$f(yf(z)) = zf(y) \tag{3}$$

Putting $y = 1$ in (3), we get

$$f(f(z)) = zf(1) \quad \forall\, z \in \mathbf{R} \tag{4}$$

Now using (2) and (4), we obtain

$$f(xf(x)) = f(f(x^2)) = x^2 f(1) \tag{5}$$

Put $y = z = x$ in (3) also given

$$f(xf(x)) = xf(x) \tag{6}$$

Comparing (5) and (6), it follows that $x^2 f(1) = xf(x)$. If $x \neq 0$, then $f(x) = cx$, for some constant c. Since $f(0) = 0$, we have $f(x) = cx$ for $x = 0$ as well. Substituting this in (1), we see that

$$c(x^2 + cyz) = cx^2 + cyz$$

or

$$c^2 yz = cyz \quad \forall\, y, z \in \mathbf{R}.$$

This implies that $c^2 = c$. Hence $c = 0$ or 1. We obtain $f(x) = 0$ for all x or $f(x) = x$ for all x. It is easy to verify that these two are solutions of the given equation.

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INMO–2006

1. In a nonequilateral triangle ABC, the sides a, b, c form an arithmetic progression. Let I and O denote the incentre and circumcentre of the triangle respectively.

   (i) Prove that IO is perpendicular to BI.

   (ii) Suppose BI extended meets AC in K, and D, E are the mid-points of BC, BA respectively. Prove that I is the circumcentre of triangle DKE.

2. Prove that for every positive integer n there exists a **unique** ordered pair (a, b) of positive integers such that

$$n = \frac{1}{2}(a+b-1)(a+b-2) + a.$$

3. Let X denote the set of all triples (a, b, c) of integers. Define a function $f : X \to X$ by

$$f(a, b, c) = (a+b+c, ab+bc+ca, abc).$$

   Find all triples (a, b, c) in X such that $f(f(a, b, c)) = (a, b, c)$.

4. Some 46 squares are randomly chosen from a 9×9 chess board and are coloured red. Show that there exists a 2×2 block of 4 squares of which at least three are coloured red.

5. In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $\angle ABC = 120°$, and $\angle ABD = 30°$. Prove that

   (i) $c \geq a + b$;

   (ii) $|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}$.

6. (a) Prove that if n is a positive integer such that $n \geq 4011^2$, then there exists an integer l such that $n < l^2 < (1 + \frac{1}{2005})n$.

   (b) Find the smallest positive integer M for which whenever an integer n is such that $n \geq M$, there exists an integer l, such that $n < l^2 < (1 + \frac{1}{2005})n$.

INMO 2006: Problems and Solutions

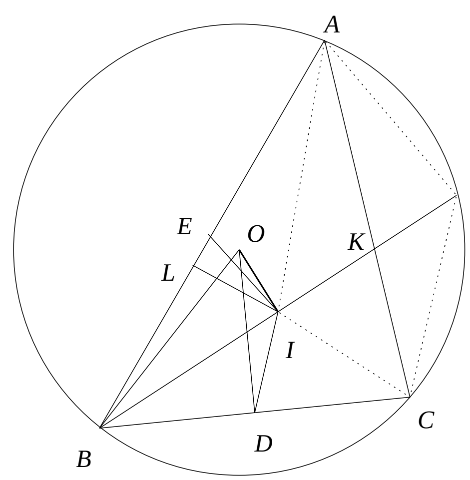
1. In a non-equilateral triangle ABC, the sides a, b, c form an arithmetic progression. Let I and O denote the incentre and circumcentre of the triangle respectively.

   (i) Prove that IO is perpendicular to BI.

   (ii) Suppose BI extended meets AC in K, and D, E are the midpoints of BC, BA respectively. Prove that I is the circumcentre of triangle DKE.

Solution:

(i) Extend BI to meet the circumcircle in F. Then we know that $FA = FI = FC$. (See Figure)



Let $BI : IF = \lambda : \mu$. Applying Stewart's theorem to triangle BAF, we get

$$\lambda AF^2 + \mu AB^2 = (\lambda + \mu)(AI^2 + BI \cdot IF).$$

Similarly, Stewart's theorem to triangle BCF gives

$$\lambda CF^2 + \mu BC^2 = (\lambda + \mu)(CI^2 + BI \cdot IF).$$

Since $CF = AF$, subtraction gives

$$\mu(AB^2 - BC^2) = (\lambda + \mu)(AI^2 - CI^2).$$

Using the standard notations $AB = c$, $BC = a$, $CA = b$ and $s = (a + b + c)/2$, we get $AI^2 = r^2 + (s - a)^2$ and $CI^2 = r^2 + (s - c)^2$ where r is the in-radius of ABC. Thus

$$\mu(c^2 - a^2) = (\lambda + \mu)\big((s - a)^2 - (s - c)^2\big) = (\lambda + \mu)(c - a)b.$$

It follows that either $c = a$ or $\mu(c+a) = (\lambda+\mu)b$. But $c = a$ implies that $a = b = c$ since a, b, c are in arithmetic progression. However, we have taken a non-equilateral triangle ABC. Thus $c \neq a$ and we have $\mu(c + a) = (\lambda + \mu)b$. But $c + a = 2b$ and we obtain

$2b\mu = (\lambda+\mu)b$. We conclude that $\lambda = \mu$. This in turn tells that I is the mid-point of BF. Since $OF = OB$, we conclude that OI is perpendicular to BF.

Alternatively

Applying Ptolemy's theorem to the cyclic quadrilateral $ABCF$, we get

$$AB \cdot CF + AF \cdot BC = BF \cdot CA.$$

Since $CF = AF$, we get $CF(c+a) = BF \cdot b = BF(c+a)/2$. This gives $BF = 2CF = 2IF$. Hence I is the mid-point of BF and as earlier we conclude that OI is perpendicular to BF.

Alternatively

Join BO.We have to prove that $\angle BIO = 90^\circ$, which is equivalent to $BI^2 + IO^2 = BO^2$. Draw IL perpendicular to AB. Let R denote the circumradius of ABC and let Δ denote its area. Observe that $BO = R$, $IO^2 = R^2 - 2Rr$,

$$BI = \frac{BL}{\cos(B/2)} = (s-b)\sqrt{\frac{ca}{s(s-b)}}.$$

Thus we obtain

$$BI^2 = ac(s-b)/s = \frac{ac}{3},$$

since a, b, c are in arithmetic progression. Thus we need to prove that

$$\frac{ac}{3} + R^2 - 2Rr = R^2.$$

This reduces to proving $2Rr = ac/3$. But

$$2Rr = 2 \cdot \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} = \frac{abc}{2s} = \frac{abc}{a+b+c} = \frac{ac}{3},$$

using $a + c = 2b$. This proves the claim.

(ii) Join ID. Note that $\angle BIO = \angle BDO = 90^\circ$. Hence B, D, I, O are concyclic and hence $\angle BID = \angle BOD = A$. Since $\angle DBI = \angle KBA = B/2$, it follows that triangles BAK and BID are similar. This gives

$$\frac{BA}{BI} = \frac{BK}{BD} = \frac{AK}{ID}.$$

However, we have seen earlier that $BI = ac/3$. Moreover $AK = bc/(a+c)$. Thus we obtain

$$BK = \frac{BA \cdot BD}{BI} = \frac{1}{2}\sqrt{3ac}, \quad ID = \frac{AK \cdot BI}{BA} = \frac{1}{2}\sqrt{\frac{ac}{3}}.$$

By symmetry, we must have $IE = \frac{1}{2}\sqrt{\frac{ac}{3}}$. Finally

$$IK = \frac{b}{a+b+c} \cdot BK = \frac{1}{3}BK = \frac{1}{2}\sqrt{\frac{ac}{3}}.$$

Thus $ID = IE = IK$ and I is the circumcentre of DKE.

Alternatively

Observe that $AK = bc/(a+c) = c/2 = AE$. Since AI bisects angle A, we see that AIE is congruent to AIK. This gives $IE = IK$. Similarly CID is congruent to CIK giving $ID = IK$. We conclude that $ID = IK = IE$.

2. Prove that for every positive integer n there exists a **unique** ordered pair (a, b) of positive integers such that
$$n = \frac{1}{2}(a + b - 1)(a + b - 2) + a.$$

Solution: We have to prove that $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by
$$f(a, b) = \frac{1}{2}(a + b - 1)(a + b - 2) + a, \quad \forall a, b \in \mathbb{N},$$
is a bijection. (Note that the right side is a natural number.) To this end define
$$T(n) = \frac{n(n+1)}{2}, \quad n \in \mathbb{N} \cup \{0\}.$$
An idea of the proof can be obtained by looking at the following table of values of $f(a, b)$ for some small values of a, b.

a \ b	1	2	3	4	5	6
1	1	2	4	7	11	16
2	3	5	8	12	17	
3	6	9	13	18		
4	10	14	19			
5	15	20				
6	21					

We observe that the n-th diagonal runs from $(1, n)$-th position to $(n, 1)$-th position and the entries are n consecutive integers; the first entry in the n-th diagonal is one more than the last entry of the $(n-1)$-th diagonal. For example the first entry in 5-th diagonal is 11 which is one more than the last entry of 4-th diagonal which is 10. Observe that 5-th diagonal starts from 11 and ends with 15 which accounts for 5 consecutive natural numbers. Thus we see that $f(n-1, 1) + 1 = f(1, n)$. We also observe that the first n diagonals exhaust all the natural numbers from 1 to $T(n)$. (Thus a kind of visual bijection is already there. We formally prove the property.)

We first observe that
$$f(a, b) - T(a + b - 2) = a > 0,$$
and
$$T(a + b - 1) - f(a, b) = \frac{(a + b - 1)(a + b)}{2} - \frac{(a + b - 1)(a + b - 2)}{2} - a = b - 1 \geq 0.$$

Thus we have

$$T(a+b-2) < f(a,b) = \frac{(a+b-1)(a+b-2)}{2} + a \le T(a+b-1).$$

Suppose $f\big(a_1,b_1\big) = f\big(a_2,b_2\big)$. Then the previous observation shows that

$$\begin{aligned} T\big(a_1+b_1-2\big) &< f\big(a_1,b_1\big) \le T\big(a_1+b_1-1\big), \\ T\big(a_2+b_2-2\big) &< f\big(a_2,b_2\big) \le T\big(a_2+b_2-1\big). \end{aligned}$$

Since the sequence $\langle T(n)\rangle_{n=0}^{\infty}$ is strictly increasing, it follows that $a_1+b_1 = a_2+b_2$. But then the relation $f\big(a_1,b_1\big) = f\big(a_2,b_2\big)$ implies that $a_1 = a_2$ and $b_1 = b_2$. Hence f is one-one.

Let n be any natural number. Since the sequence $\langle T(n)\rangle_{n=0}^{\infty}$ is strictly increasing, we can find a natural number k such that

$$T(k-1) < n \le T(k).$$

Equivalently,

$$\frac{(k-1)k}{2} < n \le \frac{k(k+1)}{2}. \tag{1}$$

Now set $a = n - \dfrac{k(k-1)}{2}$ and $b = k-a+1$. Observe that $a > 0$. Now (1) shows that

$$a = n - \frac{k(k-1)}{2} \le \frac{k(k+1)}{2} - \frac{k(k-1)}{2} = k.$$

Hence $b = k-a+1 \ge 1$. Thus a and b are both positive integers and

$$f(a,b) = \frac{1}{2}(a+b-1)(a+b-2) + a = \frac{k(k-1)}{2} + a = n.$$

This shows that every natural number is in the range of f. Thus f is also onto. We conclude that f is a bijection.

3. Let X denote the set of all triples (a,b,c) of integers. Define a function $f : X \to X$ by

$$f(a,b,c) = (a+b+c,\ ab+bc+ca,\ abc).$$

Find all triples (a,b,c) in X such that $f\big(f(a,b,c)\big) = (a,b,c)$.

Solution: We show that the solutionset consists of $\{(t,0,0)\ ;\ t \in \mathbb{Z}\} \cup \{(-1,-1,1)\}$. Let us put $a+b+c = d$, $ab+bc+ca = e$ and $abc = f$. The given condition $f\big(f(a,b,c)\big) = (a,b,c)$ implies that

$$d+e+f = a, \quad de+ef+fd = b, def = c.$$

Thus $abcdef = fc$ and hence either $cf = 0$ or $abde = 1$.

Case I: Suppose $cf = 0$. Then either $c = 0$ or $f = 0$. However $c = 0$ implies $f = 0$ and vice-versa. Thus we obtain $a+b = d$, $d+e = a$, $ab = e$ and $de = b$. The first two relations give $b = -e$. Thus $e = ab = -ae$ and $de = b = -e$. We get either $e = 0$ or $a = d = -1$.

If $e = 0$, then $b = 0$ and $a = d = t$, say. We get the triple $(a,b,c) = (t,0,0)$, where $t \in \mathbb{Z}$. If $e \ne 0$, then $a = d = -1$. But then $d+e+f = a$ implies that $-1+e+0 = -1$ forcing $e = 0$. Thus we get the solution family $(a,b,c) = (t,0,0)$, where $t \in \mathbb{Z}$.

Case II: Suppose $cf \neq 0$. In this case $abde = 1$. Hence either all are equal to 1; or two equal to 1 and the other two equal to -1; or all equal to -1.

Suppose $a = b = d = e = 1$. Then $a + b + c = d$ shows that $c = -1$. Similarly $f = -1$. Hence $e = ab + bc + ca = 1 - 1 - 1 = -1$ contradicting $e = 1$.

Suppose $a = b = 1$ and $d = e = -1$. Then $a + b + c = d$ gives $c = -3$ and $d + e + f = a$ gives $f = 3$. But then $f = abc = 1 \cdot 1 \cdot (-3) = -3$, a contradiction. Similarly $a = b = -1$ and $d = e = 1$ is not possible.

If $a = 1$, $b = -1$, $d = 1$, $e = -1$, then $a + b + c = d$ gives $c = 1$. Similarly $f = 1$. But then $f = abc = 1 \cdot 1 \cdot (-1) = -1$ a contradiction. If $a = 1$, $b = -1$, $d = -1$, $e = 1$, then $c = -1$ and $e = ab + bc + ca = -1 + 1 - 1 = -1$ and a contradiction to $e = 1$. The symmetry between (a, b, c) and (d, e, f) shows that $a = -1$, $b = 1$, $d = 1$, $e = -1$ is not possible. Finally if $a = -1$, $b = 1$, $d = -1$ and $e = 1$, then $c = -1$ and $f = -1$. But then $f = abc$ is not satisfied.

The only case left is that of a, b, d, e being all equal to -1 . Then $c = 1$ and $f = 1$. It is easy to check that $(-1, -1, 1)$ is indeed a solution.

Alternatively

$cf \neq 0$ implies that $|c| \geq 1$ and $|f| \geq 1$. Observe that

$$d^2 - 2e = a^2 + b^2 + c^2, \quad a^2 - 2b = d^2 + e^2 + f^2.$$

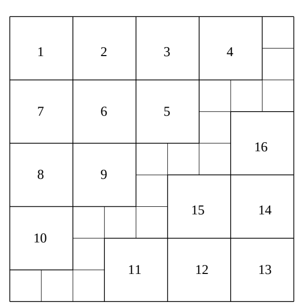
Adding these two, we get $-2(b + e) = b^2 + c^2 + e^2 + f^2$. This may be written in the form

$$(b + 1)^2 + (e + 1)^2 + c^2 + f^2 - 2 = 0.$$

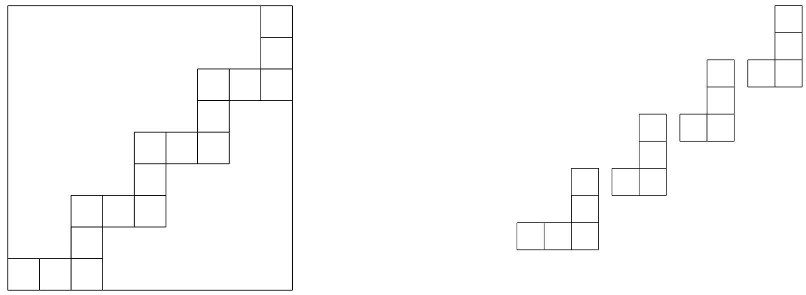
We conclude that $c^2 + f^2 \leq 2$. Using $|c| \geq 1$ and $|f| \geq 1$, we obtain $|c| = 1$ and $|f| = 1$, $b + 1 = 0$ and $e + 1 = 0$. Thus $b = e = -1$. Now $a + d = d + e + f + a + b + c$ and this gives $b + c + e + f = 0$. It follows that $c = f = 1$ and finally $a = d = -1$.

4. Some 46 squares are randomly chosen from a 9×9 chess board and are coloured red. Show that there exists a 2×2 block of 4 squares of which at least three are coloured red.

Solution: Consider a partition of 9×9 chess board using sixteen 2×2 block of 4 squares each and remaining seventeen single squares as shown in the figure below.



If any one of these 16 big squares contain 3 red squares then we are done. On the contrary, each may contain at most 2 red squares and these account for at most $16 \cdot 2 = 32$ red squares. Then there are 17 single squares connected in zig-zag fashion. It looks as follows:



We split this again in to several mirror images of L-shaped figures as shown above. There are four such forks. If all the five unit squares of the first fork are red, then we can get a 2×2 square having three red squares. Hence there can be at most four unit squares having red colour. Similarly, there can be at most three red squares from each of the remaining three forks. Together we get $4 + 3 \cdot 3 = 13$ red squares. These together with 32 from the big squares account for only 45 red squares. But we know that 46 squares have red colour. The conclusion follows.

5. In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $\angle ABC = 120°$, and $\angle ABD = 30°$. Prove that

(i) $c \geq a + b$;

(ii) $\left|\sqrt{c+a} - \sqrt{c+b}\right| = \sqrt{c-a-b}$.

Solution:

Applying cosine rule to triangle ABC, we get

$$AC^2 = a^2 + b^2 - 2ab\cos 120° = a^2 + b^2 + ab.$$

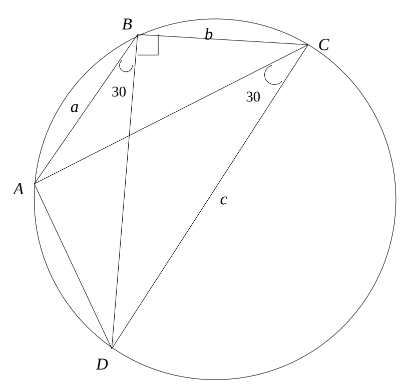
Observe that $\angle DAC = \angle DBC = 120° - 30° = 90°$. Thus we get

$$c^2 = \frac{AC^2}{\cos^2 30°} = \frac{4}{3}\left(a^2 + b^2 + ab\right).$$

So

$$c^2 - (a+b)^2 = \frac{4}{3}\left(a^2 + b^2 + ab\right) - \left(a^2 + b^2 + 2ab\right) = \frac{(a-b)^2}{3} \geq 0.$$

This proves $c \geq a + b$ and thus (i) is true.



For proving (ii), consider the product

$$Q = \big(\alpha+\beta+\gamma\big)\big(\alpha-\beta-\gamma\big)\big(\alpha+\beta-\gamma\big)\big(\alpha-\beta+\gamma\big),$$

where $\alpha = \sqrt{c+a}$, $\beta = \sqrt{c+b}$ and $\gamma = \sqrt{c-a-b}$. Expanding the product, we get

$$\begin{aligned} Q &= (c+a)^2 + (c+b)^2 + (c-a-b)^2 - 2(c+a)(c+b) - 2(c+a)(c-a-b) - 2(c+b)(c-a-b) \\ &= -3c^2 + 4a^2 + 4b^2 + 4ab \\ &= 0. \end{aligned}$$

Thus at least one of the factors must be equal to 0. Since $\alpha+\beta+\gamma > 0$ and $\alpha+\beta-\gamma > 0$, it follows that the product of the remaining two factors is 0. This gives

$$\sqrt{c+a} - \sqrt{c+b} = \sqrt{c-a-b} \text{ or } \sqrt{c+a} - \sqrt{c+b} = -\sqrt{c-a-b}.$$

We conclude that

$$\left|\sqrt{c+a} - \sqrt{c+b}\right| = \sqrt{c-a-b}.$$

6. (a) Prove that if n is a positive integer such that $n \geq 4011^2$, then there exists an integer l such that $n < l^2 < \left(1 + \dfrac{1}{2005}\right)n$.

(b) Find the smallest positive integer M for which whenever an integer n is such that $n \geq M$, there exists an integer l, such that $n < l^2 < \left(1 + \dfrac{1}{2005}\right)n$.

Solution:

(a) Let $n \geq 4011^2$ and $m \in \mathbb{N}$ be such that $m^2 \leq n < (m+1)^2$. Then

$$\begin{aligned} \left(1+\frac{1}{2005}\right)n - (m+1)^2 &\geq \left(1+\frac{1}{2005}\right)m^2 - (m+1)^2 \\ &= \frac{m^2}{2005} - 2m - 1 \\ &= \frac{1}{2005}\left(m^2 - 4010m - 2005\right) \\ &= \frac{1}{2005}\Big((m-2005)^2 - 2005^2 - 2005\Big) \\ &\geq \frac{1}{2005}\Big((4011-2005)^2 - 2005^2 - 2005\Big) \\ &= \frac{1}{2005}\Big(2006^2 - 2005^2 - 2005\Big) \\ &= \frac{1}{2005}(4011 - 2005) = \frac{2006}{2005} > 0. \end{aligned}$$

Thus we get

$$n < (m+1)^2 < \left(1+\frac{1}{2005}\right)n,$$

and $l^2 = (m+1)^2$ is the desired square.

(b) We show that $M = 4010^2 + 1$ is the required least number. Suppose $n \geq M$. Write $n = 4010^2 + k$, where k is a positive integer. Note that we may assume $n < 4011^2$ by part (a). Now

$$\begin{aligned}\left(1+\frac{1}{2005}\right)n - 4011^2 &= \left(1+\frac{1}{2005}\right)\left(4010^2+k\right) - 4011^2\\ &= 4010^2 + 2\cdot 4010 + k + \frac{k}{2005} - 4011^2\\ &= (4010+1)^2 + (k-1) + \frac{k}{2005} - 4011^2\\ &= (k-1) + \frac{k}{2005} > 0.\end{aligned}$$

Thus we obtain

$$4010^2 < n < 4011^2 < \left(1+\frac{1}{2005}\right)n.$$

We check that $M = 4010^2$ will not work. For suppose $n = 4010^2$. Then

$$\left(1+\frac{1}{2005}\right)4010^2 = 4010^2 + 2\cdot 4010 = 4011^2 - 1 < 4011^2.$$

Thus there is no square integer between n and $\left(1+\frac{1}{2005}\right)n$.

This proves (b).

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INMO–2007

1. In a triangle ABC right-angled at C, the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that

$$\frac{5}{2} < \frac{AB}{BC} < 3.$$

2. Let n be a natural number such that $n = a^2 + b^2 + c^2$, for some natural numbers a, b, c. Prove that

$$9n = (p_1 a + q_1 b + r_1 c)^2 + (p_2 a + q_2 b + r_2 c)^2 + (p_3 a + q_3 b + r_3 c)^2,$$

where p_j's, q_j's, r_j's are all **nonzero** integers. Further, of 3 does **not** divide at least one of a, b, c, prove that $9n$ can be expressed in the form $x^2 + y^2 + z^2$, where x, y, z are natural numbers **none** of which is divisible by 3.

3. Let m and n be positive integers such that the equation $x^2 - mx + n = 0$ has real roots α and β. Prove that α and β are integers if and only if $[m\alpha] + [m\beta]$ is the square of an integer. (Here $[x]$ denotes the largest integer not exceeding x.)

4. Let $\sigma = (a_1, a_2, a_3, \ldots, a_n)$ be a permutation of $(1, 2, 3, \ldots, n)$. A pair (a_i, a_j) is said to correspond to an inversion of σ, if $i < j$ but $a_i > a_j$. (Example: In the permutation $(2, 4, 5, 3, 1)$, there are 6 inversions corresponding to the pairs $(2, 1)$, $(4, 3)$, $(4, 1)$, $(5, 3)$, $(5, 1)$, $(3, 1)$.) How many permutations of $(1, 2, 3, \ldots, n)$, $(n \geq 3)$, have exactly **two** inversions.

5. Let ABC be a triangle in which $AB = AC$. Let D be the mid-point of BC and P be a point on AD. Suppose E is the foot of the perpendicular from P on AC. If $\dfrac{AP}{PD} = \dfrac{BP}{PE} = \lambda$, $\dfrac{BD}{AD} = m$ and $z = m^2(1 + \lambda)$, prove that

$$z^2 - (\lambda^3 - \lambda^2 - 2)z + 1 = 0.$$

Hence show that $\lambda \geq 2$ and $\lambda = 2$ if and only if ABC is equilateral.

6. If x, y, z are positive real numbers, prove that

$$(x+y+z)^2(yz+zx+xy)^2 \leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2).$$

Problems and Solutions of INMO-2007

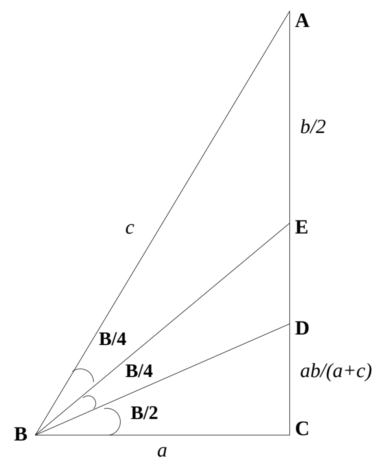
1. In a triangle ABC right-angled at C, the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that

$$\frac{5}{2} < \frac{AB}{BC} < 3.$$

Solution 1:

Since E is the mid-point of AC, we have $AE = EC = b/2$. Since BD bisects $\angle ABC$, we also know that $CD = ab/(a+c)$. Since BE bisects $\angle ABD$, we also have

$$\frac{BD^2}{BA^2} = \frac{DE^2}{EA^2}.$$



However,

$$\begin{aligned} BD^2 &= BC^2 + CD^2 = a^2 + \frac{a^2b^2}{(a+c)^2}, \\ DE^2 &= \left(\frac{b}{2} - \frac{ab}{a+c}\right)^2. \end{aligned}$$

Using these in the above expression and simplifying, we get

$$a^2\left\{(a+c)^2 + b^2\right\} = c^2(c-a)^2.$$

Using $c^2 = a^2 + b^2$ and eliminating b, we obtain

$$c^3 - 2ac^2 - a^2c - 2a^3 = 0.$$

Introducing $t = c/a$, this reduces to a cubic equation;

$$t^3 - 2t^2 - t - 2 = 0.$$

Consider the function $f(t) = t^3 - 2t^2 - t - 2$ for $t > 0$ (as c/a is positive). For $0 < t \le 2$, we see that $f(t) = t^2(t-2) - t - 2 < 0$. We also observe that $f(t) = (t-2)(t^2-1) - 4$ is strictly increasing on $(2, \infty)$. It is easy to compute

$$f\big(5/2\big) = -\frac{11}{8} < 0, \quad \text{and} \quad f(3) = 4 > 0.$$

Hence there is a unique value of t in the interval $(5/2, 3)$ such that $f(t) = 0$. We conclude that

$$\frac{5}{2} < \frac{c}{a} < 3.$$

Solution 2: Let us take $\angle B/4 = \theta$. Then $\angle EBC = \angle DBE = \theta$ and $\angle CBD = 2\theta$.Using sine rule in triangles BEA and BEC, we get

$$\begin{aligned} \frac{BE}{\sin A} &= \frac{AE}{\sin\theta}, \\ \frac{BE}{\sin 90^\circ} &= \frac{CE}{\sin 3\theta}. \end{aligned}$$

Since $AE = CE$, we obtain $\sin 3\theta \sin A = \sin\theta$. However $A = 90° - 4\theta$. Thus we get $\sin 3\theta \cos 4\theta = \sin\theta$. Note that

$$\frac{c}{a} = \frac{1}{\cos 4\theta} = \frac{\sin 3\theta}{\sin\theta} = 3 - 4\sin^2\theta.$$

This shows that $c/a < 3$. Using $c/a = 3 - 4\sin^2\theta$, it is easy to compute $\cos 2\theta = \big((c/a) - 1\big)/2$. Hence

$$\frac{a}{c} = \cos 4\theta = \frac{1}{2}\left(\frac{c}{a} - 1\right)^2 - 1.$$

Suppose $c/a \le 5/2$. Then $\big((c/a) - 1\big)^2 \le 9/4$ and $a/c \ge 2/5$. Thus

$$\frac{2}{5} \le \frac{a}{c} = \frac{1}{2}\left(\frac{c}{a} - 1\right)^2 - 1 \le \frac{9}{8} - 1 = \frac{1}{8},$$

which is absurd. We conclude that $c/a > 5/2$.

2. Let n be a natural number such that $n = a^2 + b^2 + c^2$, for some natural numbers a, b, c. Prove that

$$9n = \big(p_1 a + q_1 b + r_1 c\big)^2 + \big(p_2 a + q_2 b + r_2 c\big)^2 + \big(p_3 a + q_3 b + r_3 c\big)^2,$$

where p_j's, q_j's, r_j's are all **nonzero** integers. Further, if 3 does **not** divide at least one of a, b, c, prove that $9n$ can be expressed in the form $x^2 + y^2 + z^2$, where x, y, z are natural numbers **none** of which is divisible by 3.

Solution: It can be easily seen that

$$9n = (2b + 2c - a)^2 + (2c + 2a - b)^2 + (2a + 2b - c)^2.$$

Thus we can take $p_1 = p_2 = p_3 = 2$, $q_1 = q_2 = q_3 = 2$ and $r_1 = r_2 = r_3 = -1$. Suppose 3 does not divide $\gcd(a, b, c)$. Then 3 does divide at least one of a, b, c; say 3 does not divide a. Note that each of $2b+2c-a$, $2c+2a-b$ and $2a+2b-c$ is either divisible by 3 or none of them is divisible by 3, as the difference of any two sums is always divisible by 3. If 3 does not divide $2b+2c-a$, then we have the required representation. If 3 divides $2b + 2c - a$, then 3 does not divide $2b + 2c + a$. On the other hand, we also note that

$$9n = (2b + 2c + a)^2 + (2c - 2a - b)^2 + (-2a + 2b - c)^2 = x^2 + y^2 + z^2,$$

where $x = 2b+2c+a$, $y = 2c-2a-b$ and $z = -2a+2b-c$. Since $x-y = 3(b+a)$ and 3 does not divide x, it follows that 3 does not divide y as well. Similarly, we conclude that 3 does not divide z.

3. Let m and n be positive integers such that the equation $x^2 - mx + n = 0$ has real roots α and β. Prove that α and β are integers if and only if $[m\alpha] + [m\beta]$ is the square of an integer. (Here $[x]$ denotes the largest integer not exceeding x.)

Solution: If α and β are both integers, then

$$[m\alpha] + [m\beta] = m\alpha + m\beta = m\big(\alpha + \beta\big) = m^2.$$

This proves one implication.

Observe that $\alpha + \beta = m$ and $\alpha\beta = n$. We use the property of integer function: $x - 1 < [x] \le x$ for any real number x. Thus

$$m^2 - 2 = m(\alpha + \beta) - 2 = m\alpha - 1 + m\beta - 1 < [m\alpha] + [m\beta] \le m(\alpha + \beta) = m^2.$$

Since m and n are positive integers, both α and β must be positive. If $m \ge 2$, we observe that there is no square between $m^2 - 2$ and m^2. Hence, either $m = 1$ or $[m\alpha] + [m\beta] = m^2$. If $m = 1$, then $\alpha + \beta = 1$ implies that both α and β are positive reals smaller than 1. Hence $n = \alpha\beta$ cannot be a positive integer. We conclude that $[m\alpha] + [m\beta] = m^2$. Putting $m = \alpha + \beta$ in this relation, we get

$$[\alpha^2 + n] + [\beta^2 + n] = (\alpha + \beta)^2.$$

Using $[x + k] = [x] + k$ for any real number x and integer k, this reduces to

$$[\alpha^2] + [\beta^2] = \alpha^2 + \beta^2.$$

This shows that α^2 and β^2 are both integers. On the other hand,

$$\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = m(\alpha - \beta).$$

Thus

$$(\alpha - \beta) = \frac{\alpha^2 - \beta^2}{m},$$

is a rational number. Since $\alpha + \beta = m$ is a rational number, it follows that both α and β are rational numbers. However, both α^2 and β^2 are integers. Hence each of α and β is an integer.

4. Let $\sigma = (a_1, a_2, a_3, \ldots, a_n)$ be a permutation of $(1, 2, 3, \ldots, n)$. A pair (a_i, a_j) is said to correspond to an inversion of σ, if $i < j$ but $a_i > a_j$. (Example: In the permutation $(2, 4, 5, 3, 1)$, there are 6 inversions corresponding to the pairs $(2, 1)$, $(4, 3)$, $(4, 1)$, $(5, 3)$, $(5, 1)$, $(3, 1)$.) How many permutations of $(1, 2, 3, \ldots n)$, $(n \ge 3)$, have exactly **two** inversions?

Solution: In a permutation of $(1, 2, 3, \ldots, n)$, two inversions can occur in only one of the following two ways:

(A) Two disjoint consecutive pairs are interchanged:

$$\begin{aligned}&(1, 2, 3, j-1, j, j+1, j+2 \ldots k-1, k, k+1, k+2, \ldots, n)\\ &\quad\longrightarrow (1, 2, \ldots j-1, j+1, j, j+2, \ldots, k-1, k+1, k, k+2, \ldots, n).\end{aligned}$$

(B) Each block of three consecutive integers can be permuted in any of the following 2 ways;

$$\begin{aligned}(1, 2, 3, \ldots k, k+1, k+2, \ldots, n) &\longrightarrow (1, 2, \ldots, k+2, k, k+1, \ldots, n);\\ (1, 2, 3, \ldots k, k+1, k+2, \ldots, n) &\longrightarrow (1, 2, \ldots, k+1, k+2, k, \ldots, n).\end{aligned}$$

Consider case (A). For $j = 1$, there are $n - 3$ possible values of k; for $j = 2$, there are $n - 4$ possibilities for k and so on. Thus the number of permutations with two inversions of this type is

$$1 + 2 + \cdots + (n - 3) = \frac{(n - 3)(n - 2)}{2}.$$

In case (B), we see that there are $n-2$ permutations of each type, since k can take values from 1 to $n-2$. Hence we get $2(n-2)$ permutations of this type.

Finally, the number of permutations with **two** inversions is

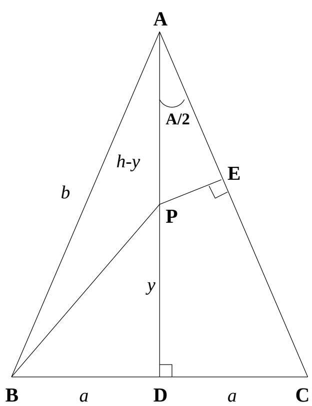
$$\frac{(n-3)(n-2)}{2} + 2(n-2) = \frac{(n+1)(n-2)}{2}.$$

5. Let ABC be a triangle in which $AB = AC$. Let D be the mid-point of BC and P be a point on AD. Suppose E is the foot of perpendicular from P on AC. If $\frac{AP}{PD} = \frac{BP}{PE} = \lambda$, $\frac{BD}{AD} = m$ and $z = m^2(1+\lambda)$, prove that

$$z^2 - \left(\lambda^3 - \lambda^2 - 2\right)z + 1 = 0.$$

Hence show that $\lambda \geq 2$ and $\lambda = 2$ if and only if ABC is equilateral.

Solution:



Let $AD = h$, $PD = y$ and $BD = DC = a$. We observe that $BP^2 = a^2 + y^2$. Moreover,

$$PE = PA \sin \angle DAC = (h-y)\frac{DC}{AC} = \frac{a(h-y)}{b},$$

where $b = AC = AB$. Using $AP/PD = (h-y)/y$, we obtain $y = h/(1+\lambda)$. Thus

$$\lambda^2 = \frac{BP^2}{PE^2} = \frac{(a^2+y^2)b^2}{(h-y)^2a^2}.$$

But $(h-y) = \lambda y = \lambda h/(1+\lambda)$ and $b^2 = a^2 + h^2$. Thus we obtain

$$\lambda^4 = \frac{\left(a^2(1+\lambda)^2 + h^2\right)(a^2+h^2)}{a^2h^2}.$$

Using $m = a/h$ and $z = m^2(1+\lambda)$, this simplifies to

$$z^2 - z(\lambda^3 - \lambda^2 - 2) + 1 = 0.$$

Dividing by z, this gives

$$z + \frac{1}{z} = \lambda^3 - \lambda^2 - 2.$$

However $z + (1/z) \geq 2$ for any positive real number z. Thus $\lambda^3 - \lambda^2 - 4 \geq 0$. This may be written in the form $\left(\lambda - 2\right)\left(\lambda^2 + \lambda + 2\right) \geq 0$. But $\lambda^2 + \lambda + 2 > 0$. (For example, one may check that its discriminant is negative.) Hence $\lambda \geq 2$. If $\lambda = 2$, then $z + (1/z) = 2$ and hence $z = 1$. This gives $m^2 = 1/3$ or $\tan(A/2) = m = 1/\sqrt{3}$. Thus $A = 60°$ and hence ABC is equilateral.

Conversely, if triangle ABC is equilateral, then $m = \tan(A/2) = 1/\sqrt{3}$ and hence $z = (1+\lambda)/3$. Substituting this in the equation satisfied by z, we obtain

$$\left(1+\lambda\right)^2 - 3\left(1+\lambda\right)\left(\lambda^3 - \lambda^2 - 2\right) + 9 = 0.$$

This may be written in the form $\left(\lambda - 2\right)\left(3\lambda^3 + 6\lambda^2 + 8\lambda + 8\right) = 0$. Here the second factor is positive because $\lambda > 0$. We conclude that $\lambda = 2$.

6. If x, y, z are positive real numbers, prove that

$$(x+y+z)^2\big(yz+zx+xy\big)^2 \le 3\big(y^2+yz+z^2\big)\big(z^2+zx+x^2\big)\big(x^2+xy+y^2\big).$$

Solution 1: We begin with the observation that

$$x^2+xy+y^2 = \frac{3}{4}(x+y)^2 + \frac{1}{4}(x-y)^2 \ge \frac{3}{4}(x+y)^2,$$

and similar bounds for y^2+yz+z^2, z^2+zx+x^2. Thus

$$3\big(x^2+xy+y^2\big)\big(y^2+yz+z^2\big)\big(z^2+zx+x^2\big) \ge \frac{81}{64}\big(x+y\big)^2\big(y+z\big)^2\big(z+x\big)^2.$$

Thus it is sufficient to prove that

$$\big(x+y+z\big)\big(xy+yz+zx\big) \le \frac{9}{8}\big(x+y\big)\big(y+z\big)\big(z+x\big).$$

Equivalently, we need to prove that

$$8\big(x+y+z\big)\big(xy+yz+zx\big) \le 9\big(x+y\big)\big(y+z\big)\big(z+x\big).$$

However, we note that

$$\big(x+y\big)\big(y+z\big)\big(z+x\big) = \big(x+y+z\big)\big(yz+zx+xy\big) - xyz.$$

Thus the required inequality takes the form

$$\big(x+y\big)\big(y+z\big)\big(z+x\big) \ge 8xyz.$$

This follows from AM-GM inequalities;

$$x+y \ge 2\sqrt{xy}, \quad y+z \ge 2\sqrt{yz}, \quad z+x \ge 2\sqrt{zx}.$$

Solution 2: Let us introduce $x+y=c$, $y+z=a$ and $z+x=b$. Then a, b, c are the sides of a triangle. If $s=(a+b+c)/2$, then it is easy to calculate $x=s-a$, $y=s-b$, $z=s-c$ and $x+y+z=s$. We also observe that

$$x^2+xy+y^2 = (x+y)^2 - xy = c^2 - \frac{1}{4}(c+a-b)(c+b-a) = \frac{3}{4}c^2 + \frac{1}{4}(a-b)^2 \ge \frac{3}{4}c^2.$$

Moreover, $xy+yz+zx = (s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)$. Thus it si sufficient to prove that

$$s\sum(s-a)(s-b) \le \frac{9}{8}abc.$$

But, $\sum(s-a)(s-b) = r(4R+r)$, where r, R are respectively the in-radius, the circum-radius of the triangle whose sides are a, b, c, and $abc = 4Rrs$. Thus the inequality reduces to

$$r(4R+r) \le \frac{9}{2}Rr.$$

This is simply $2r \le R$. This follows from $IO^2 = R(R-2r)$, where I is the incentre and O the circumcentre.

Solution 3: If we set $x = \lambda a$, $y = \lambda b$, $z = \lambda c$, then the inequality changes to

$$(a+b+c)^2(ab+bc+ca)^2 \leq 3(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2).$$

This shows that we may assume $x+y+z=1$. Let $\alpha = xy+yz+zx$. We see that

$$\begin{aligned} x^2+xy+y^2 &= (x+y)^2 - xy \\ &= (x+y)(1-z) - xy \\ &= x+y-\alpha = 1-z-\alpha. \end{aligned}$$

Thus

$$\begin{aligned} \prod(x^2+xy+y^2) &= (1-\alpha-z)(1-\alpha-x)(1-\alpha-y) \\ &= (1-\alpha)^3 - (1-\alpha)^2 + (1-\alpha)\alpha - xyz \\ &= \alpha^2 - \alpha^3 - xyz. \end{aligned}$$

Thus we need to prove that $\alpha^2 \leq 3(\alpha^2 - \alpha^3 - xyz)$. This reduces to

$$3xyz \leq \alpha^2(2-3\alpha).$$

However

$$3\alpha = 3(xy+yz+zx) \leq (x+y+z)^2 = 1,$$

so that $2 - 3\alpha \geq 1$. Thus it suffices to prove that $3xyz \leq \alpha^2$. But

$$\begin{aligned} \alpha^2 - 3xyz &= (xy+yz+zx)^2 - 3xyz(x+y+z) \\ &= \sum_{\text{cyclic}} x^2y^2 - xyz(x+y+z) \\ &= \frac{1}{2}\sum_{\text{cyclic}} (xy-yz)^2 \geq 0. \end{aligned}$$

INMO–2008

1. Let ABC be a triangle, I its in-centre; A_1, B_1, C_1 be the reflections of I in BC, CA, AB respectively. Suppose the circum-circle of triangle $A_1B_1C_1$ passes through A. Prove that B_1, C_1, I, I_1 are concyclic, where I_1 is the in-centre of triangle $A_1B_1C_1$.

2. Find all triples (p, x, y) such that $p^x = y^4 + 4$, where p is a prime and x, y are natural numbers.

3. Let A be a set of real numbers such that A has at least four elements. Suppose A has the property that $a^2 + bc$ is a rational number for all distinct numbers a, b, c in A. Prove that there exists a positive integer M such that $a\sqrt{M}$ is a rational number for every a in A.

4. All the points with integer coordinates in the xy-plane are coloured using three colours, red, blue and green, each colour being used at least once. It is known that the point $(0, 0)$ is coloured red and the point $(0, 1)$ is coloured blue. Prove that there exist three points with integer coordinates of distinct colours which form the vertices of a **right-angled** triangle.

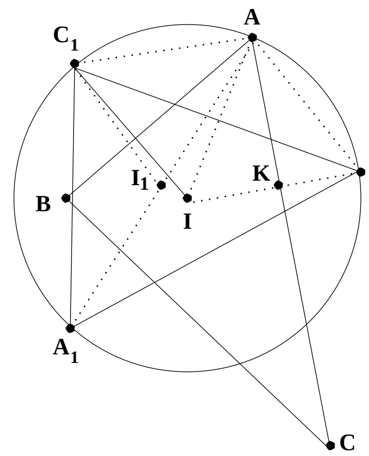
5. Let ABC be a triangle; Γ_A, Γ_B, Γ_C be three equal, disjoint circles inside ABC such that Γ_A touches AB and AC; Γ_B touches AB and BC; and Γ_C touches BC and CA. Let Γ be a circle touching circles Γ_A, Γ_B, Γ_C externally. Prove that the line joining the circum-centre O and the in-centre I of triangle ABC passes through the centre of Γ.

6. Let $P(x)$ be a given polynomial with integer coefficients. Prove that there exist two polynomials $Q(x)$ and $R(x)$, again with integer coefficients, such that (i) $P(x)Q(x)$ is a polynomial in x^2; and (ii) $P(x)R(x)$ is a polynomial in x^3.

Problems and Solutions of INMO-2008

1. Let ABC be a triangle, I its in-centre; A_1, B_1, C_1 be the reflections of I in BC, CA, AB respectively. Suppose the circum-circle of triangle $A_1B_1C_1$ passes through A. Prove that B_1, C_1, I, I_1 are concyclic, where I_1 is the in-centre of triangle $A_1B_1C_1$.

Solution:



Note that $IA_1 = IB_1 = IC_1 = 2r$, where r is the in-radius of the triangle ABC. Hence I is the circum-centre of the triangle $A_1B_1C_1$.

Let K be the point of intersection of IB_1 and AC. Then $IK = r$, $IA = 2r$ and $\angle IKA = 90°$. It follows that $\angle IAK = 30°$ and hence $\angle IAB_1 = 60°$. Thus AIB_1 is an equilateral triangle. Similarly triangle AIC_1 is also equilateral. We hence obtain $AB_1 = AC_1 = AI = IB_1 = IC_1 = 2r$.

We also observe that $\angle B_1IC_1 = 120°$ and IB_1AC_1 is a rhombus. Thus $\angle B_1AC_1 = 120°$ and by concyclicity $\angle A_1 = 60°$. Since $AB_1 = AC_1$, A is the midpoint of the arc B_1AC_1. It follows that A_1A bisects $\angle A_1$ and I_1 lies on the line A_1A. This implies that

$$\angle B_1I_1C_1 = 90° + \angle A_1/2 = 90° + 30° = 120°.$$

Since $\angle B_1IC_1 = 120°$, we conclude that B_1, I, I_1, C_1 are concyclic. (Further A is the centre.)

2. Find all triples (p, x, y) such that $p^x = y^4 + 4$, where p is a prime and x, y are natural numbers.

Solution: We begin with the standard factorisation

$$y^4 + 4 = (y^2 - 2y + 2)(y^2 + 2y + 2).$$

Thus we have $y^2 - 2y + 2 = p^m$ and $y^2 + 2y + 2 = p^n$ for some positive integers m and n such that $m + n = x$. Since $y^2 - 2y + 2 < y^2 + 2y + 2$, we have $m < n$ so that p^m divides p^n. Thus $y^2 - 2y + 2$ divides $y^2 + 2y + 2$. Writing $y^2 + 2y + 2 = y^2 - 2y + 2 + 4y$, we infer that $y^2 - 2y + 2$ divides $4y$ and hence $y^2 - 2y + 2$ divides $4y^2$. But

$$4y^2 = 4(y^2 - 2y + 2) + 8(y - 1).$$

Thus $y^2 - 2y + 2$ divides $8(y - 1)$. Since $y^2 - 2y + 2$ divides both $4y$ and $8(y - 1)$, we conclude that it also divides 8. This gives $y^2 - 2y + 2 = 1$, 2, 4 or 8.

If $y^2 - 2y + 2 = 1$, then $y = 1$ and $y^4 + 4 = 5$, giving $p = 5$ and $x = 1$. If $y^2 - 2y + 2 = 2$, then $y^2 - 2y = 0$ giving $y = 2$. But then $y^4 + 4 = 20$ is not the power of a prime. The equations $y^2 - 2y + 2 = 4$ and $y^2 - 2y + 2 = 8$ have no integer solutions. We conclude that $(p, x, y) = (5, 1, 1)$ is the only solution.

Alternatively, using $y^2 - 2y + 2 = p^m$ and $y^2 + 2y + 2 = p^n$, we may get

$$4y = p^m\left(p^{n-m} - 1\right).$$

If $m > 0$, then p divides 4 or y. If p divides 4, then $p = 2$. If p divides y, then $y^2 - 2y + 2 = p^m$ shows that p divides 2 and hence $p = 2$. But then $2^x = y^4 + 4$, which shows that y is even. Taking $y = 2z$, we get $2^{x-2} = 4z^4 + 1$. This implies that $z = 0$ and hence $y = 0$, which is a contradiction. Thus $m = 0$ and $y^2 - 2y + 2 = 1$. This gives $y = 1$ and hence $p = 5$, $x = 1$.

3. Let A be a set of real numbers such that A has at least four elements. Suppose A has the property that $a^2 + bc$ is a rational number for all distinct numbers a, b, c in A. Prove that there exists a positive integer M such that $a\sqrt{M}$ is a rational number for every a in A.

Solution: Suppose $0 \in A$. Then $a^2 = a^2 + 0 \times b$ is rational and $ab = 0^2 + ab$ is also rational for all a, b in A, $a \neq 0$, $b \neq 0$, $a \neq b$. Hence $a = a_1\sqrt{M}$ for some rational a_1 and natural number M. For any $b \neq 0$, we have

$$b\sqrt{M} = \frac{ab}{a_1}.$$

which is a rational number.

Hence we may assume 0 is not in A. If there is a number a in A such that $-a$ is also in A, then again we can get the conclusion as follows. Consider two other elements c, d in A. Then $c^2 + da$ is rational and $c^2 - da$ is also rational. It follows that c^2 is rational and da is rational. Similarly, d^2 and ca are also rationals. Thus $d/c = (da)/(ca)$ is rational. Note that we can vary d over A with $d \neq c$ and $d \neq a$. Again c^2 is rational implies that $c = c_1\sqrt{M}$ for some rational c_1 and natural number M. We observe that $c\sqrt{M} = c_1 M$ is rational, and

$$a\sqrt{M} = \frac{ca}{c_1},$$

so that $a\sqrt{M}$ is a rational number. Similarly is the case with $-a\sqrt{M}$. For any other element d,

$$b\sqrt{M} = Mc_1\frac{d}{c}$$

is a rational number.

Thus we may now assume that 0 is not in A and $a + b \neq 0$ for any a, b in A. Let a, b, c, d be four distinct elements of A. We may assume $|a| > |b$. Then $d^2 + ab$ and $d^2 + bc$ are rational numbers and so is their difference $ab - bc$. Writing $a^2 + ab = a^2 + bc + (ab - bc)$, and using the facts $a^2 + bc$, $ab - bc$ are rationals, we conclude that $a^2 + ab$ is also a rational number. Similarly, $b^2 + ab$ is also a rational number.

Consider

$$q = \frac{a}{b} = \frac{a^2 + ab}{b^2 + ab}.$$

Note that $a^2 + ab > 0$. Thus q is a rational number and $a = bq$. This gives $a^2 + ab = b^2(q^2 + q)$. Let us take $b^2(q^2 + q) = l$. Then

$$|b| = \sqrt{\frac{l}{q^2 + q}} = \sqrt{\frac{x}{y}},$$

where x and y are natural numbers. Take $M = xy$. Then $|b|\sqrt{M} = x$ is a rational number. Finally, for any c in A, we have

$$c\sqrt{M} = b\sqrt{M}\frac{c}{b},$$

is also a rational number.

4. All the points with integer coordinates in the xy-plane are coloured using three colours, red, blue and green, each colour being used at least once. It is known that the point $(0, 0)$ is coloured red and the point $(0, 1)$ is coloured blue. Prove that there exist three points with integer coordinates of distinct colours which form the vertices of a **right-angled** triangle.

Solution: Consider the lattice points(points with integer coordinates) on the lines $y = 0$ and $y = 1$, other than $(0, 0)$ and $(0, 1)$, If one of them, say $A = (p, 1)$, is coloured green, then we have a right-angled triangle with $(0, 0)$, $(0, 1)$ and A as vertices, all having different colours. (See Figures 1 and 2.)

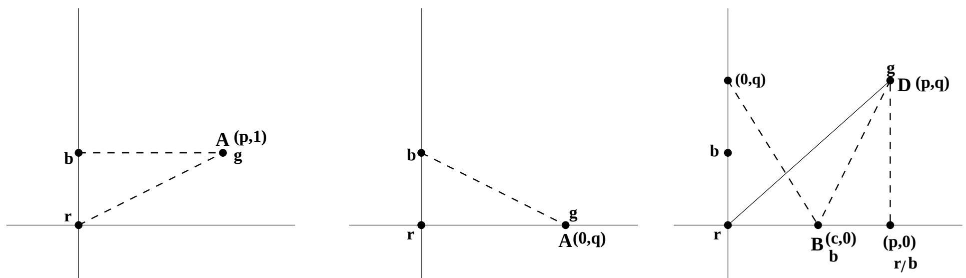


Fig-1 Fig-2 Fig-3

If not, the lattice points on $y = 0$ and $y = 1$ are all red or blue. We consider three different cases.

Case 1. Suppose a point $B = (c, 0)$ is blue. Consider a green point $D = (p, q)$ in the plane. Suppose $p \neq 0$. If its projection $(p, 0)$ on the x-axis is red, then (p, q), $(p, 0)$ and $(c, 0)$ are the vertices of a required type of right-angled triangle. If $(p, 0)$ is blue, then we can consider the triangle whose vertices are $(0, 0)$, $(p, 0)$ and (p, q). If $p = 0$, then the points D, $(0, 0)$ and $(c, 0)$ will work.(Figure 3.)

Case 2. A point $D = (c, 1)$, on the line $y = 1$, is red. A similar argument works in this case.

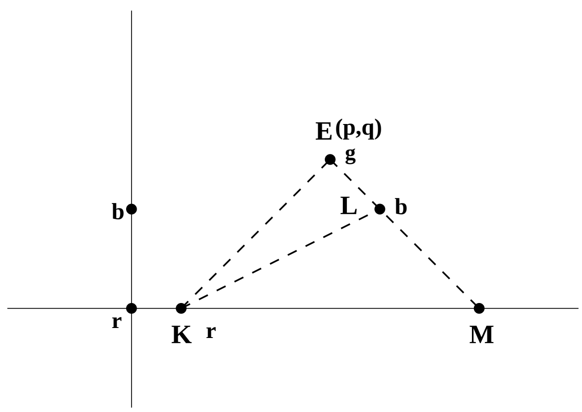
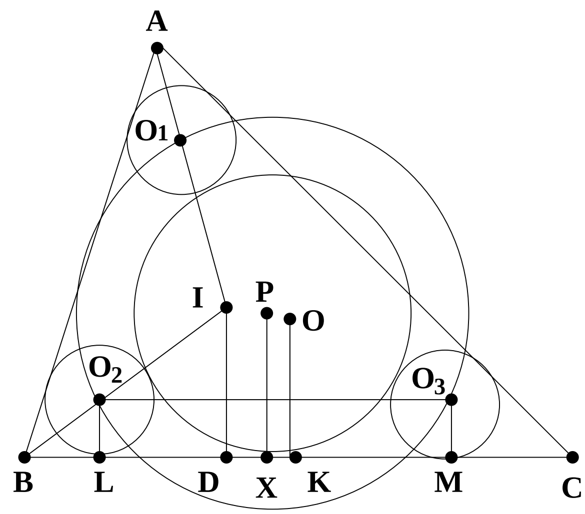


Fig-4

Case 3. Suppose all the lattice points on the line $y = 0$ are red and all on the line $y = 1$ are blue points. Consider a green point $E = (p, q)$, where $q \neq 0$ and $q \neq 1$.(See Figure 4.) Consider an isosceles right-angled triangle EKM with $\angle E = 90°$ such that the hypotenuse KM is a part of the x-axis. Let EM intersect $y =$ in L. Then K is a red point and L is a blue point. Hence EKL is a desired triangle.

5. Let ABC be a triangle; Γ_A, Γ_B, Γ_C be three equal, disjoint circles inside ABC such that Γ_A touches AB and AC; Γ_B touches AB; and BC, and Γ_C touches BC and CA. Let Γ be a circle touching circles Γ_A, Γ_B, Γ_C externally. Prove that the line joining the circum-centre O and the in-centre I of triangle ABC passes through the centre of Γ.

Solution: Let O_1, O_2, O_3 be the centres of the circles Γ_A, Γ_B, Γ_C respectively, and let P be the circum-centre of the triangle $O_1O_2O_3$. Let x denote the common radius of three circles Γ_A, Γ_B, Γ_C. Note that P is also the centre of the circle Γ, as O_1P, O_2P, O_3P each exceed the radius of Γ by x. Let D, X, K, L, M be respectively the projections of I, P, O, O_1, O_2 on BC .



From $\dfrac{BL}{BD} = \dfrac{LO_2}{DI}$, we get $BL = x(s-b)/r$, as $ID = r$ and $BD = (s-b)$. Similarly, $CM = x(s-c)/r$. Therefore, $LM = a - \dfrac{x}{r}(s-b+s-c) = \dfrac{a}{r}(r-x)$. Since O_2LMO_3 is a rectangle and PX is the perpendicular bisector of O_2O_3, it is perpendicular bisector of LM as well. Thus

$$\begin{aligned}
LX &= \frac{1}{2}LM = \frac{a}{2r}(r-x);\\
BX &= BL + LX = \frac{x}{r}(s-b) + \frac{a}{2r}(r-x) = \frac{a}{2} - \frac{x(b-c)}{2r};\\
DK &= BK - BD = \frac{a}{2} - (s-b) = \frac{b-c}{2};\\
XK &= BK - BX = \frac{a}{2} - \frac{a}{2} + \frac{x(b-c)}{2r} = \frac{x(b-c)}{2r}.
\end{aligned}$$

Hence we get

$$\frac{XK}{DK} = \frac{x}{r}.$$

We observe that the sides of triangle $O_1O_2O_3$ are

$$O_2O_3 = LM = \frac{a}{r}(r-x), \quad O_3O_1 = \frac{b}{r}(r-x), \quad O_1O_2 = \frac{c}{r}(r-x).$$

Thus the sides of $O_1O_2O_3$ and those of ABC are in the ratio $(r-x)/r$. Further, as the sides of $O_1O_2O_3$ are parallel to those of ABC, we see that I is the in-centre of $O_1O_2O_3$ as well. This gives $IP/IO = (r-x)/r$, and hence $PO/IO = x/r$. Thus we obtain

$$\frac{XK}{DK} = \frac{PO}{IO}.$$

It follows that I, P, O are collinear.

Alternately, we also infer that I is the centre of homothety which takes the figure $O_1O_2O_3$ to ABC. Hence it takes P to O. It follows that I, P, O are collinear

6. Let $P(x)$ be a given polynomial with integer coefficients. Prove that there exist two polynomials $Q(x)$ and $R(x)$, again with integer coefficients, such that (i) $P(x)Q(x)$ is a polynomial in x^2; and (ii) $P(x)R(x)$ is a polynomial in x^3.

Solution: Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial with integer coefficients.
Part (i) We may write

$$P(x) = a_0 + a_2x^2 + a_4x^4 + \cdots + x\big(a_1 + a_3x^2 + a_5x^5 + \cdots\big).$$

Define

$$Q(x) = a_0 + a_2x^2 + a_4x^4 + \cdots - x\big(a_1 + a_3x^2 + a_5x^5 + \cdots\big).$$

Then $Q(x)$ is also a polynomial with integer coefficients and

$$P(x)Q(x) = \left(a_0 + a_2x^2 + a_4x^4 + \cdots\right)^2 - x^2\left(a_1 + a_3x^2 + a_5x^5 + \cdots\right)^2$$

is a polynomial in x^2.

Part (ii) We write again

$$P(x) = A(x) + xB(x) + x^2C(x),$$

where

$$\begin{aligned} A(x) &= a_0 + a_3x^3 + a_6x^6 + \cdots, \\ B(x) &= a_1 + a_4x^3 + a_7x^6 + \cdots, \\ C(x) &= a_2 + a_5x^3 + a_8x^6 + \cdots. \end{aligned}$$

Note that $A(x)$, $B(x)$ and $C(x)$ are polynomials with integer coefficients and each of these is a polynomial in x^3. We may introduce

$$\begin{aligned} S(x) &= A(x) + \omega xB(x) + \omega^2x^2C(x), \\ T(x) &= A(x) + \omega^2xB(x) + \omega x^2C(x), \end{aligned}$$

where ω is an imaginary cube-root of unity. Then

$$S(x)T(x) = \left(A(x)\right)^2 + x^2\left(B(x)\right)^2 + x^4\left(C(x)\right)^2 \\ - xA(x)B(x) - x^3B(x)C(x) - x^2C(x)A(x)$$

since $\omega^3 = 1$ and $\omega + \omega^2 = -1$. Taking $R(x) = S(x)T(x)$, we obtain

$$P(x)R(x) = \left(A(x)\right)^3 + x^3\left(B(x)\right)^3 + x^6\left(C(x)\right)^3 - 3x^3A(x)B(x)C(x),$$

which is a polynomial in x^3. This follows from the identity

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc.$$

Alternately, $R(x)$ may be directly defined by

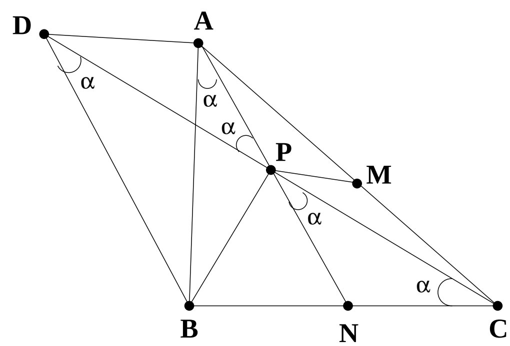
$$R(x) = \left(A(x)\right)^2 + x^2\left(B(x)\right)^2 + x^4\left(C(x)\right)^2 \\ - xA(x)B(x) - x^3B(x)C(x) - x^2C(x)A(x).$$

24th Indian National Mathematical Olympiad, 2009

Problems and Solutions

1. Let ABC be a triangle and let P be an interior point such that $\angle BPC = 90°$, $\angle BAP = \angle BCP$. Let M, N be the mid-points of AC, BC respectively. Suppose $BP = 2PM$. Prove that A, P, N are collinear.

Solution:
Extend CP to D such that $CP = PD$. Let $\angle BCP = \alpha = \angle BAP$. Observe that BP is the perpendicular bisector of CD. Hence $BC = BD$ and BCD is an isosceles triangle. Thus $\angle BDP = \alpha$. But then $\angle BDP = \alpha = \angle BAP$. This implies that B, P, A, D all lie on a circle. In turn, we conclude that $\angle DAB = \angle DPB = 90°$. Since P is the mid-point of CP(by construction) and M is the mid-point of CA(given), it follows that PM is parallel to DA and $DA = 2PM = BP$. Thus $DBPA$ is an isosceles trapezium and DB is parallel to PA.



We hence get

$$\angle DPA = \angle BAP = \angle BCP = \angle NPC;$$

the last equality follows from the fact that $\angle BPC = 90°$, and N is the mid-point of CB so that $NP = NC = NB$ for the right-angled triangle BPC. It follows that A, P, N are collinear.

Alternate Solution:
We use coordinate geometry. Let us take $P = (0, 0)$, and the coordinate axes along PC and PB; We take $C = (c, 0)$ and $B = (0, b)$. Let $A = (u, v)$. We see that $N = (c/2, b/2)$ and $M = \big((u + c)/2, v/2\big)$. The condition $PB = 2PM$ translates to

$$(u + c)^2 + v^2 = b^2.$$

We observe that the slope of $CP = 0$; that of CB is $-b/c$; that of PA is v/u; and that of BA is $(v - b)/u$. Taking proper signs, we can convert $\angle PCB = \angle PAB$, via *tan* function, to the following relation:

$$u^2 + v^2 - vb = -cu.$$

Thus we obtain

$$u(u + c) = v(b - v), \quad c(c + u) = b(b - v).$$

It follows that $v/u = b/c$. But then we get that the slope of AP and PN are the same. We conclude that A, P, N are collinear.

2. Define a sequence $\langle a_n \rangle_{n=1}^{\infty}$ as follows:

$$a_n = \begin{cases} 0, & \text{if the number of positive divisors of } n \text{ is } odd, \\ 1, & \text{if the number of positive divisors of } n \text{ is } even. \end{cases}$$

(The positive divisors of n include 1 as well as n.) Let $x = 0.a_1a_2a_3\dots$ be the real number whose decimal expansion contains a_n in the n-th place, $n \geq 1$. Determine, with proof, whether x is rational or irrational.

Solution:

We show that x is irrational. Suppose that x is rational. Then the sequence $\langle a_n \rangle_{n=1}^{\infty}$ is periodic after some stage; there exist natural numbers k, l such that $a_n = a_{n+l}$ for all $n \geq k$. Choose m such that $ml \geq k$ and ml is a perfect square. Let

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}, \quad l = p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r},$$

be the prime decompositions of m, l so that $\alpha_j + \beta_j$ is even for $1 \leq j \leq r$. Now take a prime p different from $p_1, p_2, \dots, p_r$. Consider ml and pml. Since $pml - ml$ is divisible by l, we have $a_{pml} = a_{ml}$. Hence $d(pml)$ and $d(ml)$ have same parity. But $d(pml) = 2d(ml)$, since $\gcd(p, ml) = 1$ and p is a prime. Since ml is a square, $d(ml)$ is odd. It follows that $d(pml)$ is even and hence $a_{pml} \neq a_{ml}$. This contradiction implies that x is irrational.

Alternative Solution: As earlier, assume that x is rational and choose natural numbers k, l such that $a_n = a_{n+l}$ for all $n \geq k$. Consider the numbers $a_{m+1}, a_{m+2}, \dots, a_{m+l}$, where $m \geq k$ is any number. This must contain at least one 0. Otherwise $a_n = 1$ for all $n \geq k$. But $a_r = 0$ if and only if r is a square. Hence it follows that there are no squares for $n > k$, which is absurd. Thus every l consecutive terms of the sequence $\langle a_n \rangle$ must contain a 0 after certain stage. Let $t = \max\{k, l\}$, and consider t^2 and $(t+1)^2$. Since there are no squares between t^2 and $(t+1)^2$, we conclude that $a_{t^2+j} = 1$ for $1 \leq j \leq 2t$. But then, we have $2t (> l)$ consecutive terms of the sequence $\langle a_n \rangle$ which miss 0, contradicting our earlier observation.

3. Find all real numbers x such that

$$[x^2 + 2x] = [x]^2 + 2[x].$$

(Here $[x]$ denotes the largest integer not exceeding x.)

Solution:

Adding 1 both sides, the equation reduces to

$$[(x+1)^2] = ([x+1])^2;$$

we have used $[x] + m = [x + m]$ for every integer m. Suppose $x + 1 \leq 0$. Then $[x+1] \leq x + 1 \leq 0$. Thus

$$([x+1])^2 \geq (x+1)^2 \geq [(x+1)^2] = ([x+1])^2.$$

Thus equality holds everywhere. This gives $[x+1] = x + 1$ and thus $x + 1$ is an integer. Using $x + 1 \leq 0$, we conclude that

$$x \in \{-1, -2, -3, \dots\}.$$

Suppose $x + 1 > 0$. We have

$$(x+1)^2 \geq [(x+1)^2] = ([x+1])^2.$$

Moreover, we also have

$$(x+1)^2 \leq 1 + [(x+1)^2] = 1 + ([x+1])^2.$$

Thus we obtain

$$[x] + 1 = [x + 1] \le (x + 1) < \sqrt{1 + \left([x + 1]\right)^2} = \sqrt{1 + \left([x] + 1\right)^2}.$$

This shows that

$$x \in \left[n, \sqrt{1 + (n + 1)^2} - 1\right),$$

where $n \ge -1$ is an integer. Thus the solution set is

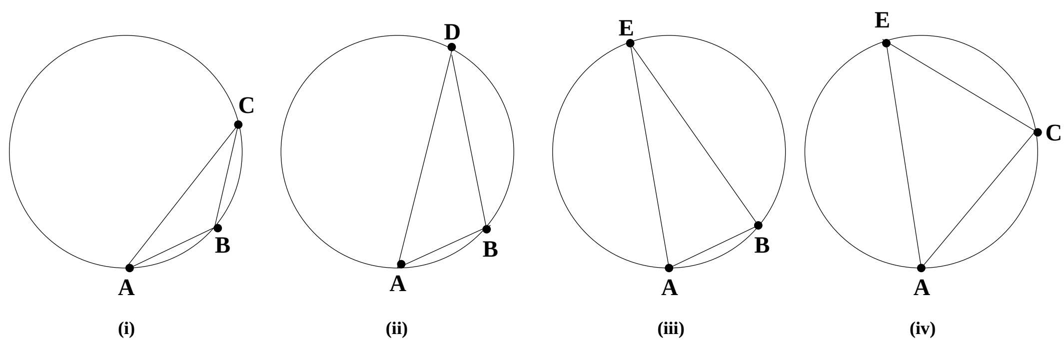
$$\{-1, -2, -3, \dots\} \cup \left\{ \cup_{n=-1}^{\infty} \left[n, \sqrt{1 + (n + 1)^2} - 1\right) \right\}.$$

It is easy verify that all the real numbers in this set indeed satisfy the given equation.

4. All the points in the plane are coloured using three colours. Prove that there exists a triangle with vertices having the same colour such that *either* it is isosceles *or* its angles are in geometric progression.

Solution:

Consider a circle of positive radius in the plane and inscribe a regular heptagon $ABCDEFG$ in it. Since the seven vertices of this heptagon are coloured by three colours, some three vertices have the same colour, by pigeon-hole principle. Consider the triangle formed by these three vertices. Let us call the part of the circumference separated by any two consecutive vertices of the heptagon an *arc*. The three vertices of the same colour are separated by *arcs* of length l, m, n as we move, say counter-clockwise, along the circle, starting from a fixed vertex among these three, where $l + m + n = 7$. Since, the order of l, m, n does not matter for a triangle, there are four possibilities: 1+1+5=7; 1+2+4=7; 1+3+3=7; 2+2+3=7. In the first, third and fourth cases, we have isosceles triangles. In the second case, we have a triangle whose angles are in geometric progression. The four corresponding figures are shown below.

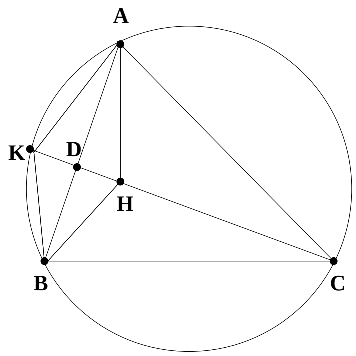


In (i), $AB = BC$; in (iii), $AE = BE$; in (iv), $AC = CE$; and in (ii) we see that $\angle D = \pi/7$, $\angle A = 2\pi/7$ and $\angle B = 4\pi/7$ which are in geometric progression.

5. Let ABC be an acute-angled triangle and let H be its ortho-centre. Let $h_{\max}$ denote the largest altitude of the triangle ABC. Prove that

$$AH + BH + CH \le 2h_{\max}.$$

Solution:



Let $\angle C$ be the smallest angle, so that $CA \geq AB$ and $CB \geq AB$. In this case the altitude through C is the longest one. Let the altitude through C meet AB in D and let H be the ortho-centre of ABC. Let CD extended meet the circum-circle of ABC in K. We have $CD = h_{\max}$ so that the inequality to be proved is

$$AH + BH + CH \leq 2CD.$$

Using $CD = CH + HD$, this reduces to $AH + BH \leq CD + HD$. However, we observe that $AH = AK$, $BH = BK$ and $HD = DK$.(For example $BH = BK$ and $DH = DK$ follow from the congruency of the right-angled triangles DBK and DBH.)

Thus we need to prove that $AK + BK \leq CK$. Applying Ptolemy's theorem to the cyclic quadrilateral $BCAK$, we get

$$AB \cdot CK = AC \cdot BK + BC \cdot AK \geq AB \cdot BK + AB \cdot AK.$$

This implies that $CK \geq AK + BK$, which is precisely what we are looking for.

There were other beautiful solutions given by students who participated in INMO-2009. We record them here.

1. Let AD, BE, CF be the altitudes and H be the ortho-centre. Observe that

$$\frac{AH}{AD} = \frac{[AHB]}{[ADB]} = \frac{[AHC]}{[ADC]}.$$

This gives

$$\frac{AH}{AD} = \frac{[AHB] + [AHC]}{[ADB] + [ADC]} = 1 - \frac{[BHC]}{[ABC]}.$$

Similar expressions for the ratios BH/BE and CH/CF may be obtained. Adding, we get

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

Suppose AD is the largest altitude. We get

$$\frac{AH}{AD} + \frac{BH}{AD} + \frac{CH}{AD} \leq \frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

This gives the result.

2. Let O be the circum-centre and let L, M, N be the mid-points of BC, CA, AB respectively. Then we know that $AH = 2OL$, $BH = 2OM$ and $CH = 2ON$. As earlier, assume AD is the largest altitude. Then BC is the least side. We have

$$\begin{aligned} 4[ABC] = 4[BOC] + 4[COA] + 4[AOB] &= BC \times 2OL + CA \times 2OM + AB \times 2ON \\ &= BC \times AH + CA \times BH + AB \times CH \\ &\geq AB(AH + BH + CH). \end{aligned}$$

Thus

$$AH + BH + CH \le \frac{4[ABC]}{AB} = 2AD.$$

3. We make use of the fact that $AH = 2R\cos\angle A$, $BH = 2R\cos\angle B$, $CH = 2R\cos\angle C$ and $AD = 2R\sin\angle B\sin\angle C$, where R is the circum-radius of ABC. We are assuming that AD is the largest altitude so that $\angle A$ is the least angle. Thus we have to prove that

$$\cos\angle A + \cos\angle B + \cos\angle C \le 2\sin\angle B\angle C,$$

under the assumption $\angle A \le \angle B$ and $\angle A \le \angle C$. On multiplying this by $2\sin\angle A$, this is equivalent to

$$2(\sin\angle A\cos\angle A + \sin\angle A\cos\angle B + \sin\angle A\cos\angle C)$$
$$\le 4\sin\angle A\sin\angle B\angle C = \sin 2A + \sin 2B + \sin 2C.$$

This is equivalent to

$$\cos\angle B(\sin\angle A - \sin\angle B) + \cos\angle C(\sin\angle A - \sin\angle C) \le 0.$$

Since ABC is acute-angled and A is the least angle, the result follows.

6. Let a, b, c be positive real numbers such that $a^3 + b^3 = c^3$. Prove that

$$a^2 + b^2 - c^2 > 6(c-a)(c-b).$$

Solution:

The given inequality may be written in the form

$$7c^2 - 6(a+b)c - (a^2 + b^2 - 6ab) < 0.$$

Putting $x = 7c^2$, $y = -6(a+b)c$, $z = -(a^2 + b^2 - 6ab)$, we have to prove that $x + y + z < 0$. Observe that x, y, z are not all equal($x > 0$, $y < 0$). Using the identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)\big[(x-y)^2 + (y-z)^2 + (z-x)^2\big],$$

we infer that it is sufficient to prove $x^3 + y^3 + z^3 - 3xyz < 0$. Substituting the values of x, y, z, we see that this is equivalent to

$$343c^6 - 216(a+b)^3c^3 - (a^2 + b^2 - 6ab)^3 - 126c^3(a+b)(a^2 + b^2 - 6ab) < 0.$$

Using $c^3 = a^3 + b^3$, this reduces to

$$343\big(a^3+b^3\big)^2 - 216(a+b)^3(a^3+b^3) - (a^2+b^2-6ab)^3 - 126((a^3+b^3)(a+b)(a^2+b^2-6ab) < 0.$$

This may be simplified (after some tedious calculations) to,

$$-a^2b^2(129a^2 - 254ab + 129b^2) < 0.$$

But $129a^2 - 254ab + 129b^2 = 129(a-b)^2 + 4ab > 0$. Hence the result follows.

Remark: The best constant θ in the inequality $a^2 + b^2 - c^2 \ge \theta(c-a)(c-b)$, where a, b, c

are positive reals such that $a^3+b^3=c^3$, is $\theta = 2\big(1+2^{1/3}+2^{-1/3}\big)$.

Here again, there were some beautiful solutions given by students.

1. We have

$$a^3 = c^3 - b^3 = (c-b)(c^2+cb+b^2),$$

which is same as

$$\frac{a^2}{c-b} = \frac{c^2+cb+b^2}{a}.$$

Similarly, we get

$$\frac{b^2}{c-a} = \frac{c^2+ca+a^2}{b}.$$

We observe that

$$\frac{a^2}{c-b} + \frac{b^2}{c-a} = \frac{c(a^2+b^2)-a^3-b^3}{(c-a)(c-b)} = \frac{c(a^2+b^2-c^2)}{(c-a)(c-b)}.$$

This shows that

$$\frac{a^2+b^2-c^2}{(c-a)(c-b)} = \frac{c^2+cb+b^2}{ca} + \frac{c^2+ca+a^2}{cb}.$$

Thus it is sufficient to prove that

$$\frac{c^2+cb+b^2}{ca} + \frac{c^2+ca+a^2}{cb} \geq 6.$$

However, we have $c^2+b^2 \geq 2cb$ and $c^2+a^2 \geq 2ca$. Hence

$$\frac{c^2+cb+b^2}{ca} + \frac{c^2+ca+a^2}{cb} \geq 3\left(\frac{b}{a}+\frac{a}{b}\right) \geq 3\times 2 = 6.$$

We have used AM-GM inequality.

2. Let us set $x=a/c$ and $y=b/c$. Then $x^3+y^3=1$ and the inequality to be proved is $x^2+y^2-1 > 6(1-x)(1-y)$. This reduces to

$$(x+y)^2 + 6(x+y) - 8xy - 7 > 0. \qquad (1)$$

But

$$1 = x^3+y^3 = (x+y)(x^2-xy+y^2),$$

which gives $xy = \big((x+y)^3-1\big)/3(x+y)$. Substituting this in (1) and introducing $x+y=t$, the inequality takes the form

$$t^2 + 6t - \frac{8}{3}\frac{(t^3-1)}{t} - 7 > 0. \qquad (2)$$

This may be simplified to $-5t^3+18t^2-2t+8>0$. Equivalently

$$-(5t-8)(t-1)^2 > 0.$$

Thus we need to prove that $5t<8$. Observe that $(x+y)^3 > x^3+y^3 = 1$, so that $t>1$. We also have

$$\left(\frac{x+y}{2}\right) \leq \frac{x^3+y^3}{2} = \frac{1}{2}.$$

This shows that $t^3 \le 4$. Thus

$$\left(\frac{5t}{8}\right)^3 \le \frac{125 \times 4}{512} = \frac{500}{512} < 1.$$

Hence $5t < 8$, which proves the given inequality.

3. We write $b^3 = c^3 - a^3$ and $a^3 = c^3 - b^3$ so that

$$c - a = \frac{b^3}{c^2 - ca + a^2}, \quad c - b = \frac{a^3}{c^2 - cb + b^2}.$$

Thus the inequality reduces to

$$a^2 + b^2 - c^2 > 6\frac{a^3b^3}{(c^2 - ca + a^2)(c^2 - cb + b^2)}.$$

This simplifies(after some lengthy calculations) to

$$-c^6 - (a+b)c^5 - abc^4 + (a^3 + b^3)c^3 + (a^4 + a^3b + a^2b^2 + ab^3 + b^4)c^2$$
$$(a^2b + ab^2 + a^3 + b^3)abc + (a^4b^2 - 6a^3b^3 + a^2b^4) > 0.$$

Substituting

$$c^3 = a^3 + b^3, \quad c^4 = c(a^3 + b^3), \quad c^5 = c^2(a^3 + b^3), \quad c^6 = (a^3 + b^3)^2,$$

the inequality further reduces to

$$a^2b^2(a^2 + b^2 + c^2 + ac + bc - 6ab) > 0.$$

Thus we need to prove that $a^2 + b^2 + c^2 + ac + bc - 6ab > 0$. Since $a^2 + b^2 \ge 2ab$, it is enough to prove that $c^2 + c(a+b) - 4ab > 0$. Multiplying this by c and using $a^3 + b^3 = c^3$, we need to prove that

$$a^3 + b^3 + c^2a + c^2b > 4abc.$$

Using AM-GM inequality to these 4 terms and using $c > a$, $c > b$ we get

$$a^3 + b^3 + c^2a + c^2b > 4(a^3b^3c^2ac^2b)^{1/4} = 4abc,$$

which proves the inequality.

INMO-2010 Problems and Solutions

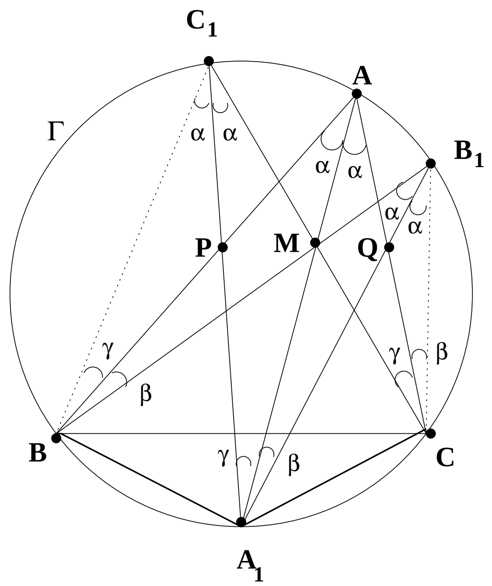
1. Let ABC be a triangle with circum-circle Γ. Let M be a point in the interior of triangle ABC which is also on the bisector of $\angle A$. Let AM, BM, CM meet Γ in A_1, B_1, C_1 respectively. Suppose P is the point of intersection of A_1C_1 with AB; and Q is the point of intersection of A_1B_1 with AC. Prove that PQ is parallel to BC.

   Solution: Let $A = 2\alpha$. Then $\angle A_1AC = \angle BAA_1 = \alpha$. Thus

$$\angle A_1B_1C = \alpha = \angle BB_1A_1 = \angle A_1C_1C = \angle BC_1A_1.$$

We also have $\angle B_1CQ = \angle AA_1B_1 = \beta$, say. It follows that triangles MA_1B_1 and QCB_1 are similar and hence

$$\frac{QC}{MA_1} = \frac{B_1C}{B_1A_1}.$$



Similarly, triangles ACM and C_1A_1M are similar and we get

$$\frac{AC}{AM} = \frac{C_1A_1}{C_1M}.$$

Using the point P, we get similar ratios:

$$\frac{PB}{MA_1} = \frac{C_1B}{A_1C_1}, \quad \frac{AB}{AM} = \frac{A_1B_1}{MB_1}.$$

Thus,

$$\frac{QC}{PB} = \frac{A_1C_1 \cdot B_1C}{C_1B \cdot B_1A_1},$$

and

$$\begin{aligned} \frac{AC}{AB} &= \frac{MB_1 \cdot C_1A_1}{A_1B_1 \cdot C_1M} \\ &= \frac{MB_1}{C_1M}\frac{C_1A_1}{A_1B_1} = \frac{MB_1}{C_1M}\frac{C_1B \cdot QC}{PB \cdot B_1C}. \end{aligned}$$

However, triangles C_1BM and B_1CM are similar, which gives

$$\frac{B_1C}{C_1B} = \frac{MB_1}{MC_1}.$$

Putting this in the last expression, we get

$$\frac{AC}{AB} = \frac{QC}{PB}.$$

We conclude that PQ is parallel to BC.

2. Find all natural numbers $n > 1$ such that n^2 **does not** divide $(n-2)!$.

Solution: Suppose $n = pqr$, where $p < q$ are primes and $r > 1$. Then $p \geq 2$, $q \geq 3$ and $r \geq 2$, not necessarily a prime. Thus we have

$$\begin{aligned} n-2 &\geq n-p = pqr - p \geq 5p > p, \\ n-2 &\geq n-q = q(pr-1) \geq 3q > q, \\ n-2 &\geq n-pr = pr(q-1) \geq 2pr > pr, \\ n-2 &\geq n-qr = qr(p-1) \geq qr. \end{aligned}$$

Observe that p, q, pr, qr are all distinct. Hence their product divides $(n-2)!$. Thus $n^2 = p^2q^2r^2$ divides $(n-2)!$ in this case. We conclude that either $n = pq$ where p, q are distinct primes or $n = p^k$ for some prime p.

Case 1. Suppose $n = pq$ for some primes p, q, where $2 < p < q$. Then $p \geq 3$ and $q \geq 5$. In this case

$$\begin{aligned} n-2 &> n-p = p(q-1) \geq 4p, \\ n-2 &> n-q = q(p-1) \geq 2q. \end{aligned}$$

Thus $p, q, 2p, 2q$ are all distinct numbers in the set $\{1, 2, 3, \ldots, n-2\}$. We see that $n^2 = p^2q^2$ divides $(n-2)!$. We conclude that $n = 2q$ for some prime $q \geq 3$. Note that $n - 2 = 2q - 2 < 2q$ in this case so that n^2 does not divide $(n-2)!$.

Case 2. Suppose $n = p^k$ for some prime p. We observe that $p, 2p, 3p, \ldots (p^{k-1}-1)p$ all lie in the set $\{1, 2, 3, \ldots, n-2\}$. If $p^{k-1} - 1 \geq 2k$, then there are at least $2k$ multiples of p in the set $\{1, 2, 3, \ldots, n-2\}$. Hence $n^2 = p^{2k}$ divides $(n-2)!$. Thus $p^{k-1} - 1 < 2k$.

If $k \geq 5$, then $p^{k-1} - 1 \geq 2^{k-1} - 1 \geq 2k$, which may be proved by an easy induction. Hence $k \leq 4$. If $k = 1$, we get $n = p$, a prime. If $k = 2$, then $p - 1 < 4$ so that $p = 2$ or 3; we get $n = 2^2 = 4$ or $n = 3^2 = 9$. For $k = 3$, we have $p^2 - 1 < 6$ giving $p = 2$; $n = 2^3 = 8$ in this case. Finally, $k = 4$ gives $p^3 - 1 < 8$. Again $p = 2$ and $n = 2^4 = 16$. However $n^2 = 2^8$ divides $14!$ and hence is not a solution.

Thus $n = p, 2p$ for some prime p or $n = 8, 9$. It is easy to verify that these satisfy the conditions of the problem.

3. Find all non-zero real numbers x, y, z which satisfy the system of equations:

$$\begin{aligned} (x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) &= xyz, \\ (x^4+x^2y^2+y^4)(y^4+y^2z^2+z^4)(z^4+z^2x^2+x^4) &= x^3y^3z^3. \end{aligned}$$

Solution: Since $xyz \neq 0$, We can divide the second relation by the first. Observe that

$$x^4 + x^2y^2 + y^4 = (x^2+xy+y^2)(x^2-xy+y^2),$$

holds for any x, y. Thus we get

$$(x^2-xy+y^2)(y^2-yz+z^2)(z^2-zx+x^2) = x^2y^2z^2.$$

However, for any real numbers x, y, we have

$$x^2 - xy + y^2 \geq |xy|.$$

Since $x^2y^2z^2 = |xy|\,|yz|\,|zx|$, we get

$$|xy|\,|yz|\,|zx| = (x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) \geq |xy|\,|yz|\,|zx|.$$

This is possible only if

$$x^2 - xy + y^2 = |xy|, \quad y^2 - yz + z^2 = |yz|, \quad z^2 - zx + x^2 = |zx|,$$

hold simultaneously. However $|xy| = \pm xy$. If $x^2 - xy + y^2 = -xy$, then $x^2 + y^2 = 0$ giving $x = y = 0$. Since we are looking for nonzero x, y, z, we conclude that $x^2 - xy + y^2 = xy$ which is same as $x = y$. Using the other two relations, we also get $y = z$ and $z = x$. The first equation now gives $27x^6 = x^3$. This gives $x^3 = 1/27$(since $x \neq 0$), or $x = 1/3$. We thus have $x = y = z = 1/3$. These also satisfy the second relation, as may be verified.

4. How many 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ are there such that each of $a_1, a_2, a_3, a_4, a_5, a_6$ is from the set $\{1, 2, 3, 4\}$ and the six expressions

$$a_j^2 - a_j a_{j+1} + a_{j+1}^2$$

for $j = 1, 2, 3, 4, 5, 6$(where a_7 is to be taken as a_1) are all equal to one another?

Solution: Without loss of generality, we may assume that a_1 is the largest among $a_1, a_2, a_3, a_4, a_5, a_6$. Consider the relation

$$a_1^2 - a_1a_2 + a_2^2 = a_2^2 - a_2a_3 + a_3^2.$$

This leads to

$$(a_1 - a_3)(a_1 + a_3 - a_2) = 0.$$

Observe that $a_1 \geq a_2$ and $a_3 > 0$ together imply that the second factor on the left side is positive. Thus $a_1 = a_3 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}$. Using this and the relation

$$a_3^2 - a_3a_4 + a_4^2 = a_4^2 - a_4a_5 + a_5^2,$$

we conclude that $a_3 = a_5$ as above. Thus we have

$$a_1 = a_3 = a_5 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}.$$

Let us consider the other relations. Using

$$a_2^2 - a_2a_3 + a_3^2 = a_3^2 - a_3a_4 + a_4^2,$$

we get $a_2 = a_4$ or $a_2 + a_4 = a_3 = a_1$. Similarly, two more relations give either $a_4 = a_6$ or $a_4 + a_6 = a_5 = a_1$; and either $a_6 = a_2$ or $a_6 + a_2 = a_1$. Let us give values to a_1 and count the number of six-tuples in each case.

(A) Suppose $a_1 = 1$. In this case all a_j's are equal and we get only one six-tuple $(1, 1, 1, 1, 1, 1)$.

(B) If $a_1 = 2$, we have $a_3 = a_5 = 2$. We observe that $a_2 = a_4 = a_6 = 1$ or $a_2 = a_4 = a_6 = 2$. We get two more six-tuples: $(2, 1, 2, 1, 2, 1)$, $(2, 2, 2, 2, 2, 2)$.

(C) Taking $a_1 = 3$, we see that $a_3 = a_5 = 3$. In this case we get nine possibilities for (a_2, a_4, a_6);

$$(1,1,1), (2,2,2), (3,3,3), (1,1,2), (1,2,1), (2,1,1), (1,2,2), (2,1,2), (2,2,1).$$

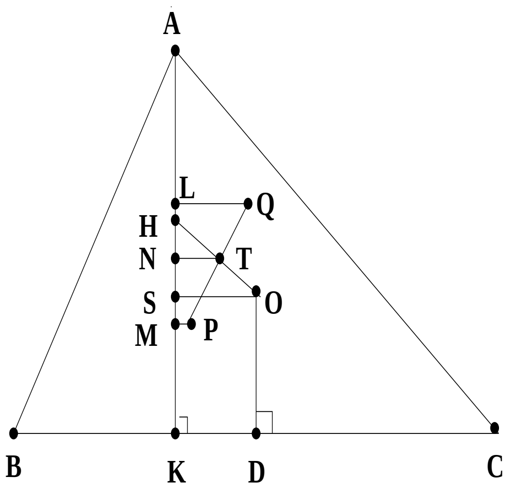
(D) In the case $a_1 = 4$, we have $a_3 = a_5 = 4$ and

$$(a_2, a_4, a_6) = (2,2,2), (4,4,4), (1,1,1), (3,3,3), \\ (1,1,3), (1,3,1), (3,1,1), (1,3,3), (3,1,3), (3,3,1).$$

Thus we get $1 + 2 + 9 + 10 = 22$ solutions. Since (a_1, a_3, a_5) and (a_2, a_4, a_6) may be interchanged, we get 22 more six-tuples. However there are 4 common among these, namely, $(1,1,1,1,1,1)$, $(2,2,2,2,2,2)$, $(3,3,3,3,3,3)$ and $(4,4,4,4,4,4)$. Hence the total number of six-tuples is $22 + 22 - 4 = 40$.

5. Let ABC be an acute-angled triangle with altitude AK. Let H be its ortho-centre and O be its circum-centre. Suppose KOH is an acute-angled triangle and P its circum-centre. Let Q be the reflection of P in the line HO. Show that Q lies on the line joining the mid-points of AB and AC.

Solution: Let D be the mid-point of BC; M that of HK; and T that of OH. Then PM is perpendicular to HK and PT is perpendicular to OH. Since Q is the reflection of P in HO, we observe that P, T, Q are collinear, and $PT = TQ$. Let QL, TN and OS be the perpendiculars drawn respectively from Q, T and O on to the altitude AK.(See the figure.)



We have $LN = NM$, since T is the mid-point of QP; $HN = NS$, since T is the mid-point of OH; and $HM = MK$, as P is the circum-centre of KHO. We obtain

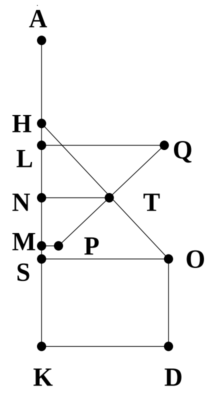
$$LH + HN = LN = NM = NS + SM,$$

which gives $LH = SM$. We know that $AH = 2OD$. Thus

$$AL = AH - LH = 2OD - LH = 2SK - SM = SK + (SK - SM) = SK + MK \\ = SK + HM = SK + HS + SM = SK + HS + LH = SK + LS = LK.$$

This shows that L is the mid-point of AK and hence lies on the line joining the midpoints of AB and AC. We observe that the line joining the mid-points of AB and AC is also perpendicular to AK. Since QL is perpendicular to AK, we conclude that Q also lies on the line joining the mid-points of AB and AC.

Remark: It may happen that H is above L as in the adjoining figure, but the result remains true here as well. We have $HN = NS$, $LN = NM$, and $HM = MK$ as earlier. Thus $HN = HL + LN$ and $NS = SM + NM$ give $HL = SM$. Now $AL = AH + HL = 2OD + SM = 2SK + SM = SK + (SK + SM) = SK + MK = SK + HM = SK + HL + LM = SK + SM + LM = LK$. The conclusion that Q lies on the line joining the mid-points of AB and AC follows as earlier.



6. Define a sequence $\langle a_n \rangle_{n \geq 0}$ by $a_0 = 0$, $a_1 = 1$ and

$$a_n = 2a_{n-1} + a_{n-2},$$

for $n \geq 2$.

(a) For every $m > 0$ and $0 \leq j \leq m$, prove that $2a_m$ divides $a_{m+j} + (-1)^j a_{m-j}$.

(b) Suppose 2^k divides n for some natural numbers n and k. Prove that 2^k divides a_n.

Solution:

(a) Consider $f(j) = a_{m+j} + (-1)^j a_{m-j}$, $0 \leq j \leq m$, where m is a natural number. We observe that $f(0) = 2a_m$ is divisible by $2a_m$. Similarly,

$$f(1) = a_{m+1} - a_{m-1} = 2a_m$$

is also divisible by $2a_m$. Assume that $2a_m$ divides $f(j)$ for all $0 \leq j < l$, where $l \leq m$. We prove that $2a_m$ divides $f(l)$. Observe

$$\begin{aligned} f(l-1) &= a_{m+l-1} + (-1)^{l-1} a_{m-l+1}, \\ f(l-2) &= a_{m+l-2} + (-1)^{l-2} a_{m-l+2}. \end{aligned}$$

Thus we have

$$\begin{aligned} a_{m+l} &= 2a_{m+l-1} + a_{m+l-2} \\ &= 2f(l-1) - 2(-1)^{l-1} a_{m-l+1} + f(l-2) - (-1)^{l-2} a_{m-l+2} \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1} \big(a_{m-l+2} - 2a_{m-l+1} \big) \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1} a_{m-l}. \end{aligned}$$

This gives

$$f(l) = 2f(l-1) + f(l-2).$$

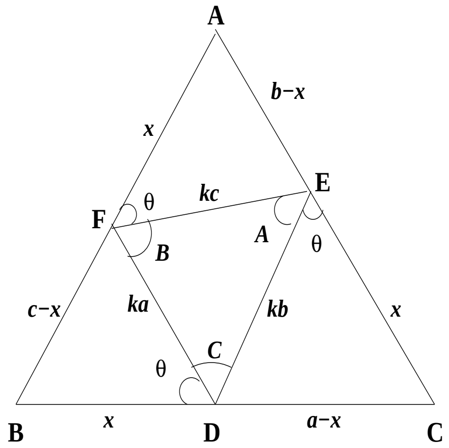
By induction hypothesis $2a_m$ divides $f(l-1)$ and $f(l-2)$. Hence $2a_m$ divides $f(l)$. We conclude that $2a_m$ divides $f(j)$ for $0 \leq j \leq m$.

(b) We see that $f(m) = a_{2m}$. Hence $2a_m$ divides a_{2m} for all natural numbers m. Let $n = 2^k l$ for some $l \geq 1$. Taking $m = 2^{k-1} l$, we see that $2a_m$ divides a_n. Using an easy induction , we conclude that $2^k a_l$ divides a_n. In particular 2^k divides a_n.

Problems and Solutions, INMO-2011

1. Let D, E, F be points on the sides BC, CA, AB respectively of a triangle ABC such that $BD = CE = AF$ and $\angle BDF = \angle CED = \angle AFE$. Prove that ABC is equilateral.

Solution 1:



Let $BD = CE = AF = x$; $\angle BDF = \angle CED = \angle AFE = \theta$. Note that $\angle AFD = B + \theta$, and hence $\angle DFE = B$. Similarly, $\angle EDF = C$ and $\angle FED = A$. Thus the triangle EFD is similar to ABC. We may take $FD = ka$, $DE = kb$ and $EF = kc$, for some positive real constant k. Applying sine rule to triangle BFD, we obtain

$$\frac{c-x}{\sin\theta} = \frac{ka}{\sin B} = \frac{2Rka}{b},$$

where R is the circum-radius of ABC. Thus we get $2Rk\sin\theta = b(c-x)/a$. Similarly, we obtain $2Rk\sin\theta = c(a-x)/b$ and $2Rk\sin\theta = a(b-x)/c$. We therefore get

$$\frac{b(c-x)}{a} = \frac{c(a-x)}{b} = \frac{a(b-x)}{c}. \tag{1}$$

If some two sides are equal, say, $a = b$, then $a(c-x) = c(a-x)$ giving $a = c$; we get $a = b = c$ and ABC is equilateral. Suppose no two sides of ABC are equal. We may assume a is the least. Since (1) is cyclic in a, b, c, we have to consider two cases: $a < b < c$ and $a < c < b$.

Case 1. <u>$a < b < c$.</u>

In this case $a < c$ and hence $b(c-x) < a(b-x)$, from (1). Since $b > a$ and $c - x > b - x$, we get $b(c-x) > a(b-x)$, which is a contradiction.

Case 2. <u>$a < c < b$.</u>

We may write (1) in the form

$$\frac{(c-x)}{a/b} = \frac{(a-x)}{b/c} = \frac{(b-x)}{c/a}. \tag{2}$$

Now $a < c$ gives $a - x < c - x$ so that $\dfrac{b}{c} < \dfrac{a}{b}$. This gives $b^2 < ac$. But $b > a$ and $b > c$, so that $b^2 > ac$, which again leads to a contradiction

Thus Case 1 and Case 2 cannot occur. We conclude that $a = b = c$.

Solution 2. We write (1) in the form (2), and start from there. The case of two equal sides is dealt as in Solution 1. We assume no two sides are equal. Using ratio properties in (2), we obtain

$$\frac{a-b}{(ab-c^2)/ca} = \frac{b-c}{(bc-a^2)/ab}.$$

This may be written as $c(a-b)(bc-a^2) = b(b-c)(ab-c^2)$. Further simplification gives $ab^3 + bc^3 + ca^3 = abc(a+b+c)$. This may be further written in the form

$$ab^2(b-c) + bc^2(c-a) + ca^2(a-b) = 0. \tag{3}$$

If $a < b < c$, we write (3) in the form

$$0 = ab^2(b-c) + bc^2(c-b+b-a) + ca^2(a-b) = b(c-b)(c^2-ab) + c(b-a)(bc-a^2).$$

Since $c > b$, $c^2 > ab$, $b > a$ and $bc > a^2$, this is impossible. If $a < c < b$, we write (3), as in previous case, in the form

$$0 = a(b-c)(b^2-ca) + c(c-a)(bc-a^2),$$

which again is impossible.

One can also use inequalities: we can show that $ab^3 + bc^3 + ca^3 \geq abc(a+b+c)$, and equality holds if and only if $a = b = c$. Here are some ways of deriving it:

(i) We can write the inequality in the form

$$\frac{b^2}{c}+\frac{c^2}{a}+\frac{a^2}{b}\geq a+b+c.$$

Adding $a+b+c$ both sides, this takes the form

$$\frac{b^2}{c}+c+\frac{c^2}{a}+a+\frac{a^2}{b}+b\geq 2(a+b+c).$$

But AM-GM inequality gives

$$\frac{b^2}{c}+c\geq 2b,\quad \frac{c^2}{a}+a\geq 2a,\quad \frac{a^2}{b}+b\geq 2a.$$

Hence the inequality follows and equality holds if and only if $a=b=c$.

(ii) Again we write the inequality in the form

$$\frac{b^2}{c}+\frac{c^2}{a}+\frac{a^2}{b}\geq a+b+c.$$

We use b/c with weight b, c/a with weight c and a/b with weight a, and apply weighted AM-HM inequality:

$$b\cdot\frac{b}{c}+c\cdot\frac{c}{a}+a\cdot\frac{a}{b}\geq\frac{(a+b+c)^2}{b\cdot\frac{c}{b}+c\cdot\frac{a}{c}+a\cdot\frac{b}{a}},$$

which reduces to $a+b+c$. Again equality holds if and only if $a=b=c$.

Solution 3. Here is a pure geometric solution given by a student. Consider the triangle BDF, CED and AFE with BD, CE and AF as bases. The sides DF, ED and FE make equal angles θ with the bases of respective triangles. If $B\geq C\geq A$, then it is easy to see that $FD\geq DE\geq EF$. Now using the triangle FDE, we see that $B\geq C\geq A$ gives $DE\geq EF\geq FD$. Combining, you get $FD=DE=EF$ and hence $A=B=C=60°$.

2. Call a natural number n *faithful*, if there exist natural numbers $a<b<c$ such that a divides b, b divides c and $n=a+b+c$.

   (i) Show that all but a finite number of natural numbers are faithful.

   (ii) Find the sum of all natural numbers which are **not** faithful.

Solution 1: Suppose $n\in\mathbb{N}$ is faithful. Let $k\in\mathbb{N}$ and consider kn. Since $n=a+b+c$, with $a>b>c$, $c\big|b$ and $b\big|a$, we see that $kn=ka+kb+kc$ which shows that kn is faithful.

Let $p>5$ be a prime. Then p is odd and $p=(p-3)+2+1$ shows that p is faithful. If $n\in\mathbb{N}$ contains a prime factor $p>5$, then the above observation shows that n is faithful. This shows that a number which is not faithful must be of the form $2^\alpha 3^\beta 5^\gamma$. We also observe that $2^4=16=12+3+1$, $3^2=9=6+2+1$ and $5^2=25=22+2+1$, so that 2^4, 3^2 and 5^2 are faithful. Hence $n\in\mathbb{N}$ is also faithful if it contains a factor of the form 2^α where $\alpha\geq 4$; a factor of the form 3^β where $\beta\geq 2$; or a factor of the form 5^γ where $\gamma\geq 2$. Thus the numbers which are not faithful are of the form $2^\alpha 3^\beta 5^\gamma$, where $\alpha\leq 3$, $\beta\leq 1$ and $\gamma\leq 1$. We may enumerate all such numbers:

$$1,2,3,4,5,6,8,10,12,15,20,24,30,40,60,120.$$

Among these $120=112+7+1$, $60=48+8+4$, $40=36+3+1$, $30=18+9+3$, $20=12+6+2$, $15=12+2+1$, and $10=6+3+1$. It is easy to check that the other numbers cannot be written in the required form. Hence the only numbers which are not faithful are

$$1,2,3,4,5,6,8,12,24.$$

Their sum is 65.

Solution 2: If $n=a+b+c$ with $a<b<c$ is faithful, we see that $a\geq 1$, $b\geq 2$ and $c\geq 4$. Hence $n\geq 7$. Thus $1,2,3,4,5,6$ are not faithful. As observed earlier, kn is faithful whenever

n is. We also notice that for odd $n \geq 7$, we can write $n = 1 + 2 + (n-3)$ so that all odd $n \geq 7$ are faithful. Consider $2n$, $4n$, $8n$, where $n \geq 7$ is odd. By observation, they are all faithful. Let us list a few of them:

$$\begin{aligned} 2n &: \; 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, \ldots \\ 4n &: \; 28, 36, 44, 52, 60, 68, \ldots \\ 8n &: \; 56, 72, \ldots, \end{aligned}$$

We observe that $16 = 12 + 3 + 1$ and hence it is faithful. Thus all multiples of 16 are also faithful. Thus we see that 16, 32, 48, 64, ... are faithful. Any even number which is not a multiple of 16 must be either an odd multiple of 2, or that of 4, or that of 8. Hence, the only numbers not covered by this process are 8, 10, 12, 20, 24, 40. Of these, we see that

$$10 = 1 + 3 + 6, \quad 20 = 2 \times 10, \quad 40 = 4 \times 10,$$

so that 10,20,40 are faithful. Thus the only numbers which are not faithful are

$$1, 2, 3, 4, 5, 6, 8, 12, 24.$$

Their sum is 65.

3. Consider two polynomials $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $Q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ with integer coefficients such that $a_n - b_n$ is a prime, $a_{n-1} = b_{n-1}$ and $a_n b_0 - a_0 b_n \neq 0$. Suppose there exists a rational number r such that $P(r) = Q(r) = 0$. Prove that r is an integer.

   Solution: Let $r = u/v$ where $\gcd(u, v) = 1$. Then we get

$$\begin{aligned} a_n u^n + a_{n-1}u^{n-1}v + \cdots + a_1 uv^{n-1} + a_0 v^n &= 0, \\ b_n u^n + b_{n-1}u^{n-1}v + \cdots + b_1 uv^{n-1} + b_0 v^n &= 0. \end{aligned}$$

   Subtraction gives

$$(a_n - b_n)u^n + (a_{n-2} - b_{n-2})u^{n-2}v^2 + \cdots + (a_1 - b_1)uv^{n-1} + (a_0 - b_0)v^n = 0,$$

   since $a_{n-1} = b_{n-1}$. This shows that v divides $(a_n - b_n)u^n$ and hence it divides $a_n - b_n$. Since $a_n - b_n$ is a prime, either $v = 1$ or $v = a_n - b_n$. Suppose the latter holds. The relation takes the form

$$u^n + (a_{n-2} - b_{n-2})u^{n-2}v + \cdots + (a_1 - b_1)uv^{n-2} + (a_0 - b_0)v^{n-1} = 0.$$

   (Here we have divided through-out by v.) If $n > 1$, this forces $v \big| u$, which is impossible since $\gcd(v, u) = 1$ ($v > 1$ since it is equal to the prime $a_n - b_n$). If $n = 1$, then we get two equations:

$$\begin{aligned} a_1 u + a_0 v &= 0, \\ b_1 u + b_0 v &= 0. \end{aligned}$$

   This forces $a_1 b_0 - a_0 b_1 = 0$ contradicting $a_n b_0 - a_0 b_n \neq 0$. (Note: The condition $a_n b_0 - a_0 b_n \neq 0$ is extraneous. The condition $a_{n-1} = b_{n-1}$ forces that for $n = 1$, we have $a_0 = b_0$. Thus we obtain, after subtraction

$$(a_1 - b_1)u = 0.$$

   This implies that $u = 0$ and hence $r = 0$ is an integer.)

4. Suppose five of the nine vertices of a regular nine-sided polygon are arbitrarily chosen. Show that one can select four among these five such that they are the vertices of a trapezium.

   Solution 1: Suppose four distinct points P, Q, R, S(in that order on the circle) among these five are such that $\widehat{PQ} = \widehat{RS}$. Then $PQRS$ is an isosceles trapezium, with $PS \parallel QR$. We use this in our argument.

   - If four of the five points chosen are adjacent, then we are through as observed earlier. (In this case four points A, B, C, D are such that $\widehat{AB} = \widehat{BC} = \widehat{CD}$.) See Fig 1.

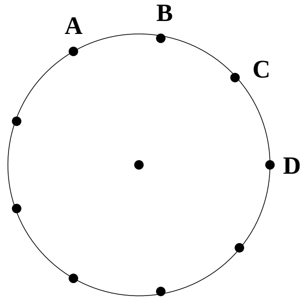


Fig 1.

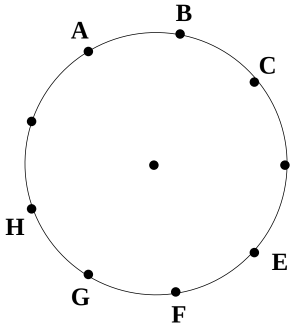


Fig 2.

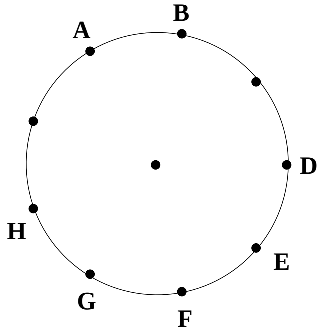


Fig 3.

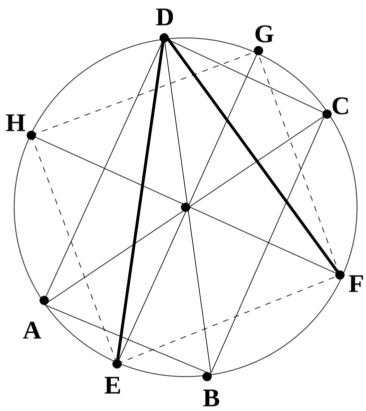
- Suppose only three of the vertices are adjacent, say A, B, C(see Fig 2.) Then the remaining two must be among E, F, G, H. If these two are adjacent vertices, we can pair them with A, B or B, C to get equal arcs. If they are not adjacent, then they must be either E, G or F, H or E, H. In the first two cases, we can pair them with A, C to get equal arcs. In the last case, we observe that $\widehat{HA} = \widehat{CE}$ and $AHEC$ is an isosceles trapezium.
- Suppose only two among the five are adjacent, say A, B. Then the remaining three are among D, E, F, G, H. (See Fig 3.) If any two of these are adjacent, we can combine them with A, B to get equal arcs. If no two among these three vertices are adjacent, then they must be D, F, H. In this case $\widehat{HA} = \widehat{BD}$ and $AHDB$ is an isosceles trapezium.

  Finally, if we choose 5 among the 9 vertices of a regular nine-sided polygon, then some two must be adjacent. Thus any choice of 5 among 9 must fall in to one of the above three possibilities.

Solution 2: Here is another solution used by many students. Suppose you join the vertices of the nine-sided regular polygon. You get $\binom{9}{2} = 36$ line segments. All these fall in to 9 sets of parallel lines. Now using any 5 points, you get $\binom{5}{2} = 10$ line segments. By pigeon-hole principle, two of these must be parallel. But, these parallel lines determine a trapezium.

5. Let $ABCD$ be a quadrilateral inscribed in a circle Γ. Let E, F, G, H be the midpoints of the arcs AB, BC, CD, DA of the circle Γ. Suppose $AC \cdot BD = EG \cdot FH$. Prove that AC, BD, EG, FH are concurrent.

Solution:



Let R be the radius of the circle Γ. Observe that $\angle EDF = \frac{1}{2}\angle D$. Hence $EF = 2R\sin\frac{D}{2}$. Similarly, $HG = 2R\sin\frac{B}{2}$. But $\angle B = 180^\circ - \angle D$. Thus $HG = 2R\cos\frac{D}{2}$. We hence get

$$EF \cdot GH = 4R^2 \sin\frac{D}{2}\cos\frac{D}{2} = 2R^2 \sin D = R \cdot AC.$$

Similarly, we obtain $EH \cdot FG = R \cdot BD$.

Therefore

$$R(AC + BD) = EF \cdot GH + EH \cdot FG = EG \cdot FH,$$

by Ptolemy's theorem. By the given hypothesis, this gives $R(AC + BD) = AC \cdot BD$. Thus

$$AC \cdot BD = R(AC + BD) \geq 2R\sqrt{AC \cdot BD},$$

using AM-GM inequality. This implies that $AC \cdot BD \geq 4R^2$. But AC and BD are the chords of Γ, so that $AC \leq 2R$ and $BD \leq 2R$. We obtain $AC \cdot BD \leq 4R^2$. It follows that $AC \cdot BD = 4R^2$, implying that $AC = BD = 2R$. Thus AC and BD are two diameters of Γ. Using $EG \cdot FH = AC \cdot BD$, we conclude that EG and FH are also two diameters of Γ. Hence AC, BD, EG and FH all pass through the centre of Γ.

6. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x+y)f(x-y) = \big(f(x)+f(y)\big)^2 - 4x^2 f(y), \tag{1}$$

for all $x, y \in \mathbf{R}$, where $\mathbf{R}$ denotes the set of all real numbers.

Solution 1.: Put $x = y = 0$; we get $f(0)^2 = 4f(0)^2$ and hence $f(0) = 0$.
Put $x = y$: we get $4f(x)^2 - 4x^2 f(x) = 0$ for all x. Hence for each x, either $f(x) = 0$ or $f(x) = x^2$.
Suppose $f(x) \not\equiv 0$. Then we can find $x_0 \neq 0$ such that $f(x_0) \neq 0$. Then $f(x_0) = x_0^2 \neq 0$. Assume that there exists some $y_0 \neq 0$ such that $f(y_0) = 0$. Then

$$f(x_0+y_0)f(x_0-y_0) = f(x_0)^2.$$

Now $f(x_0+y_0)f(x_0-y_0) = 0$ or $f(x_0+y_0)f(x_0-y_0) = (x_0+y_0)^2(x_0-y_0)^2$. If $f(x_0+y_0)f(x_0-y_0) = 0$, then $f(x_0) = 0$, a contradiction. Hence it must be the latter so that

$$(x_0^2 - y_0^2)^2 = x_0^4.$$

This reduces to $y_0^2\big(y_0^2 - 2x_0^2\big) = 0$. Since $y_0 \neq 0$, we get $y_0 = \pm\sqrt{2}x_0$.
Suppose $y_0 = \sqrt{2}x_0$. Put $x = \sqrt{2}x_0$ and $y = x_0$ in (1); we get

$$f\big((\sqrt{2}+1)x_0\big)f\big((\sqrt{2}-1)x_0\big) = \big(f(\sqrt{2}x_0) + f(x_0)\big)^2 - 4\big(2x_0^2\big)f(x_0).$$

But $f(\sqrt{2}x_0) = f(y_0) = 0$. Thus we get

$$\begin{aligned} f\big((\sqrt{2}+1)x_0\big)f\big((\sqrt{2}-1)x_0\big) &= f(x_0)^2 - 8x_0^2 f(x_0) \\ &= x_0^4 - 8x_0^4 = -7x_0^4. \end{aligned}$$

Now if LHS is equal to 0, we get $x_0 = 0$, a contradiction. Otherwise LHS is equal to $(\sqrt{2}+1)^2(\sqrt{2}-1)^2x_0^4$ which reduces to x_0^4. We obtain $x_0^4 = -7x_0^4$ and this forces again $x_0 = 0$. Hence there is no $y \neq 0$ such that $f(y) = 0$. We conclude that $f(x) = x^2$ for all x.
Thus there are two solutions: $f(x) = 0$ for all x or $f(x) = x^2$, for all x. It is easy to verify that both these satisfy the functional equation.

Solution 2: As earlier, we get $f(0) = 0$. Putting $x = 0$, we will also get

$$f(y)\big(f(y) - f(-y)\big) = 0.$$

As earlier, we may conclude that either $f(y) = 0$ or $f(y) = f(-y)$ for each $y \in \mathbb{R}$. Replacing y by $-y$, we may also conclude that $f(-y)\big(f(-y) - f(y)\big) = 0$. If $f(y) = 0$ and $f(-y) \neq 0$ for some y, then we must have $f(-y) = f(y) = 0$, a contradiction. Hence either $f(y) = f(-y) = 0$ or $f(y) = f(-y)$ for each y. This forces f is an even function.
Taking $y = 1$ in (1), we get

$$f(x+1)f(x-1) = (f(x)+f(1))^2 - 4x^2 f(1).$$

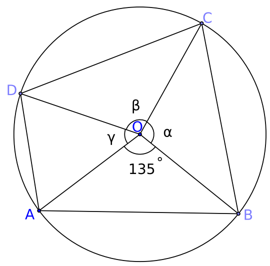
Replacing y by x and x by 1, you also get

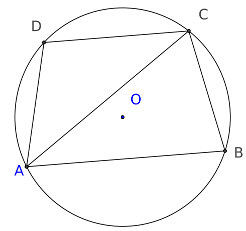
$$f(1+x)f(1-x) = (f(1)+f(x))^2 - 4f(x).$$

Comparing these two using the even nature of f, we get $f(x) = cx^2$, where $c = f(1)$. Putting $x = y = 1$ in (1), you get $4c^2 - 4c = 0$. Hence $c = 0$ or 1. We get $f(x) = 0$ for all x or $f(x) = x^2$ for all x.

Problems and Solutions: INMO-2012

1. Let $ABCD$ be a quadrilateral inscribed in a circle. Suppose $AB = \sqrt{2+\sqrt{2}}$ and AB subtends $135°$ at the centre of the circle. Find the maximum possible area of $ABCD$.





Solution: Let O be the centre of the circle in which $ABCD$ is inscribed and let R be its radius. Using cosine rule in triangle AOB, we have

$$2+\sqrt{2} = 2R^2(1-\cos 135°) = R^2(2+\sqrt{2}).$$

Hence $R = 1$.

Consider quadrilateral $ABCD$ as in the second figure above. Join AC. For $[ADC]$ to be maximum, it is clear that D should be the mid-point of the arc AC so that its distance from the segment AC is maximum. Hence $AD = DC$ for $[ABCD]$ to be maximum. Similarly, we conclude that $BC = CD$. Thus $BC = CD = DA$ which fixes the quadrilateral $ABCD$. Therefore each of the sides BC, CD, DA subtends equal angles at the centre O.

Let $\angle BOC = \alpha$, $\angle COD = \beta$ and $\angle DOA = \gamma$. Observe that

$$[ABCD] = [AOB]+[BOC]+[COD]+[DOA] = \frac{1}{2}\sin 135°+\frac{1}{2}(\sin\alpha+\sin\beta+\sin\gamma).$$

Now $[ABCD]$ has maximum area if and only if $\alpha = \beta = \gamma = (360° - 135°)/3 = 75°$. Thus

$$[ABCD] = \frac{1}{2}\sin 135° + \frac{3}{2}\sin 75° = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + 3\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{5+3\sqrt{3}}{4\sqrt{2}}.$$

Alternatively, we can use Jensen's inequality. Observe that α, β, γ are all less than $180°$. Since $\sin x$ is concave on $(0,\pi)$, Jensen's inequality gives

$$\frac{\sin\alpha+\sin\beta+\sin\gamma}{3} \le \sin\left(\frac{\alpha+\beta+\gamma}{3}\right) = \sin 75°.$$

Hence

$$[ABCD] \le \frac{1}{2\sqrt{2}} + \frac{3}{2}\sin 75° = \frac{5+3\sqrt{3}}{4\sqrt{2}},$$

with equality if and only if $\alpha = \beta = \gamma = 75°$.

2. Let $p_1 < p_2 < p_3 < p_4$ and $q_1 < q_2 < q_3 < q_4$ be two sets of prime numbers such that $p_4 - p_1 = 8$ and $q_4 - q_1 = 8$. Suppose $p_1 > 5$ and $q_1 > 5$. Prove that 30 divides $p_1 - q_1$.

Solution: Since $p_4 - p_1 = 8$, and no prime is even, we observe that $\{p_1, p_2, p_3, p_4\}$ is a subset of $\{p_1, p_1+2, p_1+4, p_1+6, p_1+8\}$. Moreover p_1 is larger than 3. If $p_1 \equiv 1 \pmod 3$, then p_1+2 and p_1+8 are divisible by 3. Hence we do not get 4 primes in the set $\{p_1, p_1+2, p_1+4, p_1+6, p_1+8\}$. Thus $p_1 \equiv 2 \pmod 3$ and p_1+4 is not a prime. We get $p_2 = p_1+2, p_3 = p_1+6, p_4 = p_1+8$.

Consider the remainders of p_1, p_1+2, p_1+6, p_1+8 when divided by 5. If $p_1 \equiv 2 \pmod 5$, then p_1+8 is divisible by 5 and hence is not a prime. If $p_1 \equiv 3 \pmod 5$, then p_1+2 is divisible by 5. If $p_1 \equiv 4 \pmod 5$, then p_1+6 is divisible by 5. Hence the only possibility is $p_1 \equiv 1 \pmod 5$.

Thus we see that $p_1 \equiv 1 \pmod 2$, $p_1 \equiv 2 \pmod 3$ and $p_1 \equiv 1 \pmod 5$. We conclude that $p_1 \equiv 11 \pmod{30}$.

Similarly $q_1 \equiv 11 \pmod{30}$. It follows that 30 divides $p_1 - q_1$.

3. Define a sequence $\langle f_0(x), f_1(x), f_2(x), \ldots \rangle$ of functions by

$$f_0(x) = 1, \quad f_1(x) = x, \quad \big(f_n(x)\big)^2 - 1 = f_{n+1}(x) f_{n-1}(x), \text{ for } n \geq 1.$$

Prove that each $f_n(x)$ is a polynomial with integer coefficients.

Solution: Observe that

$$f_n^2(x) - f_{n-1}(x) f_{n+1}(x) = 1 = f_{n-1}^2(x) - f_{n-2}(x) f_n(x).$$

This gives

$$f_n(x)\Big(f_n(x) + f_{n-2}(x)\Big) = f_{n-1}\Big(f_{n-1}(x) + f_{n+1}(x)\Big).$$

We write this as

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{f_{n-2}(x) + f_n(x)}{f_{n-1}(x)}.$$

Using induction, we get

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{f_0(x) + f_2(x)}{f_1(x)}.$$

Observe that

$$f_2(x) = \frac{f_1^2(x) - 1}{f_0(x)} = x^2 - 1.$$

Hence

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{1 + (x^2 - 1)}{x} = x.$$

Thus we obtain

$$f_{n+1}(x) = x f_n(x) - f_{n-1}(x).$$

Since $f_0(x)$, $f_1(x)$ and $f_2(x)$ are polynomials with integer coefficients, induction again shows that $f_n(x)$ is a polynomial with integer coefficients.

Note: We can get $f_n(x)$ explicitly:

$$f_n(x) = x^n - \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} - \binom{n-3}{3}x^{n-6} + \cdots$$

4. Let ABC be a triangle. An interior point P of ABC is said to be **good** if we can find exactly 27 rays emanating from P intersecting the sides of the triangle ABC such that the triangle is divided by these rays into 27 *smaller triangles of equal area*. Determine the number of **good** points for a given triangle ABC.

Solution: Let P be a good point. Let l, m, n be respetively the number of parts the sides BC, CA, AB are divided by the rays starting from P. Note that a ray must pass through each of the vertices the triangle ABC; otherwise we get some quadrilaterals.

Let h_1 be the distance of P from BC. Then h_1 is the height for all the triangles with their bases on BC. Equality of areas implies that all these bases have equal length. If we denote this by x, we get $lx = a$. Similarly, taking y and z as the lengths of the bases of triangles on CA and AB respectively, we get $my = b$ and $nz = c$. Let h_2 and h_3 be the distances of P from CA and AB respectively. Then

$$h_1 x = h_2 y = h_3 z = \frac{2\Delta}{27},$$

where Δ denotes the area of the triangle ABC. These lead to

$$h_1 = \frac{2\Delta}{27}\frac{l}{a}, \quad h_1 = \frac{2\Delta}{27}\frac{m}{b}, \quad h_1 = \frac{2\Delta}{27}\frac{n}{c}.$$

But

$$\frac{2\Delta}{a} = h_a, \quad \frac{2\Delta}{b} = h_b, \quad \frac{2\Delta}{c} = h_c.$$

Thus we get

$$\frac{h_1}{h_a} = \frac{l}{27}, \quad \frac{h_2}{h_b} = \frac{m}{27}, \quad \frac{h_3}{h_c} = \frac{n}{27}.$$

However, we also have

$$\frac{h_1}{h_a} = \frac{[PBC]}{\Delta}, \quad \frac{h_2}{h_b} = \frac{[PCA]}{\Delta}, \quad \frac{h_3}{h_c} = \frac{[PAB]}{\Delta}.$$

Adding these three relations,

$$\frac{h_1}{h_a} + \frac{h_2}{h_b} + \frac{h_3}{h_c} = 1.$$

Thus

$$\frac{l}{27} + \frac{m}{27} + \frac{n}{27} = \frac{h_1}{h_a} + \frac{h_2}{h_b} + \frac{h_3}{h_c} = 1.$$

We conclude that $l+m+n=27$. Thus every **good** point P determines a partition (l,m,n) of **27** such that there are l, m, n equal segments respectively on BC, CA, AB.

Conversely, take any partition (l,m,n) of **27**. Divide BC, CA, AB respectively in to l, m, n equal parts. Define

$$h_1 = \frac{2l\Delta}{27a}, \quad h_2 = \frac{2m\Delta}{27b}.$$

Draw a line parallel to BC at a distance h_1 from BC; draw another line parallel to CA at a distance h_2 from CA. Both lines are drawn such that they intersect at a point P inside the triangle ABC. Then

$$[PBC] = \frac{1}{2}ah_1 = \frac{l\Delta}{27}, \quad [PCA] = \frac{m\Delta}{27}.$$

Hence

$$[PAB] = \frac{n\Delta}{27}.$$

This shows that the distance of P from AB is

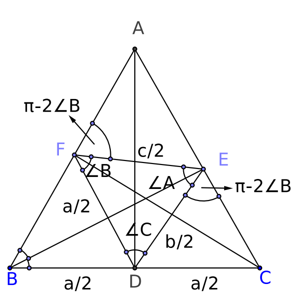
$$h_3 = \frac{2n\Delta}{27c}.$$

Therefore each traingle with base on CA has area $\frac{\Delta}{27}$. We conclude that all the triangles which partitions ABC have equal areas. Hence P is a **good** point.

Thus the number of **good** points is equal to the number of positive integral solutions of the equation $l+m+n=27$. This is equal to

$$\binom{26}{2} = 325.$$

5. Let ABC be an acute-angled triangle, and let D, E, F be points on BC, CA, AB respectively such that AD is the median, BE is the internal angle bisector and CF is the altitude. Suppose $\angle FDE = \angle C$, $\angle DEF = \angle A$ and $\angle EFD = \angle B$. Prove that ABC is equilateral.



Solution: Since ΔBFC is right-angled at F, we have $FD = BD = CD = a/2$. Hence $\angle BFD = \angle B$. Since $\angle EFD = \angle B$, we have $\angle AFE = \pi - 2\angle B$. Since $\angle DEF = \angle A$, we also get $\angle CED = \pi - 2\angle B$. Applying sine rule in ΔDEF, we have

$$\frac{DF}{\sin A} = \frac{FE}{\sin C} = \frac{DE}{\sin B}.$$

Thus we get $FE = c/2$ and $DE = b/2$. Sine rule in ΔCED gives

$$\frac{DE}{\sin C} = \frac{CD}{\sin(\pi - 2B)}.$$

Thus $(b/\sin C) = (a/2\sin B\cos B)$. Solving for $\cos B$, we have

$$\cos B = \frac{a\sin c}{2b\sin B} = \frac{ac}{2b^2}.$$

Similarly, sine rule in ΔAEF gives

$$\frac{EF}{\sin A} = \frac{AE}{\sin(\pi - 2B)}.$$

This gives (since $AE = bc/(a+c)$), as earlier,

$$\cos B = \frac{a}{a+c}.$$

Comparing the two values of $\cos B$, we get $2b^2 = c(a+c)$. We also have

$$c^2 + a^2 - b^2 = 2ca\cos B = \frac{2a^2c}{a+c}.$$

Thus

$$4a^2c = (a+c)(2c^2 + 2a^2 - 2b^2) = (a+c)(2c^2 + 2a^2 - c(a+c)).$$

This reduces to $2a^3 - 3a^2c + c^3 = 0$. Thus $(a-c)^2(2a+c) = 0$. We conclude that $a = c$. Finally

$$2b^2 = c(a+c) = 2c^2.$$

We thus get $b = c$ and hence $a = c = b$. This shows that ΔABC is equilateral.

6. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function satisfying $f(0) \neq 0$, $f(1) = 0$ and

   (i) $f(xy) + f(x)f(y) = f(x) + f(y)$;

   (ii) $\big(f(x-y) - f(0)\big)f(x)f(y) = 0$,

   for all $x, y \in \mathbb{Z}$, simultaneously.

   (a) Find the set of all possible values of the function f.

   (b) If $f(10) \neq 0$ and $f(2) = 0$, find the set of all integers n such that $f(n) \neq 0$.

Solution: Setting $y = 0$ in the condition (ii), we get

$$\big(f(x) - f(0)\big)f(x) = 0,$$

for all x (since $f(0) \neq 0$). Thus either $f(x) = 0$ or $f(x) = f(0)$, for all $x \in \mathbb{Z}$. Now taking $x = y = 0$ in (i), we see that $f(0) + f(0)^2 = 2f(0)$. This shows that $f(0) = 0$ or $f(0) = 1$. Since $f(0) \neq 0$, we must have $f(0) = 1$. We conclude that

$$\text{either } f(x) = 0 \text{ or } f(x) = 1 \text{ for each } x \in \mathbb{Z}.$$

This shows that the set of all possible value of $f(x)$ is $\{0, 1\}$. This completes (a).

Let $S = \{n \in \mathbb{Z} | f(n) \neq 0\}$. Hence we must have $S = \{n \in \mathbb{Z} | f(n) = 1\}$ by (a). Since $f(1) = 0$, 1 is not in S. And $f(0) = 1$ implies that $0 \in S$. Take any $x \in \mathbb{Z}$ and $y \in S$. Using (ii), we get

$$f(xy) + f(x) = f(x) + 1.$$

This shows that $xy \in S$. If $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ are such that $xy \in S$, then (ii) gives

$$1 + f(x)f(y) = f(x) + f(y).$$

Thus $\big(f(x) - 1\big)\big(f(y) - 1\big) = 0$. It follows that $f(x) = 1$ or $f(y) = 1$; i.e., either $x \in S$ or $y \in S$. We also observe from (ii) that $x \in S$ and $y \in S$ implies that $f(x - y) = 1$ so that $x - y \in S$. Thus S has the properties:

(A) $x \in \mathbb{Z}$ and $y \in S$ implies $xy \in S$;

(B) $x, y \in \mathbb{Z}$ and $xy \in S$ implies $x \in S$ or $y \in S$;

(C) $x, y \in S$ implies $x - y \in S$.

Now we know that $f(10) \neq 0$ and $f(2) = 0$. Hence $f(10) = 1$ and $10 \in S$; and $2 \notin S$. Writing $10 = 2 \times 5$ and using (B), we conclude that $5 \in S$ and $f(5) = 1$. Hence $f(5k) = 1$ for all $k \in \mathbb{Z}$ by (A).

Suppose $f(5k + l) = 1$ for some l, $1 \leq l \leq 4$. Then $5k + l \in S$. Choose $u \in \mathbb{Z}$ such that $lu \equiv 1 \pmod 5$. We have $(5k + l)u \in S$ by (A). Moreover, $lu = 1 + 5m$ for some $m \in \mathbb{Z}$ and

$$(5k + l)u = 5ku + lu = 5ku + 5m + 1 = 5(ku + m) + 1.$$

This shows that $5(ku+m)+1 \in S$. However, we know that $5(ku+m) \in S$. By (C), $1 \in S$ which is a contradiction. We conclude that $5k + l \notin S$ for any l, $1 \leq l \leq 4$. Thus

$$S = \{5k | k \in \mathbb{Z}\}.$$

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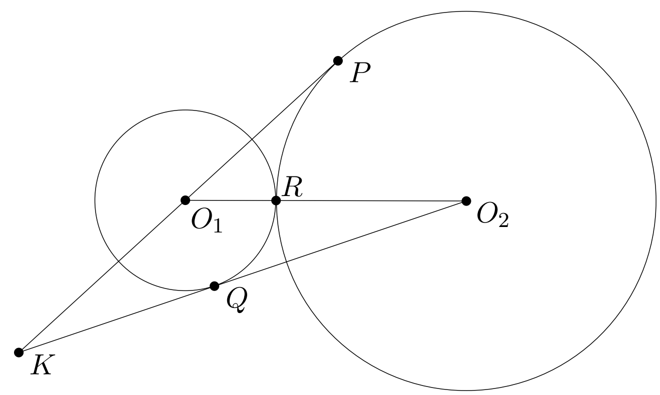
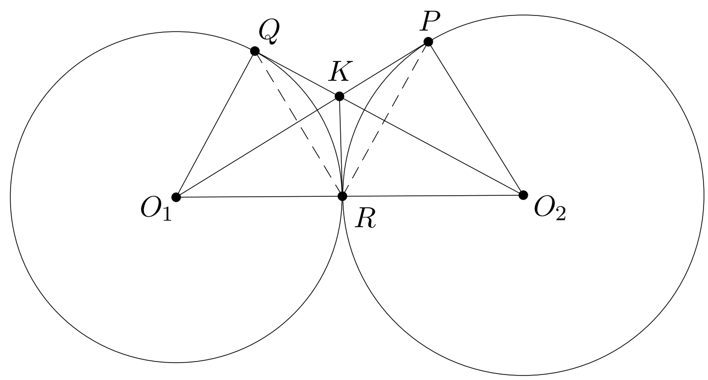
Problems and solutions: INMO 2013

Problem 1. *Let Γ_1 and Γ_2 be two circles touching each other externally at R. Let l_1 be a line which is tangent to Γ_2 at P and passing through the center O_1 of Γ_1. Similarly, let l_2 be a line which is tangent to Γ_2 at Q and passing through the center O_2 of Γ_2. Suppose l_1 and l_2 are not parallel and interesct at K. If $KP = KQ$, prove that the triangle PQR is equilateral.*

Solution. Suppose that P and Q lie on the opposite sides of line joining O_1 and O_2. By symmetry we may assume that the configuration is as shown in the figure below. Then we have $KP > KO_1 > KQ$ since KO_1 is the hypotenuse of triangle KQO_1. This is a contradiction to the given assumption, and therefore P and Q lie on the same side of the line joining O_1 and O_2.



Since $KP = KQ$ it follows that K lies on the radical axis of the given circles, which is the common tangent at R. Therefore $KP = KQ = KR$ and hence K is the cirumcenter of $\triangle PQR$.



On the other hand, $\triangle KQO_1$ and $\triangle KRO_1$ are both right-angled triangles with $KQ = KR$ and $QO_1 = RO_1$, and hence the two triangles are congruent. Therefore $\widehat{QKO_1} = \widehat{RKO_1}$, so KO_1, and hence PK is perpendicular to QR. Similarly, QK is perpendicular to PR, so it follows that K is the orthocenter of $\triangle PQR$. Hence we have that $\triangle PQR$ is equilateral.

□

Alternate solution. We again rule out the possibility that P and Q are on the opposite side of the line joining O_1O_2, and assume that they are on the same side.

Observe that $\triangle KPO_2$ is congruent to $\triangle KQO_1$ (since $KP = KQ$). Therefore $O_1P = O_2Q = r$ (say). In $\triangle O_1O_2Q$, we have $\widehat{O_1QO_2} = \pi/2$ and R is the midpoint of the hypotenuse, so $RQ = RO_1 = r$. Therefore $\triangle O_1RQ$ is equilateral, so $\widehat{QRO_1} = \pi/3$. Similarly, $PR = r$ and $\widehat{PRO_2} = \pi/3$, hence $\widehat{PRQ} = \pi/3$. Since $PR = QR$ it follows that $\triangle PQR$ is equilateral. □

Problem 2. *Find all positive integers m, n, and primes $p \geq 5$ such that*

$$m(4m^2 + m + 12) = 3(p^n - 1)\,.$$

Solution. Rewriting the given equation we have

$$4m^3 + m^2 + 12m + 3 = 3p^n\,.$$

The left hand side equals $(4m + 1)(m^2 + 3)$.

Suppose that $(4m+1, m^2+3) = 1$. Then $(4m+1, m^2+3) = (3p^n, 1), (3, p^n), (p^n, 3)$ or $(1, 3p^n)$, a contradiction since $4m + 1, m^2 + 3 \geq 4$. Therefore $(4m + 1, m^2 + 3) > 1$.

Since $4m + 1$ is odd we have $(4m + 1, m^2 + 3) = (4m + 1, 16m^2 + 48) = (4m + 1, 49) = 7$ or 49. This proves that $p = 7$, and $4m+1 = 3\cdot 7^k$ or 7^k for some natural number k. If $(4m+1, 49) = 7$ then we have $k = 1$ and $4m+1 = 21$ which does not lead to a solution. Therefore $(4m+1, m^2+3) = 49$. If 7^3 divides $4m + 1$ then it does not divide $m^2 + 3$, so we get $m^2 + 3 \leq 3 \cdot 7^2 < 7^3 \leq 4m + 1$. This implies $(m - 2)^2 < 2$, so $m \leq 3$, which does not lead to a solution. Therefore we have $4m + 1 = 49$ which implies $m = 12$ and $n = 4$. Thus $(m, n, p) = (12, 4, 7)$ is the only solution. □

Problem 3. *Let a, b, c, d be positive integers such that $a \geq b \geq c \geq d$. Prove that the equation $x^4 - ax^3 - bx^2 - cx - d = 0$ has no integer solution.*

Solution. Suppose that m is an integer root of $x^4 - ax^3 - bx^2 - cx - d = 0$. As $d \neq 0$, we have $m \neq 0$. Suppose now that $m > 0$. Then $m^4 - am^3 = bm^2 + cm + d > 0$ and hence $m > a \geq d$. On the other hand $d = m(m^3 - am^2 - bm - c)$ and hence m divides d, so $m \leq d$, a contradiction. If $m < 0$, then writing $n = -m > 0$ we have $n^4 + an^3 - bn^2 + cn - d = n^4 + n^2(an - b) + (cn - d) > 0$, a contradiction. This proves that the given polynomial has no integer roots. □

Problem 4. *Let n be a positive integer. Call a nonempty subset S of $\{1, 2, \ldots, n\}$* good *if the arithmetic mean of the elements of S is also an integer. Further let t_n denote the number of good subsets of $\{1, 2, \ldots, n\}$. Prove that t_n and n are both odd or both even.*

Solution. We show that $T_n - n$ is even. Note that the subsets $\{1\}, \{2\}, \cdots, \{n\}$ are good. Among the other good subsets, let A be the collection of subsets with an integer average which belongs to the subset, and let B be the collection of subsets with an integer average which is not a member of the subset. Then there is a bijection between A and B, because removing the average takes a member of A to a member of B; and including the average in a member of B takes it to its inverse. So $T_n - n = |A| + |B|$ is even. □

Alternate solution. Let $S = \{1, 2, \ldots, n\}$. For a subset A of S, let $\overline{A} = \{n + 1 - a | a \in A\}$. We call a subset A symmetric if $\overline{A} = A$. Note that the arithmetic mean of a symmetric subset is $(n+1)/2$. Therefore, if n is even, then there are no symmetric good subsets, while if n is odd then every symmetric subset is good.

If A is a proper good subset of S, then so is $\overline{A}$. Therefore, all the good subsets that are not symmetric can be paired. If n is even then this proves that t_n is even. If n is odd, we have to show that there are odd number of symmetric subsets. For this, we note that a symmetric subset contains the element $(n + 1)/2$ if and only if it has odd number of elements. Therefore, for any natural number k, the number of symmetric subsets of size $2k$ equals the number of symmetric subsets of size $2k + 1$. The result now follows since there is exactly one symmetric subset with only one element. □

Problem 5. *In an acute triangle ABC, O is the circumcenter, H is the orthocenter and G is the centroid. Let OD be perpendicular to BC and HE be perpendicular to CA, with D on BC and E on CA. Let F be the midpoint of AB. Suppose the areas of triangles ODC, HEA and GFB are equal. Find all the possible values of $\widehat{C}$.*

Solution. Let R be the circumradius of $\triangle ABC$ and Δ its area. We have $OD = R\cos A$ and $DC = \frac{a}{2}$, so

$$[ODC] = \frac{1}{2} \cdot OD \cdot DC = \frac{1}{2} \cdot R\cos A \cdot R\sin A = \frac{1}{2}R^2 \sin A \cos A\,. \qquad (1)$$

Again $HE = 2R\cos C \cos A$ and $EA = c\cos A$. Hence

$$[HEA] = \frac{1}{2} \cdot HE \cdot EA = \frac{1}{2} \cdot 2R\cos C\cos A \cdot c\cos A = 2R^2 \sin C \cos C \cos^2 A\,. \qquad (2)$$

Further

$$[GFB] = \frac{\Delta}{6} = \frac{1}{6} \cdot 2R^2 \sin A \sin B \sin C = \frac{1}{3}R^2 \sin A \sin B \sin C\,. \qquad (3)$$

Equating (1) and (2) we get $\tan A = 4 \sin C \cos C$. And equating (1) and (3), and using this relation we get

$$\begin{aligned} 3\cos A &= 2\sin B \sin C = 2\sin(C + A)\sin C \\ &= 2(\sin C + \cos C \tan A)\sin C \cos A \\ &= 2\sin^2 C(1 + 4\cos^2 C)\cos A\,. \end{aligned}$$

Since $\cos A \neq 0$ we get $3 = 2t(-4t + 5)$ where $t = \sin^2 C$. This implies $(4t - 3)(2t - 1) = 0$ and therefore, since $\sin C > 0$, we get $\sin C = \sqrt{3}/2$ or $\sin C = 1/\sqrt{2}$. Because $\triangle ABC$ is acute, it follows that $\widehat{C} = \pi/3$ or $\pi/4$.

We observe that the given conditions are satisfied in an equilateral triangle, so $\widehat{C} = \pi/3$ is a possibility. Also, the conditions are satisfied in a triangle where $\widehat{C} = \pi/4, \widehat{A} = \tan^{-1} 2$ and $\widehat{B} = \tan^{-1} 3$. Therefore $\widehat{C} = \pi/4$ is also a possibility.

Thus the two possible values of $\widehat{C}$ are $\pi/3$ and $\pi/4$. □

Problem 6. *Let a, b, c, x, y, z be positive real numbers such that $a+b+c = x+y+z$ and $abc = xyz$. Further, suppose that $a \leq x < y < z \leq c$ and $a < b < c$. Prove that $a = x, b = y$ and $c = z$.*

Solution. Let

$$f(t) = (t - x)(t - y)(t - z) - (t - a)(t - b)(t - c)\,.$$

Then $f(t) = kt$ for some constant k. Note that $ka = f(a) = (a - x)(a - y)(a - z) \leq 0$ and hence $k \leq 0$. Similarly, $kc = f(c) = (c - x)(c - y)(c - z) \geq 0$ and hence $k \geq 0$. Combining the two, it follows that $k = 0$ and that $f(a) = f(c) = 0$. These equalities imply that $a = x$ and $c = z$, and then it also follows that $b = y$. □

29^{th} Indian National Mathematical Olympiad-2014

February 02, 2014

1. In a triangle ABC, let D be a point on the segment BC such that $AB + BD = AC + CD$. Suppose that the points B, C and the centroids of triangles ABD and ACD lie on a circle. Prove that $AB = AC$.

   Solution. Let G_1, G_2 denote the centroids of triangles ABD and ACD. Then G_1, G_2 lie on the line parallel to BC that passes through the centriod of triangle ABC. Therefore BG_1G_2C is an isosceles trapezoid. Therefore it follows that $BG_1 = CG_2$. This proves that $AB^2+BD^2 = AC^2+CD^2$. Hence it follows that $AB\cdot BD = AC\cdot CD$. Therefore the sets $\{AB, BD\}$ and $\{AC, CD\}$ are the same (since they are both equal to the set of roots of the same polynomial). Note that if $AB = CD$ then $AC = BD$ and then $AB + AC = BC$, a contradiction. Therefore it follows that $AB = AC$.

2. Let n be a natural number. Prove that

$$\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots \left[\frac{n}{n}\right] + \left[\sqrt{n}\right]$$

   is **even**. (Here $[x]$ denotes the largest integer smaller than or equal to x.)

   Solution. Let $f(n)$ denote the given equation. Then $f(1) = 2$ which is even. Now suppose that $f(n)$ is even for some $n \geq 1$. Then

$$\begin{aligned} f(n+1) &= \left[\frac{n+1}{1}\right] + \left[\frac{n+1}{2}\right] + \left[\frac{n+1}{3}\right] + \cdots \left[\frac{n+1}{n+1}\right] + \left[\sqrt{n+1}\right] \\ &= \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots \left[\frac{n}{n}\right] + \left[\sqrt{n+1}\right] + \sigma(n+1)\,, \end{aligned}$$

   where $\sigma(n+1)$ denotes the number of positive divisors of $n+1$. This follows from $\left[\frac{n+1}{k}\right] = \left[\frac{n}{k}\right] + 1$ if k divides $n+1$, and $\left[\frac{n+1}{k}\right] = \left[\frac{n}{k}\right]$ otherwise. Note that $\left[\sqrt{n+1}\right] = \left[\sqrt{n}\right]$ unless $n+1$ is a square, in which case $\left[\sqrt{n+1}\right] = \left[\sqrt{n}\right] + 1$. On the other hand $\sigma(n+1)$ is odd if and only if $n+1$ is a square. Therefore it follows that $f(n+1) = f(n) + 2l$ for some integer l. This proves that $f(n+1)$ is even.

   Thus it follows by induction that $f(n)$ is even for all natural number n.

3. Let a, b be natural numbers with $ab > 2$. Suppose that the sum of their greatest common divisor and least common multiple is divisible by $a+b$. Prove that the quotient is at most $(a+b)/4$. When is this quotient exactly equal to $(a+b)/4$?

   Solution. Let g and l denote the greatest common divisor and the least common multiple, respectively, of a and b. Then $gl = ab$. Therefore $g + l \leq ab + 1$. Suppose that $(g+l)/(a+b) > (a+b)/4$. Then we have $ab+1 > (a+b)^2/4$, so we get $(a-b)^2 < 4$. Assuming, $a \geq b$ we either have $a = b$ or $a = b+1$. In the former case, $g = l = a$ and the quotient is $(g+l)/(a+b) = 1 \leq (a+b)/4$. In the latter case, $g = 1$ and $l = b(b+1)$ so we get that $2b+1$ divides b^2+b+1. Therefore $2b+1$ divides $4(b^2+b+1) - (2b+1)^2 = 3$ which implies that $b = 1$ and $a = 2$, a contradiction to the given assumption that $ab > 2$. This shows that $(g+l)/(a+b) \leq (a+b)/4$. Note that for the equality to hold, we need that either $a = b = 2$ or, $(a-b)^2 = 4$ and $g = 1, l = ab$. The latter case happens if and only if a and b are two consecutive odd numbers. (If $a = 2k+1$ and $b = 2k-1$ then $a+b = 4k$ divides $ab+1 = 4k^2$ and the quotient is precisely $(a+b)/4$.)

4. Written on a blackboard is the polynomial $x^2 + x + 2014$. Calvin and Hobbes take turns alternatively (starting with Calvin) in the following game. During his turn, Calvin should either increase or decrease the coefficient of x by 1. And during his turn, Hobbes should either increase or decrease the constant coefficient by 1. Calvin wins if at any point of time the polynomial on the blackboard at that instant has integer roots. Prove that Calvin has a winning strategy.

   Solution. For $i \geq 0$, let $f_i(x)$ denote the polynomial on the blackboard after Hobbes' i-th turn. We let Calvin decrease the coefficient of x by 1. Therefore $f_{i+1}(2) = f_i(2) - 1$ or $f_{i+1}(2) = f_i(2) - 3$ (depending on whether Hobbes increases or decreases the constant term). So for some i, we have $0 \leq f_i(2) \leq 2$. If $f_i(2) = 0$ then Calvin has won the game. If $f_i(2) = 2$ then Calvin wins the game by reducing the coefficient of x by 1. If $f_i(2) = 1$ then $f_{i+1}(2) = 0$ or $f_{i+1}(2) = -2$. In the former case, Calvin has won the game and in the latter case Calvin wins the game by increasing the coefficient of x by 1.

5. In an acute-angled triangle ABC, a point D lies on the segment BC. Let O_1, O_2 denote the circumcentres of triangles ABD and ACD, respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC.

   Solution. Without loss of generality assume that $\angle ADC \geq 90°$. Let O denote the circumcenter of triangle ABC and K the orthocentre of triangle O_1O_2D. We shall first show that the points O and K lie on the circumcircle of triangle AO_1O_2. Note that circumcircles of triangles ABD and ACD pass through the points A and D, so AD is perpendicular to O_1O_2 and, triangle AO_1O_2 is congruent to triangle DO_1O_2. In particular, $\angle AO_1O_2 = \angle O_2O_1D = \angle B$ since O_2O_1 is the perpendicular bisector of AD. On the other hand since OO_2 is the perpendicular bisector of AC it follows that $\angle AOO_2 = \angle B$. This shows that O lies on the circumcircle of triangle AO_1O_2. Note also that, since AD is perpendicular to O_1O_2, we have $\angle O_2KA = 90° - \angle O_1O_2K = \angle O_2O_1D = \angle B$. This proves that K also lies on the circumcircle of triangle AO_1O_2.

   Therefore $\angle AKO = 180° - \angle AO_2O = \angle ADC$ and hence OK is parallel to BC.

   Remark. The result is true even for an obtuse-angled triangle.

6. Let n be a natural number and $X = \{1, 2, \ldots, n\}$. For subsets A and B of X we define $A\Delta B$ to be the set of all those elements of X which belong to exactly one of A and B. Let $\mathcal{F}$ be a collection of subsets of X such that for any two distinct elements A and B in $\mathcal{F}$ the set $A\Delta B$ has at least two elements. Show that $\mathcal{F}$ has at most 2^{n-1} elements. Find all such collections $\mathcal{F}$ with 2^{n-1} elements.

   Solution. For each subset A of $\{1, 2, \ldots, n-1\}$, we pair it with $A \cup \{n\}$. Note that for any such pair (A, B) not both A and B can be in $\mathcal{F}$. Since there are 2^{n-1} such pairs it follows that $\mathcal{F}$ can have at most 2^{n-1} elements.

   We shall show by induction on n that if $|\mathcal{F}| = 2^{n-1}$ then $\mathcal{F}$ contains either all the subsets with odd number of elements or all the subsets with even number of elements. The result is easy to see for $n = 1$. Suppose that the result is true for $n = m - 1$. We now consider the case $n = m$. Let $\mathcal{F}_1$ be the set of those elements in $\mathcal{F}$ which contain m and $\mathcal{F}_2$ be the set of those elements which do not contain m. By induction, $\mathcal{F}_2$ can have at most 2^{m-2} elements. Further, for each element A of $\mathcal{F}_1$ we consider $A \setminus \{m\}$. This new collection also satisfies the required property, so it follows that $\mathcal{F}_1$ has at most 2^{m-2} elements. Thus, if $|\mathcal{F}| = 2^{m-1}$ then it follows that $|\mathcal{F}_1| = |\mathcal{F}_2| = 2^{m-2}$. Further, by induction hypothesis, $\mathcal{F}_2$ contains all those subsets of $\{1, 2, \ldots, m-1\}$ with (say) even number of elements. It then follows that $\mathcal{F}_1$ contains all those subsets of $\{1, 2, \ldots, m\}$ which contain the element m and which contains an even number of elements. This proves that $\mathcal{F}$ contains either all the subsets with odd number of elements or all the subsets by even number of elements.

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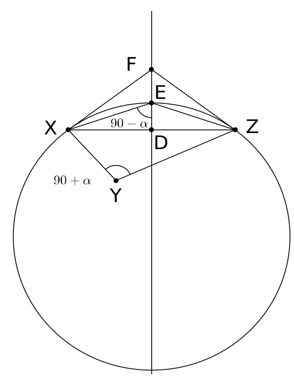
Problems and Solutions: INMO-2015

1. Let ABC be a right-angled triangle with $\angle B = 90°$. Let BD be the altitude from B on to AC. Let P, Q and I be the incentres of triangles ABD, CBD and ABC respectively. Show that the circumcentre of of the triangle PIQ lies on the hypotenuse AC.

Solution: We begin with the following lemma:

Lemma: Let XYZ be a triangle with $\angle XYZ = 90 + \alpha$. Construct an isosceles triangle XEZ, externally on the side XZ, with base angle α. Then E is the circumcentre of $\triangle XYZ$.

Proof of the Lemma: Draw $ED \perp XZ$. Then DE is the perpendicular bisector of XZ. We also observe that $\angle XED = \angle ZED = 90 - \alpha$. Observe that E is on the perpendicular bisector of XZ. Construct the circumcircle of XYZ. Draw perpendicular bisector of XY and let it meet DE in F. Then F is the circumcentre of $\triangle XYZ$. Join XF. Then $\angle XFD = 90-\alpha$. But we know that $\angle XED = 90 - \alpha$. Hence $E = F$.



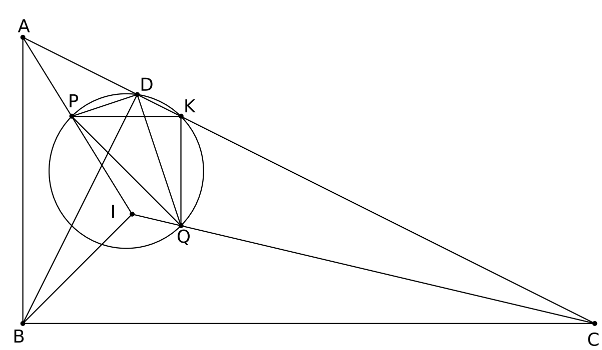
Let r_1, r_2 and r be the inradii of the triangles ABD, CBD and ABC respectively. Join PD and DQ. Observe that $\angle PDQ = 90°$. Hence

$$PQ^2 = PD^2 + DQ^2 = 2r_1^2 + 2r_2^2.$$

Let $s_1 = (AB + BD + DA)/2$. Observe that $BD = ca/b$ and $AD = \sqrt{AB^2 - BD^2} = \sqrt{c^2 - \left(\frac{ca}{b}\right)^2} = c^2/b$. This gives $s_1 = cs/b$. But $r_1 = s_1 - c = (c/b)(s-b) = cr/b$. Similarly, $r_2 = ar/b$. Hence

$$PQ^2 = 2r^2\left(\frac{c^2+a^2}{b^2}\right) = 2r^2.$$

Consider $\triangle PIQ$. Observe that $\angle PIQ = 90+(B/2) = 135$. Hence PQ subtends $90°$ on the circumference of the circumcircle of $\triangle PIQ$. But we have seen that $\angle PDQ = 90°$. Now construct a circle with PQ as diameter. Let it cut AC again in K. It follows that $\angle PKQ = 90°$ and the points P, D, K, Q are concyclic. We also notice $\angle KPQ = \angle KDQ = 45°$ and $\angle PQK = \angle PDA = 45°$.

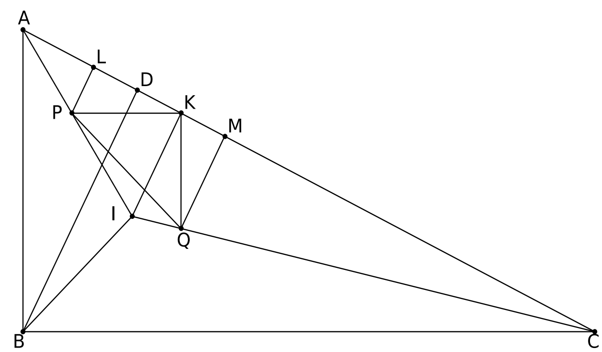


Thus PKQ is an isosceles right-angled triangle with $KP = KQ$. Therfore $KP^2 + KQ^2 = PQ^2 = 2r^2$ and hence $KP = KQ = r$.

Now $\angle PIQ = 90 + 45$ and $\angle PKQ = 2 \times 45^\circ = 90^\circ$ with $KP = KQ = r$. Hence K is the circumcentre of $\triangle PIQ$.

(Incidentally, This also shows that $KI = r$ and hence K is the point of contact of the incircle of $\triangle ABC$ with AC.)

Solution 2: Here we use computation to prove that the point of contact K of the incircle with AC is the circumcentre of $\triangle PIQ$. We show that $KP = KQ = r$. Let r_1 and r_2 be the inradii of triangles ABD and CBD respectively. Draw $PL \perp AC$ and $QM \perp AC$. If s_1 is the semiperimeter of $\triangle ABD$, then $AL = s_1 - BD$.



But

$$s_1 = \frac{AB + BD + DA}{2}, \quad BD = \frac{ca}{b}, \quad AD = \frac{c^2}{b}$$

Hence $s_1 = cs/b$. This gives $r_1 = s_1 - c = cr/b$, $AL = s_1 - BD = c(s-a)/b$. Hence $KL = AK - AL = (s-a) - \frac{c(s-a)}{b} = \frac{(b-c)(s-a)}{b}$. We observe that

$$2r^2 = \frac{(c+a-b)^2}{2} = \frac{c^2 + a^2 + b^2 - 2bc - 2ab + 2ca}{2} = (b^2 - ba - bc + ac) = (b-c)(b-a).$$

This gives

$$\begin{aligned}(s-a)(b-c) &= (s-b+b-a)(b-c) = r(b-c) + (b-a)(b-c)\\ &= r(b-c) + 2r^2 = r(b-c+c+a-b) = ra.\end{aligned}$$

Thus $KL = ra/b$. Finally,

$$KP^2 = KL^2 + LP^2 = \frac{r^2a^2}{b^2} + \frac{r^2 + c^2}{b^2} = r^2.$$

Thus $KP = r$. Similarly, $KQ = r$. This gives $KP = KI = KQ = r$ and therefore K is the circumcentre of $\triangle KIQ$.

(Incidentally, this also shows that $KL = ca/b = r_2$ and $KM = r_1$.)

2. For any natural number $n > 1$, write the infinite decimal expansion of $1/n$ (for example, we write $1/2 = 0.4\bar{9}$ as its infinite decimal expansion, not 0.5). Determine the length of the non-periodic part of the (infinite) decimal expansion of $1/n$.

Solution: For any prime p, let $\nu_p(n)$ be the maximum power of p dividing n; ie $p^{\nu_p(n)}$ divides n but not higher power. Let r be the

length of the non-periodic part of the infinite decimal expansion of $1/n$.
Write

$$\frac{1}{n} = 0.a_1a_2\cdots a_r\overline{b_1b_2\cdots b_s}.$$

We show that $r = \max(\nu_2(n), \nu_5(n))$.
Let a and b be the numbers $a_1a_2\cdots a_r$ and $b = b_1b_2\cdots b_s$ respectively. (Here a_1 and b_1 can be both 0.) Then

$$\frac{1}{n} = \frac{1}{10^r}\left(a + \sum_{k\geq 1}\frac{b}{(10^s)^k}\right) = \frac{1}{10^r}\left(a + \frac{b}{10^s - 1}\right).$$

Thus we get $10^r(10^s - 1) = n\big((10^s - 1)a + b\big)$. It shows that $r \geq \max(\nu_2(n), \nu_5(n))$. Suppose $r > \max(\nu_2(n), \nu_5(n))$. Then 10 divides $b - a$. Hence the last digits of a and b are equal: $a_r = b_s$. This means

$$\frac{1}{n} = 0.a_1a_2\cdots a_{r-1}\overline{b_sb_1b_2\cdots b_{s-1}}.$$

This contradicts the definition of r. Therefore $r = \max(\nu_2(n), \nu_5(n))$.

3. Find all real functions f from $\mathbb{R} \to \mathbb{R}$ satisfying the relation

$$f(x^2 + yf(x)) = xf(x + y).$$

Solution: Put $x = 0$ and we get $f\big(yf(0)\big) = 0$. If $f(0) \neq 0$, then $yf(0)$ takes all real values when y varies over real line. We get $f(x) \equiv 0$. Suppose $f(0) = 0$. Taking $y = -x$, we get $f\big(x^2 - xf(x)\big) = 0$ for all real x.
Suppose there exists $x_0 \neq 0$ in $\mathbb{R}$ such that $f(x_0) = 0$. Putting $x = x_0$ in the given relation we get

$$f\big(x_0^2\big) = x_0f(x_0 + y),$$

for all $y \in \mathbb{R}$. Now the left side is a constant and hence it follows that f is a constant function. But the only constant function which satisfies the equation is identically zero function, which is already obtained. Hence we may consider the case where $f(x) \neq 0$ for all $x \neq 0$.
Since $f\big(x^2 - xf(x)\big) = 0$, we conclude that $x^2 - xf(x) = 0$ for all $x \neq 0$. This implies that $f(x) = x$ for all $x \neq 0$. Since $f(0) = 0$, we conclude that $f(x) = x$ for all $x \in R$.
Thus we have two functions: $f(x) \equiv 0$ and $f(x) = x$ for all $x \in \mathbb{R}$.

4. There are four basket-ball players A, B, C, D. Initially, the ball is with A. The ball is always passed from one person to a different person. In how many ways can the ball come back to A after **seven** passes? (For example $A \to C \to B \to D \to A \to B \to C \to A$ and

$A \to D \to A \to D \to C \to A \to B \to A$ are two ways in which the ball can come back to A after seven passes.)

Solution: Let x_n be the number of ways in which A can get back the ball after n passes. Let y_n be the number of ways in which the ball goes back to a fixed person other than A after n passes. Then

$$x_n = 3y_{n-1},$$

and

$$y_n = x_{n-1} + 2y_{n-1}.$$

We also have $x_1 = 0$, $x_2 = 3$, $y_1 = 1$ and $y_2 = 2$.

Eliminating y_n and y_{n-1}, we get $x_{n+1} = 3x_{n-1} + 2x_n$. Thus

$$\begin{aligned}
x_3 &= 3x_1 + 2x_2 = 2 \times 3 = 6; \\
x_4 &= 3x_2 + 2x_3 = (3 \times 3) + (2 \times 6) = 9 + 12 = 21; \\
x_5 &= 3x_3 + 2x_4 = (3 \times 6) + (2 \times 21) = 18 + 42 = 60; \\
x_6 &= 3x_4 + 2x_5 = (3 \times 21) + (2 \times 60) = 63 + 120 = 183; \\
x_7 &= 3x_5 + 2x_6 = (3 \times 60) + (2 \times 183) = 180 + 366 = 546.
\end{aligned}$$

Alternate solution: Since the ball goes back to one of the other 3 persons, we have

$$x_n + 3y_n = 3^n,$$

since there are 3^n ways of passing the ball in n passes. Using $x_n = 3y_{n-1}$, we obtain

$$x_{n-1} + x_n = 3^{n-1},$$

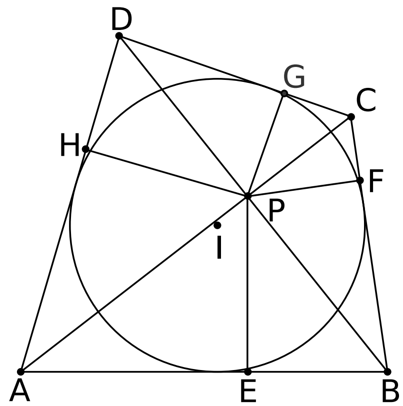
with $x_1 = 0$. Thus

$$\begin{aligned}
x_7 = 3^6 - x_6 &= 3^6 - 3^5 + x_5 = 3^6 - 3^5 + 3^4 - x_4 = 3^6 - 3^5 + 3^4 - 3^3 + x_3 \\
&= 3^6 - 3^5 + 3^4 - 3^3 + 3^2 - x_2 = 3^6 - 3^5 + 3^4 - 3^3 + 3^2 - 3 \\
&= (2 \times 3^5) + (2 \times 3^3) + (2 \times 3) = 486 + 54 + 6 = 546.
\end{aligned}$$

5. Let $ABCD$ be a convex quadrilateral. Let the diagonals AC and BD intersect in P. Let PE, PF, PG and PH be the altitudes from P on to the sides AB, BC, CD and DA respectively. Show that $ABCD$ has an incircle if and only if

$$\frac{1}{PE} + \frac{1}{PG} = \frac{1}{PF} + \frac{1}{PH}.$$

Solution: Let $AP = p$, $BP = q$, $CP = r$, $DP = s$; $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Let $\angle APB = \angle CPD = \theta$. Then $\angle BPC = \angle DPA = \pi - \theta$. Let us also write $PE = h_1$, $PF = h_2$, $PG = h_3$ and $PH = h_4$.



Observe that

$$h_1 a = pq\sin\theta, \quad h_2 b = qr\sin\theta, \quad h_3 c = rs\sin\theta, \quad h_4 d = sp\sin\theta.$$

Hence

$$\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}.$$

is equivalent to

$$\frac{a}{pq} + \frac{c}{rs} = \frac{b}{qr} + \frac{d}{sp}.$$

This is the same as

$$ars + cpq = bsp + dqr.$$

Thus we have to prove that $a+c = b+d$ if and only if $ars+cpq = bsp+dqr$. Now we can write $a + c = b + d$ as

$$a^2 + c^2 + 2ac = b^2 + d^2 + 2bd.$$

But we know that

$$a^2 = p^2 + q^2 - 2pq\cos\theta, \quad c^2 = r^2 + s^2 - 2rs\cos\theta$$
$$b^2 = q^2 + r^2 + 2qr\cos\theta, \quad d^2 = p^2 + s^2 + 2ps\cos\theta,$$

Hence $a + c = b + d$ is equivalent to

$$-pq\cos\theta + -rs\cos\theta + ac = ps\cos\theta + qr\cos\theta + bd.$$

Similarly, by squaring $ars + cpq = bsp + dqr$ we can show that it is equivalent to

$$-pq\cos\theta + -rs\cos\theta + ac = ps\cos\theta + qr\cos\theta + bd.$$

We conclude that $a + c = b + d$ is equivalent to $cpq + ars = bps + dqr$. Hence $ABCD$ has an in circle if and only if

$$\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}.$$

6. From a set of 11 square integers, show that one can choose 6 numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that

$$a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}.$$

Solution: The first observation is that we can find 5 pairs of squares such that the two numbers in a pair have the same parity. We can see this as follows:

Odd numbers	Even numbers	Odd pairs	Even pairs	Total pairs
0	11	0	5	5
1	10	0	5	5
2	9	1	4	5
3	8	1	4	5
4	7	2	3	5
5	6	2	3	5
6	5	3	2	5
7	4	3	2	5
8	3	4	1	5
9	2	4	1	5
10	1	5	0	5
11	0	5	0	5

Let us take such 5 pairs: say $(x_1^2, y_1^2), (x_2^2, y_2^2), \ldots, (x_5^2, y_5^2)$. Then $x_j^2 - y_j^2$ is divisible by 4 for $1 \leq j \leq 5$. Let r_j be the remainder when $x_j^2 - y_j^2$ is divisible by 3, $1 \leq j \leq 3$. We have 5 remainders r_1, r_2, r_3, r_4, r_5. But these can be 0, 1 or 2. Hence either one of the remainders occur 3 times or each of the remainders occur once. If, for example $r_1 = r_2 = r_3$, then 3 divides $r_1 + r_2 + r_3$; if $r_1 = 0, r_2 = 1$ and $r_3 = 2$, then again 3 divides $r_1 + r_2 + r_3$. Thus we can always find three remainders whose sum is divisible by 3. This means we can find 3 pairs, say, $(x_1^2, y_1^2), (x_2^2, y_2^2), (x_3^2, y_3^2)$ such that 3 divides $(x_1^2 - y_1^2) + (x_2^2 - y_2^2) + (x_3^2 - y_3^2)$. Since each difference is divisible by 4, we conclude that we can find 6 numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that

$$a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}.$$

31^{st} Indian National Mathematical Olympiad-2016

Time: 4 hours **January 17, 2016**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions. Maximum marks: 100.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let ABC be triangle in which $AB = AC$. Suppose the orthocentre of the triangle lies on the incircle. Find the ratio AB/BC.

2. For positive real numbers a, b, c, which of the following statements necessarily implies $a = b = c$: (I) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$, (II) $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$? Justify your answer.

3. Let $\mathbb{N}$ denote the set of all natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k+1) = 2k+2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

   (i) Show that for each $n \in \mathbb{N}$, there exists k such that $T^k(n) = 1$.

   (ii) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \geq 1$.

4. Suppose 2016 points of the circumference of a circle are coloured red and the remaining points are coloured blue. Given any natural number $n \geq 3$, prove that there is a regular n-sided polygon all of whose vertices are blue.

5. Let ABC be a right-angled triangle with $\angle B = 90°$. Let D be a point on AC such that the inradii of the triangles ABD and CBD are equal. If this common value is r' and if r is the inradius of triangle ABC, prove that

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

6. Consider a nonconstant arithmetic progression $a_1, a_2, \ldots, a_n, \ldots$. Suppose there exist relatively prime positive integers $p > 1$ and $q > 1$ such that a_1^2, a_{p+1}^2 and a_{q+1}^2 are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers.

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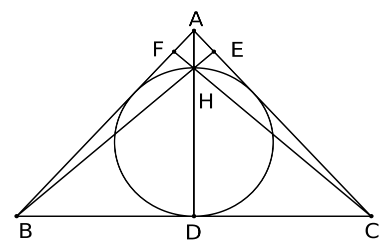
INMO-2016 problems and solutions

1. Let ABC be triangle in which $AB = AC$. Suppose the orthocentre of the triangle lies on the in-circle. Find the ratio AB/BC.

   Solution: Since the triangle is isosceles, the orthocentre lies on the perpendicular AD from A on to BC. Let it cut the in-circle at H. Now we are given that H is the orthocentre of the triangle. Let $AB = AC = b$ and $BC = 2a$. Then $BD = a$. Observe that $b > a$ since b is the hypotenuse and a is a leg of a right-angled triangle. Let BH meet AC in E and CH meet AB in F. By Pythagoras theorem applied to $\triangle BDH$, we get



$$BH^2 = HD^2 + BD^2 = 4r^2 + a^2,$$

where r is the in-radius of ABC. We want to compute BH in another way. Since A, F, H, E are con-cyclic, we have

$$BH \cdot BE = BF \cdot BA.$$

But $BF \cdot BA = BD \cdot BC = 2a^2$, since A, F, D, C are con-cyclic. Hence $BH^2 = 4a^4/BE^2$. But

$$BE^2 = 4a^2 - CE^2 = 4a^2 - BF^2 = 4a^2 - \left(\frac{2a^2}{b}\right)^2 = \frac{4a^2(b^2 - a^2)}{b^2}.$$

This leads to

$$BH^2 = \frac{a^2 b^2}{b^2 - a^2}.$$

Thus we get

$$\frac{a^2 b^2}{b^2 - a^2} = a^2 + 4r^2.$$

This simplifies to $(a^4/(b^2 - a^2)) = 4r^2$. Now we relate a, b, r in another way using area. We know that $[ABC] = rs$, where s is the semi-perimeter of ABC. We have $s = (b + b + 2a)/2 = b + a$. On the other hand area can be calculated using Heron's formula::

$$[ABC]^2 = s(s - 2a)(s - b)(s - b) = (b + a)(b - a)a^2 = a^2(b^2 - a^2).$$

Hence

$$r^2 = \frac{[ABC]^2}{s^2} = \frac{a^2(b^2 - a^2)}{(b + a)^2}.$$

Using this we get

$$\frac{a^4}{b^2 - a^2} = 4\left(\frac{a^2(b^2 - a^2)}{(b + a)^2}\right).$$

Therefore $a^2 = 4(b - a)^2$, which gives $a = 2(b - a)$ or $2b = 3a$. Finally,

$$\frac{AB}{BC} = \frac{b}{2a} = \frac{3}{4}.$$

Alternate Solution 1:

We use the known facts $BH = 2R\cos B$ and $r = 4R\sin(A/2)\sin(B/2)\sin(C/2)$, where R is the circumradius of $\triangle ABC$ and r its in-radius. Therefore

$$HD = BH\sin\angle HBD = 2R\cos B\sin\left(\frac{\pi}{2} - C\right) = 2R\cos^2 B,$$

since $\angle C = \angle B$. But $\angle B = (\pi - \angle A)/2$, since ABC is isosceles. Thus we obtain

$$HD = 2R\cos^2\left(\frac{\pi}{2} - \frac{A}{2}\right).$$

However HD is also the diameter of the in circle. Therefore $HD = 2r$. Thus we get

$$2R\cos^2\left(\frac{\pi}{2} - \frac{A}{2}\right) = 2r = 8R\sin(A/2)\sin^2((\pi - A)/4).$$

This reduces to

$$\sin(A/2) = 2\left(1 - \sin(A/2)\right).$$

Therefore $\sin(A/2) = 2/3$. We also observe that $\sin(A/2) = BD/AB$. Finally

$$\frac{AB}{BC} = \frac{AB}{2BD} = \frac{1}{2\sin(A/2)} = \frac{3}{4}.$$

Alternate Solution 2:

Let D be the mid-point of BC. Extend AD to meet the circumcircle in L. Then we know that $HD = DL$. But $HD = 2r$. Thus $DL = 2r$. Therefore $IL = ID + DL = r + 2r = 3r$. We also know that $LB = LI$. Therefore $LB = 3r$. This gives

$$\frac{BL}{LD} = \frac{3r}{2r} = \frac{3}{2}.$$

But $\triangle BLD$ is similar to $\triangle ABD$. So

$$\frac{AB}{BD} = \frac{BL}{LD} = \frac{3}{2}.$$

Finally,

$$\frac{AB}{BC} = \frac{AB}{2BD} = \frac{3}{4}.$$

Alternate Solution 3:

Let D be the mid-point of BC and E be the mid-point of DC. Since $DI = IH(= r)$ and $DE = EC$, the mid-point theorem implies that $IE \parallel CH$. But $CH \perp AB$. Therefore $EI \perp AB$. Let EI meet AB in F. Then F is the point of tangency of the incircle of $\triangle ABC$ with AB. Since the incircle is also tangent to BC at D, we have $BF = BD$. Observe that $\triangle BFE$ is similar to $\triangle BDA$. Hence

$$\frac{AB}{BD} = \frac{BE}{BF} = \frac{BE}{BD} = \frac{BD + DE}{BD} = 1 + \frac{DE}{BD} = \frac{3}{2}.$$

This gives

$$\frac{AB}{BC} = \frac{3}{4}.$$

2. For positive real numbers a, b, c, which of the following statements necessarily implies $a = b = c$: (I) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$, (II) $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$? Justify your answer.

Solution: We show that (I) need not imply that $a = b = c$ where as (II) always implies $a = b = c$.

Observe that $a(b^3 + c^3) = b(c^3 + a^3)$ gives $c^3(a - b) = ab(a^2 - b^2)$. This gives either $a = b$ or $ab(a + b) = c^3$. Similarly, $b = c$ or $bc(b + c) = a^3$. If $a \neq b$ and $b \neq c$, we obtain

$$ab(a + b) = c^3, \quad bc(b + c) = a^3.$$

Therefore

$$b(a^2 - c^2) + b^2(a - c) = c^3 - a^3.$$

This gives $(a - c)(a^2 + b^2 + c^2 + ab + bc + ca) = 0$. Since a, b, c are positive, the only possibility is $a = c$. We have therefore 4 possibilities: $a = b = c$; $a \neq b$, $b \neq c$ and $c = a$; $b \neq c$, $c \neq a$ and $a = b$; $c \neq a$, $a \neq b$ and $b = c$.

Suppose $a = b$ and $b, a \neq c$. Then $b(c^3 + a^3) = c(a^3 + b^3)$ gives $ac^3 + a^4 = 2ca^3$. This implies that $a(a - c)(a^2 - ac - c^2) = 0$. Therefore $a^2 - ac - c^2 = 0$. Putting $a/c = x$, we get the quadratic equation $x^2 - x - 1 = 0$. Hence $x = (1 + \sqrt{5})/2$. Thus we get

$$a = b = \left(\frac{1 + \sqrt{5}}{2}\right) c, \quad c \text{ arbitrary positive real number.}$$

Similarly, we get other two cases:

$$b = c = \left(\frac{1 + \sqrt{5}}{2}\right) a, \quad a \text{ arbitrary positive real number;}$$

$$c = a = \left(\frac{1 + \sqrt{5}}{2}\right) b, \quad b \text{ arbitrary positive real number.}$$

And $a = b = c$ is the fourth possibility.

Consider (II): $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$. Suppose a, b, c are mutually distinct. We may assume $a = \max\{a, b, c\}$. Hence $a > b$ and $a > c$. Using $a > b$, we get from the first relation that $a^3 + b^3 < b^3 + c^3$. Therefore $a^3 < c^3$ forcing $a < c$. This contradicts $a > c$. We conclude that a, b, c cannot be mutually distinct. This means some two must be equal. If $a = b$, the equality of the first two expressions give $a^3 + b^3 = b^3 + c^3$ so that $a = c$. Similarly, we can show that $b = c$ implies $b = a$ and $c = a$ gives $c = b$.

Alternate for (II) by a contestant: We can write

$$\begin{aligned}
\frac{a^3}{c} + \frac{b^3}{c} &= \frac{c^3}{a} + a^2, \\
\frac{b^3}{a} + \frac{c^3}{a} &= \frac{a^3}{b} + b^2, \\
\frac{c^3}{b} + \frac{a^3}{b} &= \frac{b^3}{c} + c^2.
\end{aligned}$$

Adding, we get

$$\frac{a^3}{c} + \frac{b^3}{a} + \frac{c^3}{b} = a^2 + b^2 + c^2.$$

Using C-S inequality, we have

$$\begin{aligned}(a^2+b^2+c^2)^2 &= \left(\frac{\sqrt{a^3}}{\sqrt{c}}\cdot\sqrt{ac}+\frac{\sqrt{b^3}}{\sqrt{a}}\cdot\sqrt{ba}+\frac{\sqrt{c^3}}{\sqrt{b}}\cdot\sqrt{cb}\right)^2\\ &\le \left(\frac{a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}\right)(ac+ba+cb)\\ &= (a^2+b^2+c^2)(ab+bc+ca).\end{aligned}$$

Thus we obtain

$$a^2+b^2+c^2 \le ab+bc+ca.$$

However this implies $(a-b)^2+(b-c)^2+(c-a)^2 \le 0$ and hence $a=b=c$.

3. Let $\mathbb{N}$ denote the set of all natural numbers. Define a function $T:\mathbb{N}\to\mathbb{N}$ by $T(2k)=k$ and $T(2k+1)=2k+2$. We write $T^2(n)=T(T(n))$ and in general $T^k(n)=T^{k-1}(T(n))$ for any $k>1$.

   (i) Show that for each $n\in\mathbb{N}$, there exists k such that $T^k(n)=1$.

   (ii) For $k\in\mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n)=1\}$. Prove that $c_{k+2}=c_{k+1}+c_k$, for $k\ge 1$.

   Solution:

   (i) For $n=1$, we have $T(1)=2$ and $T^2(1)=T(2)=1$. Hence we may assume that $n>1$.

   Suppose $n>1$ is even. Then $T(n)=n/2$. We observe that $(n/2)\le n-1$ for $n>1$.

   Suppose $n>1$ is odd so that $n\ge 3$. Then $T(n)=n+1$ and $T^2(n)=(n+1)/2$. Again we see that $(n+1)/2\le(n-1)$ for $n\ge 3$.

   Thus we see that in at most $2(n-1)$ steps T sends n to 1. Hence $k\le 2(n-1)$. (Here $2(n-1)$ is only a bound. In reality, less number of steps will do.)

   (ii) We show that $c_n=f_{n+1}$, where f_n is the n-th Fibonacci number.

   Let $n\in\mathbb{N}$ and let $k\in\mathbb{N}$ be such that $T^k(n)=1$. Here n can be odd or even. If n is even, it can be either of the form $4d+2$ or of the form $4d$.

   If n is odd, then $1=T^k(n)=T^{k-1}(n+1)$. (Observe that $k>1$; otherwise we get $n+1=1$ which is impossible since $n\in\mathbb{N}$.) Here $n+1$ is even.

   If $n=4d+2$, then again $1=T^k(4d+2)=T^{k-1}(2d+1)$. Here $2d+1=n/2$ is odd.

   Thus each solution of $T^{k-1}(m)=1$ produces exactly one solution of $T^k(n)=1$ and n is either odd or of the form $4d+2$.

   If $n=4d$, we see that $1=T^k(4d)=T^{k-1}(2d)=T^{k-2}(d)$. This shows that each solution of $T^{k-2}(m)=1$ produces exactly one solution of $T^k(n)=1$ of the form $4d$.

   Thus the number of solutions of $T^k(n)=1$ is equal to the number of solutions of $T^{k-1}(m)=1$ and the number of solutions of $T^{k-2}(l)=1$ for $k>2$. This shows that $c_k=c_{k-1}+c_{k-2}$ for $k>2$. We also observe that 2 is the only number which goes to 1 in one step and 4 is the only number which goes to 1 in two steps. Hence $c_1=1$ and $c_2=2$. This proves that $c_n=f_{n+1}$ for all $n\in\mathbb{N}$.

4. Suppose 2016 points of the circumference of a circle are coloured red and the remaining points are coloured blue. Given any natural number $n\ge 3$, prove that there is a regular n-sided polygon all of whose vertices are blue.

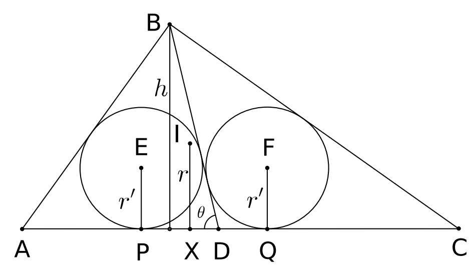
   Solution: Let $A_1, A_2, \ldots, A_{2016}$ be 2016 points on the circle which are coloured *red* and the remaining blue. Let $n \geq 3$ and let $B_1, B_2, \ldots, B_n$ be a regular n-sided polygon inscribed in this circle with the vertices chosen in anti-clock-wise direction. We place B_1 at A_1. (It is possible, in this position, some other B's also coincide with some other A's.) Rotate the polygon in anti-clock-wise direction gradually till some B's coincide with (an equal number of) A's second time. We again rotate the polygon in the same direction till some B's coincide with an equal number of A's third time, and so on until we return to the original position, i.e., B_1 at A_1. We see that the number of rotations will not be more than $2016 \times n$, that is, at most these many times some B's would have coincided with an equal number of A's. Since the interval $(0, 360°)$ has infinitely many points, we can find a value $\alpha^{\circ} \in (0, 360°)$ through which the polygon can be rotated from its initial position such that no B coincides with any A. This gives a n-sided regular polygon having only blue vertices.

Alternate Solution: Consider a regular $2017 \times n$-gon on the circle; say, $A_1A_2A_3 \cdots A_{2017n}$. For each j, $1 \leq j \leq 2017$, consider the points $\{A_k \ : \ k \equiv j \pmod{2017}\}$. These are the vertices of a regular n-gon, say S_j. We get 2017 regular n-gons; $S_1, S_2, \ldots, S_{2017}$. Since there are only 2016 red points, by pigeon-hole principle there must be some n-gon among these 2017 which does not contain any red point. But then it is a blue n-gon.

5. Let ABC be a right-angled triangle with $\angle B = 90°$. Let D be a point on AC such that the in-radii of the triangles ABD and CBD are equal. If this common value is r' and if r is the in-radius of triangle ABC, prove that

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

Solution: Let E and F be the incentres of triangles ABD and CBD respectively. Let the in-circles of triangles ABD and CBD touch AC in P and Q respectively. If $\angle BDA = \theta$, we see that



$$r' = PD\tan(\theta/2) = QD\cot(\theta/2).$$

Hence

$$PQ = PD + QD = r'\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right) = \frac{2r'}{\sin\theta}.$$

But we observe that

$$DP = \frac{BD + DA - AB}{2}, \quad DQ = \frac{BD + DC - BC}{2}.$$

Thus $PQ = (b - c - a + 2BD)/2$. We also have

$$\frac{ac}{2} = [ABC] = [ABD] + [CBD] = r'\frac{(AB + BD + DA)}{2} + r'\frac{(CB + BD + DC)}{2}$$
$$= r'\frac{(c + a + b + 2BD)}{2} = r'(s + BD).$$

But

$$r' = \frac{PQ\sin\theta}{2} = \frac{PQ \cdot h}{2BD},$$

where h is the altitude from B on to AC. But we know that $h = ac/b$. Thus we get

$$ac = 2 \times r'(s + BD) = 2 \times \frac{PQ \cdot h}{2 \times BD}(s + BD) = \frac{(b - c - a + 2BD)ca(s + BD)}{2 \times BD \times b}.$$

Thus we get

$$2 \times BD \times b = 2 \times \big(BD - (s-b)\big)(s+BD).$$

This gives $BD^2 = s(s-b)$. Since ABC is a right-angled triangle $r = s-b$. Thus we get $BD^2 = rs$. On the other hand, we also have $[ABC] = r'(s+BD)$. Thus we get

$$rs = [ABC] = r'(s+BD).$$

Hence

$$\frac{1}{r'} = \frac{1}{r} + \frac{BD}{rs} = \frac{1}{r} + \frac{1}{BD}.$$

Alternate Solution 1: Observe that

$$\frac{r'}{r} = \frac{AP}{AX} = \frac{CQ}{CX} = \frac{AP+CQ}{AC},$$

where X is the point at which the incircle of ABC touches the side AC. If s_1 and s_2 are respectively the semi-perimeters of triangles ABD and CBD, we know $AP = s_1 - BD$ and $CQ = s_2 - BD$. Therefore

$$\frac{r'}{r} = \frac{(s_1 - BD) + (s_2 - BD)}{AC} = \frac{s_1 + s_2 - 2BD}{b}.$$

But

$$s_1 + s_2 = \frac{AD + BD + c}{2} + \frac{CD + BD + a}{2} = \frac{(a+b+c) + 2BD}{2} = \frac{s+BD}{2}.$$

This gives

$$\frac{r'}{r} = \frac{s + BD - 2BD}{b} = \frac{s - BD}{b}.$$

We also have

$$r' = \frac{[ABD]}{s_1} = \frac{[CBD]}{s_2} = \frac{[ABD]+[CBD]}{s_1+s_2} = \frac{[ABC]}{s+BD} = \frac{rs}{s+BD}.$$

This implies that

$$\frac{r'}{r} = \frac{s}{s+BD}.$$

Comparing the two expressions for r'/r, we see that

$$\frac{s-BD}{b} = \frac{s}{s+BD}.$$

Therefore $s^2 - BD^2 = bs$, or $BD^2 = s(s-b)$. Thus we get $BD = \sqrt{s(s-b)}$.

We know now that

$$\frac{r'}{r} = \frac{s}{s+BD} = \frac{s-BD}{b} = \frac{BD}{(s-b)+BD} = \frac{\sqrt{s(s-b)}}{(s-b)+\sqrt{s(s-b)}} = \frac{\sqrt{s}}{\sqrt{s-b}+\sqrt{s}}.$$

Therefore

$$\frac{r}{r'} = 1 + \sqrt{\frac{s-b}{s}}.$$

This gives

$$\frac{1}{r'} = \frac{1}{r} + \left(\sqrt{\frac{s-b}{s}}\right)\frac{1}{r}.$$

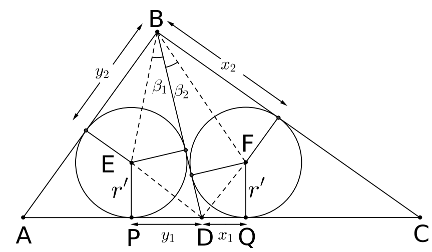
But

$$\left(\sqrt{\frac{s-b}{s}}\right)\frac{1}{r} = \left(\frac{s-b}{\sqrt{s(s-b)}}\right)\frac{1}{r} = \left(\frac{s-b}{BD}\right)\frac{1}{r}.$$

If $\angle B = 90°$, we know that $r = s-b$. Therfore we get

$$\frac{1}{r'} = \frac{1}{r} + \left(\frac{s-b}{BD}\right)\frac{1}{r} = \frac{1}{r} + \frac{1}{BD}.$$

Alternate Solution 2 by a contestant: Observe that $\angle EDF = 90°$. Hence $\triangle EDP$ is similar to $\triangle DFQ$. Therefore $DP \cdot DQ = EP \cdot FQ$. Taking $DP = y_1$ and $DQ = x_1$, we get $x_1 y_1 = (r')^2$. We also observe that $BD = x_1 + x_2 = y_1 + y_2$. Since $\angle EBF = 45°$, we get



$$1 = \tan 45° = \tan(\beta_1 + \beta_2) = \frac{\tan\beta_1 + \tan\beta_2}{1 - \tan\beta_1 \tan\beta_2}.$$

But $\tan\beta_1 = r'/y_2$ and $\tan\beta_2 = r'/x_2$. Hence we obtain

$$1 = \frac{(r'/y_2) + (r'/x_2)}{1 - (r')^2/x_2 y_2}.$$

Solving for r', we get

$$r' = \frac{x_2 y_2 - x_1 y_1}{x_2 + y_2}.$$

We also know

$$r = \frac{AB + BC - AC}{2} = \frac{x_2 + y_2 - (x_1 + y_1)}{2} = \frac{(x_2 - x_1) + (y_2 - y_1)}{2}.$$

Finally,

$$\begin{aligned} \frac{1}{r} + \frac{1}{BD} &= \frac{2}{(x_2 - x_1) + (y_2 - y_1)} + \frac{1}{x_1 + x_2} \\ &= \frac{2x_1 + 2x_2 + (x_2 - x_1) + (y_2 - y_1)}{(x_1 + x_2)((x_2 - x_1) + (y_2 - y_1))}. \end{aligned}$$

But we can write

$$2x_1 + 2x_2 + (x_2 - x_1) + (y_2 - y_1) = (x_1 + x_2 + x_2 - x_1) + (y_1 + y_2 + y_2 - y_1) = 2(x_2 + y_2),$$

and

$$\begin{aligned} (x_1 + x_2)((x_2 - x_1) + (y_2 - y_1)) &= 2(x_1 + x_2)(x_2 - y_1) \\ &= 2(x_2(x_2 + x_1 - y_1) - x_1 y_1) = 2(x_2 y_2 - x_1 y_1). \end{aligned}$$

Therefore

$$\frac{1}{r} + \frac{1}{BD} = \frac{2(x_2 + y_2)}{2(x_2 y_2 - x_1 y_1)} = \frac{1}{r'}.$$

Remark: One can also choose $B = (0,0)$, $A = (0,a)$ and $C = (1,0)$ and the coordinate geometry proof gets reduced considerbly.

6. Consider a non-constant arithmetic progression $a_1, a_2, \ldots, a_n, \ldots$. Suppose there exist relatively prime positive integers $p > 1$ and $q > 1$ such that a_1^2, a_{p+1}^2 and a_{q+1}^2 are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers.

Solution: Let us take $a_1 = a$. We have

$$a^2 = a + kd, \quad (a + pd)^2 = a + ld, \quad (a + qd)^2 = a + md.$$

Thus we have

$$a + ld = (a + pd)^2 = a^2 + 2pad + p^2d^2 = a + kd + 2pad + p^2d^2.$$

Since we have non-constant AP, we see that $d \neq 0$. Hence we obtain $2pa + p^2d = l - k$. Similarly, we get $2qa + q^2d = m - k$. Observe that $p^2q - pq^2 \neq 0$. Otherwise $p = q$ and $\gcd(p, q) = p > 1$ which is a contradiction to the given hypothesis that $\gcd(p, q) = 1$. Hence we can solve the two equations for a, d:

$$a = \frac{p^2(m-k) - q^2(l-k)}{2(p^2q - pq^2)}, \quad d = \frac{q(l-k) - p(m-k)}{p^2q - pq^2}.$$

It follows that a, d are rational numbers. We also have

$$p^2a^2 = p^2a + kp^2d.$$

But $p^2d = l - k - 2pa$. Thus we get

$$p^2a^2 = p^2a + k(l - k - 2pa) = (p - 2k)pa + k(l - k).$$

This shows that pa satisfies the equation

$$x^2 - (p - 2k)x - k(l - k) = 0.$$

Since a is rational, we see that pa is rational. Write $pa = w/z$, where w is an integer and z is a natural numbers such that $\gcd(w, z) = 1$. Substituting in the equation, we obtain

$$w^2 - (p - 2k)wz - k(l - k)z^2 = 0.$$

This shows z divides w. Since $\gcd(w, z) = 1$, it follows that $z = 1$ and $pa = w$ an integer. (In fact any rational solution of a monic polynomial with integer coefficients is necessarily an integer.) Similarly, we can prove that qa is an integer. Since $\gcd(p, q) = 1$, there are integers u and v such that $pu + qv = 1$. Therefore $a = (pa)u + (qa)v$. It follows that a is an integer.
But $p^2d = l - k - 2pa$. Hence p^2d is an integer. Similarly, q^2d is also an integer. Since $\gcd(p^2, q^2) = 1$, it follows that d is an integer. Combining these two, we see that all the terms of the AP are integers.

Alternatively, we can prove that a and d are integers in another way. We have seen that a and d are rationals; and we have three relations:

$$a^2 = a + kd, \quad p^2d + 2pa = n_1, \quad q^2d + 2qa = n_2,$$

where $n_1 = l - k$ and $n_2 = m - k$. Let $a = u/v$ and $d = x/y$ where u, x are integers and v, y are natural numbers, and $\gcd(u, v) = 1$, $\gcd(x, y) = 1$. Putting this in these relations, we obtain

$$u^2y = uvy + kxv^2, \quad (1)$$
$$2puy + p^2vx = vyn_1, \quad (2)$$
$$2quy + q^2vx = vyn_2. \quad (3)$$

Now (1) shows that $v|u^2y$. Since $\gcd(u,v) = 1$, it follows that $v|y$. Similarly (2) shows that $y|p^2vx$. Using $\gcd(y,x) = 1$, we get that $y|p^2v$. Similarly, (3) shows that $y|q^2v$. Therefore y divides $\gcd(p^2v, q^2v) = v$. The two results $v|y$ and $y|v$ imply $v = y$, since both v, y are positive.

Substitute this in (1) to get

$$u^2 = uv + kxv.$$

This shows that $v|u^2$. Since $\gcd(u,v) = 1$, it follows that $v = 1$. This gives $v = y = 1$. Finally $a = u$ and $d = x$ which are integers.

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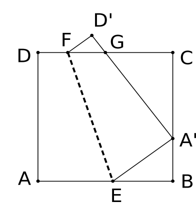
32^{nd} Indian National Mathematical Olympiad-2017

Time: 4 hours **January 15, 2017**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer all the questions.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. In the given figure, $ABCD$ is a square sheet of paper. It is folded along EF such that A goes to a point A' different from B and C, on the side BC and D goes to D'. The line $A'D'$ cuts CD in G. Show that the inradius of the triangle GCA' is the sum of the inradii of the triangles $GD'F$ and $A'BE$.



2. Suppose $n \geq 0$ is an integer and all the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$ are integers. Find all possible values of α.

3. Find the number of triples (x, a, b) where x is a real number and a, b belong to the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that
$$x^2 - a\{x\} + b = 0,$$
where $\{x\}$ denotes the fractional part of the real number x. (For example $\{1.1\} = 0.1 = \{-0.9\}$.)

4. Let $ABCDE$ be a convex pentagon in which $\angle A = \angle B = \angle C = \angle D = 120°$ and side lengths are five *consecutive integers* in some order. Find all possible values of $AB + BC + CD$.

5. Let ABC be a triangle with $\angle A = 90°$ and $AB < AC$. Let AD be the altitude from A on to BC. Let P, Q and I denote respectively the incentres of triangles ABD, ACD and ABC. Prove that AI is perpendicular to PQ and $AI = PQ$.

6. Let $n \geq 1$ be an integer and consider the sum
$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \binom{n}{4} 2^{n-4} \cdot 3^2 + \cdots \quad .$$
Show that $2x - 1, 2x, 2x + 1$ form the sides of a triangle whose area and inradius are also integers.

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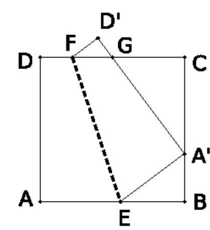
32वाँ भारतीय राष्ट्रीय गणित ओलिंपियाड – 2017

समय: 4 घंटा **जनवरी 15, 2017**

निर्देश :

- किसी भी तरह के गणक (calculators) तथा चांदा (protractors) के प्रयोग की अनुमति नहीं है.
- पैमाना (rulers) तथा परकार (compasses) के प्रयोग की अनुमति है.
- सभी प्रश्नों के अंक एकसमान हैं. अधिकतम अंक : 102.
- सभी प्रश्नों के उत्तर दीजिये.
- प्रत्येक प्रश्न का उत्तर नए पेज से प्रारंभ कीजिये. प्रश्न क्रमांक स्पष्ट रूप से इंगित कीजिये.

1. दिए हुए चित्र में $ABCD$ एक वर्गाकार कागज है. इसे EF के परित: इस प्रकार मोड़ा जाता है कि A, बिंदु B तथा C से भिन्न भुजा BC पर बिंदु A' पर आ जाता है तथा D, बिंदु D' पर जाता है. रेखा $A'D'$, CD को G पर काटती है. दिखाईये कि त्रिभुज GCA' की अंत:त्रिज्या त्रिभुजों $GD'F$ तथा $A'BE$ की अंत: त्रिज्याओं के योग के बराबर है.



2. मान लीजिये कि $n \geq 0$ एक पूर्णांक है तथा $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$ के सभी मूल पूर्णांक हैं. α के सभी संभावित मान ज्ञात कीजिये.
3. उन सभी त्रियुग्मों (x, a, b) की संख्याएं ज्ञात कीजिये जिसमें कि x एक वास्तविक संख्या है तथा a, b समुच्चय $\{1,2,3,4,5,6,7,8,9\}$ से इस प्रकार संबंधित है कि
$$x^2 - a\{x\} + b = 0$$
जहाँ $\{x\}$ वास्तविक संख्या x का भिन्नात्मक भाग है. (उदाहरण के लिए $\{1.1\} = 0.1 = \{-0.9\}$.)
4. मान लीजिये कि $ABCDE$ एक उत्तल पंचभुज (convex pentagon) है जिसमें $\angle A = \angle B = \angle C = \angle D = 120°$ है तथा भुजाओं की लम्बाई किसी क्रम में पांच *क्रमागत पूर्णांक* (consecutive integer) हैं. $AB + BC + CD$ के सभी संभव मान ज्ञात कीजिये.
5. मान लीजिये कि ABC एक त्रिभुज है जिसमें $\angle A = 90°$ तथा $AB < AC$. मान लीजिये कि AD, A से BC पर शीर्षलम्ब है. मान लीजिये कि P, Q तथा I क्रमश: त्रिभुज ABD, ACD तथा ABC के अंत:केंद्रों को निरुपित करते हैं. सिद्ध कीजिये कि AI, PQ के लम्बवत है तथा $AI = PQ$.
6. मान लीजिये कि $n \geq 1$ एक पूर्णांक है. निम्न योग पर विचार कीजिये
$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \binom{n}{4} 2^{n-4} \cdot 3^2 + \cdots .$$
दिखाइए कि $2x - 1,\ 2x,\ 2x + 1$ उस त्रिभुज की भुजाओं को बनाते हैं जिसका क्षेत्रफल तथा अंत:त्रिज्या भी एक पूर्णांक हैं.

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33^{rd} Indian National Mathematical Olympiad-2018

Problems and brief solutions

1. Let ABC be a non-equilateral triangle with integer sides. Let D and E be respectively the mid-points BC and CA; let G be the centroid of triangle ABC. Suppose D, C, E, G are concyclic. Find the least possible perimeter of triangle ABC.

   Solution: Let $m_b = BE$. Then $BG = 2m_b/3$. Since D, C, E, G are concyclic, we know that $BD \cdot BC = BG \cdot BE$. This along with Appolonius' theorem gives

   $$a^2 + b^2 = 2c^2$$

   Since a, b are integers, this implies that a, b must have same parity. This gives

   $$\left(\frac{a-b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 = c^2.$$

   Thus $\Big((a-b)/2, (a+b)/2, c\Big)$ is a Pythagorean triplet. Consider the first triplet $(3, 4, 5)$. This gives $a = 7, b = 1$ and $c = 5$. But a, b, c are not the sides of a triangle. The next triple is $(5, 12, 13)$. We obtain $a = 17, b = 7$ and $c = 13$. In this case we get a triangle and its perimeter is $17 + 7 + 13 = 37$. Since $2c < a + b + c < 3c$, it is sufficient to verify up to $c = 19$.

2. For any natural number n, consider a $1 \times n$ rectangular board made up of n unit squares. This is covered by three types of tiles: 1×1 red tile, 1×1 green tile and 1×2 blue domino. Let t_n denote the number of ways of covering $1 \times n$ rectangular board by these three types of tiles. Prove that t_n divides t_{2n+1}.

   Solution: Consider a $1 \times (2n + 1)$ board and imagine the board to be placed horizontally. Let us label the squares of the board as

   $$C_{-n}, C_{-(n-1)}, \ldots, C_{-2}, C_{-1}, C_0, C_1, C_2, \ldots, C_{n-1}, C_n$$

   from left to right. The 1×1 tiles will be referred to as tiles, and the blue 1×2 tile will be referred to as a domino.

   Let us consider the different ways in which the centre square C_0 can be covered. There are four distinct ways in which this can be done:

   (a) There is a blue domino covering the squares C_{-1}, C_0. In this case, there is a $1 \times (n - 1)$ board remaining on the left of this domino which can be covered in t_{n-1} ways, and there is a $1 \times n$ board remaining on the right of the domino which can be covered in t_n ways.

   (b) There is a blue domino covering the squares C_0, C_1. In this case, there is a $1 \times n$ board remaining on the left of this domino which can be covered in t_n ways, and there is a $1 \times (n - 1)$ board remaining on the right of the domino which can be covered in $t_(n - 1)$ ways.

(c) There is a red tile covering the square C_0. In this case, there is a $1 \times n$ board remaining on both sides of this tile, each of which can be covered in t_n ways.

(d) There is a green tile covering the square C_0. In this case, there is a $1 \times n$ board remaining on both sides of this tile, each of which can be covered in t_n ways.

Putting all the possibilities mentioned above together, we get that

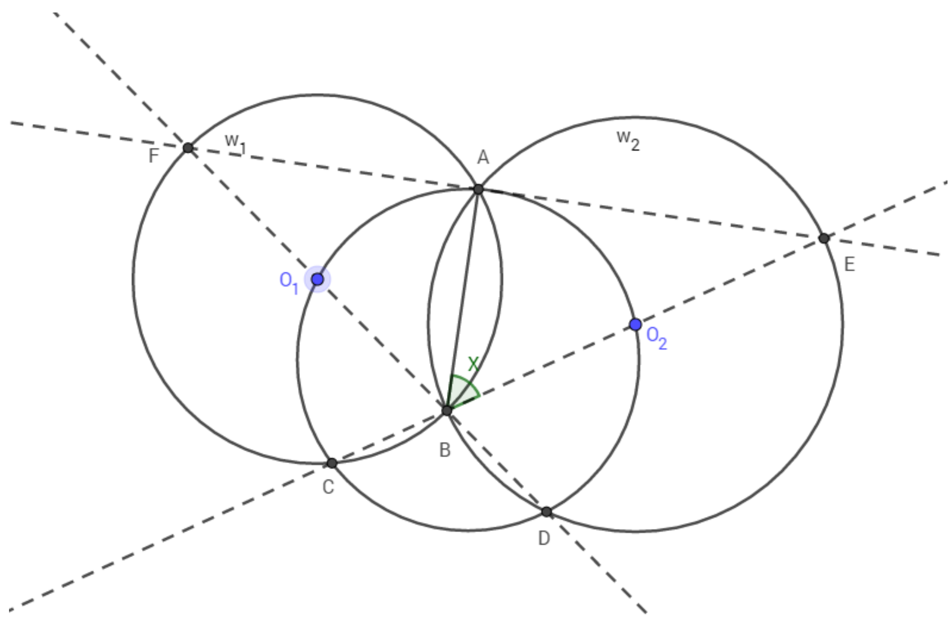
$$t_{2n+1} = 2t_{n-1}t_n + 2t_n^2 = t_n(2t_{n-1} + 2t_n)$$

which implies that t_n divides t_{2n+1}.

3. Let Γ_1 and Γ_2 be two circles with respective centres O_1 and O_2 intersecting in two distinct points A and B such that $\angle O_1AO_2$ is an obtuse angle. Let the circumcircle of triangle O_1AO_2 intersect Γ_1 and Γ_2 respectively in points C and D. Let the line CB intersect Γ_2 in E; let the line DB intersect Γ_1 in F. Prove that the points C, D, E, F are concyclic.

Solution: We will first prove that C, B, O_2, E are collinear; and this line is the bisector of $\angle ACD$:
Let $\angle ABO_2 = x$. Then by angle-chasing based on the given circles, we get

$$\angle AO_2B = (180 - 2x).$$



Hence $\angle AO_2O_1 = (90 - x)$. Since A, O_1, C, O_2 are concyclic, we obtain $\angle AO_2O_1 = \angle ACO_1 = (90 - x)$. Therefore $\angle AO_1C = 2x$. From this, we get $\angle AFC = x$ and $\angle ABC = 180 - x$. Thus, $\angle ABC$ and $\angle ABO_2$ are supplementary, implying C, B, O_2, E are collinear. Finally, we note that $O_2A = O_2D$ implies that O_2 is the midpoint of arc AO_2D; hence CO_2 is the bisector of $\angle ACD$, as required.

Similarly we obtain that D, B, O_1, F are collinear.
Hence, BE and BF are diameters of the respective circles. This shows that $\angle BAE = \angle BAF = 90°$; and hence F, A, E are collinear.
Finally, using all the above properties, we get:

$$\angle ECD = \angle BCD = \angle ACB = \angle AFB = \angle EFD.$$

Therefore C, D, E, F are concyclic, as required.

4. Find all polynomials with real coefficients $P(x)$ such that $P(x^2+x+1)$ divides $P(x^3-1)$.

Solution: We show that $P(x) = ax^n$ for some real number a and positive integer n. We prove that the only root of $P(x) = 0$ is 0. Suppose there is a root α_1 with $|\alpha_1| > 0$. Let β_1 and β_2 be the roots of $x^2 + x + 1 = \alpha_1$. Then $\beta_1 + \beta_2 = -1$. The given hypothesis shows that

$$P(\beta_1^3 - 1) = 0, \quad P(\beta_2^3 - 1) = 0.$$

We also see that

$$\beta_1^3 - 1 + \beta_2^3 - 1 = \alpha_1(\beta_1 + \beta_2 - 2).$$

Thus we have

$$|\beta_1^3 - 1| + |\beta_2^3 - 1| \geq |\beta_1^3 - 1 + \beta_2^3 - 1| = |\alpha_1||\beta_1 + \beta_2 - 2| = 3|\alpha_1|.$$

This shows that the absolute value of at least one of $\beta_1^3 - 1$ and $\beta_2^3 - 1$ is not less than $3|\alpha_1|/2$. If we take this as α_2, we have

$$|\alpha_2| > |\alpha_1|.$$

Now α_2 is a root of $P(x) = 0$ and we repeat the argument with α_2 in place of α_1. We get an infinite sequence of distinct roots of $P(x) = 0$. This contradiction proves $P(x) = ax^n$.

5. There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbors combined, the teacher takes away one apple from that girl and gives one apple each to her neighbors. Prove that this process stops after a finite number of steps. (Assume that the teacher has an abundant supply of apples.)

Solution: Let $a_1, a_2, \ldots, a_n$ denote the number of apples with these girls at any given time, all taken in a circular way. Consider two quantities associated with this distribution: $s = a_1 + a_2 + \cdots + a_n$ and $t = a_1^2 + a_2^2 + \cdots + a_n^2$. Using Cauchy-Schwarz inequality, we see that

$$nt = n(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 + a_2 + \cdots + a_n)^2 = s^2.$$

Therefore $t \geq s^2/n$ at any stage of the above process. Whenever teacher makes a move, s increases by 1. Suppose the girl with a_j apples has more than the sum of her neighbors. Then the change in t equals

$$(a_j-1)^2+(a_{j-1}+1)^2+(a_{j+1}+1)^2-a_j^2-a_{j-1}^2-a_{j+1}^2 = 2(a_{j+1}+a_{j-1}-a_j)+3 \leq 3+2(-1) = 1.$$

If s_1 and t_1 denote the corresponding sums after one move, we see that

$$s_1 = s + 1, \quad t_1 \leq t + 1.$$

Thus after teacher performs k moves, if the corresponding sums are t_k and s_k, we obtain

$$t + k \geq t_k \geq \frac{s_k^2}{n} = \frac{(s+k)^2}{n}.$$

This leads to a quadratic inequality in k:

$$k^2 + k(2s - n) + (s^2 - nt) \leq 0.$$

Since this cannot hold for large k, we see that the process must stop at some stage.

6. Let $\mathbb{N}$ denote the set of all natural numbers and let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

   (a) $f(mn) = f(m)f(n)$ for all m, n in $\mathbb{N}$;

   (b) $m + n$ divides $f(m) + f(n)$ for all m, n in $\mathbb{N}$.

   Prove that there exists an odd natural number k such that $f(n) = n^k$ for all n in $\mathbb{N}$.

   Solution: Taking $m = n = 1$ in (a), we get $f(1) = 1$. Observe $f(2n) = f(2)f(n)$. Hence $2n+1$ divides $f(2n)+f(1) = f(2)f(n)+1$. This shows that $\gcd(f(2), 2n+1) = 1$ for all n. This means $f(2) = 2^k$ for some natural number k. Since $3 = 1 + 2$ divides $f(1)+f(2) = 1+2^k$, k is odd. Now take any arbitray power of 2, say 2^m, and an arbitray integer n. By (b), 2^m+n divides $f(2^m)+f(n)$. But (a) gives $f(2^m) = \big(f(2)\big)^m = 2^{km}$. Thus $2^m + n$ divides $2^{km} + f(n)$. But

$$2^{km} + f(n) = \big(2^{km} + n^k\big) + \big(f(n) - n^k\big) = M\big(2^m + n\big) + \big(f(n) - n^k\big),$$

   since k is odd. It follows that $2^m + n$ divides $f(n) - n^k$. By Varying m over $\mathbb{N}$, we conclude that $f(n) - n^k = 0$. Therefore $f(n) = n^k$.

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34^{th} Indian National Mathematical Olympiad-2019

Problems and Solutions

1. Let ABC be a triangle with $\angle BAC > 90^{\circ}$. Let D be a point on the segment BC and E be a point on the line AD such that AB is tangent to the circumcircle of triangle ACD at A and BE is perpendicular to AD. Given that $CA = CD$ and $AE = CE$, determine $\angle BCA$ in degrees.

   Solution: Let $\angle C = 2\alpha$. Then $\angle CAD = \angle CDA = 90° - \alpha$. Moreover $\angle BAD = 2\alpha$ as BA is tangent to the circumcircle of $\triangle CAD$. Since $AE = AD$, it gives $\angle AEC = 2\alpha$. Thus $\triangle AEC$ is similar to $\triangle ACD$. Hence
$$\frac{AE}{AC} = \frac{AC}{AD}.$$
   But the condition that $BE \perp AD$ gives $AE = AB\cos 2\alpha = c\cos 2\alpha$. It is easy to see that $\angle B = 90° - 3\alpha$. Using sine rule in triangle ADC, we get
$$\frac{AD}{\sin 2\alpha} = \frac{AC}{\sin(90 - \alpha)}.$$
   This gives $AD = 2b\sin\alpha$. Thus we get
$$b^2 = AC^2 = AE \cdot AD = (c\cos 2\alpha)\cdot 2b\sin\alpha.$$
   Using $b = 2R\sin B$ and $c = 2R\sin C$, this leads to
$$\cos 3\alpha = 2\sin 2\alpha\cos 2\alpha\sin\alpha = \sin 4\alpha\sin\alpha.$$
   Writing $\cos 3\alpha = \cos(4\alpha - \alpha)$ and expanding, we get $\cos 4\alpha\cos\alpha = 0$. Therefore $\alpha = 90°$ or $4\alpha = 90°$. But $\alpha = 90°$ is not possible as $\angle C = 2\alpha$. Therefore $4\alpha = 90°$ which gives $\angle C = 2\alpha = 45°$.

2. Let $A_1B_1C_1D_1E_1$ be a regular pentagon. For $2 \le n \le 11$,
   let $A_nB_nC_nD_nE_n$ be the pentagon whose vertices are the midpoints of the sides of the pentagon $A_{n-1}B_{n-1}C_{n-1}D_{n-1}E_{n-1}$. All the 5 vertices of each of the 11 pentagons are arbitrarily coloured red or blue. Prove that four points among these 55 points have the same colour and form the vertices of a cyclic quadrilateral.

   Solution: We first observe that all the eleven pentagons are regular. Moreover, there are 5 fixed directions and all the 55 sides are in one of these directions. If we consider any two sides which are parallel, they are the parallel sides of an isosceles trapezium, which is cyclic.

   If we consider any pentagon, its two adjacent vertices have the same colour. Consider all such 11 sides whose end points are of the same colour. These are in 5 fixed directions. By pigeon-hole principle, there are 3 sides which are in the same directions and therefore parallel to each other. Among these three sides, two must have end points having one colour (again by P-H principle). Thus there are two parallel sides among the 55 and the end points of these have one fixed colour. But these two sides are parallel sides of an isosceles trapezium. Hence the four end points are concyclic.

3. Let m, n be distinct positive integers. Prove that
$$\gcd(m, n) + \gcd(m + 1, n + 1) + \gcd(m + 2, n + 2) \le 2|m - n| + 1.$$
   Further, determine when equality holds.

Solution: Observe that

$$\gcd(m+j, n+j) = \gcd(m+j, |m-n|),$$

for $j = 0, 1, 2$. Hence we can find positive integers a, b, c such that

$$\gcd(m,n) = \frac{|m-n|}{a}, \quad \gcd(m+1,n+1) = \frac{|m-n|}{b}, \quad \gcd(m+2,n+2) = \frac{|m-n|}{c}.$$

It follows that $|m-n|$ divides ma, $(m+1)b$ and $(m+2)c$. Hence we can see that $|m-n|$ divides ab and bc. We get $|m-n| \le ab$ and $|m-n| \le bc$. This leads to

$$b \ge \frac{|m-n|}{a}, \quad b \ge \frac{|m-n|}{c}.$$

Thus

$$\begin{aligned}\gcd(m,n) &+ \gcd(m+1,n+1) + \gcd(m+2,n+2)\\ &= \frac{|m-n|}{a} + \frac{|m-n|}{b} + \frac{|m-n|}{c} \le 2b + \frac{|m-n|}{b}.\end{aligned}$$

We have to prove that

$$2b + \frac{|m-n|}{b} \le 2|m-n| + 1.$$

Taking $|m-n| = K$, we have to show that $2b^2 + K \le b(2K+1)$. This reduces to $(b-K)(2b-1) \le 0$. However

$$K = |m-n| \ge b \ge 1 > \frac{1}{2}.$$

Equality holds only when $(m,n) = (k, k+1)$ or $(2k, 2k+2)$ or permutations of these for some k.

4. Let n and M be positive integers such that $M > n^{n-1}$. Prove that there are n distinct primes $p_1, p_2, p_3, \ldots, p_n$ such that p_j divides $M+j$ for $1 \le j \le n$.

Solution: If some number $M+k$, $1 \le k \le n$, has at least n distinct prime factors, then we can associate a prime factor of $M+k$ with the number $M+k$ which is not associated with any of the remaining $n-1$ numbers.

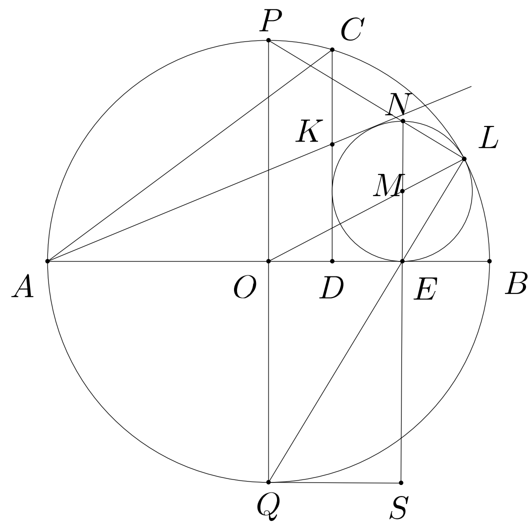
Suppose $m+j$ has less than n distinct prime factors. Write

$$M + j = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \quad r < n.$$

But $M + j > n^{n-1}$. Hence there exist t, $1 \le t \le r$ such that $p_t^{\alpha_t} > n$. Associate p_t with this $M+j$. Suppose p_t is associated with some $M+l$. Let $p_t^{\beta_t}$ be the largest power of p_t dividing $M+l$. Then $p_t^{\beta_t} > n$. Let $T = \gcd\left(p_t^{\alpha_t}, p_t^{\beta_t}\right)$. Then $T > n$. Since $T|(M+j)$ and $T|(M+l)$, it follows that $T|(|j-l|)$. But $|j-l| < n$ and $T > n$, and we get a contradiction. This shows that p_t cannot be associated with any other $M+l$. Thus each $M+j$ is associated with different primes.

5. Let AB be a diameter of a circle Γ and let C be a point on Γ different from A and B. Let D be the foot of perpendicular from C on to AB. Let K be a point of the segment CD such that AC is equal to the semiperimeter of the triangle ADK. Show that the excircle of triangle ADK opposite A is tangent to Γ.

Solution: Draw another diameter $PQ \perp AB$. Let E be the point at which the excircle Γ_1 touches the line AD. Join QE and extend it to meet Γ in L. Draw the diameter EN of Γ_1 and draw $QS \perp NE$ (extended). See the figure. We also observe that $DE = EM = EN/2$.



Since AE is equal to the semiperimeter of $\triangle ADK$, we have $AC = AE$. Hence $AE^2 = AC^2 = AD \cdot AB$ (as ACB is a right-angle triangle). Thus

$$AD(AD + DE + EB) = (AD + DE)^2 = AD^2 + 2AD \cdot DE + DE^2.$$

Simplification gives

$$\begin{aligned} AD \cdot EB &= AD \cdot DE + DE^2 \\ &= DE(AD + DE) \\ &= DE \cdot AE \\ &= DE(AB - BE). \end{aligned}$$

Therefore

$$DE \cdot AB = EB(AD + DE) = EB \cdot AE.$$

But

$$DE \cdot AB = DE \cdot PQ = 2DE \cdot OQ = EN \cdot ES,$$

and $EB \cdot AE = QE \cdot EL$. Therefore we get

$$QE \cdot EL = EN \cdot ES.$$

It follows that Q, S, L, N are concyclic. Since $\angle QSE = 90^\circ$, we get $\angle ELN = 90^\circ$. Since EN is a diameter, this implies that L also lies on Γ_1. But $\angle QLP = 90^\circ$. Therefore L, N, P are collinear. Since $NM \parallel PO$ and

$$\frac{NM}{PO} = \frac{NE}{PQ} = \frac{LN}{LP},$$

it follows that L, M, O are collinear. Hence Γ_1 is tangent to Γ at L.

Alternate solution: Let R be the radius the circle Γ and r be that of the circle Γ_1. Let O be the centre of Γ and M be that of the circle Γ_1. Let E be the point of contact of Γ_1 with AB. Then $ME = DE = r$. Observe that AE is the semiperimeter of $\triangle ADE$. We are given that $AC = AE$. Using that $\angle ACB = 90^\circ$, we also get $AC^2 = AD \cdot AB$. Hence $AE^2 = AD \cdot AB$. We have to show that $R - r = OM$ for proving that Γ_1 is tangent to Γ. We have

$$\begin{aligned} OM^2 - (R-r)^2 = OE^2 + r^2 - (R-r)^2 &= (AD + DE - AO)^2 + r^2 - (R-r)^2 \\ &= (AD - (R-r))^2 + r^2 - (R-r)^2 = AD^2 - 2AD \cdot (R-r) + r^2 \\ &= (AD^2 + 2AD \cdot r + r^2) - 2AD \cdot R = (AD + r)^2 - AD \cdot AB \\ &= (AD + DE)^2 - AD \cdot AB = AE^2 - AD \cdot AB = 0. \end{aligned}$$

Hence $OM = R - r$ and therefore Γ_1 is tangent to Γ.

6. Let f be function defined from the set $\{(x,y) : x, y \text{ reals}, xy \neq 0\}$ in to the set of all positive real numbers such that

$$\begin{array}{rl} (i) & f(xy,z) = f(x,z)f(y,z), \text{ for all } x, y \neq 0; \\ (ii) & f(x,yz) = f(x,y)f(x,z), \text{ for all } x, y \neq 0; \\ (iii) & f(x, 1-x) = 1, \text{ for all } x \neq 0, 1. \end{array}$$

Prove that

$$\begin{array}{rl} \text{(a)} & f(x,x) = f(x,-x) = 1, \text{ for all } x \neq 0; \\ \text{(b)} & f(x,y)f(y,x) = 1, \text{ for all } x, y \neq 0. \end{array}$$

Solution: (The condition (ii) was inadvertently left out in the paper. We give the solution with condition (ii).)
Taking $x = y = 1$ in (ii), weget $f(1,z)^2 = f(1,z)$ so that $f(1,z) = 1$ for all $z \neq 0$. Similarly, $x = y = -1$ gives $f(-1,z) = 1$ for all $z \neq 0$. Using the second condition, we also get $f(z,1) = f(z,-1) = 1$ for all $z \neq 0$. Observe

$$f\left(\frac{1}{x}, y\right) f(x,y) = f(1,y) = 1 = f(x,1) = f\left(x, \frac{1}{y}\right) f(x,y).$$

Therefore

$$f\left(x, \frac{1}{y}\right) = f\left(\frac{1}{x}, y\right) = \frac{1}{f(x,y)},$$

for all $x, y \neq 0$. Now for $x \neq 0, 1$, condition (iii) gives

$$1 = f\left(\frac{1}{x}, 1 - \frac{1}{x}\right) = f\left(x, \frac{1}{1 - \frac{1}{x}}\right).$$

Multiplying by $1 = f(x, 1-x)$, we get

$$1 = f(x, 1-x) f\left(x, \frac{1}{1 - \frac{1}{x}}\right) = f\left(x, \frac{1-x}{1 - \frac{1}{x}}\right) = f(x, -x),$$

for all $x \neq 0, 1$. But $f(x, -1) = 1$ for all $x \neq 0$ gives

$$f(x,x) = f(x,-x)f(x,-1) = f(x,-x) = 1$$

for all $x \neq 0, 1$. Observe $f(1,1) = f(1,-1) = 1$. Hence

$$f(x,x) = f(x,-x) = 1$$

for all $x \neq 0$, which proves (a).
We have

$$1 = f(xy, xy) = f(x, xy)f(y, xy) = f(x,x)f(x,y)f(y,x)f(y,y) = f(x,y)f(y,x),$$

for all $x, y \neq 0$, which proves (b).

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35^{th} Indian National Mathematical Olympiad-2020

Time: 4 hours **January 19, 2020**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer all the questions.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let Γ_1 and Γ_2 be two circles of unequal radii, with centres O_1 and O_2 respectively, in the plane intersecting in two distinct points A and B. Assume that the centre of each of the circles Γ_1 and Γ_2 is outside the other. The tangent to Γ_1 at B intersects Γ_2 again in C, different from B; the tangent to Γ_2 at B intersects Γ_1 again in D, different from B. The bisectors of $\angle DAB$ and $\angle CAB$ meet Γ_1 and Γ_2 again in X and Y, respectively, different from A. Let P and Q be the circumcentres of triangles ACD and XAY, respectively. Prove that PQ is the perpendicular bisector of the line segment O_1O_2.

2. Suppose $P(x)$ is a polynomial with real coefficients satisfying the condition

$$P(\cos\theta + \sin\theta) = P(\cos\theta - \sin\theta),$$

for every real θ. Prove that $P(x)$ can be expressed in the form

$$P(x) = a_0 + a_1(1-x^2)^2 + a_2(1-x^2)^4 + \cdots + a_n(1-x^2)^{2n},$$

for some real numbers $a_0, a_1, a_2, \ldots, a_n$ and nonnegative integer n.

3. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $S \subseteq X$ be such that any positive integer n can be written as $p + q$ where the non-negative integers p, q have all their digits in S. Find the smallest possible number of elements in S.

4. Let $n \geq 3$ be an integer and let $1 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ be n real numbers such that $a_1 + a_2 + a_3 + \cdots + a_n = 2n$. Prove that

$$a_1a_2\cdots a_{n-1} + a_1a_2\cdots a_{n-2} + \cdots + a_1a_2 + a_1 + 2 \leq a_1a_2\cdots a_n.$$

5. Infinitely many equidistant parallel lines are drawn in the plane. A positive integer $n \geq 3$ is called *frameable* if it is possible to draw a regular polygon with n sides all whose vertices lie on these lines and no line contains more than one vertex of the polygon.

   (a) Show that $3, 4, 6$ are *frameable.*

   (b) Show that any integer $n \geq 7$ is not frameable.

   (c) Determine whether 5 is *frameable.*

6. A *stromino* is a 3×1 rectangle. Show that a 5×5 board divided into twenty-five 1×1 squares cannot be covered by 16 *strominos* such that each *stromino* covers exactly three unit squares of the board and every unit square is covered by either one or two *strominos.* (A *stromino* can be placed either horizontally or vertically on the board.)

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INMO 2021

Official Solutions

Problem 1. Suppose $r \geq 2$ is an integer, and let $m_1, n_1, m_2, n_2, \cdots, m_r, n_r$ be $2r$ integers such that

$$|m_i n_j - m_j n_i| = 1$$

for any two integers i and j satisfying $1 \leq i < j \leq r$. Determine the maximum possible value of r.

Solution. Let m_1, n_1, m_2, n_2 be integers satisfying $m_1 n_2 - m_2 n_1 = \pm 1$. By changing the signs of m_2, n_2 if need be, we may assume that

$$m_1 n_2 - m_2 n_1 = 1.$$

If m_3, n_3 are integers satisfying $m_1 n_3 - m_3 n_1 = \pm 1$, again we may assume (by changing their signs if necessary) that

$$m_1 n_3 - m_3 n_1 = 1.$$

So $m_1(n_2 - n_3) = n_1(m_2 - m_3)$.

As m_1, n_1 are relatively prime, m_1 divides $m_2 - m_3$; say, $m_2 - m_3 = m_1 a$ for some integer a. Thus, we get $n_2 - n_3 = n_1 a$. In other words,

$$m_3 = m_2 - m_1 a \ , \ n_3 = n_2 - n_1 a.$$

Now, if $m_2 n_3 - n_2 m_3 = \pm 1$, we get

$$\pm 1 = m_2(n_2 - n_1 a) - n_2(m_2 - m_1 a) = (m_1 n_2 - m_2 n_1)a = a.$$

Thus, $m_3 = m_2 - m_1 a = m_2 \pm m_1, n_3 = n_2 - n_1 a = n_2 \pm n_1$.

Now if we were to have another pair of integers m_4, n_4 such that

$$m_1 n_4 - n_1 m_4 = \pm 1,$$

we may assume that $m_1 n_4 - n_1 m_4 = 1$. As seen above, $m_4 = m_2 \mp m_1, n_4 = n_2 \mp n_1$. But then

$$m_3 n_4 - n_3 m_4 = (m_2 \pm m_1)(n_2 \mp n_1) - (n_2 \pm n_1)(m_2 \mp m_1) = \pm 2.$$

Therefore, there can be only 3 pairs of such integers.

That there do exist many sets of 3 pairs is easy to see; for instance, $(1,0)$, $(1,1)$,$(0,1)$ is such a triple. □

Alternate Solution. It is clear that r can be 3 due to the valid solution $m_1 = 1, n_1 = 1, m_2 = 1, n_2 = 2, m_3 = 2, n_3 = 3$.

If possible, let $r > 3$. We observe that:

$$m_1 n_2 n_3 - m_2 n_1 n_3 = \pm n_3$$
$$m_2 n_3 n_1 - m_3 n_2 n_1 = \pm n_1$$
$$m_3 n_1 n_2 - m_1 n_3 n_2 = \pm n_2$$

Adding, we get $\pm n_1 \pm n_2 \pm n_3 = 0$; which forces at least one of n_1, n_2, n_3 to be even; WLOG let n_1 be even.

Repeating the argument for indices $2, 3, 4$, we deduce that at least one of n_2, n_3, n_4 is even; WLOG let n_2 be even. This leads to a contradiction, since $|m_1 n_2 - m_2 n_1| = 1$ cannot be even. Hence $r > 3$ is not possible, as claimed. □

Problem 2. Find all pairs of integers (a, b) so that each of the two cubic polynomials

$$x^3 + ax + b \text{ and } x^3 + bx + a$$

has all the roots to be integers.

Solution. The only such pair is $(0,0)$, which clearly works. To prove this is the only one, let us prove an auxiliary result first.

Lemma If α, β, γ are reals so that $\alpha+\beta+\gamma=0$ and $|\alpha|,|\beta|,|\gamma| \geq 2$, then

$$|\alpha\beta+\beta\gamma+\gamma\alpha| < |\alpha\beta\gamma|.$$

Proof. Some two of these reals have the same sign; WLOG, suppose $\alpha\beta > 0$. Then $\gamma = -(\alpha+\beta)$, so by substituting this,

$$|\alpha\beta+\beta\gamma+\gamma\alpha| = |\alpha^2+\beta^2+\alpha\beta|, \; |\alpha\beta\gamma| = |\alpha\beta(\alpha+\beta)|.$$

So we simply need to show $|\alpha\beta(\alpha+\beta)| > |\alpha^2+\beta^2+\alpha\beta|$. Since $|\alpha| \geq 2$ and $|\beta| \geq 2$, we have

$$\begin{aligned}|\alpha\beta(\alpha+\beta)| =& |\alpha||\beta(\alpha+\beta)| \geq 2|\beta(\alpha+\beta)|,\\ |\alpha\beta(\alpha+\beta)| =& |\beta||\alpha(\alpha+\beta)| \geq 2|\alpha(\alpha+\beta)|.\end{aligned}$$

Adding these and using triangle inequality,

$$\begin{aligned}2|\alpha\beta(\alpha+\beta)| &\geq 2|\beta(\alpha+\beta)| + 2|\alpha(\alpha+\beta)| \geq 2|\beta(\alpha+\beta)+\alpha(\alpha+\beta)|\\ &\geq 2(\alpha^2+\beta^2+2\alpha\beta) > 2(\alpha^2+\beta^2+\alpha\beta)\\ &= 2|\alpha^2+\beta^2+\alpha\beta|.\end{aligned}$$

Here we have used the fact that $\alpha^2+\beta^2+2\alpha\beta = (\alpha+\beta)^2$ and $\alpha^2+\beta^2+\alpha\beta = \left(\alpha+\frac{\beta}{2}\right)^2 + \frac{3\beta^2}{4}$ are both nonnegative. This proves our claim. □

For our main problem, suppose the roots of x^3+ax+b are the integers r_1, r_2, r_3 and the roots of x^3+bx+a are the integers s_1, s_2, s_3. By Vieta's relations, we have

$$\begin{aligned}r_1+r_2+r_3 = 0 = s_1+s_2+s_3\\ r_1r_2+r_2r_3+r_3r_1 = a = -s_1s_2s_3\\ s_1s_2+s_2s_3+s_3s_1 = b = -r_1r_2r_3\end{aligned}$$

If all six of these roots had an absolute value of at least 2, by our lemma, we would have

$$|b| = |s_1s_2+s_2s_3+s_3s_1| < |s_1s_2s_3| = |r_1r_2+r_2r_3+r_3r_1| < |r_1r_2r_3| = |b|,$$

which is absurd. Thus at least one of them is in the set $\{0, 1, -1\}$; WLOG, suppose it's r_1.

1. If $r_1 = 0$, then $r_2 = -r_3$, so $b = 0$. Then the roots of $x^3+bx+a = x^3+a$ are precisely the cube roots of $-a$, and these are all real only for $a = 0$. Thus $(a,b) = (0,0)$, which is a solution.
2. If $r_1 = \pm 1$, then $\pm 1 \pm a + b = 0$, so a and b can't both be even. If $a = -s_1s_2s_3$ is odd, then s_1, s_2, s_3 are all odd, so $s_1+s_2+s_3$ cannot be zero. Similarly, if b is odd, we get a contradiction.

The proof is now complete. □

Alternate Solution. The only such pair is $(0,0)$, which clearly works. Let us prove this is the only one. In what follows, we use $\nu_2(n)$ to denote the largest integer k so that $2^k | n$ for any non-zero $n \in \mathbb{Z}$.

If one of the cubics has 0 as a root, say the first one, then $0^3 + 0 \cdot a + b = 0$, so $b = 0$. Then the roots of $x^3+bx+a = x^3+a$ are precisely the cube roots of $-a$, and these are all real only for $a = 0$. Thus $(a,b) = (0,0)$.

So suppose none of the roots are zero. Take the cubic x^3+ax+b, and suppose its roots are x, y, z. We cannot have $\nu_2(x) = \nu_2(y) = \nu_2(z)$; indeed, if we had $x = 2^k x_1, y = 2^k y_1, z = 2^k z_1$ for odd x_1, y_1, z_1, then

$$0 = x+y+z = 2^k(x_1+y_1+z_1).$$

But $x_1+y_1+z_1$ is odd, and hence non-zero, so this cannot happen.

Thus we can assume WLOG that $\nu_2(x) > \nu_2(y)$. Then the third root is $-(x+y)$. Similarly, the three roots of x^3+bx+a can be written as $p, q, -(p+q)$ where $\nu_2(p) > \nu_2(q)$. By Vieta's relations,

$$\begin{aligned}xy - x(x+y) - y(x+y) = -(x^2+xy+y^2) = a = pq(p+q)\\ pq - p(p+q) - q(p+q) = -(p^2+pq+q^2) = b = xy(x+y)\end{aligned}$$

Suppose $x = 2^k x_1$ and $y = 2^\ell y_1$ for odd x_1, y_1 and $k > \ell$; in particular $k > 0$. Then

$$xy(x+y) = 2^k x_1 \cdot 2^\ell y_1 \cdot (2^k x_1 + 2^\ell y_1) = 2^{k+2\ell} x_1 y_1 (2^{k-\ell} x_1 + y_1).$$

Here $x_1 y_1 (2^{k-\ell} x_1 + y_1)$ is clearly odd, so $\nu_2(xy(x+y)) = k + 2\ell$.

Also,

$$x^2 + xy + y^2 = 2^{2k} x_1^2 + 2^k x_1 \cdot 2^\ell y_1 + 2^{2\ell} y_1^2 = 2^{2\ell}\left(2^{2k-2\ell} x_1^2 + 2^{k-\ell} x_1 y_1 + y_1^2\right).$$

Again, all the terms in the second factor are even except y_1^2, so the entire factor is odd. This means $\nu_2(x^2 + xy + y^2) = 2\ell$. Therefore

$$\nu_2(xy(x+y)) > \nu_2(x^2 + xy + y^2).$$

Similarly, one may show

$$\nu_2(pq(p+q)) > \nu_2(p^2 + pq + q^2).$$

But then

$$\nu_2(b) = \nu_2(xy(x+y)) > \nu_2(x^2 + xy + y^2) = \nu_2(pq(p+q)) > \nu_2(p^2 + pq + q^2) = \nu_2(b).$$

Here we have used the fact that $\nu_2(n) = \nu_2(-n)$ for any integer n. But this is a contradiction, proving our claim. □

Problem 3. Betal marks 2021 points on the plane such that no three are collinear, and draws all possible line segments joining these. He then chooses any 1011 of these line segments, and marks their midpoints. Finally, he chooses a line segment whose midpoint is not marked yet, and challenges Vikram to construct its midpoint using **only** a straightedge. Can Vikram always complete this challenge?

Note: A straightedge is an infinitely long ruler without markings, which can only be used to draw the line joining any two given distinct points.

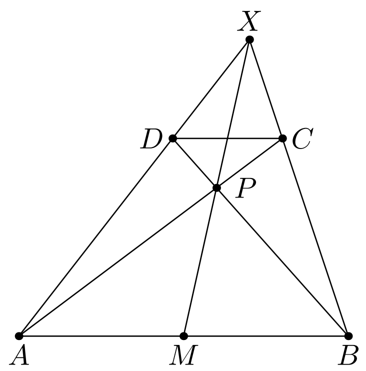
Solution. The answer is 'yes'. To prove this, we will first prove two lemmas:

Lemma 1 Given any two points A, B, their midpoint M, and any point C, Vikram can draw a line parallel to AB through C.

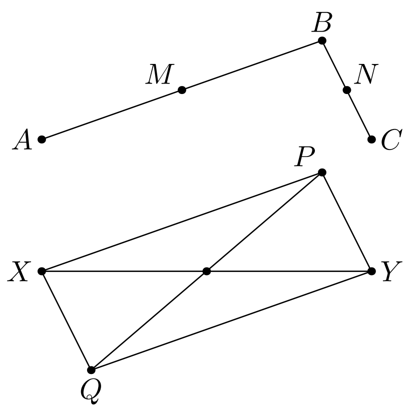
Proof. If C is on line AB we are already done. If not, extend BC to X as shown, draw $P = AC \cap XM$, and then draw $D = BP \cap AX$. We claim CD is the desired line. Indeed, using Ceva's theorem on triangle ABX and the fact $AM = MB$, we see that

$$\frac{AM}{MB} \cdot \frac{BC}{CX} \cdot \frac{XD}{DA} = 1 \implies \frac{XC}{CB} = \frac{XD}{DA}.$$

This means $CD \parallel AB$. □



Lemma 1



Lemma 2

Lemma 2 Given two non-parallel segments AB, BC and their midpoints M, N, Vikram can draw the midpoint of any other segment XY.

Proof. Assume first XY is not parallel to AB or BC. Using lemma 1, draw lines ℓ_1 and ℓ_2 through X parallel to AB and BC respectively, and similarly draw m_1 and m_2 through Y parallel to AB and BC respectively. If we draw $P = \ell_1 \cap m_2$ and $Q = \ell_2 \cap m_1$, then $XPYQ$ is a parallelogram, so intersecting PQ and XY gives the midpoint of XY.

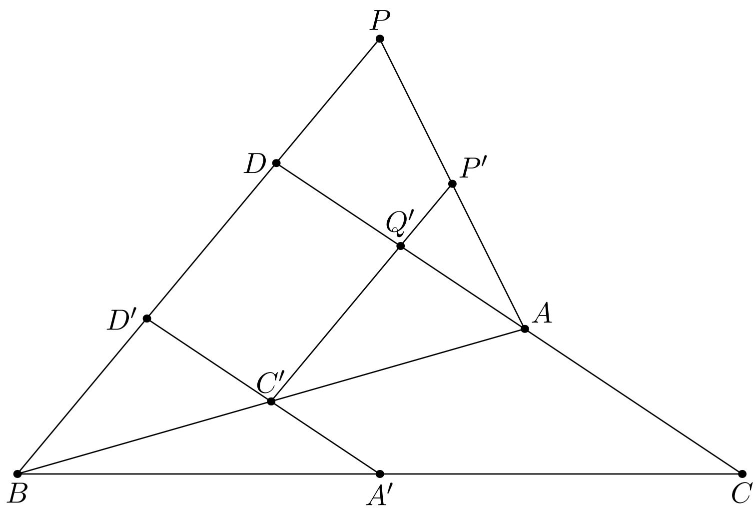
As for the remaining case, one can draw AC and construct the midpoint P of AC by the construction described above. Since XY can be parallel to at most one of the sides AB, BC and AC, we can pick the two non-parallel sides, and use the above construction to draw the midpoint of XY. □

Now for the main problem, note that if no two of the 1011 chosen segments share an endpoint, then we have at least $2 \cdot 1011 = 2022$ distinct endpoints, a contradiction. Thus there must be two segments AB and BC which have their midpoints marked. Since no three of the chosen 2021 points were collinear, AB and BC are not parallel, so using lemma 2, Vikram can construct the midpoint of any other segment, in particular, the segment chosen by Betal. □

Alternate Solution As in the previous solution, note that there exist AB and AC whose midpoints C' and B' are marked. Using the straightedge, Vikram can draw the two medians AC' and AB' and obtain their intersection, the centroid G of $\triangle ABC$. Now intersecting AG with BC gives A', the midpoint of BC.

Lemma Given a point P not on AB, AC, Vikram can draw the midpoint of PA.

Proof. If $PB || AC$ and $PC || AB$, then $PBAC$ is a parallelogram, in which case A' constructed above is the midpoint of PA. Without loss of generality, we may assume $PB \nparallel AC$.



Using the straightedge, one can mark the points $D = PB \cap AC$ and $PB \cap A'C' = D'$. Since $CA || A'C'$, we have

$$\frac{BD'}{D'D} = \frac{BC'}{C'A} = 1,$$

so D' is the midpoint of BD. Now in $\triangle ABD$, two midpoints C' and D' are known, so the midpoint of Q' of AD can be constructed using the centroid construction outlined before. Let $P' = C'Q' \cap PA$; this exists as $C'Q' || BP \nparallel AP$. As before, $C'P' || BP$, so

$$\frac{AP'}{P'P} = \frac{AC'}{C'B} = 1,$$

which means P' is the desired midpoint of PA. □

Now suppose we need to find the midpoint of PQ. If P, Q are different points from A, then one can draw the midpoints of AP and AQ using the lemma. Then by using the centroid of $\triangle APQ$, one can find the midpoint of PQ as we did for BC. If P or Q is A, the above lemma immediately yields the required midpoint. □

Problem 4 A Magician and a Detective play a game. The Magician lays down cards numbered from 1 to 52 face-down on a table. On each move, the Detective can point to two cards and inquire if the numbers on them are consecutive. The Magician replies truthfully. After a finite number of moves the Detective points to two cards. She wins if the numbers on these two cards are consecutive, and loses otherwise.

Prove that the Detective can guarantee a win if and only if she is allowed to ask at least 50 questions.

Solution. *Strategy for the Detective:* Pick a card A and compare against all others except one. If he ever gets a “Yes", that pair works; else the remaining card is consecutive with A. This process takes at most 50 queries.

Strategy for the Magician: We show that it is not always possible to obtain a “Yes" in 50 turns, hence showing that 49 turns are not enough to figure out a consecutive pair. It is enough to conjure a labelling of cards for which denying all 50 inquiries is valid.

Replace 52 by any $N > 3$. Think of the cards as vertices of a complete graph K_N. Delete all edges joining vertices which correspond to pairs of cards the Detective inquired about. We will show that deleting any $N-2$ edges of K_N still leaves a graph that admits a path containing all the vertices. Labelling all cards along this path as 1 to N would finish. Several proofs of this claim are possible. We present three of them.

Proof 1.

For any two vertices a and b, since $\deg a + \deg b \geq 2(N-1) - (N-2) = N$, they share a common neighbour. Hence the graph is connected.

Pick the longest path $\mathcal{P} : u = u_0 \to u_1 \to \cdots \to u_k = v$. All neighbours of u and v must remain within the path, else we could get a longer path. Let u have x neighbours $\{u_{i_1}, u_{i_2}, \ldots, u_{i_x}\}$ with $1 = i_1 < i_2 < \cdots < i_x \leq k$. Let v have y neighbours $\{u_{j_1}, \ldots, u_{j_y}\}$. Since $x + y \geq n$, we see that $i_s = j_r + 1$ for some r and s. Thus there exists i such that $u \to u_{i+1}$ and $u_i \to v$ are edges. Thus the path is a cycle

$$\mathcal{C} = u_{i+1} \to u_0 \to u_1 \cdots \to u_i \to v \to u_{k-1} \cdots \to u_{i+1}.$$

Suppose a vertex w is not in the path $\mathcal{P}$. By connectedness, we have a path $\mathcal{P}'$ from w to some vertex of $\mathcal{P}$. Continue along this path via the cycle $\mathcal{C}$ to obtain a path longer than $\mathcal{P}$; contradiction! Thus the graph has a path of length $N-1$, as desired. □

Proof 2.

Pick the longest cycle $\mathcal{C} = v_1 \to \cdots \to v_k \to v_1$. Note that any vertex w not in the cycle can be incident to no more than $\frac{k}{2}$ of the vertices in it; else there exists i such that wv_i and wv_{i+1} (indices mod k) are edges, so we can put w in to get a longer cycle. Thus our graph is missing at least $\frac{1}{2}k(N-k)$ edges. So $2(N-2) \geq k(N-k)$. Clearly $k > 2$ so we see that $k \in \{N-2, N-1\}$.

Case 1. $k = N-1$. Pick the leftover w outside $\mathcal{C}$. Not all edges from w to the cycle are missing (since only $N-2$ are missing in total), so follow an edge from w to $\mathcal{C}$ and continue along $\mathcal{C}$ to get a path of length $N-1$.

Case 2. $k = N-2$. Pick the leftover a, b outside $\mathcal{C}$. It is clear that both of them have edges to the cycle and ab is also an edge (since $k(N-k) = 2(N-2)$ in this case). So starting at a, going to b, to some vertex of $\mathcal{C}$ and following along $\mathcal{C}$ gives us a path of length $N-1$.

The proof is complete. □

Proof 3.

The idea is to prove the stronger claim by induction on $N \geq 3$: a graph on N vertices with $\binom{N-1}{2} + 2$ edges has a cycle of length N. Deleting the extra edge will give a path of length $N-1$ through all the vertices.

The base case $N = 3$ is trivial. Suppose it holds for all $k \leq N$, we prove it for $N+1$. Since $\frac{2\left(2+\binom{N}{2}\right)}{N+1} > N-2$ we see that some vertex v has degree either $N-1$ or N.

Case 1. If degree of v is $N-1$. Then we have an edge $e = uv$ missing among all the edges through v. Delete v along with all the edges through it in the graph. The induced graph has a cycle of length N. Pick two consecutive vertices that are not u, and append v between them.

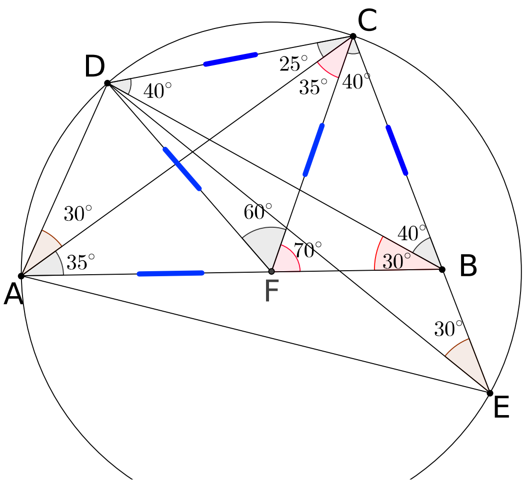
Case 2. If degree of v is N. Delete v along with all its edges. Add an arbitrarily chosen extra edge to the graph so obtained. By induction hypothesis, this resulting graph has a cycle of length N. If removing the extra edge does not disrupt the cycle, append v anywhere between two consecutive vertices. If it does break the cycle, use v to connect the vertices it joined.

The induction is complete. □

Problem 5 In a convex quadrilateral $ABCD$, $\angle ABD = 30°$, $\angle BCA = 75°$, $\angle ACD = 25°$ and $CD = CB$. Extend CB to meet the circumcircle of triangle DAC at E. Prove that $CE = BD$.

Solution. First we show that $\angle DEC = 30°$. Choose a point F on AB such that $CF = CB$. Join FC and FD. Observe that $\angle DCB = 75° + 25° = 100°$. Since $CD = CB$, we have $\angle CDB = \angle CBD = 40°$. Therefore $\angle CBF = 40° + 30° = 70°$. This gives $\angle CFB = 70°$.

Since $CD = CB = CF$, we have the isosceles triangle CDF. But $\angle BCF = 40°$. Hence $\angle FCD = 60°$. Therefore we have an equilateral triangle CDF. This means $FD = FC = CD$ and $\angle DFC = 60°$.

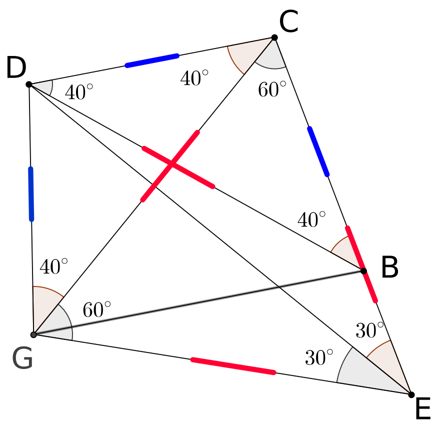


Observe that $\angle AFC = 110°$ and $\angle FCA = 35°$. Hence $\angle FAC = 35°$. This means $FA = FC = FD$. Thus F is the circumcentre of $\triangle ADC$. This implies that

$$\angle CAD = \frac{\angle CFD}{2} = 30°.$$

Therefore $\angle DEC = \angle DAC = 30°$. Now concentrate on triangle DCE.

Construct an equilateral triangle ECG with CE as base, on the side of B. Join GD.



We have $\angle CGE = \angle GCE = \angle CEG = 60°$ and $CE = EG = GC$. Since $\angle CED = 30°$, we get $\angle GED = 30°$. Thus ED is the angle bisector of the isosceles triangle GEC. This implies that ED is also the pependicular bisector of GC. Thus D is on the pependicular bisector of GC. Therefore $DC = DG$ and hence $\angle DGC = \angle DCG$.

But $\angle DCG = 100° - 60° = 40°$. This implies that $\angle DGC = 40°$ and hence $\angle CDG = 100°$.

Consider the quadrilateral $GBCD$. We have $DG = DC = CB$, $\angle GDC = 100° = \angle DCB$. It is an isosceles trapezium. (or we can show that $\triangle GDC \cong \triangle BCD$.) Therefore $DB = GC$. But $GC = CE$. Thus we get $DB = CE$. □

Alternate Solution As in the previous solution, one shows that F is the circumcenter of $\triangle ADC$. since E lies on this circumcircle, this means FE is equal to all of the sides FA, FD, FC and thus also to CD and CB. Now CDB and FCE are both isosceles triangles with base angles 40°, and they have $CD = FC$, so they are in fact congruent. This directly implies $CE = BD$, as required. □

Problem 6. Let $\mathbb{R}[x]$ be the set of all polynomials with real coefficients, and let $\deg P$ denote the degree of a nonzero polynomial P. Find all functions $f : \mathbb{R}[x] \to \mathbb{R}[x]$ satisfying the following conditions:

- f maps the zero polynomial to itself,
- for any non-zero polynomial $P \in \mathbb{R}[x]$, $\deg f(P) \le 1 + \deg P$, and
- for any two polynomials $P, Q \in \mathbb{R}[x]$, the polynomials $P - f(Q)$ and $Q - f(P)$ have the same set of real roots.

Solution.

Answer

We have $f(p) = p \forall p \in \mathbb{R}[x]$, or $f(p) = -p \forall p \in \mathbb{R}[x]$. These clearly satisfy the given conditions.

Proof

Claim 1 For all $p \in \mathbb{R}[x]$, $f(f(p)) = p$.

Proof. Using condition 3 on the polynomials p and $f(p)$, we see that $p - f(f(p))$ has the same set of real roots as $f(p) - f(p) = 0$, which is $\mathbb{R}$. Therefore $p - f(f(p))$ is identically zero. □

Note that this implies f is bijective. In what follows, $p \sim q$ will mean that p and q have the same set of real roots. Note that putting $f(q)$ for q in condition 2 gives $p - q \sim f(p) - f(q)$ for all p, q (call this statement $(\star)$). In particular, putting $q = 0$ here, $p \sim f(p)$ for all p (call this $(\star\star)$).

Claim 2 For all non-zero $p \in \mathbb{R}[x]$, $\deg p - 1 \le \deg f(p) \le \deg p + 1$.

Proof. The right inequality is simply condition 2. Now using condition 2 on the polynomial $f(p)$, we see that $\deg f(f(p)) \le \deg f(p) + 1$ which gives $\deg f(p) \ge \deg p - 1$ because of claim 1. □

Claim 3 For all $p \in \mathbb{R}[x]$, $\deg f(p) = \deg p$.

Proof. Note that nonzero constant polynomials have no root, so by $(\star\star)$, their image must have no root. This is impossible if that image has degree 1; so by condition 2, the image has degree 0, i.e., is a constant polynomial. First consider the case when $\deg p$ is even; assume for now the leading coefficient of p is positive. That means $p(x) \to \infty$ for $x \to \pm\infty$, so it has a global minimum, say C. Then the polynomial $p + k$ ($k > C$) has no real roots. Using $(\star)$ on p and the constant polynomial $-k$, we see that $f(p) - f(-k)$ has no roots. But this is impossible if $\deg f(p)$ is odd (since $f(-k)$ is a constant), so by claim 2, we must $\deg f(p) = \deg p$. A similar argument holds if p has negative leading coefficient.

Now if $\deg p$ is odd, then $\deg f(p)$ cannot be even, otherwise $q = f(p)$ would be an even degree polynomial whose image $f(q) = f(f(p)) = p$ has odd degree, contradicting the last paragraph. Thus $\deg f(p)$ is odd, and using claim 2, we infer than $\deg f(p) = \deg p$. □

We call a polynomial p *ninth-grade* if all $\deg p$ roots of p are real and distinct. Clearly for any ninth-grade p, p and $f(p)$ have the roots and same degree, so $f(p) = c_p p$ for some non-zero $c_p \in \mathbb{R}$.

Claim 4 Given any non-constant $q \in \mathbb{R}[x]$, we can choose r with degree bigger than q so that both r and $q - r$ are ninth-grade.

Proof. Assume that all real roots of q are inside the interval $[a, b]$. Now choose a number n which has the same parity as $\deg q$ and is bigger than $\deg q$, and choose numbers $c_1 = a < c_2 < \cdots < c_{n-1} < c_n = b$. Consider the polynomial $p = k(x - c_1)(x - c_2) \cdots (x - c_n)$, so that k has the same sign as the leading coefficient of q (value of k will be chosen later). Clearly p has alternating signs on the intervals $(-\infty, c_1)$, $(c_1, c_2), \cdots, (c_{n-1}, c_n)$, (c_n, ∞), and has the same sign as q outside $[a, b]$. Let $k_1, k_2, \cdots, k_{n-1}$ be the extrema of p on the intervals $[c_1, c_2], \cdots [c_{n-1}, c_n]$ in that order, and suppose they are attained at $x_1, \cdots x_{n-1}$. Make $|k|$ large enough so that $|k_i| > \max_{x \in [a,b]} |q(x)|$ for all i. Then $p + q$ has degree n, and has alternating signs at $a - \epsilon, x_1, \cdots, x_n, b + \epsilon$ for $\epsilon > 0$, so it has exactly n distinct roots. Now it is enough to take $r = -p$. $\square$

Claim 5 For any $q \in \mathbb{R}[x]$, $f(q) = c_q q$ for some non-zero real c_q.

Proof. We have already proved this for ninth-grade polynomials. Take ninth-grade r so that $q - r$is ninth grade and $n = \deg(q - r) > \deg q$. Then $q - r \sim f(q) - f(r) = f(q) - c_r r$. Since $q - r$ is ninth-grade and has the same degree as $f(q) - c_r r$, $q - r = c(f(q) - c_r r) = cf(q) - c_1 r$ for non-zero reals c, c_1. Comparing the leading term (which belongs to r) on both sides, $c_1 = 1$, therefore $q = cf(q) \implies f(q) = c_q q$. $\square$

Claim 6 For any $p, q \in \mathbb{R}[x]$, $c_p = c_q$.

Proof. We note that for any two polynomials p, q if $p - q$ has a real root which is not a root of p, then $c_p = c_q$. Indeed, if s is a root of $p - q$ (meaning $p(s) = q(s) \neq 0$), then it's also a root of $f(p) - f(q) = c_p p - c_q q$, so that $c_p p(s) = c_q q(s) \implies c_p = c_q$.

Now for any two p, q, choose odd N such that $N > \max\{\deg p, \deg q\}$. Then the polynomial $r = x^N$ is such that $r - p$ and $r - q$ both have real roots, so $c_q = c_r = c_p$. $\square$

Claim 6 clearly means there is $c \in \mathbb{R}$ so that $f(p) = cp$ for all $p \in \mathbb{R}[x]$. Using the fact $f(f(p)) = p$, we see that the only possibilities are $c = 1$ or $c = -1$, completing the proof. $\square$