COMPENDIUM IMO

International Mathematical Olympiads

1959 – 2023

Gerard Romo Garrido

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Toomates Coolección

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La Olimpiada Matemática Internacional (IMO).

Es el campeonato mundial de matemáticas para estudiantes de secundaria, y se desarrolla anualmente en un país distinto. La primera OIM tuvo lugar en 1959 en Rumanía, con la participación de 7 países. Poco a poco ha ido creciendo hasta sobrepasar los 100 países de los 5 continentes. El Consejo de la OIM garantiza que la Olimpiada se celebre cada año y que el país anfitrión respete el reglamento y las tradiciones olímpicas.

La competición consta de dos cuestionarios con tres problemas cada uno. Cada pregunta da una puntuación máxima de 7 puntos, con una puntuación máxima total de 42 puntos. La prueba se desarrolla en dos días, en cada uno de los cuales el concursante dispone de cuatro horas y media para resolver tres problemas.

La Lista Larga.

Cada país envía mediante su representante oficial una o más propuestas que considere adecuadas para la prueba, asegurándose de hacerlo de manera confidencial. De aquí se forma lo que extraoficialmente se conoce como la «Lista Larga», una lista con todos los problemas propuestos. En los últimos años han sido alrededor de 130. Antes esta lista era pública, pero ahora ya no. Ahora sólamente la conocen los miembros del Comité de Selección de Problemas, que es un grupo de matemáticos y ex-olímpicos destacados elegidos por el país sede.

La Lista Corta ("ShortList"),

El Comité de Selección de Problemas prepara lo que se conoce como la «Lista Corta de la IMO». Esta es una colección de alrededor de 32 problemas, clasificados en las cuatro áreas olímpicas clásicas: Álgebra, Combinatoria, Geometría y Teoría de Números. Esta es quizás la colección de problemas olímpicos más bella que se elabora cada año: tiene problemas novedosos, creativos y cuya dificultad varía desde los problemas de IMO fáciles, hasta problemas que quedan por encima del nivel de la competencia.

Los problemas de la prueba IMO.

De entre la lista de problemas "ShortList" se escogen los 6 problemas que finalmente constituirán la prueba IMO.

Fuente: http://blog.nekomath.com/detalles-imo/

Índice.

Nota: Los enunciados de la competición son siempre problemas seleccionados de la "Shortlist", por lo tanto, a partir de 1967, las soluciones de los problemas de la competención hay que buscarlas dentro de las soluciones de la "Shortlist" del respectivo año.

49	2008	57	390		955		2130	1987
50	2009	59	392		1008		2138	1989
51	2010	61	394		1098		2148	1991
52	2011	63	396		1175		2158	1993
53	2012	65	398		1252		2168	1995
54	2013	67	400		1304		2177	1997
55	2014	69	402		1372		2189	1999
56	2015	71	404		1459		2201	2001
57	2016	73	406		1541		2212	2003
58	2017	75	408		1632		2224	2005
59	2018	77	410		1728		2237	2007
60	2019	1800	1805	1807	1841	1825	2253	2009
61	2020	1947	1949		2308		2026	2011
62	2021	1951	1953		2404		2282	2013
63	2022	1955	1957				2293	2015
64	2023	2304	2306					2017

Fuentes.

Compendiums y materiales diversos en pdf de Internet, agrupados en un único archivo "pdf" mediante las aplicación online https://www.ilovepdf.com/

6 de Julio de 1983

Primera sesión: 4 h 30 min

24 IMO 1. Hallar todas las funciones f definida en el conjunto de los números reales, que toman valores reales positivos y que satisfacen las condiciones

1) $f\big(xf(y)\big) = yf(x)$ para todo x, y positivos,

2) $f(x) \to 0$ si $x \to \infty$.

24 IMO 2. Sea A uno de los dos puntos de intersección distintos de dos círculos distintos C_1, C_2 de centros O_1, O_2, respectivamente.
Una de las tangentes comunes a los dos círculos toca a C_1 en P_1 y a C_2 en P_2, mientras que la otra toca a C_1 en Q_1 y a C_2 en Q_2. Sea M_1 el punto medio de P_1Q_1 y M_2 el punto medio de P_2Q_2. Demostrar que $\widehat{O_1AO_2} = \widehat{M_1AM_2}$.

24 IMO 3. Sean a, b, c enteros positivos, dos a dos primos entre si. Demostrar que $2abc-ab-bc-ca$ es el mayor entero que no puede expresarse en la forma $xbc+yca+zab$, donde x, y, y z son enteros no negativos.

7 de Julio de 1983

Segunda sesión: 4 h 30 min

24 IMO 4. Sea ABC un triángulo equilátero, y $\mathcal{E}$ el conjunto de todos los puntos contenidos en los tres segmentos AB, BC y CA (con A, B y C incluidos). Determinar si es cierto que para cada partición de $\mathcal{E}$ en dos conjuntos disjuntos, por lo menos uno de los dos conjuntos contiene los vértices de un triángulo rectángulo. Justificar la respuesta.

24 IMO 5. Decir si es posible elegir 1983 enteros positivos distintos, todos menores o iguales que 10^5, de forma que tres cualesquiera de ellos no sean términos consecutivos de una progresión aritmética. Justificar la respuesta.

24 IMO 6. Sean a, b y c las longitudes de los lados de un triángulo. Demostrar que

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determinar en qué casos se cumple la igualdad.

4 de Julio de 1984

Primera sesión: 4 h 30 min

25 IMO 1. Demostrar que

$$0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27},$$

donde x, y, z son números reales no negativos que cumplen $x + y + z = 1$.

25 IMO 2. Hallar un par de enteros positivos a y b tales que

1) $ab(a+b)$ no es divisible por 7;

2) $(a+b)^7 - a^7 - b^7$ es divisible por 7^7.

Justificar la respuesta.

25 IMO 3. Tenemos en el plano dos puntos diferentes, A y O. Para cada punto X del plano distinto de O, denotamos por $\alpha(X)$ la medida del ángulo entre OA y OX, en radianes, y contado en sentido antihorario desde OA. ($0 \leq \alpha(X) < 2\pi$).

Sea $C(X)$ la circunferencia de centro O y radio de longitud $OX + \dfrac{\alpha(X)}{OX}$.

Tenemos un número finito de colores y coloreamos cada uno de los puntos del plano con ellos.

Demostrar que existe un punto Y tal que $\alpha(Y) > 0$ y tal que su color aparece sobre la circunferencia de $C(Y)$.

5 de Julio de 1984
Segunda sesión: 4 h 30 min

25 IMO 4. Sea $ABCD$ un cuadrilátero convexo tal que la recta CD es tangente al círculo de diámetro AB. Demostrar que la recta AB es tangente al círculo de diámetro CD si y sólo si las rectas BC y AD son paralelas.

25 IMO 5. Sea d la suma de las longitudes de todas las diagonales de un polígono convexo plano de n vértices ($n > 3$), y sea p su perímetro. Demostrar que

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2,$$

siendo $[x]$ la parte entera de x.

25 IMO 6. Sean a, b, c y d enteros impares tales que $0 < a < b < c < d$ y $ad = bc$. Demostrar que si $a + d = 2^k$ y $b + c = 2^m$ para ciertos enteros k y m, entonces $a = 1$.

4 de Julio de 1985

Primera sesión: 4 h 30 min

26 IMO 1. Un círculo tiene el centro sobre el lado AB del cuadrilátero inscriptible $ABCD$. Los otros tres lados son tangentes al círculo. Demostrar que $AD+BC=AB$.

26 IMO 2. Sean, n y k dos números naturales primos entre si, con $0<k<n$. Cada número del conjunto $\mathcal{M}=\{1,2,\ldots,n-1\}$ se colorea o bien en azul, o bien en blanco. Se sabe que

1) Para cada $i\in\mathcal{M}$, los elementos i y $n-i$ tienen el mismo color.

2) Para cada $i\in\mathcal{M}$, $i\neq k$, los elementos i y $|i-k|$ tienen el mismo color.

Demostrar que todos los elementos de $\mathcal{M}$ tienen el mismo color.

26 IMO 3. Dado un polinomio $P(x)=a_0+a_1x+a_2x^2+\cdots+a_kx^k$ con coeficientes enteros, denotamos por $w(P)$ el número de coeficientes impares de P. Sea $Q_i(x)=(1+x)^i$, para $i=0,1,\ldots$. Demostrar que si i_1, i_2, $\ldots$, i_n son enteros tales que $0\le i_1<i_2<\cdots<i_n$, entonces

$$w\big(Q_{i_1}+Q_{i_2}+\cdots+Q_{i_n}\big)\ge w(Q_{i_1}).$$

5 de Julio de 1985

Segunda sesión: 4 h 30 min

26 IMO 4. Sea $\mathcal{M}$ un conjunto de 1985 enteros positivos distintos, ninguno de los cuales tiene divisores primos mayores que 26. Demostrar que $\mathcal{M}$ contiene como mínimo un subconjunto de cuatro elementos distintos, cuyo producto es la cuarta potencia de un entero.

26 IMO 5. Una circunferencia de centro O pasa por los vértices A y C de un triángulo ABC y corta otra vez los segmentos AB y BC en los puntos distintos K y N, respectivamente. Las circunferencias circunscritas a los triángulos ABC y KBN se cortan exactamente en dos puntos distintos B y M. Demostrar que el ángulo $\widehat{OMB}$ es un ángulo recto.

26 IMO 6. Para cada número real x_1, se construye la sucesión $x_1, x_2, \ldots, x_n, \ldots$ haciendo

$$x_{n+1} = x_n\left(x_n + \frac{1}{n}\right) \quad \text{para cada} \quad n \geq 1.$$

Demostrar que existe exactamente un valor de x_1 para el cual $0 < x_n < x_{n+1} < 1$ para cada n.

9 de Julio de 1986

Primera sesión: 4 h 30 min

27 IMO 1. Sea d un entero positivo distinto de 2, 5 y 13. Demostrar que se pueden encontrar elementos distintos a, b en el conjunto $\{2,5,13,d\}$, de manera que $ab-1$ no sea un cuadrado perfecto.

27 IMO 2. Tenemos en el plano un punto P_0 y un triángulo $A_1A_2A_3$. Definimos $A_s = A_{s-3}$ para todo $s \geq 4$. Construimos una sucesión de puntos P_1, P_2, P_3, ..., de forma que P_{k+1} es la imagen de P_k por la rotación de centro A_{k+1} y ángulo 120° en sentido horario, para $k = 0,1,2\ldots$ Demostrar que si $P_{1986} = P_0$, entonces el triángulo $A_1A_2A_3$ es equilátero.

27 IMO 3. A cada vértice de un pentágono le asignamos un número entero, de forma que la suma de los cinco enteros sea positiva. Si tres vértices consecutivos tienen números asignados x, y, z, respectivamente, y es $y < 0$, entonces se permite hacer la siguiente operación: los números x, y, z se sustituyen respectivamente por $x+y$, $-y$, $z+y$. Esta operación se puede hacer repetidamente mientras al menos uno de los cinco números sea negativo. Determinar si este proceso acaba *necesariamente* con un número finito de pasos.

10 de Julio de 1986
Segunda sesión: 4 h 30 min

27 IMO 4. Sean A, B vértices adyacentes de un n-ágono regular ($n \geq 5$) del plano que tiene centro en O. Un triángulo XYZ que es congruente con OAB e inicialmente coincide con él, se mueve en el plano de forma que Y y Z describan la frontera del polígono, dejando X en el interior. Hallar el lugar geométrico de X.

27 IMO 5. Hallar todas las funciones f definidas en el conjunto de los números reales no negativos y que toman valores reales no negativos, tales que

1) $f\big(x\,f(y)\big)\,f(y) = f(x+y)$ para todo $x, y \geq 0$,

2) $f(2) = 0$,

3) $f(x) \neq 0$ para $0 \leq x < 2$.

27 IMO 6. Tenemos un conjunto finito de puntos del plano, cada uno con coordenadas enteras. Se pregunta si es posible colorear algunos puntos del conjunto en rojo y los restantes en blanco de forma que toda recta L paralela a uno de los ejes de coordenadas contenga puntos rojos y blancos en cantidades cuya diferencia en valor absoluto sea 1 como máximo. Justificar la respuesta.

10 de Julio de 1987

Primera sesión: 4 h 30 min

28 IMO 1. Sea $p_n(k)$ el número de permutaciones del conjunto $\{1, 2, \ldots, n\}$, $n \geq 1$, que tienen exactamente k puntos fijos. Demostrar que

$$\sum_{k=0}^{n} k\ p_n(k) = n!.$$

(*Nota:* Una permutación f de un conjunto S es una aplicación biyectiva de S sobre si mismo. Un elemento i de S se llama punto fijo de la permutación f si $f(i) = i$.)

28 IMO 2. En un triángulo acutángulo ABC la bisectriz interior del ángulo A corta a BC en L y corta la circunferencia circunscrita de ABC de nuevo en N. Trazamos perpendiculares desde L a AB y AC, con pies K y M, respectivamente. Demostrar que el cuadrilátero $AKNM$ y el triángulo ABC tienen la misma área.

28 IMO 3. Sean x_1, x_2, ..., x_n, números reales que cumplen $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Demostrar que para cada entero $k \geq 2$ existen enteros no todos nulos a_1, a_2, ..., a_n, tales que $|a_i| \leq k - 1$ para todo i y

$$|a_1 x_1 + a_2 x_2 + \cdots + a_n x_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

11 de Julio de 1987

Segunda sesión: 4 h 30 min

28 IMO 4. Demostrar que no existe ninguna función del conjunto de enteros no negativos en él mismo tal que, para todo n, $f\big(f(n)\big) = n + 1987$.

28 IMO 5. Sea n un entero mayor o igual que 3. Demostrar que existe un conjunto de n puntos del plano tal que la distancia entre dos puntos cualesquiera del conjunto es irracional, y tal que cada subconjunto de tres puntos determina un triángulo no degenerado de área racional.

28 IMO 6. Sea n un entero mayor o igual que 2. Demostrar que si $k^2 + k + n$ es primo para todos los enteros k tales que $0 \leq k \leq \sqrt{n/3}$, entonces $k^2 + k + n$ es primo para todos los enteros k tales que $0 \leq k \leq n - 2$.

15 de Julio de 1988

Primera sesión: 4 h 30 min

29 IMO 1. Consideremos dos círculos coplanarios de radios R y r $(R > r)$ con mismo centro. Sea P un punto fijo del círculo menor y B un punto variable sobre el círculo mayor. La recta BP corta al círculo mayor de nuevo en C. La perpendicular l a BP por P corta al círculo menor otra vez en A. (Si l es tangente al círculo en P, entonces $A = P$).

1) Determinar el conjunto de valores tomados por $BC^2 + CA^2 + AB^2$.

2) Hallar el lugar geométrico del punto medio de AB.

29 IMO 2. Sea n un entero positivo y sean A_1, A_2, ..., A_{2n+1} subconjuntos de un conjunto B. Supongamos que

a) Cada A_i tiene exactamente $2n$ elementos.

b) Cada $A_i \cap A_j$, $(1 \leq i < j \leq 2n+1)$ contiene exactamente un elemento.

c) Cada elemento de B pertenece como mínimo a dos de los A_i.

Determinar los valores de n para los cuales se puede asignar a cada elemento de B un valor 0 o 1, de tal manera que cada A_i tenga el 0 asignado a exactamente n de sus elementos.

29 IMO 3. Una función f se define sobre los enteros positivos por

$$f(1) = 1, \qquad f(3) = 3,$$
$$f(2n) = f(n),$$
$$f(4n+1) = 2f(2n+1) - f(n),$$
$$f(4n+3) = 3f(2n+1) - 2f(n),$$

para todo entero positivo n. Determinar el número de enteros positivos n, menores o iguales que 1988, para los cuales $f(n) = n$.

16 de Julio de 1988

Segunda sesión: 4 h 30 min

29 IMO 4. Demostrar que el conjunto de números reales x que satisfacen la desigualdad

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

es la unión de intervalos disjuntos cuyas longitudes suman 1988.

29 IMO 5. Sea ABC un triángulo rectángulo en A, y D el pie de la altura desde A. La recta que une los incentros de los triángulos ABD y ACD, interseca los lados AB y AC en los puntos K y L, respectivamente. Si S y T denotan las áreas de los triángulos ABC y AKL, respectivamente, demostrar que $S \geq 2T$.

29 IMO 6. Sean a y b enteros positivos tales que $ab+1$ divide a a^2+b^2. Demostrar que $\dfrac{a^2+b^2}{ab+1}$ es el cuadrado de un entero.

18 de Julio de 1989

Primera sesión: 4 h 30 min

30 IMO 1. Demostrar que el conjunto $\{1, 2, \ldots, 1989\}$ puede expresarse como unión disjunta de subconjuntos A_i $(i = 1, 2, \ldots, 117)$ tales que

1) Cada A_i contiene 17 elementos,

2) La suma de todos los elementos de de cada A_i es la misma.

30 IMO 2. En un triángulo acutángulo ABC, la bisectriz interior del ángulo A corta a la circunferencia circunscrita de nuevo en A_1. Los puntos B_1 y C_1 se definen análogamente. Sea A_0 el punto de intersección dela recta AA_1 con las bisectrices exteriores de los ángulos B y C. Los puntos B_0 y C_0 se definen análogamente. Demostrar que

1) El área del triángulo $A_0B_0C_0$ es el doble del área del hexágono $AC_1BA_1CB_1$.

2) El área del triángulo $A_0B_0C_0$ es mayor o igual que 4 veces el área de ABC.

30 IMO 3. Sean n y k enteros positivos, y $\mathcal{S}$ un conjunto de n puntos del plano tales que

1) Tres puntos cualesquiera de $\mathcal{S}$ no están alineados.

2) Para cada punto P de $\mathcal{S}$ hay al menos k puntos de $\mathcal{S}$ que equidistan de P.

Demostrar que $k < \frac{1}{2} + \sqrt{2n}$.

19 de Julio de 1989

Segunda sesión: 4 h 30 min

30 IMO 4. Sea $ABCD$ un cuadrilátero convexo tal que los lados AB, AD y BC satisfacen $AB = AD + BC$. Existe un punto P dentro del cuadrilátero a la distancia h de la recta CD tal que $AP = AD + h$ y $BP = BC + h$. Demostrar que

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

30 IMO 5. Demostrar que para cada entero positivo n existen n enteros positivos consecutivos, ninguno de los cuales es una potencia entera de un número primo.

30 IMO 6. Una permutación $(x_1, x_2, \ldots, x_{2n})$ del conjunto $\{1, 2, \ldots, 2n\}$, donde n es un entero positivo, se dice que tiene la propiedad P si

$$|x_i - x_{i+1}| = n$$

para al menos un i en $\{1, 2, \ldots, 2n-1\}$. Demostrar que, para cada n, hay más permutaciones con la propiedad P que sin ella.

12 de Julio de 1990
Primera sesión: 4 h 30 min

31 IMO 1. Las cuerdas AB y CD de una circunferencia se cortan en el punto E dentro del círculo. Sea M un punto interior del segmento EB. La recta tangente en E a la circunferencia que pasa por D, E y M corta las rectas BC y AC en F y G, respectivamente. Si $\dfrac{AM}{AB} = t$, hallar $\dfrac{EG}{EF}$ en función de t.

31 IMO 2. Sea $n \geq 3$ y consideremos un conjunto $\mathcal{E}$ de $2n-1$ puntos distintos sobre una circunferencia. Supongamos que exactamente k de estos puntos se colorean de negro. Tal coloración es "buena" si existe al menos un par de puntos negros de forma que el interior de al menos uno de los arcos entre ellos contiene exactamente n puntos de E. Hallar el mínimo valor de k para que cualquier coloración de este tipo de k puntos sea buena.

31 IMO 3. Determinar todos los enteros $n > 1$ tales que $\dfrac{2^n+1}{n^2}$ sea un entero.

13 de Julio de 1990

Segunda sesión: 4 h 30 min

31 IMO 4. Sea $\mathbb{Q}^+$ el conjunto de los números racionales positivos. Construir una función $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ tal que, para todo x, y en $\mathbb{Q}^+$, cumpla

$$f\big(xf(y)\big) = \frac{f(x)}{y}.$$

31 IMO 5. Dado un entero inicial $n_0 > 1$, dos jugadores, A y B, eligen enteros n_1, n_2, n_3, ..., alternativamente, según las reglas siguientes:

1) Conociendo n_{2k}, A elige cualquier entero n_{2k+1} tal que $n_{2k} \le n_{2k+1} \le n_{2k}^2$.

2) Conociendo n_{2k+1}, B elige cualquier entero n_{2k+2} tal que $\dfrac{n_{2k+1}}{n_{2k+2}}$ sea un primo elevado a una potencia entera positiva.

El jugador A gana el juego eligiendo el número 1990; el jugador B gana eligiendo el número 1.

Determinar el valor inicial n_0 que permita que:

a) A tiene una estrategia ganadora.

b) B tiene una estrategia ganadora.

c) Ningún jugador tiene estrategia ganadora.

31 IMO 6. Demostrar que existe un polígono convexo de 1990 lados con las dos siguientes propiedades:

a) Todos los ángulos son iguales.

b) Las longitudes de los lados son los números 1^2, 2^2, 3^2, ..., 1990^2 en un cierto orden.

17 de Julio de 1991

Primera sesión: 4 h 30 min

32 IMO 1. Dado un triángulo ABC, sea I el centro del círculo inscrito. Las bisectrices internas de los ángulo A, B, C cortan a los lados opuestos en A', B', C', respectivamente. Demostrar que

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

32 IMO 2. Sea $n > 6$ un entero y a_1, a_2, ..., a_k números naturales menores o iguales que n y primos con n. Si

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

demostrar que n tiene que ser primo o bien una potencia de 2.

32 IMO 3. Sea $\mathcal{S} = \{1, 2, 3, \ldots, 280\}$. Hallar el menor entero n tal que cada subconjunto de $\mathcal{S}$ de n elementos contiene cinco números que son dos a dos primos entre si.

18 de Julio de 1991

Segunda sesión: 4 h 30 min

32 IMO 4. Sea G un grafo conexo de k aristas. Demostrar que es posible etiquetar las aristas $1, 2, \ldots, k$ de tal manera que en cada vértice en que concurran dos o más aristas, el máximo común divisor de los valores de las etiquetas de dichas aristas sea 1.

[Un *grafo* consiste en un conjunto de puntos, llamados *vértices*, junto con un conjunto de *aristas* que unen ciertos pares de vértices distintos. Cada par de vértices u, v pertenece a lo sumo a una arista. El grafo G es *conexo* si para cada par de vértices distintos x, y, existe una sucesión de vértices $x = v_0, v_1, v_2, \ldots, v_m = y$ tal que cada par $v_i v_{i+1}$ $(0 \le i < m)$ está unido por una arista de G.]

32 IMO 5. Sea ABC un triángulo y P un punto interior de ABC. Demostrar que al menos uno de los ángulos $\widehat{PAB}$, $\widehat{PBC}$, $\widehat{PCA}$ es menor o igual que 30°.

32 IMO 6. Una sucesión infinita $x_0, x_1, x_2, \ldots$ de números reales se llama *acotada* si existe una constante C tal que $|x_i| \le C$ para todo $i \ge 0$. Dado un número real $a > 1$, construir una sucesión infinita acotada $x_0, x_1, x_2, \ldots$ tal que

$$|x_i - x_j|\,|i - j|^a \ge 1$$

para todo par de enteros no negativos distintos i, j.

15 de Julio de 1992

Primera sesión: 4 h 30 min

33 IMO 1. Hallar todos los enteros a, b, c, con $1 < a < b < c$ tales que

$$(a-1)(b-1)(c-1)$$

es un divisor de $abc - 1$.

33 IMO 2. Sea $\mathbb{R}$ el conjunto de los números reales. Hallar una función $f : \mathbb{R} \to \mathbb{R}$ tal que

$$f\big(x^2 + f(y)\big) = y + \big(f(x)\big)^2 \quad \text{para todo} \quad x, y \in \mathbb{R}.$$

33 IMO 3. Consideremos nueve puntos en el espacio, de forma que cuatro cualesquiera de ellos no sean coplanarios. Cada par de puntos se une con una arista (es decir, un segmento) y cada arista o bien se colorea de color azul, o bien se colorea de color rojo, o bien se deja sin colorear. Hallar el mínimo valor de n de forma que cuando se colorean exactamente n aristas, en este conjunto de aristas coloreadas hay necesariamente un triángulo con las aristas del mismo color.

16 de Julio de 1992

Segunda sesión: 4 h 30 min

33 IMO 4. Sea C un círculo del plano, L una recta tangente al círculo C, y M un punto de L. Hallar el lugar geométrico de los puntos P con la propiedad siguiente: existen dos puntos Q, R sobre L tal que M es el punto medio de QR y C es la circunferencia inscrita del triángulo PQR.

33 IMO 5. Sea S un conjunto finito de puntos del espacio tridimensional. Sean S_x, S_y, S_z conjuntos formados por las proyecciones ortogonales de los puntos de S sobre el plano yz, sobre el plano zx y sobre el plano xy, respectivamente. Demostrar que

$$|S|^2 \leq |S_x|\,|S_y|\,|S_z|,$$

siendo $|A|$ el número de elementos del conjunto finito A. (Nota: La proyección ortogonal de un punto sobre un plano es el pie de la perpendicular trazada desde el punto hasta el plano.)

33 IMO 6. Para cada entero positivo n, sea $S(n)$ el máximo entero tal que, para cada entero positivo $k \leq S(n)$, n^2 puede escribirse como suma de k cuadrados positivos.

1) Demostrar que $S(n) \leq n^2 - 14$ para cada $n \geq 4$.

2) Hallar un entero n tal que $S(n) = n^2 - 14$.

3) Demostrar que existen infinitos enteros n tales que $S(n) = n^2 - 14$.

18 de Julio de 1993

Primera sesión: 4 h 30 min

34 IMO 1. Sea $f(x) = x^n + 5x^{n-1} + 3$ con $n > 1$ entero. Demostrar que $f(x)$ no puede expresarse como producto de dos polinomios con coeficientes enteros y de grado mayor o igual que 1.

34 IMO 2. Sea D un punto interior de un triángulo acutángulo ABC tal que $\widehat{ADB} = \widehat{ACB} + 90^\circ$ y $AC \cdot BD = AD \cdot BC$.

1) Calcular el valor de la razón $\dfrac{AB \cdot CD}{AC \cdot BD}$.

2) Demostrar que las tangentes por C a las circunferencias circunscritas a los triángulos ACD y BCD son perpendiculares.

34 IMO 3. En un tablero infinito se juega el juego que se describe a continuación. Al principio, se colocan n^2 fichas en el tablero, formando un bloque $n \times n$ de casillas adyacentes, con una ficha en cada casilla. Un movimiento del juego es un salto en dirección horizontal o vertical sobre una casilla adyacente ocupada y que va a una casilla desocupada inmediata adyacente. La ficha sobre la que se ha saltado se retira del tablero. Hallar los valores de n para los cuales el juego puede terminar con una sola ficha en el tablero.

19 de Julio de 1993

Segunda sesión: 4 h 30 min

34 IMO 4. Dados tres puntos del plano P, Q, R, definimos $m(PQR)$ como el mínimo de las longitudes de las alturas del triángulo PQR (donde $m(PQR) = 0$ si P, Q, R están alineados.) Sean A, B, C puntos dados del plano. Demostrar que, para todo punto X del plano se cumple

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

34 IMO 5. Sea $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determinar si existe o no una función $f : \mathbb{N} \to \mathbb{N}$ tal que $f(1) = 2$, $f\big(f(n)\big) = f(n) + n$ y $f(n) < (f(n+1)$, para todo $n \in \mathbb{N}$.

34 IMO 6. Sea $n > 1$ un entero. Tenemos n lámparas L_0, L_1, ..., L_{n-1} situadas alrededor de un círculo. Cada lámpara puede estar encendida (`ON`) o apagada (`OFF`). Realizamos una sucesión de acciones S_0, S_1, S_2, ... sobre las lámparas. La acción S_j afecta solamente el estado de la lámpara L_j (dejando el estado de las demás inalteradas) de la forma siguiente: si L_{j-1} está en estado `ON`, S_j cambia el estado de L_j de `ON` a `OFF` o de `OFF` a `ON`; si L_{j-1} está en `OFF`, S_j deja inalterado el estado de L_j. Las lámparas están etiquetadas módulo n, es decir, $L_{-1} = L_{n-1}$, $L_0 = L_n$, $L_1 = L_{n+1}$, y así sucesivamente. Inicialmente todas las lámparas están en `ON`. Demostrar que

1) Existe un entero positivo $M(n)$ tal que después de $M(n)$ acciones, todas las lámparas vuelven a estar `ON`.

2) Si n es de la forma 2^k, entonces todas las lámparas están `ON` después de $n^2 - 1$ acciones.

3) Si n es de la forma $2^k + 1$, entonces todas las lámparas están `ON` después de $n^2 - n + 1$ acciones.

13 de Julio de 1994

Primera sesión: 4 h 30 min

35 IMO 1. Sean M y N enteros positivos. Sean a_1, a_2, ... , a_m elementos distintos de $\{1, 2, \ldots n\}$ tales que cuando $a_i + a_j \leq n$ para algún i, j, $1 \leq i \leq j \leq m$, entonces existe k, $1 \leq k \leq m$, con $a_i + a_j = a_k$. Demostrar que

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \geq \frac{n+1}{2}.$$

35 IMO 2. Sea ABC un triángulo isósceles con $AB = AC$. Supongamos que

1) M es el punto medio de BC y O es el punto de la recta AM tal que OB es perpendicular a AB.

2) Q es un punto arbitrario en el segmento BC distinto de B y de C.

3) E está sobre la recta AB y F está sobre la recta AC de manera que E, Q y F son distintos y están alineados.

Demostrar que OQ es perpendicular a EF si y sólo si $QE = QF$.

35 IMO 3. Para cualquier positivo k, sea $f(k)$ el número de elementos del conjunto $\{k+1, k+2, \ldots, 2k\}$ cuya representación en base 2 tiene exactamente tres unos.

1) Demostrar que, para cada entero positivo m, existe al menos un entero positivo k tal que $f(k) = m$.

2) Determinar todos los enteros positivos m para los cuales existe exactamente un k con $f(k) = m$.

14 de Julio de 1994

Segunda sesión: 4 h 30 min

35 IMO 4. Determinar todos los pares ordenados (m, n) de enteros positivos tales que

$$\frac{n^3+1}{mn-1}$$

es un entero.

35 IMO 5. Sea $\mathcal{S}$ el conjunto de los números reales estrictamente mayores que -1. Hallar todas las funciones $f : \mathcal{S} \to \mathcal{S}$ que satisface las dos condiciones:

1) $f\big(x + f(y) + x\,f(y)\big) = y + f(x) + y\,f(x)$ para todo $x, y \in \mathcal{S}$.

2) $\dfrac{f(x)}{x}$ es estrictamente creciente en cada uno de los intervalos $-1 < x < 0$ y $0 < x$.

35 IMO 6. Demostrar que existe un conjunto A de enteros positivos con la propiedad siguiente: Para todo conjunto infinito S de primos, existen dos enteros positivos $m \in A$ y $n \notin A$, cada uno de los cuales es un producto de k elementos distintos de S, para algún $k \geq 2$.

19 de Julio de 1995

Primera sesión: 4 h 30 min

36 IMO 1. Sean A, B, C, D cuatro puntos distintos sobre una recta, en este orden. Las circunferencias de diámetros AC y BD se cortan en X e Y. La recta XY corta a BC en Z. Sea P un punto sobre la recta XY, distinto de Z. La recta CP corta la circunferencia de diámetro AC en C y M, y la recta BP corta la circunferencia de diámetro BD en B y N. Demostrar que las rectas AM, DN y XY son concurrentes.

36 IMO 2. Sean a, b, c números reales positivos tales que $abc = 1$. Demostrar que

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

36 IMO 3. Determinar todos los enteros $n > 3$ para los cuales existen n puntos A_1, A_2, ..., A_n, no alineados tres a tres, y números reales r_1, r_2, ..., r_n, tales que para $1 \leq i < j < k \leq n$, el área del triángulo $A_iA_jA_k$ es $r_i + r_j + r_k$.

20 de Julio de 1995

Segunda sesión: 4 h 30 min

36 IMO 4. Hallar el máximo valor de x_0 para el cual existe una sucesión finita x_0, $x_1, \ldots, x_{1995}$ de números reales, con $x_0 = x_{1995}$, y tal que, para $i = 1, 2, \ldots, 1995$ se cumple

$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}.$$

36 IMO 5. Sea $ABCDEF$ un hexágono convexo con $AB = BC = CD$ y $DE = EF = FA$, tal que $\widehat{BCD} = \widehat{EFA} = \pi/3$. Supongamos que G y H son puntos en el interior del hexágono tales que $\widehat{AGB} = \widehat{DHE} = 2\pi/3$. Demostrar que $AG + GB + GH + DH + HE \geq CF$.

36 IMO 6. Sea p un número primo impar. Hallar el número de subconjuntos A del conjunto $\{1, 2, \ldots, 2p\}$ tales que

1) A tiene exactamente p elementos.

2) La suma de todos los elementos de A es divisible por p.

10 de Julio de 1996

Primera sesión: 4 h 30 min

37 IMO 1. Nos dan un entero positivo r y un tablero rectangular $ABCD$ de dimensiones $|AB| = 20$, $|BC| = 12$. El rectángulo está dividido en 20×12 casillas cuadradas de lado unidad. Se permiten movimientos de una casilla a otra sólo si la distancia entre los centros de los dos cuadrados es $\sqrt{r}$. Se trata de determinar una sucesión de movimientos que nos lleven del cuadrado que tiene a A como vértice al cuadrado que tiene a B como vértice.

1) Demostrar que no es posible hacerlo si r es divisible por 2 o por 3.

2) Demostrar que es posible si $r = 73$.

3) ¿Hay solución si $r = 97$?

37 IMO 2. Sea P un punto dentro de un triángulo ABC tal que

$$\widehat{APB} - \widehat{ACB} = \widehat{APC} - \widehat{ABC}.$$

Sean D y E los incentros de los triángulos APB y APC, respectivamente. Demostrar que AP, BD y CE se cortan en un punto.

37 IMO 3. Sea S el conjunto de los enteros no negativos. Hallar todas las funciones f definidas en S y que toman valores en S tales que

$$f\big(m + f(n)\big) = f\big(f(m)\big) + f(n), \quad \forall m, n \in S.$$

11 de Julio de 1996

Segunda sesión: 4 h 30 min

37 IMO 4. Los enteros positivos A y B son tales que los números $15a+16b$ y $16a-15b$ son ambos cuadrados de enteros positivos. ¿Cual es el menor valor que puede tomar el menor de dichos cuadrados?

37 IMO 5. Sea $ABCDEF$ un hexágono convexo tal que AB es paralelo a DE, BC es paralelo a EF y CD es paralelo a FA. Sean R_A, R_C, R_E los radios de las circunferencias circunscritas a los triángulos FAB, BCD, DEF, respectivamente. Sea p el perímetro del hexágono. Demostrar que

$$R_A + R_C + R_E \geq \frac{p}{2}.$$

37 IMO 6. Sean p, q, n enteros positivos con $p+q<n$. Sea $(x_0, x_1, \ldots, x_n)$ una $(n+1)$-pla de enteros que satisfacen las condiciones siguentes

1) $x_0 = x_n = 0$.

2) Para cada i con $1 \leq i \leq n$, o bien $x_i - x_{i-1} = p$, o bien $x_i - x_{i-1} = -q$.

Demostrar que existen índices $i<j$ con $(i,j) \neq (0,n)$ tales que $x_i = x_j$.

24 de Julio de 1997

Primera sesión: 4 h 30 min

38 IMO 1. Los puntos de coordenadas enteras del plano son los vértices de cuadrados unidad. Los cuadrados se colorean alternativamente blancos y negros, como en un tablero de ajedrez. Para todo par de enteros positivos m y n, consideremos un triángulo rectángulo cuyos vértices tienen coordenadas enteras y cuyos catetos, de longitudes m y n están sobre lados de los cuadrados. Sea S_1 el área total de la parte negra del triángulo y S_2 el área total de la parte blanca. Sea

$$f(m,n) = |S_1 - S_2|.$$

1) Calcular $f(m,n)$ para todos los enteros positivos m y n que son a la vez pares o impares.

2) Demostrar que $f(m,n) \leq \frac{1}{2}\max\{m,n\}$, para todo m, n.

3) Demostrar que no existe una constante C tal que $f(m,n) < C$ para todo m, n.

38 IMO 2. El ángulo A es el menor del triángulo ABC. Los puntos B y C dividen la circunferencia circunscrita del triángulo en dos arcos. Sea U un punto interior del arco entre B y C que no contiene a A. Las mediatrices de AB y AC cortan a la recta AU en V y W, respectivamente. Las rectas BV y CW se cortan en T. Demostrar que

$$AU = TB + TC.$$

38 IMO 3. Sean x_1, x_2, ..., x_n números reales que cumplen las condiciones

$$|x_1 + x_2 + \cdots + x_n| = 1 \quad \text{y} \quad |x_i| \leq \frac{n+1}{2} \quad \text{para} \quad i = 1, 2, \ldots, n.$$

Demostrar que existe una permutación y_1, y_2, ..., y_n de x_1, x_2, ..., x_n tal que

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

25 de Julio de 1997

Segunda sesión: 4 h 30 min

38 IMO 4. Una matriz $n \times n$ cuyos elementos toman valores en el conjunto $\mathcal{S} = \{1, 2, \ldots, 2n-1\}$ se llama matriz *plateada* si, para cada $i = 1, 2, \ldots, n$, la i-ésima fila y la i-ésima columna contienen, entre las dos, todos los elementos de $\mathcal{S}$. Demostrar que:

1) No exiten matrices plateadas para $n = 1997$.

2) Existen matrices plateadas para infinitos valores de n.

38 IMO 5. Hallar tods los pares (a, b) de enteros positivos que satisfacen la ecuación

$$a^{b^2} = b^a.$$

38 IMO 6. Para cada entero psoitivo n, sea $f(n)$ el número de maneras de representar n como suma de potencias de 2 con exponentes enteros no negativos. Las representaciones que difieren solamente en el orden de los sumandos se consideran la misma. Por ejemplo, $f(4) = 4$, ya que el número 4 puede representarse de las cuatro formas siguientes: 4; $2+2$; $2+1+1$; $1+1+1+1$. Demostrar que para cada entero $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

15 de Julio de 1998
Primera sesión: 4 h 30 min

39 IMO 1. En el cuadrilátero convexo $ABCD$, las diagonales AC y BD son perpendiculares y los lados opuestos AB y DC no son paralelos. Supongamos que el punto P de intersección de las mediatrices de AB y DC, está dentro de $ABCD$. Demostrar que $ABCD$ es un cuadrilátero inscriptible si y sólo si los dos triángulos ABP y CDP tienen la misma área.

39 IMO 2. En una competición hay a participantes y b jueces, donde $b \geq 3$ es un entero impar. Cada juez califica cada competidor como *apto* o *no apto*. Supongamos que k es un número tal que, para cada par de jueces, sus calificaciones coinciden en a lo sumo k participantes. Demostrar que

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

39 IMO 3. Designemos con $d(n)$ el número de divisores positivos del entero positivo n (con 1 y n incluidos). Determinar todos los enteros positivos k tales que

$$\frac{d(n^2)}{d(n)} = k$$

para algún n.

16 de Julio de 1998
Segunda sesión: 4 h 30 min

39 IMO 4. Determinar todos los pares (a, b) de enteros positivos tales que ab^2+b+7 divide a a^2b+a+b.

39 IMO 5. Sea I el incentro del triángulo ABC. Sean K, L, M los puntos de contacto de la circunferencia inscrita a ABC con los lados BC, CA, AB, respectivamente. Demostrar que el ángulo $\widehat{RIS}$ es agudo.

39 IMO 6. Consideremos todas las funciones f del conjunto $\mathbb{N}$ de los enteros positivos en él mismo que satisfacen $f\big(t^2 f(s)\big) = s\,\big(f(t)\big)^2$, para todo s y t en $\mathbb{N}$. Determinar el mínimo valor posible de $f(1998)$.

16 de Julio de 1999

Primera sesión: 4 h 30 min

40 IMO 1. Hallar todos los conjuntos finitos $\mathcal{S}$ de al menos tres puntos del plano tales que, para todo par de puntos distintos A, B de $\mathcal{S}$, la mediatriz de AB sea un eje de simetría de $\mathcal{S}$.

40 IMO 2. Sea $n \geq 2$ un entero fijo.

1) Hallar la mínima constante C tal que para todo conjunto de n números reales no negativos x_1, x_2, ... ,x_n, se cumpla

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{i=1}^{n} x_i \right)^4 .$$

2) Para este valor de C, determinar en qué condiciones se cumple la igualdad.

40 IMO 3. Tenemos un tablero cuadrado $n \times n$, con n par. Dos cuadrados distintos del tablero se llaman adyacentes si comparten un lado común. (Un cuadrado no es adyacente a sí mismo). Hallar el número mínimo de cuadrados que se pueden marcar de forma que todo cuadrado, marcado o no, sea adyacente al menos a un cuadrado marcado.

17 de Julio de 1999

Segunda sesión: 4 h 30 min

40 IMO 4. Hallar los pares (n, p) de enteros positivos tales que

1) p es primo.

2) $n \leq 2p$.

3) $(p-1)^n + 1$ es divisible por n^{p-1}.

40 IMO 5. Los círculos Γ_1 y Γ_2 están dentro del círculo Γ, y son tangentes a él en M y N, respectivamente. Sabemos que Γ_1 pasa por el centro de Γ_2. La cuerda común de Γ_1 y Γ_2, extendida, corta a Γ en A y B. Las rectas MA y MB cortan a Γ_1 de nuevo en C y D. Demostrar que la recta CD es tangente a Γ_2.

40 IMO 6. Determinar todas las funciones $f : \mathbb{R} \to \mathbb{R}$ tales que

$$f\big(x - f(y)\big) = f\big(f(y)\big) + x\,f(y) + f(x) - 1,$$

para todo $x, y \in \mathbb{R}$.

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Taejon, Corea del Sur

19 de Julio de 2000

Primera sesión: 4 h 30 min

41 IMO 1. Dos circunferencias Γ_1 y Γ_2 se cortan en M y N. Sea l la tangente común a Γ_1 y Γ_2 tal que M está más cerca de l que N. La recta l es tangente a Γ_1 en A y a Γ_2 en B. La recta paralela a l que pasa por M corta de nuevo a Γ_1 en C y a Γ_2 en D. Las rectas CA y DB se cortan en E; las rectas AN y CD se cortan en P; las rectas BN y CD se cortan en Q. Demostrar que $EP = EQ$.

41 IMO 2. Sean a, b, c números reales positivos tales que $abc = 1$. Demostrar que

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

41 IMO 3. Sea $n \geq 2$ un número entero positivo. Inicialmente hay n pulgas en una recta horizontal, y no todas están en el mismo punto. Para un número real positivo λ, definimos un *salto* como sigue: Se eligen dos pulgas cualesquiera situadas en puntos A y B, con A a la izquierda de B; luego, la pulga situada en A salta hasta el punto C de la recta, situado a la derecha de B, y tal que $\frac{BC}{AB} = \lambda$.

Determinar todos los valores de λ tales que, para cualquier punto M de la recta y cualesquiera posiciones iniciales de las n pulgas, existe una sucesión finita de saltos que permite situar a todas las pulgas a la derecha de M.

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Taejon, Corea del Sur

20 de Julio de 2000

Segunda sesión: 4 h 30 min

41 IMO 4. Un mago tiene cien tarjetas numeradas desde 1 hasta 100. Las coloca en tres cajas: una roja, una blanca y una azul, de modo que cada caja contiene por lo menos una tarjeta. Una persona del público selecciona dos de las tres cajas, elige una tarjeta de cada una y anuncia a la audiencia la suma de los números de las dos tarjetas elegidas. Al conocer esta suma, el mago identifica la caja de la que no se eligió ninguna tarjeta.

¿De cuántas maneras se pueden distribuir todas las tarjetas en las cajas de modo que este truco siempre funcione?

(Dos maneras de distribuir se consideran distintas, si al menos hay una tarjeta que es colocada en una caja diferente en cada distribución).

41 IMO 5. Determinar si existe un entero positivo n tal que exactamente 2000 números primos dividen a n, y n divide a $2^n + 1$.

41 IMO 6. Sean AH_1, BH_2 y CH_3 las alturas de un triángulo acutángulo ABC. La circunferencia inscrita al triángulo ABC es tangente a los lados BC, CA y AB en los puntos T_1, T_2 y T_3, respectivamente. Sea l_1 la recta simétrica de H_2H_3 con respecto a T_2T_3; l_2 la recta simétrica de H_3H_1 con respecto a T_3T_1, y l_3 la recta simétrica de H_1H_2 respecto a T_1T_2.

Demostrar que l_1, l_2, l_3 determinan un triángulo cuyos vértices son puntos de la circunferencia inscrita en el triángulo ABC.

2001 XLII Olimpiada Matemática Internacional 42

Washington DC, USA

8 de Julio de 2001

Primera sesión: 4 h 30 min

42 IMO 1. Sea ABC un triángulo acutángulo con circuncentro O. Sea P sobre BC el pie de la altura por A. Supongamos que $\widehat{BCA} \geq \widehat{ABC} + 30^\circ$.
Demostrar que $\widehat{CAB} + \widehat{COP} < 90^\circ$.

42 IMO 2. Demostrar que, cualesquiera que sean los números reales positivos a, b, c, se cumple

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^{+}8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

42 IMO 3. En un concurso matemático hay 21 chicas y 21 chicos. Sabemos que

1) Cada participante ha resuleto a lo sumo seis problemas.

2) Para cada chica y cada chico, al menos hay un problema resuelto por ambos.

Demostrar que hay un problema que al menos ha sido resuelto por tres chicas y al menos por tres chicos.

9 de Julio de 2001

Segunda sesión: 4 h 30 min

42 IMO 4. Sea n un entero mayor que 1, y sean k_1, k_2, ..., k_n enteros dados. Para cada una de las $n!$ permutaciones $a = (a_1, a_2, \dots, a_n)$ de $1, 2, \dots, n$, sea

$$S(a) = \sum_{i=1}^{n} k_i a_i.$$

Demostrar que existen dos permutaciones b y c, $b \neq c$, tal que $n!$ es un divisor de $S(b) - S(c)$.

42 IMO 5. En un triángulo ABC, sea AP la bisectriz de $\widehat{BAC}$, con P sobre BC; y sea BQ la bisectriz de $\widehat{ABC}$ con Q sobre CA. Sabemos que $\widehat{BAC} = 60^\circ$ y que $AB + BP = AQ + QB$. ¿Cuáles son los posibles ángulos del triángulo ABC?

42 IMO 6. Sean a, b, c, d enteros con $a > b > c > d > 0$. Supongamos que

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Demostrar que $ab + cd$ no es primo.

24 de Julio de 2002

Primera sesión: 4 h 30 min

43 IMO 1. Sea n un entero positivo. Sea T el conjunto de puntos (x, y) del plano tales que x e y son enteros no negativos con $x + y < n$. Cada punto de T se colorea de azul o rojo. Si un punto (x, y) es rojo, entonces también son rojos todos los puntos (x', y') de T tales que $x' \leq x$ y $y' \leq y$. Se dice que un conjunto de n puntos azules es de tipo X si las coordenadas x de sus puntos son todas distintas. Se dice que un conjunto de n puntos azules es de tipo Y si las coordenadas y de sus puntos son todas distintas. Demostrar que el número de conjuntos de tipo X es igual al número de conjuntos de tipo Y.

43 IMO 2. Sea BC un diámetro de la circunferencia Γ de centro O. Sea A un punto de Γ tal que $0° < \widehat{AOB} < 120°$. Sea D el punto medio del arco AB que no contiene a C. La recta que pasa por O y es paralela a DA intersecta a la recta AC en J. La mediatriz de OA intersecta a Γ en E y en F. Demostrar que J es el incentro del triángulo CEF.

43 IMO 3. Hallar todas las parejas de enteros $m, n \geq 3$ para las cuales existen infinitos enteros positivos a tales que

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

es entero.

25 de Julio de 2002

Segunda sesión: 4 h 30 min

43 IMO 4. Sea n un entero mayor que 1. Los enteros positivos divisores de n son d_1, d_2, ..., d_k con

$$1 = d_1 < d_2 < \cdots < d_k = n.$$

Se define $D = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$.

a) Demostrar que $D < n^2$.

b) Determinar todos los números n tales que D es un divisor de n^2.

43 IMO 5. Sea $\mathbb{R}$ el conjunto de los números reales. Hallar todas las funciones f de $\mathbb{R}$ en $\mathbb{R}$ tales que

$$\big(f(x) + f(z)\big)\big(f(y) + f(t)\big) = f(xy - zt) + f(xt + yz)$$

para todos los x, y, z, t en $\mathbb{R}$.

43 IMO 6. En el plano, sean Γ_1, Γ_2, ..., Γ_n circunferencias de radio 1, donde $n \geq 3$. Sean sus centros O_1, O_2, ..., O_n respectivamente. Supongamos que ninguna recta del plano intersecta a más de dos de las circunferencias dadas. Demostrar que

$$\sum_{1\leq i<j\leq n} \frac{1}{O_iO_j} < \frac{(n-1)\pi}{4}.$$

Version: Spanish.

PRIMER DIA
Tokio, 13 de julio de 2003.

Problema 1. Sea A un subconjunto del conjunto $S = \{1, 2, \ldots, 1000000\}$ con 101 elementos exactamente. Demostrar que existen números $t_1, t_2, \ldots, t_{100}$ en S tales que los conjuntos

$$A_j = \{x + t_j \mid x \in A\} \quad \text{para} \quad j = 1, 2, \ldots, 100$$

son disjuntos dos a dos.

Problema 2. Determinar todas las parejas de enteros positivos (a, b) tales que

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

es un entero positivo.

Problema 3. Consideremos un hexágono convexo tal que para cualesquiera dos lados opuestos se verifica la siguiente propiedad: la distancia entre sus puntos medios es igual a $\sqrt{3}/2$ multiplicado por la suma de sus longitudes. Demostrar que todos los ángulos del hexágono son iguales.

(Un hexágono convexo $ABCDEF$ tiene tres parejas de lados opuestos: AB y DE, BC y EF, CD y FA.)

Tiempo: 4 horas y media.
Cada problema vale 7 puntos.

Version: Spanish.

SEGUNDO DIA
Tokio, 14 de julio de 2003.

Problema 4. Sea $ABCD$ un cuadrilátero convexo cuyos vértices están sobre una circunferencia. Sean P, Q y R los pies de las perpendiculares trazadas desde D a las rectas BC, CA y AB respectivamente. Demostrar que $PQ = QR$ si y sólo si las bisectrices de los ángulos $\angle ABC$ y $\angle ADC$ se cortan sobre la recta AC.

Problema 5. Sea n un entero positivo, y $x_1, x_2, \ldots, x_n$ números reales tales que $x_1 \leq x_2 \leq \cdots \leq x_n$.

(a) Demostrar que

$$\left(\sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j| \right)^2 \leq \frac{2(n^2-1)}{3} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2 .$$

(b) Demostrar que se cumple la igualdad si y sólo si $x_1, x_2, \ldots, x_n$ forman una progresión aritmética.

Problema 6. Sea p un número primo. Demostrar que existe un número primo q tal que, para todo entero n, el número $n^p - p$ no es divisible por q.

Tiempo: 4 horas y media.
Cada problema vale 7 puntos.

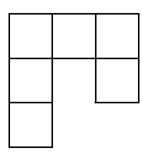
Problema 1. Sea ABC un triángulo acutángulo con $AB \neq AC$. La circunferencia de diámetro BC corta a los lados AB y AC en M y N , respectivamente. Sea O el punto medio de BC. Las bisectrices de los ángulos $\angle BAC$ y $\angle MON$ se cortan en R . Demostrar que las circunferencias circunscritas de los triángulos BMR y CNR tienen un punto común que pertenece al lado BC.

Problema 2. Encontrar todos los polinomios $P(x)$ con coeficientes reales que satisfacen la igualdad

$$P(a-b)+P(b-c)+P(c-a)=2P(a+b+c)$$

para todos los números reales a,b,c tales que $ab+bc+ca=0$.

Problema 3. Un *gancho* es una figura formada por seis cuadrados unitarios como se muestra en el diagrama



o cualquiera de las figuras que se obtienen de ésta rotándola o reflejándola. Determinar todos los rectángulos $m \times n$ que pueden cubrirse con ganchos de modo que

- el rectángulo se cubre sin huecos y sin superposiciones;
- ninguna parte de ningún gancho sobresale del rectángulo.

Problema 4. Sea $n \geq 3$ un entero. Sean $t_1, t_2, \ldots, t_n$ números reales positivos tales que

$$n^2+1>(t_1+t_2+\cdots+t_n)\left(\frac{1}{t_1}+\frac{1}{t_2}+\cdots+\frac{1}{t_n}\right).$$

Demostrar que t_i, t_j, t_k son las medidas de los lados de un triángulo para todos los i, j, k con $1 \leq i < j < k \leq n$.

Problema 5. En un cuadrilátero convexo $ABCD$ la diagonal BD no es la bisectriz ni del ángulo ABC ni del ángulo CDA. Un punto P en el interior de $ABCD$ verifica

$$\angle PBC = \angle DBA \text{ y } \angle PDC = \angle BDA.$$

Demostrar que los vértices del cuadrilátero $ABCD$ pertenecen a una misma circunferencia si y solo si $AP = CP$.

Problema 6. Un entero positivo es *alternante* si en su representación decimal en toda pareja de dígitos consecutivos uno es par y el otro es impar.

Encontrar todos los enteros positivos n tales que n tiene un múltiplo que es alternante.

46^{a} Olimpiada Internacional de Matemáticas

Mérida, México

Primer Día

Miércoles 13 de julio de 2005

Language: Spanish

Problema 1. Se eligen seis puntos en los lados de un triángulo equilátero ABC: A_1 y A_2 en BC, B_1 y B_2 en CA, C_1 y C_2 en AB. Estos puntos son los vértices de un hexágono convexo $A_1A_2B_1B_2C_1C_2$ cuyos lados son todos iguales. Demuestre que las rectas A_1B_2, B_1C_2 y C_1A_2 son concurrentes.

Problema 2. Sea $a_1, a_2, \ldots$ una sucesión de enteros que tiene infinitos términos positivos e infinitos términos negativos. Supongamos que para cada entero positivo n, los números $a_1, a_2, \ldots, a_n$ tienen n restos distintos al ser divididos entre n. Demuestre que cada entero se encuentra exactamente una vez en la sucesión.

Problema 3. Sean x, y, z números reales positivos tales que $xyz \geq 1$. Demuestre que

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

Problema 4. Consideremos la sucesión infinita $a_1, a_2, \ldots$ definida por

$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \ldots).$$

Determine todos los enteros positivos que son primos relativos (coprimos) con todos los términos de la sucesión.

Problema 5. Sea $ABCD$ un cuadrilátero convexo que tiene los lados BC y AD iguales y no paralelos. Sean E y F puntos en los lados BC y AD, respectivamente, que satisfacen $BE = DF$. Las rectas AC y BD se cortan en P, las rectas BD y EF se cortan en Q, las rectas EF y AC se cortan en R. Consideremos todos los triángulos PQR que se forman cuando E y F varían. Demuestre que las circunferencias circunscritas a esos triángulos tienen en común otro punto además de P.

Problema 6. En una competencia de matemáticas se propusieron 6 problemas a los estudiantes. Cada par de problemas fue resuelto por más de $\frac{2}{5}$ de los estudiantes. Nadie resolvió los 6 problemas. Demuestre que hay al menos 2 estudiantes tales que cada uno tiene exactamente 5 problemas resueltos.

12 de julio de 2006

Problema 1. Sea ABC un triángulo y sea I el centro de su circunferencia inscrita. Sea P un punto en el interior del triángulo tal que

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Demuestre que $AP \geq AI$ y que vale la igualdad si y sólo si $P = I$.

Problema 2. Decimos que una diagonal de un polígono regular P de 2006 lados es un *segmento bueno* si sus extremos dividen al borde de P en dos partes, cada una de ellas formada por un número impar de lados. Los lados de P también se consideran *segmentos buenos*.

Supongamos que P se ha dividido en triángulos trazando 2003 diagonales de modo que ningún par de ellas se corta en el interior de P. Encuentre el máximo número de triángulos isósceles que puede haber tales que dos de sus lados son *segmentos buenos*.

Problema 3. Determine el menor número real M tal que la desigualdad

$$|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)| \leq M \left(a^2 + b^2 + c^2\right)^2$$

se cumple para todos los números reales a, b, c.

Tiempo permitido: 4 horas 30 minutos
Cada problema vale 7 puntos.

13 de julio de 2006

Problema 4. Determine todas las parejas de enteros (x, y) tales que

$$1 + 2^x + 2^{2x+1} = y^2.$$

Problema 5. Sea $P(x)$ un polinomio de grado $n > 1$ con coeficientes enteros y sea k un entero positivo. Considere el polinomio $Q(x) = P(P(\ldots P(P(x))\ldots))$, donde P aparece k veces. Demuestre que hay a lo sumo n enteros t tales que $Q(t) = t$.

Problema 6. Asignamos a cada lado b de un polígono convexo P el área máxima que puede tener un triángulo que tiene a b como uno de sus lados y que está contenido en P. Demuestre que la suma de las áreas asignadas a los lados de P es mayor o igual que el doble del área de P.

Tiempo permitido: 4 horas 30 minutos
Cada problema vale 7 puntos.

Spanish version

Primer día

25 de julio de 2007

Problema 1. Sean $a_1, a_2, \ldots, a_n$ números reales. Para cada i $(1 \leq i \leq n)$ se define

$$d_i = \max\{a_j : 1 \leq j \leq i\} - \min\{a_j : i \leq j \leq n\}$$

y sea

$$d = \max\{d_i : 1 \leq i \leq n\}.$$

(a) Demostrar que para cualesquiera números reales $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{d}{2}. \qquad (*)$$

(b) Demostrar que existen números reales $x_1 \leq x_2 \leq \cdots \leq x_n$ para los cuales se cumple la igualdad en $(*)$.

Problema 2. Se consideran cinco puntos A, B, C, D y E tales que $ABCD$ es un paralelogramo y $BCED$ es un cuadrilátero cíclico y convexo. Sea ℓ una recta que pasa por A. Supongamos que ℓ corta al segmento DC en un punto interior F y a la recta BC en G. Supongamos también que $EF = EG = EC$.
Demostrar que ℓ es la bisectriz del ángulo DAB.

Problema 3. En una competencia de matemáticas algunos participantes son amigos. La amistad es siempre recíproca. Decimos que un grupo de participantes es una *clique* si dos cualesquiera de ellos son amigos. (En particular, cualquier grupo con menos de dos participantes es una clique). Al número de elementos de una clique se le llama *tamaño.* Se sabe que en esta competencia el mayor de los tamaños de las cliques es par.
Demostrar que los participantes pueden distribuirse en dos aulas, de manera que el mayor de los tamaños de las cliques contenidas en un aula sea igual al mayor de los tamaños de las cliques contenidas en la otra.

Problema 4. En un triángulo ABC la bisectriz del ángulo BCA corta a la circunferencia circunscrita en R $(R \neq C)$, a la mediatriz de BC en P y a la mediatriz de AC en Q. El punto medio de BC es K y el punto medio de AC es L.
Demostrar que los triángulos RPK y RQL tienen áreas iguales.

Problema 5. Sean a y b enteros positivos tales que $4ab-1$ divide a $(4a^2-1)^2$. Demostrar que $a=b$.

Problema 6. Sea n un entero positivo. Se considera

$$S=\{(x,y,z)\ :\ x,y,z\in\{0,1,\ldots,n\},\ x+y+z>0\}$$

como un conjunto de $(n+1)^3-1$ puntos en el espacio tridimensional.
Determinar el menor número posible de planos cuya unión contiene todos los puntos de S pero no incluye a $(0,0,0)$.

Tiempo : 4 horas 30 minutos
Cada problema vale 7 puntos

49th INTERNATIONAL MATHEMATICAL OLYMPIAD
MADRID (SPAIN), JULY 10-22, 2008

Miércoles 16 de julio de 2008

Problema 1. Un triángulo acutángulo ABC tiene ortocentro H. La circunferencia con centro en el punto medio de BC que pasa por H corta a la recta BC en A_1 y A_2. La circunferencia con centro en el punto medio de CA que pasa por H corta a la recta CA en B_1 y B_2. La circunferencia con centro en el punto medio de AB que pasa por H corta a la recta AB en C_1 y C_2. Demostrar que A_1, A_2, B_1, B_2, C_1, C_2 están sobre una misma circunferencia.

Problema 2. (a) Demostrar que

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1 \qquad (*)$$

para todos los números reales x, y, z, distintos de 1, con $xyz = 1$.

(b) Demostrar que existen infinitas ternas de números racionales x, y, z, distintos de 1, con $xyz = 1$ para los cuales la expresión $(*)$ es una igualdad.

Problema 3. Demostrar que existen infinitos números enteros positivos n tales que $n^2 + 1$ tiene un divisor primo mayor que $2n + \sqrt{2n}$.

Tiempo : 4 horas y 30 minutos
Cada Problema vale 7 puntos

49th INTERNATIONAL MATHEMATICAL OLYMPIAD
MADRID (SPAIN), JULY 10-22, 2008

Jueves 17 de julio de 2008

Problema 4. Hallar todas las funciones $f : (0, \infty) \to (0, \infty)$ (es decir, las funciones f de los números reales positivos en los números reales positivos) tales que

$$\frac{\big(f(w)\big)^2 + \big(f(x)\big)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

para todos los números reales positivos w, x, y, z, que satisfacen $wx = yz$.

Problema 5. Sean n y k enteros positivos tales que $k \geq n$ y $k - n$ es par. Se tienen $2n$ lámparas numeradas 1, 2, $\dots$, $2n$, cada una de las cuales puede estar encendida o apagada. Inicialmente todas las lámparas están apagadas. Se consideran sucesiones de *pasos*: en cada paso se selecciona exactamente una lámpara y se cambia su estado (si está apagada se enciende, si está encendida se apaga).

Sea N el número de sucesiones de k pasos al cabo de los cuales las lámparas 1,2, $\dots$, n quedan todas encendidas, y las lámparas $n+1$, $\dots$, $2n$ quedan todas apagadas.

Sea M el número de sucesiones de k pasos al cabo de los cuales las lámparas 1,2, $\dots$, n quedan todas encendidas, y las lámparas $n+1$, $\dots$, $2n$ quedan todas apagadas sin haber sido nunca encendidas.

Calcular la razón N/M.

Problema 6. Sea $ABCD$ un cuadrilátero convexo tal que las longitudes de los lados BA y BC son diferentes. Sean ω_1 y ω_2 las circunferencias inscritas dentro de los triángulos ABC y ADC respectivamente. Se supone que existe una circunferencia ω tangente a la prolongación del segmento BA a continuación de A y tangente a la prolongación del segmento BC a continuación de C, la cual también es tangente a las rectas AD y CD. Demostrar que el punto de intersección de las tangentes comunes exteriores de ω_1 y ω_2 está sobre ω.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Miércoles 15 de julio de 2009

Problema 1. Sea n un entero positivo y sean $a_1, \ldots, a_k$ ($k \geq 2$) enteros distintos del conjunto $\{1, \ldots, n\}$, tales que n divide a $a_i(a_{i+1} - 1)$, para $i = 1, \ldots, k - 1$. Demostrar que n no divide a $a_k(a_1 - 1)$.

Problema 2. Sea ABC un triángulo con circuncentro O. Sean P y Q puntos interiores de los lados CA y AB, respectivamente. Sean K, L y M los puntos medios de los segmentos BP, CQ y PQ, respectivamente, y Γ la circunferencia que pasa por K, L y M. Se sabe que la recta PQ es tangente a la circunferencia Γ. Demostrar que $OP = OQ$.

Problema 3. Sea $s_1, s_2, s_3, \ldots$ una sucesión estrictamente creciente de enteros positivos tal que las subsucesiones

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \qquad \text{y} \qquad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

son ambas progresiones aritméticas. Demostrar que la sucesión $s_1, s_2, s_3, \ldots$ es también una progresión aritmética.

Jueves 16 de julio de 2009

Problema 4. Sea ABC un triángulo con $AB = AC$. Las bisectrices de los ángulos $\angle CAB$ y $\angle ABC$ cortan a los lados BC y CA en D y E, respectivamente. Sea K el incentro del triángulo ADC. Supongamos que el ángulo $\angle BEK = 45^\circ$. Determinar todos los posibles valores de $\angle CAB$.

Problema 5. Determinar todas las funciones f del conjunto de los enteros positivos en el conjunto de los enteros positivos tales que, para todos los enteros positivos a y b, existe un triángulo no degenerado cuyos lados miden

$$a, \; f(b) \; \text{ y } \; f(b + f(a) - 1).$$

(Un triángulo es *no degenerado* si sus vértices no están alineados).

Problema 6. Sean $a_1, a_2, \ldots, a_n$ enteros positivos distintos y M un conjunto de $n-1$ enteros positivos que no contiene al número $s = a_1 + a_2 + \cdots + a_n$. Un saltamontes se dispone a saltar a lo largo de la recta real. Empieza en el punto 0 y da n saltos hacia la derecha de longitudes $a_1, a_2, \ldots, a_n$, en algún orden. Demostrar que el saltamontes puede organizar los saltos de manera que nunca caiga en un punto de M.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Miércoles 7 de julio de 2010

Problema 1. Determine todas las funciones $f\colon \mathbb{R} \to \mathbb{R}$ tales que

$$f\big(\lfloor x \rfloor y\big) = f(x)\big\lfloor f(y) \big\rfloor$$

para todos los números $x, y \in \mathbb{R}$. ($\lfloor z \rfloor$ denota el mayor entero que es menor o igual que z.)

Problema 2. Sea ABC un triángulo, I su incentro y Γ su circunferencia circunscrita. La recta AI corta de nuevo a Γ en D. Sean E un punto en el arco $\widehat{BDC}$ y F un punto en el lado BC tales que

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$

Sea G el punto medio del segmento IF. Demuestre que las rectas DG y EI se cortan sobre Γ.

Problema 3. Sea $\mathbb{N}$ el conjunto de los enteros positivos. Determine todas las funciones $g\colon \mathbb{N} \to \mathbb{N}$ tales que

$$\big(g(m) + n\big)\big(m + g(n)\big)$$

es un cuadrado perfecto para todo $m, n \in \mathbb{N}$.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Jueves 8 de julio de 2010

Problema 4. Sea Γ la circunferencia circunscrita al triángulo ABC y P un punto en el interior del triángulo. Las rectas AP, BP y CP cortan de nuevo a Γ en los puntos K, L y M, respectivamente. La recta tangente a Γ en C corta a la recta AB en S. Si se tiene que $SC = SP$, demuestre que $MK = ML$.

Problema 5. En cada una de las seis cajas $B_1, B_2, B_3, B_4, B_5, B_6$ hay inicialmente sólo una moneda. Se permiten dos tipos de operaciones:

Tipo 1: Elegir una caja no vacía B_j, con $1 \leq j \leq 5$. Retirar una moneda de B_j y añadir dos monedas a B_{j+1}.

Tipo 2: Elegir una caja no vacía B_k, con $1 \leq k \leq 4$. Retirar una moneda de B_k e intercambiar los contenidos de las cajas (posiblemente vacías) B_{k+1} y B_{k+2}.

Determine si existe una sucesión finita de estas operaciones que deja a las cajas B_1, B_2, B_3, B_4, B_5 vacías y a la caja B_6 con exactamente $2010^{2010^{2010}}$ monedas. (Observe que $a^{b^c} = a^{(b^c)}$.)

Problema 6. Sea $a_1, a_2, a_3, \ldots$ una sucesión de números reales positivos. Se tiene que para algún entero positivo s,

$$a_n = \max\{a_k + a_{n-k} \text{ tal que } 1 \leq k \leq n-1\}$$

para todo $n > s$. Demuestre que existen enteros positivos ℓ y N, con $\ell \leq s$, tales que $a_n = a_\ell + a_{n-\ell}$ para todo $n \geq N$.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Language: **Spanish**

Day: **1**

Lunes, 18 de julio de 2011

Problema 1. Para cualquier conjunto $A = \{a_1, a_2, a_3, a_4\}$ de cuatro enteros positivos distintos se denota la suma $a_1 + a_2 + a_3 + a_4$ por s_A. Sea n_A el número de parejas (i, j) con $1 \leq i < j \leq 4$ para las cuales $a_i + a_j$ divide a s_A. Encontrar todos los conjuntos A de cuatro enteros positivos distintos para los cuales se alcanza el mayor valor posible de n_A.

Problema 2. Sea $\mathcal{S}$ un conjunto finito de dos o más puntos del plano. En $\mathcal{S}$ no hay tres puntos colineales. Un *remolino* es un proceso que empieza con una recta ℓ que pasa por un único punto P de $\mathcal{S}$. Se rota ℓ en el sentido de las manecillas del reloj con centro en P hasta que la recta encuentre por primera vez otro punto de $\mathcal{S}$ al cual llamaremos Q. Con Q como nuevo centro se sigue rotando la recta en el sentido de las manecillas del reloj hasta que la recta encuentre otro punto de $\mathcal{S}$. Este proceso continúa indefinidamente.
Demostrar que se puede elegir un punto P de $\mathcal{S}$ y una recta ℓ que pasa por P tales que el remolino que resulta usa cada punto de $\mathcal{S}$ como centro de rotación un número infinito de veces.

Problema 3. Sea f una función del conjunto de los números reales en si mismo que satisface

$$f(x + y) \leq y f(x) + f(f(x))$$

para todo par de números reales x, y. Demostrar que $f(x) = 0$ para todo $x \leq 0$.

Martes, 19 de julio de 2011

Problema 4. Sea $n > 0$ un entero. Se dispone de una balanza de dos platillos y de n pesas cuyos pesos son $2^0, 2^1, \ldots, 2^{n-1}$. Debemos colocar cada una de las n pesas en la balanza, una tras otra, de manera tal que el platillo de la derecha nunca sea más pesado que el platillo de la izquierda. En cada paso, elegimos una de las pesas que no ha sido colocada en la balanza, y la colocamos ya sea en el platillo de la izquierda o en el platillo de la derecha, hasta que todas las pesas hayan sido colocadas. Determinar el número de formas en las que esto se puede hacer.

Problema 5. Sea f una función del conjunto de los enteros al conjunto de los enteros positivos. Se supone que para cualesquiera dos enteros m y n, la diferencia $f(m) - f(n)$ es divisible por $f(m-n)$. Demostrar que para todos los enteros m y n con $f(m) \leq f(n)$, el número $f(n)$ es divisible por $f(m)$.

Problema 6. Sea ABC un triángulo acutángulo cuya circunferencia circunscrita es Γ. Sea ℓ una recta tangente a Γ, y sean ℓ_a, ℓ_b y ℓ_c las rectas que se obtienen al reflejar ℓ con respecto a las rectas BC, CA y AB, respectivamente. Demostrar que la circunferencia circunscrita del triángulo determinado por las rectas ℓ_a, ℓ_b y ℓ_c es tangente a la circunferencia Γ.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Martes, 10 de julio de 2012

Problema 1. Dado un triángulo ABC, el punto J es el centro del excírculo opuesto al vértice A. Este excírculo es tangente al lado BC en M, y a las rectas AB y AC en K y L, respectivamente. Las rectas LM y BJ se cortan en F, y las rectas KM y CJ se cortan en G. Sea S el punto de intersección de las rectas AF y BC, y sea T el punto de intersección de las rectas AG y BC.

Demostrar que M es el punto medio de ST.

(El *excírculo* de ABC opuesto al vértice A es la circunferencia que es tangente al segmento BC, a la prolongación del lado AB más allá de B, y a la prolongación del lado AC más allá de C.)

Problema 2. Sea $n \geq 3$ un entero, y sean $a_2, a_3, \ldots, a_n$ números reales positivos tales que $a_2 a_3 \cdots a_n = 1$. Demostrar que

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n \; > \; n^n.$$

Problema 3. El *juego de la adivinanza del mentiroso* es un juego para dos jugadores A y B. Las reglas del juego dependen de dos enteros positivos k y n conocidos por ambos jugadores.

Al principio del juego, el jugador A elige enteros x y N con $1 \leq x \leq N$. El jugador A mantiene x en secreto, y le dice a B el verdadero valor de N. A continuación, el jugador B intenta obtener información acerca de x formulando preguntas a A de la siguiente manera: en cada pregunta, B especifica un conjunto arbitrario S de enteros positivos (que puede ser uno de los especificados en alguna pregunta anterior), y pregunta a A si x pertenece a S. El jugador B puede hacer tantas preguntas de ese tipo como desee. Después de cada pregunta, el jugador A debe responderla inmediatamente con *sí* o *no*, pero puede mentir tantas veces como quiera. La única restricción es que entre cualesquiera $k+1$ respuestas consecutivas, al menos una debe ser verdadera.

Cuando B haya formulado tantas preguntas como haya deseado, debe especificar un conjunto X de a lo más n enteros positivos. Si x pertenece a X entonces gana B; en caso contrario, pierde. Demostrar que:

1. Si $n \geq 2^k$, entonces B puede asegurarse la victoria.

2. Para todo k suficientemente grande, existe un entero $n \geq 1,99^k$ tal que B no puede asegurarse la victoria.

53rd International Mathematical Olympiad

MAR DEL PLATA - ARGENTINA

Language: **Spanish**

Day: **2**

Miércoles, 11 de julio de 2012

Problema 4. Hallar todas las funciones $f\colon \mathbb{Z} \to \mathbb{Z}$ que cumplen la siguiente igualdad:

$$f(a)^2 + f(b)^2 + f(c)^2 \;=\; 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a),$$

para todos los enteros a, b, c que satisfacen $a + b + c = 0$.

($\mathbb{Z}$ denota el conjunto de los números enteros.)

Problema 5. Sea ABC un triángulo tal que $\angle BCA = 90°$, y sea D el pie de la altura desde C. Sea X un punto interior del segmento CD. Sea K el punto en el segmento AX tal que $BK = BC$. Análogamente, sea L el punto en el segmento BX tal que $AL = AC$. Sea M el punto de intersección de AL y BK.

Demostrar que $MK = ML$.

Problema 6. Hallar todos los enteros positivos n para los cuales existen enteros no negativos $a_1, a_2, \ldots, a_n$ tales que

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} \;=\; \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} \;=\; 1.$$

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Martes 23 de julio de 2013

Problema 1. Demostrar que para cualquier par de enteros positivos k y n, existen k enteros positivos $m_1, m_2, \ldots, m_k$ (no necesariamente distintos) tales que

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right)\ldots\left(1 + \frac{1}{m_k}\right).$$

Problema 2. Una configuración de 4027 puntos del plano, de los cuales 2013 son rojos y 2014 azules, y no hay tres de ellos que sean colineales, se llama *colombiana*. Trazando algunas rectas, el plano queda dividido en varias regiones. Una colección de rectas es *buena* para una configuración colombiana si se cumplen las dos siguientes condiciones:

- ninguna recta pasa por ninguno de los puntos de la configuración;
- ninguna región contiene puntos de ambos colores.

Hallar el menor valor de k tal que para cualquier configuración colombiana de 4027 puntos hay una colección buena de k rectas.

Problema 3. Supongamos que el excírculo del triángulo ABC opuesto al vértice A es tangente al lado BC en el punto A_1. Análogamente, se definen los puntos B_1 en CA y C_1 en AB, utilizando los excírculos opuestos a B y C respectivamente. Supongamos que el circuncentro del triángulo $A_1B_1C_1$ pertenece a la circunferencia que pasa por los vértices A, B y C. Demostrar que el triángulo ABC es rectángulo.

El excírculo *del triángulo ABC opuesto al vértice A es la circunferencia que es tangente al segmento BC, a la prolongación del lado AB más allá de B, y a la prolongación del lado AC más allá de C. Análogamente se definen los excírculos opuestos a los vértices B y C.*

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Miércoles 24 de julio de 2013

Problema 4. Sea ABC un triángulo acutángulo con ortocentro H, y sea W un punto sobre el lado BC, estrictamente entre B y C. Los puntos M y N son los pies de las alturas trazadas desde B y C respectivamente. Se denota por ω_1 la circunferencia que pasa por los vértices del triángulo BWN, y por X el punto de ω_1 tal que WX es un diámetro de ω_1. Análogamente, se denota por ω_2 la circunferencia que pasa por los vértices del triángulo CWM, y por Y el punto de ω_2 tal que WY es un diámetro de ω_2. Demostrar que los puntos X, Y y H son colineales.

Problema 5. Sea $\mathbb{Q}_{>0}$ el conjunto de los números racionales mayores que cero. Sea $f\colon \mathbb{Q}_{>0} \to \mathbb{R}$ una función que satisface las tres siguientes condiciones:

(i) $f(x)f(y) \geq f(xy)$ para todos los $x, y \in \mathbb{Q}_{>0}$;

(ii) $f(x+y) \geq f(x) + f(y)$ para todos los $x, y \in \mathbb{Q}_{>0}$;

(iii) existe un número racional $a > 1$ tal que $f(a) = a$.

Demostrar que $f(x) = x$ para todo $x \in \mathbb{Q}_{>0}$.

Problema 6. Sea $n \geq 3$ un número entero. Se considera una circunferencia en la que se han marcado $n+1$ puntos igualmente espaciados. Cada punto se etiqueta con uno de los números $0, 1, \ldots, n$ de manera que cada número se usa exactamente una vez. Dos distribuciones de etiquetas se consideran la misma si una se puede obtener de la otra por una rotación de la circunferencia. Una distribución de etiquetas se llama *bonita* si, para cualesquiera cuatro etiquetas $a < b < c < d$, con $a + d = b + c$, la cuerda que une los puntos etiquetados a y d no corta la cuerda que une los puntos etiquetados b y c.

Sea M el número de distribuciones bonitas y N el número de pares ordenados (x, y) de enteros positivos tales que $x + y \leq n$ y $\operatorname{mcd}(x, y) = 1$. Demostrar que

$$M = N + 1.$$

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Martes 8 de julio de 2014

Problema 1. Sea $a_0 < a_1 < a_2 < \cdots$ una sucesión infinita de números enteros positivos. Demostrar que existe un único entero $n \geq 1$ tal que

$$a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}.$$

Problema 2. Sea $n \geq 2$ un entero. Consideremos un tablero de tamaño $n \times n$ formado por n^2 cuadrados unitarios. Una configuración de n fichas en este tablero se dice que es *pacífica* si en cada fila y en cada columna hay exactamente una ficha. Hallar el mayor entero positivo k tal que, para cada configuración pacífica de n fichas, existe un cuadrado de tamaño $k \times k$ sin fichas en sus k^2 cuadrados unitarios.

Problema 3. En el cuadrilátero convexo $ABCD$, se tiene $\angle ABC = \angle CDA = 90°$. La perpendicular a BD desde A corta a BD en el punto H. Los puntos S y T están en los lados AB y AD, respectivamente, y son tales que H está dentro del triángulo SCT y

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Demostrar que la recta BD es tangente a la circunferencia circunscrita del triángulo TSH.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Miércoles 9 de julio de 2014

Problema 4. Los puntos P y Q están en el lado BC del triángulo acutángulo ABC de modo que $\angle PAB = \angle BCA$ y $\angle CAQ = \angle ABC$. Los puntos M y N están en las rectas AP y AQ, respectivamente, de modo que P es el punto medio de AM, y Q es el punto medio de AN. Demostrar que las rectas BM y CN se cortan en la circunferencia circunscrita del triángulo ABC.

Problema 5. Para cada entero positivo n, el Banco de Ciudad del Cabo produce monedas de valor $\frac{1}{n}$. Dada una colección finita de tales monedas (no necesariamente de distintos valores) cuyo valor total no supera $99+\frac{1}{2}$, demostrar que es posible separar esta colección en 100 o menos montones, de modo que el valor total de cada montón sea como máximo 1.

Problema 6. Un conjunto de rectas en el plano está en *posición general* si no hay dos que sean paralelas ni tres que pasen por el mismo punto. Un conjunto de rectas en posición general separa el plano en regiones, algunas de las cuales tienen área finita; a estas las llamamos sus *regiones finitas*. Demostrar que para cada n suficientemente grande, en cualquier conjunto de n rectas en posición general es posible colorear de azul al menos $\sqrt{n}$ de ellas de tal manera que ninguna de sus regiones finitas tenga todos los lados de su frontera azules.

Nota: A las soluciones que reemplacen $\sqrt{n}$ por $c\sqrt{n}$ se les otorgarán puntos dependiendo del valor de c.

Viernes 10 de julio de 2015

Problema 1. Decimos que un conjunto finito $\mathcal{S}$ de puntos del plano es *equilibrado* si para cada dos puntos distintos A y B en $\mathcal{S}$ hay un punto C en $\mathcal{S}$ tal que $AC = BC$. Decimos que $\mathcal{S}$ es *libre de centros* si para cada tres puntos distintos A, B, C en $\mathcal{S}$ no existe ningún punto P en $\mathcal{S}$ tal que $PA = PB = PC$.

(a) Demostrar que para todo $n \geq 3$ existe un conjunto de n puntos equilibrado.

(b) Determinar todos los enteros $n \geq 3$ para los que existe un conjunto de n puntos equilibrado y libre de centros.

Problema 2. Determinar todas las ternas (a, b, c) de enteros positivos tales que cada uno de los números

$$ab - c, \quad bc - a, \quad ca - b$$

es una potencia de 2.

(*Una potencia de* 2 *es un entero de la forma* 2^n, *donde* n *es un entero no negativo.*)

Problema 3. Sea ABC un triángulo acutángulo con $AB > AC$. Sea Γ su circunferencia circunscrita, H su ortocentro, y F el pie de la altura desde A. Sea M el punto medio del segmento BC. Sea Q el punto de Γ tal que $\angle HQA = 90°$ y sea K el punto de Γ tal que $\angle HKQ = 90°$. Supongamos que los puntos A, B, C, K y Q son todos distintos y están sobre Γ en este orden.

Demostrar que la circunferencia circunscrita al triángulo KQH es tangente a la circunferencia circunscrita al triángulo FKM.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Sábado 11 de julio de 2015

Problema 4. El triángulo ABC tiene circunferencia circunscrita Ω y circuncentro O. Una circunferencia Γ de centro A corta al segmento BC en los puntos D y E tales que B, D, E y C son todos diferentes y están en la recta BC en este orden. Sean F y G los puntos de intersección de Γ y Ω, tales que A, F, B, C y G están sobre Ω en este orden. Sea K el segundo punto de intersección de la circunferencia circunscrita al triángulo BDF y el segmento AB. Sea L el segundo punto de intersección de la circunferencia circunscrita al triángulo CGE y el segmento CA.

Supongamos que las rectas FK y GL son distintas y se cortan en el punto X. Demostrar que X está en la recta AO.

Problema 5. Sea $\mathbb{R}$ el conjunto de los números reales. Determinar todas las funciones $f : \mathbb{R} \to \mathbb{R}$ que satisfacen la ecuación

$$f\big(x + f(x+y)\big) + f(xy) \;=\; x + f(x+y) + yf(x)$$

para todos los números reales x, y.

Problema 6. La sucesión de enteros $a_1, a_2, \ldots$ satisface las siguientes condiciones:

(i) $1 \leq a_j \leq 2015$ para todo $j \geq 1$;

(ii) $k + a_k \neq \ell + a_\ell$ para todo $1 \leq k < \ell$.

Demostrar que existen dos enteros positivos b y N tales que

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leq 1007^2$$

para todos los enteros m y n que satisfacen $n > m \geq N$.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Lunes, 11 de julio de 2016

Problema 1. El triángulo BCF es rectángulo en B. Sea A el punto de la recta CF tal que $FA = FB$ y F está entre A y C. Se elige el punto D de modo que $DA = DC$ y AC es bisectriz del ángulo $\angle DAB$. Se elige el punto E de modo que $EA = ED$ y AD es bisectriz del ángulo $\angle EAC$. Sea M el punto medio de CF. Sea X el punto tal que $AMXE$ es un paralelogramo (con $AM \parallel EX$ y $AE \parallel MX$). Demostrar que las rectas BD, FX, y ME son concurrentes.

Problema 2. Hallar todos los enteros positivos n para los que en cada casilla de un tablero $n \times n$ se puede escribir una de las letras I, M y O de manera que:

- en cada fila y en cada columna, un tercio de las casillas tiene I, un tercio tiene M y un tercio tiene O; y
- en cualquier línea diagonal compuesta por un número de casillas divisible por 3, exactamente un tercio de las casillas tienen I, un tercio tiene M y un tercio tiene O.

Nota: Las filas y las columnas del tablero $n \times n$ se numeran desde 1 hasta n , en su orden natural. Así, cada casilla corresponde a un par de enteros positivos (i, j) con $1 \leqslant i, j \leqslant n$. Para $n > 1$, el tablero tiene $4n - 2$ *líneas diagonales* de dos tipos. Una línea diagonal del primer tipo se compone de todas las casillas (i, j) para las que $i + j$ es una constante, mientras que una línea diagonal del segundo tipo se compone de todas las casillas (i, j) para las que $i - j$ es una constante.

Problema 3. Sea $P = A_1A_2 \ldots A_k$ un polígono convexo en el plano. Los vértices A_1, A_2, $\ldots$, A_k tienen coordenadas enteras y se encuentran sobre una circunferencia. Sea S el área de P. Sea n un entero positivo impar tal que los cuadrados de las longitudes de los lados de P son todos números enteros divisibles por n. Demostrar que $2S$ es un entero divisible por n.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Martes, 12 de julio de 2016

Problema 4. Un conjunto de números enteros positivos se llama *fragante* si contiene al menos dos elementos, y cada uno de sus elementos tiene algún factor primo en común con al menos uno de los elementos restantes. Sea $P(n) = n^2 + n + 1$. Determinar el menor número entero positivo b para el cual existe algún número entero no negativo a tal que el conjunto

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

es fragante.

Problema 5. En la pizarra está escrita la ecuación

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

que tiene 2016 factores lineales en cada lado. Determinar el menor valor posible de k para el cual pueden borrarse exactamente k de estos 4032 factores lineales, de modo que al menos quede un factor en cada lado y la ecuación que resulte no tenga soluciones reales.

Problema 6. Se tienen $n \geqslant 2$ segmentos en el plano tales que cada par de segmentos se intersecan en un punto interior a ambos, y no hay tres segmentos que tengan un punto en común. Mafalda debe elegir uno de los extremos de cada segmento y colocar sobre él una rana mirando hacia el otro extremo. Luego silbará $n-1$ veces. En cada silbido, cada rana saltará inmediatamente hacia adelante hasta el siguiente punto de intersección sobre su segmento. Las ranas nunca cambian las direcciones de sus saltos. Mafalda quiere colocar las ranas de tal forma que nunca dos de ellas ocupen al mismo tiempo el mismo punto de intersección.

(a) Demostrar que si n es impar, Mafalda siempre puede lograr su objetivo.

(b) Demostrar que si n es par, Mafalda nunca logrará su objetivo.

Martes, 18 de julio de 2017

Problema 1. Para cada entero $a_0 > 1$, se define la sucesión $a_0, a_1, a_2, \ldots$ tal que para cada $n \geqslant 0$:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{si } \sqrt{a_n} \text{ es entero,} \\ a_n + 3 & \text{en otro caso.} \end{cases}$$

Determinar todos los valores de a_0 para los que existe un número A tal que $a_n = A$ para infinitos valores de n.

Problema 2. Sea $\mathbb{R}$ el conjunto de los números reales. Determinar todas las funciones $f\colon \mathbb{R} \to \mathbb{R}$ tales que, para cualesquiera números reales x e y,

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

Problema 3. Un conejo invisible y un cazador juegan como sigue en el plano euclídeo. El punto de partida A_0 del conejo, y el punto de partida B_0 del cazador son el mismo. Después de $n-1$ rondas del juego, el conejo se encuentra en el punto A_{n-1} y el cazador se encuentra en el punto B_{n-1}. En la n-ésima ronda del juego, ocurren tres hechos en el siguiente orden:

(i) El conejo se mueve de forma invisible a un punto A_n tal que la distancia entre A_{n-1} y A_n es exactamente 1.

(ii) Un dispositivo de rastreo reporta un punto P_n al cazador. La única información segura que da el dispositivo al cazador es que la distancia entre P_n y A_n es menor o igual que 1.

(iii) El cazador se mueve de forma visible a un punto B_n tal que la distancia entre B_{n-1} y B_n es exactamente 1.

¿Es siempre posible que, cualquiera que sea la manera en que se mueva el conejo y cualesquiera que sean los puntos que reporte el dispositivo de rastreo, el cazador pueda escoger sus movimientos de modo que después de 10^9 rondas el cazador pueda garantizar que la distancia entre él mismo y el conejo sea menor o igual que 100?

Miércoles, 19 de julio de 2017

Problema 4. Sean R y S puntos distintos sobre la circunferencia Ω tales que RS no es un diámetro de Ω. Sea ℓ la recta tangente a Ω en R. El punto T es tal que S es el punto medio del segmento RT. El punto J se elige en el menor arco RS de Ω de manera que Γ, la circunferencia circunscrita al triángulo JST, intersecta a ℓ en dos puntos distintos. Sea A el punto común de Γ y ℓ más cercano a R. La recta AJ corta por segunda vez a Ω en K. Demostrar que la recta KT es tangente a Γ.

Problema 5. Sea $N \geqslant 2$ un entero dado. Los $N(N+1)$ jugadores de un grupo de futbolistas, todos de distinta estatura, se colocan en fila. El técnico desea quitar $N(N-1)$ jugadores de esta fila, de modo que la fila resultante formada por los $2N$ jugadores restantes satisfaga las N condiciones siguientes:

(1) Que no quede nadie ubicado entre los dos jugadores más altos.

(2) Que no quede nadie ubicado entre el tercer jugador más alto y el cuarto jugador más alto.

$\vdots$

(N) Que no quede nadie ubicado entre los dos jugadores de menor estatura.

Demostrar que esto siempre es posible.

Problema 6. Un par ordenado (x, y) de enteros es *un punto primitivo* si el máximo común divisor de x e y es 1. Dado un conjunto finito S de puntos primitivos, demostrar que existen un entero positivo n y enteros a_0, a_1, $\ldots$, a_n tales que, para cada (x, y) de S, se cumple:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

Lunes 9 de julio de 2018

Problema 1. Sea Γ la circunferencia circunscrita al triángulo acutángulo ABC. Los puntos D y E están en los segmentos AB y AC, respectivamente, y son tales que $AD = AE$. Las mediatrices de BD y CE cortan a los arcos menores AB y AC de Γ en los puntos F y G, respectivamente. Demostrar que las rectas DE y FG son paralelas (o son la misma recta).

Problema 2. Hallar todos los enteros $n \geq 3$ para los que existen números reales $a_1, a_2, \ldots, a_{n+2}$, tales que $a_{n+1} = a_1$ y $a_{n+2} = a_2$, y

$$a_i a_{i+1} + 1 = a_{i+2}$$

para $i = 1, 2, \ldots, n$.

Problema 3. Un *triángulo anti-Pascal* es una disposición de números en forma de triángulo equilátero de tal manera que cada número, excepto los de la última fila, es el valor absoluto de la diferencia de los dos números que están inmediatamente debajo de él. Por ejemplo, la siguiente disposición es un triángulo anti-Pascal con cuatro filas que contiene todos los enteros desde 1 hasta 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Determinar si existe un triángulo anti-Pascal con 2018 filas que contenga todos los enteros desde 1 hasta $1 + 2 + \cdots + 2018$.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Martes 10 de julio de 2018

Problema 4. Un *lugar* es un punto (x, y) en el plano tal que x, y son ambos enteros positivos menores o iguales que 20.

Al comienzo, cada uno de los 400 lugares está vacío. Ana y Beto colocan piedras alternadamente, comenzando con Ana. En su turno, Ana coloca una nueva piedra roja en un lugar vacío tal que la distancia entre cualesquiera dos lugares ocupados por piedras rojas es distinto de $\sqrt{5}$. En su turno, Beto coloca una nueva piedra azul en cualquier lugar vacío. (Un lugar ocupado por una piedra azul puede estar a cualquier distancia de cualquier otro lugar ocupado.) Ellos paran cuando alguno de los dos no pueda colocar una piedra.

Hallar el mayor K tal que Ana pueda asegurarse de colocar al menos K piedras rojas, sin importar cómo Beto coloque sus piedras azules.

Problema 5. Sea $a_1, a_2, \ldots$ una sucesión infinita de enteros positivos. Supongamos que existe un entero $N > 1$ tal que para cada $n \geq N$ el número

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

es entero. Demostrar que existe un entero positivo M tal que $a_m = a_{m+1}$ para todo $m \geq M$.

Problema 6. Un cuadrilátero convexo $ABCD$ satisface $AB \cdot CD = BC \cdot DA$. El punto X en el interior de $ABCD$ es tal que

$$\angle XAB = \angle XCD \quad \text{y} \quad \angle XBC = \angle XDA.$$

Demostrar que $\angle BXA + \angle DXC = 180^\circ$.

Language: Spanish

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

3

Problems

3.1 The First IMO
Bucharest–Brasov, Romania, July 23–31, 1959

3.1.1 Contest Problems

First Day

1. **(POL)** For every integer n prove that the fraction $\frac{21n+4}{14n+3}$ cannot be reduced any further.

2. **(ROM)** For which real numbers x do the following equations hold:

$$\begin{aligned}&\text{(a)}\ \sqrt{x+\sqrt{2x-1}}+\sqrt{x+\sqrt{2x-1}}=\sqrt{2}\ ,\\ &\text{(b)}\ \sqrt{x+\sqrt{2x-1}}+\sqrt{x+\sqrt{2x-1}}=1\ ,\\ &\text{(c)}\ \sqrt{x+\sqrt{2x-1}}+\sqrt{x+\sqrt{2x-1}}=2\ ?\end{aligned}$$

3. **(HUN)** Let x be an angle and let the real numbers a, b, c, $\cos x$ satisfy the following equation:

$$a\cos^2 x + b\cos x + c = 0\ .$$

Write the analogous quadratic equation for a, b, c, $\cos 2x$. Compare the given and the obtained equality for $a = 4$, $b = 2$, $c = -1$.

Second Day

4. **(HUN)** Construct a right-angled triangle whose hypotenuse c is given if it is known that the median from the right angle equals the geometric mean of the remaining two sides of the triangle.

5. **(ROM)** A segment AB is given and on it a point M. On the same side of AB squares $AMCD$ and $BMFE$ are constructed. The circumcircles of the two squares, whose centers are P and Q, intersect in M and another point N.

(a) Prove that lines FA and BC intersect at N.
(b) Prove that all such constructed lines MN pass through the same point S, regardless of the selection of M.
(c) Find the locus of the midpoints of all segments PQ, as M varies along the segment AB.

6. **(CZS)** Let α and β be two planes intersecting at a line p. In α a point A is given and in β a point C is given, neither of which lies on p. Construct B in α and D in β such that $ABCD$ is an equilateral trapezoid, $AB \parallel CD$, in which a circle can be inscribed.

3.2 The Second IMO
Bucharest–Sinaia, Romania, July 18–25, 1960

3.2.1 Contest Problems

First Day

1. **(BUL)** Find all the three-digit numbers for which one obtains, when dividing the number by 11, the sum of the squares of the digits of the initial number.

2. **(HUN)** For which real numbers x does the following inequality hold:
$$\frac{4x^2}{(1-\sqrt{1+2x})^2} < 2x+9\ ?$$

3. **(ROM)** A right-angled triangle ABC is given for which the hypotenuse BC has length a and is divided into n equal segments, where n is odd. Let α be the angle with which the point A sees the segment containing the middle of the hypotenuse. Prove that
$$\tan\alpha = \frac{4nh}{(n^2-1)a},$$
where h is the height of the triangle.

Second Day

4. **(HUN)** Construct a triangle ABC whose lengths of heights h_a and h_b (from A and B, respectively) and length of median m_a (from A) are given.

5. **(CZS)** A cube $ABCDA'B'C'D'$ is given.
   (a) Find the locus of all midpoints of segments XY, where X is any point on segment AC and Y any point on segment $B'D'$.
   (b) Find the locus of all points Z on segments XY such that $\overrightarrow{ZY} = 2\overrightarrow{XZ}$.

6. **(BUL)** An isosceles trapezoid with bases a and b and height h is given.
   (a) On the line of symmetry construct the point P such that both (non-base) sides are seen from P with an angle of $90°$.
   (b) Find the distance of P from one of the bases of the trapezoid.
   (c) Under what conditions for a, b, and h can the point P be constructed (analyze all possible cases)?

7. **(GDR)** A sphere is inscribed in a regular cone. Around the sphere a cylinder is circumscribed so that its base is in the same plane as the base of the cone. Let V_1 be the volume of the cone and V_2 the volume of the cylinder.
   (a) Prove that $V_1 = V_2$ is impossible.
   (b) Find the smallest k for which $V_1 = kV_2$, and in this case construct the angle at the vertex of the cone.

3.3 The Third IMO
Budapest–Veszprem, Hungary, July 6–16, 1961

3.3.1 Contest Problems

First Day

1. **(HUN)** Solve the following system of equations:
$$\begin{aligned} x+y+z &= a, \\ x^2+y^2+z^2 &= b^2, \\ xy &= z^2, \end{aligned}$$
where a and b are given real numbers. What conditions must hold on a and b for the solutions to be positive and distinct?

2. **(POL)** Let a, b, and c be the lengths of a triangle whose area is S. Prove that
$$a^2+b^2+c^2 \ge 4S\sqrt{3}\ .$$
In what case does equality hold?

3. **(BUL)** Solve the equation $\cos^n x - \sin^n x = 1$, where n is a given positive integer.

Second Day

4. **(GDR)** In the interior of $\triangle P_1P_2P_3$ a point P is given. Let Q_1, Q_2, and Q_3 respectively be the intersections of PP_1, PP_2, and PP_3 with the opposing edges of $\triangle P_1P_2P_3$. Prove that among the ratios PP_1/PQ_1, PP_2/PQ_2, and PP_3/PQ_3 there exists at least one not larger than 2 and at least one not smaller than 2.

5. **(CZS)** Construct a triangle ABC if the following elements are given: $AC = b$, $AB = c$, and $\measuredangle AMB = \omega$ ($\omega < 90^o$), where M is the midpoint of BC. Prove that the construction has a solution if and only if
$$b\tan\frac{\omega}{2} \le c < b\ .$$
In what case does equality hold?

6. **(ROM)** A plane ϵ is given and on one side of the plane three noncollinear points A, B, and C such that the plane determined by them is not parallel to ϵ. Three arbitrary points A', B', and C' in ϵ are selected. Let L, M, and N be the midpoints of AA', BB', and CC', and G the centroid of $\triangle LMN$. Find the locus of all points obtained for G as A', B', and C' are varied (independently of each other) across ϵ.

3.4 The Fourth IMO
Prague–Hluboka, Czechoslovakia, July 7–15, 1962

3.4.1 Contest Problems

First Day

1. **(POL)** Find the smallest natural number n with the following properties:
   (a) In decimal representation it ends with 6.
   (b) If we move this digit to the front of the number, we get a number 4 times larger.

2. **(HUN)** Find all real numbers x for which
$$\sqrt{3-x}-\sqrt{x+1}>\frac{1}{2}\ .$$

3. **(CZS)** A cube $ABCDA'B'C'D'$ is given. The point X is moving at a constant speed along the square $ABCD$ in the direction from A to B. The point Y is moving with the same constant speed along the square $BCC'B'$ in the direction from B' to C'. Initially, X and Y start out from A and B' respectively. Find the locus of all the midpoints of XY.

Second Day

4. **(ROM)** Solve the equation
$$\cos^2 x+\cos^2 2x+\cos^2 3x=1\ .$$

5. **(BUL)** On the circle k three points A, B, and C are given. Construct the fourth point on the circle D such that one can inscribe a circle in $ABCD$.

6. **(GDR)** Let ABC be an isosceles triangle with circumradius r and inradius ρ. Prove that the distance d between the circumcenter and incenter is given by
$$d=\sqrt{r(r-2\rho)}\ .$$

7. **(USS)** Prove that a tetrahedron $SABC$ has five different spheres that touch all six lines determined by its edges if and only if it is regular.

3.5 The Fifth IMO
Wroclaw, Poland, July 5–13, 1963

3.5.1 Contest Problems

First Day

1. **(CZS)** Determine all real solutions of the equation $\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$, where p is a real number.

2. **(USS)** Find the locus of points in space that are vertices of right angles of which one ray passes through a given point and the other intersects a given segment.

3. **(HUN)** Prove that if all the angles of a convex n-gon are equal and the lengths of consecutive edges $a_1, \dots, a_n$ satisfy $a_1 \ge a_2 \ge \cdots \ge a_n$, then $a_1 = a_2 = \cdots = a_n$.

Second Day

4. **(USS)** Find all solutions $x_1, \dots, x_5$ to the system of equations

$$\begin{cases} x_5 + x_2 = yx_1, \\ x_1 + x_3 = yx_2, \\ x_2 + x_4 = yx_3, \\ x_3 + x_5 = yx_4, \\ x_4 + x_1 = yx_5, \end{cases}$$

where y is a real parameter.

5. **(GDR)** Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$.

6. **(HUN)** Five students A, B, C, D, and E have taken part in a certain competition. Before the competition, two persons X and Y tried to guess the rankings. X thought that the ranking would be A, B, C, D, E; and Y thought that the ranking would be D, A, E, C, B. At the end, it was revealed that X didn't guess correctly any rankings of the participants, and moreover, didn't guess any of the orderings of pairs of consecutive participants. On the other hand, Y guessed the correct rankings of two participants and the correct ordering of two pairs of consecutive participants. Determine the rankings of the competition.

3.6 The Sixth IMO
Moscow, Soviet Union, June 30–July 10, 1964

3.6.1 Contest Problems

First Day

1. **(CZS)** (a) Find all natural numbers n such that the number $2^n - 1$ is divisible by 7.
   (b) Prove that for all natural numbers n the number $2^n + 1$ is not divisible by 7.
2. **(HUN)** Denote by a, b, c the lengths of the sides of a triangle. Prove that
$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leq 3abc.$$
3. **(YUG)** The incircle is inscribed in a triangle ABC with sides a, b, c. Three tangents to the incircle are drawn, each of which is parallel to one side of the triangle ABC. These tangents form three smaller triangles (internal to $\triangle ABC$) with the sides of $\triangle ABC$. In each of these triangles an incircle is inscribed. Determine the sum of areas of all four incircles.

Second Day

4. **(HUN)** Each of 17 students talked with every other student. They all talked about three different topics. Each pair of students talked about one topic. Prove that there are three students that talked about the same topic among themselves.
5. **(ROM)** Five points are given in the plane. Among the lines that connect these five points, no two coincide and no two are parallel or perpendicular. Through each point we construct an altitude to each of the other lines. What is the maximal number of intersection points of these altitudes (excluding the initial five points)?
6. **(POL)** Given a tetrahedron $ABCD$, let D_1 be the centroid of the triangle ABC and let A_1, B_1, C_1 be the intersection points of the lines parallel to DD_1 and passing through the points A, B, C with the opposite faces of the tetrahedron. Prove that the volume of the tetrahedron $ABCD$ is one-third the volume of the tetrahedron $A_1B_1C_1D_1$. Does the result remain true if the point D_1 is replaced with any point inside the triangle ABC?

3.7 The Seventh IMO
Berlin, DR Germany, July 3–13, 1965

3.7.1 Contest Problems

First Day

1. **(YUG)** Find all real numbers $x \in [0, 2\pi]$ such that

$$2\cos x \le |\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}| \le \sqrt{2}.$$

2. **(POL)** Consider the system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0, \end{cases}$$

whose coefficients satisfy the following conditions:
(a) a_{11}, a_{22}, a_{33} are positive real numbers;
(b) all other coefficients are negative;
(c) in each of the equations the sum of the coefficients is positive.
Prove that $x_1 = x_2 = x_3 = 0$ is the only solution to the system.

3. **(CZS)** A tetrahedron $ABCD$ is given. The lengths of the edges AB and CD are a and b, respectively, the distance between the lines AB and CD is d, and the angle between them is equal to ω. The tetrahedron is divided into two parts by the plane π parallel to the lines AB and CD. Calculate the ratio of the volumes of the parts if the ratio between the distances of the plane π from AB and CD is equal to k.

Second Day

4. **(USS)** Find four real numbers x_1, x_2, x_3, x_4 such that the sum of any of the numbers and the product of other three is equal to 2.

5. **(ROM)** Given a triangle OAB such that $\angle AOB = \alpha < 90°$, let M be an arbitrary point of the triangle different from O. Denote by P and Q the feet of the perpendiculars from M to OA and OB, respectively. Let H be the orthocenter of the triangle OPQ. Find the locus of points H when:
(a) M belongs to the segment AB;
(b) M belongs to the interior of $\triangle OAB$.

6. **(POL)** We are given $n \ge 3$ points in the plane. Let d be the maximal distance between two of the given points. Prove that the number of pairs of points whose distance is equal to d is less than or equal to n.

3.8 The Eighth IMO
Sofia, Bulgaria, July 3–13, 1966

3.8.1 Contest Problems

First Day

1. **(USS)** Three problems A, B, and C were given on a mathematics olympiad. All 25 students solved at least one of these problems. The number of students who solved B and not A is twice the number of students who solved C and not A. The number of students who solved only A is greater by 1 than the number of students who along with A solved at least one other problem. Among the students who solved only one problem, half solved A. How many students solved only B?

2. **(HUN)** If a, b, and c are the sides and α, β, and γ the respective angles of the triangle for which $a + b = \tan\frac{\gamma}{2}(a\tan\alpha + b\tan\beta)$, prove that the triangle is isosceles.

3. **(BUL)** Prove that the sum of distances from the center of the circumsphere of the regular tetrahedron to its four vertices is less than the sum of distances from any other point to the four vertices.

Second Day

4. **(YUG)** Prove the following equality:

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \frac{1}{\sin 8x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x,$$

where $n \in \mathbb{N}$ and $x \notin \pi\mathbb{Z}/2^k$ for every $k \in \mathbb{N}$.

5. **(CZS)** Solve the following system of equations:

$$\begin{aligned}
|a_1 - a_2|x_2 + |a_1 - a_3|x_3 + |a_1 - a_4|x_4 &= 1,\\
|a_2 - a_1|x_1 + |a_2 - a_3|x_3 + |a_2 - a_4|x_4 &= 1,\\
|a_3 - a_1|x_1 + |a_3 - a_2|x_2 + |a_3 - a_4|x_4 &= 1,\\
|a_4 - a_1|x_1 + |a_4 - a_2|x_2 + |a_4 - a_3|x_3 &= 1,
\end{aligned}$$

where a_1, a_2, a_3, and a_4 are mutually distinct real numbers.

6. **(POL)** Let M, K, and L be points on (AB), (BC), and (CA), respectively. Prove that the area of at least one of the three triangles $\triangle MAL$, $\triangle KBM$, and $\triangle LCK$ is less than or equal to one-fourth the area of $\triangle ABC$.

3.8.2 Some Longlisted Problems 1959–1966

1. **(CZS)** We are given $n > 3$ points in the plane, no three of which lie on a line. Does there necessarily exist a circle that passes through at least three of the given points and contains none of the other given points in its interior?

2. **(GDR)** Given n positive real numbers $a_1, a_2, \dots, a_n$ such that $a_1 a_2 \cdots a_n = 1$, prove that
$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n.$$

3. **(BUL)** A regular triangular prism has height h and a base of side length a. Both bases have small holes in the centers, and the inside of the three vertical walls has a mirror surface. Light enters through the small hole in the top base, strikes each vertical wall once and leaves through the hole in the bottom. Find the angle at which the light enters and the length of its path inside the prism.

4. **(POL)** Five points in the plane are given, no three of which are collinear. Show that some four of them form a convex quadrilateral.

5. **(USS)** Prove the inequality
$$\tan \frac{\pi \sin x}{4 \sin \alpha} + \tan \frac{\pi \cos x}{4 \cos \alpha} > 1$$
for any x, α with $0 \leq x \leq \pi/2$ and $\pi/6 < y < \pi/3$.

6. **(USS)** A convex planar polygon $\mathcal{M}$ with perimeter l and area S is given. Let $M(R)$ be the set of all points in space that lie a distance at most R from a point of $\mathcal{M}$. Show that the volume $V(R)$ of this set equals
$$V(R) = \frac{4}{3}\pi R^3 + \frac{\pi}{2} l R^2 + 2SR.$$

7. **(USS)** For which arrangements of two infinite circular cylinders does their intersection lie in a plane?

8. **(USS)** We are given a bag of sugar, a two-pan balance, and a weight of 1 gram. How do we obtain 1 kilogram of sugar in the smallest possible number of weighings?

9. **(ROM)** Find x such that
$$\frac{\sin 3x \cos(60^\circ - 4x) + 1}{\sin(60^\circ - 7x) - \cos(30^\circ + x) + m} = 0,$$
where m is a fixed real number.

10. **(GDR)** How many real solutions are there to the equation $x = 1964 \sin x - 189$?

11. **(CZS)** Does there exist an integer z that can be written in two different ways as $z = x! + y!$, where x, y are natural numbers with $x \le y$?

12. **(BUL)** Find digits x, y, z such that the equality
$$\sqrt{\underbrace{\overline{xx\cdots x}}_{2n} - \underbrace{\overline{yy\cdots y}}_{n}} = \underbrace{\overline{zz\cdots z}}_{n}$$
holds for at least two values of $n \in \mathbb{N}$, and in that case find all n for which this equality is true.

13. **(YUG)** Let $a_1, a_2, \dots, a_n$ be positive real numbers. Prove the inequality
$$\binom{n}{2}\sum_{i<j}\frac{1}{a_i a_j} \ge 4\left(\sum_{i<j}\frac{1}{a_i + a_j}\right)^2$$
and find the conditions on the numbers a_i for equality to hold.

14. **(POL)** Compute the largest number of regions into which one can divide a disk by joining n points on its circumference.

15. **(POL)** Points A, B, C, D lie on a circle such that AB is a diameter and CD is not. If the tangents at C and D meet at P while AC and BD meet at Q, show that PQ is perpendicular to AB.

16. **(CZS)** We are given a circle K with center S and radius 1 and a square Q with center M and side 2. Let XY be the hypotenuse of an isosceles right triangle XYZ. Describe the locus of points Z as X varies along K and Y varies along the boundary of Q.

17. **(ROM)** Suppose $ABCD$ and $A'B'C'D'$ are two parallelograms arbitrarily arranged in space, and let points M, N, P, Q divide the segments AA', BB', CC', DD' respectively in equal ratios.
    (a) Show that $MNPQ$ is a parallelogram;
    (b) Find the locus of $MNPQ$ as M varies along the segment AA'.

18. **(HUN)** Solve the equation $\frac{1}{\sin x} + \frac{1}{\cos x} = \frac{1}{p}$, where p is a real parameter. Discuss for which values of p the equation has at least one real solution and determine the number of solutions in $[0, 2\pi)$ for a given p.

19. **(HUN)** Construct a triangle given the three exradii.

20. **(HUN)** We are given three equal rectangles with the same center in three mutually perpendicular planes, with the long sides also mutually perpendicular. Consider the polyhedron with vertices at the vertices of these rectangles.
    (a) Find the volume of this polyhedron;
    (b) can this polyhedron be regular, and under what conditions?

21. **(BUL)** Prove that the volume V and the lateral area S of a right circular cone satisfy the inequality $\left(\frac{6V}{\pi}\right)^2 \le \left(\frac{2S}{\pi\sqrt{3}}\right)^3$. When does equality occur?

22. **(BUL)** Assume that two parallelograms P, P' of equal areas have sides a, b and a', b' respectively such that $a' \le a \le b \le b'$ and a segment of length b' can be placed inside P. Prove that P and P' can be partitioned into four pairwise congruent parts.

23. **(BUL)** Three faces of a tetrahedron are right triangles, while the fourth is not an obtuse triangle.
    (a) Prove that a necessary and sufficient condition for the fourth face to be a right triangle is that at some vertex exactly two angles are right.
    (b) Prove that if all the faces are right triangles, then the volume of the tetrahedron equals one -sixth the product of the three smallest edges not belonging to the same face.

24. **(POL)** There are $n \ge 2$ people in a room. Prove that there exist two among them having equal numbers of friends in that room. (Friendship is always mutual.)

25. **(GDR)** Show that $\tan 7^\circ 30' = \sqrt{6} + \sqrt{2} - \sqrt{3} - 2$.

26. **(CZS)** (a) Prove that $(a_1 + a_2 + \cdots + a_k)^2 \le k(a_1^2 + \cdots + a_k^2)$, where $k \ge 1$ is a natural number and $a_1, \ldots, a_k$ are arbitrary real numbers.
    (b) If real numbers $a_1, \ldots, a_n$ satisfy

$$a_1 + a_2 + \cdots + a_n \ge \sqrt{(n-1)(a_1^2 + \cdots + a_n^2)},$$

    show that they are all nonnegative.

27. **(GDR)** We are given a circle K and a point P lying on a line g. Construct a circle that passes through P and touches K and g.

28. **(CZS)** Let there be given a circle with center S and radius 1 in the plane, and let ABC be an arbitrary triangle circumscribed about the circle such that $SA \le SB \le SC$. Find the loci of the vertices A, B, C.

29. **(ROM)** (a) Find the number of ways 500 can be represented as a sum of consecutive integers.
    (b) Find the number of such representations for $N = 2^\alpha 3^\beta 5^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$. Which of these representations consist only of natural numbers?
    (c) Determine the number of such representations for an arbitrary natural number N.

30. **(ROM)** If n is a natural number, prove that
    (a) $\log_{10}(n+1) > \frac{3}{10n} + \log_{10} n$;
    (b) $\log n! > \frac{3n}{10}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - 1\right)$.

31. **(ROM)** Solve the equation $|x^2 - 1| + |x^2 - 4| = mx$ as a function of the parameter m. Which pairs (x, m) of integers satisfy this equation?

32. **(BUL)** The sides a, b, c of a triangle ABC form an arithmetic progression; the sides of another triangle $A_1B_1C_1$ also form an arithmetic progression.

Suppose that $\angle A = \angle A_1$. Prove that the triangles ABC and $A_1B_1C_1$ are similar.

33. **(BUL)** Two circles touch each other from inside, and an equilateral triangle is inscribed in the larger circle. From the vertices of the triangle one draws segments tangent to the smaller circle. Prove that the length of one of these segments equals the sum of the lengths of the other two.

34. **(BUL)** Determine all pairs of positive integers (x, y) satisfying the equation $2^x = 3^y + 5$.

35. **(POL)** If a, b, c, d are integers such that ad is odd and bc is even, prove that at least one root of the polynomial $ax^3 + bx^2 + cx + d$ is irrational.

36. **(POL)** Let $ABCD$ be a cyclic quadrilateral. Show that the centroids of the triangles ABC, CDA, BCD, DAB lie on a circle.

37. **(POL)** Prove that the perpendiculars drawn from the midpoints of the sides of a cyclic quadrilateral to the opposite sides meet at one point.

38. **(ROM)** Two concentric circles have radii R and r respectively. Determine the greatest possible number of circles that are tangent to both these circles and mutually nonintersecting. Prove that this number lies between $\frac{3}{2} \cdot \frac{\sqrt{R}+\sqrt{r}}{\sqrt{R}-\sqrt{r}} - 1$ and $\frac{63}{20} \cdot \frac{R+r}{R-r}$.

39. **(ROM)** In a plane, a circle with center O and radius R and two points A, B are given.
    (a) Draw a chord CD parallel to AB so that AC and BD intersect at a point P on the circle.
    (b) Prove that there are two possible positions of point P, say P_1, P_2, and find the distance between them if $OA = a$, $OB = b$, $AB = d$.

40. **(CZS)** For a positive real number p, find all real solutions to the equation
$$\sqrt{x^2 + 2px - p^2} - \sqrt{x^2 - 2px - p^2} = 1.$$

41. **(CZS)** If $A_1A_2 \dots A_n$ is a regular n-gon ($n \geq 3$), how many different obtuse triangles $A_iA_jA_k$ exist?

42. **(CZS)** Let $a_1, a_2, \dots, a_n$ ($n \geq 2$) be a sequence of integers. Show that there is a subsequence $a_{k_1}, a_{k_2}, \dots, a_{k_m}$, where $1 \leq k_1 < k_2 < \dots < k_m \leq n$, such that $a_{k_1}^2 + a_{k_2}^2 + \dots + a_{k_m}^2$ is divisible by n.

43. **(CZS)** Five points in a plane are given, no three of which are collinear. Every two of them are joined by a segment, colored either red or gray, so that no three segments form a triangle colored in one color.
    (a) Prove that (1) every point is a vertex of exactly two red and two gray segments, and (2) the red segments form a closed path that passes through each point.
    (b) Give an example of such a coloring.

44. **(YUG)** What is the greatest number of balls of radius $1/2$ that can be placed within a rectangular box of size $10 \times 10 \times 1$?

45. **(YUG)** An alphabet consists of n letters. What is the maximal length of a word, if
    (i) two neighboring letters in a word are always different, and
    (ii) no word $abab$ ($a \neq b$) can be obtained by omitting letters from the given word?

46. **(YUG)** Let
$$f(a,b,c) = \left| \frac{|b-a|}{|ab|} + \frac{b+a}{ab} - \frac{2}{c} \right| + \frac{|b-a|}{|ab|} + \frac{b+a}{ab} + \frac{2}{c}.$$
Prove that $f(a,b,c) = 4\max\{1/a, 1/b, 1/c\}$.

47. **(ROM)** Find the number of lines dividing a given triangle into two parts of equal area which determine the segment of minimum possible length inside the triangle. Compute this minimum length in terms of the sides a, b, c of the triangle.

48. **(USS)** Find all positive numbers p for which the equation $x^2+px+3p=0$ has integral roots.

49. **(USS)** Two mirror walls are placed to form an angle of measure α. There is a candle inside the angle. How many reflections of the candle can an observer see?

50. **(USS)** Given a quadrangle of sides a, b, c, d and area S, show that $S \leq \frac{a+c}{2} \cdot \frac{b+d}{2}$.

51. **(USS)** In a school, n children numbered 1 to n are initially arranged in the order $1, 2, \dots, n$. At a command, every child can either exchange its position with any other child or not move. Can they rearrange into the order $n, 1, 2, \dots, n-1$ after two commands?

52. **(USS)** A figure of area 1 is cut out from a sheet of paper and divided into 10 parts, each of which is colored in one of 10 colors. Then the figure is turned to the other side and again divided into 10 parts (not necessarily in the same way). Show that it is possible to color these parts in the 10 colors so that the total area of the portions of the figure both of whose sides are of the same color is at least 0.1.

53. **(USS, 1966)** Prove that in every convex hexagon of area S one can draw a diagonal that cuts off a triangle of area not exceeding $\frac{1}{6}S$.

54. **(USS, 1966)** Find the last two digits of a sum of eighth powers of 100 consecutive integers.

55. **(USS, 1966)** Given the vertex A and the centroid M of a triangle ABC, find the locus of vertices B such that all the angles of the triangle lie in the interval $[40°, 70°]$.

56. **(USS**, 1966**)** Let $ABCD$ be a tetrahedron such that $AB \perp CD$, $AC \perp BD$, and $AD \perp BC$. Prove that the midpoints of the edges of the tetrahedron lie on a sphere.

57. **(USS**, 1966**)** Is it possible to choose a set of 100 (or 200) points on the boundary of a cube such that this set is fixed under each isometry of the cube into itself? Justify your answer.

3.9 The Ninth IMO
Cetinje, Yugoslavia, July 2–13, 1967

3.9.1 Contest Problems

First Day (July 5)

1. $ABCD$ is a parallelogram; $AB = a$, $AD = 1$, α is the size of $\angle DAB$, and the three angles of the triangle ABD are acute. Prove that the four circles K_A, K_B, K_C, K_D, each of radius 1, whose centers are the vertices A, B, C, D, cover the parallelogram if and only if $a \le \cos\alpha + \sqrt{3}\sin\alpha$.
2. Exactly one side of a tetrahedron is of length greater than 1. Show that its volume is less than or equal to $1/8$.
3. Let k, m, and n be positive integers such that $m + k + 1$ is a prime number greater than $n + 1$. Write c_s for $s(s + 1)$. Prove that the product $(c_{m+1} - c_k)(c_{m+2} - c_k)\cdots(c_{m+n} - c_k)$ is divisible by the product $c_1c_2\cdots c_n$.

Second Day (July 6)

4. The triangles $A_0B_0C_0$ and $A'B'C'$ have all their angles acute. Describe how to construct one of the triangles ABC, similar to $A'B'C'$ and circumscribing $A_0B_0C_0$ (so that A, B, C correspond to A', B', C', and AB passes through C_0, BC through A_0, and CA through B_0). Among these triangles ABC describe, and prove, how to construct the triangle with the maximum area.
5. Consider the sequence (c_n):
$$\begin{array}{lcl} c_1 & = & a_1 + a_2 + \cdots + a_8, \\ c_2 & = & a_1^2 + a_2^2 + \cdots + a_8^2, \\ \cdots & & \cdots\cdots\cdots \\ c_n & = & a_1^n + a_2^n + \cdots + a_8^n, \\ \cdots & & \cdots\cdots\cdots \end{array}$$
where $a_1, a_2, \ldots, a_8$ are real numbers, not all equal to zero. Given that among the numbers of the sequence (c_n) there are infinitely many equal to zero, determine all the values of n for which $c_n = 0$.
6. In a sports competition lasting n days there are m medals to be won. On the first day, one medal and $1/7$ of the remaining $m - 1$ medals are won. On the second day, 2 medals and $1/7$ of the remainder are won. And so on. On the nth day exactly n medals are won. How many days did the competition last and what was the total number of medals?

3.9.2 Longlisted Problems

1. **(BUL 1)** Prove that all numbers in the sequence

$$\frac{107811}{3}, \quad \frac{110778111}{3}, \quad \frac{111077781111}{3}, \quad \ldots$$

are perfect cubes.

2. **(BUL 2)** Prove that $\frac{1}{3}n^2 + \frac{1}{2}n + \frac{1}{6} \ge (n!)^{2/n}$ (n is a positive integer) and that equality is possible only in the case $n = 1$.

3. **(BUL 3)** Prove the trigonometric inequality $\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{16}$, where $x \in (0, \pi/2)$.

4. **(BUL 4)** Suppose medians m_a and m_b of a triangle are orthogonal. Prove that:
   (a) The medians of that triangle correspond to the sides of a right-angled triangle.
   (b) The inequality
   $$5(a^2 + b^2 - c^2) \ge 8ab$$
   is valid, where a, b, and c are side lengths of the given triangle.

5. **(BUL 5)** Solve the system
$$\begin{aligned} x^2 + x - 1 &= y, \\ y^2 + y - 1 &= z, \\ z^2 + z - 1 &= x. \end{aligned}$$

6. **(BUL 6)** Solve the system
$$\begin{aligned} |x + y| + |1 - x| &= 6, \\ |x + y + 1| + |1 - y| &= 4. \end{aligned}$$

7. **(CZS 1)** Find all real solutions of the system of equations
$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= a, \\ x_1^2 + x_2^2 + \cdots + x_n^2 &= a^2, \\ \cdots\cdots\cdots\cdots\cdots & \quad \cdots \\ x_1^n + x_2^n + \cdots + x_n^n &= a^n. \end{aligned}$$

8. **(CZS 2)**$^{\text{IMO1}}$ $ABCD$ is a parallelogram; $AB = a$, $AD = 1$, α is the size of $\angle DAB$, and the three angles of the triangle ABD are acute. Prove that the four circles K_A, K_B, K_C, K_D, each of radius 1, whose centers are the vertices A, B, C, D, cover the parallelogram if and only if $a \le \cos\alpha + \sqrt{3}\sin\alpha$.

9. **(CZS 3)** The circle k and its diameter AB are given. Find the locus of the centers of circles inscribed in the triangles having one vertex on AB and two other vertices on k.

10. **(CZS 4)** The square $ABCD$ is to be decomposed into n triangles (nonoverlapping) all of whose angles are acute. Find the smallest integer n for which there exists a solution to this problem and construct at

least one decomposition for this n. Answer whether it is possible to ask additionally that (at least) one of these triangles has a perimeter less than an arbitrarily given positive number.

11. **(CZS 5)** Let n be a positive integer. Find the maximal number of noncongruent triangles whose side lengths are integers less than or equal to n.

12. **(CZS 6)** Given a segment AB of the length 1, define the set M of points in the following way: it contains the two points A, B, and also all points obtained from A, B by iterating the following rule: $(*)$ for every pair of points X, Y in M, the set M also contains the point Z of the segment XY for which $YZ = 3XZ$.
    (a) Prove that the set M consists of points X from the segment AB for which the distance from the point A is either
    $$AX = \frac{3k}{4^n} \quad \text{or} \quad AX = \frac{3k-2}{4^n},$$
    where n, k are nonnegative integers.
    (b) Prove that the point X_0 for which $AX_0 = 1/2 = X_0B$ does not belong to the set M.

13. **(GDR 1)** Find whether among all quadrilaterals whose interiors lie inside a semicircle of radius r there exists one (or more) with maximal area. If so, determine their shape and area.

14. **(GDR 2)** Which fraction p/q, where p, q are positive integers less than 100, is closest to $\sqrt{2}$? Find all digits after the decimal point in the decimal representation of this fraction that coincide with digits in the decimal representation of $\sqrt{2}$ (without using any tables).

15. **(GDR 3)** Suppose $\tan\alpha = p/q$, where p and q are integers and $q \neq 0$. Prove that the number $\tan\beta$ for which $\tan 2\beta = \tan 3\alpha$ is rational only when $p^2 + q^2$ is the square of an integer.

16. **(GDR 4)** Prove the following statement: If r_1 and r_2 are real numbers whose quotient is irrational, then any real number x can be approximated arbitrarily well by numbers of the form $z_{k_1,k_2} = k_1r_1 + k_2r_2$, k_1, k_2 integers; i.e., for every real number x and every positive real number p two integers k_1 and k_2 can be found such that $|x - (k_1r_1 + k_2r_2)| < p$.

17. **(GBR 1)**$^{\text{IMO3}}$ Let k, m, and n be positive integers such that $m+k+1$ is a prime number greater than $n+1$. Write c_s for $s(s+1)$. Prove that the product $(c_{m+1} - c_k)(c_{m+2} - c_k)\cdots(c_{m+n} - c_k)$ is divisible by the product $c_1c_2\cdots c_n$.

18. **(GBR 5)** If x is a positive rational number, show that x can be uniquely expressed in the form
    $$x = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots,$$

where $a_1, a_2, \dots$ are integers, $0 \le a_n \le n-1$ for $n > 1$, and the series terminates.
Show also that x can be expressed as the sum of reciprocals of different integers, each of which is greater than 10^6.

19. **(GBR 6)** The n points $P_1, P_2, \dots, P_n$ are placed inside or on the boundary of a disk of radius 1 in such a way that the minimum distance d_n between any two of these points has its largest possible value D_n. Calculate D_n for $n = 2$ to 7 and justify your answer.

20. **(HUN 1)** In space, n points ($n \ge 3$) are given. Every pair of points determines some distance. Suppose all distances are different. Connect every point with the nearest point. Prove that it is impossible to obtain a polygonal line in such a way. [1]

21. **(HUN 2)** Without using any tables, find the exact value of the product
$$P = \cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}.$$

22. **(HUN 3)** The distance between the centers of the circles k_1 and k_2 with radii r is equal to r. Points A and B are on the circle k_1, symmetric with respect to the line connecting the centers of the circles. Point P is an arbitrary point on k_2. Prove that
$$PA^2 + PB^2 \ge 2r^2.$$
When does equality hold?

23. **(HUN 4)** Prove that for an arbitrary pair of vectors f and g in the plane, the inequality
$$af^2 + bfg + cg^2 \ge 0$$
holds if and only if the following conditions are fulfilled: $a \ge 0$, $c \ge 0$, $4ac \ge b^2$.

24. **(HUN 5)**$^{\text{IMO6}}$ Father has left to his children several identical gold coins. According to his will, the oldest child receives one coin and one-seventh of the remaining coins, the next child receives two coins and one-seventh of the remaining coins, the third child receives three coins and one-seventh of the remaining coins, and so on through the youngest child. If every child inherits an integer number of coins, find the number of children and the number of coins.

25. **(HUN 6)** Three disks of diameter d are touching a sphere at their centers. Moreover, each disk touches the other two disks. How do we choose the radius R of the sphere so that the axis of the whole figure makes an angle

[1] The statement so formulated is false. It would be trivially true under the additional assumption that the polygonal line is closed. However, from the offered solution, which is not clear, it does not seem that the proposer had this in mind.

of 60° with the line connecting the center of the sphere with the point on the disks that is at the largest distance from the axis? (The axis of the figure is the line having the property that rotation of the figure through 120° about that line brings the figure to its initial position. The disks are all on one side of the plane, pass through the center of the sphere, and are orthogonal to the axes.)

26. **(ITA 1)** Let $ABCD$ be a regular tetrahedron. To an arbitrary point M on one edge, say CD, corresponds the point $P = P(M)$, which is the intersection of two lines AH and BK, drawn from A orthogonally to BM and from B orthogonally to AM. What is the locus of P as M varies?

27. **(ITA 2)** Which regular polygons can be obtained (and how) by cutting a cube with a plane?

28. **(ITA 3)** Find values of the parameter u for which the expression
$$y = \frac{\tan(x-u) + \tan x + \tan(x+u)}{\tan(x-u)\tan x \tan(x+u)}$$
does not depend on x.

29. **(ITA 4)**$^{\mathrm{IMO4}}$ The triangles $A_0B_0C_0$ and $A'B'C'$ have all their angles acute. Describe how to construct one of the triangles ABC, similar to $A'B'C'$ and circumscribing $A_0B_0C_0$ (so that A, B, C correspond to A', B', C', and AB passes through C_0, BC through A_0, and CA through B_0). Among these triangles ABC, describe, and prove, how to construct the triangle with the maximum area.

30. **(MON 1)** Given $m+n$ numbers a_i $(i = 1, 2, \dots, m)$, b_j $(j = 1, 2, \dots, n)$, determine the number of pairs (a_i, b_j) for which $|i - j| \ge k$, where k is a nonnegative integer.

31. **(MON 2)** An urn contains balls of k different colors; there are n_i balls of the ith color. Balls are drawn at random from the urn, one by one, without replacement. Find the smallest number of draws necessary for getting m balls of the same color.

32. **(MON 3)** Determine the volume of the body obtained by cutting the ball of radius R by the trihedron with vertex in the center of that ball if its dihedral angles are α, β, γ.

33. **(MON 4)** In what case does the system
$$\begin{aligned} x + y + mz &= a, \\ x + my + z &= b, \\ mx + y + z &= c, \end{aligned}$$
have a solution? Find the conditions under which the unique solution of the above system is an arithmetic progression.

34. **(MON 5)** The faces of a convex polyhedron are six squares and eight equilateral triangles, and each edge is a common side for one triangle and one square. All dihedral angles obtained from the triangle and square with a common edge are equal. Prove that it is possible to circumscribe a sphere around this polyhedron and compute the ratio of the squares of the volumes of the polyhedron and of the ball whose boundary is the circumscribed sphere.

35. **(MON 6)** Prove the identity

$$\sum_{k=0}^{n}\binom{n}{k}\left(\tan\frac{x}{2}\right)^{2k}\left[1+2^k\frac{1}{\left(1-\tan^2(x/2)\right)^k}\right]=\sec^{2n}\frac{x}{2}+\sec^n x.$$

36. **(POL 1)** Prove that the center of the sphere circumscribed around a tetrahedron $ABCD$ coincides with the center of a sphere inscribed in that tetrahedron if and only if $AB = CD$, $AC = BD$, and $AD = BC$.

37. **(POL 2)** Prove that for arbitrary positive numbers the following inequality holds:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{a^8+b^8+c^8}{a^3b^3c^3}.$$

38. **(POL 3)** Does there exist an integer such that its cube is equal to $3n^2 + 3n + 7$, where n is integer?

39. **(POL 4)** Show that the triangle whose angles satisfy the equality

$$\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\cos^2 A+\cos^2 B+\cos^2 C}=2$$

is a right-angled triangle.

40. **(POL 5)**$^{\mathrm{IMO2}}$ Exactly one side of a tetrahedron is of length greater than 1. Show that its volume is less than or equal to $1/8$.

41. **(POL 6)** A line l is drawn through the intersection point H of the altitudes of an acute-angled triangle. Prove that the symmetric images l_a, l_b, l_c of l with respect to sides BC, CA, AB have one point in common, which lies on the circumcircle of ABC.

42. **(ROM 1)** Decompose into real factors the expression $1-\sin^5 x-\cos^5 x$.

43. **(ROM 2)** The equation

$$x^5+5\lambda x^4-x^3+(\lambda\alpha-4)x^2-(8\lambda+3)x+\lambda\alpha-2=0$$

is given.

(a) Determine α such that the given equation has exactly one root independent of λ.

(b) Determine α such that the given equation has exactly two roots independent of λ.

44. **(ROM 3)** Suppose p and q are two different positive integers and x is a real number. Form the product $(x+p)(x+q)$.
   (a) Find the sum $S(x,n) = \sum (x+p)(x+q)$, where p and q take values from 1 to n.
   (b) Do there exist integer values of x for which $S(x,n) = 0$?

45. **(ROM 4)** (a) Solve the equation

$$\sin^3 x + \sin^3\left(\frac{2\pi}{3} + x\right) + \sin^3\left(\frac{4\pi}{3} + x\right) + \frac{3}{4}\cos 2x = 0.$$

   (b) Suppose the solutions are in the form of arcs AB of the trigonometric circle (where A is the beginning of arcs of the trigonometric circle), and P is a regular n-gon inscribed in the circle with one vertex at A.
      (1) Find the subset of arcs with the endpoint B at a vertex of the regular dodecagon.
      (2) Prove that the endpoint B cannot be at a vertex of P if $2, 3 \nmid n$ or n is prime.

46. **(ROM 5)** If x, y, z are real numbers satisfying the relations $x+y+z=1$ and $\arctan x + \arctan y + \arctan z = \pi/4$, prove that

$$x^{2n+1} + y^{2n+1} + z^{2n+1} = 1$$

for all positive integers n.

47. **(ROM 6)** Prove the inequality

$$x_1 x_2 \cdots x_k \left(x_1^{n-1} + x_2^{n-1} + \cdots + x_k^{n-1}\right) \le x_1^{n+k-1} + x_2^{n+k-1} + \cdots + x_k^{n+k-1},$$

where $x_i > 0$ $(i = 1, 2, \dots, k)$, $k \in N$, $n \in N$.

48. **(SWE 1)** Determine all positive roots of the equation $x^x = 1/\sqrt{2}$.

49. **(SWE 2)** Let n and k be positive integers such that $1 \le n \le N+1$, $1 \le k \le N+1$. Show that

$$\min_{n \neq k} |\sin n - \sin k| < \frac{2}{N}.$$

50. **(SWE 3)** The function $\varphi(x,y,z)$, defined for all triples (x,y,z) of real numbers, is such that there are two functions f and g defined for all pairs of real numbers such that

$$\varphi(x,y,z) = f(x+y,z) = g(x,y+z)$$

for all real x, y, and z. Show that there is a function h of one real variable such that

$$\varphi(x,y,z) = h(x+y+z)$$

for all real x, y, and z.

51. **(SWE 4)** A subset S of the set of integers $0, \dots, 99$ is said to have property A if it is impossible to fill a crossword puzzle with 2 rows and 2 columns with numbers in S (0 is written as 00, 1 as 01, and so on). Determine the maximal number of elements in sets S with property A.

52. **(SWE 5)** In the plane a point O and a sequence of points $P_1, P_2, P_3, \dots$ are given. The distances $OP_1, OP_2, OP_3, \dots$ are $r_1, r_2, r_3, \dots$, where $r_1 \le r_2 \le r_3 \le \cdots$. Let α satisfy $0 < \alpha < 1$. Suppose that for every n the distance from the point P_n to any other point of the sequence is greater than or equal to r_n^α. Determine the exponent β, as large as possible, such that for some C independent of n,[2]

$$r_n \ge Cn^\beta, \quad n = 1, 2, \dots .$$

53. **(SWE 6)** In making Euclidean constructions in geometry it is permitted to use a straightedge and compass. In the constructions considered in this question, no compasses are permitted, but the straightedge is assumed to have two parallel edges, which can be used for constructing two parallel lines through two given points whose distance is at least equal to the breadth of the ruler. Then the distance between the parallel lines is equal to the breadth of the straightedge. Carry through the following constructions with such a straightedge. Construct:
(a) The bisector of a given angle.
(b) The midpoint of a given rectilinear segment.
(c) The center of a circle through three given noncollinear points.
(d) A line through a given point parallel to a given line.

54. **(USS 1)** Is it possible to put 100 (or 200) points on a wooden cube such that by all rotations of the cube the points map into themselves? Justify your answer.

55. **(USS 2)** Find all x for which for all n,

$$\sin x + \sin 2x + \sin 3x + \cdots + \sin nx \le \frac{\sqrt{3}}{2}.$$

56. **(USS 3)** In a group of interpreters each one speaks one or several foreign languages; 24 of them speak Japanese, 24 Malay, 24 Farsi. Prove that it is possible to select a subgroup in which exactly 12 interpreters speak Japanese, exactly 12 speak Malay, and exactly 12 speak Farsi.

57. **(USS 4)**$^{\text{IMO5}}$ Consider the sequence (c_n):

$$\begin{array}{lcl} c_1 &=& a_1 + a_2 + \cdots + a_8, \\ c_2 &=& a_1^2 + a_2^2 + \cdots + a_8^2, \\ \dots & & \dots\dots\dots \\ c_n &=& a_1^n + a_2^n + \cdots + a_8^n, \\ \dots & & \dots\dots\dots \end{array}$$

[2] This problem is not elementary. The solution offered by the proposer, which is not quite clear and complete, only shows that if such a β exists, then $\beta \ge \frac{1}{2(1-\alpha)}$.

where $a_1, a_2, \dots, a_8$ are real numbers, not all equal to zero. Given that among the numbers of the sequence (c_n) there are infinitely many equal to zero, determine all the values of n for which $c_n = 0$.

58. **(USS 5)** A linear binomial $l(z) = Az + B$ with complex coefficients A and B is given. It is known that the maximal value of $|l(z)|$ on the segment $-1 \le x \le 1$ $(y = 0)$ of the real line in the complex plane $(z = x + iy)$ is equal to M. Prove that for every z

$$|l(z)| \le M\rho,$$

where ρ is the sum of distances from the point $P = z$ to the points Q_1: $z = 1$ and Q_3: $z = -1$.

59. **(USS 6)** On the circle with center O and radius 1 the point A_0 is fixed and points $A_1, A_2, \dots, A_{999}$, A_{1000} are distributed in such a way that $\angle A_0OA_k = k$ (in radians). Cut the circle at points $A_0, A_1, \dots, A_{1000}$. How many arcs with different lengths are obtained?

3.10 The Tenth IMO
Moscow–Leningrad, Soviet Union, July 5–18, 1968

3.10.1 Contest Problems

First Day

1. Prove that there exists a unique triangle whose side lengths are consecutive natural numbers and one of whose angles is twice the measure of one of the others.

2. Find all positive integers x for which $p(x) = x^2 - 10x - 22$, where $p(x)$ denotes the product of the digits of x.

3. Let a, b, c be real numbers. Prove that the system of equations

$$\begin{cases} ax_1^2 + bx_1 + c = x_2, \\ ax_2^2 + bx_2 + c = x_3, \\ \cdots\cdots\cdots\cdots \\ ax_{n-1}^2 + bx_{n-1} + c = x_n, \\ ax_n^2 + bx_n + c = x_1, \end{cases}$$

   (a) has no real solutions if $(b-1)^2 - 4ac < 0$;
   (b) has a unique real solution if $(b-1)^2 - 4ac = 0$;
   (c) has more than one real solution if $(b-1)^2 - 4ac > 0$.

Second Day

4. Prove that in any tetrahedron there is a vertex such that the lengths of its sides through that vertex are sides of a triangle.

5. Let $a > 0$ be a real number and $f(x)$ a real function defined on all of $\mathbb{R}$, satisfying for all $x \in \mathbb{R}$,

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2}.$$

   (a) Prove that the function f is periodic; i.e., there exists $b > 0$ such that for all x, $f(x+b) = f(x)$.
   (b) Give an example of such a nonconstant function for $a = 1$.

6. Let $[x]$ denote the integer part of x, i.e., the greatest integer not exceeding x. If n is a positive integer, express as a simple function of n the sum

$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \cdots + \left[\frac{n+2^i}{2^{i+1}}\right] + \cdots.$$

3.10.2 Shortlisted Problems

1. **(SWE 2)** Two ships sail on the sea with constant speeds and fixed directions. It is known that at 9:00 the distance between them was 20 miles; at 9:35, 15 miles; and at 9:55, 13 miles. At what moment were the ships the smallest distance from each other, and what was that distance?

2. **(ROM 5)**$^{\text{IMO1}}$ Prove that there exists a unique triangle whose side lengths are consecutive natural numbers and one of whose angles is twice the measure of one of the others.

3. **(POL 4)**$^{\text{IMO4}}$ Prove that in any tetrahedron there is a vertex such that the lengths of its sides through that vertex are sides of a triangle.

4. **(BUL 2)**$^{\text{IMO3}}$ Let a, b, c be real numbers. Prove that the system of equations

$$\begin{cases} ax_1^2 + bx_1 + c = x_2, \\ ax_2^2 + bx_2 + c = x_3, \\ \cdots\cdots\cdots\cdots \\ ax_{n-1}^2 + bx_{n-1} + c = x_n, \\ ax_n^2 + bx_n + c = x_1, \end{cases}$$

has a unique real solution if and only if $(b-1)^2 - 4ac = 0$.

Remark. It is assumed that $a \neq 0$.

5. **(BUL 5)** Let h_n be the apothem (distance from the center to one of the sides) of a regular n-gon ($n \geq 3$) inscribed in a circle of radius r. Prove the inequality

$$(n+1)h_{n+1} - nh_n > r.$$

Also prove that if r on the right side is replaced with a greater number, the inequality will not remain true for all $n \geq 3$.

6. **(HUN 1)** If a_i ($i = 1, 2, \dots, n$) are distinct non-zero real numbers, prove that the equation

$$\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \cdots + \frac{a_n}{a_n - x} = n$$

has at least $n - 1$ real roots.

7. **(HUN 5)** Prove that the product of the radii of three circles exscribed to a given triangle does not exceed $\frac{3\sqrt{3}}{8}$ times the product of the side lengths of the triangle. When does equality hold?

8. **(ROM 2)** Given an oriented line Δ and a fixed point A on it, consider all trapezoids $ABCD$ one of whose bases AB lies on Δ, in the positive direction. Let E, F be the midpoints of AB and CD respectively.
Find the loci of vertices B, C, D of trapezoids that satisfy the following:
(i) $|AB| \leq a$ (a fixed);
(ii) $|EF| = l$ (l fixed);

(iii) the sum of squares of the nonparallel sides of the trapezoid is constant.

Remark. The constants are chosen so that such trapezoids exist.

9. **(ROM 3)** Let ABC be an arbitrary triangle and M a point inside it. Let d_a, d_b, d_c be the distances from M to sides BC, CA, AB; a, b, c the lengths of the sides respectively, and S the area of the triangle ABC. Prove the inequality
$$abd_ad_b + bcd_bd_c + cad_cd_a \le \frac{4S^2}{3}.$$
Prove that the left-hand side attains its maximum when M is the centroid of the triangle.

10. **(ROM 4)** Consider two segments of length a, b $(a > b)$ and a segment of length $c = \sqrt{ab}$.
(a) For what values of a/b can these segments be sides of a triangle?
(b) For what values of a/b is this triangle right-angled, obtuse-angled, or acute-angled?

11. **(ROM 6)** Find all solutions $(x_1, x_2, \dots, x_n)$ of the equation
$$1 + \frac{1}{x_1} + \frac{x_1+1}{x_1x_2} + \frac{(x_1+1)(x_2+1)}{x_1x_2x_3} + \cdots + \frac{(x_1+1)\cdots(x_{n-1}+1)}{x_1x_2\cdots x_n} = 0.$$

12. **(POL 1)** If a and b are arbitrary positive real numbers and m an integer, prove that
$$\left(1 + \frac{a}{b}\right)^m + \left(1 + \frac{b}{a}\right)^m \ge 2^{m+1}.$$

13. **(POL 5)** Given two congruent triangles $A_1A_2A_3$ and $B_1B_2B_3$ ($A_iA_k = B_iB_k$), prove that there exists a plane such that the orthogonal projections of these triangles onto it are congruent and equally oriented.

14. **(BUL 5)** A line in the plane of a triangle ABC intersects the sides AB and AC respectively at points X and Y such that $BX = CY$. Find the locus of the center of the circumcircle of triangle XAY.

15. **(GBR 1)**[IMO6] Let $[x]$ denote the integer part of x, i.e., the greatest integer not exceeding x. If n is a positive integer, express as a simple function of n the sum
$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \cdots + \left[\frac{n+2^i}{2^{i+1}}\right] + \cdots.$$

16. **(GBR 3)** A polynomial $p(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$ with integer coefficients is said to be divisible by an integer m if $p(x)$ is divisible by m for all integers x. Prove that if $p(x)$ is divisible by m, then $k!a_0$ is also divisible by m. Also prove that if a_0, k, m are nonnegative integers for which $k!a_0$ is divisible by m, there exists a polynomial $p(x) = a_0x^k + \cdots + a_k$ divisible by m.

17. **(GBR 4)** Given a point O and lengths x, y, z, prove that there exists an equilateral triangle ABC for which $OA = x$, $OB = y$, $OC = z$, if and only if $x+y \geq z$, $y+z \geq x$, $z+x \geq y$ (the points O, A, B, C are coplanar).

18. **(ITA 2)** If an acute-angled triangle ABC is given, construct an equilateral triangle $A'B'C'$ in space such that lines AA', BB', CC' pass through a given point.

19. **(ITA 5)** We are given a fixed point on the circle of radius 1, and going from this point along the circumference in the positive direction on curved distances $0, 1, 2, \ldots$ from it we obtain points with abscisas $n = 0, 1, 2, \ldots$ respectively. How many points among them should we take to ensure that some two of them are less than the distance $1/5$ apart?

20. **(CZS 1)** Given n ($n \geq 3$) points in space such that every three of them form a triangle with one angle greater than or equal to $120°$, prove that these points can be denoted by $A_1, A_2, \ldots, A_n$ in such a way that for each i, j, k, $1 \leq i < j < k \leq n$, angle $A_iA_jA_k$ is greater than or equal to $120°$.

21. **(CZS 2)** Let $a_0, a_1, \ldots, a_k$ ($k \geq 1$) be positive integers. Find all positive integers y such that
$$a_0 \mid y; \quad (a_0 + a_1) \mid (y + a_1); \quad \ldots; \quad (a_0 + a_n) \mid (y + a_n).$$

22. **(CZS 3)**[IMO2] Find all positive integers x for which $p(x) = x^2 - 10x - 22$, where $p(x)$ denotes the product of the digits of x.

23. **(CZS 4)** Find all complex numbers m such that polynomial
$$x^3 + y^3 + z^3 + mxyz$$
can be represented as the product of three linear trinomials.

24. **(MON 1)** Find the number of all n-digit numbers for which some fixed digit stands only in the ith ($1 < i < n$) place and the last j digits are distinct.[3]

25. **(MON 2)** Given k parallel lines and a few points on each of them, find the number of all possible triangles with vertices at these given points.[4]

26. **(GDR)**[IMO5] Let $a > 0$ be a real number and $f(x)$ a real function defined on all of $\mathbb{R}$, satisfying for all $x \in \mathbb{R}$,
$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2}.$$
   (a) Prove that the function f is periodic; i.e., there exists $b > 0$ such that for all x, $f(x+b) = f(x)$.
   (b) Give an example of such a nonconstant function for $a = 1$.

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[3] The problem is unclear. Presumably n, i, j and the ith digit are fixed.

[4] The problem is unclear. The correct formulation could be the following:

Given k parallel lines $l_1, \ldots, l_k$ and n_i points on the line l_i, $i = 1, 2, \ldots, k$, find the maximum possible number of triangles with vertices at these points.

3.11 The Eleventh IMO
Bucharest, Romania, July 5–20, 1969

3.11.1 Contest Problems

First Day (July 10)

1. Prove that there exist infinitely many natural numbers a with the following property: the number $z = n^4 + a$ is not prime for any natural number n.

2. Let $a_1, a_2, \dots, a_n$ be real constants and

$$y(x) = \cos(a_1 + x) + \frac{\cos(a_2 + x)}{2} + \frac{\cos(a_3 + x)}{2^2} + \cdots + \frac{\cos(a_n + x)}{2^{n-1}}.$$

If x_1, x_2 are real and $y(x_1) = y(x_2) = 0$, prove that $x_1 - x_2 = m\pi$ for some integer m.

3. Find conditions on the positive real number a such that there exists a tetrahedron k of whose edges ($k = 1, 2, 3, 4, 5$) have length a, and the other $6 - k$ edges have length 1.

Second Day (July 11)

4. Let AB be a diameter of a circle γ. A point C different from A and B is on the circle γ. Let D be the projection of the point C onto the line AB. Consider three other circles γ_1, γ_2, and γ_3 with the common tangent AB: γ_1 inscribed in the triangle ABC, and γ_2 and γ_3 tangent to both (the segment) CD and γ. Prove that γ_1, γ_2, and γ_3 have two common tangents.

5. Given n points in the plane such that no three of them are collinear, prove that one can find at least $\binom{n-3}{2}$ convex quadrilaterals with their vertices at these points.

6. Under the conditions $x_1, x_2 > 0$, $x_1 y_1 > z_1^2$, and $x_2 y_2 > z_2^2$, prove the inequality

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \le \frac{1}{x_1 y_1 - z_1^2} + \frac{1}{x_2 y_2 - z_2^2}.$$

3.11.2 Longlisted Problems

1. **(BEL 1)** A parabola P_1 with equation $x^2 - 2py = 0$ and parabola P_2 with equation $x^2 + 2py = 0$, $p > 0$, are given. A line t is tangent to P_2. Find the locus of pole M of the line t with respect to P_1.

2. **(BEL 2)** (a) Find the equations of regular hyperbolas passing through the points $A(\alpha, 0)$, $B(\beta, 0)$, and $C(0, \gamma)$.
   (b) Prove that all such hyperbolas pass through the orthocenter H of the triangle ABC.

(c) Find the locus of the centers of these hyperbolas.

(d) Check whether this locus coincides with the nine-point circle of the triangle ABC.

3. **(BEL 3)** Construct the circle that is tangent to three given circles.

4. **(BEL 4)** Let O be a point on a nondegenerate conic. A right angle with vertex O intersects the conic at points A and B. Prove that the line AB passes through a fixed point located on the normal to the conic through the point O.

5. **(BEL 5)** Let G be the centroid of the triangle OAB.

   (a) Prove that all conics passing through the points O, A, B, G are hyperbolas.

   (b) Find the locus of the centers of these hyperbolas.

6. **(BEL 6)** Evaluate $(\cos(\pi/4) + i\sin(\pi/4))^{10}$ in two different ways and prove that
$$\binom{10}{1} - \binom{10}{3} + \frac{1}{2}\binom{10}{5} = 2^4.$$

7. **(BUL 1)** Prove that the equation $\sqrt{x^3 + y^3 + z^3} = 1969$ has no integral solutions.

8. **(BUL 2)** Find all functions f defined for all x that satisfy the condition
$$xf(y) + yf(x) = (x + y)f(x)f(y),$$
for all x and y. Prove that exactly two of them are continuous.

9. **(BUL 3)** One hundred convex polygons are placed on a square with edge of length 38 cm. The area of each of the polygons is smaller than π cm^2, and the perimeter of each of the polygons is smaller than 2π cm. Prove that there exists a disk with radius 1 in the square that does not intersect any of the polygons.

10. **(BUL 4)** Let M be the point inside the right-angled triangle ABC ($\angle C = 90^\circ$) such that
$$\angle MAB = \angle MBC = \angle MCA = \varphi.$$
Let ψ be the acute angle between the medians of AC and BC. Prove that $\frac{\sin(\varphi+\psi)}{\sin(\varphi-\psi)} = 5$.

11. **(BUL 5)** Let Z be a set of points in the plane. Suppose that there exists a pair of points that cannot be joined by a polygonal line not passing through any point of Z. Let us call such a pair of points *unjoinable*. Prove that for each real $r > 0$ there exists an unjoinable pair of points separated by distance r.

12. **(CZS 1)** Given a unit cube, find the locus of the centroids of all tetrahedra whose vertices lie on the sides of the cube.

13. **(CZS 2)** Let p be a prime odd number. Is it possible to find $p-1$ natural numbers $n+1, n+2, \dots, n+p-1$ such that the sum of the squares of these numbers is divisible by the sum of these numbers?

14. **(CZS 3)** Let a and b be two positive real numbers. If x is a real solution of the equation $x^2+px+q=0$ with real coefficients p and q such that $|p| \le a$, $|q| \le b$, prove that
$$|x| \le \frac{1}{2}\left(a+\sqrt{a^2+4b}\right). \tag{1}$$
Conversely, if x satisfies (1), prove that there exist real numbers p and q with $|p| \le a$, $|q| \le b$ such that x is one of the roots of the equation $x^2+px+q=0$.

15. **(CZS 4)** Let $K_1, \dots, K_n$ be nonnegative integers. Prove that
$$K_1!K_2!\cdots K_n! \ge [K/n]!^n,$$
where $K = K_1 + \cdots + K_n$.

16. **(CZS 5)** A convex quadrilateral $ABCD$ with sides $AB=a$, $BC=b$, $CD=c$, $DA=d$ and angles $\alpha = \angle DAB$, $\beta = \angle ABC$, $\gamma = \angle BCD$, and $\delta = \angle CDA$ is given. Let $s=(a+b+c+d)/2$ and P be the area of the quadrilateral. Prove that
$$P^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{\alpha+\gamma}{2}.$$

17. **(CZS 6)** Let d and p be two real numbers. Find the first term of an arithmetic progression $a_1, a_2, a_3, \dots$ with difference d such that $a_1a_2a_3a_4 = p$. Find the number of solutions in terms of d and p.

18. **(FRA 1)** Let a and b be two nonnegative integers. Denote by $H(a,b)$ the set of numbers n of the form $n = pa + qb$, where p and q are positive integers. Determine $H(a) = H(a,a)$. Prove that if $a \ne b$, it is enough to know all the sets $H(a,b)$ for coprime numbers a,b in order to know all the sets $H(a,b)$. Prove that in the case of coprime numbers a and b, $H(a,b)$ contains all numbers greater than or equal to $\omega = (a-1)(b-1)$ and also $\omega/2$ numbers smaller than ω.

19. **(FRA 2)** Let n be an integer that is not divisible by any square greater than 1. Denote by x_m the last digit of the number x^m in the number system with base n. For which integers x is it possible for x_m to be 0? Prove that the sequence x_m is periodic with period t independent of x. For which x do we have $x_t = 1$. Prove that if m and x are relatively prime, then $0_m, 1_m, \dots, (n-1)_m$ are different numbers. Find the minimal period t in terms of n. If n does not meet the given condition, prove that it is possible to have $x_m = 0 \ne x_1$ and that the sequence is periodic starting only from some number $k > 1$.

20. **(FRA 3)** A polygon (not necessarily convex) with vertices in the lattice points of a rectangular grid is given. The area of the polygon is S. If I is the number of lattice points that are strictly in the interior of the polygon and B the number of lattice points on the border of the polygon, find the number $T = 2S - B - 2I + 2$.

21. **(FRA 4)** A right-angled triangle OAB has its right angle at the point B. An arbitrary circle with center on the line OB is tangent to the line OA. Let AT be the tangent to the circle different from OA (T is the point of tangency). Prove that the median from B of the triangle OAB intersects AT at a point M such that $MB = MT$.

22. **(FRA 5)** Let $\alpha(n)$ be the number of pairs (x, y) of integers such that $x + y = n$, $0 \leq y \leq x$, and let $\beta(n)$ be the number of triples (x, y, z) such that $x + y + z = n$ and $0 \leq z \leq y \leq x$. Find a simple relation between $\alpha(n)$ and the integer part of the number $\frac{n+2}{2}$ and the relation among $\beta(n)$, $\beta(n-3)$ and $\alpha(n)$. Then evaluate $\beta(n)$ as a function of the residue of n modulo 6. What can be said about $\beta(n)$ and $1 + \frac{n(n+6)}{12}$? And what about $\frac{(n+3)^2}{6}$?
   Find the number of triples (x, y, z) with the property $x + y + z \leq n$, $0 \leq z \leq y \leq x$ as a function of the residue of n modulo 6. What can be said about the relation between this number and the number $\frac{(n+6)(2n^2+9n+12)}{72}$?

23. **(FRA 6)** Consider the integer $d = \frac{a^b - 1}{c}$, where a, b, and c are positive integers and $c \leq a$. Prove that the set G of integers that are between 1 and d and relatively prime to d (the number of such integers is denoted by $\varphi(d)$) can be partitioned into n subsets, each of which consists of b elements. What can be said about the rational number $\frac{\varphi(d)}{b}$?

24. **(GBR 1)** The polynomial $P(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$, where $a_0, \dots, a_k$ are integers, is said to be divisible by an integer m if $P(x)$ is a multiple of m for every integral value of x. Show that if $P(x)$ is divisible by m, then $a_0 \cdot k!$ is a multiple of m. Also prove that if a, k, m are positive integers such that $ak!$ is a multiple of m, then a polynomial $P(x)$ with leading term ax^k can be found that is divisible by m.

25. **(GBR 2)** Let a, b, x, y be positive integers such that a and b have no common divisor greater than 1. Prove that the largest number not expressible in the form $ax + by$ is $ab - a - b$. If $N(k)$ is the largest number not expressible in the form $ax + by$ in only k ways, find $N(k)$.

26. **(GBR 3)** A smooth solid consists of a right circular cylinder of height h and base-radius r, surmounted by a hemisphere of radius r and center O. The solid stands on a horizontal table. One end of a string is attached to a point on the base. The string is stretched (initially being kept in the vertical plane) over the highest point of the solid and held down at the point P on the hemisphere such that OP makes an angle α with

the horizontal. Show that if α is small enough, the string will slacken if slightly displaced and no longer remain in a vertical plane. If then pulled tight through P, show that it will cross the common circular section of the hemisphere and cylinder at a point Q such that $\angle SOQ = \phi$, S being where it initially crossed this section, and $\sin\phi = \frac{r\tan\alpha}{h}$.

27. **(GBR 4)** The segment AB perpendicularly bisects CD at X. Show that, subject to restrictions, there is a right circular cone whose axis passes through X and on whose surface lie the points A, B, C, D. What are the restrictions?

28. **(GBR 5)** Let us define $u_0 = 0$, $u_1 = 1$ and for $n \ge 0$, $u_{n+2} = au_{n+1} + bu_n$, a and b being positive integers. Express u_n as a polynomial in a and b. Prove the result. Given that b is prime, prove that b divides $a(u_b - 1)$.

29. **(GDR 1)** Find all real numbers λ such that the equation

$$\sin^4 x - \cos^4 x = \lambda(\tan^4 x - \cot^4 x)$$

(a) has no solution,
(b) has exactly one solution,
(c) has exactly two solutions,
(d) has more than two solutions (in the interval $(0, \pi/4)$).

30. **(GDR 2)**[IMO1] Prove that there exist infinitely many natural numbers a with the following property: The number $z = n^4 + a$ is not prime for any natural number n.

31. **(GDR 3)** Find the number of permutations $a_1, \dots, a_n$ of the set $\{1, 2, \dots, n\}$ such that $|a_i - a_{i+1}| \neq 1$ for all $i = 1, 2, \dots, n-1$. Find a recurrence formula and evaluate the number of such permutations for $n \le 6$.

32. **(GDR 4)** Find the maximal number of regions into which a sphere can be partitioned by n circles.

33. **(GDR 5)** Given a ring G in the plane bounded by two concentric circles with radii R and $R/2$, prove that we can cover this region with 8 disks of radius $2R/5$. (A region is covered if each of its points is inside or on the border of some disk.)

34. **(HUN 1)** Let a and b be arbitrary integers. Prove that if k is an integer not divisible by 3, then $(a+b)^{2k} + a^{2k} + b^{2k}$ is divisible by $a^2 + ab + b^2$.

35. **(HUN 2)** Prove that

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{5}{4}.$$

36. **(HUN 3)** In the plane 4000 points are given such that each line passes through at most 2 of these points. Prove that there exist 1000 disjoint quadrilaterals in the plane with vertices at these points.

37. **(HUN 4)**$^{\text{IMO2}}$ If $a_1, a_2, \dots, a_n$ are real constants, and if

$$y = \cos(a_1 + x) + 2\cos(a_2 + x) + \cdots + n\cos(a_n + x)$$

has two zeros x_1 and x_2 whose difference is not a multiple of π, prove that $y \equiv 0$.

38. **(HUN 5)** Let r and m ($r \le m$) be natural numbers and $A_k = \frac{2k-1}{2m}\pi$. Evaluate

$$\frac{1}{m^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\sin(rA_k)\sin(rA_l)\cos(rA_k - rA_l).$$

39. **(HUN 6)** Find the positions of three points A, B, C on the boundary of a unit cube such that $\min\{AB, AC, BC\}$ is the greatest possible.

40. **(MON 1)** Find the number of five-digit numbers with the following properties: there are two pairs of digits such that digits from each pair are equal and are next to each other, digits from different pairs are different, and the remaining digit (which does not belong to any of the pairs) is different from the other digits.

41. **(MON 2)** Given two numbers x_0 and x_1, let α and β be coefficients of the equation $1 - \alpha y - \beta y^2 = 0$. Under the given conditions, find an expression for the solution of the system

$$x_{n+2} - \alpha x_{n+1} - \beta x_n = 0, \qquad n = 0, 1, 2, \dots .$$

42. **(MON 3)** Let A_k ($1 \le k \le h$) be n-element sets such that each two of them have a nonempty intersection. Let A be the union of all the sets A_k, and let B be a subset of A such that for each k ($1 \le k \le h$) the intersection of A_k and B consists of exactly two different elements a_k and b_k. Find all subsets X of the set A with r elements satisfying the condition that for at least one index k, both elements a_k and b_k belong to X.

43. **(MON 4)** Let p and q be two prime numbers greater than 3. Prove that if their difference is 2^n, then for any two integers m and n, the number $S = p^{2m+1} + q^{2m+1}$ is divisible by 3.

44. **(MON 5)** Find the radius of the circle circumscribed about the isosceles triangle whose sides are the solutions of the equation $x^2 - ax + b = 0$.

45. **(MON 6)**$^{\text{IMO5}}$ Given n points in the plane such that no three of them are collinear, prove that one can find at least $\binom{n-3}{2}$ convex quadrilaterals with their vertices at these points.

46. **(NET 1)** The vertices of an $(n+1)$-gon are placed on the edges of a regular n-gon so that the perimeter of the n-gon is divided into equal parts. How does one choose these $n+1$ points in order to obtain the $(n+1)$gon with
(a) maximal area;

(b) minimal area?

47. **(NET 2)**[IMO4] Let A and B be points on the circle γ. A point C, different from A and B, is on the circle γ. Let D be the projection of the point C onto the line AB. Consider three other circles γ_1, γ_2, and γ_3 with the common tangent AB: γ_1 inscribed in the triangle ABC, and γ_2 and γ_3 tangent to both (the segment) CD and γ. Prove that γ_1, γ_2, and γ_3 have two common tangents.

48. **(NET 3)** Let x_1, x_2, x_3, x_4, and x_5 be positive integers satisfying

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 &= 1000,\\
x_1 - x_2 + x_3 - x_4 + x_5 &> 0,\\
x_1 + x_2 - x_3 + x_4 - x_5 &> 0,\\
-x_1 + x_2 + x_3 - x_4 + x_5 &> 0,\\
x_1 - x_2 + x_3 + x_4 - x_5 &> 0,\\
-x_1 + x_2 - x_3 + x_4 + x_5 &> 0.
\end{aligned}$$

(a) Find the maximum of $(x_1 + x_3)^{x_2+x_4}$.
(b) In how many different ways can we choose $x_1, \dots, x_5$ to obtain the desired maximum?

49. **(NET 4)** A boy has a set of trains and pieces of railroad track. Each piece is a quarter of circle, and by concatenating these pieces, the boy obtained a closed railway. The railway does not intersect itself. In passing through this railway, the train sometimes goes in the clockwise direction, and sometimes in the opposite direction. Prove that the train passes an even number of times through the pieces in the clockwise direction and an even number of times in the counterclockwise direction. Also, prove that the number of pieces is divisible by 4.

50. **(NET 5)** The bisectors of the exterior angles of a pentagon $B_1B_2B_3B_4B_5$ form another pentagon $A_1A_2A_3A_4A_5$. Construct $B_1B_2B_3B_4B_5$ from the given pentagon $A_1A_2A_3A_4A_5$.

51. **(NET 6)** A curve determined by

$$y = \sqrt{x^2 - 10x + 52}, \qquad 0 \le x \le 100,$$

is constructed in a rectangular grid. Determine the number of squares cut by the curve.

52. **(POL 1)** Prove that a regular polygon with an odd number of edges cannot be partitioned into four pieces with equal areas by two lines that pass through the center of polygon.

53. **(POL 2)** Given two segments AB and CD not in the same plane, find the locus of points M such that

$$MA^2 + MB^2 = MC^2 + MD^2.$$

54. **(POL 3)** Given a polynomial $f(x)$ with integer coefficients whose value is divisible by 3 for three integers k, $k+1$, and $k+2$, prove that $f(m)$ is divisible by 3 for all integers m.

55. **(POL 4)**$^{\text{IMO3}}$ Find the conditions on the positive real number a such that there exists a tetrahedron k of whose edges $(k = 1, 2, 3, 4, 5)$ have length a, and the other $6-k$ edges have length 1.

56. **(POL 5)** Let a and b be two natural numbers that have an equal number n of digits in their decimal expansions. The first m digits (from left to right) of the numbers a and b are equal. Prove that if $m > n/2$, then

$$a^{1/n} - b^{1/n} < \frac{1}{n}.$$

57. **(POL 6)** On the sides AB and AC of triangle ABC two points K and L are given such that $\frac{KB}{AK} + \frac{LC}{AL} = 1$. Prove that KL passes through the centroid of ABC.

58. **(SWE 1)** Six points $P_1, \dots, P_6$ are given in 3-dimensional space such that no four of them lie in the same plane. Each of the line segments P_jP_k is colored black or white. Prove that there exists one triangle $P_jP_kP_l$ whose edges are of the same color.

59. **(SWE 2)** For each λ ($0 < \lambda < 1$ and $\lambda \neq 1/n$ for all $n = 1, 2, 3, \dots$) construct a continuous function f such that there do not exist x, y with $0 < \lambda < y = x + \lambda \le 1$ for which $f(x) = f(y)$.

60. **(SWE 3)** Find the natural number n with the following properties:
    (1) Let $S = \{p_1, p_2, \dots\}$ be an arbitrary finite set of points in the plane, and r_j the distance from P_j to the origin O. We assign to each P_j the closed disk D_j with center P_j and radius r_j. Then some n of these disks contain all points of S.
    (2) n is the smallest integer with the above property.

61. **(SWE 4)** Let a_0, a_1, a_2 be determined with $a_0 = 0$, $a_{n+1} = 2a_n + 2^n$. Prove that if n is power of 2, then so is a_n.

62. **(SWE 5)** Which natural numbers can be expressed as the difference of squares of two integers?

63. **(SWE 6)** Prove that there are infinitely many positive integers that cannot be expressed as the sum of squares of three positive integers.

64. **(USS 1)** Prove that for a natural number $n > 2$,

$$(n!)! > n[(n-1)!]^{n!}.$$

65. **(USS 2)** Prove that for $a > b^2$,

$$\sqrt{a - b\sqrt{a + b\sqrt{a - b\sqrt{a + \cdots}}}} = \sqrt{a - \frac{3}{4}b^2} - \frac{1}{2}b.$$

66. **(USS 3)** (a) Prove that if $0 \le a_0 \le a_1 \le a_2$, then

$$(a_0 + a_1 x - a_2 x^2)^2 \le (a_0 + a_1 + a_2)^2 \left(1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{2}x^3 + x^4\right).$$

(b) Formulate and prove the analogous result for polynomials of third degree.

67. **(USS 4)**[IMO6] Under the conditions $x_1, x_2 > 0$, $x_1 y_1 > z_1^2$, and $x_2 y_2 > z_2^2$, prove the inequality

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \le \frac{1}{x_1 y_1 - z_1^2} + \frac{1}{x_2 y_2 - z_2^2}.$$

68. **(USS 5)** Given 5 points in the plane, no three of which are collinear, prove that we can choose 4 points among them that form a convex quadrilateral.

69. **(YUG 1)** Suppose that positive real numbers x_1, x_2, x_3 satisfy

$$x_1 x_2 x_3 > 1, \qquad x_1 + x_2 + x_3 < \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}.$$

Prove that:
(a) None of x_1, x_2, x_3 equals 1.
(b) Exactly one of these numbers is less than 1.

70. **(YUG 2)** A park has the shape of a convex pentagon of area $5\sqrt{3}$ ha ($= 50000\sqrt{3}$ m^2). A man standing at an interior point O of the park notices that he stands at a distance of at most 200 m from each vertex of the pentagon. Prove that he stands at a distance of at least 100 m from each side of the pentagon.

71. **(YUG 3)** Let four points A_i ($i = 1, 2, 3, 4$) in the plane determine four triangles. In each of these triangles we choose the smallest angle. The sum of these angles is denoted by S. What is the exact placement of the points A_i if $S = 180°$?

3.12 The Twelfth IMO
Budapest–Keszthely, Hungary, July 8–22, 1970

3.12.1 Contest Problems

First Day (July 13)

1. Given a point M on the side AB of the triangle ABC, let r_1 and r_2 be the radii of the inscribed circles of the triangles ACM and BCM respectively while ρ_1 and ρ_2 are the radii of the excircles of the triangles ACM and BCM at the sides AM and BM respectively. Let r and ρ denote the respective radii of the inscribed circle and the excircle at the side AB of the triangle ABC. Prove that
$$\frac{r_1}{\rho_1}\frac{r_2}{\rho_2} = \frac{r}{\rho}\,.$$

2. Let a and b be the bases of two number systems and let
$$\begin{aligned} A_n &= \overline{x_1x_2\dots x_n}^{(a)}, & A_{n+1} &= \overline{x_0x_1x_2\dots x_n}^{(a)}, \\ B_n &= \overline{x_1x_2\dots x_n}^{(b)}, & B_{n+1} &= \overline{x_0x_1x_2\dots x_n}^{(b)}, \end{aligned}$$
be numbers in the number systems with respective bases a and b, so that $x_0, x_1, x_2, \dots, x_n$ denote digits in the number system with base a as well as in the number system with base b. Suppose that neither x_0 nor x_1 is zero. Prove that $a > b$ if and only if
$$\frac{A_n}{A_{n+1}} < \frac{B_n}{B_{n+1}}\,.$$

3. Let $1 = a_0 \le a_1 \le a_2 \le \cdots \le a_n \le \cdots$ be a sequence of real numbers. Consider the sequence $b_1, b_2, \dots$ defined by
$$b_n = \sum_{k=1}^{n}\left(1 - \frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}}\,.$$
Prove that:
(a) For all natural numbers n, $0 \le b_n < 2$.
(b) Given an arbitrary $0 \le b < 2$, there is a sequence $a_0, a_1, \dots, a_n, \dots$ of the above type such that $b_n > b$ is true for an infinity of natural numbers n.

Second Day (July 14)

4. For what natural numbers n can the product of some of the numbers $n, n+1, n+2, n+3, n+4, n+5$ be equal to the product of the remaining ones?

5. In the tetrahedron $ABCD$, the edges BD and CD are mutually perpendicular, and the projection of the vertex D to the plane ABC is the intersection of the altitudes of the triangle ABC. Prove that

$$(AB + BC + CA)^2 \le 6(DA^2 + DB^2 + DC^2)\ .$$

For which tetrahedra does equality hold?

6. Given 100 points in the plane, no three of which are on the same line, consider all triangles that have all their vertices chosen from the 100 given points. Prove that at most 70% of these triangles are acute-angled.

3.12.2 Longlisted Problems

1. **(AUT 1)** Prove that

$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \le \frac{1}{2}(a+b+c) \qquad (a, b, c > 0).$$

2. **(AUT 2)** Prove that the two last digits of 9^{9^9} and $9^{9^{9^9}}$ in decimal representation are equal.
3. **(AUT 3)** Prove that for $a, b \in \mathbb{N}$, $a!b!$ divides $(a+b)!$.
4. **(AUT 4)** Solve the system of equations

$$\begin{aligned} x^2 + xy &= a^2 + ab \\ y^2 + xy &= a^2 - ab, \end{aligned} \qquad a, b \text{ real},\ a \ne 0.$$

5. **(AUT 5)** Prove that $\sqrt[n]{\frac{1}{n+1} + \frac{2}{n+1} + \cdots + \frac{n}{n+1}} \ge 1$ for $n \ge 2$.
6. **(BEL 1)** Prove that the equation in x

$$\sum_{i=1}^{n} \frac{b_i}{x - a_i} = c, \qquad b_i > 0,\ \ a_1 < a_2 < a_3 < \cdots < a_n,$$

has $n-1$ roots $x_1, x_2, x_3, \dots, x_{n-1}$ such that $a_1 < x_1 < a_2 < x_2 < a_3 < x_3 < \cdots < x_{n-1} < a_n$.

7. **(BEL 2)** Let $ABCD$ be any quadrilateral. A square is constructed on each side of the quadrilateral, all in the same manner (i.e., outward or inward). Denote the centers of the squares by M_1, M_2, M_3, and M_4. Prove:
(a) $M_1M_3 = M_2M_4$;
(b) M_1M_3 is perpendicular to M_2M_4.
8. **(BEL 3)** (SL70-1).

9. **(BEL 4)** If n is even, prove that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{n}=2\left(\frac{1}{n+2}+\frac{1}{n+4}+\frac{1}{n+6}+\cdots+\frac{1}{2n}\right).$$

10. **(BEL 5)** Let A, B, C be angles of a triangle. Prove that

$$1<\cos A+\cos B+\cos C \leq \frac{3}{2}.$$

11. **(BEL 6)** Let $ABCD$ and $A'B'C'D'$ be two squares in the same plane and oriented in the same direction. Let A'', B'', C'', and D'' be the midpoints of AA', BB', CC', and DD'. Prove that $A''B''C''D''$ is also a square.

12. **(BUL 1)** Let $x_1, x_2, x_3, x_4, x_5, x_6$ be given integers, not divisible by 7. Prove that at least one of the expressions of the form

$$\pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 \pm x_6$$

is divisible by 7, where the signs are selected in all possible ways. (Generalize the statement to every prime number!)

13. **(BUL 2)** A triangle ABC is given. Each side of ABC is divided into equal parts, and through each of the division points are drawn lines parallel to AB, BC, and CA, thus cutting ABC into small triangles. To each of the vertices of these triangles is assigned 1, 2, or 3, so that:
(1) to A, B, C are assigned 1, 2 and 3 respectively;
(2) points on AB are marked by 1 or 2;
(3) points on BC are marked by 2 or 3;
(4) points on CA are marked by 3 or 1.
Prove that there must exist a small triangle whose vertices are marked by 1, 2, and 3.

14. **(BUL 3)** Let $\alpha+\beta+\gamma=\pi$. Prove that

$$\begin{aligned}\sin 2\alpha+\sin 2\beta+\sin 2\gamma = {} & 2(\sin\alpha+\sin\beta+\sin\gamma)(\cos\alpha+\cos\beta+\cos\gamma)\\ & -2(\sin\alpha+\sin\beta+\sin\gamma).\end{aligned}$$

15. **(BUL 4)** Given a triangle ABC, let R be the radius of its circumcircle, O_1, O_2, O_3 the centers of its exscribed circles, and q the perimeter of $\triangle O_1O_2O_3$. Prove that $q \leq 6\sqrt{3}\,R$.

16. **(BUL 5)** Show that the equation

$$\sqrt{2-x^2}+\sqrt[3]{3-x^3}=0$$

has no real roots.

17. **(BUL 6)** (SL70-3).
Original formulation. In a triangular pyramid $SABC$ one of the angles at S is right and the projection of S onto the base ABC is the orthocenter of ABC. Let r be the radius of the circle inscribed in the base, $SA = m$, $SB = n$, $SC = p$, H the height of the pyramid (through S), and r_1, r_2, r_3 the radii of the circles inscribed in the intersections of the pyramid with the planes determined by the altitude of the pyramid and the lines SA, SB, SC respectively. Prove that:
(a) $m^2 + n^2 + p^2 \geq 18r^2$;
(b) the ratios r_1/H, r_2/H, r_3/H lie in the interval $[0.4, 0.5]$.

18. **(CZS 1)** (SL70-4).

19. **(CZS 2)** Let $n > 1$ be a natural number, $a \geq 1$ a real number, and $x_1, x_2, \dots, x_n$ numbers such that $x_1 = 1$, $\frac{x_{k+1}}{x_k} = a + \alpha_k$ for $k = 1, 2, \dots, n-1$, where α_k are real numbers with $\alpha_k \leq \frac{1}{k(k+1)}$. Prove that

$$\sqrt[n-1]{x_n} < a + \frac{1}{n-1}.$$

20. **(CZS 3)** (SL70-5).

21. **(CZS 4)** Find necessary and sufficient conditions on given positive numbers u, v for the following claim to be valid: there exists a right-angled triangle $\triangle ABC$ with $CD = u$, $CE = v$, where D, E are points of the segments AB such that $AD = DE = EB = \frac{1}{3}AB$.

22. **(FRA 1)** (SL70-6).

23. **(FRA 2)** Let E be a finite set, $\mathcal{P}_E$ the family of its subsets, and f a mapping from $\mathcal{P}_E$ to the set of nonnegative real numbers such that for any two disjoint subsets A, B of E,

$$f(A \cup B) = f(A) + f(B).$$

Prove that there exists a subset F of E such that if with each $A \subset E$ we associate a subset A' consisting of elements of A that are not in F, then $f(A) = f(A')$, and $f(A)$ is zero if and only if A is a subset of F.

24. **(FRA 3)** Let n and p be two integers such that $2p \leq n$. Prove the inequality

$$\frac{(n-p)!}{p!} \leq \left(\frac{n+1}{2}\right)^{n-2p}.$$

For which values does equality hold?

25. **(FRA 4)** Suppose that f is a real function defined for $0 \leq x \leq 1$ having the first derivative f' for $0 \leq x \leq 1$ and the second derivative f'' for $0 < x < 1$. Prove that if

$$f(0) = f'(0) = f'(1) = f(1) - 1 = 0,$$

there exists a number $0 < y < 1$ such that $|f''(y)| \geq 4$.

26. **(FRA 5)** Consider a finite set of vectors in space $\{a_1, a_2, \dots, a_n\}$ and the set E of all vectors of the form $x = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n$, where λ_i are nonnegative numbers. Let F be the set consisting of all the vectors in E and vectors parallel to a given plane P. Prove that there exists a set of vectors $\{b_1, b_2, \dots, b_p\}$ such that F is the set of all vectors y of the form

$$y = \mu_1 b_1 + \mu_2 b_2 + \cdots + \mu_p b_p,$$

where the μ_j are nonnegative.

27. **(FRA 6)** Find a natural number n such that for all prime numbers p, n is divisible by p if and only if n is divisible by $p - 1$.

28. **(GDR 1)** A set G with elements $u, v, w, \dots$ is a group if the following conditions are fulfilled:
   (1) There is a binary algebraic operation $\circ$ defined on G such that for all $u, v \in G$ there is a $w \in G$ with $u \circ v = w$.
   (2) This operation is associative; i.e., for all $u, v, w \in G$, $(u \circ v) \circ w = u \circ (v \circ w)$.
   (3) For any two elements $u, v \in G$ there exists an element $x \in G$ such that $u \circ x = v$, and an element $y \in G$ such that $y \circ u = v$.

   Let K be a set of all real numbers greater than 1. On K is defined an operation by
$$a \circ b = ab + \sqrt{(a^2 - 1)(b^2 - 1)}.$$
   Prove that K is a group.

29. **(GDR 2)** Prove that the equation $4^x + 6^x = 9^x$ has no rational solutions.

30. **(GDR 3)** (SL70-9).

31. **(GDR 4)** Prove that for any triangle with sides a, b, c and area P the following inequality holds:

$$P \leq \frac{\sqrt{3}}{4}(abc)^{2/3}.$$

   Find all triangles for which equality holds.

32. **(NET 1)** Let there be given an acute angle $\angle AOB = 3\alpha$, where $\overline{OA} = \overline{OB}$. The point A is the center of a circle with radius $\overline{OA}$. A line s parallel to OA passes through B. Inside the given angle a variable line t is drawn through O. It meets the circle in O and C and the given line s in D, where $\angle AOC = x$. Starting from an arbitrarily chosen position t_0 of t, the series $t_0, t_1, t_2, \dots$ is determined by defining $\overline{BD_{i+1}} = \overline{OC_i}$ for each i (in which C_i and D_i denote the positions of C and D, corresponding to t_i). Making use of the graphical representations of BD and OC as functions of x, determine the behavior of t_i for $i \to \infty$.

33. **(NET 2)** The vertices of a given square are clockwise lettered A, B, C, D. On the side AB is situated a point E such that $AE = AB/3$.
Starting from an arbitrarily chosen point P_0 on segment AE and going clockwise around the perimeter of the square, a series of points $P_0, P_1, P_2, \dots$ is marked on the perimeter such that $P_iP_{i+1} = AB/3$ for each i. It will be clear that when P_0 is chosen in A or in E, then some P_i will coincide with P_0. Does this possibly also happen if P_0 is chosen otherwise?

34. **(NET 3)** In connection with a convex pentagon $ABCDE$ we consider the set of ten circles, each of which contains three of the vertices of the pentagon on its circumference. Is it possible that none of these circles contains the pentagon? Prove your answer.

35. **(NET 4)** Find for every value of n a set of numbers p for which the following statement is true: Any convex n-gon can be divided into p isosceles triangles.

    Alternative version. The same about division into p polygons with axis of symmetry.

36. **(NET 5)** Let x, y, z be nonnegative real numbers satisfying

$$x^2 + y^2 + z^2 = 5 \quad \text{and} \quad yz + zx + xy = 2.$$

Which values can the greatest of the numbers $x^2 - yz$, $y^2 - xz$, $z^2 - xy$ have?

37. **(NET 6)** Solve the set of simultaneous equations

$$\begin{aligned}
v^2 + w^2 + x^2 + y^2 &= 6 - 2u,\\
u^2 + w^2 + x^2 + y^2 &= 6 - 2v,\\
u^2 + v^2 + x^2 + y^2 &= 6 - 2w,\\
u^2 + v^2 + w^2 + y^2 &= 6 - 2x,\\
u^2 + v^2 + w^2 + x^2 &= 6 - 2y.
\end{aligned}$$

38. **(POL 1)** Find the greatest integer A for which in any permutation of the numbers $1, \dots, 100$ there exist ten consecutive numbers whose sum is at least A.

39. **(POL 2)** (SL70-8).

40. **(POL 5)** Let ABC be a triangle with angles α, β, γ commensurable with π. Starting from a point P interior to the triangle, a ball reflects on the sides of ABC, respecting the law of reflection that the angle of incidence is equal to the angle of reflection.
Prove that, supposing that the ball never reaches any of the vertices A, B, C, the set of all directions in which the ball will move through time is finite. In other words, its path from the moment 0 to infinity consists of segments parallel to a finite set of lines.

41. **(POL 6)** Let a cube of side 1 be given. Prove that there exists a point A on the surface S of the cube such that every point of S can be joined to A by a path on S of length not exceeding 2. Also prove that there is a point of S that cannot be joined with A by a path on S of length less than 2.

42. **(ROM 1)** (SL70-2).

43. **(ROM 2)** Prove that the equation

$$x^3 - 3\tan\frac{\pi}{12}x^2 - 3x + \tan\frac{\pi}{12} = 0$$

has one root $x_1 = \tan\frac{\pi}{36}$, and find the other roots.

44. **(ROM 3)** If a, b, c are side lengths of a triangle, prove that

$$(a+b)(b+c)(c+a) \geq 8(a+b-c)(b+c-a)(c+a-b).$$

45. **(ROM 4)** Let M be an interior point of tetrahedron $VABC$. Denote by A_1, B_1, C_1 the points of intersection of lines MA, MB, MC with the planes VBC, VCA, VAB, and by A_2, B_2, C_2 the points of intersection of lines VA_1, VB_1, VC_1 with the sides BC, CA, AB.
    (a) Prove that the volume of the tetrahedron $VA_2B_2C_2$ does not exceed one-fourth of the volume of $VABC$.
    (b) Calculate the volume of the tetrahedron $V_1A_1B_1C_1$ as a function of the volume of $VABC$, where V_1 is the point of intersection of the line VM with the plane ABC, and M is the barycenter of $VABC$.

46. **(ROM 5)** Given a triangle ABC and a plane π having no common points with the triangle, find a point M such that the triangle determined by the points of intersection of the lines MA, MB, MC with π is congruent to the triangle ABC.

47. **(ROM 6)** Given a polynomial

$$\begin{aligned} P(x) = {} & ab(a-c)x^3 + (a^3 - a^2c + 2ab^2 - b^2c + abc)x^2 \\ & + (2a^2b + b^2c + a^2c + b^3 - abc)x + ab(b+c), \end{aligned}$$

where $a, b, c \neq 0$, prove that $P(x)$ is divisible by

$$Q(x) = abx^2 + (a^2 + b^2)x + ab$$

and conclude that $P(x_0)$ is divisible by $(a+b)^3$ for $x_0 = (a+b+1)^n$, $n \in \mathbb{N}$.

48. **(ROM 7)** Let a polynomial $p(x)$ with integer coefficients take the value 5 for five different integer values of x. Prove that $p(x)$ does not take the value 8 for any integer x.

49. **(SWE 1)** For $n \in \mathbb{N}$, let $f(n)$ be the number of positive integers $k \leq n$ that do not contain the digit 9. Does there exist a positive real number p such that $\frac{f(n)}{n} \geq p$ for all positive integers n?

50. **(SWE 2)** The area of a triangle is S and the sum of the lengths of its sides is L. Prove that $36S \leq L^2\sqrt{3}$ and give a necessary and sufficient condition for equality.

51. **(SWE 3)** Let p be a prime number. A rational number x, with $0 < x < 1$, is written in lowest terms. The rational number obtained from x by adding p to both the numerator and the denominator differs from x by $1/p^2$. Determine all rational numbers x with this property.

52. **(SWE 4)** (SL70-10).

53. **(SWE 5)** A square $ABCD$ is divided into $(n-1)^2$ congruent squares, with sides parallel to the sides of the given square. Consider the grid of all n^2 corners obtained in this manner. Determine all integers n for which it is possible to construct a nondegenerate parabola with its axis parallel to one side of the square and that passes through exactly n points of the grid.

54. **(SWE 6)** (SL70-11).

55. **(USS 1)** A turtle runs away from an UFO with a speed of 0.2 m/s. The UFO flies 5 meters above the ground, with a speed of 20 m/s. The UFO's path is a broken line, where after flying in a straight path of length ℓ (in meters) it may turn through for any acute angle α such that $\tan\alpha < \frac{\ell}{1000}$. When the UFO's center approaches within 13 meters of the turtle, it catches the turtle. Prove that for any initial position the UFO can catch the turtle.

56. **(USS 2)** A square hole of depth h whose base is of length a is given. A dog is tied to the center of the square at the bottom of the hole by a rope of length $L > \sqrt{2a^2 + h^2}$, and walks on the ground around the hole. The edges of the hole are smooth, so that the rope can freely slide along it. Find the shape and area of the territory accessible to the dog (whose size is neglected).

57. **(USS 3)** Let the numbers $1, 2, \ldots, n^2$ be written in the cells of an $n \times n$ square board so that the entries in each column are arranged increasingly. What are the smallest and greatest possible sums of the numbers in the kth row? (k a positive integer, $1 \leq k \leq n$.)

58. **(USS 4)** (SL70-12).

59. **(USS 5)** (SL70-7).

3.12.3 Shortlisted Problems

1. **(BEL 3)** Consider a regular $2n$-gon and the n diagonals of it that pass through its center. Let P be a point of the inscribed circle and let $a_1, a_2, \dots, a_n$ be the angles in which the diagonals mentioned are visible from the point P. Prove that

$$\sum_{i=1}^{n} \tan^2 a_i = 2n \frac{\cos^2 \frac{\pi}{2n}}{\sin^4 \frac{\pi}{2n}}.$$

2. **(ROM 1)**$^{\mathrm{IMO2}}$ Let a and b be the bases of two number systems and let

$$\begin{aligned} A_n &= \overline{x_1 x_2 \dots x_n}^{(a)}, & A_{n+1} &= \overline{x_0 x_1 x_2 \dots x_n}^{(a)}, \\ B_n &= \overline{x_1 x_2 \dots x_n}^{(b)}, & B_{n+1} &= \overline{x_0 x_1 x_2 \dots x_n}^{(b)}, \end{aligned}$$

be numbers in the number systems with respective bases a and b, so that $x_0, x_1, x_2, \dots, x_n$ denote digits in the number system with base a as well as in the number system with base b. Suppose that neither x_0 nor x_1 is zero. Prove that $a > b$ if and only if

$$\frac{A_n}{A_{n+1}} < \frac{B_n}{B_{n+1}} \;.$$

3. **(BUL 6)**$^{\mathrm{IMO5}}$ In the tetrahedron $SABC$ the angle BSC is a right angle, and the projection of the vertex S to the plane ABC is the intersection of the altitudes of the triangle ABC. Let z be the radius of the inscribed circle of the triangle ABC. Prove that

$$SA^2 + SB^2 + SC^2 \geq 18z^2.$$

4. **(CZS 1)**$^{\mathrm{IMO4}}$ For what natural numbers n can the product of some of the numbers $n, n+1, n+2, n+3, n+4, n+5$ be equal to the product of the remaining ones?

5. **(CZS 3)** Let M be an interior point of the tetrahedron $ABCD$. Prove that

$$\begin{aligned} &\overrightarrow{MA}\,\mathrm{vol}(MBCD) + \overrightarrow{MB}\,\mathrm{vol}(MACD) \\ &\quad + \overrightarrow{MC}\,\mathrm{vol}(MABD) + \overrightarrow{MD}\,\mathrm{vol}(MABC) = 0 \end{aligned}$$

(vol($PQRS$) denotes the volume of the tetrahedron $PQRS$).

6. **(FRA 1)** In the triangle ABC let B' and C' be the midpoints of the sides AC and AB respectively and H the foot of the altitude passing through the vertex A. Prove that the circumcircles of the triangles $AB'C'$, $BC'H$, and $B'CH$ have a common point I and that the line HI passes through the midpoint of the segment $B'C'$.

7. **(USS 5)** For which digits a do exist integers $n \geq 4$ such that each digit of $\frac{n(n+1)}{2}$ equals a?

8. **(POL 2)**$^{\text{IMO1}}$ Given a point M on the side AB of the triangle ABC, let r_1 and r_2 be the radii of the inscribed circles of the triangles ACM and BCM respectively and let ρ_1 and ρ_2 be the radii of the excircles of the triangles ACM and BCM at the sides AM and BM respectively. Let r and ρ denote the radii of the inscribed circle and the excircle at the side AB of the triangle ABC respectively. Prove that
$$\frac{r_1}{\rho_1}\frac{r_2}{\rho_2} = \frac{r}{\rho}.$$

9. **(GDR 3)** Let $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$ be real numbers. Prove that
$$1 + \sum_{i=1}^{n}(u_i + v_i)^2 \leq \frac{4}{3}\left(1 + \sum_{i=1}^{n} u_i^2\right)\left(1 + \sum_{i=1}^{n} v_i^2\right).$$
In what case does equality hold?

10. **(SWE 4)**$^{\text{IMO3}}$ Let $1 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$ be a sequence of real numbers. Consider the sequence $b_1, b_2, \dots$ defined by:
$$b_n = \sum_{k=1}^{n}\left(1 - \frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}}.$$
Prove that:
(a) For all natural numbers n, $0 \leq b_n < 2$.
(b) Given an arbitrary $0 \leq b < 2$, there is a sequence $a_0, a_1, \dots, a_n, \dots$ of the above type such that $b_n > b$ is true for infinitely many natural numbers n.

11. **(SWE 6)** Let P, Q, R be polynomials and let $S(x) = P(x^3) + xQ(x^3) + x^2R(x^3)$ be a polynomial of degree n whose roots $x_1, \dots, x_n$ are distinct. Construct with the aid of the polynomials P, Q, R a polynomial T of degree n that has the roots $x_1^3, x_2^3, \dots, x_n^3$.

12. **(USS 4)**$^{\text{IMO6}}$ We are given 100 points in the plane, no three of which are on the same line. Consider all triangles that have all vertices chosen from the 100 given points. Prove that at most 70% of these triangles are acute angled.

3.13 The Thirteenth IMO Bratislava–Zilina, Czechoslovakia, July 10–21, 1971

3.13.1 Contest Problems

First Day (July 13)

1. Prove that the following statement is true for $n = 3$ and for $n = 5$, and false for all other $n > 2$:
For any real numbers $a_1, a_2, \dots, a_n$,
$$(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n) + (a_2 - a_1)(a_2 - a_3)\cdots(a_2 - a_n) + \dots$$
$$+(a_n - a_1)(a_n - a_2)\cdots(a_n - a_{n-1}) \geq 0.$$
2. Given a convex polyhedron P_1 with 9 vertices $A_1, \dots, A_9$, let us denote by $P_2, P_3, \dots, P_9$ the images of P_1 under the translations mapping the vertex A_1 to $A_2, A_3, \dots, A_9$, respectively. Prove that among the polyhedra $P_1, \dots, P_9$ at least two have a common interior point.
3. Prove that the sequence $2^n - 3$ $(n > 1)$ contains a subsequence of numbers relatively prime in pairs.

Second Day (July 14)

4. Given a tetrahedron $ABCD$ all of whose faces are acute-angled triangles, set
$$\sigma = \measuredangle DAB + \measuredangle BCD - \measuredangle ABC - \measuredangle CDA.$$
Consider all closed broken lines $XYZTX$ whose vertices X, Y, Z, T lie in the interior of segments AB, BC, CD, DA respectively. Prove that:
(a) if $\sigma \neq 0$, then there is no broken line $XYZT$ of minimal length;
(b) if $\sigma = 0$, then there are infinitely many such broken lines of minimal length. That length equals $2AC\sin(\alpha/2)$, where
$$\alpha = \measuredangle BAC + \measuredangle CAD + \measuredangle DAB.$$
5. Prove that for every natural number $m \geq 1$ there exists a finite set S_m of points in the plane satisfying the following condition: If A is any point in S_m, then there are exactly m points in S_m whose distance to A equals 1.
6. Consider the $n \times n$ array of nonnegative integers
$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$
with the following property: If an element a_{ij} is zero, then the sum of the elements of the ith row and the jth column is greater than or equal to n. Prove that the sum of all the elements is greater than or equal to $\geq \frac{1}{2}n^2$.

3.13.2 Longlisted Problems

1. **(AUT 1)** The points $S(i,j)$ with integer Cartesian coordinates $0 < i \le n$, $0 < j \le m$, $m \le n$, form a lattice. Find the number of:
   (a) rectangles with vertices on the lattice and sides parallel to the coordinate axes;
   (b) squares with vertices on the lattice and sides parallel to the coordinate axes;
   (c) squares in total, with vertices on the lattice.

2. **(AUT 2)** Let us denote by $s(n) = \sum_{d|n} d$ the sum of divisors of a natural number n (1 and n included). If n has at most 5 distinct prime divisors, prove that $s(n) < \frac{77}{16}n$. Also prove that there exists a natural number n for which $s(n) > \frac{76}{16}n$ holds.

3. **(AUT 3)** Let a, b, c be positive real numbers, $0 < a \le b \le c$. Prove that for any positive real numbers x, y, z the following inequality holds:
$$(ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) \le (x + y + z)^2 \frac{(a + c)^2}{4ac}.$$

4. **(BUL 1)** Let $x_n = 2^{2^n} + 1$ and let m be the least common multiple of $x_2, x_3, \dots, x_{1971}$. Find the last digit of m.

5. **(BUL 2)** (SL71-1).
   Original formulation. Consider a sequence of polynomials $X_0(x)$, $X_1(x)$, $X_2(x), \dots, X_n(x), \dots$, where $X_0(x) = 2$, $X_1(x) = x$, and for every $n \ge 1$ the following equality holds:
$$X_n(x) = \frac{1}{x}\left(X_{n+1}(x) + X_{n-1}(x)\right).$$
   Prove that $(x^2 - 4)[X_n^2(x) - 4]$ is a square of a polynomial for all $n \ge 0$.

6. **(BUL 3)** Let squares be constructed on the sides BC, CA, AB of a triangle ABC, all to the outside of the triangle, and let A_1, B_1, C_1 be their centers. Starting from the triangle $A_1B_1C_1$ one analogously obtains a triangle $A_2B_2C_2$. If S, S_1, S_2 denote the areas of triangles $ABC, A_1B_1C_1, A_2B_2C_2$, respectively, prove that $S = 8S_1 - 4S_2$.

7. **(BUL 4)** In a triangle ABC, let H be its orthocenter, O its circumcenter, and R its circumradius. Prove that:
   (a) $|OH| = R\sqrt{1 - 8\cos\alpha\cos\beta\cos\gamma}$, where α, β, γ are angles of the triangle ABC;
   (b) $O \equiv H$ if and only if ABC is equilateral.

8. **(BUL 5)** (SL71-2).
   Original formulation. Prove that for every natural number $n \ge 1$ there exists an infinite sequence $M_1, M_2, \dots, M_k, \dots$ of distinct points in the plane such that for all i, exactly n among these points are at distance 1 from M_i.

9. **(BUL 6)** The base of an inclined prism is a triangle ABC. The perpendicular projection of B_1, one of the top vertices, is the midpoint of BC. The dihedral angle between the lateral faces through BC and AB is α, and the lateral edges of the prism make an angle β with the base. If r_1, r_2, r_3 are exradii of a perpendicular section of the prism, assuming that in ABC, $\cos^2 A + \cos^2 B + \cos^2 C = 1$, $\angle A < \angle B < \angle C$, and $BC = a$, calculate $r_1r_2 + r_1r_3 + r_2r_3$.

10. **(CUB 1)** In how many different ways can three knights be placed on a chessboard so that the number of squares attacked would be maximal?

11. **(CUB 2)** Prove that $n!$ cannot be the square of any natural number.

12. **(CUB 3)** A system of n numbers $x_1, x_2, \dots, x_n$ is given such that

$$x_1 = \log_{x_{n-1}} x_n, \qquad x_2 = \log_{x_n} x_1, \qquad \dots \qquad , \qquad x_n = \log_{x_{n-2}} x_{n-1}.$$

Prove that $\prod_{k=1}^{n} x_k = 1$.

13. **(CUB 4)** One Martian, one Venusian, and one Human reside on Pluton. One day they make the following conversation:

Martian : I have spent 1/12 of my life on Pluton.
Human : I also have.
Venusian : Me too.
Martian : But Venusian and I have spend much more time here than you, Human.
Human : That is true. However, Venusian and I are of the same age.
Venusian : Yes, I have lived 300 Earth years.
Martian : Venusian and I have been on Pluton for the past 13 years.

It is known that Human and Martian together have lived 104 Earth years. Find the ages of Martian, Venusian, and Human.[5]

14. **(GBR 1)** Note that $8^3 - 7^3 = 169 = 13^2$ and $13 = 2^2 + 3^2$. Prove that if the difference between two consecutive cubes is a square, then it is the square of the sum of two consecutive squares.

15. **(GBR 2)** Let $ABCD$ be a convex quadrilateral whose diagonals intersect at O at an angle θ. Let us set $OA = a$, $OB = b$, $OC = c$, and $OD = d$, $c > a > 0$, and $d > b > 0$.
Show that if there exists a right circular cone with vertex V, with the properties:
(1) its axis passes through O, and
(2) its curved surface passes through A, B, C and D, then

$$OV^2 = \frac{d^2b^2(c+a)^2 - c^2a^2(d+b)^2}{ca(d-b)^2 - db(c-a)^2}.$$

[5] The numbers in the problem are not necessarily in base 10.

Show also that if $\frac{c+a}{d+b}$ lies between $\frac{ca}{db}$ and $\sqrt{\frac{ca}{db}}$, and $\frac{c-a}{d-b} = \frac{ca}{db}$, then for a suitable choice of θ, a right circular cone exists with properties (1) and (2).

16. **(GBR 3)** (SL71-4).
Original formulation. Two (intersecting) circles are given and a point P through which it is possible to draw a straight line on which the circles intercept two equal chords. Describe a construction by straightedge and compass for the straight line and prove the validity of your construction.

17. **(GDR 1)** (SL71-3).
Original formulation. Find all solutions of the system
$$\begin{aligned} x+y+z &= 3, \\ x^3+y^3+z^3 &= 15, \\ x^5+y^5+z^5 &= 83. \end{aligned}$$

18. **(GDR 2)** Let $a_1, a_2, \dots, a_n$ be positive numbers, $m_g = (a_1 a_2 \cdots a_n)^{1/n}$ their geometric mean, and $m_a = (a_1 + a_2 + \cdots + a_n)/n$ their arithmetic mean. Prove that
$$(1+m_g)^n \le (1+a_1)\cdots(1+a_n) \le (1+m_a)^n.$$

19. **(GDR 3)** In a triangle $P_1P_2P_3$ let P_iQ_i be the altitude from P_i for $i = 1,2,3$ (Q_i being the foot of the altitude). The circle with diameter P_iQ_i meets the two corresponding sides at two points different from P_i. Denote the length of the segment whose endpoints are these two points by l_i. Prove that $l_1 = l_2 = l_3$.

20. **(GDR 4)** Let M be the circumcenter of a triangle ABC. The line through M perpendicular to CM meets the lines CA and CB at Q and P respectively. Prove that
$$\frac{\overline{CP}}{\overline{CM}}\frac{\overline{CQ}}{\overline{CM}}\frac{\overline{AB}}{\overline{PQ}} = 2.$$

21. **(HUN 1)** (SL71-5).

22. **(HUN 2)** We are given an $n \times n$ board, where n is an odd number. In each cell of the board either $+1$ or -1 is written. Let a_k and b_k denote the products of numbers in the kth row and in the kth column respectively. Prove that the sum $a_1 + a_2 + \cdots + a_n + b_1 + b_2 + \cdots + b_n$ cannot be equal to zero.

23. **(HUN 3)** Find all integer solutions of the equation
$$x^2 + y^2 = (x-y)^3.$$

24. **(HUN 4)** Let A, B, and C denote the angles of a triangle. If $\sin^2 A + \sin^2 B + \sin^2 C = 2$, prove that the triangle is right-angled.

25. **(HUN 5)** Let $ABC, AA_1A_2, BB_1B_2, CC_1C_2$ be four equilateral triangles in the plane satisfying only that they are all positively oriented (i.e., in the counterclockwise direction). Denote the midpoints of the segments A_2B_1, B_2C_1, C_2A_1 by P, Q, R in this order. Prove that the triangle PQR is equilateral.

26. **(HUN 6)** An infinite set of rectangles in the Cartesian coordinate plane is given. The vertices of each of these rectangles have coordinates $(0,0), (p,0), (p,q), (0,q)$ for some positive integers p, q. Show that there must exist two among them one of which is entirely contained in the other.

27. **(HUN 7)** (SL71-6).

28. **(NET 1)** (SL71-7).
Original formulation. A tetrahedron $ABCD$ is given. The sum of angles of the tetrahedron at the vertex A (namely $\angle BAC, \angle CAD, \angle DAB$) is denoted by α, and β, γ, δ are defined analogously. Let P, Q, R, S be variable points on edges of the tetrahedron: P on AD, Q on BD, R on BC, and S on AC, none of them at some vertex of $ABCD$. Prove that:
(a) if $\alpha + \beta \neq 2\pi$, then $PQ + QR + RS + SP$ attains no minimal value;
(b) if $\alpha + \beta = 2\pi$, then

$$AB \sin \frac{\alpha}{2} = CD \sin \frac{\gamma}{2} \quad \text{and} \quad PQ + QR + RS + SP \geq 2AB \sin \frac{\alpha}{2}.$$

29. **(NET 2)** A rhombus with its incircle is given. At each vertex of the rhombus a circle is constructed that touches the incircle and two edges of the rhombus. These circles have radii r_1, r_2, while the incircle has radius r. Given that r_1 and r_2 are natural numbers and that $r_1 r_2 = r$, find r_1, r_2, and r.

30. **(NET 3)** Prove that the system of equations

$$\begin{aligned} 2yz + x - y - z &= a, \\ 2xz - x + y - z &= a, \\ 2xy - x - y + z &= a, \end{aligned}$$

a being a parameter, cannot have five distinct solutions. For what values of a does this system have four distinct integer solutions?

31. **(NET 4)** (SL71-8).

32. **(NET 5)** Two half-lines a and b, with the common endpoint O, make an acute angle α. Let A on a and B on b be points such that $OA = OB$, and let b' be the line through A parallel to b. Let β be the circle with center B and radius BO. We construct a sequence of half-lines $c_1, c_2, c_3, \ldots$, all lying inside the angle α, in the following manner:
(i) c_1 is given arbitrarily;

(ii) for every natural number k, the circle β intercepts on c_k a segment that is of the same length as the segment cut on b' by a and c_{k+1}.

Prove that the angle determined by the lines c_k and b has a limit as k tends to infinity and find that limit.

33. **(NET 6)** A square $2n \times 2n$ grid is given. Let us consider all possible paths along grid lines, going from the center of the grid to the border, such that (1) no point of the grid is reached more than once, and (2) each of the squares homothetic to the grid having its center at the grid center is passed through only once.
   (a) Prove that the number of all such paths is equal to $4\prod_{i=2}^{n}(16i-9)$.
   (b) Find the number of pairs of such paths that divide the grid into two congruent figures.
   (c) How many quadruples of such paths are there that divide the grid into four congruent parts?
34. **(POL 1)** (SL71-9).
35. **(POL 2)** (SL71-10).
36. **(POL 3)** (SL71-11).
37. **(POL 4)** Let S be a circle, and $\alpha = \{A_1, \dots, A_n\}$ a family of open arcs in S. Let $N(\alpha) = n$ denote the number of elements in α. We say that α is a covering of S if $\bigcup_{k=1}^{n} A_k \supset S$.
   Let $\alpha = \{A_1, \dots, A_n\}$ and $\beta = \{B_1, \dots, B_m\}$ be two coverings of S. Show that we can choose from the family of all sets $A_i \cap B_j$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, a covering γ of S such that $N(\gamma) \le N(\alpha) + N(\beta)$.
38. **(POL 5)** Let A, B, C be three points with integer coordinates in the plane and K a circle with radius R passing through A, B, C. Show that $AB \cdot BC \cdot CA \ge 2R$, and if the center of K is in the origin of the coordinates, show that $AB \cdot BC \cdot CA \ge 4R$.
39. **(POL 6)** (SL71-12).
40. **(SWE 1)** Prove that
$$\left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{4^3}\right)\cdots\left(1-\frac{1}{n^3}\right) > \frac{1}{2}, \qquad n = 2, 3, \dots.$$
41. **(SWE 2)** Consider the set of grid points (m, n) in the plane, m, n integers. Let σ be a finite subset and define
$$S(\sigma) = \sum_{(m,n)\in\sigma} (100 - |m| - |n|).$$
   Find the maximum of S, taken over the set of all such subsets σ.
42. **(SWE 3)** Let L_i, $i = 1, 2, 3$, be line segments on the sides of an equilateral triangle, one segment on each side, with lengths l_i, $i = 1, 2, 3$. By L_i^* we

denote the segment of length l_i with its midpoint on the midpoint of the corresponding side of the triangle. Let $M(L)$ be the set of points in the plane whose orthogonal projections on the sides of the triangle are in L_1, L_2, and L_3, respectively; $M(L^*)$ is defined correspondingly. Prove that if $l_1 \geq l_2 + l_3$, we have that the area of $M(L)$ is less than or equal to the area of $M(L^*)$.

43. **(SWE 4)** Show that for nonnegative real numbers a, b and integers $n \geq 2$,

$$\frac{a^n + b^n}{2} \geq \left(\frac{a+b}{2}\right)^n .$$

When does equality hold?

44. **(SWE 5)** (SL71-13).

45. **(SWE 6)** Let m and n denote integers greater than 1, and let $\nu(n)$ be the number of primes less than or equal to n. Show that if the equation $\frac{n}{\nu(n)} = m$ has a solution, then so does the equation $\frac{n}{\nu(n)} = m - 1$.

46. **(USS 1)** (SL71-14).

47. **(USS 2)** (SL71-15).

48. **(USS 3)** A sequence of real numbers $x_1, x_2, \dots, x_n$ is given such that $x_{i+1} = x_i + \frac{1}{30000}\sqrt{1 - x_i^2}$, $i = 1, 2, \dots$, and $x_1 = 0$. Can n be equal to 50000 if $x_n < 1$?

49. **(USS 4)** Diagonals of a convex quadrilateral $ABCD$ intersect at a point O. Find all angles of this quadrilateral if $\measuredangle OBA = 30^\circ, \measuredangle OCB = 45^\circ, \measuredangle ODC = 45^\circ$, and $\measuredangle OAD = 30^\circ$.

50. **(USS 5)** (SL71-16).

51. **(USS 6)** Suppose that the sides AB and DC of a convex quadrilateral $ABCD$ are not parallel. On the sides BC and AD, pairs of points (M, N) and (K, L) are chosen such that $BM = MN = NC$ and $AK = KL = LD$. Prove that the areas of triangles OKM and OLN are different, where O is the intersection point of AB and CD.

52. **(YUG 1)** (SL71-17).

53. **(YUG 2)** Denote by $x_n(p)$ the multiplicity of the prime p in the canonical representation of the number $n!$ as a product of primes. Prove that $\frac{x_n(p)}{n} < \frac{1}{p-1}$ and $\lim_{n\to\infty} \frac{x_n(p)}{n} = \frac{1}{p-1}$.

54. **(YUG 3)** A set M is formed of $\binom{2n}{n}$ men, $n = 1, 2, \dots$. Prove that we can choose a subset P of the set M consisting of $n+1$ men such that one of the following conditions is satisfied:
(1) every member of the set P knows every other member of the set P;
(2) no member of the set P knows any other member of the set P.

55. **(YUG 4)** Prove that the polynomial $x^4 + \lambda x^3 + \mu x^2 + \nu x + 1$ has no real roots if λ, μ, ν are real numbers satisfying

$$|\lambda| + |\mu| + |\nu| \le \sqrt{2}.$$

3.13.3 Shortlisted Problems

1. **(BUL 2)** Consider a sequence of polynomials $P_0(x), P_1(x), P_2(x), \dots, P_n(x), \dots$, where $P_0(x) = 2$, $P_1(x) = x$ and for every $n \ge 1$ the following equality holds:

$$P_{n+1}(x) + P_{n-1}(x) = xP_n(x).$$

Prove that there exist three real numbers a, b, c such that for all $n \ge 1$,

$$(x^2 - 4)[P_n^2(x) - 4] = [aP_{n+1}(x) + bP_n(x) + cP_{n-1}(x)]^2. \qquad (1)$$

2. **(BUL 5)**$^{\text{IMO5}}$ Prove that for every natural number $m \ge 1$ there exists a finite set S_m of points in the plane satisfying the following condition: If A is any point in S_m, then there are exactly m points in S_m whose distance to A equals 1.

3. **(GDR 1)** Knowing that the system

$$\begin{aligned} x + y + z &= 3, \\ x^3 + y^3 + z^3 &= 15, \\ x^4 + y^4 + z^4 &= 35, \end{aligned}$$

has a real solution x, y, z for which $x^2 + y^2 + z^2 < 10$, find the value of $x^5 + y^5 + z^5$ for that solution.

4. **(GBR 3)** We are given two mutually tangent circles in the plane, with radii r_1, r_2. A line intersects these circles in four points, determining three segments of equal length. Find this length as a function of r_1 and r_2 and the condition for the solvability of the problem.

5. **(HUN 1)**$^{\text{IMO1}}$ Let a, b, c, d, e be real numbers. Prove that the expression

$$(a-b)(a-c)(a-d)(a-e)+(b-a)(b-c)(b-d)(b-e)+(c-a)(c-b)(c-d)(c-e)$$
$$+ (d-a)(d-b)(d-c)(d-e) + (e-a)(e-b)(e-c)(e-d)$$

is nonnegative.

6. **(HUN 7)** Let $n \ge 2$ be a natural number. Find a way to assign natural numbers to the vertices of a regular 2^n-gon such that the following conditions are satisfied:
(1) only digits 1 and 2 are used;
(2) each number consists of exactly n digits;
(3) different numbers are assigned to different vertices;

(4) the numbers assigned to two neighboring vertices differ at exactly one digit.

7. **(NET 1)**IMO4 Given a tetrahedron $ABCD$ whose all faces are acute-angled triangles, set

$$\sigma = \measuredangle DAB + \measuredangle BCD - \measuredangle ABC - \measuredangle CDA.$$

Consider all closed broken lines $XYZTX$ whose vertices X, Y, Z, T lie in the interior of segments AB, BC, CD, DA respectively. Prove that:
(a) if $\sigma \neq 0$, then there is no broken line $XYZT$ of minimal length;
(b) if $\sigma = 0$, then there are infinitely many such broken lines of minimal length. That length equals $2AC \sin(\alpha/2)$, where

$$\alpha = \measuredangle BAC + \measuredangle CAD + \measuredangle DAB.$$

8. **(NET 4)** Determine whether there exist distinct real numbers a, b, c, t for which:
(i) the equation $ax^2 + btx + c = 0$ has two distinct real roots x_1, x_2,
(ii) the equation $bx^2 + ctx + a = 0$ has two distinct real roots x_2, x_3,
(iii) the equation $cx^2 + atx + b = 0$ has two distinct real roots x_3, x_1.

9. **(POL 1)** Let $T_k = k - 1$ for $k = 1, 2, 3, 4$ and

$$T_{2k-1} = T_{2k-2} + 2^{k-2}, \qquad T_{2k} = T_{2k-5} + 2^k \qquad (k \geq 3).$$

Show that for all k,

$$1 + T_{2n-1} = \left[\frac{12}{7} 2^{n-1}\right] \quad \text{and} \quad 1 + T_{2n} = \left[\frac{17}{7} 2^{n-1}\right],$$

where $[x]$ denotes the greatest integer not exceeding x.

10. **(POL 2)**IMO3 Prove that the sequence $2^n - 3$ $(n > 1)$ contains a subsequence of numbers relatively prime in pairs.

11. **(POL 3)** The matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

satisfies the inequality $\sum_{j=1}^{n} |a_{j1}x_1 + \dots + a_{jn}x_n| \leq M$ for each choice of numbers x_i equal to ± 1. Show that

$$|a_{11} + a_{22} + \dots + a_{nn}| \leq M.$$

12. **(POL 6)** Two congruent equilateral triangles ABC and $A'B'C'$ in the plane are given. Show that the midpoints of the segments AA', BB', CC' either are collinear or form an equilateral triangle.

13. **(SWE 5)**$^{\text{IMO6}}$ Consider the $n \times n$ array of nonnegative integers

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

with the following property: If an element a_{ij} is zero, then the sum of the elements of the ith row and the jth column is greater than or equal to n. Prove that the sum of all the elements is greater than or equal to $\frac{1}{2}n^2$.

14. **(USS 1)** A broken line $A_1A_2\dots A_n$ is drawn in a 50×50 square, so that the distance from any point of the square to the broken line is less than 1. Prove that its total length is greater than 1248.

15. **(USS 2)** Natural numbers from 1 to 99 (not necessarily distinct) are written on 99 cards. It is given that the sum of the numbers on any subset of cards (including the set of all cards) is not divisible by 100. Show that all the cards contain the same number.

16. **(USS 5)**$^{\text{IMO2}}$ Given a convex polyhedron P_1 with 9 vertices $A_1, \dots, A_9$, let us denote by $P_2, P_3, \dots, P_9$ the images of P_1 under the translations mapping the vertex A_1 to $A_2, A_3, \dots, A_9$ respectively. Prove that among the polyhedra $P_1, \dots, P_9$ at least two have a common interior point.

17. **(YUG 1)** Prove the inequality

$$\frac{a_1+a_3}{a_1+a_2}+\frac{a_2+a_4}{a_2+a_3}+\frac{a_3+a_1}{a_3+a_4}+\frac{a_4+a_2}{a_4+a_1} \ge 4,$$

where $a_i > 0$, $i = 1, 2, 3, 4$.

3.14 The Fourteenth IMO
Warsaw–Toruna, Poland, July 5–17, 1972

3.14.1 Contest Problems

First Day (July 10)

1. A set of 10 positive integers is given such that the decimal expansion of each of them has two digits. Prove that there are two disjoint subsets of the set with equal sums of their elements.
2. Prove that for each $n \geq 4$ every cyclic quadrilateral can be decomposed into n cyclic quadrilaterals.
3. Let m and n be nonnegative integers. Prove that $\frac{(2m)!(2n)!}{m!n!(m+n)!}$ is an integer $(0! = 1)$.

Second Day (July 11)

4. Find all solutions in positive real numbers x_i $(i = 1, 2, 3, 4, 5)$ of the following system of inequalities:

$$\begin{aligned}
(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) &\leq 0 && \text{(i)}\\
(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) &\leq 0 && \text{(ii)}\\
(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) &\leq 0 && \text{(iii)}\\
(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) &\leq 0 && \text{(iv)}\\
(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) &\leq 0. && \text{(v)}
\end{aligned}$$

5. Let f and φ be real functions defined in the interval $(-\infty, \infty)$ satisfying the functional equation

$$f(x + y) + f(x - y) = 2\varphi(y)f(x),$$

   for arbitrary real x, y (give examples of such functions). Prove that if $f(x)$ is not identically 0 and $|f(x)| \leq 1$ for all x, then $|\varphi(x)| \leq 1$ for all x.
6. Given four distinct parallel planes, show that a regular tetrahedron exists with a vertex on each plane.

3.14.2 Longlisted Problems

1. **(BUL 1)** Find all integer solutions of the equation

$$1 + x + x^2 + x^3 + x^4 = y^4.$$

2. **(BUL 2)** Find all real values of the parameter a for which the system of equations

$$x^4 = yz - x^2 + a,$$
$$y^4 = zx - y^2 + a,$$
$$z^4 = xy - z^2 + a,$$

has at most one real solution.

3. **(BUL 3)** On a line a set of segments is given of total length less than n. Prove that every set of n points of the line can be translated in some direction along the line for a distance smaller than $n/2$ so that none of the points remain on the segments.
4. **(BUL 4)** Given a triangle, prove that the points of intersection of three pairs of trisectors of the inner angles at the sides lying closest to those sides are vertices of an equilateral triangle.
5. **(BUL 5)** Given a pyramid whose base is an n-gon inscribable in a circle, let H be the projection of the top vertex of the pyramid to its base. Prove that the projections of H to the lateral edges of the pyramid lie on a circle.
6. **(BUL 6)** Prove the inequality

$$(n+1)\cos\frac{\pi}{n+1} - n\cos\frac{\pi}{n} > 1$$

for all natural numbers $n \geq 2$.

7. **(BUL 7)** (SL72-1).
8. **(CZS 1)** (SL72-2).
9. **(CZS 2)** Given natural numbers k and n, $k \leq n$, $n \geq 3$, find the set of all values in the interval $(0, \pi)$ that the kth-largest among the interior angles of a convex $n$gon can take.
10. **(CZS 3)** Given five points in the plane, no three of which are collinear, prove that there can be found at least two obtuse-angled triangles with vertices at the given points. Construct an example in which there are exactly two such triangles.
11. **(CZS 4)** (SL72-3).
12. **(CZS 5)** A circle $k = (S, r)$ is given and a hexagon $AA'BB'CC'$ inscribed in it. The lengths of sides of the hexagon satisfy $AA' = A'B$, $BB' = B'C$, $CC' = C'A$. Prove that the area P of triangle ABC is not greater than the area P' of triangle $A'B'C'$. When does $P = P'$ hold?
13. **(CZS 6)** Given a sphere K, determine the set of all points A that are vertices of some parallelograms $ABCD$ that satisfy $AC \leq BD$ and whose entire diagonal BD is contained in K.
14. **(GBR 1)** (SL72-7).
15. **(GBR 2)** (SL72-8).

16. **(GBR 3)** Consider the set S of all the different odd positive integers that are not multiples of 5 and that are less than $30m$, m being a positive integer. What is the smallest integer k such that in any subset of k integers from S there must be two integers one of which divides the other? Prove your result.

17. **(GBR 4)** A solid right circular cylinder with height h and base-radius r has a solid hemisphere of radius r resting upon it. The center of the hemisphere O is on the axis of the cylinder. Let P be any point on the surface of the hemisphere and Q the point on the base circle of the cylinder that is furthest from P (measuring along the surface of the combined solid). A string is stretched over the surface from P to Q so as to be as short as possible. Show that if the string is not in a plane, the straight line PO when produced cuts the curved surface of the cylinder.

18. **(GBR 5)** We have p players participating in a tournament, each player playing against every other player exactly once. A point is scored for each victory, and there are no draws. A sequence of nonnegative integers $s_1 \le s_2 \le s_3 \le \cdots \le s_p$ is given. Show that it is possible for this sequence to be a set of final scores of the players in the tournament if and only if

$$\text{(i)} \ \sum_{i=1}^{p} s_i = \frac{1}{2}p(p-1) \quad \text{and} \quad \text{(ii) for all } k < p, \ \sum_{i=1}^{k} s_i \ge \frac{1}{2}k(k-1).$$

19. **(GBR 6)** Let S be a subset of the real numbers with the following properties:
   (i) If $x \in S$ and $y \in S$, then $x - y \in S$;
   (ii) If $x \in S$ and $y \in S$, then $xy \in S$;
   (iii) S contains an exceptional number x' such that there is no number y in S satisfying $x'y + x' + y = 0$;
   (iv) If $x \in S$ and $x \ne x'$, there is a number y in S such that $xy + x + y = 0$.
   Show that
   (a) S has more than one number in it;
   (b) $x' \ne -1$ leads to a contradiction;
   (c) $x \in S$ and $x \ne 0$ implies $1/x \in S$.

20. **(GDR 1)** (SL72-4).

21. **(GDR 2)** (SL72-5).

22. **(GDR 3)** (SL72-6).

23. **(MON 1)** Does there exist a $2n$-digit number $\overline{a_{2n}a_{2n-1}\dots a_1}$ (for an arbitrary n) for which the following equality holds:

$$\overline{a_{2n}\dots a_1} = (\overline{a_n \dots a_1})^2?$$

24. **(MON 2)** The diagonals of a convex 18-gon are colored in 5 different colors, each color appearing on an equal number of diagonals. The diagonals of one color are numbered $1, 2, \dots$. One randomly chooses one-fifth

of all the diagonals. Find the number of possibilities for which among the chosen diagonals there exist exactly n pairs of diagonals of the same color and with fixed indices i, j.

25. **(NET 1)** We consider n real variables x_i ($1 \le i \le n$), where n is an integer and $n \ge 2$. The product of these variables will be denoted by p, their sum by s, and the sum of their squares by S. Furthermore, let α be a positive constant. We now study the inequality $ps \le S^\alpha$. Prove that it holds for every n-tuple (x_i) if and only if $\alpha = \frac{n+1}{2}$.

26. **(NET 2)** (SL72-9).

27. **(NET 3)** (SL72-10).

28. **(NET 4)** The lengths of the sides of a rectangle are given to be odd integers. Prove that there does not exist a point within that rectangle that has integer distances to each of its four vertices.

29. **(NET 5)** Let A, B, C be points on the sides B_1C_1, C_1A_1, A_1B_1 of a triangle $A_1B_1C_1$ such that A_1A, B_1B, C_1C are the bisectors of angles of the triangle. We have that $AC = BC$ and $A_1C_1 \ne B_1C_1$.
(a) Prove that C_1 lies on the circumcircle of the triangle ABC.
(b) Suppose that $\measuredangle BAC_1 = \pi/6$; find the form of triangle ABC.

30. **(NET 6)** (SL72-11).

31. **(ROM 1)** Find values of $n \in \mathbb{N}$ for which the fraction $\frac{3^n-2}{2^n-3}$ is reducible.

32. **(ROM 2)** If $n_1, n_2, \dots, n_k$ are natural numbers and $n_1+n_2+\cdots+n_k = n$, show that
$$\max_{n_1+\cdots+n_k=n} n_1n_2\cdots n_k = (t+1)^r t^{k-r},$$
where $t = [n/k]$ and r is the remainder of n upon division by k; i.e., $n = tk + r$, $0 \le r \le k-1$.

33. **(ROM 3)** A rectangle $ABCD$ is given whose sides have lengths 3 and $2n$, where n is a natural number. Denote by $U(n)$ the number of ways in which one can cut the rectangle into rectangles of side lengths 1 and 2.
(a) Prove that $U(n+1) + U(n-1) = 4U(n)$;
(b) Prove that $U(n) = \frac{1}{2\sqrt{3}}[(\sqrt{3}+1)(2+\sqrt{3})^n + (\sqrt{3}-1)(2-\sqrt{3})^n]$.

34. **(ROM 4)** If p is a prime number greater than 2 and a, b, c integers not divisible by p, prove that the equation
$$ax^2 + by^2 = pz + c$$
has an integer solution.

35. **(ROM 5)** (a) Prove that for $a, b, c, d \in \mathbb{R}$, $m \in [1, +\infty)$ with $am + b = -cm + d = m$,
(i) $\sqrt{a^2+b^2} + \sqrt{c^2+d^2} + \sqrt{(a-c)^2+(b-d)^2} \ge \frac{4m^2}{1+m^2}$, and

(ii) $2 \le \frac{4m^2}{1+m^2} < 4$.

(b) Express a, b, c, d as functions of m so that there is equality in (1).

36. **(ROM 6)** A finite number of parallel segments in the plane are given with the property that for any three of the segments there is a line intersecting each of them. Prove that there exists a line that intersects all the given segments.

37. **(SWE 1)** On a chessboard (8×8 squares with sides of length 1) two diagonally opposite corner squares are taken away. Can the board now be covered with nonoverlapping rectangles with sides of lengths 1 and 2?

38. **(SWE 2)** Congruent rectangles with sides m (cm) and n (cm) are given (m, n positive integers). Characterize the rectangles that can be constructed from these rectangles (in the fashion of a jigsaw puzzle). (The number of rectangles is unbounded.)

39. **(SWE 3)** How many tangents to the curve $y = x^3 - 3x$ ($y = x^3 + px$) can be drawn from different points in the plane?

40. **(SWE 4)** Prove the inequalities

$$\frac{u}{v} \le \frac{\sin u}{\sin v} \le \frac{\pi}{2}\frac{u}{v}, \qquad \text{for } 0 \le u < v \le \frac{\pi}{2}.$$

41. **(SWE 5)** The ternary expansion $x = 0.10101010\ldots$ is given. Give the binary expansion of x.
Alternatively, transform the binary expansion $y = 0.110110110\ldots$ into a ternary expansion.

42. **(SWE 6)** The decimal number 13^{101} is given. It is instead written as a ternary number. What are the two last digits of this ternary number?

43. **(USS 1)** A fixed point A inside a circle is given. Consider all chords XY of the circle such that $\angle XAY$ is a right angle, and for all such chords construct the point M symmetric to A with respect to XY. Find the locus of points M.

44. **(USS 2)** (SL72-12).

45. **(USS 3)** Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD intersect at point O. Let a line through O intersect segment AB at M and segment CD at N. Prove that the segment MN is not longer than at least one of the segments AC and BD.

46. **(USS 4)** Numbers $1, 2, \ldots, 16$ are written in a 4×4 square matrix so that the sum of the numbers in every row, every column, and every diagonal is the same and furthermore that the numbers 1 and 16 lie in opposite corners. Prove that the sum of any two numbers symmetric with respect to the center of the square equals 17.

3.14.3 Shortlisted Problems

1. **(BUL 7)**$^{\mathrm{IMO5}}$ Let f and φ be real functions defined on the set $\mathbb{R}$ satisfying the functional equation
$$f(x+y)+f(x-y)=2\varphi(y)f(x), \tag{1}$$
for arbitrary real x, y (give examples of such functions). Prove that if $f(x)$ is not identically 0 and $|f(x)| \le 1$ for all x, then $|\varphi(x)| \le 1$ for all x.

2. **(CZS 1)** We are given $3n$ points $A_1, A_2, \dots, A_{3n}$ in the plane, no three of them collinear. Prove that one can construct n disjoint triangles with vertices at the points A_i.

3. **(CZS 4)** Let $x_1, x_2, \dots, x_n$ be real numbers satisfying $x_1+x_2+\cdots+x_n = 0$. Let m be the least and M the greatest among them. Prove that
$$x_1^2+x_2^2+\cdots+x_n^2 \le -nmM.$$

4. **(GDR 1)** Let n_1, n_2 be positive integers. Consider in a plane E two disjoint sets of points M_1 and M_2 consisting of $2n_1$ and $2n_2$ points, respectively, and such that no three points of the union $M_1 \cup M_2$ are collinear. Prove that there exists a straightline g with the following property: Each of the two half-planes determined by g on E (g not being included in either) contains exactly half of the points of M_1 and exactly half of the points of M_2.

5. **(GDR 2)** Prove the following assertion: The four altitudes of a tetrahedron $ABCD$ intersect in a point if and only if
$$AB^2+CD^2=BC^2+AD^2=CA^2+BD^2.$$

6. **(GDR 3)** Show that for any $n \not\equiv 0 \pmod{10}$ there exists a multiple of n not containing the digit 0 in its decimal expansion.

7. **(GBR 1)**$^{\mathrm{IMO6}}$ (a) A plane π passes through the vertex O of the regular tetrahedron $OPQR$. We define p, q, r to be the signed distances of P, Q, R from π measured along a directed normal to π. Prove that
$$p^2+q^2+r^2+(q-r)^2+(r-p)^2+(p-q)^2=2a^2,$$
where a is the length of an edge of a tetrahedron.
(b) Given four parallel planes not all of which are coincident, show that a regular tetrahedron exists with a vertex on each plane.

8. **(GBR 2)**$^{\mathrm{IMO3}}$ Let m and n be nonnegative integers. Prove that $m!n!(m+n)!$ divides $(2m)!(2n)!$.

9. **(NET 2)**$^{\mathrm{IMO4}}$ Find all solutions in positive real numbers x_i ($i = 1, 2, 3, 4, 5$) of the following system of inequalities:

$$\begin{aligned}
(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) &\le 0, && \text{(i)}\\
(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) &\le 0, && \text{(ii)}\\
(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) &\le 0, && \text{(iii)}\\
(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) &\le 0, && \text{(iv)}\\
(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) &\le 0. && \text{(v)}
\end{aligned}$$

10. **(NET 3)**[IMO2] Prove that for each $n \ge 4$ every cyclic quadrilateral can be decomposed into n cyclic quadrilaterals.

11. **(NET 6)** Consider a sequence of circles $K_1, K_2, K_3, K_4, \dots$ of radii $r_1, r_2, r_3, r_4, \dots$, respectively, situated inside a triangle ABC. The circle K_1 is tangent to AB and AC; K_2 is tangent to K_1, BA, and BC; K_3 is tangent to K_2, CA, and CB; K_4 is tangent to K_3, AB, and AC; etc.
    (a) Prove the relation

$$r_1 \cot \frac{1}{2}A + 2\sqrt{r_1 r_2} + r_2 \cot \frac{1}{2}B = r\left(\cot \frac{1}{2}A + \cot \frac{1}{2}B\right),$$

    where r is the radius of the incircle of the triangle ABC. Deduce the existence of a t_1 such that

$$r_1 = r \cot \frac{1}{2}B \cot \frac{1}{2}C \sin^2 t_1.$$

    (b) Prove that the sequence of circles $K_1, K_2, \dots$ is periodic.

12. **(USS 2)**[IMO1] A set of 10 positive integers is given such that the decimal expansion of each of them has two digits. Prove that there are two disjoint subsets of the set with equal sums of their elements.

3.15 The Fifteenth IMO
Moscow, Soviet Union, July 5–16, 1973

3.15.1 Contest Problems

First Day (July 9)

1. Let O be a point on the line l and $\overrightarrow{OP_1}, \overrightarrow{OP_2}, \dots, \overrightarrow{OP_n}$ unit vectors such that points $P_1, P_2, \dots, P_n$ and line l lie in the same plane and all points P_i lie in the same half-plane determined by l. Prove that if n is odd, then
$$\left\|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \cdots + \overrightarrow{OP_n}\right\| \geq 1.$$
($\left\|\overrightarrow{OM}\right\|$ is the length of vector $\overrightarrow{OM}$).

2. Does there exist a finite set M of points in space, not all in the same plane, such that for each two points $A, B \in M$ there exist two other points $C, D \in M$ such that lines AB and CD are parallel but not equal?

3. Determine the minimum of $a^2 + b^2$ if a and b are real numbers for which the equation
$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$
has at least one real solution.

Second Day (July 10)

4. A soldier has to investigate whether there are mines in an area that has the form of equilateral triangle. The radius of his detector's range is equal to one-half the altitude of the triangle. The soldier starts from one vertex of the triangle. Determine the smallest path through which the soldier has to pass in order to check the entire region.

5. Let G be the set of functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$. Suppose that G satisfies the following conditions:
(1) If $f, g \in G$, then $g \circ f \in G$, where $(g \circ f)(x) = g[f(x)]$.
(2) If $f \in G$ and $f(x) = ax + b$, then the inverse f^{-1} of f belongs to G ($f^{-1}(x) = (x - b)/a$).
(3) For each $f \in G$ there exists a number $x_f \in \mathbb{R}$ such that $f(x_f) = x_f$.
Prove that there exists a number $k \in \mathbb{R}$ such that $f(k) = k$ for all $f \in G$.

6. Let $a_1, a_2, \dots, a_n$ be positive numbers and q a given real number, $0 < q < 1$. Find n real numbers $b_1, b_2, \dots, b_n$ that satisfy:
(1) $a_k < b_k$ for all $k = 1, 2, \dots, n$;
(2) $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for all $k = 1, 2, \dots, n-1$;
(3) $b_1 + b_2 + \cdots + b_n < \frac{1+q}{1-q}(a_1 + a_2 + \cdots + a_n)$.

3.15.2 Shortlisted Problems

1. **(BUL 6)** Let a tetrahedron $ABCD$ be inscribed in a sphere S. Find the locus of points P inside the sphere S for which the equality
$$\frac{AP}{PA_1}+\frac{BP}{PB_1}+\frac{CP}{PC_1}+\frac{DP}{PD_1}=4$$
holds, where A_1, B_1, C_1, and D_1 are the intersection points of S with the lines AP, BP, CP, and DP, respectively.

2. **(CZS 1)** Given a circle K, find the locus of vertices A of parallelograms $ABCD$ with diagonals $AC \le BD$, such that BD is inside K.

3. **(CZS 6)**$^{\text{IMO1}}$ Prove that the sum of an odd number of unit vectors passing through the same point O and lying in the same half-plane whose border passes through O has length greater than or equal to 1.

4. **(GBR 1)** Let P be a set of 7 different prime numbers and C a set of 28 different composite numbers each of which is a product of two (not necessarily different) numbers from P. The set C is divided into 7 disjoint four-element subsets such that each of the numbers in one set has a common prime divisor with at least two other numbers in that set. How many such partitions of C are there?

5. **(FRA 2)** A circle of radius 1 is located in a right-angled trihedron and touches all its faces. Find the locus of centers of such circles.

6. **(POL 2)**$^{\text{IMO2}}$ Does there exist a finite set M of points in space, not all in the same plane, such that for each two points $A, B \in M$ there exist two other points $C, D \in M$ such that lines AB and CD are parallel?

7. **(POL 3)** Given a tetrahedron $ABCD$, let $x = AB \cdot CD$, $y = AC \cdot BD$, and $z = AD \cdot BC$. Prove that there exists a triangle with edges x, y, z.

8. **(ROM 1)** Prove that there are exactly $\binom{k}{[k/2]}$ arrays $a_1, a_2, \dots, a_{k+1}$ of nonnegative integers such that $a_1 = 0$ and $|a_i - a_{i+1}| = 1$ for $i = 1, 2, \dots, k$.

9. **(ROM 2)** Let Ox, Oy, Oz be three rays, and G a point inside the trihedron $Oxyz$. Consider all planes passing through G and cutting Ox, Oy, Oz at points A, B, C, respectively. How is the plane to be placed in order to yield a tetrahedron $OABC$ with minimal perimeter?

10. **(SWE 3)**$^{\text{IMO6}}$ Let $a_1, a_2, \dots, a_n$ be positive numbers and q a given real number, $0 < q < 1$. Find n real numbers $b_1, b_2, \dots, b_n$ that satisfy:
(1) $a_k < b_k$ for all $k = 1, 2, \dots, n$;
(2) $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for all $k = 1, 2, \dots, n-1$;
(3) $b_1 + b_2 + \dots + b_n < \frac{1+q}{1-q}(a_1 + a_2 + \dots + a_n)$.

11. **(SWE 4)**$^{\text{IMO3}}$ Determine the minimum of $a^2 + b^2$ if a and b are real numbers for which the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

has at least one real solution.

12. **(SWE 6)** Consider the two square matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

with entries 1 and -1. The following operations will be called *elementary*:
(1) Changing signs of all numbers in one row;
(2) Changing signs of all numbers in one column;
(3) Interchanging two rows (two rows exchange their positions);
(4) Interchanging two columns.
Prove that the matrix B cannot be obtained from the matrix A using these operations.

13. **(YUG 4)** Find the sphere of maximal radius that can be placed inside every tetrahedron that has all altitudes of length greater than or equal to 1.

14. **(YUG 5)**$^{\text{IMO4}}$ A soldier has to investigate whether there are mines in an area that has the form of an equilateral triangle. The radius of his detector is equal to one-half of an altitude of the triangle. The soldier starts from one vertex of the triangle. Determine the shortest path that the soldier has to traverse in order to check the whole region.

15. **(CUB 1)** Prove that for all $n \in \mathbb{N}$ the following is true:

$$2^n \prod_{k=1}^{n} \sin \frac{k\pi}{2n+1} = \sqrt{2n+1}.$$

16. **(CUB 2)** Given $a, \theta \in \mathbb{R}$, $m \in \mathbb{N}$, and $P(x) = x^{2m} - 2|a|^m x^m \cos\theta + a^{2m}$, factorize $P(x)$ as a product of m real quadratic polynomials.

17. **(POL 1)**$^{\text{IMO5}}$ Let $\mathcal{F}$ be a nonempty set of functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$. Suppose that $\mathcal{F}$ satisfies the following conditions:
(1) If $f, g \in \mathcal{F}$, then $g \circ f \in \mathcal{F}$, where $(g \circ f)(x) = g[f(x)]$.
(2) If $f \in \mathcal{F}$ and $f(x) = ax + b$, then the inverse f^{-1} of f belongs to $\mathcal{F}$ $(f^{-1}(x) = (x - b)/a)$.
(3) None of the functions $f(x) = x + c$, for $c \neq 0$, belong to $\mathcal{F}$.
Prove that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$ for all $f \in \mathcal{F}$.

3.16 The Sixteenth IMO
Erfurt–Berlin, DR Germany, July 4–17, 1974

3.16.1 Contest Problems

First Day (July 8)

1. Alice, Betty, and Carol took the same series of examinations. There was one grade of A, one grade of B, and one grade of C for each examination, where A, B, C are different positive integers. The final test scores were

Alice	Betty	Carol
20	10	9

If Betty placed first in the arithmetic examination, who placed second in the spelling examination?

2. Let $\triangle ABC$ be a triangle. Prove that there exists a point D on the side AB such that CD is the geometric mean of AD and BD if and only if

$$\sqrt{\sin A \sin B} \le \sin \frac{C}{2}.$$

3. Prove that there does not exist a natural number n for which the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

is divisible by 5.

Second Day (July 9)

4. Consider a partition of an 8×8 chessboard into p rectangles whose interiors are disjoint such that each rectangle contains an equal number of white and black cells. Assume that $a_1 < a_2 < \cdots < a_p$, where a_i denotes the number of white cells in the ith rectangle. Find the maximal p for which such a partition is possible and for that p determine all possible corresponding sequences $a_1, a_2, \dots, a_p$.

5. If a, b, c, d are arbitrary positive real numbers, find all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}.$$

6. Let $P(x)$ be a polynomial with integer coefficients. If $n(P)$ is the number of (distinct) integers k such that $P^2(k) = 1$, prove that $n(P) - \deg(P) \le 2$, where $\deg(P)$ denotes the degree of the polynomial P.

3.16.2 Longlisted Problems

1. **(BUL 1)** (SL74-11).

2. **(BUL 2)** Let $\{u_n\}$ be the Fibonacci sequence, i.e., $u_0 = 0$, $u_1 = 1$, $u_n = u_{n-1} + u_{n-2}$ for $n > 1$. Prove that there exist infinitely many prime numbers p that divide u_{p-1}.

3. **(BUL 3)** Let $ABCD$ be an arbitrary quadrilateral. Let squares ABB_1A_2, BCC_1B_2, CDD_1C_2, DAA_1D_2 be constructed in the exterior of the quadrilateral. Furthermore, let AA_1PA_2 and CC_1QC_2 be parallelograms. For any arbitrary point P in the interior of $ABCD$, parallelograms $RASC$ and $RPTQ$ are constructed. Prove that these two parallelograms have two vertices in common.

4. **(BUL 4)** Let K_a, K_b, K_c with centers O_a, O_b, O_c be the excircles of a triangle ABC, touching the interiors of the sides BC, CA, AB at points T_a, T_b, T_c respectively.
   Prove that the lines O_aT_a, O_bT_b, O_cT_c are concurrent in a point P for which $PO_a = PO_b = PO_c = 2R$ holds, where R denotes the circumradius of ABC. Also prove that the circumcenter O of ABC is the midpoint of the segment PJ, where J is the incenter of ABC.

5. **(BUL 5)** A straight cone is given inside a rectangular parallelepiped B, with the apex at one of the vertices, say T, of the parallelepiped, and the base touching the three faces opposite to T. Its axis lies at the long diagonal through T. If V_1 and V_2 are the volumes of the cone and the parallelepiped respectively, prove that
$$V_1 \le \frac{\sqrt{3}\pi V_2}{27}.$$

6. **(CUB 1)** Prove that the product of two natural numbers with their sum cannot be the third power of a natural number.

7. **(CUB 2)** Let P be a prime number and n a natural number. Prove that the product
$$N = \frac{1}{p^{n^2}} \prod_{i=1;\ 2\nmid i}^{2n-1} \left[((p-1)i)! \binom{p^2 i}{pi} \right]$$
   is a natural number that is not divisible by p.

8. **(CUB 3)** (SL74-9).

9. **(CZS 1)** Solve the following system of linear equations with unknown $x_1, \dots, x_n$ $(n \ge 2)$ and parameters $c_1, \dots, c_n$:

$$\begin{array}{rrrrrrcl}
2x_1 & -x_2 & & & & & = & c_1; \\
-x_1 & +2x_2 & -x_3 & & & & = & c_2; \\
 & -x_2 & +2x_3 & -x_4 & & & = & c_3; \\
 & \cdots & & \cdots & & \cdots & & \cdots \\
 & & & -x_{n-2} & +2x_{n-1} & -x_n & = & c_{n-1}; \\
 & & & & -x_{n-1} & +2x_n & = & c_n.
\end{array}$$

10. **(CZS 2)** A regular octagon P is given whose incircle k has diameter 1. About k is circumscribed a regular 16-gon, which is also inscribed in P, cutting from P eight isosceles triangles. To the octagon P, three of these triangles are added so that exactly two of them are adjacent and no two of them are opposite to each other. Every 11-gon so obtained is said to be P'.
    Prove the following statement: Given a finite set M of points lying in P such that every two points of this set have a distance not exceeding 1, one of the 11-gons P' contains all of M.
11. **(CZS 3)** Given a line p and a triangle $\triangle$ in the plane, construct an equilateral triangle one of whose vertices lies on the line p, while the other two halve the perimeter of $\triangle$.
12. **(CZS 4)** A circle K with radius r, a point D on K, and a convex angle with vertex S and rays a and b are given in the plane. Construct a parallelogram $ABCD$ such that A and B lie on a and b respectively, $SA + SB = r$, and C lies on K.
13. **(FIN 1)** Prove that $2^{147} - 1$ is divisible by 343.
14. **(FIN 2)** Let n and k be natural numbers and $a_1, a_2, \dots, a_n$ positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = 1$. Prove that
    $$a_1^{-k} + a_2^{-k} + \cdots + a_n^{-k} \geq n^{k+1}.$$
15. **(FIN 3)** (SL74-10).
16. **(GBR 1)** A pack of $2n$ cards contains n different pairs of cards. Each pair consists of two identical cards, either of which is called the *twin* of the other. A game is played between two players A and B. A third person called the *dealer* shuffles the pack and deals the cards one by one face upward onto the table. One of the players, called the *receiver*, takes the card dealt, provided he does not have already its twin. If he does already have the twin, his opponent takes the dealt card and becomes the receiver. A is initially the receiver and takes the first card dealt. The player who first obtains a complete set of n different cards wins the game. What fraction of all possible arrangements of the pack lead to A winning? Prove the correctness of your answer.
17. **(GBR 2)** Show that there exists a set S of 15 distinct circles on the surface of a sphere, all having the same radius and such that 5 touch exactly 5 others, 5 touch exactly 4 others, and 5 touch exactly 3 others.

18. **(GBR 3)** (SL74-5).

19. **(GBR 4)** (Alternative to GBR 2) Prove that there exists, for $n \geq 4$, a set S of $3n$ equal circles in spacethat can be partitioned into three subsets s_5, s_4, and s_3, each containing n circles, such that each circle in s_r touches exactly r circles in S.

20. **(NET 1)** For which natural numbers n do there exist n natural numbers a_i $(1 \leq i \leq n)$ such that $\sum_{i=1}^{n} a_i^{-2} = 1$?

21. **(NET 2)** Let M be a nonempty subset of $\mathbb{Z}^+$ such that for every element x in M, the numbers $4x$ and $[\sqrt{x}]$ also belong to M. Prove that $M = \mathbb{Z}^+$.

22. **(NET 3)** (SL74-8).

23. **(POL 1)** (SL74-2).

24. **(POL 2)** (SL74-7).

25. **(POL 3)** Let $f : \mathbb{R} \to \mathbb{R}$ be of the form $f(x) = x + \varepsilon \sin x$, where $0 < |\varepsilon| \leq 1$. Define for any $x \in \mathbb{R}$,
$$x_n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}(x).$$
Show that for every $x \in \mathbb{R}$ there exists an integer k such that $\lim_{n\to\infty} x_n = k\pi$.

26. **(POL 4)** Let $g(k)$ be the number of partitions of a k-element set M, i.e., the number of families $\{A_1, A_2, \dots, A_s\}$ of nonempty subsets of M such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} A_i = M$. Prove that
$$n^n \leq g(2n) \leq (2n)^{2n} \qquad \text{for every } n.$$

27. **(ROM 1)** Let C_1 and C_2 be circles in the same plane, P_1 and P_2 arbitrary points on C_1 and C_2 respectively, and Q the midpoint of segment P_1P_2. Find the locus of points Q as P_1 and P_2 go through all possible positions.

    Alternative version. Let C_1, C_2, C_3 be three circles in the same plane. Find the locus of the centroid of triangle $P_1P_2P_3$ as P_1, P_2, and P_3 go through all possible positions on C_1, C_2, and C_3 respectively.

28. **(ROM 2)** Let M be a finite set and $P = \{M_1, M_2, \dots, M_k\}$ a partition of M (i.e., $\bigcup_{i=1}^{k} M_i = M$, $M_i \neq \emptyset$, $M_i \cap M_j = \emptyset$ for all $i, j \in \{1, 2, \dots, k\}$, $i \neq j$). We define the following elementary operation on P:

    Choose $i, j \in \{1, 2, \dots, k\}$, such that $i \neq j$ and M_i has a elements and M_j has b elements such that $a \geq b$. Then take b elements from M_i and place them into M_j, i.e., M_j becomes the union of itself unifies and a b-element subset of M_i, while the same subset is subtracted from M_i (if $a = b$, M_i is thus removed from the partition).

    Let a finite set M be given. Prove that the property "for every partition P of M there exists a sequence $P = P_1, P_2, \dots, P_r$ such that P_{i+1} is obtained from P_i by an elementary operation and $P_r = \{M\}$" is equivalent to "the number of elements of M is a power of 2."

29. **(ROM 3)** Let A, B, C, D be points in space. If for every point M on the segment AB the sum

$$\text{area}(AMC)+\text{area}(CMD)+\text{area}(DMB)$$

is constant show that the points A, B, C, D lie in the same plane.

30. **(ROM 4)** (SL74-6).

31. **(ROM 5)** Let $y^\alpha = \sum_{i=1}^n x_i^\alpha$, where $\alpha \neq 0$, $y > 0$, $x_i > 0$ are real numbers, and let $\lambda \neq \alpha$ be a real number. Prove that $y^\lambda > \sum_{i=1}^n x_i^\lambda$ if $\alpha(\lambda - \alpha) > 0$, and $y^\lambda < \sum_{i=1}^n x_i^\lambda$ if $\alpha(\lambda - \alpha) < 0$.

32. **(SWE 1)** Let $a_1, a_2, \dots, a_n$ be n real numbers such that $0 < a \leq a_k \leq b$ for $k = 1, 2, \dots, n$. If

$$m_1 = \frac{1}{n}(a_1 + a_2 + \cdots + a_n) \qquad \text{and} \qquad m_2 = \frac{1}{n}(a_1^2 + a_2^2 + \cdots + a_n^2),$$

prove that $m_2 \leq \frac{(a+b)^2}{4ab} m_1^2$ and find a necessary and sufficient condition for equality.

33. **(SWE 2)** Let a be a real number such that $0 < a < 1$, and let n be a positive integer. Define the sequence $a_0, a_1, a_2, \dots, a_n$ recursively by

$$a_0 = a; \qquad a_{k+1} = a_k + \frac{1}{n} a_k^2 \qquad \text{for } k = 0, 1, \dots, n-1.$$

Prove that there exists a real number A, depending on a but independent of n, such that

$$0 < n(A - a_n) < A^3.$$

34. **(SWE 3)** (SL74-3).

35. **(SWE 4)** If p and q are distinct prime numbers, then there are integers x_0 and y_0 such that $1 = px_0 + qy_0$. Determine the maximum value of $b - a$, where a and b are positive integers with the following property: If $a \leq t \leq b$, and t is an integer, then there are integers x and y with $0 \leq x \leq q - 1$ and $0 \leq y \leq p - 1$ such that $t = px + qy$.

36. **(SWE 5)** Consider infinite diagrams

$$D = \begin{array}{|llll} \vdots & \vdots & \vdots & \\ n_{20} & n_{21} & n_{22} & \cdots \\ n_{10} & n_{11} & n_{12} & \cdots \\ n_{00} & n_{01} & n_{02} & \cdots \\ \hline \end{array}$$

where all but a finite number of the integers n_{ij}, $i = 0, 1, 2, \dots$, $j = 0, 1, 2, \dots$, are equal to 0. Three elements of a diagram are called *adjacent* if there are integers i and j with $i \geq 0$ and $j \geq 0$ such that the three elements are

(i) n_{ij}, $n_{i,j+1}$, $n_{i,j+2}$, or
(ii) n_{ij}, $n_{i+1,j}$, $n_{i+2,j}$, or
(iii) $n_{i+2,j}$, $n_{i+1,j+1}$, $n_{i,j+2}$.

An elementary operation on a diagram is an operation by which three adjacent elements n_{ij} are changed into n'_{ij} in such a way that $|n_{ij} - n'_{ij}| = 1$. Two diagrams are called equivalent if one of them can be changed into the other by a finite sequence of elementary operations. How many inequivalent diagrams exist?

37. **(USA 1)** Let a, b, and c denote the three sides of a billiard table in the shape of an equilateral triangle. A ball is placed at the midpoint of side a and then propelled toward side b with direction defined by the angle θ. For what values of θ will the ball strike the sides b, c, a in that order?

38. **(USA 2)** Consider the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $(k = 1, 2, \dots, n-1)$. Determine all positive integers n for which $\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}$ are all even numbers.

39. **(USA 3)** Let n be a positive integer, $n \geq 2$, and consider the polynomial equation
$$x^n - x^{n-2} - x + 2 = 0.$$
For each n, determine all complex numbers x that satisfy the equation and have modulus $|x| = 1$.

40. **(USA 4)** (SL74-1).

41. **(USA 5)** Through the circumcenter O of an arbitrary acute-angled triangle, chords A_1A_2, B_1B_2, C_1C_2 are drawn parallel to the sides BC, CA, AB of the triangle respectively. If R is the radius of the circumcircle, prove that
$$A_1O \cdot OA_2 + B_1O \cdot OB_2 + C_1O \cdot OC_2 = R^2.$$

42. **(USS 1)** (SL74-12).

43. **(USS 2)** An $(n^2+n+1) \times (n^2+n+1)$ matrix of zeros and ones is given. If no four ones are vertices of a rectangle, prove that the number of ones does not exceed $(n+1)(n^2+n+1)$.

44. **(USS 3)** We are given n mass points of equal mass in space. We define a sequence of points $O_1, O_2, O_3, \dots$ as follows: O_1 is an arbitrary point (within the unit distance of at least one of the n points); O_2 is the center of gravity of all the n given points that are inside the unit sphere centered at O_1; O_3 is the center of gravity of all of the n given points that are inside the unit sphere centered at O_2; etc. Prove that starting from some m, all points $O_m, O_{m+1}, O_{m+2}, \dots$ coincide.

45. **(USS 4)** (SL74-4).

46. **(USS 5)** Outside an arbitrary triangle ABC, triangles ADB and BCE are constructed such that $\angle ADB = \angle BEC = 90^\circ$ and $\angle DAB = \angle EBC = 30^\circ$. On the segment AC the point F with $AF = 3FC$ is chosen. Prove that
$$\angle DFE = 90^\circ \quad \text{and} \quad \angle FDE = 30^\circ.$$

47. **(VIE 1)** Given two points A, B outside of a given plane P, find the positions of points M in the plane P for which the ratio $\frac{MA}{MB}$ takes a minimum or maximum.

48. **(VIE 2)** Let a be a number different from zero. For all integers n define $S_n = a^n + a^{-n}$. Prove that if for some integer k both S_k and S_{k+1} are integers, then for each integer n the number S_n is an integer.

49. **(VIE 3)** Determine an equation of third degree with integral coefficients having roots $\sin\frac{\pi}{14}$, $\sin\frac{5\pi}{14}$ and $\sin\frac{-3\pi}{14}$.

50. **(YUG 1)** Let m and n be natural numbers with $m > n$. Prove that
$$2(m-n)^2(m^2-n^2+1) \geq 2m^2 - 2mn + 1.$$

51. **(YUG 2)** There are n points on a flat piece of paper, any two of them at a distance of at least 2 from each other. An inattentive pupil spills ink on a part of the paper such that the total area of the damaged part equals 3/2. Prove that there exist two vectors of equal length less than 1 and with their sum having a given direction, such that after a translation by either of these two vectors no points of the given set remain in the damaged area.

52. **(YUG 3)** A fox stands in the center of the field which has the form of an equilateral triangle, and a rabbit stands at one of its vertices. The fox can move through the whole field, while the rabbit can move only along the border of the field. The maximal speeds of the fox and rabbit are equal to u and v, respectively. Prove that:
    (a) If $2u > v$, the fox can catch the rabbit, no matter how the rabbit moves.
    (b) If $2u \leq v$, the rabbit can always run away from the fox.

3.16.3 Shortlisted Problems

1. **I 1 (USA 4)**$^{\text{IMO1}}$ Alice, Betty, and Carol took the same series of examinations. There was one grade of A, one grade of B, and one grade of C for each examination, where A, B, C are different positive integers. The final test scores were

Alice	Betty	Carol
20	10	9

If Betty placed first in the arithmetic examination, who placed second in the spelling examination?

2. **I 2 (POL 1)** Prove that the squares with sides $1/1, 1/2, 1/3, \dots$ may be put into the square with side $3/2$ in such a way that no two of them have any interior point in common.

3. **I 3 (SWE 3)**$^{\text{IMO6}}$ Let $P(x)$ be a polynomial with integer coefficients. If $n(P)$ is the number of (distinct) integers k such that $P^2(k) = 1$, prove that
$$n(P) - \deg(P) \le 2,$$
where $\deg(P)$ denotes the degree of the polynomial P.

4. **I 4 (USS 4)** The sum of the squares of five real numbers a_1, a_2, a_3, a_4, a_5 equals 1. Prove that the least of the numbers $(a_i - a_j)^2$, where $i, j = 1, 2, 3, 4, 5$ and $i \ne j$, does not exceed $1/10$.

5. **I 5 (GBR 3)** Let A_r, B_r, C_r be points on the circumference of a given circle S. From the triangle $A_rB_rC_r$, called $\triangle_r$, the triangle $\triangle_{r+1}$ is obtained by constructing the points $A_{r+1}, B_{r+1}, C_{r+1}$ on S such that $A_{r+1}A_r$ is parallel to B_rC_r, $B_{r+1}B_r$ is parallel to C_rA_r, and $C_{r+1}C_r$ is parallel to A_rB_r. Each angle of $\triangle_1$ is an integer number of degrees and those integers are not multiples of 45. Prove that at least two of the triangles $\triangle_1, \triangle_2, \dots, \triangle_{15}$ are congruent.

6. **I 6 (ROM 4)**$^{\text{IMO3}}$ Does there exist a natural number n for which the number
$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$
is divisible by 5?

7. **II 1 (POL 2)** Let a_i, b_i be coprime positive integers for $i = 1, 2, \dots, k$, and m the least common multiple of $b_1, \dots, b_k$. Prove that the greatest common divisor of $a_1\frac{m}{b_1}, \dots, a_k\frac{m}{b_k}$ equals the greatest common divisor of $a_1, \dots, a_k$.

8. **II 2 (NET 3)**$^{\text{IMO5}}$ If a, b, c, d are arbitrary positive real numbers, find all possible values of
$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}.$$

9. **II 3 (CUB 3)** Let x, y, z be real numbers each of whose absolute value is different from $1/\sqrt{3}$ such that $x + y + z = xyz$. Prove that
$$\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{3x - x^3}{1 - 3x^2} \cdot \frac{3y - y^3}{1 - 3y^2} \cdot \frac{3z - z^3}{1 - 3z^2}.$$

10. **II 4 (FIN 3)**$^{\text{IMO2}}$ Let $\triangle ABC$ be a triangle. Prove that there exists a point D on the side AB such that CD is the geometric mean of AD and BD if and only if $\sqrt{\sin A \sin B} \le \sin \frac{C}{2}$.

11. **II 5 (BUL 1)**IMO4 Consider a partition of an 8×8 chessboard into p rectangles whose interiors are disjoint such that each of them has an equal number of white and black cells. Assume that $a_1 < a_2 < \cdots < a_p$, where a_i denotes the number of white cells in the ith rectangle. Find the maximal p for which such a partition is possible and for that p determine all possible corresponding sequences $a_1, a_2, \dots, a_p$.

12. **II 6 (USS 1)** In a certain language words are formed using an alphabet of three letters. Some words of two or more letters are not allowed, and any two such distinct words are of different lengths. Prove that one can form a word of arbitrary length that does not contain any nonallowed word.

3.17 The Seventeenth IMO Burgas–Sofia, Bulgaria, 1975

3.17.1 Contest Problems

First Day (July 7)

1. Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ be two n-tuples of numbers. Prove that
$$\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2$$
is true when $z_1, z_2, \ldots, z_n$ denote $y_1, y_2, \ldots, y_n$ taken in another order.

2. Let $a_1, a_2, a_3, \ldots$ be any infinite increasing sequence of positive integers. (For every integer $i > 0$, $a_{i+1} > a_i$.) Prove that there are infinitely many m for which positive integers x, y, h, k can be found such that $0 < h < k < m$ and $a_m = xa_h + ya_k$.

3. On the sides of an arbitrary triangle ABC, triangles BPC, CQA, and ARB are externally erected such that
$\measuredangle PBC = \measuredangle CAQ = 45^\circ$,
$\measuredangle BCP = \measuredangle QCA = 30^\circ$,
$\measuredangle ABR = \measuredangle BAR = 15^\circ$.
Prove that $\measuredangle QRP = 90^\circ$ and $QR = RP$.

Second Day (July 8)

4. Let A be the sum of the digits of the number 4444^{4444} and B the sum of the digits of the number A. Find the sum of the digits of the number B.

5. Is it possible to plot 1975 points on a circle with radius 1 so that the distance between any two of them is a rational number (distances have to be measured by chords)?

6. The function $f(x, y)$ is a homogeneous polynomial of the nth degree in x and y. If $f(1, 0) = 1$ and for all a, b, c,
$$f(a + b, c) + f(b + c, a) + f(c + a, b) = 0,$$
prove that $f(x, y) = (x - 2y)(x + y)^{n-1}$.

3.17.2 Shortlisted Problems

1. **(FRA)** There are six ports on a lake. Is it possible to organize a series of routes satisfying the following conditions:
(i) Every route includes exactly three ports;

(ii) No two routes contain the same three ports;
(iii) The series offers exactly two routes to each tourist who desires to visit two different arbitrary ports?

2. **(CZS)**^IMO1 Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ be two n-tuples of numbers. Prove that
$$\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2$$
is true when $z_1, z_2, \dots, z_n$ denote $y_1, y_2, \dots, y_n$ taken in another order.

3. **(USA)** Find the integer represented by $\left[\sum_{n=1}^{10^9} n^{-2/3}\right]$. Here $[x]$ denotes the greatest integer less than or equal to x (e.g. $[\sqrt{2}] = 1$).

4. **(SWE)** Let $a_1, a_2, \dots, a_n, \dots$ be a sequence of real numbers such that $0 \leq a_n \leq 1$ and $a_n - 2a_{n+1} + a_{n+2} \geq 0$ for $n = 1, 2, 3, \dots$. Prove that
$$0 \leq (n+1)(a_n - a_{n+1}) \leq 2 \qquad \text{for } n = 1, 2, 3, \dots.$$

5. **(SWE)** Let M be the set of all positive integers that do not contain the digit 9 (base 10). If $x_1, \dots, x_n$ are arbitrary but distinct elements in M, prove that
$$\sum_{j=1}^{n} \frac{1}{x_j} < 80.$$

6. **(USS)**^IMO4 Let A be the sum of the digits of the number 16^{16} and B the sum of the digits of the number A. Find the sum of the digits of the number B without calculating 16^{16}.

7. **(GDR)** Prove that from $x + y = 1$ $(x, y \in \mathbb{R})$ it follows that
$$x^{m+1} \sum_{j=0}^{n} \binom{m+j}{j} y^j + y^{n+1} \sum_{i=0}^{m} \binom{n+i}{i} x^i = 1 \qquad (m, n = 0, 1, 2, \dots).$$

8. **(NET)**^IMO3 On the sides of an arbitrary triangle ABC, triangles BPC, CQA, and ARB are externally erected such that
$\measuredangle PBC = \measuredangle CAQ = 45^\circ$,
$\measuredangle BCP = \measuredangle QCA = 30^\circ$,
$\measuredangle ABR = \measuredangle BAR = 15^\circ$.
Prove that $\measuredangle QRP = 90^\circ$ and $QR = RP$.

9. **(NET)** Let $f(x)$ be a continuous function defined on the closed interval $0 \leq x \leq 1$. Let $G(f)$ denote the graph of $f(x)$: $G(f) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ y = f(x)\}$. Let $G_a(f)$ denote the graph of the translated function $f(x - a)$ (translated over a distance a), defined by $G_a(f) = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq a + 1,\ y = f(x - a)\}$. Is it possible to find for every a, $0 < a < 1$, a continuous function $f(x)$, defined on $0 \leq x \leq 1$, such that $f(0) = f(1) = 0$ and $G(f)$ and $G_a(f)$ are disjoint point sets?

10. **(GBR)**$^{\text{IMO6}}$ The function $f(x, y)$ is a homogeneous polynomial of the nth degree in x and y. If $f(1, 0) = 1$ and for all a, b, c,

$$f(a + b, c) + f(b + c, a) + f(c + a, b) = 0,$$

prove that $f(x, y) = (x - 2y)(x + y)^{n-1}$.

11. **(GBR)**$^{\text{IMO2}}$ Let $a_1, a_2, a_3, \dots$ be any infinite increasing sequence of positive integers. (For every integer $i > 0$, $a_{i+1} > a_i$.) Prove that there are infinitely many m for which positive integers x, y, h, k can be found such that $0 < h < k < m$ and $a_m = xa_h + ya_k$.

12. **(GRE)** Consider on the first quadrant of the trigonometric circle the arcs $AM_1 = x_1, AM_2 = x_2, AM_3 = x_3, \dots, AM_\nu = x_\nu$, such that $x_1 < x_2 < x_3 < \dots < x_\nu$. Prove that

$$\sum_{i=0}^{\nu-1} \sin 2x_i - \sum_{i=0}^{\nu-1} \sin(x_i - x_{i+1}) < \frac{\pi}{2} + \sum_{i=0}^{\nu-1} \sin(x_i + x_{i+1}).$$

13. **(ROM)** Let $A_0, A_1, \dots, A_n$ be points in a plane such that
(i) $A_0A_1 \le \frac{1}{2}A_1A_2 \le \dots \le \frac{1}{2^{n-1}}A_{n-1}A_n$ and
(ii) $0 < \measuredangle A_0A_1A_2 < \measuredangle A_1A_2A_3 < \dots < \measuredangle A_{n-2}A_{n-1}A_n < 180^\circ$,
where all these angles have the same orientation. Prove that the segments A_kA_{k+1}, A_mA_{m+1} do not intersect for each k and n such that $0 \le k \le m - 2 < n - 2$.

14. **(YUG)** Let $x_0 = 5$ and $x_{n+1} = x_n + \frac{1}{x_n}$ $(n = 0, 1, 2, \dots)$. Prove that $45 < x_{1000} < 45,1$.

15. **(USS)**$^{\text{IMO5}}$ Is it possible to plot 1975 points on a circle with radius 1 so that the distance between any two of them is a rational number (distances have to be measured by chords)?

3.18 The Eighteenth IMO Wienna–Linz, Austria, 1976

3.18.1 Contest Problems

First Day (July 12)

1. In a convex quadrangle with area $32\,\mathrm{cm}^2$, the sum of the lengths of two nonadjacent edges and of the length of one diagonal is equal to $16\,\mathrm{cm}$. What is the length of the other diagonal?
2. Let $P_1(x) = x^2 - 2$, $P_j(x) = P_1(P_{j-1}(x))$, $j = 2, 3, \dots$. Show that for arbitrary n, the roots of the equation $P_n(x) = x$ are real and different from one another.
3. A rectangular box can be filled completely with unit cubes. If one places cubes with volume 2 in the box such that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine all possible (interior) sizes of the box.

Second Day (July 13)

4. Find the largest number obtainable as the product of positive integers whose sum is 1976.
5. Let a set of p equations be given,
$$\begin{aligned} a_{11}x_1 + \cdots + a_{1q}x_q &= 0,\\ a_{21}x_1 + \cdots + a_{2q}x_q &= 0,\\ &\vdots\\ a_{p1}x_1 + \cdots + a_{pq}x_q &= 0, \end{aligned}$$
with coefficients a_{ij} satisfying $a_{ij} = -1$, 0, or $+1$ for all $i = 1, \dots, p$ and $j = 1, \dots, q$. Prove that if $q = 2p$, there exists a solution $x_1, \dots, x_q$ of this system such that all x_j $(j = 1, \dots, q)$ are integers satisfying $|x_j| \le q$ and $x_j \ne 0$ for at least one value of j.
6. For all positive integral n, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$, $u_0 = 2$, and $u_1 = 2\frac{1}{2}$. Prove that
$$3 \log_2 [u_n] = 2^n - (-1)^n,$$
where $[x]$ is the integral part of x.

3.18.2 Longlisted Problems

1. **(BUL 1)** (SL76-1).
2. **(BUL 2)** Let P be a set of n points and S a set of l segments. It is known that:

(i) No four points of P are coplanar.
(ii) Any segment from S has its endpoints at P.
(iii) There is a point, say g, in P that is the endpoint of a maximal number of segments from S and that is not a vertex of a tetrahedron having all its edges in S.

Prove that $l \leq \frac{n^2}{3}$.

3. **(BUL 3)** (SL76-2).

4. **(BUL 4)** Find all pairs of natural numbers (m, n) for which $2^m \cdot 3^n + 1$ is the square of some integer.

5. **(BUL 5)** Let $ABCDS$ be a pyramid with four faces and with $ABCD$ as a base, and let a plane α through the vertex A meet its edges SB and SD at points M and N, respectively. Prove that if the intersection of the plane α with the pyramid $ABCDS$ is a parallelogram, then

$$SM \cdot SN > BM \cdot DN.$$

6. **(CZS 1)** For each point X of a given polytope, denote by $f(X)$ the sum of the distances of the point X from all the planes of the faces of the polytope.
Prove that if f attains its maximum at an interior point of the polytope, then f is constant.

7. **(CZS 2)** Let P be a fixed point and T a given triangle that contains the point P. Translate the triangle T by a given vector $\mathbf{v}$ and denote by T' this new triangle. Let r, R, respectively, be the radii of the smallest disks centered at P that contain the triangles T, T', respectively.
Prove that
$$r + |\mathbf{v}| \leq 3R$$
and find an example to show that equality can occur.

8. **(CZS 3)** (SL76-3).

9. **(CZS 4)** Find all (real) solutions of the system

$$\begin{aligned}
3x_1 - x_2 - x_3 \qquad\quad - x_5 \qquad\qquad\qquad\qquad &= 0,\\
-x_1 + 3x_2 \qquad - x_4 \qquad - x_6 \qquad\qquad &= 0,\\
-x_1 \qquad + 3x_3 - x_4 \qquad\qquad - x_7 \qquad &= 0,\\
-x_2 - x_3 + 3x_4 \qquad\qquad\qquad - x_8 &= 0,\\
-x_1 \qquad\qquad\qquad + 3x_5 - x_6 - x_7 \qquad &= 0,\\
-x_2 \qquad\qquad - x_5 + 3x_6 \qquad - x_8 &= 0,\\
-x_3 \qquad - x_5 \qquad + 3x_7 - x_8 &= 0,\\
-x_4 \qquad - x_6 - x_7 + 3x_8 &= 0.
\end{aligned}$$

10. **(FIN 1)** Show that the reciprocal of any number of the form $2(m^2 + m + 1)$, where m is a positive integer, can be represented as a sum of consecutive terms in the sequence $(a_j)_{j=1}^{\infty}$,

$$a_j = \frac{1}{j(j+1)(j+2)}.$$

11. **(FIN 2)** (SL76-9).

12. **(FIN 3)** Five points lie on the surface of a ball of unit radius. Find the maximum of the smallest distance between any two of them.

13. **(GBR 1a)** (SL76-4).

14. **(GBR 1b)** A sequence $\{u_n\}$ of integers is defined by

$$u_1 = 2, \quad u_2 = u_3 = 7,$$
$$u_{n+1} = u_n u_{n-1} - u_{n-2}, \quad \text{for } n \geq 3.$$

Prove that for each $n \geq 1$, u_n differs by 2 from an integral square.

15. **(GBR 2)** Let ABC and $A'B'C'$ be any two coplanar triangles. Let L be a point such that $AL \| BC$, $A'L \| B'C'$, and M, N similarly defined. The line BC meets $B'C'$ at P, and similarly defined are Q and R. Prove that PL, QM, RN are concurrent.

16. **(GBR 3)** Prove that there is a positive integer n such that the decimal representation of 7^n contains a block of at least m consecutive zeros, where m is any given positive integer.

17. **(GBR 4)** Show that there exists a convex polyhedron with all its vertices on the surface of a sphere and with all its faces congruent isosceles triangles whose ratio of sides are $\sqrt{3} : \sqrt{3} : 2$.

18. **(GDR 1)** Prove that the number $19^{1976} + 76^{1976}$:
    (a) is divisible by the (Fermat) prime number $F_4 = 2^{2^4} + 1$;
    (b) is divisible by at least four distinct primes other than F_4.

19. **(GDR 2)** For a positive integer n, let $6^{(n)}$ be the natural number whose decimal representation consists of n digits 6. Let us define, for all natural numbers m, k with $1 \leq k \leq m$,

$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{6^{(m)} \cdot 6^{(m-1)} \cdots 6^{(m-k+1)}}{6^{(1)} \cdot 6^{(2)} \cdots 6^{(k)}}.$$

Prove that for all m, k, $\begin{bmatrix} m \\ k \end{bmatrix}$ is a natural number whose decimal representation consists of exactly $k(m + k - 1) - 1$ digits.

20. **(GDR 3)** Let (a_n), $n = 0, 1, \ldots$, be a sequence of real numbers such that $a_0 = 0$ and

$$a_{n+1}^3 = \frac{1}{2}a_n^2 - 1, \qquad n = 0, 1, \ldots.$$

Prove that there exists a positive number q, $q < 1$, such that for all $n = 1, 2, \ldots,$

$$|a_{n+1} - a_n| \le q|a_n - a_{n-1}|,$$

and give one such q explicitly.

21. **(GDR 4)** Find the largest positive real number p (if it exists) such that the inequality

$$x_1^2 + x_2^2 + \cdots + x_n^2 \ge p(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n) \tag{1}$$

is satisfied for all real numbers x_i, and (a) $n = 2$; (b) $n = 5$.
Find the largest positive real number p (if it exists) such that the inequality (1) holds for all real numbers x_i and all natural numbers n, $n \ge 2$.

22. **(GDR 5)** A regular pentagon $A_1A_2A_3A_4A_5$ with side length s is given. At each point A_i a sphere K_i of radius $s/2$ is constructed. There are two spheres $K_1{}'$ and $K_2{}'$ eah of radius $s/2$ touching all the five spheres K_i. Decide whether $K_1{}'$ and $K_2{}'$ intersect each other, touch each other, or have no common points.

23. **(NET 1)** Prove that in a Euclidean plane there are infinitely many concentric circles C such that all triangles inscribed in C have at least one irrational side.

24. **(NET 2)** Let $0 \le x_1 \le x_2 \le \cdots \le x_n \le 1$. Prove that for all $A \ge 1$ there exists an interval I of length $2\sqrt[n]{A}$ such that for all $x \in I$,

$$|(x - x_1)(x - x_2)\cdots(x - x_n)| \le A.$$

25. **(NET 3)** (SL76-5).

26. **(NET 4)** (SL76-6).

27. **(NET 5)** In a plane three points P, Q, R, not on a line, are given. Let k, l, m be positive numbers. Construct a triangle ABC whose sides pass through P, Q, and R such that
    P divides the segment AB in the ratio $1 : k$,
    Q divides the segment BC in the ratio $1 : l$, and
    R divides the segment CA in the ratio $1 : m$.

28. **(POL 1a)** Let Q be a unit square in the plane: $Q = [0, 1] \times [0, 1]$. Let $T : Q \to Q$ be defined as follows:

$$T(x, y) = \begin{cases} (2x, y/2) & \text{if } 0 \le x \le 1/2; \\ (2x - 1, y/2 + 1/2) & \text{if } 1/2 < x \le 1. \end{cases}$$

Show that for every disk $D \subset Q$ there exists an integer $n > 0$ such that $T^n(D) \cap D \ne \emptyset$.

29. **(POL 1b)** (SL76-7).

30. **(POL 2)** Prove that if $P(x) = (x-a)^k Q(x)$, where k is a positive integer, a is a nonzero real number, $Q(x)$ is a nonzero polynomial, then $P(x)$ has at least $k+1$ nonzero coefficients.

31. **(POL 3)** Into every lateral face of a quadrangular pyramid a circle is inscribed. The circles inscribed into adjacent faces are tangent (have one point in common). Prove that the points of contact of the circles with the base of the pyramid lie on a circle.

32. **(POL 4)** We consider the infinite chessboard covering the whole plane. In every field of the chessboard there is a nonnegative real number. Every number is the arithmetic mean of the numbers in the four adjacent fields of the chessboard. Prove that the numbers occurring in the fields of the chessboard are all equal.

33. **(SWE 1)** A finite set of points P in the plane has the following property: Every line through two points in P contains at least one more point belonging to P. Prove that all points in P lie on a straight line.

34. **(SWE 2)** Let $\{a_n\}_0^\infty$ and $\{b_n\}_0^\infty$ be two sequences determined by the recursion formulas
$$\begin{aligned} a_{n+1} &= a_n + b_n, \\ b_{n+1} &= 3a_n + b_n, \qquad n = 0, 1, 2, \dots, \end{aligned}$$
and the initial values $a_0 = b_0 = 1$. Prove that there exists a uniquely determined constant c such that $n|ca_n - b_n| < 2$ for all nonnegative integers n.

35. **(SWE 3)** (SL76-8).

36. **(USA 1)** Three concentric circles with common center O are cut by a common chord in successive points A, B, C. Tangents drawn to the circles at the points A, B, C enclose a triangular region. If the distance from point O to the common chord is equal to p, prove that the area of the region enclosed by the tangents is equal to
$$\frac{AB \cdot BC \cdot CA}{2p}.$$

37. **(USA 2)** From a square board 11 squares long and 11 squares wide, the central square is removed. Prove that the remaining 120 squares cannot be covered by 15 strips each 8 units long and one unit wide.

38. **(USA 3)** Let $x = \sqrt{a} + \sqrt{b}$, where a and b are natural numbers, x is not an integer, and $x < 1976$. Prove that the fractional part of x exceeds $10^{-19.76}$.

39. **(USA 4)** In $\triangle ABC$, the inscribed circle is tangent to side BC at X. Segment AX is drawn. Prove that the line joining the midpoint of segment

AX to the midpoint of side BC passes through the center I of the inscribed circle.

40. **(USA 5)** Let $g(x)$ be a fixed polynomial and define $f(x)$ by $f(x) = x^2 + xg(x^3)$. Show that $f(x)$ is not divisible by $x^2 - x + 1$.

41. **(USA 6)** (SL76-10).

42. **(USS 1)** For a point O inside a triangle ABC, denote by A_1, B_1, C_1 the respective intersection points of AO, BO, CO with the corresponding sides. Let $n_1 = \frac{AO}{A_1O}$, $n_2 = \frac{BO}{B_1O}$, $n_3 = \frac{CO}{C_1O}$. What possible values of n_1, n_2, n_3 can all be positive integers?

43. **(USS 2)** Prove that if for a polynomial $P(x, y)$ we have
$$P(x-1, y-2x+1) = P(x, y),$$
then there exists a polynomial $\Phi(x)$ with $P(x, y) = \Phi(y - x^2)$.

44. **(USS 3)** A circle of radius 1 rolls around a circle of radius $\sqrt{2}$. Initially, the tangent point is colored red. Afterwards, the red points map from one circle to another by contact. How many red points will be on the bigger circle when the center of the smaller one has made n circuits around the bigger one?

45. **(USS 4)** We are given n ($n \geq 5$) circles in a plane. Suppose that every three of them have a common point. Prove that all n circles have a common point.

46. **(USS 5)** For $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, prove the inequality
$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2.$$

47. **(VIE 1)** (SL76-11).

48. **(VIE 2)** (SL76-12).

49. **(VIE 3)** Determine whether there exist 1976 nonsimilar triangles with angles α, β, γ, each of them satisfying the relations
$$\frac{\sin\alpha + \sin\beta + \sin\gamma}{\cos\alpha + \cos\beta + \cos\gamma} = \frac{12}{7} \quad \text{and} \quad \sin\alpha\sin\beta\sin\gamma = \frac{12}{25}.$$

50. **(VIE 4)** Find a function $f(x)$ defined for all real values of x such that for all x,
$$f(x+2) - f(x) = x^2 + 2x + 4,$$
and if $x \in [0, 2)$, then $f(x) = x^2$.

51. **(YUG 1)** Four swallows are catching a fly. At first, the swallows are at the four vertices of a tetrahedron, and the fly is in its interior. Their maximal speeds are equal. Prove that the swallows can catch the fly.

3.18.3 Shortlisted Problems

1. **(BUL 1)** Let ABC be a triangle with bisectors AA_1, BB_1, CC_1 ($A_1 \in BC$, etc.) and M their common point. Consider the triangles MB_1A, $MC_1A, MC_1B, MA_1B, MA_1C, MB_1C$, and their inscribed circles. Prove that if four of these six inscribed circles have equal radii, then $AB = BC = CA$.

2. **(BUL 3)** Let $a_0, a_1, \dots, a_n, a_{n+1}$ be a sequence of real numbers satisfying the following conditions:

$$a_0 = a_{n+1} = 0,$$
$$|a_{k-1} - 2a_k + a_{k+1}| \le 1 \qquad (k = 1, 2, \dots, n).$$

Prove that $|a_k| \le \frac{k(n+1-k)}{2}$ $(k = 0, 1, \dots, n+1)$.

3. **(CZS 3)**$^{\text{IMO1}}$ In a convex quadrangle with area $32\,\text{cm}^2$, the sum of the lengths of two nonadjacent edges and of the length of one diagonal is equal to $16\,\text{cm}$.
   (a) What is the length of the other diagonal?
   (b) What are the lengths of the edges of the quadrangle if the perimeter is a minimum?
   (c) Is it possible to choose the edges in such a way that the perimeter is a maximum?

4. **(GBR 1a)**$^{\text{IMO6}}$ For all positive integral n, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$, $u_0 = 2$, and $u_1 = 5/2$. Prove that

$$3 \log_2 [u_n] = 2^n - (-1)^n,$$

where $[x]$ is the integral part of x.

5. **(NET 3)**$^{\text{IMO5}}$ Let a set of p equations be given,

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1q}x_q &= 0, \\ a_{21}x_1 + \cdots + a_{2q}x_q &= 0, \\ &\vdots \\ a_{p1}x_1 + \cdots + a_{pq}x_q &= 0, \end{aligned}$$

with coefficients a_{ij} satisfying $a_{ij} = -1$, 0, or $+1$ for all $i = 1, \dots, p$ and $j = 1, \dots, q$. Prove that if $q = 2p$, there exists a solution $x_1, \dots, x_q$ of this system such that all x_j $(j = 1, \dots, q)$ are integers satisfying $|x_j| \le q$ and $x_j \ne 0$ for at least one value of j.

6. **(NET 4)**$^{\text{IMO3}}$ A rectangular box can be filled completely with unit cubes. If one places cubes with volume 2 in the box such that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine all possible (interior) sizes of the box.

7. **(POL 1b)** Let $I = (0, 1]$ be the unit interval of the real line. For a given number $a \in (0, 1)$ we define a map $T : I \to I$ by the formula

$$T(x, y) = \begin{cases} x + (1 - a) & \text{if } 0 < x \le a, \\ x - a & \text{if } a < x \le 1. \end{cases}$$

Show that for every interval $J \subset I$ there exists an integer $n > 0$ such that $T^n(J) \cap J \neq \emptyset$.

8. **(SWE 3)** Let P be a polynomial with real coefficients such that $P(x) > 0$ if $x > 0$. Prove that there exist polynomials Q and R with nonnegative coefficients such that $P(x) = \frac{Q(x)}{R(x)}$ if $x > 0$.

9. **(FIN 2)**$^{\text{IMO2}}$ Let $P_1(x) = x^2 - 2$, $P_j(x) = P_1(P_{j-1}(x))$, $j = 2, 3, \dots$. Show that for arbitrary n the roots of the equation $P_n(x) = x$ are real and different from one another.

10. **(USA 6)**$^{\text{IMO4}}$ Find the largest number obtainable as the product of positive integers whose sum is 1976.

11. **(VIE 1)** Prove that there exist infinitely many positive integers n such that the decimal representation of 5^n contains a block of 1976 consecutive zeros.

12. **(VIE 2)** The polynomial $1976(x + x^2 + \cdots + x^n)$ is decomposed into a sum of polynomials of the form $a_1x + a_2x^2 + \ldots + a_nx^n$, where $a_1, a_2, \cdots, a_n$ are distinct positive integers not greater than n. Find all values of n for which such a decomposition is possible.

3.19 The Nineteenth IMO
Belgrade–Arandjelovac, Yugoslavia, July 1–13, 1977

3.19.1 Contest Problems

First Day (July 6)

1. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments AK, BK, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon.
2. In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.
3. Let n be a given integer greater than 2, and let V_n be the set of integers $1 + kn$, where $k = 1, 2, \dots$. A number $m \in V_n$ is called indecomposable in V_n if there do not exist numbers $p, q \in V_n$ such that $pq = m$. Prove that there exists a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Expressions that differ only in order of the elements of V_n will be considered the same.)

Second Day (July 7)

4. Let a, b, A, B be given constant real numbers and
$$f(x) = 1 - a\cos x - b\sin x - A\cos 2x - B\sin 2x.$$
Prove that if $f(x) \ge 0$ for all real x, then
$$a^2 + b^2 \le 2 \quad \text{and} \quad A^2 + B^2 \le 1.$$
5. Let a and b be natural numbers and let q and r be the quotient and remainder respectively when $a^2 + b^2$ is divided by $a + b$. Determine the numbers a and b if $q^2 + r = 1977$.
6. Let $f : \mathbb{N} \to \mathbb{N}$ be a function that satisfies the inequality $f(n+1) > f(f(n))$ for all $n \in \mathbb{N}$. Prove that $f(n) = n$ for all natural numbers n.

3.19.2 Longlisted Problems

1. **(BUL 1)** A pentagon $ABCDE$ inscribed in a circle for which $BC < CD$ and $AB < DE$ is the base of a pyramid with vertex S. If AS is the longest edge starting from S, prove that $BS > CS$.
2. **(BUL 2)** (SL77-1).

3. **(BUL 3)** In a company of n persons, each person has no more than d acquaintances, and in that company there exists a group of k persons, $k \geq d$, who are not acquainted with each other. Prove that the number of acquainted pairs is not greater than $[n^2/4]$.

4. **(BUL 4)** We are given n points in space. Some pairs of these points are connected by line segments so that the number of segments equals $[n^2/4]$, and a connected triangle exists. Prove that any point from which the maximal number of segments starts is a vertex of a connected triangle.

5. **(CZS 1)** (SL77-2).

6. **(CZS 2)** Let $x_1, x_2, \dots, x_n$ $(n \geq 1)$ be real numbers such that $0 \leq x_j \leq \pi$, $j = 1, 2, \dots, n$. Prove that if $\sum_{j=1}^{n}(\cos x_j + 1)$ is an odd integer, then $\sum_{j=1}^{n} \sin x_j \geq 1$.

7. **(CZS 3)** Prove the following assertion: If $c_1, c_2, \dots, c_n$ $(n \geq 2)$ are real numbers such that
$$(n-1)(c_1^2 + c_2^2 + \cdots + c_n^2) = (c_1 + c_2 + \cdots + c_n)^2,$$
then either all these numbers are nonnegative or all these numbers are nonpositive.

8. **(CZS 4)** A hexahedron $ABCDE$ is made of two regular congruent tetrahedra $ABCD$ and $ABCE$. Prove that there exists only one isometry $\mathcal{Z}$ that maps points A, B, C, D, E onto B, C, A, E, D, respectively. Find all points X on the surface of hexahedron whose distance from $\mathcal{Z}(X)$ is minimal.

9. **(CZS 5)** Let $ABCD$ be a regular tetrahedron and $\mathcal{Z}$ an isometry mapping A, B, C, D into B, C, D, A, respectively. Find the set $\mathcal{M}$ of all points X of the face ABC whose distance from $\mathcal{Z}(X)$ is equal to a given number t. Find necessary and sufficient conditions for the set $\mathcal{M}$ to be nonempty.

10. **(FRG 1)** (SL77-3).

11. **(FRG 2)** Let n and z be integers greater than 1 and $(n, z) = 1$. Prove:
    (a) At least one of the numbers $z_i = 1 + z + z^2 + \cdots + z^i$, $i = 0, 1, \dots, n-1$, is divisible by n.
    (b) If $(z-1, n) = 1$, then at least one of the numbers z_i, $i = 0, 1, \dots, n-2$, is divisible by n.

12. **(FRG 3)** Let z be an integer > 1 and let M be the set of all numbers of the form $z_k = 1 + z + \cdots + z^k$, $k = 0, 1, \dots$. Determine the set T of divisors of at least one of the numbers z_k from M.

13. **(FRG 4)** (SL77-4).

14. **(FRG 5)** (SL77-5).

15. **(GDR 1)** Let n be an integer greater than 1. In the Cartesian coordinate system we consider all squares with integer vertices (x, y) such that $1 \le x, y \le n$. Denote by p_k $(k = 0, 1, 2, \dots)$ the number of pairs of points that are vertices of exactly k such squares. Prove that $\sum_k (k-1)p_k = 0$.

16. **(GDR 2)** (SL77-6).

17. **(GDR 3)** A ball K of radius r is touched from the outside by mutually equal balls of radius R. Two of these balls are tangent to each other. Moreover, for two balls K_1 and K_2 tangent to K and tangent to each other there exist two other balls tangent to K_1, K_2 and also to K. How many balls are tangent to K? For a given r determine R.

18. **(GDR 4)** Given an isosceles triangle ABC with a right angle at C, construct the center M and radius r of a circle cutting on segments AB, BC, CA the segments DE, FG, and HK, respectively, such that $\angle DME + \angle FMG + \angle HMK = 180^\circ$ and $DE : FG : HK = AB : BC : CA$.

19. **(GBR 1)** Given any integer $m > 1$ prove that there exist infinitely many positive integers n such that the last m digits of 5^n are a sequence $a_m, a_{m-1}, \dots, a_1 = 5$ $(0 \le a_j < 10)$ in which each digit except the last is of opposite parity to its successor (i.e., if a_i is even, then a_{i-1} is odd, and if a_i is odd, then a_{i-1} is even).

20. **(GBR 2)** (SL77-7).

21. **(GBR 3)** Given that $x_1+x_2+x_3 = y_1+y_2+y_3 = x_1y_1+x_2y_2+x_3y_3 = 0$, prove that
$$\frac{x_1^2}{x_1^2+x_2^2+x_3^2} + \frac{y_1^2}{y_1^2+y_2^2+y_3^2} = \frac{2}{3}.$$

22. **(GBR 4)** (SL77-8).

23. **(HUN 1)** (SL77-9).

24. **(HUN 2)** Determine all real functions $f(x)$ that are defined and continuous on the interval $(-1, 1)$ and that satisfy the functional equation
$$f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)} \qquad (x, y, x+y \in (-1, 1)).$$

25. **(HUN 3)** Prove the identity
$$(z+a)^n = z^n + a\sum_{k=1}^{n} \binom{n}{k} (a-kb)^{k-1}(z+kb)^{n-k}.$$

26. **(NET 1)** Let p be a prime number greater than 5. Let V be the collection of all positive integers n that can be written in the form $n = kp+1$ or $n = kp-1$ $(k = 1, 2, \dots)$. A number $n \in V$ is called *indecomposable* in V if it is impossible to find $k, l \in V$ such that $n = kl$. Prove that there exists

a number $N \in V$ that can be factorized into indecomposable factors in V in more than one way.

27. **(NET 2)** (SL77-10).

28. **(NET 3)** (SL77-11).

29. **(NET 4)** (SL77-12).

30. **(NET 5)** A triangle ABC with $\angle A = 30^\circ$ and $\angle C = 54^\circ$ is given. On BC a point D is chosen such that $\angle CAD = 12^\circ$. On AB a point E is chosen such that $\angle ACE = 6^\circ$. Let S be the point of intersection of AD and CE. Prove that $BS = BC$.

31. **(POL 1)** Let f be a function defined on the set of pairs of nonzero rational numbers whose values are positive real numbers. Suppose that f satisfies the following conditions:
(1) $f(ab, c) = f(a, c)f(b, c)$, $f(c, ab) = f(c, a)f(c, b)$;
(2) $f(a, 1 - a) = 1$.
Prove that $f(a, a) = f(a, -a) = 1$, $f(a, b)f(b, a) = 1$.

32. **(POL 2)** In a room there are nine men. Among every three of them there are two mutually acquainted. Prove that some four of them are mutually acquainted.

33. **(POL 3)** A circle K centered at $(0, 0)$ is given. Prove that for every vector (a_1, a_2) there is a positive integer n such that the circle K translated by the vector $n(a_1, a_2)$ contains a lattice point (i.e., a point both of whose coordinates are integers).

34. **(POL 4)** (SL77-13).

35. **(ROM 1)** Find all numbers $N = \overline{a_1 a_2 \dots a_n}$ for which $9 \times \overline{a_1 a_2 \dots a_n} = \overline{a_n \dots a_2 a_1}$ such that at most one of the digits $a_1, a_2, \dots, a_n$ is zero.

36. **(ROM 2)** Consider a sequence of numbers $(a_1, a_2, \dots, a_{2^n})$. Define the operation

$$S((a_1, a_2, \dots, a_{2^n})) = (a_1 a_2, a_2 a_3, \dots, a_{2^n - 1} a_{2^n}, a_{2^n} a_1).$$

Prove that whatever the sequence $(a_1, a_2, \dots, a_{2^n})$ is, with $a_i \in \{-1, 1\}$ for $i = 1, 2, \dots, 2^n$, after finitely many applications of the operation we get the sequence $(1, 1, \dots, 1)$.

37. **(ROM 3)** Let $A_1, A_2, \dots, A_{n+1}$ be positive integers such that $(A_i, A_{n+1}) = 1$ for every $i = 1, 2, \dots, n$. Show that the equation

$$x_1^{A_1} + x_2^{A_2} + \dots + x_n^{A_n} = x_{n+1}^{A_{n+1}}$$

has an infinite set of solutions $(x_1, x_2, \dots, x_{n+1})$ in positive integers.

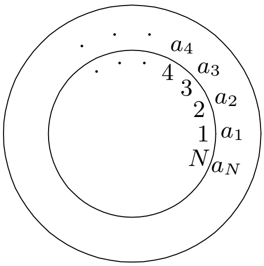
38. **(ROM 4)** Let $m_j > 0$ for $j = 1, 2, \dots, n$ and $a_1 \le \dots \le a_n < b_1 \le \dots \le b_n < c_1 \le \dots \le c_n$ be real numbers. Prove:

$$\left[\sum_{j=1}^{n} m_j(a_j + b_j + c_j)\right]^2 > 3\left(\sum_{j=1}^{n} m_j\right)\left[\sum_{j=1}^{n} m_j(a_jb_j + b_jc_j + c_ja_j)\right].$$

39. **(ROM 5)** Consider 37 distinct points in space, all with integer coordinates. Prove that we may find among them three distinct points such that their barycenter has integers coordinates.

40. **(SWE 1)** The numbers $1, 2, 3, \dots, 64$ are placed on a chessboard, one number in each square. Consider all squares on the chessboard of size 2×2. Prove that there are at least three such squares for which the sum of the 4 numbers contained exceeds 100.

41. **(SWE 2)** A wheel consists of a fixed circular disk and a mobile circular ring. On the disk the numbers $1, 2, 3, \dots, N$ are marked, and on the ring N integers $a_1, a_2, \dots, a_N$ of sum 1 are marked (see the figure). The ring can be turned into N different positions in which the numbers on the disk and on the ring match each other. Multiply every number on the ring with the corresponding number on the disk and form the sum of N products. In this way a sum is obtained for every position of the ring. Prove that the N sums are different.



42. **(SWE 3)** The sequence $a_{n,k}$, $k = 1, 2, 3, \dots, 2^n$, $n = 0, 1, 2, \dots$, is defined by the following recurrence formula:

$$a_1 = 2, \qquad a_{n,k} = 2a_{n-1,k}^3, \qquad a_{n,k+2^{n-1}} = \frac{1}{2}a_{n-1,k}^3$$
$$\text{for } \ k = 1, 2, 3, \dots, 2^{n-1}, \ n = 0, 1, 2, \dots .$$

Prove that the numbers $a_{n,k}$ are all different.

43. **(FIN 1)** Evaluate

$$S = \sum_{k=1}^{n} k(k+1)\cdots(k+p),$$

where n and p are positive integers.

44. **(FIN 2)** Let E be a finite set of points in space such that E is not contained in a plane and no three points of E are collinear. Show that E contains the vertices of a tetrahedron $T = ABCD$ such that $T \cap E = \{A, B, C, D\}$ (including interior points of T) and such that the projection of A onto the plane BCD is inside a triangle that is similar to the triangle BCD and whose sides have midpoints B, C, D.

45. **(FIN 2′)** (SL77-14).

46. **(FIN 3)** Let f be a strictly increasing function defined on the set of real numbers. For x real and t positive, set

$$g(x,t)=\frac{f(x+t)-f(x)}{f(x)-f(x-t)}.$$

Assume that the inequalities

$$2^{-1}<g(x,t)<2$$

hold for all positive t if $x=0$, and for all $t\le|x|$ otherwise.
Show that

$$14^{-1}<g(x,t)<14$$

for all real x and positive t.

47. **(USS 1)** A square $ABCD$ is given. A line passing through A intersects CD at Q. Draw a line parallel to AQ that intersects the boundary of the square at points M and N such that the area of the quadrilateral $AMNQ$ is maximal.

48. **(USS 2)** The intersection of a plane with a regular tetrahedron with edge a is a quadrilateral with perimeter P. Prove that $2a\le P\le 3a$.

49. **(USS 3)** Find all pairs of integers (p,q) for which all roots of the trinomials x^2+px+q and x^2+qx+p are integers.

50. **(USS 4)** Determine all positive integers n for which there exists a polynomial $P_n(x)$ of degree n with integer coefficients that is equal to n at n different integer points and that equals zero at zero.

51. **(USS 5)** Several segments, which we shall call white, are given, and the sum of their lengths is 1. Several other segments, which we shall call black, are given, and the sum of their lengths is 1. Prove that every such system of segments can be distributed on the segment that is 1.51 long in the following way: Segments of the same color are disjoint, and segments of different colors are either disjoint or one is inside the other. Prove that there exists a system that cannot be distributed in that way on the segment that is 1.49 long.

52. **(USA 1)** Two perpendicular chords are drawn through a given interior point P of a circle with radius R. Determine, with proof, the maximum and the minimum of the sum of the lengths of these two chords if the distance from P to the center of the circle is kR.

53. **(USA 2)** Find all pairs of integers a and b for which

$$7a+14b=5a^2+5ab+5b^2.$$

54. **(USA 3)** If $0 \le a \le b \le c \le d$, prove that
$$a^b b^c c^d d^a \ge b^a c^b d^c a^d.$$

55. **(USA 4)** Through a point O on the diagonal BD of a parallelogram $ABCD$, segments MN parallel to AB, and PQ parallel to AD, are drawn, with M on AD, and Q on AB. Prove that diagonals AO, BP, DN (extended if necessary) will be concurrent.

56. **(USA 5)** The four circumcircles of the four faces of a tetrahedron have equal radii. Prove that the four faces of the tetrahedron are congruent triangles.

57. **(VIE 1)** (SL77-15).

58. **(VIE 2)** Prove that for every triangle the following inequality holds:
$$\frac{ab+bc+ca}{4S} \ge \cot\frac{\pi}{6},$$
where a, b, c are lengths of the sides and S is the area of the triangle.

59. **(VIE 3)** (SL77-16).

60. **(VIE 4)** Suppose $x_0, x_1, \dots, x_n$ are integers and $x_0 > x_1 > \cdots > x_n$. Prove that at least one of the numbers $|F(x_0)|, |F(x_1)|, |F(x_2)|, \dots, |F(x_n)|$, where
$$F(x) = x^n + a_1 x^{n-1} + \cdots + a_n, \qquad a_i \in \mathbb{R}, \;\; i = 1, \dots, n,$$
is greater than $\frac{n!}{2^n}$.

3.19.3 Shortlisted Problems

1. **(BUL 2)**$^{\text{IMO6}}$ Let $f : \mathbb{N} \to \mathbb{N}$ be a function that satisfies the inequality $f(n+1) > f(f(n))$ for all $n \in \mathbb{N}$. Prove that $f(n) = n$ for all natural numbers n.

2. **(CZS 1)** A lattice point in the plane is a point both of whose coordinates are integers. Each lattice point has four neighboring points: upper, lower, left, and right. Let k be a circle with radius $r \ge 2$, that does not pass through any lattice point. An interior boundary point is a lattice point lying inside the circle k that has a neighboring point lying outside k. Similarly, an exterior boundary point is a lattice point lying outside the circle k that has a neighboring point lying inside k. Prove that there are four more exterior boundary points than interior boundary points.

3. **(FRG 1)**$^{\text{IMO5}}$ Let a and b be natural numbers and let q and r be the quotient and remainder respectively when $a^2 + b^2$ is divided by $a + b$. Determine the numbers a and b if $q^2 + r = 1977$.

4. **(FRG 4)** Describe all closed bounded figures Φ in the plane any two points of which are connectable by a semicircle lying in Φ.

5. **(FRG 5)** There are 2^n words of length n over the alphabet $\{0, 1\}$. Prove that the following algorithm generates the sequence $w_0, w_1, \dots, w_{2^n-1}$ of all these words such that any two consecutive words differ in exactly one digit.
   (1) $w_0 = 00\dots0$ (n zeros).
   (2) Suppose $w_{m-1} = a_1a_2\dots a_n$, $a_i \in \{0, 1\}$. Let $e(m)$ be the exponent of 2 in the representation of n as a product of primes, and let $j = 1 + e(m)$. Replace the digit a_j in the word w_{m-1} by $1 - a_j$. The obtained word is w_m.

6. **(GDR 2)** Let n be a positive integer. How many integer solutions (i, j, k, l), $1 \le i, j, k, l \le n$, does the following system of inequalities have:

$$\begin{aligned} 1 \le & -j+k+l \le n \\ 1 \le & \quad i-k+l \le n \\ 1 \le & \quad i-j+l \le n \\ 1 \le & \quad i+j-k \le n\ ? \end{aligned}$$

7. **(GBR 2)**$^{\text{IMO4}}$ Let a, b, A, B be given constant real numbers and

$$f(x) = 1 - a\cos x - b\sin x - A\cos 2x - B\sin 2x.$$

Prove that if $f(x) \ge 0$ for all real x, then

$$a^2 + b^2 \le 2 \qquad \text{and} \qquad A^2 + B^2 \le 1.$$

8. **(GBR 4)** Let S be a convex quadrilateral $ABCD$ and O a point inside it. The feet of the perpendiculars from O to AB, BC, CD, DA are A_1, B_1, C_1, D_1 respectively. The feet of the perpendiculars from O to the sides of S_i, the quadrilateral $A_iB_iC_iD_i$, are $A_{i+1}B_{i+1}C_{i+1}D_{i+1}$, where $i = 1, 2, 3$. Prove that S_4 is similar to S.

9. **(HUN 1)** For which positive integers n do there exist two polynomials f and g with integer coefficients of n variables $x_1, x_2, \dots, x_n$ such that the following equality is satisfied:

$$\left(\sum_{i=1}^{n} x_i\right) f(x_1, x_2, \dots, x_n) = g(x_1^2, x_2^2, \dots, x_n^2)?$$

10. **(NET 2)**$^{\text{IMO3}}$ Let n be an integer greater than 2. Define $V = \{1 + kn \mid k = 1, 2, \dots\}$. A number $p \in V$ is called *indecomposable* in V if it is not possible to find numbers $q_1, q_2 \in V$ such that $q_1q_2 = p$. Prove that there exists a number $N \in V$ that can be factorized into indecomposable factors in V in more than one way.

11. **(NET 3)** Let n be an integer greater than 1. Define

$$x_1 = n,\ \ y_1 = 1, \qquad x_{i+1} = \left[\frac{x_i + y_i}{2}\right],\ \ y_{i+1} = \left[\frac{n}{x_{i+1}}\right] \qquad \text{for } i = 1, 2, \dots,$$

where $[z]$ denotes the largest integer less than or equal to z. Prove that

$$\min\{x_1, x_2, \dots x_n\} = [\sqrt{n}].$$

12. **(NET 4)**IMO1 On the sides of a square $ABCD$ one constructs inwardly equilateral triangles ABK, BCL, CDM, DAN. Prove that the midpoints of the four segments KL, LM, MN, NK, together with the midpoints of the eight segments AK, BK, BL, CL, CM, DM, DN, AN, are the 12 vertices of a regular dodecagon.

13. **(POL 4)** Let B be a set of k sequences each having n terms equal to 1 or -1. The product of two such sequences $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ is defined as $(a_1b_1, a_2b_2, \dots, a_nb_n)$. Prove that there exists a sequence $(c_1, c_2, \dots, c_n)$ such that the intersection of B and the set containing all sequences from B multiplied by $(c_1, c_2, \dots, c_n)$ contains at most $k^2/2^n$ sequences.

14. **(FIN 2‘)** Let E be a finite set of points such that E is not contained in a plane and no three points of E are collinear. Show that at least one of the following alternatives holds:
    (i) E contains five points that are vertices of a convex pyramid having no other points in common with E;
    (ii) some plane contains exactly three points from E.

15. **(VIE 1)**IMO2 The length of a finite sequence is defined as the number of terms of this sequence. Determine the maximal possible length of a finite sequence that satisfies the following condition: The sum of each seven successive terms is negative, and the sum of each eleven successive terms is positive.

16. **(VIE 3)** Let E be a set of n points in the plane $(n \geq 3)$ whose coordinates are integers such that any three points from E are vertices of a nondegenerate triangle whose centroid doesn't have both coordinates integers. Determine the maximal n.

3.20 The Twentieth IMO Bucharest, Romania, 1978

3.20.1 Contest Problems

First Day (July 6)

1. Let $n > m \geq 1$ be natural numbers such that the groups of the last three digits in the decimal representation of $1978^m, 1978^n$ coincide. Find the ordered pair (m, n) of such m, n for which $m + n$ is minimal.
2. Given any point P in the interior of a sphere with radius R, three mutually perpendicular segments PA, PB, PC are drawn terminating on the sphere and having one common vertex in P. Consider the rectangular parallelepiped of which PA, PB, PC are coterminal edges. Find the locus of the point Q that is diagonally opposite P in the parallelepiped when P and the sphere are fixed.
3. Let $\{f(n)\}$ be a strictly increasing sequence of positive integers: $0 < f(1) < f(2) < f(3) < \dots$. Of the positive integers not belonging to the sequence, the nth in order of magnitude is $f(f(n)) + 1$. Determine $f(240)$.

Second day (July 7)

4. In a triangle ABC we have $AB = AC$. A circle is tangent internally to the circumcircle of ABC and also to the sides AB, AC, at P, Q respectively. Prove that the midpoint of PQ is the center of the incircle of ABC.
5. Let $\varphi : \{1, 2, 3, \dots\} \to \{1, 2, 3, \dots\}$ be injective. Prove that for all n,
$$\sum_{k=1}^{n} \frac{\varphi(k)}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$
6. An international society has its members in 6 different countries. The list of members contains 1978 names, numbered $1, 2, \dots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two, not necessarily distinct, of his compatriots.

3.20.2 Longlisted Problems

1. **(BUL 1)** (SL78-1).
2. **(BUL 2)** If
$$f(x) = (x + 2x^2 + \cdots + nx^n)^2 = a_2x^2 + a_3x^3 + \cdots + a_{2n}x^{2n},$$
prove that

$$a_{n+1} + a_{n+2} + \cdots + a_{2n} = \binom{n+1}{2} \frac{5n^2 + 5n + 2}{12}.$$

3. **(BUL 3)** Find all numbers α for which the equation

$$x^2 - 2x[x] + x - \alpha = 0$$

has two nonnegative roots. ($[x]$ denotes the largest integer less than or equal to x.)

4. **(BUL 4)** (SL78-2).

5. **(CUB 1)** Prove that for any triangle ABC there exists a point P in the plane of the triangle and three points A', B', and C' on the lines BC, AC, and AB respectively such that

$$AB \cdot PC' = AC \cdot PB' = BC \cdot PA' = 0.3M^2,$$

where $M = \max\{AB, AC, BC\}$.

6. **(CUB 2)** Prove that for all $X > 1$ there exists a triangle whose sides have lengths $P_1(X) = X^4 + X^3 + 2X^2 + X + 1$, $P_2(X) = 2X^3 + X^2 + 2X + 1$, and $P_3(X) = X^4 - 1$. Prove that all these triangles have the same greatest angle and calculate it.

7. **(CUB 3)** (SL78-3).

8. **(CZS 1)** For two given triangles $A_1A_2A_3$ and $B_1B_2B_3$ with areas Δ_A and Δ_B, respectively, $A_iA_k \geq B_iB_k$, $i, k = 1, 2, 3$. Prove that $\Delta_A \geq \Delta_B$ if the triangle $A_1A_2A_3$ is not obtuse-angled.

9. **(CZS 2)** (SL78-4).

10. **(CZS 3)** Show that for any natural number n there exist two prime numbers p and q, $p \neq q$, such that n divides their difference.

11. **(CZS 4)** Find all natural numbers $n < 1978$ with the following property: If m is a natural number, $1 < m < n$, and $(m, n) = 1$ (i.e., m and n are relatively prime), then m is a prime number.

12. **(FIN 1)** The equation $x^3 + ax^2 + bx + c = 0$ has three (not necessarily distinct) real roots t, u, v. For which a, b, c do the numbers t^3, u^3, v^3 satisfy the equation $x^3 + a^3x^2 + b^3x + c^3 = 0$?

13. **(FIN 2)** The satellites A and B circle the Earth in the equatorial plane at altitude h. They are separated by distance $2r$, where r is the radius of the Earth. For which h can they be seen in mutually perpendicular directions from some point on the equator?

14. **(FIN 3)** Let $p(x, y)$ and $q(x, y)$ be polynomials in two variables such that for $x \geq 0$, $y \geq 0$ the following conditions hold:
    (i) $p(x, y)$ and $q(x, y)$ are increasing functions of x for every fixed y.

(ii) $p(x, y)$ is an increasing and $q(x)$ is a decreasing function of y for every fixed x.

(iii) $p(x, 0) = q(x, 0)$ for every x and $p(0, 0) = 0$.

Show that the simultaneous equations $p(x, y) = a$, $q(x, y) = b$ have a unique solution in the set $x \geq 0$, $y \geq 0$ for all a, b satisfying $0 \leq b \leq a$ but lack a solution in the same set if $a < b$.

15. **(FRA 1)** Prove that for every positive integer n coprime to 10 there exists a multiple of n that does not contain the digit 1 in its decimal representation.

16. **(FRA 2)** (SL78-6).

17. **(FRA 3)** (SL78-17).

18. **(FRA 4)** Given a natural number n, prove that the number $M(n)$ of points with integer coordinates inside the circle $(O(0, 0), \sqrt{n})$ satisfies
$$\pi n - 5\sqrt{n} + 1 < M(n) < \pi n + 4\sqrt{n} + 1.$$

19. **(FRA 5)** (SL78-7).

20. **(GBR 1)** Let O be the center of a circle. Let OU, OV be perpendicular radii of the circle. The chord PQ passes through the midpoint M of UV. Let W be a point such that $PM = PW$, where U, V, M, W are collinear. Let R be a point such that $PR = MQ$, where R lies on the line PW. Prove that $MR = UV$.

    Alternative version: A circle S is given with center O and radius r. Let M be a point whose distance from O is $\frac{r}{\sqrt{2}}$. Let PMQ be a chord of S. The point N is defined by $\overrightarrow{PN} = \overrightarrow{MQ}$. Let R be the reflection of N by the line through P that is parallel to OM. Prove that $MR = \sqrt{2}r$.

21. **(GBR 2)** A circle touches the sides AB, BC, CD, DA of a square at points K, L, M, N respectively, and BU, KV are parallel lines such that U is on DM and V on DN. Prove that UV touches the circle.

22. **(GBR 3)** Two nonzero integers x, y (not necessarily positive) are such that $x + y$ is a divisor of $x^2 + y^2$, and the quotient $\frac{x^2+y^2}{x+y}$ is a divisor of 1978. Prove that $x = y$.

23. **(GBR 4)** (SL78-8).

24. **(GBR 5)** (SL78-9).

25. **(GDR 1)** Consider a polynomial $P(x) = ax^2 + bx + c$ with $a > 0$ that has two real roots x_1, x_2. Prove that the absolute values of both roots are less than or equal to 1 if and only if $a + b + c \geq 0$, $a - b + c \geq 0$, and $a - c \geq 0$.

26. **(GDR 2)** (SL78-5).

27. **(GDR 3)** Determine the sixth number after the decimal point in the number $\left(\sqrt{1978}+\left[\sqrt{1978}\right]\right)^{20}$.

28. **(GDR 4)** Let c, s be real functions defined on $\mathbb{R}\backslash\{0\}$ that are nonconstant on any interval and satisfy

$$c\left(\frac{x}{y}\right)=c(x)c(y)-s(x)s(y) \qquad \text{for any } x\neq 0,\ y\neq 0.$$

Prove that:
(a) $c(1/x)=c(x)$, $s(1/x)=-s(x)$ for any $x\neq 0$, and also $c(1)=1$, $s(1)=s(-1)=0$;
(b) c and s are either both even or both odd functions (a function f is even if $f(x)=f(-x)$ for all x, and odd if $f(x)=-f(-x)$ for all x).
Find functions c, s that also satisfy $c(x)+s(x)=x^n$ for all x, where n is a given positive integer.

29. **(GDR 5)** (*Variant of GDR 4*) Given a nonconstant function $f:\mathbb{R}^+\to\mathbb{R}$ such that $f(xy)=f(x)f(y)$ for any $x,y>0$, find functions $c,s:\mathbb{R}^+\to\mathbb{R}$ that satisfy $c(x/y)=c(x)c(y)-s(x)s(y)$ for all $x,y>0$ and $c(x)+s(x)=f(x)$ for all $x>0$.

30. **(NET 1)** (SL78-10).

31. **(NET 2)** Let the polynomials

$$\begin{aligned}P(x)&=x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0,\\Q(x)&=x^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0,\end{aligned}$$

be given satisfying the identity $P(x)^2=(x^2-1)Q(x)^2+1$. Prove the identity

$$P'(x)=nQ(x).$$

32. **(NET 3)** Let $\mathcal{C}$ be the circumcircle of the square with vertices $(0,0)$, $(0,1978)$, $(1978,0)$, $(1978,1978)$ in the Cartesian plane. Prove that $\mathcal{C}$ contains no other point for which both coordinates are integers.

33. **(SWE 1)** A sequence $(a_n)_0^\infty$ of real numbers is called *convex* if $2a_n\leq a_{n-1}+a_{n+1}$ for all positive integers n. Let $(b_n)_0^\infty$ be a sequence of positive numbers and assume that the sequence $(\alpha^n b_n)_0^\infty$ is convex for any choice of $\alpha>0$. Prove that the sequence $(\log b_n)_0^\infty$ is convex.

34. **(SWE 2)** (SL78-11).

35. **(SWE 3)** A sequence $(a_n)_0^N$ of real numbers is called *concave* if $2a_n\geq a_{n-1}+a_{n+1}$ for all integers n, $1\leq n\leq N-1$.
(a) Prove that there exists a constant $C>0$ such that

$$\left(\sum_{n=0}^{N}a_n\right)^2\geq C(N-1)\sum_{n=0}^{N}a_n^2 \qquad (1)$$

for all concave positive sequences $(a_n)_0^N$.

(b) Prove that (1) holds with $C = 3/4$ and that this constant is best possible.

36. **(TUR 1)** The integers 1 through 1000 are located on the circumference of a circle in natural order. Starting with 1, every fifteenth number (i.e., $1, 16, 31, \dots$) is marked. The marking is continued until an already marked number is reached. How many of the numbers will be left unmarked?

37. **(TUR 2)** Simplify
$$\frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)},$$
where a, b, c are positive real numbers.

38. **(TUR 3)** Given a circle, construct a chord that is trisected by two given noncollinear radii.

39. **(TUR 4)** A is a $2m$-digit positive integer each of whose digits is 1. B is an m-digit positive integer each of whose digits is 4. Prove that $A+B+1$ is a perfect square.

40. **(TUR 5)** If $C_n^p = \frac{n!}{p!(n-p)!}$ $(p \geq 1)$, prove the identity
$$C_n^p = C_{n-1}^{p-1} + C_{n-2}^{p-1} + \cdots + C_p^{p-1} + C_{p-1}^{p-1}$$
and then evaluate the sum
$$S = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + 97 \cdot 98 \cdot 99.$$

41. **(USA 1)** (SL78-12).

42. **(USA 2)** A, B, C, D, E are points on a circle O with radius equal to r. Chords AB and DE are parallel to each other and have length equal to x. Diagonals AC, AD, BE, CE are drawn. If segment XY on O meets AC at X and EC at Y, prove that lines BX and DY meet at Z on the circle.

43. **(USA 3)** If p is a prime greater than 3, show that at least one of the numbers $\frac{3}{p^2}, \frac{4}{p^2}, \dots, \frac{p-2}{p^2}$ is expressible in the form $\frac{1}{x} + \frac{1}{y}$, where x and y are positive integers.

44. **(USA 4)** In $\triangle ABC$ with $\angle C = 60^o$, prove that $\frac{c}{a} + \frac{c}{b} \geq 2$.

45. **(USA 5)** If $r > s > 0$ and $a > b > c$, prove that
$$a^r b^s + b^r c^s + c^r a^s \geq a^s b^r + b^s c^r + c^s a^r.$$

46. **(USA 6)** (SL78-13).

47. **(VIE 1)** Given the expression
$$P_n(x) = \frac{1}{2^n}\left[\left(x + \sqrt{x^2-1}\right)^n + \left(x - \sqrt{x^2-1}\right)^n\right],$$
prove:

(a) $P_n(x)$ satisfies the identity $P_n(x) - xP_{n-1}(x) + \frac{1}{4}P_{n-2}(x) \equiv 0$.
(b) $P_n(x)$ is a polynomial in x of degree n.

48. **(VIE 2)** (SL78-14).

49. **(VIE 3)** Let A, B, C, D be four arbitrary distinct points in space.
   (a) Prove that using the segments $AB + CD$, $AC + BD$ and $AD + BC$ it is always possible to construct a triangle T that is nondegenerate and has no obtuse angle.
   (b) What should these four points satisfy in order for the triangle T to be right-angled?

50. **(VIE 4)** A variable tetrahedron $ABCD$ has the following properties: Its edge lengths can change as well as its vertices, but the opposite edges remain equal ($BC = DA$, $CA = DB$, $AB = DC$); and the vertices A, B, C lie respectively on three fixed spheres with the same center P and radii $3, 4, 12$. What is the maximal length of PD?

51. **(VIE 5)** Find the relations among the angles of the triangle ABC whose altitude AH and median AM satisfy $\angle BAH = \angle CAM$.

52. **(YUG 1)** (SL78-15).

53. **(YUG 2)** (SL78-16).

54. **(YUG 3)** Let p, q and r be three lines in space such that there is no plane that is parallel to all three of them. Prove that there exist three planes α, β, and γ, containing p, q, and r respectively, that are perpendicular to each other ($\alpha \perp \beta$, $\beta \perp \gamma$, $\gamma \perp \alpha$).

3.20.3 Shortlisted Problems

1. **(BUL 1)** The set $M = \{1, 2, \dots, 2n\}$ is partitioned into k nonintersecting subsets $M_1, M_2, \dots, M_k$, where $n \geq k^3 + k$. Prove that there exist even numbers $2j_1, 2j_2, \dots, 2j_{k+1}$ in M that are in one and the same subset M_i ($1 \leq i \leq k$) such that the numbers $2j_1 - 1, 2j_2 - 1, \dots, 2j_{k+1} - 1$ are also in one and the same subset M_j ($1 \leq j \leq k$).

2. **(BUL 4)** Two identically oriented equilateral triangles, ABC with center S and $A'B'C$, are given in the plane. We also have $A' \neq S$ and $B' \neq S$. If M is the midpoint of $A'B$ and N the midpoint of AB', prove that the triangles $SB'M$ and $SA'N$ are similar.

3. **(CUB 3)**$^{\mathrm{IMO1}}$ Let $n > m \geq 1$ be natural numbers such that the groups of the last three digits in the decimal representation of $1978^m, 1978^n$ coincide. Find the ordered pair (m, n) of such m, n for which $m + n$ is minimal.

4. **(CZS 2)** Let T_1 be a triangle having a, b, c as lengths of its sides and let T_2 be another triangle having u, v, w as lengths of its sides. If P, Q are the areas of the two triangles, prove that

$$16PQ \le a^2(-u^2+v^2+w^2)+b^2(u^2-v^2+w^2)+c^2(u^2+v^2-w^2).$$

When does equality hold?

5. **(GDR 2)** For every integer $d \ge 1$, let M_d be the set of all positive integers that cannot be written as a sum of an arithmetic progression with difference d, having at least two terms and consisting of positive integers. Let $A = M_1$, $B = M_2 \setminus \{2\}$, $C = M_3$. Prove that every $c \in C$ may be written in a unique way as $c = ab$ with $a \in A, b \in B$.

6. **(FRA 2)**$^{\text{IMO5}}$ Let $\varphi : \{1,2,3,\dots\} \to \{1,2,3,\dots\}$ be injective. Prove that for all n,
$$\sum_{k=1}^{n} \frac{\varphi(k)}{k^2} \ge \sum_{k=1}^{n} \frac{1}{k}.$$

7. **(FRA 5)** We consider three distinct half-lines Ox, Oy, Oz in a plane. Prove the existence and uniqueness of three points $A \in Ox$, $B \in Oy$, $C \in Oz$ such that the perimeters of the triangles OAB, OBC, OCA are all equal to a given number $2p > 0$.

8. **(GBR 4)** Let S be the set of all the odd positive integers that are not multiples of 5 and that are less than $30m$, m being an arbitrary positive integer. What is the smallest integer k such that in any subset of k integers from S there must be two different integers, one of which divides the other?

9. **(GBR 5)**$^{\text{IMO3}}$ Let $\{f(n)\}$ be a strictly increasing sequence of positive integers: $0 < f(1) < f(2) < f(3) < \cdots$. Of the positive integers not belonging to the sequence, the nth in order of magnitude is $f(f(n))+1$. Determine $f(240)$.

10. **(NET 1)**$^{\text{IMO6}}$ An international society has its members in 6 different countries. The list of members contains 1978 names, numbered $1,2,\dots,$ 1978. Prove that there is at least one member whose number is the sum of the numbers of two, not necessarily distinct, of his compatriots.

11. **(SWE 2)** A function $f : I \to \mathbb{R}$, defined on an interval I, is called *concave* if $f(\theta x + (1-\theta)y) \ge \theta f(x) + (1-\theta) f(y)$ for all $x, y \in I$ and $0 \le \theta \le 1$. Assume that the functions $f_1, \dots, f_n$, having all nonnegative values, are concave. Prove that the function $(f_1 f_2 \dots f_n)^{1/n}$ is concave.

12. **(USA 1)**$^{\text{IMO4}}$ In a triangle ABC we have $AB = AC$. A circle is tangent internally to the circumcircle of ABC and also to the sides AB, AC, at P, Q respectively. Prove that the midpoint of PQ is the center of the incircle of ABC.

13. **(USA 6)**$^{\text{IMO2}}$ Given any point P in the interior of a sphere with radius R, three mutually perpendicular segments PA, PB, PC are drawn terminating on the sphere and having one common vertex in P. Consider the rectangular parallelepiped of which PA, PB, PC are coterminal

edges. Find the locus of the point Q that is diagonally opposite P in the parallelepiped when P and the sphere are fixed.

14. **(VIE 2)** Prove that it is possible to place $2n(2n+1)$ parallelepipedic (rectangular) pieces of soap of dimensions $1 \times 2 \times (n+1)$ in a cubic box with edge $2n+1$ if and only if n is even or $n = 1$.

    Remark. It is assumed that the edges of the pieces of soap are parallel to the edges of the box.

15. **(YUG 1)** Let p be a prime and $A = \{a_1, \dots, a_{p-1}\}$ an arbitrary subset of the set of natural numbers such that none of its elements is divisible by p. Let us define a mapping f from $\mathcal{P}(A)$ (the set of all subsets of A) to the set $P = \{0, 1, \dots, p-1\}$ in the following way:
    (i) if $B = \{a_{i_1}, \dots, a_{i_k}\} \subset A$ and $\sum_{j=1}^{k} a_{i_j} \equiv n \pmod{p}$, then $f(B) = n$,
    (ii) $f(\emptyset) = 0$, $\emptyset$ being the empty set.
    Prove that for each $n \in P$ there exists $B \subset A$ such that $f(B) = n$.

16. **(YUG 2)** Determine all the triples (a, b, c) of positive real numbers such that the system

$$\begin{aligned} ax + by - cz &= 0, \\ a\sqrt{1-x^2} + b\sqrt{1-y^2} - c\sqrt{1-z^2} &= 0, \end{aligned}$$

    is compatible in the set of real numbers, and then find all its real solutions.

17. **(FRA 3)** Prove that for any positive integers x, y, z with $xy - z^2 = 1$ one can find nonnegative integers a, b, c, d such that $x = a^2 + b^2$, $y = c^2 + d^2$, $z = ac + bd$.
    Set $z = (2q)!$ to deduce that for any prime number $p = 4q + 1$, p can be represented as the sum of squares of two integers.

3.21 The Twenty-First IMO London, United Kingdom, 1979

3.21.1 Contest Problems

First Day (July 2)

1. Given that
$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{1318}+\frac{1}{1319}=\frac{p}{q},$$
where p and q are natural numbers having no common factor, prove that p is divisible by 1979.

2. A pentagonal prism $A_1A_2\ldots A_5B_1B_2\ldots B_5$ is given. The edges, the diagonals of the lateral walls, and the internal diagonals of the prism are each colored either red or green in such a way that no triangle whose vertices are vertices of the prism has its three edges of the same color. Prove that all edges of the bases are of the same color.

3. There are two circles in the plane. Let a point A be one of the points of intersection of these circles. Two points begin moving simultaneously with constant speeds from the point A, each point along its own circle. The two points return to the point A at the same time.
Prove that there is a point P in the plane such that at every moment of time the distances from the point P to the moving points are equal.

Second Day (July 3)

4. Given a point P in a given plane π and also a given point Q not in π, determine all points R in π such that $\frac{QP+PR}{QR}$ is a maximum.

5. The nonnegative real numbers $x_1, x_2, x_3, x_4, x_5, a$ satisfy the following relations:
$$\sum_{i=1}^{5} ix_i = a, \qquad \sum_{i=1}^{5} i^3x_i = a^2, \qquad \sum_{i=1}^{5} i^5x_i = a^3.$$
What are the possible values of a?

6. Let S and F be two opposite vertices of a regular octagon. A counter starts at S and each second is moved to one of the two neighboring vertices of the octagon. The direction is determined by the toss of a coin. The process ends when the counter reaches F. We define a_n to be the number of distinct paths of duration n seconds that the counter may take to reach F from S. Prove that for $n = 1, 2, 3, \ldots,$
$$a_{2n-1} = 0, \qquad a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1}-y^{n-1}), \qquad \text{where } x = 2+\sqrt{2},\ y = 2-\sqrt{2}.$$

3.21.2 Longlisted Problems

1. **(BEL 1)** (SL79-1).
2. **(BEL 2)** For a finite set E of cardinality $n \geq 3$, let $f(n)$ denote the maximum number of 3-element subsets of E, any two of them having exactly one common element. Calculate $f(n)$.
3. **(BEL 3)** Is it possible to partition 3-dimensional Euclidean space into 1979 mutually isometric subsets?
4. **(BEL 4)** (SL79-2).
5. **(BEL 5)** Describe which natural numbers do not belong to the set
$$E = \{[n + \sqrt{n} + 1/2] \mid n \in \mathbb{N}\}.$$
6. **(BEL 6)** Prove that $\frac{1}{2}\sqrt{4\sin^2 36^\circ - 1} = \cos 72^\circ$.
7. **(BRA 1)** $M = (a_{i,j})$, $i,j = 1,2,3,4$, is a square matrix of order four. Given that:
   (i) for each $i = 1,2,3,4$ and for each $k = 5,6,7$,
$$\begin{aligned} a_{i,k} &= a_{i,k-4};\\ P_i &= a_{1,i} + a_{2,i+1} + a_{3,i+2} + a_{4,i+3};\\ S_i &= a_{4,i} + a_{3,i+1} + a_{2,i+2} + a_{1,i+3};\\ L_i &= a_{i,1} + a_{i,2} + a_{i,3} + a_{i,4};\\ C_i &= a_{1,i} + a_{2,i} + a_{3,i} + a_{4,i}, \end{aligned}$$
   (ii) for each $i,j = 1,2,3,4$, $P_i = P_j$, $S_i = S_j$, $L_i = L_j$, $C_i = C_j$, and
   (iii) $a_{1,1} = 0$, $a_{1,2} = 7$, $a_{2,1} = 11$, $a_{2,3} = 2$, and $a_{3,3} = 15$;
   find the matrix M.
8. **(BRA 2)** The sequence (a_n) of real numbers is defined as follows:
$$a_1 = 1, \qquad a_2 = 2 \qquad \text{and} \qquad a_n = 3a_{n-1} - a_{n-2}, \quad n \geq 3.$$
   Prove that for $n \geq 3$, $a_n = \left[\frac{a_{n-1}^2}{a_{n-2}}\right] + 1$, where $[x]$ denotes the integer p such that $p \leq x < p + 1$.
9. **(BRA 3)** The real numbers $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are positive. Let us denote by $h = \frac{n}{1/\alpha_1 + 1/\alpha_2 + \cdots + 1/\alpha_n}$ the harmonic mean, $g = \sqrt[n]{\alpha_1\alpha_2\cdots\alpha_n}$ the geometric mean, $a = \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n}$ the arithmetic mean. Prove that $h \leq g \leq a$, and that each of the equalities implies the other one.
10. **(BUL 1)** (SL79-3).
11. **(BUL 2)** Prove that a pyramid $A_1A_2\dots A_{2k+1}S$ with equal lateral edges and equal space angles between adjacent lateral walls is regular.

Variant. Prove that a pyramid $A_1 \dots A_{2k+1}S$ with equal space angles between adjacent lateral walls is regular if there exists a sphere tangent to all its edges.

12. **(BUL 3)** (SL79-4).

13. **(BUL 4)** The plane is divided into equal squares by parallel lines; i.e., a square net is given. Let M be an arbitrary set of n squares of this net. Prove that it is possible to choose no fewer than $n/4$ squares of M in such a way that no two of them have a common point.

14. **(CZS 1)** Let S be a set of $n^2 + 1$ closed intervals (n a positive integer). Prove that at least one of the following assertions holds:
    (i) There exists a subset S' of $n+1$ intervals from S such that the intersection of the intervals in S' is nonempty.
    (ii) There exists a subset S'' of $n+1$ intervals from S such that any two of the intervals in S'' are disjoint.

15. **(CZS 2)** (SL79-5).

16. **(CZS 3)** Let Q be a square with side length 6. Find the smallest integer n such that in Q there exists a set S of n points with the property that any square with side 1 completely contained in Q contains in its interior at least one point from S.

17. **(CZS 4)** (SL79-6).

18. **(FIN 1)** Show that for no integers $a \ge 1$, $n \ge 1$ is the sum
$$1 + \frac{1}{1+a} + \frac{1}{1+2a} + \cdots + \frac{1}{1+na}$$
an integer.

19. **(FIN 2)** For $k = 1, 2, \dots$ consider the k-tuples $(a_1, a_2, \dots, a_k)$ of positive integers such that
$$a_1 + 2a_2 + \cdots + ka_k = 1979.$$
Show that there are as many such k-tuples with odd k as there are with even k.

20. **(FIN 3)** (SL79-10).

21. **(FRA 1)** Let E be the set of all bijective mappings from $\mathbb{R}$ to $\mathbb{R}$ satisfying
$$(\forall t \in \mathbb{R}) \qquad f(t) + f^{-1}(t) = 2t,$$
where f^{-1} is the mapping inverse to f. Find all elements of E that are monotonic mappings.

22. **(FRA 2)** Consider two quadrilaterals $ABCD$ and $A'B'C'D'$ in an affine Euclidian plane such that $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, and $DA = D'A'$. Prove that the following two statements are true:

(a) If the diagonals BD and AC are mutually perpendicular, then the diagonals $B'D'$ and $A'C'$ are also mutually perpendicular.
(b) If the perpendicular bisector of BD intersects AC at M, and that of $B'D'$ intersects $A'C'$ at M', then $\frac{\overline{MA}}{\overline{MC}} = \frac{\overline{M'A'}}{\overline{M'C'}}$ (if $\overline{MC} = 0$ then $\overline{M'C'} = 0$).

23. **(FRA 3)** Consider the set E consisting of pairs of integers (a, b), with $a \geq 1$ and $b \geq 1$, that satisfy in the decimal system the following properties:
(i) b is written with three digits, as $\overline{\alpha_2\alpha_1\alpha_0}$, $\alpha_2 \neq 0$;
(ii) a is written as $\overline{\beta_p \dots \beta_1\beta_0}$ for some p;
(iii) $(a + b)^2$ is written as $\overline{\beta_p \dots \beta_1\beta_0\alpha_2\alpha_1\alpha_0}$.
Find the elements of E.

24. **(FRA 4)** Let a and b be coprime integers, greater than or equal to 1. Prove that all integers n greater than or equal to $(a - 1)(b - 1)$ can be written in the form:

$$n = ua + vb, \qquad \text{with } (u, v) \in \mathbb{N} \times \mathbb{N}.$$

25. **(FRG 1)** (SL79-7).

26. **(FRG 2)** Let n be a natural number. If $4^n + 2^n + 1$ is a prime, prove that n is a power of three.

27. **(FRG 3)** (SL79-8).

28. **(FRG 4)** (SL79-9).

29. **(GDR 1)** (SL79-11).

30. **(GDR 2)** Let M be a set of points in a plane with at least two elements. Prove that if M has two axes of symmetry g_1 and g_2 intersecting at an angle $\alpha = q\pi$, where q is irrational, then M must be infinite.

31. **(GDR 3)** (SL79-12).

32. **(GDR 4)** Let $n, k \geq 1$ be natural numbers. Find the number $A(n, k)$ of solutions in integers of the equation

$$|x_1| + |x_2| + \cdots + |x_k| = n.$$

33. **(GRE 1)** (SL79-13).

34. **(GRE 2)** Notice that in the fraction $\frac{16}{64}$ we can perform a simplification as $\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$ obtaining a correct equality. Find all fractions whose numerators and denominators are two-digit positive integers for which such a simplification is correct.

35. **(GRE 3)** Given a sequence (a_n), with $a_1 = 4$ and $a_{n+1} = a_n^2 - 2$ $(\forall n \in \mathbb{N})$, prove that there is a triangle with side lengths $a_n - 1, a_n, a_n + 1$, and that its area is equal to an integer.

36. **(GRE 4)** A regular tetrahedron $A_1B_1C_1D_1$ is inscribed in a regular tetrahedron $ABCD$, where A_1 lies in the plane BCD, B_1 in the plane ACD, etc. Prove that $A_1B_1 \geq AB/3$.

37. **(GRE 5)** (SL79-14).

38. **(HUN 1)** Prove the following statement: If a polynomial $f(x)$ with real coefficients takes only nonnegative values, then there exists a positive integer n and polynomials $g_1(x), g_2(x), \dots, g_n(x)$ such that
$$f(x) = g_1(x)^2 + g_2(x)^2 + \cdots + g_n(x)^2.$$

39. **(HUN 2)** A desert expedition camps at the border of the desert, and has to provide one liter of drinking water for another member of the expedition, residing on the distance of n days of walking from the camp, under the following conditions:
    (i) Each member of the expedition can pick up at most 3 liters of water.
    (ii) Each member must drink one liter of water every day spent in the desert.
    (iii) All the members must return to the camp.
    How much water do they need (at least) in order to do that?

40. **(HUN 3)** A polynomial $P(x)$ has degree at most $2k$, where $k = 0, 1, 2, \dots$. Given that for an integer i, the inequality $-k \leq i \leq k$ implies $|P(i)| \leq 1$, prove that for all real numbers x, with $-k \leq x \leq k$, the following inequality holds:
$$|P(x)| < (2k+1)\binom{2k}{k}.$$

41. **(HUN 4)** Prove the following statement: There does not exist a pyramid with square base and congruent lateral faces for which the measures of all edges, total area, and volume are integers.

42. **(HUN 5)** Let a quadratic polynomial $g(x) = ax^2 + bx + c$ be given and an integer $n \geq 1$. Prove that there exists at most one polynomial $f(x)$ of nth degree such that $f(g(x)) = g(f(x))$.

43. **(ISR 1)** Let a, b, c denote the lengths of the sides BC, CA, AB, respectively, of a triangle ABC. If P is any point on the circumference of the circle inscribed in the triangle, show that $aPA^2 + bPB^2 + cPC^2$ is constant.

44. **(ISR 2)** (SL79-15).

45. **(ISR 3)** For any positive integer n we denote by $F(n)$ the number of ways in which n can be expressed as the sum of three different positive integers, without regard to order. Thus, since $10 = 7+2+1 = 6+3+1 = 5+4+1 = 5+3+2$, we have $F(10) = 4$. Show that $F(n)$ is even if $n \equiv 2$ or 4 (mod 6), but odd if n is divisible by 6.

46. **(ISR 4)** (SL79-16).

47. **(NET 1)** (SL79-17).

48. **(NET 2)** In the plane a circle C of unit radius is given. For any line l a number $s(l)$ is defined in the following way: If l and C intersect in two points, $s(l)$ is their distance; otherwise, $s(l) = 0$.
Let P be a point at distance r from the center of C. One defines $M(r)$ to be the maximum value of the sum $s(m) + s(n)$, where m and n are variable mutually orthogonal lines through P. Determine the values of r for which $M(r) > 2$.

49. **(NET 3)** Let there be given two sequences of integers $f_i(1), f_i(2), \dots$ $(i = 1, 2)$ satisfying:
(i) $f_i(nm) = f_i(n)f_i(m)$ if $\gcd(n, m) = 1$;
(ii) for every prime P and all $k = 2, 3, 4, \dots,$ $f_i(P^k) = f_i(P)f_i(P^{k-1}) - P^2 f(P^{k-2})$.
Moreover, for every prime P:
(iii) $f_1(P) = 2P$,
(iv) $f_2(P) < 2P$.
Prove that $|f_2(n)| < f_1(n)$ for all n.

50. **(POL 1)** (SL79-18).

51. **(POL 2)** Let ABC be an arbitrary triangle and let $S_1, S_2, \dots, S_7$ be circles satisfying the following conditions:
S_1 is tangent to CA and AB,
S_2 is tangent to S_1, AB, and BC,
S_3 is tangent to S_2, BC, and CA,
....
S_7 is tangent to S_6, CA and AB.
Prove that the circles S_1 and S_7 coincide.

52. **(POL 3)** Let a real number $\lambda > 1$ be given and a sequence (n_k) of positive integers such that $\frac{n_{k+1}}{n_k} > \lambda$ for $k = 1, 2, \dots$. Prove that there exists a positive integer c such that no positive integer n can be represented in more than c ways in the form $n = n_k + n_j$ or $n = n_r - n_s$.

53. **(POL 4)** An infinite increasing sequence of positive integers n_j $(j = 1, 2, \dots)$ has the property that for a certain c, $\frac{1}{N}\sum_{n_j \le N} n_j \le c$, for every $N > 0$
Prove that there exist finitely many sequences $m_j^{(i)}$ $(i = 1, 2, \dots, k)$ such that
$$\{n_1, n_2, \dots\} = \textstyle\bigcup_{i=1}^{k}\{m_1^{(i)}, m_2^{(i)}, \dots\} \qquad \text{and}$$
$$m_{j+1}^{(i)} > 2m_j^{(i)} \qquad (1 \le i \le k,\ j = 1, 2, \dots).$$

54. **(ROM 1)** (SL79-19).

55. **(ROM 2)** Let a, b be coprime integers. Show that the equation $ax^2 + by^2 = z^3$ has an infinite set of solutions (x, y, z) with $x, y, z \in \mathbb{Z}$ and x, y mutually coprime (in each solution).

56. **(ROM 3)** Show that for every natural number n, $n\sqrt{2} - [n\sqrt{2}] > \frac{1}{2n\sqrt{2}}$ and that for every $\varepsilon > 0$ there exists a natural number n with $n\sqrt{2} - [n\sqrt{2}] < \frac{1}{2n\sqrt{2}} + \varepsilon$.

57. **(ROM 4)** Let M be a set, and A, B, C given subsets of M. Find a necessary and sufficient condition for the existence of a set $X \subset M$ for which $(X \cup A) \setminus (X \cap B) = C$. Describe all such sets X.

58. **(ROM 5)** Prove that there exists a natural number k_0 such that for every natural number $k > k_0$ we may find a finite number of lines in the plane, not all parallel to one of them, that divide the plane exactly in k regions. Find k_0.

59. **(SWE 1)** Determine the maximum value of $x^2y^2z^2w$ when $x, y, z, w \geq 0$ and
$$2x + xy + z + yzw = 1.$$

60. **(SWE 2)** (SL79-20).

61. **(SWE 3)** Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two sequences such that $\sum_{k=1}^{m} a_k \geq \sum_{k=1}^{m} b_k$ for all $m \leq n$ with equality for $m = n$. Let f be a convex function defined on the real numbers. Prove that
$$\sum_{k=1}^{n} f(a_k) \leq \sum_{k=1}^{n} f(b_k).$$

62. **(SWE 4)** T is a given triangle with vertices P_1, P_2, P_3. Consider an arbitrary subdivision of T into finitely many subtriangles such that no vertex of a subtriangle lies strictly between two vertices of another subtriangle. To each vertex V of the subtriangles there is assigned a number $n(V)$ according to the following rules:
(i) If $V = P_i$, then $n(V) = i$.
(ii) If V lies on the side P_iP_j of T, then $n(V) = i$ or j.
(iii) If V lies inside the triangle T, then $n(V)$ is any of the numbers 1,2,3.
Prove that there exists at least one subtriangle whose vertices are numbered 1, 2, and 3.

63. **(USA 1)** If $a_1, a_2, \ldots, a_n$ denote the lengths of the sides of an arbitrary n-gon, prove that
$$2 \geq \frac{a_1}{s - a_1} + \frac{a_2}{s - a_2} + \cdots + \frac{a_n}{s - a_n} \geq \frac{n}{n-1},$$
where $s = a_1 + a_2 + \cdots + a_n$.

64. **(USA 2)** From point P on arc BC of the circumcircle about triangle ABC, PX is constructed perpendicular to BC, PY is perpendicular to AC, and PZ perpendicular to AB (all extended if necessary). Prove that
$$\frac{BC}{PX} = \frac{AC}{PY} + \frac{AB}{PZ}.$$

65. **(USA 3)** Given $f(x) \le x$ for all real x and

$$f(x+y) \le f(x) + f(y) \quad \text{for all real } x, y,$$

prove that $f(x) = x$ for all x.

66. **(USA 4)** (SL79-23).

67. **(USA 5)** (SL79-24).

68. **(USA 6)** (SL79-25).

69. **(USS 1)** (SL79-21).

70. **(USS 2)** There are 1979 equilateral triangles: $T_1, T_2, \dots, T_{1979}$. A side of triangle T_k is equal to $1/k$, $k = 1, 2, \dots, 1979$. At what values of a number a can one place all these triangles into the equilateral triangle with side length a so that they don't intersect (points of contact are allowed)?

71. **(USS 3)** (SL79-22).

72. **(VIE 1)** Let $f(x)$ be a polynomial with integer coefficients. Prove that if $f(x)$ equals 1979 for four different integer values of x, then $f(x)$ cannot be equal to 2×1979 for any integral value of x.

73. **(VIE 2)** In a plane a finite number of equal circles are given. These circles are mutually nonintersecting (they may be externally tangent). Prove that one can use at most four colors for coloring these circles so that two circles tangent to each other are of different colors. What is the smallest number of circles that requires four colors?

74. **(VIE 3)** Given an equilateral triangle ABC of side a in a plane, let M be a point on the circumcircle of the triangle. Prove that the sum $s = MA^4 + MB^4 + MC^4$ is independent of the position of the point M on the circle, and determine that constant value as a function of a.

75. **(VIE 4)** Given an equilateral triangle ABC, let M be an arbitrary point in space.
    (a) Prove that one can construct a triangle from the segments MA, MB, MC.
    (b) Suppose that P and Q are two points symmetric with respect to the center O of ABC. Prove that the two triangles constructed from the segments PA, PB, PC and QA, QB, QC are of equal area.

76. **(VIE 5)** Suppose that a triangle whose sides are of integer lengths is inscribed in a circle of diameter 6.25. Find the sides of the triangle.

77. **(YUG 1)** By $h(n)$, where n is an integer greater than 1, let us denote the greatest prime divisor of the number n. Are there infinitely many numbers n for which $h(n) < h(n+1) < h(n+2)$ holds?

78. **(YUG 2)** By $\omega(n)$, where n is an integer greater than 1, let us denote the number of different prime divisors of the number n. Prove that there

exist infinitely many numbers n for which $\omega(n) < \omega(n+1) < \omega(n+2)$ holds.

79. **(YUG 3)** Let S be a unit circle and K a subset of S consisting of several closed arcs. Let K satisfy the following properties:
   (i) K contains three points A, B, C, that are the vertices of an acute-angled triangle;
   (ii) for every point A that belongs to K its diametrically opposite point A' and all points B on an arc of length $1/9$ with center A' do not belong to K.

   Prove that there are three points E, F, G on S that are vertices of an equilateral triangle and that do not belong to K.

80. **(YUG 4)** (SL79-26).

81. **(YUG 5)** Let $\mathcal{P}$ be the set of rectangular parallelepipeds that have at least one edge of integer length. If a rectangular parallelepiped P_0 can be decomposed into parallelepipeds $P_1, P_2, \dots, P_n \in \mathcal{P}$, prove that $P_0 \in \mathcal{P}$.

3.21.3 Shortlisted Problems

1. **(BEL 1)** Prove that in the Euclidean plane every regular polygon having an even number of sides can be dissected into lozenges. (A lozenge is a quadrilateral whose four sides are all of equal length).

2. **(BEL 4)** From a bag containing 5 pairs of socks, each pair a different color, a random sample of 4 single socks is drawn. Any complete pairs in the sample are discarded and replaced by a new pair draw from the bag. The process continues until the bag is empty or there are 4 socks of different colors held outside the bag. What is the probability of the latter alternative?

3. **(BUL 1)** Find all polynomials $f(x)$ with real coefficients for which

$$f(x)f(2x^2) = f(2x^3 + x).$$

4. **(BUL 3)**$^{\text{IMO2}}$ A pentagonal prism $A_1A_2 \dots A_5B_1B_2 \dots B_5$ is given. The edges, the diagonals of the lateral walls and the internal diagonals of the prism are each colored either red or green in such a way that no triangle whose vertices are vertices of the prism has its three edges of the same color. Prove that all edges of the bases are of the same color.

5. **(CZS 2)** Let $n \ge 2$ be an integer. Find the maximal cardinality of a set M of pairs (j, k) of integers, $1 \le j < k \le n$, with the following property: If $(j, k) \in M$, then $(k, m) \notin M$ for any m.

6. **(CZS 4)** Find the real values of p for which the equation

$$\sqrt{2p + 1 - x^2} + \sqrt{3x + p + 4} = \sqrt{x^2 + 9x + 3p + 9}$$

in x has exactly two real distinct roots ($\sqrt{t}$ means the positive square root of t).

7. **(FRG 1)**$^{\text{IMO1}}$ Given that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319} = \frac{p}{q}$, where p and q are natural numbers having no common factor, prove that p is divisible by 1979.

8. **(FRG 3)** For all rational x satisfying $0 \le x < 1$, f is defined by
$$f(x) = \begin{cases} f(2x)/4, & \text{for } 0 \le x < 1/2, \\ 3/4 + f(2x-1)/4, & \text{for } 1/2 \le x < 1. \end{cases}$$
Given that $x = 0.b_1 b_2 b_3 \dots$ is the binary representation of x, find $f(x)$.

9. **(FRG 4)**$^{\text{IMO6}}$ Let S and F be two opposite vertices of a regular octagon. A counter starts at S and each second is moved to one of the two neighboring vertices of the octagon. The direction is determined by the toss of a coin. The process ends when the counter reaches F. We define a_n to be the number of distinct paths of duration n seconds that the counter may take to reach F from S. Prove that for $n = 1, 2, 3, \dots$,
$$a_{2n-1} = 0, \qquad a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}), \qquad \text{where } x = 2 + \sqrt{2},\ y = 2 - \sqrt{2}.$$

10. **(FIN 3)** Show that for any vectors a, b in Euclidean space,
$$|a \times b|^3 \le \frac{3\sqrt{3}}{8} |a|^2 |b|^2 |a - b|^2.$$
Remark. Here $\times$ denotes the vector product.

11. **(GDR 1)** Given real numbers $x_1, x_2, \dots, x_n$ $(n \ge 2)$, with $x_i \ge 1/n$ $(i = 1, 2, \dots, n)$ and with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, find whether the product $P = x_1 x_2 x_3 \cdots x_n$ has a greatest and/or least value and if so, give these values.

12. **(GDR 3)** Let R be a set of exactly 6 elements. A set F of subsets of R is called an *S-family over* R if and only if it satisfies the following three conditions:
 (i) For no two sets X, Y in F is $X \subseteq Y$;
 (ii) For any three sets X, Y, Z in F, $X \cup Y \cup Z \ne R$,
 (iii) $\bigcup_{X \in F} X = R$.
 We define $|F|$ to be the number of elements of F (i.e., the number of subsets of R belonging to F). Determine, if it exists, $h = \max |F|$, the maximum being taken over all S-families over R.

13. **(GRE 1)** Show that $\frac{20}{60} < \sin 20^\circ < \frac{21}{60}$.

14. **(GRE 5)** Find all bases of logarithms in which a real positive number can be equal to its logarithm or prove that none exist.

15. **(ISR 2)**$^{\text{IMO5}}$ The nonnegative real numbers $x_1, x_2, x_3, x_4, x_5, a$ satisfy the following relations:

$$\sum_{i=1}^{5} i x_i = a, \qquad \sum_{i=1}^{5} i^3 x_i = a^2, \qquad \sum_{i=1}^{5} i^5 x_i = a^3.$$

What are the possible values of a?

16. **(ISR 4)** Let K denote the set $\{a, b, c, d, e\}$. F is a collection of 16 different subsets of K, and it is known that any three members of F have at least one element in common. Show that all 16 members of F have exactly one element in common.

17. **(NET 1)** Inside an equilateral triangle ABC one constructs points P, Q and R such that

$$\angle QAB = \angle PBA = 15^\circ,$$
$$\angle RBC = \angle QCB = 20^\circ,$$
$$\angle PCA = \angle RAC = 25^\circ.$$

Determine the angles of triangle PQR.

18. **(POL 1)** Let m positive integers $a_1, \dots, a_m$ be given. Prove that there exist fewer than 2^m positive integers $b_1, \dots, b_n$ such that all sums of distinct b_k's are distinct and all a_i $(i \le m)$ occur among them.

19. **(ROM 1)** Consider the sequences (a_n), (b_n) defined by

$$a_1 = 3, \qquad b_1 = 100, \qquad\quad a_{n+1} = 3^{a_n}, \qquad b_{n+1} = 100^{b_n}.$$

Find the smallest integer m for which $b_m > a_{100}$.

20. **(SWE 2)** Given the integer $n > 1$ and the real number $a > 0$ determine the maximum of $\sum_{i=1}^{n-1} x_i x_{i+1}$ taken over all nonnegative numbers x_i with sum a.

21. **(USS 1)** Let N be the number of integral solutions of the equation

$$x^2 - y^2 = z^3 - t^3$$

satisfying the condition $0 \le x, y, z, t \le 10^6$, and let M be the number of integral solutions of the equation

$$x^2 - y^2 = z^3 - t^3 + 1$$

satisfying the condition $0 \le x, y, z, t \le 10^6$. Prove that $N > M$.

22. **(USS 3)**$^{\text{IMO3}}$ There are two circles in the plane. Let a point A be one of the points of intersection of these circles. Two points begin moving simultaneously with constant speeds from the point A, each point along its own circle. The two points return to the point A at the same time. Prove that there is a point P in the plane such that at every moment of time the distances from the point P to the moving points are equal.

23. **(USA 4)** Find all natural numbers n for which $2^8 + 2^{11} + 2^n$ is a perfect square.

24. **(USA 5)** A circle O with center O on base BC of an isosceles triangle ABC is tangent to the equal sides AB, AC. If point P on AB and point Q on AC are selected such that $PB \times CQ = (BC/2)^2$, prove that line segment PQ is tangent to circle O, and prove the converse.

25. **(USA 6)**$^{\mathrm{IMO4}}$ Given a point P in a given plane π and also a given point Q not in π, show how to determine a point R in π such that $\frac{QP+PR}{QR}$ is a maximum.

26. **(YUG 4)** Prove that the functional equations

$$\begin{aligned} & f(x+y) = f(x) + f(y), \\ \text{and} \quad & f(x+y+xy) = f(x) + f(y) + f(xy) \quad (x, y \in \mathbb{R}) \end{aligned}$$

are equivalent.

3.22 The Twenty-Second IMO Washington DC, United States of America, July 8–20, 1981

3.22.1 Contest Problems

First Day (July 13)

1. Find the point P inside the triangle ABC for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is minimal, where PD, PE, PF are the perpendiculars from P to BC, CA, AB respectively.

2. Let $f(n,r)$ be the arithmetic mean of the minima of all r-subsets of the set $\{1,2,\ldots,n\}$. Prove that $f(n,r) = \frac{n+1}{r+1}$.

3. Determine the maximum value of $m^2 + n^2$ where m and n are integers satisfying

$$m, n \in \{1, 2, \ldots, 1981\} \quad \text{and} \quad (n^2 - mn - m^2)^2 = 1.$$

Second Day (July 14)

4. (a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set in the set is a divisor of the least common multiple of the remaining $n-1$ numbers?
   (b) For which values of $n > 2$ is there a unique set having the stated property?

5. Three equal circles touch the sides of a triangle and have one common point O. Show that the center of the circle inscribed in and of the circle circumscribed about the triangle ABC and the point O are collinear.

6. Assume that $f(x,y)$ is defined for all positive integers x and y, and that the following equations are satisfied:

$$\begin{aligned} f(0,y) &= y+1, \\ f(x+1,0) &= f(x,1), \\ f(x+1,y+1) &= f(x, f(x+1,y)). \end{aligned}$$

Determine $f(4, 1981)$.

3.22.2 Shortlisted Problems

1. **(BEL)**IMO4 (a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers?
   (b) For which values of $n > 2$ is there a unique set having the stated property?

2. **(BUL)** A sphere S is tangent to the edges AB, BC, CD, DA of a tetrahedron $ABCD$ at the points E, F, G, H respectively. The points E, F, G, H are the vertices of a square. Prove that if the sphere is tangent to the edge AC, then it is also tangent to the edge BD.

3. **(CAN)** Find the minimum value of
$$\max(a + b + c, b + c + d, c + d + e, d + e + f, e + f + g)$$
subject to the constraints
$$\text{(i)}\ \ a, b, c, d, e, f, g \geq 0, \qquad \text{(ii)}\ \ a + b + c + d + e + f + g = 1.$$

4. **(CAN)** Let $\{f_n\}$ be the Fibonacci sequence $\{1, 1, 2, 3, 5, \dots\}$.
   (a) Find all pairs (a, b) of real numbers such that for each n, $af_n + bf_{n+1}$ is a member of the sequence.
   (b) Find all pairs (u, v) of positive real numbers such that for each n, $uf_n^2 + vf_{n+1}^2$ is a member of the sequence.

5. **(COL)** A cube is assembled with 27 white cubes. The larger cube is then painted black on the outside and disassembled. A blind man reassembles it. What is the probability that the cube is now completely black on the outside? Give an approximation of the size of your answer.

6. **(CUB)** Let $P(z)$ and $Q(z)$ be complex-variable polynomials, with degree not less than 1. Let
$$P_k = \{z \in \mathbb{C} \mid P(z) = k\}, \qquad Q_k = \{z \in \mathbb{C} \mid Q(z) = k\}.$$
Let also $P_0 = Q_0$ and $P_1 = Q_1$. Prove that $P(z) \equiv Q(z)$.

7. **(FIN)**IMO6 Assume that $f(x, y)$ is defined for all positive integers x and y, and that the following equations are satisfied:
$$\begin{aligned} f(0, y) &= y + 1, \\ f(x + 1, 0) &= f(x, 1), \\ f(x + 1, y + 1) &= f(x, f(x + 1, y)). \end{aligned}$$
Determine $f(2, 2), f(3, 3)$ and $f(4, 4)$.
Alternative version: Determine $f(4, 1981)$.

8. **(FRG)**$^{\text{IMO2}}$ Let $f(n,r)$ be the arithmetic mean of the minima of all r-subsets of the set $\{1,2,\dots,n\}$. Prove that $f(n,r)=\frac{n+1}{r+1}$.

9. **(FRG)** A sequence (a_n) is defined by means of the recursion
$$a_1=1, \qquad a_{n+1}=\frac{1+4a_n+\sqrt{1+24a_n}}{16}.$$
Find an explicit formula for a_n.

10. **(FRA)** Determine the smallest natural number n having the following property: For every integer p, $p\ge n$, it is possible to subdivide (partition) a given square into p squares (not necessarily equal).

11. **(NET)** On a semicircle with unit radius four consecutive chords AB, BC, CD, DE with lengths a,b,c,d, respectively, are given. Prove that
$$a^2+b^2+c^2+d^2+abc+bcd<4.$$

12. **(NET)**$^{\text{IMO3}}$ Determine the maximum value of m^2+n^2 where m and n are integers satisfying
$$m,n\in\{1,2,\dots,100\} \quad \text{and} \quad (n^2-mn-m^2)^2=1.$$

13. **(ROM)** Let P be a polynomial of degree n satisfying
$$P(k)=\binom{n+1}{k}^{-1} \qquad \text{for } k=0,1,\dots,n.$$
Determine $P(n+1)$.

14. **(ROM)** Prove that a convex pentagon (a five-sided polygon) $ABCDE$ with equal sides and for which the interior angles satisfy the condition $\angle A\ge\angle B\ge\angle C\ge\angle D\ge\angle E$ is a regular pentagon.

15. **(GBR)**$^{\text{IMO1}}$ Find the point P inside the triangle ABC for which
$$\frac{BC}{PD}+\frac{CA}{PE}+\frac{AB}{PF}$$
is minimal, where PD, PE, PF are the perpendiculars from P to BC, CA, AB respectively.

16. **(GBR)** A sequence of real numbers $u_1,u_2,u_3,\dots$ is determined by u_1 and the following recurrence relation for $n\ge 1$:
$$4u_{n+1}=\sqrt[3]{64u_n+15}.$$
Describe, with proof, the behavior of u_n as $n\to\infty$.

17. **(USS)**$^{\text{IMO5}}$ Three equal circles touch the sides of a triangle and have one common point O. Show that the center of the circle inscribed in and of the circle circumscribed about the triangle ABC and the point O are collinear.

18. **(USS)** Several equal spherical planets are given in outer space. On the surface of each planet there is a set of points that is invisible from any of the remaining planets. Prove that the sum of the areas of all these sets is equal to the area of the surface of one planet.

19. **(YUG)** A finite set of unit circles is given in a plane such that the area of their union U is S. Prove that there exists a subset of mutually disjoint circles such that the area of their union is greater that $\frac{2S}{9}$.

3.23 The Twenty-Third IMO
Budapest, Hungary, July 5–14, 1982

3.23.1 Contest Problems

First Day (July 9)

1. The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n,
$$f(m+n) - f(m) - f(n) = 0 \quad \text{or} \quad 1;$$
$$f(2) = 0, \quad f(3) > 0, \quad \text{and} \quad f(9999) = 3333.$$
Determine $f(1982)$.

2. A nonisosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite to A_i). For all $i = 1, 2, 3$, M_i is the midpoint of side a_i, T_i is the point where the incircle touches side a_i, and the reflection of T_i in the interior bisector of A_i yields the point S_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent.

3. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:
$$x_0 = 1 \quad \text{and for all} \quad i \geq 0, \ \ x_{i+1} \leq x_i.$$
(a) Prove that for every such sequence there is an $n \geq 1$ such that
$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$
(b) Find such a sequence for which $\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4$ for all n.

Second Day (July 10)

4. Prove that if n is a positive integer such that the equation $x^3 - 3xy^2 + y^3 = n$ has a solution in integers (x, y), then it has at least three such solutions. Show that the equation has no solution in integers when $n = 2891$.

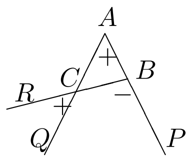
5. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that $\frac{AM}{AC} = \frac{CN}{CE} = r$. Determine r if B, M, and N are collinear.

6. Let S be a square with sides of length 100 and let L be a path within S that does not meet itself and that is composed of linear segments $A_0A_1, A_1A_2, \ldots, A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $\frac{1}{2}$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1 and the length of the part of L that lies between X and Y is not smaller than 198.

3.23.2 Longlisted Problems

1. **(AUS 1)** It is well known that the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, $0 \le k \le n$, are positive integers. The factorial $n!$ is defined inductively by $0! = 1$, $n! = n \cdot (n-1)!$ for $n \ge 1$.
   (a) Prove that $\frac{1}{n+1}\binom{2n}{n}$ is an integer for $n \ge 0$.
   (b) Given a positive integer k, determine the smallest integer C_k with the property that $\frac{C_k}{n+k+1}\binom{2n}{n+k}$ is an integer for all $n \ge k$.

2. **(AUS 2)** Given a finite number of angular regions $A_1, \dots, A_k$ in a plane, each A_i being bounded by two half-lines meeting at a vertex and provided with a $+$ or $-$ sign, we assign to each point P of the plane and not on a bounding half-line the number $k - l$, where k is the number of $+$ regions and l the number of $-$ regions that contain P. (Note that the boundary of A_i does not belong to A_i.)
   For instance, in the figure we have two $+$ regions QAP and RCQ, and one $-$ region RBP. Every point inside $\triangle ABC$ receives the number



   $+1$, while every point not inside $\triangle ABC$ and not on a boundary halfline the number 0. We say that the interior of $\triangle ABC$ is represented as a sum of the signed angular regions QAP, RBP, and RCQ.
   (a) Show how to represent the interior of any convex planar polygon as a sum of signed angular regions.
   (b) Show how to represent the interior of a tetrahedron as a sum of signed solid angular regions, that is, regions bounded by three planes intersecting at a vertex and provided with a $+$ or $-$ sign.

3. **(AUS 3)** Given n points $X_1, X_2, \dots, X_n$ in the interval $0 \le X_i \le 1$, $i = 1, 2, \dots, n$, show that there is a point y, $0 \le y \le 1$, such that
$$\frac{1}{n}\sum_{i=1}^{n} |y - X_i| = \frac{1}{2}.$$

4. **(AUS 4)** (SL82-14).
   Original formulation. Let $ABCD$ be a convex planar quadrilateral and let A_1 denote the circumcenter of $\triangle BCD$. Define B_1, C_1, D_1 in a corresponding way.
   (a) Prove that either all of A_1, B_1, C_1, D_1 coincide in one point, or they are all distinct. Assuming the latter case, show that A_1, C_1 are on opposite sides of the line B_1D_1, and similarly, B_1, D_1 are on opposite sides of the line A_1C_1. (This establishes the convexity of the quadrilateral $A_1B_1C_1D_1$.)
   (b) Denote by A_2 the circumcenter of $B_1C_1D_1$, and define B_2, C_2, D_2 in an analogous way. Show that the quadrilateral $A_2B_2C_2D_2$ is similar to the quadrilateral $ABCD$.

(c) If the quadrilateral $A_1B_1C_1D_1$ was obtained from the quadrilateral $ABCD$ by the above process, what condition must be satisfied by the four points A_1, B_1, C_1, D_1? Assuming that the four points A_1, B_1, C_1, D_1 satisfying this condition are given, describe a construction by straightedge and compass to obtain the original quadrilateral $ABCD$. (It is not necessary to actually perform the construction).

5. **(BEL 1)** Among all triangles with a given perimeter, find the one with the maximal radius of its incircle.

6. **(BEL 2)** On the three distinct lines a, b, and c three points A, B, and C are given, respectively. Construct three collinear points X, Y, Z on lines a, b, c, respectively, such that $\frac{BY}{AX} = 2$ and $\frac{CZ}{AX} = 3$.

7. **(BEL 3)** Find all solutions $(x, y) \in \mathbb{Z}^2$ of the equation
$$x^3 - y^3 = 2xy + 8.$$

8. **(BRA 1)** (SL82-10).

9. **(BRA 2)** Let n be a natural number, $n \geq 2$, and let ϕ be Euler's function; i.e., $\phi(n)$ is the number of positive integers not exceeding n and coprime to n. Given any two real numbers α and β, $0 \leq \alpha < \beta \leq 1$, prove that there exists a natural number m such that
$$\alpha < \frac{\phi(m)}{m} < \beta.$$

10. **(BRA 3)** Let $r_1, \dots, r_n$ be the radii of n spheres. Call $S_1, S_2, \dots, S_n$ the areas of the set of points of each sphere from which one cannot see any point of any other sphere. Prove that
$$\frac{S_1}{r_1^2} + \frac{S_2}{r_2^2} + \cdots + \frac{S_n}{r_n^2} = 4\pi.$$

11. **(BRA 4)** A rectangular pool table has a hole at each of three of its corners. The lengths of sides of the table are the real numbers a and b. A billiard ball is shot from the fourth corner along its angle bisector. The ball falls in one of the holes. What should the relation between a and b be for this to happen?

12. **(BRA 5)** Let there be 3399 numbers arbitrarily chosen among the first 6798 integers $1, 2, \dots, 6798$ in such a way that none of them divides another. Prove that there are exactly 1982 numbers in $\{1, 2, \dots, 6798\}$ that must end up being chosen.

13. **(BUL 1)** A regular n-gonal truncated pyramid is circumscribed around a sphere. Denote the areas of the base and the lateral surfaces of the pyramid by S_1, S_2, and S, respectively. Let σ be the area of the polygon whose vertices are the tangential points of the sphere and the lateral faces of the pyramid. Prove that

$$\sigma S = 4S_1S_2\cos^2\frac{\pi}{n}.$$

14. **(BUL 2)** (SL82-4).
15. **(CAN 1)** Show that the set S of natural numbers n for which $3/n$ cannot be written as the sum of two reciprocals of natural numbers ($S = \{n \mid 3/n \neq 1/p + 1/q$ for any $p, q \in \mathbb{N}\}$) is not the union of finitely many arithmetic progressions.
16. **(CAN 2)** (SL82-7).
17. **(CAN 3)** (SL82-11).
18. **(CAN 4)** You are given an algebraic system admitting addition and multiplication for which all the laws of ordinary arithmetic are valid except commutativity of multiplication. Show that
$$(a + ab^{-1}a)^{-1} + (a + b)^{-1} = a^{-1},$$
where x^{-1} is the element for which $x^{-1}x = xx^{-1} = e$, where e is the element of the system such that for all a the equality $ea = ae = a$ holds.
19. **(CAN 5)** (SL82-15).
20. **(CZS 1)** Consider a cube C and two planes σ, τ, which divide Euclidean space into several regions. Prove that the interior of at least one of these regions meets at least three faces of the cube.
21. **(CZS 2)** All edges and all diagonals of regular hexagon $A_1A_2A_3A_4A_5A_6$ are colored blue or red such that each triangle $A_jA_kA_m,\ 1 \le j < k < m \le 6$ has at least one red edge. Let R_k be the number of red segments A_kA_j, $(j \neq k)$. Prove the inequality
$$\sum_{k=1}^{6}(2R_k - 7)^2 \le 54.$$
22. **(CZS 3)** (SL82-19).
23. **(FIN 1)** Determine the sum of all positive integers whose digits (in base ten) form either a strictly increasing or a strictly decreasing sequence.
24. **(FIN 2)** Prove that if a person a has infinitely many descendants (children, their children, etc.), then a has an infinite sequence $a_0, a_1, \dots$ of descendants (i.e., $a = a_0$ and for all $n \ge 1$, a_{n+1} is always a child of a_n). It is assumed that no-one can have infinitely many children.
Variant 1. Prove that if a has infinitely many ancestors, then a has an infinite descending sequence of ancestors (i.e., $a_0, a_1, \dots$ where $a = a_0$ and a_n is always a child of a_{n+1}).
Variant 2. Prove that if someone has infinitely many ancestors, then all people cannot descend from $A(dam)$ and $E(ve)$.

25. **(FIN 3)** (SL82-12).

26. **(FRA 1)** Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of natural numbers. Determine whether there exists a pair (p,q) of natural numbers that satisfy
$$p < q \qquad \text{and} \qquad a_p \leq a_q, \quad b_p \leq b_q.$$

27. **(FRA 2)** (SL82-18).

28. **(FRA 3)** Let $(u_1, \dots, u_n)$ be an ordered $n$tuple. For each k, $1 \leq k \leq n$, define $v_k = \sqrt[k]{u_1 u_2 \cdots u_k}$. Prove that
$$\sum_{k=1}^{n} v_k \leq e \cdot \sum_{k=1}^{n} u_k.$$
(e is the base of the natural logarithm).

29. **(FRA 4)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that the restriction of f to the set of irrational numbers is injective. What can we say about f? Answer the analogous question if f is restricted to rationals.

30. **(GBR 1)** (SL82-9).

31. **(GBR 2)** (SL82-16).

32. **(GBR 3)** (SL82-1).

33. **(GBR 4)** A sequence (u_n) of integers is defined for $n \geq 0$ by $u_0 = 0$, $u_1 = 1$, and $u_n - 2u_{n-1} + (1-c)u_{n-2} = 0$ $(n \geq 2)$, where c is a fixed integer independent of n. Find the least value of c for which both of the following statements are true:
    (i) If p is a prime less than or equal to P, then p divides u_p.
    (ii) If p is a prime greater than P, then p does not divide u_p.

34. **(GDR 1)** Let M be the set of all functions f with the following properties:
    (i) f is defined for all real numbers and takes only real values.
    (ii) For all $x, y \in \mathbb{R}$ the following equality holds: $f(x)f(y) = f(x+y) + f(x-y)$.
    (iii) $f(0) \neq 0$.
    Determine all functions $f \in M$ such that
    (a) $f(1) = 5/2$;
    (b) $f(1) = \sqrt{3}$.

35. **(GDR 2)** If the inradius of a triangle is half of its circumradius, prove that the triangle is equilateral.

36. **(NET 1)** (SL82-13).

37. **(NET 2)** (SL82-5).

38. **(POL 1)** Numbers $u_{n,k}$ $(1 \le k \le n)$ are defined as follows:

$$u_{1,1} = 1, \qquad u_{n,k} = \binom{n}{k} - \sum_{d|n,\ d|k,\ d>1} u_{n/d,k/d}$$

(the empty sum is defined to be equal to zero). Prove that $n \mid u_{n,k}$ for every natural number n and for every k $(1 \le k \le n)$.

39. **(POL 2)** Let S be the unit circle with center O and let $P_1, P_2, \dots, P_n$ be points of S such that the sum of vectors $v_i = \overrightarrow{OP_i}$ is the zero vector. Prove that the inequality $\sum_{i=1}^{n} XP_i \ge n$ holds for every point X.

40. **(POL 3)** We consider a game on an infinite chessboard similar to that of solitaire: If two adjacent fields are occupied by pawns and the next field is empty (the three fields lie on a vertical or horizontal line), then we may remove these two pawns and put one of them on the third field. Prove that if in the initial position pawns fill a $3k \times n$ rectangle, then it is impossible to reach a position with only one pawn on the board.

41. **(POL 4)** (SL82-8).

42. **(POL 5)** Let $\mathcal{F}$ be the family of all k-element subsets of the set $\{1, 2, \dots, 2k+1\}$. Prove that there exists a bijective function $f : \mathcal{F} \to \mathcal{F}$ such that for every $A \in \mathcal{F}$, the sets A and $f(A)$ are disjoint.

43. **(TUN 1)** (a) What is the maximal number of acute angles in a convex polygon?
    (b) Consider m points in the interior of a convex n-gon. The n-gon is partitioned into triangles whose vertices are among the $n+m$ given points (the vertices of the n-gon and the given points). Each of the m points in the interior is a vertex of at least one triangle. Find the number of triangles obtained.

44. **(TUN 2)** Let A and B be positions of two ships M and N, respectively, at the moment when N saw M moving with constant speed v following the line Ax. In search of help, N moves with speed kv $(k < 1)$ along the line By in order to meet M as soon as possible. Denote by C the point of meeting of the two ships, and set

$$AB = d, \;\; \angle BAC = \alpha, \;\; 0 \le \alpha < \frac{\pi}{2}.$$

Determine the angle $\angle ABC = \beta$ and time t that N needs in order to meet M.

45. **(TUN 3)** (SL82-20).

46. **(USA 1)** Prove that if a diagonal is drawn in a quadrilateral inscribed in a circle, the sum of the radii of the circles inscribed in the two triangles thus formed is the same, no matter which diagonal is drawn.

47. **(USA 2)** Evaluate $\sec'' \frac{\pi}{4} + \sec'' \frac{3\pi}{4} + \sec'' \frac{5\pi}{4} + \sec'' \frac{7\pi}{4}$. (Here $\sec''$ means the second derivative of sec.)

48. **(USA 3)** Given a finite sequence of complex numbers $c_1, c_2, \dots, c_n$, show that there exists an integer k $(1 \le k \le n)$ such that for every finite sequence $a_1, a_2, \dots, a_n$ of real numbers with $1 \ge a_1 \ge a_2 \ge \dots \ge a_n \ge 0$, the following inequality holds:

$$\left| \sum_{m=1}^{n} a_m c_m n \right| \le \left| \sum_{m=1}^{n} c_m \right| .$$

49. **(USA 4)** Simplify

$$\sum_{k=0}^{n} \frac{(2n)!}{(k!)^2((n-k)!)^2} .$$

50. **(USS 1)** Let O be the midpoint of the axis of a right circular cylinder. Let A and B be diametrically opposite points of one base, and C a point of the other base circle that does not belong to the plane OAB. Prove that the sum of dihedral angles of the trihedral $OABC$ is equal to 2π.

51. **(USS 2)** Let n numbers $x_1, x_2, \dots, x_n$ be chosen in such a way that $1 \ge x_1 \ge x_2 \ge \dots \ge x_n \ge 0$. Prove that

$$(1 + x_1 + x_2 + \dots + x_n)^\alpha \le 1 + x_1^\alpha + 2^{\alpha-1} x_2^\alpha + \dots + n^{\alpha-1} x_n^\alpha$$

if $0 \le \alpha \le 1$.

52. **(USS 3)** We are given $2n$ natural numbers

$$1, 1, 2, 2, 3, 3, \dots, n-1, n-1, n, n.$$

Find all n for which these numbers can be arranged in a row such that for each $k \le n$, there are exactly k numbers between the two numbers k.

53. **(USS 4)** (SL82-3).

54. **(USS 5)** (SL82-17).

55. **(VIE 1)** (SL82-6).

56. **(VIE 2)** Let $f(x) = ax^2 + bx + c$ and $g(x) = cx^2 + bx + a$. If $|f(0)| \le 1$, $|f(1)| \le 1$, $|f(-1)| \le 1$, prove that for $|x| \le 1$,
(a) $|f(x)| \le 5/4$,
(b) $|g(x)| \le 2$.

57. **(YUG 1)** (SL82-2).

3.23.3 Shortlisted Problems

1. **A1 (GBR 3)**[IMO1] The function $f(n)$ is defined for all positive integers n and takes on nonnegative integer values. Also, for all m, n,

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1;$$

$$f(2) = 0, \quad f(3) > 0, \quad \text{and} \quad f(9999) = 3333.$$

Determine $f(1982)$.

2. **A2 (YUG 1)** Let K be a convex polygon in the plane and suppose that K is positioned in the coordinate system in such a way that

$$\text{area } (K \cap Q_i) = \frac{1}{4} \text{ area } K \quad (i = 1, 2, 3, 4,),$$

where the Q_i denote the quadrants of the plane. Prove that if K contains no nonzero lattice point, then the area of K is less than 4.

3. **A3 (USS 4)**$^{\mathrm{IMO3}}$ Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1 \quad \text{and for all} \quad i \geq 0, \quad x_{i+1} \leq x_i.$$

(a) Prove that for every such sequence there is an $n \geq 1$ such that $\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999$.

(b) Find such a sequence for which $\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4$ for all n.

4. **A4 (BUL 2)** Determine all real values of the parameter a for which the equation

$$16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$$

has exactly four distinct real roots that form a geometric progression.

5. **A5 (NET 2)**$^{\mathrm{IMO5}}$ Let $A_1A_2A_3A_4A_5A_6$ be a regular hexagon. Each of its diagonals $A_{i-1}A_{i+1}$ is divided into the same ratio $\frac{\lambda}{1-\lambda}$, where $0 < \lambda < 1$, by a point B_i in such a way that A_i, B_i, and B_{i+2} are collinear ($i \equiv 1, \ldots, 6 \pmod 6$). Compute λ.

6. **A6 (VIE 1)**$^{\mathrm{IMO6}}$ Let S be a square with sides of length 100 and let L be a path within S that does not meet itself and that is composed of linear segments $A_0A_1, A_1A_2, \ldots, A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $\frac{1}{2}$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1 and the length of that part of L that lies between X and Y is not smaller than 198.

7. **B1 (CAN 2)** Let $p(x)$ be a cubic polynomial with integer coefficients with leading coefficient 1 and with one of its roots equal to the product of the other two. Show that $2p(-1)$ is a multiple of $p(1)+p(-1)-2(1+p(0))$.

8. **B2 (POL 4)** A convex, closed figure lies inside a given circle. The figure is seen from every point of the circumference at a right angle (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the center of the circle is a center of symmetry of the figure.

9. **B3 (GBR 1)** Let ABC be a triangle, and let P be a point inside it such that $\measuredangle PAC = \measuredangle PBC$. The perpendiculars from P to BC and CA meet these lines at L and M, respectively, and D is the midpoint of AB. Prove that $DL = DM$.

10. **B4 (BRA 1)** A box contains p white balls and q black balls. Beside the box there is a pile of black balls. Two balls are taken out of the box. If they have the same color, a black ball from the pile is put into the box. If they have different colors, the white ball is put back into the box. This procedure is repeated until the last two balls are removed from the box and one last ball is put in. What is the probability that this last ball is white?

11. **B5 (CAN 3)** (a) Find the rearrangement $\{a_1, \dots, a_n\}$ of $\{1, 2, \dots, n\}$ that maximizes

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 = Q.$$

   (b) Find the rearrangement that minimizes Q.

12. **B6 (FIN 3)** Four distinct circles C, C_1, C_2, C_3 and a line L are given in the plane such that C and L are disjoint and each of the circles C_1, C_2, C_3 touches the other two, as well as C and L. Assuming the radius of C to be 1, determine the distance between its center and L.

13. **C1 (NET 1)**[IMO2] A scalene triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite to A_i). For all $i = 1, 2, 3$, M_i is the midpoint of side a_i, T_i is the point where the incircle touches side a_i, and the reflection of T_i in the interior bisector of A_i yields the point S_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent.

14. **C2 (AUS 4)** Let $ABCD$ be a convex plane quadrilateral and let A_1 denote the circumcenter of $\triangle BCD$. Define B_1, C_1, D_1 in a corresponding way.
   (a) Prove that either all of A_1, B_1, C_1, D_1 coincide in one point, or they are all distinct. Assuming the latter case, show that A_1, C_1 are on opposite sides of the line B_1D_1, and similarly, B_1, D_1 are on opposite sides of the line A_1C_1. (This establishes the convexity of the quadrilateral $A_1B_1C_1D_1$.)
   (b) Denote by A_2 the circumcenter of $B_1C_1D_1$, and define B_2, C_2, D_2 in an analogous way. Show that the quadrilateral $A_2B_2C_2D_2$ is similar to the quadrilateral $ABCD$.

15. **C3 (CAN 5)** Show that

$$\frac{1 - s^a}{1 - s} \le (1 + s)^{a-1}$$

   holds for every $1 \neq s > 0$ real and $0 < a \le 1$ rational.

16. **C4 (GBR 2)**$^{\text{IMO4}}$ Prove that if n is a positive integer such that the equation $x^3 - 3xy^2 + y^3 = n$ has a solution in integers (x, y), then it has at least three such solutions. Show that the equation has no solution in integers when $n = 2891$.

17. **C5 (USS 5)** The right triangles ABC and AB_1C_1 are similar and have opposite orientation. The right angles are at C and C_1, and we also have $\measuredangle CAB = \measuredangle C_1AB_1$. Let M be the point of intersection of the lines BC_1 and B_1C. Prove that if the lines AM and CC_1 exist, they are perpendicular.

18. **C6 (FRA 2)** Let O be a point of three-dimensional space and let l_1, l_2, l_3 be mutually perpendicular straight lines passing through O. Let S denote the sphere with center O and radius R, and for every point M of S, let S_M denote the sphere with center M and radius R. We denote by P_1, P_2, P_3 the intersection of S_M with the straight lines l_1, l_2, l_3, respectively, where we put $P_i \neq O$ if l_i meets S_M at two distinct points and $P_i = O$ otherwise ($i = 1, 2, 3$). What is the set of centers of gravity of the (possibly degenerate) triangles $P_1P_2P_3$ as M runs through the points of S?

19. **C7 (CZS 3)** Let M be the set of real numbers of the form $\frac{m+n}{\sqrt{m^2+n^2}}$, where m and n are positive integers. Prove that for every pair $x \in M$, $y \in M$ with $x < y$, there exists an element $z \in M$ such that $x < z < y$.

20. **C8 (TUN 3)** Let $ABCD$ be a convex quadrilateral and draw regular triangles ABM, CDP, BCN, ADQ, the first two outward and the other two inward. Prove that $MN = AC$. What can be said about the quadrilateral $MNPQ$?

3.24 The Twenty-Fourth IMO
Paris, France, July 1–12, 1983

3.24.1 Contest Problems

First Day (July 6)

1. Find all functions f defined on the positive real numbers and taking positive real values that satisfy the following conditions:
(i) $f(xf(y)) = yf(x)$ for all positive real x, y;
(ii) $f(x) \to 0$ as $x \to +\infty$.
2. Let K be one of the two intersection points of the circles W_1 and W_2. Let O_1 and O_2 be the centers of W_1 and W_2. The two common tangents to the circles meet W_1 and W_2 respectively in P_1 and P_2, the first tangent, and Q_1 and Q_2 the second tangent. Let M_1 and M_2 be the midpoints of P_1Q_1 and P_2Q_2, respectively. Prove that $\angle O_1KO_2 = \angle M_1KM_2$.
3. Let a, b, c be positive integers satisfying $(a, b) = (b, c) = (c, a) = 1$. Show that $2abc - ab - bc - ca$ is the largest integer not representable as
$$xbc + yca + zab$$
with nonnegative integers x, y, z.

Second Day (July 7)

4. Let ABC be an equilateral triangle. Let E be the set of all points from segments AB, BC, and CA (including A, B, and C). Is it true that for any partition of the set E into two disjoint subsets, there exists a right-angled triangle all of whose vertices belong to the same subset in the partition?
5. Prove or disprove the following statement: In the set $\{1, 2, 3, \dots, 10^5\}$ a subset of 1983 elements can be found that does not contain any three consecutive terms of an arithmetic progression.
6. If a, b, and c are sides of a triangle, prove that
$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$$
and determine when there is equality.

3.24.2 Longlisted Problems

1. **(AUS 1)** (SL83-1).
2. **(AUS 2)** Seventeen cities are served by four airlines. It is noted that there is direct service (without stops) between any two cities and that all airline schedules offer round-trip flights. Prove that at least one of the airlines can offer a round trip with an odd number of landings.

3. **(AUS 3)** (a) Given a tetrahedron $ABCD$ and its four altitudes (i.e., lines through each vertex, perpendicular to the opposite face), assume that the altitude dropped from D passes through the orthocenter H_4 of ΔABC. Prove that this altitude DH_4 intersects all the other three altitudes.
   (b) If we further know that a second altitude, say the one from vertex A to the face BCD, also passes through the orthocenter H_1 of ΔBCD, then prove that all four altitudes are concurrent and each one passes through the orthocenter of the respective triangle.
4. **(BEL 1)** (SL83-2).
5. **(BEL 2)** Consider the set $\mathbb{Q}^2$ of points in $\mathbb{R}^2$, both of whose coordinates are rational.
   (a) Prove that the union of segments with vertices from $\mathbb{Q}^2$ is the entire set $\mathbb{R}^2$.
   (b) Is the convex hull of $\mathbb{Q}^2$ (i.e., the smallest convex set in $\mathbb{R}^2$ that contains $\mathbb{Q}^2$) equal to $\mathbb{R}^2$?
6. **(BEL 3)** (SL83-3).
7. **(BEL 4)** Find all numbers $x \in \mathbb{Z}$ for which the number
$$x^4 + x^3 + x^2 + x + 1$$
   is a perfect square.
8. **(BEL 5)** (SL83-4).
9. **(BRA 1)** (SL83-5).
10. **(BRA 2)** Which of the numbers $1, 2, \dots, 1983$ has the largest number of divisors?
11. **(BRA 3)** A boy at point A wants to get water at a circular lake and carry it to point B. Find the point C on the lake such that the distance walked by the boy is the shortest possible given that the line AB and the lake are exterior to each other.
12. **(BRA 4)** The number 0 or 1 is to be assigned to each of the n vertices of a regular polygon. In how many different ways can this be done (if we consider two assignments that can be obtained one from the other through rotation in the plane of the polygon to be identical)?
13. **(BUL 1)** Let p be a prime number and $a_1, a_2, \dots, a_{(p+1)/2}$ different natural numbers less than or equal to p. Prove that for each natural number r less than or equal to p, there exist two numbers (perhaps equal) a_i and a_j such that
$$p \equiv a_i a_j (\text{mod } r).$$

14. **(BUL 2)** Let l be tangent to the circle k at B. Let A be a point on k and P the foot of perpendicular from A to l. Let M be symmetric to P with respect to AB. Find the set of all such points M.

15. **(CAN 1)** Find all possible finite sequences $\{n_0, n_1, n_2, \dots, n_k\}$ of integers such that for each i, i appears in the sequence n_i times $(0 \le i \le k)$.

16. **(CAN 2)** (SL83-6).

17. **(CAN 3)** In how many ways can $1, 2, \dots, 2n$ be arranged in a $2 \times n$ rectangular array $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$ for which:
    (i) $a_1 < a_2 < \cdots < a_n$,
    (ii) $b_1 < b_2 < \cdots < b_n$,
    (iii) $a_1 < b_1$, $a_2 < b_2$, $\dots$, $a_n < b_n$?

18. **(CAN 4)** Let $b \ge 2$ be a positive integer.
    (a) Show that for an integer N, written in base b, to be equal to the sum of the squares of its digits, it is necessary either that $N = 1$ or that N have only two digits.
    (b) Give a complete list of all integers not exceeding 50 that, relative to some base b, are equal to the sum of the squares of their digits.
    (c) Show that for any base b the number of two-digit integers that are equal to the sum of the squares of their digits is even.
    (d) Show that for any odd base b there is an integer other than 1 that is equal to the sum of the squares of its digits.

19. **(CAN 5)** (SL83-7).

20. **(COL 1)** Let f and g be functions from the set A to the same set A. We define f to be *a functional nth root of g* (n is a positive integer) if $f^n(x) = g(x)$, where $f^n(x) = f^{n-1}(f(x))$.
    (a) Prove that the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 1/x$ has an infinite number of nth functional roots for each positive integer n.
    (b) Prove that there is a bijection from $\mathbb{R}$ onto $\mathbb{R}$ that has no nth functional root for each positive integer n.

21. **(COL 2)** Prove that there are infinitely many positive integers n for which it is possible for a knight, starting at one of the squares of an $n \times n$ chessboard, to go through each of the squares exactly once.

22. **(CUB 1)** Does there exist an infinite number of sets C consisting of 1983 consecutive natural numbers such that each of the numbers is divisible by some number of the form a^{1983}, with $a \in \mathbb{N}$, $a \neq 1$?

23. **(FIN 1)** (SL83-10).

24. **(FIN 2)** Every x, $0 \le x \le 1$, admits a unique representation $x = \sum_{j=0}^{\infty} a_j 2^{-j}$, where all the a_j belong to $\{0, 1\}$ and infinitely many of them are 0. If $b(0) = \frac{1+c}{2+c}$, $b(1) = \frac{1}{2+c}$, $c > 0$, and

$$f(x) = a_0 + \sum_{j=0}^{\infty} b(a_0)\cdots b(a_j)a_{j+1},$$

show that $0 < f(x) - x < c$ for every x, $0 < x < 1$.
(FIN 2′) (SL83-11).

25. **(FRG 1)** How many permutations $a_1, a_2, \dots, a_n$ of $\{1, 2, \dots, n\}$ are sorted into increasing order by at most three repetitions of the following operation: Move from left to right and interchange a_i and a_{i+1} whenever $a_i > a_{i+1}$ for i running from 1 up to $n-1$?

26. **(FRG 2)** Let a, b, c be positive integers satisfying $(a, b) = (b, c) = (c, a) = 1$. Show that $2abc - ab - bc - ca$ cannot be represented as $bcx + cay + abz$ with nonnegative integers x, y, z.

27. **(FRG 3)** (SL83-18).

28. **(GBR 1)** Show that if the sides a, b, c of a triangle satisfy the equation
$$2(ab^2 + bc^2 + ca^2) = a^2b + b^2c + c^2a + 3abc,$$
then the triangle is equilateral. Show also that the equation can be satisfied by positive real numbers that are not the sides of a triangle.

29. **(GBR 2)** Let O be a point outside a given circle. Two lines OAB, OCD through O meet the circle at A, B, C, D, where A, C are the midpoints of OB, OD, respectively. Additionally, the acute angle θ between the lines is equal to the acute angle at which each line cuts the circle. Find $\cos\theta$ and show that the tangents at A, D to the circle meet on the line BC.

30. **(GBR 3)** Prove the existence of a unique sequence $\{u_n\}$ $(n = 0, 1, 2 \dots)$ of positive integers such that
$$u_n^2 = \sum_{r=0}^{n} \binom{n+r}{r} u_{n-r} \qquad \text{for all } n \ge 0,$$
where $\binom{m}{r}$ is the usual binomial coefficient.

31. **(GBR 4)** (SL83-12).

32. **(GBR 5)** Let a, b, c be positive real numbers and let $[x]$ denote the greatest integer that does not exceed the real number x. Suppose that f is a function defined on the set of nonnegative integers n and taking real values such that $f(0) = 0$ and
$$f(n) \le an + f([bn]) + f([cn]), \qquad \text{for all } n \ge 1.$$
Prove that if $b + c < 1$, there is a real number k such that
$$f(n) \le kn \qquad \text{for all } n, \tag{1}$$

while if $b+c=1$, there is a real number K such that $f(n) \le Kn\log_2 n$ for all $n \ge 2$. Show that if $b+c=1$, there may not be a real number k that satisfies (1).

33. **(GDR 1)** (SL83-16).

34. **(GDR 2)** In a plane are given n points P_i $(i=1,2,\dots,n)$ and two angles α and β. Over each of the segments $P_iP_{i=1}$ $(P_{n+1}=P_1)$ a point Q_i is constructed such that for all i:
 (i) upon moving from P_i to P_{i+1}, Q_i is seen on the same side of P_iP_{i+1},
 (ii) $\angle P_{i+1}P_iQ_i = \alpha$,
 (iii) $\angle P_iP_{i+1}Q_i = \beta$.
 Furthermore, let g be a line in the same plane with the property that all the points P_i, Q_i lie on the same side of g. Prove that
$$\sum_{i=1}^{n} d(P_i,g) = \sum_{i=1}^{n} d(Q_i,g),$$
 where $d(M,g)$ denotes the distance from point M to line g.

35. **(GDR 3)** (SL83-17).

36. **(ISR 1)** The set X has 1983 members. There exists a family of subsets $\{S_1,S_2,\dots,S_k\}$ such that:
 (i) the union of any three of these subsets is the entire set X, while
 (ii) the union of any two of them contains at most 1979 members.
 What is the largest possible value of k?

37. **(ISR 2)** The points $A_1,A_2,\dots,A_{1983}$ are set on the circumference of a circle and each is given one of the values ± 1. Show that if the number of points with the value $+1$ is greater than 1789, then at least 1207 of the points will have the property that the partial sums that can be formed by taking the numbers from them to any other point, in either direction, are strictly positive.

38. **(KUW 1)** Let $\{u_n\}$ be the sequence defined by its first two terms u_0,u_1 and the recursion formula
$$u_{n+2} = u_n - u_{n+1}.$$
 (a) Show that u_n can be written in the form $u_n = \alpha a^n + \beta b^n$, where a,b,α,β are constants independent of n that have to be determined.
 (b) If $S_n = u_0+u_1+\dots+u_n$, prove that S_n+u_{n-1} is a constant independent of n. Determine this constant.

39. **(KUW 2)** If α is the real root of the equation
$$E(x) = x^3-5x-50 = 0$$
 such that $x_{n+1} = (5x_n+50)^{1/3}$ and $x_1 = 5$, where n is a positive integer, prove that:

(a) $x_{n+1}^3 - \alpha^3 = 5(x_n - \alpha)$
(b) $\alpha < x_{n+1} < x_n$

40. **(LUX 1)** Four faces of tetrahedron $ABCD$ are congruent triangles whose angles form an arithmetic progression. If the lengths of the sides of the triangles are $a < b < c$, determine the radius of the sphere circumscribed about the tetrahedron as a function on a, b, and c. What is the ratio c/a if $R = a$?

41. **(LUX 2)** (SL83-13).

42. **(LUX 3)** Consider the square $ABCD$ in which a segment is drawn between each vertex and the midpoints of both opposite sides. Find the ratio of the area of the octagon determined by these segments and the area of the square $ABCD$.

43. **(LUX 4)** Given a square $ABCD$, let P, Q, R, and S be four variable points on the sides AB, BC, CD, and DA, respectively. Determine the positions of the points P, Q, R, and S for which the quadrilateral $PQRS$ is a parallelogram, a rectangle, a square, or a trapezoid.

44. **(LUX 5)** We are given twelve coins, one of which is a fake with a different mass from the other eleven. Determine that coin with three weighings and whether it is heavier or lighter than the others.

45. **(LUX 6)** Let two glasses, numbered 1 and 2, contain an equal quantity of liquid, milk in glass 1 and coffee in glass 2. One does the following: Take one spoon of mixture from glass 1 and pour it into glass 2, and then take the same spoon of the new mixture from glass 2 and pour it back into the first glass. What happens after this operation is repeated n times, and what as n tends to infinity?

46. **(LUX 7)** Let f be a real-valued function defined on $I = (0, +\infty)$ and having no zeros on I. Suppose that
$$\lim_{x \to +\infty} \frac{f'(x)}{f(x)} = +\infty.$$
For the sequence $u_n = \ln \left| \frac{f(n+1)}{f(n)} \right|$, prove that $u_n \to +\infty$ $(n \to +\infty)$.

47. **(NET 1)** In a plane, three pairwise intersecting circles C_1, C_2, C_3 with centers M_1, M_2, M_3 are given. For $i = 1, 2, 3$, let A_i be one of the points of intersection of C_j and C_k ($\{i, j, k\} = \{1, 2, 3\}$). Prove that if $\angle M_3 A_1 M_2 = \angle M_1 A_2 M_3 = \angle M_2 A_3 M_1 = \pi/3$ (directed angles), then $M_1 A_1$, $M_2 A_2$, and $M_3 A_3$ are concurrent.

48. **(NET 2)** Prove that in any parallelepiped the sum of the lengths of the edges is less than or equal to twice the sum of the lengths of the four diagonals.

49. **(POL 1)** Given positive integers k, m, n with $km \le n$ and nonnegative real numbers $x_1, \dots, x_k$, prove that

$$n\left(\prod_{i=1}^{k} x_i^m - 1\right) \le m \sum_{i=1}^{k} (x_i^n - 1).$$

50. **(POL 2)** (SL83-14).

51. **(POL 3)** (SL83-15).

52. **(ROM 1)** (SL83-19).

53. **(ROM 2)** Let $a \in \mathbb{R}$ and let $z_1, z_2, \dots, z_n$ be complex numbers of modulus 1 satisfying the relation

$$\sum_{k=1}^{n} z_k^3 = 4(a + (a-n)i) - 3\sum_{k=1}^{n} \overline{z_k}.$$

Prove that $a \in \{0, 1, \dots, n\}$ and $z_k \in \{1, i\}$ for all k.

54. **(ROM 3)** (SL83-20).

55. **(ROM 4)** For every $a \in \mathbb{N}$ denote by $M(a)$ the number of elements of the set

$$\{b \in \mathbb{N} \mid a + b \text{ is a divisor of } ab\}.$$

Find $\max_{a \le 1983} M(a)$.

56. **(ROM 5)** Consider the expansion

$$(1 + x + x^2 + x^3 + x^4)^{496} = a_0 + a_1 x + \dots + a_{1984} x^{1984}.$$

(a) Determine the greatest common divisor of the coefficients $a_3, a_8, a_{13}, \dots, a_{1983}$.
(b) Prove that $10^{340} < a^{992} < 10^{347}$.

57. **(SPA 1)** In the system of base $n^2 + 1$ find a number N with n different digits such that:
(i) N is a multiple of n. Let $N = nN'$.
(ii) The number N and N' have the same number n of different digits in base $n^2 + 1$, none of them being zero.
(iii) If $s(C)$ denotes the number in base $n^2 + 1$ obtained by applying the permutation s to the n digits of the number C, then for each permutation s, $s(N) = ns(N')$.

58. **(SPA 2)** (SL83-8).

59. **(SPA 3)** Solve the equation

$$\tan^2(2x) + 2\tan(2x) \cdot \tan(3x) - 1 = 0.$$

60. **(SWE 1)** (SL83-21).

61. **(SWE 2)** Let a and b be integers. Is it possible to find integers p and q such that the integers $p+na$ and $q+nb$ have no common prime factor no matter how the integer n is chosen.

62. **(SWE 3)** A circle γ is drawn and let AB be a diameter. The point C on γ is the midpoint of the line segment BD. The line segments AC and DO, where O is the center of γ, intersect at P. Prove that there is a point E on AB such that P is on the circle with diameter AE.

63. **(SWE 4)** (SL83-22).

64. **(USA 1)** The sum of all the face angles about all of the vertices except one of a given polyhedron is 5160. Find the sum of all of the face angles of the polyhedron.

65. **(USA 2)** Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD intersect in a point P. Prove that

$$\frac{AP}{PC}=\frac{\cot\angle BAC+\cot\angle DAC}{\cot\angle BCA+\cot\angle DCA}.$$

66. **(USA 3)** (SL83-9).

67. **(USA 4)** The altitude from a vertex of a given tetrahedron intersects the opposite face in its orthocenter. Prove that all four altitudes of the tetrahedron are concurrent.

68. **(USA 5)** Three of the roots of the equation $x^4-px^3+qx^2-rx+s=0$ are $\tan A$, $\tan B$, and $\tan C$, where A, B, and C are angles of a triangle. Determine the fourth root as a function only of p, q, r, and s.

69. **(USS 1)** (SL83-23).

70. **(USS 2)** (SL83-24).

71. **(USS 3)** (SL83-25).

72. **(USS 4)** Prove that for all $x_1,x_2,\dots,x_n\in\mathbb{R}$ the following inequality holds:

$$\sum_{n\ge i>j\ge 1}\cos^2(x_i-x_j)\ge\frac{n(n-2)}{4}.$$

73. **(VIE 1)** Let ABC be a nonequilateral triangle. Prove that there exist two points P and Q in the plane of the triangle, one in the interior and one in the exterior of the circumcircle of ABC, such that the orthogonal projections of any of these two points on the sides of the triangle are vertices of an equilateral triangle.

74. **(VIE 2)** In a plane we are given two distinct points A,B and two lines a,b passing through B and A respectively ($a\ni B, b\ni A$) such that the line AB is equally inclined to a and b. Find the locus of points M in the plane such that the product of distances from M to A and a equals the

product of distances from M to B and b (i.e., $MA \cdot MA' = MB \cdot MB'$, where A' and B' are the feet of the perpendiculars from M to a and b respectively).

75. **(VIE 3)** Find the sum of the fiftieth powers of all sides and diagonals of a regular 100-gon inscribed in a circle of radius R.

3.24.3 Shortlisted Problems

1. **(AUS 1)** The localities $P_1, P_2, \dots, P_{1983}$ are served by ten international airlines $A_1, A_2, \dots, A_{10}$. It is noticed that there is direct service (without stops) between any two of these localities and that all airline schedules offer round-trip flights. Prove that at least one of the airlines can offer a round trip with an odd number of landings.

2. **(BEL 1)** Let n be a positive integer. Let $\sigma(n)$ be the sum of the natural divisors d of n (including 1 and n). We say that an integer $m \geq 1$ is *superabundant* (P.Erdös, 1944) if $\forall k \in \{1, 2, \dots, m-1\}$, $\frac{\sigma(m)}{m} > \frac{\sigma(k)}{k}$. Prove that there exists an infinity of superabundant numbers.

3. **(BEL 3)**$^{\text{IMO4}}$ We say that a set E of points of the Euclidian plane is "Pythagorean" if for any partition of E into two sets A and B, at least one of the sets contains the vertices of a right-angled triangle. Decide whether the following sets are Pythagorean:
   (a) a circle;
   (b) an equilateral triangle (that is, the set of three vertices and the points of the three edges).

4. **(BEL 5)** On the sides of the triangle ABC, three similar isosceles triangles ABP ($AP = PB$), AQC ($AQ = QC$), and BRC ($BR = RC$) are constructed. The first two are constructed externally to the triangle ABC, but the third is placed in the same half-plane determined by the line BC as the triangle ABC. Prove that $APRQ$ is a parallelogram.

5. **(BRA 1)** Consider the set of all strictly decreasing sequences of n natural numbers having the property that in each sequence no term divides any other term of the sequence. Let $A = (a_j)$ and $B = (b_j)$ be any two such sequences. We say that A precedes B if for some k, $a_k < b_k$ and $a_i = b_i$ for $i < k$. Find the terms of the first sequence of the set under this ordering.

6. **(CAN 2)** Suppose that $\{x_1, x_2, \dots, x_n\}$ are positive integers for which $x_1 + x_2 + \cdots + x_n = 2(n+1)$. Show that there exists an integer r with $0 \leq r \leq n-1$ for which the following $n-1$ inequalities hold:

$$\begin{aligned} x_{r+1} + \cdots + x_{r+i} &\leq 2i + 1 & \forall i,\ 1 \leq i \leq n-r; \\ x_{r+1} + \cdots + x_n + x_1 + \cdots + x_i &\leq 2(n-r+i) + 1 & \forall i,\ 1 \leq i \leq r-1. \end{aligned}$$

   Prove that if all the inequalities are strict, then r is unique and that otherwise there are exactly two such r.

7. **(CAN 5)** Let a be a positive integer and let $\{a_n\}$ be defined by $a_0 = 0$ and

$$a_{n+1} = (a_n + 1)a + (a+1)a_n + 2\sqrt{a(a+1)a_n(a_n+1)} \qquad (n = 1, 2 \dots).$$

Show that for each positive integer n, a_n is a positive integer.

8. **(SPA 2)** In a test, $3n$ students participate, who are located in three rows of n students in each. The students leave the test room one by one. If $N_1(t)$, $N_2(t)$, $N_3(t)$ denote the numbers of students in the first, second, and third row respectively at time t, find the probability that for each t during the test,

$$|N_i(t) - N_j(t)| < 2, \qquad i \neq j, \qquad i, j = 1, 2, \dots .$$

9. **(USA 3)**$^{\text{IMO6}}$ If a, b, and c are sides of a triangle, prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determine when there is equality.

10. **(FIN 1)** Let p and q be integers. Show that there exists an interval I of length $1/q$ and a polynomial P with integral coefficients such that

$$\left| P(x) - \frac{p}{q} \right| < \frac{1}{q^2}$$

for all $x \in I$.

11. **(FIN 2′)** Let $f : [0,1] \to \mathbb{R}$ be continuous and satisfy:

$$\begin{aligned} bf(2x) &= f(x), & 0 \leq x \leq 1/2; \\ f(x) &= b + (1-b)f(2x-1), & 1/2 \leq x \leq 1, \end{aligned}$$

where $b = \frac{1+c}{2+c}$, $c > 0$. Show that $0 < f(x) - x < c$ for every $x,\ 0 < x < 1$.

12. **(GBR 4)**$^{\text{IMO1}}$ Find all functions f defined on the positive real numbers and taking positive real values that satisfy the following conditions:
(i) $f(xf(y)) = yf(x)$ for all positive real x, y.
(ii) $f(x) \to 0$ as $x \to +\infty$.

13. **(LUX 2)** Let E be the set of 1983^3 points of the space $\mathbb{R}^3$ all three of whose coordinates are integers between 0 and 1982 (including 0 and 1982). A coloring of E is a map from E to the set $\{\text{red}, \text{blue}\}$. How many colorings of E are there satisfying the following property: The number of red vertices among the 8 vertices of any right-angled parallelepiped is a multiple of 4?

14. **(POL 2)**$^{\text{IMO5}}$ Prove or disprove: From the interval $[1, \dots, 30000]$ one can select a set of 1000 integers containing no arithmetic triple (three consecutive numbers of an arithmetic progression).

15. **(POL 3)** Decide whether there exists a set M of natural numbers satisfying the following conditions:
   (i) For any natural number $m > 1$ there are $a, b \in M$ such that $a+b = m$.
   (ii) If $a, b, c, d \in M$, $a, b, c, d > 10$ and $a + b = c + d$, then $a = c$ or $a = d$.

16. **(GDR 1)** Let $F(n)$ be the set of polynomials $P(x) = a_0+a_1x+\cdots+a_nx^n$, with $a_0, a_1, \dots, a_n \in \mathbb{R}$ and $0 \le a_0 = a_n \le a_1 = a_{n-1} \le \cdots \le a_{[n/2]} = a_{[(n+1)/2]}$. Prove that if $f \in F(m)$ and $g \in F(n)$, then $fg \in F(m+n)$.

17. **(GDR 3)** Let $P_1, P_2, \dots, P_n$ be distinct points of the plane, $n \ge 2$. Prove that
$$\max_{1\le i<j\le n} P_iP_j > \frac{\sqrt{3}}{2}(\sqrt{n}-1) \min_{1\le i<j\le n} P_iP_j.$$

18. **(FRG 3)**IMO3 Let a, b, c be positive integers satisfying $(a, b) = (b, c) = (c, a) = 1$. Show that $2abc - ab - bc - ca$ is the largest integer not representable as
$$xbc + yca + zab$$
with nonnegative integers x, y, z.

19. **(ROM 1)** Let $(F_n)_{n\ge1}$ be the Fibonacci sequence $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ $(n \ge 1)$, and $P(x)$ the polynomial of degree 990 satisfying
$$P(k) = F_k, \qquad \text{for } k = 992, \dots, 1982.$$
Prove that $P(1983) = F_{1983} - 1$.

20. **(ROM 3)** Solve the system of equations
$$\begin{aligned} x_1|x_1| &= x_2|x_2| + (x_1 - a)|x_1 - a|, \\ x_2|x_2| &= x_3|x_3| + (x_2 - a)|x_2 - a|, \\ &\cdots \\ x_n|x_n| &= x_1|x_1| + (x_n - a)|x_n - a|, \end{aligned}$$
in the set of real numbers, where $a > 0$.

21. **(SWE 1)** Find the greatest integer less than or equal to $\sum_{k=1}^{2^{1983}} k^{1/1983-1}$.

22. **(SWE 4)** Let n be a positive integer having at least two different prime factors. Show that there exists a permutation $a_1, a_2, \dots, a_n$ of the integers $1, 2, \dots, n$ such that
$$\sum_{k=1}^{n} k \cdot \cos \frac{2\pi a_k}{n} = 0.$$

23. **(USS 1)**IMO2 Let K be one of the two intersection points of the circles W_1 and W_2. Let O_1 and O_2 be the centers of W_1 and W_2. The two common tangents to the circles meet W_1 and W_2 respectively in P_1 and P_2, the first tangent, and Q_1 and Q_2, the second tangent. Let M_1 and M_2 be the midpoints of P_1Q_1 and P_2Q_2, respectively. Prove that
$$\angle O_1KO_2 = \angle M_1KM_2.$$

24. **(USS 2)** Let d_n be the last nonzero digit of the decimal representation of $n!$. Prove that d_n is aperiodic; that is, there do not exist T and n_0 such that for all $n \geq n_0$, $d_{n+T} = d_n$.

25. **(USS 3)** Prove that every partition of 3-dimensional space into three disjoint subsets has the following property: One of these subsets contains all possible distances; i.e., for every $a \in \mathbb{R}_+$, there are points M and N inside that subset such that distance between M and N is exactly a.

3.25 The Twenty-Fifth IMO
Prague, Czechoslovakia, June 29–July 10, 1984

3.25.1 Contest Problems

First Day (July 4)

1. Let x, y, z be nonnegative real numbers with $x + y + z = 1$. Show that
$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

2. Find two positive integers a, b such that none of the numbers $a, b, a+b$ is divisible by 7 and $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

3. In a plane two different points O and A are given. For each point $X \neq O$ of the plane denote by $\alpha(X)$ the angle AOX measured in radians $(0 \leq \alpha(X) < 2\pi)$ and by $C(X)$ the circle with center O and radius $OX + \frac{\alpha(X)}{OX}$. Suppose each point of the plane is colored by one of a finite number of colors. Show that there exists a point X with $\alpha(X) > 0$ such that its color appears somewhere on the circle $C(X)$.

Second Day (July 5)

4. Let $ABCD$ be a convex quadrilateral for which the circle of diameter AB is tangent to the line CD. Show that the circle of diameter CD is tangent to the line AB if and only if the lines BC and AD are parallel.

5. Let d be the sum of the lengths of all diagonals of a convex polygon of n $(n > 3)$ vertices, and let p be its perimeter. Prove that
$$\frac{n-3}{2} < \frac{d}{p} < \frac{1}{2}\left(\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2\right).$$

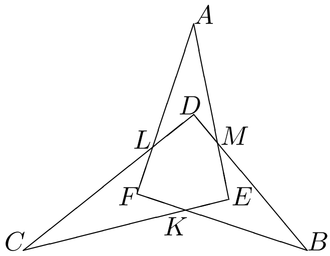
6. Let a, b, c, d be odd positive integers such that $a < b < c < d$, $ad = bc$, and $a + d = 2^k$, $b + c = 2^m$ for some integers k and m. Prove that $a = 1$.

3.25.2 Longlisted Problems

1. **(AUS 1)** The fraction $\frac{3}{10}$ can be written as the sum of two positive fractions with numerator 1 as follows: $\frac{3}{10} = \frac{1}{5} + \frac{1}{10}$ and also $\frac{3}{10} = \frac{1}{4} + \frac{1}{20}$. There are the only two ways in which this can be done.
In how many ways can $\frac{3}{1984}$ be written as the sum of two positive fractions with numerator 1?
Is there a positive integer n, not divisible by 3, such that $\frac{3}{n}$ can be written as the sum of two positive fractions with numerator 1 in exactly 1984 ways?

2. **(AUS 2)** Given a regular convex $2m$-sided polygon P, show that there is a $2m$-sided polygon π with the same vertices as P (but in different order) such that π has exactly one pair of parallel sides.

3. **(AUS 3)** The opposite sides of the reentrant hexagon $AFBDCE$ intersect at the points K, L, M (as shown in the figure). It is given that $AL = AM = a$, $BM = BK = b$, $CK = CL = c$, $LD = DM = d$, $ME = EK = e$, $FK = FL = f$.
   (a) Given length a and the three angles α, β, and γ at the vertices A, B, and C, respectively, satisfying the condition $\alpha + \beta + \gamma < 180°$, show that all the angles and sides of the hexagon are thereby uniquely determined.
   (b) Prove that
   $$\frac{1}{a} + \frac{1}{e} = \frac{1}{b} + \frac{1}{d}.$$
   Easier version of (b). Prove that
   $$(a + f)(b + d)(c + e) = (a + e)(b + f)(c + d).$$



4. **(BEL 1)** Given a triangle ABC, three equilateral triangles AEB, BFC, and CGA are constructed in the exterior of ABC. Prove that:
   (a) $CE = AF = BG$;
   (b) CE, AF, and BG have a common point.

5. **(BEL 2)** For a real number x, let $[x]$ denote the greatest integer not exceeding x. If $m \geq 3$, prove that
$$\left[\frac{m(m+1)}{2(2m-1)}\right] = \left[\frac{m+1}{4}\right].$$

6. **(BEL 3)** Let P, Q, R be the polynomials with real or complex coefficients such that at least one of them is not constant. If $P^n + Q^n + R^n = 0$, prove that $n < 3$.

7. **(BUL 1)** Prove that for any natural number n, the number $\binom{2n}{n}$ divides the least common multiple of the numbers $1, 2, \dots, 2n-1, 2n$.

8. **(BUL 2)** In the plane of a given triangle $A_1A_2A_3$ determine (with proof) a straight line l such that the sum of the distances from A_1, A_2, and A_3 to l is the least possible.

9. **(BUL 3)** The circle inscribed in the triangle $A_1A_2A_3$ is tangent to its sides A_1A_2, A_2A_3, A_3A_1 at points T_1, T_2, T_3, respectively. Denote by M_1, M_2, M_3 the midpoints of the segments A_2A_3, A_3A_1, A_1A_2, respectively. Prove that the perpendiculars through the points M_1, M_2, M_3 to the lines T_2T_3, T_3T_1, T_1T_2 meet at one point.

10. **(BUL 4)** Assume that the bisecting plane of the dihedral angle at edge AB of the tetrahedron $ABCD$ meets the edge CD at point E. Denote by S_1, S_2, S_3, respectively the areas of the triangles ABC, ABE, and ABD. Prove that no tetrahedron exists for which S_1, S_2, S_3 (in this order) form an arithmetic or geometric progression.

11. **(BUL 5)** (SL84-13).

12. **(CAN 1)** (SL84-11).
Original formulation. Suppose that $a_1, a_2, \dots, a_{2n}$ are distinct integers such that
$$(x - a_1)(x - a_2)\cdots(x - a_{2n}) + (-1)^{n-1}(n!)^2 = 0$$
has an integer solution r. Show that $r = \frac{a_1+a_2+\cdots+a_{2n}}{2n}$.

13. **(CAN 2)** (SL84-2).
Original formulation. Let m, n be nonzero integers. Show that $4mn-m-n$ can be a square infinitely many times, but that this never happens when either m or n is positive.
Alternative formulation. Let m, n be positive integers. Show that $4mn - m - n$ can be 1 less than a perfect square infinitely often, but can never be a square.

14. **(CAN 3)** (SL84-6).

15. **(CAN 4)** Consider all the sums of the form
$$\sum_{k=1}^{1985} e_k k^5 = \pm 1^5 \pm 2^5 \pm \cdots \pm 1985^5,$$
where $e_k = \pm 1$. What is the smallest nonnegative value attained by a sum of this type?

16. **(CAN 5)** (SL84-19).

17. **(FRA 1)** (SL84-1).

18. **(FRA 2)** Let c be the inscribed circle of the triangle ABC, d a line tangent to c which does not pass through the vertices of triangle ABC. Prove the existence of points A_1, B_1, C_1, respectively, on the lines BC, CA, AB satisfying the following two properties:
(i) Lines AA_1, BB_1, and CC_1 are parallel.
(ii) Lines AA_1, BB_1, and CC_1 meet d respectively at points A', B', and C' such that
$$\frac{\overline{A'A_1}}{\overline{A'A}} = \frac{\overline{B'B_1}}{\overline{B'B}} = \frac{\overline{C'C_1}}{\overline{C'C}}.$$

19. **(FRA 3)** Let ABC be an isosceles triangle with right angle at point A. Find the minimum of the function F given by
$$F(M) = BM + CM - \sqrt{3}AM.$$

20. **(FRG 1)** (SL84-5).

21. **(FRG 2)**
    (1) Start with a white balls and b black balls.
    (2) Draw one ball at random.
    (3) If the ball is white, then stop. Otherwise, add two black balls and go to step 2.

    Let S be the number of draws before the process terminates. For the cases $a = b = 1$ and $a = b = 2$ only, find $a_n = P(S = n)$, $b_n = P(S \leq n)$, $\lim_{n\to\infty} b_n$, and the expectation value of the number of balls drawn: $E(S) = \sum_{n\geq 1} na_n$.

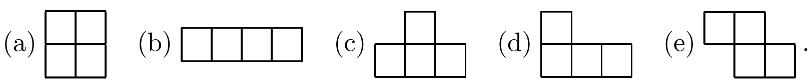
22. **(FRG 3)** (SL84-17).
    Original formulation. In a permutation $(x_1, x_2, \dots, x_n)$ of the set $1, 2, \dots, n$ we call a pair (x_i, x_j) *discordant* if $i < j$ and $x_i > x_j$. Let $d(n, k)$ be the number of such permutations with exactly k discordant pairs.
    (a) Find $d(n, 2)$.
    (b) Show that

    $$d(n, k) = d(n, k - 1) + d(n - 1, k) - d(n - 1, k - 1)$$

    with $d(n, k) = 0$ for $k < 0$ and $d(n, 0) = 1$ for $n \geq 1$. Compute with this recursion a table of $d(n, k)$ for $n = 1$ to 6.

23. **(FRG 4)** A $2 \times 2 \times 12$ box fixed in space is to be filled with twenty-four $1 \times 1 \times 2$ bricks. In how many ways can this be done?

24. **(FRG 5)** (SL84-7).
    Original formulation. Consider several types of 4-cell figures:



    Find, with proof, for which of these types of figures it is not possible to number the fields of the 8×8 chessboard using the numbers $1, 2, \dots, 64$ in such a way that the sum of the four numbers in each of its parts congruent to the given figure is divisible by 4.

25. **(GBR 1)** (SL84-10).

26. **(GBR 2)** A cylindrical container has height 6 cm and radius 4 cm. It rests on a circular hoop, also of radius 4 cm, fixed in a horizontal plane with its axis vertical and with each circular rim of the cylinder touching the hoop at two points.
    The cylinder is now moved so that each of its circular rims still touches the hoop in two points. Find with proof the locus of one of the cylinder's vertical ends.

27. **(GBR 3)** The function $f(n)$ is defined on the nonnegative integers n by: $f(0) = 0$, $f(1) = 1$,

$$f(n) = f\left(n - \frac{1}{2}m(m-1)\right) - f\left(\frac{1}{2}m(m+1) - n\right),$$

for $\frac{1}{2}m(m-1) < n \le \frac{1}{2}m(m+1)$, $m \ge 2$. Find the smallest integer n for which $f(n) = 5$.

28. **(GBR 4)** A "number triangle" (t_{nk}) $(0 \le k \le n)$ is defined by $t_{n,0} = t_{n,n} = 1$ $(n \ge 0)$,

$$t_{n+1,m} = \left(2 - \sqrt{3}\right)^m t_{n,m} + \left(2 + \sqrt{3}\right)^{n-m+1} t_{n,m-1} \qquad (1 \le m \le n).$$

Prove that all $t_{n,m}$ are integers.

29. **(GDR 1)** Let $S_n = \{1, \dots, n\}$ and let f be a function that maps every subset of S_n into a positive real number and satisfies the following condition: For all $A \subseteq S_n$ and $x, y \in S_n$, $x \ne y$, $f(A \cup \{x\}) f(A \cup \{y\}) \le f(A \cup \{x, y\}) f(A)$.
Prove that for all $A, B \subseteq S_n$ the following inequality holds:

$$f(A) \cdot f(B) \le f(A \cup B) \cdot f(A \cap B).$$

30. **(GDR 2)** Decide whether it is possible to color the 1984 natural numbers $1, 2, 3, \dots, 1984$ using 15 colors so that no geometric sequence of length 3 of the same color exists.

31. **(LUX 1)** Let $f_1(x) = x^3 + a_1x^2 + b_1x + c_1 = 0$ be an equation with three positive roots $\alpha > \beta > \gamma > 0$. From the equation $f_1(x) = 0$ one constructs the equation $f_2(x) = x^3 + a_2x^2 + b_2x + c_2 = x(x + b_1)^2 - (a_1x + c_1)^2 = 0$. Continuing this process, we get equations $f_3, \dots, f_n$. Prove that

$$\lim_{n\to\infty} \sqrt[2^{n-1}]{-a_n} = \alpha.$$

32. **(LUX 2)** (SL84-15).

33. **(MON 1)** (SL84-4).

34. **(MON 2)** One country has n cities and every two of them are linked by a railroad. A railway worker should travel by train exactly once through the entire railroad system (reaching each city exactly once). If it is impossible for worker to travel by train between two cities, he can travel by plane. What is the minimal number of flights that the worker will have to use?

35. **(MON 3)** Prove that there exist distinct natural numbers $m_1, m_2, \dots, m_k$ satisfying the conditions

$$\pi^{-1984} < 25 - \left(\frac{1}{m_1} + \frac{1}{m_2} + \dots + \frac{1}{m_k}\right) < \pi^{-1960}$$

where π is the ratio between circle and its diameter.

36. **(MON 4)** The set $\{1, 2, \dots, 49\}$ is divided into three subsets. Prove that at least one of these subsets contains three different numbers a, b, c such that $a + b = c$.

37. **(MOR 1)** Denote by $[x]$ the greatest integer not exceeding x. For all real $k > 1$, define two sequences:

$$a_n(k) = [nk] \quad \text{and} \quad b_n(k) = \left[\frac{nk}{k-1}\right].$$

If $A(k) = \{a_n(k) : n \in \mathbb{N}\}$ and $B(k) = \{b_n(k) : n \in \mathbb{N}\}$, prove that $A(k)$ and $B(k)$ form a partition of $\mathbb{N}$ if and only if k is irrational.

38. **(MOR 2)** Determine all continuous functions f such that

$$\left(\forall (x, y) \in \mathbb{R}^2\right) \qquad f(x+y)f(x-y) = \left(f(x)f(y)\right)^2.$$

39. **(MOR 3)** Let ABC be an isosceles triangle, $AB = AC$, $\angle A = 20^\circ$. Let D be a point on AB, and E a point on AC such that $\angle ACD = 20^\circ$ and $\angle ABE = 30^\circ$. What is the measure of the angle $\angle CDE$?

40. **(NET 1)** (SL84-12).

41. **(NET 2)** Determine positive integers p, q, and r such that the diagonal of a block consisting of $p \times q \times r$ unit cubes passes through exactly 1984 of the unit cubes, while its length is minimal. (The diagonal is said to pass through a unit cube if it has more than one point in common with the unit cube.)

42. **(NET 3)** Triangle ABC is given for which $BC = AC + \frac{1}{2}AB$. The point P divides AB such that $RP : PA = 1 : 3$. Prove that $\angle CAP = 2\angle CPA$.

43. **(POL 1)** (SL84-16).

44. **(POL 2)** (SL84-9).

45. **(POL 3)** Let X be an arbitrary nonempty set contained in the plane and let sets $A_1, A_2, \dots, A_m$ and $B_1, B_2, \dots, B_n$ be its images under parallel translations. Let us suppose that

$$A_1 \cup A_2 \cup \cdots \cup A_m \subset B_1 \cup B_2 \cup \cdots \cup B_n$$

and that the sets $A_1, A_2, \dots, A_m$ are disjoint. Prove that $m \le n$.

46. **(ROM 1)** Let $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ be two sequences of natural numbers such that $a_{n+1} = na_n + 1$, $b_{n+1} = nb_n - 1$ for every $n \ge 1$. Show that these two sequences can have only a finite number of terms in common.

47. **(ROM 2)** (SL84-8).

48. **(ROM 3)** Let ABC be a triangle with interior angle bisectors AA_1, BB_1, CC_1 and incenter I. If $\sigma[IA_1B] + \sigma[IB_1C] + \sigma[IC_1A] = \frac{1}{2}\sigma[ABC]$, where $\sigma[ABC]$ denotes the area of ABC, show that ABC is isosceles.

49. **(ROM 4)** Let $n > 1$ and $x_i \in \mathbb{R}$ for $i = 1, \dots, n$. Set $S_k = x_1^k + x_2^k + \cdots + x_n^k$ for $k \geq 1$. If $S_1 = S_2 = \cdots = S_{n+1}$, show that $x_i \in \{0, 1\}$ for every $i = 1, 2, \dots, n$.

50. **(ROM 5)** (SL84-14).

51. **(SPA 1)** Two cyclists leave simultaneously a point P in a circular runway with constant velocities v_1, v_2 ($v_1 > v_2$) and in the same sense. A pedestrian leaves P at the same time, moving with velocity $v_3 = \frac{v_1+v_2}{12}$. If the pedestrian and the cyclists move in opposite directions, the pedestrian meets the second cyclist 91 seconds after he meets the first. If the pedestrian moves in the same direction as the cyclists, the first cyclist overtakes him 187 seconds before the second does. Find the point where the first cyclist overtakes the second cyclist the first time.

52. **(SPA 2)** Construct a scalene triangle such that
$$a(\tan B - \tan C) = b(\tan A - \tan C).$$

53. **(SPA 3)** Find a sequence of natural numbers a_i such that $a_i = \sum_{r=1}^{i+4} d_r$, where $d_r \neq d_s$ for $r \neq s$ and d_r divides a_i.

54. **(SPA 4)** Let P be a convex planar polygon with equal angles. Let $l_1, \dots, l_n$ be its sides. Show that a necessary and sufficient condition for P to be regular is that the sum of the ratios $\frac{l_i}{l_{i+1}}$ ($i = 1, \dots, n$; $l_{n+1} = l_1$) equals the number of sides.

55. **(SPA 5)** Let a, b, c be natural numbers such that $a+b+c = 2pq(p^{30}+q^{30})$, $p > q$ being two given positive integers.
(a) Prove that $k = a^3 + b^3 + c^3$ is not a prime number.
(b) Prove that if $a \cdot b \cdot c$ is maximum, then 1984 divides k.

56. **(SWE 1)** Let a, b, c be nonnegative integers such that $a \leq b \leq c$, $2b \neq a + c$ and $\frac{a+b+c}{3}$ is an integer. Is it possible to find three nonnegative integers d, e, and f such that $d \leq e \leq f$, $f \neq c$, and such that $a^2+b^2+c^2 = d^2 + e^2 + f^2$?

57. **(SWE 2)** Let a, b, c, d be a permutation of the numbers $1, 9, 8, 4$ and let $n = (10a + b)^{10c+d}$. Find the probability that $1984!$ is divisible by n.

58. **(SWE 3)** Let $(a_n)_1^\infty$ be a sequence such that $a_n \leq a_{n+m} \leq a_n + a_m$ for all positive integers n and m. Prove that $\frac{a_n}{n}$ has a limit as n approaches infinity.

59. **(USA 1)** Determine the smallest positive integer m such that $529^n + m \cdot 132^n$ is divisible by 262417 for all odd positive integers n.

60. **(USA 2)** (SL84-20).

61. **(USA 3)** A fair coin is tossed repeatedly until there is a run of an odd number of heads followed by a tail. Determine the expected number of tosses.

62. **(USA 4)** From a point P exterior to a circle K, two rays are drawn intersecting K in the respective pairs of points A, A' and B, B'. For any other pair of points C, C' on K, let D be the point of intersection of the circumcircles of triangles PAC and $PB'C'$ other than point P. Similarly, let D' be the point of intersection of the circumcircles of triangles $PA'C'$ and PBC other than point P. Prove that the points P, D, and D' are collinear.

63. **(USA 5)** (SL84-18).

64. **(USS 1)** For a matrix (p_{ij}) of the format $m \times n$ with real entries, set

$$a_i = \sum_{j=1}^{n} p_{ij} \text{ for } i = 1, \dots, m \quad \text{and} \quad b_j = \sum_{i=1}^{m} p_{ij} \text{ for } j = 1, \dots, n. \qquad (1)$$

By *integering* a real number we mean replacing the number with the integer closest to it.
Prove that integering the numbers a_i, b_j, p_{ij} can be done in such a way that (1) still holds.

65. **(USS 2)** A tetrahedron is inscribed in a sphere of radius 1 such that the center of the sphere is inside the tetrahedron.
Prove that the sum of lengths of all edges of the tetrahedron is greater than 6.

66. **(USS 3)** (SL84-3).
Original formulation. All the divisors of a positive integer n arranged in increasing order are $x_1 < x_2 < \cdots < x_k$. Find all such numbers n for which $x_5^2 + x_6^2 - 1 = n$.

67. **(USS 4)** With the medians of an acute-angled triangle another triangle is constructed. If R and R_m are the radii of the circles circumscribed about the first and the second triangle, respectively, prove that

$$R_m > \frac{5}{6} R.$$

68. **(USS 5)** In the Martian language every finite sequence of letters of the Latin alphabet letters is a word. The publisher "Martian Words" makes a collection of all words in many volumes. In the first volume there are only one-letter words, in the second, two-letter words, etc., and the numeration of the words in each of the volumes continues the numeration of the previous volume. Find the word whose numeration is equal to the sum of numerations of the words *Prague*, *Olympiad*, *Mathematics*.

3.25.3 Shortlisted Problems

1. **(FRA 1)** Find all solutions of the following system of n equations in n variables:

$$\begin{aligned}
x_1|x_1| - (x_1 - a)|x_1 - a| &= x_2|x_2|, \\
x_2|x_2| - (x_2 - a)|x_2 - a| &= x_3|x_3|, \\
&\cdots \\
x_n|x_n| - (x_n - a)|x_n - a| &= x_1|x_1|,
\end{aligned}$$

where a is a given number.

2. **(CAN 2)** Prove:
   (a) There are infinitely many triples of positive integers m, n, p such that $4mn - m - n = p^2 - 1$.
   (b) There are no positive integers m, n, p such that $4mn - m - n = p^2$.

3. **(USS 3)** Find all positive integers n such that

$$n = d_6^2 + d_7^2 - 1,$$

where $1 = d_1 < d_2 < \cdots < d_k = n$ are all positive divisors of the number n.

4. **(MON 1)**$^{\text{IMO5}}$ Let d be the sum of the lengths of all diagonals of a convex polygon of n $(n > 3)$ vertices and let p be its perimeter. Prove that

$$\frac{n-3}{2} < \frac{d}{p} < \frac{1}{2}\left(\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2\right).$$

5. **(FRG 1)**$^{\text{IMO1}}$ Let x, y, z be nonnegative real numbers with $x+y+z = 1$. Show that

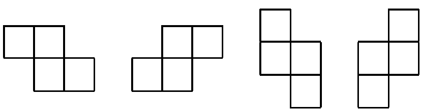
$$0 \le xy + yz + zx - 2xyz \le \frac{7}{27}.$$

6. **(CAN 3)** Let c be a positive integer. The sequence $\{f_n\}$ is defined as follows:

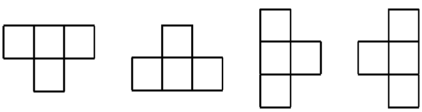
$$f_1 = 1, \quad f_2 = c, \quad f_{n+1} = 2f_n - f_{n-1} + 2 \quad (n \ge 2).$$

Show that for each $k \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that $f_k f_{k+1} = f_r$.

7. **(FRG 5)**
   (a) Decide whether the fields of the 8×8 chessboard can be numbered by the numbers $1, 2, \dots, 64$ in such a way that the sum of the four numbers in each of its parts of one of the forms



   is divisible by four.
   (b) Solve the analogous problem for



8. **(ROM 2)**IMO3 In a plane two different points O and A are given. For each point $X \neq O$ of the plane denote by $\alpha(X)$ the angle AOX measured in radians ($0 \leq \alpha(X) < 2\pi$) and by $C(X)$ the circle with center O and radius $OX + \frac{\alpha(X)}{OX}$. Suppose each point of the plane is colored by one of a finite number of colors. Show that there exists a point X with $\alpha(X) > 0$ such that its color appears somewhere on the circle $C(X)$.

9. **(POL 2)** Let a, b, c be positive numbers with $\sqrt{a} + \sqrt{b} + \sqrt{c} = \frac{\sqrt{3}}{2}$. Prove that the system of equations

$$\begin{aligned}\sqrt{y-a} + \sqrt{z-a} &= 1,\\ \sqrt{z-b} + \sqrt{x-b} &= 1,\\ \sqrt{x-c} + \sqrt{y-c} &= 1,\end{aligned}$$

has exactly one solution (x, y, z) in real numbers.

10. **(GBR 1)** Prove that the product of five consecutive positive integers cannot be the square of an integer.

11. **(CAN 1)** Let n be a natural number and $a_1, a_2, \dots, a_{2n}$ mutually distinct integers. Find all integers x satisfying

$$(x - a_1) \cdot (x - a_2) \cdots (x - a_{2n}) = (-1)^n (n!)^2.$$

12. **(NET 1)**IMO2 Find two positive integers a, b such that none of the numbers $a, b, a+b$ is divisible by 7 and $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

13. **(BUL 5)** Prove that the volume of a tetrahedron inscribed in a right circular cylinder of volume 1 does not exceed $\frac{2}{3\pi}$.

14. **(ROM 5)**IMO4 Let $ABCD$ be a convex quadrilateral for which the circle with diameter AB is tangent to the line CD. Show that the circle with diameter CD is tangent to the line AB if and only if the lines BC and AD are parallel.

15. **(LUX 2)** Angles of a given triangle ABC are all smaller than $120°$. Equilateral triangles AFB, BDC and CEA are constructed in the exterior of $\triangle ABC$.
    (a) Prove that the lines AD, BE, and CF pass through one point S.
    (b) Prove that $SD + SE + SF = 2(SA + SB + SC)$.

16. **(POL 1)**IMO6 Let a, b, c, d be odd positive integers such that $a < b < c < d$, $ad = bc$, and $a + d = 2^k$, $b + c = 2^m$ for some integers k and m. Prove that $a = 1$.

17. **(FRG 3)** In a permutation $(x_1, x_2, \dots, x_n)$ of the set $1, 2, \dots, n$ we call a pair (x_i, x_j) *discordant* if $i < j$ and $x_i > x_j$. Let $d(n, k)$ be the number of such permutations with exactly k discordant pairs. Find $d(n, 2)$ and $d(n, 3)$.

18. **(USA 5)** Inside triangle ABC there are three circles k_1, k_2, k_3 each of which is tangent to two sides of the triangle and to its incircle k. The radii of k_1, k_2, k_3 are 1, 4, and 9. Determine the radius of k.

19. **(CAN 5)** The triangular array $(a_{n,k})$ of numbers is given by $a_{n,1} = 1/n$, for $n = 1, 2, \ldots$, $a_{n,k+1} = a_{n-1,k} - a_{n,k}$, for $1 \le k \le n-1$. Find the harmonic mean of the 1985th row.

20. **(USA 2)** Determine all pairs (a, b) of positive real numbers with $a \neq 1$ such that
$$\log_a b < \log_{a+1}(b+1).$$

3.26 The Twenty-Sixth IMO
Joutsa, Finland, June 29–July 11, 1985

3.26.1 Contest Problems

First Day (July 4)

1. A circle whose center is on the side ED of the cyclic quadrilateral $BCDE$ touches the other three sides. Prove that $EB + CD = ED$.

2. Each of the numbers in the set $N = \{1, 2, 3, \dots, n-1\}$, where $n \geq 3$, is colored with one of two colors, say red or black, so that:
   (i) i and $n - i$ always receive the same color, and
   (ii) for some $j \in N$ relatively prime to n, i and $|j - i|$ receive the same color for all $i \in N$, $i \neq j$.
   Prove that all numbers in N must receive the same color.

3. The weight $w(p)$ of a polynomial p, $p(x) = \sum_{i=0}^{n} a_i x^i$, with integer coefficients a_i is defined as the number of its odd coefficients. For $i = 0, 1, 2, \dots$, let $q_i(x) = (1+x)^i$. Prove that for any finite sequence $0 \leq i_1 < i_2 < \cdots < i_n$ the inequality
$$w(q_{i_1} + \cdots + q_{i_n}) \geq w(q_{i_1})$$
holds.

Second Day (July 5)

4. Given a set M of 1985 positive integers, none of which has a prime divisor larger than 26, prove that M has four distinct elements whose geometric mean is an integer.

5. A circle with center O passes through points A and C and intersects the sides AB and BC of the triangle ABC at points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at two distinct points B and M. Prove that $\measuredangle OMB = 90^\circ$.

6. The sequence $f_1, f_2, \dots, f_n, \dots$ of functions is defined for $x > 0$ recursively by
$$f_1(x) = x, \qquad f_{n+1}(x) = f_n(x)\left(f_n(x) + \frac{1}{n}\right).$$
Prove that there exists one and only one positive number a such that $0 < f_n(a) < f_{n+1}(a) < 1$ for all integers $n \geq 1$.

3.26.2 Longlisted Problems

1. **(AUS 1)** (SL85-4).

2. **(AUS 2)** We are given a triangle ABC and three rectangles R_1, R_2, R_3 with sides parallel to two fixed perpendicular directions and such that their union covers the sides AB, BC, and CA; i.e., each point on the perimeter of ABC is contained in or on at least one of the rectangles. Prove that all points inside the triangle are also covered by the union of R_1, R_2, R_3.

3. **(AUS 3)** A function f has the following property: If $k > 1$, $j > 1$, and $(k, j) = m$, then $f(kj) = f(m)\,(f(k/m) + f(j/m))$. What values can $f(1984)$ and $f(1985)$ take?

4. **(BEL 1)** Let x, y, and z be real numbers satisfying $x + y + z = xyz$. Prove that

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) = 4xyz.$$

5. **(BEL 2)** (SL85-16).

6. **(BEL 3)** On a one-way street, an unending sequence of cars of width a, length b passes with velocity v. The cars are separated by the distance c. A pedestrian crosses the street perpendicularly with velocity w, without paying attention to the cars.
   (a) What is the probability that the pedestrian crosses the street uninjured?
   (b) Can he improve this probability by crossing the road in a direction other than perpendicular?

7. **(BRA 1)** A convex quadrilateral is inscribed in a circle of radius 1. Prove that the difference between its perimeter and the sum of the lengths of its diagonals is greater than zero and less than 2.

8. **(BRA 2)** Let K be a convex set in the xy-plane, symmetric with respect to the origin and having area greater than 4. Prove that there exists a point $(m, n) \neq (0, 0)$ in K such that m and n are integers.

9. **(BRA 3)** (SL85-2).

10. **(BUL 1)** (SL85-13).

11. **(BUL 2)** Let a and b be integers and n a positive integer. Prove that

$$\frac{b^{n-1}a(a+b)(a+2b)\cdots(a+(n-1)b)}{n!}$$

is an integer.

12. **(CAN 1)** Find the maximum value of

$$\sin^2\theta_1 + \sin^2\theta_2 + \cdots + \sin^2\theta_n$$

subject to the restrictions $0 \le \theta_i \le \pi$, $\theta_1 + \theta_2 + \cdots + \theta_n = \pi$.

13. **(CAN 2)** Find the average of the quantity

$$(a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_{n-1} - a_n)^2$$

taken over all permutations $(a_1, a_2, \dots, a_n)$ of $(1, 2, \dots, n)$.

14. **(CAN 3)** Let k be a positive integer. Define $u_0 = 0$, $u_1 = 1$, and $u_n = ku_{n-1} - u_{n-2}$, $n \geq 2$. Show that for each integer n, the number $u_1^3 + u_2^3 + \cdots + u_n^3$ is a multiple of $u_1 + u_2 + \cdots + u_n$.

15. **(CAN 4)** *Superchess* is played on on a 12×12 board, and it uses *superknights*, which move between opposite corner cells of any 3×4 subboard. Is it possible for a superknight to visit every other cell of a superchessboard exactly once and return to its starting cell?

16. **(CAN 5)** (SL85-18).

17. **(CUB 1)** Set
$$A_n = \sum_{k=1}^{n} \frac{k^6}{2^k}.$$
Find $\lim_{n \to \infty} A_n$.

18. **(CYP 1)** The circles (R, r) and (P, ρ), where $r > \rho$, touch externally at A. Their direct common tangent touches (R, r) at B and (P, ρ) at C. The line RP meets the circle (P, ρ) again at D and the line BC at E. If $|BC| = 6|DE|$, prove that:
(a) the lengths of the sides of the triangle RBE are in an arithmetic progression, and
(b) $|AB| = 2|AC|$.

19. **(CYP 2)** Solve the system of simultaneous equations
$$\begin{aligned} \sqrt{x} - 1/y - 2w + 3z &= 1, \\ x + 1/y^2 - 4w^2 - 9z^2 &= 3, \\ x\sqrt{x} - 1/y^3 - 8w^3 + 27z^3 &= -5, \\ x^2 + 1/y^4 - 16w^4 - 81z^4 &= 15. \end{aligned}$$

20. **(CZS 1)** Let T be the set of all lattice points (i.e., all points with integer coordinates) in three-dimensional space. Two such points (x, y, z) and (u, v, w) are called *neighbors* if $|x - u| + |y - v| + |z - w| = 1$. Show that there exists a subset S of T such that for each $p \in T$, there is exactly one point of S among p and its neighbors.

21. **(CZS 2)** Let A be a set of positive integers such that for any two elements x, y of A, $|x - y| \geq \frac{xy}{25}$. Prove that A contains at most nine elements. Give an example of such a set of nine elements.

22. **(CZS 3)** (SL85-7).

23. **(CZS 4)** Let $\mathbb{N} = \{1, 2, 3, \dots\}$. For real x, y, set $S(x, y) = \{s \mid s = [nx + y], n \in \mathbb{N}\}$. Prove that if $r > 1$ is a rational number, there exist real numbers u and v such that

$$S(r,0) \cap S(u,v) = \emptyset, \ S(r,0) \cup S(u,v) = \mathbb{N}.$$

24. **(FRA 1)** Let $d \geq 1$ be an integer that is not the square of an integer. Prove that for every integer $n \geq 1$,
$$(n\sqrt{d}+1)|\sin(n\pi\sqrt{d})| \geq 1.$$
25. **(FRA 2)** Find eight positive integers $n_1, n_2, \dots, n_8$ with the following property: For every integer k, $-1985 \leq k \leq 1985$, there are eight integers $\alpha_1, \alpha_2, \dots, \alpha_8$, each belonging to the set $\{-1, 0, 1\}$, such that $k = \sum_{i=1}^{8} \alpha_i n_i$.
26. **(FRA 3)** (SL85-15).
27. **(FRA 4)** Let O be a point on the oriented Euclidean plane and $(\mathbf{i}, \mathbf{j})$ a directly oriented orthonormal basis. Let C be the circle of radius 1, centered at O. For every real number t and nonnegative integer n let M_n be the point on C for which $\langle \mathbf{i}, \overrightarrow{OM_n} \rangle = \cos 2^n t$ (or $\overrightarrow{OM_n} = \cos 2^n t\mathbf{i} + \sin 2^n t\mathbf{j}$). Let $k \geq 2$ be an integer. Find all real numbers $t \in [0, 2\pi)$ that satisfy
   (i) $M_0 = M_k$, and
   (ii) if one starts from M_0 and goes once around C in the positive direction, one meets successively the points $M_0, M_1, \dots, M_{k-2}, M_{k-1}$, in this order.
28. **(FRG 1)** Let M be the set of the lengths of an octahedron whose sides are congruent quadrangles. Prove that M has at most three elements.
   (FRG 1a) Let an octahedron whose sides are congruent quadrangles be given. Prove that each of these quadrangles has two equal sides meeting at a common vertex.
29. **(FRG 2)** Call a four-digit number $(xyzt)_B$ in the number system with base B *stable* if $(xyzt)_B = (dcba)_B - (abcd)_B$, where $a \leq b \leq c \leq d$ are the digits of $(xyzt)_B$ in ascending order. Determine all stable numbers in the number system with base B.
   (FRG 2a) The same problem with $B = 1985$.
   (FRG 2b) With assumptions as in FRG 2, determine the number of bases $B \leq 1985$ such that there is a stable number with base B.
30. **(GBR 1)** A plane rectangular grid is given and a "rational point" is defined as a point (x, y) where x and y are both rational numbers. Let A, B, A', B' be four distinct rational points. Let P be a point such that $\frac{A'B'}{AB} = \frac{B'P}{BC} = \frac{PA'}{PA}$. In other words, the triangles ABP, $A'B'P$ are directly or oppositely similar. Prove that P is in general a rational point and find the exceptional positions of A' and B' relative to A and B such that there exists a P that is not a rational point.
31. **(GBR 2)** Let E_1, E_2, and E_3 be three mutually intersecting ellipses, all in the same plane. Their foci are respectively $F_2, F_3; F_3, F_1$; and F_1, F_2. The three foci are not on a straight line. Prove that the common chords of each pair of ellipses are concurrent.

32. **(GBR 3)** A collection of $2n$ letters contains 2 each of n different letters. The collection is partitioned into n pairs, each pair containing 2 letters, which may be the same or different. Denote the number of distinct partitions by u_n. (Partitions differing in the order of the pairs in the partition or in the order of the two letters in the pairs are not considered distinct.) Prove that $u_{n+1} = (n+1)u_n - \frac{n(n-1)}{2}u_{n-2}$.
(GBR 3a) A pack of n cards contains n pairs of 2 identical cards. It is shuffled and 2 cards are dealt to each of n different players. Let p_n be the probability that every one of the n players is dealt two identical cards. Prove that $\frac{1}{p_{n+1}} = \frac{n+1}{p_n} - \frac{n(n-1)}{2p_{n-2}}$.

33. **(GBR 4)** (SL85-12).

34. **(GBR 5)** (SL85-20).

35. **(GDR 1)** We call a coloring f of the elements in the set $M = \{(x, y) \mid x = 0, 1, \dots, kn-1;\ y = 0, 1, \dots, ln-1\}$ with n colors *allowable* if every color appears exactly k and l times in each row and column and there are no rectangles with sides parallel to the coordinate axes such that all the vertices in M have the same color. Prove that every allowable coloring f satisfies $kl \le n(n+1)$.

36. **(GDR 2)** Determine whether there exist 100 distinct lines in the plane having exactly 1985 distinct points of intersection.

37. **(GDR 3)** Prove that a triangle with angles α, β, γ, circumradius R, and area A satisfies
$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \le \frac{9R^2}{4A}.$$

38. **(IRE 1)** (SL85-21).

39. **(IRE 2)** Given a triangle ABC and external points X, Y, and Z such that $\measuredangle BAZ = \measuredangle CAY$, $\measuredangle CBX = \measuredangle ABZ$, and $\measuredangle ACY = \measuredangle BCX$, prove that AX, BY, and CZ are concurrent.

40. **(IRE 3)** Each of the numbers $x_1, x_2, \dots, x_n$ equals 1 or -1 and
$$\begin{aligned} &x_1x_2x_3x_4 + x_2x_3x_4x_5 + \cdots + x_{n-3}x_{n-2}x_{n-1}x_n \\ &\quad + x_{n-2}x_{n-1}x_nx_1 + x_{n-1}x_nx_1x_2 + x_nx_1x_2x_3 = 0. \end{aligned}$$
Prove that n is divisible by 4.

41. **(IRE 4)** (SL85-14).

42. **(ISR 1)** Prove that the product of two sides of a triangle is always greater than the product of the diameters of the inscribed circle and the circumscribed circle.

43. **(ISR 2)** Suppose that 1985 points are given inside a unit cube. Show that one can always choose 32 of them in such a way that every (possibly

degenerate) closed polygon with these points as vertices has a total length of less than $8\sqrt{3}$.

44. **(ISR 3)** (SL85-19).

45. **(ITA 1)** Two persons, X and Y, play with a die. X wins a game if the outcome is 1 or 2; Y wins in the other cases. A player wins a match if he wins two consecutive games. For each player determine the probability of winning a match within 5 games. Determine the probabilities of winning in an unlimited number of games. If X bets 1, how much must Y bet for the game to be fair?

46. **(ITA 2)** Let C be the curve determined by the equation $y = x^3$ in the rectangular coordinate system. Let t be the tangent to C at a point P of C; t intersects C at another point Q. Find the equation of the set L of the midpoints M of PQ as P describes C. Is the correspondence associating P and M a bijection of C on L? Find a similarity that transforms C into L.

47. **(ITA 3)** Let F be the correspondence associating with every point $P = (x, y)$ the point $P' = (x', y')$ such that

$$x' = ax + b, \qquad y' = ay + 2b. \tag{1}$$

Show that if $a \neq 1$, all lines PP' are concurrent. Find the equation of the set of points corresponding to $P = (1, 1)$ for $b = a^2$. Show that the composition of two mappings of type (1) is of the same type.

48. **(ITA 4)** In a given country, all inhabitants are knights or knaves. A knight never lies; a knave always lies. We meet three persons, A, B, and C. Person A says, "If C is a knight, B is a knave." Person C says, "A and I are different; one is a knight and the other is a knave." Who are the knights, and who are the knaves?

49. **(MON 1)** (SL85-1).

50. **(MON 2)** From each of the vertices of a regular n-gon a car starts to move with constant speed along the perimeter of the n-gon in the same direction. Prove that if all the cars end up at a vertex A at the same time, then they never again meet at any other vertex of the n-gon. Can they meet again at A?

51. **(MON 3)** Let $f_1 = (a_1, a_2, \dots, a_n)$, $n > 2$, be a sequence of integers. From f_1 one constructs a sequence f_k of sequences as follows: if $f_k = (c_1, c_2, \dots, c_n)$, then $f_{k+1} = (c_{i_1}, c_{i_2}, c_{i_3} + 1, c_{i_4} + 1, \dots, c_{i_n} + 1)$, where $(c_{i_1}, c_{i_2}, \dots, c_{i_n})$ is a permutation of $(c_1, c_2, \dots, c_n)$. Give a necessary and sufficient condition for f_1 under which it is possible for f_k to be a constant sequence $(b_1, b_2, \dots, b_n)$, $b_1 = b_2 = \dots = b_n$, for some k.

52. **(MON 4)** In the triangle ABC, let B_1 be on AC, E on AB, G on BC, and let EG be parallel to AC. Furthermore, let EG be tangent to the

inscribed circle of the triangle ABB_1 and intersect BB_1 at F. Let r, r_1, and r_2 be the inradii of the triangles ABC, ABB_1, and BFG, respectively. Prove that $r = r_1 + r_2$.

53. **(MON 5)** For each P inside the triangle ABC, let $A(P)$, $B(P)$, and $C(P)$ be the points of intersection of the lines AP, BP, and CP with the sides opposite to A, B, and C, respectively. Determine P in such a way that the area of the triangle $A(P)B(P)C(P)$ is as large as possible.

54. **(MOR 1)** Set $S_n = \sum_{p=1}^{n}(p^5 + p^7)$. Determine the greatest common divisor of S_n and S_{3n}.

55. **(MOR 2)** The points A, B, C are in this order on line D, and $AB = 4BC$. Let M be a variable point on the perpendicular to D through C. Let MT_1 and MT_2 be tangents to the circle with center A and radius AB. Determine the locus of the orthocenter of the triangle MT_1T_2.

56. **(MOR 3)** Let $ABCD$ be a rhombus with angle $\angle A = 60^\circ$. Let E be a point, different from D, on the line AD. The lines CE and AB intersect at F. The lines DF and BE intersect at M. Determine the angle $\measuredangle BMD$ as a function of the position of E on AD.

57. **(NET 1)** The solid S is defined as the intersection of the six spheres with the six edges of a regular tetrahedron T, with edge length 1, as diameters. Prove that S contains two points at a distance $\frac{1}{\sqrt{6}}$.
(NET 1a) Using the same assumptions, prove that no pair of points in S has a distance larger than $\frac{1}{\sqrt{6}}$.

58. **(NET 2)** Prove that there are infinitely many pairs (k, N) of positive integers such that $1 + 2 + \cdots + k = (k + 1) + (k + 2) + \cdots + N$.

59. **(NET 3)** (SL85-3).

60. **(NOR 1)** The sequence (s_n), where $s_n = \sum_{k=1}^{n} \sin k$, $n = 1, 2, \ldots$, is bounded. Find an upper and lower bound.

61. **(NOR 2)** Consider the set $A = \{0, 1, 2, \ldots, 9\}$ and let $(B_1, B_2, \ldots, B_k)$ be a collection of nonempty subsets of A such that $B_i \cap B_j$ has at most two elements for $i \neq j$. What is the maximal value of k?

62. **(NOR 3)** A "large" circular disk is attached to a vertical wall. It rotates clockwise with one revolution per minute. An insect lands on the disk and immediately starts to climb vertically upward with constant speed $\frac{\pi}{3}$ cm per second (relative to the disk). Describe the path of the insect
(a) relative to the disk;
(b) relative to the wall.

63. **(POL 1)** (SL85-6).

64. **(POL 2)** Let p be a prime. For which k can the set $\{1, 2, \ldots, k\}$ be partitioned into p subsets with equal sums of elements?

65. **(POL 3)** Define the functions $f,\ F : \mathbb{N} \to \mathbb{N}$, by

$$f(n) = \left[\frac{3-\sqrt{5}}{2}n\right], \quad F(k) = \min\{n \in \mathbb{N} | f^k(n) > 0\},$$

where $f^k = f \circ \cdots \circ f$ is f iterated n times. Prove that $F(k+2) = 3F(k+1) - F(k)$ for all $k \in \mathbb{N}$.

66. **(ROM 1)** (SL85-5).

67. **(ROM 2)** Let $k \geq 2$ and $n_1, n_2, \dots, n_k \geq 1$ natural numbers having the property $n_2 \mid 2^{n_1} - 1$, $n_3 \mid 2^{n_2} - 1, \dots, n_k \mid 2^{n_{k-1}} - 1$, and $n_1 \mid 2^{n_k} - 1$. Show that $n_1 = n_2 = \cdots = n_k = 1$.

68. **(ROM 3)** Show that the sequence $\{a_n\}_{n\geq 1}$ defined by $a_n = [n\sqrt{2}]$ contains an infinite number of integer powers of 2. ($[x]$ is the integer part of x.)

69. **(ROM 4)** Let A and B be two finite disjoint sets of points in the plane such that any three distinct points in $A \cup B$ are not collinear. Assume that at least one of the sets A, B contains at least five points. Show that there exists a triangle all of whose vertices are contained in A or in B that does not contain in its interior any point from the other set.

70. **(ROM 5)** Let C be a class of functions $f : \mathbb{N} \to \mathbb{N}$ that contains the functions $S(x) = x + 1$ and $E(x) = x - [\sqrt{x}]^2$ for every $x \in \mathbb{N}$. ($[x]$ is the integer part of x.) If C has the property that for every $f, g \in C$, $f + g, fg, f \circ g \in C$, show that the function $\max(f(x) - g(x), 0)$ is in C.

71. **(ROM 6)** For every integer $r > 1$ find the smallest integer $h(r) > 1$ having the following property: For any partition of the set $\{1, 2, \dots, h(r)\}$ into r classes, there exist integers $a \geq 0$, $1 \leq x \leq y$ such that the numbers $a + x, a + y, a + x + y$ are contained in the same class of the partition.

72. **(SPA 1)** Construct a triangle ABC given the side AB and the distance OH from the circumcenter O to the orthocenter H, assuming that OH and AB are parallel.

73. **(SPA 2)** Let A_1A_2, B_1B_2, C_1C_2 be three equal segments on the three sides of an equilateral triangle. Prove that in the triangle formed by the lines B_2C_1, C_2A_1, A_2B_1, the segments B_2C_1, C_2A_1, A_2B_1 are proportional to the sides in which they are contained.

74. **(SPA 3)** Find the triples of positive integers x, y, z satisfying

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{5}.$$

75. **(SPA 4)** Let $ABCD$ be a rectangle, $AB = a$, $BC = b$. Consider the family of parallel and equidistant straight lines (the distance between two consecutive lines being d) that are at an the angle ϕ, $0 \leq \phi \leq 90°$,

with respect to AB. Let L be the sum of the lengths of all the segments intersecting the rectangle. Find:
(a) how L varies,
(b) a necessary and sufficient condition for L to be a constant, and
(c) the value of this constant.

76. **(SWE 1)** Are there integers m and n such that
$$5m^2 - 6mn + 7n^2 = 1985?$$

77. **(SWE 2)** Two equilateral triangles are inscribed in a circle with radius r. Let A be the area of the set consisting of all points interior to both triangles. Prove that $2A \geq r^2\sqrt{3}$.

78. **(SWE 3)** (SL85-17).

79. **(SWE 4)** Let a, b, and c be real numbers such that
$$\frac{1}{bc-a^2} + \frac{1}{ca-b^2} + \frac{1}{ab-c^2} = 0.$$
Prove that
$$\frac{a}{(bc-a^2)^2} + \frac{b}{(ca-b^2)^2} + \frac{c}{(ab-c^2)^2} = 0.$$

80. **(TUR 1)** Let $E = \{1, 2, \dots, 16\}$ and let M be the collection of all 4×4 matrices whose entries are distinct members of E. If a matrix $A = (a_{ij})_{4\times 4}$ is chosen randomly from M, compute the probability $p(k)$ of $\max_i \min_j a_{ij} = k$ for $k \in E$. Furthermore, determine $l \in E$ such that $p(l) = \max\{p(k) \mid k \in E\}$.

81. **(TUR 2)** Given the side a and the corresponding altitude h_a of a triangle ABC, find a relation between a and h_a such that it is possible to construct, with straightedge and compass, triangle ABC such that the altitudes of ABC form a right triangle admitting h_a as hypotenuse.

82. **(TUR 3)** Find all cubic polynomials $x^3 + ax^2 + bx + c$ admitting the rational numbers a, b, and c as roots.

83. **(TUR 4)** Let Γ_i, $i = 0, 1, 2, \dots$, be a circle of radius r_i inscribed in an angle of measure 2α such that each Γ_i is externally tangent to Γ_{i+1} and $r_{i+1} < r_i$. Show that the sum of the areas of the circles Γ_i is equal to the area of a circle of radius $r = \frac{1}{2}r_0(\sqrt{\sin\alpha} + \sqrt{\csc\alpha})$.

84. **(TUR 5)** (SL85-8).

85. **(USA 1)** Let CD be a diameter of circle K. Let AB be a chord that is parallel to CD. The line segment AE, with E on K, is parallel to CB; F is the point of intersection of line segments AB and DE. The line segment FG, with G on DC, extended is parallel to CB. Is GA tangent to K at point A?

86. **(USA 2)** Let l denote the length of the smallest diagonal of all rectangles inscribed in a triangle T. (By inscribed, we mean that all four vertices of the rectangle lie on the boundary of T.) Determine the maximum value of $\frac{l^2}{S(T)}$ taken over all triangles ($S(T)$ denotes the area of triangle T).

87. **(USA 3)** (SL85-9).

88. **(USA 4)** Determine the range of $w(w+x)(w+y)(w+z)$, where x, y, z, and w are real numbers such that
$$x+y+z+w = x^7+y^7+z^7+w^7 = 0.$$

89. **(USA 5)** Given that n elements $a_1, a_2, \dots, a_n$ are organized into n pairs $P_1, P_2, \dots, P_n$ in such a way that two pairs P_i, P_j share exactly one element when (a_i, a_j) is one of the pairs, prove that every element is in exactly two of the pairs.

90. **(USS 1)** Decompose the number $5^{1985}-1$ into a product of three integers, each of which is larger than 5^{100}.

91. **(USS 2)** Thirty-four countries participated in a jury session of the IMO, each represented by the leader and the deputy leader of the team. Before the meeting, some participants exchanged handshakes, but no team leader shook hands with his deputy. After the meeting, the leader of the Illyrian team asked every other participant the number of people they had shaken hands with, and all the answers she got were different. How many people did the deputy leader of the Illyrian team greet?

92. **(USS 3)** (SL85-11).
(USS 3a) Given six numbers, find a method of computing by using not more than 15 additions and 14 multiplications the following five numbers: the sum of the numbers, the sum of products of the numbers taken two at a time, and the sums of the products of the numbers taken three, four, and five at a time.

93. **(USS 4)** The sphere inscribed in tetrahedron $ABCD$ touches the sides ABD and DBC at points K and M, respectively. Prove that $\measuredangle AKB = \measuredangle DMC$.

94. **(USS 5)** (SL85-22).

95. **(VIE 1)** (SL85-10).
(VIE 1a) Prove that for each point M on the edges of a regular tetrahedron there is one and only one point M' on the surface of the tetrahedron such that there are at least three curves joining M and M' on the surface of the tetrahedron of minimal length among all curves joining M and M' on the surface of the tetrahedron. Denote this minimal length by d_M. Determine the positions of M for which d_M attains an extremum.

96. **(VIE 2)** Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following two conditions:

(a) $f(x+y) + f(x-y) = 2f(x)f(y)$ for all $x, y \in \mathbb{R}$,
(b) $\lim_{x\to\infty} f(x) = 0$.

97. **(VIE 3)** In a plane a circle with radius R and center w and a line Λ are given. The distance between w and Λ is d, $d > R$. The points M and N are chosen on Λ in such a way that the circle with diameter MN is externally tangent to the given circle. Show that there exists a point A in the plane such that all the segments MN are seen in a constant angle from A.

3.26.3 Shortlisted Problems

Proposals of the Problem Selection Committee.

1. **(MON 1)**$^{\mathrm{IMO4}}$ Given a set M of 1985 positive integers, none of which has a prime divisor larger than 26, prove that the set has four distinct elements whose geometric mean is an integer.

2. **(BRA 3)** A polyhedron has 12 faces and is such that:
   (i) all faces are isosceles triangles,
   (ii) all edges have length either x or y,
   (iii) at each vertex either 3 or 6 edges meet, and
   (iv) all dihedral angles are equal.
   Find the ratio x/y.

3. **(NET 3)**$^{\mathrm{IMO3}}$ The *weight* $w(p)$ of a polynomial p, $p(x) = \sum_{i=0}^{n} a_i x^i$, with integer coefficients a_i is defined as the number of its odd coefficients. For $i = 0, 1, 2, \dots,$ let $q_i(x) = (1+x)^i$. Prove that for any finite sequence $0 \le i_1 < i_2 < \cdots < i_n$, the inequality
$$w(q_{i_1} + \cdots + q_{i_n}) \ge w(q_{i_1})$$
   holds.

4. **(AUS 1)**$^{\mathrm{IMO2}}$ Each of the numbers in the set $N = \{1, 2, 3, \dots, n-1\}$, where $n \ge 3$, is colored with one of two colors, say red or black, so that:
   (i) i and $n - i$ always receive the same color, and
   (ii) for some $j \in N$, relatively prime to n, i and $|j - i|$ receive the same color for all $i \in N$, $i \ne j$.
   Prove that all numbers in N must receive the same color.

5. **(ROM 1)** Let D be the interior of the circle C and let $A \in C$. Show that the function $f : D \to \mathbb{R}$, $f(M) = \frac{|MA|}{|MM'|}$, where $M' = (AM \cap C$, is strictly convex; i.e., $f(P) < \frac{f(M_1)+f(M_2)}{2}$, $\forall M_1, M_2 \in D$, $M_1 \ne M_2$, where P is the midpoint of the segment M_1M_2.

6. **(POL 1)** Let $x_n = \sqrt[2]{2 + \sqrt[3]{3 + \dots + \sqrt[n]{n}}}$. Prove that

$$x_{n+1} - x_n < \frac{1}{n!}, \qquad n = 2, 3, \dots .$$

Alternatives

7. **1a.(CZS 3)** The positive integers $x_1, \dots, x_n$, $n \geq 3$, satisfy $x_1 < x_2 < \cdots < x_n < 2x_1$. Set $P = x_1 x_2 \cdots x_n$. Prove that if p is a prime number, k a positive integer, and P is divisible by p^k, then $\frac{P}{p^k} \geq n!$.
8. **1b.(TUR 5)** Find the smallest positive integer n such that
   (i) n has exactly 144 distinct positive divisors, and
   (ii) there are ten consecutive integers among the positive divisors of n.
9. **2a.(USA 3)** Determine the radius of a sphere S that passes through the centroids of each face of a given tetrahedron T inscribed in a unit sphere with center O. Also, determine the distance from O to the center of S as a function of the edges of T.
10. **2b.(VIE 1)** Prove that for every point M on the surface of a regular tetrahedron there exists a point M' such that there are at least three different curves on the surface joining M to M' with the smallest possible length among all curves on the surface joining M to M'.
11. **3a.(USS 3)** Find a method by which one can compute the coefficients of $P(x) = x^6 + a_1 x^5 + \cdots + a_6$ from the roots of $P(x) = 0$ by performing not more than 15 additions and 15 multiplications.
12. **3b.(GBR 4)** A sequence of polynomials $P_m(x, y, z)$, $m = 0, 1, 2, \dots$, in x, y, and z is defined by $P_0(x, y, z) = 1$ and by

    $$P_m(x, y, z) = (x + z)(y + z)P_{m-1}(x, y, z + 1) - z^2 P_{m-1}(x, y, z)$$

    for $m > 0$. Prove that each $P_m(x, y, z)$ is symmetric, in other words, is unaltered by any permutation of x, y, z.
13. **4a.(BUL 1)** Let m boxes be given, with some balls in each box. Let $n < m$ be a given integer. The following operation is performed: choose n of the boxes and put 1 ball in each of them. Prove:
    (a) If m and n are relatively prime, then it is possible, by performing the operation a finite number of times, to arrive at the situation that all the boxes contain an equal number of balls.
    (b) If m and n are not relatively prime, there exist initial distributions of balls in the boxes such that an equal distribution is not possible to achieve.
14. **4b.(IRE 4)** A set of 1985 points is distributed around the circumference of a circle and each of the points is marked with 1 or -1. A point is called "good" if the partial sums that can be formed by starting at that point and proceeding around the circle for any distance in either direction are

all strictly positive. Show that if the number of points marked with -1 is less than 662, there must be at least one good point.

15. **5a.(FRA 3)** Let K and K' be two squares in the same plane, their sides of equal length. Is it possible to decompose K into a finite number of triangles $T_1, T_2, \dots, T_p$ with mutually disjoint interiors and find translations $t_1, t_2, \dots, t_p$ such that
$$K' = \bigcup_{i=1}^{p} t_i(T_i)?$$

16. **5b.(BEL 2)** If possible, construct an equilateral triangle whose three vertices are on three given circles.

17. **6a.(SWE 3)**IMO6 The sequence $f_1, f_2, \dots, f_n, \dots$ of functions is defined for $x > 0$ recursively by
$$f_1(x) = x, \qquad f_{n+1}(x) = f_n(x)\left(f_n(x) + \frac{1}{n}\right).$$
Prove that there exists one and only one positive number a such that $0 < f_n(a) < f_{n+1}(a) < 1$ for all integers $n \geq 1$.

18. **6b.(CAN 5)** Let $x_1, x_2, \dots, x_n$ be positive numbers. Prove that
$$\frac{x_1^2}{x_1^2 + x_2x_3} + \frac{x_2^2}{x_2^2 + x_3x_4} + \cdots + \frac{x_{n-1}^2}{x_{n-1}^2 + x_nx_1} + \frac{x_n^2}{x_n^2 + x_1x_2} \leq n - 1.$$

Supplementary Problems

19. **(ISR 3)** For which integers $n \geq 3$ does there exist a regular n-gon in the plane such that all its vertices have integer coordinates in a rectangular coordinate system?

20. **(GBR 5)**IMO1 A circle whose center is on the side ED of the cyclic quadrilateral $BCDE$ touches the other three sides. Prove that $EB + CD = ED$.

21. **(IRE 1)** The tangents at B and C to the circumcircle of the acute-angled triangle ABC meet at X. Let M be the midpoint of BC. Prove that
(a) $\angle BAM = \angle CAX$, and
(b) $\frac{AM}{AX} = \cos\angle BAC$.

22. **(USS 5)**IMO5 A circle with center O passes through points A and C and intersects the sides AB and BC of the triangle ABC at points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at two distinct points B and M. Prove that $\angle OMB = 90^\circ$.

3.27 The Twenty-Seventh IMO
Warsaw, Poland, July 4–15, 1986

3.27.1 Contest Problems

First Day (July 9)

1. The set $S = \{2, 5, 13\}$ has the property that for every $a, b \in S$, $a \neq b$, the number $ab - 1$ is a perfect square. Show that for every positive integer d not in S, the set $S \cup \{d\}$ does not have the above property.

2. Let A, B, C be fixed points in the plane. A man starts from a certain point P_0 and walks directly to A. At A he turns his direction by 60° to the left and walks to P_1 such that $P_0A = AP_1$. After he performs the same action 1986 times successively around the points $A, B, C, A, B, C, \dots$, he returns to the starting point. Prove that ABC is an equilateral triangle, and that the vertices A, B, C are arranged counterclockwise.

3. To each vertex P_i $(i = 1, \dots, 5)$ of a pentagon an integer x_i is assigned, the sum $s = \sum x_i$ being positive. The following operation is allowed, provided at least one of the x_i's is negative: Choose a negative x_i, replace it by $-x_i$, and add the former value of x_i to the integers assigned to the two neighboring vertices of P_i (the remaining two integers are left unchanged).
   This operation is to be performed repeatedly until all negative integers disappear. Decide whether this procedure must eventually terminate.

Second Day (July 10)

4. Let A, B be adjacent vertices of a regular n-gon in the plane and let O be its center. Now let the triangle ABO glide around the polygon in such a way that the points A and B move along the whole circumference of the polygon. Describe the figure traced by the vertex O.

5. Find, with proof, all functions f defined on the nonnegative real numbers and taking nonnegative real values such that
   (i) $f[xf(y)]f(y) = f(x + y)$,
   (ii) $f(2) = 0$ but $f(x) \neq 0$ for $0 \leq x < 2$.

6. Prove or disprove: Given a finite set of points with integer coefficients in the plane, it is possible to color some of these points red and the remaining ones white in such a way that for any straight line L parallel to one of the coordinate axes, the number of red colored points and the number of white colored points on L differ by at most 1.

3.27.2 Longlisted Problems

1. **(AUS 1)** Let k be one of the integers $2, 3, 4$ and let $n = 2^k - 1$. Prove the inequality
$$1 + b^k + b^{2k} + \cdots + b^{nk} \geq (1 + b^n)^k$$
for all real $b \geq 0$.

2. **(AUS 2)** Let $ABCD$ be a convex quadrilateral. DA and CB meet at F and AB and DC meet at E. The bisectors of the angles DFC and AED are perpendicular. Prove that these angle bisectors are parallel to the bisectors of the angles between the lines AC and BD.

3. **(AUS 3)** A line parallel to the side BC of a triangle ABC meets AB in F and AC in E. Prove that the circles on BE and CF as diameters intersect in a point lying on the altitude of the triangle ABC dropped from A to BC.

4. **(BEL 1)** Find the last eight digits of the binary development of 27^{1986}.

5. **(BEL 2)** Let ABC and DEF be acute-angled triangles. Write $d = EF$, $e = FD$, $f = DE$. Show that there exists a point P in the interior of ABC for which the value of the expression $d \cdot AP + e \cdot BP + f \cdot CP$ attains a minimum.

6. **(BEL 3)** In an urn there are one ball marked 1, two balls marked 2, and so on, up to n balls marked n. Two balls are randomly drawn without replacement. Find the probability that the two balls are assigned the same number.

7. **(BUL 1)** (SL86-11).

8. **(BUL 2)** (SL86-19).

9. **(CAN 1)** In a triangle ABC, $\angle BAC = 100^\circ$, $AB = AC$. A point D is chosen on the side AC such that $\angle ABD = \angle CBD$. Prove that $AD + DB = BC$.

10. **(CAN 2)** A set of n standard dice are shaken and randomly placed in a straight line. If $n < 2r$ and $r < s$, then the probability that there will be a string of at least r, but not more than s, consecutive 1's can be written as $P/6^{s+2}$. Find an explicit expression for P.

11. **(CAN 3)** (SL86-20).

12. **(CHN 1)** Let O be an interior point of a tetrahedron $A_1A_2A_3A_4$. Let S_1, S_2, S_3, S_4 be spheres with centers A_1, A_2, A_3, A_4, respectively, and let U, V be spheres with centers at O. Suppose that for $i, j = 1, 2, 3, 4$, $i \neq j$, the spheres S_i and S_j are tangent to each other at a point B_{ij} lying on A_iA_j. Suppose also that U is tangent to all edges A_iA_j and V is tangent to the spheres S_1, S_2, S_3, S_4. Prove that $A_1A_2A_3A_4$ is a regular tetrahedron.

13. **(CHN 2)** Let $N = \{1, 2, \dots, n\}$, $n \geq 3$. To each pair i, j of elements of N, $i \neq j$, there is assigned a number $f_{ij} \in \{0, 1\}$ such that $f_{ij} + f_{ji} = 1$. Let $r(i) = \sum_{j \neq i} f_{ij}$ and write $M = \max_{i \in N} r(i)$, $m = \min_{i \in N} r(i)$. Prove that for any $w \in N$ with $r(w) = m$ there exist $u, v \in N$ such that $r(u) = M$ and $f_{uv} f_{vw} = 1$.

14. **(CHN 3)** (SL86-17).

15. **(CHN 4)** Let $\mathbb{N} = B_1 \cup \cdots \cup B_q$ be a partition of the set $\mathbb{N}$ of all positive integers and let an integer $l \in \mathbb{N}$ be given. Prove that there exist a set $X \subset \mathbb{N}$ of cardinality l, an infinite set $T \subset \mathbb{N}$, and an integer k with $1 \leq k \leq q$ such that for any $t \in T$ and any finite set $Y \subset X$, the sum $t + \sum_{y \in Y} y$ belongs to B_k.

16. **(CZS 1)** Given a positive integer k, find the least integer n_k for which there exist five sets S_1, S_2, S_3, S_4, S_5 with the following properties:

$$|S_j| = k \quad \text{for } j = 1, \dots, 5, \quad \left| \bigcup_{j=1}^{5} S_j \right| = n_k;$$
$$|S_i \cap S_{i+1}| = 0 = |S_5 \cap S_1|, \quad \text{for } i = 1, \dots, 4.$$

17. **(CZS 2)** We call a tetrahedron right-faced if each of its faces is a right-angled triangle.
    (a) Prove that every orthogonal parallelepiped can be partitioned into six right-faced tetrahedra.
    (b) Prove that a tetrahedron with vertices A_1, A_2, A_3, A_4 is fight-faced if and only if there exist four distinct real numbers c_1, c_2, c_3, and c_4 such that the edges $A_j A_k$ have lengths $A_j A_k = \sqrt{|c_j - c_k|}$ for $1 \leq j < k \leq 4$.

18. **(CZS 3)** (SL86-4).

19. **(FIN 1)** Let $f : [0, 1] \to [0, 1]$ satisfy $f(0) = 0$, $f(1) = 1$ and

$$f(x + y) - f(x) = f(x) - f(x - y)$$

for all $x, y \geq 0$ with $x - y, x + y \in [0, 1]$. Prove that $f(x) = x$ for all $x \in [0, 1]$.

20. **(FIN 2)** For any angle α with $0 < \alpha < 180^\circ$, we call a closed convex planar set an *α-set* if it is bounded by two circular arcs (or an arc and a line segment) whose angle of intersection is α. Given a (closed) triangle T, find the greatest α such that any two points in T are contained in an α-set $S \subset T$.

21. **(FRA 1)** Let AB be a segment of unit length and let C, D be variable points of this segment. Find the maximum value of the product of the lengths of the six distinct segments with endpoints in the set $\{A, B, C, D\}$.

22. **(FRA 2)** Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of integers defined recursively by $a_0 = 0$, $a_1 = 1$, $a_{n+2} = 4a_{n+1} + a_n$ for $n \geq 0$. Find the common divisors of a_{1986} and a_{6891}.

23. **(FRA 3)** Let I and J be the centers of the incircle and the excircle in the angle BAC of the triangle ABC. For any point M in the plane of the triangle, not on the line BC, denote by I_M and J_M the centers of the incircle and the excircle (touching BC) of the triangle BCM. Find the locus of points M for which II_MJJ_M is a rectangle.

24. **(FRA 4)** Two families of parallel lines are given in the plane, consisting of 15 and 11 lines, respectively. In each family, any two neighboring lines are at a unit distance from one another; the lines of the first family are perpendicular to the lines of the second family. Let V be the set of 165 intersection points of the lines under consideration. Show that there exist not fewer than 1986 distinct squares with vertices in the set V.

25. **(FRA 5)** (SL86-7).

26. **(FRG 1)** (SL86-5).

27. **(FRG 2)** In an urn there are n balls numbered $1, 2, \dots, n$. They are drawn at random one by one one without replacement and the numbers are recorded. What is the probability that the resulting random permutation has only one local maximum?
    A term in a sequence is a local maximum if it is greater than all its neighbors.

28. **(FRG 3)** (SL86-13).

29. **(FRG 4)** We define a binary operation $\star$ in the plane as follows: Given two points A and B in the plane, $C = A \star B$ is the third vertex of the equilateral triangle ABC oriented positively. What is the relative position of three points I, M, O in the plane if $I \star (M \star O) = (O \star I) \star M$ holds?

30. **(FRG 5)** Prove that a convex polyhedron all of whose faces are equilateral triangles has at most 30 edges.

31. **(GBR 1)** Let P and Q be distinct points in the plane of a triangle ABC such that $AP : AQ = BP : BQ = CP : CQ$. Prove that the line PQ passes through the circumcenter of the triangle.

32. **(GBR 2)** Find, with proof, all solutions of the equation $\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1$ in positive integers x, y, z.

33. **(GBR 3)** (SL86-1).

34. **(GBR 4)** For each nonnegative integer n, $F_n(x)$ is a polynomial in x of degreee n. Prove that if the identity

$$F_n(2x) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} 2^r F_r(x)$$

holds for each n, then

$$F_n(tx) = \sum_{r=0}^{n} \binom{n}{r} t^r (1-t)^{n-r} F_r(x)$$

for each n and all t.

35. **(GBR 5)** Establish the maximum and minimum values that the sum $|a| + |b| + |c|$ can have if a, b, c are real numbers such that the maximum value of $|ax^2 + bx + c|$ is 1 for $-1 \le x \le 1$.

36. **(GDR 1)** (SL86-9).

37. **(GDR 2)** Prove that the set $\{1, 2, \dots, 1986\}$ can be partitioned into 27 disjoint sets so that no one of these sets contains an arithmetic triple (i.e., three distinct numbers in an arithmetic progression).

38. **(GDR 3)** (SL86-12).

39. **(GRE 1)** Let S be a k-element set.
   (a) Find the number of mappings $f : S \to S$ such that

$$\text{(i) } f(x) \ne x \text{ for } x \in S, \quad \text{(ii) } f(f(x)) = x \text{ for } x \in S.$$

   (b) The same with the condition (i) left out.

40. **(GRE 2)** Find the maximum value that the quantity $2m + 7n$ can have such that there exist distinct positive integers x_i $(1 \le i \le m)$, y_j $(1 \le j \le n)$ such that the x_i's are even, the y_j's are odd, and $\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j = 1986$.

41. **(GRE 3)** Let M, N, P be the midpoints of the sides BC, CA, AB of a triangle ABC. The lines AM, BN, CP intersect the circumcircle of ABC at points A', B', C', respectively. Show that if $A'B'C'$ is an equilateral triangle, then so is ABC.

42. **(HUN 1)** The integers $1, 2, \dots, n^2$ are placed on the fields of an $n \times n$ chessboard $(n > 2)$ in such a way that any two fields that have a common edge or a vertex are assigned numbers differing by at most $n + 1$. What is the total number of such placements?

43. **(HUN 2)** (SL86-10).

44. **(IRE 1)** (SL86-14).

45. **(IRE 2)** Given n real numbers $a_1 \le a_2 \le \cdots \le a_n$, define

$$M_1 = \frac{1}{n} \sum_{i=1}^{n} a_i, \quad M_2 = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} a_i a_j, \quad Q = \sqrt{M_1^2 - M_2}.$$

Prove that

$$a_1 \le M_1 - Q \le M_1 + Q \le a_n$$

and that equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

46. **(IRE 3)** We wish to construct a matrix with 19 rows and 86 columns, with entries $x_{ij} \in \{0, 1, 2\}$ $(1 \le i \le 19,\ 1 \le j \le 86)$, such that:
  (i) in each column there are exactly k terms equal to 0;
  (ii) for any distinct $j, k \in \{1, \dots, 86\}$ there is $i \in \{1, \dots, 19\}$ with $x_{ij} + x_{ik} = 3$.

  For what values of k is this possible?

47. **(ISR 1)** (SL86-16).

48. **(ISR 2)** Let P be a convex 1986-gon in the plane. Let A, D be interior points of two distinct sides of P and let B, C be two distinct interior points of the line segment AD. Starting with an arbitrary point Q_1 on the boundary of P, define recursively a sequence of points Q_n as follows: given Q_n extend the directed line segment Q_nB to meet the boundary of P in a point R_n and then extend R_nC to meet the boundary of P again in a point, which is defined to be Q_{n+1}. Prove that for all n large enough the points Q_n are on one of the sides of P containing A or D.

49. **(ISR 3)** Let C_1, C_2 be circles of radius $1/2$ tangent to each other and both tangent internally to a circle C of radius 1. The circles C_1 and C_2 are the first two terms of an infinite sequence of distinct circles C_n defined as follows: C_{n+2} is tangent externally to C_n and C_{n+1} and internally to C. Show that the radius of each C_n is the reciprocal of an integer.

50. **(LUX 1)** Let D be the point on the side BC of the triangle ABC such that AD is the bisector of $\angle CAB$. Let I be the incenter of $\triangle ABC$.
  (a) Construct the points P and Q on the sides AB and AC, respectively, such that PQ is parallel to BC and the perimeter of the triangle APQ is equal to $k \cdot BC$, where k is a given rational number.
  (b) Let R be the intersection point of PQ and AD. For what value of k does the equality $AR = RI$ hold?
  (c) In which case do the equalities $AR = RI = ID$ hold?

51. **(MON 1)** Let a, b, c, d be the lengths of the sides of a quadrilateral circumscribed about a circle and let S be its area. Prove that $S \le \sqrt{abcd}$ and find conditions for equality.

52. **(MON 2)** Solve the system of equations

$$\begin{aligned} \tan x_1 + \cot x_1 &= 3 \tan x_2, \\ \tan x_2 + \cot x_2 &= 3 \tan x_3, \\ &\cdots \cdots \\ \tan x_n + \cot x_n &= 3 \tan x_1. \end{aligned}$$

53. **(MON 3)** For given positive integers r, v, n let $S(r, v, n)$ denote the number of n-tuples of nonnegative integers $(x_1, \dots, x_n)$ satisfying the equation $x_1 + \dots + x_n = r$ and such that $x_i \le v$ for $i = 1, \dots, n$. Prove that

$$S(r, v, n) = \sum_{k=0}^{m} (-1)^k \binom{n}{k} \binom{r - (v+1)k + n - 1}{n - 1},$$

where $m = \min\left\{n, \left[\frac{r}{v+1}\right]\right\}$.

54. **(MON 4)** Find the least integer n with the following property: For any set V of 8 points in the plane, no three lying on a line, and for any set E of n line segments with endpoints in V, one can find a straight line intersecting at least 4 segments in E in interior points.

55. **(MON 5)** Given an integer $n \geq 2$, determine all n-digit numbers $M_0 = \overline{a_1 a_2 \dots a_n}$ $(a_i \neq 0,\ i = 1, 2, \dots, n)$ divisible by the numbers $M_1 = \overline{a_2 a_3 \dots a_n a_1}$, $M_2 = \overline{a_3 a_4 \dots a_n a_1 a_2}$, $\dots$, $M_{n-1} = \overline{a_n a_1 a_2 \dots a_{n-1}}$.

56. **(MOR 1)** Let $A_1A_2A_3A_4A_5A_6$ be a hexagon inscribed into a circle with center O. Consider the circular arc with endpoints A_1, A_6 not containing A_2. For any point M of that arc denote by h_i the distance from M to the line A_iA_{i+1} $(1 \leq i \leq 5)$. Construct M such that the sum $h_1 + \dots + h_5$ is maximal.

57. **(MOR 2)** In a triangle ABC, the incircle touches the sides BC, CA, AB in the points A', B', C', respectively; the excircle in the angle A touches the lines containing these sides in A_1, B_1, C_1, and similarly, the excircles in the angles B and C touch these lines in A_2, B_2, C_2 and A_3, B_3, C_3. Prove that the triangle ABC is right-angled if and only if one of the point triples (A', B_3, C'), (A_3, B', C_3), (A', B', C_2), (A_2, B_2, C'), (A_2, B_1, C_2), (A_3, B_3, C_1), (A_1, B_2, C_1), (A_1, B_1, C_3) is collinear.

58. **(NET 1)** (SL86-6).

59. **(NET 2)** (SL86-15).

60. **(NET 3)** Prove the inequality

$$(-a+b+c)^2(a-b+c)^2(a+b-c)^2 \geq (-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)$$

for all real numbers a, b, c.

61. **(ROM 1)** Given a positive integer n, find the greatest integer p with the property that for any function $f : \mathbb{P}(X) \to C$, where X and C are sets of cardinality n and p, respectively, there exist two distinct sets $A, B \in \mathbb{P}(X)$ such that $f(A) = f(B) = f(A \cup B)$. ($\mathbb{P}(X)$ is the family of all subsets of X.)

62. **(ROM 2)** Determine all pairs of positive integers (x, y) satisfying the equation $p^x - y^3 = 1$, where p is a given prime number.

63. **(ROM 3)** Let AA', BB', CC' be the bisectors of the angles of a triangle ABC $(A' \in BC,\ B' \in CA,\ C' \in AB)$. Prove that each of the lines $A'B'$, $B'C'$, $C'A'$ intersects the incircle in two points.

64. **(ROM 4)** Let $(a_n)_{n \in \mathbb{N}}$ be the sequence of integers defined recursively by $a_1 = a_2 = 1$, $a_{n+2} = 7a_{n+1} - a_n - 2$ for $n \geq 1$. Prove that a_n is a perfect square for every n.

65. **(ROM 5)** Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle C. Show that there is a point M on C such that $MA_1 - MA_2 + MA_3 - MA_4 = 0$.

66. **(SWE 1)** One hundred red points and one hundred blue points are chosen in the plane, no three of them lying on a line. Show that these points can be connected pairwise, red ones with blue ones, by disjoint line segments.

67. **(SWE 2)** (SL86-2).

68. **(SWE 3)** Consider the equation $x^4 + ax^3 + bx^2 + ax + 1 = 0$ with real coefficients a, b. Determine the number of distinct real roots and their multiplicities for various values of a and b. Display your result graphically in the (a, b) plane.

69. **(TUR 1)** (SL86-18).

70. **(TUR 2)** (SL86-21).

71. **(TUR 3)** Two straight lines perpendicular to each other meet each side of a triangle in points symmetric with respect to the midpoint of that side. Prove that these two lines intersect in a point on the nine-point circle.

72. **(TUR 4)** A one-person game with two possible outcomes is played as follows: After each play, the player receives either a or b points, where a and b are integers with $0 < b < a < 1986$. The game is played as many times as one wishes and the total score of the game is defined as the sum of points received after successive plays. It is observed that every integer $x \geq 1986$ can be obtained as the total score whereas 1985 and 663 cannot. Determine a and b.

73. **(TUR 5)** Let $(a_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers such that $\lim_{i\to\infty} a_i = +\infty$ and $a_{i+1}/a_i \leq 10$ for each i. Prove that for every positive integer k there are infinitely many pairs (i, j) with $10^k \leq a_i/a_j \leq 10^{k+1}$.

74. **(USA 1)** (SL86-8).
Alternative formulation. Let A be a set of n points in space. From the family of all segments with endpoints in A, q segments have been selected and colored yellow. Suppose that all yellow segments are of different length. Prove that there exists a polygonal line composed of m yellow segments, where $m \geq \frac{2q}{n}$, arranged in order of increasing length.

75. **(USA 2)** The incenter of a triangle is the midpoint of the line segment of length 4 joining the centroid and the orthocenter of the triangle. Determine the maximum possible area of the triangle.

76. **(USA 3)** (SL86-3).

77. **(USS 1)** Find all integers x, y, z that satisfy

$$x^3 + y^3 + z^3 = x + y + z = 8.$$

78. **(USS 2)** If T and T_1 are two triangles with angles x, y, z and x_1, y_1, z_1, respectively, prove the inequality
$$\frac{\cos x_1}{\sin x} + \frac{\cos y_1}{\sin y} + \frac{\cos z_1}{\sin z} \le \cot x + \cot y + \cot z.$$

79. **(USS 3)** Let AA_1, BB_1, CC_1 be the altitudes in an acute-angled triangle ABC. K and M are points on the line segments A_1C_1 and B_1C_1 respectively. Prove that if the angles MAK and CAA_1 are equal, then the angle C_1KM is bisected by AK.

80. **(USS 4)** Let $ABCD$ be a tetrahedron and O its incenter, and let the line OD be perpendicular to AD. Find the angle between the planes DOB and DOC.

3.27.3 Shortlisted Problems

1. **(GBR 3)**IMO5 Find, with proof, all functions f defined on the nonnegative real numbers and taking nonnegative real values such that
   (i) $f[xf(y)]f(y) = f(x+y)$,
   (ii) $f(2) = 0$ but $f(x) \neq 0$ for $0 \le x < 2$.

2. **(SWE 2)** Let $f(x) = x^n$ where n is a fixed positive integer and $x = 1, 2, \dots$. Is the decimal expansion $a = 0.f(1)f(2)f(3)\dots$ rational for any value of n?
   The decimal expansion of a is defined as follows: If $f(x) = d_1(x)d_2(x)\dots$ $\dots d_{r(x)}(x)$ is the decimal expansion of $f(x)$, then $a = 0.1d_1(2)d_2(2)\dots$ $\dots d_{r(2)}(2)d_1(3)\dots d_{r(3)}(3)d_1(4)\dots$.

3. **(USA 3)** Let A, B, and C be three points on the edge of a circular chord such that B is due west of C and ABC is an equilateral triangle whose side is 86 meters long. A boy swam from A directly toward B. After covering a distance of x meters, he turned and swam westward, reaching the shore after covering a distance of y meters. If x and y are both positive integers, determine y.

4. **(CZS 3)** Let n be a positive integer and let p be a prime number, $p > 3$. Find at least $3(n+1)$ [easier version: $2(n+1)$] sequences of positive integers x, y, z satisfying
$$xyz = p^n(x+y+z)$$
   that do not differ only by permutation.

5. **(FRG 1)**IMO1 The set $S = \{2, 5, 13\}$ has the property that for every $a, b \in S$, $a \neq b$, the number $ab - 1$ is a perfect square. Show that for every positive integer d not in S, the set $S \cup \{d\}$ does not have the above property.

6. **(NET 1)** Find four positive integers each not exceeding 70000 and each having more than 100 divisors.

7. **(FRA 5)** Let real numbers $x_1, x_2, \dots, x_n$ satisfy $0 < x_1 < x_2 < \cdots < x_n < 1$ and set $x_0 = 0$, $x_{n+1} = 1$. Suppose that these numbers satisfy the following system of equations:
$$\sum_{j=0,\, j\neq i}^{n+1} \frac{1}{x_i - x_j} = 0 \quad \text{where } i = 1, 2, \dots, n. \tag{1}$$
Prove that $x_{n+1-i} = 1 - x_i$ for $i = 1, 2, \dots, n$.

8. **(USA 1)** From a collection of n persons q distinct two-member teams are selected and ranked $1, \dots, q$ (no ties). Let m be the least integer larger than or equal to $2q/n$. Show that there are m distinct teams that may be listed so that (i) each pair of consecutive teams on the list have one member in common and (ii) the chain of teams on the list are in rank order.

   Alternative formulation. Given a graph with n vertices and q edges numbered $1, \dots, q$, show that there exists a chain of m edges, $m \geq \frac{2q}{n}$, each two consecutive edges having a common vertex, arranged monotonically with respect to the numbering.

9. **(GDR 1)**$^{\mathrm{IMO6}}$ Prove or disprove: Given a finite set of points with integer coordinates in the plane, it is possible to color some of these points red and the remaining ones white in such a way that for any straight line L parallel to one of the coordinate axes, the number of red colored points and the number of white colored points on L differ by at most 1.

10. **(HUN 2)** Three persons A, B, C, are playing the following game: A k-element subset of the set $\{1, \dots, 1986\}$ is randomly chosen, with an equal probability of each choice, where k is a fixed positive integer less than or equal to 1986. The winner is A, B or C, respectively, if the sum of the chosen numbers leaves a remainder of 0, 1, or 2 when divided by 3. For what values of k is this game a fair one? (A game is fair if the three outcomes are equally probable.)

11. **(BUL 1)** Let $f(n)$ be the least number of distinct points in the plane such that for each $k = 1, 2, \dots, n$ there exists a straight line containing exactly k of these points. Find an explicit expression for $f(n)$.
    Simplified version. Show that $f(n) = \left[\frac{n+1}{2}\right]\left[\frac{n+2}{2}\right]$ ($[x]$ denoting the greatest integer not exceeding x).

12. **(GDR 3)**$^{\mathrm{IMO3}}$ To each vertex P_i $(i = 1, \dots, 5)$ of a pentagon an integer x_i is assigned, the sum $s = \sum x_i$ being positive. The following operation is allowed, provided at least one of the x_i's is negative: Choose a negative x_i, replace it by $-x_i$, and add the former value of x_i to the integers assigned to the two neighboring vertices of P_i (the remaining two integers are left unchanged).
    This operation is to be performed repeatedly until all negative integers disappear. Decide whether this procedure must eventually terminate.

13. **(FRG 3)** A particle moves from $(0,0)$ to (n,n) directed by a fair coin. For each head it moves one step east and for each tail it moves one step north. At (n,y), $y<n$, it stays there if a head comes up and at (x,n), $x<n$, it stays there if a tail comes up. Let k be a fixed positive integer. Find the probability that the particle needs exactly $2n+k$ tosses to reach (n,n).

14. **(IRE 1)** The circle inscribed in a triangle ABC touches the sides BC, CA, AB in D, E, F, respectively, and X, Y, Z are the midpoints of EF, FD, DE, respectively. Prove that the centers of the inscribed circle and of the circles around XYZ and ABC are collinear.

15. **(NET 2)** Let $ABCD$ be a convex quadrilateral whose vertices do not lie on a circle. Let $A'B'C'D'$ be a quadrangle such that A', B', C', D' are the centers of the circumcircles of triangles BCD, ACD, ABD, and ABC. We write $T(ABCD) = A'B'C'D'$. Let us define $A''B''C''D'' = T(A'B'C'D') = T(T(ABCD))$.
    (a) Prove that $ABCD$ and $A''B''C''D''$ are similar.
    (b) The ratio of similitude depends on the size of the angles of $ABCD$. Determine this ratio.

16. **(ISR 1)**$^{\text{IMO4}}$ Let A, B be adjacent vertices of a regular n-gon in the plane and let O be its center. Now let the triangle ABO glide around the polygon in such a way that the points A and B move along the whole circumference of the polygon. Describe the figure traced by the vertex O.

17. **(CHN 3)**$^{\text{IMO2}}$ Let A, B, C be fixed points in the plane. A man starts from a certain point P_0 and walks directly to A. At A he turns his direction by 60° to the left and walks to P_1 such that $P_0A = AP_1$. After he does the same action 1986 times successively around the points $A, B, C, A, B, C, \dots$, he returns to the starting point. Prove that $\triangle ABC$ is equilateral and that the vertices A, B, C are arranged counterclockwise.

18. **(TUR 1)** Let AX, BY, CZ be three cevians concurrent at an interior point D of a triangle ABC. Prove that if two of the quadrangles $DYAZ, DZBX, DXCY$ are circumscribable, so is the third.

19. **(BUL 2)** A tetrahedron $ABCD$ is given such that $AD = BC = a$; $AC = BD = b$; $AB \cdot CD = c^2$. Let $f(P) = AP + BP + CP + DP$, where P is an arbitrary point in space. Compute the least value of $f(P)$.

20. **(CAN 3)** Prove that the sum of the face angles at each vertex of a tetrahedron is a straight angle if and only if the faces are congruent triangles.

21. **(TUR 2)** Let $ABCD$ be a tetrahedron having each sum of opposite sides equal to 1. Prove that
$$r_A + r_B + r_C + r_D \le \frac{\sqrt{3}}{3},$$
where r_A, r_B, r_C, r_D are the inradii of the faces, equality holding only if $ABCD$ is regular.

3.28 The Twenty-Eighth IMO
Havana, Cuba, July 5–16, 1987

3.28.1 Contest Problems

First Day (July 10)

1. Let S be a set of n elements. We denote the number of all permutations of S that have exactly k fixed points by $p_n(k)$. Prove that
$$\sum_{k=0}^{n} kp_n(k) = n!.$$

2. The prolongation of the bisector AL ($L \in BC$) in the acute-angled triangle ABC intersects the circumscribed circle at point N. From point L to the sides AB and AC are drawn the perpendiculars LK and LM respectively. Prove that the area of the triangle ABC is equal to the area of the quadrilateral $AKNM$.

3. Suppose $x_1, x_2, \dots, x_n$ are real numbers with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for any integer $k > 1$ there are integers e_i not all 0 and with $|e_i| < k$ such that
$$|e_1x_1 + e_2x_2 + \cdots + e_nx_n| \le \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

Second Day (July 11)

4. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$, such that $f(f(n)) = n + 1987$ for every natural number n?

5. Prove that for every natural number $n \ge 3$ it is possible to put n points in the Euclidean plane such that the distance between each pair of points is irrational and each three points determine a nondegenerate triangle with rational area.

6. Let $f(x) = x^2 + x + p$, $p \in \mathbb{N}$. Prove that if the numbers $f(0), f(1), \dots,$ $f([\sqrt{p/3}\,])$ are primes, then all the numbers $f(0), f(1), \dots, f(p-2)$ are primes.

3.28.2 Longlisted Problems

1. **(AUS 1)** Let $x_1, x_2, \dots, x_n$ be n integers. Let $n = p + q$, where p and q are positive integers. For $i = 1, 2, \dots, n$, put
$$S_i = x_i + x_{i+1} + \cdots + x_{i+p-1} \text{ and } T_i = x_{i+p} + x_{i+p+1} + \cdots + x_{i+n-1}$$
(it is assumed that $x_{i+n} = x_i$ for all i). Next, let $m(a, b)$ be the number of indices i for which S_i leaves the remainder a and T_i leaves the remainder b on division by 3, where $a, b \in \{0, 1, 2\}$. Show that $m(1, 2)$ and $m(2, 1)$ leave the same remainder when divided by 3.

2. **(AUS 2)** Suppose we have a pack of $2n$ cards, in the order $1, 2, \dots, 2n$. A perfect shuffle of these cards changes the order to $n+1, 1, n+2, 2, \dots, n-1, 2n, n$; i.e., the cards originally in the first n positions have been moved to the places $2, 4, \dots, 2n$, while the remaining n cards, in their original order, fill the odd positions $1, 3, \dots, 2n-1$.
   Suppose we start with the cards in the above order $1, 2, \dots, 2n$ and then successively apply perfect shuffles. What conditions on the number n are necessary for the cards eventually to return to their original order? Justify your answer.

   Remark. This problem is trivial. Alternatively, it may be required to find the least number of shuffles after which the cards will return to the original order.

3. **(AUS 3)** A town has a road network that consists entirely of one-way streets that are used for bus routes. Along these routes, bus stops have been set up. If the one-way signs permit travel from bus stop X to bus stop $Y \neq X$, then we shall say Y *can be reached from* X.
   We shall use the phrase Y *comes after* X when we wish to express that every bus stop from which the bus stop X can be reached is a bus stop from which the bus stop Y can be reached, and every bus stop that can be reached from Y can also be reached from X. A visitor to this town discovers that if X and Y are any two different bus stops, then the two sentences "Y can be reached from X" and "Y comes after X" have exactly the same meaning in this town.
   Let A and B be two bus stops. Show that of the following two statements, exactly one is true: (i) B can be reached from A; (ii) A can be reached from B.

4. **(AUS 4)** Let $a_1, a_2, a_3, b_1, b_2, b_3$ be positive real numbers. Prove that

$$\begin{aligned}&(a_1b_2 + a_2b_1 + a_1b_3 + a_3b_1 + a_2b_3 + a_3b_2)^2\\ &\qquad \geq 4(a_1a_2 + a_2a_3 + a_3a_1)(b_1b_2 + b_2b_3 + b_3b_1)\end{aligned}$$

   and show that the two sides of the inequality are equal if and only if $a_1/b_1 = a_2/b_2 = a_3/b_3$.

5. **(AUS 5)** Let there be given three circles K_1, K_2, K_3 with centers O_1, O_2, O_3 respectively, which meet at a common point P. Also, let $K_1 \cap K_2 = \{P, A\}$, $K_2 \cap K_3 = \{P, B\}$, $K_3 \cap K_1 = \{P, C\}$. Given an arbitrary point X on K_1, join X to A to meet K_2 again in Y, and join X to C to meet K_3 again in Z.
   (a) Show that the points Z, B, Y are collinear.
   (b) Show that the area of triangle XYZ is less than or equal to 4 times the area of triangle $O_1O_2O_3$.

6. **(AUS 6)** (SL87-1).

7. **(BEL 1)** Let $f : (0, +\infty) \to \mathbb{R}$ be a function having the property that $f(x) = f(1/x)$ for all $x > 0$. Prove that there exists a function $u : [1, +\infty) \to \mathbb{R}$ satisfying $u\left(\frac{x+1/x}{2}\right) = f(x)$ for all $x > 0$.

8. **(BEL 2)** Determine the least possible value of the natural number n such that $n!$ ends in exactly 1987 zeros.

9. **(BEL 3)** In the set of 20 elements $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 0, A, B, C, D, J, K, L, U, X, Y, Z\}$ we have made a random sequence of 28 throws. What is the probability that the sequence *CUBA JULY* 1987 appears in this order in the sequence already thrown?

10. **(FIN 1)** In a Cartesian coordinate system, the circle C_1 has center $O_1(-2, 0)$ and radius 3. Denote the point $(1, 0)$ by A and the origin by O. Prove that there is a constant $c > 0$ such that for every X that is exterior to C_1,
$$OX - 1 \geq c \min\{AX, AX^2\}.$$
Find the largest possible c.

11. **(FIN 2)** Let $S \subset [0, 1]$ be a set of 5 points with $\{0, 1\} \subset S$. The graph of a real function $f : [0, 1] \to [0, 1]$ is continuous and increasing, and it is linear on every subinterval I in $[0, 1]$ such that the endpoints but no interior points of I are in S. We want to compute, using a computer, the extreme values of $g(x, t) = \frac{f(x+t)-f(x)}{f(x)-f(x-t)}$ for $x - t, x + t \in [0, 1]$. At how many points (x, t) is it necessary to compute $g(x, t)$ with the computer?

12. **(FIN 3)** (SL87-3).

13. **(FIN 4)** A be an infinite set of positive integers such that every $n \in A$ is the product of at most 1987 prime numbers. Prove that there is an infinite set $B \subset A$ and a number p such that the greatest common divisor of any two distinct numbers in B is b.

14. **(FRA 1)** Given n real numbers $0 < t_1 \leq t_2 \leq \cdots \leq t_n < 1$, prove that
$$(1 - t_n^2)\left(\frac{t_1}{(1 - t_1^2)^2} + \frac{t_2^2}{(1 - t_2^3)^2} + \cdots + \frac{t_n^n}{(1 - t_n^{n+1})^2}\right) < 1.$$

15. **(FRA 2)** Let $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ be nine strictly positive real numbers. We set
$$\begin{aligned} S_1 &= a_1b_2c_3, & S_2 &= a_2b_3c_1, & S_3 &= a_3b_1c_2; \\ T_1 &= a_1b_3c_2, & T_2 &= a_2b_1c_3, & T_3 &= a_3b_2c_1. \end{aligned}$$
Suppose that the set $\{S_1, S_2, S_3, T_1, T_2, T_3\}$ has at most two elements. Prove that
$$S_1 + S_2 + S_3 = T_1 + T_2 + T_3.$$

16. **(FRA 3)** Let ABC be a triangle. For every point M belonging to segment BC we denote by B' and c' the orthogonal projections of M on the straight lines AC and BC. Find points M for which the length of segment $B'C'$ is a minimum.

17. **(FRA 4)** Consider the number α obtained by writing one after another the decimal representations of $1, 1987, 1987^2, \dots$ to the right the decimal point. Show that α is irrational.

18. **(FRA 5)** (SL87-4).

19. **(FRG 1)** (SL87-14).

20. **(FRG 2)** (SL87-15).

21. **(FRG 3)** (SL87-16).

22. **(GBR 1)** (SL87-5).

23. **(GBR 2)** A lampshade is part of the surface of a right circular cone whose axis is vertical. Its upper and lower edges are two horizontal circles. Two points are selected on the upper smaller circle and four points on the lower larger circle. Each of these six points has three of the others that are its nearest neighbors at a distance d from it. By distance is meant the shortest distance measured over the curved survace of the lampshade. Prove that the area of the lampshade if $d^2(2\theta + \sqrt{3})$, where $\cot\frac{\theta}{2} = \frac{3}{\theta}$.

24. **(GBR 3)** Prove that if the equation $x^4 + ax^3 + bx + c = 0$ has all its roots real, then $ab \le 0$.

25. **(GBR 4)** Numbers $d(n,m)$, with m, n integers, $0 \le m \le n$, ae defined by $d(n,0) = d(n,n) = 0$ for all $n \ge 0$ and
$$md(n,m) = md(n-1,m) + (2n-m)d(n-1,m-1) \qquad \text{for all } 0 < m < n.$$
Prove that all the $d(n,m)$ are integers.

26. **(GBR 5)** Prove that if x, y, z are real numbers such that $x^2+y^2+z^2 = 2$, then
$$x + y + z \le xyz + 2.$$

27. **(GBR 6)** Find, with proof, the smallest real number C with the following property: For every infinite sequence $\{x_i\}$ of positive real numbers such that $x_1 + x_2 + \cdots + x_n \le x_{n+1}$ for $n = 1, 2, 3, \dots$, we have
$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \le c\sqrt{x_1 + x_2 + \cdots + x_n} \quad \text{for } n = 1, 2, 3, \dots.$$

28. **(GDR 1)** In a chess tournament there are $n \ge 5$ players, and they have already played $\left[\frac{n^2}{4}\right] + 2$ games (each pair have played each other at most once).
    (a) Prove that there are five players a, b, c, d, e for which the pairs ab, ac, bc, ad, ae, de have already played.

(b) Is the statement also valid for the $\left[\frac{n^2}{4}\right]+1$ games played?
Make the proof by induction over n.

29. **(GDR 2)** (SL87-13).

30. **(GRE 1)** Consider the regular 1987-gon $A_1A_2\dots A_{1987}$ with center O. Show that the sum of vectors belonging to any proper subset of $M = \{OA_j \mid j = 1, 2, \dots, 1987\}$ is nonzero.

31. **(GRE 2)** Construct a triangle ABC given its side $a = BC$, its circumradius R ($2R \ge a$), and the difference $1/k = 1/c - 1/b$, where $c = AB$ and $b = AC$.

32. **(GRE 3)** Solve the equation $28^x = 19^y + 87^z$, where x, y, z are integers.

33. **(GRE 4)** (SL87-6).

34. **(HUN 1)** (SL87-8).

35. **(HUN 2)** (SL87-9).

36. **(ICE 1)** A game consists in pushing a flat stone along a sequence of squares $S_0, S_1, S_2, \dots$ that are arranged in linear order. The stone is initially placed on square S_0. When the stone stops on a square S_k it is pushed again in the same direction and so on until it reaches S_{1987} or goes beyond it; then the game stops. Each time the stone is pushed, the probability that it will advance exactly n squares is $1/2^n$. Determine the probability that the stone will stop exactly on square S_{1987}.

37. **(ICE 2)** Five distinct numbers are drawn successively and at random from the set $\{1, \dots, n\}$. Show that the probability of a draw in which the first three numbers as well as all five numbers can be arranged to form an arithmetic progression is greater than $\frac{6}{(n-2)^3}$.

38. **(ICE 3)** (SL87-10).

39. **(LUX 1)** Let A be a set of polynomials with real coefficients and let them satisfy the following conditions:
   (i) if $f \in A$ and $\deg f \le 1$, then $f(x) = x - 1$;
   (ii) if $f \in A$ and $\deg f \ge 2$, then either there exists $g \in A$ such that $f(x) = x^{2+\deg g} + xg(x) - 1$ or there exist $g, h \in A$ such that $f(x) = x^{1+\deg g} g(x) + h(x)$;
   (iii) for every $f, g \in A$, both $x^{2+\deg f} + xf(x) - 1$ and $x^{1+\deg f} f(x) + g(x)$ belong to A.

   Let $R_n(f)$ be the remainder of the Euclidean division of the polynomial $f(x)$ by x^n. Prove that for all $f \in A$ and for all natural numbers $n \ge 1$ we have
$$R_n(f)(1) \le 0 \quad \text{and} \quad R_n(f)(1) = 0 \Rightarrow R_n(f) \in A.$$

40. **(MON 1)** The perpendicular line issued from the center of the circumcircle to the bisector of angle C in a triangle ABC divides the segment of

the bisector inside ABC into two segments with ratio of lengths λ. Given $b = AC$ and $a = BC$, find the length of side c.

41. **(MON 2)** Let n points be given arbitrarily in the plane, no three of them collinear. Let us draw segments between pairs of these points. What is the minimum number of segments that can be colored red in such a way that among any four points, three of them are connected by segments that form a red triangle?

42. **(MON 3)** Find the integer solutions of the equation

$$\left[\sqrt{2}\, m\right] = \left[(2+\sqrt{2})n\right].$$

43. **(MON 4)** Let $2n+3$ points be given in the plane in such a way that no three lie on a line and no four lie on a circle. Prove that the number of circles that pass through three of these points and contain exactly n interior points is not less than $\frac{1}{3}\binom{2n+3}{2}$.

44. **(MOR 1)** Let $\theta_1, \theta_2, \dots, \theta_n$ be real numbers such that $\sin\theta_1 + \cdots + \sin\theta_n = 0$. Prove that

$$|\sin\theta_1 + 2\sin\theta_2 + \cdots + n\sin\theta_n| \le \left[\frac{n^2}{4}\right].$$

45. **(MOR 2)** Let us consider a variable polygon with $2n$ sides ($n \in \mathbb{N}$) in a fixed circle such that $2n-1$ of its sides pass through $2n-1$ fixed points lying on a straight line Δ. Prove that the last side also passes through a fixed point lying on Δ.

46. **(NET 1)** (SL87-7).

47. **(NET 2)** Through a point P within a triangle ABC the lines l, m, and n perpendicular respectively to AP, BP, CP are drawn. Prove that if l intersects the line BC in Q, m intersects AC in R, and n intersects AB in S, then the points Q, R, and S are collinear.

48. **(POL 1)** (SL87-11).

49. **(POL 2)** In the coordinate system in the plane we consider a convex polygon W and lines given by equations $x = k$, $y = m$, where k and m are integers. The lines determine a tiling of the plane with unit squares. We say that the boundary of W intersects a square if the boundary contains an interior point of the square. Prove that the boundary of W intersects at most $4\lceil d\rceil$ unit squares, where d is the maximal distance of points belonging to W (i.e., the diameter of W) and $\lceil d\rceil$ is the least integer not less than d.

50. **(POL 3)** Let P, Q, R be polynomials with real coefficients, satisfying $P^4 + Q^4 = R^2$. Prove that there exist real numbers p, q, r and a polynomial S such that $P = pS$, $Q = qS$ and $R = rS^2$.

Variants: (1) $P^4+Q^4=R^4$; (2) $\gcd(P,Q)=1$; (3) $\pm P^4+Q^4=R^2$ or R^4.

51. **(POL 4)** The function F is a one-to-one transformation of the plane into itself that maps rectangles into rectangles (rectangles are closed; continuity is not assumed). Prove that F maps squares into squares.

52. **(POL 5)** (SL87-12).

53. **(ROM 1)** (SL87-17).

54. **(ROM 2)** Let n be a natural number. Solve in integers the equation
$$x^n+y^n=(x-y)^{n+1}.$$

55. **(ROM 3)** Two moving bodies M_1, M_2 are displaced uniformly on two coplanar straight lines. Describe the union of all straight lines M_1M_2.

56. **(ROM 4)** (SL87-18).

57. **(ROM 5)** The bisectors of the angles B, C of a triangle ABC intersect the opposite sides in B', C' respectively. Prove that the straight line $B'C'$ intersects the inscribed circle in two different points.

58. **(SPA 1)** Find, with argument, the integer solutions of the equation
$$3z^2=2x^3+385x^2+256x-58195.$$

59. **(SPA 2)** It is given that a_{11}, a_{22} are real numbers, that $x_1, x_2, a_{12}, b_1, b_2$ are complex numbers, and that $a_{11}a_{22}=a_{12}\overline{a_{12}}$ (where $\overline{a_{12}}$ is the conjugate of a_{12}). We consider the following system in x_1, x_2:
$$\begin{aligned}\overline{x_1}(a_{11}x_1+a_{12}x_2)&=b_1,\\ \overline{x_2}(a_{12}x_1+a_{22}x_2)&=b_2.\end{aligned}$$
(a) Give one condition to make the system consistent.
(b) Give one condition to make $\arg x_1-\arg x_2=98^\circ$.

60. **(TUR 1)** It is given that $x=-2272$, $y=10^3+10^2c+10b+a$, and $z=1$ satisfy the equation $ax+by+cz=1$, where a, b, c are positive integers with $a<b<c$. Find y.

61. **(TUR 2)** Let PQ be a line segment of constant length λ taken on the side BC of a triangle ABC with the order B, P, Q, C, and let the lines through P and Q parallel to the lateral sides meet AC at P_1 and Q_1 and AB at P_2 and Q_2 respectively. Prove that the sum of the areas of the trapezoids PQQ_1P_1 and PQQ_2P_2 is independent of the position of PQ on BC.

62. **(TUR 3)** Let l, l' be two lines in 3-space and let A, B, C be three points taken on l with B as midpoint of the segment AC. If a, b, c are the distances of A, B, C from l', respectively, show that $b\le\sqrt{\frac{a^2+c^2}{2}}$, equality holding if l, l' are parallel.

63. **(TUR 4)** Compute $\sum_{k=0}^{2n}(-1)^k a_k^2$, where a_k are the coefficients in the expansion

$$(1-\sqrt{2}x+x^2)^n=\sum_{k=0}^{2n} a_k x^k.$$

64. **(USA 1)** Let $r>1$ be a real number, and let n be the largest integer smaller than r. Consider an arbitrary real number x with $0\le x\le \frac{n}{r-1}$. By a *base-r expansion* of x we mean a representation of x in the form

$$x=\frac{a_1}{r}+\frac{a_2}{r^2}+\frac{a_3}{r^3}+\cdots,$$

where the a_i are integers with $0\le a_i<r$.
You may assume without proof that every number x with $0\le x\le\frac{n}{r-1}$ has at least one base-r expansion.
Prove that if r is not an integer, then there exists a number p, $0\le p\le\frac{n}{r-1}$, which has infinitely many distinct base-r expansions.

65. **(USA 2)** The *runs* of a decimal number are its increasing or decreasing blocks of digits. Thus 024379 has three runs: 024, 43, and 379. Determine the average number of runs for a decimal number in the set $\{d_1d_2\dots d_n \mid d_k\neq d_{k+1},\ k=1,2,\dots,n-1\}$, where $n\ge 2$.

66. **(USA 3)** (SL87-2).

67. **(USS 1)** If a,b,c,d are real numbers such that $a^2+b^2+c^2+d^2\le 1$, find the maximum of the expression

$$(a+b)^4+(a+c)^4+(a+d)^4+(b+c)^4+(b+d)^4+(c+d)^4.$$

68. **(USS 2)** (SL87-19).
Original formulation. Let there be given positive real numbers α,β,γ such that $\alpha+\beta+\gamma<\pi$, $\alpha+\beta>\gamma$, $\beta+\gamma>\alpha$, $\gamma+\alpha>\beta$. Prove that it is possible to draw a triangle with the lengths of the sides $\sin\alpha$, $\sin\beta$, $\sin\gamma$. Moreover, prove that its area is less than

$$\frac{1}{8}(\sin 2\alpha+\sin 2\beta+\sin 2\gamma).$$

69. **(USS 3)** (SL87-20).

70. **(USS 4)** (SL87-21).

71. **(USS 5)** To every natural number k, $k\ge 2$, there corresponds a sequence $a_n(k)$ according to the following rule:

$$a_0=k,\qquad a_n=\tau(a_{n-1})\ \text{ for } n\ge 1,$$

in which $\tau(a)$ is the number of different divisors of a. Find all k for which the sequence $a_n(k)$ does not contain the square of an integer.

72. **(VIE 1)** Is it possible to cover a rectangle of dimensions $m \times n$ with bricks that have the trimino angular shape (an arrangement of three unit squares forming the letter L) if:
(a) $m \times n = 1985 \times 1987$;
(b) $m \times n = 1987 \times 1989$?

73. **(VIE 2)** Let $f(x)$ be a periodic function of period $T > 0$ defined over $\mathbb{R}$. Its first derivative is continuous on $\mathbb{R}$. Prove that there exist $x, y \in [0, T)$ such that $x \neq y$ and
$$f(x)f'(y) = f(y)f'(x).$$

74. **(VIE 3)** (SL87-22).

75. **(VIE 4)** Let a_k be positive numbers such that $a_1 \geq 1$ and $a_{k+1} - a_k \geq 1$ $(k = 1, 2, \dots)$. Prove that for every $n \in \mathbb{N}$,
$$\sum_{k=1}^{n} \frac{1}{a_{k+1} \sqrt[1987]{a_k}} < 1987.$$

76. **(VIE 5)** Given two sequences of positive numbers $\{a_k\}$ and $\{b_k\}$ $(k \in \mathbb{N})$ such that
(i) $a_k < b_k$,
(ii) $\cos a_k x + \cos b_k x \geq -\frac{1}{k}$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$,
prove the existence of $\lim_{k\to\infty} \frac{a_k}{b_k}$ and find this limit.

77. **(YUG 1)** Find the least natural number k such that for any $n \in [0, 1]$ and any natural number n,
$$a^k(1 - a)^n < \frac{1}{(n + 1)^3}.$$

78. **(YUG 2)** (SL87-23).

3.28.3 Shortlisted Problems

1. **(AUS 6)** Let f be a function that satisfies the following conditions:
(i) If $x > y$ and $f(y) - y \geq v \geq f(x) - x$, then $f(z) = v + z$, for some number z between x and y.
(ii) The equation $f(x) = 0$ has at least one solution, and among the solutions of this equation, there is one that is not smaller than all the other solutions;
(iii) $f(0) = 1$.
(iv) $f(1987) \leq 1988$.
(v) $f(x)f(y) = f(xf(y) + yf(x) - xy)$.
Find $f(1987)$.

2. **(USA 3)** At a party attended by n married couples, each person talks to everyone else at the party except his or her spouse. The conversations involve sets of persons or cliques $C_1, C_2, \dots, C_k$ with the following property: no couple are members of the same clique, but for every other pair of persons there is exactly one clique to which both members belong. Prove that if $n \geq 4$, then $k \geq 2n$.

3. **(FIN 3)** Does there exist a second-degree polynomial $p(x, y)$ in two variables such that every nonnegative integer n equals $p(k, m)$ for one and only one ordered pair (k, m) of nonnegative integers?

4. **(FRA 5)** Let $ABCDEFGH$ be a parallelepiped with $AE \| BF \| CG \| DH$. Prove the inequality

$$AF + AH + AC \leq AB + AD + AE + AG.$$

In what cases does equality hold?

5. **(GBR 1)** Find, with proof, the point P in the interior of an acute-angled triangle ABC for which $BL^2 + CM^2 + AN^2$ is a minimum, where L, M, N are the feet of the perpendiculars from P to BC, CA, AB respectively.

6. **(GRE 4)** Show that if a, b, c are the lengths of the sides of a triangle and if $2S = a + b + c$, then

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \left(\frac{2}{3}\right)^{n-2} S^{n-1}, \qquad n \geq 1.$$

7. **(NET 1)** Given five real numbers u_0, u_1, u_2, u_3, u_4, prove that it is always possible to find five real numbers v_0, v_1, v_2, v_3, v_4 that satisfy the following conditions:
   (i) $u_i - v_i \in \mathbb{N}$.
   (ii) $\sum_{0 \leq i < j \leq 4} (v_i - v_j)^2 < 4$.

8. **(HUN 1)** (a) Let $(m, k) = 1$. Prove that there exist integers $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_k$ such that each product $a_i b_j$ ($i = 1, 2, \dots, m$; $j = 1, 2, \dots, k$) gives a different residue when divided by mk.
   (b) Let $(m, k) > 1$. Prove that for any integers $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_k$ there must be two products $a_i b_j$ and $a_s b_t$ ($(i, j) \neq (s, t)$) that give the same residue when divided by mk.

9. **(HUN 2)** Does there exist a set M in usual Euclidean space such that for every plane λ the intersection $M \cap \lambda$ is finite and nonempty?

10. **(ICE 3)** Let S_1 and S_2 be two spheres with distinct radii that touch externally. The spheres lie inside a cone C, and each sphere touches the cone in a full circle. Inside the cone there are n additional solid spheres arranged in a ring in such a way that each solid sphere touches the cone C, both of the spheres S_1 and S_2 externally, as well as the two neighboring solid spheres. What are the possible values of n?

11. **(POL 1)** Find the number of partitions of the set $\{1, 2, \dots, n\}$ into three subsets A_1, A_2, A_3, some of which may be empty, such that the following conditions are satisfied:
    (i) After the elements of every subset have been put in ascending order, every two consecutive elements of any subset have different parity.
    (ii) If A_1, A_2, A_3 are all nonempty, then in exactly one of them the minimal number is even.

12. **(POL 5)** Given a nonequilateral triangle ABC, the vertices listed counterclockwise, find the locus of the centroids of the equilateral triangles $A'B'C'$ (the vertices listed counterclockwise) for which the triples of points A, B', C'; A', B, C'; and A', B', C are collinear.

13. **(GDR 2)**$^{\text{IMO5}}$ Is it possible to put 1987 points in the Euclidean plane such that the distance between each pair of points is irrational and each three points determine a nondegenerate triangle with rational area?

14. **(FRG 1)** How many words with n digits can be formed from the alphabet $\{0, 1, 2, 3, 4\}$, if neighboring digits must differ by exactly one?

15. **(FRG 2)**$^{\text{IMO3}}$ Suppose $x_1, x_2, \dots, x_n$ are real numbers with $x_1^2 + x_2^2 + \dots + x_n^2 = 1$. Prove that for any integer $k > 1$ there are integers e_i not all 0 and with $|e_i| < k$ such that

$$|e_1x_1 + e_2x_2 + \dots + e_nx_n| \le \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

16. **(FRG 3)**$^{\text{IMO1}}$ Let S be a set of n elements. We denote the number of all permutations of S that have exactly k fixed points by $p_n(k)$. Prove:
    (a) $\sum_{k=0}^{n} kp_n(k) = n!$;
    (b) $\sum_{k=0}^{n} (k-1)^2 p_n(k) = n!$.

17. **(ROM 1)** Prove that there exists a four-coloring of the set $M = \{1, 2, \dots, 1987\}$ such that any arithmetic progression with 10 terms in the set M is not monochromatic.

    Alternative formulation. Let $M = \{1, 2, \dots, 1987\}$. Prove that there is a function $f : M \to \{1, 2, 3, 4\}$ that is not constant on every set of 10 terms from M that form an arithmetic progression.

18. **(ROM 4)** For any integer $r \ge 1$, determine the smallest integer $h(r) \ge 1$ such that for any partition of the set $\{1, 2, \dots, h(r)\}$ into r classes, there are integers $a \ge 0, 1 \le x \le y$, such that $a + x, a + y, a + x + y$ belong to the same class.

19. **(USS 2)** Let α, β, γ be positive real numbers such that $\alpha + \beta + \gamma < \pi$, $\alpha + \beta > \gamma$, $\beta + \gamma > \alpha$, $\gamma + \alpha > \beta$. Prove that with the segments of lengths $\sin\alpha$, $\sin\beta$, $\sin\gamma$ we can construct a triangle and that its area is not greater than

$$\frac{1}{8}(\sin 2\alpha + \sin 2\beta + \sin 2\gamma).$$

20. **(USS 3)**$^{\text{IMO6}}$ Let $f(x) = x^2 + x + p$, $p \in \mathbb{N}$. Prove that if the numbers $f(0), f(1), \dots, f([\sqrt{p/3}\,])$ are primes, then all the numbers $f(0), f(1), \dots,$ $f(p-2)$ are primes.

21. **(USS 4)**$^{\text{IMO2}}$ The prolongation of the bisector AL ($L \in BC$) in the acute-angled triangle ABC intersects the circumscribed circle at point N. From point L to the sides AB and AC are drawn the perpendiculars LK and LM respectively. Prove that the area of the triangle ABC is equal to the area of the quadrilateral $AKNM$.

22. **(VIE 3)**$^{\text{IMO4}}$ Does there exist a function $f : \mathbb{N} \to \mathbb{N}$, such that $f(f(n)) = n + 1987$ for every natural number n?

23. **(YUG 2)** Prove that for every natural number k ($k \geq 2$) there exists an irrational number r such that for every natural number m,

$$[r^m] \equiv -1 \pmod{k}.$$

Remark. An easier variant: Find r as a root of a polynomial of second degree with integer coefficients.

3.29 The Twenty-Ninth IMO Canberra, Australia, July 9–21, 1988

3.29.1 Contest Problems

First Day (July 15)

1. Consider two concentric circles of radii R and r $(R > r)$ with center O. Fix P on the small circle and consider the variable chord PA of the small circle. Points B and C lie on the large circle; B, P, C are collinear and BC is perpendicular to AP.
   (a) For which value(s) of $\angle OPA$ is the sum $BC^2 + CA^2 + AB^2$ extremal?
   (b) What are the possible positions of the midpoints U of BA and V of AC as $\angle OPA$ varies?

2. Let n be an even positive integer. Let $A_1, A_2, \ldots, A_{n+1}$ be sets having n elements each such that any two of them have exactly one element in common, while every element of their union belongs to at least two of the given sets. For which n can one assign to every element of the union one of the numbers 0 and 1 in such a manner that each of the sets has exactly $n/2$ zeros?

3. A function f defined on the positive integers (and taking positive integer values) is given by
$$\begin{aligned} f(1) = 1, \quad & f(3) = 3, \\ f(2n) &= f(n), \\ f(4n+1) &= 2f(2n+1) - f(n), \\ f(4n+3) &= 3f(2n+1) - 2f(n), \end{aligned}$$
for all positive integers n. Determine with proof the number of positive integers less than or equal to 1988 for which $f(n) = n$.

Second Day (July 16)

4. Show that the solution set of the inequality
$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$
is the union of disjoint half-open intervals with the sum of lengths 1988.

5. In a right-angled triangle ABC let AD be the altitude drawn to the hypotenuse and let the straight line joining the incenters of the triangles ABD, ACD intersect the sides AB, AC at the points K, L respectively. If E and E_1 denote the areas of the triangles ABC and AKL respectively, show that $\frac{E}{E_1} \geq 2$.

6. Let a and b be two positive integers such that $ab+1$ divides a^2+b^2. Show that $\frac{a^2+b^2}{ab+1}$ is a perfect square.

3.29.2 Longlisted Problems

1. **(BUL 1)** (SL88-1).
2. **(BUL 2)** Let $a_n = \left[\sqrt{(n+1)^2 + n^2}\right]$, $n = 1, 2, \dots$, where $[x]$ denotes the integer part of x. Prove that
   (a) there are infinitely many positive integers m such that $a_{m+1} - a_m > 1$;
   (b) there are infinitely many positive integers m such that $a_{m+1} - a_m = 1$.
3. **(BUL 3)** (SL88-2).
4. **(CAN 1)** (SL88-3).
5. **(CUB 1)** Let k be a positive integer and M_k the set of all the integers that are between $2k^2 + k$ and $2k^2 + 3k$, both included. Is it possible to partition M_k into two subsets A and B such that
$$\sum_{x \in A} x^2 = \sum_{x \in B} x^2?$$
6. **(CZS 1)** (SL88-4).
7. **(CZS 2)** (SL88-5).
8. **(CZS 3)** (SL88-6).
9. **(FRA 1)** If a_0 is a positive real number, consider the sequence $\{a_n\}$ defined by
$$a_{n+1} = \frac{a_n^2 - 1}{n+1} \qquad \text{for } n \geq 0.$$
   Show that there exists a real number $a > 0$ such that:
   (i) for all real $a_0 \geq a$, the sequence $\{a_n\} \to +\infty$ $(n \to \infty)$;
   (ii) for all real $a_0 < a$, the sequence $\{a_n\} \to 0$.
10. **(FRA 2)** (SL88-7).
11. **(FRA 3)** (SL88-8).
12. **(FRA 4)** Show that there do not exist more than 27 half-lines (or rays) emanating from the origin in 3-dimensional space such that the angle between each pair of rays is greater than of equal to $\pi/4$.
13. **(FRA 5)** Let T be a triangle with inscribed circle C. A square with sides of length a is circumscribed about the same circle C. Show that the total length of the parts of the edges of the square interior to the triangle T is at least $2a$.
14. **(FRG 1)** (SL88-9).
15. **(FRG 2)** Let $1 \leq k < n$. Consider all finite sequences of positive integers with sum n. Find $T(n, k)$, the total number of terms of size k in all of these sequences.

16. **(FRG 3)** Show that if n runs through all positive integers, $f(n) = \left[n + \sqrt{n/3} + 1/2\right]$ runs through all positive integers skipping the terms of the sequence $a_n = 3n^2 - 2n$.

17. **(FRG 4)** Show that if n runs through all positive integers, $f(n) = \left[n + \sqrt{3n} + 1/2\right]$ runs through all positive integers skipping the terms of the sequence $a_n = \left[\frac{n^2+2n}{3}\right]$.

18. **(GBR 1)** (SL88-25).

19. **(GBR 2)** (SL88-26).

20. **(GBR 3)** It is proposed to partition the set of positive integers into two disjoint subsets A and B subject to the following conditions:
    (i) 1 is in A;
    (ii) no two distinct members of A have a sum of the form $2^k + 2$ ($k = 0, 1, 2, \ldots$); and
    (iii) no two distinct members of B have a sum of that form.
    Show that this partitioning can be carried out in a unique manner and determine the subsets to which 1987, 1988, and 1989 belong.

21. **(GBR 4)** (SL88-27).

22. **(GBR 5)** (SL88-28).

23. **(GDR 1)** (SL88-10).

24. **(GDR 2)** Let $Z_{m,n}$ be the set of all ordered pairs (i, j) with $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Also let $a_{m,n}$ be the number of all those subsets of $Z_{m,n}$ that contain no two ordered pairs $(i_1, j_1), (i_2, j_2)$ with $|i_1 - i_2| + |j_1 - j_2| = 1$. Show that for all positive integers m and k,
$$a_{m,2k}^2 \le a_{m,2k-1}a_{m,2k+1}.$$

25. **(GDR 3)** (SL88-11).

26. **(GRE 1)** Let AB and CD be two perpendicular chords of a circle with center O and radius r, and let X, Y, Z, W denote in cyclical order the four parts into which the disk is thus divided. Find the maximum and minimum of the quantity $\frac{A(Z)}{A(Y)+A(W)}$, where $A(U)$ denotes the area of U.

27. **(GRE 2)** (SL88-12).

28. **(GRE 3)** (SL88-13).

29. **(GRE 4)** Find positive integers $x_1, x_2, \ldots, x_{29}$, at least one of which is greater than 1988, such that
$$x_1^2 + x_2^2 + \cdots + x_{29}^2 = 29x_1x_2\ldots x_{29}.$$

30. **(HKG 1)** Find the total number of different integers that the function

$$f(x) = [x] + [2x] + \left[\frac{5x}{3}\right] + [3x] + [4x]$$

takes for $0 \le x \le 100$.

31. **(HKG 2)** The circle $x^2 + y^2 = r^2$ meets the coordinate axes at $A = (r, 0)$, $B = (-r, 0)$, $C = (0, r)$, and $D = (0, -r)$. Let $P = (u, v)$ and $Q = (-u, v)$ be two points on the circumference of the circle. Let N be the point of intersection of PQ and the y-axis, and let M be the foot of the perpendicular drawn from P to the x-axis. If r^2 is odd, $u = p^m > q^n = v$, where p and q are prime numbers, and m and n are natural numbers, show that

$$|AM| = 1, \quad |BM| = 9, \quad |DN| = 8, \quad |PQ| = 8.$$

32. **(HKG 3)** Assuming that the roots of $x^3 + px^2 + qx + r = 0$ are all real and positive, find a relation between p, q, and r that gives a necessary condition for the roots to be exactly the cosines of three angles of a triangle.

33. **(HKG 4)** Find a necessary and sufficient condition on the natural number n for the equation $x^n + (2 + x)^n + (2 - x)^n = 0$ to have a real root.

34. **(HKG 5)** Express the number 1988 as the sum of some positive integers in such a way that the product of these positive integers is maximal.

35. **(HKG 6)** In the triangle ABC, let D, E, and F be the midpoints of the three sides, X, Y, and Z the feet of the three altitudes, H the orthocenter, and P, Q, and R the midpoints of the line segments joining H to the three vertices. Show that the nine points $D, E, F, P, Q, R, X, Y, Z$ lie on a circle.

36. **(HUN 1)** (SL88-14).

37. **(HUN 2)** Let n points be given on the surface of a sphere. Show that the surface can be divided into n congruent regions such that each of them contains exactly one of the given points.

38. **(HUN 3)** In a multiple choice test there were 4 questions and 3 possible answers for each question. A group of students was tested and it turned out that for any 3 of them there was a question that the three students answered differently. What is the maximal possible number of students tested?

39. **(ICE 1)** (SL88-15).

40. **(ICE 2)** A sequence of numbers a_n, $n = 1, 2, \ldots$, is defined as follows: $a_1 = 1/2$, and for each $n \ge 2$,

$$a_n = \left(\frac{2n-3}{2n}\right) a_{n-1}.$$

Prove that $\sum_{k=1}^{n} a_k < 1$ for all $n \ge 1$.

41. **(INA 1)**
   (a) Let ABC be a triangle with $AB = 12$ and $AC = 16$. Suppose M is the midpoint of side BC and points E and F are chosen on sides AC and AB respectively, and suppose that the lines EF and AM intersect at G. If $AE = 2AF$ then find the ratio EG/GF.
   (b) Let E be a point external to a circle and suppose that two chords EAB and ECD meet at an angle of $40°$. If $AB = BC = CD$, find the size of $\angle ACD$.

42. **(INA 2)**
   (a) Four balls of radius 1 are mutually tangent, three resting an the floor and the fourth resting on the others. A tetrahedron, each of whose edges has length s, is circumscribed around the balls. Find the value of s.
   (b) Suppose that $ABCD$ and $EFGH$ are opposite faces of a rectangular solid, with $\angle DHC = 45°$ and $\angle FHB = 60°$. Find the cosine of $\angle BHD$.

43. **(INA 3)**
   (a) The polynomial $x^{2k} + 1 + (x+1)^{2k}$ is not divisible by $x^2 + x + 1$. Find the value of k.
   (b) If p, q, and r are distinct roots of $x^3 - x^2 + x - 2 = 0$, find the value of $p^3 + q^3 + r^3$.
   (c) If r is the remainder when each of the numbers 1059, 1417, and 2312 is divided by d, where d is an integer greater than one, find the value of $d - r$.
   (d) What is the smallest positive odd integer n such that the product of $2^{1/7}, 2^{3/7}, \dots, 2^{(2n+1)/7}$ is greater than 1000?

44. **(INA 4)**
   (a) Let $g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. What is the remainder when the polynomial $g(x^{12})$ is divided by the polynomial $g(x)$?
   (b) If k is a positive integer and f is a function such that for every positive number x, $f(x^2+1)^{\sqrt{x}} = k$, find the value of $f\left(\frac{9+y^2}{y^2}\right)^{\sqrt{12/y}}$ for every positive number y.
   (c) The function f satisfies the functional equation $f(x) + f(y) = f(x + y) - xy - 1$ for every pair x, y of real numbers. If $f(1) = 1$, find the number of integers n for which $f(n) = n$.

45. **(INA 5)**
   (a) Consider a circle K with diameter AB, a circle L tangent to AB and to K, and a circle M tangent to circle K, circle L, and AB. Calculate the ratio of the area of circle K to the area of circle M.
   (b) In triangle ABC, $AB = AC$ and $\measuredangle CAB = 80°$. If points D, E, and F lie on sides BC, AC, and AB, respectively, and $CE = CD$ and $BF = BD$, find the measure of $\measuredangle EDF$.

46. **(INA 6)**
   (a) Calculate $x = \frac{(11+6\sqrt{2})\sqrt{11-6\sqrt{2}}-(11-6\sqrt{2})\sqrt{11+6\sqrt{2}}}{(\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2})-(\sqrt{\sqrt{5}+1})}$.
   (b) For each positive number x, let $k = \frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$. Calculate the minimum value of k.

47. **(IRE 1)** (SL88-16).

48. **(IRE 2)** Find all plane triangles whose sides have integer length and whose incircles have unit radius.

49. **(IRE 3)** Let $-1 < x < 1$. Show that

$$\sum_{k=0}^{6} \frac{1-x^2}{1-2x\cos(2\pi k/7)+x^2} = \frac{7(1+x^7)}{1-x^7}.$$

   Deduce that

$$\csc^2\frac{\pi}{7} + \csc^2\frac{2\pi}{7} + \csc^2\frac{3\pi}{7} = 8.$$

50. **(IRE 4)** Let $g(n)$ be defined as follows:

$$g(1) = 0, \qquad g(2) = 1,$$
$$g(n+2) = g(n) + g(n+1) + 1 \qquad (n \geq 1).$$

   Prove that if $n > 5$ is a prime, then n divides $g(n)(g(n)+1)$.

51. **(ISR 1)** Let $A_1, A_2, \dots, A_{29}$ be 29 different sequences of positive integers. For $1 \leq i < j \leq 29$ and any natural number x, we define $N_i(x)$ to be the number of elements of the sequence A_i that are less than or equal to x, and $N_{ij}(x)$ to be the number of elements of the intersection $A_i \cap A_j$ that are less than or equal to x.
   It is given that for all $1 \leq i \leq 29$ and every natural number x,

$$N_i(x) \geq \frac{x}{e}, \qquad \text{where } e = 2.71828\dots.$$

   Prove that there exists at least one pair i, j ($1 \leq i < j \leq 29$) such that $N_{ij}(1988) > 200$.

52. **(ISR 2)** (SL88-17).

53. **(KOR 1)** Let $x = p$, $y = q$, $z = r$, $w = s$ be the unique solution of the system of linear equations

$$x + a_i y + a_i^2 z + a_i^3 w = a_i^4, \qquad i = 1, 2, 3, 4.$$

   Express the solution of the following system in terms of p, q, r, and s:

$$x + a_i^2 y + a_i^4 z + a_i^6 w = a_i^8, \qquad i = 1, 2, 3, 4.$$

   Assume the uniqueness of the solution.

54. **(KOR 2)** (SL88-22).

55. **(KOR 3)** Find all positive integers x such that the product of all digits of x is given by $x^2 - 10x - 22$.

56. **(KOR 4)** The Fibonacci sequence is defined by

$$a_{n+1} = a_n + a_{n-1} \quad (n \geq 1), \quad a_0 = 0,\ a_1 = a_2 = 1.$$

Find the greatest common divisor of the 1960th and 1988th terms of the Fibonacci sequence.

57. **(KOR 5)** Let C be a cube with edges of length 2. Construct a solid with fourteen faces by cutting off all eight corners of C, keeping the new faces perpendicular to the diagonals of the cube and keeping the newly formed faces identical. If at the conclusion of this process the fourteen faces so formed have the same area, find the area of each face of the new solid.

58. **(KOR 6)** For each pair of positive integers k and n, let $S_k(n)$ be the base-k digit sum of n. Prove that there are at most two primes p less than 20,000 for which $S_{31}(p)$ is a composite number.

59. **(LUX 1)** (SL88-18).

60. **(MEX 1)** (SL88-19).

61. **(MEX 2)** Prove that the numbers A, B, and C are equal, where we define A as the number of ways that we can cover a $2 \times n$ rectangle with 2×1 rectangles, B as the number of sequences of ones and twos that add up to n, and C as

$$\begin{cases} \binom{m}{0} + \binom{m+1}{2} + \cdots + \binom{2m}{2m} & \text{if } n = 2m, \\ \binom{m+1}{1} + \binom{m+2}{3} + \cdots + \binom{2m+1}{2m+1} & \text{if } n = 2m+1. \end{cases}$$

62. **(MON 1)** The positive integer n has the property that in any set of n integers chosen from the integers $1, 2, \ldots, 1988$, twenty-nine of them form an arithmetic progression. Prove that $n > 1788$.

63. **(MON 2)** Let $ABCD$ be a quadrilateral. Let $A'BCD'$ be the reflection of $ABCD$ in BC, while $A''B'CD'$ is the reflection of $A'BCD'$ in CD' and $A''B''C'D'$ is the reflection of $A''B'CD'$ in $D'A''$. Show that if the lines AA'' and BB'' are parallel, then $ABCD$ is a cyclic quadrilateral.

64. **(MON 3)** Given n points $A_1, A_2, \ldots, A_n$, no three collinear, show that the n-gon $A_1A_2 \ldots A_n$ can be inscribed in a circle if and only if

$$\begin{aligned} &A_1A_2 \cdot A_3A_n \cdots A_{n-1}A_n + A_2A_3 \cdot A_4A_n \cdots A_{n-1}A_n \cdot A_1A_n + \cdots \\ &+ A_{n-1}A_{n-2} \cdot A_1A_n \cdots A_{n-3}A_n = A_1A_{n-1} \cdot A_2A_n \cdots A_{n-2}A_n. \end{aligned}$$

65. **(MON 4)** (SL88-20).

66. **(MON 5)** Suppose $\alpha_i > 0$, $\beta_i > 0$ for $1 \le i \le n$ $(n > 1)$ and that $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = \pi$. Prove that

$$\sum_{i=1}^{n} \frac{\cos \beta_i}{\sin \alpha_i} \le \sum_{i=1}^{n} \cot \alpha_i.$$

67. **(NET 1)** Given a set of 1988 points in the plane, no three points of the set collinear, the points of a subset with 1788 points are colored blue, and the remaining 200 are colored red. Prove that there exists a line in the plane such that each of the two parts into which the line divides the plane contains 894 blue points and 100 red points.

68. **(NET 2)** Let S be the set of all sequences $\{a_i \mid 1 \le i \le 7, a_i = 0 \text{ or } 1\}$. The distance between two elements $\{a_i\}$ and $\{b_i\}$ of S is defined as $\sum_{i=1}^7 |a_i - b_i|$. Let T be a subset of S in which any two elements have a distance apart greater than or equal to 3. Prove that T contains at most 16 elements. Give an example of such a subset with 16 elements.

69. **(POL 1)** For a convex polygon P in the plane let P' denote the convex polygon with vertices at the midpoints of the sides of P. Given an integer $n \ge 3$, determine sharp bounds for the ratio $\dfrac{\operatorname{area}(P')}{\operatorname{area}(P)}$ over all convex n-gons P.

70. **(POL 2)** In 3-dimensional space a point O is given and a finite set A of segments with the sum of the lengths equal to 1988. Prove that there exists a plane disjoint from A such that the distance from it to O does not exceed 574.

71. **(POL 3)** Given integers $a_1, \dots, a_{10}$, prove that there exists a nonzero sequence $(x_1, \dots, x_{10})$ such that all x_i belong to $\{-1, 0, 1\}$ and the number $\sum_{i=1}^{10} x_i a_i$ is divisible by 1001.

72. **(POL 4)** (SL88-21).

73. **(SIN 1)** In a group of n people each one knows exactly three others. They are seated around a table. We say that the seating is *perfect* if everyone knows the two sitting by their sides. Show that if there is a perfect seating S for the group, then there is always another perfect seating that cannot be obtained from S by rotation or reflection.

74. **(SIN 2)** (SL88-23).

75. **(SPA 1)** Let ABC be a triangle with inradius r and circumradius R. Show that

$$\sin\frac{A}{2}\sin\frac{B}{2} + \sin\frac{B}{2}\sin\frac{C}{2} + \sin\frac{C}{2}\sin\frac{A}{2} \le \frac{5}{8} + \frac{r}{4R}.$$

76. **(SPA 2)** The quadrilateral $A_1A_2A_3A_4$ is cyclic and its sides are $a_1 = A_1A_2$, $a_2 = A_2A_3$, $a_3 = A_3A_4$, and $a_4 = A_4A_1$. The respective circles

with centers I_i and radii ρ_i are tangent externally to each side a_i and to the sides a_{i+1} and a_{i-1} extended ($a_0 = a_4$). Show that

$$\prod_{i=1}^{4} \frac{a_i}{\rho_i} = 4(\csc A_1 + \csc A_2)^2.$$

77. **(SPA 3)** Consider $h+1$ chessboards. Number the squares of each board from 1 to 64 in such a way that when the perimeters of any two boards of the collection are brought into coincidence in any possible manner, no two squares in the same position have the same number. What is the maximum value of h?

78. **(SWE 1)** A two-person game is played with nine boxes arranged in a 3×3 square, initially empty, and with white and black stones. At each move a player puts three stones, not necessarily of the same color, in three boxes in either a horizontal or a vertical row. No box can contain stones of different colors: If, for instance, a player puts a white stone in a box containing black stones, the white stone and one of the black stones are removed from the box. The game is over when the center box and the corner boxes each contain one black stone and the other boxes are empty. At one stage of the game x boxes contained one black stone each and the other boxes were empty. Determine all possible values of x.

79. **(SWE 2)** (SL88-24).

80. **(SWE 3)** Let S be an infinite set of integers containing zero and such that the distance between successive numbers never exceeds a given fixed number. Consider the following procedure: Given a set X of integers, we construct a new set consisting of all numbers $x \pm s$, where x belongs to X and s belongs to S.
Starting from $S_0 = \{0\}$ we successively construct sets $S_1, S_2, S_3, \dots$ using this procedure. Show that after a finite number of steps we do not obtain any new sets; i.e., $S_k = S_{k_0}$ for $k \geq k_0$.

81. **(USA 1)** There are $n \geq 3$ job openings at a factory, ranked 1 to n in order of increasing pay. There are n job applicants, ranked 1 to n in order of increasing ability. Applicant i is qualified for job j if and only if $i \geq j$. The applicants arrive one at a time in random order. Each in turn is hired to the highest-ranking job for which he or she is qualified and that is lower in rank than any job already filled. (Under these rules, job 1 is always filled and hiring terminates thereafter.)
Show that applicants n and $n-1$ have the same probability of being hired.

82. **(USA 2)** The triangle ABC has a right angle at C. The point P is located on segment AC such that triangles PBA and PBC have congruent inscribed circles. Express the length $x = PC$ in terms of $a = BC$, $b = CA$, and $c = AB$.

83. **(USA 3)** (SL88-29).

84. **(USS 1)** (SL88-30).

85. **(USS 2)** (SL88-31).

86. **(USS 3)** Let a, b, c be integers different from zero. It is known that the equation $ax^2 + by^2 + cz^2 = 0$ has a solution (x, y, z) in integers different from the solution $x = y = z = 0$. Prove that the equation $ax^2+by^2+cz^2 = 1$ has a solution in rational numbers.

87. **(USS 4)** All the irreducible positive rational numbers such that the product of the numerator and the denominator is less than 1988 are written in increasing order. Prove that any two adjacent fractions a/b and c/d, $a/b < c/d$, satisfy the equation $bc - ad = 1$.

88. **(USS 5)** There are six circles inside a fixed circle, each tangent to the fixed circle and tangent to the two adjacent smaller circles. If the points of contact between the six circles and the larger circle are, in order, A_1, A_2, A_3, A_4, A_5, and A_6, prove that
$$A_1A_2 \cdot A_3A_4 \cdot A_5A_6 = A_2A_3 \cdot A_4A_5 \cdot A_6A_1.$$

89. **(VIE 1)** We match sets $\mathcal{M}$ of points in the coordinate plane to sets $\mathcal{M}^*$ according to the rule that (x^*, y^*) belongs to $\mathcal{M}^*$ if and only if $xx^*+yy^* \leq 1$ whenever $(x, y) \in \mathcal{M}$. Find all triangles $\mathcal{Y}$ such that $\mathcal{Y}^*$ is the reflection of $\mathcal{Y}$ at the origin.

90. **(VIE 2)** Does there exist a number α $(0 < \alpha < 1)$ such that there is an infinite sequence $\{a_n\}$ of positive numbers satisfying
$$1 + a_{n+1} \leq a_n + \frac{\alpha}{n} a_n, \qquad n = 1, 2, \ldots?$$

91. **(VIE 3)** A regular 14-gon with side length a is inscribed in a circle of radius one. Prove that
$$\frac{2-a}{2a} > \sqrt{3 \cos \frac{\pi}{7}}.$$

92. **(VIE 4)** Let $p \geq 2$ be a natural number. Prove that there exists an integer n_0 such that
$$\sum_{i=1}^{n_0} \frac{1}{i \sqrt[p]{i+1}} > p.$$

93. **(VIE 5)** Given a natural number n, find all polynomials $P(x)$ of degree less than n satisfying the following condition:
$$\sum_{i=0}^{n} P(i)(-1)^i \binom{n}{i} = 0.$$

94. **(VIE 6)** Let $n + 1$ $(n \geq 1)$ positive integers be given such that for each integer, the set of all prime numbers dividing this integer is a subset of

the set of n given prime numbers. Prove that among these $n+1$ integers one can find numbers (possibly one number) whose product is a perfect square.

3.29.3 Shortlisted Problems

1. **(BUL 1)** An integer sequence is defined by
$$a_n = 2a_{n-1} + a_{n-2} \quad (n > 1), \qquad a_0 = 0, \ \ a_1 = 1.$$
Prove that 2^k divides a_n if and only if 2^k divides n.

2. **(BUL 3)** Let n be a positive integer. Find the number of odd coefficients of the polynomial
$$u_n(x) = (x^2 + x + 1)^n.$$

3. **(CAN 1)** The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B, and C meet the circle again at A', B', and C' respectively. Prove that the area of triangle $A'B'C'$ is greater than or equal to the area of triangle ABC.

4. **(CZS 1)** An $n \times n$ chessboard ($n \geq 2$) is numbered by the numbers $1, 2, \dots, n^2$ (every number occurs once). Prove that there exist two neighboring (which share a common edge) squares such that their numbers differ by at least n.

5. **(CZS 2)**$^{\mathrm{IMO2}}$ Let n be an even positive integer. Let $A_1, A_2, \dots, A_{n+1}$ be sets having n elements each such that any two of them have exactly one element in common while every element of their union belongs to at least two of the given sets. For which n can one assign to every element of the union one of the numbers 0 and 1 in such a manner that each of the sets has exactly $n/2$ zeros?

6. **(CZS 3)** In a given tetrahedron $ABCD$ let K and L be the centers of edges AB and CD respectively. Prove that every plane that contains the line KL divides the tetrahedron into two parts of equal volume.

7. **(FRA 2)** Let a be the greatest positive root of the equation $x^3-3x^2+1 = 0$. Show that $[a^{1788}]$ and $[a^{1988}]$ are both divisible by 17. ($[x]$ denotes the integer part of x.)

8. **(FRA 3)** Let $u_1, u_2, \dots, u_m$ be m vectors in the plane, each of length less than or equal to 1, which add up to zero. Show that one can rearrange $u_1, u_2, \dots, u_m$ as a sequence $v_1, v_2, \dots, v_m$ such that each partial sum $v_1, v_1 + v_2, v_1 + v_2 + v_3, \dots, v_1 + v_2 + \cdots + v_m$ has length less than or equal to $\sqrt{5}$.

9. **(FRG 1)**$^{\mathrm{IMO6}}$ Let a and b be two positive integers such that $ab+1$ divides $a^2 + b^2$. Show that $\frac{a^2+b^2}{ab+1}$ is a perfect square.

10. **(GDR 1)** Let $N = \{1, 2, \ldots, n\}$, $n \geq 2$. A collection $F = \{A_1, \ldots, A_t\}$ of subsets $A_i \subseteq N$, $i = 1, \ldots, t$, is said to be *separating* if for every pair $\{x, y\} \subseteq N$, there is a set $A_i \in F$ such that $A_i \cap \{x, y\}$ contains just one element. A collection F is said to be *covering* if every element of N is contained in at least one set $A_i \in F$. What is the smallest value $f(n)$ of t such that there is a set $F = \{A_1, \ldots, A_t\}$ that is simultaneously separating and covering?

11. **(GDR 3)** The lock on a safe consists of three wheels, each of which may be set in eight different positions. Due to a defect in the safe mechanism the door will open if any two of the three wheels are in the correct position. What is the smallest number of combinations that must be tried if one is to guarantee being able to open the safe (assuming that the "right combination" is not known)?

12. **(GRE 2)** In a triangle ABC, choose any points $K \in BC$, $L \in AC$, $M \in AB$, $N \in LM$, $R \in MK$, and $F \in KL$. If E_1, E_2, E_3, E_4, E_5, E_6, and E denote the areas of the triangles AMR, CKR, BKF, ALF, BNM, CLN, and ABC respectively, show that

$$E \geq 8\sqrt[6]{E_1 E_2 E_3 E_4 E_5 E_6}.$$

Remark. Points K, L, M, N, R, F lie on segments BC, AC, AB, LM, MK, KL respectively.

13. **(GRE 3)**$^{\text{IMO5}}$ In a right-angled triangle ABC, let AD be the altitude drawn to the hypotenuse and let the straight line joining the incenters of the triangles ABD, ACD intersect the sides AB, AC at the points K, L respectively. If E and E_1 denote the areas of the triangles ABC and AKL respectively, show that $\frac{E}{E_1} \geq 2$.

14. **(HUN 1)** For what values of n does there exist an $n \times n$ array of entries $-1, 0$, or 1 such that the $2n$ sums obtained by summing the elements of the rows and the columns are all different?

15. **(ICE 1)** Let ABC be an acute-angled triangle. Three lines L_A, L_B, and L_C are constructed through the vertices A, B, and C respectively according to the following prescription: Let H be the foot of the altitude drawn from the vertex A to the side BC; let S_A be the circle with diameter AH; let S_A meet the sides AB and AC at M and N respectively, where M and N are distinct from A; then L_A is the line through A perpendicular to MN. The lines L_B and L_C are constructed similarly. Prove that L_A, L_B, and L_C are concurrent.

16. **(IRE 1)**$^{\text{IMO4}}$ Show that the solution set of the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals the sum of whose lengths is 1988.

17. **(ISR 2)** In the convex pentagon $ABCDE$, the sides BC, CD, DE have the same length. Moreover, each diagonal of the pentagon is parallel to a side (AC is parallel to DE, BD is parallel to AE, etc.). Prove that $ABCDE$ is a regular pentagon.

18. **(LUX 1)**$^{\text{IMO1}}$ Consider two concentric circles of radii R and r $(R > r)$ with center O. Fix P on the small circle and consider the variable chord PA of the small circle. Points B and C lie on the large circle; B, P, C are collinear and BC is perpendicular to AP.
    (a) For what value(s) of $\angle OPA$ is the sum $BC^2 + CA^2 + AB^2$ extremal?
    (b) What are the possible positions of the midpoints U of BA and V of AC as $\angle OPA$ varies?

19. **(MEX 1)** Let $f(n)$ be a function defined on the set of all positive integers and having its values in the same set. Suppose that $f(f(m)+f(n)) = m+n$ for all positive integers n, m. Find all possible values for $f(1988)$.

20. **(MON 4)** Find the least natural number n such that if the set $\{1, 2, \dots, n\}$ is arbitrarily divided into two nonintersecting subsets, then one of the subsets contains three distinct numbers such that the product of two of them equals the third.

21. **(POL 4)** Forty-nine students solve a set of three problems. The score for each problem is a whole number of points from 0 to 7. Prove that there exist two students A and B such that for each problem, A will score at least as many points as B.

22. **(KOR 2)** Let p be the product of two consecutive integers greater than 2. Show that there are no integers $x_1, x_2, \dots, x_p$ satisfying the equation
$$\sum_{i=1}^{p} x_i^2 - \frac{4}{4p+1}\left(\sum_{i=1}^{p} x_i\right)^2 = 1.$$
Alternative formulation. Show that there are only two values of p for which there are integers $x_1, x_2, \dots, x_p$ satisfying the above inequality.

23. **(SIN 2)** Let Q be the center of the inscribed circle of a triangle ABC. Prove that for any point P,
$$a(PA)^2+b(PB)^2+c(PC)^2 = a(QA)^2+b(QB)^2+c(QC)^2+(a+b+c)(QP)^2,$$
where $a = BC$, $b = CA$, and $c = AB$.

24. **(SWE 2)** Let $\{a_k\}_1^\infty$ be a sequence of nonnegative real numbers such that $a_k - 2a_{k+1} + a_{k+2} \ge 0$ and $\sum_{j=1}^{k} a_j \le 1$ for all $k = 1, 2, \dots$. Prove that $0 \le (a_k - a_{k+1}) < \frac{2}{k^2}$ for all $k = 1, 2, \dots$.

25. **(GBR 1)** A positive integer is called a *double number* if its decimal representation consists of a block of digits, not commencing with 0, followed immediately by an identical block. For instance, 360360 is a double number, but 36036 is not. Show that there are infinitely many double numbers that are perfect squares.

26. **(GBR 2)**$^{\text{IMO3}}$ A function f defined on the positive integers (and taking positive integer values) is given by

$$\begin{aligned} f(1) = 1, \quad & f(3) = 3, \\ f(2n) &= f(n), \\ f(4n+1) &= 2f(2n+1) - f(n), \\ f(4n+3) &= 3f(2n+1) - 2f(n), \end{aligned}$$

for all positive integers n. Determine with proof the number of positive integers less than or equal to 1988 for which $f(n) = n$.

27. **(GBR 4)** The triangle ABC is acute-angled. Let L be any line in the plane of the triangle and let u, v, w be the lengths of the perpendiculars from A, B, C respectively to L. Prove that

$$u^2 \tan A + v^2 \tan B + w^2 \tan C \geq 2\Delta,$$

where Δ is the area of the triangle, and determine the lines L for which equality holds.

28. **(GBR 5)** The sequence $\{a_n\}$ of integers is defined by $a_1 = 2$, $a_2 = 7$, and

$$-\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leq \frac{1}{2}, \qquad \text{for } n \geq 2.$$

Prove that a_n is odd for all $n > 1$.

29. **(USA 3)** A number of signal lights are equally spaced along a one-way railroad track, labeled in order $1, 2, \dots, N$ $(N \geq 2)$. As a safety rule, a train is not allowed to pass a signal if any other train is in motion on the length of track between it and the following signal. However, there is no limit to the number of trains that can be parked motionless at a signal, one behind the other. (Assume that the trains have zero length.)
A series of K freight trains must be driven from Signal 1 to Signal N. Each train travels at a distinct but constant speed (i.e., the speed is fixed and different from that of each of the other trains) at all times when it is not blocked by the safety rule. Show that regardless of the order in which the trains are arranged, the same time will elapse between the first train's departure from Signal 1 and the last train's arrival at Signal N.

30. **(USS 1)** A point M is chosen on the side AC of the triangle ABC in such a way that the radii of the circles inscribed in the triangles ABM and BMC are equal. Prove that

$$BM^2 = \Delta \cot \frac{B}{2},$$

where Δ is the area of the triangle ABC.

31. **(USS 2)** Around a circular table an even number of persons have a discussion. After a break they sit again around the circular table in a different order. Prove that there are at least two people such that the number of participants sitting between them before and after the break is the same.

3.30 The Thirtieth IMO Braunschweig–Niedersachen, FR Germany, July 13–24, 1989

3.30.1 Contest Problems

First Day (July 18)

1. Prove that the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of 17 subsets $A_1, A_2, \ldots, A_{17}$ such that:
   (i) each A_i contains the same number of elements;
   (ii) the sum of all elements of each A_i is the same for $i = 1, 2, \ldots, 17$.

2. Let ABC be a triangle. The bisector of angle A meets the circumcircle of triangle ABC in A_1. Points B_1 and C_1 are defined similarly. Let AA_1 meet the lines that bisect the two external angles at B and C in point A^0. Define B^0 and C^0 similarly. If $S_{X_1X_2\ldots X_n}$ denotes the area of the polygon $X_1X_2\ldots X_n$, prove that
$$S_{A^0B^0C^0} = 2S_{AC_1BA_1CB_1} \geq 4S_{ABC}.$$

3. Given a set S in the plane containing n points and satisfying the conditions
   (i) no three points of S are collinear,
   (ii) for every point P of S there exist at least k points in S that have the same distance to P,

   prove that the following inequality holds:
$$k < \frac{1}{2} + \sqrt{2n}.$$

Second Day (July 19)

4. The quadrilateral $ABCD$ has the following properties:
   (i) $AB = AD + BC$;
   (ii) there is a point P inside it at a distance x from the side CD such that $AP = x + AD$ and $BP = x + BC$.

   Show that
$$\frac{1}{\sqrt{x}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

5. For which positive integers n does there exist a positive integer N such that none of the integers $1 + N, 2 + N, \ldots, n + N$ is the power of a prime number?

6. We consider permutations $(x_1, \ldots, x_{2n})$ of the set $\{1, \ldots, 2n\}$ such that $|x_i - x_{i+1}| = n$ for at least one $i \in \{1, \ldots, 2n - 1\}$. For every natural number n, find out whether permutations with this property are more or less numerous than the remaining permutations of $\{1, \ldots, 2n\}$.

3.30.2 Longlisted Problems

1. **(AUS 1)** In the set $S_n = \{1, 2, \dots, n\}$ a new multiplication $a*b$ is defined with the following properties:
   (i) $c = a * b$ is in S_n for any $a \in S_n, b \in S_n$.
   (ii) If the ordinary product $a \cdot b$ is less than or equal to n, then $a * b = a \cdot b$.
   (iii) The ordinary rules of multiplication hold for $*$, i.e.,
   (1) $a * b = b * a$ (commutativity)
   (2) $(a * b) * c = a * (b * c)$ (associativity)
   (3) If $a * b = a * c$ then $b = c$ (cancellation law).

   Find a suitable multiplication table for the new product for $n = 11$ and $n = 12$.

2. **(AUS 2)** (SL89-1).

3. **(AUS 3)** (SL89-2).

4. **(AUS 4)** (SL89-3).

5. **(BUL 1)** The sequences $a_0, a_1, \dots$ and $b_0, b_1, \dots$ are defined by the equalities
$$a_0 = \frac{\sqrt{2}}{2}, \qquad a_{n+1} = \frac{\sqrt{2}}{2}\sqrt{1 - \sqrt{1 - a_n^2}}, \qquad n = 0, 1, 2, \dots$$
   and
$$b_0 = 1, \qquad b_{n+1} = \frac{\sqrt{1 + b_n^2} - 1}{b_n}, \qquad n = 0, 1, 2, \dots .$$
   Prove the inequalities
$$2^{n+2} a_n < \pi < 2^{n+2} b_n, \qquad \text{for every } n = 0, 1, 2, \dots .$$

6. **(BUL 2)** The circles c_1 and c_2 are tangent at the point A. A straight line l through A intersects c_1 and c_2 at points C_1 and C_2 respectively. A circle c, which contains C_1 and C_2, meets c_1 and c_2 at points B_1 and B_2 respectively. Let κ be the circle circumscribed around triangle AB_1B_2. The circle k tangent to κ at the point A meets c_1 and c_2 at the points D_1 and D_2 respectively. Prove that
   (a) the points C_1, C_2, D_1, D_2 are concyclic or collinear;
   (b) the points B_1, B_2, D_1, D_2 are concyclic if and only if AC_1 and AC_2 are diameters of c_1 and c_2.

7. **(BUL 3)** (SL89-4).

8. **(COL 1)** (SL89-5).

9. **(COL 2)** Let m be a positive integer and define $f(m)$ to be the number of factors of 2 in $m!$ (that is, the greatest positive integer k such that $2^k \mid m!$). Prove that there are infinitely many positive integers m such that $m - f(m) = 1989$.

10. **(CUB 1)** Given the equation

$$4x^3 + 4x^2y - 15xy^2 - 18y^3 - 12x^2 + 6xy + 36y^2 + 5x - 10y = 0,$$

find all positive integer solutions.

11. **(CUB 2)** Given the equation

$$y^4 + 4y^2x - 11y^2 + 4xy - 8y + 8x^2 - 40x + 52 = 0,$$

find all real solutions.

12. **(CUB 3)** Let $P(x)$ be a polynomial such that the following inequalities are satisfied:

$$\begin{aligned} &P(0) > 0;\\ &P(1) > P(0);\\ &P(2) > 2P(1) - P(0);\\ &P(3) > 3P(2) - 3P(1) + P(0); \end{aligned}$$

and also for every natural number n, $P(n+4) > 4P(n+3) - 6P(n+2) + 4P(n+1) - P(n)$. Prove that for every positive natural number n, $P(n)$ is positive.

13. **(CUB 4)** Let n be a natural number not greater than 44. Prove that for any function f defined over $\mathbb{N}^2$ whose images are in the set $\{1, 2, \dots, n\}$, there are four ordered pairs $(i, j), (i, k), (l, j)$, and (l, k) such that $f(i, j) = f(i, k) = f(l, j) = f(l, k)$, where i, j, k, l are chosen in such a way that there are natural numbers n, p that satisfy

$$1989m \le i < l < 1989 + 1989m, \qquad 1989p \le j < k < 1989 + 1989p.$$

14. **(CZS 1)** (SL89-6).

15. **(CZS 2)** A sequence $a_1, a_2, a_3, \dots$ is defined recursively by $a_1 = 1$ and $a_{2^k+j} = -a_j$ $(j = 1, 2, \dots, 2^k)$. Prove that this sequence is not periodic.

16. **(FIN 1)** (SL89-7).

17. **(FIN 2)** Let a, $0 < a < 1$, be a real number and f a continuous function on $[0, 1]$ satisfying $f(0) = 0$, $f(1) = 1$, and

$$f\left(\frac{x+y}{2}\right) = (1-a)f(x) + af(y)$$

for all $x, y \in [0, 1]$ with $x \le y$. Determine $f(1/7)$.

18. **(FIN 3)** There are some boys and girls sitting in an $n \times n$ quadratic array. We know the number of girls in every column and row and every line parallel to the diagonals of the array. For which n is this information sufficient to determine the exact positions of the girls in the array? For which seats can we say for sure that a girl sits there or not?

19. **(FRA 1)** Let $a_1, \dots, a_n$ be distinct positive integers that do not contain a 9 in their decimal representations. Prove that
$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} \le 30.$$

20. **(FRA 2)** (SL89-8).

21. **(FRA 2b)** Same problem as previous, but with a rectangular parallelepiped having at least one integral side.

22. **(FRA 3)** Let ABC be an equilateral triangle with side length equal to a natural number N. Consider the set S of all points M inside the triangle ABC such that $\overrightarrow{AM} = \frac{1}{N}(n\overrightarrow{AB} + m\overrightarrow{AC})$, where m, n are integers and $0 \le m, n, m+n \le N$. Every point of S is colored in one of the three colors blue, white, red such that no point on AB is colored blue, no point on AC is colored white, and no point on BC is colored red. Prove that there exists an equilateral triangle with vertices in S and side length 1 whose three vertices are colored blue, white, and red.

23. **(FRA 3b)** Like the previous problem, but with a regular tetrahedron and four different colors used.

24. **(FRA 4)** (SL89-9).

25. **(GBR 1)** Let ABC be a triangle. Prove that there is a unique point U in the plane of ABC such that there exist real numbers $\lambda, \mu, \nu, \kappa$, not all zero, such that
$$\lambda PL^2 + \mu PM^2 + \nu PN^2 - \kappa UP^2$$
is constant for all points P of the plane, where L, M, N are the feet of the perpendiculars from P to BC, CA, AB respectively.

26. **(GBR 2)** Let a, b, c, d be positive integers such that $ab = cd$ and $a + b = c - d$.
Prove that there exists a right-angled triangle the measures of whose sides (in some unit) are integers and whose area measure is ab square units.

27. **(GBR 3)** Integers $c_{m,n}$ $(m \ge 0, n \ge 0)$ are defined by $c_{m,0} = 1$ for all $m \ge 0$, $c_{0,n} = 1$ for all $n \ge 0$, and $c_{m,n} = c_{m-1,n} - nc_{m-1,n-1}$ for all $m > 0, n > 0$. Prove that $c_{m,n} = c_{n,m}$ for all $m \ge 0, n \ge 0$.

28. **(GBR 4)** Let $b_1, b_2, \dots, b_{1989}$ be positive real numbers such that the equations
$$x_{r-1} - 2x_r + x_{r+1} + b_r x_r = 0 \qquad (1 \le r \le 1989)$$
have a solution with $x_0 = x_{1990} = 0$ but not all of $x_1, \dots, x_{1989}$ are equal to zero. Prove that
$$b_1 + b_2 + \cdots + b_{1989} \ge \frac{2}{995}.$$

29. **(GRE 1)** Let L denote the set of all lattice points of the plane (points with integral coordinates). Show that for any three points A, B, C of L there is a fourth point D, different from A, B, C, such that the interiors of the segments AD, BD, CD contain no points of L. Is the statement true if one considers four points of L instead of three?

30. **(GRE 2)** In a triangle ABC for which $6(a+b+c)r^2 = abc$, we consider a point M on the inscribed circle and the projections D, E, F of M on the sides BC, AC, and AB respectively. Let S, S_1 denote the areas of the triangles ABC and DEF respectively. Find the maximum and minimum values of the quotient $\frac{S}{S_1}$ (here r denotes the inradius of ABC and, as usual, $a = BC$, $b = AC$, $c = AB$).

31. **(GRE 3)** (SL89-10).

32. **(HKG 1)** Let ABC be an equilateral triangle. Let D, E, F, M, N, and P bee the mid-points of BC, CA, AB, FD, FB, and DC respectively.
   (a) Show that the line segments AM, EN, and FP are concurrent.
   (b) Let O be the point of intersection of AM, EN, and FP. Find $OM : OF : ON : OE : OP : OA$.

33. **(HKG 2)** Let n be a positive integer. Show that $(\sqrt{2}+1)^n = \sqrt{m} + \sqrt{m-1}$ for some positive integer m.

34. **(HKG 3)** Given an acute triangle find a point inside the triangle such that the sum of the distances from this point to the three vertices is the least.

35. **(HKG 4)** Find all square numbers S_1 and S_2 such that $S_1 - S_2 = 1989$.

36. **(HKG 5)** Prove the identity

$$1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\cdots+\frac{1}{478}+\frac{1}{479}-\frac{2}{480} = 2\sum_{k=0}^{159}\frac{641}{(161+k)(480-k)}.$$

37. **(HUN 1)** (SL89-11).

38. **(HUN 2)** Connecting the vertices of a regular n-gon we obtain a closed (not necessarily convex) n-gon. Show that if n is even, then there are two parallel segments among the connecting segments and if n is odd then there cannot be exactly two parallel segments.

39. **(HUN 3)** (SL89-12).

40. **(ICE 1)** A sequence of real numbers $x_0, x_1, x_2, \ldots$ is defined as follows: $x_0 = 1989$ and for each $n \geq 1$

$$x_n = -\frac{1989}{n}\sum_{k=0}^{n-1} x_k.$$

Calculate the value of $\sum_{n=0}^{1989} 2^n x_n$.

41. **(ICE 2)** Alice has two urns. Each urn contains four balls and on each ball a natural number is written. She draws one ball from each urn at random, notes the sum of the numbers written on them, and replaces the balls in the urns from which she took them. This she repeats a large number of times. Bill, on examining the numbers recorded, notices that the frequency with which each sum occurs is the same as if it were the sum of two natural numbers drawn at random from the range 1 to 4. What can he deduce about the numbers on the balls?

42. **(ICE 3)** (SL89-13).

43. **(INA 1)** Let $f(x) = a\sin^2 x + b\sin x + c$, where a, b, and c are real numbers. Find all values of a, b, and c such that the following three conditions are satisfied simultaneously:
   (i) $f(x) = 381$ if $\sin x = 1/2$.
   (ii) The absolute maximum of $f(x)$ is 444.
   (iii) The absolute minimum of $f(x)$ is 364.

44. **(INA 2)** Let A and B be fixed distinct points on the X axis, none of which coincides with the origin $O(0,0)$, and let C be a point on the Y axis of an orthogonal Cartesian coordinate system. Let g be a line through the origin $O(0,0)$ and perpendicular to the line AC. Find the locus of the point of intersection of the lines g and BC as C varies along the Y axis. (Give an equation and a description of the locus.)

45. **(INA 3)** The expressions $a+b+c$, $ab+ac+bc$, and abc are called the elementary symmetric expressions on the three letters a,b,c; symmetric because if we interchange any two letters, say a and c, the expressions remain algebraically the same. The common degree of its terms is called the order of the expression.
   Let $S_k(n)$ denote the elementary expression on k different letters of order n; for example $S_4(3) = abc + abd + acd + bcd$. There are four terms in $S_4(3)$. How many terms are there in $S_{9891}(1989)$? (Assume that we have 9891 different letters.)

46. **(INA 4)** Given two distinct numbers b_1 and b_2, their product can be formed in two ways: $b_1 \times b_2$ and $b_2 \times b_1$. Given three distinct numbers, b_1, b_2, b_3, their product can be formed in twelve ways: $b_1 \times (b_2 \times b_3)$; $(b_1 \times b_2) \times b_3$; $b_1 \times (b_3 \times b_2)$; $(b_1 \times b_3) \times b_2$; $b_2 \times (b_1 \times b_3)$; $(b_2 \times b_1) \times b_3$; $b_2 \times (b_3 \times b_1)$; $(b_2 \times b_3) \times b_1$; $b_3 \times (b_1 \times b_2)$; $(b_3 \times b_1) \times b_2$; $b_3 \times (b_2 \times b_1)$; $(b_3 \times b_2) \times b_1$. In how many ways can the product of n distinct letters be formed?

47. **(INA 5)** Let $\log_2^2 x - 4\log_2 x - m^2 - 2m - 13 = 0$ be an equation in x. Prove:
   (a) For any real value of m the equation has has two distinct solutions.
   (b) The product of the solutions of the equation does not depend on m.

(c) One of the solutions of the equation is less than 1, while the other solution is greater than 1.

Find the minimum value of the larger solution and the maximum value of the smaller solution.

48. **(INA 6)** Let S be the point of intersection of the two lines $l_1 : 7x - 5y + 8 = 0$ and $l_2 : 3x + 4y - 13 = 0$. Let $P = (3, 7)$, $Q = (11, 13)$, and let A and B be points on the line PQ such that P is between A and Q, and B is between P and Q, and such that $PA/AQ = PB/BQ = 2/3$. Without finding the coordinates of B find the equations of the lines SA and SB.

49. **(IND 1)** Let A, B denote two distinct fixed points in space. Let X, P denote variable points (in space), while K, N, n denote positive integers. Call (X, K, N, P) *admissible* if $(N - K)PA + K \cdot PB \geq N \cdot PX$. Call (X, K, N) admissible if (X, K, N, P) is admissible for all choices of P. Call (X, N) admissible if (X, K, N) is admissible for some choice of K in the interval $0 < K < N$. Finally, call X admissible if (X, N) is admissible for some choice of N $(N > 1)$. Determine:
(a) the set of admissible X;
(b) the set of X for which $(X, 1989)$ is admissible but not (X, n), $n < 1989$.

50. **(IND 2)** (SL89-14).

51. **(IND 3)** Let $t(n)$, for $n = 3, 4, 5, \ldots$, represent the number of distinct, incongruent, integer-sided triangles whose perimeter is n; e.g., $t(3) = 1$. Prove that
$$t(2n-1) - t(2n) = \left[\frac{n}{6}\right] \text{ or } \left[\frac{n}{6} + 1\right].$$

52. **(IRE 1)** (SL89-15).

53. **(IRE 2)** Let $f(x) = (x - a_1)(x - a_2)\cdots(x - a_n) - 2$, where $n \geq 3$ and $a_1, a_2, \ldots, a_n$ are distinct integers. Suppose that $f(x) = g(x)h(x)$, where $g(x), h(x)$ are both nonconstant polynomials with integer coefficients. Prove that $n = 3$.

54. **(IRE 3)** Let f be a function from the real numbers to the real numbers such that $f(1) = 1$, $f(a+b) = f(a) + f(b)$ for all a, b, and $f(x)f(1/x) = 1$ for all $x \neq 0$.
Prove that $f(x) = x$ for all real numbers x.

55. **(IRE 4)** Let $[x]$ denote the greatest integer less than or equal to x. Let α be the positive root of the equation $x^2 - 1989x - 1 = 0$. Prove that there exist infinitely many natural numbers n that satisfy the equation
$$[\alpha n + 1989\alpha[\alpha n]] = 1989n + (1989^2 + 1)[\alpha n].$$

56. **(IRE 5)** Let $n = 2k - 1$, where $k \geq 6$ is an integer. Let T be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where x_i is 0 or 1 $(i = 1, 2, \ldots, n)$. For $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in T, let $d(\mathbf{x}, \mathbf{y})$ denote the number of integers j with $1 \leq j \leq n$ such that $x_j \neq y_j$. (In particular $d(\mathbf{x}, \mathbf{x}) = 0$.)

Suppose that there exists a subset S of T with 2^k elements that has the following property: Given any element $\mathbf{x}$ in T, there is a unique element $\mathbf{y}$ in S with $d(\mathbf{x},\mathbf{y}) \le 3$. Prove that $n = 23$.

57. **(ISR 1)** (SL89-16).

58. **(ISR 2)** Let $P_1(x), P_2(x), \dots, P_n(x)$ be polynomials with real coefficients. Show that there exist real polynomials $A_r(x), B_r(x)$ $(r = 1, 2, 3)$ such that

$$\begin{aligned}\textstyle\sum_{s=1}^{n}(P_s(x))^2 &= (A_1(x))^2 + (B_1(x))^2\\ &= (A_2(x))^2 + x(B_2(x))^2\\ &= (A_3(x))^2 - x(B_3(x))^2.\end{aligned}$$

59. **(ISR 3)** Let $v_1, v_2, \dots, v_{1989}$ be a set of coplanar vectors with $|v_r| \le 1$ for $1 \le r \le 1989$. Show that it is possible to find ϵ_r $(1 \le r \le 1989)$, each equal to ± 1, such that

$$\left|\sum_{r=1}^{1989} \epsilon_r v_r\right| \le \sqrt{3}.$$

60. **(KOR 1)** A real-valued function f on $\mathbb{Q}$ satisfies the following conditions for arbitrary $\alpha, \beta \in \mathbb{Q}$:

(i) $f(0) = 0$, (ii) $f(\alpha) > 0$ if $\alpha \neq 0$,
(iii) $f(\alpha\beta) = f(\alpha)f(\beta)$, (iv) $f(\alpha+\beta) \le f(\alpha) + f(\beta)$,
(v) $f(m) \le 1989$ for all $m \in \mathbb{Z}$.

Prove that $f(\alpha+\beta) = \max\{f(\alpha), f(\beta)\}$ if $f(\alpha) \neq f(\beta)$.
Here, $\mathbb{Z}, \mathbb{Q}$ denote the sets of integers and rational numbers, respectively.

61. **(KOR 2)** Let A be a set of positive integers such that no positive integer greater than 1 divides all the elements of A. Prove that any sufficiently large positive integer can be written as a sum of elements of A. (Elements may occur several times in the sum.)

62. **(KOR 3)** (SL89-25).

63. **(KOR 4)** (SL89-26).

64. **(KOR 5)** Let a regular $(2n+1)$-gon be inscribed in a circle of radius r. We consider all the triangles whose vertices are from those of the regular $(2n+1)$-gon.
(a) How many triangles among them contain the center of the circle in their interior?
(b) Find the sum of the areas of all those triangles that contain the center of the circle in their interior.

65. **(LUX 1)** A regular n-gon $A_1A_2A_3\dots A_k\dots A_n$ inscribed in a circle of radius R is given. If S is a point on the circle, calculate $T = SA_1^2 + SA_2^2 + \dots + SA_n^2$.

66. **(MON 1)** (SL89-17).

67. **(MON 2)** A family of sets $A_1, A_2, \dots, A_n$ has the following properties:
(i) Each A_i contains 30 elements.
(ii) $A_i \cap A_j$ contains exactly one element for all i, j, $1 \le i < j \le 30$.
Find the largest possible n if the intersection of all these sets is empty.

68. **(MON 3)** If $0 < k \le 1$ and a_i are positive real numbers, $i = 1, 2, \dots, n$, prove that

$$\left(\frac{a_1}{a_2 + \cdots + a_n}\right)^k + \cdots + \left(\frac{a_n}{a_1 + \cdots + a_{n-1}}\right)^k \ge \frac{n}{(n-1)^k}.$$

69. **(MON 4)** (SL89-18).

70. **(MON 5)** Three mutually nonparallel lines l_i ($i = 1, 2, 3$) are given in a plane. The lines l_i determine a triangle and reflections f_i with axes on lines l_i. Prove that for every point of the plane, there exists a finite composition of the reflections f_i that maps that point to a point interior to the triangle.

71. **(MON 6)** (SL89-19).

72. **(MOR 1)** Let $ABCD$ be a quadrilateral inscribed in a circle with diameter AB such that $BC = a$, $CD = 2a$, $DA = \frac{3\sqrt{5}-1}{2}a$. For each point M on the semicircle AB not containing C and D, denote by h_1, h_2, h_3 the distances from M to the sides BC, CD, and DA. Find the maximum of $h_1 + h_2 + h_3$.

73. **(NET 1)** (SL89-20).

74. **(NET 2)** (SL89-21).

75. **(PHI 1)** (SL89-22).

76. **(PHI 2)** Let k and s be positive integers. For sets of real numbers $\{\alpha_1, \alpha_2, \dots, \alpha_s\}$ and $\{\beta_1, \beta_2, \dots, \beta_s\}$ that satisfy $\sum_{i=1}^{s} \alpha_i^j = \sum_{i=1}^{s} \beta_i^j$ for each $j = 1, 2, \dots, k$, we write

$$\{\alpha_1, \alpha_2, \dots, \alpha_s\} =_k \{\beta_1, \beta_2, \dots, \beta_s\}.$$

Prove that if $\{\alpha_1, \alpha_2, \dots, \alpha_s\} =_k \{\beta_1, \beta_2, \dots, \beta_s\}$ and $s \le k$, then there exists a permutation π of $\{1, 2, \dots, s\}$ such that $\beta_i = \alpha_{\pi(i)}$ for $i = 1, 2, \dots, s$.

77. **(POL 1)** Given that

$$\frac{\cos x + \cos y + \cos z}{\cos(x+y+z)} = \frac{\sin x + \sin y + \sin z}{\sin(x+y+z)} = a,$$

show that

$$\cos(y+z) + \cos(z+x) + \cos(x+y) = a.$$

78. **(POL 2)** (SL89-23).

Alternative formulation. Two identical packs of n different cards are shuffled together; all arrangements are equiprobable. The cards are then laid face up, one at a time. For every natural number n, find out which is more probable, that at least one pair of identical cards will appear in immediate succession or that there will be no such pair.

79. **(POL 3)** To each pair (x, y) of distinct elements of a finite set X a number $f(x, y)$ equal to 0 or 1 is assigned in such a way that $f(x, y) \neq f(y, x)$ for all x, y $(x \neq y)$. Prove that exactly one of the following situations occurs:
   (i) X is the union of two disjoint nonempty subsets U, V such that $f(u, v) = 1$ for every $u \in U, v \in V$.
   (ii) The elements of X can be labeled $x_1, \dots, x_n$ so that $f(x_1, x_2) = f(x_2, x_3) = \cdots = f(x_{n-1}, x_n) = f(x_n, x_1) = 1$.

   Alternative formulation. In a tournament of n participants, each pair plays one game (no ties). Prove that exactly one of the following situations occurs:
   (i) The league can be partitioned into two nonempty groups such that each player in one of these groups has won against each player of the other.
   (ii) All participants can be ranked 1 through n so that ith player wins the game against the $(i + 1)$st and the nth player wins against the first.

80. **(POL 4)** We are given a finite collection of segments in the plane, of total length 1. Prove that there exists a line ℓ such that the sum of the lengths of the projections of the given segments to the line ℓ is less than $2/\pi$.

81. **(POL 5)** (SL89-24).

82. **(POR 1)** Solve in the set of real numbers the equation $3x^3 - [x] = 3$, where $[x]$ denotes the integer part of x.

83. **(POR 2)** Poldavia is a strange kingdom. Its currency unit is the bourbaki and there exist only two types of coins: gold ones and silver ones. Each gold coin is worth n bourbakis and each silver coin is worth m bourbakis (n and m are positive integers). Using gold and solver coins, it is possible to obtain sums such as 10000 bourbakis, 1875 bourbakis, 3072 bourbakis, and so on. But Poldavia's monetary system is not as strange as it seems:
   (a) Prove that it is possible to buy anything that costs an integral number of bourbakis, as long as one can receive change.
   (b) Prove that any payment above $mn-2$ bourbakis can be made without the need to receive change.

84. **(POR 3)** Let a, b, c, r, and s be real numbers. Show that if r is a root of $ax^2+bx+c = 0$ and s is a root of $-ax^2+bx+c = 0$, then $\frac{a}{2}x^2+bx+c = 0$ has a root between r and s.

85. **(POR 4)** Let $P(x)$ be a polynomial with integer coefficients such that $P(m_1) = P(m_2) = P(m_3) = P(m_4) = 7$ for given distinct integers m_1, m_2, m_3, and m_4. Show that there is no integer m such that $P(m) = 14$.

86. **(POR 5)** Given two natural numbers w and n, the tower of n w's is the natural number $T_n(w)$ defined by
$$T_n(w) = w^{w^{\cdot^{\cdot^{\cdot^{w}}}}},$$
with n w's on the right side. More precisely, $T_1(w) = w$ and $T_{n+1}(w) = w^{T_n(w)}$. For example, $T_3(2) = 2^{2^2} = 16$, $T_4(2) = 2^{16} = 65536$, and $T_2(3) = 3^3 = 27$.
Find the smallest tower of 3's that exceeds the tower of 1989 2's. In other words, find the smallest value of n such that $T_n(3) > T_{1989}(2)$. Justify your answer.

87. **(POR 6)** A balance has a left pan, a right pan, and a pointer that moves along a graduated ruler. Like many other grocer balances, this one works as follows: An object of weight L is placed in the left pan and another of weight R in the right pan, the pointer stops at the number $R - L$ on the graduated ruler.
There are n (≥ 2) bags of coins, each containing $\frac{n(n-1)}{2} + 1$ coins. All coins look the same (shape, color, and so on). Of the bags, $n-1$ contain genuine coins, all with the same weight. The remaining bag (we don't know which one it is) contains counterfeit coins. All counterfeit coins have the same weight, and this weight is different from the weight of the genuine coins. A legal weighing consists of placing a certain number of coins in one of the pans, putting a certain number of coins in the other pan, and reading the number given by the pointer in the graduated ruler. With just two legal weighings it is possible to identify the bag containing counterfeit coins. Find a way to do this and explain it.

88. **(ROM 1)** (SL89-27).

89. **(ROM 2)** (SL89-28).

90. **(ROM 3)** Prove that the sequence $(a_n)_{n\geq 0}$, $a_n = [n\sqrt{2}]$, contains an infinite number of perfect squares.

91. **(ROM 4)** (SL89-29).

92. **(ROM 5)** Find the set of all $a \in \mathbb{R}$ for which there is no infinite sequence $(x_n)_{n\geq 0} \subset \mathbb{R}$ satisfying $x_0 = a$, $x_{n+1} = \frac{x_n+\alpha}{\beta x_n + 1}$, $n = 0, 1, \dots$, where $\alpha\beta > 0$.

93. **(ROM 6)** For $\Phi : \mathbb{N} \to \mathbb{Z}$ let us define $M_\Phi = \{f : \mathbb{N} \to \mathbb{Z}; f(x) > F(\Phi(x)), \forall x \in \mathbb{N}\}$.
    (a) Prove that if $M_{\Phi_1} = M_{\Phi_2} \neq \emptyset$, then $\Phi_1 = \Phi_2$.
    (b) Does this property remain true if $M_\Phi = \{f : \mathbb{N} \to \mathbb{N}; f(x) > F(\Phi(x)), \forall x \in \mathbb{N}\}$?

94. **(SWE 1)** Prove that $a < b$ implies that $a^3 - 3a \le b^3 - 3b + 4$. When does equality occur?

95. **(SWE 2)** (SL89-30).

96. **(SWE 3)** (SL89-31).

97. **(THA 1)** Let n be a positive integer, $X = \{1, 2, \dots, n\}$, and k a positive integer such that $n/2 \le k \le n$. Determine, with proof, the number of all functions $f : X \to X$ that satisfy the following conditions:
   (i) $f^2 = f$;
   (ii) the number of elements in the image of f is k;
   (iii) for each y in the image of f, the number of all points x in X such that $f(x)=y$ is at most 2.

98. **(THA 2)** Let $f : \mathbb{N} \to \mathbb{N}$ be such that
   (i) f is strictly increasing;
   (ii) $f(mn) = f(m)f(n)\ \forall m, n \in \mathbb{N}$; and
   (iii) if $m \ne n$ and $m^n = n^m$, then $f(m) = n$ or $f(n) = m$.
   Determine $f(30)$.

99. **(THA 3)** An arithmetic function is a real-valued function whose domain is the set of positive integers. Define the convolution product of two arithmetic functions f and g to be the arithmetic function $f \star g$, where $(f \star g)(n) = \sum_{ij=n} f(i)g(i)$, and $f^{\star k} = f \star f \star \cdots \star f$ (k times).
   We say that two arithmetic functions f and g are dependent if there exists a nontrivial polynomial of two variables $P(x, y) = \sum_{i,j} a_{ij} x^i y^j$ with real coefficients such that
   $$P(f, g) = \sum_{i,j} a_{ij} f^{\star i} \star g^{\star j} = 0,$$
   and say that they are independent if they are not dependent. Let p and q be two distinct primes and set
   $$f_1(n) = \begin{cases} 1 \text{ if } n = p, \\ 0 \text{ otherwise;} \end{cases} \qquad f_2(n) = \begin{cases} 1 \text{ if } n = q, \\ 0 \text{ otherwise.} \end{cases}$$
   Prove that f_1 and f_2 are independent.

100. **(THA 4)** Let A be an $n \times n$ matrix whose elements are nonnegative real numbers. Assume that A is a nonsingular matrix and all elements of A^{-1} are nonnegative real numbers. Prove that every row and every column of A has exactly one nonzero element.

101. **(TUR 1)** Let ABC be an equilateral triangle and Γ the semicircle drawn exteriorly to the triangle, having BC as diameter. Show that if a line passing through A trisects BC, it also trisects the arc Γ.

102. **(TUR 2)** If in a convex quadrilateral $ABCD$, E and F are the midpoints of the sides BC and DA respectively. Show that the sum of the areas of the triangles EDA and FBC is equal to the area of the quadrangle.

103. **(USA 1)** An accurate 12-hour analog clock has an hour hand, a minute hand, and a second hand that are aligned at 12:00 o'clock and make one revolution in 12 hours, 1 hour, and 1 minute, respectively. It is well known, and not difficult to prove, that there is no time when the three hands are equally spaced around the clock, with each separating angle $2\pi/3$. Let $f(t), g(t), h(t)$ be the respective absolute deviations of the separating angles from $2\pi/3$ at t hours after 12:00 o'clock. What is the minimum value of $\max\{f(t), g(t), h(t)\}$?

104. **(USA 2)** For each nonzero complex number z, let $\arg z$ be the unique real number t such that $-\pi < t \le \pi$ and $z = |z|(\cos t + \imath \sin t)$. Given a real number $c > 0$ and a complex number $z \neq 0$ with $\arg z \neq \pi$, define

$$B(c, z) = \{b \in \mathbb{R} \mid \; |w - z| < b \Rightarrow |\arg w - \arg z| < c\}.$$

Determine necessary and sufficient conditions, in terms of c and z, such that $B(c, z)$ has a maximum element, and determine what this maximum element is in this case.

105. **(USA 3)** (SL89-32).

106. **(USA 4)** Let $n > 1$ be a fixed integer. Define functions $f_0(x) = 0$, $f_1(x) = 1 - \cos x$, and for $k > 0$,

$$f_{k+1}(x) = 2f_k(x)\cos x - f_{k-1}(x).$$

If $F(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$, prove that
(a) $0 < F(x) < 1$ for $0 < x < \frac{\pi}{n+1}$, and
(b) $F(x) > 1$ for $\frac{\pi}{n+1} < x < \frac{\pi}{n}$.

107. **(VIE 1)** Let E be the set of all triangles whose only points with integer coordinates (in the Cartesian coordinate system in space), in its interior or on its sides, are its three vertices, and let f be the function of area of a triangle. Determine the set of values $f(E)$ of f.

108. **(VIE 2)** For every sequence $(x_1, x_2, \ldots, x_n)$ of the numbers $\{1, 2, \ldots, n\}$ arranged in any order, denote by $f(s)$ the sum of absolute values of the differences between two consecutive members of s. Find the maximum value of $f(s)$ (where s runs through the set of all such sequences).

109. **(VIE 3)** Let Ax, By be two noncoplanar rays with AB as a common perpendicular, and let M, N be two mobile points on Ax and By respectively such that $AM + BN = MN$.

First version. Prove that there exist infinitely many lines coplanar with each of the lines MN.

Second version. Prove that there exist infinitely many rotations around a fixed axis Δ mapping the line Ax onto a line coplanar with each of the lines MN.

110. **(VIE 4)** Do there exist two sequences of real numbers $\{a_i\}, \{b_i\}$, $i \in \mathbb{N} = \{1, 2, 3, \dots\}$, satisfying the following conditions:

$$\frac{3\pi}{2} \le a_i \le b_i, \qquad \cos a_i x + \cos b_i x \ge -\frac{1}{i}$$

for all $i \in \mathbb{N}$ and all x, $0 < x < 1$?

111. **(VIE 5)** Find the greatest number c such that for all natural numbers n, $\{n\sqrt{2}\} \ge \frac{c}{n}$ (where $\{n\sqrt{2}\} = n\sqrt{2} - [n\sqrt{2}]$; $[x]$ is the integer part of x). For this number c, find all natural numbers n for which $\{n\sqrt{2}\} = \frac{c}{n}$.

3.30.3 Shortlisted Problems

1. **(AUS 2)**$^{\mathrm{IMO2}}$ Let ABC be a triangle. The bisector of angle A meets the circumcircle of triangle ABC in A_1. Points B_1 and C_1 are defined similarly. Let AA_1 meet the lines that bisect the two external angles at B and C in point A^0. Define B^0 and C^0 similarly. If $S_{X_1X_2\dots X_n}$ denotes the area of the polygon $X_1X_2\dots X_n$, prove that

$$S_{A^0B^0C^0} = 2S_{AC_1BA_1CB_1} \ge 4S_{ABC}.$$

2. **(AUS 3)** Ali Barber, the carpet merchant, has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. If the two rooms have dimensions of 38 feet by 55 feet and 50 feet by 55 feet, what are the carpet dimensions?

3. **(AUS 4)** Ali Barber, the carpet merchant, has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. He knows that the sides of the carpet are integral numbers of feet and that his two storerooms have the same (unknown) length, but widths of 38 feet and 50 feet respectively. What are the carpet dimensions?

4. **(BUL 3)** Prove that for every integer $n > 1$ the equation

$$\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + \frac{x^2}{2!} + \frac{x}{1!} + 1 = 0$$

has no rational roots.

5. **(COL 1)** Consider the polynomial $p(x) = x^n + nx^{n-1} + a_2x^{n-2} + \cdots + a_n$ having all real roots. If $r_1^{16} + r_2^{16} + \cdots + r_n^{16} = n$, where the r_j are the roots of $p(x)$, find all such roots.

6. **(CZS 1)** For a triangle ABC, let k be its circumcircle with radius r. The bisectors of the inner angles A, B, and C of the triangle intersect respectively the circle k again at points A', B', and C'. Prove the inequality
$$16Q^3 \geq 27r^4P,$$
where Q and P are the areas of the triangles $A'B'C'$ and ABC respectively.

7. **(FIN 1)** Show that any two points lying inside a regular n-gon E can be joined by two circular arcs lying inside E and meeting at an angle of at least $\left(1-\frac{2}{n}\right)\pi$.

8. **(FRA 2)** Let R be a rectangle that is the union of a finite number of rectangles R_i, $1 \leq i \leq n$, satisfying the following conditions:
   (i) The sides of every rectangle R_i are parallel to the sides of R.
   (ii) The interiors of any two different R_i are disjoint.
   (iii) Every R_i has at least one side of integral length.
   Prove that R has at least one side of integral length.

9. **(FRA 4)** For all integers n, $n \geq 0$, there exist uniquely determined integers a_n, b_n, c_n such that
$$\left(1+4\sqrt[3]{2}-4\sqrt[3]{4}\right)^n = a_n + b_n\sqrt[3]{2} + c_n\sqrt[3]{4}.$$
Prove that $c_n = 0$ implies $n = 0$.

10. **(GRE 3)** Let $g : \mathbb{C} \to \mathbb{C}$, $w \in \mathbb{C}$, $a \in \mathbb{C}$, $w^3 = 1$ ($w \neq 1$). Show that there is one and only one function $f : \mathbb{C} \to \mathbb{C}$ such that
$$f(z) + f(wz + a) = g(z), \qquad z \in \mathbb{C}.$$
Find the function f.

11. **(HUN 1)** Define sequence a_n by $\sum_{d|n} a_d = 2^n$. Show that $n|a_n$.

12. **(HUN 3)** At n distinct points of a circular race course there are n cars ready to start. Each car moves at a constant speed and covers the circle in an hour. On hearing the initial signal, each of them selects a direction and starts moving immediately. If two cars meet, both of them change directions and go on without loss of speed.
Show that at a certain moment each car will be at its starting point.

13. **(ICE 3)**[IMO4] The quadrilateral $ABCD$ has the following properties:
   (i) $AB = AD + BC$;
   (ii) there is a point P inside it at a distance x from the side CD such that $AP = x + AD$ and $BP = x + BC$.
   Show that
$$\frac{1}{\sqrt{x}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

14. **(IND 2)** A *bicentric* quadrilateral is one that is both inscribable in and circumscribable about a circle. Show that for such a quadrilateral, the centers of the two associated circles are collinear with the point of intersection of the diagonals.

15. **(IRE 1)** Let a, b, c, d, m, n be positive integers such that $a^2+b^2+c^2+d^2 = 1989$, $a+b+c+d = m^2$, and the largest of a, b, c, d is n^2. Determine, with proof, the values of m and n.

16. **(ISR 1)** The set $\{a_0, a_1, \dots, a_n\}$ of real numbers satisfies the following conditions:
    (i) $a_0 = a_n = 0$;
    (ii) for $1 \le k \le n-1$,
$$a_k = c + \sum_{i=k}^{n-1} a_{i-k}(a_i + a_{i+1}).$$
    Prove that $c \le \frac{1}{4n}$.

17. **(MON 1)** Given seven points in the plane, some of them are connected by segments so that:
    (i) among any three of the given points, two are connected by a segment;
    (i) the number of segments is minimal.
    How many segments does a figure satisfying (i) and (ii) contain? Give an example of such a figure.

18. **(MON 4)** Given a convex polygon $A_1A_2\dots A_n$ with area S, and a point M in the same plane, determine the area of polygon $M_1M_2\dots M_n$, where M_i is the image of M under rotation $\mathcal{R}_{A_i}^{\alpha}$ around A_i by α, $i = 1, 2, \dots, n$.

19. **(MON 6)** A positive integer is written in each square of an $m \times n$ board. The allowed move is to add an integer k to each of two adjacent numbers in such a way that no negative numbers are obtained. (Two squares are adjacent if they have a common side.) Find a necessary and sufficient condition for it to be possible for all the numbers to be zero by a finite sequence of moves.

20. **(NET 1)**^IMO3 Given a set S in the plane containing n points and satisfying the conditions:
    (i) no three points of S are collinear,
    (ii) for every point P of S there exist at least k points in S that have the same distance to P,
    prove that the following inequality holds:
$$k < \frac{1}{2} + \sqrt{2n}.$$

21. **(NET 2)** Prove that the intersection of a plane and a regular tetrahedron can be an obtuse-angled triangle and that the obtuse angle in any such triangle is always smaller than 120°.

22. **(PHI 1)**$^{\text{IMO1}}$ Prove that the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of 17 subsets $A_1, A_2, \ldots, A_{17}$ such that:
 (i) each A_i contains the same number of elements;
 (ii) the sum of all elements of each A_i is the same for $i = 1, 2, \ldots, 17$.

23. **(POL 2)**$^{\text{IMO6}}$ We consider permutations $(x_1, \ldots, x_{2n})$ of the set $\{1, \ldots, 2n\}$ such that $|x_i - x_{i+1}| = n$ for at least one $i \in \{1, \ldots, 2n-1\}$. For every natural number n, find out whether permutations with this property are more or less numerous than the remaining permutations of $\{1, \ldots, 2n\}$.

24. **(POL 5)** For points $A_1, \ldots, A_5$ on the sphere of radius 1, what is the maximum value that $\min_{1 \le i,j \le 5} A_iA_j$ can take? Determine all configurations for which this maximum is attained. (Or: determine the diameter of any set $\{A_1, \ldots, A_5\}$ for which this maximum is attained.)

25. **(KOR 3)** Let a, b be integers that are not perfect squares. Prove that if
$$x^2 - ay^2 - bz^2 + abw^2 = 0$$
has a nontrivial solution in integers, then so does
$$x^2 - ay^2 - bz^2 = 0.$$

26. **(KOR 4)** Let n be a positive integer and let a, b be given real numbers. Determine the range of x_0 for which
$$\sum_{i=0}^{n} x_i = a \quad \text{and} \quad \sum_{i=0}^{n} x_i^2 = b,$$
where $x_0, x_1, \ldots, x_n$ are real variables.

27. **(ROM 1)** Let m be a positive odd integer, $m \ge 2$. Find the smallest positive integer n such that 2^{1989} divides $m^n - 1$.

28. **(ROM 2)** Consider in a plane Π the points O, A_1, A_2, A_3, A_4 such that $\sigma(OA_iA_j) \ge 1$ for all $i, j = 1, 2, 3, 4$, $i \ne j$. Prove that there is at least one pair $i_0, j_0 \in \{1, 2, 3, 4\}$ such that $\sigma(OA_{i_0}A_{j_0}) \ge \sqrt{2}$.
(We have denoted by $\sigma(OA_iA_j)$ the area of triangle OA_iA_j.)

29. **(ROM 4)** A flock of 155 birds sit down on a circle C. Two birds P_i, P_j are mutually visible if $m(P_iP_j) \le 10°$. Find the smallest number of mutually visible pairs of birds. (One assumes that a position (point) on C can be occupied simultaneously by several birds.)

30. **(SWE 2)**$^{\text{IMO5}}$ For which positive integers n does there exist a positive integer N such that none of the integers $1 + N, 2 + N, \ldots, n + N$ is the power of a prime number?

31. **(SWE 3)** Let $a_1 \ge a_2 \ge a_3$ be given positive integers and let $N(a_1, a_2, a_3)$ be the number of solutions (x_1, x_2, x_3) of the equation

$$\frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3} = 1,$$

where x_1, x_2, and x_3 are positive integers. Show that

$$N(a_1, a_2, a_3) \leq 6a_1a_2(3 + \ln(2a_1)).$$

32. **(USA 3)** The vertex A of the acute triangle ABC is equidistant from the circumcenter O and the orthocenter H. Determine all possible values for the measure of angle A.

3.31 The Thirty-First IMO
Beijing, China, July 8–19, 1990

3.31.1 Contest Problems

First Day (July 12)

1. Given a circle with two chords AB, CD that meet at E, let M be a point of chord AB other than E. Draw the circle through D, E, and M. The tangent line to the circle DEM at E meets the lines BC, AC at F, G, respectively. Given $\frac{AM}{AB} = \lambda$, find $\frac{GE}{EF}$.
2. On a circle, $2n-1$ $(n \geq 3)$ different points are given. Find the minimal natural number N with the property that whenever N of the given points are colored black, there exist two black points such that the interior of one of the corresponding arcs contains exactly n of the given $2n-1$ points.
3. Find all positive integers n having the property that $\frac{2^n+1}{n^2}$ is an integer.

Second Day (July 13)

4. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that
$$f(xf(y)) = \frac{f(x)}{y}, \qquad \text{for all } x, y \text{ in } \mathbb{Q}^+.$$
5. Two players A and B play a game in which they choose numbers alternately according to the following rule: At the beginning, an initial natural number $n_0 > 1$ is given. Knowing n_{2k}, player A may choose any $n_{2k+1} \in \mathbb{N}$ such that
$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$
Then player B chooses a number $n_{2k+2} \in \mathbb{N}$ such that
$$\frac{n_{2k+1}}{n_{2k+2}} = p^r,$$
where p is a prime number and $r \in \mathbb{N}$.
It is stipulated that player A wins the game if he (she) succeeds in choosing the number 1990, and player B wins if he (she) succeeds in choosing 1. For which natural numbers n_0 can player A manage to win the game, for which n_0 can player B manage to win, and for which n_0 can players A and B each force a tie?
6. Is there a 1990-gon with the following properties (i) and (ii)?
   (i) All angles are equal;
   (ii) The lengths of the 1990 sides are a permutation of the numbers $1^2, 2^2, \ldots, 1989^2, 1990^2$.

3.31.2 Shortlisted Problems

1. **(AUS 3)** The integer 9 can be written as a sum of two consecutive integers: $9 = 4+5$. Moreover, it can be written as a sum of (more than one) consecutive positive integers in exactly two ways: $9 = 4+5 = 2+3+4$. Is there an integer that can be written as a sum of 1990 consecutive integers and that can be written as a sum of (more than one) consecutive positive integers in exactly 1990 ways?

2. **(CAN 1)** Given n countries with three representatives each, m committees $A(1), A(2), \dots A(m)$ are called *a cycle* if
   (i) each committee has n members, one from each country;
   (ii) no two committees have the same membership;
   (iii) for $i = 1, 2, \dots, m$, committee $A(i)$ and committee $A(i+1)$ have no member in common, where $A(m+1)$ denotes $A(1)$;
   (iv) if $1 < |i-j| < m-1$, then committees $A(i)$ and $A(j)$ have at least one member in common.

   Is it possible to have a cycle of 1990 committees with 11 countries?

3. **(CZS 1)**$^{\text{IMO2}}$ On a circle, $2n-1$ $(n \ge 3)$ different points are given. Find the minimal natural number N with the property that whenever N of the given points are colored black, there exist two black points such that the interior of one of the corresponding arcs contains exactly n of the given $2n-1$ points.

4. **(CZS 2)** Assume that the set of all positive integers is decomposed into r (disjoint) subsets $A_1 \cup A_2 \cup \cdots A_r = \mathbb{N}$. Prove that one of them, say A_i, has the following property: There exists a positive m such that for any k one can find numbers $a_1, a_2, \dots, a_k$ in A_i with $0 < a_{j+1} - a_j \le m$ $(1 \le j \le k-1)$.

5. **(FRA 1)** Given $\triangle ABC$ with no side equal to another side, let G, K, and H be its centroid, incenter, and orthocenter, respectively. Prove that $\angle GKH > 90°$.

6. **(FRG 2)**$^{\text{IMO5}}$ Two players A and B play a game in which they choose numbers alternately according to the following rule: At the beginning, an initial natural number $n_0 > 1$ is given. Knowing n_{2k}, player A may choose any $n_{2k+1} \in \mathbb{N}$ such that
$$n_{2k} \le n_{2k+1} \le n_{2k}^2.$$
   Then player B chooses a number $n_{2k+2} \in \mathbb{N}$ such that
$$\frac{n_{2k+1}}{n_{2k+2}} = p^r,$$
   where p is a prime number and $r \in \mathbb{N}$.
   It is stipulated that player A wins the game if he (she) succeeds in choosing the number 1990, and player B wins if he (she) succeeds in choosing 1.

For which natural numbers n_0 can player A manage to win the game, for which n_0 can player B manage to win, and for which n_0 can players A and B each force a tie?

7. **(GRE 2)** Let $f(0) = f(1) = 0$ and

$$f(n+2) = 4^{n+2} f(n+1) - 16^{n+1} f(n) + n \cdot 2^{n^2}, \qquad n = 0, 1, 2, 3, \dots.$$

Show that the numbers $f(1989), f(1990), f(1991)$ are divisible by 13.

8. **(HUN 1)** For a given positive integer k denote the square of the sum of its digits by $f_1(k)$ and let $f_{n+1}(k) = f_1(f_n(k))$.
Determine the value of $f_{1991}(2^{1990})$.

9. **(HUN 3)** The incenter of the triangle ABC is K. The midpoint of AB is C_1 and that of AC is B_1. The lines C_1K and AC meet at B_2, the lines B_1K and AB at C_2. If the areas of the triangles AB_2C_2 and ABC are equal, what is the measure of angle $\angle CAB$?

10. **(ICE 2)** A plane cuts a right circular cone into two parts. The plane is tangent to the circumference of the base of the cone and passes through the midpoint of the altitude. Find the ratio of the volume of the smaller part to the volume of the whole cone.

11. **(IND 3′)**$^{\text{IMO1}}$ Given a circle with two chords AB, CD that meet at E, let M be a point of chord AB other than E. Draw the circle through D, E, and M. The tangent line to the circle DEM at E meets the lines BC, AC at F, G, respectively. Given $\frac{AM}{AB} = \lambda$, find $\frac{GE}{EF}$.

12. **(IRE 1)** Let ABC be a triangle and L the line through C parallel to the side AB. Let the internal bisector of the angle at A meet the side BC at D and the line L at E and let the internal bisector of the angle at B meet the side AC at F and the line L at G. If $GF = DE$, prove that $AC = BC$.

13. **(IRE 2)** An eccentric mathematician has a ladder with n rungs that he always ascends and descends in the following way: When he ascends, each step he takes covers a rungs of the ladder, and when he descends, each step he takes covers b rungs of the ladder, where a and b are fixed positive integers. By a sequence of ascending and descending steps he can climb from ground level to the top rung of the ladder and come back down to ground level again. Find, with proof, the minimum value of n, expressed in terms of a and b.

14. **(JAP 2)** In the coordinate plane a rectangle with vertices $(0, 0), (m, 0)$, $(0, n)$, (m, n) is given where both m and n are odd integers. The rectangle is partitioned into triangles in such a way that
   (i) each triangle in the partition has at least one side (to be called a "good" side) that lies on a line of the form $x = j$ or $y = k$, where j and k are integers, and the altitude on this side has length 1;

(ii) each "bad" side (i.e., a side of any triangle in the partition that is not a "good" one) is a common side of two triangles in the partition.

Prove that there exist at least two triangles in the partition each of which has two good sides.

15. **(MEX 2)** Determine for which positive integers k the set

$$X = \{1990, 1990+1, 1990+2, \dots, 1990+k\}$$

can be partitioned into two disjoint subsets A and B such that the sum of the elements of A is equal to the sum of the elements of B.

16. **(NET 1)**$^{\mathrm{IMO6}}$ Is there a 1990-gon with the following properties (i) and (ii)?
    (i) All angles are equal;
    (ii) The lengths of the 1990 sides are a permutation of the numbers $1^2, 2^2, \dots, 1989^2, 1990^2$.

17. **(NET 3)** Unit cubes are made into beads by drilling a hole through them along a diagonal. The beads are put on a string in such a way that they can move freely in space under the restriction that the vertices of two neighboring cubes are touching. Let A be the beginning vertex and B be the end vertex. Let there be $p \times q \times r$ cubes on the string $(p, q, r \geq 1)$.
    (a) Determine for which values of p, q, and r it is possible to build a block with dimensions p, q, and r. Give reasons for your answers.
    (b) The same question as (a) with the extra condition that $A = B$.

18. **(NOR)** Let a, b be natural numbers with $1 \leq a \leq b$, and $M = \left[\frac{a+b}{2}\right]$. Define the function $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} n + a, & \text{if } n < M, \\ n - b, & \text{if } n \geq M. \end{cases}$$

Let $f^1(n) = f(n)$, $f^{i+1}(n) = f(f^i(n))$, $i = 1, 2, \dots$. Find the smallest natural number k such that $f^k(0) = 0$.

19. **(POL 1)** Let P be a point inside a regular tetrahedron T of unit volume. The four planes passing through P and parallel to the faces of T partition T into 14 pieces. Let $f(P)$ be the joint volume of those pieces that are neither a tetrahedron nor a parallelepiped (i.e., pieces adjacent to an edge but not to a vertex). Find the exact bounds for $f(P)$ as P varies over T.

20. **(POL 3)** Prove that every integer k greater than 1 has a multiple that is less than k^4 and can be written in the decimal system with at most four different digits.

21. **(ROM 1′)** Let n be a composite natural number and p a proper divisor of n. Find the binary representation of the smallest natural number N such that $\frac{(1+2^p+2^{n-p})N-1}{2^n}$ is an integer.

22. **(ROM 4)** Ten localities are served by two international airlines such that there exists a direct service (without stops) between any two of these localities and all airline schedules offer round-trip service between the cities they serve. Prove that at least one of the airlines can offer two disjoint round trips each containing an odd number of landings.

23. **(ROM 5)**$^{\text{IMO3}}$ Find all positive integers n having the property that $\frac{2^n+1}{n^2}$ is an integer.

24. **(THA 2)** Let a, b, c, d be nonnegative real numbers such that $ab + bc + cd + da = 1$. Show that
$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

25. **(TUR 4)**$^{\text{IMO4}}$ Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that
$$f(xf(y)) = \frac{f(x)}{y}, \qquad \text{for all } x, y \text{ in } \mathbb{Q}^+.$$

26. **(USA 2)** Let P be a cubic polynomial with rational coefficients, and let $q_1, q_2, q_3, \dots$ be a sequence of rational numbers such that $q_n = P(q_{n+1})$ for all $n \geq 1$. Prove that there exists $k \geq 1$ such that for all $n \geq 1$, $q_{n+k} = q_n$.

27. **(USS 1)** Find all natural numbers n for which every natural number whose decimal representation has $n-1$ digits 1 and one digit 7 is prime.

28. **(USS 3)** Prove that on the coordinate plane it is impossible to draw a closed broken line such that
    (i) the coordinates of each vertex are rational;
    (ii) the length each of its edges is 1;
    (iii) the line has an odd number of vertices.

3.32 The Thirty-Second IMO Sigtuna, Sweden, July 12–23, 1991

3.32.1 Contest Problems

First Day (July 17)

1. Prove for each triangle ABC the inequality
$$\frac{1}{4} < \frac{IA \cdot IB \cdot IC}{l_A l_B l_C} \le \frac{8}{27},$$
where I is the incenter and l_A, l_B, l_C are the lengths of the angle bisectors of ABC.
2. Let $n > 6$ and let $a_1 < a_2 < \dots < a_k$ be all natural numbers that are less than n and relatively prime to n. Show that if $a_1, a_2, \dots, a_k$ is an arithmetic progression, then n is a prime number or a natural power of two.
3. Let $S = \{1, 2, 3, \dots, 280\}$. Find the minimal natural number n such that in any n-element subset of S there are five numbers that are pairwise relatively prime.

Second Day (July 18)

4. Suppose G is a connected graph with n edges. Prove that it is possible to label the edges of G from 1 to n in such a way that in every vertex v of G with two or more incident edges, the set of numbers labeling those edges has no common divisor greater than 1.
5. Let ABC be a triangle and M an interior point in ABC. Show that at least one of the angles $\measuredangle MAB, \measuredangle MBC$, and $\measuredangle MCA$ is less than or equal to 30°.
6. Given a real number $a > 1$, construct an infinite and bounded sequence $x_0, x_1, x_2, \dots$ such that for all natural numbers i and j, $i \ne j$, the following inequality holds:
$$|x_i - x_j||i - j|^a \ge 1.$$

3.32.2 Shortlisted Problems

1. **(PHI 3)** Let ABC be any triangle and P any point in its interior. Let P_1, P_2 be the feet of the perpendiculars from P to the two sides AC and BC. Draw AP and BP, and from C drop perpendiculars to AP and BP. Let Q_1 and Q_2 be the feet of these perpendiculars. Prove that the lines Q_1P_2, Q_2P_1, and AB are concurrent.

2. **(JAP 5)** For an acute triangle ABC, M is the midpoint of the segment BC, P is a point on the segment AM such that $PM = BM$, H is the foot of the perpendicular line from P to BC, Q is the point of intersection of segment AB and the line passing through H that is perpendicular to PB, and finally, R is the point of intersection of the segment AC and the line passing through H that is perpendicular to PC.
   Show that the circumcircle of $\triangle QHR$ is tangent to the side BC at point H.

3. **(PRK 1)** Let S be any point on the circumscribed circle of $\triangle PQR$. Then the feet of the perpendiculars from S to the three sides of the triangle lie on the same straight line. Denote this line by $l(S, PQR)$. Suppose that the hexagon $ABCDEF$ is inscribed in a circle. Show that the four lines $l(A, BDF), l(B, ACE), l(D, ABF)$, and $l(E, ABC)$ intersect at one point if and only if $CDEF$ is a rectangle.

4. **(FRA 2)**[IMO5] Let ABC be a triangle and M an interior point in ABC. Show that at least one of the angles $\measuredangle MAB, \measuredangle MBC$, and $\measuredangle MCA$ is less than or equal to 30°.

5. **(SPA 4)** In the triangle ABC, with $\measuredangle A = 60^\circ$, a parallel IF to AC is drawn through the incenter I of the triangle, where F lies on the side AB. The point P on the side BC is such that $3BP = BC$. Show that $\measuredangle BFP = \measuredangle B/2$.

6. **(USS 4)**[IMO1] Prove for each triangle ABC the inequality
$$\frac{1}{4} < \frac{IA \cdot IB \cdot IC}{l_A l_B l_C} \leq \frac{8}{27},$$
   where I is the incenter and l_A, l_B, l_C are the lengths of the angle bisectors of ABC.

7. **(CHN 2)** Let O be the center of the circumsphere of a tetrahedron $ABCD$. Let L, M, N be the midpoints of BC, CA, AB respectively, and assume that $AB+BC = AD+CD$, $BC+CA = BD+AD$, and $CA+AB = CD + BD$. Prove that $\angle LOM = \angle MON = \angle NOL$.

8. **(NET 1)** Let S be a set of n points in the plane. No three points of S are collinear. Prove that there exists a set P containing $2n - 5$ points satisfying the following condition: In the interior of every triangle whose three vertices are elements of S lies a point that is an element of P.

9. **(FRA 3)** In the plane we are given a set E of 1991 points, and certain pairs of these points are joined with a path. We suppose that for every point of E, there exist at least 1593 other points of E to which it is joined by a path. Show that there exist six points of E every pair of which are joined by a path.

   Alternative version. Is it possible to find a set E of 1991 points in the plane and paths joining certain pairs of the points in E such that every point of E is joined with a path to at least 1592 other points of E, and in every subset of six points of E there exist at least two points that are not joined?

10. **(USA 5)**$^{\text{IMO4}}$ Suppose G is a connected graph with n edges. Prove that it is possible to label the edges of G from 1 to n in such a way that in every vertex v of G with two or more incident edges, the set of numbers labeling those edges has no common divisor greater than 1.

11. **(AUS 4)** Prove that

$$\sum_{m=0}^{995} \frac{(-1)^m}{1991-m}\binom{1991-m}{m} = \frac{1}{1991}.$$

12. **(CHN 3)**$^{\text{IMO3}}$ Let $S = \{1, 2, 3, \dots, 280\}$. Find the minimal natural number n such that in any n-element subset of S there are five numbers that are pairwise relatively prime.

13. **(POL 4)** Given any integer $n \geq 2$, assume that the integers $a_1, a_2, \dots, a_n$ are not divisible by n and, moreover, that n does not divide $a_1 + a_2 + \cdots + a_n$. Prove that there exist at least n different sequences $(e_1, e_2, \cdots, e_n)$ consisting of zeros or ones such that $e_1a_1 + e_2a_2 + \cdots + e_na_n$ is divisible by n.

14. **(POL 3)** Let a, b, c be integers and p an odd prime number. Prove that if $f(x) = ax^2 + bx + c$ is a perfect square for $2p - 1$ consecutive integer values of x, then p divides $b^2 - 4ac$.

15. **(USS 2)** Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence $a_1, a_2, \dots, a_n, \dots$ become periodic after a finite number of terms?

16. **(ROM 1)**$^{\text{IMO2}}$ Let $n > 6$ and $a_1 < a_2 < \cdots < a_k$ be all natural numbers that are less than n and relatively prime to n. Show that if $a_1, a_2, \dots, a_k$ is an arithmetic progression, then n is a prime number or a natural power of two.

17. **(HKG 4)** Find all positive integer solutions x, y, z of the equation $3^x + 4^y = 5^z$.

18. **(BUL 1)** Find the highest degree k of 1991 for which 1991^k divides the number

$$1990^{1991^{1992}} + 1992^{1991^{1990}}.$$

19. **(IRE 5)** Let a be a rational number with $0 < a < 1$ and suppose that

$$\cos 3\pi a + 2\cos 2\pi a = 0.$$

(Angle measurements are in radians.) Prove that $a = 2/3$.

20. **(IRE 3)** Let α be the positive root of the equation $x^2 = 1991x + 1$. For natural numbers m, n define

$$m * n = mn + [\alpha m][\alpha n],$$

where $[x]$ is the greatest integer not exceeding x. Prove that for all natural numbers p, q, r,

$$(p * q) * r = p * (q * r).$$

21. **(HKG 6)** Let $f(x)$ be a monic polynomial of degree 1991 with integer coefficients. Define $g(x) = f^2(x) - 9$. Show that the number of distinct integer solutions of $g(x) = 0$ cannot exceed 1995.

22. **(USA 4)** Real constants a, b, c are such that there is exactly one square all of whose vertices lie on the cubic curve $y = x^3 + ax^2 + bx + c$. Prove that the square has sides of length $\sqrt[4]{72}$.

23. **(IND 2)** Let f and g be two integer-valued functions defined on the set of all integers such that
(a) $f(m + f(f(n))) = -f(f(m+1) - n$ for all integers m and n;
(b) g is a polynomial function with integer coefficients and $g(n) = g(f(n))$ for all integers n.
Determine $f(1991)$ and the most general form of g.

24. **(IND 1)** An odd integer $n \geq 3$ is said to be "nice" if there is at least one permutation $a_1, a_2, \dots, a_n$ of $1, 2, \dots, n$ such that the n sums $a_1 - a_2 + a_3 - \cdots - a_{n-1} + a_n, a_2 - a_3 + a_4 - \cdots - a_n + a_1, a_3 - a_4 + a_5 - \cdots - a_1 + a_2, \dots, a_n - a_1 + a_2 - \cdots - a_{n-2} + a_{n-1}$ are all positive. Determine the set of all "nice" integers.

25. **(USA 1)** Suppose that $n \geq 2$ and $x_1, x_2, \dots, x_n$ are real numbers between 0 and 1 (inclusive). Prove that for some index i between 1 and $n-1$ the inequality

$$x_i(1 - x_{i+1}) \geq \frac{1}{4} x_1(1 - x_n)$$

holds.

26. **(CZS 1)** Let $n \geq 2$ be a natural number and let the real numbers $p, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ satisfy $1/2 \leq p \leq 1$, $0 \leq a_i$, $0 \leq b_i \leq p$, $i = 1, \dots, n$, and $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$. Prove the inequality

$$\sum_{i=1}^n b_i \prod_{\substack{j=1 \\ j \neq i}}^n a_j \leq \frac{p}{(n-1)^{n-1}}.$$

27. **(POL 2)** Determine the maximum value of the sum

$$\sum_{i<j} x_i x_j (x_i + x_j)$$

over all n-tuples $(x_1, \dots, x_n)$, satisfying $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$.

28. **(NET 1)**[IMO6] Given a real number $a > 1$, construct an infinite and bounded sequence $x_0, x_1, x_2, \ldots$ such that for all natural numbers i and j, $i \neq j$, the following inequality holds:

$$|x_i - x_j||i - j|^a \geq 1.$$

29. **(FIN 2)** We call a set S on the real line $\mathbb{R}$ *superinvariant* if for any stretching A of the set by the transformation taking x to $A(x) = x_0 + a(x - x_0)$ there exists a translation B, $B(x) = x + b$, such that the images of S under A and B agree; i.e., for any $x \in S$ there is a $y \in S$ such that $A(x) = B(y)$ and for any $t \in S$ there is a $u \in S$ such that $B(t) = A(u)$. Determine all superinvariant sets.

    Remark. It is assumed that $a > 0$.

30. **(BUL 3)** Two students A and B are playing the following game: Each of them writes down on a sheet of paper a positive integer and gives the sheet to the referee. The referee writes down on a blackboard two integers, one of which is the sum of the integers written by the players. After that, the referee asks student A: "Can you tell the integer written by the other student?" If A answers "no," the referee puts the same question to student B. If B answers "no," the referee puts the question back to A, and so on. Assume that both students are intelligent and truthful. Prove that after a finite number of questions, one of the students will answer "yes."

3.33 The Thirty-Third IMO
Moscow, Russia, July 10–21, 1992

3.33.1 Contest Problems

First Day (July 15)

1. Find all integer triples (p, q, r) such that $1 < p < q < r$ and $(p-1)(q-1)(r-1)$ is a divisor of $(pqr - 1)$.
2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 + f(y)) = y + f(x)^2 \text{ for all } x, y \text{ in } \mathbb{R}.$$
3. Given nine points in space, no four of which are coplanar, find the minimal natural number n such that for any coloring with red or blue of n edges drawn between these nine points there always exists a triangle having all edges of the same color.

Second Day (July 16)

4. In the plane, let there be given a circle C, a line l tangent to C, and a point M on l. Find the locus of points P that has the following property: There exist two points Q and R on l such that M is the midpoint of QR and C is the incircle of PQR.
5. Let V be a finite subset of Euclidean space consisting of points (x, y, z) with integer coordinates. Let S_1, S_2, S_3 be the projections of V onto the yz, xz, xy planes, respectively. Prove that
$$|V|^2 \le |S_1||S_2||S_3|$$
($|X|$ denotes the number of elements of X).
6. For each positive integer n, denote by $s(n)$ the greatest integer such that for all positive integer $k \le s(n)$, n^2 can be expressed as a sum of squares of k positive integers.
   (a) Prove that $s(n) \le n^2 - 14$ for all $n \ge 4$.
   (b) Find a number n such that $s(n) = n^2 - 14$.
   (c) Prove that there exist infinitely many positive integers n such that $s(n) = n^2 - 14$.

3.33.2 Longlisted Problems

1. **(AUS 1)** Points D and E are chosen on the sides AB and AC of the triangle ABC in such a way that if F is the intersection point of BE and CD, then $AE + EF = AD + DF$. Prove that $AC + CF = AB + BF$.

2. **(AUS 2)** (SL92-1).
   Original formulation. Let m be a positive integer and x_0, y_0 integers such that x_0, y_0 are relatively prime, y_0 divides $x_0^2 + m$, and x_0 divides $y_0^2 + m$. Prove that there exist positive integers x and y such that x and y are relatively prime, y divides $x^2 + m$, x divides $y^2 + m$, and $x + y \leq m + 1$.

3. **(AUS 3)** Let ABC be a triangle, O its circumcenter, S its centroid, and H its orthocenter. Denote by A_1, B_1, and C_1 the centers of the circles circumscribed about the triangles CHB, CHA, and AHB, respectively. Prove that the triangle ABC is congruent to the triangle $A_1B_1C_1$ and that the nine-point circle of $\triangle ABC$ is also the nine-point circle of $\triangle A_1B_1C_1$.

4. **(CAN 1)** Let p, q, and r be the angles of a triangle, and let $a = \sin 2p$, $b = \sin 2q$, and $c = \sin 2r$. If $s = (a + b + c)/2$, show that
   $$s(s-a)(s-b)(s-c) \geq 0.$$
   When does equality hold?

5. **(CAN 2)** Let I, H, O be the incenter, centroid, and circumcenter of the nonisosceles triangle ABC. Prove that $AI \| HO$ if and only if $\angle BAC = 120°$.

6. **(CAN 3)** Suppose that n numbers $x_1, x_2, \ldots, x_n$ are chosen randomly from the set $\{1, 2, 3, 4, 5\}$. Prove that the probability that $x_1^2 + x_2^2 + \cdots + x_n^2 \equiv 0 \pmod 5$ is at least $1/5$.

7. **(CAN 4)** Let X be a bounded, nonempty set of points in the Cartesian plane. Let $f(X)$ be the set of all points that are at a distance of at most 1 from some point in X. Let $f^n(X) = f(f(\ldots(f(X))\ldots))$ (n times). Show that $f^n(X)$ becomes "more circular" as n gets larger. In other words, if $r_n = \sup\{\text{radii of circles contained in } f^n(X)\}$ and $R_n = \inf\{\text{radii of circles containing } f^n(X)\}$, then show that R_n/r_n gets arbitrarily close to 1 as n becomes arbitrarily large.

8. **(CHN 1)** (SL92-2).

9. **(CHN 2)** (SL92-3).

10. **(CHN 3)** (SL92-4).

11. **(COL 1)** Let $\phi(n, m)$, $m \neq 1$, be the number of positive integers less than or equal to n that are coprime with m. Clearly, $\phi(m, m) = \phi(m)$, where $\phi(m)$ is Euler's phi function. Find all integers m that satisfy the following inequality:
    $$\frac{\phi(n, m)}{n} \geq \frac{\phi(m)}{m}$$
    for every positive integer n.

12. **(COL 2)** Given a triangle ABC such that the circumcenter is in the interior of the incircle, prove that the triangle ABC is acute-angled.

13. **(COL 3)** (SL92-5).

14. **(FIN 1)** Integers $a_1, a_2, \dots, a_n$ satisfy $|a_k| = 1$ and
$$\sum_{k=1}^{n} a_k a_{k+1} a_{k+2} a_{k+3} = 2,$$
where $a_{n+j} = a_j$. Prove that $n \neq 1992$.

15. **(FIN 2)** Prove that there exist 78 lines in the plane such that they have exactly 1992 points of intersection.

16. **(FIN 3)** Find all triples (x, y, z) of integers such that
$$\frac{1}{x^2} + \frac{2}{y^2} + \frac{3}{z^2} = \frac{2}{3}.$$

17. **(FRA 1)** (SL92-20).

18. **(FRG 1)** Fibonacci numbers are defined as follows: $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, $n \geq 1$. Let a_n be the number of words that consist of n letters 0 or 1 and contain no two letters 1 at distance two from each other. Express a_n in terms of Fibonacci numbers.

19. **(FRG 2)** Denote by a_n the greatest number that is not divisible by 3 and that divides n. Consider the sequence $s_0 = 0$, $s_n = a_1 + a_2 + \cdots + a_n$, $n \in \mathbb{N}$. Denote by $A(n)$ the number of all sums s_k $(0 \leq k \leq 3^n,\ k \in \mathbb{N}_0)$ that are divisible by 3. Prove the formula
$$A(n) = 3^{n-1} + 2 \cdot 3^{(n/2)-1} \cos(n\pi/6), \qquad n \in \mathbb{N}_0.$$

20. **(FRG 3)** Let X and Y be two sets of points in the plane and M be a set of segments connecting points from X and Y. Let k be a natural number. Prove that the segments from M can be painted using k colors in such a way that for any point $x \in X \cup Y$ and two colors α and β $(\alpha \neq \beta)$, the difference between the number of α-colored segments and the number of β-colored segments originating in X is less than or equal to 1.

21. **(GBR 1)** Prove that if $x, y, z > 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, then
$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

22. **(GBR 2)** (SL92-21).

23. **(HKG 1)** An *Egyptian number* is a positive integer that can be expressed as a sum of positive integers, not necessarily distinct, such that the sum of their reciprocals is 1. For example, $32 = 2 + 3 + 9 + 18$ is Egyptian because $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = 1$. Prove that all integers greater than 23 are Egyptian.

24. **(ICE 1)** Let $\mathbb{Q}^+$ denote the set of nonnegative rational numbers. Show that there exists exactly one function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the following conditions:
   (i) if $0 < q < \frac{1}{2}$, then $f(q) = 1 + f\left(\frac{q}{1-2q}\right)$;
   (ii) if $1 < q \le 2$, then $f(q) = 1 + f(q+1)$;
   (iii) $f(q)f(1/q) = 1$ for all $q \in \mathbb{Q}^+$.
   Find the smallest rational number $q \in \mathbb{Q}^+$ such that $f(q) = 19/92$.

25. **(IND 1)** (a) Show that the set $\mathbb{N}$ of all natural numbers can be partitioned into three disjoint subsets A, B, and C satisfying the following conditions:
$$\begin{aligned} A^2 &= A, & B^2 &= C, & C^2 &= B, \\ AB &= B, & AC &= C, & BC &= A, \end{aligned}$$
   where HK stands for $\{hk \mid h \in H,\ k \in K\}$ for any two subsets H, K of $\mathbb{N}$, and H^2 denotes HH.
   (b) Show that for every such partition of $\mathbb{N}$, $\min\{n \in \mathbb{N} \mid n \in A \text{ and } n + 1 \in A\}$ is less than or equal to 77.

26. **(IND 2)** (SL92-6).

27. **(IND 3)** Let ABC be an arbitrary scalene triangle. Define Σ to be the set of all circles y that have the following properties:
   (i) y meets each side of $\triangle ABC$ in two (possibly coincident) points;
   (ii) if the points of intersection of y with the sides of the triangle are labeled by P, Q, R, S, T, U, with the points occurring on the sides in orders $\mathcal{B}(B, P, Q, C)$, $\mathcal{B}(C, R, S, A)$, $\mathcal{B}(A, T, U, B)$, then the following relations of parallelism hold: $TS \| BC$; $PU \| CA$; $RQ \| AB$. (In the limiting cases, some of the conditions of parallelism will hold vacuously; e.g., if A lies on the circle y, then T, S both coincide with A and the relation $TS \| BC$ holds vacuously.)
   (a) Under what circumstances is Σ nonempty?
   (b) Assuming that Σ is nonempty, show how to construct the locus of centers of the circles in the set Σ.
   (c) Given that the set Σ has just one element, deduce the size of the largest angle of $\triangle ABC$.
   (d) Show how to construct the circles in Σ that have, respectively, the largest and the smallest radii.

28. **(IND 4)** (SL92-7).
   Alternative formulation. Two circles G_1 and G_2 are inscribed in a segment of a circle G and touch each other externally at a point W. Let A be a point of intersection of a common internal tangent to G_1 and G_2 with the arc of the segment, and let B and C be the endpoints of the chord. Prove that W is the incenter of the triangle ABC.

29. **(IND 5)** (SL92-8).

30. **(IND 6)** Let $P_n = (19+92)(19^2+92^2)\cdots(19^n+92^n)$ for each positive integer n. Determine, with proof, the least positive integer m, if it exists, for which P_m is divisible by 33^{33}.

31. **(IRE 1)** (SL92-19).

32. **(IRE 2)** Let $S_n = \{1, 2, \dots, n\}$ and $f_n : S_n \to S_n$ be defined inductively as follows: $f_1(1) = 1$, $f_n(2j) = j$ $(j = 1, 2, \dots, [n/2])$ and
 (i) if $n = 2k$ $(k \geq 1)$, then $f_n(2j-1) = f_k(j) + k$ $(j = 1, 2, \dots, k)$;
 (ii) if $n = 2k+1$ $(k \geq 1)$, then $f_n(2k+1) = k + f_{k+1}(1)$, $f_n(2j-1) = k + f_{k+1}(j+1)$ $(j = 1, 2, \dots, k)$.
 Prove that $f_n(x) = x$ if and only if x is an integer of the form
$$\frac{(2n+1)(2^d-1)}{2^{d+1}-1}$$
 for some positive integer d.

33. **(IRE 3)** Let a, b, c be positive real numbers and p, q, r complex numbers. Let S be the set of all solutions (x, y, z) in $\mathbb{C}$ of the system of simultaneous equations
$$\begin{aligned} ax + by + cz &= p, \\ ax^2 + by^2 + cz^2 &= q, \\ ax^3 + bx^3 + cx^3 &= r. \end{aligned}$$
 Prove that S has at most six elements.

34. **(IRE 4)** Let a, b, c be integers. Prove that there are integers p_1, q_1, r_1, p_2, q_2, r_2 such that
$$a = q_1r_2 - q_2r_1, \quad b = r_1p_2 - r_2p_1, \quad c = p_1q_2 - p_2q_1.$$

35. **(IRN 1)** (SL92-9).

36. **(IRN 2)** Find all rational solutions of
$$\begin{aligned} a^2 + c^2 + 17(b^2 + d^2) &= 21, \\ ab + cd &= 2. \end{aligned}$$

37. **(IRN 3)** Let the circles C_1, C_2, and C_3 be orthogonal to the circle C and intersect each other inside C forming acute angles of measures A, B, and C. Show that $A + B + C < \pi$.

38. **(ITA 1)** (SL92-10).

39. **(ITA 2)** Let $n \geq 2$ be an integer. Find the minimum k for which there exists a partition of $\{1, 2, \dots, k\}$ into n subsets $X_1, X_2, \dots, X_n$ such that the following condition holds: for any i, j, $1 \leq i < j \leq n$, there exist $x_1 \in X_1$, $x_2 \in X_2$ such that $|x_i - x_j| = 1$.

40. **(ITA 3)** The colonizers of a spherical planet have decided to build N towns, each having area $1/1000$ of the total area of the planet. They also decided that any two points belonging to different towns will have different latitude and different longitude. What is the maximal value of N?

41. **(JAP 1)** Let S be a set of positive integers $n_1, n_2, \dots, n_6$ and let $n(f)$ denote the number $n_1 n_{f(1)} + n_2 n_{f(2)} + \cdots + n_6 n_{f(6)}$, where f is a permutation of $\{1, 2, \dots, 6\}$. Let

$$\Omega = \{n(f) \mid f \text{ is a permutation of } \{1, 2, \dots, 6\}\}.$$

Give an example of positive integers $n_1, \dots, n_6$ such that Ω contains as many elements as possible and determine the number of elements of Ω.

42. **(JAP 2)** (SL92-11).

43. **(KOR 1)** Find the number of positive integers n satisfying $\phi(n) \mid n$ such that

$$\sum_{m=1}^{\infty} \left(\frac{n}{m} - \frac{n-1}{m} \right) = 1992.$$

What is the largest number among them? As usual, $\phi(n)$ is the number of positive integers less than or equal to n and relatively prime to n.[6]

44. **(KOR 2)** (SL92-16).

45. **(KOR 3)** Let n be a positive integer. Prove that the number of ways to express n as a sum of distinct positive integers (up to order) and the number of ways to express n as a sum of odd positive integers (up to order) are the same.

46. **(KOR 4)** Prove that the sequence $5, 12, 19, 26, 33, \dots$ contains no term of the form $2^n - 1$.

47. **(KOR 5)** Find the largest integer not exceeding $\prod_{n=1}^{1992} \frac{3n+2}{3n+1}$.

48. **(MON 1)** Find all the functions $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying the identity

$$f(x)f(y) = y^\alpha \cdot f\left(\frac{x}{2}\right) + x^\beta \cdot f\left(\frac{y}{2}\right), \quad x, y \in \mathbb{R}^+,$$

where α, β are given real numbers.

49. **(MON 2)** Given real numbers x_i $(i = 1, 2, \dots, 4x + 2)$ such that

$$\sum_{i=1}^{4x+2} (-1)^{i+1} x_i x_{i+1} = 4m \quad (x_1 = x_{4k+3}),$$

prove that it is possible to choose numbers $x_{k_1}, \dots, x_{k_6}$ such that

[6] The problem in this formulation is senseless. The correct formulation could be, "Find … such that $\sum_{m=1}^{\infty} \left(\left[\frac{n}{m}\right] - \left[\frac{n-1}{m}\right] \right) = 1992$ … ."

$$\sum_{i=1}^{6}(-1)^6 x_{k_1} x_{k_{k+1}} > m \quad (x_{k_1} = x_{k_7}).$$

50. **(MON 3)** Let N be a point inside the triangle ABC. Through the midpoints of the segments AN, BN, and CN the lines parallel to the opposite sides of $\triangle ABC$ are constructed. Let A_N, B_N, and C_N be the intersection points of these lines. If N is the orthocenter of the triangle ABC, prove that the nine-point circles of $\triangle ABC$ and $\triangle A_N B_N C_N$ coincide.

    Remark. The statement of the original problem was that the nine-point circles of the triangles $A_N B_N C_N$ and $A_M B_M C_M$ coincide, where N and M are the orthocenter and the centroid of $\triangle ABC$. This statement is false.

51. **(NET 1)** (SL92-12).

52. **(NET 2)** Let n be an integer > 1. In a circular arrangement of n lamps $L_0, \dots, L_{n-1}$, each one of which can be either ON or OFF, we start with the situation that all lamps are ON, and then carry out a sequence of steps, $Step_0, Step_1, \dots$. If L_{j-1} (j is taken $\mod n$) is ON, then $Step_j$ changes the status of L_j (it goes from ON to OFF or from OFF to ON) but does not change the status of any of the other lamps. If L_{j-1} is OFF, then $Step_j$ does not change anything at all. Show that:
    (a) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again.
    (b) If n has the form 2^k, then all lamps are ON after $n^2 - 1$ steps.
    (c) If n has the form $2^k + 1$, then all lamps are ON after $n^2 - n + 1$ steps.

53. **(NZL 1)** (SL92-13).

54. **(POL 1)** Suppose that $n > m \geq 1$ are integers such that the string of digits 143 occurs somewhere in the decimal representation of the fraction m/n. Prove that $n > 125$

55. **(POL 2)** (SL92-14).

56. **(POL 3)** A directed graph (any two distinct vertices joined by at most one directed line) has the following property: If x, u, and v are three distinct vertices such that $x \to u$ and $x \to v$, then $u \to w$ and $v \to w$ for some vertex w. Suppose that $x \to u \to y \to \cdots \to z$ is a path of length n, that cannot be extended to the right (no arrow goes away from z). Prove that every path beginning at x arrives after n steps at z.

57. **(POL 4)** For positive numbers a, b, c define $A = (a+b+c)/3$, $G = (abc)^{1/3}$, $H = 3/(a^{-1} + b^{-1} + c^{-1})$. Prove that

$$\left(\frac{A}{G}\right)^3 \geq \frac{1}{4} + \frac{3}{4} \cdot \frac{A}{H},$$

for every $a, b, c > 0$.

58. **(POR 1)** Let ABC be a triangle. Denote by a, b, and c the lengths of the sides opposite to the angles A, B, and C, respectively. Prove that[7]

$$\frac{bc}{a+b+c} = \frac{\sin A + \sin B + \sin C}{\cos(A/2)\sin(B/2)\sin(C/2)}.$$

59. **(PRK 1)** Let a regular 7-gon $A_0A_1A_2A_3A_4A_5A_6$ be inscribed in a circle. Prove that for any two points P, Q on the arc A_0A_6 the following equality holds:

$$\sum_{i=0}^{6}(-1)^i PA_i = \sum_{i=0}^{6}(-1)^i QA_i.$$

60. **(PRK 2)** (SL92-15).

61. **(PRK 3)** There are a board with $2n\cdot 2n$ $(=4n^2)$ squares and $4n^2-1$ cards numbered with different natural numbers. These cards are put one by one on each of the squares. One square is empty. We can move a card to an empty square from one of the adjacent squares (two squares are adjacent if they have a common edge). Is it possible to exchange two cards on two adjacent squares of a column (or a row) in a finite number of movements?

62. **(ROM 1)** Let $c_1, \dots, c_n$ $(n \ge 2)$ be real numbers such that $0 \le \sum c_i \le n$. Prove that there exist integers $x_1, \dots, x_n$ such that $\sum k_i = 0$ and $1-n \le c_i + nk_i \le n$ for every $i = 1, \dots, n$.

63. **(ROM 2)** Let a and b be integers. Prove that $\frac{2a^2-1}{b^2+2}$ is not an integer.

64. **(ROM 3)** For any positive integer n consider all representations $n = a_1 + \cdots + a_k$, where $a_1 > a_2 > \cdots > a_k > 0$ are integers such that for all $i \in \{1, 2, \dots, k-1\}$, the number a_i is divisible by a_{i+1}. Find the longest such representation of the number 1992.

65. **(SAF 1)** If A, B, C, and D are four distinct points in space, prove that there is a plane P on which the orthogonal projections of A, B, C, and D form a parallelogram (possibly degenerate).

66. **(SPA 1)** A circle of radius ρ is tangent to the sides AB and AC of the triangle ABC, and its center K is at a distance p from BC.
    (a) Prove that $a(p-\rho) = 2s(r-\rho)$, where r is the inradius and $2s$ the perimeter of ABC.
    (b) Prove that if the circle intersect BC at D and E, then

$$DE = \frac{4\sqrt{rr_1(\rho - r)(r_1 - \rho)}}{(r_1 - r)},$$

    where r_1 is the exradius corresponding to the vertex A.

[7] The statement of the problem is obviously wrong, and the authors couldn't determine a suitable alteration of the formulation which would make the problem correct. We put it here only for completeness of the problem set.

67. **(SPA 2)** In a triangle, a symmedian is a line through a vertex that is symmetric to the median with the respect to the internal bisector (all relative to the same vertex). In the triangle ABC, the median m_a meets BC at A' and the circumcircle again at A_1. The symmedian s_a meets BC at M and the circumcircle again at A_2. Given that the line A_1A_2 contains the circumcenter O of the triangle, prove that:
(a) $\dfrac{AA'}{AM} = \dfrac{b^2+c^2}{2bc}$;
(b) $1 + 4b^2c^2 = a^2(b^2+c^2)$.

68. **(SPA 3)** Show that the numbers $\tan(r\pi/15)$, where r is a positive integer less than 15 and relatively prime to 15, satisfy

$$x^8 - 92x^6 + 134x^4 - 28x^2 + 1 = 0.$$

69. **(SWE 1)** (SL92-17).

70. **(THA 1)** Let two circles A and B with unequal radii r and R, respectively, be tangent internally at the point A_0. If there exists a sequence of distinct circles (C_n) such that each circle is tangent to both A and B, and each circle C_{n+1} touches circle C_n at the point A_n, prove that

$$\sum_{n=1}^{\infty} |A_{n+1}A_n| < \frac{4\pi Rr}{R+r}.$$

71. **(THA 2)** Let $P_1(x,y)$ and $P_2(x,y)$ be two relatively prime polynomials with complex coefficients. Let $Q(x,y)$ and $R(x,y)$ be polynomials with complex coefficients and each of degree not exceeding d. Prove that there exist two integers A_1, A_2 not simultaneously zero with $|A_i| \le d+1$ $(i = 1,2)$ and such that the polynomial $A_1P_1(x,y) + A_2P_2(x,y)$ is coprime to $Q(x,y)$ and $R(x,y)$.

72. **(TUR 1)** In a school six different courses are taught: mathematics, physics, biology, music, history, geography. The students were required to rank these courses according to their preferences, where equal preferences were allowed. It turned out that:
(i) mathematics was ranked among the most preferred courses by all students;
(ii) no student ranked music among the least preferred ones;
(iii) all students preferred history to geography and physics to biology; and
(iv) no two rankings were the same.
Find the greatest possible value for the number of students in this school.

73. **(TUR 2)** Let $\{A_n \mid n = 1,2,\dots\}$ be a set of points in the plane such that for each n, the disk with center A_n and radius 2^n contains no other point A_j. For any given positive real numbers $a < b$ and R, show that there is a subset G of the plane satisfying:
(i) the area of G is greater than or equal to R;

(ii) for each point P in G, $a < \sum_{n=1}^{\infty} \frac{1}{|A_n P|} < b$.

74. **(TUR 3)** Let $S = \left\{ \frac{\pi^n}{1992^m} \mid n, m \in \mathbb{Z} \right\}$. Show that every real number $x \geq 0$ is an accumulation point of S.

75. **(TWN 1)** A sequence $\{a_n\}$ of positive integers is defined by

$$a_n = \left[n + \sqrt{n} + \frac{1}{2} \right], \quad n \in \mathbb{N}.$$

Determine the positive integers that occur in the sequence.

76. **(TWN 2)** Given any triangle ABC and any positive integer n, we say that n is a *decomposable* number for triangle ABC if there exists a decomposition of the triangle ABC into n subtriangles with each subtriangle similar to $\triangle ABC$. Determine the positive integers that are decomposable numbers for every triangle.

77. **(TWN 3)** Show that if 994 integers are chosen from $1, 2, \ldots, 1992$ and one of the chosen integers is less than 64, then there exist two among the chosen integers such that one of them is a factor of the other.

78. **(USA 1)** Let F_n be the nth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. Let A_0, A_1, $A_2, \ldots$ be a sequence of points on a circle of radius 1 such that the minor arc from A_{k-1} to A_k runs clockwise and such that

$$\mu(A_{k-1} A_k) = \frac{4F_{2k+1}}{F_{2k+1}^2 + 1}$$

for $k \geq 1$, where $\mu(XY)$ denotes the radian measure of the arc XY in the clockwise direction. What is the limit of the radian measure of arc $A_0 A_n$ as n approaches infinity?

79. **(USA 2)** (SL92-18).

80. **(USA 3)** Given a graph with n vertices and a positive integer m that is less than n, prove that the graph contains a set of $m + 1$ vertices in which the difference between the largest degree of any vertex in the set and the smallest degree of any vertex in the set is at most $m - 1$.

81. **(USA 4)** Suppose that points X, Y, Z are located on sides BC, CA, and AB, respectively, of $\triangle ABC$ in such a way that $\triangle XYZ$ is similar to $\triangle ABC$. Prove that the orthocenter of $\triangle XYZ$ is the circumcenter of $\triangle ABC$.

82. **(VIE 1)** Let $f(x) = x^m + a_1 x^{m-1} + \cdots + a_{m-1} x + a_m$ and $g(x) = x^n + b_1 x^{n-1} + \cdots + b_{n-1} + b_n$ be two polynomials with real coefficients such that for each real number x, $f(x)$ is the square of an integer if and only if so is $g(x)$. Prove that if $n + m > 0$, then there exists a polynomial $h(x)$ with real coefficients such that $f(x) \cdot g(x) = (h(x))^2$.

3.33.3 Shortlisted Problems

1. **(AUS 2)** Prove that for any positive integer m there exist an infinite number of pairs of integers (x, y) such that (i) x and y are relatively prime; (ii) y divides $x^2 + m$; (iii) x divides $y^2 + m$.

2. **(CHN 1)** Let $\mathbb{R}^+$ be the set of all nonnegative real numbers. Given two positive real numbers a and b, suppose that a mapping $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the functional equation
$$f(f(x)) + af(x) = b(a + b)x.$$
Prove that there exists a unique solution of this equation.

3. **(CHN 2)** The diagonals of a quadrilateral $ABCD$ are perpendicular: $AC \perp BD$. Four squares, $ABEF, BCGH, CDIJ, DAKL$, are erected externally on its sides. The intersection points of the pairs of straight lines CL, DF; DF, AH; AH, BJ; BJ, CL are denoted by P_1, Q_1, R_1, S_1, respectively, and the intersection points of the pairs of straight lines AI, BK; BK, CE; CE, DG; DG, AI are denoted by P_2, Q_2, R_2, S_2, respectively. Prove that $P_1Q_1R_1S_1 \cong P_2Q_2R_2S_2$.

4. **(CHN 3)**$^{\text{IMO3}}$ Given nine points in space, no four of which are coplanar, find the minimal natural number n such that for any coloring with red or blue of n edges drawn between these nine points there always exists a triangle having all edges of the same color.

5. **(COL 3)** Let $ABCD$ be a convex quadrilateral such that $AC = BD$. Equilateral triangles are constructed on the sides of the quadrilateral. Let O_1, O_2, O_3, O_4 be the centers of the triangles constructed on AB, BC, CD, DA respectively. Show that O_1O_3 is perpendicular to O_2O_4.

6. **(IND 2)**$^{\text{IMO2}}$ Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 + f(y)) = y + f(x)^2 \quad \text{for all } x, y \text{ in } \mathbb{R}.$$

7. **(IND 4)** Circles G, G_1, G_2 are three circles related to each other as follows: Circles G_1 and G_2 are externally tangent to one another at a point W and both these circles are internally tangent to the circle G. Points A, B, C are located on the circle G as follows: Line BC is a direct common tangent to the pair of circles G_1 and G_2, and line WA is the transverse common tangent at W to G_1 and G_2, with W and A lying on the same side of the line BC. Prove that W is the incenter of the triangle ABC.

8. **(IND 5)** Show that in the plane there exists a convex polygon of 1992 sides satisfying the following conditions:
   (i) its side lengths are $1, 2, 3, \ldots, 1992$ in some order;
   (ii) the polygon is circumscribable about a circle.

Alternative formulation. Does there exist a 1992-gon with side lengths $1, 2, 3, \dots, 1992$ circumscribed about a circle? Answer the same question for a 1990-gon.

9. **(IRN 1)** Let $f(x)$ be a polynomial with rational coefficients and α be a real number such that $\alpha^3 - \alpha = f(\alpha)^3 - f(\alpha) = 33^{1992}$. Prove that for each $n \ge 1$,
$$(f^{(n)}(\alpha))^3 - f^{(n)}(\alpha) = 33^{1992},$$
where $f^{(n)}(x) = f(f(\dots f(x)))$, and n is a positive integer.

10. **(ITA 1)**$^{\text{IMO5}}$ Let V be a finite subset of Euclidean space consisting of points (x, y, z) with integer coordinates. Let S_1, S_2, S_3 be the projections of V onto the yz, xz, xy planes, respectively. Prove that
$$|V|^2 \le |S_1||S_2||S_3|$$
($|X|$ denotes the number of elements of X).

11. **(JAP 2)** In a triangle ABC, let D and E be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with the sides AC, AB, respectively. Determine the angles $\angle A, \angle B, \angle C$ if
$$\measuredangle BDE = 24^\circ, \qquad \measuredangle CED = 18^\circ.$$

12. **(NET 1)** Let f, g, and a be polynomials with real coefficients, f and g in one variable and a in two variables. Suppose
$$f(x) - f(y) = a(x, y)(g(x) - g(y)) \qquad \text{for all } x, y \in \mathbb{R}.$$
Prove that there exists a polynomial h with $f(x) = h(g(x))$ for all $x \in \mathbb{R}$.

13. **(NZL 1)**$^{\text{IMO1}}$ Find all integer triples (p, q, r) such that $1 < p < q < r$ and $(p-1)(q-1)(r-1)$ is a divisor of $(pqr - 1)$.

14. **(POL 2)** For any positive integer x define
$$\begin{aligned} g(x) &= \text{ greatest odd divisor of } x, \\ f(x) &= \begin{cases} x/2 + x/g(x), & \text{if } x \text{ is even;} \\ 2^{(x+1)/2}, & \text{if } x \text{ is odd.} \end{cases} \end{aligned}$$
Construct the sequence $x_1 = 1, x_{n+1} = f(x_n)$. Show that the number 1992 appears in this sequence, determine the least n such that $x_n = 1992$, and determine whether n is unique.

15. **(PRK 2)** Does there exist a set M with the following properties?
    (i) The set M consists of 1992 natural numbers.
    (ii) Every element in M and the sum of any number of elements have the form m^k $(m, k \in \mathbb{N},\ k \ge 2)$.

16. **(KOR 2)** Prove that $N = \frac{5^{125}-1}{5^{25}-1}$ is a composite number.

17. **(SWE 1)** Let $\alpha(n)$ be the number of digits equal to one in the binary representation of a positive integer n. Prove that:
(a) the inequality $\alpha(n^2) \le \frac{1}{2}\alpha(n)(\alpha(n)+1)$ holds;
(b) the above inequality is an equality for infinitely many positive integers;
(c) there exists a sequence $(n_i)_1^\infty$ such that $\alpha(n_i^2)/\alpha(n_i) \to 0$ as $i \to \infty$.

Alternative parts: Prove that there exists a sequence $(n_i)_1^\infty$ such that $\alpha(n_i^2)/\alpha(n_i)$ tends to
(d) ∞;
(e) an arbitrary real number $\gamma \in (0,1)$;
(f) an arbitrary real number $\gamma \ge 0$.

18. **(USA 2)** Let $[x]$ denote the greatest integer less than or equal to x. Pick any x_1 in $[0,1)$ and define the sequence $x_1, x_2, x_3, \dots$ by $x_{n+1} = 0$ if $x_n = 0$ and $x_{n+1} = 1/x_n - [1/x_n]$ otherwise. Prove that

$$x_1 + x_2 + \cdots + x_n < \frac{F_1}{F_2} + \frac{F_2}{F_3} + \cdots + \frac{F_n}{F_{n+1}},$$

where $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \ge 1$.

19. **(IRE 1)** Let $f(x) = x^8 + 4x^6 + 2x^4 + 28x^2 + 1$. Let $p > 3$ be a prime and suppose there exists an integer z such that p divides $f(z)$. Prove that there exist integers $z_1, z_2, \dots, z_8$ such that if

$$g(x) = (x - z_1)(x - z_2)\cdots(x - z_8),$$

then all coefficients of $f(x) - g(x)$ are divisible by p.

20. **(FRA 1)**$^{\mathrm{IMO4}}$ In the plane, let there be given a circle C, a line l tangent to C, and a point M on l. Find the locus of points P that have the following property: There exist two points Q and R on l such that M is the midpoint of QR and C is the incircle of PQR.

21. **(GBR 2)**$^{\mathrm{IMO6}}$ For each positive integer n, denote by $s(n)$ the greatest integer such that for all positive integers $k \le s(n)$, n^2 can be expressed as a sum of squares of k positive integers.
(a) Prove that $s(n) \le n^2 - 14$ for all $n \ge 4$.
(b) Find a number n such that $s(n) = n^2 - 14$.
(c) Prove that there exist infinitely many positive integers n such that $s(n) = n^2 - 14$.

3.34 The Thirty-Fourth IMO
Istanbul, Turkey, July 13–24, 1993

3.34.1 Contest Problems

First Day (July 18)

1. Let $n > 1$ be an integer and let $f(x) = x^n + 5x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that $f(x) = g(x)h(x)$.

2. A, B, C, D are four points in the plane, with C, D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\measuredangle ADB = 90° + \measuredangle ACB$. Find the ratio
$$\frac{AB \cdot CD}{AC \cdot BD},$$
and prove that circles ACD, BCD are orthogonal. (Intersecting circles are said to be orthogonal if at either common point their tangents are perpendicular.)

3. On an infinite chessboard, a solitaire game is played as follows: At the start, we have n^2 pieces occupying n^2 squares that form a square of side n. The only allowed move is a jump horizontally or vertically over an occupied square to an unoccupied one, and the piece that has been jumped over is removed. For what positive integers n can the game end with only one piece remaining on the board?

Second Day (July 19)

4. For three points A, B, C in the plane we define $m(ABC)$ to be the smallest length of the three altitudes of the triangle ABC, where in the case of A, B, C collinear, $m(ABC) = 0$. Let A, B, C be given points in the plane. Prove that for any point X in the plane,
$$m(ABC) \le m(ABX) + m(AXC) + m(XBC).$$

5. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine whether there exists a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ with the following properties:
$$f(1) = 2; \tag{1}$$
$$f(f(n)) = f(n) + n \qquad (n \in \mathbb{N}). \tag{2}$$

6. Let n be an integer greater than 1. In a circular arrangement of n lamps $L_0, \ldots, L_{n-1}$, each one of that can be either ON or OFF, we start with the situation where all lamps are ON, and then carry out a sequence of steps, $Step_0$,$Step_1$,... . If L_{j-1} (j is taken $\bmod n$) is ON, then $Step_j$ changes the status of L_j (it goes from ON to OFF or from OFF to ON) but does not change the status of any of the other lamps. If L_{j-1} is OFF, then $Step_j$ does not change anything at all. Show that:

(a) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again.
(b) If n has the form 2^k, then all lamps are ON after $n^2 - 1$ steps.
(c) If n has the form $2^k + 1$, then all lamps are ON after $n^2 - n + 1$ steps.

3.34.2 Shortlisted Problems

1. **(BRA 1)** Show that there exists a finite set $A \subset \mathbb{R}^2$ such that for every $X \in A$ there are points $Y_1, Y_2, \dots, Y_{1993}$ in A such that the distance between X and Y_i is equal to 1, for every i.

2. **(CAN 2)** Let triangle ABC be such that its circumradius R is equal to 1. Let r be the inradius of ABC and let p be the inradius of the orthic triangle $A'B'C'$ of triangle ABC.
Prove that $p \le 1 - \frac{1}{3}(1+r)^2$.

   Remark. The *orthic* triangle is the triangle whose vertices are the feet of the altitudes of ABC.

3. **(SPA 1)** Consider the triangle ABC, its circumcircle k with center O and radius R, and its incircle with center I and radius r. Another circle k_c is tangent to the sides CA, CB at D, E, respectively, and it is internally tangent to k.
Show that the incenter I is the midpoint of DE.

4. **(SPA 2)** In the triangle ABC, let D, E be points on the side BC such that $\angle BAD = \angle CAE$. If M, N are, respectively, the points of tangency with BC of the incircles of the triangles ABD and ACE, show that
$$\frac{1}{MB} + \frac{1}{MD} = \frac{1}{NC} + \frac{1}{NE}.$$

5. **(FIN 3)**$^{\text{IMO3}}$ On an infinite chessboard, a solitaire game is played as follows: At the start, we have n^2 pieces occupying n^2 squares that form a square of side n. The only allowed move is a jump horizontally or vertically over an occupied square to an unoccupied one, and the piece that has been jumped over is removed. For what positive integers n can the game end with only one piece remaining on the board?

6. **(GER 1)**$^{\text{IMO5}}$ Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Determine whether there exists a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ with the following properties:
$$f(1) = 2; \tag{1}$$
$$f(f(n)) = f(n) + n \qquad (n \in \mathbb{N}). \tag{2}$$

7. **(GEO 3)** Let a, b, c be given integers $a > 0$, $ac - b^2 = P = P_1 \cdots P_m$ where $P_1, \dots, P_m$ are (distinct) prime numbers. Let $M(n)$ denote the number of pairs of integers (x, y) for which
$$ax^2 + 2bxy + cy^2 = n.$$
Prove that $M(n)$ is finite and $M(n) = M(P^k \cdot n)$ for every integer $k \ge 0$.

8. **(IND 1)** Define a sequence $\langle f(n)\rangle_{n=1}^{\infty}$ of positive integers by $f(1)=1$ and
$$f(n)=\begin{cases} f(n-1)-n, & \text{if } f(n-1)>n;\\ f(n-1)+n, & \text{if } f(n-1)\le n,\end{cases}$$
for $n\ge 2$. Let $S=\{n\in\mathbb{N}\mid f(n)=1993\}$.
(a) Prove that S is an infinite set.
(b) Find the least positive integer in S.
(c) If all the elements of S are written in ascending order as $n_1<n_2<n_3<\cdots$, show that
$$\lim_{i\to\infty}\frac{n_{i+1}}{n_i}=3.$$

9. **(IND 4)**
(a) Show that the set $\mathbb{Q}^+$ of all positive rational numbers can be partitioned into three disjoint subsets A,B,C satisfying the following conditions:
$$BA=B,\qquad B^2=C,\qquad BC=A,$$
where HK stands for the set $\{hk\mid h\in H,k\in K\}$ for any two subsets H,K of $\mathbb{Q}^+$ and H^2 stands for HH.
(b) Show that all positive rational cubes are in A for such a partition of $\mathbb{Q}^+$.
(c) Find such a partition $\mathbb{Q}^+=A\cup B\cup C$ with the property that for no positive integer $n\le 34$ are both n and $n+1$ in A; that is,
$$\min\{n\in\mathbb{N}\mid n\in A,n+1\in A\}>34.$$

10. **(IND 5)** A natural number n is said to have the property P if whenever n divides a^n-1 for some integer a, n^2 also necessarily divides a^n-1.
(a) Show that every prime number has property P.
(b) Show that there are infinitely many composite numbers n that possess property P.

11. **(IRE 1)**$^{\text{IMO1}}$ Let $n>1$ be an integer and let $f(x)=x^n+5x^{n-1}+3$. Prove that there do not exist polynomials $g(x),h(x)$, each having integer coefficients and degree at least one, such that $f(x)=g(x)h(x)$.

12. **(IRE 2)** Let n,k be positive integers with $k\le n$ and let S be a set containing n distinct real numbers. Let T be the set of all real numbers of the form $x_1+x_2+\cdots+x_k$, where $x_1,x_2,\dots,x_k$ are distinct elements of S. Prove that T contains at least $k(n-k)+1$ distinct elements.

13. **(IRE 3)** Let S be the set of all pairs (m,n) of relatively prime positive integers m,n with n even and $m<n$. For $s=(m,n)\in S$ write $n=2^kn_0$, where k,n_0 are positive integers with n_0 odd and define $f(s)=(n_0,m+n-n_0)$.
Prove that f is a function from S to S and that for each $s=(m,n)\in S$, there exists a positive integer $t\le\frac{m+n+1}{4}$ such that $f^t(s)=s$, where

$$f^t(s) = \underbrace{(f \circ f \circ \cdots \circ f)}_{t \text{ times}}(s).$$

If $m+n$ is a prime number that does not divide 2^k-1 for $k = 1, 2, \dots, m+n-2$, prove that the smallest value of t that satisfies the above conditions is $\left[\frac{m+n+1}{4}\right]$, where $[x]$ denotes the greatest integer less than or equal to x.

14. **(ISR 1)** The vertices D, E, F of an equilateral triangle lie on the sides BC, CA, AB respectively of a triangle ABC. If a, b, c are the respective lengths of these sides, and S the area of ABC, prove that

$$DE \geq \frac{2\sqrt{2}S}{\sqrt{a^2 + b^2 + c^2 + 4\sqrt{3}S}}.$$

15. **(MCD 1)**$^{\text{IMO4}}$ For three points A, B, C in the plane we define $m(ABC)$ to be the smallest length of the three altitudes of the triangle ABC, where in the case of A, B, C collinear, $m(ABC) = 0$. Let A, B, C be given points in the plane. Prove that for any point X in the plane,

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

16. **(MCD 3)** Let $n \in \mathbb{N}$, $n \geq 2$, and $A_0 = (a_{01}, a_{02}, \dots, a_{0n})$ be any n-tuple of natural numbers such that $0 \leq a_{0i} \leq i-1$, for $i = 1, \dots, n$. The n-tuples $A_1 = (a_{11}, a_{12}, \dots, a_{1n})$, $A_2 = (a_{21}, a_{22}, \dots, a_{2n}), \dots$ are defined by

$$a_{i+1,j} = \operatorname{Card}\{a_{i,l} \mid 1 \leq l \leq j-1, a_{i,l} \geq a_{i,j}\}, \quad \text{for } i \in \mathbb{N} \text{ and } j = 1, \dots, n.$$

Prove that there exists $k \in \mathbb{N}$, such that $A_{k+2} = A_k$.

17. **(NET 2)**$^{\text{IMO6}}$ Let n be an integer greater than 1. In a circular arrangement of n lamps $L_0, \dots, L_{n-1}$, each one of that can be either ON or OFF, we start with the situation where all lamps are ON, and then carry out a sequence of steps, $Step_0$,$Step_1$,... . If L_{j-1} (j is taken $\bmod n$) is ON, then $Step_j$ changes the status of L_j (it goes from ON to OFF or from OFF to ON) but does not change the status of any of the other lamps. If L_{j-1} is OFF, then $Step_j$ does not change anything at all. Show that:
(a) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again.
(b) If n has the form 2^k, then all lamps are ON after $n^2 - 1$ steps.
(c) If n has the form $2^k + 1$, then all lamps are ON after $n^2 - n + 1$ steps.

18. **(POL 1)** Let S_n be the number of sequences $(a_1, a_2, \dots, a_n)$, where $a_i \in \{0, 1\}$, in which no six consecutive blocks are equal. Prove that $S_n \to \infty$ as $n \to \infty$.

19. **(ROM 2)** Let a, b, n be positive integers, $b > 1$ and $b^n - 1 \mid a$. Show that the representation of the number a in the base b contains at least n digits different from zero.

20. **(ROM 3)** Let $c_1, \dots, c_n \in \mathbb{R}$ $(n \ge 2)$ such that $0 \le \sum_{i=1}^n c_i \le n$. Show that we can find integers $k_1, \dots, k_n$ such that $\sum_{i=1}^n k_i = 0$ and

$$1 - n \le c_i + nk_i \le n \qquad \text{for every } i = 1, \dots, n.$$

21. **(GBR 1)** A circle S is said to cut a circle Σ *diametrally* if their common chord is a diameter of Σ.
Let S_A, S_B, S_C be three circles with distinct centers A, B, C respectively. Prove that A, B, C are collinear if and only if there is no unique circle S that cuts each of S_A, S_B, S_C diametrally. Prove further that if there exists more than one circle S that cuts each of S_A, S_B, S_C diametrally, then all such circles pass through two fixed points. Locate these points in relation to the circles S_A, S_B, S_C.

22. **(GBR 2)**$^{\text{IMO2}}$ A, B, C, D are four points in the plane, with C, D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\measuredangle ADB = 90^\circ + \measuredangle ACB$. Find the ratio

$$\frac{AB \cdot CD}{AC \cdot BD},$$

and prove that circles ACD, BCD are orthogonal. (Intersecting circles are said to be orthogonal if at either common point their tangents are perpendicular.)

23. **(GBR 3)** A finite set of (distinct) positive integers is called a "DS-set" if each of the integers divides the sum of them all. Prove that every finite set of positive integers is a subset of some DS-set.

24. **(USA 3)** Prove that

$$\frac{a}{b + 2c + 3d} + \frac{b}{c + 2d + 3a} + \frac{c}{d + 2a + 3b} + \frac{d}{a + 2b + 3c} \ge \frac{2}{3}$$

for all positive real numbers a, b, c, d.

25. **(VIE 1)** Solve the following system of equations, in which a is a given number satisfying $|a| > 1$:

$$\begin{aligned} x_1^2 &= ax_2 + 1, \\ x_2^2 &= ax_3 + 1, \\ \cdots & \quad \cdots \\ x_{999}^2 &= ax_{1000} + 1, \\ x_{1000}^2 &= ax_1 + 1. \end{aligned}$$

26. **(VIE 2)** Let a, b, c, d be four nonnegative numbers satisfying $a+b+c+d = 1$. Prove the inequality

$$abc + bcd + cda + dab \le \frac{1}{27} + \frac{176}{27} abcd.$$

3.35 The Thirty-Fifth IMO
Hong Kong, July 9–22, 1994

3.35.1 Contest Problems

First Day (July 13)

1. Let m and n be positive integers. The set $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of $1, 2, \ldots, n$. Whenever $a_i + a_j \le n$, $1 \le i \le j \le m$, $a_i + a_j$ also belongs to A. Prove that
$$\frac{a_1 + a_2 + \cdots + a_m}{m} \ge \frac{n+1}{2}.$$
2. N is an arbitrary point on the bisector of $\angle BAC$. P and O are points on the lines AB and AN, respectively, such that $\measuredangle ANP = 90^\circ = \measuredangle APO$. Q is an arbitrary point on NP, and an arbitrary line through Q meets the lines AB and AC at E and F respectively. Prove that $\measuredangle OQE = 90^\circ$ if and only if $QE = QF$.
3. For any positive integer k, A_k is the subset of $\{k+1, k+2, \ldots, 2k\}$ consisting of all elements whose digits in base 2 contain exactly three 1's. Let $f(k)$ denote the number of elements in A_k.
   (a) Prove that for any positive integer m, $f(k) = m$ has at least one solution.
   (b) Determine all positive integers m for which $f(k) = m$ has a unique solution.

Second Day (July 14)

4. Determine all pairs (m, n) of positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.
5. Let S be the set of real numbers greater than -1. Find all functions $f : S \to S$ such that
$$f(x + f(y) + xf(y)) = y + f(x) + yf(x) \qquad \text{for all } x \text{ and } y \text{ in } S,$$
and $f(x)/x$ is strictly increasing for $-1 < x < 0$ and for $0 < x$.
6. Find a set A of positive integers such that for any infinite set P of prime numbers, there exist positive integers $m \in A$ and $n \notin A$, both the product of the same number (at least two) of distinct elements of P.

3.35.2 Shortlisted Problems

1. **A1 (USA)** Let $a_0 = 1994$ and $a_{n+1} = \frac{a_n^2}{a_n+1}$ for each nonnegative integer n. Prove that $1994 - n$ is the greatest integer less than or equal to a_n, $0 \le n \le 998$.

2. **A2 (FRA)**$^{\text{IMO1}}$ Let m and n be positive integers. The set $A = \{a_1, a_2, \dots, a_m\}$ is a subset of $\{1, 2, \dots, n\}$. Whenever $a_i + a_j \le n$, $1 \le i \le j \le m$, $a_i + a_j$ also belongs to A. Prove that
$$\frac{a_1 + a_2 + \cdots + a_m}{m} \ge \frac{n+1}{2}.$$

3. **A3 (GBR)**$^{\text{IMO5}}$ Let S be the set of real numbers greater than -1. Find all functions $f : S \to S$ such that
$$f(x + f(y) + xf(y)) = y + f(x) + yf(x) \qquad \text{for all } x \text{ and } y \text{ in } S,$$
and $f(x)/x$ is strictly increasing for $-1 < x < 0$ and for $0 < x$.

4. **A4 (MON)** Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}^+$ the subset of all positive ones. Let α and β be given elements in $\mathbb{R}$, not necessarily distinct. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that
$$f(x)f(y) = y^\alpha f\left(\frac{x}{2}\right) + x^\beta f\left(\frac{y}{2}\right) \qquad \text{for all } x \text{ and } y \text{ in } \mathbb{R}^+.$$

5. **A5 (POL)** Let $f(x) = \frac{x^2+1}{2x}$ for $x \ne 0$. Define $f^{(0)}(x) = x$ and $f^{(n)}(x) = f(f^{(n-1)}(x))$ for all positive integers n and $x \ne 0$. Prove that for all nonnegative integers n and $x \ne -1, 0$, or 1,
$$\frac{f^{(n)}(x)}{f^{(n+1)}(x)} = 1 + \frac{1}{f\left(\left(\frac{x+1}{x-1}\right)^{2^n}\right)}.$$

6. **C1 (UKR)** On a 5×5 board, two players alternately mark numbers on empty cells. The first player always marks 1's, the second 0's. One number is marked per turn, until the board is filled. For each of the nine 3×3 squares the sum of the nine numbers on its cells is computed. Denote by A the maximum of these sums. How large can the first player make A, regardless of the responses of the second player?

7. **C2 (COL)** In a certain city, age is reckoned in terms of real numbers rather than integers. Every two citizens x and x' either know each other or do not know each other. Moreover, if they do not, then there exists a chain of citizens $x = x_0, x_1, \dots, x_n = x'$ for some integer $n \ge 2$ such that x_{i-1} and x_i know each other. In a census, all male citizens declare their ages, and there is at least one male citizen. Each female citizen provides only the information that her age is the average of the ages of all the citizens she knows. Prove that this is enough to determine uniquely the ages of all the female citizens.

8. **C3 (MCD)** Peter has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled.
   (a) Prove that Peter can always transfer all his money into two accounts.

(b) Can Peter always transfer all his money into one account?

9. **C4 (EST)** There are $n+1$ fixed positions in a row, labeled 0 to n in increasing order from right to left. Cards numbered 0 to n are shuffled and dealt, one in each position. The object of the game is to have card i in the ith position for $0 \le i \le n$. If this has not been achieved, the following move is executed. Determine the smallest k such that the kth position is occupied by a card $l > k$. Remove this card, slide all cards from the $(k+1)$st to the lth position one place to the right, and replace the card l in the lth position.
   (a) Prove that the game lasts at most $2^n - 1$ moves.
   (b) Prove that there exists a unique initial configuration for which the game lasts exactly $2^n - 1$ moves.

10. **C5 (SWE)** At a round table are 1994 girls, playing a game with a deck of n cards. Initially, one girl holds all the cards. In each turn, if at least one girl holds at least two cards, one of these girls must pass a card to each of her two neighbors. The game ends when and only when each girl is holding at most one card.
    (a) Prove that if $n \ge 1994$, then the game cannot end.
    (b) Prove that if $n < 1994$, then the game must end.

11. **C6 (FIN)** On an infinite square grid, two players alternately mark symbols on empty cells. The first player always marks X's, the second O's. One symbol is marked per turn. The first player wins if there are 11 consecutive X's in a row, column, or diagonal. Prove that the second player can prevent the first from winning.

12. **C7 (BRA)** Prove that for any integer $n \ge 2$, there exists a set of 2^{n-1} points in the plane such that no 3 lie on a line and no $2n$ are the vertices of a convex $2n$-gon.

13. **G1 (FRA)** A semicircle Γ is drawn on one side of a straight line l. C and D are points on Γ. The tangents to Γ at C and D meet l at B and A respectively, with the center of the semicircle between them. Let E be the point of intersection of AC and BD, and F the point on l such that EF is perpendicular to l. Prove that EF bisects $\angle CFD$.

14. **G2 (UKR)** $ABCD$ is a quadrilateral with BC parallel to AD. M is the midpoint of CD, P that of MA and Q that of MB. The lines DP and CQ meet at N. Prove that N is not outside triangle ABM.[8]

15. **G3 (RUS)** A circle ω is tangent to two parallel lines l_1 and l_2. A second circle ω_1 is tangent to l_1 at A and to ω externally at C. A third circle ω_2 is tangent to l_2 at B, to ω externally at D, and to ω_1 externally at E. AD intersects BC at Q. Prove that Q is the circumcenter of triangle CDE.

[8] This problem is false. However, it is true if "not outside ABM" is replaced by "not outside $ABCD$".

16. **G4 (AUS-ARM)**$^{\text{IMO2}}$ N is an arbitrary point on the bisector of $\angle BAC$. P and O are points on the lines AB and AN, respectively, such that $\measuredangle ANP = 90^\circ = \measuredangle APO$. Q is an arbitrary point on NP, and an arbitrary line through Q meets the lines AB and AC at E and F respectively. Prove that $\measuredangle OQE = 90^\circ$ if and only if $QE = QF$.

17. **G5 (CYP)** A line l does not meet a circle ω with center O. E is the point on l such that OE is perpendicular to l. M is any point on l other than E. The tangents from M to ω touch it at A and B. C is the point on MA such that EC is perpendicular to MA. D is the point on MB such that ED is perpendicular to MB. The line CD cuts OE at F. Prove that the location of F is independent of that of M.

18. **N1 (BUL)** M is a subset of $\{1, 2, 3, \dots, 15\}$ such that the product of any three distinct elements of M is not a square. Determine the maximum number of elements in M.

19. **N2 (AUS)**$^{\text{IMO4}}$ Determine all pairs (m, n) of positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.

20. **N3 (FIN)**$^{\text{IMO6}}$ Find a set A of positive integers such that for any infinite set P of prime numbers, there exist positive integers $m \in A$ and $n \notin A$, both the product of the same number of distinct elements of P.

21. **N4 (FRA)** For any positive integer x_0, three sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ are defined as follows:
    (i) $y_0 = 4$ and $z_0 = 1$;
    (ii) if x_n is even for $n \ge 0$, $x_{n+1} = \frac{x_n}{2}$, $y_{n+1} = 2y_n$, and $z_{n+1} = z_n$;
    (iii) if x_n is odd for $n \ge 0$, $x_{n+1} = x_n - \frac{y_n}{2} - z_n$, $y_{n+1} = y_n$, and $z_{n+1} = y_n + z_n$.

    The integer x_0 is said to be *good* if $x_n = 0$ for some $n \ge 1$. Find the number of good integers less than or equal to 1994.

22. **N5 (ROM)**$^{\text{IMO3}}$ For any positive integer k, A_k is the subset of $\{k+1, k+2, \dots, 2k\}$ consisting of all elements whose digits in base 2 contain exactly three 1's. Let $f(k)$ denote the number of elements in A_k.
    (a) Prove that for any positive integer m, $f(k) = m$ has at least one solution.
    (b) Determine all positive integers m for which $f(k) = m$ has a unique solution.

23. **N6 (LAT)** Let x_1 and x_2 be relatively prime positive integers. For $n \ge 2$, define $x_{n+1} = x_n x_{n-1} + 1$.
    (a) Prove that for every $i > 1$, there exists $j > i$ such that x_i^i divides x_j^j.
    (b) Is it true that x_1 must divide x_j^j for some $j > 1$?

24. **N7 (GBR)** A *wobbly* number is a positive integer whose digits in base 10 are alternately nonzero and zero, the units digit being nonzero. Determine all positive integers that do not divide any wobbly number.

3.36 The Thirty-Sixth IMO
Toronto, Canada, July 13–25, 1995

3.36.1 Contest Problems

First Day (July 19)

1. Let A, B, C, and D be distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. O is an arbitrary point on the line XY but not on AD. CO intersects the circle with diameter AC again at M, and BO intersects the other circle again at N. Prove that the lines AM, DN, and XY are concurrent.

2. Let a, b, and c be positive real numbers such that $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ such that there are n points $A_1, A_2, \dots, A_n$ in the plane that satisfy the following two conditions simultaneously:
(a) No three lie on the same line.
(b) There exist real numbers $p_1, p_2, \dots, p_n$ such that the area of $\triangle A_iA_jA_k$ is equal to $p_i + p_j + p_k$, for $1 \leq i < j < k \leq n$.

Second Day (July 20)

4. The positive real numbers $x_0, x_1, \dots, x_{1995}$ satisfy $x_0 = x_{1995}$ and
$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$$
for $i = 1, 2, \dots, 1995$. Find the maximum value that x_0 can have.

5. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$, and $\angle BCD = \angle EFA = \pi/3$ (that is, $60°$). Let G and H be two points interior to the hexagon, such that angles AGB and DHE are both $2\pi/3$ (that is, $120°$). Prove that $AG + GB + GH + DH + HE \geq CF$.

6. Let p be an odd prime. Find the number of p-element subsets A of $\{1, 2, \dots, 2p\}$ such that the sum of all elements of A is divisible by p.

3.36.2 Shortlisted Problems

1. **A1 (RUS)**$^{\text{IMO2}}$ Let a, b, and c be positive real numbers such that $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

2. **A2 (SWE)** Let a and b be nonnegative integers such that $ab \geq c^2$, where c is an integer. Prove that there is a number n and integers $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ such that

$$\sum_{i=1}^{n} x_i^2 = a, \qquad \sum_{i=1}^{n} y_i^2 = b, \qquad \text{and} \qquad \sum_{i=1}^{n} x_i y_i = c.$$

3. **A3 (UKR)** Let n be an integer, $n \geq 3$. Let $a_1, a_2, \dots, a_n$ be real numbers such that $2 \leq a_i \leq 3$ for $i = 1, 2, \dots, n$. If $s = a_1 + a_2 + \cdots + a_n$, prove that

$$\frac{a_1^2 + a_2^2 - a_3^2}{a_1 + a_2 - a_3} + \frac{a_2^2 + a_3^2 - a_4^2}{a_2 + a_3 - a_4} + \cdots + \frac{a_n^2 + a_1^2 - a_2^2}{a_n + a_1 - a_2} \leq 2s - 2n.$$

4. **A4 (USA)** Let a, b, and c be given positive real numbers. Determine all positive real numbers x, y, and z such that

$$x + y + z = a + b + c$$

and

$$4xyz - (a^2 x + b^2 y + c^2 z) = abc.$$

5. **A5 (UKR)** Let $\mathbb{R}$ be the set of real numbers. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ that simultaneously satisfies the following three conditions?
(a) There is a positive number M such that $-M \leq f(x) \leq M$ for all x.
(b) $f(1) = 1$.
(c) If $x \neq 0$, then

$$f\left(x + \frac{1}{x^2}\right) = f(x) + \left[f\left(\frac{1}{x}\right)\right]^2.$$

6. **A6 (JAP)** Let n be an integer, $n \geq 3$. Let $x_1, x_2, \dots, x_n$ be real numbers such that $x_i < x_{i+1}$ for $1 \leq i \leq n-1$. Prove that

$$\frac{n(n-1)}{2} \sum_{i<j} x_i x_j > \left(\sum_{i=1}^{n-1} (n-i) x_i\right) \left(\sum_{j=2}^{n} (j-1) x_j\right).$$

7. **G1 (BUL)**[IMO1] Let A, B, C, and D be distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. O is an arbitrary point on the line XY but not on AD. CO intersects the circle with diameter AC again at M, and BO intersects the other circle again at N. Prove that the lines AM, DN, and XY are concurrent.

8. **G2 (GER)** Let A, B, and C be noncollinear points. Prove that there is a unique point X in the plane of ABC such that $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2$.

9. **G3 (TUR)** The incircle of ABC touches BC, CA, and AB at D, E, and F respectively. X is a point inside ABC such that the incircle of XBC touches BC at D also, and touches CX and XB at Y and Z, respectively. Prove that $EFZY$ is a cyclic quadrilateral.

10. **G4 (UKR)** An acute triangle ABC is given. Points A_1 and A_2 are taken on the side BC (with A_2 between A_1 and C), B_1 and B_2 on the side AC (with B_2 between B_1 and A), and C_1 and C_2 on the side AB (with C_2 between C_1 and B) such that

$$\angle AA_1A_2 = \angle AA_2A_1 = \angle BB_1B_2 = \angle BB_2B_1 = \angle CC_1C_2 = \angle CC_2C_1.$$

The lines AA_1, BB_1, and CC_1 form a triangle, and the lines AA_2, BB_2, and CC_2 form a second triangle. Prove that all six vertices of these two triangles lie on a single circle.

11. **G5 (NZL)**$^{\mathrm{IMO5}}$ Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$, and $\measuredangle BCD = \measuredangle EFA = \pi/3$ (that is, $60°$). Let G and H be two points interior to the hexagon such that angles AGB and DHE are both $2\pi/3$ (that is, $120°$). Prove that $AG + GB + GH + DH + HE \geq CF$.

12. **G6 (USA)** Let $A_1A_2A_3A_4$ be a tetrahedron, G its centroid, and A_1', A_2', A_3', and A_4' the points where the circumsphere of $A_1A_2A_3A_4$ intersects GA_1, GA_2, GA_3, and GA_4, respectively. Prove that

$$GA_1 \cdot GA_2 \cdot GA_3 \cdot GA_4 \leq GA_1' \cdot GA_2' \cdot GA_3' \cdot GA_4'$$

and

$$\frac{1}{GA_1'} + \frac{1}{GA_2'} + \frac{1}{GA_3'} + \frac{1}{GA_4'} \leq \frac{1}{GA_1} + \frac{1}{GA_2} + \frac{1}{GA_3} + \frac{1}{GA_4}.$$

13. **G7 (LAT)** O is a point inside a convex quadrilateral $ABCD$ of area S. K, L, M, and N are interior points of the sides AB, BC, CD, and DA respectively. If $OKBL$ and $OMDN$ are parallelograms, prove that $\sqrt{S} \geq \sqrt{S_1} + \sqrt{S_2}$, where S_1 and S_2 are the areas of $ONAK$ and $OLCM$ respectively.

14. **G8 (COL)** Let ABC be a triangle. A circle passing through B and C intersects the sides AB and AC again at C' and B', respectively. Prove that BB', CC', and HH' are concurrent, where H and H' are the orthocenters of triangles ABC and $AB'C'$ respectively.

15. **N1 (ROM)** Let k be a positive integer. Prove that there are infinitely many perfect squares of the form $n2^k - 7$, where n is a positive integer.

16. **N2 (RUS)** Let $\mathbb{Z}$ denote the set of all integers. Prove that for any integers A and B, one can find an integer C for which $M_1 = \{x^2 + Ax + B : x \in \mathbb{Z}\}$ and $M_2 = \{2x^2 + 2x + C : x \in \mathbb{Z}\}$ do not intersect.

17. **N3 (CZE)**IMO3 Determine all integers $n > 3$ such that there are n points $A_1, A_2, \ldots, A_n$ in the plane that satisfy the following two conditions simultaneously:
    (a) No three lie on the same line.
    (b) There exist real numbers $p_1, p_2, \ldots, p_n$ such that the area of $\triangle A_i A_j A_k$ is equal to $p_i + p_j + p_k$, for $1 \le i < j < k \le n$.

18. **N4 (BUL)** Find all positive integers x and y such that $x + y^2 + z^3 = xyz$, where z is the greatest common divisor of x and y.

19. **N5 (IRE)** At a meeting of $12k$ people, each person exchanges greetings with exactly $3k+6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting?

20. **N6 (POL)**IMO6 Let p be an odd prime. Find the number of p-element subsets A of $\{1, 2, \ldots, 2p\}$ such that the sum of all elements of A is divisible by p.

21. **N7 (BLR)** Does there exist an integer $n > 1$ that satisfies the following condition?
    The set of positive integers can be partitioned into n nonempty subsets such that an arbitrary sum of $n - 1$ integers, one taken from each of any $n - 1$ of the subsets, lies in the remaining subset.

22. **N8 (GER)** Let p be an odd prime. Determine positive integers x and y for which $x \le y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is nonnegative and as small as possible.

23. **S1 (UKR)** Does there exist a sequence $F(1), F(2), F(3), \ldots$ of nonnegative integers that simultaneously satisfies the following three conditions?
    (a) Each of the integers $0, 1, 2, \ldots$ occurs in the sequence.
    (b) Each positive integer occurs in the sequence infinitely often.
    (c) For any $n \ge 2$,
    $$F\left(F\left(n^{163}\right)\right) = F(F(n)) + F(F(361)).$$

24. **S2 (POL)**IMO4 The positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfy $x_0 = x_{1995}$ and
    $$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$$
    for $i = 1, 2, \ldots, 1995$. Find the maximum value that x_0 can have.

25. **S3 (POL)** For an integer $x \ge 1$, let $p(x)$ be the least prime that does not divide x, and define $q(x)$ to be the product of all primes less than $p(x)$. In particular, $p(1) = 2$. For x such that $p(x) = 2$, define $q(x) = 1$. Consider the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = 1$ and
    $$x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$$
    for $n \ge 0$. Find all n such that $x_n = 1995$.

26. **S4 (NZL)** Suppose that $x_1, x_2, x_3, \dots$ are positive real numbers for which

$$x_n^n = \sum_{j=0}^{n-1} x_n^j$$

for $n = 1, 2, 3, \dots$. Prove that for all n,

$$2 - \frac{1}{2^{n-1}} \le x_n < 2 - \frac{1}{2^n}.$$

27. **S5 (FIN)** For positive integers n, the numbers $f(n)$ are defined inductively as follows: $f(1) = 1$, and for every positive integer n, $f(n+1)$ is the greatest integer m such that there is an arithmetic progression of positive integers $a_1 < a_2 < \cdots < a_m = n$ for which

$$f(a_1) = f(a_2) = \cdots = f(a_m).$$

Prove that there are positive integers a and b such that $f(an+b) = n+2$ for every positive integer n.

28. **S6 (IND)** Let $\mathbb{N}$ denote the set of all positive integers. Prove that there exists a unique function $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(m + f(n)) = n + f(m + 95)$$

for all m and n in $\mathbb{N}$. What is the value of $\sum_{k=1}^{19} f(k)$?

3.37 The Third-Seventh IMO Mumbai, India, July 5–17, 1996

3.37.1 Contest Problems

First Day (July 10)

1. We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20$, $|BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: One can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square corresponding to vertex A to the square corresponding to vertex B.
   (a) Show that the task cannot be done if r is divisible by 2 or 3.
   (b) Prove that the task is possible when $r = 73$.
   (c) Is there a solution when $r = 97$?

2. Let P be a point inside $\triangle ABC$ such that
$$\angle APB - \angle C = \angle APC - \angle B.$$
Let D, E be the incenters of $\triangle APB, \triangle APC$ respectively. Show that AP, BD, and CE meet in a point.

3. Let $\mathbb{N}_0$ denote the set of nonnegative integers. Find all functions f from $\mathbb{N}_0$ into itself such that
$$f(m + f(n)) = f(f(m)) + f(n), \qquad \forall m, n \in \mathbb{N}_0.$$

Second Day (July 11)

4. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

5. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to AF. Let R_A, R_C, R_E be the circumradii of triangles FAB, BCD, DEF respectively, and let P denote the perimeter of the hexagon. Prove that
$$R_A + R_C + R_E \geq \frac{P}{2}.$$

6. Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, \ldots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:
   (i) $x_0 = x_n = 0$.
   (ii) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.
   Show that there exists a pair (i, j) of distinct indices with $(i, j) \neq (0, n)$ such that $x_i = x_j$.

3.37.2 Shortlisted Problems

1. **A1 (SLO)** Let a, b, and c be positive real numbers such that $abc = 1$. Prove that
$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \le 1.$$
When does equality hold?

2. **A2 (IRE)** Let $a_1 \ge a_2 \ge \cdots \ge a_n$ be real numbers such that
$$a_1^k + a_2^k + \cdots + a_n^k \ge 0$$
for all integers $k > 0$. Let $p = \max\{|a_1|, \dots, |a_n|\}$. Prove that $p = a_1$ and that
$$(x-a_1)(x-a_2)\cdots(x-a_n) \le x^n - a_1^n$$
for all $x > a_1$.

3. **A3 (GRE)** Let $a > 2$ be given, and define recursively
$$a_0 = 1, \qquad a_1 = a, \qquad a_{n+1} = \left(\frac{a_n^2}{a_{n-1}^2} - 2\right) a_n.$$
Show that for all $k \in \mathbb{N}$, we have
$$\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} < \frac{1}{2}\left(2 + a - \sqrt{a^2-4}\right).$$

4. **A4 (KOR)** Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers, not all zero.
   (a) Prove that $x^n - a_1 x^{n-1} - \cdots - a_{n-1}x - a_n = 0$ has precisely one positive real root.
   (b) Let $A = \sum_{j=1}^n a_j$, $B = \sum\limits_{j=1}^{n} j a_j$, and let R be the positive real root of the equation in part (a). Prove that
$$A^A \le R^B.$$

5. **A5 (ROM)** Let $P(x)$ be the real polynomial function $P(x) = ax^3 + bx^2 + cx + d$. Prove that if $|P(x)| \le 1$ for all x such that $|x| \le 1$, then
$$|a| + |b| + |c| + |d| \le 7.$$

6. **A6 (IRE)** Let n be an even positive integer. Prove that there exists a positive integer k such that
$$k = f(x)(x+1)^n + g(x)(x^n+1)$$
for some polynomials $f(x), g(x)$ having integer coefficients. If k_0 denotes the least such k, determine k_0 as a function of n.

A6′ Let n be an even positive integer. Prove that there exists a positive integer k such that

$$k = f(x)(x+1)^n + g(x)(x^n+1)$$

for some polynomials $f(x), g(x)$ having integer coefficients. If k_0 denotes the least such k, show that $k_0 = 2^q$, where q is the odd integer determined by $n = q2^r$, $r \in \mathbb{N}$.

A6″ Prove that for each positive integer n, there exist polynomials $f(x), g(x)$ having integer coefficients such that

$$f(x)(x+1)^{2^n} + g(x)(x^{2^n}+1) = 2.$$

7. **A7 (ARM)** Let f be a function from the set of real numbers $\mathbb{R}$ into itself such that for all $x \in \mathbb{R}$, we have $|f(x)| \le 1$ and

$$f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right).$$

Prove that f is a periodic function (that is, there exists a nonzero real number c such that $f(x+c) = f(x)$ for all $x \in \mathbb{R}$).

8. **A8 (ROM)**$^{\mathrm{IMO3}}$ Let $\mathbb{N}_0$ denote the set of nonnegative integers. Find all functions f from $\mathbb{N}_0$ into itself such that

$$f(m + f(n)) = f(f(m)) + f(n), \qquad \forall m, n \in \mathbb{N}_0.$$

9. **A9 (POL)** Let the sequence $a(n)$, $n = 1, 2, 3, \ldots$, be generated as follows: $a(1) = 0$, and for $n > 1$,

$$a(n) = a([n/2]) + (-1)^{\frac{n(n+1)}{2}}. \qquad \text{(Here } [t] = \text{the greatest integer} \le t.\text{)}$$

(a) Determine the maximum and minimum value of $a(n)$ over $n \le 1996$ and find all $n \le 1996$ for which these extreme values are attained.
(b) How many terms $a(n)$, $n \le 1996$, are equal to 0?

10. **G1 (GBR)** Let triangle ABC have orthocenter H, and let P be a point on its circumcircle, distinct from A, B, C. Let E be the foot of the altitude BH, let $PAQB$ and $PARC$ be parallelograms, and let AQ meet HR in X. Prove that EX is parallel to AP.

11. **G2 (CAN)**$^{\mathrm{IMO2}}$ Let P be a point inside $\triangle ABC$ such that

$$\angle APB - \angle C = \angle APC - \angle B.$$

Let D, E be the incenters of $\triangle APB, \triangle APC$ respectively. Show that AP, BD and CE meet in a point.

12. **G3 (GBR)** Let ABC be an acute-angled triangle with $BC > CA$. Let O be the circumcenter, H its orthocenter, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$.
Possible second part: What happens if $|BC| \le |CA|$ (the triangle still being acute-angled)?

13. **G4 (USA)** Let $\triangle ABC$ be an equilateral triangle and let P be a point in its interior. Let the lines AP, BP, CP meet the sides BC, CA, AB in the points A_1, B_1, C_1 respectively. Prove that
$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \ge A_1B \cdot B_1C \cdot C_1A.$$

14. **G5 (ARM)**$^{\mathrm{IMO5}}$ Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to AF. Let R_A, R_C, R_E be the circumradii of triangles FAB, BCD, DEF respectively, and let P denote the perimeter of the hexagon. Prove that
$$R_A + R_C + R_E \ge \frac{P}{2}.$$

15. **G6 (ARM)** Let the sides of two rectangles be $\{a, b\}$ and $\{c, d\}$ with $a < c \le d < b$ and $ab < cd$. Prove that the first rectangle can be placed within the second one if and only if
$$(b^2 - a^2)^2 \le (bd - ac)^2 + (bc - ad)^2.$$

16. **G7 (GBR)** Let ABC be an acute-angled triangle with circumcenter O and circumradius R. Let AO meet the circle BOC again in A', let BO meet the circle COA again in B', and let CO meet the circle AOB again in C'. Prove that
$$OA' \cdot OB' \cdot OC' \ge 8R^3.$$
When does equality hold?

17. **G8 (RUS)** Let $ABCD$ be a convex quadrilateral, and let R_A, R_B, R_C, and R_D denote the circumradii of the triangles DAB, ABC, BCD, and CDA respectively. Prove that $R_A + R_C > R_B + R_D$ if and only if
$$\angle A + \angle C > \angle B + \angle D.$$

18. **G9 (UKR)** In the plane are given a point O and a polygon $\mathcal{F}$ (not necessarily convex). Let P denote the perimeter of $\mathcal{F}$, D the sum of the distances from O to the vertices of $\mathcal{F}$, and H the sum of the distances from O to the lines containing the sides of $\mathcal{F}$. Prove that
$$D^2 - H^2 \ge \frac{P^2}{4}.$$

19. **N1 (UKR)** Four integers are marked on a circle. At each step we simultaneously replace each number by the difference between this number and the next number on the circle, in a given direction (that is, the numbers a, b, c, d are replaced by $a-b, b-c, c-d, d-a$). Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc-ad|, |ac-bd|, |ab-cd|$ are primes?

20. **N2 (RUS)**$^{\mathrm{IMO4}}$ The positive integers a and b are such that the numbers $15a+16b$ and $16a-15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

21. **N3 (BUL)** A finite sequence of integers $a_0, a_1, \dots, a_n$ is called *quadratic* if for each $i \in \{1, 2, \dots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$.
    (a) Prove that for any two integers b and c, there exist a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$.
    (b) Find the smallest natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

22. **N4 (BUL)** Find all positive integers a and b for which

$$\left[\frac{a^2}{b}\right] + \left[\frac{b^2}{a}\right] = \left[\frac{a^2+b^2}{ab}\right] + ab$$

where as usual, $[t]$ refers to greatest integer that is less than or equal to t.

23. **N5 (ROM)** Let $\mathbb{N}_0$ denote the set of nonnegative integers. Find a bijective function f from $\mathbb{N}_0$ into $\mathbb{N}_0$ such that for all $m, n \in \mathbb{N}_0$,

$$f(3mn+m+n) = 4f(m)f(n) + f(m) + f(n).$$

24. **C1 (FIN)**$^{\mathrm{IMO1}}$ We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20$, $|BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: One can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square corresponding to vertex A to the square corresponding to vertex B.
    (a) Show that the task cannot be done if r is divisible by 2 or 3.
    (b) Prove that the task is possible when $r = 73$.
    (c) Is there a solution when $r = 97$?

25. **C2 (UKR)** An $(n-1) \times (n-1)$ square is divided into $(n-1)^2$ unit squares in the usual manner. Each of the n^2 vertices of these squares is to be colored red or blue. Find the number of different colorings such that each unit square has exactly two red vertices. (Two coloring schemes are regarded as different if at least one vertex is colored differently in the two schemes.)

26. **C3 (USA)** Let k, m, n be integers such that $1 < n \le m - 1 \le k$. Determine the maximum size of a subset S of the set $\{1, 2, 3, \dots, k\}$ such that no n distinct elements of S add up to m.

27. **C4 (FIN)** Determine whether or not there exist two disjoint infinite sets $\mathcal{A}$ and $\mathcal{B}$ of points in the plane satisfying the following conditions:
    (i) No three points in $\mathcal{A} \cup \mathcal{B}$ are collinear, and the distance between any two points in $\mathcal{A} \cup \mathcal{B}$ is at least 1.
    (ii) There is a point of $\mathcal{A}$ in any triangle whose vertices are in $\mathcal{B}$, and there is a point of $\mathcal{B}$ in any triangle whose vertices are in $\mathcal{A}$.

28. **C5 (FRA)**[IMO6] Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, \dots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:
    (i) $x_0 = x_n = 0$.
    (ii) For each i with $1 \le i \le n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.
    Show that there exists a pair (i, j) of distinct indices with $(i, j) \ne (0, n)$ such that $x_i = x_j$.

29. **C6 (CAN)** A finite number of beans are placed on an infinite row of squares. A sequence of moves is performed as follows: At each stage a square containing more than one bean is chosen. Two beans are taken from this square; one of them is placed on the square immediately to the left, and the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one bean on each square. Given some initial configuration, show that any legal sequence of moves will terminate after the same number of steps and with the same final configuration.

30. **C7 (IRE)** Let U be a finite set and let f, g be bijective functions from U onto itself. Let
    $$S = \{w \in U : f(f(w)) = g(g(w))\}, \qquad T = \{w \in U : f(g(w)) = g(f(w))\},$$
    and suppose that $U = S \cup T$. Prove that for $w \in U$, $f(w) \in S$ if and only if $g(w) \in S$.

3.38 The Thirty-Eighth IMO
Mar del Plata, Argentina, July 18–31, 1997

3.38.1 Contest Problems

First Day (July 24)

1. An infinite square grid is colored in the chessboard pattern. For any pair of positive integers m, n consider a right-angled triangle whose vertices are grid points and whose legs, of lengths m and n, run along the lines of the grid. Let S_b be the total area of the black part of the triangle and S_w the total area of its white part. Define the function $f(m, n) = |S_b - S_w|$.
   (a) Calculate $f(m, n)$ for all m, n that have the same parity.
   (b) Prove that $f(m, n) \le \frac{1}{2}\max(m, n)$.
   (c) Show that $f(m, n)$ is not bounded from above.

2. In triangle ABC the angle at A is the smallest. A line through A meets the circumcircle again at the point U lying on the arc BC opposite to A. The perpendicular bisectors of CA and AB meet AU at V and W, respectively, and the lines CV, BW meet at T. Show that $AU = TB + TC$.

3. Let $x_1, x_2, \dots, x_n$ be real numbers satisfying the conditions
$$|x_1 + x_2 + \cdots + x_n| = 1 \qquad \text{and} \qquad |x_i| \le \frac{n+1}{2} \quad \text{for} \quad i = 1, 2, \dots, n.$$
   Show that there exists a permutation $y_1, \dots, y_n$ of the sequence $x_1, \dots, x_n$ such that
$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

Second Day (July 25)

4. An $n \times n$ matrix with entries from $\{1, 2, \dots, 2n-1\}$ is called a *silver matrix* if for each i the union of the ith row and the ith column contains $2n-1$ distinct entries. Show that:
   (a) There exist no silver matrices for $n = 1997$.
   (b) Silver matrices exist for infinitely many values of n.

5. Find all pairs of integers $x, y \ge 1$ satisfying the equation $x^{y^2} = y^x$.

6. For a positive integer n, let $f(n)$ denote the number of ways to represent n as the sum of powers of 2 with nonnegative integer exponents. Representations that differ only in the ordering in their summands are not considered to be distinct. (For instance, $f(4) = 4$ because the number 4 can be represented in the following four ways: 4; 2+2; 2+1+1; 1+1+1+1.) Prove that the inequality
$$2^{n^2/4} < f(2^n) < 2^{n^2/2}$$
   holds for any integer $n \ge 3$.

3.38.2 Shortlisted Problems

1. **(BLR)**[IMO1] An infinite square grid is colored in the chessboard pattern. For any pair of positive integers m, n consider a right-angled triangle whose vertices are grid points and whose legs, of lengths m and n, run along the lines of the grid. Let S_b be the total area of the black part of the triangle and S_w the total area of its white part. Define the function $f(m,n) = |S_b - S_w|$.
   (a) Calculate $f(m,n)$ for all m, n that have the same parity.
   (b) Prove that $f(m,n) \leq \frac{1}{2}\max(m,n)$.
   (c) Show that $f(m,n)$ is not bounded from above.

2. **(CAN)** Let $R_1, R_2, \dots$ be the family of finite sequences of positive integers defined by the following rules: $R_1 = (1)$, and if $R_{n-1} = (x_1, \dots, x_s)$, then
$$R_n = (1, 2, \dots, x_1, 1, 2, \dots, x_2, \dots, 1, 2, \dots, x_s, n).$$
   For example, $R_2 = (1,2), R_3 = (1,1,2,3), R_4 = (1,1,1,2,1,2,3,4)$. Prove that if $n > 1$, then the kth term from the left in R_n is equal to 1 if and only if the kth term from the right in R_n is different from 1.

3. **(GER)** For each finite set U of nonzero vectors in the plane we define $l(U)$ to be the length of the vector that is the sum of all vectors in U. Given a finite set V of nonzero vectors in the plane, a subset B of V is said to be maximal if $l(B)$ is greater than or equal to $l(A)$ for each nonempty subset A of V.
   (a) Construct sets of 4 and 5 vectors that have 8 and 10 maximal subsets respectively.
   (b) Show that for any set V consisting of $n \geq 1$ vectors, the number of maximal subsets is less than or equal to $2n$.

4. **(IRN)**[IMO4] An $n \times n$ matrix with entries from $\{1, 2, \dots, 2n-1\}$ is called a *coveralls matrix* if for each i the union of the ith row and the ith column contains $2n-1$ distinct entries. Show that:
   (a) There exist no coveralls matrices for $n = 1997$.
   (b) Coveralls matrices exist for infinitely many values of n.

5. **(ROM)** Let $ABCD$ be a regular tetrahedron and M, N distinct points in the planes ABC and ADC respectively. Show that the segments MN, BN, MD are the sides of a triangle.

6. **(IRE)** (a) Let n be a positive integer. Prove that there exist distinct positive integers x, y, z such that
$$x^{n-1} + y^n = z^{n+1}.$$
   (b) Let a, b, c be positive integers such that a and b are relatively prime and c is relatively prime either to a or to b. Prove that there exist

infinitely many triples (x, y, z) of distinct positive integers x, y, z such that

$$x^a + y^b = z^c.$$

Original formulation: Let a, b, c, n be positive integers such that n is odd and ac is relatively prime to $2b$. Prove that there exist distinct positive integers x, y, z such that
(i) $x^a + y^b = z^c$, and
(ii) xyz is relatively prime to n.

7. **(RUS)** Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$, $EF = FA$. Prove that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

When does equality occur?

8. **(GBR)**$^{\text{IMO2}}$ Four different points A, B, C, D are chosen on a circle Γ such that the triangle BCD is not right-angled. Prove that:
(a) The perpendicular bisectors of AB and AC meet the line AD at certain points W and V, respectively, and that the lines CV and BW meet at a certain point T.
(b) The length of one of the line segments AD, BT, and CT is the sum of the lengths of the other two.
Original formulation. In triangle ABC the angle at A is the smallest. A line through A meets the circumcircle again at the point U lying on the arc BC opposite to A. The perpendicular bisectors of CA and AB meet AU at V and W, respectively, and the lines CV, BW meet at T. Show that $AU = TB + TC$.

9. **(USA)** Let $A_1A_2A_3$ be a nonisosceles triangle with incenter I. Let C_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (the addition of indices being mod 3). Let B_i, $i = 1, 2, 3$, be the second point of intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles $A_1B_1I, A_2B_2I, A_3B_3I$ are collinear.

10. **(CZE)** Find all positive integers k for which the following statement is true:
If $F(x)$ is a polynomial with integer coefficients satisfying the condition

$$0 \leq F(c) \leq k \quad \text{for each } c \in \{0, 1, \dots, k+1\},$$

then $F(0) = F(1) = \cdots = F(k+1)$.

11. **(NET)** Let $P(x)$ be a polynomial with real coefficients such that $P(x) > 0$ for all $x \geq 0$. Prove that there exists a positive integer n such that $(1 + x)^n P(x)$ is a polynomial with nonnegative coefficients.

12. **(ITA)** Let p be a prime number and let $f(x)$ be a polynomial of degree d with integer coefficients such that:

(i) $f(0) = 0$, $f(1) = 1$;
(ii) for every positive integer n, the remainder of the division of $f(n)$ by p is either 0 or 1.

Prove that $d \geq p - 1$.

13. **(IND)** In town A, there are n girls and n boys, and each girl knows each boy. In town B, there are n girls $g_1, g_2, \dots, g_n$ and $2n - 1$ boys $b_1, b_2, \dots, b_{2n-1}$. The girl g_i, $i = 1, 2, \dots, n$, knows the boys $b_1, b_2, \dots, b_{2i-1}$, and no others. For all $r = 1, 2, \dots, n$, denote by $A(r), B(r)$ the number of different ways in which r girls from town A, respectively town B, can dance with r boys from their own town, forming r pairs, each girl with a boy she knows. Prove that $A(r) = B(r)$ for each $r = 1, 2, \dots, n$.

14. **(IND)** Let b, m, n be positive integers such that $b > 1$ and $m \neq n$. Prove that if $b^m - 1$ and $b^n - 1$ have the same prime divisors, then $b + 1$ is a power of 2.

15. **(RUS)** An infinite arithmetic progression whose terms are positive integers contains the square of an integer and the cube of an integer. Show that it contains the sixth power of an integer.

16. **(BLR)** In an acute-angled triangle ABC, let AD, BE be altitudes and AP, BQ internal bisectors. Denote by I and O the incenter and the circumcenter of the triangle, respectively. Prove that the points D, E, and I are collinear if and only if the points P, Q, and O are collinear.

17. **(CZE)**$^{\mathrm{IMO5}}$ Find all pairs of integers $x, y \geq 1$ satisfying the equation $x^{y^2} = y^x$.

18. **(GBR)** The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F, respectively. The line through D parallel to EF meets the lines AC and AB at Q and R, respectively. The line EF meets BC at P. Prove that the circumcircle of the triangle PQR passes through the midpoint of BC.

19. **(IRE)** Let $a_1 \geq \cdots \geq a_n \geq a_{n+1} = 0$ be a sequence of real numbers. Prove that
$$\sqrt{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n} \sqrt{k}\left(\sqrt{a_k} - \sqrt{a_{k+1}}\right).$$

20. **(IRE)** Let D be an internal point on the side BC of a triangle ABC. The line AD meets the circumcircle of ABC again at X. Let P and Q be the feet of the perpendiculars from X to AB and AC, respectively, and let γ be the circle with diameter XD. Prove that the line PQ is tangent to γ if and only if $AB = AC$.

21. **(RUS)**$^{\mathrm{IMO3}}$ Let $x_1, x_2, \dots, x_n$ be real numbers satisfying the conditions
$$|x_1 + x_2 + \cdots + x_n| = 1 \qquad \text{and} \qquad |x_i| \leq \frac{n+1}{2} \quad \text{for} \quad i = 1, 2, \dots, n.$$

Show that there exists a permutation $y_1, \dots, y_n$ of the sequence $x_1, \dots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

22. **(UKR)** (a) Do there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \qquad \text{and} \qquad g(f(x)) = x^3 \qquad \text{for all } x \in \mathbb{R}?$$

(b) Do there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \qquad \text{and} \qquad g(f(x)) = x^4 \qquad \text{for all } x \in \mathbb{R}?$$

23. **(GBR)** Let $ABCD$ be a convex quadrilateral and O the intersection of its diagonals AC and BD. If

$$OA \sin \angle A + OC \sin \angle C = OB \sin \angle B + OD \sin \angle D,$$

prove that $ABCD$ is cyclic.

24. **(LIT)**^IMO6^ For a positive integer n, let $f(n)$ denote the number of ways to represent n as the sum of powers of 2 with nonnegative integer exponents. Representations that differ only in the ordering in their summands are not considered to be distinct. (For instance, $f(4) = 4$ because the number 4 can be represented in the following four ways: 4; 2+2; 2+1+1; 1+1+1+1.) Prove that the inequality

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}$$

holds for any integer $n \ge 3$.

25. **(POL)** The bisectors of angles A, B, C of a triangle ABC meet its circumcircle again at the points K, L, M, respectively. Let R be an internal point on the side AB. The points P and Q are defined by the following conditions: RP is parallel to AK, and BP is perpendicular to BL; RQ is parallel to BL, and AQ is perpendicular to AK. Show that the lines KP, LQ, MR have a point in common.

26. **(ITA)** For every integer $n \ge 2$ determine the minimum value that the sum $a_0 + a_1 + \cdots + a_n$ can take for nonnegative numbers $a_0, a_1, \dots, a_n$ satisfying the condition

$$a_0 = 1, \qquad a_i \le a_{i+1} + a_{i+2} \qquad \text{for } i = 0, \dots, n-2.$$

3.39 The Thirty-Ninth IMO
Taipei, Taiwan, July 10–21, 1998

3.39.1 Contest Problems

First Day (July 15)

1. A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of AB and CD meet at a unique point P inside $ABCD$. Prove that $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

2. In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that
$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

3. For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that $\frac{\tau(n^2)}{\tau(n)} = m$.

Second Day (July 16)

4. Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

5. Let I be the incenter of triangle ABC. Let K, L, and M be the points of tangency of the incircle of ABC with AB, BC, and CA, respectively. The line t passes through B and is parallel to KL. The lines MK and ML intersect t at the points R and S. Prove that $\angle RIS$ is acute.

6. Determine the least possible value of $f(1998)$, where f is a function from the set $\mathbb{N}$ of positive integers into itself such that for all $m, n \in \mathbb{N}$,
$$f(n^2 f(m)) = m[f(n)]^2.$$

3.39.2 Shortlisted Problems

1. **(LUX)**$^{\text{IMO1}}$ A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of AB and CD meet at a unique point P inside $ABCD$. Prove that $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

2. **(POL)** Let $ABCD$ be a cyclic quadrilateral. Let E and F be variable points on the sides AB and CD, respectively, such that $AE : EB = CF : FD$. Let P be the point on the segment EF such that $PE : PF = AB : CD$. Prove that the ratio between the areas of triangles APD and BPC does not depend on the choice of E and F.

3. **(UKR)**$^{\text{IMO5}}$ Let I be the incenter of triangle ABC. Let K, L, and M be the points of tangency of the incircle of ABC with AB, BC, and CA, respectively. The line t passes through B and is parallel to KL. The lines MK and ML intersect t at the points R and S. Prove that $\angle RIS$ is acute.

4. **(ARM)** Let M and N be points inside triangle ABC such that
$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$
Prove that
$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

5. **(FRA)** Let ABC be a triangle, H its orthocenter, O its circumcenter, and R its circumradius. Let D be the reflection of A across BC, E that of B across CA, and F that of C across AB. Prove that D, E, and F are collinear if and only if $OH = 2R$.

6. **(POL)** Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and
$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$
Prove that
$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

7. **(GBR)** Let ABC be a triangle such that $\angle ACB = 2\angle ABC$. Let D be the point on the side BC such that $CD = 2BD$. The segment AD is extended to E so that $AD = DE$. Prove that
$$\angle ECB + 180^\circ = 2\angle EBC.$$

8. **(IND)** Let ABC be a triangle such that $\angle A = 90^\circ$ and $\angle B < \angle C$. The tangent at A to its circumcircle ω meets the line BC at D. Let E be the reflection of A across BC, X the foot of the perpendicular from A to BE, and Y the midpoint of AX. Let the line BY meet ω again at Z. Prove that the line BD is tangent to the circumcircle of triangle ADZ.

9. **(MON)** Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n < 1$. Prove that
$$\frac{a_1 a_2 \cdots a_n[1 - (a_1 + a_2 + \cdots + a_n)]}{(a_1 + a_2 + \cdots + a_n)(1 - a_1)(1 - a_2)\cdots(1 - a_n)} \le \frac{1}{n^{n+1}}.$$

10. **(AUS)** Let $r_1, r_2, \ldots, r_n$ be real numbers greater than or equal to 1. Prove that
$$\frac{1}{r_1+1}+\frac{1}{r_2+1}+\cdots+\frac{1}{r_n+1} \geq \frac{n}{\sqrt[n]{r_1 r_2 \cdots r_n}+1}.$$

11. **(RUS)** Let x, y, and z be positive real numbers such that $xyz = 1$. Prove that
$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

12. **(POL)** Let $n \geq k \geq 0$ be integers. The numbers $c(n,k)$ are defined as follows:
$$\begin{aligned} c(n,0) = c(n,n) &= 1 && \text{for all } n \geq 0; \\ c(n+1,k) &= 2^k c(n,k) + c(n,k-1) && \text{for } n \geq k \geq 1. \end{aligned}$$
Prove that $c(n,k) = c(n,n-k)$ for all $n \geq k \geq 0$.

13. **(BUL)**$^{\text{IMO6}}$ Determine the least possible value of $f(1998)$, where f is a function from the set $\mathbb{N}$ of positive integers into itself such that for all $m, n \in \mathbb{N}$,
$$f(n^2 f(m)) = m[f(n)]^2.$$

14. **(GBR)**$^{\text{IMO4}}$ Determine all pairs (x,y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

15. **(AUS)** Determine all pairs (a,b) of real numbers such that $a\lfloor bn\rfloor = b\lfloor an\rfloor$ for all positive integers n. (Note that $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.)

16. **(UKR)** Determine the smallest integer $n \geq 4$ for which one can choose four different numbers a, b, c, and d from any n distinct integers such that $a+b-c-d$ is divisible by 20.

17. **(GBR)** A sequence of integers $a_1, a_2, a_3, \ldots$ is defined as follows: $a_1 = 1$, and for $n \geq 1$, a_{n+1} is the smallest integer greater than a_n such that $a_i + a_j \neq 3a_k$ for any i, j, k in $\{1, 2, \ldots, n+1\}$, not necessarily distinct. Determine a_{1998}.

18. **(BUL)** Determine all positive integers n for which there exists an integer m such that $2^n - 1$ is a divisor of $m^2 + 9$.

19. **(BLR)**$^{\text{IMO3}}$ For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that $\frac{\tau(n^2)}{\tau(n)} = m$.

20. **(ARG)** Prove that for each positive integer n, there exists a positive integer with the following properties:
    (i) It has exactly n digits.
    (ii) None of the digits is 0.

(iii) It is divisible by the sum of its digits.

21. **(CAN)** Let $a_0, a_1, a_2, \ldots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j, k are not necessarily distinct. Determine a_{1998}.

22. **(UKR)** A rectangular array of numbers is given. In each row and each column, the sum of all numbers is an integer. Prove that each nonintegral number x in the array can be changed into either $\lceil x \rceil$ or $\lfloor x \rfloor$ so that the row sums and column sums remain unchanged. (Note that $\lceil x \rceil$ is the least integer greater than or equal to x, while $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

23. **(BLR)** Let n be an integer greater than 2. A positive integer is said to be *attainable* if it is 1 or can be obtained from 1 by a sequence of operations with the following properties:
 (i) The first operation is either addition or multiplication.
 (ii) Thereafter, additions and multiplications are used alternately.
 (iii) In each addition one can choose independently whether to add 2 or n.
 (iv) In each multiplication, one can choose independently whether to multiply by 2 or by n.

 A positive integer that cannot be so obtained is said to be *unattainable*.
 (a) Prove that if $n \geq 9$, there are infinitely many unattainable positive integers.
 (b) Prove that if $n = 3$, all positive integers except 7 are attainable.

24. **(SWE)** Cards numbered 1 to 9 are arranged at random in a row. In a move, one may choose any block of consecutive cards whose numbers are in ascending or descending order, and switch the block around. For example, 91<u>653</u>2748 may be changed to 91<u>356</u>2748. Prove that in at most 12 moves, one can arrange the 9 cards so that their numbers are in ascending or descending order.

25. **(NZL)** Let $U = \{1, 2, \ldots, n\}$, where $n \geq 3$. A subset S of U is said to be *split* by an arrangement of the elements of U if an element not in S occurs in the arrangement somewhere between two elements of S. For example, 13542 splits $\{1, 2, 3\}$ but not $\{3, 4, 5\}$. Prove that for any $n - 2$ subsets of U, each containing at least 2 and at most $n - 1$ elements, there is an arrangement of the elements of U that splits all of them.

26. **(IND)**$^{\text{IMO2}}$ In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that $\frac{k}{m} \geq \frac{n-1}{2n}$.

27. **(BLR)** Ten points such that no three of them lie on a line are marked in the plane. Each pair of points is connected with a segment. Each of these segments is painted with one of k colors in such a way that for any k of

the ten points, there are k segments each joining two of them with no two being painted the same color. Determine all integers k, $1 \le k \le 10$, for which this is possible.

28. **(IRN)** A solitaire game is played on an $m \times n$ rectangular board, using mn markers that are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner square, which contains a marker with its black side up. In each move, one can take away one marker with its black side up, but must then turn over all markers that are in squares having an edge in common with the square of the removed marker. Determine all pairs (m, n) of positive integers such that all markers can be removed from the board.

3.40 The Fortieth IMO
Bucharest, Romania, July 10–22, 1999

3.40.1 Contest Problems

First Day (July 16)

1. A set S of points in the plane will be called *completely symmetric* if it has at least three elements and satisfies the following condition: For every two distinct points A, B from S the perpendicular bisector of the segment AB is an axis of symmetry for S.
   Prove that if a completely symmetric set is finite, then it consists of the vertices of a regular polygon.
2. Let $n \geq 2$ be a fixed integer. Find the least constant C such that the inequality
$$\sum_{i<j} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_i x_i \right)^4$$
   holds for every $x_1, \dots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant C, characterize the instances of equality.
3. Let n be an even positive integer. We say that two different cells of an $n \times n$ board are *neighboring* if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that every cell (marked or not marked) has a marked neighboring cell.

Second Day (July 17)

4. Find all pairs of positive integers (x, p) such that p is a prime, $x \leq 2p$, and x^{p-1} is a divisor of $(p-1)^x + 1$.
5. Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N, and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersect Ω_1 in C and D. Prove that Ω_2 is tangent to CD.
6. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$
   for all $x, y \in \mathbb{R}$.

3.40.2 Shortlisted Problems

1. **N1 (TWN)**$^{\text{IMO4}}$ Find all pairs of positive integers (x, p) such that p is a prime, $x \leq 2p$, and x^{p-1} is a divisor of $(p-1)^x + 1$.

2. **N2 (ARM)** Prove that every positive rational number can be represented in the form $\dfrac{a^3+b^3}{c^3+d^3}$, where a, b, c, d are positive integers.

3. **N3 (RUS)** Prove that there exist two strictly increasing sequences (a_n) and (b_n) such that $a_n(a_n+1)$ divides b_n^2+1 for every natural number n.

4. **N4 (FRA)** Denote by S the set of all primes p such that the decimal representation of $1/p$ has its fundamental period divisible by 3. For every $p \in S$ such that $1/p$ has its fundamental period $3r$ one may write $1/p = 0.a_1a_2\dots a_{3r}a_1a_2\dots a_{3r}\dots$, where $r = r(p)$; for every $p \in S$ and every integer $k \geq 1$ define $f(k,p)$ by

$$f(k,p) = a_k + a_{k+r(p)} + a_{k+2r(p)}.$$

   (a) Prove that S is infinite.
   (b) Find the highest value of $f(k,p)$ for $k \geq 1$ and $p \in S$.

5. **N5 (ARM)** Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer m that is divisible by n and the sum of whose digits in decimal representation is k.

6. **N6 (BLR)** Prove that for every real number M there exists an infinite arithmetic progression such that:
   (i) each term is a positive integer and the common difference is not divisible by 10;
   (ii) the sum of the digits of each term (in decimal representation) exceeds M.

7. **G1 (ARM)** Let ABC be a triangle and M an interior point. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + AC + BC.$$

8. **G2 (JAP)** A circle is called a *separator* for a set of five points in a plane if it passes through three of these points, it contains a fourth point in its interior, and the fifth point is outside the circle.
   Prove that every set of five points such that no three are collinear and no four are concyclic has exactly four separators.

9. **G3 (EST)**[IMO1] A set S of points in space will be called *completely symmetric* if it has at least three elements and satisfies the following condition: For every two distinct points A, B from S the perpendicular bisector of the segment AB is an axis of symmetry for S.
   Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, a regular tetrahedron, or a regular octahedron.

10. **G4 (GBR)** For a triangle $T = ABC$ we take the point X on the side (AB) such that $AX/XB = 4/5$, the point Y on the segment (CX) such that $CY = 2YX$, and, if possible, the point Z on the ray $(CA$ such that

$\measuredangle CXZ = 180^\circ - \measuredangle ABC$. We denote by Σ the set of all triangles T for which $\measuredangle XYZ = 45^\circ$.
Prove that all the triangles from Σ are similar and find the measure of their smallest angle.

11. **G5 (FRA)** Let ABC be a triangle, Ω its incircle and $\Omega_a, \Omega_b, \Omega_c$ three circles three circles orthogonal to Ω passing through B and C, A and C, and A and B respectively. The circles Ω_a, Ω_b meet again in C'; in the same way we obtain the points B' and A'. Prove that the radius of the circumcircle of $A'B'C'$ is half the radius of Ω.

12. **G6 (RUS)**[IMO5] Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N, and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersect Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

13. **G7 (ARM)** The point M inside the convex quadrilateral $ABCD$ is such that
$$MA = MC, \quad \angle AMB = \angle MAD + \angle MCD, \quad \angle CMD = \angle MCB + \angle MAB.$$
Prove that $AB \cdot CM = BC \cdot MD$ and $BM \cdot AD = MA \cdot CD$.

14. **G8 (RUS)** Points A, B, C divide the circumcircle Ω of the triangle ABC into three arcs. Let X be a variable point on the arc AB, and let O_1, O_2 be the incenters of the triangles CAX and CBX. Prove that the circumcircle of the triangle XO_1O_2 intersects Ω in a fixed point.

15. **A1 (POL)**[IMO2] Let $n \geq 2$ be a fixed integer. Find the least constant C such that the inequality
$$\sum_{i<j} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_i x_i \right)^4$$
holds for every $x_1, \dots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant C, characterize the instances of equality.

16. **A2 (RUS)** The numbers from 1 to n^2 are randomly arranged in the cells of a $n \times n$ square ($n \geq 2$). For any pair of numbers situated in the same row or in the same column, the ratio of the greater number to the smaller one is calculated.
Let us call the *characteristic* of the arrangement the smallest of these $n^2(n-1)$ fractions. What is the highest possible value of the characteristic?

17. **A3 (FIN)** A game is played by n girls ($n \geq 2$), everybody having a ball. Each of the $\binom{n}{2}$ pairs of players, in an arbitrary order, exchange the balls they have at that moment. The game is called *nice* if at the end nobody has her own ball, and it is called *tiresome* if at the end everybody has her initial ball. Determine the values of n for which there exists a nice game and those for which there exists a tiresome game.

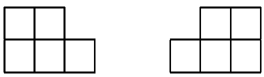
18. **A4 (BLR)** Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that for any two integers x, y taken from two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.

19. **A5 (JAP)**$^{\text{IMO6}}$ Find all the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$
for all $x, y \in \mathbb{R}$.

20. **A6 (SWE)** For $n \geq 3$ and $a_1 \leq a_2 \leq \cdots \leq a_n$ given real numbers we have the following instructions:
(1) place the numbers in some order in a circle;
(2) delete one of the numbers from the circle;
(3) if just two numbers are remaining in the circle, let S be the sum of these two numbers. Otherwise, if there are more than two numbers in the circle, replace $(x_1, x_2, x_3, \dots, x_{p-1}, x_p)$ with $(x_1 + x_2, x_2 + x_3, \dots, x_{p-1} + x_p, x_p + x_1)$. Afterwards, start again with step (2).
Show that the largest sum S that can result in this way is given by the formula
$$S_{\max} = \sum_{k=2}^{n} \binom{n-2}{\left[\frac{k}{2}\right] - 1} a_k.$$

21. **C1 (IND)** Let $n \geq 1$ be an integer. A *path* from $(0, 0)$ to (n, n) in the xy plane is a chain of consecutive unit moves either to the right (move denoted by E) or upwards (move denoted by N), all the moves being made inside the half-plane $x \geq y$. A *step* in a path is the occurrence of two consecutive moves of the form EN.
Show that the number of paths from $(0, 0)$ to (n, n) that contain exactly s steps ($n \geq s \geq 1$) is
$$\frac{1}{s}\binom{n-1}{s-1}\binom{n}{s-1}.$$

22. **C2 (CAN)** (a) If a $5 \times n$ rectangle can be tiled using n pieces like those shown in the diagram, prove that n is even.



(b) Show that there are more than $2 \cdot 3^{k-1}$ ways to tile a fixed $5 \times 2k$ rectangle ($k \geq 3$) with $2k$ pieces. (Symmetric constructions are considered to be different.)

23. **C3 (GBR)** A biologist watches a chameleon. The chameleon catches flies and rests after each catch. The biologist notices that:
(i) the first fly is caught after a resting period of one minute;
(ii) the resting period before catching the $2m$th fly is the same as the resting period before catching the mth fly and one minute shorter than the resting period before catching the $(2m + 1)$th fly;

(iii) when the chameleon stops resting, he catches a fly instantly.

(a) How many flies were caught by the chameleon before his first resting period of 9 minutes?

(b) After how many minutes will the chameleon catch his 98th fly?

(c) How many flies were caught by the chameleon after 1999 minutes passed?

24. **C4 (GBR)** Let A be a set of N residues (mod N^2). Prove that there exists a set B of N residues (mod N^2) such that the set $A+B=\{a+b \mid a \in A, b \in B\}$ contains at least half of all residues (mod N^2).

25. **C5 (BLR)**[IMO3] Let n be an even positive integer. We say that two different cells of an $n \times n$ board are *neighboring* if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that every cell (marked or not marked) has a marked neighboring cell.

26. **C6 (GBR)** Suppose that every integer has been given one of the colors red, blue, green, yellow. Let x and y be odd integers such that $|x| \neq |y|$. Show that there are two integers of the same color whose difference has one of the following values: $x, y, x+y, x-y$.

27. **C7 (IRE)** Let $p>3$ be a prime number. For each nonempty subset T of $\{0,1,2,3,\ldots,p-1\}$ let $E(T)$ be the set of all $(p-1)$-tuples $(x_1,\ldots,x_{p-1})$, where each $x_i \in T$ and $x_1+2x_2+\cdots+(p-1)x_{p-1}$ is divisible by p and let $|E(T)|$ denote the number of elements in $E(T)$.
Prove that
$$|E(\{0,1,3\})| \geq |E(\{0,1,2\})|,$$
with equality if and only if $p=5$.

3.41 The Forty-First IMO
Taejon, South Korea, July 13–25, 2000

3.41.1 Contest Problems

First day (July 18)

1. Two circles G_1 and G_2 intersect at M and N. Let AB be the line tangent to these circles at A and B, respectively, such that M lies closer to AB than N. Let CD be the line parallel to AB and passing through M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

2. Let a, b, c be positive real numbers with product 1. Prove that
$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

3. Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move of choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C such that $BC/AB = \lambda$.
Determine all values of λ such that for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

Second Day (July 19)

4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one, and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn. How many ways are there to put the cards in the three boxes so that the trick works?

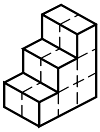
5. Does there exist a positive integer n such that n has exactly 2000 prime divisors and $2^n + 1$ is divisible by n?

6. $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i, and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.

3.41.2 Shortlisted Problems

1. **C1 (HUN)**^IMO4^ A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one, and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn. How many ways are there to put the cards in the three boxes so that the trick works?

2. **C2 (ITA)** A brick staircase with three steps of width 2 is made of twelve unit cubes. Determine all integers n for which it is possible to build a cube of side n using such bricks.



3. **C3 (COL)** Let $n \geq 4$ be a fixed positive integer. Given a set $S = \{P_1, P_2, \ldots, P_n\}$ of points in the plane such that no three are collinear and no four concyclic, let a_t, $1 \leq t \leq n$, be the number of circles $P_iP_jP_k$ that contain P_t in their interior, and let
$$m(S) = a_1 + a_2 + \cdots + a_n.$$
Prove that there exists a positive integer $f(n)$, depending only on n, such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

4. **C4 (CZE)** Let n and k be positive integers such that $n/2 < k \leq 2n/3$. Find the least number m for which it is possible to place m pawns on m squares of an $n \times n$ chessboard so that no column or row contains a block of k adjacent unoccupied squares.

5. **C5 (RUS)** In the plane we have n rectangles with parallel sides. The sides of distinct rectangles lie on distinct lines. The boundaries of the rectangles cut the plane into connected regions. A region is *nice* if it has at least one of the vertices of the n rectangles on its boundary. Prove that the sum of the numbers of the vertices of all nice regions is less than $40n$. (There can be nonconvex regions as well as regions with more than one boundary curve.)

6. **C6 (FRA)** Let p and q be relatively prime positive integers. A subset S of $\{0, 1, 2, \ldots\}$ is called *ideal* if $0 \in S$ and for each element $n \in S$, the integers $n + p$ and $n + q$ belong to S. Determine the number of ideal subsets of $\{0, 1, 2 \ldots\}$.

7. **A1 (USA)**^IMO2^ Let a, b, c be positive real numbers with product 1. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1.$$

8. **A2 (GBR)** Let a, b, c be positive integers satisfying the conditions $b > 2a$ and $c > 2b$. Show that there exists a real number t with the property that all the three numbers ta, tb, tc have their fractional parts lying in the interval $(1/3, 2/3]$.

9. **A3 (BLR)** Find all pairs of functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+g(y)) = xf(y) - yf(x) + g(x) \qquad \text{for all } x, y \in R.$$

10. **A4 (GBR)** The function F is defined on the set of nonnegative integers and takes nonnegative integer values satisfying the following conditions: For every $n \geq 0$,
(i) $F(4n) = F(2n) + F(n)$;
(ii) $F(4n+2) = F(4n) + 1$;
(iii) $F(2n+1) = F(2n) + 1$.
Prove that for each positive integer m, the number of integers n with $0 \leq n < 2^m$ and $F(4n) = F(3n)$ is $F(2^{m+1})$.

11. **A5 (BLR)**$^{\text{IMO3}}$ Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move of choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C such that $BC/AB = \lambda$.
Determine all values of λ such that for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

12. **A6 (IRE)** A nonempty set A of real numbers is called a B_3-set if the conditions $a_1, a_2, a_3, a_4, a_5, a_6 \in A$ and $a_1+a_2+a_3 = a_4+a_5+a_6$ imply that the sequences (a_1, a_2, a_3) and (a_4, a_5, a_6) are identical up to a permutation. Let $A = \{a_0 = 0 < a_1 < a_2 < \cdots\}$, $B = \{b_0 = 0 < b_1 < b_2 < \cdots\}$ be infinite sequences of real numbers with $D(A) = D(B)$, where, for a set X of real numbers, $D(X)$ denotes the difference set $\{|x-y| \mid x, y \in X\}$. Prove that if A is a B_3-set, then $A = B$.

13. **A7 (RUS)** For a polynomial P of degree 2000 with distinct real coefficients let $M(P)$ be the set of all polynomials that can be produced from P by permutation of its coefficients. A polynomial P will be called n-independent if $P(n) = 0$ and we can get from any Q in $M(P)$ a polynomial Q_1 such that $Q_1(n) = 0$ by interchanging at most one pair of coefficients of Q. Find all integers n for which n-independent polynomials exist.

14. **N1 (JAP)** Determine all positive integers $n \geq 2$ that satisfy the following condition: For all integers a, b relatively prime to n,

$$a \equiv b \pmod{n} \qquad \text{if and only if} \qquad ab \equiv 1 \pmod{n}.$$

15. **N2 (FRA)** For a positive integer n, let $d(n)$ be the number of all positive divisors of n. Find all positive integers n such that $d(n)^3 = 4n$.

16. **N3 (RUS)**$^{\mathrm{IMO5}}$ Does there exist a positive integer n such that n has exactly 2000 prime divisors and $2^n + 1$ is divisible by n?

17. **N4 (BRA)** Determine all triples of positive integers (a, m, n) such that $a^m + 1$ divides $(a + 1)^n$.

18. **N5 (BUL)** Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semiperimeter and the inradius of a triangle with integer side lengths.

19. **N6 (ROM)** Show that the set of positive integers that cannot be represented as a sum of distinct perfect squares is finite.

20. **G1 (NET)** In the plane we are given two circles intersecting at X and Y. Prove that there exist four points A, B, C, D with the following property: For every circle touching the two given circles at A and B, and meeting the line XY at C and D, each of the lines AC, AD, BC, BD passes through one of these points.

21. **G2 (RUS)**$^{\mathrm{IMO1}}$ Two circles G_1 and G_2 intersect at M and N. Let AB be the line tangent to these circles at A and B, respectively, such that M lies closer to AB than N. Let CD be the line parallel to AB and passing through M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

22. **G3 (IND)** Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that $OD + DH = OE + EH = OF + FH$ and the lines AD, BE, and CF are concurrent.

23. **G4 (RUS)** Let $A_1A_2 \dots A_n$ be a convex polygon, $n \geq 4$. Prove that $A_1A_2 \dots A_n$ is cyclic if and only if to each vertex A_j one can assign a pair (b_j, c_j) of real numbers, $j = 1, 2, \dots n$, such that
$$A_iA_j = b_jc_i - b_ic_j \qquad \text{for all } i, j \text{ with } 1 \leq i \leq j \leq n.$$

24. **G5 (GBR)** The tangents at B and A to the circumcircle of an acute-angled triangle ABC meet the tangent at C at T and U respectively. AT meets BC at P, and Q is the midpoint of AP; BU meets CA at R, and S is the midpoint of BR. Prove that $\angle ABQ = \angle BAS$. Determine, in terms of ratios of side lengths, the triangles for which this angle is a maximum.

25. **G6 (ARG)** Let $ABCD$ be a convex quadrilateral with AB not parallel to CD, let X be a point inside $ABCD$ such that $\measuredangle ADX = \measuredangle BCX < 90^\circ$ and $\measuredangle DAX = \measuredangle CBX < 90^\circ$. If Y is the point of intersection of the perpendicular bisectors of AB and CD, prove that $\measuredangle AYB = 2\measuredangle ADX$.

26. **G7 (IRN)** Ten gangsters are standing on a flat surface, and the distances between them are all distinct. At twelve o'clock, when the church bells start chiming, each of them fatally shoots the one among the other nine gangsters who is the nearest. At least how many gangsters will be killed?

27. **G8 (RUS)**$^{\mathrm{IMO6}}$ $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i, and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3, and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.

3.42 The Forty-Second IMO Washington DC, United States of America, July 1–14, 2001

3.42.1 Contest Problems

First Day (July 8)

1. In acute triangle ABC with circumcenter O and altitude AP, $\measuredangle C \geq \measuredangle B + 30^\circ$. Prove that $\measuredangle A + \measuredangle COP < 90^\circ$.
2. Prove that for all positive real numbers a, b, c,
$$\frac{a}{\sqrt{a^2+8bc}} + \frac{a}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$
3. Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that
   (i) each contestant solved at most six problems, and
   (ii) for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy.
   Show that there is a problem that was solved by at least three girls and at least three boys.

Second Day (July 9)

4. Let n be an odd integer greater than 1 and let $c_1, c_2, \dots, c_n$ be integers. For each permutation $a = (a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist permutations $a \neq b$ of $\{1, 2, \dots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.
5. Let ABC be a triangle with $\measuredangle BAC = 60^\circ$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?
6. Let $a > b > c > d$ be positive integers and suppose
$$ac + bd = (b + d + a - c)(b + d - a + c).$$
   Prove that $ab + cd$ is not prime.

3.42.2 Shortlisted Problems

1. **A1 (IND)** Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ such that
$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}\,(f(p+1,q-1,r) + f(p-1,q+1,r) & \\ \qquad +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ \qquad +f(p,q+1,r-1) + f(p,q-1,r+1)) & \text{otherwise.} \end{cases}$$

2. **A2 (POL)** Let $a_0, a_1, a_2, \ldots$ be an arbitrary infinite sequence of positive numbers. Show that the inequality $1 + a_n > a_{n-1}\sqrt[n]{2}$ holds for infinitely many positive integers n.

3. **A3 (ROM)** Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality
$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

4. **A4 (LIT)** Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(xy)(f(x) - f(y)) = (x - y)f(x)f(y)$$
for all x, y.

5. **A5 (BUL)** Find all positive integers $a_1, a_2, \ldots, a_n$ such that
$$\frac{99}{100} = \frac{a_0}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n},$$
where $a_0 = 1$ and $(a_{k+1} - 1)a_{k-1} \ge a_k^2(a_k - 1)$ for $k = 1, 2, \ldots, n-1$.

6. **A6 (KOR)**$^{\mathrm{IMO2}}$ Prove that for all positive real numbers a, b, c,
$$\frac{a}{\sqrt{a^2+8bc}} + \frac{a}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \ge 1.$$

7. **C1 (COL)** Let $A = (a_1, a_2, \ldots, a_{2001})$ be a sequence of positive integers. Let m be the number of 3-element subsequences (a_i, a_j, a_k) with $1 \le i < j < k \le 2001$ such that $a_j = a_i + 1$ and $a_k = a_j + 1$. Considering all such sequences A, find the greatest value of m.

8. **C2 (CAN)**$^{\mathrm{IMO4}}$ Let n be an odd integer greater than 1 and let $c_1, c_2, \ldots, c_n$ be integers. For each permutation $a = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist permutations $a \neq b$ of $\{1, 2, \ldots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

9. **C3 (RUS)** Define a *k-clique* to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

10. **C4 (NZL)** A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called *historic* if $\{z - y, y - x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the union of disjoint historic sets.

11. **C5 (FIN)** Find all finite sequences $(x_0, x_1, \ldots, x_n)$ such that for every j, $0 \le j \le n$, x_j equals the number of times j appears in the sequence.

12. **C6 (CAN)** For a positive integer n define a sequence of zeros and ones to be *balanced* if it contains n zeros and n ones. Two balanced sequences a and b are *neighbors* if you can move one of the $2n$ symbols of a to another position to form b. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbors because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set S of at most $\frac{1}{n+1}\binom{2n}{n}$ balanced sequences such that every balanced sequence is equal to or is a neighbor of at least one sequence in S.

13. **C7 (FRA)** A pile of n pebbles is placed in a vertical column. This configuration is modified according to the following rules. A pebble can be moved if it is at the top of a column that contains at least two more pebbles than the column immediately to its right. (If there are no pebbles to the right, think of this as a column with 0 pebbles.) At each stage, choose a pebble from among those that can be moved (if there are any) and place it at the top of the column to its right. If no pebbles can be moved, the configuration is called a *final configuration*. For each n, show that no matter what choices are made at each stage, the final configuration is unique. Describe that configuration in terms of n.

14. **C8 (GER)**$^{\text{IMO3}}$ Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that
    (i) each contestant solved at most six problems, and
    (ii) for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy.

    Show that there is a problem that was solved by at least three girls and at least three boys.

15. **G1 (UKR)** Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1, C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

16. **G2 (KOR)**$^{\text{IMO1}}$ In acute triangle ABC with circumcenter O and altitude AP, $\measuredangle C \geq \measuredangle B + 30°$. Prove that $\measuredangle A + \measuredangle COP < 90°$.

17. **G3 (GBR)** Let ABC be a triangle with centroid G. Determine, with proof, the position of the point P in the plane of ABC such that
$$AP \cdot AG + BP \cdot BG + CP \cdot CG$$
is a minimum, and express this minimum value in terms of the side lengths of ABC.

18. **G4 (FRA)** Let M be a point in the interior of triangle ABC. Let A' lie on BC with MA' perpendicular to BC. Define B' on CA and C' on AB

similarly. Define

$$p(M) = \frac{MA' \cdot MB' \cdot MC'}{MA \cdot MB \cdot MC}.$$

Determine, with proof, the location of M such that $p(M)$ is maximal. Let $\mu(ABC)$ denote the maximum value. For which triangles ABC is the value of $\mu(ABC)$ maximal?

19. **G5 (GRE)** Let ABC be an acute triangle. Let DAC, EAB, and FBC be isosceles triangles exterior to ABC, with $DA = DC$, $EA = EB$, and $FB = FC$ such that

$$\angle ADC = 2\angle BAC, \qquad \angle BEA = 2\angle ABC, \qquad \angle CFB = 2\angle ACB.$$

Let D' be the intersection of lines DB and EF, let E' be the intersection of EC and DF, and let F' be the intersection of FA and DE. Find, with proof, the value of the sum

$$\frac{DB}{DD'} + \frac{EC}{EE'} + \frac{FA}{FF'}.$$

20. **G6 (IND)** Let ABC be a triangle and P an exterior point in the plane of the triangle. Suppose AP, BP, CP meet the sides BC, CA, AB (or extensions thereof) in D, E, F, respectively. Suppose further that the areas of triangles PBD, PCE, PAF are all equal. Prove that each of these areas is equal to the area of triangle ABC itself.

21. **G7 (BUL)** Let O be an interior point of acute triangle ABC. Let A_1 lie on BC with OA_1 perpendicular to BC. Define B_1 on CA and C_1 on AB similarly. Prove that O is the circumcenter of ABC if and only if the perimeter of $A_1B_1C_1$ is not less than any one of the perimeters of AB_1C_1, BC_1A_1, and CA_1B_1.

22. **G8 (ISR)**$^{\text{IMO5}}$ Let ABC be a triangle with $\measuredangle BAC = 60°$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?

23. **N1 (AUS)** Prove that there is no positive integer n such that for $k = 1, 2, \dots, 9$, the leftmost digit (in decimal notation) of $(n+k)!$ equals k.

24. **N2 (COL)** Consider the system

$$\begin{aligned} x + y &= z + u, \\ 2xy &= zu. \end{aligned}$$

Find the greatest value of the real constant m such that $m \le x/y$ for every positive integer solution x, y, z, u of the system with $x \ge y$.

25. **N3 (GBR)** Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, \qquad n \ge 4.$$

Determine $a_{14^{14}}$.

26. **N4 (VIE)** Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p-2$ such that neither $a^{p-1}-1$ nor $(a+1)^{p-1}-1$ is divisible by p^2.

27. **N5 (BUL)**$^{\mathrm{IMO6}}$ Let $a > b > c > d$ be positive integers and suppose

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

28. **N6 (RUS)** Is it possible to find 100 positive integers not exceeding 25,000 such that all pairwise sums of them are different?

3.43 The Forty-Third IMO Glasgow, United Kingdom, July 19–30, 2002

3.43.1 Contest Problems

First Day (July 24)

1. Let n be a positive integer. Each point (x, y) in the plane, where x and y are nonnegative integers with $x + y = n$, is colored red or blue, subject to the following condition: If a point (x, y) is red, then so are all points (x', y') with $x' \le x$ and $y' \le y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

2. The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\measuredangle AOB < 120°$. Let D be the midpoint of the arc AB that does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.

3. Find all pairs of positive integers $m, n \ge 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

Second Day (July 25)

4. Let $n \ge 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \cdots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.

5. Find all functions f from the reals to the reals such that

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real x, y, z, t.

6. Let $n \ge 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centers $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \le i < j \le n} \frac{1}{O_iO_j} \le \frac{(n-1)\pi}{4}.$$

3.43.2 Shortlisted Problems

1. **N1 (UZB)** What is the smallest positive integer t such that there exist integers $x_1, x_2, \ldots, x_t$ with
$$x_1^3 + x_2^3 + \cdots + x_t^3 = 2002^{2002}?$$

2. **N2 (ROM)**$^{\text{IMO4}}$ Let $n \ge 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \cdots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.

3. **N3 (MON)** Let $p_1, p_2, \ldots, p_n$ be distinct primes greater than 3. Show that $2^{p_1p_2\cdots p_n} + 1$ has at least 4^n divisors.

4. **N4 (GER)** Is there a positive integer m such that the equation
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$
has infinitely many solutions in positive integers a, b, c?

5. **N5 (IRN)** Let $m, n \ge 2$ be positive integers, and let $a_1, a_2, \ldots, a_n$ be integers, none of which is a multiple of m^{n-1}. Show that there exist integers $e_1, e_2, \ldots, e_n$, not all zero, with $|e_i| < m$ for all i, such that $e_1a_1 + e_2a_2 + \cdots + e_na_n$ is a multiple of m^n.

6. **N6 (ROM)**$^{\text{IMO3}}$ Find all pairs of positive integers $m, n \ge 3$ for which there exist infinitely many positive integers a such that
$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$
is itself an integer.

7. **G1 (FRA)** Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcenter of triangle BCD lies on the circumcircle of triangle ABC.

8. **G2 (KOR)** Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E respectively. Prove that
$$AB + AC \ge 4DE.$$

9. **G3 (KOR)**$^{\text{IMO2}}$ The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\measuredangle AOB < 120°$. Let D be the midpoint of the arc AB that does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.

10. **G4 (RUS)** Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that as A_1 and B_1 vary, the circumcenters of triangles A_1A_2C all lie on one fixed circle.

11. **G5 (AUS)** For any set S of five points in the plane, no three of which are collinear, let $M(S)$ and $m(S)$ denote the greatest and smallest areas, respectively, of triangles determined by three points from S. What is the minimum possible value of $M(S)/m(S)$?

12. **G6 (UKR)**[IMO6] Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centers $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$

13. **G7 (BUL)** The incircle Ω of the acute-angled triangle ABC is tangent to BC at K. Let AD be an altitude of triangle ABC and let M be the midpoint of AD. If N is the other common point of Ω and KM, prove that Ω and the circumcircle of triangle BCN are tangent at N.

14. **G8 (ARM)** Let S_1 and S_2 be circles meeting at the points A and B. A line through A meets S_1 at C and S_2 at D. Points M, N, K lie on the line segments CD, BC, BD respectively, with MN parallel to BD and MK parallel to BC. Let E and F be points on those arcs BC of S_1 and BD of S_2 respectively that do not contain A. Given that EN is perpendicular to BC and FK is perpendicular to BD, prove that $\measuredangle EMF = 90°$.

15. **A1 (CZE)** Find all functions f from the reals to the reals such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all real x, y.

16. **A2 (YUG)** Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers for which there exists a real number c with $0 \leq a_i \leq c$ for all i such that

$$|a_i - a_j| \geq \frac{1}{i+j} \qquad \text{for all } i, j \text{ with } i \neq j.$$

Prove that $c \geq 1$.

17. **A3 (POL)** Let P be a cubic polynomial given by $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are integers and $a \neq 0$. Suppose that $xP(x) = yP(y)$ for infinitely many pairs x, y of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

18. **A4 (IND)**[IMO5] Find all functions f from the reals to the reals such that

$$(f(x)+f(z))(f(y)+f(t)) = f(xy-zt)+f(xt+yz)$$

for all real x, y, z, t.

19. **A5 (IND)** Let n be a positive integer that is not a perfect cube. Define real numbers a, b, c by

$$a = \sqrt[3]{n}, \qquad b = \frac{1}{a-[a]}, \qquad c = \frac{1}{b-[b]},$$

where $[x]$ denotes the integer part of x. Prove that there are infinitely many such integers n with the property that there exist integers r, s, t, not all zero, such that $ra+sb+tc=0$.

20. **A6 (IRN)** Let A be a nonempty set of positive integers. Suppose that there are positive integers $b_1, \dots, b_n$ and $c_1, \dots, c_n$ such that
(i) for each i the set $b_iA+c_i = \{b_ia+c_i \mid a \in A\}$ is a subset of A, and
(ii) the sets b_iA+c_i and b_jA+c_j are disjoint whenever $i \neq j$.
Prove that
$$\frac{1}{b_1}+\cdots+\frac{1}{b_n} \le 1.$$

21. **C1 (COL)**[IMO1] Let n be a positive integer. Each point (x, y) in the plane, where x and y are nonnegative integers with $x+y \le n$, is colored red or blue, subject to the following condition: If a point (x, y) is red, then so are all points (x', y') with $x' \le x$ and $y' \le y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

22. **C2 (ARM)** For n an odd positive integer, the unit squares of an $n \times n$ chessboard are colored alternately black and white, with the four corners colored black. A *tromino* is an L-shape formed by three connected unit squares. For which values of n is it possible to cover all the black squares with nonoverlapping trominos? When it is possible, what is the minimum number of trominos needed?

23. **C3 (COL)** Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called *full* if it satisfies the following condition: For each positive integer $k \ge 2$, if the number k appears in the sequence, then so does the number $k-1$, and moreover, the first occurrence of $k-1$ comes before the last occurrence of k. For each n, how many full sequences are there?

24. **C4 (BUL)** Let T be the set of ordered triples (x, y, z), where x, y, z are integers with $0 \le x, y, z \le 9$. Players A and B play the following guessing game: Player A chooses a triple (x, y, z) in T, and Player B has to discover A's triple in as few moves as possible. A *move* consists of the following: B gives A a triple (a, b, c) in T, and A replies by giving B the number

$|x+y-a-b|+|y+z-b-c|+|z+x-c-a|$. Find the minimum number of moves that B needs to be sure of determining A's triple.

25. **C5 (BRA)** Let $r \geq 2$ be a fixed positive integer, and let $\mathcal{F}$ be an infinite family of sets, each of size r, no two of which are disjoint. Prove that there exists a set of size $r-1$ that meets each set in $\mathcal{F}$.

26. **C6 (POL)** Let n be an even positive integer. Show that there is a permutation $x_1, x_2, \ldots, x_n$ of $1, 2, \ldots, n$ such that for every $1 \leq i \leq n$ the number x_{i+1} is one of $2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1$ (where we take $x_{n+1} = x_1$).

27. **C7 (NZL)** Among a group of 120 people, some pairs are friends. A *weak quartet* is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets?

3.44 The Forty-Fourth IMO
Tokyo, Japan, July 7–19, 2003

3.44.1 Contest Problems

First Day (July 13)

1. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets
$$A_j = \{x + t_j | x \in A\}, \qquad j = 1, 2, \ldots, 100,$$
are pairwise disjoint.
2. Determine all pairs (a, b) of positive integers such that
$$\frac{a^2}{2ab^2 - b^3 + 1}$$
is a positive integer.
3. Each pair of opposite sides of a convex hexagon has the following property: The distance between their midpoints is equal to $\sqrt{3}/2$ times the sum of their lengths.
Prove that all the angles of the hexagon are equal.

Second Day (July 14)

4. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.
5. Let n be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers.
(a) Prove that
$$\left(\sum_{i,j=1}^{n} |x_i - x_j| \right)^2 \le \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$
(b) Show that equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic progression.
6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

3.44.2 Shortlisted Problems

1. **A1 (USA)** Let a_{ij}, $i = 1, 2, 3$, $j = 1, 2, 3$, be real numbers such that a_{ij} is positive for $i = j$ and negative for $i \neq j$.
   Prove that there exist positive real numbers c_1, c_2, c_3 such that the numbers
   $$a_{11}c_1 + a_{12}c_2 + a_{13}c_3, \qquad a_{21}c_1 + a_{22}c_2 + a_{23}c_3, \qquad a_{31}c_1 + a_{32}c_2 + a_{33}c_3$$
   are all negative, all positive, or all zero.

2. **A2 (AUS)** Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that
   (i) $f(0) = 0$, $f(1) = 1$;
   (ii) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a, b such that $a < 1 < b$.

3. **A3 (GEO)** Consider pairs of sequences of positive real numbers $a_1 \geq a_2 \geq a_3 \geq \cdots$, $b_1 \geq b_2 \geq b_3 \geq \cdots$ and the sums $A_n = a_1 + \cdots + a_n$, $B_n = b_1 + \cdots + b_n$, $n = 1, 2, \ldots$. For any pair define $c_i = \min\{a_i, b_i\}$ and $C_n = c_1 + \cdots + c_n$, $n = 1, 2, \ldots$.
   (a) Does there exist a pair $(a_i)_{i\geq 1}, (b_i)_{i\geq 1}$ such that the sequences $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are unbounded while the sequence $(C_n)_{n\geq 1}$ is bounded?
   (b) Does the answer to question (1) change by assuming additionally that $b_i = 1/i$, $i = 1, 2, \ldots$?
   Justify your answer.

4. **A4 (IRE)**[IMO5] Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers.
   (a) Prove that
   $$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$
   (b) Show that equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic progession.

5. **A5 (KOR)** Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions:
   (i) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$.
   (ii) $f(x) < f(y)$ for all $1 \leq x < y$.

6. **A6 (USA)** Let n be a positive integer and let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, z_3, \ldots, z_{2n})$ is a sequence of positive real numbers such that
   $$z_{i+j}^2 \geq x_i y_j \qquad \text{for all } 1 \leq i, j \leq n.$$
   Let $M = \max\{z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M+z_2+\cdots+z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1+\cdots+x_n}{n}\right)\left(\frac{y_1+\cdots+y_n}{n}\right).$$

7. **C1 (BRA)**IMO1 Let A be a 101-element subset of the set $S=\{1,2,\dots,1000000\}$. Prove that there exist numbers $t_1,t_2,\dots,t_{100}$ in S such that the sets
$$A_j=\{x+t_j \mid x\in A\},\quad j=1,2,\dots,100,$$
are pairwise disjoint.

8. **C2 (GEO)** Let $D_1,\dots,D_n$ be closed disks in the plane. (A closed disk is a region bounded by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 disks D_i. Prove that there exists disk D_k that intersects at most $7\cdot 2003-1$ other disks D_i.

9. **C3 (LIT)** Let $n\geq 5$ be a given integer. Determine the largest integer k for which there exists a polygon with n vertices (convex or not, with non-self-intersecting boundary) having k internal right angles.

10. **C4 (IRN)** Let $x_1,\dots,x_n$ and $y_1,\dots,y_n$ be real numbers. Let $A=(a_{ij})_{1\leq i,j\leq n}$ be the matrix with entries
$$a_{ij}=\begin{cases}1, & \text{if } x_i+y_j\geq 0;\\ 0, & \text{if } x_i+y_j<0.\end{cases}$$
Suppose that B is an $n\times n$ matrix whose entries are 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A=B$.

11. **C5 (ROM)** Every point with integer coordinates in the plane is the center of a disk with radius $1/1000$.
    (a) Prove that there exists an equilateral triangle whose vertices lie in different disks.
    (b) Prove that every equilateral triangle with vertices in different disks has side length greater than 96.

12. **C6 (SAF)** Let $f(k)$ be the number of integers n that satisfy the following conditions:
    (i) $0\leq n<10^k$, so n has exactly k digits (in decimal notation), with leading zeros allowed;
    (ii) the digits of n can be permuted in such a way that they yield an integer divisible by 11.
    Prove that $f(2m)=10f(2m-1)$ for every positive integer m.

13. **G1 (FIN)**IMO4 Let $ABCD$ be a cyclic quadrilateral. Let P,Q,R be the feet of the perpendiculars from D to the lines BC,CA,AB, respectively. Show that $PQ=QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

14. **G2 (GRE)** Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose center does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

15. **G3 (IND)** Let ABC be a triangle and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, and AB, respectively. Suppose that
$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$
Denote by I_A, I_B, I_C the excenters of the triangle ABC. Prove that P is the circumcenter of the triangle $I_AI_BI_C$.

16. **G4 (ARM)** Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that
$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

17. **G5 (KOR)** Let ABC be an isosceles triangle with $AC = BC$, whose incenter is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

18. **G6 (POL)**$^{\text{IMO3}}$ Each pair of opposite sides of a convex hexagon has the following property: The distance between their midpoints is equal to $\sqrt{3}/2$ times the sum of their lengths.
Prove that all the angles of the hexagon are equal.

19. **G7 (SAF)** Let ABC be a triangle with semiperimeter s and inradius r. The semicircles with diameters BC, CA, AB are drawn outside of the triangle ABC. The circle tangent to all three semicircles has radius t. Prove that
$$\frac{s}{2} < t \le \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$

20. **N1 (POL)** Let m be a fixed integer greater than 1. The sequence $x_0, x_1, x_2, \dots$ is defined as follows:
$$x_i = \begin{cases} 2^i, & \text{if } 0 \le i \le m-1; \\ \sum_{j=1}^{m} x_{i-j}, & \text{if } i \ge m. \end{cases}$$
Find the greatest k for which the sequence contains k consecutive terms divisible by m.

21. **N2 (USA)** Each positive integer a undergoes the following procedure in order to obtain the number $d = d(a)$:
(1) move the last digit of a to the first position to obtain the number b;
(2) square b to obtain the number c;
(3) move the first digit of c to the end to obtain the number d.
(All the numbers in the problem are considered to be represented in base 10.) For example, for $a = 2003$, we have $b = 3200$, $c = 10240000$, and $d = 02400001 = 2400001 = d(2003)$.
Find all numbers a for which $d(a) = a^2$.

22. **N3 (BUL)**[IMO2] Determine all pairs (a, b) of positive integers such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

23. **N4 (ROM)** Let b be an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{11\dots1}_{n-1}\underbrace{22\dots2}_{n}5,$$

written in base b. Prove that the following condition holds if and only if $b = 10$: There exists a positive integer M such that for every integer n greater than M, the number x_n is a perfect square.

24. **N5 (KOR)** An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property: m can be represented in infinitely many ways as a sum of three distinct good integers whose product is the square of an odd integer.

25. **N6 (FRA)**[IMO6] Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

26. **N7 (BRA)** The sequence $a_0, a_1, a_2, \dots$ is defined as follows:

$$a_0 = 2, \qquad a_{k+1} = 2a_k^2 - 1 \quad \text{for } k \geq 0.$$

Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

27. **N8 (IRN)** Let p be a prime number and let A be a set of positive integers that satisfies the following conditions:
(i) the set of prime divisors of the elements in A consists of $p-1$ elements;
(ii) for any nonempty subset of A, the product of its elements is not a perfect pth power.
What is the largest possible number of elements in A?

3.45 The Forty-Fifth IMO
Athens, Greece, July 7–19, 2004

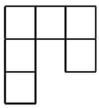
3.45.1 Contest Problems

First Day (July 12)

1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the line segment BC.

2. Find all polynomials $P(x)$ with real coefficients that satisfy the equality
$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c)$$
for all triples a, b, c of real numbers such that $ab + bc + ca = 0$.

3. Determine all $m \times n$ rectangles that can be covered with *hooks* made up of 6 unit squares, as in the figure:



Rotations and reflections of hooks are allowed. The rectangle must be covered without gaps and overlaps. No part of a hook may cover area outside the rectangle.

Second Day (July 13)

4. Let $n \geq 3$ be an integer and $t_1, t_2, \ldots, t_n$ positive real numbers such that
$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$
Show that t_i, t_j, t_k are the side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

5. In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies
$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$
Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

6. We call a positive integer *alternate* if its decimal digits are alternately odd and even. Find all positive integers n such that n has an alternate multiple.

3.45.2 Shortlisted Problems

1. **A1 (KOR)**$^{\text{IMO4}}$ Let $n \geq 3$ be an integer and $t_1, t_2, \ldots, t_n$ positive real numbers such that
$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$
Show that t_i, t_j, t_k are the side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

2. **A2 (ROM)** An infinite sequence $a_0, a_1, a_2, \ldots$ of real numbers satisfies the condition
$$a_n = |a_{n+1} - a_{n+2}| \text{ for every } n \geq 0$$
with a_0 and a_1 positive and distinct. Can this sequence be bounded?

3. **A3 (CAN)** Does there exist a function $s\colon \mathbb{Q} \to \{-1, 1\}$ such that if x and y are distinct rational numbers satisfying $xy = 1$ or $x + y \in \{0, 1\}$, then $s(x)s(y) = -1$? Justify your answer.

4. **A4 (KOR)**$^{\text{IMO2}}$ Find all polynomials $P(x)$ with real coefficients that satisfy the equality
$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c)$$
for all triples a, b, c of real numbers such that $ab + bc + ca = 0$.

5. **A5 (THA)** Let $a, b, c > 0$ and $ab + bc + ca = 1$. Prove the inequality
$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

6. **A6 (RUS)** Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation
$$f\left(x^2 + y^2 + 2f(xy)\right) = (f(x+y))^2 \quad \text{for all } x, y \in \mathbb{R}.$$

7. **A7 (IRE)** Let $a_1, a_2, \ldots, a_n$ be positive real numbers, $n > 1$. Denote by g_n their geometric mean, and by $A_1, A_2, \ldots, A_n$ the sequence of arithmetic means defined by $A_k = \frac{a_1+a_2+\cdots+a_k}{k}$, $k = 1, 2, \ldots, n$. Let G_n be the geometric mean of $A_1, A_2, \ldots, A_n$. Prove the inequality
$$n\sqrt[n]{\frac{G_n}{A_n}} + \frac{g_n}{G_n} \leq n + 1$$
and establish the cases of equality.

8. **C1 (PUR)** There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

(i) Each pair of students are in exactly one club.
(ii) For each student and each society, the student is in exactly one club of the society.
(iii) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

9. **C2 (GER)** Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are distinct. Each intersection point must be colored with one of n distinct colors so that each color is used at least once, and exactly k distinct colors occur on each circle. Find all values of $n \geq 2$ and k for which such a coloring is possible.

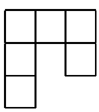
10. **C3 (AUS)** The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices (where each pair of vertices are joined by an edge).

11. **C4 (POL)** Consider a matrix of size $n \times n$ whose entries are real numbers of absolute value not exceeding 1, and the sum of all entries is 0. Let n be an even positive integer. Determine the least number C such that every such matrix necessarily has a row or a column with the sum of its entries not exceeding C in absolute value.

12. **C5 (NZL)** Let N be a positive integer. Two players A and B, taking turns, write numbers from the set $\{1, \dots, N\}$ on a blackboard. A begins the game by writing 1 on his first move. Then, if a player has written n on a certain move, his adversary is allowed to write $n+1$ or $2n$ (provided the number he writes does not exceed N). The player who writes N wins. We say that N is of type A or of type B according as A or B has a winning strategy.
(a) Determine whether $N = 2004$ is of type A or of type B.
(b) Find the least $N > 2004$ whose type is different from that of 2004.

13. **C6 (IRN)** For an $n \times n$ matrix A, let X_i be the set of entries in row i, and Y_j the set of entries in column j, $1 \leq i, j \leq n$. We say that A is *golden* if $X_1, \dots, X_n, Y_1, \dots, Y_n$ are distinct sets. Find the least integer n such that there exists a 2004×2004 golden matrix with entries in the set $\{1, 2, \dots, n\}$.

14. **C7 (EST)**$^{\text{IMO3}}$ Determine all $m \times n$ rectangles that can be covered with *hooks* made up of 6 unit squares, as in the figure:



Rotations and reflections of hooks are allowed. The rectangle must be covered without gaps and overlaps. No part of a hook may cover area outside the rectangle.

15. **C8 (POL)** For a finite graph G, let $f(G)$ be the number of triangles and $g(G)$ the number of tetrahedra formed by edges of G. Find the least constant c such that
$$g(G)^3 \le c \cdot f(G)^4 \text{ for every graph } G.$$

16. **G1 (ROM)**[IMO1] Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the line segment BC.

17. **G2 (KAZ)** The circle Γ and the line ℓ do not intersect. Let AB be the diameter of Γ perpendicular to ℓ, with B closer to ℓ than A. An arbitrary point $C \neq A, B$ is chosen on Γ. The line AC intersects ℓ at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect ℓ at F, and let AF intersect Γ at $G \neq A$. Prove that the reflection of G in AB lies on the line CF.

18. **G3 (KOR)** Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcenters of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extension points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60°$.

19. **G4 (POL)**[IMO5] In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies
$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$
Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

20. **G5 (SMN)** Let $A_1 A_2 \dots A_n$ be a regular n-gon. The points $B_1, \dots, B_{n-1}$ are defined as follows:
   (i) If $i = 1$ or $i = n-1$, then B_i is the midpoint of the side $A_i A_{i+1}$.
   (ii) If $i \neq 1$, $i \neq n-1$, and S is the intersection point of $A_1 A_{i+1}$ and $A_n A_i$, then B_i is the intersection point of the bisector of the angle $A_i S A_{i+1}$ with $A_i A_{i+1}$.

   Prove the equality
$$\angle A_1 B_1 A_n + \angle A_1 B_2 A_n + \cdots + \angle A_1 B_{n-1} A_n = 180°.$$

21. **G6 (GBR)** Let $\mathcal{P}$ be a convex polygon. Prove that there is a convex hexagon that is contained in $\mathcal{P}$ and that occupies at least 75 percent of the area of $\mathcal{P}$.

22. **G7 (RUS)** For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

23. **G8 (SMN)** A cyclic quadrilateral $ABCD$ is given. The lines AD and BC intersect at E, with C between B and E; the diagonals AC and BD intersect at F. Let M be the midpoint of the side CD, and let $N \neq M$ be a point on the circumcircle of the triangle ABM such that $AN/BN = AM/BM$. Prove that the points E, F, and N are collinear.

24. **N1 (BLR)** Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation
$$\tau(an) = n$$
does not have a positive integer solution n.

25. **N2 (RUS)** The function ψ from the set $\mathbb{N}$ of positive integers into itself is defined by the equality
$$\psi(n) = \sum_{k=1}^{n} (k, n), \qquad n \in \mathbb{N},$$
where (k, n) denotes the greatest common divisor of k and n.
    (a) Prove that $\psi(mn) = \psi(m)\psi(n)$ for every two relatively prime $m, n \in \mathbb{N}$.
    (b) Prove that for each $a \in \mathbb{N}$ the equation $\psi(x) = ax$ has a solution.
    (c) Find all $a \in \mathbb{N}$ such that the equation $\psi(x) = ax$ has a unique solution.

26. **N3 (IRN)** A function f from the set of positive integers $\mathbb{N}$ into itself is such that for all $m, n \in \mathbb{N}$ the number $(m^2 + n)^2$ is divisible by $f^2(m) + f(n)$. Prove that $f(n) = n$ for each $n \in \mathbb{N}$.

27. **N4 (POL)** Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by
$$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \ldots$$
has all of its terms relatively prime to m.

28. **N5 (IRN)**$^{\text{IMO6}}$ We call a positive integer *alternate* if its decimal digits are alternately odd and even. Find all positive integers n such that n has an alternate multiple.

29. **N6 (IRE)** Given an integer $n > 1$, denote by P_n the product of all positive integers x less than n and such that n divides $x^2 - 1$. For each $n > 1$, find the remainder of P_n on division by n.

30. **N7 (BUL)** Let p be an odd prime and n a positive integer. In the coordinate plane, eight distinct points with integer coordinates lie on a circle with diameter of length p^n. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by p^{n+1}.

46rd IMO 2005

Problem 1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Problem 2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

Problem 3. Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

Problem 4. Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

Problem 5. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R.
Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

Problem 6. In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

12 July 2006

Problem 1. Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Problem 2. Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Problem 3. Determine the least real number M such that the inequality

$$\left| ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2) \right| \leq M(a^2 + b^2 + c^2)^2$$

holds for all real numbers a, b and c.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

47th INTERNATIONAL MATHEMATICAL OLYMPIAD SLOVENIA 2006

day: **2**

language: **English**

13 July 2006

Problem 4. Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Problem 5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Problem 6. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

July 25, 2007

Problem 1. Real numbers $a_1, a_2, \ldots, a_n$ are given. For each i $(1 \leq i \leq n)$ define

$$d_i = \max\{a_j : 1 \leq j \leq i\} - \min\{a_j : i \leq j \leq n\}$$

and let

$$d = \max\{d_i : 1 \leq i \leq n\}.$$

(a) Prove that, for any real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ such that equality holds in $(*)$.

Problem 2. Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

Problem 3. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

Language: English

July 26, 2007

Problem 4. In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Problem 5. Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

Problem 6. Let n be a positive integer. Consider

$$S = \{(x, y, z) \ : \ x, y, z \in \{0, 1, \ldots, n\}, \ x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

Language: **English** Day: **1**

49th INTERNATIONAL MATHEMATICAL OLYMPIAD
MADRID (SPAIN), JULY 10-22, 2008

Wednesday, July 16, 2008

Problem 1. An acute-angled triangle ABC has orthocentre H. The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with centre the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

Problem 2. (a) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

Problem 3. Prove that there exist infinitely many positive integers n such that $n^2 + 1$ has a prime divisor which is greater than $2n + \sqrt{2n}$.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

49th INTERNATIONAL MATHEMATICAL OLYMPIAD
MADRID (SPAIN), JULY 10-22, 2008

Thursday, July 17, 2008

Problem 4. Find all functions $f : (0, \infty) \to (0, \infty)$ (so, f is a function from the positive real numbers to the positive real numbers) such that

$$\frac{\big(f(w)\big)^2 + \big(f(x)\big)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 5. Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled 1, 2, $\ldots$, $2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of *steps*: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off.

Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine the ratio N/M.

Problem 6. Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Wednesday, July 15, 2009

Problem 1. Let n be a positive integer and let $a_1, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $\{1, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

Problem 2. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Problem 3. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \qquad \text{and} \qquad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Thursday, July 16, 2009

Problem 4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

Problem 5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a,\ f(b)\ \text{ and }\ f(b+f(a)-1).$$

(A triangle is *non-degenerate* if its vertices are not collinear.)

Problem 6. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

Wednesday, July 7, 2010

Problem 1. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the equality

$$f\big(\lfloor x \rfloor y\big) = f(x)\big\lfloor f(y) \big\rfloor$$

holds for all $x, y \in \mathbb{R}$. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

Problem 2. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc $\widehat{BDC}$ and F a point on the side BC such that

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.

Problem 3. Let $\mathbb{N}$ be the set of positive integers. Determine all functions $g\colon \mathbb{N} \to \mathbb{N}$ such that

$$\big(g(m) + n\big)\big(m + g(n)\big)$$

is a perfect square for all $m, n \in \mathbb{N}$.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Thursday, July 8, 2010

Problem 4. Let P be a point inside the triangle ABC. The lines AP, BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M respectively. The tangent to Γ at C intersects the line AB at S. Suppose that $SC = SP$. Prove that $MK = ML$.

Problem 5. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed:

Type 1: Choose a nonempty box B_j with $1 \leq j \leq 5$. Remove one coin from B_j and add two coins to B_{j+1}.

Type 2: Choose a nonempty box B_k with $1 \leq k \leq 4$. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2}.

Determine whether there is a finite sequence of such operations that results in boxes B_1, B_2, B_3, B_4, B_5 being empty and box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{b^c} = a^{(b^c)}$.)

Problem 6. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers. Suppose that for some positive integer s, we have

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\}$$

for all $n > s$. Prove that there exist positive integers ℓ and N, with $\ell \leq s$ and such that $a_n = a_\ell + a_{n-\ell}$ for all $n \geq N$.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

International Mathematical Olympiad Amsterdam 2011

Language: **English**

Day: **1**

Monday, July 18, 2011

Problem 1. Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \le i < j \le 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

Problem 2. Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.
Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x + y) \le yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \le 0$.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

International Mathematical Olympiad Amsterdam 2011

Language: **English**

Day: 2

Tuesday, July 19, 2011

Problem 4. Let $n > 0$ be an integer. We are given a balance and n weights of weight 2^0, $2^1, \ldots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed.
Determine the number of ways in which this can be done.

Problem 5. Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

Problem 6. Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Tuesday, July 10, 2012

Problem 1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC.

Prove that M is the midpoint of ST.

(The *excircle* of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 2. Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n \;>\; n^n.$$

Problem 3. The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

1. If $n \geq 2^k$, then B can guarantee a win.
2. For all sufficiently large k, there exists an integer $n \geq 1.99^k$ such that B cannot guarantee a win.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Wednesday, July 11, 2012

Problem 4. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a+b+c=0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 \;=\; 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here $\mathbb{Z}$ denotes the set of integers.)

Problem 5. Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.

Show that $MK = ML$.

Problem 6. Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} \;=\; \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} \;=\; 1.$$

Tuesday, July 23, 2013

Problem 1. Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, \ldots, m_k$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right)\cdots\left(1 + \frac{1}{m_k}\right).$$

Problem 2. A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:

- no line passes through any point of the configuration;
- no region contains points of both colours.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Problem 3. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

The excircle *of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined.*

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Language: **English**

Day: **2**

Wednesday, July 24, 2013

Problem 4. Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Problem 5. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f\colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;

(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;

(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

Problem 6. Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.

Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

Tuesday, July 8, 2014

Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}.$$

Problem 2. Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90° .$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Wednesday, July 9, 2014

Problem 4. Points P and Q lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

Problem 5. For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Problem 6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

Friday, July 10, 2015

Problem 1. We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points A, B and C in $\mathcal{S}$, there is no point P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geqslant 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geqslant 3$ for which there exists a balanced centre-free set consisting of n points.

Problem 2. Determine all triples (a, b, c) of positive integers such that each of the numbers

$$ab - c, \quad bc - a, \quad ca - b$$

is a power of 2.

(*A power of* 2 *is an integer of the form* 2^n, *where* n *is a non-negative integer.*)

Problem 3. Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocentre, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$, and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different, and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Language: **English**

Day: **2**

Saturday, July 11, 2015

Problem 4. Triangle ABC has circumcircle Ω and circumcentre O. A circle Γ with centre A intersects the segment BC at points D and E, such that B, D, E and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Problem 5. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x + f(x+y)\big) + f(xy) \;=\; x + f(x+y) + yf(x)$$

for all real numbers x and y.

Problem 6. The sequence $a_1, a_2, \ldots$ of integers satisfies the following conditions:

(i) $1 \leqslant a_j \leqslant 2015$ for all $j \geqslant 1$;

(ii) $k + a_k \neq \ell + a_\ell$ for all $1 \leqslant k < \ell$.

Prove that there exist two positive integers b and N such that

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leqslant 1007^2$$

for all integers m and n satisfying $n > m \geqslant N$.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Monday, July 11, 2016

Problem 1. Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen such that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that lines BD, FX, and ME are concurrent.

Problem 2. Find all positive integers n for which each cell of an $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

- in each row and each column, one third of the entries are I, one third are M and one third are O; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Note: The rows and columns of an $n \times n$ table are each labelled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integers (i, j) with $1 \leqslant i, j \leqslant n$. For $n > 1$, the table has $4n - 2$ *diagonals* of two types. A diagonal of the first type consists of all cells (i, j) for which $i + j$ is a constant, and a diagonal of the second type consists of all cells (i, j) for which $i - j$ is a constant.

Problem 3. Let $P = A_1A_2 \ldots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \ldots, A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

Tuesday, July 12, 2016

Problem 4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible value of the positive integer b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

Problem 5. The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

Problem 6. There are $n \geqslant 2$ line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfil his wish if n is odd.

(b) Prove that Geoff can never fulfil his wish if n is even.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

58th International Mathematical Olympiad — IMO 2017 — RIO DE JANEIRO - BRAZIL

English (eng), day 1

Tuesday, July 18, 2017

Problem 1. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases} \qquad \text{for each } n \geqslant 0.$$

Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

Problem 2. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

Problem 3. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0, are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.

(i) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

(ii) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

(iii) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds she can ensure that the distance between her and the rabbit is at most 100?

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Wednesday, July 19, 2017

Problem 4. Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Problem 5. An integer $N \geqslant 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

$\vdots$

(N) no one stands between the two shortest players.

Show that this is always possible.

Problem 6. An ordered pair (x, y) of integers is a *primitive point* if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that, for each (x, y) in S, we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

Monday, July 9, 2018

Problem 1. Let Γ be the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).

Problem 2. Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_{n+2}$, such that $a_{n+1} = a_1$ and $a_{n+2} = a_2$, and

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \ldots, n$.

Problem 3. An *anti-Pascal triangle* is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + \cdots + 2018$?

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Tuesday, July 10, 2018

Problem 4. A *site* is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

Problem 5. Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Problem 6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

English (eng), day 1

Tuesday, July 16, 2019

Problem 1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

Problem 2. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Problem 3. A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

> Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Wednesday, July 17, 2019

Problem 4. Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

Problem 5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

Problem 6. Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

4

Solutions

4.1 Solutions to the Contest Problems of IMO 1959

1. The desired result $(14n + 3, 21n + 4) = 1$ follows from

$$3(14n + 3) - 2(21n + 4) = 1.$$

2. For the square roots to be real we must have $2x - 1 \geq 0 \Rightarrow x \geq 1/2$ and $x \geq \sqrt{2x-1} \Rightarrow x^2 \geq 2x - 1 \Rightarrow (x-1)^2 \geq 0$, which always holds. Then we have $\sqrt{x + \sqrt{2x-1}} + \sqrt{x - \sqrt{2x-1}} = c \iff$

$$c^2 = 2x + 2\sqrt{x^2 - \sqrt{2x-1}^2} = 2x + 2|x-1| = \begin{cases} 2, & 1/2 \leq x \leq 1, \\ 4x-2, & x \geq 1. \end{cases}$$

   (a) $c^2 = 2$. The equation holds for $1/2 \leq x \leq 1$.
   (b) $c^2 = 1$. The equation has no solution.
   (c) $c^2 = 4$. The equation holds for $4x - 2 = 4 \Rightarrow x = 3/2$.

3. Multiplying the equality by $4(a\cos^2 x - b\cos x + c)$, we obtain $4a^2\cos^4 x + 2(4ac - 2b^2)\cos^2 x + 4c^2 = 0$. Plugging in $2\cos^2 x = 1 + \cos 2x$ we obtain (after quite a bit of manipulation):

$$a^2\cos^2 2x + (2a^2 + 4ac - 2b^2)\cos 2x + (a^2 + 4ac - 2b^2 + 4c^2) = 0.$$

   For $a = 4$, $b = 2$, and $c = -1$ we get $4\cos^2 x + 2\cos x - 1 = 0$ and $16\cos^2 2x + 8\cos 2x - 4 = 0 \Rightarrow 4\cos^2 2x + 2\cos 2x - 1 = 0$.

4. *Analysis.* Let a and b be the other two sides of the triangle. From the conditions of the problem we have $c^2 = a^2 + b^2$ and $c/2 = \sqrt{ab} \Leftrightarrow 3/2c^2 = a^2 + b^2 + 2ab = (a+b)^2 \Leftrightarrow \sqrt{3/2}c = a + b$. Given a desired $\triangle ABC$ let D be a point on $(AC$ such that $CD = CB$. In that case, $AD = a + b = \sqrt{3/2}c$, and also, since $BC = CD$, it follows that $\angle ADB = 45°$.

   Construction. From a segment of length c we elementarily construct a segment AD of length $\sqrt{3/2}\,c$. We then construct a ray $(DX$ such that

$\angle ADX = 45^\circ$ and a circle $k(A,c)$ that intersects the ray at point B. Finally, we construct the perpendicular from B to AD; point C is the foot of that perpendicular.

Proof. It holds that $AB = c$, and, since $CB = CD$, it also holds that $AC + CB = AC + CD = AD = \sqrt{3/2}\,c$. From this it follows that $\sqrt{AC \cdot CB} = c/2$. Since BC is perpendicular to AD, it follows that $\measuredangle BCA = 90^\circ$. Thus ABC is the desired triangle.

Discussion. Since $AB\sqrt{2} = \sqrt{2}c > \sqrt{3/2}\,c = AD > AB$, the circle k intersects the ray DX in exactly two points, which correspond to two symmetric solutions.

5. (a) It suffices to prove that $AF \perp BC$, since then for the intersection point X we have $\angle AXC = \angle BXF = 90^\circ$, implying that X belongs to the circumcircles of both squares and thus that $X = N$. The relation $AF \perp BC$ holds because from $MA = MC$, $MF = MB$, and $\angle AMC = \angle FMB$ it follows that $\triangle AMF$ is obtained by rotating $\triangle BMC$ by 90° around M.
   (b) Since N is on the circumcircle of $BMFE$, it follows that $\angle ANM = \angle MNB = 45^\circ$. Hence MN is the bisector of $\angle ANB$. It follows that MN passes through the midpoint of the arc $\widehat{AB}$ of the circle with diameter AB (i.e., the circumcircle of $\triangle ABN$) not containing N.
   (c) Let us introduce a coordinate system such that $A = (0,0)$, $B = (b,0)$, and $M = (m,0)$. Setting in general $W = (x_W, y_W)$ for an arbitrary point W and denoting by R the midpoint of PQ, we have $y_R = (y_P + y_Q)/2 = (m+b-m)/4 = b/4$ and $x_R = (x_P+x_Q)/2 = (m+m+b)/4 = (2m+b)/4$, the parameter m varying from 0 to b. Thus the locus of all points R is the closed segment R_1R_2 where $R_1 = (b/4, b/4)$ and $R_2 = (b/4, 3b/4)$.

6. *Analysis.* For $AB \parallel CD$ to hold evidently neither must intersect p and hence constructing lines r in α through A and s in β through C, both being parallel to p, we get that $B \in r$ and $D \in s$. Hence the problem reduces to a planar problem in γ, determined by r and s. Denote by A' the foot of the perpendicular from A to s. Since $ABCD$ is isosceles and has an incircle, it follows $AD = BC = (AB + CD)/2 = A'C$. The remaining parts of the problem are now obvious.

4.2 Solutions to the Contest Problems of IMO 1960

1. Given the number $\overline{acb}$, since $11 \mid \overline{acb}$, it follows that $c = a + b$ or $c = a + b - 11$. In the first case, $a^2 + b^2 + (a + b)^2 = 10a + b$, and in the second case, $a^2 + b^2 + (a + b - 11)^2 = 10(a - 1) + b$. In the first case the LHS is even, and hence $b \in \{0, 2, 4, 6, 8\}$, while in the second case it is odd, and hence $b \in \{1, 3, 5, 7, 9\}$. Analyzing the 10 quadratic equations for a we obtain that the only valid solutions are 550 and 803.

2. The LHS term is well-defined for $x \geq -1/2$ and $x \neq 0$. Furthermore, $4x^2/(1 - \sqrt{1 + 2x})^2 = (1 + \sqrt{1 + 2x})^2$. Since $f(x) = (1 + \sqrt{1 + 2x})^2 - 2x - 9 = 2\sqrt{1 + 2x} - 7$ is increasing and since $f(45/8) = 0$, it follows that the inequality holds precisely for $-1/2 \leq x < 45/8$ and $x \neq 0$.

3. Let $B'C'$ be the middle of the $n = 2k + 1$ segments and let D be the foot of the perpendicular from A to the hypotenuse. Let us assume $\mathcal{B}(C, D, C', B', B)$. Then from $CD < BD$, $CD + BD = a$, and $CD \cdot BD = h^2$ we have $CD^2 - a \cdot CD + h^2 = 0 \implies CD = (a - \sqrt{a^2 - 4h^2})/2$. Let us define $\measuredangle DAC' = \gamma$ and $\measuredangle DAB' = \beta$; then $\tan\beta = DB'/h$ and $\tan\gamma = DC'/h$. Since $DB' = CB' - CD = (k + 1)a/(2k + 1) - (c - \sqrt{c^2 - 4h^2})/2$ and $DC' = ka/(2k + 1) - (c - \sqrt{c^2 - 4h^2})/2$, we have

$$\tan\alpha = \tan(\beta - \gamma) = \frac{\tan\beta - \tan\gamma}{1 + \tan\beta \cdot \tan\gamma} = \frac{\frac{a}{(2k+1)h}}{1 + \frac{a^2 - 4h^2}{4h^2} - \frac{a^2}{4h^2(2k+1)^2}}$$

$$= \frac{4h(2k + 1)}{4ak(k + 1)} = \frac{4nh}{(n^2 - 1)a}\,.$$

The case $\mathcal{B}(C, C', D, B', B)$ is similar.

4. *Analysis.* Let A' and B' be the feet of the perpendiculars from A and B, respectively, to the opposite sides, A_1 the midpoint of BC, and let D' be the foot of the perpendicular from A_1 to AC. We then have $AA_1 = m_a$, $AA' = h_a$, $\angle AA'A_1 = 90^\circ$, $A_1D' = h_b/2$, and $\angle AD'A_1 = 90^\circ$.

   Construction. We construct the quadrilateral $AD'A_1A'$ (starting from the circle with diameter AA_1). Then C is the intersection of $A'A_1$ and AD', and B is on the line A_1C such that $CA_1 = A_1B$ and $\mathcal{B}(B, A_1, C)$.

   Discussion. We must have $m_a \geq h_a$ and $m_a \geq h_b/2$. The number of solutions is 0 if $m_a = h_a = h_b/2$, 1 if two of $m_a, h_a, h_b/2$ are equal, and 2 otherwise.

5. (a) The locus of the points is the square $EFGH$ where these four points are the centers of the faces $ABB'A'$, $BCC'B'$, $CDD'C'$ and $DAA'D'$.
   (b) The locus of the points is the rectangle $IJKL$ where these points are on AB', CB', CD', and AD' at a distance of $AA'/3$ with respect to the plane $ABCD$.

6. Let E, F respectively be the midpoints of the bases AB, CD of the isosceles trapezoid $ABCD$.
   (a) The point P is on the intersection of EF and the circle with diameter BC.
   (b) Let $x = EP$. Since $\triangle BEP \sim \triangle PFC$, we have $x(h - x) = ab/4 \Rightarrow x_{1,2} = (h \pm \sqrt{h^2 - ab})/2$.
   (c) If $h^2 > ab$ there are two solutions, if $h^2 = ab$ there is only one solution, and if $h^2 < ab$ there are no solutions.

7. Let A be the vertex of the cone, O the center of the sphere, S the center of the base of the cone, B a point on the base circle, and r the radius of the sphere. Let $\angle SAB = \alpha$. We easily obtain $AS = r(1+\sin\alpha)/\sin\alpha$ and $SB = r(1+\sin\alpha)\tan\alpha/\sin\alpha$ and hence $V_1 = \pi SB^2 \cdot SA/3 = \pi r^3(1+\sin\alpha)^2/[3\sin\alpha(1-\sin\alpha)]$. We also have $V_2 = 2\pi r^3$ and hence

$$k = \frac{(1+\sin\alpha)^2}{6\sin\alpha(1-\sin\alpha)} \Rightarrow (1+6k)\sin^2\alpha + 2(1-3k)\sin\alpha + 1 = 0 .$$

The discriminant of this quadratic must be nonnegative: $(1-3k)^2 - (1+6k) \geq 0 \Rightarrow k \geq 4/3$. Hence we cannot have $k = 1$. For $k = 4/3$ we have $\sin\alpha = 1/3$, whose construction is elementary.

4.3 Solutions to the Contest Problems of IMO 1961

1. This is a problem solvable using elementary manipulations, so we shall state only the final solutions. For $a = 0$ we get $(x, y, z) = (0, 0, 0)$. For $a \neq 0$ we get $(x, y, z) \in \{(t_1, t_2, z_0), (t_2, t_1, z_0)\}$, where

$$z_0 = \frac{a^2 - b^2}{2a} \quad \text{and} \quad t_{1,2} = \frac{a^2 + b^2 \pm \sqrt{(3a^2 - b^2)(3b^2 - a^2)}}{4a}.$$

For the solutions to be positive and distinct the following conditions are necessary and sufficient: $3b^2 > a^2 > b^2$ and $a > 0$.

2. Using $S = bc \sin \alpha / 2$, $a^2 = b^2 + c^2 - 2bc \cos \alpha$ and $(\sqrt{3} \sin \alpha + \cos \alpha)/2 = \cos(\alpha - 60^\circ)$ we have

$$\begin{gathered} a^2 + b^2 + c^2 \geq 4S\sqrt{3} \iff b^2 + c^2 \geq bc(\sqrt{3} \sin \alpha + \cos \alpha) \iff \\ \iff (b - c)^2 + 2bc(1 - \cos(\alpha - 60^\circ)) \geq 0, \end{gathered}$$

where equality holds if and only if $b = c$ and $\alpha = 60^\circ$, i.e., if the triangle is equilateral.

3. For $n \geq 2$ we have

$$\begin{aligned} 1 = \cos^n x - \sin^n x &\leq |\cos^n x - \sin^n x| \\ &\leq |\cos^n x| + |\sin^n x| \leq \cos^2 x + \sin^2 x = 1. \end{aligned}$$

Hence $\sin^2 x = |\sin^n x|$ and $\cos^2 x = |\cos^n x|$, from which it follows that $\sin x, \cos x \in \{1, 0, -1\} \Rightarrow x \in \pi\mathbb{Z}/2$. By inspection one obtains the set of solutions

$$\{m\pi \mid m \in \mathbb{Z}\} \text{ for even } n \quad \text{and} \quad \{2m\pi, 2m\pi - \pi/2 \mid m \in \mathbb{Z}\} \text{ for odd } n.$$

For $n = 1$ we have $1 = \cos x - \sin x = -\sqrt{2} \sin(x - \pi/4)$, which yields the set of solutions

$$\{2m\pi, 2m\pi - \pi/2 \mid m \in \mathbb{Z}\}.$$

4. Let $x_i = PP_i/PQ_i$ for $i = 1, 2, 3$. For all i we have

$$\frac{1}{x_i + 1} = \frac{PQ_i}{P_iQ_i} = \frac{S_{PP_jP_k}}{S_{P_1P_2P_3}},$$

where the indices j and k are distinct and different from i. Hence we have

$$\begin{aligned} f(x_1, x_2, x_3) &= \frac{1}{x_1 + 1} + \frac{1}{x_2 + 1} + \frac{1}{x_3 + 1} \\ &= \frac{S(PP_2P_3) + S(PP_1P_3) + S(PP_2P_3)}{S(P_1P_2P_3)} = 1. \end{aligned}$$

It follows that $1/(x_i + 1) \geq 1/3$ for some i and $1/(x_j + 1) \leq 1/3$ for some j. Consequently, $x_i \leq 2$ and $x_j \geq 2$.

5. *Analysis.* Let C_1 be the midpoint of AB. In $\triangle AMB$ we have $MC_1 = b/2$, $AB = c$, and $\angle AMB = \omega$. Thus, given $AB = c$, the point M is at the intersection of the circle $k(C', b/2)$ and the set of points e that view AB at an angle of ω. The construction of ABC is now obvious.

   Discussion. It suffices to establish the conditions for which k and e intersect. Let E be the midpoint of one of the arcs that make up e. A necessary and sufficient condition for k to intersect e is

$$\frac{c}{2} = C'A \le \frac{b}{2} \le C'E = \frac{c}{2}\cot\frac{\omega}{2} \;\Leftrightarrow\; b\tan\frac{\omega}{2} \le c < b.$$

6. Let $h(X)$ denote the distance of a point X from ϵ, X restricted to being on the same side of ϵ as A, B, and C. Let G_1 be the (fixed) centroid of $\triangle ABC$ and G_1' the centroid of $\triangle A'B'C'$. It is trivial to prove that G is the midpoint of G_1G_1'. Hence varying G_1' across ϵ, we get that the locus of G is the plane α parallel to ϵ such that

$$X \in \alpha \Leftrightarrow h(X) = \frac{h(G_1)}{2} = \frac{h(A) + h(B) + h(C)}{6}\,.$$

4.4 Solutions to the Contest Problems of IMO 1962

1. From the conditions of the problem we have $n = 10x + 6$ and $4n = 6 \cdot 10^m + x$ for some integer x. Eliminating x from these two equations, we get $40n = 6 \cdot 10^{m+1} + n - 6 \Rightarrow n = 2(10^{m+1} - 1)/13$. Hence we must find the smallest m such that this fraction is an integer. By inspection, this happens for $m = 6$, and for this m we obtain $n = 153846$, which indeed satisfies the conditions of the problem.

2. We note that $f(x) = \sqrt{3-x} - \sqrt{x+1}$ is well-defined only for $-1 \le x \le 3$ and is decreasing (and obviously continuous) on this interval. We also note that $f(-1) = 2 > 1/2$ and $f(1 - \sqrt{31}/8) = \sqrt{(1/4 + \sqrt{31}/4)^2} - \sqrt{(1/4 - \sqrt{31}/4)^2} = 1/2$. Hence the inequality is satisfied for $-1 \le x < 1 - \sqrt{31}/8$.

3. By inspecting the four different stages of this periodic motion we easily obtain that the locus of the midpoints of XY is the edges of $MNCQ$, where M, N, and Q are the centers of $ABB'A'$, $BCC'B'$, and $ABCD$, respectively.

4. Since $\cos 2x = 1 + \cos^2 x$ and $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$, we have $\cos^2 x + \cos^2 2x + \cos^2 3x = 1 \Leftrightarrow \cos 2x + \cos 4x + 2\cos^2 3x = 2\cos 3x(\cos x + \cos 3x) = 0 \Leftrightarrow 4\cos 3x \cos 2x \cos x = 0$. Hence the solutions are $x \in \{\pi/2 + m\pi,\ \pi/4 + m\pi/2,\ \pi/6 + m\pi/3 \mid m \in \mathbb{Z}\}$.

5. *Analysis.* Let $ABCD$ be the desired quadrilateral. Let us assume w.l.o.g. that $AB > BC$ (for $AB = BC$ the construction is trivial). For a tangent quadrilateral we have $AD - DC = AB - BC$. Let X be a point on AD such that $DX = DC$. We then have $AX = AB - BC$ and $\measuredangle AXC = \measuredangle ADC + \measuredangle CDX = 180^\circ - \angle ABC/2$. Constructing X and hence D is now obvious.

6. This problem is a special case, when the triangle is isosceles, of Euler's formula, which holds for all triangles.

7. The spheres are arranged in a similar manner as in the planar case where we have one incircle and three excircles. Here we have one "insphere" and four "exspheres" corresponding to each of the four sides. Each vertex of the tetrahedron effectively has three tangent lines drawn from it to each of the five spheres. Repeatedly using the equality of the three tangent segments from a vertex (in the same vein as for tangent planar quadrilaterals) we obtain $SA + BC = SB + CA = SC + AB$ from the insphere. From the exsphere opposite of S we obtain $SA - BC = SB - CA = SC - AB$, hence $SA = SB = SC$ and $AB = BC = CA$. By symmetry, we also have $AB = AC = AS$. Hence indeed, all the edges of the tetrahedron are equal in length and thus we have shown that the tetrahedron is regular.

4.5 Solutions to the Contest Problems of IMO 1963

1. Obviously, $x \geq 0$; hence squaring the given equation yields an equivalent equation $5x^2 - p - 4 + 4\sqrt{(x^2-1)(x^2-p)} = x^2$, i.e., $4\sqrt{(x^2-1)(x^2-p)} = (p+4) - 4x^2$. If $4x^2 \leq (p+4)$, we may square the equation once again to get $-16(p+1)x^2 + 16p = -8(p+4)x^2 + (p+4)^2$, which is equivalent to $x^2 = (4-p)^2/[4(4-2p)]$, i.e., $x = (4-p)/(2\sqrt{4-2p})$. For this to be a solution we must have $p \leq 2$ and $(4-p)^2/(4-2p) = 4x^2 \leq (p+4)$. Hence $4/3 \leq p \leq 2$. Otherwise there is no solution.

2. Let A be the given point, BC the given segment, and $\mathcal{B}_1, \mathcal{B}_2$ the closed balls with the diameters AB and AC respectively. Consider one right angle $\angle AOK$ with $K \in [BC]$. If B', C' are the feet of the perpendiculars from B, C to AO respectively, then O lies on the segment $B'C'$, which implies that it lies on exactly one of the segments AB', AC'. Hence O belongs to exactly one of the balls $\mathcal{B}_1, \mathcal{B}_2$; i.e., $O \in \mathcal{B}_1 \Delta \mathcal{B}_2$. This is obviously the required locus.

3. Let $\overrightarrow{OA_1}, \overrightarrow{OA_2}, \dots, \overrightarrow{OA_n}$ be the vectors corresponding respectively to the edges $a_1, a_2, \dots, a_n$ of the polygon. By the conditions of the problem, these vectors satisfy $\overrightarrow{OA_1} + \cdots + \overrightarrow{OA_n} = \overrightarrow{0}$, $\angle A_1OA_2 = \angle A_2OA_3 = \cdots = \angle A_nOA_1 = 2\pi/n$ and $OA_1 \geq OA_2 \geq \cdots \geq OA_n$. Our task is to prove that $OA_1 = \cdots = OA_n$.
   Let l be the line through O perpendicular to OA_n, and $B_1, \dots, B_{n-1}$ the projections of $A_1, \dots, A_{n-1}$ onto l respectively. By the assumptions, the sum of the $\overrightarrow{OB_i}$'s is $\overrightarrow{0}$. On the other hand, since $OB_i \leq OB_{n-i}$ for all $i \leq n/2$, all the sums $\overrightarrow{OB_i} + \overrightarrow{OB_{n-i}}$ lie on the same side of the point O. Hence all these sums must be equal to $\overrightarrow{0}$. Consequently, $OA_i = OA_{n-i}$, from which the result immediately follows.

4. Summing up all the equations yields $2(x_1 + x_2 + x_3 + x_4 + x_5) = y(x_1 + x_2 + x_3 + x_4 + x_5)$. If $y = 2$, then the given equations imply $x_1 - x_2 = x_2 - x_3 = \cdots = x_5 - x_1$; hence $x_1 = x_2 = \cdots = x_5$, which is clearly a solution. If $y \neq 2$, then $x_1 + \cdots + x_5 = 0$, and summing the first three equalities gives $x_2 = y(x_1 + x_2 + x_3)$. Using that $x_1 + x_3 = yx_2$ we obtain $x_2 = (y^2 + y)x_2$, i.e., $(y^2 + y - 1)x_2 = 0$. If $y^2 + y - 1 \neq 0$, then $x_2 = 0$, and similarly $x_1 = \cdots = x_5 = 0$. If $y^2 + y - 1 = 0$, it is easy to prove that the last two equations are the consequence of the first three. Thus choosing any values for x_1 and x_5 will give exactly one solution for x_2, x_3, x_4.

5. The LHS of the desired identity equals $S = \cos(\pi/7) + \cos(3\pi/7) + \cos(5\pi/7)$. Now
$$S \sin\frac{\pi}{7} = \frac{\sin\frac{2\pi}{7}}{2} + \frac{\sin\frac{4\pi}{7} - \sin\frac{2\pi}{7}}{2} + \frac{\sin\frac{6\pi}{7} - \sin\frac{4\pi}{7}}{2} = \frac{\sin\frac{6\pi}{7}}{2} \Rightarrow S = \frac{1}{2}.$$

6. The result is $EDACB$.

4.6 Solutions to the Contest Problems of IMO 1964

1. Let $n = 3k + r$, where $0 \le r < 2$. Then $2^n = 2^{3k+r} = 8^k \cdot 2^r \equiv 2^r \pmod 7$. Thus the remainder of 2^n modulo 7 is $1, 2, 4$ if $n \equiv 0, 1, 2 \pmod 3$. Hence $2^n - 1$ is divisible by 7 if and only if $3 \mid n$, while $2^n + 1$ is never divisible by 7.

2. By substituting $a = x + y$, $b = y + z$, and $c = z + x$ $(x, y, z > 0)$ the given inequality becomes
$$6xyz \le x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2,$$
which follows immediately by the AM–GM inequality applied to x^2y, xy^2, x^2z, xz^2, y^2z, yz^2.

3. Let r be the radius of the incircle of $\triangle ABC$, r_a, r_b, r_c the radii of the smaller circles corresponding to A, B, C, and h_a, h_b, h_c the altitudes from A, B, C respectively. The coefficient of similarity between the smaller triangle at A and the triangle ABC is $1 - 2r/h_a$, from which we easily obtain $r_a = (h_a - 2r)r/h_a = (s - a)r/s$. Similarly, $r_b = (s - b)r/s$ and $r_c = (s - c)r/s$. Now a straightforward computation gives that the sum of areas of the four circles is given by
$$\Sigma = \frac{(b + c - a)(c + a - b)(a + b - c)(a^2 + b^2 + c^2)\pi}{(a + b + c)^3}.$$

4. Let us call the topics T_1, T_2, T_3. Consider an arbitrary student A. By the pigeonhole principle there is a topic, say T_3, he discussed with at least 6 other students. If two of these 6 students discussed T_3, then we are done. Suppose now that the 6 students discussed only T_1 and T_2 and choose one of them, say B. By the pigeonhole principle he discussed one of the topics, say T_2, with three of these students. If two of these three students also discussed T_2, then we are done. Otherwise, all the three students discussed only T_1, which completes the task.

5. Let us first compute the number of intersection points of the perpendiculars passing through two distinct points B and C. The perpendiculars from B to the lines through C other than BC meet all perpendiculars from C, which counts to $3 \cdot 6 = 18$ intersection points. Each perpendicular from B to the 3 lines not containing C can intersect at most 5 of the perpendiculars passing through C, which counts to another $3 \cdot 5 = 15$ intersection points. Thus there are $18 + 15 = 33$ intersection points corresponding to B, C.
   It follows that the required total number is at most $10 \cdot 33 = 330$. But some of these points, namely the orthocenters of the triangles with vertices at the given points, are counted thrice. There are 10 such points. Hence the maximal number of intersection points is $330 - 2 \cdot 10 = 310$.

Remark. The jury considered only the combinatorial part of the problem and didn't require an example in which 310 points appear. However, it is "easily" verified that, for instance, the set of points $A(1,1), B(e,\pi)$, $C(e^2,\pi^2), D(e^3,\pi^3), E(e^4,\pi^4)$ works.

6. We shall prove that the statement is valid in the general case, for an arbitrary point D_1 inside $\triangle ABC$. Since D_1 belongs to the plane ABC, there are real numbers a, b, c such that $(a+b+c)\overrightarrow{DD_1} = a\overrightarrow{DA}+b\overrightarrow{DB}+c\overrightarrow{DC}$. Since $AA_1 \parallel DD_1$, it holds that $\overrightarrow{AA_1} = k\overrightarrow{DD_1}$ for some $k \in \mathbb{R}$. Now it is easy to get $\overrightarrow{DA_1} = -(b\overrightarrow{DB} + c\overrightarrow{DC})/a$, $\overrightarrow{DB_1} = -(a\overrightarrow{DA} + c\overrightarrow{DC})/b$, and $\overrightarrow{DC_1} = -(a\overrightarrow{DA} + b\overrightarrow{DB})/c$. This implies

$$\overrightarrow{D_1A_1} = -\frac{a^2\overrightarrow{DA} + b(a+2b+c)\overrightarrow{DB} + c(a+b+2c)\overrightarrow{DC}}{a(a+b+c)},$$
$$\overrightarrow{D_1B_1} = -\frac{a(2a+b+c)\overrightarrow{DA} + b^2\overrightarrow{DB} + c(a+b+2c)\overrightarrow{DC}}{b(a+b+c)}, \quad \text{and}$$
$$\overrightarrow{D_1C_1} = -\frac{a(2a+b+c)\overrightarrow{DA} + b(a+2b+c)\overrightarrow{DB} + c^2\overrightarrow{DC}}{c(a+b+c)}.$$

By using

$$6V_{D_1A_1B_1C_1} = \left|\left[\overrightarrow{D_1A_1}, \overrightarrow{D_1B_1}, \overrightarrow{D_1C_1}\right]\right| \quad \text{and} \quad 6V_{DABC} = \left|\left[\overrightarrow{DA}, \overrightarrow{DB}, \overrightarrow{DC}\right]\right|$$

we get

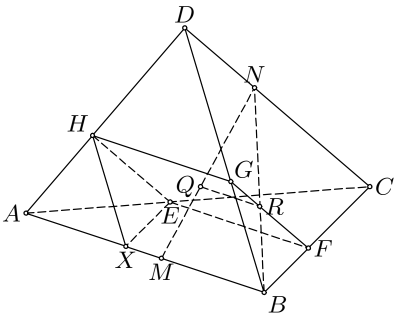
$$V_{D_1A_1B_1C_1} = \frac{\left|\begin{vmatrix} a^2 & b(a+2b+c) & c(a+b+2c) \\ a(2a+b+c) & b^2 & c(a+b+2c) \\ a(2a+b+c) & b(a+2b+c) & c^2 \end{vmatrix}\right|}{6abc(a+b+c)^3} = 3V_{DABC}.$$

4.7 Solutions to the Contest Problems of IMO 1965

1. Let us set $S = \left|\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}\right|$. Observe that $S^2 = 2 - 2\sqrt{1-\sin^2 2x} = 2 - 2|\cos 2x| \le 2$, implying $S \le \sqrt{2}$. Thus the right-hand inequality holds for all x.
   It remains to investigate the left-hand inequality. If $\pi/2 \le x \le 3\pi/2$, then $\cos x \le 0$ and the inequality trivially holds. Assume now that $\cos x > 0$. Then the inequality is equivalent to $2 + 2\cos 2x = 4\cos^2 x \le S^2 = 2 - 2|\cos 2x|$, which is equivalent to $\cos 2x \le 0$, i.e., to $x \in [\pi/4, \pi/2] \cup [3\pi/2, 7\pi/4]$. Hence the solution set is $\pi/4 \le x \le 7\pi/4$.

2. Suppose that (x_1, x_2, x_3) is a solution. We may assume w.l.o.g. that $|x_1| \ge |x_2| \ge |x_3|$. Suppose that $|x_1| > 0$. From the first equation we obtain that
$$0 = |x_1| \cdot \left|a_{11} + a_{12}\frac{x_2}{x_1} + a_{13}\frac{x_3}{x_1}\right| \ge |x_1| \cdot (a_{11} - |a_{12}| - |a_{13}|) > 0,$$
   which is a contradiction. Hence $|x_1| = 0$ and consequently $x_1 = x_2 = x_3 = 0$.

3. Let d denote the distance between the lines AB and CD. Being parallel to AB and CD, the plane π intersects the faces of the tetrahedron in a parallelogram $EFGH$. Let $X \in AB$ be a points such that $HX \parallel DB$.
   Clearly $V_{AEHBFG} = V_{AXEH} + V_{XEHBFG}$. Let MN be the common perpendicular to lines AB and CD ($M \in AB$, $N \in CD$) and let MN, BN meet the plane π at Q and R respectively. Then it holds that $BR/RN = MQ/QN = k$ and consequently $AX/XB = AE/EC = AH/HD = BF/FC = BG/GD = k$. Now we have $V_{AXEH}/V_{ABCD} = k^3/(k+1)^3$.



   Furthermore, if $h = 3V_{ABCD}/S_{ABC}$ is the height of $ABCD$ from D, then
$$V_{XEHBFG} = \frac{1}{2}S_{XBFE}\frac{k}{k+1}h \quad \text{and}$$
$$S_{XBFE} = S_{ABC} - S_{AXE} - S_{EFC} = \frac{(k+1)^2 - 1 - k^2}{(k+1)^2} = \frac{2k}{(1+k)^2}.$$
   These relations give us $V_{XEHBFG}/V_{ABCD} = 3k^2/(1+k)^3$. Finally,
$$\frac{V_{AEHBFG}}{V_{ABCD}} = \frac{k^3 + 3k^2}{(k+1)^3}.$$
   Similarly, $V_{CEFDHG}/V_{ABCD} = (3k+1)/(k+1)^3$, and hence the required ratio is $(k^3 + 3k^2)/(3k+1)$.

4. It is easy to see that all x_i are nonzero. Let $x_1x_2x_3x_4 = p$. The given system of equations can be rewritten as $x_i + p/x_i = 2$, $i = 1, 2, 3, 4$. The equation $x + p/x = 2$ has at most two real solutions, say y and z. Then each x_i is equal either to y or to z. There are three cases:
   (i) $x_1 = x_2 = x_3 = x_4 = y$. Then $y + y^3 = 2$ and hence $y = 1$.
   (ii) $x_1 = x_2 = x_3 = y$, $x_4 = z$. Then $z + y^3 = y + y^2 z = 2$. It is easy to obtain that the only possibilities for (y, z) are $(-1, 3)$ and $(1, 1)$.
   (iii) $x_1 = x_2 = y$, $x_3 = x_4$. In this case the only possibility is $y = z = 1$.

   Hence the solutions for (x_1, x_2, x_3, x_4) are $(1, 1, 1, 1)$, $(-1, -1, -1, 3)$, and the cyclic permutations.

5. (a) Let A' and B' denote the feet of the perpendiculars from A and B to OB and OA respectively. We claim that $H \in A'B'$. Indeed, since $MPHQ$ is a parallelogram, we have $B'P/B'A = BM/BA = MQ/AA' = PH/AA'$, which implies by Thales's theorem that $H \in A'B'$. It is easy to see that the locus of H is the whole segment $A'B'$.
   (b) In this case the locus of points H is obviously the interior of the triangle $OA'B'$.

6. We recall the simple statement that every two diameters of a set must have a common point.
   Consider any point B that is an endpoint of $k \geq 2$ diameters $BC_1, BC_2, \dots, BC_k$. We may assume w.l.o.g. that all the points $C_1, \dots, C_k$ lie on the arc C_1C_k, whose center is B and measure does not exceed $60°$. We observe that for $1 < i < k$ any diameter with the endpoint C_i has to intersect both the diameters C_1B and C_lB. Hence C_iB is the only diameter with an endpoint at C_i if $i = 2, \dots, k-1$. In other words, with each point that is an endpoint of $k \geq 2$ we can associate $k - 2$ points that are endpoints of exactly one diameter.
   We now assume that each A_i is an endpoint of exactly $k_i \geq 0$ diameters, and that $k_1, \dots, k_s \geq 2$, while $k_{s+1}, \dots, k_n \leq 1$. The total number D of diameters satisfies the inequality $2D \leq k_1 + k_2 + \cdots + k_s + (n - s)$. On the other hand, by the above consideration we have $(k_1 - 2) + \cdots + (k_s - 2) \leq n - s$, i.e., $k_1 + \cdots + k_s \leq n + s$. Hence $2D \leq (n + s) + (n - s) = 2n$, which proves the result.

4.8 Solutions to the Contest Problems of IMO 1966

1. Let $N_a, N_b, N_c, N_{ab}, N_{ac}, N_{bc}, N_{abc}$ denote the number of students who solved exactly the problems whose letters are stated in the index of the variable. From the conditions of the problem we have

$$N_a + N_b + N_c + N_{ab} + N_{bc} + N_{ac} + N_{abc} = 25,$$

$$N_b + N_{bc} = 2(N_c + N_{bc}), \qquad N_a - 1 = N_{ac} + N_{abc} + N_{ab}, \qquad N_a = N_b + N_c.$$

From the first and third equations we get $2N_a + N_b + N_c + N_{bc} = 26$, and from the second and fourth we get $4N_b + N_c = 26$ and thus $N_b \leq 6$. On the other hand, we have from the second equation $N_b = 2N_c + N_{bc} \Rightarrow N_c \leq N_b/2 \Rightarrow 26 \leq 9N_b/2 \Rightarrow N_b \geq 6$; hence $N_b = 6$.

2. Angles α and β are less than $90°$, otherwise if w.l.o.g. $\alpha \geq 90°$ we have $\tan(\gamma/2) \cdot (a \tan\alpha + b \tan\beta) < b \tan(\gamma/2) \tan\beta \leq b \tan(\gamma/2) \cot(\gamma/2) = b < a + b$. Since $a \geq b \Leftrightarrow \tan a \geq \tan b$, Chebyshev's inequality gives $a \tan\alpha + b \tan\beta \geq (a+b)(\tan\alpha + \tan\beta)/2$. Due to the convexity of the tan function we also have $(\tan\alpha + \tan\beta)/2 \geq \tan[(\alpha+\beta)/2] = \cot(\gamma/2)$. Hence we have

$$\begin{aligned} \tan\frac{\gamma}{2}(a\tan\alpha + b\tan\beta) &\geq \frac{1}{2}\tan\frac{\gamma}{2}(a+b)(\tan\alpha + \tan\beta) \\ &\geq \tan\frac{\gamma}{2}(a+b)\cot\frac{\gamma}{2} = a+b. \end{aligned}$$

The equalities can hold only if $a = b$. Thus the triangle is isosceles.

3. Consider a coordinate system in which the points of the regular tetrahedron are placed at $A(-a,-a,-a)$, $B(-a,a,a)$, $C(a,-a,a)$ and $D(a,a,-a)$. Then the center of the tetrahedron is at $O(0,0,0)$. For a point $X(x,y,z)$ the sum $XA + XB + XC + XD$ by the QM–AM inequality does not exceed $2\sqrt{XA^2 + XB^2 + XC^2 + XD^2}$. Now, since $XA^2 = (x+a)^2 + (y+a)^2 + (z+a)^2$ etc., we easily obtain

$$\begin{aligned} XA^2 + XB^2 + XC^2 + XD^2 &= 4(x^2+y^2+z^2) + 12a^2 \\ &\geq 12a^2 = OA^2 + OB^2 + OC^2 + OD^2. \end{aligned}$$

Hence $XA + XB + XC + XD \geq 2\sqrt{OA^2 + OB^2 + OC^2 + OD^2} = OA + OB + OC + OD$.

4. It suffices to prove $1/\sin 2^k x = \cot 2^{k-1}x - \cot 2^k x$ for any integer k and real x, i.e., $1/\sin 2x = \cot x - \cot 2x$ for all real x. We indeed have

$$\cot x - \cot 2x = \cot x - \frac{\cot^2 x - 1}{2\cot x} = \frac{\left(\frac{\cos x}{\sin x}\right)^2 + 1}{2\frac{\cos x}{\sin x}} = \frac{1}{2\sin x\cos x} = \frac{1}{\sin 2x}.$$

5. We define $L_1 = |a_1 - a_2|x_2 + |a_1 - a_3|x_3 + |a_1 - a_4|x_4$ and analogously L_2, L_3, and L_4. Let us assume w.l.o.g. that $a_1 < a_2 < a_3 < a_4$. In that case,

$$\begin{aligned}
2|a_1 - a_2||a_2 - a_3|x_2 &= |a_3 - a_2|L_1 - |a_1 - a_3|L_2 + |a_1 - a_2|L_3 \\
&= |a_3 - a_2| - |a_1 - a_3| + |a_1 - a_2| = 0, \\
2|a_2 - a_3||a_3 - a_4|x_3 &= |a_4 - a_3|L_2 - |a_2 - a_4|L_3 + |a_2 - a_3|L_4 \\
&= |a_4 - a_3| - |a_2 - a_4| + |a_2 - a_3| = 0.
\end{aligned}$$

Hence it follows that $x_2 = x_3 = 0$ and consequently $x_1 = x_4 = 1/|a_1 - a_4|$. This solution set indeed satisfies the starting equations. It is easy to generalize this result to any ordering of a_1, a_2, a_3, a_4.

6. Let S denote the area of $\triangle ABC$. Let A_1, B_1, C_1 be the midpoints of BC, AC, AB respectively. We note that $S_{A_1B_1C} = S_{A_1BC_1} = S_{AB_1C_1} = S_{A_1B_1C_1} = S/4$. Let us assume w.l.o.g. that $M \in [AC_1]$. We then must have $K \in [BA_1]$ and $L \in [CB_1]$. However, we then have $S(KLM) > S(KLC_1) > S(KB_1C_1) = S(A_1B_1C_1) = S/4$. Hence, by the pigeonhole principle one of the remaining three triangles $\triangle MAL$, $\triangle KBM$, and $\triangle LCK$ must have an area less than or equal to $S/4$. This completes the proof.

4.9 Solutions to the Longlisted Problems of IMO 1967

1. Let us denote the nth term of the given sequence by a_n. Then

$$\begin{aligned} a_n &= \frac{1}{3}\left(\frac{10^{3n+3}-10^{2n+3}}{9}+7\frac{10^{2n+2}-10^{n+1}}{9}+\frac{10^{n+2}-1}{9}\right) \\ &= \frac{1}{27}(10^{3n+3}-3\cdot 10^{2n+2}+3\cdot 10^{n+1}-1)=\left(\frac{10^{n+1}-1}{3}\right)^3. \end{aligned}$$

2. $(n!)^{2/n}=((1\cdot 2\cdots n)^{1/n})^2\le\left(\frac{1+2+\cdots+n}{n}\right)^2=\left(\frac{n+1}{2}\right)^2\le\frac{1}{3}n^2+\frac{1}{2}n+\frac{1}{6}$.

3. Consider the function $f:[0,\pi/2]\to\mathbb{R}$ defined by $f(x)=1-x^2/2+x^4/16-\cos x$.
   It is easy to calculate that $f'(0)=f''(0)=f'''(0)=0$ and $f''''(x)=3/2-\cos x$.
   Since $f''''(x)>0$, $f'''(x)$ is increasing. Together with $f'''(0)=0$, this gives $f'''(x)>0$ for $x>0$; hence $f''(x)$ is increasing, etc. Continuing in the same way we easily conclude that $f(x)>0$.

4. (a) Let $ABCD$ be a parallelogram, and K,L the midpoints of segments BC and CD respectively. The sides of $\triangle AKL$ are equal and parallel to the medians of $\triangle ABC$.
   (b) Using the formulas $4m_a^2=2b^2+2c^2-a^2$ etc., it is easy to obtain that $m_a^2+m_b^2=m_c^2$ is equivalent to $a^2+b^2=5c^2$. Then

$$5(a^2+b^2-c^2)=4(a^2+b^2)\ge 8ab.$$

5. If one of x,y,z is equal to 1 or -1, then we obtain solutions $(-1,-1,-1)$ and $(1,1,1)$. We claim that these are the only solutions to the system.
   Let $f(t)=t^2+t-1$. If among x,y,z one is greater than 1, say $x>1$, we have $x<f(x)=y<f(y)=z<f(z)=x$, which is impossible. It follows that $x,y,z\le 1$.
   Suppose now that one of x,y,z, say x, is less than -1. Since $\min_t f(t)=-5/4$, we have $x=f(z)\in[-5/4,-1)$. Also, since $f([-5/4,-1))=(-1,-11/16)\subseteq(-1,0)$ and $f((-1,0))=[-5/4,-1)$, it follows that $y=f(x)\in(-1,0)$, $z=f(y)\in[-5/4,-1)$, and $x=f(z)\in(-1,0)$, which is a contradiction. Therefore $-1\le x,y,z\le 1$.
   If $-1<x,y,z<1$, then $x>f(x)=y>f(y)=z>f(z)=x$, a contradiction. This proves our claim.

6. The given system has two solutions: $(-2,-1)$ and $(-14/3,13/3)$.

7. Let $S_k=x_1^k+x_2^k+\cdots+x_n^k$ and let σ_k, $k=1,2,\dots,n$ denote the kth elementary symmetric polynomial in $x_1,\dots,x_n$. The given system can be written as $S_k=a^k$, $k=1,\dots,n$. Using Newton's formulas

$$k\sigma_k=S_1\sigma_{k-1}-S_2\sigma_{k-2}+\cdots+(-1)^kS_{k-1}\sigma_1+(-1)^{k-1}S_k,\quad k=1,2,\dots,n,$$

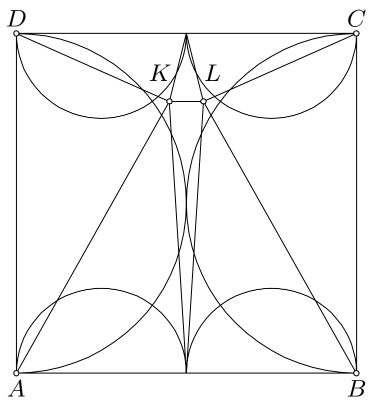
the system easily leads to $\sigma_1 = a$ and $\sigma_k = 0$ for $k = 2, \dots, n$. By Vieta's formulas, $x_1, x_2, \dots, x_n$ are the roots of the polynomial $x^n - ax^{n-1}$, i.e., $a, 0, 0, \dots, 0$ in some order.

Remark. This solution does not use the assumption that the x_j's are real.

8. The circles K_A, K_B, K_C, K_D cover the parallelogram if and only if for every point X inside the parallelogram, the length of one of the segments XA, XB, XC, XD does not exceed 1.
   Let O and r be the center and radius of the circumcircle of $\triangle ABD$. For every point X inside $\triangle ABD$, it holds that $XA \le r$ or $XB \le r$ or $XD \le r$. Similarly, for X inside $\triangle BCD$, $XB \le r$ or $XC \le r$ or $XD \le r$. Hence K_A, K_B, K_C, K_D cover the parallelogram if and only if $r \le 1$, which is equivalent to $\angle ABD \ge 30°$. However, this last is exactly equivalent to $a = AB = 2r \sin \angle ADB \le 2 \sin(\alpha + 30°) = \sqrt{3} \sin \alpha + \cos \alpha$.

9. The incenter of any such triangle lies inside the circle k. We shall show that every point S interior to the circle S is the incenter of one such triangle. If S lies on the segment AB, then it is obviously the incenter of an isosceles triangle inscribed in k that has AB as an axis of symmetry. Let us now suppose S does not lie on AB. Let X and Y be the intersection points of lines AS and BS with k, and let Z be the foot of the perpendicular from S to AB. Since the quadrilateral $BZSX$ is cyclic, we have $\angle ZXS = \angle ABS = \angle SXY$ and analogously $\angle ZYS = \angle SYX$, which implies that S is the incenter of $\triangle XYZ$.

10. Let n be the number of triangles and let b and i be the numbers of vertices on the boundary and in the interior of the square, respectively.
    Since all the triangles are acute, each of the vertices of the square belongs to at least two triangles. Additionally, every vertex on the boundary belongs to at least three, and every vertex in the interior belongs to at least five triangles. Therefore
$$3n \ge 8 + 3b + 5i. \tag{1}$$
    Moreover, the sum of angles at any vertex that lies in the interior, on the boundary, or at a vertex of the square is equal to $2\pi, \pi, \pi/2$ respectively. The sum of all angles of the triangles equals $n\pi$, which gives us $n\pi = 4 \cdot \pi/2 + b\pi + 2i\pi$, i.e., $n = 2 + b + 2i$. This relation together with (1) easily yields that $i \ge 2$. Since each of the vertices inside the square belongs to at least five triangles, and at most two contain both, it follows that $n \ge 8$.



It is shown in the figure that the square can be decomposed into eight acute triangles. Obviously one of them can have an arbitrarily small perimeter.

11. We have to find the number p_n of triples of positive integers (a,b,c) satisfying $a \le b \le c \le n$ and $a+b>c$. Let us denote by $p_n(k)$ the number of such triples with $c=k$, $k=1,2,\dots,n$. For k even, $p_n(k)=k+(k-2)+(k-4)+\cdots+2=(k^2+2k)/4$, and for k odd, $p_n(k)=(k^2+2k+1)/4$. Hence
$$p_n = p_n(1)+p_n(2)+\cdots+p_n(n) = \begin{cases} n(n+2)(2n+5)/24, & \text{for } 2 \mid n, \\ (n+1)(n+3)(2n+1)/24, & \text{for } 2 \nmid n. \end{cases}$$

12. Let us denote by M_n the set of points of the segment AB obtained from A and B by not more than n iterations of $(*)$. It can be proved by induction that
$$M_n = \left\{ X \in AB \mid AX = \frac{3k}{4^n} \text{ or } \frac{3k-2}{4^n} \text{ for some } k \in \mathbb{N} \right\}.$$
Thus (a) immediately follows from $M = \bigcup M_n$. It also follows that if $a,b \in \mathbb{N}$ and $a/b \in M$, then $3 \mid a(b-a)$. Therefore $1/2 \notin M$.

13. The maximum area is $3\sqrt{3}r^2/4$ (where r is the radius of the semicircle) and is attained in the case of a trapezoid with two vertices at the endpoints of the diameter of the semicircle and the other two vertices dividing the semicircle into three equal arcs.

14. We have that
$$\left|\frac{p}{q}-\sqrt{2}\right| = \frac{|p-q\sqrt{2}|}{q} = \frac{|p^2-2q^2|}{q(p+q\sqrt{2})} \ge \frac{1}{q(p+q\sqrt{2})}, \tag{1}$$
because $|p^2-2q^2| \ge 1$.
The greatest solution to the equation $|p^2-2q^2|=1$ with $p,q \le 100$ is $(p,q)=(99,70)$. It is easy to verify using (1) that $\frac{99}{70}$ best approximates $\sqrt{2}$ among the fractions p/q with $p,q \le 100$.

Second solution. By using some basic facts about Farey sequences one can find that $\frac{41}{29} < \sqrt{2} < \frac{99}{70}$ and that $\frac{41}{29} < \frac{p}{q} < \frac{99}{70}$ implies $p \ge 41+99 > 100$ because $99 \cdot 29 - 41 \cdot 70 = 1$. Of the two fractions $41/29$ and $99/70$, the latter is closer to $\sqrt{2}$.

15. Given that $\tan\alpha \in \mathbb{Q}$, we have that $\tan\beta$ is rational if and only if $\tan\gamma$ is rational, where $\gamma = \beta - \alpha$ and $2\gamma = \alpha$. Putting $t = \tan\gamma$ we obtain $\frac{p}{q} = \tan 2\gamma = \frac{2t}{1-t^2}$, which leads to the quadratic equation $pt^2+2qt-p=0$. This equation has rational solutions if and only if its discriminant $4(p^2+q^2)$ is a perfect square, and the result follows.

16. First let us notice that all the numbers $z_{m_1,m_2} = m_1r_1 + m_2r_2$ $(m_1,m_2 \in \mathbb{Z})$ are distinct, since r_1/r_2 is irrational. Thus for any $n \in \mathbb{N}$ the interval $[-n(|r_1|+|r_2|), n(|r_1|+|r_2|)]$ contains $(2n+1)^2$ numbers z_{m_1,m_2},

where $|m_1|, |m_2| \leq n$. Therefore some two of these $(2n+1)^2$ numbers, say z_{m_1,m_2}, z_{n_1,n_2}, differ by at most $\frac{2n(|r_1|+|r_2|)}{(2n+1)^2-1} = \frac{(|r_1|+|r_2|)}{2(n+1)}$. By taking n large enough we can achieve that

$$z_{q_1,q_2} = |z_{m_1,m_2} - z_{n_1,n_2}| \leq p.$$

If now k is the integer such that $kz_{q_1,q_2} \leq x < (k+1)z_{q_1,q_2}$, then $z_{kq_1,kq_2} = kz_{q_1,q_2}$ differs from x by at most p, as desired.

17. Using $c_r - c_s = (r-s)(r+s+1)$ we can easily get

$$\frac{(c_{m+1}-c_k)\cdots(c_{m+n}-c_k)}{c_1c_2\cdots c_n} = \frac{(m-k+n)!}{(m-k)!n!} \cdot \frac{(m+k+n+1)!}{(m+k+1)!(n+1)!}.$$

The first factor $\frac{(m-k+n)!}{(m-k)!n!} = \binom{m-k+n}{n}$ is clearly an integer. The second factor is also an integer because by the assumption, $m+k+1$ and $(m+k)!(n+1)!$ are coprime, and $(m+k+n+1)!$ is divisible by both; hence it is also divisible by their product.

18. In the first part, it is sufficient to show that each rational number of the form $m/n!$, $m, n \in \mathbb{N}$, can be written uniquely in the required form. We prove this by induction on n.
The statement is trivial for $n = 1$. Let us assume it holds for $n-1$, and let there be given a rational number $m/n!$. Let us take $a_n \in \{0, \dots, n-1\}$ such that $m - a_n = nm_1$ for some $m_1 \in \mathbb{N}$. By the inductive hypothesis, there are unique $a_1 \in \mathbb{N}_0$, $a_i \in \{0, \dots, i-1\}$ $(i = 1, \dots, n-1)$ such that $m_1/(n-1)! = \sum_{i=1}^{n-1} a_i/i!$, and then

$$\frac{m}{n!} = \frac{m_1}{(n-1)!} + \frac{a_n}{n!} = \sum_{i=1}^{n} \frac{a_i}{i!},$$

as desired. On the other hand, if $m/n! = \sum_{i=1}^{n} a_i/i!$, multiplying by $n!$ we see that $m - a_n$ must be a multiple of n, so the choice of a_n was unique and therefore the representation itself. This completes the induction.
In particular, since $a_i \mid i!$ and $i!/a_i > (i-1)! \geq (i-1)!/a_{i-1}$, we conclude that each rational q, $0 < q < 1$, can be written as the sum of different reciprocals.
Now we prove the second part. Let $x > 0$ be a rational number. For any integer $m > 10^6$, let $n > m$ be the greatest integer such that $y = x - \frac{1}{m} - \frac{1}{m+1} - \cdots - \frac{1}{n} > 0$. Then y can be written as the sum of reciprocals of different positive integers, which all must be greater than n. The result follows immediately.

19. Suppose $n \leq 6$. Let us decompose the disk by its radii into n congruent regions, so that one of the points P_j lies on the boundaries of two of these regions. Then one of these regions contains two of the n given points. Since the diameter of each of these regions is $2\sin\frac{\pi}{n}$, we have $d_n \leq 2\sin\frac{\pi}{n}$. This

value is attained if P_i are the vertices of a regular n-gon inscribed in the boundary circle. Hence $D_n = 2\sin\frac{\pi}{n}$.
For $n = 7$ we have $D_7 \le D_6 = 1$. This value is attained if six of the seven points form a regular hexagon inscribed in the boundary circle and the seventh is at the center. Hence $D_7 = 1$.

20. The statement so formulated is false. It would be true under the additional assumption that the polygonal line is closed. However, from the offered solution, which is not clear, it does not seem that the proposer had this in mind.

21. Using the formula

$$\cos x \cos 2x \cos 4x \cdots \cos 2^{n-1}x = \frac{\sin 2^n x}{2^n \sin x},$$

which is shown by simple induction, we obtain

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15} = -\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15} = \frac{1}{16},$$

$$\cos\frac{3\pi}{15}\cos\frac{6\pi}{15} = \frac{1}{4}, \qquad \cos\frac{5\pi}{15} = \frac{1}{2}.$$

Multiplying these equalities, we get that the required product P equals $1/128$.

22. Let O_1 and O_2 be the centers of circles k_1 and k_2 and let C be the midpoint of the segment AB. Using the well-known relation for elements of a triangle, we obtain

$$PA^2 + PB^2 = 2PC^2 + 2CA^2 \ge 2O_1C^2 + 2CA^2 = 2O_1A^2 = 2r^2.$$

Equality holds if P coincides with O_1 or if A and B coincide with O_2.

23. Suppose that $a \ge 0$, $c \ge 0$, $4ac \ge b^2$. If $a = 0$, then $b = 0$, and the inequality reduces to the obvious $cg^2 \ge 0$. Also, if $a > 0$, then

$$af^2 + bfg + cg^2 = a\left(f + \frac{b}{2a}g\right)^2 + \frac{4ac - b^2}{4a}g^2 \ge 0.$$

Suppose now that $af^2+bfg+cg^2 \ge 0$ holds for an arbitrary pair of vectors f, g. Substituting f by tg ($t \in \mathbb{R}$) we get that $(at^2 + bt + c)g^2 \ge 0$ holds for any real number t. Therefore $a \ge 0$, $c \ge 0$, $4ac \ge b^2$.

24. Let the kth child receive x_k coins. By the condition of the problem, the number of coins that remain after him was $6(x_k - k)$. This gives us a recurrence relation

$$x_{k+1} = k + 1 + \frac{6(x_k - k) - k - 1}{7} = \frac{6}{7}x_k + \frac{6}{7},$$

which, together with the condition $x_1 = 1 + (m-1)/7$, yields

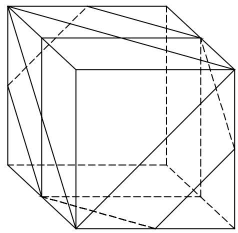
$$x_k = \frac{6^{k-1}}{7^k}(m-36) + 6 \quad \text{for} \quad 1 \le k \le n.$$

Since we are given $x_n = n$, we obtain $6^{n-1}(m-36) = 7^n(n-6)$. It follows that $6^{n-1} \mid n-6$, which is possible only for $n = 6$. Hence, $n = 6$ and $m = 36$.

25. The answer is $R = (4+\sqrt{3})d/6$.

26. Let L be the midpoint of the edge AB. Since P is the orthocenter of $\triangle ABM$ and ML is its altitude, P lies on ML and therefore belongs to the triangular area LCD. Moreover, from the similarity of triangles ALP and MLB we have $LP \cdot LM = LA \cdot LB = a^2/4$, where a is the side length of tetrahedron $ABCD$. It easily follows that the locus of P is the image of the segment CD under the inversion of the plane LCD with center L and radius $a/2$. This locus is the arc of a circle with center L and endpoints at the orthocenters of triangles ABC and ABD.

27. Regular polygons with 3, 4, and 6 sides can be obtained by cutting a cube with a plane, as shown in the figure. A polygon with more than 6 sides cannot be obtained in such a way, for a cube has 6 faces. Also, if a pentagon is obtained by cutting a cube with a plane, then its sides lying on opposite faces are parallel; hence it cannot be regular.



28. The given expression can be transformed into

$$y = \frac{4\cos 2u + 2}{\cos 2u - \cos 2x} - 3.$$

It does not depend on x if and only if $\cos 2u = -1/2$, i.e., $u = \pm\pi/3 + k\pi$ for some $k \in \mathbb{Z}$.

29. Let arc l_a be the locus of points A lying on the opposite side from A_0 with respect to the line B_0C_0 such that $\angle B_0AC_0 = \angle A'$. Let k_a be the circle containing l_a, and let S_a be the center of k_a. We similarly define $l_b, l_c, k_b, k_c, S_b, S_c$. It is easy to show that circles k_a, k_b, k_c have a common point S inside $\triangle ABC$. Let A_1, B_1, C_1 be the points on the arcs l_a, l_b, l_c diametrically opposite to S with respect to S_a, S_b, S_c respectively. Then $A_0 \in B_1C_1$ because $\angle B_1A_0S = \angle C_1A_0S = 90^\circ$; similarly, $B_0 \in A_1C_1$ and $C_0 \in A_1B_1$. Hence the triangle $A_1B_1C_1$ is circumscribed about $\triangle A_0B_0C_0$ and similar to $\triangle A'B'C'$.
Moreover, we claim that $\triangle A_1B_1C_1$ is the triangle ABC with the desired properties having the maximum side BC and hence the maximum area.

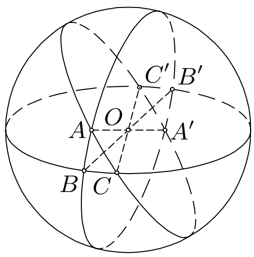
Indeed, if ABC is any other such triangle and S_b', S_c' are the projections of S_b and S_c onto the line BC, it holds that $BC = 2S_b'S_c' \le 2S_bS_c = B_1C_1$, which proves the maximality of B_1C_1.

30. We assume w.l.o.g. that $m \le n$. Let r and s be the numbers of pairs for which $i - j \ge k$ and of those for which $j - i \ge k$. The desired number is $r + s$. We easily find that

$$r = \begin{cases} (m-k)(m-k+1)/2, & k < m, \\ 0, & k \ge m, \end{cases}$$
$$s = \begin{cases} m(2n-2k-m+1)/2, & k < n-m, \\ (n-k)(n-k+1)/2, & n-m \le k < n, \\ 0, & k \ge n. \end{cases}$$

31. Suppose that $n_1 \le n_2 \le \cdots \le n_k$. If $n_k < m$, there is no solution. Otherwise, the solution is $1 + (m-1)(k-s+1) + \sum_{i<s} n_i$, where s is the smallest i for which $m \le n_i$ holds.

32. Let us denote by V the volume of the given body, and by V_a, V_b, V_c the volumes of the parts of the given ball that lie inside the dihedra of the given trihedron. It holds that $V_a = 2R^3\alpha/3$, $V_b = 2R^3\beta/3$, $V_c = 2R^3\gamma/3$. It is easy to see that $2(V_a+V_b+V_c) = 4V + 4\pi R^3/3$, from which it follows that



$$V = \frac{1}{3}R^3(\alpha + \beta + \gamma - \pi).$$

33. If $m \notin \{-2, 1\}$, the system has the unique solution

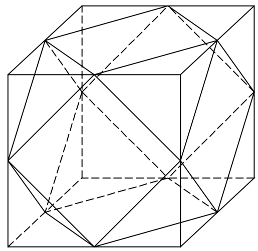
$$x = \frac{b + a - (1+m)c}{(2+m)(1-m)}, \quad y = \frac{a + c - (1+m)b}{(2+m)(1-m)}, \quad z = \frac{b + c - (1+m)a}{(2+m)(1-m)}.$$

The numbers x, y, z form an arithmetic progression if and only if a, b, c do so.

For $m = 1$ the system has a solution if and only if $a = b = c$, while for $m = -2$ it has a solution if and only if $a + b + c = 0$. In both these cases it has infinitely many solutions.

34. Each vertex of the polyhedron is a vertex of exactly two squares and triangles (more than two is not possible; otherwise, the sum of angles at a vertex exceeds 360°). By using the condition that the trihedral angles are equal it is easy to see that such a polyhedron is uniquely determined by its side length.

The polyhedron obtained from a cube by "cutting" its vertices, as shown in the figure, satisfies the conditions.



Now it is easy to calculate that the ratio of the squares of volumes of that polyhedron and of the ball whose boundary is the circumscribed sphere is equal to $25/(8\pi^2)$.

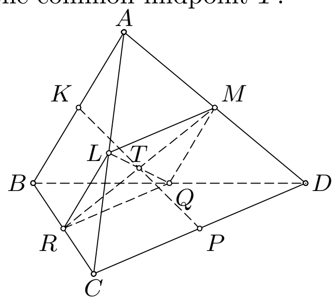
35. The given sum can be rewritten as

$$\sum_{k=0}^{n}\binom{n}{k}\left(\tan^2\frac{x}{2}\right)^k+\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right)^k.$$

Since $\frac{2\tan^2(x/2)}{1-\tan^2(x/2)}=\frac{1-\cos x}{\cos x}$, the above sum is transformed using the binomial formula into

$$\left(1+\tan^2\frac{x}{2}\right)^n+\left(1+\frac{1-\cos x}{\cos x}\right)^n=\sec^{2n}\frac{x}{2}+\sec^n x.$$

36. Suppose that the skew edges of the tetrahedron $ABCD$ are equal. Let K, L, M, P, Q, R be the midpoints of edges AB, AC, AD, CD, DB, BC respectively. Segments KP, LQ, MR have the common midpoint T.



We claim that the lines KP, LQ and MR are axes of symmetry of the tetrahedron $ABCD$. From $LM \parallel CD \parallel RQ$ and similarly $LR \parallel MQ$ and $LM = CD/2 = AB/2 = LR$ it follows that $LMQR$ is a rhombus and therefore $LQ \perp MR$. We similarly show that KP is perpendicular to LQ and MR, and thus it is perpendicular to the plane $LMQR$. Since the lines AB and CD are parallel to the plane $LMQR$, they are perpendicular to KP. Hence the points A and C are symmetric to B and D with respect to the line KP, which means that KP is an axis of symmetry of the tetrahedron $ABCD$. Similarly, so are the lines LQ and MR.

The centers of circumscribed and inscribed spheres of tetrahedron $ABCD$ must lie on every axis of symmetry of the tetrahedron, and hence both coincide with T.

Conversely, suppose that the centers of circumscribed and inscribed spheres of the tetrahedron $ABCD$ coincide with some point T. Then the orthogonal projections of T onto the faces ABC and ABD are the circumcenters O_1 and O_2 of these two triangles, and moreover, $TO_1 = TO_2$.

Pythagoras's theorem gives $AO_1 = AO_2$, which by the law of sines implies $\angle ACB = \angle ADB$. Now it easily follows that the sum of the angles at one vertex of the tetrahedron is equal to 180°. Let D', D'', and D''' be the points in the plane ABC lying outside $\triangle ABC$ such that $\triangle D'BC \cong \triangle DBC$, $\triangle D''CA \cong \triangle DCA$, and $\triangle D'''AB \cong \triangle DAB$. The angle $D''AD'''$ is then straight, and hence A, B, C are midpoints of the segments $D''D''', D'''D', D'D''$ respectively. Hence $AD = D''D'''/2 = BC$, and analogously $AB = CD$ and $AC = BD$.

37. Using the A–G mean inequality we obtain

$$\begin{aligned} 8a^2b^3c^3 &\le 2a^8 + 3b^8 + 3c^8, \\ 8a^3b^2c^3 &\le 3a^8 + 2b^8 + 3c^8, \\ 8a^3b^3c^2 &\le 3a^8 + 3b^8 + 2c^8. \end{aligned}$$

By adding these inequalities and dividing by $3a^3b^3c^3$ we obtain the desired one.

38. Suppose that there exist integers n and m such that $m^3 = 3n^2 + 3n + 7$. Then from $m^3 \equiv 1 \pmod 3$ it follows that $m = 3k + 1$ for some $k \in \mathbb{Z}$. Substituting into the initial equation we obtain $3k(3k^2 + 3k + 1) = n^2 + n + 2$. It is easy to check that $n^2 + n + 2$ cannot be divisible by 3, and so this equality cannot be true. Therefore our equation has no solutions in integers.

39. Since $\sin^2 A + \sin^2 B + \sin^2 C + \cos^2 A + \cos^2 B + \cos^2 C = 3$, the given equality is equivalent to $\cos^2 A + \cos^2 B + \cos^2 C = 1$, which by multiplying by 2 is transformed into

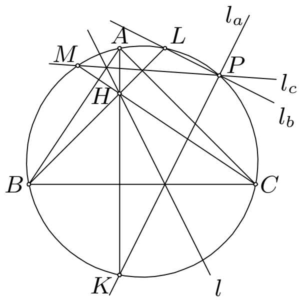
$$\begin{aligned} 0 &= \cos 2A + \cos 2B + 2\cos^2 C = 2\cos(A+B)\cos(A-B) + 2\cos^2 C \\ &= 2\cos C(\cos(A-B) - \cos C). \end{aligned}$$

It follows that either $\cos C = 0$ or $\cos(A - B) = \cos C$. In both cases the triangle is right-angled.

40. Suppose CD is the longest edge of the tetrahedron $ABCD$, $AB = a$, CK and DL are the altitudes of the triangles ABC and ABD respectively, and DM is the altitude of the tetrahedron $ABCD$. Then $CK^2 \le 1 - a^2/4$, since CK is a leg of the right triangle whose other leg has length not less than $a/2$ and whose hypotenuse has length not greater than 1 (AKC or BKC). In the similar way we can show that $DL^2 \le 1 - a^2/4$. Since $DM \le DL$, then $DM^2 \le 1 - a^2/4$. It follows that

$$\begin{aligned} V &= \frac{1}{3}\left(\frac{a}{2}CK\right)DM \le \frac{1}{6}a\left(1 - \frac{a^2}{4}\right) = \frac{1}{24}a(2-a)(2+a) \\ &= \frac{1}{24}[1 - (a-1)^2](2+a) \le \frac{1}{24}\cdot 1 \cdot 3 = \frac{1}{8}. \end{aligned}$$

41. It is well known that the points K, L, M, symmetric to H with respect to BC, CA, AB respectively, lie on the circumcircle k of the triangle ABC. For K, this follows from an elementary calculation of angles of triangles HBC and noting that $\measuredangle KBC = \measuredangle HBC = \measuredangle KAC$. For other points the proof is analogous. Since the lines l_a, l_b pass through K and L and l_b is obtained from l_a by rotation about C for an angle $2\gamma = \angle LCK$, it follows that the intersection point P of l_a and l_b is at the circumcircle of KLC, that is, k. Similarly, l_b and l_c meet at a point on k; hence they must pass through the same point P.



42. $E = (1-\sin x)(1-\cos x)[3+2(\sin x+\cos x)+2\sin x\cos x+\sin x\cos x(\sin x+\cos x)]$.

43. We can write the given equation in the form

$$x^5 - x^3 - 4x^2 - 3x - 2 + \lambda(5x^4 + \alpha x^2 - 8x + \alpha) = 0.$$

A root of this equation is independent of λ if and only if it is a common root of the equations

$$x^5 - x^3 - 4x^2 - 3x - 2 = 0 \quad \text{and} \quad 5x^4 + \alpha x^2 - 8x + \alpha = 0.$$

The first of these two equations is equivalent to $(x-2)(x^2+x+1)^2 = 0$ and has three different roots: $x_1 = 2$, $x_{2,3} = (-1 \pm i\sqrt{3})/2$.
(a) For $\alpha = -64/5$, $x_1 = 2$ is the unique root independent of λ.
(b) For $\alpha = -3$ there are two roots independent of λ: $x_1 = \omega$ and $x_2 = \omega^2$.

44. (a) $S(x,n) = n(n-1)\left[x^2 + (n+1)x + (n+1)(3n+2)/12\right]$.
(b) It is easy to see that the equation $S(x,n) = 0$ has two roots $x_{1,2} = \left(-(n+1) \pm \sqrt{(n+1)/3}\right)/2$. They are integers if and only if $n = 3k^2 - 1$ for some $k \in \mathbb{N}$.

45. (a) Using the formula $4\sin^3 x = 3\sin x - \sin 3x$ one can easily reduce the given equation to $\sin 3x = \cos 2x$. Its solutions are given by $x = (4k+1)\pi/10$, $k \in \mathbb{Z}$.
(b) (1) The point B corresponding to the solution $x = (4k+1)\pi/10$ is a vertex of the regular dodecagon if and only if $(4k+1)\pi/10 = 2m\pi/12$, i.e., $3(4k+1) = 5m$ for some $m \in \mathbb{Z}$. This is possible if and only if $5 \mid 4k+1$, i.e., $k \equiv 1 \pmod 5$.
(2) Similarly, if the point B corresponding to $x = (4k+1)\pi/10$ is a vertex of a polygon P, then $(4k+1)n = 20m$ for some $m \in \mathbb{N}$, which implies that $4 \mid n$.

46. Let us set $\arctan x = a$, $\arctan y = b$, $\arctan z = c$. Then $\tan(a+b) = \frac{x+y}{1-xy}$ and $\tan(a+b+c) = \frac{x+y+z-xyz}{1-yz-zx-xy} = 1$, which implies that

$$(x-1)(y-1)(z-1) = xyz - xy - yz - zx + x + y + z - 1 = 0.$$

One of x, y, z is equal to 1, say $z = 1$, and consequently $x + y = 0$. Therefore

$$x^{2n+1} + y^{2n+1} + z^{2n+1} = x^{2n+1} + (-x)^{2n+1} + 1^{2n+1} = 1.$$

47. Using the A–G mean inequality we get

$$\begin{aligned} (n+k-1)x_1^n x_2 \cdots x_k &\le nx_1^{n+k-1} + x_2^{n+k-1} + \cdots + x_k^{n+k-1}, \\ (n+k-1)x_1 x_2^n \cdots x_k &\le x_1^{n+k-1} + nx_2^{n+k-1} + \cdots + x_k^{n+k-1}, \\ \cdots\cdots & \qquad \cdots\cdots\cdots \\ (n+k-1)x_1 x_2 \cdots x_k^n &\le x_1^{n+k-1} + x_2^{n+k-1} + \cdots + nx_k^{n+k-1}. \end{aligned}$$

By adding these inequalities and dividing by $n + k - 1$ we obtain the desired one.

Remark. This is also an immediate consequence of Muirhead's inequality.

48. Put $f(x) = x \ln x$. The given equation is equivalent to $f(x) = f(1/2)$, which has the solutions $x_1 = 1/2$ and $x_2 = 1/4$. Since the function f is decreasing on $(0, 1/e)$, and increasing on $(1/e, +\infty)$, this equation has no other solutions.

49. Since $\sin 1, \sin 2, \dots, \sin(N+1) \in (-1, 1)$, two of these $N+1$ numbers have distance less than $2/N$. Therefore $|\sin n - \sin k| < 2/N$ for some integers $1 \le k, n \le N+1$, $n \ne k$.

50. Since $\varphi(x,y,z) = f(x+y,z) = \varphi(0,x+y,z) = g(0,x+y+z)$, it is enough to put $h(t) = g(0,t)$.

51. If there exist two numbers $\overline{ab}, \overline{bc} \in S$, then one can fill a crossword puzzle as $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. The converse is obvious. Hence the set S has property A if and only if the set of first digits and the set of second digits of numbers in S are disjoint. Thus the maximum size of S is 25.

52. This problem is not elementary. The solution offered by the proposer was not quite clear and complete (the existence was not proved).

53. (a) We can construct two lines parallel to the rays of the angle, at equal distances from the rays. The intersection of these two lines lies on the bisector of the angle.
    (b) If the length of a segment AB exceeds the breadth of the ruler, we can construct parallel lines through A and B in two different ways. The diagonal in the resulting rhombus is the perpendicular bisector of the segment AB.

If the segment AB is too short, we can construct a line l parallel to AB and centrally project AB onto l from a point C chosen sufficiently close to the segment, thus obtaining an arbitrarily long segment $A'B' \parallel AB$. Then we construct the midpoint D' of $A'B'$ as above. The line $D'C$ intersects the segment AB at its midpoint D. By means of lines parallel to DC the segment AB can be prolonged symmetrically, and then the perpendicular bisector can be found as above.

(c) follows immediately from part (b).

(d) Let there be given a point P and a line l. We draw an arbitrary line through P that intersects l at A, and two lines l_1 and l_2 parallel to AP, at equal distances from AP and on either side of AP. Line l_1 intersects l at B. We can construct the midpoint C of AP. If BC intersects l_2 at D, then PD is parallel to l.

54. Let S be the given set of points on the cube. Let x, y, z denote the numbers of points from S lying at a vertex, at the midpoint of an edge, at the midpoint of a face of the cube, respectively, and let u be the number of all other points from S.
Either there are no points from S at the vertices of the cube, or there is a point from S at each vertex. Hence x is either 0 or 8. Similarly, y is either 0 or 12, and z is either 0 or 6. Any other point of S has 24 possible images under rotations of the cube. Hence u is divisible by 24. Since $n = x + y + z + u$ and $6 \mid y, z, u$, it follows that either $6 \mid n$ or $6 \mid n - 8$, i.e., $n \equiv 0$ or $n \equiv 2 \pmod 6$. Thus $n = 200$ is possible, while $n = 100$ is not, because $n \equiv 4 \pmod 6$.

55. It is enough to find all x from $(0, 2\pi]$ such that the given inequality holds for all n.
Suppose $0 < x < 2\pi/3$. If n is the maximum integer for which $nx \leq 2\pi/3$, we have $\pi/3 < nx \leq 2\pi/3$, and consequently $\sin nx \geq \sqrt{3}/2$. Thus $\sin x + \sin 2x + \cdots + \sin nx > \sqrt{3}/2$.
Suppose now that $2\pi/3 \leq x < 2\pi$. We have

$$\sin x + \cdots + \sin nx = \frac{\cos\frac{x}{2} - \cos\frac{2n+1}{2}x}{2\sin\frac{x}{2}} \leq \frac{\cos\frac{x}{2} + 1}{2\sin\frac{x}{2}} = \frac{\cot\frac{x}{4}}{2} \leq \frac{\sqrt{3}}{2}.$$

For $x = 2\pi$ the given inequality clearly holds for all n. Hence, the inequality holds for all n if and only if $2\pi/3 + 2k\pi \leq x \leq 2\pi + 2k\pi$ for some integer k.

56. We shall prove by induction on n the following statement: If in some group of interpreters exactly n persons, $n \geq 2$, speak each of the three languages, then it is possible to select a subgroup in which each language is spoken by exactly two persons.
The statement of the problem easily follows from this: it suffices to select six such groups.

The case $n = 2$ is trivial. Let us assume $n \geq 2$, and let N_j, N_m, N_f, N_{jm}, N_{jf}, N_{mf}, N_{jmf} be the sets of those interpreters who speak only Japanese, only Malay, only Farsi, only Japanese and Malay, only Japanese and Farsi, only Malay and Farsi, and all the three languages, respectively, and n_j, n_m, n_f, $n_{jm}, n_{jf}, n_{mf}, n_{jmf}$ the cardinalities of these sets, respectively. By the condition of the problem, $n_j+n_{jm}+n_{jf}+n_{jmf} = n_m+n_{jm}+n_{mf}+n_{jmf} = n_f + n_{jf} + n_{mf} + n_{jmf} = 24$, and consequently

$$n_j - n_{mf} = n_m - n_{jf} = n_f - n_{jm} = c.$$

Now if $c < 0$, then $n_{jm}, n_{jf}, n_{mf} > 0$, and it is enough to select one interpreter from each of the sets N_{jm}, N_{jf}, N_{mf}. If $c > 0$, then $n_j, n_m, n_f > 0$, and it is enough to select one interpreter from each of the sets N_j, N_m, N_f and then use the inductive assumption. Also, if $c = 0$, then w.l.o.g. $n_j = n_{mf} > 0$, and it is enough to select one interpreter from each of the sets N_j, N_{mf} and then use the inductive hypothesis. This completes the induction.

57. Obviously $c_n > 0$ for all even n. Thus $c_n = 0$ is possible only for an odd n. Let us assume $a_1 \leq a_2 \leq \cdots \leq a_8$: in particular, $a_1 \leq 0 \leq a_8$.
If $|a_1| < |a_8|$, then there exists n_0 such that for every odd $n > n_0$, $7|a_1|^n < a_8^n \Rightarrow a_1^n + \cdots + a_7^n + a_8^n > 7a_1^n + a_8^n > 0$, contradicting the condition that $c_n = 0$ for infinitely many n. Similarly $|a_1| > |a_8|$ is impossible, and we conclude that $a_1 = -a_8$.
Continuing in the same manner we can show that $a_2 = -a_7$, $a_3 = -a_6$ and $a_4 = -a_5$. Hence $c_n = 0$ for every odd n.

58. The following sequence of equalities and inequalities gives an even stronger estimate than needed.

$$\begin{aligned} |l(z)| = |Az + B| &= \frac{1}{2}|(z+1)(A+B) + (z-1)(A-B)| \\ &= \frac{1}{2}|(z+1)f(1) + (z-1)f(-1)| \\ &\leq \frac{1}{2}(|z+1| \cdot |f(1)| + |z-1| \cdot |f(-1)|) \\ &\leq \frac{1}{2}(|z+1| + |z-1|)M = \frac{1}{2}\rho M. \end{aligned}$$

59. By the arc AB we shall always mean the positive arc AB. We denote by $|AB|$ the length of arc AB. Let a *basic* arc be one of the $n+1$ arcs into which the circle is partitioned by the points $A_0, A_1, \dots, A_n$, where $n \in \mathbb{N}$. Suppose that A_pA_0 and A_0A_q are the basic arcs with an endpoint at A_0, and that x_n, y_n are their lengths, respectively. We show by induction on n that for each n the length of a basic arc is equal to x_n, y_n or $x_n + y_n$.
The statement is trivial for $n = 1$. Assume that it holds for n, and let $A_iA_{n+1}, A_{n+1}A_j$ be basic arcs. We shall prove that these two arcs have lengths x_n, y_n, or x_n+y_n. If i, j are both strictly positive, then $|A_iA_{n+1}| =$

$|A_{i-1}A_n|$ and $|A_{n+1}A_j| = |A_nA_{j-1}|$ are equal to x_n, y_n, or $x_n + y_n$ by the inductive hypothesis.

Let us assume now that $i = 0$, i.e., that A_pA_{n+1} and $A_{n+1}A_0$ are basic arcs. Then $|A_pA_{n+1}| = |A_0A_{n+1-p}| \geq |A_0A_q| = y_n$ and similarly $|A_{n+1}A_q| \geq x_n$, but $|A_pA_q| = x_n + y_n$, from which it follows that $|A_pA_{n+1}| = |A_0A_q| = y_n$ and consequently $n+1 = p+q$. Also, $x_{n+1} = |A_{n+1}A_0| = y_n - x_n$ and $y_{n+1} = y_n$. Now, all basic arcs have lengths $y_n - x_n$, x_n, y_n, $x_n + y_n$. A presence of a basic arc of length $x_n + y_n$ would spoil our inductive step. However, if any basic arc A_kA_l has length $x_n + y_n$, then we must have $l - q = k - p$ because 2π is irrational, and therefore the arc A_kA_l contains either the point A_{k-p} (if $k \geq p$) or the point A_{k+q} (if $k < p$), which is impossible; hence, the proof is complete for $i = 0$. The proof for $j = 0$ is analogous. This completes the induction.

It can be also seen from the above considerations that the basic arcs take only two distinct lengths if and only if $n = p + q - 1$. If we denote by n_k the sequence of n's for which this holds, and by p_k, q_k the sequences of the corresponding p, q, we have $p_1 = q_1 = 1$ and

$$(p_{k+1}, q_{k+1}) = \begin{cases} (p_k + q_k, q_k), \text{ if } \{p_k/(2\pi)\} + \{q_k/(2\pi)\} > 1, \\ (p_k, p_k + q_k), \text{ if } \{p_k/(2\pi)\} + \{q_k/(2\pi)\} < 1. \end{cases}$$

It is now "easy" to calculate that $p_{19} = p_{20} = 333$, $q_{19} = 377$, $q_{20} = 710$, and thus $n_{19} = 709 < 1000 < 1042 = n_{20}$. It follows that the lengths of the basic arcs for $n = 1000$ take exactly three different values.

4.10 Solutions to the Shortlisted Problems of IMO 1968

1. Since the ships are sailing with constant speeds and directions, the second ship is sailing at a constant speed and direction in reference to the first ship. Let A be the constant position of the first ship in this frame. Let B_1, B_2, B_3, and B on line b defining the trajectory of the ship be positions of the second ship with respect to the first ship at 9:00, 9:35, 9:55, and at the moment the two ships were closest. Then we have the following equations for distances (in miles):

$$AB_1 = 20, \quad AB_2 = 15, \quad AB_3 = 13,$$
$$B_1B_2 : B_2B_3 = 7 : 4, \quad AB_i^2 = AB^2 + BB_i^2.$$

Since $BB_1 > BB_2 > BB_3$, it follows that $\mathcal{B}(B_3, B, B_2, B_1)$ or $\mathcal{B}(B, B_3, B_2, B_1)$. We get a system of three quadratic equations with three unknowns: AB, BB_3 and B_3B_2 (BB_3 being negative if $\mathcal{B}(B_3, B, B_1, B_2)$, positive otherwise). This can be solved by eliminating AB and then BB_3. The unique solution ends up being

$$AB = 12, \quad BB_3 = 5, \quad B_3B_2 = 4,$$

and consequently, the two ships are closest at 10:20 when they are at a distance of 12 miles.

2. The sides a, b, c of a triangle ABC with $\angle ABC = 2\angle BAC$ satisfy $b^2 = a(a + c)$ (this statement is the lemma in (SL98-7)). Taking into account the remaining condition that a, b, c are consecutive integers with $a < b$, we obtain three cases:
   (i) $a = n$, $b = n+1$, $c = n+2$. We get the equation $(n+1)^2 = n(2n+2)$, giving us $(a, b, c) = (1, 2, 3)$, which is not a valid triangle.
   (ii) $a = n$, $b = n + 2$, $c = n + 1$. We get $(n + 2)^2 = n(2n + 1) \Rightarrow (n - 4)(n + 1) = 0$, giving us the triangle $(a, b, c) = (4, 6, 5)$.
   (iii) $a = n + 1$, $b = n + 2$, $c = n$. We get $(n + 2)^2 = (n + 1)(2n + 1) \Rightarrow n^2 - n - 3 = 0$, which has no positive integer solutions for n.

   Hence, the only solution is the triangle with sides of lengths 4, 5, and 6.

3. A triangle cannot be formed out of three lengths if and only if one of them is larger than the sum of the other two. Let us assume this is the case for all triplets of edges out of each vertex in a tetrahedron $ABCD$. Let w.l.o.g. AB be the largest edge of the tetrahedron. Then $AB \geq AC + AD$ and $AB \geq BC + BD$, from which it follows that $2AB \geq AC + AD + BC + BD$. This implies that either $AB \geq AC + BC$ or $AB \geq AD + BD$, contradicting the triangle inequality. Hence the three edges coming out of at least one of the vertices A and B form a triangle.

   Remark. The proof can be generalized to prove that in a polyhedron with only triangular surfaces there is a vertex such that the edges coming out of this vertex form a triangle.

4. We will prove the equivalence in the two directions separately:
   ($\Rightarrow$) Suppose $\{x_1, \dots, x_n\}$ is the unique solution of the equation. Since $\{x_n, x_1, x_2 \dots, x_{n-1}\}$ is also a solution, it follows that $x_1 = x_2 = \cdots = x_n = x$ and the system of equations reduces to a single equation $ax^2 + (b-1)x + c = 0$. For the solution for x to be unique the discriminant $(b-1)^2 - 4ac$ of this quadratic equation must be 0.
   ($\Leftarrow$) Assume $(b-1)^2 - 4ac = 0$. Adding up the equations, we get
$$\sum_{i=1}^{n} f(x_i) = 0, \qquad \text{where} \qquad f(x) = ax^2 + (b-1)x + c.$$
   But by the assumed condition, $f(x) = a\left(x + \frac{b-1}{2a}\right)^2$. Hence we must have $f(x_i) = 0$ for all i, and $x_i = -\frac{b-1}{2a}$, which is indeed a solution.

5. We have $h_k = r\cos(\pi/k)$ for all $k \in \mathbb{N}$. Using $\cos x = 1 - 2\sin^2(x/2)$ and $\cos x = 2/(1+\tan^2(x/2)) - 1$ and $\tan x > x > \sin x$ for all $0 < x < \pi/2$, it suffices to prove
$$\begin{aligned} &(n+1)\left(1 - 2\frac{\pi^2}{4(n+1)^2}\right) - n\left(\frac{2}{1+\pi^2/(4n^2)} - 1\right) > 1 \\ \Leftrightarrow\ & 1 + 2n\left(1 - \frac{1}{1+\pi^2/(4n^2)}\right) - \frac{\pi^2}{2(n+1)} > 1 \\ \Leftrightarrow\ & 1 + \frac{\pi^2}{2}\left(\frac{1}{n+\pi^2/(4n)} - \frac{1}{n+1}\right) > 1\,, \end{aligned}$$
   where the last inequality holds because $\pi^2 < 4n$. It is also apparent that as n tends to infinity the term in parentheses tends to 0, and hence it is not possible to strengthen the bound. This completes the proof.

6. We define $f(x) = \frac{a_1}{a_1-x} + \frac{a_2}{a_2-x} + \cdots + \frac{a_n}{a_n-x}$. Let us assume w.l.o.g. $a_1 < a_2 < \cdots < a_n$. We note that for all $1 \le i < n$ the function f is continuous in the interval (a_i, a_{i+1}) and satisfies $\lim_{x\to a_i} f(x) = -\infty$ and $\lim_{x\to a_{i+1}} f(x) = \infty$. Hence the equation $f(x) = n$ will have a real solution in each of the $n-1$ intervals (a_i, a_{i+1}).

   Remark. In fact, this equation has exactly n solutions, and hence they are all real. Moreover, the solutions are distinct if all a_i are of the same sign, since $x = 0$ is an evident solution.

7. Let r_a, r_b, r_c denote the radii of the exscribed circles corresponding to the sides of lengths a, b, c respectively, and R, p and S denote the circumradius, semiperimeter, and area of the given triangle. It is well-known that $r_a(p-a) = r_b(p-b) = r_c(p-c) = S = \sqrt{p(p-a)(p-b)(p-c)} = \frac{abc}{4R}$. Hence, the desired inequality $r_a r_b r_c \le \frac{3\sqrt{3}}{8}abc$ reduces to $p \le \frac{3\sqrt{3}}{2}R$, which is by the law of sines equivalent to
$$\sin\alpha + \sin\beta + \sin\gamma \le \frac{3\sqrt{3}}{2}.$$

This inequality immediately follows from Jensen's inequality, since the sine is concave on $[0, \pi]$. Equality holds if and only if the triangle is equilateral.

8. Let G be the point such that $BCDG$ is a parallelogram and let H be the midpoint of AG. Obviously $HEFD$ is also a parallelogram, and thus $DH = EF = l$. If $AD^2 + BC^2 = m^2$ is fixed, then from the Stewart theorem we have
$$DH^2 = \frac{2DA^2 + 2DG^2 - AG^2}{4} = \frac{2m^2 - AG^2}{4},$$
which is fixed.
Thus G and H are fixed points, and from here the locus of D is a circle with center H and radius l. The locus of B is the segment $(GI]$, where $I \in \Delta$ is a point in the positive direction such that $AI = a$. Finally, the locus of C is a region of the plane consisting of a rectangle sandwiched between two semicircles of radius l centered at points H and H', where H' is a point such that $\overrightarrow{IH'} = \overrightarrow{GH}$.

9. We note that $S_a = ad_a/2$, $S_b = bd_b/2$, and $S_c = cd_c/2$ are the areas of the triangles MBC, MCA, and MAB respectively. The desired inequality now follows from
$$S_aS_b + S_bS_c + S_cS_a \le \frac{1}{3}(S_a + S_b + S_c)^2 = \frac{S^2}{3}.$$
Equality holds if and only if $S_a = S_b = S_c$, which is equivalent to M being the centroid of the triangle.

10. (a) Let us set $k = a/b > 1$. Then $a = kb$ and $c = \sqrt{k}b$, and $a > c > b$. The segments a, b, c form a triangle if and only if $k < \sqrt{k} + 1$, which holds if and only if $1 < k < \frac{3+\sqrt{5}}{2}$.
    (b) The triangle is right-angled if and only if $a^2 = b^2 + c^2 \Leftrightarrow k^2 = k + 1 \Leftrightarrow k = \frac{1+\sqrt{5}}{2}$. Also, it is acute-angled if and only if $k^2 < k + 1 \Leftrightarrow 1 < k < \frac{1+\sqrt{5}}{2}$ and obtuse-angled if $\frac{1+\sqrt{5}}{2} < k < \frac{3+\sqrt{5}}{2}$.

11. Introducing $y_i = \frac{1}{x_i}$, we transform our equation to
$$\begin{aligned} 0 &= 1 + y_1 + (1 + y_1)y_2 + \cdots + (1 + y_1)\cdots(1 + y_{n-1})y_n \\ &= (1 + y_1)(1 + y_2)\cdots(1 + y_n). \end{aligned}$$
The solutions are n-tuples $(y_1, \dots, y_n)$ with $y_i \ne 0$ for all i and $y_j = -1$ for at least one index j. Returning to x_i, we conclude that the solutions are all the n-tuples $(x_1, \dots, x_n)$ with $x_i \ne 0$ for all i, and $x_j = -1$ for at least one index j.

12. The given inequality is equivalent to $(a + b)^m/b^m + (a + b)^m/a^m \ge 2^{m+1}$, which can be rewritten as
$$\frac{1}{2}\left(\frac{1}{a^m} + \frac{1}{b^m}\right) \ge \left(\frac{2}{a + b}\right)^m.$$

Since $f(x) = 1/x^m$ is a convex function for every $m \in \mathbb{Z}$, the last inequality immediately follows from Jensen's inequality $(f(a) + f(b))/2 \geq f((a+b)/2)$.

13. Translating one of the triangles if necessary, we may assume w.l.o.g. that $B_1 \equiv A_1$. We also assume that $B_2 \not\equiv A_2$ and $B_3 \not\equiv A_3$, since the result is obvious otherwise.
There exists a plane π through A_1 that is parallel to both A_2B_2 and A_3B_3. Let A_2', A_3', B_2', B_3' denote the orthogonal projections of A_2, A_3, B_2, B_3 onto π, and let h_2, h_3 denote the distances of A_2, B_2 and of A_3, B_3 from π. By the Pythagorean theorem, ${A_2'A_3'}^2 = A_2A_3^2 - (h_2+h_3)^2 = B_2B_3^2 - (h_2+h_3)^2 = {B_2'B_3'}^2$, and similarly $A_1A_2' = A_1B_2'$ and $A_1A_3' = A_1B_3'$; hence $\triangle A_1A_2'A_3'$ and $\triangle A_1B_2'B_3'$ are congruent. If these two triangles are equally oriented, then we have finished. Otherwise, they are symmetric with respect to some line a passing through A_1, and consequently the projections of the triangles $A_1A_2A_3$ and $A_1B_2B_3$ onto the plane through a perpendicular to π coincide.

14. Let O, D, E be the circumcenter of $\triangle ABC$ and the midpoints of AB and AC, and given arbitrary $X \in AB$ and $Y \in AC$ such that $BX = CY$, let O_1, D_1, E_1 be the circumcenter of $\triangle AXY$ and the midpoints of AX and AY, respectively. Since $AD = AB/2$ and $AD_1 = AX/2$, it follows that $DD_1 = BX/2$ and similarly $EE_1 = CY/2$. Hence O_1 is at the same distance $BX/2 = CY/2$ from the lines OD and OE and lies on the half-line bisector l of $\angle DOE$.
If we let X, Y vary along the segments AB and AC, we obtain that the locus of O_1 is the segment OP, where $P \in l$ is a point at distance $\min(AB, AC)/2$ from OD and OE.

15. Set
$$f(n) = \left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \cdots + \left[\frac{n+2^i}{2^{i+1}}\right] + \cdots.$$
We prove by induction that $f(n) = n$. This obviously holds for $n = 1$. Let us assume that $f(n-1) = n-1$. Define
$$g(i,n) = \left[\frac{n+2^i}{2^{i+1}}\right] - \left[\frac{n-1+2^i}{2^{i+1}}\right].$$
We have that $f(n) - f(n+1) = \sum_{i=0}^{\infty} g(i,n)$. We also note that $g(i,n) = 1$ if and only if $2^{i+1} \mid n + 2^i$; otherwise, $g(i,n) = 0$. The divisibility $2^{i+1} \mid n + 2^i$ is equivalent to $2^i \mid n$ and $2^{i+1} \nmid n$, which for a given n holds for exactly one $i \in \mathbb{N}_0$. Thus it follows that $f(n) - f(n-1) = 1 \Rightarrow f(n) = n$. The proof by induction is now complete.
Second solution. It is easy to show that $[x + 1/2] = [2x] - [x]$ for $x \in \mathbb{R}$. Now $f(x) = ([x] - [x/2]) + ([x/2] - [x/4]) + \cdots = [x]$. Hence, $f(n) = n$ for all $n \in \mathbb{N}$.

16. We shall prove the result by induction on k. It trivially holds for $k = 0$. Assume that the statement is true for some $k-1$, and let $p(x)$ be a polynomial of degree k. Let us set $p_1(x) = p(x+1) - p(x)$. Then $p_1(x)$ is a polynomial of degree $k-1$ with leading coefficient ka_0. Also, $m \mid p_1(x)$ for all $x \in \mathbb{Z}$ and hence by the inductive assumption $m \mid (k-1)! \cdot ka_0 = k!a_0$, which completes the induction.
    On the other hand, for any a_0, k and $m \mid k!a_0$, $p(x) = k!a_0\binom{x}{k}$ is a polynomial with leading coefficient a_0 that is divisible by m.

17. Let there be given an equilateral triangle ABC and a point O such that $OA = x$, $OB = y$, $OC = z$. Let X be the point in the plane such that $\triangle CXB$ and $\triangle COA$ are congruent and equally oriented. Then $BX = x$ and the triangle XOC is equilateral, which implies $OX = z$. Thus we have a triangle OBX with $BX = x$, $BO = y$, and $OX = z$.
    Conversely, given a triangle OBX with $BX = x$, $BO = y$ and $OX = z$ it is easy to construct the triangle ABC.

18. The required construction is not feasible. In fact, let us consider the special case $\angle BOC = 135°$, $\angle AOC = 120°$, $\angle AOB = 90°$, where $AA' \cap BB' \cap CC' = \{O\}$. Denoting OA', OB', OC' by a, b, c respectively we obtain the system of equations $a^2+b^2 = a^2+c^2+ac = b^2+c^2+\sqrt{2}bc$. Assuming w.l.o.g. $c = 1$ we easily obtain $a^3 - a^2 - a - 1 = 0$, which is an irreducible equation of third degree. By a known theorem, its solution a is not constructible by ruler and compass.

19. We shall denote by d_n the shortest curved distance from the initial point to the nth point in the positive direction. The sequence d_n goes as follows: $0, 1, 2, 3, 4, 5, 6, 0.72, 1.72, \dots, 5.72, 0.43, 1.43, \dots, 5.43, 0.15 = d_{19}$. Hence the required number of points is 20.

20. Let us denote the points $A_1, A_2, \dots, A_n$ in such a manner that A_1A_n is a diameter of the set of given points, and $A_1A_2 \le A_1A_3 \le \cdots \le A_1A_n$.
    Since for each $1 < i < n$ it holds that $A_1A_i < A_1A_n$, we have $\angle A_iA_1A_n < 120°$ and hence $\angle A_iA_1A_n < 60°$ (otherwise, all angles in $\triangle A_1A_iA_n$ are less than $120°$). It follows that for all $1 < i < j \le n$, $\angle A_iA_1A_j < 120°$. Consequently, the angle in the triangle $A_1A_iA_j$ that is at least $120°$ must be $\angle A_1A_iA_j$. Moreover, for any $1 < i < j < k \le n$ it holds that $\angle A_iA_jA_k \ge \angle A_1A_jA_k - \angle A_1A_jA_i > 120° - 60° = 60°$ (because $\angle A_1A_jA_i < 60°$); hence $\angle A_iA_jA_k \ge 120°$. This proves that the denotation is correct.

    Remark. It is easy to show that the diameter is unique. Hence the denotation is also unique.

21. The given conditions are equivalent to $y - a_0$ being divisible by $a_0, a_0 + a_1, a_0 + a_2, \dots, a_0 + a_n$, i.e., to $y = k[a_0, a_0 + a_1, \dots, a_0 + a_n] + a_0$, $k \in \mathbb{N}_0$.

22. It can be shown by induction on the number of digits of x that $p(x) \le x$ for all $x \in \mathbb{N}$. It follows that $x^2 - 10x - 22 \le x$, which implies $x \le 12$.

Since $0 < x^2 - 10x - 22 = (x-12)(x+2) + 2$, one easily obtains $x \geq 12$. Now one can directly check that $x = 12$ is indeed a solution, and thus the only one.

23. We may assume w.l.o.g. that in all the factors the coefficient of x is 1. Suppose that $x + ay + bz$ is one of the linear factors of $p(x,y,z) = x^3 + y^3 + z^3 + mxyz$. Then $p(x)$ is 0 at every point (x,y,z) with $z = -ax - by$. Hence $x^3 + y^3 + (-ax-by)^3 + mxy(-ax-by) = (1-a^3)x^3 - (3ab + m)(ax+by)xy + (1-b^3)y^3 \equiv 0$. This is obviously equivalent to $a^3 = b^3 = 1$ and $m = -3ab$, from which it follows that $m \in \{-3, -3\omega, -3\omega^2\}$, where $\omega = \frac{-1+i\sqrt{3}}{2}$. Conversely, for each of the three possible values for m there are exactly three possibilities (a,b). Hence $-3, -3\omega, -3\omega^2$ are the desired values.

24. If the ith digit is 0, then the result is $9^{k-j}9!/(10-j)!$ if $i > k-j$ and $9^{k-j-1}9!/(9-j)!$ otherwise. If the ith digit is not 0, then the above results are multiplied by 8.

25. The answer is

$$\sum_{1 \leq p < q < r \leq k} n_p n_q n_r + \sum_{1 \leq p < q \leq k} \left[n_p \binom{n_q}{2} + n_q \binom{n_p}{2} \right].$$

26. (a) We shall show that the period of f is $2a$. From $\left(f(x+a) - 1/2\right)^2 = f(x) - f(x)^2$ we obtain

$$\left(f(x) - f(x)^2\right) + \left(f(x+a) - f(x+a)^2\right) = \frac{1}{4}.$$

Subtracting the above relation for $x + a$ in place of x we get $f(x) - f(x)^2 = f(x+2a) - f(x+2a)^2$, which implies $\left(f(x) - 1/2\right)^2 = \left(f(x+2a) - 1/2\right)^2$. Since $f(x) \geq 1/2$ holds for all x by the condition of the problem, we conclude that $f(x+2a) = f(x)$.

(b) The following function, as is directly verified, satisfies the conditions:

$$f(x) = \begin{cases} 1/2 & \text{if } 2n \leq x < 2n+1, \\ 1 & \text{if } 2n+1 \leq x < 2n+2, \end{cases} \quad \text{for } n = 0, 1, 2, \ldots .$$

4.11 Solutions to the Contest Problems of IMO 1969

1. Set $a = 4m^4$, where $m \in \mathbb{N}$ and $m > 1$. We then have $z = n^4 + 4m^4 = (n^2 + 2m^2)^2 - (2mn)^2 = (n^2 + 2m^2 + 2mn)(n^2 + 2m^2 - 2mn)$. Since $n^2 + 2m^2 - 2mn = (n - m)^2 + m^2 \geq m^2 > 1$, it follows that z must be composite. Thus we have found infinitely many a that satisfy the condition of the problem.

2. Using $\cos(a + x) = \cos a \cos x - \sin a \sin x$, we obtain $f(x) = A \sin x + B \cos x$ where $A = -\sin a_1 - \sin a_2/2 - \cdots - \sin a_n/2^{n-1}$ and $B = \cos a_1 + \cos a_2/2 + \cdots + \cos a_n/2^{n-1}$. Numbers A and B cannot both be equal to 0, for otherwise f would be identically equal to 0, while on the other hand, we have $f(-a_1) = \cos(a_1 - a_1) + \cos(a_2 - a_1)/2 + \cdots + \cos(a_n - a_1)/2^{n-1} \geq 1 - 1/2 - \cdots - 1/2^{n-1} = 1/2^{n-1} > 0$. Setting $A = C \cos \phi$ and $B = C \sin \phi$, where $C \neq 0$ (such C and ϕ always exist), we get $f(x) = C \sin(x + \phi)$. It follows that the zeros of f are of the form $x_0 \in -\phi + \pi\mathbb{Z}$, from which $f(x_1) = f(x_2) \Rightarrow x_1 - x_2 = m\pi$ immediately follows.

3. We have several cases:

   1° $k = 1$. W.l.o.g. let $AB = a$ and the remaining segments have length 1. Let M be the midpoint of CD. Then $AM = BM = \sqrt{3}/2$ ($\triangle CDA$ and $\triangle CDB$ are equilateral) and $0 < AB < AM + BM = \sqrt{3}$, i.e., $0 < a < \sqrt{3}$. It is evident that all values of a within this interval are realizable.

   2° $k = 2$. We have two subcases.
   First, let $AC = AD = a$. Let M be the midpoint of CD. We have $CD = 1$, $AM = \sqrt{a^2 - 1/4}$, and $BM = \sqrt{3}/2$. Then we have $1 - \sqrt{3}/2 = AB - BM < AM < AB + BM = 1 + \sqrt{3}/2$, which gives us $\sqrt{2 - \sqrt{3}} < a < \sqrt{2 + \sqrt{3}}$.
   Second, let $AB = CD = a$. Let M be the midpoint of CD. From $\triangle MAB$ we get $a < \sqrt{2}$.
   Thus, from $\sqrt{2 - \sqrt{3}} < \sqrt{2} < \sqrt{2 + \sqrt{3}}$ it follows that the required condition in this case is $0 < a < \sqrt{2 + \sqrt{3}}$. All values for a in this range are realizable.

   3° $k = 3$. We show that such a tetrahedron exists for all a. Assume $a > 1$. Assume $AB = AC = AD = a$. Varying A along the line perpendicular to the plane BCD and through the center of $\triangle BCD$ we achieve all values of $a > 1/\sqrt{3}$. For $a < 1/\sqrt{3}$ we can observe a similar tetrahedron with three edges of length $1/a$ and three of length 1 and proceed as before.

   4° $k = 4$. By observing the similar tetrahedron we reduce this case to $k = 2$ with length $1/a$ instead of a. Thus we get $a > \sqrt{2 - \sqrt{3}}$.

   5° $k = 5$. We reduce to $k = 1$ and get $a > 1/\sqrt{3}$.

4. Let O be the midpoint of AB, i.e., the center of γ. Let O_1, O_2, and O_3 respectively be the centers of γ_1, γ_2, and γ_3 and let r_1, r_2, r_3 respectively

be the radii of γ_1, γ_2 and γ_3. Let C_1, C_2, and C_3 respectively be the points of tangency of γ_1, γ_2 and γ_3 with AB. Let D_2 and D_3 respectively be the points of tangency of γ_2 and γ_3 with CD. Finally, let G_2 and G_3 respectively be the points of tangency of γ_2 and γ_3 with γ. We have $\mathcal{B}(G_2, O_2, O)$, $G_2O_2 = O_2D_2$, and $G_2O = OB$. Hence, G_2, D_2, B are collinear. Similarly, G_3, D_3, A are collinear. It follows that AG_2D_2D and BG_3D_3D are cyclic, since $\angle AG_2D_2 = \angle D_2DA = \angle D_3DB = \angle BG_3D_3 = 90°$. Hence $BC_2^2 = BD_2 \cdot BG_2 = BD \cdot BA = BC^2 \Rightarrow BC_2 = BC$ and hence $AC_2 = AB - BC$. Similarly, $AC_3 = AC$. We thus have $AC_1 = (AC + AB - BC)/2 = (AC_3 + AC_2)/2$. Hence, C_1 is the midpoint of C_2C_3. We also have $r_2 + r_3 = C_2C_3 = AC + BC - AB = 2r_1$, from which it follows that O_1, O_2, O_3 are collinear.

Second solution. We shall prove the statement for arbitrary points A, B, C on γ.

Let us apply the inversion ψ with respect to the circle γ_1. We denote by $\widehat{X}$ the image of an object X under ψ. Also, ψ maps lines BC, CA, AB onto circles $\widehat{a}, \widehat{b}, \widehat{c}$, respectively. Circles $\widehat{a}, \widehat{b}, \widehat{c}$ pass through the center O_1 of γ_1 and have radii equal to the radius of $\widehat{\gamma}$. Let P, Q, R be the centers of $\widehat{a}, \widehat{b}, \widehat{c}$ respectively.

The line CD maps onto a circle k through $\widehat{C}$ and O_1 that is perpendicular to $\widehat{c}$. Therefore its center K lies in the intersection of the tangent t to $\widehat{c}$ and the line PQ (which bisects $\widehat{C}O_1$). Let O be a point such that RO_1KO is a parallelogram and γ_2', γ_3' the circles centered at O tangent to k. It is easy to see that γ_2' and γ_3' are also tangent to $\widehat{c}$, since OR and OK have lengths equal to the radii of k and $\widehat{c}$. Hence γ_2' and γ_3' are the images of γ_2 and γ_3 under ψ. Moreover, since $Q\widehat{A}OK$ and $P\widehat{B}OK$ are parallelograms and Q, P, K are collinear, it follows that $\widehat{A}, \widehat{B}, O$ are also collinear. Hence the centers of $\gamma_1, \gamma_2, \gamma_3$ are collinear, lying on the line O_1O, and the statement follows.

Third solution. Moreover, the statement holds for an arbitrary point $D \in BC$. Let E, F, G, H be the points of tangency of γ_2 with AB, CD and of γ_3 with AB, CD, respectively. Let O_i be the center of γ_i, $i = 1, 2, 3$. As is shown in the third solution of (SL93-3), EF and GH meet at O_1. Hence the problem of proving the collinearity of O_1, O_2, O_3 reduces to the following simple problem:

> Let D, E, F, G, H be points such that $D \in EG$, $F \in DH$ and $DE = DF$, $DG = DH$. Let O_1, O_2, O_3 be points such that $\angle O_2ED = \angle O_2FD = 90°$, $\angle O_3GD = \angle O_3HD = 90°$, and $O_1 = EF \cap GH$. Then O_1, O_2, O_3 are collinear.

Let $K_2 = DO_2 \cap EF$ and $K_3 = DO_3 \cap GH$. Then $O_2K_2/O_2D = DK_3/DO_3 = K_2O_1/DO_3$ and hence by Thales' theorem $O_1 \in O_2O_3$.

5. We first prove the following lemma.

   Lemma. If of five points in a plane no three belong to a single line, then there exist four that are the vertices of a convex quadrilateral.

Proof. If the convex hull of the five points A, B, C, D, E is a pentagon or a quadrilateral, the statement automatically holds. If the convex hull is a triangle, then w.l.o.g. let $\triangle ABC$ be that triangle and D, E points in its interior. Let the line DE w.l.o.g. intersect $[AB]$ and $[AC]$. Then B, C, D, E form the desired quadrilateral.

We now observe each quintuplet of points within the set. There are $\binom{n}{5}$ such quintuplets, and for each of them there is at least one quadruplet of points forming a convex quadrilateral. Each quadruplet, however, will be counted up to $n-4$ times. Hence we have found at least $\frac{1}{n-4}\binom{n}{5}$ quadruplets. Since $\frac{1}{n-4}\binom{n}{5} \geq \binom{n-3}{2} \Leftrightarrow (n-5)(n-6)(n+8) \geq 0$, which always holds, it follows that we have found at least $\binom{n-3}{2}$ desired quadruplets of points.

6. Define $u_1 = \sqrt{x_1y_1} + z_1$, $u_2 = \sqrt{x_2y_2} + z_2$, $v_1 = \sqrt{x_1y_1} - z_1$, and $v_2 = \sqrt{x_2y_2} - z_2$. By expanding both sides of the equation we can easily verify $(x_1+x_2)(y_1+y_2) - (z_1+z_2)^2 = (u_1+u_2)(v_1+v_2) + (\sqrt{x_1y_2} - \sqrt{x_2y_1})^2 \geq (u_1+u_2)(v_1+v_2)$. Since $x_iy_i - z_i^2 = u_iv_i$ for $i = 1, 2$, it suffices to prove

$$\frac{8}{(u_1+u_2)(v_1+v_2)} \leq \frac{1}{u_1v_1} + \frac{1}{u_2v_2}$$
$$\Leftrightarrow 8u_1u_2v_1v_2 \leq (u_1+u_2)(v_1+v_2)(u_1v_1+u_2v_2),$$

which trivially follows from the AM–GM inequalities $2\sqrt{u_1u_2} \leq u_1+u_2$, $2\sqrt{v_1v_2} \leq v_1+v_2$ and $2\sqrt{u_1v_1u_2v_2} \leq u_1v_1+u_2v_2$.

Equality holds if and only if $x_1y_2 = x_2y_1$, $u_1 = u_2$ and $v_1 = v_2$, i.e. if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.

Second solution. Let us define $f(x,y,z) = 1/(xy-z^2)$. The problem actually states that

$$2f\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) \leq f(x_1,y_1,z_1) + f(x_2,y_2,z_2),$$

i.e., that the function f is convex on the set $D = \{(x,y,z) \in \mathbb{R}^2 \mid xy - z^2 > 0\}$. It is known that a twice continuously differentiable function $f(t_1, t_2, \ldots, t_n)$ is convex if and only if its Hessian $[f''_{ij}]_{i,j=1}^n$ is positive semidefinite, or equivalently, if its principal minors $D_k = \det[f''_{ij}]_{i,j=1}^k$, $k = 1, 2, \ldots, n$, are nonnegative. In the case of our f this is directly verified: $D_1 = 2y^2/(xy-z^2)^3$, $D_2 = 3xy + z^2/(xy-z^2)^5$, $D_3 = 6/(xy-z^2)^6$ are obviously positive.

4.12 Solutions to the Shortlisted Problems of IMO 1970

1. Denote respectively by R and r the radii of the circumcircle and incircle, by $A_1, \dots, A_n, B_1, \dots, B_n$,the vertices of the $2n$-gon and by O its center. Let P' be the point symmetric to P with respect to O. Then $A_iP'B_iP$ is a parallelogram, and applying cosine theorem on triangles A_iB_iP and $PP'B_i$ yields

$$\begin{aligned} 4R^2 &= PA_i^2 + PB_i^2 - 2PA_i \cdot PB_i \cos a_i \\ 4r^2 &= PB_i^2 + P'B_i^2 - 2PB_i \cdot P'B_i \cos \angle PB_iP'. \end{aligned}$$

Since $A_iP'B_iP$ is a parallelogram, we have that $P'B_i = PA_i$ and $\angle PB_iP' = \pi - a_i$. Subtracting the expression for $4r^2$ from the one for $4R^2$ yields $4(R^2 - r^2) = -4PA_i \cdot PB_i \cos a_i = -8S_{\triangle A_iB_iP} \cot a_i$, hence we conclude that

$$\tan^2 a_i = \frac{4S^2_{\triangle A_iB_iP}}{(R^2 - r^2)^2}. \tag{1}$$

Denote by M_i the foot of the perpendicular from P to A_iB_i and let $m_i = PM_i$. Then $S_{\triangle A_iB_iP} = Rm_i$. Substituting this into (1) and adding up these relations for $i = 1, 2, \dots, n$, we obtain

$$\sum_{i=1}^{n} \tan^2 a_i = \frac{4R^2}{(R^2 - r^2)^2} \left(\sum_{i=1}^{n} m_i^2 \right).$$

Note that all the points M_i lie on a circle with diameter OP and form a regular n-gon. Denote its center by F. We have that $m_i^2 = \|\overrightarrow{PM_i}\|^2 = \|\overrightarrow{FM_i} - \overrightarrow{FP}\|^2 = \|\overrightarrow{FM_i}^2\| + \|\overrightarrow{FP}^2\| - 2\langle \overrightarrow{FM_i}, \overrightarrow{FP} \rangle = r^2/2 - 2\langle \overrightarrow{FM_i}, \overrightarrow{FP} \rangle$. From this it follows that $\sum_{i=1}^{n} m_i^2 = 2n(r/2)^2 - 2\sum_{i=1}^{n} \langle \overrightarrow{FM_i}, \overrightarrow{FP} \rangle = 2n(r/2)^2 - 2\langle \sum_{i=1}^{n} \overrightarrow{FM_i}, \overrightarrow{FP} \rangle = 2n(r/2)^2$, because $\sum_{i=1}^{n} \overrightarrow{FM_i} = \overrightarrow{0}$. Thus

$$\sum_{i=1}^{n} \tan^2 a_i = \frac{4R^2}{(R^2 - r^2)^2} 2n \left(\frac{r}{2} \right)^2 = 2n \frac{(r/R)^2}{\left(1 - (r/R)^2\right)^2} = 2n \frac{\cos^2 \frac{\pi}{2n}}{\sin^4 \frac{\pi}{2n}}.$$

Remark. For $n = 1$ there is no regular 2-gon. However, if we think of a 2-gon as a line segment, the statement will remain true.

2. Suppose that $a > b$. Consider the polynomial $P(X) = x_1X^{n-1} + x_2X^{n-2} + \dots + x_{n-1}X + x_n$. We have $A_n = P(a)$, $B_n = P(b)$, $A_{n+1} = x_0a^n + P(a)$, and $B_{n+1} = x_0b^n + P(b)$. Now $A_n/A_{n+1} < B_n/B_{n+1}$ becomes $P(a)/(x_0a^n + P(a)) < P(b)/(x_0b^n + P(b))$, i.e.,

$$b^nP(a) < a^nP(b).$$

Since $a > b$, we have that $a^i > b^i$ and hence $x_ia^nb^{n-i} \geq x_ib^na^{n-i}$ (also, for $i \geq 1$ the inequality is strict). Summing up all these inequalities for $i = 1, \dots, n$ we get $a^nP(b) > b^nP(a)$, which completes the proof for $a > b$.

On the other hand, for $a < b$ we analogously obtain the opposite inequality $A_n/A_{n+1} > B_n/B_{n+1}$, while for $a = b$ we have equality. Thus $A_n/A_{n+1} < B_n/B_{n+1} \Leftrightarrow a > b$.

3. We shall use the following lemma

   Lemma. If an altitude of a tetrahedron passes through the orthocenter of the opposite side, then each of the other altitudes possesses the same property.

   Proof. Denote the tetrahedron by $SABC$ and let $a = BC$, $b = CA$, $c = AB$, $m = SA$, $n = SB$, $p = SC$. It is enough to prove that an altitude passes through the orthocenter of the opposite side if and only if $a^2 + m^2 = b^2 + n^2 = c^2 + p^2$.

   Suppose that the foot S' of the altitude from S is the orthocenter of ABC. Then $SS' \perp ABC \Rightarrow SB^2 - SC^2 = S'B^2 - S'C^2$. But from $AS' \perp BC$ it follows that $AB^2 - AC^2 = S'B^2 - S'C^2$. From these two equalities it can be concluded that $n^2 - p^2 = c^2 - b^2$, or equivalently, $n^2 + b^2 = c^2 + p^2$. Analogously, $a^2 + m^2 = n^2 + b^2$, so we have proved the first part of the equivalence.

   Now suppose that $a^2 + m^2 = b^2 + n^2 = c^2 + p^2$. Defining S' as before, we get $n^2 - p^2 = S'B^2 - S'C^2$. From the condition $n^2 - p^2 = c^2 - b^2$ ($\Leftrightarrow b^2 + n^2 = c^2 + p^2$) we conclude that $AS' \perp BC$. In the same way $CS' \perp AB$, which proves that S' is the orthocenter of $\triangle ABC$. The lemma is thus proven.

   Now using the lemma it is easy to see that if one of the angles at S is right, than so are the others. Indeed, suppose that $\angle ASB = \pi/2$. From the lemma we have that the altitude from C passes through the orthocenter of $\triangle ASB$, which is S, so $CS \perp ASB$ and $\angle CSA = \angle CSB = \pi/2$. Therefore $m^2 + n^2 = c^2$, $n^2 + p^2 = a^2$, and $p^2 + m^2 = b^2$, so it follows that $m^2 + n^2 + p^2 = (a^2 + b^2 + c^2)/2$. By the inequality between the arithmetic and quadric means, we have that $(a^2 + b^2 + c^2)/2 \geq 2s^2/3$, where s denotes the semiperimeter of $\triangle ABC$. It remains to be shown that $2s^2/3 \geq 18r^2$. Since $S_{\triangle ABC} = sr$, this is equivalent to $2s^4/3 \geq 18S_{ABC}^2 = 18s(s-a)(s-b)(s-c)$ by Heron's formula. This reduces to $s^3 \geq 27(s-a)(s-b)(s-c)$, which is an obvious consequence of the AM–GM mean inequality.

   Remark. In the place of the lemma one could prove that the opposite edges of the tetrahedron are mutually perpendicular and proceed in the same way.

4. Suppose that n is such a natural number. If a prime number p divides any of the numbers $n, n+1, \dots, n+5$, then it must divide another one of them, so the only possibilities are $p = 2, 3, 5$. Moreover, $n+1, n+2, n+3, n+4$ have no prime divisors other than 2 and 3 (if some prime number greater than 3 divides one of them, then none of the remaining numbers can have that divisor). Since two of these numbers are odd, they must be powers of

3 (greater than 1). However, there are no two powers of 3 whose difference is 2. Therefore there is no such natural number n.

Second solution. Obviously, none of $n, n+1, \dots, n+5$ is divisible by 7; hence they form a reduced system of residues. We deduce that $n(n+1)\cdots(n+5) \equiv 1\cdot 2\cdots 6 \equiv -1 \pmod 7$. If $\{n, \dots, n+5\}$ can be partitioned into two subsets with the same products, both congruent to, say, p modulo 7, then $p^2 \equiv -1 \pmod 7$, which is impossible.

Remark. Erdős has proved that a set $n, n+1, \dots, n+m$ of consecutive natural numbers can never be partitioned into two subsets with equal products of elements.

5. Denote respectively by A_1, B_1, C_1 and D_1 the points of intersection of the lines AM, BM, CM, and DM with the opposite sides of the tetrahedron. Since $\operatorname{vol}(MBCD) = \operatorname{vol}(ABCD)\overrightarrow{MA_1}/\overrightarrow{AA_1}$, the relation we have to prove is equivalent to

$$\overrightarrow{MA}\cdot\frac{\overrightarrow{MA_1}}{\overrightarrow{AA_1}} + \overrightarrow{MB}\cdot\frac{\overrightarrow{MB_1}}{\overrightarrow{BB_1}} + \overrightarrow{MC}\cdot\frac{\overrightarrow{MC_1}}{\overrightarrow{CC_1}} + \overrightarrow{MD}\cdot\frac{\overrightarrow{MD_1}}{\overrightarrow{DD_1}} = 0. \qquad (1)$$

There exist unique real numbers α, β, γ, and δ such that $\alpha+\beta+\gamma+\delta = 1$ and for every point O in space

$$\overrightarrow{OM} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB} + \gamma\overrightarrow{OC} + \delta\overrightarrow{OD}. \qquad (2)$$

(This follows easily from $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + k\overrightarrow{AB} + l\overrightarrow{AC} + m\overrightarrow{AD} = \overrightarrow{AB} + k(\overrightarrow{OB} - \overrightarrow{OA}) + l(\overrightarrow{OC} - \overrightarrow{OA}) + m(\overrightarrow{OD} - \overrightarrow{OA})$ for some $k, l, m \in \mathbb{R}$.) Further, from the condition that A_1 belongs to the plane BCD we obtain for every O in space the following equality for some β', γ', δ':

$$\overrightarrow{OA_1} = \beta'\overrightarrow{OB} + \gamma'\overrightarrow{OC} + \delta'\overrightarrow{OD}. \qquad (3)$$

However, for $\lambda = \overrightarrow{MA_1}/\overrightarrow{AA_1}$, $\overrightarrow{OM} = \lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OA_1}$; hence substituting (2) and (3) in this expression and equating coefficients for $\overrightarrow{OA}$ we obtain $\lambda = \overrightarrow{MA_1}/\overrightarrow{AA_1} = \alpha$. Analogously, $\beta = \overrightarrow{MB_1}/\overrightarrow{BB_1}$, $\gamma = \overrightarrow{MC_1}/\overrightarrow{CC_1}$, and $\delta = \overrightarrow{MD_1}/\overrightarrow{DD_1}$; hence (1) follows immediately for $O = M$.

Remark. The statement of the problem actually follows from the fact that M is the center of mass of the system with masses $\operatorname{vol}(MBCD)$, $\operatorname{vol}(MACD)$, $\operatorname{vol}(MABD)$, $\operatorname{vol}(MABC)$ at A, B, C, D respectively. Our proof is actually a formal verification of this fact.

6. Let F be the midpoint of $B'C'$, A' the midpoint of BC, and I the intersection point of the line HF and the circle circumscribed about $\triangle BHC'$. Denote by M the intersection point of the line AA' with the circumscribed circle about the triangle ABC. Triangles $HB'C'$ and ABC are similar. Since $\angle C'IF = \angle ABC = \angle A'MC$, $\angle C'FI = \angle AA'B = \angle MA'C$,

$2C'F = C'B'$, and $2A'C = CB$, it follows that $\triangle C'IB' \sim \triangle CMB$, hence $\angle FIB' = \angle A'MB = \angle ACB$. Now one concludes that I belongs to the circumscribed circles of $\triangle AB'C'$ (since $\angle C'IB' = 180^\circ - \angle C'AB'$) and $\triangle HCB'$.

Second Solution. We denote the angles of $\triangle ABC$ by α, β, γ. Evidently $\triangle ABC \sim \triangle HC'B'$. Within $\triangle HC'B'$ there exists a unique point I such that $\angle HIB' = 180^\circ - \gamma$, $\angle HIC' = 180^\circ - \beta$, and $\angle C'IB' = 180^\circ - \alpha$, and all three circles must contain this point. Let HI and $B'C'$ intersect in F. It remains to show that $FB' = FC'$. From $\angle HIB' + \angle HB'F = 180^\circ$ we obtain $\angle IHB' = \angle IB'F$. Similarly, $\angle IHC' = \angle IC'F$. Thus circles around $\triangle IHC'$ and $\triangle IHB'$ are both tangent to $B'C'$, giving us $FB'^2 = FI \cdot FH = FC'^2$.

7. For $a = 5$ one can take $n = 10$, while for $a = 6$ one takes $n = 11$. Now assume $a \notin \{5, 6\}$.
If there exists an integer n such that each digit of $n(n+1)/2$ is equal to a, then there is an integer k such that $n(n+1)/2 = (10^k - 1)a/9$. After multiplying both sides of the equation by 72, one obtains $36n^2 + 36n = 8a \cdot 10^k - 8a$, which is equivalent to

$$9(2n+1)^2 = 8a \cdot 10^k - 8a + 9. \tag{1}$$

So $8a \cdot 10^k - 8a + 9$ is the square of some odd integer. This means that its last digit is 1, 5, or 9. Therefore $a \in \{1, 3, 5, 6, 8\}$.
If $a = 3$ or $a = 8$, the number on the RHS of (1) is divisible by 5, but not by 25 (for $k \geq 2$), and thus cannot be a square. It remains to check the case $a = 1$. In that case, (1) becomes $9(2n+1)^2 = 8 \cdot 10^k + 1$, or equivalently $[3(2n+1)-1][3(2n+1)+1] = 8 \cdot 10^k \Rightarrow (3n+1)(3n+2) = 2 \cdot 10^k$. Since the factors $3n+1, 3n+2$ are relatively prime, this implies that one of them is 2^{k+1} and the other one is 5^k. It is directly checked that their difference really equals 1 only for $k = 1$ and $n = 1$, which is excluded. Hence, the desired n exists only for $a \in \{5, 6\}$.

8. Let $AC = b, BC = a, AM = x, BM = y, CM = l$. Denote by I_1 the incenter and by S_1 the center of the excircle of ΔAMC. Suppose that P_1 and Q_1 are feet of perpendiculars from I_1 and S_1, respectively, to the line AC. Then $\triangle I_1CP_1 \sim \triangle S_1CQ_1$, hence $r_1/\rho_1 = CP_1/CQ_1$. We have $CP_1 = (AC + MC - AM)/2 = (b + l - x)/2$ and $CQ_1 = (AC + MC + AM)/2 = (b + l + x)/2$. Hence

$$\frac{r_1}{\rho_1} = \frac{b+l-x}{b+l+x}.$$

We similarly obtain

$$\frac{r_2}{\rho_2} = \frac{b+l-y}{b+l+y} \quad \text{and} \quad \frac{r}{\rho} = \frac{a+b-x-y}{a+b+x+y}.$$

What we have to prove is now equivalent to

$$\frac{(b+l-x)(a+l-y)}{(b+l+x)(a+l+y)} = \frac{a+b-x-y}{a+b+x+y}. \tag{1}$$

Multiplying both sides of (1) by $(a+l+y)(b+l+x)(a+b+x+y)$ we obtain an expression that reduces to $l^2x + l^2y + x^2y + xy^2 = b^2y + a^2x$. Dividing both sides by $c = x + y$, we get that (1) is equivalent to $l^2 = b^2y/(x+y) + a^2x/(x+y) - xy$, which is exactly Stewart's theorem for l. This finally proves the desired result.

9. Let us set $a = \sqrt{\sum_{i=1}^{n} u_i^2}$ and $b = \sqrt{\sum_{i=1}^{n} v_i^2}$. By Minkowski's inequality (for $p = 2$) we have $\sum_{i=1}^{n}(u_i + v_i)^2 \le (a+b)^2$. Hence the LHS of the desired inequality is not greater than $1 + (a+b)^2$, while the RHS is equal to $4(1+a^2)(1+b^2)/3$. Now it is sufficient to prove that

$$3 + 3(a+b)^2 \le 4(1+a^2)(1+b^2).$$

The last inequality can be reduced to the trivial $0 \le (a-b)^2 + (2ab-1)^2$. The equality in the initial inequality holds if and only if $u_i/v_i = c$ for some $c \in \mathbb{R}$ and $a = b = 1/\sqrt{2}$.

10. (a) Since $a_{n-1} < a_n$, we have

$$\begin{aligned}\left(1 - \frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}} &= \frac{a_k - a_{k-1}}{a_k^{3/2}} \\ &\le \frac{2(\sqrt{a_k} - \sqrt{a_{k-1}})\sqrt{a_k}}{a_k\sqrt{a_{k-1}}} = 2\left(\frac{1}{\sqrt{a_{k-1}}} - \frac{1}{\sqrt{a_k}}\right).\end{aligned}$$

Summing up all these inequalities for $k = 1, 2, \dots, n$ we obtain

$$b_n \le 2\left(\frac{1}{\sqrt{a_0}} - \frac{1}{\sqrt{a_n}}\right) < 2.$$

(b) Choose a real number $q > 1$, and let $a_k = q^k$, $k = 1, 2, \dots$. Then $\left(1 - a_{k-1}/a_k\right)/\sqrt{a_k} = (1 - 1/q)/q^{k/2}$, and consequently

$$b_n = \left(1 - \frac{1}{q}\right)\sum_{k=1}^{n}\frac{1}{q^{k/2}} = \frac{\sqrt{q}+1}{q}\left(1 - \frac{1}{q^{n/2}}\right).$$

Since $(\sqrt{q}+1)/q$ can be arbitrarily close to 2, one can set q such that $(\sqrt{q}+1)/q > b$. Then $b_n \ge b$ for all sufficiently large n.

Second solution.

(a) Note that

$$b_n = \sum_{k=1}^{n}\left(1 - \frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}} = \sum_{k=1}^{n}(a_k - a_{k-1}) \cdot \frac{1}{a_k^{3/2}};$$

hence b_n represents exactly the lower Darboux sum for the function $f(x) = x^{-3/2}$ on the interval $[a_0, a_n]$. Then $b_n \le \int_{a_0}^{a_n} x^{-3/2}dx < \int_1^{+\infty} x^{-3/2}dx = 2$.

(b) For each $b < 2$ there exists a number $\alpha > 1$ such that $\int_1^{\alpha} x^{-3/2}dx > b + (2-b)/2$. Now, by Darboux's theorem, there exists an array $1 = a_0 \le a_1 \le \cdots \le a_n = \alpha$ such that the corresponding Darboux sums are arbitrarily close to the value of the integral. In particular, there is an array $a_0, \dots, a_n$ with $b_n > b$.

11. Let $S(x) = (x - x_1)(x - x_2)\cdots(x - x_n)$. We have $x^3 - x_i^3 = (x - x_i)(\omega x - x_i)(\omega^2 x - x_i)$, where ω is a primitive third root of 1. Multiplying these equalities for $i = 1, \dots, n$ we obtain

$$T(x^3) = (x^3 - x_1^3)(x^3 - x_2^3)\cdots(x^3 - x_n^3) = S(x)S(\omega x)S(\omega^2 x).$$

Since $S(\omega x) = P(x^3) + \omega x Q(x^3) + \omega^2 x^2 R(x^3)$ and $S(\omega^2 x) = P(x^3) + \omega^2 x Q(x^3) + \omega x^2 R(x^3)$, the above expression reduces to

$$T(x^3) = P^3(x^3) + x^3 Q^3(x^3) + x^6 R^3(x^3) - 3P(x^3)Q(x^3)R(x^3).$$

Therefore the zeros of the polynomial

$$T(x) = P^3(x) + xQ^3(x) + x^2 R^3(x) - 3P(x)Q(x)R(x)$$

are exactly $x_1^3, \dots, x_n^3$. It is easily verified that $\deg T = \deg S = n$, and hence T is the desired polynomial.

12. *Lemma.* Five points are given in the plane such that no three of them are collinear. Then there are at least three triangles with vertices at these points that are not acute-angled.

Proof. We consider three cases, according to whether the convex hull of these points is a triangle, quadrilateral, or pentagon.

(i) Let a triangle ABC be the convex hull and two other points D and E lie inside the triangle. At least two of the triangles ADB, BDC and CDA have obtuse angles at the point D. Similarly, at least two of the triangles AEB, BEC and CEA are obtuse-angled. Thus there are at least four non-acute-angled triangles.

(ii) Suppose that $ABCD$ is the convex hull and that E is a point of its interior. At least one angle of the quadrilateral is not acute, determining one non-acute-angled triangle. Also, the point E lies in the interior of either $\triangle ABC$ or $\triangle CDA$ hence, as in the previous case, it determines another two obtuse-angled triangles.

(iii) It is easy to see that at least two of the angles of the pentagon are not acute. We may assume that these two angles are among the angles corresponding to vertices A, B, and C. Now consider the quadrilateral $ACDE$. At least one its angles is not acute. Hence, there are at least three triangles that are not acute-angled.

Now we consider all combinations of 5 points chosen from the given 100. There are $\binom{100}{5}$ such combinations, and for each of them there are at least three non-acute-angled triangles with vertices in it. On the other hand, vertices of each of the triangles are counted $\binom{97}{2}$ times. Hence there are at least $3\binom{100}{5}/\binom{97}{2}$ non-acute-angled triangles with vertices in the given 100 points. Since the number of all triangles with vertices in the given points is $\binom{100}{3}$, the ratio between the number of acute-angled triangles and the number of all triangles cannot be greater than

$$1 - \frac{3\binom{100}{5}}{\binom{97}{2}\binom{100}{3}} = 0.7.$$

4.13 Solutions to the Shortlisted Problems of IMO 1971

1. Assuming that a, b, c in (1) exist, let us find what their values should be. Since $P_2(x) = x^2 - 2$, equation (1) for $n = 1$ becomes $(x^2 - 4)^2 = [a(x^2 - 2) + bx + 2c]^2$. Therefore, there are two possibilities for (a, b, c): $(1, 0, -1)$ and $(-1, 0, 1)$. In both cases we must prove that

$$(x^2 - 4)[P_n(x)^2 - 4] = [P_{n+1}(x) - P_{n-1}(x)]^2. \qquad (2)$$

It suffices to prove (2) for all x in the interval $[-2, 2]$. In this interval we can set $x = 2\cos t$ for some real t. We prove by induction that

$$P_n(x) = 2\cos nt \quad \text{for all } n. \qquad (3)$$

This is trivial for $n = 0, 1$. Assume (3) holds for some $n - 1$ and n. Then $P_{n+1}(x) = 4\cos t \cos nt - 2\cos(n - 1)t = 2\cos(n + 1)t$ by the additive formula for the cosine. This completes the induction.
Now (2) reduces to the obviously correct equality

$$16\sin^2 t \sin^2 nt = (2\cos(n + 1)t - 2\cos(n - 1)t)^2.$$

Second solution. If x is fixed, the linear recurrence relation $P_{n+1}(x) + P_{n-1}(x) = xP_n(x)$ can be solved in the standard way. The characteristic polynomial $t^2 - xt + 1$ has zeros $t_{1,2}$ with $t_1 + t_2 = x$ and $t_1t_2 = 1$; hence, the general $P_n(x)$ has the form $at_1^n + bt_2^n$ for some constants a, b. From $P_0 = 2$ and $P_1 = x$ we obtain that

$$P_n(x) = t_1^n + t_2^n.$$

Plugging in these values and using $t_1t_2 = 1$ one easily verifies (2).

2. We will construct such a set S_m of 2^m points.
Take vectors $u_1, \dots, u_m$ in a given plane, such that $|u_i| = 1/2$ and $0 \neq |c_1u_1 + c_2u_2 + \cdots + c_nu_n| \neq 1/2$ for any choice of numbers c_i equal to 0 or ± 1. Such vectors are easily constructed by induction on m: For $u_1, \dots, u_{m-1}$ fixed, there are only finitely many vector values u_m that violate the upper condition, and we may set u_m to be any other vector of length $1/2$.
Let S_m be the set of all points $M_0 + \varepsilon_1u_1 + \varepsilon_2u_2 + \cdots + \varepsilon_mu_m$, where M_0 is any fixed point in the plane and $\varepsilon_i = \pm 1$ for $i = 1, \dots, m$. Then S_m obviously satisfies the condition of the problem.

3. Let x, y, z be a solution of the given system with $x^2 + y^2 + z^2 = \alpha < 10$. Then

$$xy + yz + zx = \frac{(x + y + z)^2 - (x^2 + y^2 + z^2)}{2} = \frac{9 - \alpha}{2}.$$

Furthermore, $3xyz = x^3 + y^3 + z^3 - (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$, which gives us $xyz = 3(9 - \alpha)/2 - 4$. We now have

$$\begin{aligned}35 = x^4+y^4+z^4 &= (x^3+y^3+z^3)(x+y+z)\\ &\quad -(x^2+y^2+z^2)(xy+yz+zx)+xyz(x+y+z)\\ &= 45-\frac{\alpha(9-\alpha)}{2}+\frac{9(9-\alpha)}{2}-12.\end{aligned}$$

The solutions in α are $\alpha=7$ and $\alpha=11$. Therefore $\alpha=7$, $xyz=-1$, $xy+xz+yz=1$, and

$$\begin{aligned}x^5+y^5+z^5 &= (x^4+y^4+z^4)(x+y+z)\\ &\quad -(x^3+y^3+z^3)(xy+xz+yz)+xyz(x^2+y^2+z^2)\\ &= 35\cdot 3-15\cdot 1+7\cdot(-1)=83.\end{aligned}$$

4. In the coordinate system in which the x-axis passes through the centers of the circles and the y-axis is their common tangent, the circles have equations

$$x^2+y^2+2r_1x=0,\qquad x^2+y^2-2r_2x=0.$$

Let p be the desired line with equation $y=ax+b$. The abscissas of points of intersection of p with both circles satisfy one of

$$(1+a^2)x^2+2(ab+r_1)x+b^2=0,\qquad (1+a^2)x^2+2(ab-r_2)x+b^2=0.$$

Let us denote the lengths of the chords and their projections onto the x-axis by d and d_1, respectively. From these equations it follows that

$$d_1^2=\frac{4(ab+r_1)^2}{(1+a^2)^2}-\frac{4b^2}{1+a^2}=\frac{4(ab-r_2)^2}{(1+a^2)^2}-\frac{4b^2}{1+a^2}. \tag{1}$$

Consider the point of intersection of p with the y-axis. This point has equal powers with respect to both circles. Hence, if that point divides the segment determined on p by the two circles on two segments of lengths x and y, this power equals $x(x+d)=y(y+d)$, which implies $x=y=d/2$. Thus each of the equations in (1) has two roots, one of which is thrice the other. This fact gives us $(ab+r_1)^2=4(1+a^2)b^2/3$. From (1) and this we obtain

$$ab=\frac{r_2-r_1}{2},\qquad 4b^2+a^2b^2=3[(ab+r_1)^2-a^2b^2]=3r_1r_2;$$

$$a^2=\frac{4(r_2-r_1)^2}{14r_1r_2-r_1^2-r_2^2},\qquad b^2=\frac{14r_1r_2-r_1^2-r_2^2}{16};$$

$$d_1^2=\frac{(14r_1r_2-r_1^2-r_2^2)^2}{36(r_1+r_2)^2}.$$

Finally, since $d^2=d_1^2(1+a^2)$, we conclude that

$$d^2=\frac{1}{12}(14r_1r_2-r_1^2-r_2^2),$$

and that the problem is solvable if and only if $7-4\sqrt{3}\le\frac{r_1}{r_2}\le 7+4\sqrt{3}$.

5. Without loss of generality, we may assume that $a \geq b \geq c \geq d \geq e$. Then $a - b = -(b - a) \geq 0$, $a - c \geq b - c \geq 0$, $a - d \geq b - d \geq 0$ and $a - e \geq b - e \geq 0$, and hence

$$(a-b)(a-c)(a-d)(a-e) + (b-a)(b-c)(b-d)(b-e) \geq 0.$$

Analogously, $(d-a)(d-b)(d-c)(d-e) + (e-a)(e-b)(e-c)(e-d) \geq 0$. Finally, $(c-a)(c-b)(c-d)(c-e) \geq 0$ as a product of two nonnegative numbers, from which the inequality stated in the problem follows.

Remark. The problem in an alternative formulation, accepted for the IMO, asked to prove that the analogous inequality

$$(a_1 - a_2)(a_1 - a_2)\cdots(a_1 - a_n) + (a_2 - a_1)(a_2 - a_3)\cdots(a_2 - a_n) + \cdots \\ + (a_n - a_1)(a_n - a_2)\cdots(a_n - a_{n-1}) \geq 0$$

holds for arbitrary real numbers a_i if and only if $n = 3$ or $n = 5$.
The case $n = 3$ is analogous to $n = 5$. For $n = 4$, a counterexample is $a_1 = 0$, $a_2 = a_3 = a_4 = 1$, while for $n > 5$ one can take $a_1 = a_2 = \cdots = a_{n-4} = 0$, $a_{n-3} = a_{n-2} = a_{n-1} = 2$, $a_n = 1$ as a counterexample.

6. The proof goes by induction on n. For $n = 2$, the following numeration satisfies the conditions (a)–(d): $C_1 = 11, C_2 = 12, C_3 = 22, C_4 = 21$. Suppose that $n > 2$, and that the numeration $C_1, C_2, \ldots, C_{2^{n-1}}$ of a regular 2^{n-1}-gon, in cyclical order, satisfies (i)–(iv). Then one can assign to the vertices of a 2^n-gon cyclically the following numbers:

$$\overline{1C_1}, \overline{1C_2}, \ldots, \overline{1C_{2^{n-1}}}, \overline{2C_{2^{n-1}}}, \ldots, \overline{2C_2}, \overline{2C_1}.$$

The conditions (i), (ii) obviously hold, while (iii) and (iv) follow from the inductive assumption.

7. (a) Suppose that X, Y, Z are fixed on segments AB, BC, CD. It is proven in a standard way that if $\angle ATX \neq \angle ZTD$, then $ZT + TX$ can be reduced. It follows that if there exists a broken line $XYZTX$ of minimal length, then the following conditions hold:

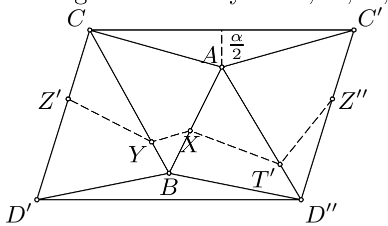
$$\begin{aligned} \angle DAB &= \pi - \angle ATX - \angle AXT, \\ \angle ABC &= \pi - \angle BXY - \angle BYX = \pi - \angle AXT - \angle CYZ, \\ \angle BCD &= \pi - \angle CYZ - \angle CZY, \\ \angle CDA &= \pi - \angle DTZ - \angle DZT = \pi - \angle ATX - \angle CZY. \end{aligned}$$

Thus $\sigma = 0$.

(b) Now let $\sigma = 0$. Let us cut the surface of the tetrahedron along the edges AC, CD, and DB and set it down into a plane. Consider the plane figure $\mathcal{S} = ACD'BD''C'$ thus obtained made up of triangles BCD', ABC, ABD'', and $AC'D''$, with Z', T', Z'' respectively on $CD', AD'', C'D''$ (here C' corresponds to C, etc.). Since

$\angle C'D''A + \angle D''AB + \angle ABC + \angle BCD' = 0$ as an oriented angle (because $\sigma = 0$), the lines CD' and $C'D''$ are parallel and equally oriented; i.e., $CD'D''C'$ is a parallelogram.

The broken line $XYZTX$ has minimal length if and only if Z'', T', X, Y, Z' are collinear (where $Z'Z'' \parallel CC'$), and then this length equals $Z'Z'' = CC' = 2AC\sin(\alpha/2)$. There is an infinity of such lines, one for every line $Z'Z''$ parallel to CC' that meets the interiors of all the segments CB, BA, AD''. Such



$Z'Z''$ exist. Indeed, the triangles CAB and $D''AB$ are acute-angled, and thus the segment AB has a common interior point with the parallelogram $CD'D''C'$. Therefore the desired result follows.

8. Suppose that a, b, c, t satisfy all the conditions. Then $abc \neq 0$ and

$$x_1x_2 = \frac{c}{a}, \qquad x_2x_3 = \frac{a}{b}, \qquad x_3x_1 = \frac{b}{c}.$$

Multiplying these equations, we obtain $x_1^2x_2^2x_3^2 = 1$, and hence $x_1x_2x_3 = \varepsilon = \pm 1$. From (1) we get $x_1 = \varepsilon b/a$, $x_2 = \varepsilon c/b$, $x_3 = \varepsilon a/c$. Substituting x_1 in the first equation, we get $ab^2/a^2 + t\varepsilon b^2/a + c = 0$, which gives us

$$b^2(1 + t\varepsilon) = -ac. \tag{1}$$

Analogously, $c^2(1+t\varepsilon) = -ab$ and $a^2(1+t\varepsilon) = -bc$, and therefore $(1 + t\varepsilon)^3 = -1$; i.e., $1 + t\varepsilon = -1$, since it is real. This also implies together with (1) that $b^2 = ac$, $c^2 = ab$, and $a^2 = bc$, and consequently

$$a = b = c.$$

Thus the three equations in the problem are equal, which is impossible. Hence, such a, b, c, t do not exist.

9. We use induction. Since $T_1 = 0$, $T_2 = 1$, $T_3 = 2$, $T_4 = 3$, $T_5 = 5$, $T_6 = 8$, the statement is true for $n = 1, 2, 3$. Suppose that both formulas from the problem hold for some $n \geq 3$. Then

$$T_{2n+1} = 1 + T_{2n} + 2^{n-1} = \left[\frac{17}{7}2^{n-1} + 2^{n-1}\right] = \left[\frac{12}{7}2^n\right],$$
$$T_{2n+2} = 1 + T_{2n-3} + 2^{n+1} = \left[\frac{12}{7}2^{n-2} + 2^{n+1}\right] = \left[\frac{17}{7}2^n\right].$$

Therefore the formulas hold for $n + 1$, which completes the proof.

10. We use induction. Suppose that every two of the numbers $a_1 = 2^{n_1} - 3, a_2 = 2^{n_2} - 3, \dots, a_k = 2^{n_k} - 3$, where $2 = n_1 < n_2 < \cdots < n_k$, are coprime. Then one can construct $a_{k+1} = 2^{n_{k+1}} - 3$ in the following way:

Set $s = a_1 a_2 \dots a_k$. Among the numbers $2^0, 2^1, \dots, 2^s$, two give the same residue upon division by s, say $s \mid 2^\alpha - 2^\beta$. Since s is odd, it can be assumed w.l.o.g. that $\beta = 0$ (this is actually a direct consequence of Euler's theorem). Let $2^\alpha - 1 = qs$, $q \in \mathbb{N}$. Since $2^{\alpha+2} - 3 = 4qs + 1$ is then coprime to s, it is enough to take $n_{k+1} = \alpha + 2$. We obviously have $n_{k+1} > n_k$.

11. We use induction. The statement for $n = 1$ is trivial. Suppose that it holds for $n = k$ and consider $n = k + 1$. From the given condition, we have

$$\begin{aligned} &\sum_{j=1}^{k} |a_{j,1}x_1 + \cdots + a_{j,k}x_k + a_{j,k+1}| \\ &+ |a_{k+1,1}x_1 + \cdots + a_{k+1,k}x_k + a_{k+1,k+1}| \leq M, \\ &\sum_{j=1}^{k} |a_{j,1}x_1 + \cdots + a_{j,k}x_k - a_{j,k+1}| \\ &+ |a_{k+1,1}x_1 + \cdots + a_{k+1,k}x_k - a_{k+1,k+1}| \leq M \end{aligned}$$

for each choice of $x_i = \pm 1$. Since $|a + b| + |a - b| \geq 2|a|$ for all a, b, we obtain

$$\begin{aligned} 2\sum_{j=1}^{k} |a_{j1}x_1 + \cdots + a_{jk}x_k| + 2|a_{k+1,k+1}| \leq 2M, \quad \text{that is,} \\ \sum_{j=1}^{k} |a_{j1}x_1 + \cdots + a_{jk}x_k| \leq M - |a_{k+1,k+1}|. \end{aligned}$$

Now by the inductive assumption $\sum_{j=1}^{k} |a_{jj}| \leq M - |a_{k+1,k+1}|$, which is equivalent to the desired inequality.

12. Let us start with the case $A = A'$. If the triangles ABC and $A'B'C'$ are oppositely oriented, then they are symmetric with respect to some axis, and the statement is true. Suppose that they are equally oriented. There is a rotation around A by 60° that maps ABB' onto ACC'. This rotation also maps the midpoint B_0 of BB' onto the midpoint C_0 of CC', hence the triangle AB_0C_0 is equilateral.
In the general case, when $A \neq A'$, let us denote by T the translation that maps A onto A'. Let X' be the image of a point X under the (unique) isometry mapping ABC onto $A'B'C'$, and X'' the image of X under T. Furthermore, let X_0, X_0' be the midpoints of segments $XX', X'X''$. Then X_0 is the image of X_0' under the translation $-(1/2)T$. However, since it has already been proven that the triangle $A_0'B_0'C_0'$ is equilateral, its image $A_0B_0C_0$ under $(1/2)T$ is also equilateral. The statement of the problem is thus proven.

13. Let p be the least of all the sums of elements in one row or column. If $p \geq n/2$, then the sum of all elements of the array is $s \geq np \geq n^2/2$.

Now suppose that $p < n/2$. Without loss of generality, one can assume that the sum of elements in the first row is p, and that exactly the first q elements of it are different from zero. Then the sum of elements in the last $n-q$ columns is greater than or equal to $(n-p)(n-q)$. Furthermore, the sum of elements in the first q columns is greater than or equal to pq. This implies that the sum of all elements in the array is

$$s \geq (n-p)(n-q) + pq = \frac{1}{2}n^2 + \frac{1}{2}(n-2p)(n-2q) \geq \frac{1}{2}n^2,$$

since $n \geq 2p \geq 2q$.

14. Denote by V the figure made by a circle of radius 1 whose center moves along the broken line. From the condition of the problem, V contains the whole 50×50 square, and thus the area $S(V)$ of V is not less than 2500. Let L be the length of the broken line. We shall show that $S(V) \leq 2L + \pi$, from which it will follow that $L \geq 1250 - \pi/2 > 1248$. For each segment $l_i = A_iA_{i+1}$ of the broken line, consider the figure V_i obtained by a circle of radius 1 whose center moves along it, and let $\overline{V_i}$ be obtained by cutting off the circle of radius 1 with center at the starting point of l_i. The area of $\overline{V_i}$ is equal to $2A_iA_{i+1}$. It is clear that the union of all the figures $\overline{V_i}$ together with a semicircle with center in A_1 and a semicircle with center in A_n contains V completely. Therefore

$$S(V) \leq \pi + 2A_1A_2 + 2A_2A_3 + \cdots + 2A_{n-1}A_n = \pi + 2L.$$

This completes the proof.

15. Assume the opposite. Then one can numerate the cards 1 to 99, with a number n_i written on the card i, so that $n_{98} \neq n_{99}$. Denote by x_i the remainder of $n_1 + n_2 + \cdots + n_i$ upon division by 100, for $i = 1, 2, \ldots, 99$. All x_i must be distinct: Indeed, if $x_i = x_j$, $i < j$, then $n_{i+1} + \cdots + n_j$ is divisible by 100, which is impossible. Also, no x_i can be equal to 0. Thus, the numbers $x_1, x_2, \ldots, x_{99}$ take exactly the values $1, 2, \ldots, 99$ in some order.
Let x be the remainder of $n_1 + n_2 + \cdots + n_{97} + n_{99}$ upon division by 100. It is not zero; hence it must be equal to x_k for some $k \in \{1, 2, \ldots, 99\}$. There are three cases:
(i) $x = x_k$, $k \leq 97$. Then $n_{k+1} + n_{k+2} + \cdots + n_{97} + n_{99}$ is divisible by 100, a contradiction;
(ii) $x = x_{98}$. Then $n_{98} = n_{99}$, a contradiction;
(iii) $x = x_{99}$. Then n_{98} is divisible by 100, a contradiction.
Therefore, all the cards contain the same number.

16. Denote by P' the polyhedron defined as the image of P under the homothety with center at A_1 and coefficient of similarity 2. It is easy to see that all P_i, $i = 1, \ldots, 9$, are contained in P' (indeed, if $M \in P_k$, then $\frac{1}{2}\overrightarrow{A_1M} = \frac{1}{2}(\overrightarrow{A_1A_k} + \overrightarrow{A_1M'})$ for some $M' \in P$, and the claim follows from

the convexity of P). But the volume of P' is exactly 8 times the volume of P, while the volumes of P_i add up to 9 times that volume. We conclude that not all P_i have disjoint interiors.

17. We use the following obvious consequences of $(a+b)^2 \geq 4ab$:

$$\frac{1}{(a_1+a_2)(a_3+a_4)} \geq \frac{4}{(a_1+a_2+a_3+a_4)^2},$$

$$\frac{1}{(a_1+a_4)(a_2+a_3)} \geq \frac{4}{(a_1+a_2+a_3+a_4)^2}.$$

Now we have

$$\begin{aligned}
&\frac{a_1+a_3}{a_1+a_2}+\frac{a_2+a_4}{a_2+a_3}+\frac{a_3+a_1}{a_3+a_4}+\frac{a_4+a_2}{a_4+a_1} \\
&= \frac{(a_1+a_3)(a_1+a_2+a_3+a_4)}{(a_1+a_2)(a_3+a_4)}+\frac{(a_2+a_4)(a_1+a_2+a_3+a_4)}{(a_1+a_4)(a_2+a_3)} \\
&\geq \frac{4(a_1+a_3)}{a_1+a_2+a_3+a_4}+\frac{4(a_2+a_4)}{a_1+a_2+a_3+a_4}=4.
\end{aligned}$$

4.14 Solutions to the Shortlisted Problems of IMO 1972

1. Suppose that $f(x_0) \neq 0$ and for a given y define the sequence x_k by the formula
$$x_{k+1} = \begin{cases} x_k + y, & \text{if } |f(x_k+y)| \geq |f(x_k-y)|; \\ x_k - y, & \text{otherwise.} \end{cases}$$
It follows from (1) that $|f(x_{k+1})| \geq |\varphi(y)||f(x_k)|$; hence by induction, $|f(x_k)| \geq |\varphi(y)|^k|f(x_0)|$. Since $|f(x_k)| \leq 1$ for all k, we obtain $|\varphi(y)| \leq 1$.
Second solution. Let $M = \sup f(x) \leq 1$, and x_k any sequence, possibly constant, such that $f(x_k) \to M$, $k \to \infty$. Then for all k,
$$|\varphi(y)| = \frac{|f(x_k+y)+f(x_k-y)|}{2|f(x_k)|} \leq \frac{2M}{2|f(x_k)|} \to 1, \qquad k \to \infty.$$

2. We use induction. For $n = 1$ the assertion is obvious. Assume that it is true for a positive integer n. Let $A_1, A_2, \dots, A_{3n+3}$ be given $3n+3$ points, and let w.l.o.g. $A_1A_2\dots A_m$ be their convex hull.
Among all the points A_i distinct from A_1, A_2, we choose the one, say A_k, for which the angle $\angle A_kA_1A_2$ is minimal (this point is uniquely determined, since no three points are collinear). The line A_1A_k separates the plane into two half-planes, one of which contains A_2 only, and the other one all the remaining $3n$ points. By the inductive hypothesis, one can construct n disjoint triangles with vertices in these $3n$ points. Together with the triangle $A_1A_2A_k$, they form the required system of disjoint triangles.

3. We have for each $k = 1, 2, \dots, n$ that $m \leq x_k \leq M$, which gives $(M - x_k)(m - x_k) \leq 0$. It follows directly that
$$0 \geq \sum_{k=1}^{n}(M-x_k)(m-x_k) = nmM - (m+M)\sum_{k=1}^{n} x_k + \sum_{k=1}^{n} x_k^2.$$
But $\sum_{k=1}^{n} x_k = 0$, implying the required inequality.

4. Choose in E a half-line s beginning at a point O. For every α in the interval $[0, 180°]$, denote by $s(\alpha)$ the line obtained by rotation of s about O by α, and by $g(\alpha)$ the oriented line containing $s(\alpha)$ on which $s(\alpha)$ defines the positive direction. For each P in M_i, $i = 1, 2$, let $P(\alpha)$ be the foot of the perpendicular from P to $g(\alpha)$, and $l_P(\alpha)$ the oriented (positive, negative or zero) distance of $P(\alpha)$ from O. Then for $i = 1, 2$ one can arrange the $l_P(\alpha)$ $(P \in M_i)$ in ascending order, as $l_1(\alpha), l_2(\alpha), \dots, l_{2n_i}(\alpha)$. Call $J_i(\alpha)$ the interval $[l_{n_i}(\alpha), l_{n_i+1}(\alpha)]$. It is easy to see that any line perpendicular to $g(\alpha)$ and passing through the point with the distance l in the interior of $J_i(\alpha)$ from O, will divide the set M_i into two subsets of equal cardinality. Therefore it remains to show that for some α, the interiors of intervals $J_1(\alpha)$ and $J_2(\alpha)$ have a common point. If this holds for $\alpha = 0$, then

we have finished. Suppose w.l.o.g. that $J_1(0)$ lies on $g(0)$ to the left of $J_2(0)$; then $J_1(180°)$ lies to the right of $J_2(180°)$. Note that J_1 and J_2 cannot simultaneously degenerate to a point (otherwise, we would have four collinear points in $M_1 \cup M_2$); also, each of them degenerates to a point for only finitely many values of α. Since $J_1(\alpha)$ and $J_2(\alpha)$ move continuously, there exists a subinterval I of $[0, 180°]$ on which they are not disjoint. Thus, at some point of I, they are both nondegenerate and have a common interior point, as desired.

5. *Lemma.* If X, Y, Z, T are points in space, then the lines XZ and YT are perpendicular if and only if $XY^2 + ZT^2 = YZ^2 + TX^2$.

   Proof. Consider the plane π through XZ parallel to YT. If Y', T' are the feet of the perpendiculars to π from Y, T respectively, then

$$\begin{aligned} & XY^2 + ZT^2 = XY'^2 + ZT'^2 + 2YY'^2, \\ \text{and} \quad & YZ^2 + TX^2 = Y'Z^2 + T'X^2 + 2YY'^2. \end{aligned}$$

   Since by the Pythagorean theorem $XY'^2 + ZT'^2 = Y'Z^2 + T'X^2$, i.e., $XY'^2 - Y'Z^2 = XT'^2 - T'Z^2$, if and only if $Y'T' \perp XZ$, the statement follows.

   Assume that the four altitudes intersect in a point P. Then we have $DP \perp ABC \Rightarrow DP \perp AB$ and $CP \perp ABD \Rightarrow CP \perp AB$, which implies that $CDP \perp AB$, and $CD \perp AB$. By the lemma, $AC^2 + BD^2 = AD^2 + BC^2$. Using the same procedure we obtain the relation $AD^2 + BC^2 = AB^2 + CD^2$.

   Conversely, assume that $AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2$. The lemma implies that $AB \perp CD$, $AC \perp BD$, $AD \perp BC$. Let π be the plane containing CD that is perpendicular to AB, and let h_D be the altitude from D to ABC. Since $\pi \perp AB$, we have $\pi \perp ABC \Rightarrow h_D \subset \pi$ and $\pi \perp ABD \Rightarrow h_C \subset \pi$. The altitudes h_D and h_C are not parallel; thus they have an intersection point P_{CD}. Analogously, $h_B \cap h_C = \{P_{BC}\}$ and $h_B \cap h_D = \{P_{BD}\}$, where both these points belong to π. On the other hand, h_B doesn't belong to π; otherwise, it would be perpendicular to both ACD and $AB \subset \pi$, i.e. $AB \subset ACD$, which is impossible. Hence, h_B can have at most one common point with π, implying $P_{BD} = P_{CD}$. Analogously, $P_{AB} = P_{BD} = P_{CD} = P_{ABCD}$.

6. Let $n = 2^\alpha 5^\beta m$, where $\alpha = 0$ or $\beta = 0$. These two cases are analogous, and we treat only $\alpha = 0$, $n = 5^\beta m$. The case $m = 1$ is settled by the following lemma.

   Lemma. For any integer $\beta \geq 1$ there exists a multiple M_β of 5^β with β digits in decimal expansion, all different from 0.

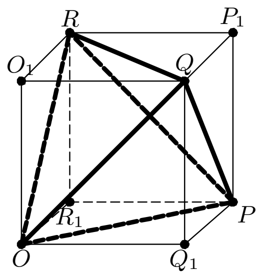
   Proof. For $\beta = 1$, $M_1 = 5$ works. Assume that the lemma is true for $\beta = k$. There is a positive integer $C_k \leq 5$ such that $C_k 2^k + m_k \equiv 0 \pmod 5$, where $5^k m_k = M_k$, i.e. $C_k 10^k + M_k \equiv 0 \pmod{5^{k+1}}$. Then $M_{k+1} = C_k 10^k + M_k$ satisfies the conditions, and proves the lemma.

In the general case, consider, the sequence $1, 10^{\beta}, 10^{2\beta}, \dots$. It contains two numbers congruent modulo $(10^{\beta}-1)m$, and therefore for some $k > 0$, $10^{k\beta} \equiv 1 \pmod{(10^{\beta}-1)m}$ (this is in fact a consequence of Fermat's theorem). The number

$$\frac{10^{k\beta}-1}{10^{\beta}-1}M_{\beta} = 10^{(k-1)\beta}M_{\beta} + 10^{(k-2)\beta}M_{\beta} + \cdots + M_{\beta}$$

is a multiple of $n = 5^{\beta}m$ with the required property.

7. (i) Consider the circumscribing cube $OQ_1PR_1O_1QP_1R$ (that is, the cube in which the edges of the tetrahedron are small diagonals), of side $b = a\sqrt{2}/2$. The left-hand side is the sum of squares of the projections of the edges of the tetrahedron onto a perpendicular l to π. On the other hand, if l forms angles $\varphi_1, \varphi_2, \varphi_3$ with OO_1, OQ_1, OR_1 respectively, then the projections of OP and QR onto l have lengths $b(\cos\varphi_2 + \cos\varphi_3)$ and $b|\cos\varphi_2 - \cos\varphi_3|$. Summing up all these expressions, we obtain



$$4b^2(\cos^2\varphi_1 + \cos^2\varphi_2 + \cos^2\varphi_3) = 4b^2 = 2a^2.$$

(ii) We construct a required tetrahedron of edge length a given in (i). Take O arbitrarily on π_0, and let p, q, r be the distances of O from π_1, π_2, π_3. Since $a > p, q, r, |p-q|$, we can choose P on π_1 anywhere at distance a from O, and Q at one of the two points on π_2 at distance a from both O and P. Consider the fourth vertex of the tetrahedron: its distance from π_0 will satisfy the equation from (i); i.e., there are two values for this distance; clearly, one of them is r, putting R on π_3.

8. Let $f(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}$. Then it is directly shown that

$$f(m,n) = 4f(m,n-1) - f(m+1,n-1),$$

and thus n may be successively reduced until one obtains $f(m,n) = \sum_r c_r f(r,0)$. Now $f(r,0)$ is a simple binomial coefficient, and the c_r's are integers.

Second solution. For each prime p, the greatest exponents of p that divide the numerator $(2m)!(2n)!$ and denominator $m!n!(m+n)!$ are respectively

$$\sum_{k>0}\left(\left[\frac{2m}{p^k}\right] + \left[\frac{2n}{p^k}\right]\right) \quad \text{and} \quad \sum_{k>0}\left(\left[\frac{m}{p^k}\right] + \left[\frac{n}{p^k}\right] + \left[\frac{m+n}{p^k}\right]\right);$$

hence it suffices to show that the first exponent is not less than the second one for every p. This follows from the fact that for each real x, $[2x]+[2y] \geq$

$[x]+[y]+[x+y]$, which is straightforward to prove (for example, using $[2x]=[x]+[x+1/2]$).

9. Clearly $x_1=x_2=x_3=x_4=x_5$ is a solution. We shall show that this describes all solutions.
Suppose that not all x_i are equal. Then among x_3,x_5,x_2,x_4,x_1 two consecutive are distinct: Assume w.l.o.g. that $x_3\neq x_5$. Moreover, since $(1/x_1,\dots,1/x_5)$ is a solution whenever $(x_1,\dots,x_5)$ is, we may assume that $x_3<x_5$.
Consider first the case $x_1\le x_2$. We infer from (i) that $x_1\le\sqrt{x_3x_5}<x_5$ and $x_2\ge\sqrt{x_3x_5}>x_3$. Then $x_5^2>x_1x_3$, which together with (iv) gives $x_4^2\le x_1x_3<x_3x_5$; but we also have $x_3^2\le x_5x_2$; hence by (iii), $x_4^2\ge x_5x_2>x_5x_3$, a contradiction.
Consider next the case $x_1>x_2$. We infer from (i) that $x_1\ge\sqrt{x_3x_5}>x_3$ and $x_2\le\sqrt{x_3x_5}<x_5$. Then by (ii) and (v),
$$x_1x_4\le\max(x_2^2,x_3^2)\le x_3x_5\qquad\text{and}\qquad x_2x_4\ge\min(x_1^2,x_5^2)\ge x_3x_5,$$
which contradicts the assumption $x_1>x_2$.
Second solution.
$$\begin{aligned}0\ge L_1&=(x_1^2-x_3x_5)(x_2^2-x_3x_5)=x_1^2x_2^2+x_3^2x_5^2-(x_1^2+x_2^2)x_3x_5\\&\ge x_1^2x_2^2+x_3^2x_5^2-\frac{1}{2}(x_1^2x_3^2+x_1^2x_5^2+x_2^2x_3^2+x_2^2x_5^2),\end{aligned}$$
and analogously for $L_2,\dots,L_5$. Therefore $L_1+L_2+L_3+L_4+L_5\ge 0$, with the only case of equality $x_1=x_2=x_3=x_4=x_5$.

10. Consider first a triangle. It can be decomposed into $k=3$ cyclic quadrilaterals by perpendiculars from some interior point of it to the sides; also, it can be decomposed into a cyclic quadrilateral and a triangle, and it follows by induction that this decomposition is possible for every k. Since every triangle can be cut into two triangles, the required decomposition is possible for each $n\ge 6$. It remains to treat the cases $n=4$ and $n=5$.
$n=4$. If the center O of the circumcircle is inside a cyclic quadrilateral $ABCD$, then the required decomposition is effected by perpendiculars from O to the four sides. Otherwise, let C and D be the vertices of the obtuse angles of the quadrilateral. Draw the perpendiculars at C and D to the lines BC and AD respectively, and choose points P and Q on them such that $PQ\parallel AB$. Then the required decomposition is effected by CP,PQ,QD and the perpendiculars from P and Q to AB.
$n=5$. If $ABCD$ is an isosceles trapezoid with $AB\parallel CD$ and $AD=BC$, then it is trivially decomposed by lines parallel to AB. Otherwise, $ABCD$ can be decomposed into a cyclic quadrilateral and a trapezoid; this trapezoid can be cut into an isosceles trapezoid and a triangle, which can further be cut into three cyclic quadrilaterals and an isosceles trapezoid.

Remark. It can be shown that the assertion is not true for $n = 2$ and $n = 3$.

11. Let $\angle A = 2x$, $\angle B = 2y$, $\angle C = 2z$.
    (a) Denote by M_i the center of K_i, $i = 1, 2, \dots$. If N_1, N_2 are the projections of M_1, M_2 onto AB, we have $AN_1 = r_1 \cot x$, $N_2B = r_2 \cot y$, and $N_1N_2 = \sqrt{(r_1 + r_2)^2 - (r_1 - r_2)^2} = 2\sqrt{r_1 r_2}$. The required relation between r_1, r_2 follows from $AB = AN_1 + N_1N_2 + N_2B$.
    If this relation is further considered as a quadratic equation in $\sqrt{r_2}$, then its discriminant, which equals
    $$\Delta = 4\left(r(\cot x + \cot y)\cot y - r_1(\cot x \cot y - 1)\right),$$
    must be nonnegative, and therefore $r_1 \le r \cot y \cot z$. Then $t_1, t_2, \dots$ exist, and we can assume that $t_i \in [0, \pi/2]$.
    (b) Substituting $r_1 = r \cot y \cot z \sin^2 t_1$, $r_2 = r \cot z \cot x \sin^2 t_2$ in the relation of (a) we obtain that $\sin^2 t_1 + \sin^2 t_2 + k^2 + 2k \sin t_1 \sin t_2 = 1$, where we set $k = \sqrt{\tan x \tan y}$. It follows that $(k + \sin t_1 \sin t_2)^2 = (1 - \sin^2 t_1)(1 - \sin^2 t_2) = \cos^2 t_1 \cos^2 t_2$, and hence
    $$\cos(t_1 + t_2) = \cos t_1 \cos t_2 - \sin t_1 \sin t_2 = k = \sqrt{\tan x \tan y},$$
    which is constant. Writing the analogous relations for each t_i, t_{i+1} we conclude that $t_1 + t_2 = t_4 + t_5$, $t_2 + t_3 = t_5 + t_6$, and $t_3 + t_4 = t_6 + t_7$. It follows that $t_1 = t_7$, i.e., $K_1 = K_7$.

12. First we observe that it is not essential to require the subsets to be disjoint (if they aren't, one simply excludes their intersection). There are $2^{10} - 1 = 1023$ different subsets and at most 990 different sums. By the pigeonhole principle there are two different subsets with equal sums.

4.15 Solutions to the Shortlisted Problems of IMO 1973

1. The condition of the point P can be written in the form $\frac{AP^2}{AP\cdot PA_1}+\frac{BP^2}{BP\cdot PB_1}+\frac{CP^2}{CP\cdot PC_1}+\frac{DP^2}{DP\cdot PD_1}=4$. All the four denominators are equal to R^2-OP^2, i.e., to the power of P with respect to S. Thus the condition becomes
$$AP^2+BP^2+CP^2+DP^2=4(R^2-OP^2). \tag{1}$$
Let M and N be the midpoints of segments AB and CD respectively, and G the midpoint of MN, or the centroid of $ABCD$. By Stewart's formula, an arbitrary point P satisfies
$$\begin{aligned} AP^2+BP^2+CP^2+DP^2 &= 2MP^2+2NP^2+\frac{1}{2}AB^2+\frac{1}{2}CD^2 \\ &= 4GP^2+MN^2+\frac{1}{2}(AB^2+CD^2). \end{aligned}$$
Particularly, for $P\equiv O$ we get $4R^2=4OG^2+MN^2+\frac{1}{2}(AB^2+CD^2)$, and the above equality becomes
$$AP^2+BP^2+CP^2+DP^2=4GP^2+4R^2-4OG^2.$$
Therefore (1) is equivalent to $OG^2=OP^2+GP^2 \Leftrightarrow \angle OPG=90^\circ$. Hence the locus of points P is the sphere with diameter OG. Now the converse is easy.

2. Let D' be the reflection of D across A. Since $BCAD'$ is then a parallelogram, the condition $BD\ge AC$ is equivalent to $BD\ge BD'$, which is in turn equivalent to $\angle BAD\ge\angle BAD'$, i.e. to $\angle BAD\ge 90^\circ$. Thus the needed locus is actually the locus of points A for which there exist points B,D inside K with $\angle BAD=90^\circ$. Such points B,D exist if and only if the two tangents from A to K, say AP and AQ, determine an obtuse angle. Then if $P,Q\in K$, we have $\angle PAO=\angle QAO=\varphi>45^\circ$; hence $OA=\frac{OP}{\sin\varphi}<OP\sqrt{2}$. Therefore the locus of A is the interior of the circle K' with center O and radius $\sqrt{2}$ times the radius of K.

3. We use induction on odd numbers n. For $n=1$ there is nothing to prove. Suppose that the result holds for $n-2$ vectors, and let us be given vectors $v_1,v_2,\dots,v_n$ arranged clockwise. Set $v'=v_2+v_3+\cdots+v_{n-1}$, $u=v_1+v_n$, and $v=v_1+v_2+\cdots+v_n=v'+u$. By the inductive hypothesis we have $|v'|\ge 1$. Now if the angles between v' and the vectors v_1, v_n are α and β respectively, then the angle between u and v' is $|\alpha-\beta|/2\le 90^\circ$. Hence $|v'+u|\ge|v'|\ge 1$.

   Second solution. Again by induction, it can be easily shown that all possible values of the sum $v=v_1+v_2+\cdots+v_n$, for n vectors $v_1,\dots,v_n$ in the upper half-plane (with $y\ge 0$), are those for which $|v|\le n$ and $|v-ke|\ge 1$ for every integer k for which $n-k$ is odd, where e is the unit vector on the x axis.

4. Each of the subsets must be of the form $\{a^2, ab, ac, ad\}$ or $\{a^2, ab, ac, bc\}$. It is now easy to count up the partitions. The result is 26460.

5. Let O be the vertex of the trihedron, Z the center of a circle k inscribed in the trihedron, and A, B, C points in which the plane of the circle meets the edges of the trihedron. We claim that the distance OZ is constant. Set $OA = x$, $OB = y$, $OC = z$, $BC = a$, $CA = b$, $AB = c$, and let S and $r = 1$ be the area and inradius of $\triangle ABC$. Since Z is the incenter of ABC, we have $(a+b+c)\overrightarrow{OZ} = a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}$. Hence
$$(a+b+c)^2 OZ^2 = (a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC})^2 = a^2x^2 + b^2y^2 + c^2z^2. \quad (1)$$
But since $y^2 + z^2 = a^2$, $z^2 + x^2 = b^2$ and $x^2 + y^2 = c^2$, we obtain $x^2 = \frac{-a^2+b^2+c^2}{2}$, $y^2 = \frac{a^2-b^2+c^2}{2}$, $z^2 = \frac{a^2+b^2-c^2}{2}$. Substituting these values in (1) yields
$$\begin{aligned}(a+b+c)^2 OZ^2 &= \frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{2} \\ &= 8S^2 = 2(a+b+c)^2 r^2.\end{aligned}$$
Hence $OZ = r\sqrt{2} = \sqrt{2}$, and Z belongs to a sphere σ with center O and radius $\sqrt{2}$.
Moreover, the distances of Z from the faces of the trihedron do not exceed 1; hence Z belongs to a part of σ that lies inside the unit cube with three faces lying on the faces of the trihedron. It is easy to see that this part of σ is exactly the required locus.

6. Yes. Take for $\mathcal{M}$ the set of vertices of a cube $ABCDEFGH$ and two points I, J symmetric to the center O of the cube with respect to the laterals $ABCD$ and $EFGH$.

   Remark. We prove a stronger result: Given an arbitrary finite set of points $\mathcal{S}$, then there is a finite set $\mathcal{M} \supset \mathcal{S}$ with the described property.
   Choose a point $A \in \mathcal{S}$ and any point O such that $AO \parallel BC$ for some two points $B, C \in \mathcal{S}$. Now let X' be the point symmetric to X with respect to O, and $\mathcal{S}' = \{X, X' \mid X \in S\}$. Finally, take $\mathcal{M} = \{X, \overline{X} \mid X \in S'\}$, where $\overline{X}$ denotes the point symmetric to X with respect to A. This $\mathcal{M}$ has the desired property: If $X, Y \in \mathcal{M}$ and $Y \neq \overline{X}$, then $XY \parallel \overline{XY}$; otherwise, $X\overline{X}$, i.e., XA is parallel to $X'A'$ if $X \neq A'$, or to BC otherwise.

7. The result follows immediately from Ptolemy's inequality.

8. Let f_n be the required total number, and let $f_n(k)$ denote the number of sequences $a_1, \dots, a_n$ of nonnegative integers such that $a_1 = 0$, $a_n = k$, and $|a_i - a_{i+1}| = 1$ for $i = 1, \dots, n-1$. In particular, $f_1(0) = 1$ and $f_n(k) = 0$ if $k < 0$ or $k \geq n$. Since a_{n-1} is either $k-1$ or $k+1$, we have
$$f_n(k) = f_{n-1}(k+1) + f_{n-1}(k-1) \quad \text{for } k \geq 1. \quad (1)$$

By successive application of (1) we obtain

$$f_n(k) = \sum_{i=0}^{r} \left[\binom{r}{i} - \binom{r}{i-k-1} \right] f_{n-r}(k+r-2i). \tag{2}$$

This can be verified by direct induction. Substituting $r = n-1$ in (2), we get at most one nonzero summand, namely the one for which $i = \frac{k+n-1}{2}$. Therefore $f_n(n-1-2j) = \binom{n-1}{j} - \binom{n-1}{j-1}$. Adding up these equalities for $j = 0, 1, \dots, \left[\frac{n-1}{2}\right]$ we obtain $f_n = \binom{n-1}{[\frac{n-1}{2}]}$, as required.

9. Let a, b, c be vectors going along Ox, Oy, Oz, respectively, such that $\overrightarrow{OG} = a + b + c$. Now let $A \in Ox$, $B \in Oy$, $C \in Oz$ and let $\overrightarrow{OA} = \alpha a$, $\overrightarrow{OB} = \beta b$, $\overrightarrow{OC} = \gamma c$, where $\alpha, \beta, \gamma > 0$. Point G belongs to a plane ABC with $A \in Ox$, $B \in Oy$, $C \in Oz$ if and only if there exist positive real numbers λ, μ, ν with sum 1 such that $\lambda\overrightarrow{OA} + \mu\overrightarrow{OB} + \nu\overrightarrow{OC} = \overrightarrow{OG}$, which is equivalent to $\lambda\alpha = \mu\beta = \nu\gamma = 1$. Such λ, μ, ν exist if and only if
$$\alpha, \beta, \gamma > 0 \quad \text{and} \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.$$
Since the volume of $OABC$ is proportional to the product $\alpha\beta\gamma$, it is minimized when $\frac{1}{\alpha} \cdot \frac{1}{\beta} \cdot \frac{1}{\gamma}$ is maximized, which occurs when $\alpha = \beta = \gamma = 3$ and G is the centroid of $\triangle ABC$.

10. Let
$$b_k = a_1 q^{k-1} + \cdots + a_{k-1} q + a_k + a_{k+1} q + \cdots + a_n q^{n-k}, \qquad k = 1, 2, \dots, n.$$
We show that these numbers satisfy the required conditions. Obviously $b_k \ge a_k$. Further,
$$b_{k+1} - q b_k = -[(q^2-1)a_{k+1} + \cdots + q^{n-k-1}(q^2-1)a_n] > 0\,;$$
we analogously obtain $q b_{k+1} - b_k < 0$. Finally,
$$\begin{aligned} b_1 + b_2 + \cdots + b_n &= a_1(q^{n-1} + \cdots + q + 1) + \dots \\ &\quad + a_k(q^{n-k} + \cdots + q + 1 + q + \cdots + q^{k-1}) + \dots \\ &\le (a_1 + a_2 + \cdots + a_n)(1 + 2q + 2q^2 + \cdots + 2q^{n-1}) \\ &< \frac{1+q}{1-q}(a_1 + \cdots + a_n). \end{aligned}$$

11. Putting $x + \frac{1}{x} = t$ we also get $x^2 + \frac{1}{x^2} = t^2 - 2$, and the given equation reduces to $t^2 + at + b - 2 = 0$. Since $x = \frac{t \pm \sqrt{t^2-4}}{2}$, x will be real if and only if $|t| \ge 2$, $t \in \mathbb{R}$. Thus we need the minimum value of $a^2 + b^2$ under the condition $at + b = -(t^2 - 2)$, $|t| \ge 2$.
However, by the Cauchy–Schwarz inequality we have

$$(a^2+b^2)(t^2+1) \geq (at+b)^2 = (t^2-2)^2.$$

It follows that $a^2+b^2 \geq h(t) = \frac{(t^2-2)^2}{t^2+1}$. Since $h(t) = (t^2+1) + \frac{9}{t^2+1} - 6$ is increasing for $t \geq 2$, we conclude that $a^2+b^2 \geq h(2) = \frac{4}{5}$.
The cases of equality are easy to examine: These are $a = \pm\frac{4}{5}$ and $b = -\frac{2}{5}$.
Second solution. In fact, there was no need for considering $x = t+1/t$. By the Cauchy–Schwarz inequality we have $(a^2+2b^2+a^2)(x^6+x^4/2+x^2) \geq (ax^3+bx^2+ax)^2 = (x^4+1)^2$. Hence

$$a^2+b^2 \geq \frac{(x^4+1)^2}{2x^6+x^4+2x^2} \geq \frac{4}{5},$$

with equality for $x=1$.

12. Observe that the absolute values of the determinants of the given matrices are invariant under all the admitted operations. The statement follows from $\det A = 16 \neq \det B = 0$.

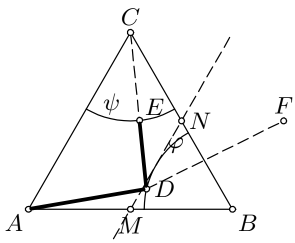
13. Let S_1, S_2, S_3, S_4 denote the areas of the faces of the tetrahedron, V its volume, h_1, h_2, h_3, h_4 its altitudes, and r the radius of its inscribed sphere. Since

$$3V = S_1h_1 = S_2h_2 = S_3h_3 = S_4h_4 = (S_1+S_2+S_3+S_4)r,$$

it follows that
$$\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}+\frac{1}{h_4} = \frac{1}{r}.$$

In our case, $h_1, h_2, h_3, h_4 \geq 1$, hence $r \geq 1/4$. On the other hand, it is clear that a sphere of radius greater than $1/4$ cannot be inscribed in a tetrahedron all of whose altitudes have length equal to 1. Thus the answer is $1/4$.

14. Suppose that the soldier starts at the vertex A of the equilateral triangle ABC of side length a. Let φ, ψ be the arcs of circles with centers B and C and radii $a\sqrt{3}/4$ respectively, that lie inside the triangle. In order to check the vertices B, C, he must visit some points $D \in \varphi$ and $E \in \psi$.



Thus his path cannot be shorter than the path ADE (or AED) itself. The length of the path ADE is $AD+DE \geq AD+DC-a\sqrt{3}/4$. Let F be the reflection of C across the line MN, where M, N are the midpoints of AB and BC. Then $DC \geq DF$ and hence $AD+DC \geq AD+DF \geq AF$. Consequently $AD+DE \geq AF - a\frac{\sqrt{3}}{4} = a\left(\frac{\sqrt{7}}{2} - \frac{\sqrt{3}}{4}\right)$, with equality if and only if D is the midpoint of arc φ and $E = (CD) \cap \psi$.

Moreover, it is easy to verify that, in following the path ADE, the soldier will check the whole region. Therefore this path (as well as the one symmetric to it) is shortest possible path that the soldier can take in order to check the entire field.

15. If $z = \cos\theta + i\sin\theta$, then $z - z^{-1} = 2i\sin\theta$. Now put $z = \cos\frac{\pi}{2n+1} + i\sin\frac{\pi}{2n+1}$. Using de Moivre's formula we transform the required equality into

$$A = \prod_{k=1}^{n}(z^k - z^{-k}) = i^n\sqrt{2n+1}. \tag{1}$$

On the other hand, the complex numbers z^{2k} $(k = -n, -n+1, \dots, n)$ are the roots of $x^{2n+1} - 1$, and hence

$$\prod_{k=1}^{n}(x - z^{2k})(x - z^{-2k}) = \frac{x^{2n+1} - 1}{x - 1} = x^{2n} + \cdots + x + 1. \tag{2}$$

Now we go back to proving (1). We have

$$(-1)^n z^{n(n+1)/2}A = \prod_{k=1}^{n}(1 - z^{2k}) \quad \text{and} \quad z^{-n(n+1)/2}A = \prod_{k=1}^{n}(1 - z^{-2k}).$$

Multiplying these two equalities, we obtain $(-1)^n A^2 = \prod_{k=1}^{n}(1 - z^{2k})(1 - z^{-2k}) = 2n+1$, by (2). Therefore $A = \pm i^{-n}\sqrt{2n+1}$. This actually implies that the required product is $\pm\sqrt{2n+1}$, but it must be positive, since all the sines are, and the result follows.

16. First, we have $P(x) = Q(x)R(x)$ for $Q(x) = x^m - |a|^m e^{i\theta}$ and $R(x) = x^m - |a|^m e^{-i\theta}$, where $e^{i\varphi}$ means of course $\cos\varphi + i\sin\varphi$. It remains to factor both Q and R. Suppose that $Q(x) = (x - q_1)\cdots(x - q_m)$ and $R(x) = (x - r_1)\cdots(x - r_m)$.
Considering $Q(x)$, we see that $|q_k^m| = |a|^m$ and also $|q_k| = |a|$ for $k = 1, \dots, m$. Thus we may put $q_k = |a|e^{i\beta_k}$ and obtain by de Moivre's formula $q_k^m = |a|^m e^{im\beta_k}$. It follows that $m\beta_k = \theta + 2j\pi$ for some $j \in \mathbb{Z}$, and we have exactly m possibilities for β_k modulo 2π: $\beta_k = \frac{\theta + 2(k-1)\pi}{m}$ for $k = 1, 2, \dots, m$.
Thus $q_k = |a|e^{i\beta_k}$; analogously we obtain for $R(x)$ that $r_k = |a|e^{-i\beta_k}$. Consequently,

$$x^m - |a|^m e^{i\theta} = \prod_{k=1}^{m}(x - |a|e^{i\beta_k}) \quad \text{and} \quad x^m - |a|^m e^{-i\theta} = \prod_{k=1}^{m}(x - |a|e^{-i\beta_k}).$$

Finally, grouping the kth factors of both polynomials, we get

$$\begin{aligned} P(x) &= \prod_{k=1}^{m}(x - |a|e^{i\beta_k})(x - |a|e^{-i\beta_k}) = \prod_{k=1}^{m}(x^2 - 2|a|x\cos\beta_k + a^2) \\ &= \prod_{k=1}^{m}\left(x^2 - 2|a|x\cos\frac{\theta + 2(k-1)\pi}{m} + a^2\right). \end{aligned}$$

17. Let $f_1(x) = ax+b$ and $f_2(x) = cx+d$ be two functions from $\mathcal{F}$. We define

$$g(x) = f_1 \circ f_2(x) = acx+(ad+b) \quad \text{and} \quad h(x) = f_2 \circ f_1(x) = acx+(bc+d).$$

By the condition for $\mathcal{F}$, both $g(x)$ and $h(x)$ belong to $\mathcal{F}$. Moreover, there exists $h^{-1}(x) = \frac{x-(bc+d)}{ac}$, and

$$h^{-1} \circ g(x) = \frac{acx + (ad+b) - (bc+d)}{ac} = x + \frac{(ad+b) - (bc+d)}{ac}$$

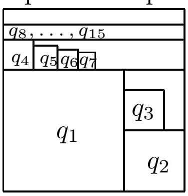
belongs to $\mathcal{F}$. Now it follows that we must have $ad+b = bc+d$ for every $f_1, f_2 \in \mathcal{F}$, which is equivalent to $\frac{b}{a-1} = \frac{d}{c-1} = k$. But these formulas exactly describe the fixed points of f_1 and f_2: $f_1(x) = ax+b = x \Rightarrow x = \frac{b}{a-1}$. Hence all the functions in $\mathcal{F}$ fix the point k.

4.16 Solutions to the Shortlisted Problems of IMO 1974

1. Denote by n the number of exams. We have $n(A+B+C) = 20+10+9 = 39$, and since A, B, C are distinct, their sum is at least 6; therefore $n = 3$ and $A + B + C = 13$.
   Assume w.l.o.g. that $A > B > C$. Since Betty gained A points in arithmetic, but fewer than 13 points in total, she had C points in both remaining exams (in spelling as well). Furthermore, Carol also gained fewer than 13 points, but with at least B points on two examinations (on which Betty scored C), including spelling. If she had A in spelling, then she would have at least $A + B + C = 13$ points in total, a contradiction. Hence, Carol scored B and placed second in spelling.

   Remark. Moreover, it follows that Alice, Betty, and Carol scored $B+A+A$, $A + C + C$, and $C + B + B$ respectively, and that $A = 8$, $B = 4$, $C = 1$.

2. We denote by q_i the square with side $\frac{1}{i}$. Let us divide the big square into rectangles r_i by parallel lines, where the size of r_i is $\frac{3}{2} \times \frac{1}{2^i}$ for $i = 2, 3, \dots$ and $\frac{3}{2} \times 1$ for $i = 1$ (this can be done because $1 + \sum_{i=2}^{\infty} \frac{1}{2^i} = \frac{3}{2}$). In rectangle r_1, one can put the squares q_1, q_2, q_3, as is done on the figure. Also, since $\frac{1}{2^i} + \cdots + \frac{1}{2^{i+1}-1} < 2^i \cdot \frac{1}{2^i} = 1 < \frac{3}{2}$, in each r_i, $i \geq 2$, one can put $q_{2^i}, \dots, q_{2^{i+1}-1}$. This completes the proof.



   Remark. It can be shown that the squares q_1, q_2 cannot fit in any square of side less than $\frac{3}{2}$.

3. For $\deg(P) \leq 2$ the statement is obvious, since $n(P) \leq \deg(P^2) = 2\deg(P) \leq \deg(P) + 2$.
   Suppose now that $\deg(P) \geq 3$ and $n(P) > \deg(P) + 2$. Then there is at least one integer b for which $P(b) = -1$, and at least one x with $P(x) = 1$. We may assume w.l.o.g. that $b = 0$ (if necessary, we consider the polynomial $P(x + b)$ instead). If $k_1, \dots, k_m$ are all integers for which $P(k_i) = 1$, then $P(x) = Q(x)(x - k_1)\cdots(x - k_m) + 1$ for some polynomial $Q(x)$ with integer coefficients. Setting $x = 0$ we obtain $(-1)^m Q(0)k_1 \cdots k_m = 1 - P(0) = 2$. It follows that $k_1 \cdots k_m \mid 2$, and hence m is at most 3. The same holds for the polynomial $-P(x)$, and thus $P(x) = -1$ also has at most 3 integer solutions. This counts for 6 solutions of $P^2(x) = 1$ in total, implying the statement for $\deg(P) \geq 4$.
   It remains to verify the statement for $n = 3$. If $\deg(P) = 3$ and $n(P) = 6$, then it follows from the above consideration that $P(x)$ is either $-(x^2 - 1)(x - 2) + 1$ or $(x^2 - 1)(x + 2) + 1$. It is directly checked that $n(P)$ equals only 4 in both cases.

4. Assume w.l.o.g. that $a_1 \le a_2 \le a_3 \le a_4 \le a_5$. If m is the least value of $|a_i - a_j|$, $i \ne j$, then $a_{i+1} - a_i \ge m$ for $i = 1, 2, \dots, 5$, and consequently $a_i - a_j \ge (i-j)m$ for any $i, j \in \{1, \dots, 5\}$, $i > j$. Then it follows that

$$\sum_{i>j}(a_i - a_j)^2 \ge m^2 \sum_{i>j}(i-j)^2 = 50m^2.$$

On the other hand, by the condition of the problem,

$$\sum_{i>j}(a_i - a_j)^2 = 5\sum_{i=1}^{5} a_i^2 - (a_1 + \cdots + a_5)^2 \le 5.$$

Therefore $50m^2 \le 5$; i.e., $m^2 \le \frac{1}{10}$.

5. All the angles are assumed to be oriented and measured modulo $180°$. Denote by $\alpha_i, \beta_i, \gamma_i$ the angles of triangle $\triangle_i$, at A_i, B_i, C_i respectively. Let us determine the angles of $\triangle_{i+1}$. If D_i is the intersection of lines B_iB_{i+1} and C_iC_{i+1}, we have $\angle B_{i+1}A_{i+1}C_{i+1} = \angle D_iB_iC_{i+1} = \angle B_iD_iC_{i+1} + \angle D_iC_{i+1}B_i = \angle B_iD_iC_i - \angle B_iC_{i+1}C_i = -2\angle B_iA_iC_i$. We conclude that

$$\alpha_{i+1} = -2\alpha_i, \qquad \text{and analogously} \qquad \beta_{i+1} = -2\beta_i, \qquad \gamma_{i+1} = -2\gamma_i.$$

Therefore $\alpha_{r+t} = (-2)^t\alpha_r$. However, since $(-2)^{12} \equiv 1 \pmod{45}$ and consequently $(-2)^{14} \equiv (-2)^2 \pmod{180}$, it follows that $\alpha_{15} = \alpha_3$, since all values are modulo $180°$. Analogously, $\beta_{15} = \beta_3$ and $\gamma_{15} = \gamma_3$, and moreover, $\triangle_3$ and $\triangle_{15}$ are inscribed in the same circle; hence $\triangle_3 \cong \triangle_{15}$.

6. We set

$$x = \sum_{k=0}^{n}\binom{2n+1}{2k+1}2^{3k} = \frac{1}{\sqrt{8}}\sum_{k=0}^{n}\binom{2n+1}{2k+1}\sqrt{8}^{2k+1},$$
$$y = \sum_{k=0}^{n}\binom{2n+1}{2k}2^{3k} = \sum_{k=0}^{n}\binom{2n+1}{2k}\sqrt{8}^{2k}.$$

Both x and y are positive integers. Also, from the binomial formula we obtain

$$y + x\sqrt{8} = \sum_{i=0}^{2n+1}\binom{2n+1}{i}\sqrt{8}^{i} = (1+\sqrt{8})^{2n+1},$$
$$\text{and similarly} \quad y - x\sqrt{8} = (1-\sqrt{8})^{2n+1}.$$

Multiplying these equalities, we get $y^2 - 8x^2 = (1+\sqrt{8})^{2n+1}(1-\sqrt{8})^{2n+1} = -7^{2n+1}$. Reducing modulo 5 gives us

$$3x^2 - y^2 \equiv 2^{2n+1} \equiv 2 \cdot (-1)^n.$$

Now we see that if x is divisible by 5, then $y^2 \equiv \pm 2 \pmod 5$, which is impossible. Therefore x is never divisible by 5.

Second solution. Another standard way is considering recurrent formulas. If we set

$$x_m = \sum_k \binom{m}{2k+1} 8^k, \qquad y_m = \sum_k \binom{m}{2k} 8^k,$$

then since $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$, it follows that $x_{m+1} = x_m + y_m$ and $y_{m+1} = 8x_m + y_m$; therefore $x_{m+1} = 2x_m + 7x_{m-1}$. We need to show that none of x_{2n+1} are divisible by 5. Considering the sequence $\{x_m\}$ modulo 5, we get that $x_m = 0, 1, 2, 1, 1, 4, 0, 3, 1, 3, 3, 2, 0, 4, 3, 4, 4, 1, \dots$. Zeros occur in the initial position of blocks of length 6, where each subsequent block is obtained by multiplying the previous one by 3 (modulo 5). Consequently, x_m is divisible by 5 if and only if m is a multiple of 6, which cannot happen if $m = 2n + 1$.

7. Consider an arbitrary prime number p. If $p \mid m$, then there exists b_i that is divisible by the same power of p as m. Then p divides neither $a_i \frac{m}{b_i}$ nor a_i, because $(a_i, b_i) = 1$. If otherwise $p \nmid m$, then $\frac{m}{b_i}$ is not divisible by p for any i, hence p divides a_i and $a_i \frac{m}{b_i}$ to the same power. Therefore $(a_1, \dots, a_k)$ and $\left(a_1 \frac{m}{b_1}, \dots, a_k \frac{m}{b_k}\right)$ have the same factorization; hence they are equal.

   Second solution. For $k = 2$ we easily verify the formula $\left(m\frac{a_1}{b_1}, m\frac{a_2}{b_2}\right) = \frac{m}{b_1 b_2}(a_1 b_2, a_2 b_1) = \frac{1}{b_1 b_2}[b_1, b_2](a_1, a_2)(b_1, b_2) = (a_1, a_2)$, since $[b_1, b_2] \cdot (b_1, b_2) = b_1 b_2$. We proceed by induction:

$$\begin{aligned}\left(a_1 \frac{m}{b_1}, \dots, a_k \frac{m}{b_k}, a_{k+1} \frac{m}{b_{k+1}}\right) &= \left(\frac{m}{[b_1, \dots, b_k]}(a_1, \dots, a_k), a_{k+1} \frac{m}{b_{k+1}}\right) \\ &= (a_1, \dots, a_k, a_{k+1}).\end{aligned}$$

8. It is clear that

$$\frac{a}{a+b+c+d} + \frac{b}{a+b+c+d} + \frac{c}{a+b+c+d} + \frac{d}{a+b+c+d} < S$$
$$\text{and} \quad S < \frac{a}{a+b} + \frac{b}{a+b} + \frac{c}{c+d} + \frac{d}{c+d},$$

   or equivalently, $1 < S < 2$.
   On the other hand, all values from $(1, 2)$ are attained. Since $S = 1$ for $(a, b, c, d) = (0, 0, 1, 1)$ and $S = 2$ for $(a, b, c, d) = (0, 1, 0, 1)$, due to continuity all the values from $(1, 2)$ are obtained, for example, for $(a, b, c, d) = (x(1 - x), x, 1 - x, 1)$, where x goes through $(0, 1)$.

   Second solution. Set

$$S_1 = \frac{a}{a+b+d} + \frac{c}{b+c+d} \quad \text{and} \quad S_2 = \frac{b}{a+b+c} + \frac{d}{a+c+d}.$$

We may assume without loss of generality that $a+b+c+d=1$. Putting $a+c=x$ and $b+d=y$ (then $x+y=1$), we obtain that the set of values of

$$S_1 = \frac{a}{1-c} + \frac{c}{1-a} = \frac{2ac + x - x^2}{ac + 1 - x}$$

is $\left(x, \frac{2x}{2-x}\right)$. Having the analogous result for S_2 in mind, we conclude that the values that $S = S_1 + S_2$ can take are $\left(x+y, \frac{2x}{2-x} + \frac{2y}{2-y}\right]$. Since $x+y=1$ and

$$\frac{2x}{2-x} + \frac{2y}{2-y} = \frac{4-4xy}{2+xy} \le 2$$

with equality for $xy=0$, the desired set of values for S is $(1,2)$.

9. There exist real numbers a,b,c with $\tan a = x$, $\tan b = y$, $\tan c = z$. Then using the additive formula for tangents we obtain

$$\tan(a+b+c) = \frac{x+y+z-xyz}{1-xy-xz-yz}.$$

We are given that $xyz = x+y+z$. In this case $xy+yz+zx = 1$ is impossible; otherwise, x,y,z would be the zeros of a cubic polynomial $t^3 - \lambda t^2 + t - \lambda = (t^2+1)(t-\lambda)$ (where $\lambda = xyz$), which has only one real root. It follows that

$$x+y+z = xyz \iff \tan(a+b+c) = 0. \tag{1}$$

Hence $a+b+c = k\pi$ for some $k \in \mathbb{Z}$. We note that $\frac{3x-x^3}{1-3x^2}$ actually expresses $\tan 3a$. Since $3a+3b+3c = 3k\pi$, the result follows from (1) for the numbers $\frac{3x-x^3}{1-3x^2}$, $\frac{3y-y^3}{1-3y^2}$, $\frac{3z-z^3}{1-3z^2}$.

10. If we set $\angle ACD = \gamma_1$ and $\angle BCD = \gamma_2$ for a point D on the segment AB, then by the sine theorem,

$$f(D) = \frac{CD^2}{AD \cdot BD} = \frac{CD}{AD} \cdot \frac{CD}{BD} = \frac{\sin\alpha \sin\beta}{\sin\gamma_1 \sin\gamma_2}.$$

The denominator of the last fraction is

$$\begin{aligned} \sin\gamma_1 \sin\gamma_2 &= \frac{1}{2}\left(\cos(\gamma_1 - \gamma_2) - \cos(\gamma_1 + \gamma_2)\right) \\ &= \frac{1}{2}\left(\cos(\gamma_1 - \gamma_2) - \cos\gamma\right) \le \frac{1-\cos\gamma}{2} = \sin^2\frac{\gamma}{2}, \end{aligned}$$

from which we deduce that the set of values of $f(D)$ is the interval $\left[\frac{\sin\alpha\sin\beta}{\sin^2\frac{\gamma}{2}}, +\infty\right)$. Hence $f(D) = 1$ (equivalently, $CD^2 = AD \cdot BD$) is possible if and only if $\sin\alpha\sin\beta \le \sin^2\frac{\gamma}{2}$, i.e.,

$$\sqrt{\sin\alpha\sin\beta} \le \sin\frac{\gamma}{2}.$$

Second solution. Let E be the second point of intersection of the line CD with the circumcircle k of ABC. Since $AD \cdot BD = CD \cdot ED$ (power of D with respect to k), $CD^2 = AD \cdot BD$ ie equivalent to $ED \geq CD$. Clearly the ratio $\frac{ED}{CD}$ ($D \in AB$) takes a minimal value when E is the midpoint of the arc AB not containing C. (This follows from $ED : CD = E'D : C'D$ when C' and E' are respectively projections from C and E onto AB.) On the other hand, it is directly shown that in this case

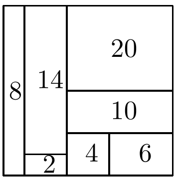
$$\frac{ED}{CD} = \frac{\sin^2 \frac{\gamma}{2}}{\sin\alpha \sin\beta},$$

and the assertion follows.

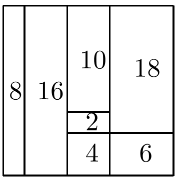
11. First, we notice that $a_1 + a_2 + \cdots + a_p = 32$. The numbers a_i are distinct, and consequently $a_i \geq i$ and $a_1 + \cdots + a_p \geq p(p+1)/2$. Therefore $p \leq 7$. The number 32 can be represented as a sum of 7 mutually distinct positive integers in the following ways:

$$\begin{array}{ll} (1) & 32 = 1+2+3+4+5+6+11; \\ (2) & 32 = 1+2+3+4+5+7+10; \\ (3) & 32 = 1+2+3+4+5+8+9; \\ (4) & 32 = 1+2+3+4+6+7+9; \\ (5) & 32 = 1+2+3+5+6+7+8. \end{array}$$

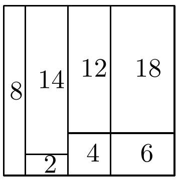
The case (1) is eliminated because there is no rectangle with 22 cells on an 8×8 chessboard. In the other cases the partitions are realized as below.



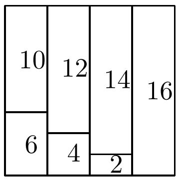
Case (2)



Case (3)



Case (4)



Case (5)

12. We say that a word is *good* if it doesn't contain any nonallowed word. Let a_n be the number of good words of length n. If we prolong any good word of length n by adding one letter to its end (there are $3a_n$ words that can be so obtained), we get either
 (i) a good word of length $n+1$, or
 (ii) an $(n+1)$-letter word of the form XY, where X is a good word and Y a nonallowed word.

The number of words of type (ii) with word Y of length k is exactly a_{n+1-k}; hence the total number of words of kind (ii) doesn't exceed $a_{n-1} + \cdots + a_1 + a_0$ (where $a_0 = 1$). Hence

$$a_{n+1} \geq 3a_n - (a_{n-1} + \cdots + a_1 + a_0), \qquad a_0 = 1,\ a_1 = 3. \tag{1}$$

We prove by induction that $a_{n+1} > 2a_n$ for all n. For $n = 1$ the claim is trivial. If it holds for $i \leq n$, then $a_i \leq 2^{i-n}a_n$; thus we obtain from (1)

$$a_{n+1} > a_n \left(3 - \frac{1}{2} - \frac{1}{2^2} - \cdots - \frac{1}{2^n}\right) > 2a_n.$$

Therefore $a_n \geq 2^n$ for all n (moreover, one can show from (1) that $a_n \geq (n+2)2^{n-1}$); hence there exist good words of length n.

Remark. If there are two nonallowed words (instead of one) of each length greater than 1, the statement of the problem need not remain true.

4.17 Solutions to the Shortlisted Problems of IMO 1975

1. First, we observe that there cannot exist three routes of the form (A,B,C), (A,B,D), (A,C,D), for if E,F are the remaining two ports, there can be only one route covering A,E, namely, (A,E,F). Thus if (A,B,C), (A,B,D) are two routes, the one covering A,C must be w.l.o.g. (A,C,E). The other roots are uniquely determined: These are (A,D,F), (A,E,F), (B,D,E), (B,E,F), (B,C,F), (C,D,E), (C,D,F).

2. Since there are finitely many arrangements of the z_i's, assume that $z_1,\dots,z_n$ is the one for which $\sum_{i=1}^n (x_i-z_i)^2$ is minimal. We claim that in this case $i<j \Rightarrow z_i \ge z_j$, from which the claim of the problem directly follows.
   Indeed, otherwise we would have
$$\begin{aligned}(x_i-z_j)^2+(x_j-z_i)^2 &= (x_i-z_i)^2+(x_j-z_j)^2\\ &\quad +2(x_iz_i+x_jz_j-x_iz_j-x_jz_i)\\ &= (x_i-z_i)^2+(x_j-z_j)^2+2(x_i-x_j)(z_i-z_j)\\ &\le (x_i-z_i)^2+(x_j-z_j)^2,\end{aligned}$$
   contradicting the assumption.

3. From $\left((k+1)^{2/3}+(k+1)^{1/3}k^{1/3}+k^{2/3}\right)\left((k+1)^{1/3}-k^{1/3}\right) = 1$ and $3k^{2/3} < (k+1)^{2/3} + (k+1)^{1/3}k^{1/3}+k^{2/3} < 3(k+1)^{2/3}$ we obtain
$$3\left((k+1)^{1/3}-k^{1/3}\right) < k^{-2/3} < 3\left(k^{1/3}-(k-1)^{1/3}\right).$$
   Summing from 1 to n we get
$$1+3\left((n+1)^{1/3}-2^{1/3}\right) < \sum_{k=1}^{n} k^{-2/3} < 1+3(n^{1/3}-1).$$
   In particular, for $n=10^9$ this inequality gives
$$2997 < 1+3\left((10^9+1)^{1/3}-2^{1/3}\right) < \sum_{k=1}^{10^9} k^{-2/3} < 2998.$$
   Therefore $\left[\sum_{k=1}^{10^9} k^{-2/3}\right] = 2997$.

4. Put $\Delta a_n = a_n - a_{n+1}$. By the imposed condition, $\Delta a_n > \Delta a_{n+1}$. Suppose that for some n, $\Delta a_n < 0$: Then for each $k \ge n$, $\Delta a_k < \Delta a_n$; hence $a_n - a_{n+m} = \Delta a_n + \cdots + \Delta a_{n+m-1} < m\Delta a_n$. Thus for sufficiently large m it holds that $a_n - a_{n+m} < -1$, which is impossible. This proves the first part of the inequality.
   Next one observes that
$$n \ge \sum_{k=1}^{n} a_k = na_{n+1} + \sum_{k=1}^{n} k\Delta a_k \ge (1+2+\cdots+n)\Delta a_n = \frac{n(n+1)}{2}\Delta a_n.$$
   Hence $(n+1)\Delta a_n \le 2$.

5. There are exactly $8 \cdot 9^{k-1}$ k-digit numbers in M (the first digit can be chosen in 8 ways, while any other position admits 9 possibilities). The least of them is 10^k, and hence

$$\sum_{x_j<10^k} \frac{1}{x_j} = \sum_{i=1}^{k} \sum_{10^{i-1}\le x_j<10^i} \frac{1}{x_j} < \sum_{i=1}^{k} \sum_{10^{i-1}\le x_j<10^i} \frac{1}{10^{i-1}}$$
$$= \sum_{i=1}^{k} \frac{8 \cdot 9^{i-1}}{10^{i-1}} = 80\left(1-\frac{9^k}{10^k}\right) < 80.$$

6. Let us denote by C the sum of digits of B. We know that $16^{16} \equiv A \equiv B \equiv C \pmod 9$. Since $16^{16} = 2^{64} = 2^{6\cdot 10+4} \equiv 2^4 \equiv 7$, we get $C \equiv 7 \pmod 9$. Moreover, $16^{16} < 100^{16} = 10^{32}$, hence A cannot exceed $9 \cdot 32 = 288$; consequently, B cannot exceed 19 and C is at most 10. Therefore $C = 7$.

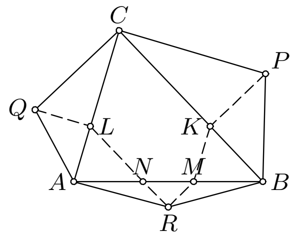
7. We use induction on m. Denote by S_m the left-hand side of the equality to be proved. First $S_0 = (1-y)(1+y+\cdots+y^n)+y^{n+1} = 1$, since $x = 1-y$. Furthermore,

$$\begin{aligned}
&S_{m+1} - S_m \\
&= \binom{m+n+1}{m+1} x^{m+1}y^{n+1} + x^{m+1}\sum_{j=0}^{n}\left(\binom{m+1+j}{j}xy^j - \binom{m+j}{j}y^j\right) \\
&= \binom{m+n+1}{m+1} x^{m+1}y^{n+1} \\
&\quad + x^{m+1}\sum_{j=0}^{n}\left(\binom{m+1+j}{j}y^j - \binom{m+j}{j}y^j - \binom{m+1+j}{j}y^{j+1}\right) \\
&= x^{m+1}\left[\binom{m+n+1}{n}y^{n+1} + \sum_{j=0}^{n}\left(\binom{m+j}{j-1}y^j - \binom{m+j+1}{j}y^{j+1}\right)\right] \\
&= 0;
\end{aligned}$$

i.e., $S_{m+1} = S_m = 1$ for every m.

Second solution. Let us be given an unfair coin that, when tossed, shows heads with probability x and tails with probability y. Note that $x^{m+1}\binom{m+j}{j}y^j$ is the probability that until the moment when the $(m+1)$th head appears, exactly j tails ($j < n+1$) have appeared. Similarly, $y^{n+1}\binom{n+i}{i}x^i$ is the probability that exactly i heads will appear before the $(n+1)$th tail occurs. Therefore, the above sum is the probability that either $m+1$ heads will appear before $n+1$ tails, or vice versa, and this probability is clearly 1.

8. Let K and L be the feet of perpendiculars from P and Q to BC and AC respectively.



Let M, N be points on AB (ordered $A - N - M - B$) such that RMN is a right isosceles triangle with $\angle R = 90^\circ$. By sine theorem we have $\frac{BM}{BA} = \frac{BM}{BR} \cdot \frac{BR}{BA} = \frac{\sin 15^\circ}{\sin 45^\circ}$. Since $\frac{BK}{BC} = \frac{\sin 45^\circ \sin 30^\circ}{\cos 15^\circ} = \frac{\sin 15^\circ}{\sin 45^\circ}$, we deduce that $MK \parallel AC$ and $MK = AL$. Similarly, $NL \parallel BC$ and $NL = BK$. It follows that the vectors $\overrightarrow{RN}, \overrightarrow{NL}, \overrightarrow{LQ}$ are the images of $\overrightarrow{RM}, \overrightarrow{KP}, \overrightarrow{MK}$ respectively under a rotation of 90°, and consequently the same holds for their sums $\overrightarrow{RQ}$ and $\overrightarrow{RP}$. Therefore, $QR = RP$ and $\angle QRP = 90^\circ$.

Second solution. Let ABS be the equilateral triangle constructed in the exterior of $\triangle ABC$. Obviously, the triangles BPC, BRS, ARS, AQC are similar. Let f be the rotational homothety centered at B that maps P onto C, and let g be the rotational homothety about A that maps C onto Q. The composition $h = g \circ f$ is also a rotational homothety; its angle is $\angle PBC + \angle CAQ = 90^\circ$, and the coefficient is $\frac{BC}{BP} \cdot \frac{AQ}{AC} = 1$. Moreover, R is a fixed point of h because $f(R) = S$ and $g(S) = R$. Hence R is the center of h, and the statement follows from $h(P) = Q$.

Remark. There are two more possible approaches: One includes using complex numbers and the other one is mere calculating of RP, RQ, PQ by the cosine theorem.

Second remark. The problem allows a generalization: Given that $\angle CBP = \angle CAQ = \alpha$, $\angle BCP = \angle ACQ = \beta$, and $\angle RAB = \angle RBA = 90^\circ - \alpha - \beta$, show that $RP = RQ$ and $\angle PRQ = 2\alpha$.

9. Suppose n is the natural number with $na \le 1 < (n+1)a$. If a function f with the desired properties exists, then $f_a(a) = 0$ and let w.l.o.g. $f(a) > 0$, or equivalently, let the graph of f_a lie below the graph of f. In this case also $f(2a) > f(a)$, since otherwise, the graphs of f and f_a would intersect between a and $2a$. Continuing in this way we are led to $0 = f(0) < f(a) < f(2a) < \cdots < f(na)$. Thus if $na = 1$, i.e., $a = 1/n$, such an f does not exist. On the other hand, if $a \ne 1/n$, then we similarly obtain $f(1) > f(1-a) > f(1-2a) > \cdots > f(1-na)$. Choosing values of f at $ia, 1 - ia$, $i = 1, \dots, n$, so that they satisfy $f(1-na) < \cdots < f(1-a) < 0 < f(a) < \cdots < f(na)$, we can extend f to other values of $[0,1]$ by linear interpolation. A function obtained this way has the desired property.

10. We shall prove that for all x, y with $x+y = 1$ it holds that $f(x,y) = x-2y$. In this case $f(x,y) = f(x, 1-x)$ can be regarded as a polynomial in $z = x - 2y = 3x - 2$, say $f(x, 1-x) = F(z)$. Putting in the given relation $a = b = x/2$, $c = 1 - x$, we obtain $f(x, 1-x) + 2f(1 - x/2, x/2) = 0$; hence $F(z) + 2F(-z/2) = 0$. Now $F(1) = 1$, and we get that for all k,

$F((-2)^k) = (-2)^k$. Thus $F(z) = z$ for infinitely many values of z; hence $F(z) \equiv z$. Consequently $f(x,y) = x - 2y$ if $x + y = 1$.
For general x, y with $x+y \neq 0$, since f is homogeneous ,we have $f(x,y) = (x+y)^n f\left(\frac{x}{x+y}, \frac{y}{x+y}\right) = (x+y)^n \left(\frac{x}{x+y} - 2\frac{y}{x+y}\right) = (x+y)^{n-1}(x-2y)$. The same is true for $x + y = 0$, because f is a polynomial.

11. Let (a_{k_i}) be the subsequence of (a_k) consisting of all a_k's that give remainder r upon division by a_1. For every $i > 1$, $a_{k_i} \equiv a_{k_1} \pmod{a_1}$; hence $a_{k_i} = a_{k_1} + ya_1$ for some integer $y > 0$. It follows that for every $r = 0, 1, \dots, a_1 - 1$ there is exactly one member of the corresponding $(a_{k_i})_{i \geq 1}$ that cannot be represented as $xa_l + ya_m$, and hence at most $a_1 + 1$ members of (a_k) in total are not representable in the given form.

12. Since $\sin 2x_i = 2 \sin x_i \cos x_i$ and $\sin(x_i + x_{i+1}) + \sin(x_i - x_{i+1}) = 2 \sin x_i \cos x_{i+1}$, the inequality from the problem is equivalent to

$$\begin{aligned} (\cos x_1 - \cos x_2) \sin x_1 + (\cos x_2 - \cos x_3) \sin x_2 + \cdots \\ \cdots + (\cos x_{\nu-1} - \cos x_\nu) \sin x_{\nu-1} < \frac{\pi}{4}. \end{aligned} \tag{1}$$

Consider the unit circle with center at $O(0,0)$ and points $M_i(\cos x_i, \sin x_i)$ on it. Also, choose the points $N_i(\cos x_i, 0)$ and $M_i'(\cos x_{i+1}, \sin x_i)$. It is clear that $(\cos x_i - \cos x_{i+1}) \sin x_i$ is equal to the area of the rectangle $M_i N_i N_{i+1} M_i'$. Since all these rectangles are disjoint and lie inside the quarter circle in the first quadrant whose area is $\frac{\pi}{4}$, inequality (1) follows.

13. Suppose that $A_k A_{k+1} \cap A_m A_{m+1} \neq \emptyset$ for some k, $m > k + 1$. Without loss of generality we may suppose that $k = 0$, $m = n - 1$ and that no two segments $A_k A_{k+1}$ and $A_m A_{m+1}$ intersect for $0 \leq k < m - 1 < n - 1$ except for $k = 0$, $m = n - 1$. Also, shortening $A_0 A_1$, we may suppose that $A_0 \in A_{n-1} A_n$. Finally, we may reduce the problem to the case that $A_0 \dots A_{n-1}$ is convex: Otherwise, the segment $A_{n-1} A_n$ can be prolonged so that it intersects some $A_k A_{k+1}$, $0 < k < n - 2$.
If $n = 3$, then $A_1 A_2 \geq 2 A_0 A_1$ implies $A_0 A_2 > A_0 A_1$, hence $\angle A_0 A_1 A_2 > \angle A_1 A_2 A_3$, a contradiction.
Let $n = 4$. From $A_3 A_2 > A_1 A_2$ we conclude that $\angle A_3 A_1 A_2 > \angle A_1 A_3 A_2$. Using the inequality $\angle A_0 A_3 A_2 > \angle A_0 A_1 A_2$ we obtain that $\angle A_0 A_3 A_1 > \angle A_0 A_1 A_3$ implying $A_0 A_1 > A_0 A_3$. Now we have $A_2 A_3 < A_3 A_0 + A_0 A_1 + A_1 A_2 < 2 A_0 A_1 + A_1 A_2 \leq 2 A_1 A_2 \leq A_2 A_3$, which is not possible.
Now suppose $n \geq 5$. If α_i is the exterior angle at A_i, then $\alpha_1 > \cdots > \alpha_{n-1}$; hence $\alpha_{n-1} < \frac{360^\circ}{n-1} \leq 90^\circ$. Consequently $\angle A_{n-2} A_{n-1} A_0 \geq 90^\circ$ and $A_0 A_{n-2} > A_{n-1} A_{n-2}$. On the other hand, $A_0 A_{n-2} < A_0 A_1 + A_1 A_2 + \cdots + A_{n-3} A_{n-2} < \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \cdots + \frac{1}{2}\right) A_{n-1} A_{n-2} < A_{n-1} A_{n-2}$, which contradicts the previous relation.

14. We shall prove that for every $n \in \mathbb{N}$, $\sqrt{2n+25} \leq x_n \leq \sqrt{2n+25} + 0.1$. Note that for $n = 1000$ this gives us exactly the desired inequalities.

First, notice that the recurrent relation is equivalent to

$$2x_k(x_{k+1} - x_k) = 2. \tag{1}$$

Since $x_0 < x_1 < \cdots < x_k < \cdots$, from (1) we get $x_{k+1}^2 - x_k^2 = (x_{k+1} + x_k)(x_{k+1} - x_k) > 2$. Adding these up we obtain $x_n^2 \geq x_0^2 + 2n$, which proves the first inequality.

On the other hand, $x_{k+1} = x_k + \frac{1}{x_k} \leq x_k + 0.2$ (for $x_k \geq 5$), and one also deduces from (1) that $x_{k+1}^2 - x_k^2 - 0.2(x_{k+1} - x_k) = (x_{k+1} + x_k - 0.2)(x_{k+1} - x_k) \leq 2$. Again, adding these inequalities up, $(k = 0, \dots, n-1)$ yields

$$x_n^2 \leq 2n + x_0^2 + 0.2(x_n - x_0) = 2n + 24 + 0.2x_n.$$

Solving the corresponding quadratic equation, we obtain $x_n < 0.1 + \sqrt{2n + 24.01} < 0.1 + +\sqrt{2n + 25}$.

15. Assume that the center of the circle is at the origin $O(0, 0)$, and that the points $A_1, A_2, \dots, A_{1975}$ are arranged on the upper half-circle so that $\angle A_iOA_1 = \alpha_i$ $(\alpha_1 = 0)$. The distance A_iA_j equals $2\sin\frac{\alpha_j - \alpha_i}{2} = 2\sin\frac{\alpha_j}{2}\cos\frac{\alpha_i}{2} - \cos\frac{\alpha_j}{2}\sin\frac{\alpha_i}{2}$, and it will be rational if all $\sin\frac{\alpha_k}{2}, \cos\frac{\alpha_k}{2}$ are rational.

    Finally, observe that there exist infinitely many angles α such that both $\sin\alpha$, $\cos\alpha$ are rational, and that such α can be arbitrarily small. For example, take α so that $\sin\alpha = \frac{2t}{t^2+1}$ and $\cos\alpha = \frac{t^2-1}{t^2+1}$ for any $t \in \mathbb{Q}$.

4.18 Solutions to the Shortlisted Problems of IMO 1976

1. Let r denote the common inradius. Some two of the four triangles with the inradii ρ have cross angles at M: Suppose these are $\triangle AMB_1$ and $\triangle BMA_1$. We shall show that $\triangle AMB_1 \cong \triangle BMA_1$. Indeed, the altitudes of these two triangles are both equal to r, the inradius of $\triangle ABC$, and their interior angles at M are equal to some angle φ. If P is the point of tangency of the incircle of $\triangle A_1MB$ with MB, then $\frac{r}{\rho} = \frac{A_1M+BM+A_1B}{A_1B}$, which also implies $\frac{r-2\rho}{\rho} = \frac{A_1M+BM-A_1B}{A_1B} = \frac{2MP}{A_1B} = \frac{2r\cot(\varphi/2)}{A_1B}$. Since similarly $\frac{r-2\rho}{\rho} = \frac{2r\cot(\varphi/2)}{B_1A}$, we obtain $A_1B = B_1A$ and consequently $\triangle AMB_1 \cong \triangle BMA_1$. Thus $\angle BAC = \angle ABC$ and $CC_1 \perp AB$. There are two alternatives for the other two incircles:
   (i) If the inradii of AMC_1 and AMB_1 are equal to r, it is easy to obtain that $\triangle AMC_1 \cong \triangle AMB_1$. Hence $\angle AB_1M = \angle AC_1M = 90°$, and $\triangle ABC$ is equilateral.
   (ii) The inradii of AMB_1 and CMB_1 are equal to r. Put $x = \angle MAC_1 = \angle MBC_1$. In this case $\varphi = 2x$ and $\angle B_1MC = 90° - x$. Now we have $\frac{AB_1}{CB_1} = \frac{S_{AMB_1}}{S_{CMB_1}} = \frac{AM+MB_1+AB_1}{CM+MB_1+CB_1} = \frac{AM+MB_1-AB_1}{CM+MB_1-CB_1} = \frac{\cot x}{\cot(45°-x/2)}$. On the other hand, $\frac{AB_1}{CB_1} = \frac{AB}{BC} = 2\cos 2x$. Thus we have an equation for x: $\tan(45° - x/2) = 2\cos 2x \tan x$, or equivalently

$$2\tan\left(45° - \frac{x}{2}\right)\sin\left(45° - \frac{x}{2}\right)\cos\left(45° - \frac{x}{2}\right) = 2\cos 2x \sin x.$$

   Hence $\sin 3x - \sin x = 2\sin^2\left(45° - \frac{x}{2}\right) = 1 - \sin x$, implying $\sin 3x = 1$, i.e., $x = 30°$. Therefore $\triangle ABC$ is equilateral.

2. Let us put $b_i = i(n+1-i)/2$, and let $c_i = a_i - b_i$, $i = 0, 1, \dots, n+1$. It is easy to verify that $b_0 = b_{n+1} = 0$ and $b_{i-1} - 2b_i + b_{i+1} = -1$. Subtracting this inequality from $a_{i-1} - 2a_i + a_{i+1} \ge -1$, we obtain $c_{i-1} - 2c_i + c_{i+1} \ge 0$, i.e., $2c_i \le c_{i-1} + c_{i+1}$. We also have $c_0 = c_{n+1} = 0$.
   Suppose that there exists $i \in \{1, \dots, n\}$ for which $c_i > 0$, and let c_k be the maximal such c_i. Assuming w.l.o.g. that $c_{k-1} < c_k$, we obtain $c_{k-1} + c_{k+1} < 2c_k$, which is a contradiction. Hence $c_i \le 0$ for all i; i.e., $a_i \le b_i$.
   Similarly, considering the sequence $c_i' = a_i + b_i$ one can show that $c_i' \ge 0$, i.e., $a_i \ge -b_i$ for all i. This completes the proof.

3. (a) Let $ABCD$ be a quadrangle with $16 = d = AB + CD + AC$, and let S be its area. Then $S \le (AC \cdot AB + AC \cdot CD)/2 = AC(d - AC)/2 \le d^2/8 = 32$, where equality occurs if and only if $AB \perp AC \perp CD$ and $AC = AB + CD = 8$. In this case $BD = 8\sqrt{2}$.
   (b) Let A' be the point with $\overrightarrow{DA'} = \overrightarrow{AC}$. The triangular inequality implies $AD + BC \ge AA' = 8\sqrt{5}$. Thus the perimeter attains its minimum for $AB = CD = 4$.

(c) Let us assume w.l.o.g. that $CD \le AB$. Then C lies inside $\triangle BDA'$ and hence $BC + AD = BC + CA' < BD + DA'$. The maximal value $BD + DA'$ of $BC + AD$ is attained when C approaches D, making a degenerate quadrangle.

4. The first few values are easily verified to be $2^{r_n} + 2^{-r_n}$, where $r_0 = 0$, $r_1 = r_2 = 1$, $r_3 = 3$, $r_4 = 5$, $r_5 = 11$, $\dots$. Let us put $u_n = 2^{r_n} + 2^{-r_n}$ (we will show that r_n exists and is integer for each n). A simple calculation gives us $u_n(u_{n-1}^2 - 2) = 2^{r_n+2r_{n-1}} + 2^{-r_n-2r_{n-1}} + 2^{r_n-2r_{n-1}} + 2^{-r_n+2r_{n-1}}$. If an array q_n, with $q_0 = 0$ and $q_1 = 1$, is set so as to satisfy the linear recurrence $q_{n+1} = q_n + 2q_{n-1}$, then it also satisfies $q_n - 2q_{n-1} = -(q_{n-1} - 2q_{n-2}) = \cdots = (-1)^{n-1}(q_1 - 2q_0) = (-1)^{n-1}$. Assuming inductively up to n $r_i = q_i$, the expression for $u_n(u_{n-1}^2 - 2) = u_{n+1} + u_1$ reduces to $2^{q_{n+1}} + 2^{-q_{n+1}} + u_1$. Therefore, $r_{n+1} = q_{n+1}$. The solution to this linear recurrence with $r_0 = 0$, $r_1 = 1$ is $r_n = q_n = \frac{2^n - (-1)^n}{3}$, and since $[u_n] = 2^{r_n}$ for $n \ge 0$, the result follows.

Remark. One could simply guess that $u_n = 2^{r_n} + 2^{-r_n}$ for $r_n = \frac{2^n-(-1)^n}{3}$, and then prove this result by induction.

5. If one substitutes an integer q-tuple $(x_1, \dots, x_q)$ satisfying $|x_i| \le p$ for all i in an equation of the given system, the absolute value of the right-hand member never exceeds pq. So for the right-hand member of the system there are $(2pq + 1)^p$ possibilities There are $(2p + 1)^q$ possible q-tuples $(x_1, \dots, x_q)$. Since $(2p + 1)^q \ge (2pq + 1)^p$, there are at least two q-tuples $(y_1, \dots, y_q)$ and $(z_1, \dots, z_q)$ giving the same right-hand members in the given system. The difference $(x_1, \dots, x_q) = (y_1 - z_1, \dots, y_q - z_q)$ thus satisfies all the requirements of the problem.

6. Suppose $a_1 \le a_2 \le a_3$ are the dimensions of the box. If we set $b_i = [a_i/\sqrt[3]{2}]$, the condition of the problem is equivalent to $\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} = 5$. We list some values of a, $b = [a/\sqrt[3]{2}]$ and a/b:

a	2	3	4	5	6	7	8	9	10
b	1	2	3	3	4	5	6	7	7
a/b	2	1.5	1.33	1.67	1.5	1.4	1.33	1.29	1.43

We note that if $a > 2$, then $a/b \le 5/3$, and if $a > 5$, then $a/b \le 3/2$. If $a_1 > 2$, then $\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} < (5/3)^3 < 5$, a contradiction. Hence $a_1 = 2$. If also $a_2 = 2$, then $a_3/b_3 = 5/4 \le \sqrt[3]{2}$, which is impossible. Also, if $a_2 \ge 6$, then $\frac{a_2}{b_2} \cdot \frac{a_3}{b_3} \le (1.5)^2 < 2.5$, again a contradiction. We thus have the following cases:

(i) $a_1 = 2, a_2 = 3$, then $a_3/b_3 = 5/3$, which holds only if $a_3 = 5$;
(ii) $a_1 = 2, a_2 = 4$, then $a_3/b_3 = 15/8$, which is impossible;
(iii) $a_1 = 2, a_2 = 5$, then $a_3/b_3 = 3/2$, which holds only if $a_3 = 6$.

The only possible sizes of the box are therefore $(2, 3, 5)$ and $(2, 5, 6)$.

7. The map T transforms the interval $(0,a]$ onto $(1-a,1]$ and the interval $(a,1]$ onto $(0,1-a]$. Clearly T preserves the measure. Since the measure of the interval $[0,1]$ is finite, there exist two positive integers k, $l>k$ such that $T^k(J)$ and $T^l(J)$ are not disjoint. But the map T is bijective; hence $T^{l-k}(J)$ and J are not disjoint.

8. Every polynomial with real coefficients can be factored as a product of linear and quadratic polynomials with real coefficients. Thus it suffices to prove the result only for a quadratic polynomial $P(x)=x^2-2ax+b^2$, with $a>0$ and $b^2>a^2$.
   Using the identity
   $$(x^2+b^2)^{2n}-(2ax)^{2n}=(x^2-2ax+b^2)\sum_{k=0}^{2n-1}(x^2+b^2)^k(2ax)^{2n-k-1}$$
   we have solved the problem if we can choose n such that $b^{2n}\binom{2n}{n}>2^{2n}a^{2n}$. However, it is is easy to show that $2n\binom{2n}{n}<2^{2n}$; hence it is enough to take n such that $(b/a)^{2n}>2n$. Since $\lim_{n\to\infty}(2n)^{1/(2n)}=1<b/a$, such an n always exists.

9. The equation $P_n(x)=x$ is of degree 2^n, and has at most 2^n distinct roots. If $x>2$, then by simple induction $P_n(x)>x$ for all n. Similarly, if $x<-1$, then $P_1(x)>2$, which implies $P_n(x)>2$ for all n. It follows that all real roots of the equation $P_n(x)=x$ lie in the interval $[-2,2]$, and thus have the form $x=2\cos t$.
   Now we observe that $P_1(2\cos t)=4\cos^2 t-2=2\cos 2t$, and in general $P_n(2\cos t)=2\cos 2^n t$. Our equation becomes
   $$\cos 2^n t=\cos t,$$
   which indeed has 2^n different solutions $t=\frac{2\pi m}{2^n-1}$ $(m=0,1,\dots,2^{n-1}-1)$ and $t=\frac{2\pi m}{2^n+1}$ $(m=1,2,\dots,2^{n-1})$.

10. Let $a_1<a_2<\cdots<a_n$ be positive integers whose sum is 1976. Let M denote the maximal value of $a_1a_2\cdots a_n$. We make the following observations:
    (1) $a_1=1$ does not yield the maximum, since replacing $1,a_2$ by $1+a_2$ increases the product.
    (2) $a_j-a_i\ge 2$ does not yield the maximal value, since replacing a_i,a_j by a_i+1,a_j-1 increases the product.
    (3) $a_i\ge 5$ does not yield the maximal value, since $2(a_i-2)=2a_i-4>a_i$.
    Since $4=2^2$, we may assume that all a_i are either 2 or 3, and $M=2^k3^l$, where $2k+3l=1976$.
    (4) $k\ge 3$ does not yield the maximal value, since $2\cdot 2\cdot 2<3\cdot 3$.
    Hence $k\le 2$ and $2k\equiv 1976 \pmod 3$ gives us $k=1$, $l=658$ and $M=2\cdot 3^{658}$.

11. We shall show by induction that $5^{2^k} - 1 = 2^{k+2}q_k$ for each $k = 0, 1, \dots,$ where $q_k \in \mathbb{N}$. Indeed, the statement is true for $k = 0$, and if it holds for some k then $5^{2^{k+1}} - 1 = \left(5^{2^k} + 1\right)\left(5^{2^k} - 1\right) = 2^{k+3}d_{k+1}$ where $d_{k+1} = \left(5^{2^k} + 1\right) d_k/2$ is an integer by the inductive hypothesis.

    Let us now choose $n = 2^k + k + 2$. We have $5^n = 10^{k+2}q_k + 5^{k+2}$. It follows from $5^4 < 10^3$ that 5^{k+2} has at most $[3(k+2)/4]+2$ nonzero digits, while $10^{k+2}q_k$ ends in $k+2$ zeros. Hence the decimal representation of 5^n contains at least $[(k+2)/4] - 2$ consecutive zeros. Now it suffices to take $k > 4 \cdot 1978$.

12. Suppose the decomposition into k polynomials is possible. The sum of coefficients of each polynomial $a_1x + a_2x^2 + \cdots + a_nx^n$ equals $1 + \cdots + n = n(n+1)/2$ while the sum of coefficients of $1976(x + x^2 + \cdots + x^n)$ is $1976n$. Hence we must have $1976n = kn(n+1)/2$, which reduces to $(n+1) \mid 3952 = 2^4 \cdot 13 \cdot 19$. In other words, n is of the form $n = 2^\alpha 13^\beta 19^\gamma - 1$, with $0 \le \alpha \le 4$, $0 \le \beta \le 1$, $0 \le \gamma \le 1$. We can immediately eliminate the values $n = 0$ and $n = 3951$ that correspond to $\alpha = \beta = \gamma = 0$ and $\alpha = 4,\ \beta = \gamma = 1$.

    We claim that all other values n are permitted. There are two cases.

    $\alpha \le 3$. In this case $k = 3952/(n+1)$ is even. The simple choice of the polynomials $P = x + 2x^2 + \cdots + nx^n$ and $P' = nx + (n-1)x^2 + \cdots + x^n$ suffices, since $k(P + P')/2 = 1976(x + x^2 + \cdots + x^n)$.

    $\alpha = 4$. Then k is odd. Consider $(k-3)/2$ pairs (P, P') of the former case and

$$\begin{aligned} P_1 &= \left[nx + (n-1)x^3 + \cdots + \tfrac{n+1}{2}x^n\right] \\ &\quad + \left[\tfrac{n-1}{2}x^2 + \tfrac{n-3}{2}x^4 + \cdots + x^{n-1}\right]; \\ P_2 &= \left[\tfrac{n+1}{2}x + \tfrac{n-1}{2}x^3 + \cdots + x^n\right] \\ &\quad + \left[nx^2 + (n-1)x^4 + \cdots + \tfrac{n+3}{2}x^{n-1}\right]. \end{aligned}$$

    Then $P + P_1 + P_2 = 3(n+1)(x + x^2 + \cdots + x^n)/2$ and therefore $(k-3)(P+P')/2 + (P + P_1 + P_2) = 1976(x + x^2 + \cdots + x^n)$.

    It follows that the desired decomposition is possible if and only if $1 < n < 3951$ and $n + 1 \mid 2 \cdot 1976$.

4.19 Solutions to the Longlisted Problems of IMO 1977

1. Let P be the projection of S onto the plane $ABCDE$. Obviously $BS > CS$ is equivalent to $BP > CP$. The conditions of the problem imply that $PA > PB$ and $PA > PE$. The locus of such points P is the region of the plane that is determined by the perpendicular bisectors of segments AB and AE and that contains the point diametrically opposite A. But since $AB < DE$, the whole of this region lies on one side of the perpendicular bisector of BC. The result follows immediately.

   Remark. The assumption $BC < CD$ is redundant.

2. We shall prove by induction on n that $f(x) > f(n)$ whenever $x > n$. The case $n = 0$ is trivial. Suppose that $n \geq 1$ and that $x > k$ implies $f(x) > f(k)$ for all $k < n$. It follows that $f(x) \geq n$ holds for all $x \geq n$. Let $f(m) = \min_{x \geq n} f(x)$. If we suppose that $m > n$, then $m - 1 \geq n$ and consequently $f(m-1) \geq n$. But in this case the inequality $f(m) > f(f(m-1))$ contradicts the minimality property of m. The inductive proof is thus completed.
   It follows that f is strictly increasing, so $f(n+1) > f(f(n))$ implies that $n+1 > f(n)$. But since $f(n) \geq n$ we must have $f(n) = n$.

3. Let $v_1, v_2, \dots, v_k$ be k persons who are not acquainted with each other. Let us denote by m the number of acquainted couples and by d_j the number of acquaintances of person v_j. Then

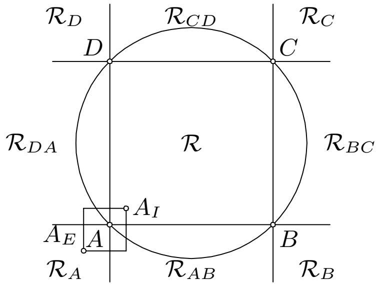
$$m \leq d_{k+1} + d_{k+2} + \cdots + d_n \leq d(n-k) \leq k(n-k) \leq \left(\frac{k+(n-k)}{2}\right)^2 = \frac{n^2}{4}.$$

4. Consider any vertex v_n from which the maximal number d of segments start, and suppose it is not a vertex of a triangle. Let $\mathcal{A} = \{v_1, v_2, \dots, v_d\}$ be the set of points that are connected to v_n, and let $\mathcal{B} = \{v_{d+1}, v_{d+2}, \dots, v_n\}$ be the set of the other points. Since v_n is not a vertex of a triangle, there is no segment both of whose vertices lie in $\mathcal{A}$; i.e., each segment has an end in $\mathcal{B}$. Thus, if d_j denotes the number of segments at v_j and m denotes the total number of segments, we have

$$m \leq d_{d+1} + d_{d+2} + \cdots + d_n \leq d(n-d) \leq \left[\frac{n^2}{4}\right] = m.$$

   This means that each inequality must be equality, implying that each point in $\mathcal{B}$ is a vertex of d segments, and each of these segments has the other end in $\mathcal{A}$. Then there is no triangle at all, which is a contradiction.

5. Let us denote by I and E the sets of interior boundary points and exterior boundary points. Let $ABCD$ be the square inscribed in the circle k with sides parallel to the coordinate axes. Lines AB, BC, CD, DA divide the



plane into 9 regions: $\mathcal{R}$, $\mathcal{R}_A$, $\mathcal{R}_B$, $\mathcal{R}_C$, $\mathcal{R}_D$, $\mathcal{R}_{AB}$, $\mathcal{R}_{BC}$, $\mathcal{R}_{CD}$, $\mathcal{R}_{DA}$. There is a unique pair of lattice points $A_I \in \mathcal{R}$, $A_E \in \mathcal{R}_A$ that are opposite vertices of a unit square. We similarly define B_I, C_I, D_I, B_E, C_E, D_E. Let us form a graph G by connecting each point from E lying in $\mathcal{R}_{AB}$ (respectively $\mathcal{R}_{BC}, \mathcal{R}_{CD}, \mathcal{R}_{DA}$) to its upper (respectively left, lower, right) neighbor point (which clearly belongs to I). It is easy to see that:

(i) All vertices from I other than A_I, B_I, C_I, D_I have degree 1.
(ii) A_E is not in E if and only if $A_I \in I$ and $\deg A_I = 2$.
(iii) No other lattice points inside $\mathcal{R}_A$ belong to E.

Thus if m is the number of edges of the graph G and s is the number of points among A_E, B_E, C_E, and D_E that are in E, using (i)–(iii) we easily obtain $|E| = m + s$ and $|I| = m - (4 - s) = |E| + 4$.

6. Let $\langle y \rangle$ denote the distance from $y \in \mathbb{R}$ to the closest even integer. We claim that

$$\langle 1 + \cos x \rangle \le \sin x \qquad \text{for all } x \in [0, \pi].$$

Indeed, if $\cos x \ge 0$, then $\langle 1 + \cos x \rangle = 1 - \cos x \le 1 - \cos^2 x = \sin^2 x \le \sin x$; the proof is similar if $\cos x < 0$.
We note that $\langle x + y \rangle \le \langle x \rangle + \langle y \rangle$ holds for all $x, y \in \mathbb{R}$. Therefore

$$\sum_{j=1}^{n} \sin x_j \ge \sum_{j=1}^{n} \langle 1 + \cos x_j \rangle \ge \left\langle \sum_{j=1}^{n} (1 + \cos x_j) \right\rangle = 1.$$

7. Let us suppose that $c_1 \le c_2 \le \cdots \le c_n$ and that $c_1 < 0 < c_n$. There exists k, $1 \le k < n$, such that $c_k \le 0 < c_{k+1}$. Then we have

$$\begin{aligned}(n-1)(c_1^2 + c_2^2 + \cdots + c_n^2) &\ge k(c_1^2 + \cdots + c_k^2) + (n-k)(c_{k+1}^2 + \cdots + c_n^2)\\ &\ge (c_1 + \cdots + c_k)^2 + (c_{k+1} + \cdots + c_n)^2\\ &= (c_1 + \cdots + c_n)^2\\ &\quad -2(c_1 + \cdots + c_k)(c_{k+1} + \cdots + c_n),\end{aligned}$$

from which we obtain $(c_1 + \cdots + c_k)(c_{k+1} + \cdots + c_n) \ge 0$, a contradiction.

Second solution. By the given condition and the inequality between arithmetic and quadratic mean we have

$$\begin{aligned}(c_1 + \cdots + c_n)^2 &= (n-1)(c_1^2 + \cdots + c_{n-1}^2) + (n-1)c_n^2\\ &\ge (c_1 + \cdots + c_{n-1})^2 + (n-1)c_n^2,\end{aligned}$$

which is equivalent to $2(c_1 + c_2 + \cdots + c_n)c_n \ge nc_n^2$. Similarly, $2(c_1 + c_2 + \cdots + c_n)c_i \ge nc_i^2$ for all $i = 1, \ldots, n$. Hence all c_i are of the same sign.

8. There is exactly one point satisfying the given condition on each face of the hexahedron. Namely, on the face ABD it is the point that divides the median from D in the ratio $32:3$.

9. A necessary and sufficient condition for $\mathcal{M}$ to be nonempty is that $1/\sqrt{10} \le t \le 1$.

10. Integers a, b, q, r satisfy

$$a^2 + b^2 = (a+b)q + r, \qquad 0 \le r < a+b, \qquad q^2 + r = 1977.$$

From $q^2 \le 1977$ it follows that $q \le 44$, and consequently $a^2 + b^2 < 45(a+b)$. Having in mind the inequality $(a+b)^2 \le 2(a^2+b^2)$, we get $(a+b)^2 < 90(a+b)$, i.e., $a+b < 90$ and consequently $r < 90$. Now from $q^2 = 1977 - r > 1977 - 90 = 1887$ it follows that $q > 43$; hence $q = 44$ and $r = 41$. It remains to find positive integers a and b satisfying $a^2 + b^2 = 44(a+b) + 41$, or equivalently

$$(a-22)^2 + (b-22)^2 = 1009.$$

The only solutions to this Diophantine equation are $(|a-22|, |b-22|) \in \{(15,28), (28,15)\}$, which yield $(a,b) \in \{(7,50), (37,50), (50,7), (50,37)\}$.

11. (a) Suppose to the contrary that none of the numbers $z_0, z_1, \dots, z_{n-1}$ is divisible by n. Then two of these numbers, say z_k and z_l ($0 \le k < l \le n-1$), are congruent modulo n, and thus $n \mid z_l - z_k = z^{k+1} z_{l-k-1}$. But since $(n,z) = 1$, this implies $n \mid z_{l-k-1}$, which is a contradiction.
    (b) Again suppose the contrary, that none of $z_0, z_1, \dots, z_{n-2}$ is divisible by n. Since $(z-1, n) = 1$, this is equivalent to $n \nmid (z-1)z_j$, i.e., $z^k \not\equiv 1 \pmod{n}$ for all $k = 1, 2, \dots, n-1$. But since $(z,n) = 1$, we also have that $z^k \not\equiv 0 \pmod{n}$. It follows that there exist k, l, $1 \le k < l \le n-1$ such that $z^k \equiv z^l$, i.e., $z^{l-k} \equiv 1 \pmod{n}$, which is a contradiction.

12. According to part (a) of the previous problem we can conclude that $T = \{n \in \mathbb{N} \mid (n,z) = 1\}$.

13. The figure Φ contains two points A and B having maximum distance. Let h be the semicircle with diameter AB that lies in Φ, and let k be the circle containing h. Consider any point M inside k. The line passing through M that is orthogonal to AM meets h in some point P (because $\angle AMB > 90^\circ$). Let h' and $\overline{h'}$ be the two semicircles with diameter AP, where $M \in h'$. Since $\overline{h'}$ contains a point C such that $BC > AB$, it cannot be contained in Φ, implying that $h' \subset \Phi$. Hence M belongs to Φ. Since Φ contains no points outside the circle k, it must coincide with the disk determined by k. On the other hand, any disk has the required property.

14. We prove by induction on n that independently of the word w_0, the given algorithm generates all words of length n. This is clear for $n = 1$. Suppose now the statement is true for $n-1$, and that we are given a word $w_0 =$

$c_1c_2\dots c_n$ of length n. Obviously, the words $w_0, w_1, \dots, w_{2^{n-1}-1}$ all have the nth digit c_n, and by the inductive hypothesis these are all words whose nth digit is c_n. Similarly, by the inductive hypothesis $w_{2^{n-1}}, \dots, w_{2^n-1}$ are all words whose nth digit is $1 - c_n$, and the induction is complete.

15. Each segment is an edge of at most two squares and a diagonal of at most one square. Therefore $p_k = 0$ for $k > 3$, and we have to prove that

$$p_0 = p_2 + 2p_3. \tag{1}$$

Let us calculate the number $q(n)$ of considered squares. Each of these squares is inscribed in a square with integer vertices and sides parallel to the coordinate axes. There are $(n-s)^2$ squares of side s with integer vertices and sides parallel to the coordinate axes, and each of them circumscribes exactly s of the considered squares. It follows that $q(n) = \sum_{s=1}^{n-1}(n-s)^2 s = n^2(n^2-1)/12$. Computing the number of edges and diagonals of the considered squares in two ways, we obtain that

$$p_1 + 2p_2 + 3p_3 = 6q(n). \tag{2}$$

On the other hand, the total number of segments with endpoints in the considered integer points is given by

$$p_0 + p_1 + p_2 + p_3 = \binom{n^2}{2} = \frac{n^2(n^2-1)}{2} = 6q(n). \tag{3}$$

Now (1) follows immediately from (2) and (3).

16. For $i = k$ and $j = l$ the system is reduced to $1 \le i, j \le n$, and has exactly n^2 solutions. Let us assume that $i \ne k$ or $j \ne l$. The points $A(i, j)$, $B(k, l)$, $C(-j+k+l, i-k+l)$, $D(i-j+l, i+j-k)$ are vertices of a negatively oriented square with integer vertices lying inside the square $[1, n] \times [1, n]$, and each of these squares corresponds to exactly 4 solutions to the system. By the previous problem there are exactly $q(n) = n^2(n^2-1)/12$ such squares. Hence the number of solutions is equal to $n^2 + 4q(n) = n^2(n^2+2)/3$.

17. Centers of the balls that are tangent to K are vertices of a regular polyhedron with triangular faces, with edge length $2R$ and radius of circumscribed sphere $r + R$. Therefore the number n of these balls is 4, 6, or 20. It is straightforward to obtain that:
    (i) If $n = 4$, then $r + R = 2R(\sqrt{6}/4)$, whence $R = r(2 + \sqrt{6})$.
    (ii) If $n = 6$, then $r + R = 2R(\sqrt{2}/2)$, whence $R = r(1 + \sqrt{2})$.
    (iii) If $n = 20$, then $r + R = 2R\sqrt{5+\sqrt{5}}/8$, whence $R = r\left[\sqrt{5 - 2\sqrt{5}} + (3 - \sqrt{5})/2\right]$.

18. Let U be the midpoint of the segment AB. The point M belongs to CU and $CM = (\sqrt{5} - 1)CU/2$, $r = CU\sqrt{\sqrt{5} - 2}$.

19. We shall prove the statement by induction on m. For $m = 2$ it is trivial, since each power of 5 greater than 5 ends in 25. Suppose that the statement is true for some $m \geq 2$, and that the last m digits of 5^n alternate in parity. It can be shown by induction that the maximum power of 2 that divides $5^{2^{m-2}} - 1$ is 2^m, and consequently the difference $5^{n+2^{m-2}} - 5^n$ is divisible by 10^m but not by $2 \cdot 10^m$. It follows that the last m digits of the numbers $5^{n+2^{m-2}}$ and 5^n coincide, but the digits at the position $m+1$ have opposite parity. Hence the last $m+1$ digits of one of these two powers of 5 alternate in parity. The inductive proof is completed.

20. There exist u, v such that $a \cos x + b \sin x = r \cos(x - u)$ and $A \cos 2x + B \sin 2x = R \cos 2(x - v)$, where $r = \sqrt{a^2 + b^2}$ and $R = \sqrt{A^2 + B^2}$. Then $1 - f(x) = r \cos(x - u) + R \cos 2(x - v) \leq 1$ holds for all $x \in \mathbb{R}$.
There exists $x \in \mathbb{R}$ such that $\cos(x - u) \geq 0$ and $\cos 2(x - v) = 1$ (indeed, either $x = v$ or $x = v + \pi$ works). It follows that $R \leq 1$. Similarly, there exists $x \in \mathbb{R}$ such that $\cos(x - u) = 1/\sqrt{2}$ and $\cos 2(x - v) \geq 0$ (either $x = u - \pi/4$ or $x = u + \pi/4$ works). It follows that $r \leq \sqrt{2}$.

Remark. The proposition of this problem contained as an addendum the following, more difficult, inequality:

$$\sqrt{a^2 + b^2} + \sqrt{A^2 + B^2} \leq 2.$$

The proof follows from the existence of $x \in \mathbb{R}$ such that $\cos(x - u) \geq 1/2$ and $\cos 2(x - v) \geq 1/2$.

21. Let us consider the vectors $v_1 = (x_1, x_2, x_3)$, $v_2 = (y_1, y_2, y_3)$, $v_3 = (1, 1, 1)$ in space. The given equalities express the condition that these three vectors are mutually perpendicular. Also, $\frac{x_1^2}{x_1^2+x_2^2+x_3^2}$, $\frac{y_1^2}{y_1^2+y_2^2+y_3^2}$, and $1/3$ are the squares of the projections of the vector $(1, 0, 0)$ onto the directions of v_1, v_2, v_3, respectively. The result follows from the fact that the sum of squares of projections of a unit vector on three mutually perpendicular directions is 1.

22. Since the quadrilateral OA_1BB_1 is cyclic, $\angle OA_1B_1 = \angle OBC$. By using the analogous equalities we obtain $\angle OA_4B_4 = \angle OB_3C_3 = \angle OC_2D_2 = \angle OD_1A_1 = \angle OAB$, and similarly $\angle OB_4A_4 = \angle OBA$. Hence $\triangle OA_4B_4 \sim \triangle OAB$. Analogously, we have for the other three pairs of triangles $\triangle OB_4C_4 \sim \triangle OBC$, $\triangle OC_4D_4 \sim \triangle OCD$, $\triangle OD_4A_4 \sim \triangle ODA$, and consequently $ABCD \sim A_4B_4C_4D_4$.

23. Every polynomial $q(x_1, \dots, x_n)$ with integer coefficients can be expressed in the form $q = r_1 + x_1 r_2$, where r_1, r_2 are polynomials in $x_1, \dots, x_n$ with integer coefficients in which the variable x_1 occurs only with even exponents. Thus if $q_1 = r_1 - x_1 r_2$, the polynomial $qq_1 = r_1^2 - x_1^2 r_2^2$ contains x_1 only with even exponents. We can continue inductively constructing polynomials q_j, $j = 2, 3, \dots, n$, such that $qq_1q_2 \cdots q_j$ contains each of

variables $x_1, x_2, \ldots, x_j$ only with even exponents. Thus the polynomial $qq_1 \cdots q_n$ is a polynomial in $x_1^2, \ldots, x_n^2$.
The polynomials f and g exist for every $n \in \mathbb{N}$. In fact, it suffices to construct $q_1, \ldots, q_n$ for the polynomial $q = x_1 + \cdots + x_n$ and take $f = q_1 q_2 \cdots q_n$.

24. Setting $x = y = 0$ gives us $f(0) = 0$. Let us put $g(x) = \arctan f(x)$. The given functional equation becomes $\tan g(x+y) = \tan(g(x) + g(y))$; hence
$$g(x+y) = g(x) + g(y) + k(x,y)\pi,$$
where $k(x,y)$ is an integer function. But $k(x,y)$ is continuous and $k(0,0) = 0$, therefore $k(x,y) = 0$. Thus we obtain the classical Cauchy's functional equation $g(x+y) = g(x) + g(y)$ on the interval $(-1,1)$, all of whose continuous solutions are of the form $g(x) = ax$ for some real a. Moreover, $g(x) \in (-\pi, \pi)$ implies $|a| \le \pi/2$.
Therefore $f(x) = \tan ax$ for some $|a| \le \pi/2$, and this is indeed a solution to the given equation.

25. Let
$$f_n(z) = z^n + a \sum_{k=1}^{n} \binom{n}{k} (a - kb)^{k-1} (z + kb)^{n-k}.$$
We shall prove by induction on n that $f_n(z) = (z+a)^n$. This is trivial for $n = 1$. Suppose that the statement is true for some positive integer $n-1$. Then
$$\begin{aligned} f_n'(z) &= nz^{n-1} + a \sum_{k=1}^{n-1} \binom{n}{k} (n-k)(a-kb)^{k-1}(z+kb)^{n-k-1} \\ &= nz^{n-1} + na \sum_{k=1}^{n-1} \binom{n-1}{k} (a-kb)^{k-1}(z+kb)^{n-k-1} \\ &= nf_{n-1}(z) = n(z+a)^{n-1}. \end{aligned}$$
It remains to prove that $f_n(-a) = 0$. For $z = -a$ we have by the lemma of (SL81-13),
$$\begin{aligned} f_n(-a) &= (-a)^n + a \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} (a-kb)^{n-1} \\ &= a \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (a-kb)^{n-1} = 0. \end{aligned}$$

26. The result is an immediate consequence (for $G = \{-1, 1\}$) of the following generalization.
    (1) Let G be a proper subgroup of $\mathbb{Z}_n^*$ (the multiplicative group of residue classes modulo n coprime to n), and let V be the union of elements

of G. A number $m \in V$ is called indecomposable in V if there do not exist numbers $p, q \in V$, $p, q \notin \{-1, 1\}$, such that $pq = m$. There exists a number $r \in V$ that can be expressed as a product of elements indecomposable in V in more than one way.

First proof. We shall start by proving the following lemma.

Lemma. There are infinitely many primes not in V that do not divide n.

Proof. There is at least one such prime: In fact, any number other than ± 1 not in V must have a prime factor not in V, since V is closed under multiplication. If there were a finite number of such primes, say $p_1, p_2, \dots, p_k$, then one of the numbers $p_1 p_2 \cdots p_k + n$, $p_1^2 p_2 \cdots p_k + n$ is not in V and is coprime to n and $p_1, \dots, p_k$, which is a contradiction. [This lemma is actually a direct consequence of Dirichlet's theorem.]

Let us consider two such primes p, q that are congruent modulo n. Let p^k be the least power of p that is in V. Then p^k, q^k, $p^{k-1}q$, pq^{k-1} belong to V and are indecomposable in V. It follows that

$$r = p^k \cdot q^k = p^{k-1}q \cdot pq^{k-1}$$

has the desired property.

Second proof. Let p be any prime not in V that does not divide n, and let p^k be the least power of p that is in V. Obviously p^k is indecomposable in V. Then the number

$$r = p^k \cdot (p^{k-1} + n)(p + n) = p(p^{k-1} + n) \cdot p^{k-1}(p + n)$$

has at least two different factorizations into indecomposable factors.

27. The result is a consequence of the generalization from the previous problem for $G = \{1\}$.

Remark. There is an explicit example: $r = (n-1)^2 \cdot (2n-1)^2 = [(n-1)(2n-1)]^2$.

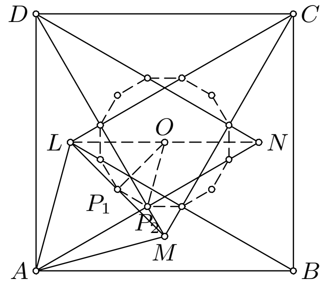
28. The recurrent relations give us that

$$x_{i+1} = \left[\frac{x_i + [n/x_i]}{2}\right] = \left[\frac{x_i + n/x_i}{2}\right] \geq [\sqrt{n}].$$

On the other hand, if $x_i > [\sqrt{n}]$ for some i, then we have $x_{i+1} < x_i$. This follows from the fact that $x_{i+1} < x_i$ is equivalent to $x_i > (x_i + n/x_i)/2$, i.e., to $x_i^2 > n$. Therefore $x_i = [\sqrt{n}]$ holds for at least one $i \leq n - [\sqrt{n}] + 1$.

Remark. If $n + 1$ is a perfect square, then $x_i = [\sqrt{n}]$ implies $x_{i+1} = [\sqrt{n}] + 1$. Otherwise, $x_i = [\sqrt{n}]$ implies $x_{i+1} = [\sqrt{n}]$.

29. Let us denote the midpoints of segments LM, AN, BL, MN, BK, CM, NK, CL, DN, KL, DM, AK by P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, respectively.



We shall prove that the dodecagon $P_1P_2P_3\dots P_{11}P_{12}$ is regular. From $BL = BA$ and $\angle ABL = 30^\circ$ it follows that $\angle BAL = 75^\circ$. Similarly $\angle DAM = 75^\circ$, and therefore $\angle LAM = 60^\circ$, which together with $AL = AM$ implies that the triangle ALM is equilateral. Now, from the triangles OLM and ALN, we get $OP_1 = LM/2$, $OP_2 = AL/2$ and $OP_2 \parallel AL$. Hence $OP_1 = OP_2$, $\angle P_1OP_2 = \angle P_1AL = 30^\circ$ and $\angle P_2OM = \angle LAD = 15^\circ$. The desired result follows from symmetry.

30. Suppose $\angle SBA = x$. By the trigonometric form of Ceva's theorem we have
$$\frac{\sin(96^\circ - x)}{\sin x}\frac{\sin 18^\circ}{\sin 12^\circ}\frac{\sin 6^\circ}{\sin 48^\circ} = 1. \tag{1}$$
We claim that $x = 12^\circ$ is a solution of this equation. To prove this, it is enough to show that $\sin 84^\circ \sin 6^\circ \sin 18^\circ = \sin 48^\circ \sin 12^\circ \sin 12^\circ$, which is equivalent to $\sin 18^\circ = 2\sin 48^\circ \sin 12^\circ = \cos 36^\circ - \cos 60^\circ$. The last equality can be checked directly.
Since the equation is equivalent to $(\sin 96^\circ \cot x - \cos 96^\circ)\sin 6^\circ \sin 18^\circ = \sin 48^\circ \sin 12^\circ$, the solution $x \in [0, \pi)$ is unique. Hence $x = 12^\circ$.

Second solution. We know that if a, b, c, a', b', c' are points on the unit circle in the complex plane, the lines aa', bb', cc' are concurrent if and only if
$$(a - b')(b - c')(c - a') = (a - c')(b - a')(c - b'). \tag{1}$$
We shall prove that $x = 12^\circ$. We may suppose that ABC is the triangle in the complex plane with vertices $a = 1$, $b = \epsilon^9$, $c = \epsilon^{14}$, where $\epsilon = \cos\frac{\pi}{15} + i\sin\frac{\pi}{15}$. If $a' = \epsilon^{12}$, $b' = \epsilon^{28}$, $c' = \epsilon$, our task is the same as proving that lines aa', bb', cc' are concurrent, or by (1) that
$$(1 - \epsilon^{28})(\epsilon^9 - \epsilon)(\epsilon^{14} - \epsilon^{12}) - (1 - \epsilon)(\epsilon^9 - \epsilon^{12})(\epsilon^{14} - \epsilon^{28}) = 0.$$
The last equality holds, since the left-hand side is divisible by the minimum polynomial of ϵ: $z^8 + z^7 - z^5 - z^4 - z^3 + z + 1$.

31. We obtain from (1) that $f(1, c) = f(1, c)f(1, c)$; hence $f(1, c) = 1$ and consequently $f(-1, c)f(-1, c) = f(1, c) = 1$, i.e. $f(-1, c) = 1$. Analogously, $f(c, 1) = f(c, -1) = 1$.
Clearly $f(1, 1) = f(-1, 1) = f(1, -1) = 1$. Now let us assume that $a \neq 1$. Observe that $f(x^{-1}, y) = f(x, y^{-1}) = f(x, y)^{-1}$. Thus by (1) and (2) we get
$$\begin{aligned} 1 &= f(a, 1 - a)f(1/a, 1 - 1/a) \\ &= f(a, 1 - a)f\left(a, \frac{1}{1 - 1/a}\right) = f\left(a, \frac{1 - a}{1 - 1/a}\right) = f(a, -a). \end{aligned}$$

We now have $f(a,a) = f(a,-1)f(a,-a) = 1 \cdot 1 = 1$ and $1 = f(ab,ab) = f(a,ab)f(b,ab) = f(a,a)f(a,b)f(b,a)f(b,b) = f(a,b)f(b,a)$.

32. It is a known result that among six persons there are 3 mutually acquainted or 3 mutually unacquainted. By the condition of the problem the last case is excluded.
If there is a man in the room who is not acquainted with four of the others, then these four men are mutually acquainted. Otherwise, each man is acquainted with at least five others, and since the sum of numbers of acquaintances of all men in the room is even, one of the men is acquainted with at least six men. Among these six there are three mutually acquainted, and they together with the first one make a group of four mutually acquainted men.

33. Let r be the radius of K and $s > \sqrt{2}/r$ an integer. Consider the points $A_k(ka_1 - [ka_1], ka_2 - [ka_2])$, where $k = 0, 1, 2, \dots, s^2$. Since all these points are in the unit square, two of them, say A_p, A_q, $q > p$, are in a small square with side $1/s$, and consequently $A_pA_q \le \sqrt{2}/s < r$. Therefore, for $n = q - p$, $m_1 = [qa_1] - [pa_1]$ and $m_2 = [qa_2] - [pa_2]$ the distance between the points $n(a_1, a_2)$ and (m_1, m_2) is less then r, i.e., the point (m_1, m_2) is in the circle $K + n(a_1, a_2)$.

34. Let A be the set of the 2^n sequences of n terms equal to ± 1. Since there are k^2 products ab with $a, b \in B$, by the pigeonhole principle there exists $c \in A$ such that $ab = c$ holds for at most $k^2/2^n$ pairs $(a,b) \in B \times B$. Then $cb \in B$ holds for at most $k^2/2^n$ values $b \in B$, which means that $|B \cap cB| \le k^2/2^n$.

35. The solutions are 0 and $N_k = 10\underbrace{99\dots9}_{k}89$, where $k = 0, 1, 2, \dots$.

Remark. If we omit the condition that at most one of the digits is zero, the solutions are numbers of the form $N_{k_1}N_{k_2}\dots N_{k_r}$, where $k_1 = k_r$, $k_2 = k_{r-1}$ etc.
The more general problem $k \cdot \overline{a_1a_2\dots a_n} = \overline{a_n\dots a_2a_1}$ has solutions only for $k = 9$ and for $k = 4$ (namely 0, $2199\dots978$ and combinations as above).

36. It can be shown by simple induction that $S^m(a_1, \dots, a_{2^n}) = (b_1, \dots, b_{2^n})$, where
$$b_k = \prod_{i=0}^{m} a_{k+i}^{\binom{m}{i}} \quad \text{(assuming that } a_{k+2^n} = a_k).$$
If we take $m = 2^n$ all the binomial coefficients $\binom{m}{i}$ apart from $i = 0$ and $i = m$ will be even, and thus $b_k = a_ka_{k+m} = 1$ for all k.

37. We look for a solution with $x_1^{A_1} = \dots = x_n^{A_n} = n^{A_1A_2\cdots A_nx}$ and $x_{n+1} = n^y$. In order for this to be a solution we must have $A_1A_2\cdots A_nx + 1 =$

$A_{n+1}y$. This equation has infinitely many solutions (x, y) in $\mathbb{N}$, since $A_1A_2\cdots A_n$ and A_{n+1} are coprime.

38. The condition says that the quadratic equation $f(x) = 0$ has distinct real solutions, where

$$f(x) = 3x^2\sum_{j=1}^{n} m_j - 2x\sum_{j=1}^{n} m_j(a_j + b_j + c_j) + \sum_{j=1}^{n} m_j(a_jb_j + b_jc_j + c_ja_j).$$

It is easy to verify that the function f is the derivative of

$$F(x) = \sum_{j=1}^{n} m_j(x - a_j)(x - b_j)(x - c_j).$$

Since $F(a_1) \le 0 \le F(a_n)$, $F(b_1) \le 0 \le F(b_n)$ and $F(c_1) \le 0 \le F(c_n)$, $F(x)$ has three distinct real roots, and hence by Rolle's theorem its derivative $f(x)$ has two distinct real roots.

39. By the pigeonhole principle, we can find 5 distinct points among the given 37 such that their x-coordinates are congruent and their y-coordinates are congruent modulo 3. Now among these 5 points either there exist three with z-coordinates congruent modulo 3, or there exist three whose z-coordinates are congruent to 0, 1, 2 modulo 3. These three points are the desired ones.

Remark. The minimum number n such that among any n integer points in space one can find three points whose barycenter is an integer point is $n = 19$. Each proof of this result seems to consist in studying a great number of cases.

40. Let us divide the chessboard into 16 squares $Q_1, Q_2, \dots, Q_{16}$ of size 2×2. Let s_k be the sum of numbers in Q_k, and let us assume that $s_1 \ge s_2 \ge \cdots \ge s_{16}$. Since $s_4 + s_5 + \cdots + s_{16} \ge 1 + 2 + \cdots + 52 = 1378$, we must have $s_4 \ge 100$ and hence $s_1, s_2, s_3 \ge 100$ as well.

41. The considered sums are congruent modulo n to $S_k = \sum_{i=1}^{N}(i + k)a_i$, $k = 0, 1, \dots, N - 1$. Since $S_k = S_0 + k(a_1 + \cdots + a_n) = S_0 + k$, all these sums give distinct residues modulo n and therefore are distinct.

42. It can be proved by induction on n that

$$\{a_{n,k} \mid 1 \le k \le 2^n\} = \{2^m \mid m = 3^n + 3^{n-1}s_1 + \cdots + 3^1 s_{n-1} + s_n \ (s_i = \pm 1)\}.$$

Thus the result is an immediate consequence of the following lemma.

Lemma. Each positive integer s can be uniquely represented in the form

$$s = 3^n + 3^{n-1}s_1 + \cdots + 3^1 s_{n-1} + s_n, \qquad \text{where } s_i \in \{-1, 0, 1\}. \quad (1)$$

Proof. Both the existence and the uniqueness can be shown by simple induction on s. The statement is trivial for $s = 1$, while for $s > 1$

there exist $q \in \mathbb{N}$, $r \in \{-1, 0, 1\}$ such that $s = 3q + r$, and q has a unique representation of the form (1).

43. Since $k(k+1)\cdots(k+p) = (p+1)!\binom{k+p}{p+1} = (p+1)!\left[\binom{k+p+1}{p+2} - \binom{k+p}{p+2}\right]$, it follows that
$$\sum_{k=1}^{n} k(k+1)\cdots(k+p) = (p+1)!\binom{n+p+1}{p+2} = \frac{n(n+1)\cdots(n+p+1)}{p+2}.$$

44. Let $d(X, \sigma)$ denote the distance from a point X to a plane σ. Let us consider the pair (A, π) where $A \in E$ and π is a plane containing some three points $B, C, D \in E$ such that $d(A, \pi)$ is the smallest possible. We may suppose that B, C, D are selected such that $\triangle BCD$ contains no other points of E. Let A' be the projection of A on π, and let l_b, l_c, l_d be lines through B, C, D parallel to CD, DB, BC respectively. If A' is in the half-plane determined by l_d not containing BC, then $d(D, ABC) \le d(A', ABC) < d(A, BCD)$, which is impossible. Similarly, A' lies in the half-planes determined by l_b, l_c that contain D, and hence A' is inside the triangle bordered by l_b, l_c, l_d. The minimality property of (A, π) and the way in which BCD was selected guarantee that $E \cap T = \{A, B, C, D\}$.

45. As in the previous problem, let us choose the pair (A, π) such that $d(A, \pi)$ is minimal. If π contains only three points of E, we are done. If not, there are four points in $E \cap P$, say A_1, A_2, A_3, A_4, such that the quadrilateral $Q = A_1A_2A_3A_4$ contains no other points of E. Suppose Q is not convex, and that w.l.o.g. A_1 is inside the triangle $A_2A_3A_4$. If A_0 is the projection of A on P, the point A_1 belongs to one of the triangles $A_0A_2A_3$, $A_0A_3A_4$, $A_0A_4A_2$, say $A_0A_2A_3$. Then $d(A_1, AA_2A_3) \le d(A_0, AA_2A_3) < AA_0$, which is impossible. Hence Q is convex. Also, by the minimality property of (A, π) the pyramid $AA_1A_2A_3A_4$ contains no other points of E.

46. We need to consider only the case $t > |x|$. There is no loss of generality in assuming $x > 0$.
To obtain the estimate from below, set
$$\begin{aligned} a_1 &= f\left(-\frac{x+t}{2}\right) - f(-(x+t)), & a_2 &= f(0) - f\left(-\frac{x+t}{2}\right), \\ a_3 &= f\left(\frac{x+t}{2}\right) - f(0), & a_4 &= f(x+t) - f\left(\frac{x+t}{2}\right). \end{aligned}$$
Since $-(x+t) < x - t$ and $x < (x+t)/2$, we have $f(x) - f(x-t) \le a_1 + a_2 + a_3$. Since $2^{-1} < a_{j+1}/a_j < 2$, it follows that
$$g(x,t) > \frac{a_4}{a_1 + a_2 + a_3} > \frac{a_3/2}{4a_3 + 2a_3 + a_3} = 14^{-1}.$$
To obtain the estimate from above, set

$$b_1 = f(0) - f\left(-\frac{x+t}{3}\right), \qquad b_2 = f\left(\frac{x+t}{3}\right) - f(0),$$
$$b_3 = f\left(\frac{2(x+t)}{3}\right) - f\left(\frac{x+t}{3}\right), \quad b_4 = f(x+t) - f\left(\frac{2(x+t)}{3}\right).$$

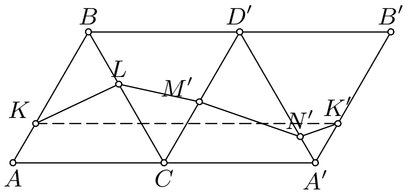
If $t < 2x$, then $x - t < -(x+t)/3$ and therefore $f(x) - f(x-t) \geq b_1$. If $t \geq 2x$, then $(x+t)/3 \leq x$ and therefore $f(x) - f(x-t) \geq b_2$. Since $2^{-1} < b_{j+1}/b_j < 2$, we get

$$g(x,t) < \frac{b_2 + b_3 + b_4}{\min\{b_1, b_2\}} < \frac{b_2 + 2b_2 + 4b_2}{b_2/2} = 14.$$

47. M lies on AB and N lies on BC. If $CQ \leq 2CD/3$, then $BM = CQ/2$. If $CQ > 2CD/3$, then N coincides with C.

48. Let a plane cut the edges AB, BC, CD, DA at points K, L, M, N respectively.

Let D', A', B' be distinct points in the plane ABC such that the triangles BCD', $CD'A'$, $D'A'B'$ are equilateral, and $M' \in [CD']$, $N' \in [D'A']$, and $K' \in [A'B']$ such that $CM' = CM$, $A'N' = AN$, and $A'K' = AK$. The perimeter P of the quadrilateral $KLMN$ is equal to the length of the polygonal line $KLM'N'K'$, which is not less than KK'. It follows that $P \geq 2a$.



Let us consider all quadrilaterals $KLMN$ that are obtained by intersecting the tetrahedron by a plane parallel to a fixed plane α. The lengths of the segments KL, LM, MN, NK are linear functions in AK, and so is P. Thus P takes its maximum at an endpoint of the interval, i.e., when the plane $KLMN$ passes through one of the vertices A, B, C, D, and it is easy to see that in this case $P \leq 3a$.

49. If one of p, q, say p, is zero, then $-q$ is a perfect square. Conversely, $(p,q) = (0, -t^2)$ and $(p,q) = (-t^2, 0)$ satisfy the conditions for $t \in \mathbb{Z}$.

We now assume that p, q are nonzero. If the trinomial $x^2 + px + q$ has two integer roots x_1, x_2, then $|q| = |x_1 x_2| \geq |x_1| + |x_2| - 1 \geq |p| - 1$. Similarly, if $x^2 + qx + p$ has integer roots, then $|p| \geq |q| - 1$ and $q^2 - 4p$ is a square. Thus we have two cases to investigate:

(i) $|p| = |q|$. Then $p^2 - 4q = p^2 \pm 4p$ is a square, so $(p,q) = (4,4)$.

(ii) $|p| = |q| \pm 1$. The solutions for (p,q) are $(t, -1-t)$ for $t \in \mathbb{Z}$ and $(5,6)$, $(6,5)$.

50. Suppose that $P_n(x) = n$ for $x \in \{x_1, x_2, \ldots, x_n\}$. Then

$$P_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n) + n.$$

From $P_n(0) = 0$ we obtain $n = |x_1 x_2 \cdots x_n| \geq 2^{n-2}$ (because at least $n-2$ factors are different from ± 1) and therefore $n \geq 2^{n-2}$. It follows that $n \leq 4$.
For each positive integer $n \leq 4$ there exists a polynomial P_n. Here is the list of such polynomials:

$$n = 1 : \pm x, \qquad n = 2 : 2x^2,\ \ x^2 \pm x,\ \ -x^2 \pm 3x,$$
$$n = 3 : \pm(x^3 - x) + 3x^2, \qquad n = 4 : -x^4 + 5x^2.$$

51. We shall use the following algorithm:

    Choose a segment of maximum length ("basic" segment) and put on it unused segments of the opposite color without overlapping, each time of the maximum possible length, as long as it is possible. Repeat the procedure with remaining segments until all the segments are used.

    Let us suppose that the last basic segment is black. Then the length of the used part of any white basic segment is greater than the free part, and consequently at least one-half of the length of the white segments has been used more than once. Therefore all basic segments have total length at most 1.5 and can be distributed on a segment of length 1.51.
    On the other hand, if we are given two white segments of lengths 0.5 and two black segments of lengths 0.999 and 0.001, we cannot distribute them on a segment of length less than 1.499.

52. The maximum and minimum are $2R\sqrt{4-2k^2}$ and $2R\left(1+\sqrt{1-k^2}\right)$ respectively.

53. The discriminant of the given equation considered as a quadratic equation in b is $196-75a^2$. Thus $75a^2 \leq 196$ and hence $-1 \leq a \leq 1$. Now the integer solutions of the given equation are easily found: $(-1,3)$, $(0,0)$, $(1,2)$.

54. We shall use the following lemma.
    Lemma. If a real function f is convex on the interval I and $x, y, z \in I$, $x \leq y \leq z$, then

    $$(y-z)f(x) + (z-x)f(y) + (x-y)f(z) \leq 0.$$

    Proof. The inequality is obvious for $x = y = z$. If $x < z$, then there exist p, r such that $p + r = 1$ and $y = px + rz$. Then by Jensen's inequality $f(px + rz) \leq pf(x) + rf(z)$, which is equivalent to the statement of the lemma.

    By applying the lemma to the convex function $-\ln x$ we obtain $x^y y^z z^x \geq y^x z^y x^z$ for any $0 < x \leq y \leq z$. Multiplying the inequalities $a^b b^c c^a \geq b^a c^b a^c$ and $a^c c^d d^a \geq c^a d^c a^d$ we get the desired inequality.

    Remark. Similarly, for $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ it holds that $a_1^{a_2} a_2^{a_3} \cdots a_n^{a_1} \geq a_2^{a_1} a_3^{a_2} \cdots a_1^{a_n}$.

55. The statement is true without the assumption that $O \in BD$. Let $BP \cap DN = \{K\}$. If we denote $\overrightarrow{AB} = a$, $\overrightarrow{AD} = b$ and $\overrightarrow{AO} = \alpha a + \beta b$ for some $\alpha, \beta \in \mathbb{R}$, $1/\alpha + 1/\beta \neq 1$, by straightforward calculation we obtain that

$$\overrightarrow{AK} = \frac{\alpha}{\alpha+\beta-\alpha\beta}a + \frac{\beta}{\alpha+\beta-\alpha\beta}b = \frac{1}{\alpha+\beta-\alpha\beta}\overrightarrow{AO}.$$

Hence A, K, O are collinear.

56. See the solution to (LL67-38).

57. Suppose that there exists a sequence of 17 terms $a_1, a_2, \dots, a_{17}$ satisfying the required conditions. Then the sum of terms in each row of the rectangular array below is positive, while the sum of terms in each column is negative, which is a contradiction.

$$\begin{array}{cccc} a_1 & a_2 & \dots & a_{11} \\ a_2 & a_3 & \dots & a_{12} \\ \vdots & \vdots & & \vdots \\ a_7 & a_8 & \dots & a_{17} \end{array}$$

On the other hand, there exist 16-term sequences with the required property. An example is $5, 5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5$ which can be obtained by solving the system of equations $\sum_{i=k}^{k+10} a_i = 1$ ($k = 1, 2, \dots, 6$) and $\sum_{i=l}^{l+6} a_i = -1$ ($l = 1, 2, \dots, 10$).

Second solution. We shall prove a stronger statement: If 7 and 11 in the question are replaced by any positive integers m, n, then the maximum number of terms is $m + n - (m, n) - 1$.

Let $a_1, a_2, \dots, a_l$ be a sequence of real numbers, and let us define $s_0 = 0$ and $s_k = a_1 + \dots + a_k$ ($k = 1, \dots, l$). The given conditions are equivalent to $s_k > s_{k+m}$ for $0 \le k \le l - m$ and $s_k < s_{k+n}$ for $0 \le k \le l - n$.

Let $d = (m, n)$ and $m = m'd$, $n = n'd$. Suppose that there exists a sequence (a_k) of length greater than or equal to $l = m + n - d$ satisfying the required conditions. Then the $m' + n'$ numbers $s_0, s_d, \dots, s_{(m'+n'-1)d}$ satisfy n' inequalities $s_{k+m} < s_k$ and m' inequalities $s_k < s_{k+n}$. Moreover, each term s_{kd} appears twice in these inequalities: once on the left-hand and once on the right-hand side. It follows that there exists a ring of inequalities $s_{i_1} < s_{i_2} < \dots < s_{i_k} < s_{i_1}$, giving a contradiction.

On the other hand, suppose that such a ring of inequalities can be made also for $l = m + n - d - 1$, say $s_{i_1} < s_{i_2} < \dots < s_{i_k} < s_{i_1}$. If there are p inequalities of the form $a_{k+m} < a_k$ and q inequalities of the form $a_{k+n} > a_k$ in the ring, then $qn = rm$, which implies $m' \mid q$, $n' \mid p$ and thus $k = p + q \ge m' + n'$. But since all $i_1, i_2, \dots, i_k$ are congruent modulo d, we have $k \le m' + n' - 1$, a contradiction. Hence there exists a sequence of length $m + n - d - 1$ with the required property.

58. The following inequality (Finsler and Hadwiger, 1938) is sharper than the one we have to prove:

$$2ab + 2bc + 2ca - a^2 - b^2 - c^2 \ge 4S\sqrt{3}. \tag{1}$$

First proof. Let us set $2x = b + c - a$, $2y = c + a - b$, $2z = a + b - c$.

Then $x, y, z > 0$ and the inequality (1) becomes

$$y^2z^2 + z^2x^2 + x^2y^2 \geq xyz(x + y + z),$$

which is equivalent to the obvious inequality $(xy - yz)^2 + (yz - zx)^2 + (zx - xy)^2 \geq 0$.

Second proof. Using the known relations for a triangle

$$\begin{aligned} a^2 + b^2 + c^2 &= 2s^2 - 2r^2 - 8rR, \\ ab + bc + ca &= s^2 + r^2 + 4rR, \\ S &= rs, \end{aligned}$$

where r and R are the radii of the incircle and the circumcircle, s the semiperimeter and S the area, we can transform (1) into

$$s\sqrt{3} \leq 4R + r.$$

The last inequality is a consequence of the inequalities $2r \leq R$ and $s^2 \leq 4R^2 + 4Rr + 3r^2$, where the last one follows from the equality $HI^2 = 4R^2 + 4Rr + 3r^2 - s^2$ (H and I being the orthocenter and the incenter of the triangle).

59. Let us consider the set R of pairs of coordinates of the points from E reduced modulo 3. If some element of R occurs thrice, then the corresponding points are vertices of a triangle with integer barycenter. Also, no three elements from E can have distinct x-coordinates and distinct y-coordinates. By an easy discussion we can conclude that the set R contains at most four elements. Hence $|E| \leq 8$.
An example of a set E consisting of 8 points that satisfies the required condition is

$$E = \{(0,0), (1,0), (0,1), (1,1), (3,6), (4,6), (3,7), (4,7)\}.$$

60. By Lagrange's interpolation formula we have

$$F(x) = \sum_{j=0}^{n} F(x_j) \frac{\prod_{i \neq j}(x - x_j)}{\prod_{i \neq j}(x_i - x_j)}.$$

Since the leading coefficient in $F(x)$ is 1, it follows that

$$1 = \sum_{j=0}^{n} \frac{F(x_j)}{\prod_{i \neq j}(x_i - x_j)} .$$

Since

$$\left| \prod_{i \neq j}(x_i - x_j) \right| = \prod_{i=0}^{j-1} |x_i - x_j| \prod_{i=j+1}^{n} |x_i - x_j| \geq j!(n-j)!,$$

we have

$$1 \le \sum_{j=0}^{n} \frac{|F(x_j)|}{\left|\prod_{i \ne j}(x_i - x_j)\right|} \le \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} |F(x_j)| \le \frac{2^n}{n!} \max |F(x_j)|.$$

Now the required inequality follows immediately.

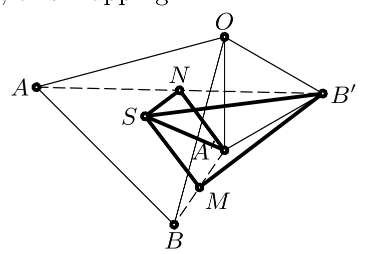
4.20 Solutions to the Shortlisted Problems of IMO 1978

1. There exists an M_s that contains at least $2n/k = 2(k^2+1)$ elements. It follows that M_s contains either at least k^2+1 even numbers or at least k^2+1 odd numbers. In the former case, consider the predecessors of those k^2+1 numbers: among them, at least $\frac{k^2+1}{k+1} > k$, i.e., at least $k+1$, belong to the same subset, say M_t. Then we choose s, t. The latter case is similar.

   Second solution. For all $i, j \in \{1, 2, \dots, k\}$, consider the set $N_{ij} = \{r \mid 2r \in M_i,\ 2r-1 \in M_j\}$. Then $\{N_{ij} \mid i, j\}$ is a partition of $\{1, 2, \dots, n\}$ into k^2 subsets. For $n \geq k^3+1$ one of these subsets contains at least $k+1$ elements, and the statement follows.

   Remark. The statement is not necessarily true when $n = k^3$.

2. Consider the transformation ϕ of the plane defined as the homothety $\mathcal{H}$ with center B and coefficient 2 followed by the rotation $\mathcal{R}$ about the center O through an angle of 60°. Being direct, this mapping must be a rotational homothety. We also see that $\mathcal{H}$ maps S into the point symmetric to S with respect to OA, and $\mathcal{R}$ takes it back to S. Hence S is a fixed point, and is consequently also the center of ϕ. Therefore ϕ is the rotational homothety about S with the angle 60°

   and coefficient 2. (In fact, this could also be seen from the fact that ϕ preserves angles of triangles and maps the segment SR onto SB, where R is the midpoint of AB.)
   Since $\phi(M) = B'$, we conclude that $\angle MSB' = 60^\circ$ and $SB'/SM = 2$. Similarly, $\angle NSA' = 60^\circ$ and $SA'/SN = 2$, so triangles MSB' and NSA' are indeed similar.

   Second solution. Probably the simplest way here is using complex numbers. Put the origin at O and complex numbers a, a' at points A, A', and denote the primitive sixth root of 1 by ω. Then the numbers at B, B', S and N are ωa, $\omega a'$, $(a+\omega a)/3$, and $(a+\omega a')/2$ respectively. Now it is easy to verify that $(n-s) = \omega(a'-s)/2$, i.e., that $\angle NSA' = 60^\circ$ and $SA'/SN = 2$.

3. What we need are m, n for which $1978^m(1978^{n-m}-1)$ is divisible by $1000 = 8 \cdot 125$. Since $1978^{n-m}-1$ is odd, it follows that 1978^m is divisible by 8, so $m \geq 3$.
   Also, $1978^{n-m}-1$ is divisible by 125, i.e., $1978^{n-m} \equiv 1 \pmod{125}$. Note that $1978 \equiv -2 \pmod 5$, and consequently also $-2^{n-m} \equiv 1$. Hence $4 \mid n-m = 4k$, $k \geq 1$. It remains to find the least k such that $1978^{4k} \equiv 1 \pmod{125}$. Since $1978^4 \equiv (-22)^4 = 484^2 \equiv (-16)^2 = 256 \equiv 6$, we reduce it to $6^k \equiv 1$. Now $6^k = (1+5)^k \equiv 1 + 5k + 25\binom{k}{2} \pmod{125}$, which

reduces to $125 \mid 5k(5k-3)$. But $5k-3$ is not divisible by 5, and so $25 \mid k$. Therefore $100 \mid n-m$, and the desired values are $m=3$, $n=103$.

4. Let γ, φ be the angles of T_1 and T_2 opposite to c and w respectively. By the cosine theorem, the inequality is transformed into

$$\begin{aligned} a^2(2v^2-2uv\cos\varphi)+b^2(2u^2-2uv\cos\varphi) \\ +2(a^2+b^2-2ab\cos\gamma)uv\cos\varphi \geq 4abuv\sin\gamma\sin\varphi. \end{aligned}$$

This is equivalent to $2(a^2v^2+b^2u^2)-4abuv(\cos\gamma\cos\varphi+\sin\gamma\sin\varphi) \geq 0$, i.e., to

$$2(av-bu)^2+4abuv(1-\cos(\gamma-\varphi)) \geq 0,$$

which is clearly satisfied. Equality holds if and only if $\gamma=\varphi$ and $a/b = u/v$, i.e., when the triangles are similar, a corresponding to u and b to v.

5. We first explicitly describe the elements of the sets M_1, M_2.

$x \notin M_1$ is equivalent to $x = a+(a+1)+\cdots+(a+n-1) = n(2a+n-1)/2$ for some natural numbers n, a, $n \geq 2$. Among n and $2a+n-1$, one is odd and the other even, and both are greater than 1; so x has an odd factor ≥ 3. On the other hand, for every x with an odd divisor $p > 3$ it is easy to see that there exist corresponding a, n. Therefore $M_1 = \{2^k \mid k = 0,1,2,\dots\}$.

$x \notin M_2$ is equivalent to $x = a+(a+2)+\cdots+(a+2(n-1)) = n(a+n-1)$, where $n \geq 2$, i.e. to x being composite. Therefore $M_2 = \{1\} \cup \{p \mid p = \text{prime}\}$.

$x \notin M_3$ is equivalent to $x = a+(a+3)+\cdots+(a+3(n-1)) = n(2a+3(n-1))/2$.

It remains to show that every $c \in M_3$ can be written as $c = 2^k p$ with p prime. Suppose the opposite, that $c = 2^k pq$, where p, q are odd and $q \geq p \geq 3$. Then there exist positive integers a, n $(n \geq 2)$ such that $c = n(2a+3(n-1))/2$ and hence $c \notin M_3$. Indeed, if $k=0$, then $n=2$ and $2a+3 = pq$ work; otherwise, setting $n=p$ one obtains $a = 2^k q - 3(p-1)/2 \geq 2q - 3(p-1)/2 \geq (p+3)/2 > 1$.

6. For fixed n and the set $\{\varphi(1),\dots,\varphi(n)\}$, there are finitely many possibilities for mapping φ to $\{1,\dots,n\}$. Suppose φ is the one among these for which $\sum_{k=1}^n \varphi(k)/k^2$ is minimal. If $i<j$ and $\varphi(i)<\varphi(j)$ for some $i,j \in \{1,\dots,n\}$, define ψ as $\psi(i)=\varphi(j)$, $\psi(j)=\varphi(i)$, and $\psi(k)=\varphi(k)$ for all other k. Then

$$\begin{aligned} \sum \frac{\varphi(k)}{k^2} - \sum \frac{\psi(k)}{k^2} &= \left(\frac{\varphi(i)}{i^2}+\frac{\varphi(j)}{j^2}\right) - \left(\frac{\varphi(i)}{j^2}+\frac{\varphi(j)}{i^2}\right) \\ &= (i-j)(\varphi(j)-\varphi(i))\frac{i+j}{i^2j^2} > 0, \end{aligned}$$

which contradicts the assumption. This shows that $\varphi(1) < \cdots < \varphi(n)$, and consequently $\varphi(k) \geq k$ for all k. Hence

$$\sum_{k=1}^{n} \frac{\varphi(k)}{k^2} \geq \sum_{k=1}^{n} \frac{k}{k^2} = \sum_{k=1}^{n} \frac{1}{k}.$$

7. Let $x = OA$, $y = OB$, $z = OC$, $\alpha = \angle BOC$, $\beta = \angle COA$, $\gamma = \angle AOB$. The conditions yield the equation $x + y + \sqrt{x^2 + y^2 - 2xy\cos\gamma} = 2p$, which transforms to $(2p - x - y)^2 = x^2 + y^2 - 2xy\cos\gamma$, i.e. $(p - x)(p - y) = xy(1 - \cos\gamma)$. Thus

$$\frac{p-x}{x} \cdot \frac{p-y}{y} = 1 - \cos\gamma,$$

and analogously $\frac{p-y}{y} \cdot \frac{p-z}{z} = 1 - \cos\alpha$, $\frac{p-z}{z} \cdot \frac{p-x}{x} = 1 - \cos\beta$. Setting $u = \frac{p-x}{x}$, $v = \frac{p-y}{y}$, $w = \frac{p-z}{z}$, the above system becomes

$$uv = 1 - \cos\gamma, \qquad vw = 1 - \cos\alpha, \qquad wu = 1 - \cos\beta.$$

This system has a unique solution in positive real numbers u, v, w: $u = \sqrt{\frac{(1-\cos\beta)(1-\cos\gamma)}{1-\cos\alpha}}$, etc. Finally, the values of x, y, z are uniquely determined from u, v, w.

Remark. It is not necessary that the three lines be in the same plane. Also, there could be any odd number of lines instead of three.

8. Take the subset $\{a_i\} = \{1, 7, 11, 13, 17, 19, 23, 29, \ldots, 30m - 1\}$ of S containing all the elements of S that are not multiples of 3. There are $8m$ such elements. Every element in S can be uniquely expressed as $3^t a_i$ for some i and $t \geq 0$. In a subset of S with $8m + 1$ elements, two of them will have the same a_i, hance one will divide the other.
On the other hand, for each $i = 1, 2, \ldots, 8m$ choose $t \geq 0$ such that $10m < b_i = 3^t a_i < 30m$. Then there are $8m$ b_i's in the interval $(10m, 30m)$, and the quotient of any two of them is less than 3, so none of them can divide any other. Thus the answer is $8m$.

9. Since the nth missing number (gap) is $f(f(n)) + 1$ and $f(f(n))$ is a member of the sequence, there are exactly $n - 1$ gaps less than $f(f(n))$. This leads to
$$f(f(n)) = f(n) + n - 1. \tag{1}$$
Since 1 is not a gap, we have $f(1) = 1$. The first gap is $f(f(1)) + 1 = 2$. Two consecutive integers cannot both be gaps (the predecessor of a gap is of the form $f(f(m))$). Now we deduce $f(2) = 3$; a repeated application of the formula above gives $f(3) = 3 + 1 = 4$, $f(4) = 4 + 2 = 6$, $f(6) = 9$, $f(9) = 14$, $f(14) = 22$, $f(22) = 35$, $f(35) = 56$, $f(56) = 90$, $f(90) = 145$, $f(145) = 234$, $f(234) = 378$.
Also, $f(f(35)) + 1 = 91$ is a gap, so $f(57) = 92$. Then by (1), $f(92) = 148$, $f(148) = 239$, $f(239) = 386$. Finally, here $f(f(148)) + 1 = 387$ is a gap, so $f(240) = 388$.

Second solution. As above, we arrive at formula (1). Then by simple induction it follows that $f(F_n+1)=F_{n+1}+1$, where F_k is the Fibonacci sequence ($F_1=F_2=1$).
We now prove by induction (on n) that $f(F_n+x)=F_{n+1}+f(x)$ for all x with $1\le x\le F_{n-1}$. This is trivially true for $n=0,1$. Supposing that it holds for $n-1$, we shall prove it for n:
(i) If $x=f(y)$ for some y, then by the inductive assumption and (1)

$$\begin{aligned} f(F_n+x) &= f(F_n+f(y)) = f(f(F_{n-1}+y)) \\ &= F_n+f(y)+F_{n-1}+y-1 = F_{n+1}+f(x). \end{aligned}$$

(ii) If $x=f(f(y))+1$ is a gap, then $f(F_n+x-1)+1=F_{n+1}+f(x-1)+1$ is a gap also:

$$\begin{aligned} F_{n+1}+f(x)+1 &= F_{n+1}+f(f(f(y)))+1 \\ &= f(F_n+f(f(y)))+1 = f(f(F_{n-1}+f(y)))+1. \end{aligned}$$

It follows that $f(F_n+x)=F_{n+1}+f(x-1)+2=F_{n+1}+f(x)$.
Now, since we know that each positive integer x is expressible as $x=F_{k_1}+F_{k_2}+\cdots+F_{k_r}$, where $0<k_r\neq 2$, $k_i\ge k_{i+1}+2$, we obtain $f(x)=F_{k_1+1}+F_{k_2+1}+\cdots+F_{k_r+1}$. Particularly, $240=233+5+2$, so $f(240)=377+8+3=388$.

Remark. It can be shown that $f(x)=[\alpha x]$, where $\alpha=(1+\sqrt{5})/2$.

10. Assume the opposite. One of the countries, say A, contains at least 330 members $a_1,a_2,\dots,a_{330}$ of the society ($6\cdot 329=1974$). Consider the differences $a_{330}-a_i, =1,2,\dots,329$: the members with these numbers are not in A, so at least 66 of them, $a_{330}-a_{i_1},\dots,a_{330}-a_{i_{66}}$, belong to the same country, say B. Then the differences $(a_{i_{66}}-a_{330})-(a_{i_j}-a_{330})=a_{i_{66}}-a_{i_j}$, $j=1,2,\dots,65$, are neither in A nor in B. Continuing this procedure, we find that 17 of these differences are in the same country, say C, then 6 among 16 differences of themselves in a country D, and 3 among 5 differences of themselves in E; finally, one among two differences of these 3 differences belong to country F, so that the difference of themselves cannot be in any country. This is a contradiction.

   Remark. The following stronger ($[6!e]=1957$) statement can be proved in the same way.

   Schurr's lemma. If n is a natural number and e the logarithm base, then for every partition of the set $\{1,2,\dots,[en!]\}$ into n subsets one of these subsets contains some two elements and their difference.

11. Set $F(x)=f_1(x)f_2(x)\cdots f_n(x)$: we must prove concavity of $F^{1/n}$. By the assumption,

$$F(\theta x+(1-\theta)y)\geq\prod_{i=1}^{n}[\theta f_i(x)+(1-\theta)f(y)]$$
$$=\sum_{k=0}^{n}\theta^k(1-\theta)^{n-k}\sum f_{i_1}(x)\dots f_{i_k}(x)f_{i_{k+1}}(y)f_{i_n}(y),$$

where the second sum goes through all $\binom{n}{k}$ k-subsets $\{i_1,\dots,i_k\}$ of $\{1,\dots,n\}$. The inequality between the arithmetic and geometric means now gives us

$$\sum f_{i_1}(x)f_{i_2}(x)\cdots f_{i_k}(x)f_{i_{k+1}}(y)f_{i_n}(y)\geq\binom{n}{k}F(x)^{k/n}F(y)^{(n-k)/n}.$$

Inserting this in the above inequality and using the binomial formula, we finally obtain

$$F(\theta x+(1-\theta)y)\geq\sum_{k=0}^{n}\theta^k(1-\theta)^{n-k}\binom{n}{k}F(x)^{k/n}F(y)^{(n-k)/n}$$
$$=\left(\theta F(x)^{1/n}+(1-\theta)F(y)^{1/n}\right)^n,$$

which proves the assertion.

12. Let O be the center of the smaller circle, T its contact point with the circumcircle of ABC, and J the midpoint of segment BC. The figure is symmetric with respect to the line through A,O,J,T.
A homothety centered at A taking T into J will take the smaller circle into the incircle of ABC, hence will take O into the incenter I. On the other hand, $\angle ABT=\angle ACT=90°$ implies that the quadrilaterals $ABTC$ and $APOQ$ are similar. Hence the above homothety also maps O to the midpoint of PQ. This finishes the proof.

Remark. The assertion is true for a nonisosceles triangle ABC as well, and this (more difficult) case is a matter of *SL93-3*.

13. *Lemma.* If $MNPQ$ is a rectangle and O any point in space, then $OM^2+OP^2=ON^2+OQ^2$.
Proof. Let O_1 be the projection of O onto $MNPQ$, and m,n,p,q denote the distances of O_1 from MN,NP,PQ,QM, respectively. Then $OM^2=OO_1^2+q^2+m^2$, $ON^2=OO_1^2+m^2+n^2$, $OP^2=OO_1^2+n^2+p^2$, $OQ^2=OO_1^2+p^2+q^2$, and the lemma follows immediately.
Now we return to the problem. Let O be the center of the given sphere S, and X the point opposite P in the face of the parallelepiped through P,A,B. By the lemma, we have $OP^2+OQ^2=OC^2+OX^2$ and $OP^2+OX^2=OA^2+OB^2$. Hence $2OP^2+OQ^2=OA^2+OB^2+OC^2=3R^2$, i.e. $OQ=\sqrt{3R^2-OP^2}>R$.
We claim that the locus of Q is the whole sphere $(O,\sqrt{3R^2-OP^2})$. Choose any point Q on this sphere. Since $OQ>R>OP$, the sphere

with diameter PQ intersects S on a circle. Let C be an arbitrary point on this circle, and X the point opposite C in the rectangle $PCQX$. By the lemma, $OP^2 + OQ^2 = OC^2 + OX^2$, hence $OX^2 = 2R^2 - OP^2 > R^2$. The plane passing through P and perpendicular to PC intersects S in a circle γ; both P, X belong to this plane, P being inside and X outside the circle, so that the circle with diameter PX intersects γ at some point B. Finally, we choose A to be the point opposite B in the rectangle $PBXA$: we deduce that $OA^2 + OB^2 = OP^2 + OX^2$, and consequently $A \in S$. By the construction, there is a rectangular parallelepiped through P, A, B, C, X, Q.

14. We label the cells of the cube by (a_1, a_2, a_3), $a_i \in \{1, 2, \dots, 2n+1\}$, in a natural way: for example, as Cartesian coordinates of centers of the cells ($(1,1,1)$ is one corner, etc.). Notice that there should be $(2n+1)^3 - 2n(2n+1) \cdot 2(n+1) = 2n+1$ void cells, i.e., those not covered by any piece of soap.

    $n = 1$. In this case, six pieces of soap $1 \times 2 \times 2$ can be placed on the following positions: $[(1,1,1),(2,2,1)]$, $[(3,1,1),(3,2,2)]$, $[(2,3,1),(3,3,2)]$ and the symmetric ones with respect to the center of the box. (Here $[A, B]$ denotes the rectangle with opposite corners at A, B.)

    n is even. Each of the $2n+1$ planes $P_k = \{(a_1, a_2, k) \mid a_i = 1, \dots, 2n+1\}$ can receive $2n$ pieces of soap: In fact, P_k can be partitioned into four $n \times (n+1)$ rectangles at the corners and the central cell, while an $n \times (n+1)$ rectangle can receive $n/2$ pieces of soap.

    n is odd, $n > 1$. Let us color a cell (a_1, a_2, a_3) blue, red, or yellow if exactly three, two or one a_i respectively is equal to $n+1$. Thus there are 1 blue, $6n$ red, and $12n^2$ yellow cells. We notice that each piece of soap must contain at least one colored cell (because $2(n+1) > 2n+1$). Also, every piece of soap contains an even number (actually, $1 \cdot 2$, $1(n+1)$, or $2(n+1)$) of cells in P_k. On the other hand, $2n+1$ cells are void, i.e., one in each plane.

    There are several cases for a piece of soap S:

    (i) S consists of 1 blue, $n+1$ red and n yellow cells;
    (ii) S consists of 2 red and $2n$ yellow cells (and no blue cells);
    (iii) S contains 1 red cell, $n+1$ yellow cells, and the are rest uncolored;
    (iv) S contains 2 yellow cells and no blue or red ones.

    From the descriptions of the last three cases, we can deduce that if S contains r red cells and no blue, then it contains exactly $2 + (n-1)r$ red ones. $(*)$

    Now, let $B_1, \dots, B_k$ be all boxes put in the cube, with a possible exception for the one covering the blue cell: thus $k = 2n(2n+1)$ if the blue cell is void, or $k = 2n(2n+1) - 1$ otherwise. Let r_i and y_i respectively be the numbers of red and yellow cells inside B_i. By $(*)$ we have $y_1 + \dots + y_k = 2k + (n-1)(r_1 + \dots + r_k)$. If the blue cell is void, then $r_1 + \dots + r_k = 6n$ and consequently $y_1 + \dots + y_k =$

$4n(2n+1)+6n(n-1) = 14n^2-2n$, which is impossible because there are only $12n^2 < 14n^2-2n$ yellow cells. Otherwise, $r_1+\cdots+r_k \geq 5n-2$ (because $n+1$ red cells are covered by the box containing the blue cell, and one can be void) and consequently $y_1+\cdots+y_k \geq 4n(2n+1)-2+(n-1)(5n-2) = 13n^2-3n$; since there are n more yellow cells in the box containing the blue one, this counts for $13n^2-2n > 12n^2$ $(n \geq 3)$, again impossible.

Remark. The following solution of the case n odd is simpler, but does not work for $n=3$. For $k=1,2,3$, let m_k be the number of pieces whose long sides are perpendicular to the plane $\pi_k(a_k = n+1)$. Each of these m_k pieces covers exactly 2 cells of π_k, while any other piece covers $n+1$, $2(n+1)$, or none. It follows that $4n^2+4n-2m_k$ is divisible by $n+1$, and so is $2m_k$. This further implies that $2m_1+2m_2+2m_3 = 4n(2n+1)$ is a multiple of $n+1$, which is impossible for each odd n except $n=1$ and $n=3$.

15. Let $C_n = \{a_1, \dots, a_n\}$ $(C_0 = \emptyset)$ and $P_n = \{f(B) \mid B \subseteq C_n\}$. We claim that P_n contains at least $n+1$ distinct elements. First note that $P_0 = \{0\}$ contains one element. Suppose that $P_{n+1} = P_n$ for some n. Since $P_{n+1} = \{a_{n+1}+r \mid r \in P_n\}$, it follows that for each $r \in P_n$, also $r+b_n \in P_n$. Then obviously $0 \in P_n$ implies $kb_n \in P_n$ for all k; therefore $P_n = P$ has at least $p \geq n+1$ elements. Otherwise, if $P_{n+1} \supset P_n$ for all n, then $|P_{n+1}| \geq |P_n|+1$ and hence $|P_n| \geq n+1$, as claimed. Consequently, $|P_{p-1}| \geq p$. (All the operations here are performed modulo p.)

16. Clearly $|x| \leq 1$. As x runs over $[-1,1]$, the vector $u = (ax, a\sqrt{1-x^2})$ runs over all vectors of length a in the plane having a nonnegative vertical component. Putting $v = (by, b\sqrt{1-y^2})$, $w = (cz, c\sqrt{1-z^2})$, the system becomes $u+v=w$, with vectors u,v,w of lengths a,b,c respectively in the upper half-plane. Then a,b,c are sides of a (possibly degenerate) triangle; i.e, $|a-b| \leq c \leq a+b$ is a necessary condition.
Conversely, if a,b,c satisfy this condition, one constructs a triangle OMN with $OM=a$, $ON=b$, $MN=c$. If the vectors $\overrightarrow{OM}, \overrightarrow{ON}$ have a positive nonnegative component, then so does their sum. For every such triangle, putting $u = \overrightarrow{OM}$, $v = \overrightarrow{ON}$, and $w = \overrightarrow{OM}+\overrightarrow{ON}$ gives a solution, and every solution is given by one such triangle. This triangle is uniquely determined up to congruence: $\alpha = \angle MON = \angle(u,v)$ and $\beta = \angle(u,w)$.
Therefore, all solutions of the system are

$$x = \cos t, \quad y = \cos(t+\alpha), \quad z = y = \cos(t+\beta), \qquad t \in [0, \pi-\alpha] \quad \text{or}$$
$$x = \cos t, \quad y = \cos(t-\alpha), \quad z = y = \cos(t-\beta), \qquad t \in [\alpha, \pi].$$

17. Let $z_0 \geq 1$ be a positive integer. Supposing that the statement is true for all triples (x,y,z) with $z < z_0$, we shall prove that it is true for $z = z_0$ too.

If $z_0 = 1$, verification is trivial, while $x_0 = y_0$ is obviously impossible. So let there be given a triple (x_0, y_0, z_0) with $z_0 > 1$ and $x_0 < y_0$, and define another triple (x, y, z) by

$$x = z_0, \qquad y = x_0 + y_0 - 2z_0, \qquad \text{and} \qquad z = z_0 - x_0.$$

Then x, y, z are positive integers. This is clear for x, z, while $y = x_0 + y_0 - 2z_0 \geq 2(\sqrt{x_0 y_0} - z_0) > 2(z_0 - z_0) = 0$. Moreover, $xy - z^2 = x_0(x_0 + y_0 - 2z_0) - (z_0 - x_0)^2 = x_0 y_0 - z_0^2 = 1$ and $z < z_0$, so that by the assumption, the statement holds for x, y, z. Thus for some nonnegative integers a, b, c, d we have

$$x = a^2 + b^2, \qquad y = c^2 + d^2, \qquad z = ac + bd.$$

But then we obtain representations of this sort for x_0, y_0, z_0 too:

$$x_0 = a^2 + b^2, \qquad y_0 = (a + c)^2 + (b + d)^2, \qquad z_0 = a(a + c) + b(b + d).$$

For the second part of the problem, we note that for $z = (2q)!$,

$$\begin{aligned} z^2 &= (2q)!(2q)(2q-1)\cdots 1 \equiv (2q)! \cdot (-(2q+1))(-(2q+2))\cdots(-4q) \\ &= (-1)^{2q}(4q)! \equiv -1 \pmod{p}, \end{aligned}$$

by Wilson's theorem. Hence $p \mid z^2 + 1 = py$ for some positive integer $y > 0$. Now it follows from the first part that there exist integers a, b such that $x = p = a^2 + b^2$.

Second solution. Another possibility is using arithmetic of Gaussian integers.

Lemma. Suppose m, n, p, q are elements of $\mathbb{Z}$ or any other unique factorization domain, with $mn = pq$. then there exist elements a, b, c, d such that $m = ab$, $n = cd$, $p = ac$, $q = bd$.

Proof is direct, for example using factorization of a, b, c, d into primes.

We now apply this lemma to the Gaussian integers in our case (because $\mathbb{Z}[i]$ has the unique factorization property), having in mind that $xy = z^2 + 1 = (z + i)(z - i)$. We obtain

$$(1) \quad x = ab, \quad (2) \quad y = cd, \quad (3) \quad z + i = ac, \quad (4) \quad z - i = bd$$

for some $a, b, c, d \in \mathbb{Z}[i]$. Let $a = a_1 + a_2 i$, etc. By (3) and (4), $\gcd(a_1, a_2) = \cdots = \gcd(d_1, d_2)$. Then (1) and (2) give us $b = \overline{a}$, $c = \overline{d}$. The statement follows at once: $x = ab = a\overline{a} = a_1^2 + a_2^2$, $y = d\overline{d} = d_1^2 + d_2^2$ and $z + i = (a_1 d_1 + a_2 d_2) + \imath(a_2 d_1 - a_1 d_2) \Rightarrow z = a_1 d_1 + a_2 d_2$.

4.21 Solutions to the Shortlisted Problems of IMO 1979

1. We prove more generally, by induction on n, that any $2n$-gon with equal edges and opposite edges parallel to each other can be dissected. For $n = 2$ the only possible such $2n$-gon is a single lozenge, so our theorem holds in this case. We will now show that it holds for general n. Assume by induction that it holds for $n-1$. Let $A_1A_2\dots A_{2n}$ be an arbitrary $2n$-gon with equal edges and opposite edges parallel to each other. Then we can construct points B_i for $i = 3, 4, \dots, n$ such that $\overrightarrow{A_iB_i} = \overrightarrow{A_2A_1} = \overrightarrow{A_{n+1}A_{n+2}}$. We set $B_2 = A_{2n+1} = A_1$ and $B_{n+1} = A_{n+2}$. It follows that $A_iB_iB_{i+1}A_{i+1}$ for $i = 2, 3, 4, \dots, n$ are all lozenges. It also follows that B_iB_{i+1} for $i = 2, 3, 4, \dots, n$ are equal to the edges of $A_1A_2\dots A_{2n}$ and parallel to A_iA_{i+1} and hence to $A_{n+i}A_{n+i+1}$. Thus $B_2\dots B_{n+1}A_{n+3}\dots A_{2n}$ is a $2(n-1)$-gon with equal edges and opposite sides parallel and hence, by the induction hypothesis, can be dissected into lozenges. We have thus provided a dissection for $A_1A_2\dots A_{2n}$. This completes the proof.

2. The only way to arrive at the latter alternative is to draw four different socks in the first drawing or to draw only one pair in the first drawing and then draw two different socks in the last drawing. We will call these probabilities respectively p_1, p_2, p_3. We calculate them as follows:

$$p_1 = \frac{\binom{5}{4}2^4}{\binom{10}{4}} = \frac{8}{21},\quad p_2 = \frac{5\binom{4}{2}2^2}{\binom{10}{4}} = \frac{4}{7},\quad p_3 = \frac{4}{\binom{6}{2}} = \frac{4}{15}\,.$$

We finally calculate the desired probability: $P = p_1 + p_2p_3 = \frac{8}{15}$.

3. An obvious solution is $f(x) = 0$. We now look for nonzero solutions. We note that plugging in $x = 0$ we get $f(0)^2 = f(0)$; hence $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$, then f is of the form $f(x) = x^k g(x)$, where $g(0) \neq 0$. Plugging this formula into $f(x)f(2x^2) = f(2x^3 + x)$ we get $2^k x^{2k} g(x)g(2x^2) = (2x^2+1)^k g(2x^3+x)$. Plugging in $x = 0$ gives us $g(0) = 0$, which is a contradiction. Hence $f(0) = 1$.
For an arbitrary root α of the polynomial f, $2\alpha^3 + \alpha$ must also be a root. Let α be a root of the largest modulus. If $|\alpha| > 1$ then $|2\alpha^3 + \alpha| > 2|\alpha|^3 - |\alpha| > |\alpha|$, which is impossible. It follows that $|\alpha| \le 1$ and hence all roots of f have modules less than or equal to 1. But the product of all roots of f is $|f(0)| = 1$, which implies that all the roots have modulus 1. Consequently, for a root α it holds that $|\alpha| = |2\alpha^3 - \alpha| = 1$. This is possible only if $\alpha = \pm\imath$. Since the coefficients of f are real it follows that f must be of the form $f(x) = (x^2+1)^k$ where $k \in \mathbb{N}_0$. These polynomials satisfy the original formula. Hence, the solutions for f are $f(x) = 0$ and $f(x) = (x^2+1)^k$, $k \in \mathbb{N}_0$.

4. Let us prove first that the edges $A_1A_2, A_2A_3, \dots, A_5A_1$ are of the same color. Assume the contrary, and let w.l.o.g. A_1A_2 be red and A_2A_3 be

green. Three of the segments A_2B_l ($l = 1, 2, 3, 4, 5$), say A_2B_i, A_2B_j, A_2B_k, have to be of the same color, let it w.l.o.g. be red. Then A_1B_i, A_1B_j, A_1B_k must be green. At least one of the sides of triangle $B_iB_jB_k$, say B_iB_j, must be an edge of the prism. Then looking at the triangles $A_1B_iB_j$ and $A_2B_iB_j$ we deduce that B_iB_j can be neither green nor red, which is a contradiction. Hence all five edges of the pentagon $A_1A_2A_3A_4A_5$ have the same color. Similarly, all five edges of $B_1B_2B_3B_4B_5$ have the same color. We now show that the two colors are the same. Assume otherwise, i.e., that w.l.o.g. the A edges are painted red and the B edges green. Let us call segments of the form A_iB_j diagonal (i and j may be equal). We now count the diagonal segments by grouping the red segments based on their A point, and the green segments based on their B point. As above, the assumption that three of A_iB_j for fixed i are red leads to a contradiction. Hence at most two diagonal segments out of each A_i may be red, which counts up to at most 10 red segments. Similarly, at most 10 diagonal segments can be green. But then we can paint at most 20 diagonal segments out of 25, which is a contradiction. Hence all edges in the pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ have the same color.

5. Let $A = \{x \mid (x, y) \in M\}$ and $B = \{y \mid (x, y) \in M$. Then A and B are disjoint and hence

$$|M| \leq |A| \cdot |B| \leq \frac{(|A| + |B|)^2}{4} \leq \left[\frac{n^2}{4}\right].$$

These cardinalities can be achieved for $M = \{(a, b) \mid a = 1, 2, \dots, [n/2], b = [n/2] + 1, \dots, n\}$.

6. Setting $q = x^2 + x - p$, the given equation becomes

$$\sqrt{(x+1)^2 - 2q} + \sqrt{(x+2)^2 - q} = \sqrt{(2x+3)^2 - 3q}. \tag{1}$$

Taking squares of both sides we get $2\sqrt{((x+1)^2 - 2q)((x+2)^2 - q)} = 2(x+1)(x+2)$. Taking squares again we get

$$q\left(2q - 2(x+2)^2 - (x+1)^2\right) = 0.$$

If $2q = 2(x+2)^2 + (x+1)^2$, at least one of the expressions under the three square roots in (1) is negative, and in that case the square root is not well-defined. Thus, we must have $q = 0$.

Now (1) is equivalent to $|x+1| + |x+2| = |2x+3|$, which holds if and only if $x \notin (-2, -1)$. The number of real solutions x of $q = x^2 + x - p = 0$ which are not in the interval $(-2, -1)$ is zero if $p < -1/4$, one if $p = -1/4$ or $0 < p < 2$, and two otherwise.

Hence, the answer is $-1/4 < p \leq 0$ or $p \geq 2$.

7. We denote the sum mentioned above by S. We have the following equalities:

$$\begin{aligned} S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319} \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{1319} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1318}\right) \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{1319} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{659}\right) \\ &= \frac{1}{660} + \frac{1}{661} + \cdots + \frac{1}{1319} \\ &= \sum_{i=660}^{989} \frac{1}{i} + \frac{1}{1979-i} = \sum_{i=660}^{989} \frac{1979}{i \cdot (1979-i)} \end{aligned}$$

Since no term in the sum contains a denominator divisible by 1979 (1979 is a prime number), it follows that when S is represented as p/q the numerator p will have to be divisible by 1979.

8. By the definition of f, it holds that $f(0.b_1b_2\dots) = 3b_1/4 + f(0.b_2b_3\dots)/4$ $= 0.b_1b_1 + f(0.b_2b_3\dots)/4$. Continuing this argument we obtain

$$f(0.b_1b_2b_3\dots) = 0.b_1b_1\dots b_nb_n + \frac{1}{2^{2n}} f(0.b_{n+1}b_{n+2}\dots). \tag{1}$$

The binary representation of every rational number is eventually periodic. Let us first determine $f(x)$ for a rational x with the periodic representation $x = 0.\overline{b_1b_2\dots b_n}$. Using (1) we obtain $f(x) = 0.b_1b_1\dots b_nb_n + f(x)/2^{2n}$, and hence $f(x) = \frac{2^n}{2^n-1} 0.b_1b_1\dots b_nb_n = 0.\overline{b_1b_1\dots b_nb_n}$.
Now let $x = 0.a_1a_2\dots a_k\overline{b_1b_2\dots b_n}$ be an arbitrary rational number. Then it follows from (1) that

$$f(x) = 0.a_1a_1\dots a_ka_k + \frac{1}{2^{2n}} f(0.\overline{b_1b_2\dots b_n}) = 0.a_1a_1\dots a_ka_k\overline{b_1b_1\dots b_nb_n}.$$

Hence $f(0.b_1b_2\dots) = 0.b_1b_1b_2b_2\dots$ for every rational number $0.b_1b_2\dots$.

9. Let us number the vertices, starting from S and moving clockwise. In that case $S = 1$ and $F = 5$. After an odd number of moves to a neighboring point we can be only on an even point, and hence it follows that $a_{2n-1} = 0$ for all $n \in \mathbb{N}$. Let us define respectively z_n and w_n as the number of paths from S to S in $2n$ moves and the number of paths from S to points 3 and 7 in $2n$ moves. We easily derive the following recurrence relations:

$$a_{2n+2} = w_n, \quad w_{n+1} = 2w_n + 2z_n, \quad z_{n+1} = 2z_n + w_n, \qquad n = 0, 1, 2, \dots .$$

By subtracting the second equation from the third we get $z_{n+1} = w_{n+1} - w_n$. By plugging this equation into the formula for w_{n+2} we get $w_{n+2} - 4w_{n+1} + 2w_n = 0$. The roots of the characteristic equation $r^2 - 4r + 2 = 0$ are $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$. From the conditions $w_0 = 0$ and $w_1 = 2$ we easily obtain $a_{2n} = w_{n-1} = (x^{n-1} - y^{n-1})/\sqrt{2}$.

10. In the cases $a = \overrightarrow{0}$, $b = \overrightarrow{0}$, and $a \parallel b$ the inequality is trivial. Otherwise, let us consider a triangle ABC such that $\overrightarrow{CB} = a$ and $\overrightarrow{CA} = b$. From this point on we shall refer to α, β, γ as angles of ABC. Since $|a \times b| = |a||b| \sin\gamma$, our inequality reduces to $|a||b| \sin^3\gamma \le 3\sqrt{3}|c|^2/8$, which is further reduced to
$$\sin\alpha \sin\beta \sin\gamma \le \frac{3\sqrt{3}}{8}$$
using the sine law. The last inequality follows immediately from Jensen's inequality applied to the function $f(x) = \ln \sin x$, which is concave for $0 < x < \pi$ because $f'(x) = \cot x$ is strictly decreasing.

11. Let us define $y_i = x_i^2$. We thus have $y_1 + y_2 + \cdots + y_n = 1$, $y_i \ge 1/n^2$, and $P = \sqrt{y_1 y_2 \dots y_n}$.
The upper bound is obtained immediately from the AM–GM inequality: $P \le 1/n^{n/2}$, where equality holds when $x_i = \sqrt{y_i} = 1/\sqrt{n}$.
For the lower bound, let us assume w.l.o.g. that $y_1 \ge y_2 \ge \cdots \ge y_n$. We note that if $a \ge b \ge 1/n^2$ and $s = a + b > 2/n^2$ is fixed, then $ab = (s^2 - (a-b)^2)/4$ is minimized when $|a - b|$ is maximized, i.e., when $b = 1/n^2$. Hence $y_1 y_2 \cdots y_n$ is minimal when $y_2 = y_3 = \cdots = y_n = 1/n^2$. Then $y_1 = (n^2 - n + 1)/n^2$ and therefore $P_{\min} = \sqrt{n^2 - n + 1}/n^n$.

12. The first criterion ensures that all sets in an S-family are distinct. Since the number of different families of subsets is finite, h has to exist. In fact, we will show that $h = 11$. First of all, if there exists $X \in F$ such that $|X| \ge 5$, then by (3) there exists $Y \in F$ such that $X \cup Y = R$. In this case $|F|$ is at most 2. Similarly, for $|X| = 4$, for the remaining two elements either there exists a subset in F that contains both, in which case we obtain the previous case, or there exist different Y and Z containing them, in which case $X \cup Y \cup Z = R$, which must not happen. Hence we can assume $|X| \le 4$ for all $X \in F$.
Assume $|X| = 1$ for some X. In that case other sets must not contain that subset and hence must be contained in the remaining 5-element subset. These elements must not be subsets of each other. From elementary combinatorics, the largest number of subsets of a 5-element set of which none is subset of another is $\binom{5}{2} = 10$. This occurs when we take all 2-element subsets. These subsets also satisfy (2). Hence $|F|_{\max} = 11$ in this case.
Otherwise, let us assume $|X| = 3$ for some X. Let us define the following families of subsets: $G = \{Z = Y \setminus X \mid Y \in F\}$ and $H = \{Z = Y \cap X \mid Y \in F\}$. Then no two sets in G must complement each other in $R \setminus X$, and G must cover this set. Hence G contains exactly the sets of each of the remaining 3 elements. For each element of G no two sets in H of which one is a subset of another may be paired with it. There can be only 3 such subsets selected within a 3-element set X. Hence the number of remaining sets is smaller than $3 \cdot 3 = 9$. Hence in this case $|F|_{\max} = 10$.

In the remaining case all subsets have two elements. There are $\binom{6}{2} = 15$ of them. But for every three that complement each other one must be discarded; hence the maximal number for F in this case is $2 \cdot 15/3 = 10$. It follows that $h = 11$.

13. From elementary trigonometry we have $\sin 3t = 3\sin t - 4\sin^3 t$. Hence, if we denote $y = \sin 20°$, we have $\sqrt{3}/2 = \sin 60° = 3y - 4y^3$. Obviously $0 < y < 1/2 = \sin 30°$. The function $f(x) = 3x - 4x^3$ is strictly increasing on $[0, 1/2)$ because $f'(x) = 3 - 12x^2 > 0$ for $0 \le x < 1/2$. Now the desired inequality $\frac{20}{60} = \frac{1}{3} < \sin 20° < \frac{21}{60} = \frac{7}{20}$ follows from
$$f\left(\frac{1}{3}\right) < \frac{\sqrt{3}}{2} < f\left(\frac{7}{20}\right),$$
which is directly verified.

14. Let us assume that $a \in \mathbb{R} \setminus \{1\}$ is such that there exist a and x such that $x = \log_a x$, or equivalently $f(x) := \ln x / x = \ln a$. Then a is a value of the function $f(x)$ for $x \in \mathbb{R}^+ \setminus \{1\}$, and the converse also holds.
First we observe that $f(x)$ tends to $-\infty$ as $x \to 0$ and $f(x)$ tends to 0 as $x \to 1$. Since $f(x) > 0$ for $x > 1$, the function $f(x)$ takes its maximum at a point x for which $f'(x) = (1 - \ln x)/x^2 = 0$. Hence
$$\max f(x) = f(e) = e^{1/e}.$$
It follows that the set of values of $f(x)$ for $x \in \mathbb{R}^+$ is the interval $(-\infty, e^{1/e})$, and consequently the desired set of bases a of logarithms is $(0, 1) \cup (1, e^{1/e}]$.

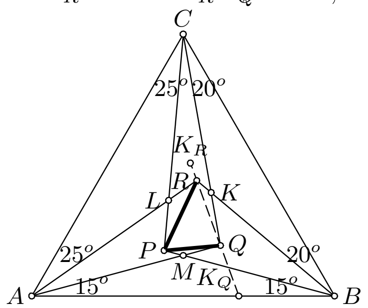
15. We note that
$$\sum_{i=1}^{5} i(a-i^2)^2 x_i = a^2 \sum_{i=1}^{5} i x_i - 2a \sum_{i=1}^{5} i^3 x_i + \sum_{i=1}^{5} i^5 x_i = a^2 \cdot a - 2a \cdot a^2 + a^3 = 0.$$
Since the terms in the sum on the left are all nonnegative, it follows that all the terms have to be 0. Thus, either $x_i = 0$ for all i, in which case $a = 0$, or $a = j^2$ for some j and $x_i = 0$ for $i \neq j$. In this case, $x_j = a/j = j$. Hence, the only possible values of a are $\{0, 1, 4, 9, 16, 25\}$.

16. Obviously, no two elements of F can be complements of each other. If one of the sets has one element, then the conclusion is trivial. If there exist two different 2-element sets, then they must contain a common element, which in turn must then be contained in all other sets. Thus we can assume that there exists at most one 2-element subset of K in F. Since there can be at most 6 subsets of more than 3 elements of a 5-element set, it follows that at least 9 out of 10 possible 3-element subsets of K belong to F. Let us assume, without loss of generality, that all sets but $\{c, d, e\}$ belong to F. Then sets $\{a, b, c\}$, $\{a, d, e\}$, and $\{b, c, d\}$ have no common element, which is a contradiction. Hence it follows that all sets have a common element.

17. Let K, L, and M be intersections of CQ and BR, AR and CP, and AQ and BP, respectively. Let $\angle X$ denote the angle of the hexagon $KQMPLR$ at the vertex X, where X is one of the six points. By an elementary calculation of angles we get

$$\angle K = 140^\circ,\ \angle L = 130^\circ, \angle M = 150^\circ, \angle P = 100^\circ, \angle Q = 95^\circ, \angle R = 105^\circ\ .$$

Since $\angle KBC = \angle KCB$, it follows that K is on the symmetry line of ABC through A. Analogous statements hold for L and M. Let K_R and K_Q be points symmetric to K with respect to AR and AQ, respectively. Since $\angle AK_QQ = \angle AK_QK_R = 70^\circ$ and $\angle AK_RR = \angle AK_RK_Q = 70^\circ$, it follows that K_R, R, Q, and K_Q are collinear. Hence $\angle QRK = 2\angle R - 180^\circ$ and $\angle RQK = 2\angle Q - 180^\circ$. We analogously get $\angle PRL = 2\angle R - 180^\circ$, $\angle RPL = 2\angle P - 180^\circ$, $\angle QPM = 2\angle P - 180^\circ$ and $\angle PQM = 2\angle Q - 180^\circ$. From these formulas we easily get $\angle RPQ = 60^\circ$, $\angle RQP = 75^\circ$, and $\angle QRP = 45^\circ$.



18. Let us write all a_i in binary representation. For $S \subseteq \{1, 2, \dots, m\}$ let us define $b(S)$ as the number in whose binary representation ones appear in exactly the slots where ones appear in all a_i where $i \subseteq S$ and don't appear in any other a_i. Some $b(S)$, including $b(\emptyset)$, will equal 0, and hence there are fewer than 2^m different positive $b(S)$. We note that no two positive $b(S_1)$ and $b(S_2)$ $(S_1 \neq S_2)$ have ones in the same decimal places. Hence sums of distinct $b(S)$'s are distinct. Moreover

$$a_i = \sum_{i \in S} b(S)$$

and hence the positive $b(S)$ are indeed the numbers $b_1, \dots, b_n$ whose existence we had to prove.

19. Let us define i_j for two positive integers i and j in the following way: $i_1 = i$ and $i_{j+1} = i^{i_j}$ for all positive integers j. Thus we must find the smallest m such that $100_m > 3_{100}$. Since $100_1 = 100 > 27 = 3_2$, we inductively have $100_j = 10^{100_{j-1}} > 3^{100_{j-1}} > 3^{3_j} = 3_{j+1}$ and hence $m \leq 99$. We now prove that $m = 99$ by proving $100_{98} < 3_{100}$. We note that $(100_1)^2 = 10^4 < 27^4 = 3^{12} < 3^{27} = 3_3$. We also note for $d > 12$ (which trivially holds for all $d = 100_i$) that if $c > d^2$, then we have

$$3^c > 3^{d^2} > 3^{12d} = (3^{12})^d > 10000^d = (100^d)^2.$$

Hence from $3_3 > (100_1)^2$ it inductively follows that $3_j > (100_{j-2})^2 > 100_{j-2}$ and hence that $100_{99} > 3_{100} > 100_{98}$. Hence $m = 99$.

20. Let $x_k = \max\{x_1, x_2, \dots, x_n\}$. Then $x_i x_{i+1} \le x_i x_k$ for $i = 1, 2, \dots, k-1$ and $x_i x_{i+1} \le x_k x_{i+1}$ for $i = k, \dots, n-1$. Summing up these inequalities for $i = 1, 2, \dots, n-1$ we obtain

$$\sum_{i=1}^{n-1} \le x_k(x_1 + \cdots + x_{k-1} + x_{k+1} + \cdots + x_n) = x_k(a - x_k) \le \frac{a^2}{4}.$$

We note that the value $a^2/4$ is attained for $x_1 = x_2 = a/2$ and $x_3 = \cdots = x_n = 0$. Hence $a^2/4$ is the required maximum.

21. Let $f(n)$ be the number of different ways $n \in \mathbb{N}$ can be expressed as $x^2 + y^3$ where $x, y \in \{0, 1, \dots, 10^6\}$. Clearly $f(n) = 0$ for $n < 0$ or $n > 10^{12} + 10^{18}$. The first equation can be written as $x^2 + t^3 = y^2 + z^3 = n$, whereas the second equation can be written as $x^2 + t^3 = n + 1$, $y^2 + z^3 = n$. Hence we obtain the following formulas for M and N:

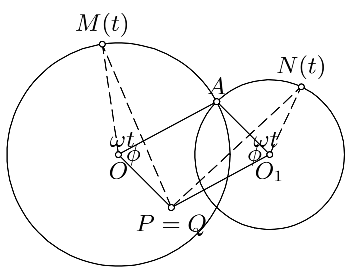
$$M = \sum_{i=0}^{m} f(i)^2, \qquad N = \sum_{i=0}^{m-1} f(i) f(i+1) .$$

Using the AM–GM inequality we get

$$\begin{aligned} N &= \sum_{i=0}^{m-1} f(i) f(i+1) \\ &\le \sum_{i=0}^{m-1} \frac{f(i)^2 + f(i+1)^2}{2} = \frac{f(0)^2}{2} + \sum_{i=1}^{m-1} f(i)^2 + \frac{f(m)^2}{2} < M . \end{aligned}$$

The last inequality is strong, since $f(0) = 1 > 0$. This completes our proof.

22. Let the centers of the two circles be denoted by O and O_1 and their respective radii by r and r_1, and let the positions of the points on the circles at time t be denoted by $M(t)$ and $N(t)$. Let Q be the point such that OAO_1Q is a parallelogram. We will show that Q is the point P we are looking for, i.e., that $QM(t) = QN(t)$ for all t. We note that $OQ = O_1A = r_1$, $O_1Q = OA = r$ and $\angle QOA = \angle QO_1A = \phi$. Since the two points return to A at the same time, it follows that $\angle M(t)OA = \angle N(t)O_1A = \omega t$. Therefore $\angle QOM(t) = \angle QO_1N(t) = \phi + \omega t$, from which it follows that $\triangle QOM(t) \cong \triangle QO_1N(t)$. Hence $QM(t) = QN(t)$, as we claimed.



23. It is easily verified that no solutions exist for $n \le 8$. Let us now assume that $n > 8$. We note that $2^8 + 2^{11} + 2^n = 2^8 \cdot (9 + 2^{n-8})$. Hence $9 + 2^{n-8}$

must also be a square, say $9 + 2^{n-8} = x^2$, $x \in \mathbb{N}$, i.e., $2^{n-8} = x^2 - 9 = (x-3)(x+3)$. Thus $x-3$ and $x+3$ are both powers of 2, which is possible only for $x = 5$ and $n = 12$. Hence, $n = 12$ is the only solution.

24. Clearly O is the midpoint of BC. Let M and N be the points of tangency of the circle with AB and AC, respectively, and let $\angle BAC = 2\varphi$. Then $\angle BOM = \angle CON = \varphi$.
Let us assume that PQ touches the circle in X. If we set $\angle POM = \angle POX = x$ and $\angle QON = \angle QOX = y$, then $2x + 2y = \angle MON = 180^\circ - 2\varphi$, i.e., $y = 90^\circ - \varphi - x$. It follows that $\angle OQC = 180^\circ - \angle QOC - \angle OCQ = 180^\circ - (\varphi + y) - (90^\circ - \varphi) = 90^\circ - y = x + \varphi = \angle BOP$. Hence the triangles BOP and CQO are similar, and consequently $BP \cdot CQ = BO \cdot CO = (BC/2)^2$.
Conversely, let $BP \cdot CQ = (BC/2)^2$ and let Q' be the point on (AC) such that PQ' is tangent to the circle. Then $BP \cdot CQ' = (BC/2)^2$, which implies $Q \equiv Q'$.

25. Let us first look for such a point R on a line l in π going through P. Let $\angle QPR = 2\theta$. Consider a point Q' on l such that $Q'P = QP$. Then we have
$$\frac{QP + PR}{QR} = \frac{RQ'}{QR} = \frac{\sin Q'QR}{\sin QQ'R}\,.$$
Since $QQ'P$ is fixed, the maximal value of the expression occurs when $\angle QQ'R = 90^\circ$. In this case $(QP + PR)/QR = 1/\sin\theta$. Looking at all possible lines l, we see that θ is minimized when l equals the projection of PQ onto π. Hence, the point R is the intersection of the projection of PQ onto π and the plane through Q perpendicular to PQ.

26. Let us assume that $f(x + y) = f(x) + f(y)$ for all reals. In this case we trivially apply the equation to get $f(x + y + xy) = f(x + y) + f(xy) = f(x) + f(y) + f(xy)$. Hence the equivalence is proved in the first direction. Now let us assume that $f(x+y+xy) = f(x)+f(y)+f(xy)$ for all reals. Plugging in $x = y = 0$ we get $f(0) = 0$. Plugging in $y = -1$ we get $f(x) = -f(-x)$. Plugging in $y = 1$ we get $f(2x + 1) = 2f(x) + f(1)$ and hence $f(2(u + v + uv) + 1) = 2f(u + v + uv) + f(1) = 2f(uv) + 2f(u) + 2f(v) + f(1)$ for all real u and v. On the other hand, plugging in $x = u$ and $y = 2v + 1$ we get $f(2(u + v + uv) + 1) = f(u + (2v + 1) + u(2v + 1)) = f(u) + 2f(v) + f(1) + f(2uv + u)$. Hence it follows that $2f(uv) + 2f(u) + 2f(v) + f(1) = f(u) + 2f(v) + f(1) + f(2uv + u)$, i.e.,
$$f(2uv + u) = 2f(uv) + f(u). \tag{1}$$
Plugging in $v = -1/2$ we get $0 = 2f(-u/2) + f(u) = -2f(u/2) + f(u)$. Hence, $f(u) = 2f(u/2)$ and consequently $f(2x) = 2f(x)$ for all reals. Now (1) reduces to $f(2uv + u) = f(2uv) + f(u)$. Plugging in $u = y$ and $x = 2uv$, we obtain $f(x) + f(y) = f(x + y)$ for all nonzero reals x and y. Since $f(0) = 0$, it trivially holds that $f(x + y) = f(x) + f(y)$ when one of x and y is 0.

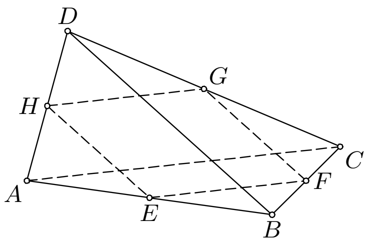
4.22 Solutions to the Shortlisted Problems of IMO 1981

1. Assume that the set $\{a-n+1, a-n+2, \dots, a\}$ of n consecutive numbers satisfies the condition $a \mid \operatorname{lcm}[a-n+1, \dots, a-1]$. Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be the canonic representation of a, where $p_1 < p_2 < \cdots < p_r$ are primes and $\alpha_1, \cdots, \alpha_r > 0$. Then for each $j = 1, 2, \dots, r$, there exists m, $m = 1, 2, \dots, n-1$, such that $p_j^{\alpha_j} \mid a - m$, i.e., such that $p_j^{\alpha_j} \mid m$. Thus $p_j^{\alpha_j} \le n-1$. If $r = 1$, then $a = p_1^{\alpha_1} \le n-1$, which is impossible. Therefore $r \ge 2$. But then there must exist two distinct prime numbers less than n; hence $n \ge 4$.
   For $n = 4$, we must have $p_1^{\alpha_1}, p_2^{\alpha_2} \le 3$, which leads to $p_1 = 2$, $p_2 = 3$, $\alpha_1 = \alpha_2 = 1$. Therefore $a = 6$, and $\{3, 4, 5, 6\}$ is a unique set satisfying the condition of the problem.
   For every $n \ge 5$ there exist at least two such sets. In fact, for $n = 5$ we easily find two sets: $\{2, 3, 4, 5, 6\}$ and $\{8, 9, 10, 11, 12\}$. Suppose that $n \ge 6$. Let r, s, t be natural numbers such that $2^r \le n - 1 < 2^{r+1}$, $3^s \le n - 1 < 3^{s+1}$, $5^t \le n - 1 < 5^{t+1}$. Taking $a = 2^r \cdot 3^s$ and $a = 2^r \cdot 5^t$ we obtain two distinct sets with the required property. Thus the answers are (a) $n \ge 4$ and (b) $n = 4$.

2. *Lemma.* Let E, F, G, H, I, and K be points on edges AB, BC, CD, DA, AC, and BD of a tetrahedron. Then there is a sphere that touches the edges at these points if and only if

$$\begin{aligned} AE = AH = AI, \quad BE = BF = BK, \\ CF = CG = CI, \quad DG = DH = DK. \end{aligned} \qquad (*)$$

   Proof. The "only if" side of the equivalence is obvious.
   We now assume $(*)$. Denote by $\epsilon, \phi, \gamma, \eta, \iota$, and κ planes through E, F, G, H, I, K perpendicular to AB, BC, CD, DA, AC and BD respectively. Since the three planes ϵ, η, and ι are not mutually parallel, they intersect in a common point O. Clearly, $\triangle AEO \cong \triangle AHO \cong \triangle AIO$; hence $OE = OH = OI = r$, and the sphere $\sigma(O, r)$ is tangent to AB, AD, AC.



   To prove that σ is also tangent to BC, CD, BD it suffices to show that planes ϕ, γ, and κ also pass through O. Without loss of generality we can prove this for just ϕ. By the conditions for E, F, I, these are exactly the points of tangency of the incircle of $\triangle ABC$ and its sides, and if S is the incenter, then $SE \perp AB$, $SF \perp BC$, $SI \perp AC$. Hence ϵ, ι, and ϕ all pass through S and are perpendicular to the plane ABC, and consequently all share the line l through S perpendicular to ABC.

Since $l = \epsilon \cap \iota$, the point O will be situated on l, and hence ϕ will also contain O. This completes our proof of the lemma.

Let $AH = AE = x$, $BE = BF = y$, $CF = CG = z$, and $DG = DH = w$. If the sphere is also tangent to AC at some point I, then $AI = x$ and $IC = z$. Using the stated lemma it suffices to prove that if $AC = x + z$, then $BD = y + w$.

Let $EF = FG = GH = HI = t$, $\angle BAD = \alpha$, $\angle ABC = \beta$, $\angle BCD = \gamma$, and $\angle ADC = \delta$. We get

$$t^2 = EH^2 = AE^2 + AH^2 - 2 \cdot AE \cdot AH \cos\alpha = 2x^2(1 - \cos\alpha).$$

We similarly conclude that $t^2 = 2y^2(1-\cos\beta) = 2z^2(1-\cos\gamma) = 2w^2(1-\cos\delta)$. Further, using that $AB = x + y$, $BC = y + z$, $\cos\beta = 1 - t^2/2y^2$, we obtain

$$AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cos\beta = (x - z)^2 + t^2 \left(\frac{x}{y} + 1\right)\left(\frac{z}{y} + 1\right).$$

Analogously, from the triangle ADC we get $AC^2 = (x - z)^2 + t^2(x/w + 1)(z/w + 1)$, which gives $(x/y + 1)(z/y + 1) = (x/w + 1)(z/w + 1)$. Since $f(s) = (x/s + 1)(z/s + 1)$ is a decreasing function in s, it follows that $y = w$; similarly $x = z$.

Hence $CF = CG = x$ and $DG = DH = y$. Hence $AC \parallel EF$ and $AC : t = AC : EF = AB : EB = (x + y) : y$; i.e., $AC = t(x + y)/y$. Similarly, from the triangle ABD, we get that $BD = t(x+y)/x$. Hence if $AC = x+z = 2x$, it follows that $2x = t(x + y)/y \Rightarrow 2xy = t(x + y) \Rightarrow BD = t(x + y)/x = 2y = y + w$. This completes the proof.

Second solution. Without loss of generality, assume that $EF = 2$. Consider the Cartesian system in which points O, E, F, G, H respectively have coordinates $(0, 0, 0)$, $(-1, -1, a)$, $(1, -1, a)$, $(1, 1, a)$, $(-1, 1, a)$. Line AH is perpendicular to OH and AE is perpendicular to OE; hence from Pythagoras's theorem $AO^2 = AH^2 + HO^2 = AE^2 + EO^2 = AE^2 + HO^2$, which implies $AH = AE$. Therefore the y-coordinate of A is zero; analogously the x-coordinates of B and D and the y-coordinate of C are 0. Let A have coordinates $(x_0, 0, z_1)$: then $\overrightarrow{EA}(x_0 + 1, 1, z_1 - a) \perp \overrightarrow{EO}(1, 1, -a)$, i.e., $\overrightarrow{EA} \cdot \overrightarrow{EO} = x_0 + 2 + a(a - z_1) = 0$. Similarly, for $B(0, y_0, z_2)$ we have $y_0 + 2 + a(a - z_2) = 0$. This gives us

$$z_1 = \frac{x_0 + a^2 + 2}{a}, \qquad z_2 = \frac{y_0 + a^2 + 2}{a}. \tag{1}$$

We haven't used yet that $A(x_0, 0, z_1)$, $E(-1, -1, a)$ and $B(0, y_0, z_2)$ are collinear, so let A', B', E' be the feet of perpendiculars from A, B, E to the plane xy. The line $A'B'$, given by $y_0x + x_0y = x_0y_0$, $z = 0$, contains the point $E'(-1, -1, 0)$, from which we obtain

$$(x_0 + 1)(y_0 + 1) = 1. \tag{2}$$

In the same way, from the points B and C we get relations similar to (1) and (2) and conclude that C has the coordinates $C(-x_0, 0, z_1)$. Similarly we get $D(0, -y_0, z_2)$. The condition that AC is tangent to the sphere $\sigma(O, OE)$ is equivalent to $z_1 = \sqrt{a^2+2}$, i.e., to $x_0 = a\sqrt{a^2+2} - (a^2+2)$. But then (2) implies that $y_0 = -a\sqrt{a^2+2} - (a^2+2)$ and $z_2 = -\sqrt{a^2+2}$, which means that the sphere σ is tangent to BD as well. This finishes the proof.

3. Denote $\max(a+b+c, b+c+d, c+d+e, d+e+f, e+f+g)$ by p. We have

$$(a+b+c) + (c+d+e) + (e+f+g) = 1 + c + e \leq 3p,$$

which implies that $p \geq 1/3$. However, $p = 1/3$ is achieved by taking $(a, b, c, d, e, f, g) = (1/3, 0, 0, 1/3, 0, 0, 1/3)$. Therefore the answer is $1/3$.

Remark. In fact, one can prove a more general statement in the same way. Given positive integers n, k, $n \geq k$, if $a_1, a_2, \dots, a_n$ are nonnegative real numbers with sum 1, then the minimum value of $\max_{i=1,\dots,n-k+1}\{a_i + a_{i+1} + \cdots + a_{i+k-1}\}$ is $1/r$, where r is the integer with $k(r-1) < n \leq kr$.

4. We shall use the known formula for the Fibonacci sequence

$$f_n = \frac{1}{\sqrt{5}}(\alpha^n - (-1)^n \alpha^{-n}), \qquad \text{where } \alpha = \frac{1+\sqrt{5}}{2}. \tag{1}$$

(a) Suppose that $af_n + bf_{n+1} = f_{k_n}$ for all n, where $k_n > 0$ is an integer depending on n. By (1), this is equivalent to $a(\alpha^n - (-1)^n\alpha^{-n}) + b(\alpha^{n+1} + (-1)^n\alpha^{-n-1}) = \alpha^{k_n} - (-1)^{k_n}\alpha^{-k_n}$, i.e.,

$$\alpha^{k_n - n} = a + b\alpha - \alpha^{-2n}(-1)^n(a - b\alpha^{-1} - (-\alpha)^{n-k_n}) \to a + b\alpha \tag{2}$$

as $n \to \infty$. Hence, since k_n is an integer, $k_n - n$ must be constant from some point on: $k_n = n + k$ and $\alpha^k = a + b\alpha$. Then it follows from (2) that $\alpha^{-k} = a - b\alpha^{-1}$, and from (1) we conclude that $af_n + bf_{n+1} = f_{k+n}$ holds for every n. Putting $n = 1$ and $n = 2$ in the previous relation and solving the obtained system of equations we get $a = f_{k-1}$, $b = f_k$. It is easy to verify that such a and b satisfy the conditions.

(b) As in (a), suppose that $uf_n^2 + vf_{n+1}^2 = f_{l_n}$ for all n. This leads to

$$\begin{aligned} u + v\alpha^2 - \sqrt{5}\alpha^{l_n - 2n} &= 2(u-v)(-1)^n\alpha^{-2n} \\ &\qquad -(u\alpha^{-4n} + v\alpha^{-4n-2} + (-1)^{l_n}\sqrt{5}\alpha^{-l_n-2n}) \\ &\to 0, \end{aligned}$$

as $n \to \infty$. Thus $u + v\alpha^2 = \sqrt{5}\alpha^{l_n - 2n}$, and $l_n - 2n = k$ is equal to a constant. Putting this into the above equation and multiplying by α^{2n} we get $u - v \to 0$ as $n \to \infty$, i.e., $u = v$. Finally, substituting $n = 1$ and $n = 2$ in $uf_n^2 + uf_{n+1}^2 = f_{l_n}$ we easily get that the only possibility is $u = v = 1$ and $k = 1$. It is easy to verify that such u and v satisfy the conditions.

5. There are four types of small cubes upon disassembling:
   (1) 8 cubes with three faces, painted black, at one corner;
   (2) 12 cubes with two black faces, both at one edge;
   (3) 6 cubes with one black face;
   (4) 1 completely white cube.

   All cubes of type (1) must go to corners, and be placed in a correct way (one of three): for this step we have $3^8 \cdot 8!$ possibilities. Further, all cubes of type (2) must go in a correct way (one of two) to edges, admitting $2^{12} \cdot 12!$ possibilities; similarly, there are $4^6 \cdot 6!$ ways for cubes of type (3), and 24 ways for the cube of type (4). Thus the total number of good reassemblings is $3^8 8! \cdot 2^{12} 12! \cdot 4^6 6! \cdot 24$, while the number of all possible reassemblings is $24^{27} \cdot 27!$. The desired probability is $\frac{3^8 8! \cdot 2^{12} 12! \cdot 4^6 6! \cdot 24}{24^{27} \cdot 27!}$. It is not necessary to calculate these numbers to find out that the blind man practically has no chance to reassemble the cube in a right way: in fact, the probability is of order $1.8 \cdot 10^{-37}$.

6. Assume w.l.o.g. that $n = \deg P \geq \deg Q$, and let $P_0 = \{z_1, z_2, \dots, z_k\}$, $P_1 = \{z_{k+1}, z_{k+2}, \dots z_{k+m}\}$. The polynomials P and Q match at $k+m$ points $z_1, z_2, \dots, z_{k+m}$; hence if we prove that $k+m > n$, the result will follow.

   By the assumption,

$$P(x) = (x - z_1)^{\alpha_1} \cdots (x - z_k)^{\alpha_k} = (x - z_{k+1})^{\alpha_{k+1}} \cdots (x - z_{k+m})^{\alpha_{k+m}} + 1$$

   for some positive integers $\alpha_1, \dots, \alpha_{k+m}$. Let us consider $P'(x)$. As we know, it is divisible by $(x - z_i)^{\alpha_i - 1}$ for $i = 1, 2, \dots, k+m$; i.e.,

$$\prod_{i=1}^{k+m} (x - z_i)^{\alpha_i - 1} \mid P'(x).$$

   Therefore $2n - k - m = \deg \prod_{i=1}^{k+m} (x - z_i)^{\alpha_i - 1} \leq \deg P' = n - 1$, i.e., $k + m \geq n + 1$, as we claimed.

7. We immediately find that $f(1,0) = f(0,1) = 2$. Then $f(1, y+1) = f(0, f(1,y)) = f(1,y) + 1$; hence $f(1,y) = y + 2$ for $y \geq 0$. Next we find that $f(2,0) = f(1,1) = 3$ and $f(2, y+1) = f(1, f(2,y)) = f(2,y) + 2$, from which $f(2,y) = 2y + 3$. Particularly, $f(2,2) = 7$. Further, $f(3,0) = f(2,1) = 5$ and $f(3, y+1) = f(2, f(3,y)) = 2f(3,y) + 3$. This gives by induction $f(3,y) = 2^{y+3} - 3$. For $y = 3$, $f(3,3) = 61$. Finally, from $f(4,0) = f(3,1) = 13$ and $f(4, y+1) = f(3, f(4,y)) = 2^{f(4,y)+3} - 3$, we conclude that

$$f(4,y) = 2^{2^{\cdot^{\cdot^{2}}}} - 3 \qquad (y + 3 \text{ twos}).$$

8. Since the number k, $k = 1, 2, \dots, n - r + 1$, is the minimum in exactly $\binom{n-k}{r-1}$ r-element subsets of $\{1, 2, \dots, n\}$, it follows that

$$f(n,r) = \frac{1}{\binom{n}{r}} \sum_{k=1}^{n-r+1} k \binom{n-k}{r-1}.$$

To calculate the sum in the above expression, using the equality $\binom{r+j}{j} = \sum_{i=0}^{j} \binom{r+i-1}{r-1}$, we note that

$$\begin{aligned}\sum_{k=1}^{n-r+1} k \binom{n-k}{r-1} &= \sum_{j=0}^{n-r} \left(\sum_{i=0}^{j} \binom{r+i-1}{r-1} \right) \\ &= \sum_{j=0}^{n-r} \binom{r+j}{r} = \binom{n+1}{r+1} = \frac{n+1}{r+1}\binom{n}{r}.\end{aligned}$$

Therefore $f(n,r) = (n+1)/(r+1)$.

9. If we put $1 + 24a_n = b_n^2$, the given recurrent relation becomes

$$\frac{2}{3} b_{n+1}^2 = \frac{3}{2} + \frac{b_n^2}{6} + b_n = \frac{2}{3}\left(\frac{3}{2} + \frac{b_n}{2}\right)^2, \qquad \text{i.e.,} \qquad b_{n+1} = \frac{3 + b_n}{2}, \quad (1)$$

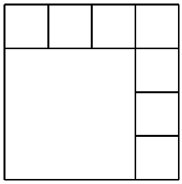
where $b_1 = 5$. To solve this recurrent equation, we set $c_n = 2^{n-1} b_n$. From (1) we obtain

$$\begin{aligned}c_{n+1} &= c_n + 3 \cdot 2^{n-1} = \cdots = c_1 + 3(1 + 2 + 2^2 + \cdots + 2^{n-1}) \\ &= 5 + 3(2^n - 1) = 3 \cdot 2^n + 2.\end{aligned}$$

Therefore $b_n = 3 + 2^{-n+2}$ and consequently

$$a_n = \frac{b_n^2 - 1}{24} = \frac{1}{3}\left(1 + \frac{3}{2^n} + \frac{1}{2^{2n-1}}\right) = \frac{1}{3}\left(1 + \frac{1}{2^{n-1}}\right)\left(1 + \frac{1}{2^n}\right).$$

10. It is easy to see that partitioning into $p = 2k$ squares is possible for $k \geq 2$ (Fig. 1). Furthermore, whenever it is possible to partition the square into p squares, there is a partition of the square into $p + 3$ squares: namely, in the partition into p squares, divide one of them into four new squares.



$p = 8$

Fig. 1

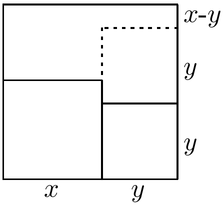


Fig. 2

This implies that both $p = 2k$ and $p = 2k + 3$ are possible if $k \geq 2$, and therefore all $p \geq 6$ are possible.

On the other hand, partitioning the square into 5 squares is not possible. Assuming it is possible, one of its sides would be covered by exactly two squares, which cannot be of the same size (Fig. 2). The rest of the big square cannot be partitioned into three squares. Hence, the answer is $n = 6$.

11. Let us denote the center of the semicircle by O, and $\angle AOB = 2\alpha$, $\angle BOC = 2\beta$, $AC = m$, $CE = n$.
We claim that $a^2+b^2+n^2+abn = 4$. Indeed, since $a = 2\sin\alpha$, $b = 2\sin\beta$, $n = 2\cos(\alpha+\beta)$, we have

$$\begin{aligned}
&a^2 + b^2 + n^2 + abn \\
&\quad = 4(\sin^2\alpha + \sin^2\beta + \cos^2(\alpha+\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)) \\
&\quad = 4 + 4\left(-\frac{\cos 2\alpha}{2} - \frac{\cos 2\beta}{2} + \cos(\alpha+\beta)\cos(\alpha-\beta)\right) \\
&\quad = 4 + 4\left(\cos(\alpha+\beta)\cos(\alpha-\beta) - \cos(\alpha+\beta)\cos(\alpha-\beta)\right) = 4.
\end{aligned}$$

Analogously, $c^2 + d^2 + m^2 + cdm = 4$. By adding both equalities and subtracting $m^2 + n^2 = 4$ we obtain

$$a^2 + b^2 + c^2 + d^2 + abn + cdm = 4.$$

Since $n > c$ and $m > b$, the desired inequality follows.

12. We will solve the contest problem (in which $m, n \in \{1, 2, \dots, 1981\}$). For $m = 1$, n can be either 1 or 2. If $m > 1$, then $n(n-m) = m^2 \pm 1 > 0$; hence $n - m > 0$. Set $p = n - m$. Since $m^2 - mp - p^2 = m^2 - p(m+p) = -(n^2 - nm - m^2)$, we see that (m, n) is a solution of the equation if and only if (p, m) is a solution too. Therefore, all the solutions of the equation are given as two consecutive members of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, \dots .$$

So the required maximum is $987^2 + 1597^2$.

13. *Lemma.* For any polynomial P of degree at most n,

$$\sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} P(i) = 0. \tag{1}$$

Proof. We shall use induction on n. For $n = 0$ it is trivial. Assume that it is true for $n = k$ and suppose that $P(x)$ is a polynomial of degree $k+1$. Then $P(x) - P(x+1)$ clearly has degree at most k; hence (1) gives

$$\begin{aligned}0 &= \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}(P(i)-P(i+1))\\&= \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}P(i)+\sum_{i=1}^{k+2}(-1)^i\binom{k+1}{i-1}P(i)\\&= \sum_{i=0}^{k+2}(-1)^i\binom{k+2}{i}P(i).\end{aligned}$$

This completes the proof of the lemma.

Now we apply the lemma to obtain the value of $P(n+1)$. Since $P(i) = \binom{n+1}{i}^{-1}$ for $i = 0, 1, \dots, n$, we have

$$0 = \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}P(i) = (-1)^{n+1}P(n+1) + \begin{cases}1, & 2 \mid n;\\ 0, & 2 \nmid n.\end{cases}$$

It follows that $P(n+1) = \begin{cases}1, & 2 \mid n;\\ 0, & 2 \nmid n.\end{cases}$

14. We need the following lemma.

Lemma. If a convex quadrilateral $PQRS$ satisfies $PS = QR$ and $\angle SPQ \geq \angle RQP$, then $\angle QRS \geq \angle PSR$.

Proof. If the lines PS and QR are parallel, then this quadrilateral is a parallelogram, and the statement is trivial. Otherwise, let X be the point of intersection of lines PS and QR.

Assume that $\angle SPQ + \angle RQP > 180°$. Then $\angle XPQ \leq \angle XQP$ implies that $XP \geq XQ$, and consequently $XS \geq XR$. Hence, $\angle QRS = \angle XRS \geq \angle XSR = \angle PSR$.

Similarly, if $\angle SPQ + \angle RQP < 180°$, then $\angle XPQ \geq \angle XQP$, from which it follows that $XP \leq XQ$, and thus $XS \leq XR$; hence $\angle QRS = 180° - \angle XRS \geq 180° - \angle XSR = \angle PSR$.

Now we apply the lemma to the quadrilateral $ABCD$. Since $\angle B \geq \angle C$ and $AB = CD$, it follows that $\angle CDA \geq \angle BAD$, which together with $\angle EDA = \angle EAD$ gives $\angle D \geq \angle A$. Thus $\angle A = \angle B = \angle C = \angle D$. Analogously, by applying the lemma to $BCDE$ we obtain $\angle E \geq \angle B$, and hence $\angle B = \angle C = \angle D = \angle E$.

15. Set $BC = a$, $CA = b$, $AB = c$, and denote the area of $\triangle ABC$ by P, and $a/PD + b/PE + c/PF$ by S. Since $a \cdot PD + b \cdot PE + c \cdot PF = 2P$, by the Cauchy–Schwarz inequality we have

$$2PS = (a \cdot PD + b \cdot PE + c \cdot PF)\left(\frac{a}{PD} + \frac{b}{PE} + \frac{c}{PF}\right) \geq (a+b+c)^2,$$

with equality if and only if $PD = PE = PF$, i.e., P is the incenter of $\triangle ABC$. In that case, S attains its minimum:

$$S_{\min} = \frac{(a+b+c)^2}{2P}.$$

16. The sequence $\{u_n\}$ is bounded, whatever u_1 is. Indeed, assume the opposite, and let u_m be the first member of the sequence such that $|u_m| > \max\{2, |u_1|\}$. Then $|u_{m-1}| = |u_m^3 - 15/64| > |u_m|$, which is impossible.

    Next, let us see for what values of u_m, u_{m+1} is greater, equal, or smaller, respectively.

    If $u_{m+1} = u_m$, then $u_m = u_{m+1}^3 - 15/64 = u_m^3 - 15/64$; i.e., u_m is a root of $x^3 - x - 15/64 = 0$. This equation factors as $(x + 1/4)(x^2 - x/4 - 15/16) = 0$, and hence u_m is equal to $x_1 = (1 - \sqrt{61})/8$, $x_2 = -1/4$, or $x_3 = (1 + \sqrt{61})/8$, and these are the only possible limits of the sequence. Each of $u_{m+1} > u_m$, $u_{m+1} < u_m$ is equivalent to $u_m^3 - u_m - 15/64 < 0$ and $u_m^3 - u_m - 15/64 > 0$ respectively. Thus the former is satisfied for u_m in the interval $I_1 = (-\infty, x_1)$ or $I_3 = (x_2, x_3)$, while the latter is satisfied for u_m in $I_2 = (x_1, x_2)$ or $I_4 = (x_3, \infty)$. Moreover, since the function $f(x) = \sqrt[3]{x + 15/64}$ is strictly increasing with fixed points x_1, x_2, x_3, it follows that u_m will never escape from the interval I_1, I_2, I_3, or I_4 to which it belongs initially. Therefore:

    (1) if u_1 is one of x_1, x_2, x_3, the sequence $\{u_m\}$ is constant;
    (2) if $u_1 \in I_1$, then the sequence is strictly increasing and tends to x_1;
    (3) if $u_1 \in I_2$, then the sequence is strictly decreasing and tends to x_1;
    (4) if $u_1 \in I_3$, then the sequence is strictly increasing and tends to x_3;
    (5) if $u_1 \in I_4$, then the sequence is strictly decreasing and tends to x_3.

17. Let us denote by S_A, S_B, S_C the centers of the given circles, where S_A lies on the bisector of $\angle A$, etc. Then $S_AS_B \parallel AB$, $S_BS_C \parallel BC$, $S_CS_A \parallel CA$, so that the inner bisectors of the angles of triangle ABC are also inner bisectors of the angles of $\triangle S_AS_BS_C$. These two triangles thus have a common incenter S, which is also the center of the homothety χ mapping $\triangle S_AS_BS_C$ onto $\triangle ABC$.

    The point O is the circumcenter of triangle $S_AS_BS_C$, and so is mapped by χ onto the circumcenter P of ABC. This means that O, P, and the center S of χ are collinear.

18. Let C be the convex hull of the set of the planets: its border consists of parts of planes, parts of cylinders, and parts of the surfaces of some planets. These parts of planets consist exactly of all the invisible points; any point on a planet that is inside C is visible. Thus it remains to show that the areas of all the parts of planets lying on the border of C add up to the area of one planet.

    As we have seen, an invisible part of a planet is bordered by some main spherical arcs, parallel two by two. Now fix any planet P, and translate these arcs onto arcs on the surface of P. All these arcs partition the surface of P into several parts, each of which corresponds to the invisible part of one of the planets. This correspondence is bijective, and therefore the statement follows.

19. Consider the partition of plane π into regular hexagons, each having inradius 2. Fix one of these hexagons, denoted by γ. For any other hexagon x in the partition, there exists a unique translation τ_x taking it onto γ. Define the mapping $\varphi : \pi \to \gamma$ as follows: If A belongs to the interior of a hexagon x, then $\varphi(A) = \tau_x(A)$ (if A is on the border of some hexagon, it does not actually matter where its image is).
The total area of the images of the union of the given circles equals S, while the area of the hexagon γ is $8\sqrt{3}$. Thus there exists a point B of γ that is covered at least $\frac{S}{8\sqrt{3}}$ times, i.e., such that $\varphi^{-1}(B)$ consists of at least $\frac{S}{8\sqrt{3}}$ distinct points of the plane that belong to some of the circles. For any of these points, take a circle that contains it. All these circles are disjoint, with total area not less than $\frac{\pi}{8\sqrt{3}}S \geq 2S/9$.

Remark. The statement becomes false if the constant $2/9$ is replaced by any number greater than $1/4$. In that case a counterexample is, for example, a set of unit circles inside a circle of radius 2 covering a sufficiently large part of its area.

4.23 Solutions to the Shortlisted Problems of IMO 1982

1. From $f(1) + f(1) \leq f(2) = 0$ we obtain $f(1) = 0$. Since $0 < f(3) \leq f(1) + f(2) + 1$, it follows that $f(3) = 1$. Note that if $f(3n) \geq n$, then $f(3n+3) \geq f(3n) + f(3) \geq n + 1$. Hence by induction $f(3n) \geq n$ holds for all $n \in \mathbb{N}$. Moreover, if the inequality is strict for some n, then it is so for all integers greater than n as well. Since $f(9999) = 3333$, we deduce that $f(3n) = n$ for all $n \leq 3333$.
   By the given condition, we have $3f(n) \leq f(3n) \leq 3f(n) + 2$. Therefore $f(n) = [f(3n)/3] = [n/3]$ for $n \leq 3333$. In particular, $f(1982) = [1982/3] = 660$.

2. Since K does not contain a lattice point other than $O(0,0)$, it is bounded by four lines u, v, w, x that pass through the points $U(1,0)$, $V(0,1)$, $W(-1,0)$, $X(0,-1)$ respectively. Let $PQRS$ be the quadrilateral formed by these lines, where $U \in SP$, $V \in PQ$, $W \in QR$, $X \in RS$.
   If one of the quadrants, say Q_1, contains no vertices of $PQRS$, then $K \cap Q_1$ is contained in $\triangle OUV$ and hence has area less than $1/2$. Consequently the area of K is less than 2.
   Let us now suppose that P, Q, R, S lie in different quadrants. One of the angles of $PQRS$ is at least 90°: let it be $\angle P$. Then $S_{UPV} \leq PU \cdot PV/2 \leq (PU^2 + PV^2)/4 \leq UV^2/4 = 1/2$, which implies that $S_{K \cap Q_1} < S_{OUPV} \leq 1$. Hence the area of K is less than 4.

3. (a) By the Cauchy–Schwarz inequality we have $\left(x_0^2/x_1 + \cdots + x_{n-1}^2/x_n\right) \cdot (x_1 + \cdots + x_n) \geq (x_0 + \cdots + x_{n-1})^2$. Let us set $X_{n-1} = x_1 + x_2 + \cdots + x_{n-1}$. Using $x_0 = 1$, the last inequality can be rewritten as
   $$\frac{x_0^2}{x_1} + \cdots + \frac{x_{n-1}^2}{x_n} \geq \frac{(1 + X_{n-1})^2}{X_{n-1} + x_n} \geq \frac{4X_{n-1}}{X_{n-1} + x_n} = \frac{4}{1 + x_n/X_{n-1}}. \quad (1)$$
   Since $x_n \leq x_{n-1} \leq \cdots \leq x_1$, it follows that $X_{n-1} \geq (n-1)x_n$. Now (1) yields $x_0^2/x_1 + \cdots + x_{n-1}^2/x_n \geq 4(n-1)/n$, which exceeds 3.999 for $n > 4000$.

   (b) The sequence $x_n = 1/2^n$ obviously satisfies the required condition.

   Second solution to part (a). For each $n \in \mathbb{N}$, let us find a constant c_n such that the inequality $x_0^2/x_1 + \cdots + x_{n-1}^2/x_n \geq c_n x_0$ holds for any sequence $x_0 \geq x_1 \geq \cdots \geq x_n > 0$.
   For $n = 1$ we can take $c_1 = 1$. Assuming that c_n exists, we have
   $$\frac{x_0^2}{x_1} + \left(\frac{x_1^2}{x_2} + \cdots + \frac{x_n^2}{x_{n+1}}\right) \geq \frac{x_0^2}{x_1} + c_n x_1 \geq 2\sqrt{x_0^2 c_n} = x_0 \cdot 2\sqrt{c_n}.$$
   Thus we can take $c_{n+1} = 2\sqrt{c_n}$. Then inductively $c_n = 2^{2-1/2^{n-2}}$, and since $c_n \to 4$ as $n \to \infty$, the result follows.

   Third solution. Since $\{x_n\}$ is decreasing, there exists $\lim_{n\to\infty} x_n = x \geq 0$. If $x > 0$, then $x_{n-1}^2/x_n \geq x_n \geq x$ holds for each n, and the result is trivial.

If otherwise $x = 0$, then we note that $x_{n-1}^2/x_n \geq 4(x_{n-1} - x_n)$ for each n, with equality if and only if $x_{n-1} = 2x_n$. Hence

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{x_{k-1}^2}{x_k} \geq \lim_{n\to\infty} \sum_{k=1}^{n} 4(x_{k-1} - x_k) = 4x_0 = 4.$$

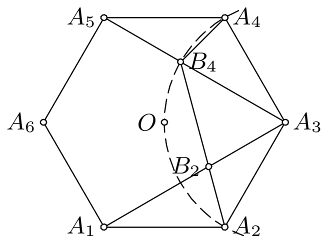
Equality holds if and only if $x_{n-1} = 2x_n$ for all n, and consequently $x_n = 1/2^n$.

4. Suppose that a satisfies the requirements of the problem and that x, qx, q^2x, q^3x are the roots of the given equation. Then $x \neq 0$ and we may assume that $|q| > 1$, so that $|x| < |qx| < |q^2x| < |q^3x|$. Since the equation is symmetric, $1/x$ is also a root and therefore $1/x = q^3x$, i.e., $q = x^{-2/3}$. It follows that the roots are $x, x^{1/3}, x^{-1/3}, x^{-1}$. Now by Vieta's formula we have $x + x^{1/3} + x^{-1/3} + x^{-1} = a/16$ and $x^{4/3} + x^{2/3} + 2 + x^{-2/3} + x^{-4/3} = (2a + 17)/16$. On setting $z = x^{1/3} + x^{-1/3}$ these equations become

$$z^3 - 2z = a/16,$$
$$(z^2 - 2)^2 + z^2 - 2 = (2a + 17)/16.$$

Substituting $a = 16(z^3 - 2z)$ in the second equation leads to $z^4 - 2z^3 - 3z^2 + 4z + 15/16 = 0$. We observe that this polynomial factors as $(z + 3/2)(z - 5/2)(z^2 - z - 1/4)$. Since $|z| = |x^{1/3} + x^{-1/3}| \geq 2$, the only viable value is $z = 5/2$. Consequently $a = 170$ and the roots are $1/8, 1/2, 2, 8$.

5. We first observe that $\triangle A_5B_4A_4 \cong \triangle A_3B_2A_2$. Since $\angle A_5A_3A_2 = 90°$, we have $\angle A_2B_4A_4 = \angle A_2B_4A_3 + \angle A_3B_4A_4 = (90° - \angle B_2A_2A_3) + (\angle B_4A_5A_4 + \angle A_5A_4B_4) = 90° + \angle B_4A_5A_4 = 120°$. Hence B_4 belongs to the circle with center A_3 and radius A_3A_4, so $A_3A_4 = A_3B_4$. Thus $\lambda = A_3B_4/A_3A_5 = A_3A_4/A_3A_5 = 1/\sqrt{3}$.



6. Denote by $d(U, V)$ the distance between points or sets of points U and V. For $P, Q \in L$ we shall denote by L_{PQ} the part of L between points P and Q and by l_{PQ} the length of this part. Let us denote by S_i $(i = 1, 2, 3, 4)$ the vertices of S and by T_i points of L such that $S_iT_i \leq 1/2$ in such a way that $l_{A_0T_1}$ is the least of the $l_{A_0T_i}$'s, S_2 and S_4 are neighbors of S_1, and $l_{A_0T_2} < l_{A_0T_4}$.
Now we shall consider the points of the segment S_1S_4. Let D and E be the sets of points defined as follows: $D = \{X \in [S_1S_4] \mid d(X, L_{A_0T_2}) \leq 1/2\}$ and $E = \{X \in [S_1S_4] \mid d(X, L_{T_2A_n}) \leq 1/2\}$. Clearly D and E are closed, nonempty (indeed, $S_1 \in D$ and $S_4 \in E$) subsets of $[S_1S_4]$. Since their union is a connected set S_1S_4, it follows that they must have a nonempty intersection. Let $P \in D \cap E$. Then there exist points $X \in L_{A_0T_2}$ and

$Y \in L_{T_2A_n}$ such that $d(P,X) \le 1/2$, $d(P,Y) \le 1/2$, and consequently $d(X,Y) \le 1$. On the other hand, T_2 lies between X and Y on L, and thus $L_{XY} = L_{XT_2} + L_{T_2Y} \ge XT_2 + T_2Y \ge (PS_2 - XP - S_2T_2) + (PS_2 - YP - S_2T_2) \ge 99 + 99 = 198$.

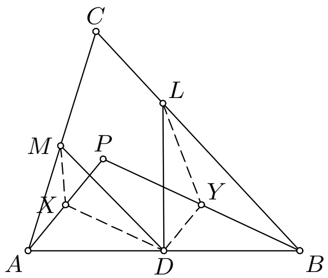
7. Let a, b, ab be the roots of the cubic polynomial $P(x) = (x-a)(x-b)(x-ab)$. Observe that

$$\begin{aligned} 2p(-1) &= -2(1+a)(1+b)(1+ab); \\ p(1) + p(-1) - 2(1+p(0)) &= -2(1+a)(1+b). \end{aligned}$$

The statement of the problem is trivial if both the expressions are equal to zero. Otherwise, the quotient $\frac{2p(-1)}{p(1)+p(-1)-2(1+p(0))} = 1 + ab$ is rational and consequently ab is rational. But since $(ab)^2 = -P(0)$ is an integer, it follows that ab is also an integer. This completes the proof.

8. Let $\mathcal{F}$ be the given figure. Consider any chord AB of the circumcircle γ that supports $\mathcal{F}$. The other supporting lines to $\mathcal{F}$ from A and B intersect γ again at D and C respectively so that $\angle DAB = \angle ABC = 90°$. Then $ABCD$ is a rectangle, and hence CD must support $\mathcal{F}$ as well, from which it follows that $\mathcal{F}$ is inscribed in the rectangle $ABCD$ touching each of its sides. We easily conclude that $\mathcal{F}$ is the intersection of all such rectangles. Now, since the center O of γ is the center of symmetry of all these rectangles, it must be so for their intersection $\mathcal{F}$ as well.

9. Let X and Y be the midpoints of the segments AP and BP. Then $DYPX$ is a parallelogram. Since X and Y are the circumcenters of $\triangle APM$ and $\triangle BPL$, it follows that $XM = XP = DY$ and $YL = YP = DX$. Furthermore, $\angle DXM = \angle DXP + \angle PXM = \angle DXP + 2\angle PAM = \angle DYP + 2\angle PBL = \angle DYP + \angle PYL = \angle DYL$. Therefore, the triangles DXM and LYD are congruent, implying $DM = DL$.



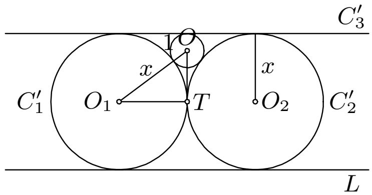
10. If the two balls taken from the box are both white, then the number of white balls decreases by two; otherwise, it remains unchanged. Hence the parity of the number of white balls does not change during the procedure. Therefore if p is even, the last ball cannot be white; the probability is 0. If p is odd, the last ball has to be white; the probability is 1.

11. (a) Suppose $\{a_1, a_2, \dots, a_n\}$ is the arrangement that yields the maximal value $Q_{\max}$ of Q. Note that the value of Q for the rearrangement $\{a_1, \dots, a_{i-1}, a_j, a_{j-1}, \dots, a_i, a_{j+1}, \dots, a_n\}$ equals $Q_{\max} - (a_i - a_j)(a_{i-1} - a_{j+1})$, where $1 < i < j < n$. Hence $(a_i - a_j)(a_{i-1} - a_{j+1}) \ge 0$ for all $1 < i < j < n$.

We may suppose w.l.o.g. that $a_1 = 1$. Let $a_i = 2$. If $2 < i < n$, then $(a_2 - a_i)(a_1 - a_{i+1}) < 0$, which is impossible. Therefore i is either 2 or n; let w.l.o.g. $a_n = 2$. Further, if $a_j = 3$ for $2 < j < n$, then $(a_1 - a_{j+1})(a_2 - a_j) < 0$, which is impossible; therefore $a_2 = 3$. Continuing this argument we obtain that $A = \{1, 3, 5, \dots, 2[(n-1)/2] + 1, 2[n/2], \dots, 4, 2\}$.

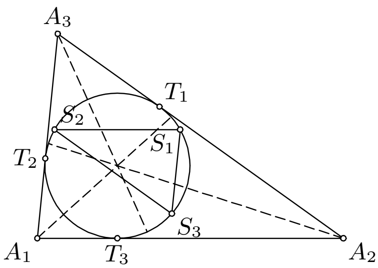
(b) A similar argument leads to the minimizing rearrangement $\{1, n, 2, n-1, \dots, [n/2] + 1\}$.

12. Let y be the line perpendicular to L passing through the center of C. It can be shown by a continuity argument that there exists a point $Y \in y$ such that an inversion Ψ centered at Y maps C and L onto two concentric circles $\widehat{C}$ and $\widehat{L}$. Let $\widehat{X}$ denote the image of an object X under Ψ. Then the circles $\widehat{C_i}$ touch $\widehat{C}$ externally and $\widehat{L}$ internally, and all have the same radius. Let us now rotate the picture around the common center Z of $\widehat{C}$ and $\widehat{L}$ so that $\widehat{C_3}$ passes through Y. Applying the inversion Ψ again on the picture thus obtained, $\widehat{C}$ and $\widehat{L}$ go back to C and L, but $\widehat{C_3}$ goes to a line C_3' parallel to L, while the images of $\widehat{C_1}$ and $\widehat{C_2}$ go to two equal circles C_1' and C_2' touching L, C_3', and C. This way we have achieved that C_3 becomes a line.



Denote by O_1, O_2, O respectively the centers of the circles C_1', C_2', C and by T the point of tangency of the circles C_1' and C_2'. If x is the common radius of the circles C_1' and C_2', then from $\triangle O_1TO$ we obtain that $(x-1)^2 + x^2 = (x+1)^2$, and thus $x = 4$. Hence the distance of O from L equals $2x - 1 = 7$.

13. Points S_1, S_2, S_3 clearly lie on the inscribed circle. Let $\widehat{XY}$ denote the oriented arc XY. The arcs $\widehat{T_2S_1}$ and $\widehat{T_1T_3}$ are equal, since they are symmetric with respect to the bisector of $\angle A_1$. Similarly, $\widehat{T_3T_2} = \widehat{S_2T_1}$. Therefore $\widehat{T_3S_1} = \widehat{T_3T_2} + \widehat{T_2S_1} = \widehat{S_2T_1} + \widehat{T_1T_3} = \widehat{S_2T_3}$. It follows that S_1S_2 is parallel to A_1A_2, and consequently $S_1S_2 \parallel M_1M_2$. Analogously $S_1S_3 \parallel M_1M_3$ and $S_2S_3 \parallel M_2M_3$.



Since the circumcircles of $\triangle M_1M_2M_3$ and $\triangle S_1S_2S_3$ are not equal, these triangles are not congruent and hence they must be homothetic. Then all the lines M_iS_i pass through the center of homothety.

Second solution. Set the complex plane so that the incenter of $\triangle A_1A_2A_3$ is the unit circle centered at the origin. Let t_i, s_i respectively denote the complex numbers of modulus 1 corresponding to T_i, S_i. Clearly $t_1\overline{t_1} =$

$t_2\overline{t_2} = t_3\overline{t_3} = 1$. Since T_2T_3 and T_1S_1 are parallel, we obtain $t_2t_3 = t_1s_1$, or $s_1 = t_2t_3\overline{t_1}$. Similarly $s_2 = t_1t_3\overline{t_2}$, $s_3 = t_1t_2\overline{t_3}$, from which it follows that $s_2 - s_3 = t_1(t_3\overline{t_2} - t_2\overline{t_3})$. Since the number in parentheses is strictly imaginary, we conclude that $OT_1 \perp S_2S_3$ and consequently $S_2S_3 \parallel A_2A_3$. We proceed as in the first solution.

14. (a) If any two of A_1, B_1, C_1, D_1 coincide, say $A_1 \equiv B_1$, then $ABCD$ is inscribed in a circle centered at A_1 and hence all A_1, B_1, C_1, D_1 coincide.
Assume now the opposite, and let w.l.o.g. $\angle DAB + \angle DCB < 180°$. Then A is outside the circumcircle of $\triangle BCD$, so $A_1A > A_1C$. Similarly, $C_1C > C_1A$. Hence the perpendicular bisector l_{AC} of AC separates points A_1 and C_1. Since B_1, D_1 lie on l_{AC}, this means that A_1 and C_1 are on opposite sides B_1D_1. Similarly one can show that B_1 and D_1 are on opposite sides of A_1C_1.
(b) Since $A_2B_2 \perp C_1D_1$ and $C_1D_1 \perp AB$, it follows that $A_2B_2 \parallel AB$. Similarly $A_2C_2 \parallel AC$, $A_2D_2 \parallel AD$, $B_2C_2 \parallel BC$, $B_2D_2 \parallel BD$, and $C_2D_2 \parallel CD$. Hence $\triangle A_2B_2C_2 \sim \triangle ABC$ and $\triangle A_2D_2C_2 \sim \triangle ADC$, and the result follows.

15. Let $a = k/n$, where $n, k \in \mathbb{N}$, $n \geq k$. Putting $t^n = s$, the given inequality becomes $\frac{1-t^k}{1-t^n} \leq (1+t^n)^{k/n-1}$, or equivalently

$$(1+t+\cdots+t^{k-1})^n(1+t^n)^{n-k} \leq (1+t+\cdots+t^{n-1})^n.$$

This is clearly true for $k = n$. Therefore it is enough to prove that the left-hand side of the above inequality is an increasing function of k. We are led to show that $(1+t+\cdots+t^{k-1})^n(1+t^n)^{n-k} \leq (1+t+\cdots+t^k)^n(1+t^n)^{n-k-1}$. This is equivalent to $1+t^n \leq A^n$, where $A = \frac{1+t+\cdots+t^k}{1+t+\cdots+t^{k-1}}$. But this easily follows, since

$$\begin{aligned} A^n - t^n &= (A-t)(A^{n-1} + A^{n-2}t + \cdots + t^{n-1}) \\ &\geq (A-t)(1+t+\cdots+t^{n-1}) = \frac{1+t+\cdots+t^{n-1}}{1+t+\cdots+t^{k-1}} \geq 1. \end{aligned}$$

Remark. The original problem asked to prove the inequality for real a.

16. It is easy to verify that whenever (x, y) is a solution of the equation $x^3 - 3xy^2 + y^3 = n$, so are the pairs $(y - x, -x)$ and $(-y, x - y)$. No two of these three solutions are equal unless $x = y = n = 0$.
Observe that $2981 \equiv 2 \pmod 9$. Since $x^3, y^3 \equiv 0, \pm 1 \pmod 9$, $x^3 - 3xy^2 + y^3$ cannot give the remainder 2 when divided by 9. Hence the above equation for $n = 2981$ has no integer solutions.

17. Let A be the origin of the Cartesian plane. Suppose that $BC : AC = k$ and that (a, b) and (a_1, b_1) are coordinates of the points C and C_1, respectively. Then the coordinates of the point B are $(a, b) + k(-b, a) = (a - kb, b + ka)$,

while the coordinates of B_1 are $(a_1, b_1) + k(b_1, -a_1) = (a + kb_1, b_1 - ka_1)$. Thus the lines BC_1 and CB_1 are given by the equations $\frac{x-a_1}{y-b_1} = \frac{x-(a-kb)}{y-(b+ka)}$ and $\frac{x-a}{y-b} = \frac{x-(a_1+kb_1)}{y-(b_1-ka_1)}$ respectively. After multiplying, these equations transform into the forms

$$\begin{aligned} BC_1: &\quad kax + kby = kaa_1 + kbb_1 + ba_1 - ab_1 - (b - b_1)x + (a - a_1)y \\ CB_1: &\quad ka_1x + kb_1y = kaa_1 + kbb_1 + ba_1 - ab_1 - (b - b_1)x + (a - a_1)y. \end{aligned}$$

The coordinates (x_0, y_0) of the point M satisfy these equations, from which we deduce that $kax_0 + kby_0 = ka_1x_0 + kb_1y_0$. This yields $\frac{x_0}{y_0} = -\frac{b_1-b}{a_1-a}$, implying that the lines CC_1 and AM are perpendicular.

18. Set the coordinate system with the axes x, y, z along the lines l_1, l_2, l_3 respectively. The coordinates (a, b, c) of M satisfy $a^2 + b^2 + c^2 = R^2$, and so S_M is given by the equation $(x-a)^2+(y-b)^2+(z-c)^2 = R^2$. Hence the coordinates of P_1 are $(x, 0, 0)$ with $(x - a)^2 + b^2 + c^2 = R^2$, implying that either $x = 2a$ or $x = 0$. Thus by the definition we obtain $x = 2a$. Similarly, the coordinates of P_2 and P_3 are $(0, 2b, 0)$ and $(0, 0, 2c)$ respectively. Now, the centroid of $\triangle P_1P_2P_3$ has the coordinates $(2a/3, 2b/3, 2c/3)$. Therefore the required locus of points is the sphere with center O and radius $2R/3$.

19. Let us set $x = m/n$. Since $f(x) = (m + n)/\sqrt{m^2 + n^2} = (x + 1)/\sqrt{1 + x^2}$ is a continuous function of x, $f(x)$ takes all values between any two values of f; moreover, the corresponding x can be rational. This completes the proof.

   Remark. Since f is increasing for $x \geq 1$, $1 \leq x < z < y$ implies $f(x) < f(z) < f(y)$.

20. Since MN is the image of AC under rotation about B for $60°$, we have $MN = AC$.
   Similarly, PQ is the image of AC under rotation about D through $60°$, from which it follows that $PQ \parallel MN$. Hence either M, N, P, Q are collinear or $MNPQ$ is a parallelogram.

4.24 Solutions to the Shortlisted Problems of IMO 1983

1. Suppose that there are n airlines $A_1, \dots, A_n$ and $N > 2^n$ cities. We shall prove that there is a round trip by at least one A_i containing an odd number of stops.

   For $n = 1$ the statement is trivial, since one airline serves at least 3 cities and hence $P_1P_2P_3P_1$ is a round trip with 3 landings. We use induction on n, and assume that $n > 1$. Suppose the contrary, that all round trips by A_n consist of an even number of stops. Then we can separate the cities into two nonempty classes $Q = \{Q_1, \dots, Q_r\}$ and $R = \{R_1, \dots, R_s\}$ (where $r+s = N$), so that each flight by A_n runs between a Q-city and an R-city. (Indeed, take any city Q_1 served by A_n; include each city linked to Q_1 by A_n in R, then include in Q each city linked by A_n to any R-city, etc. Since all round trips are even, no contradiction can arise.) At least one of r, s is larger than 2^{n-1}, say $r > 2^{n-1}$. But, only $A_1, \dots, A_{n-1}$ run between cities in $\{Q_1, \dots, Q_r\}$; hence by the induction hypothesis at least one of them flies a round trip with an odd number of landings, a contradiction. It only remains to notice that for $n = 10$, $2^n = 1024 < 1983$.

   Remark. If there are $N = 2^n$ cities, there is a schedule with n airlines that contain no odd round trip by any of the airlines. Let the cities be P_k, $k = 0, \dots, 2^n - 1$, and write k in the binary system as an n-digit number $\overline{a_1 \dots a_n}$ (e.g., $1 = (0\dots001)_2$). Link P_k and P_l by A_i if the ith digits k and l are distinct but the first $i-1$ digits are the same. All round trips under A_i are even, since the ith digit alternates.

2. By definition, $\sigma(n) = \sum_{d|n} d = \sum_{d|n} n/d = n \sum_{d|n} 1/d$, hence $\sigma(n)/n = \sum_{d|n} 1/d$. In particular, $\sigma(n!)/n! = \sum_{d|n!} 1/d \ge \sum_{k=1}^{n} 1/k$. It follows that the sequence $\sigma(n)/n$ is unbounded, and consequently there exist an infinite number of integers n such that $\sigma(n)/n$ is strictly greater than $\sigma(k)/k$ for $k < n$.

3. (a) A circle is not Pythagorean. Indeed, consider the partition into two semicircles each closed at one and open at the other end.

   (b) An equilateral triangle, call it PQR, is Pythagorean. Let P', Q', and R' be the points on QR, RP, and PQ such that $PR' : R'Q = QP' : P'R = RQ' : Q'P = 1 : 2$. Then $Q'R' \perp PQ$, etc. Suppose that PQR is not Pythagorean, and consider a partition into A, B, neither of which contains the vertices of a right-angled triangle. At least two of P', Q', and R' belong to the same class, say $P', Q' \in A$. Then $[PR] \setminus \{Q'\} \subset B$ and hence $R' \in A$ (otherwise, if R'' is the foot of the perpendicular from R' to PR, $\triangle RR'R''$ is right-angled with all vertices in B). But this implies again that $[PQ] \setminus \{R'\} \subset B$, and thus B contains vertices of a rectangular triangle. This is a contradiction.

4. The rotational homothety centered at C that sends B to R also sends A to Q; hence the triangles ABC and QRC are similar. For the same reason,

$\triangle ABC$ and $\triangle PBR$ are similar. Moreover, $BR = CR$; hence $\triangle CRQ \cong \triangle RBP$. Thus $PR = QC = AQ$ and $QR = PB = PA$, so $APQR$ is a parallelogram.

5. Each natural number p can be written uniquely in the form $p = 2^q(2r-1)$. We call $2r-1$ the odd part of p. Let $A_n = (a_1, a_2, \dots, a_n)$ be the first sequence. Clearly the terms of A_n must have different odd parts, so those parts must be at least $1, 3, \dots, 2n-1$. Being the first sequence, A_n must have the numbers $2n-1, 2n-3, \dots, 2k+1$ as terms, where $k = [n+1/3]$ (then $3(2k-1) < 2n-1 < 3(2k+1)$). Smaller odd numbers $2s+1$ (with $s < k$) obviously cannot be terms of A_n. In this way we have obtained the $n-k$ odd numbers of A_n. The other k terms must be even, and by the same reasoning as above they must be precisely the terms of $2A_k$ (twice the terms of A_k). Therefore A_n is defined recursively as

$$A_0 = \emptyset, \qquad A_1 = \{1\}, \qquad A_2 = \{3, 2\};$$
$$A_n = \{2n-1, 2n-3, \dots, 2k+1\} \cup 2A_k.$$

6. The existence of r: Let $S = \{x_1 + x_2 + \cdots + x_i - 2i \mid i = 1, 2, \dots, n\}$. Let $\max S$ be attained for the first time at r'.
If $r' = n$, then $x_1 + x_2 + \cdots + x_i - 2i < 2$ for $1 \le i \le n-1$, so one can take $r = r'$.
Suppose that $r' < n$. Then for $l < n-r'$ we have $x_{r'+1} + x_{r'+2} + \cdots + x_{r'+l} = (x_1 + \cdots + x_{r'+l} - 2(r'+l)) - (x_1 + \cdots + x_{r'} - 2r') + 2l \le 2l$; also, for $i < r'$ we have $(x_{r'+1} + \cdots + x_n) + (x_1 + \cdots + x_i - 2i) < (x_{r'+1} + \cdots + x_n) + (x_1 + \cdots + x_{r'} - 2r') = (x_1 + \cdots + x_n) - 2r' = 2(n-r') + 2 \Rightarrow x_{r'+1} + \cdots + x_n + x_1 + \cdots + x_i \le 2(n+i-r') + 1$, so we can again take $r = r'$.
For the second part of the problem, we relabel the sequence so that $r = 0$ works.
Suppose that the inequalities are strict. We have $x_1 + x_2 + \cdots + x_k \le 2k$, $k = 1, \dots, n-1$. Now, $2n + 2 = (x_1 + \cdots + x_k) + (x_{k+1} + \cdots + x_n) \le 2k + x_{k+1} + \cdots + x_n \Rightarrow x_{k+1} + \cdots + x_n \ge 2(n-k) + 2 > 2(n-k) + 1$. So we cannot begin with x_{k+1} for any $k > 0$.
Now assume that there is an equality for some k. There are two cases:
   (i) Suppose $x_1 + x_2 + \cdots + x_i \le 2i$ ($i = 1, \dots, k$) and $x_1 + \cdots + x_k = 2k+1$, $x_1 + \cdots + x_{k+l} \le 2(k+l) + 1$ ($1 \le l \le n-1-k$). For $i \le k-1$ we have $x_{i+1} + \cdots + x_n = 2(n+1) - (x_1 + \cdots + x_i) > 2(n-i) + 1$, so we cannot take $r = i$. If there is a $j \ge 1$ such that $x_1 + x_2 + \cdots + x_{k+j} \le 2(k+j)$, then also $x_{k+j+1} + \cdots + x_n > 2(n-k-j) + 1$. If $(\forall j \ge 1)$ $x_1 + \cdots + x_{k+j} = 2(k+j) + 1$, then $x_n = 3$ and $x_{k+1} = \cdots = x_{n-1} = 2$. In this case we directly verify that we cannot take $r = k+j$. However, we can also take $r = k$: for $k+l \le n-1$, $x_{k+1} + \cdots + x_{k+l} \le 2(k+l) + 1 - (2k+1) = 2l$, also $x_{k+1} + \cdots + x_n = 2(n-k) + 1$, and moreover $x_1 \le 2$, $x_1 + x_2 \le 4, \dots$.
   (ii) Suppose $x_1 + \cdots + x_i \le 2i$ ($1 \le i \le n-2$) and $x_1 + \cdots + x_{n-1} = 2n-1$. Then we can obviously take $r = n-1$. On the other hand, for any

$1 \le i \le n-2$, $x_{i+1} + \cdots + x_{n-1} + x_n = (x_1 + \cdots + x_{n-1}) - (x_1 + \cdots + x_i) + 3 > 2(n-i) + 1$, so we cannot take another $r \neq 0$.

7. Clearly, each a_n is positive and $\sqrt{a_{n+1}} = \sqrt{a_n}\sqrt{a+1} + \sqrt{a_n+1}\sqrt{a}$. Notice that $\sqrt{a_{n+1}+1} = \sqrt{a+1}\sqrt{a_n+1} + \sqrt{a}\sqrt{a_n}$. Therefore

$$\begin{aligned}
&(\sqrt{a+1} - \sqrt{a})(\sqrt{a_n+1} - \sqrt{a_n}) \\
&\quad = (\sqrt{a+1}\sqrt{a_n+1} + \sqrt{a}\sqrt{a_n}) - (\sqrt{a_n}\sqrt{a+1} + \sqrt{a_n+1}\sqrt{a}) \\
&\quad = \sqrt{a_{n+1}+1} - \sqrt{a_{n+1}}.
\end{aligned}$$

By induction, $\sqrt{a_{n+1}} - \sqrt{a_n} = \left(\sqrt{a+1} - \sqrt{a}\right)^n$. Similarly, $\sqrt{a_{n+1}} + \sqrt{a_n} = \left(\sqrt{a+1} + \sqrt{a}\right)^n$. Hence,

$$\sqrt{a_n} = \frac{1}{2}\left[\left(\sqrt{a+1} + \sqrt{a}\right)^n - \left(\sqrt{a+1} - \sqrt{a}\right)^n\right],$$

from which the result follows.

8. Situations in which the condition of the statement is fulfilled are the following:

   S_1: $N_1(t) = N_2(t) = N_3(t)$

   S_2: $N_i(t) = N_j(t) = h$, $N_k(t) = h+1$, where (i,j,k) is a permutation of the set $\{1,2,3\}$. In this case the first student to leave must be from row k. This leads to the situation S_1.

   S_3: $N_i(t) = h, N_j(t) = N_k(t) = h+1$, ($(i,j,k)$ is a permutation of the set $\{1,2,3\}$). In this situation the first student leaving the room belongs to row j (or k) and the second to row k (or j). After this we arrive at the situation S_1.

   Hence, the initial situation is S_1 and after each triple of students leaving the room the situation S_1 must recur. We shall compute the probability P_h that from a situation S_1 with $3h$ students in the room ($h \le n$) one arrives at a situation S_1 with $3(h-1)$ students in the room:

$$P_h = \frac{(3h)\cdot(2h)\cdot h}{(3h)\cdot(3h-1)\cdot(3h-2)} = \frac{3!h^3}{3h(3h-1)(3h-2)}.$$

   Since the room becomes empty after the repetition of n such processes, which are independent, we obtain for the probability sought

$$P = \prod_{h=1}^{n} P_h = \frac{(3!)^n (n!)^3}{(3n)!}.$$

9. For any triangle of sides a, b, c there exist 3 nonnegative numbers x, y, z such that $a = y+z$, $b = z+x$, $c = x+y$ (these numbers correspond to the division of the sides of a triangle by the point of contact of the incircle). The inequality becomes

$$(y+z)^2(z+x)(y-x)+(z+x)^2(x+y)(z-y)+(x+y)^2(y+z)(x-z) \geq 0.$$

Expanding, we get $xy^3 + yz^3 + zx^3 \geq xyz(x+y+z)$. This follows from Cauchy's inequality $(xy^3+yz^3+zx^3)(z+x+y) \geq \left(\sqrt{xyz}(x+y+z)\right)^2$ with equality if and only if $xy^3/z = yz^3/x = zx^3/y$, or equivalently $x = y = z$, i.e., $a = b = c$.

10. Choose $P(x) = \frac{p}{q}\left((qx-1)^{2n+1}+1\right)$, $I = [1/2q, 3/2q]$. Then all the coefficients of P are integers, and

$$\left|P(x) - \frac{p}{q}\right| = \left|\frac{p}{q}(qx-1)^{2n+1}\right| \leq \left|\frac{p}{q}\right| \frac{1}{2^{2n+1}},$$

for $x \in I$. The desired inequality follows if n is chosen large enough.

11. First suppose that the binary representation of x is finite: $x = 0, a_1 a_2 \dots a_n = \sum_{j=1}^{n} a_j 2^{-j}$, $a_i \in \{0,1\}$. We shall prove by induction on n that

$$f(x) = \sum_{j=1}^{n} b_0 \dots b_{j-1} a_j, \quad \text{where } b_k = \begin{cases} -b & \text{if } a_k = 0, \\ 1-b & \text{if } a_k = 1. \end{cases}$$

(Here $a_0 = 0$.) Indeed, by the recursion formula,
$a_1 = 0 \Rightarrow f(x) = bf(\sum_{j=1}^{n-1} a_{j+1}2^{-j}) = b\sum_{j=1}^{n-1} b_1 \dots b_j a_{j+1}$ hence $f(x) = \sum_{j=0}^{n-1} b_0 \dots b_j a_{j+1}$ as $b_0 = b_1 = b$;
$a_1 = 1 \Rightarrow f(x) = b + (1-b)f(\sum_{j=1}^{n-1} a_{j+1}2^{-j}) = \sum_{j=0}^{n-1} b_0 \dots b_j a_{j+1}$, as $b_0 = b$, $b_1 = 1-b$.
Clearly, $f(0) = 0$, $f(1) = 1$, $f(1/2) = b > 1/2$. Assume $x = \sum_{j=0}^{n} a_j 2^{-j}$, and for $k \geq 2$, $v = x + 2^{-n-k+1}$, $u = x + 2^{-n-k} = (v+x)/2$. Then $f(v) = f(x) + b_0 \dots b_n b^{k-2}$ and $f(u) = f(x) + b_0 \dots b_n b^{k-1} > (f(v)+f(x))/2$. This means that the point $(u, f(u))$ lies above the line joining $(x, f(x))$ and $(v, f(v))$. By induction, every $(x, f(x))$, where x has a finite binary expansion, lies above the line joining $(0,0)$ and $(1/2, b)$ if $0 < x < 1/2$, or above the line joining $(1/2, b)$ and $(1,1)$ if $1/2 < x < 1$. It follows immediately that $f(x) > x$. For the second inequality, observe that

$$\begin{aligned} f(x) - x &= \sum_{j=1}^{\infty} (b_0 \dots b_{j-1} - 2^{-j}) a_j \\ &< \sum_{j=1}^{\infty} (b^j - 2^{-j}) a_j < \sum_{j=1}^{\infty} (b^j - 2^{-j}) = \frac{b}{1-b} - 1 = c. \end{aligned}$$

By continuity, these inequalities also hold for x with infinite binary representations.

12. Putting $y = x$ in (1) we see that there exist positive real numbers z such that $f(z) = z$ (this is true for every $z = xf(x)$). Let a be any of them.

Then $f(a^2) = f(af(a)) = af(a) = a^2$, and by induction, $f(a^n) = a^n$. If $a > 1$, then $a^n \to +\infty$ as $n \to \infty$, and we have a contradiction with (2). Again, $a = f(a) = f(1 \cdot a) = af(1)$, so $f(1) = 1$. Then, $af(a^{-1}) = f(a^{-1}f(a)) = f(1) = 1$, and by induction, $f(a^{-n}) = a^{-n}$. This shows that $a \not< 1$. Hence, $a = 1$. It follows that $xf(x) = 1$, i.e., $f(x) = 1/x$ for all x. This function satisfies (1) and (2), so $f(x) = 1/x$ is the unique solution.

13. Given any coloring of the $3\times1983-2$ points of the axes, we prove that there is a unique coloring of E having the given property and extending this coloring. The first thing to notice is that given any rectangle R_1 parallel to a coordinate plane and whose edges are parallel to the axes, there is an even number r_1 of red vertices on R_1. Indeed, let R_2 and R_3 be two other rectangles that are translated from R_1 orthogonally to R_1 and let r_2, r_3 be the numbers of red vertices on R_2 and R_3 respectively. Then $r_1 + r_2$, r_1+r_3, and r_2+r_3 are multiples of 4, so $r_1 = (r_1+r_2+r_1+r_3-r_2-r_3)/2$ is even.
    Since any point of a coordinate plane is a vertex of a rectangle whose remaining three vertices lie on the corresponding axes, this determines uniquely the coloring of the coordinate planes. Similarly, the coloring of the inner points of the parallelepiped is completely determined. The solution is hence $2^{3\times1983-2} = 2^{5947}$.

14. Let T_n be the set of all nonnegative integers whose ternary representations consist of at most n digits and do not contain a digit 2. The cardinality of T_n is 2^n, and the greatest integer in T_n is $11\dots1 = 3^0 + 3^1 + \dots + 3^{n-1} = (3^n - 1)/2$. We claim that there is no arithmetic triple in T_n. To see this, suppose $x, y, z \in T_n$ and $2y = x + z$. Then $2y$ has only 0's and 2's in its ternary representation, and a number of this form can be the sum of two integers $x, z \in T_n$ in only one way, namely $x = z = y$. But $|T_{10}| = 2^{10} = 1024$ and $\max T_{10} = (3^{10} - 1)/2 = 29524 < 30000$. Thus the answer is yes.

15. There is no such set. Suppose M satisfies (a) and (b) and let $q_n = |\{a \in M : a \le n\}|$. Consider the differences $b - a$, where $a, b \in M$ and $10 < a < b \le k$. They are all positive and less than k, and (b) implies that they are $\binom{q_k-q_{10}}{2}$ different integers. Hence $\binom{q_k-q_{10}}{2} < k$, so $q_k \le \sqrt{2k}+10$. It follows from (a) that among the numbers of the form $a + b$, where $a, b \in M$, $a \le b \le n$, or $a \le n < b \le 2n$, there are all integers from the interval $[2, 2n+1]$. Thus $\binom{q_n+1}{2} + q_n(q_{2n} - q_n) \ge 2n$ for every $n \in \mathbb{N}$. Set $Q_k = \sqrt{2k} + 10$. We have

$$\binom{q_n+1}{2} + q_n(q_{2n}-q_n) = \frac{1}{2}q_n + \frac{1}{2}q_n(2q_{2n}-q_n)$$
$$\leq \frac{1}{2}q_n + \frac{1}{2}q_n(2Q_{2n}-q_n)$$
$$\leq \frac{1}{2}Q_n + \frac{1}{2}Q_n(2Q_{2n}-Q_n)$$
$$\leq 2(\sqrt{2}-1)n + (20+\sqrt{2}/2)\sqrt{n} + 55,$$

which is less than n for n large enough, a contradiction.

16. Set $h_{n,i}(x) = x^i + \cdots + x^{n-i}$, $2i \leq n$. The set $F(n)$ is the set of linear combinations with nonnegative coefficients of the $h_{n,i}$'s. This is a convex cone. Hence, it suffices to prove that $h_{n,i}h_{m,j} \in F(m+n)$. Indeed, setting $p = n-2i$ and $q = m-2j$ and assuming $p \leq q$ we obtain

$$h_{n,i}(x)h_{m,j}(x) = (x^i + \cdots + x^{i+p})(x^j + \cdots + x^{j+q}) = \sum_{k=i+j}^{n-i+j} h_{m+n,k},$$

which proves the claim.

17. Set $a = \min P_iP_j$, $b = \max P_iP_j$. We use the following lemma.
Lemma. There exists a disk of radius less than or equal to $b/\sqrt{3}$ containing all the P_i's.
Assuming that this is proved, the disks with center P_i and radius $a/2$ are disjoint and included in a disk of radius $b/\sqrt{3} + a/2$; hence comparing areas,

$$n\pi \cdot \frac{a^2}{4} < \pi \cdot \left(\frac{b}{\sqrt{3}} + a/2\right)^2 \quad \text{and} \quad b > \sqrt{3}/2 \cdot (\sqrt{n}-1)a.$$

Proof of the lemma. If a nonobtuse triangle with sides $a \geq b \geq c$ has a circumscribed circle of radius R, we have $R = a/(2\sin\alpha) \leq a/\sqrt{3}$. Now we show that there exists a disk D of radius R containing $A = \{P_1, \dots, P_n\}$ whose border C is such that $C \cap A$ is not included in an open semicircle, and hence contains either two diametrically opposite points and $R \leq b/2$, or an acute-angled triangle and $R \leq b/\sqrt{3}$.
Among all disks whose borders pass through three points of A and that contain all of A, let D be the one of least radius. Suppose that $C \cap A$ is contained in an arc of central angle less than $180°$, and that P_i, P_j are its endpoints. Then there exists a circle through P_i, P_j of smaller radius that contains A, a contradiction. Thus D has the required property, and the assertion follows.

18. Let (x_0, y_0, z_0) be one solution of $bcx + cay + abz = n$ (not necessarily nonnegative). By subtracting $bcx_0 + cay_0 + abz_0 = n$ we get

$$bc(x-x_0) + ca(y-y_0) + ab(z-z_0) = 0.$$

Since $(a,b)=(a,c)=1$, we must have $a|x-x_0$ or $x-x_0=as$. Substituting this in the last equation gives

$$bcs+c(y-y_0)+b(z-z_0)=0.$$

Since $(b,c)=1$, we have $b|y-y_0$ or $y-y_0=bt$. If we substitute this in the last equation we get $bcs+bct+b(z-z_0)=0$, or $cs+ct+z-z_0=0$, or $z-z_0=-c(s+t)$. In $x=x_0+as$ and $y=y_0+bt$, we can choose s and t such that $0\le x\le a-1$ and $0\le y\le b-1$. If $n>2abc-bc-ca-ab$, then $abz=n-bcx-acy>2abc-ab-bc-ca-bc(a-1)-ca(b-1)=-ab$ or $z>-1$, i.e., $z\ge 0$. Hence, it is representable as $bcx+cay+abz$ with $x,y,z\ge 0$.

Now we prove that $2abc-bc-ca-ab$ is not representable as $bcx+cay+abz$ with $x,y,z\ge 0$. Suppose that $bcx+cay+abz=2abc-ab-bc-ca$ with $x,y,z\ge 0$. Then

$$bc(x+1)+ca(y+1)+ab(z+1)=2abc$$

with $x+1,y+1,z+1\ge 1$. Since $(a,b)=(a,c)=1$, we have $a|x+1$ and thus $a\le x+1$. Similarly $b\le y+1$ and $c\le z+1$. Thus $bca+cab+abc\le 2abc$, a contradiction.

19. For all n, there exists a unique polynomial P_n of degree n such that $P_n(k)=F_k$ for $n+2\le k\le 2n+2$ and $P_n(2n+3)=F_{2n+3}-1$. For $n=0$, we have $F_1=F_2=1$, $F_3=2$, $P_0=1$. Now suppose that P_{n-1} has been constructed and let P_n be the polynomial of degree n satisfying $P_n(X+2)-P_n(X+1)=P_{n-1}(X)$ and $P_n(n+2)=F_{n+2}$. (The mapping $\mathbb{R}_n[X]\to\mathbb{R}_{n-1}[X]\times\mathbb{R}$, $P\mapsto(Q,P(n+2))$, where $Q(X)=P(X+2)-P(X+1)$, is bijective, since it is injective and those two spaces have the same dimension; clearly $\deg Q=\deg P-1$.) Thus for $n+2\le k\le 2n+2$ we have $P_n(k+1)=P_n(k)+F_{k-1}$ and $P_n(n+2)=F_{n+2}$; hence by induction on k, $P_n(k)=F_k$ for $n+2\le k\le 2n+2$ and

$$P_n(2n+3)=F_{2n+2}+P_{n-1}(2n+1)=F_{2n+3}-1.$$

Finally, P_{990} is exactly the polynomial P of the terms of the problem, for $P_{990}-P$ has degree less than or equal to 990 and vanishes at the 991 points $k=992,\dots,1982$.

20. If $(x_1,x_2,\dots,x_n)$ satisfies the system with parameter a, then $(-x_1,-x_2,\dots,-x_n)$ satisfies the system with parameter $-a$. Hence it is sufficient to consider only $a\ge 0$.

Let $(x_1,\dots,x_n)$ be a solution. Suppose $x_1\le a$, $x_2\le a,\dots,x_n\le a$. Summing the equations we get

$$(x_1-a)^2+\cdots+(x_n-a)^2=0$$

and see that $(a,a,\dots,a)$ is the only such solution. Now suppose that $x_k\ge a$ for some k. According to the kth equation,

$$x_{k+1}|x_{k+1}| = x_k^2 - (x_k - a)^2 = a(2x_k - a) \geq a^2,$$

which implies that $x_{k+1} \geq a$ as well (here $x_{n+1} = x_1$). Consequently, all $x_1, x_2, \dots, x_n$ are greater than or equal to a, and as above $(a, a, \dots, a)$ is the only solution.

21. Using the identity

$$a^n - b^n = (a - b) \sum_{m=0}^{n-1} a^{n-m-1} b^m$$

with $a = k^{1/n}$ and $b = (k-1)^{1/n}$ one obtains

$$1 < \left(k^{1/n} - (k-1)^{1/n}\right) n k^{1-1/n} \quad \text{for all integers } n > 1 \text{ and } k \geq 1.$$

This gives us the inequality $k^{1/n-1} < n\left(k^{1/n} - (k-1)^{1/n}\right)$ if $n > 1$ and $k \geq 1$. In a similar way one proves that $n\left((k+1)^{1/n} - k^{1/n}\right) < k^{1/n-1}$ if $n > 1$ and $k \geq 1$. Hence for $n > 1$ and $m > 1$ it holds that

$$\begin{aligned} n\sum_{k=1}^{m}\left((k+1)^{1/n} - k^{1/n}\right) &< \sum_{k=1}^{m} k^{1/n-1} \\ &< n\sum_{k=2}^{m}\left(k^{1/n} - (k-1)^{1/n}\right) + 1, \end{aligned}$$

or equivalently,

$$n\left((m+1)^{1/n} - 1\right) < \sum_{k=1}^{m} k^{1/n-1} < n\left(m^{1/n} - 1\right) + 1.$$

The choice $n = 1983$ and $m = 2^{1983}$ then gives

$$1983 < \sum_{k=1}^{2^{1983}} k^{1/1983-1} < 1984.$$

Therefore the greatest integer less than or equal to the given sum is 1983.

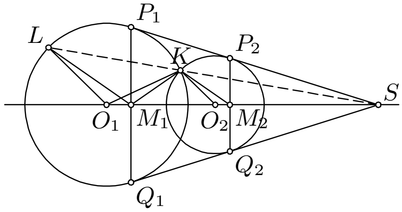
22. Decompose n into $n = st$, where the greatest common divisor of s and t is 1 and where $s > 1$ and $t > 1$. For $1 \leq k \leq n$ put $k = vs + u$, where $0 \leq v \leq t - 1$ and $1 \leq u \leq s$, and let $a_k = a_{vs+u}$ be the unique integer in the set $\{1, 2, 3, \dots, n\}$ such that $vs + ut - a_{vs+u}$ is a multiple of n. To prove that this construction gives a permutation, assume that $a_{k_1} = a_{k_2}$, where $k_i = v_i s + u_i$, $i = 1, 2$. Then $(v_1 - v_2)s + (u_1 - u_2)t$ is a multiple of $n = st$. It follows that t divides $(v_1 - v_2)$, while $|v_1 - v_2| \leq t - 1$, and that s divides $(u_1 - u_2)$, while $|u_1 - u_2| \leq s - 1$. Hence, $v_1 = v_2$, $u_1 = u_2$, and $k_1 = k_2$. We have proved that $a_1, \dots, a_n$ is a permutation of $\{1, 2, \dots, n\}$ and hence

$$\sum_{k=1}^{n} k \cos \frac{2\pi a_k}{n} = \sum_{v=0}^{t-1} \left(\sum_{u=1}^{s} (vs+u) \cos \left(\frac{2\pi v}{t} + \frac{2\pi u}{s} \right) \right).$$

Using $\sum_{u=1}^{s} \cos(2\pi u/s) = \sum_{u=1}^{s} \sin(2\pi u/s) = 0$ and the additive formulas for cosine, one finds that

$$\begin{aligned}\sum_{k=1}^{n} k \cos \frac{2\pi a_k}{n} &= \sum_{v=0}^{t-1} \left(\cos \frac{2\pi v}{t} \sum_{u=1}^{s} u \cos \frac{2\pi u}{s} - \sin \frac{2\pi v}{t} \sum_{u=1}^{s} u \sin \frac{2\pi u}{s} \right) \\ &= \left(\sum_{u=1}^{s} u \cos \frac{2\pi u}{s} \right) \left(\sum_{v=0}^{t-1} \cos \frac{2\pi v}{t} \right) \\ &\quad - \left(\sum_{u=1}^{s} u \sin \frac{2\pi u}{s} \right) \left(\sum_{v=0}^{t-1} \sin \frac{2\pi v}{t} \right) = 0.\end{aligned}$$

23. We note that $\angle O_1KO_2 = \angle M_1KM_2$ is equivalent to $\angle O_1KM_1 = \angle O_2KM_2$. Let S be the intersection point of the common tangents, and let L be the second point of intersection of SK and W_1. Since $\triangle SO_1P_1 \sim \triangle SP_1M_1$, we have $SK \cdot SL = SP_1^2 = SO_1 \cdot SM_1$ which implies that points O_1, L, K, M_1 lie on a circle. Hence $\angle O_1KM_1 = \angle O_1LM_1 = \angle O_2KM_2$.



24. See the solution of (SL91-15).

25. Suppose the contrary, that $\mathbb{R}^3 = P_1 \cup P_2 \cup P_3$ is a partition such that $a_1 \in \mathbb{R}^+$ is not realized by P_1, $a_2 \in \mathbb{R}^+$ is not realized by P_2 and $a_3 \in \mathbb{R}^+$ not realized by P_3, where w.l.o.g. $a_1 \geq a_2 \geq a_3$.
If $P_1 = \emptyset = P_2$, then $P_3 = \mathbb{R}^3$, which is impossible.
If $P_1 = \emptyset$, and $X \in P_2$, the sphere centered at X with radius a_2 is included in P_3 and $a_3 \leq a_2$ is realized, which is impossible.
If $P_1 \neq \emptyset$, let $X_1 \in P_1$. The sphere S centered in X_1, of radius a_1 is included in $P_2 \cap P_3$. Since $a_1 \geq a_3$, $S \not\subset P_3$. Let $X_2 \in P_2 \cap S$. The circle $\{Y \in S \mid d(X_2, Y) = a_2\}$ is included in P_3, but $a_2 \leq a_1$; hence it has radius $r = a_2\sqrt{1 - a_2^2/(4a_1^2)} \geq a_2\sqrt{3}/2$ and $a_3 \leq a_2 \leq a_2\sqrt{3} < 2r$; hence a_3 is realized by P_3.

4.25 Solutions to the Shortlisted Problems of IMO 1984

1. This is the same problem as (SL83-20).
2. (a) For $m = t(t-1)/2$ and $n = t(t+1)/2$ we have $4mn - m - n = (t^2-1)^2 - 1$.
   (b) Suppose that $4mn - m - n = p^2$, or equivalently, $(4m-1)(4n-1) = 4p^2+1$. The number $4m-1$ has at least one prime divisor, say q, that is of the form $4k+3$. Then $4p^2 \equiv -1 \pmod{q}$. However, by Fermat's theorem we have
   $$1 \equiv (2p)^{q-1} = \left(4p^2\right)^{\frac{q-1}{2}} \equiv (-1)^{\frac{q-1}{2}} \pmod{q},$$
   which is impossible since $(q-1)/2 = 2k+1$ is odd.
3. From the equality $n = d_6^2 + d_7^2 - 1$ we see that d_6 and d_7 are relatively prime and $d_7 \mid d_6^2 - 1 = (d_6-1)(d_6+1)$, $d_6 \mid d_7^2 - 1 = (d_7-1)(d_7+1)$. Suppose that $d_6 = ab$, $d_7 = cd$ with $1 < a < b$, $1 < c < d$. Then n has 7 divisors smaller than d_7, namely $1, a, b, c, d, ab, ac$, which is impossible. Hence, one of the two numbers d_6 and d_7 is either a prime p or the square of a prime p^2, where p is not 2. Let it be d_i, $i \in \{6, 7\}$; then $d_i \mid (d_j-1)(d_j+1)$ implies that $d_j \equiv \pm 1 \pmod{d_i}$, and consequently $(d_i^2-1)/d_j \equiv \pm 1$ as well. But either d_j or $(d_i^2-1)/d_j$ is less than d_i, and therefore equals $d_i - 1$. We thus conclude that $d_7 = d_6 + 1$. Setting $d_6 = x$, $d_7 = x+1$ we obtain that $n = x^2 + (x+1)^2 - 1 = 2x(x+1)$ is even.
   (i) Assume that one of $x, x+1$ is a prime p. The other one has at most 6 divisors and hence must be of the form $2^3, 2^4, 2^5, 2q, 2q^2, 4q$, where q is an odd prime. The numbers 2^3 and 2^4 are easily eliminated, while 2^5 yields the solution $x = 31$, $x+1 = 32$, $n = 1984$. Also, $2q$ is eliminated because $n = 4pq$ then has only 4 divisors less than x; $2q^2$ is eliminated because $n = 4pq^2$ has at least 6 divisors less than x; $4q$ is also eliminated because $n = 8pq$ has 6 divisors less than x.
   (ii) Assume that one of $x, x+1$ is p^2. The other one has at most 5 divisors (p excluded), and hence is of the form $2^3, 2^4, 2q$, where q is an odd prime. The number 2^3 yields the solution $x = 8$, $x+1 = 9$, $n = 144$, while 2^4 is easily eliminated. Also, $2q$ is eliminated because $n = 4p^2q$ has 6 divisors less than x.

   Thus there are two solutions in total: 144 and 1984.
4. Consider the convex n-gon $A_1A_2 \dots A_n$ (the indices are considered modulo n). For any diagonal A_iA_j we have $A_iA_j + A_{i+1}A_{j+1} > A_iA_{i+1} + A_jA_{j+1}$. Summing all such $n(n-3)/2$ inequalities, we obtain $2d > (n-3)p$, proving the first inequality.

   Let us now prove the second inequality. We notice that for each diagonal A_iA_{i+j} (we may assume w.l.o.g. that $j \le [n/2]$) the following relation holds:
   $$A_iA_{i+j} < A_iA_{i+1} + \cdots + A_{i+j-1}A_{i+j}. \tag{1}$$

If $n = 2k+1$, then summing the inequalities (1) for $j = 2,3,\dots,k$ and $i = 1,2,\dots,n$ yields $d < (2+3+\cdots+k)p = ([n/2]\,[n+1/2]-2)\,p/2$.
If $n = 2k$, then summing the inequalities (1) for $j = 2,3,\dots,k-1$, $i = 1,2,\dots,n$ and for $j = k$, $i = 1,2,\dots,k$ again yields $d < (2+3+\cdots+(k-1)+k/2)p = \frac{1}{2}\left([n/2]\,[n+1/2]-2\right)p$.

5. Let $f(x,y,z) = xy+yz+zx-2xyz$. The first inequality follows immediately by adding $xy \geq xyz$, $yz \geq xyz$, and $zx \geq xyz$ (in fact, a stronger inequality $xy+yz+zx-9xyz \geq 0$ holds).
Assume w.l.o.g. that z is the smallest of x,y,z. Since $xy \leq (x+y)^2/4 = (1-z)^2/4$ and $z \leq 1/2$, we have
$$\begin{aligned} xy+yz+zx-2xyz &= (x+y)z+xy(1-2z) \\ &\leq (1-z)z+\frac{(1-z)^2(1-2z)}{4} \\ &= \frac{7}{27}-\frac{(1-2z)(1-3z)^2}{108} \leq \frac{7}{27}. \end{aligned}$$

6. From the given recurrence we infer $f_{n+1}-f_n = f_n-f_{n-1}+2$. Consequently, $f_{n+1}-f_n = (f_2-f_1)+2(n-1) = c-1+2(n-1)$. Summing up for $n = 1,2,\dots,k-1$ yields the explicit formula
$$f_k = f_1+(k-1)(c-1)+(k-1)(k-2) = k^2+bk-b,$$
where $b = c-4$. Now we easily obtain $f_kf_{k+1} = k^4+2(b+1)k^3+(b^2+b+1)k^2-(b^2+b)k-b$. We are looking for an r for which the last expression equals f_r. Setting $r = k^2+pk+q$ we get by a straightforward calculation that $p = b+1$, $q = -b$, and $r = k^2+(b+1)k-b = f_k+k$. Hence $f_kf_{k+1} = f_{f_k+k}$ for all k.

7. It clearly suffices to solve the problem for the remainders modulo 4 (16 of each kind).
(a) The remainders can be placed as shown in Figure 1, so that they satisfy the conditions.

$$\begin{array}{cccccccc} 1&0&3&2&1&0&3&2\\ 2&3&0&1&2&3&0&1\\ 3&2&1&0&3&2&1&0\\ 0&1&2&3&0&1&2&3\\ 1&0&3&2&1&0&3&2\\ 2&3&0&1&2&3&0&1\\ 3&2&1&0&3&2&1&0\\ 0&1&2&3&0&1&2&3 \end{array}$$

Fig. 1

$$\begin{array}{ccc} & p & \\ q & r & s\\ & t & \end{array}$$

Fig. 2

(b) Suppose that the required numbering exists. Consider a part of the chessboard as in Figure 2. By the stated condition, all the numbers

$p+q+r+s, q+r+s+t, p+q+r+t, p+r+s+t$ give the same remainder modulo 4, and so do p, q, r, s. We deduce that all numbers on black cells of the board, except possibly the two corner cells, give the same remainder, which is impossible.

8. Suppose that the statement of the problem is false. Consider two arbitrary circles $R = (O, r)$ and $S = (O, s)$ with $0 < r < s < 1$. The point $X \in R$ with $\alpha(X) = r(s-r) < 2\pi$ satisfies that $C(X) = S$. It follows that the color of the point X does not appear on S. Consequently, the set of colors that appear on R is not the same as the set of colors that appear on S. Hence any two distinct circles with center at O and radii less than 1 have distinct sets of colors. This is a contradiction, since there are infinitely many such circles but only finitely many possible sets of colors.

9. Let us show first that the system has at most one solution. Suppose that (x, y, z) and (x', y', z') are two distinct solutions and that w.l.o.g. $x < x'$. Then the second and third equation imply that $y > y'$ and $z > z'$, but then $\sqrt{y-a} + \sqrt{z-a} > \sqrt{y'-a} + \sqrt{z'-a}$, which is a contradiction.
   We shall now prove the existence of at least one solution. Let P be an arbitrary point in the plane and K, L, M points such that $PK = \sqrt{a}$, $PL = \sqrt{b}$, $PM = \sqrt{c}$, and $\angle KPL = \angle LPM = \angle MPK = 120°$. The lines through K, L, M perpendicular respectively to PK, PL, PM form an equilateral triangle ABC, where $K \in BC$, $L \in AC$, and $M \in AB$. Since its area equals $AB^2\sqrt{3}/4 = S_{\triangle BPC} + S_{\triangle APC} + S_{\triangle APB} = AB\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)/2$, it follows that $AB = 1$. Therefore $x = PA^2$, $y = PB^2$, and $z = PC^2$ is a solution of the system (indeed, $\sqrt{y-a} + \sqrt{z-a} = \sqrt{PB^2 - PK^2} + \sqrt{PC^2 - PK^2} = BK + CK = 1$, etc.).

10. Suppose that the product of some five consecutive numbers is a square. It is easily seen that among them at least one, say n, is divisible neither by 2 nor 3. Since n is coprime to the remaining four numbers, it is itself a square of a number m of the form $6k \pm 1$. Thus $n = (6k \pm 1)^2 = 24r + 1$, where $r = k(3k \pm 1)/2$. Note that neither of the numbers $24r - 1$, $24r + 5$ is one of our five consecutive numbers because it is not a square. Hence the five numbers must be $24r, 24r + 1, \dots, 24r + 4$. However, the number $24r + 4 = (6k \pm 1)^2 + 3$ is divisible by $6r + 1$, which implies that it is a square as well. It follows that these two squares are 1 and 4, which is impossible.

11. Suppose that an integer x satisfies the equation. Then the numbers $x - a_1, x - a_2, \dots, x - a_{2n}$ are $2n$ distinct integers whose product is $1 \cdot (-1) \cdot 2 \cdot (-2) \cdots n \cdot (-n)$.
    From here it is obvious that the numbers $x - a_1, x - a_2, \dots, x - a_{2n}$ are some reordering of the numbers $-n, -n+1, \dots, -1, 1, \dots, n-1, n$. It follows that their sum is 0, and therefore $x = (a_1 + a_2 + \cdots + a_{2n})/2n$. This is

the only solution if $\{a_1, a_2, \dots, a_{2n}\} = \{x-n, \dots, x-1, x+1, \dots, x+n\}$ for some $x \in \mathbb{N}$. Otherwise there is no solution.

12. By the binomial formula we have

$$\begin{aligned}(a+b)^7 - a^7 - b^7 &= 7ab[(a^5+b^5) + 3ab(a^3+b^3) + 5a^2b^2(a+b)]\\ &= 7ab(a+b)(a^2+ab+b^2)^2.\end{aligned}$$

Thus it will be enough to find a and b such that $7 \nmid a, b$ and $7^3 \mid a^2+ab+b^2$. Such numbers must satisfy $(a+b)^2 > a^2+ab+b^2 \ge 7^3 = 343$, implying $a+b \ge 19$. Trying $a=1$ we easily find the example $(a,b) = (1,18)$.

13. Let Z be the given cylinder of radius r, altitude h, and volume $\pi r^2 h = 1$, k_1 and k_2 the circles surrounding its bases, and V the volume of an inscribed tetrahedron $ABCD$.
We claim that there is no loss of generality in assuming that A, B, C, D all lie on $k_1 \cup k_2$. Indeed, if the vertices A, B, C are fixed and D moves along a segment EF parallel to the axis of the cylinder ($E \in k_1$, $F \in k_2$), the maximum distance of D from the plane ABC (and consequently the maximum value of V) is achieved either at E or at F. Hence we shall consider only the following two cases:
   (i) $A, B \in k_1$ and $C, D \in k_2$. Let P, Q be the projections of A, B on the plane of k_2, and R, S the projections of C, D on the plane of k_1, respectively. Then V is one-third of the volume V' of the prism $ARBSCPDQ$ with bases $ARBS$ and $CPDQ$. The area of the quadrilateral $ARBS$ inscribed in k_1 does not exceed the area of the square inscribed therein, which is $2r^2$. Hence $3V = V' \le 2r^2h = 2/\pi$.
   (ii) $A, B, C \in k_1$ and $D \in k_2$. The area of the triangle ABC does not exceed the area of an equilateral triangle inscribed in k_1, which is $3\sqrt{3}r^2/4$. Consequently, $V \le \frac{\sqrt{3}}{4} r^2 h = \frac{\sqrt{3}}{4\pi} < \frac{2}{3\pi}$.

14. Let M and N be the midpoints of AB and CD, and let M', N' be their projections on CD and AB, respectively. We know that $MM' = AB/$, and hence

$$S_{ABCD} = S_{AMD} + S_{BMC} + S_{CMD} = \frac{1}{2}(S_{ABD} + S_{ABC}) + \frac{1}{4} AB \cdot CD. \quad (1)$$

The line AB is tangent to the circle with diameter CD if and only if $NN' = CD/2$, or equivalently,

$$S_{ABCD} = S_{AND} + S_{BNC} + S_{ANB} = \frac{1}{2}(S_{BCD} + S_{ACD}) + \frac{1}{4} AB \cdot CD.$$

By (1), this is further equivalent to $S_{ABC} + S_{ABD} = S_{BCD} + S_{ACD}$. But since $S_{ABC} + S_{ACD} = S_{ABD} + S_{BCD} = S_{ABCD}$, this reduces to $S_{ABC} = S_{BCD}$, i.e., to $BC \parallel AD$.

15. (a) Since rotation by 60° around A transforms the triangle CAF into $\triangle EAB$, it follows that $\measuredangle(CF, EB) = 60^\circ$. We similarly deduce that

$\measuredangle(EB, AD) = \measuredangle(AD, FC) = 60°$. Let S be the intersection point of BE and AD. Since $\measuredangle CSE = \measuredangle CAE = 60°$, it follows that $EASC$ is cyclic. Therefore $\measuredangle(AS, SC) = 60° = \measuredangle(AD, FC)$, which implies that S lies on CF as well.

(b) A rotation of $EASC$ around E by $60°$ transforms A into C and S into a point T for which $SE = ST = SC + CT = SC + SA$. Summing the equality $SE = SC + SA$ and the analogous equalities $SD = SB + SC$ and $SF = SA + SB$ yields the result.

16. From the first two conditions we can easily conclude that $a + d > b + c$ (indeed, $(d + a)^2 - (d - a)^2 = (c + b)^2 - (c - b)^2 = 4ad = 4bc$ and $d - a > c - b > 0$). Thus $k > m$.
From $d = 2^k - a$ and $c = 2^m - b$ we get $a(2^k - a) = b(2^m - b)$, or equivalently,
$$(b + a)(b - a) = 2^m(b - 2^{k-m}a). \tag{1}$$
Since $2^{k-m}a$ is even and b is odd, the highest power of 2 that divides the right-hand side of (1) is m. Hence $(b+a)(b-a)$ is divisible by 2^m but not by 2^{m+1}, which implies $b+a = 2^{m_1}p$ and $b-a = 2^{m_2}q$, where $m_1, m_2 \geq 1$, $m_1 + m_2 = m$, and p, q are odd.
Furthermore, $b = (2^{m_1}p + 2^{m_2}q)/2$ and $a = (2^{m_1}p - 2^{m_2}q)/2$ are odd, so either $m_1 = 1$ or $m_2 = 1$. Note that $m_1 = 1$ is not possible, since it would imply that $b - a = 2^{m-1}q \geq 2^{m-1}$, although $b + c = 2^m$ and $b < c$ imply that $b < 2^{m-1}$. Hence $m_2 = 1$ and $m_1 = m - 1$. Now since $a + b < b + c = 2^m$, we obtain $a + b = 2^{m-1}$ and $b - a = 2q$, where q is an odd integer. Substituting these into (1) and dividing both sides by 2^m we get
$$q = 2^{m-2} + q - 2^{k-m}a \quad \Longrightarrow \quad 2^{k-m}a = 2^{m-2}.$$
Since a is odd and $k > m$, it follows that $a = 1$.

Remark. Now it is not difficult to prove that all fourtuplets (a, b, c, d) that satisfy the given conditions are of the form $(1, 2^{m-1}-1, 2^{m-1}+1, 2^{2m-2}-1)$, where $m \in \mathbb{N}$, $m \geq 3$.

17. For any $m = 0, 1, \dots, n - 1$, we shall find the number of permutations $(x_1, x_2, \dots, x_n)$ with exactly k discordant pairs such that $x_n = n - m$. This x_n is a member of exactly m discordant pairs, and hence the permutation $(x_1, \dots, x_{n-1}$ of the set $\{1, 2, \dots, n\} \setminus \{m\}$ must have exactly $k - m$ discordant pairs: there are $d(n - 1, k - m)$ such permutations. Therefore
$$\begin{aligned} d(n, k) &= d(n - 1, k) + d(n - 1, k - 1) \cdots + d(n - 1, k - n + 1) \\ &= d(n - 1, k) + d(n, k - 1) \end{aligned}$$
(note that $d(n, k)$ is 0 if $k < 0$ or $k > \binom{n}{2}$).
We now proceed to calculate $d(n, 2)$ and $d(n, 3)$. Trivially, $d(n, 0) = 1$. It follows that $d(n, 1) = d(n - 1, 1) + d(n, 0) = d(n - 1, 1) + 1$, which yields $d(n, 1) = d(1, 1) + n - 1 = n - 1$.

Further, $d(n,2) = d(n-1,2) + d(n,1) = d(n-1,2) + n - 1 = d(2,2) + 2 + 3 + \cdots + n - 1 = (n^2 - n - 2)/2$.
Finally, using the known formula $1^2 + 2^2 + \cdots + k^2 = k(k+1)(2k+1)/6$, we have $d(n,3) = d(n-1,3) + d(n,2) = d(n-1,3) + (n^2 - n - 2)/2 = d(2,3) + \sum_{i=3}^{n}(n^2 - n - 2)/2 = (n^3 - 7n + 6)/6$.

18. Suppose that circles $k_1(O_1, r_1)$, $k_2(O_2, r_2)$, and $k_3(O_3, r_3)$ touch the edges of the angles $\angle BAC$, $\angle ABC$, and $\angle ACB$, respectively. Denote also by O and r the center and radius of the incircle. Let P be the point of tangency of the incircle with AB and let F be the foot of the perpendicular from O_1 to OP. From $\triangle O_1FO$ we obtain $\cot(\alpha/2) = 2\sqrt{rr_1}/(r - r_1)$ and analogously $\cot(\beta/2) = 2\sqrt{rr_2}/(r - r_2)$, $\cot(\gamma/2) = 2\sqrt{rr_3}/(r - r_3)$. We will now use a well-known trigonometric identity for the angles of a triangle:
$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2} \cdot \cot\frac{\beta}{2} \cdot \cot\frac{\gamma}{2}\,.$$
(This identity follows from $\tan(\gamma/2) = \cot(\alpha/2 + \beta/2)$ and the formula for the cotangent of a sum.)
Plugging in the obtained cotangents, we get
$$\begin{aligned}\frac{2\sqrt{rr_1}}{r - r_1} + \frac{2\sqrt{rr_2}}{r - r_2} + \frac{2\sqrt{rr_3}}{r - r_3} &= \frac{2\sqrt{rr_1}}{r - r_1} \cdot \frac{2\sqrt{rr_2}}{r - r_2} \cdot \frac{2\sqrt{rr_3}}{r - r_3} \Rightarrow \\ &\sqrt{r_1}(r - r_2)(r - r_3) + \sqrt{r_2}(r - r_1)(r - r_3) \\ &+ \sqrt{r_3}(r - r_1)(r - r_2) = 4r\sqrt{r_1r_2r_3}.\end{aligned}$$
For $r_1 = 1$, $r_2 = 4$, and $r_3 = 9$ we get
$$(r-4)(r-9) + 2(r-1)(r-9) + 3(r-1)(r-4) = 24r \Rightarrow 6(r-1)(r-11) = 0.$$
Clearly, $r = 11$ is the only viable value for r.

19. First, we shall prove that the numbers in the nth row are exactly the numbers
$$\frac{1}{n\binom{n-1}{0}}, \frac{1}{n\binom{n-1}{1}}, \frac{1}{n\binom{n-1}{2}}, \ldots, \frac{1}{n\binom{n-1}{n-1}}. \tag{1}$$
The proof of this fact can be done by induction. For small n, the statement can be easily verified. Assuming that the statement is true for some n, we have that the kth element in the $(n+1)$st row is, as is directly verified,
$$\frac{1}{n\binom{n-1}{k-1}} - \frac{1}{(n+1)\binom{n}{k-1}} = \frac{1}{(n+1)\binom{n}{k}}.$$
Thus (1) is proved. Now the geometric mean of the elements of the nth row becomes:
$$\frac{1}{n\sqrt[n]{\binom{n-1}{0}\cdot\binom{n-1}{1}\cdots\binom{n-1}{n-1}}} \ge \frac{1}{n\left(\frac{\binom{n-1}{0}+\binom{n-1}{1}+\cdots+\binom{n-1}{n-1}}{n}\right)} = \frac{1}{2^{n-1}}.$$
The desired result follows directly from substituting $n = 1984$.

20. Define the set $S = \mathbb{R}^+ \setminus \{1\}$. The given inequality is equivalent to $\ln b/\ln a < \ln(b+1)/\ln(a+1)$.
If $b = 1$, it is obvious that each $a \in S$ satisfies this inequality. Suppose now that b is also in S.
Let us define on S a function $f(x) = \ln(x+1)/\ln x$. Since $\ln(x+1) > \ln x$ and $1/x > 1/x+1 > 0$, we have

$$f'(x) = \frac{\frac{\ln x}{x+1} - \frac{\ln(x+1)}{x}}{\ln^2 x} < 0 \qquad \text{for all } x.$$

Hence f is always decreasing. We also note that $f(x) < 0$ for $x < 1$ and that $f(x) > 0$ for $x > 1$ (at $x = 1$ there is a discontinuity).
Let us assume $b > 1$. From $\ln b/\ln a < \ln(b+1)/\ln(a+1)$ we get $f(b) > f(a)$. This holds for $b > a$ or for $a < 1$.
Now let us assume $b < 1$. This time we get $f(b) < f(a)$. This holds for $a < b$ or for $a > 1$.
Hence all the solutions to $\log_a b < \log_{a+1}(b+1)$ are $\{b = 1, a \in S\}$, $\{a > b > 1\}$, $\{b > 1 > a\}$, $\{a < b < 1\}$, and $\{b < 1 < a\}$.

4.26 Solutions to the Shortlisted Problems of IMO 1985

1. Since there are 9 primes ($p_1 = 2 < p_2 = 3 < \cdots < p_9 = 23$) less than 26, each number $x_j \in M$ is of the form $\prod_{i=1}^{9} p_i^{a_{ij}}$, where $0 \le a_{ij}$. Now, $x_j x_k$ is a square if $a_{ij} + a_{ik} \equiv 0 \pmod 2$ for $i = 1, \dots, 9$. Since the number of distinct ninetuples modulo 2 is 2^9, any subset of M with at least 513 elements contains two elements with square product. Starting from M and eliminating such pairs, one obtains $(1985 - 513)/2 = 736 > 513$ distinct two-element subsets of M each having a square as the product of elements. Reasoning as above, we find at least one (in fact many) pair of such squares whose product is a fourth power.

2. The polyhedron has $3 \cdot 12/2 = 18$ edges, and by Euler's formula, 8 vertices. Let v_1 and v_2 be the numbers of vertices at which respectively 3 and 6 edges meet. Then $v_1 + v_2 = 8$ and $3v_1 + 6v_2 = 2 \cdot 18$, implying that $v_1 = 4$. Let A, B, C, D be the vertices at which three edges meet. Since the dihedral angles are equal, all the edges meeting at A, say AE, AF, AG, must have equal length, say x. (If $x = AE = AF \neq AG = y$, and AEF, AFG, and AGE are isosceles, $\angle EAF \neq \angle FAG$, in contradiction to the equality of the dihedral angles.) It is easy to see that at E, F, and G six edges meet. One proceeds to conclude that if H is the fourth vertex of this kind, $EFGH$ must be a regular tetrahedron of edge length y, and the other vertices A, B, C, and D are tops of isosceles pyramids based on EFG, EFH, FGH, and GEH. Let the plane through A, B, C meet EF, HF, and GF, at E', H', and G'. Then $AE'BH'CG'$ is a regular hexagon, and since $x = FA = FE'$, we have $E'G' = x$ and $AE' = x/\sqrt{3}$. From the isosceles triangles AEF and FAE' we obtain finally, with $\measuredangle EFA = \alpha$, $\frac{y}{2x} = \cos\alpha = 1 - 2\sin^2(\alpha/2)$, $x/(2x\sqrt{3}) = \sin(\alpha/2)$, and $y/x = 5/3$.

3. We shall write $P \equiv Q$ for two polynomials P and Q if $P(x) - Q(x)$ has even coefficients.
We observe that $(1+x)^{2^m} \equiv 1 + x^{2^m}$ for every $m \in \mathbb{N}$. Consequently, for every polynomial p with degree less than $k = 2^m$, $w(p \cdot q_k) = 2w(p)$.
Now we prove the inequality from the problem by induction on i_n. If $i_n \le 1$, the inequality is trivial. Assume it is true for any sequence with $i_1 < \cdots < i_n < 2^m$ $(m \ge 1)$, and let there be given a sequence with $k = 2^m \le i_n < 2^{m+1}$. Consider two cases.
(i) $i_1 \ge k$. Then $w(q_{i_1} + \cdots + q_{i_n}) = 2w(q_{i_1-k} + \cdots + q_{i_n-k}) \ge 2w(q_{i_1-k}) = w(q_{i_1})$.
(ii) $i_1 < k$. Then the polynomial $p = q_{i_1} + \cdots + q_{i_n}$ has the form

$$p = \sum_{i=0}^{k-1} a_i x^i + (1+x)^k \sum_{i=0}^{k-1} b_i x^i \equiv \sum_{i=0}^{k-1} \left[(a_i + b_i)x^i + b_i x^{i+k}\right].$$

Whenever some a_i is odd, either $a_i + b_i$ or b_i in the above sum will be odd. It follows that $w(p) \ge w(q_{i_1})$, as claimed.

The proof is complete.

4. Let $\langle x\rangle$ denote the residue of an integer x modulo n. Also, we write $a \sim b$ if a and b receive the same color. We claim that all the numbers $\langle ij\rangle$, $i = 1, 2, \dots, n-1$, are of the same color. Since j and n are coprime, this will imply the desired result.
We use induction on i. For $i = 1$ the statement is trivial. Assume now that the statement is true for $i = 1, \dots, k-1$. For $1 < k < n$ we have $\langle kj\rangle \neq j$. If $\langle kj\rangle > j$, then by (ii), $\langle kj\rangle \sim \langle kj\rangle - j = \langle (k-1)j\rangle$. If otherwise $\langle kj\rangle < j$, then by (ii) and (i), $\langle kj\rangle \sim j - \langle kj\rangle \sim n - j + \langle kj\rangle = \langle (k-1)j\rangle$. This completes the induction.

5. Let w.l.o.g. circle C have unit radius. For each $m \in \mathbb{R}$, the locus of points M such that $f(M) = m$ is the circle C_m with radius $r_m = m/(m+1)$, that is tangent to C at A. Let O_m be the center of C_m. We have to show that if $M \in C_m$ and $N \in C_n$, where $m, n > 0$, then the midpoint P of MN lies inside the circle $C_{(m+n)/2}$. This is trivial if $m = n$, so let $m \neq n$. For fixed M, P is in the image C'_n of C_n under the homothety with center M and coefficient $1/2$. The center of the circle C'_n is at the midpoint of O_nM. If we let both M and N vary, P will be on the union of circles with radius $r_n/2$ and centers in the image of C_m under the homothety with center O_n and coefficient $1/2$. Hence P is not outside the circle centered at the midpoint O_mO_n and with radius $(r_m + r_n)/2$. It remains to show that $r_{(m+n)/2} > (r_m + r_n)/2$. But this inequality is easily reduced to $(m-n)^2 > 0$, which is true.

6. Let us set

$$x_{n,i} = \sqrt[i]{i + \sqrt[i+1]{i+1+\cdots+\sqrt[n]{n}}},$$
$$y_{n,i} = x_{n+1,i}^{i-1} + x_{n+1,i}^{i-2}x_{n,i} + \cdots + x_{n,i}^{i-1}.$$

In particular, $x_{n,2} = x_n$ and $x_{n,i} = 0$ for $i > n$. We observe that for $n > i > 2$,

$$x_{n+1,i} - x_{n,i} = \frac{x_{n+1,i}^i - x_{n,i}^i}{y_{n,i}} = \frac{x_{n+1,i+1} - x_{n,i+1}}{y_{n,i}}.$$

Since $y_{n,i} > ix_{n,i}^{i-1} \geq i^{1+(i-1)/i} \geq i^{3/2}$ and $x_{n+1,n+1} - x_{n,n+1} = \sqrt[n+1]{n+1}$, simple induction gives

$$x_{n+1} - x_n \leq \frac{\sqrt[n+1]{n+1}}{(n!)^{3/2}} < \frac{1}{n!} \qquad \text{for } n > 2.$$

The inequality for $n = 2$ is directly verified.

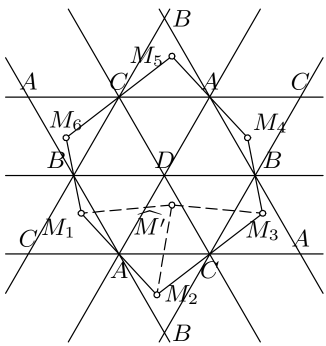
7. Let $k_i \geq 0$ be the largest integer such that $p^{k_i} \mid x_i$, $i = 1, \dots, n$, and $y_i = x_i/p^{k_i}$. We may assume that $k = k_1 + \cdots + k_n$. All the y_i must be

distinct. Indeed, if $y_i = y_j$ and $k_i > k_j$, then $x_i \geq px_j \geq 2x_i \geq 2x_1$, which is impossible. Thus $y_1 y_2 \dots y_n = P/p^k \geq n!$.
If equality holds, we must have $y_i = 1$, $y_j = 2$ and $y_k = 3$ for some i, j, k. Thus $p \geq 5$, which implies that either $y_i/y_j \leq 1/2$ or $y_i/y_j \geq 5/2$, which is impossible. Hence the inequality is strict.

8. Among ten consecutive integers that divide n, there must exist numbers divisible by 2^3, 3^2, 5, and 7. Thus the desired number has the form $n = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}7^{\alpha_4}11^{\alpha_5}\cdots$, where $\alpha_1 \geq 3$, $\alpha_2 \geq 2$, $\alpha_3 \geq 1$, $\alpha_4 \geq 1$. Since n has $(\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\cdots$ distinct factors, and $(\alpha_1+1)(\alpha_2+1)(\alpha_3+1)(\alpha_4+1) \geq 48$, we must have $(\alpha_5+1)\cdots \leq 3$. Hence at most one α_j, $j > 4$, is positive, and in the minimal n this must be α_5. Checking through the possible combinations satisfying $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_5+1) = 144$ one finds that the minimal n is $2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 110880$.

9. Let $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d}$ denote the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}$ respectively. Then $|\overrightarrow{a}| = |\overrightarrow{b}| = |\overrightarrow{c}| = |\overrightarrow{d}| = 1$. The centroids of the faces are $(\overrightarrow{b}+\overrightarrow{c}+\overrightarrow{d})/3$, $(\overrightarrow{a}+\overrightarrow{c}+\overrightarrow{d})/3$, etc., and each of these is at distance $1/3$ from $P = (\overrightarrow{a}+\overrightarrow{b}+\overrightarrow{c}+\overrightarrow{d})/3$; hence the required radius is $1/3$. To compute $|P|$ as a function of the edges of $ABCD$, observe that $AB^2 = (\overrightarrow{b}-\overrightarrow{a})^2 = 2 - 2\overrightarrow{a}\cdot\overrightarrow{b}$ etc. Now
$$P^2 = \frac{|\overrightarrow{a}+\overrightarrow{b}+\overrightarrow{c}+\overrightarrow{d}|^2}{9} = \frac{16 - 2(AB^2 + BC^2 + AC^2 + AD^2 + BD^2 + CD^2)}{9}.$$

10. If M is at a vertex of the regular tetrahedron $ABCD$ ($AB = 1$), then one can take M' at the center of the opposite face of the tetrahedron.
Let M be on the face (ABC) of the tetrahedron, excluding the vertices. Consider a continuous mapping f of $\mathbb{C}$ onto the surface S of $ABCD$ that maps $m + ne^{\imath\pi/3}$ for $m, n \in \mathbb{Z}$ onto A, B, C, D if $(m,n) \equiv (1,1), (1,0), (0,1), (0,0) \pmod 2$ respectively, and maps each unit equilateral triangle with vertices of the form $m + ne^{\imath\pi/3}$ isometrically onto the corresponding face of $ABCD$. The point M then has one preimage M_j, $j = 1, 2, \dots, 6$, in each of the six preimages of $\triangle ABC$ having two vertices on the unit circle. The M_j's form a convex centrally symmetric (possibly degenerate) hexagon. Of the triangles formed by two adjacent sides of this hexagon consider the one, say $M_1M_2M_3$, with the smallest radius of circumcircle and denote by $\widehat{M'}$



its circumcenter. Then we can choose $M' = f(\widehat{M'})$. Indeed, the images of the segments $M_1\widehat{M'}$, $M_2\widehat{M'}$, $M_3\widehat{M'}$ are three different shortest paths on S from M to M'.

11. Let $-x_1, \dots, -x_6$ be the roots of the polynomial. Let $s_{k,i}$ $(k \le i \le 6)$ denote the sum of all products of k of the numbers $x_1, \dots, x_i$. By Vieta's formula we have $a_k = s_{k,6}$ for $k = 1, \dots, 6$. Since $s_{k,i} = s_{k-1,i-1}x_i + s_{k,i-1}$, one can compute the a_k by the following scheme (the horizontal and vertical arrows denote multiplications and additions respectively):

$$\begin{array}{ccccccccccc}
x_1 & \to & s_{2,2} & \to & s_{3,3} & \to & s_{4,4} & \to & s_{5,5} & \to & a_6 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
s_{1,2} & \to & s_{2,3} & \to & s_{3,4} & \to & s_{4,5} & \to & a_5 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
s_{1,3} & \to & s_{2,4} & \to & s_{3,5} & \to & a_4 & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
s_{1,4} & \to & s_{2,5} & \to & a_3 & & & & & & \\
\downarrow & & \downarrow & & & & & & & & \\
s_{1,5} & \to & a_2 & & & & & & & & \\
\downarrow & & & & & & & & & & \\
a_1 & & & & & & & & & &
\end{array}$$

12. We shall prove by induction on m that $P_m(x, y, z)$ is symmetric and that

$$(x + y)P_m(x, z, y + 1) - (x + z)P_m(x, y, z + 1) = (y - z)P_m(x, y, z) \quad (1)$$

holds for all x, y, z. This is trivial for $m = 0$. Assume now that it holds for $m = n - 1$.

Since obviously $P_n(x, y, z) = P_n(y, x, z)$, the symmetry of P_n will follow if we prove that $P_n(x, y, z) = P_n(x, z, y)$. Using (1) we have $P_n(x, z, y) - P_n(x, y, z) = (y+z)[(x+y)P_{n-1}(x, z, y+1) - (x+z)P_{n-1}(x, y, z+1)] - (y^2 - z^2)P_{n-1}(x, y, z) = (y+z)(y-z)P_{n-1}(x, y, z) - (y^2 - z^2)P_{n-1}(x, y, z) = 0$.

It remains to prove (1) for $m = n$. Using the already established symmetry we have

$$\begin{aligned}
&(x + y)P_n(x, z, y + 1) - (x + z)P_n(x, y, z + 1) \\
&= (x + y)P_n(y + 1, z, x) - (x + z)P_n(z + 1, y, x) \\
&= (x + y)[(y + x + 1)(z + x)P_{n-1}(y + 1, z, x + 1) - x^2P_{n-1}(y + 1, z, x)] \\
&\quad -(x + z)[(z + x + 1)(y + x)P_{n-1}(z + 1, y, x + 1) - x^2P_{n-1}(z + 1, y, x)] \\
&= (x + y)(x + z)(y - z)P_{n-1}(x + 1, y, z) - x^2(y - z)P_{n-1}(x, y, z) \\
&= (y - z)P_n(z, y, x) = (y - z)P_n(x, y, z),
\end{aligned}$$

as claimed.

13. If m and n are relatively prime, there exist positive integers p, q such that $pm = qn + 1$. Thus by putting m balls in some boxes p times we can

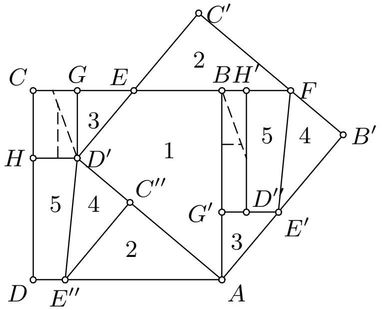
achieve that one box receives $q+1$ balls while all others receive q balls. Repeating this process sufficiently many times, we can obtain an equal distribution of the balls.

Now assume $\gcd(m,n) > 1$. If initially there is only one ball in the boxes, then after k operations the number of balls will be $1+km$, which is never divisible by n. Hence the task cannot be done.

14. It suffices to prove the existence of a good point in the case of exactly 661 -1's. We prove by induction on k that in any arrangement with $3k+2$ points k of which are -1's a good point exists. For $k=1$ this is clear by inspection. Assume that the assertion holds for all arrangements of $3n+2$ points and consider an arrangement of $3(n+1)+2$ points. Now there exists a sequence of consecutive -1's surrounded by two $+1$'s. There is a point P which is good for the arrangement obtained by removing the two $+1$'s bordering the sequence of -1's and one of these -1's. Since P is out of this sequence, clearly the removal either leaves a partial sum as it was or diminishes it by 1, so P is good for the original arrangement.

    Second solution. Denote the number on an arbitrary point by a_1, and the numbers on successive points going in the positive direction by $a_2, a_3, \dots$ (in particular, $a_{k+1985} = a_k$). We define the partial sums $s_0 = 0$, $s_n = a_1 + a_2 + \dots + a_n$ for all positive integers n; then $s_{k+1985} = s_k + s_{1985}$ and $s_{1985} \geq 663$. Since $s_{1985m} \geq 663m$ and $3 \cdot 663m > 1985(m+2)+1$ for large m, not all values $0, 1, 2, \dots 663m$ can appear thrice among the $1985(m+2)+1$ sums $s_{-1985}, s_{-1984}, \dots, s_{1985(m+1)}$ (and none of them appears out of this set). Thus there is an integral value $s > 0$ that appears at most twice as a partial sum, say $s_k = s_l = s$, $k < l$. Then either a_k or a_l is a good point. Actually, $s_i > s$ must hold for all $i > l$, and $s_i < s$ for all $i < k$ (otherwise, the sum s would appear more than twice). Also, for the same reason there cannot exist indices p, q between k and l such that $s_p > s$ and $s_q < s$; i.e., for $k < p < l$, s_p's are either all greater than or equal to s, or smaller than or equal to s. In the former case a_k is good, while in the latter a_l is good.

15. There is no loss of generality if we assume $K = ABCD$, $K' = AB'C'D'$, and that K' is obtained from K bya clockwise rotation around A by ϕ, $0 \leq \phi \leq \pi/4$. Let $C'D'$, $B'C'$, and the parallel to AB through D' meet the line BC at E, F, and G respectively. Let us now choose points $E' \in AB'$, $G' \in AB$, $C'' \in AD'$, and $E'' \in AD$ such that



the triangles $AE'G'$ and $AC''E''$ are translates of the triangles $D'EG$ and $FC'E$ respectively. Since $AE' = D'E$ and $AC'' = FC'$, we have $C''E'' = C'E = B'E'$ and $C''D' = B'F$, which imply that $\triangle E''C''D'$ is a translate of $\triangle E'B'F$, and consequently $E''D' = E'F$ and $E''D' \parallel E'F$. It follows that there exist points $H \in CD$, $H' \in BF$, and $D'' \in E'G'$ such that $E''D'HD$ is a translate of $E'FH'D''$. The remaining parts of K and K' are the rectangles $D'GCH$ and $D''H'BG'$ of equal area.

We shall now show that two rectangles with parallel sides and equal areas can be decomposed into translation invariant parts. Let the sides of the rectangles $XYZT$ and $X'Y'Z'T'$ ($XY \parallel X'Y'$) satisfy $X'Y' < XY$, $Y'Z' > YZ$, and $X'Y' \cdot Y'Z' = XY \cdot YZ$. Suppose that $2X'Y' > XY$ (otherwise, we may cut off congruent rectangles from both the original ones until we reduce them to the case of $2X'Y' > XY$). Let $U \in XY$ and $V \in ZT$ be points such that $YU = TV = X'Y'$ and $W \in XV$ be a point such that $UW \parallel XT$. Then translating $\triangle XUW$ to a triangle VZR and $\triangle XVT$ to a triangle WRS results in a rectangle $UYRS$ congruent to $X'Y'Z'T'$.

Thus we have partitioned K and K' into translation-invariant parts. Although not all the parts are triangles, we may simply triangulate them.

16. Let the three circles be $\alpha(A,a)$, $\beta(B,b)$, and $\gamma(C,c)$, and assume $c \le a, b$. We denote by $\mathcal{R}_{X,\varphi}$ the rotation around X through an angle φ. Let PQR be an equilateral triangle, say of positive orientation (the case of negatively oriented $\triangle PQR$ is analogous), with $P \in \alpha$, $Q \in \beta$, and $R \in \gamma$. Then $Q = \mathcal{R}_{P,-60^\circ}(R) \in \mathcal{R}_{P,-60^\circ}(\gamma) \cap \beta$.
Since the center of $\mathcal{R}_{P,-60^\circ}(\gamma)$ is $\mathcal{R}_{P,-60^\circ}(C) = \mathcal{R}_{C,60^\circ}(P)$ and it lies on $\mathcal{R}_{C,60^\circ}(\alpha)$, the union of circles $\mathcal{R}_{P,-60^\circ}(\gamma)$ as P varies on α is the annulus $\mathcal{U}$ with center $A' = \mathcal{R}_{C,60^\circ}(A)$ and radii $a-c$ and $a+c$. Hence there is a solution if and only if $\mathcal{U} \cap \beta$ is nonempty.

17. The statement of the problem is equivalent to the statement that there is one and only one a such that $1 - 1/n < f_n(a) < 1$ for all n. We note that each f_n is a polynomial with positive coefficients, and therefore increasing and convex in $\mathbb{R}^+$.
Define x_n and y_n by $f_n(x_n) = 1 - 1/n$ and $f_n(y_n) = 1$. Since
$$f_{n+1}(x_n) = \left(1 - \frac{1}{n}\right)^2 + \left(1 - \frac{1}{n}\right)\frac{1}{n} = 1 - \frac{1}{n}$$
and $f_{n+1}(y_n) = 1 + 1/n$, it follows that $x_n < x_{n+1} < y_{n+1} < y_n$. Moreover, the convexity of f_n together with the fact that $f_n(x) > x$ for all $x > 0$ implies that $y_n - x_n < f_n(y_n) - f_n(x_n) = 1/n$. Therefore the sequences have a common limit a, which is the only number lying between x_n and y_n for all n. By the definition of x_n and y_n, the statement immediately follows.

18. Set $y_i = \frac{x_i^2}{x_{i+1}x_{i+2}}$, where $x_{n+i} = x_i$. Then $\prod_{i=1}^n y_i = 1$ and the inequality to be proved becomes $\sum_{i=1}^n \frac{y_i}{1+y_i} \le n-1$, or equivalently

$$\sum_{i=1}^{n} \frac{1}{1+y_i} \geq 1.$$

We prove this inequality by induction on n.
Since $\frac{1}{1+y} + \frac{1}{1+y^{-1}} = 1$, the inequality is true for $n = 2$. Assume that it is true for $n-1$, and let there be given $y_1, \dots, y_n > 0$ with $\prod_{i=1}^{n} y_i = 1$. Then $\frac{1}{1+y_{n-1}} + \frac{1}{1+y_n} > \frac{1}{1+y_{n-1}y_n}$, which is equivalent to $1 + y_n y_{n-1}(1 + y_n + y_{n-1}) > 0$. Hence by the inductive hypothesis

$$\sum_{i=1}^{n} \frac{1}{1+y_i} \geq \sum_{i=1}^{n-2} \frac{1}{1+y_i} + \frac{1}{1+y_{n-1}y_n} \geq 1.$$

Remark. The constant $n-1$ is best possible (take for example $x_i = a^i$ with a arbitrarily large).

19. Suppose that for some $n > 6$ there is a regular n-gon with vertices having integer coordinates, and that $A_1A_2\dots A_n$ is the smallest such n-gon, of side length a. If O is the origin and B_i the point such that $\overrightarrow{OB_i} = \overrightarrow{A_{i-1}A_i}$, $i = 1, 2, \dots, n$ (where $A_0 = A_n$), then B_i has integer coordinates and $B_1B_2\dots B_n$ is a regular polygon of side length $2a\sin(\pi/n) < a$, which is impossible.
It remains to analyze the cases $n \leq 6$. If $\mathcal{P}$ is a regular n-gon with $n = 3, 5, 6$, then its center C has rational coordinates. We may suppose that C also has integer coordinates and then rotate $\mathcal{P}$ around C thrice through $90°$, thus obtaining a regular 12-gon or 20-gon, which is impossible. Hence we must have $n = 4$ which is indeed a solution.

20. Let O be the center of the circle touching the three sides of $BCDE$ and let $F \in (ED)$ be the point such that $EF = EB$. Then $\angle EFB = 90° - \angle E/2 = \angle C/2 = \angle OCB$, which implies that B, C, F, O lie on a circle. It follows that $\angle DFC = \angle OBC = \angle B/2 = 90° - \angle D/2$ and consequently $\angle DCF = \angle DFC$. Hence $ED = EF + FD = EB + CD$.

    Second solution. Let r be the radius of the small circle and let M, N be the points of tangency of the circle with BE and CD respectively. Then $EM = r\cot E$, $DN = r\cot D$, $MB = r\cot(\angle B/2) = r\tan(\angle D/2)$, $NC = r\tan(\angle E/2)$, and $ED = EO + OD = r/\sin D + r/\sin E$. The statement follows from the identity $\cot x + \tan(x/2) = 1/\sin x$.

21. Let B_1 and C_1 be the points on the rays AC and AB respectively such that $XB_1 = XC = XB = XC_1$. Then $\angle XB_1C = \angle XCB_1 = \angle ABC$ and $\angle XC_1B = \angle XBC_1 = \angle ACB$, which imply that B_1, X, C_1 are collinear and $\triangle AB_1C_1 \sim \triangle ABC$. Moreover, X is the midpoint of B_1C_1 because $XB_1 = XC = XB = XC_1$, from which we conclude that $\triangle AXC_1 \sim \triangle AMC$. Therefore $\angle BAX = \angle CAM$ and

$$\frac{AM}{AX} = \frac{CM}{XC_1} = \frac{CM}{XC} = \cos\alpha.$$

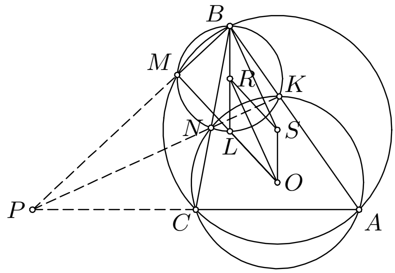
22. Assume that $\triangle ABC$ is acute (the case of an obtuse $\triangle ABC$ is similar). Let S and R be the centers of the circumcircles of $\triangle ABC$ and $\triangle KBN$, respectively. Since $\angle BNK = \angle BAC$, the triangles BNK and BAC are similar. Now we have $\angle CBR = \angle ABS = 90^\circ - \angle ACB$, which gives us $BR \perp AC$ and consequently $BR \parallel OS$. Similarly $BS \perp KN$ implies that $BS \parallel OR$. Hence $BROS$ is a parallelogram.

    Let L be the point symmetric to B with respect to R. Then $RLOS$ is also a parallelogram, and since $SR \perp BM$, we obtain $OL \perp BM$. However, we also have $LM \perp BM$, from which we conclude that O, L, M are collinear and $OM \perp BM$.

    Second solution. The lines BM, NK, and CA are the radical axes of pairs of the three circles, and hence they intersect at a single point P. Also, the quadrilateral $MNCP$ is cyclic. Let $OA = OC = OK = ON = r$. We then have

    $BM \cdot BP = BN \cdot BC = OB^2 - r^2,$
    $PM \cdot PB = PN \cdot PK = OP^2 - r^2.$

    It follows that $OB^2 - OP^2 = BP(BM - PM) = BM^2 - PM^2$, which implies that $OM \perp MB$.



4.27 Solutions to the Shortlisted Problems of IMO 1986

1. If $w > 2$, then setting in (i) $x = w - 2$, $y = 2$, we get $f(w) = f((w-2)f(w))f(2) = 0$. Thus

$$f(x) = 0 \quad \text{if and only if} \quad x \geq 2.$$

Now let $0 \leq y < 2$ and $x \geq 0$. The LHS in (i) is zero if and only if $xf(y) \geq 2$, while the RHS is zero if and only if $x + y \geq 2$. It follows that $x \geq 2/f(y)$ if and only if $x \geq 2 - y$. Therefore

$$f(y) = \begin{cases} \frac{2}{2-y} & \text{for } 0 \leq y < 2; \\ 0 & \text{for } y \geq 2. \end{cases}$$

The confirmation that f satisfies the given conditions is straightforward.

2. No. If a were rational, its decimal expansion would be periodic from some point. Let p be the number of decimals in the period. Since $f(10^{2p})$ has $2np$ zeros, it contains a full periodic part; hence the period would consist only of zeros, which is impossible.

3. Let E be the point where the boy turned westward, reaching the shore at D. Let the ray DE cut AC at F and the shore again at G. Then $EF = AE = x$ (because AEF is an equilateral triangle) and $FG = DE = y$. From $AE \cdot EB = DE \cdot EG$ we obtain $x(86 - x) = y(x + y)$. If x is odd, then $x(86 - x)$ is odd, while $y(x + y)$ is even. Hence x is even, and so y must also be even. Let $y = 2y_1$. The above equation can be rewritten as

$$(x + y_1 - 43)^2 + (2y_1)^2 = (43 - y_1)^2.$$

Since $y_1 < 43$, we have $(2y_1, 43 - y_1) = 1$, and thus $(|x + y_1 - 43|, 2y_1, 43 - y_1)$ is a primitive Pythagorean triple. Consequently there exist integers $a > b > 0$ such that $y_1 = ab$ and $43 - y_1 = a^2 + b^2$. We obtain that $a^2 + b^2 + ab = 43$, which has the unique solution $a = 6$, $b = 1$. Hence $y = 12$ and $x = 2$ or $x = 72$.

Remark. The Diophantine equation $x(86-x) = y(x+y)$ can be also solved directly. Namely, we have that $x(344 - 3x) = (2y + x)^2$ is a square, and since x is even, we have $(x, 344 - 3x) = 2$ or 4. Consequently x, $344 - 3x$ are either both squares or both two times squares. The rest is easy.

4. Let $x = p^\alpha x', y = p^\beta y', z = p^\gamma z'$ with $p \nmid x'y'z'$ and $\alpha \geq \beta \geq \gamma$. From the given equation it follows that $p^n(x + y) = z(xy - p^n)$ and consequently $z' \mid x + y$. Since also $p^\gamma \mid x + y$, we have $z \mid x + y$, i.e., $x + y = qz$. The given equation together with the last condition gives us

$$xy = p^n(q + 1) \qquad \text{and} \qquad x + y = qz. \tag{1}$$

Conversely, every solution of (1) gives a solution of the given equation.

For $q = 1$ and $q = 2$ we obtain the following classes of $n + 1$ solutions each:

$$\begin{aligned} q = 1 : (x, y, z) &= (2p^i, p^{n-i}, 2p^i + p^{n-i}) && \text{for } i = 0, 1, 2, \dots, n; \\ q = 2 : (x, y, z) &= \left(3p^j, p^{n-j}, \tfrac{3p^j + p^{n-j}}{2}\right) && \text{for } j = 0, 1, 2, \dots, n. \end{aligned}$$

For $n = 2k$ these two classes have a common solution $(2p^k, p^k, 3p^k)$; otherwise, all these solutions are distinct. One further solution is given by $(x, y, z) = \big(1, p^n(p^n + 3)/2, p^2 + 2\big)$, not included in the above classes for $p > 3$. Thus we have found $2(n + 1)$ solutions.
Another type of solution is obtained if we put $q = p^k + p^{n-k}$. This yields the solutions

$$(x, y, z) = (p^k, p^n + p^{n-k} + p^{2n-2k}, p^{n-k} + 1) \qquad \text{for } k = 0, 1, \dots, n.$$

For $k < n$ these are indeed new solutions. So far, we have found $3(n+1)-1$ or $3(n + 1)$ solutions. One more solution is given by $(x, y, z) = (p, p^n + p^{n-1}, p^{n-1} + p^{n-2} + 1)$.

5. Suppose that for every $a, b \in \{2, 5, 13, d\}$, $a \neq b$, the number $ab - 1$ is a perfect square. In particular, for some integers x, y, z we have

$$2d - 1 = x^2, \qquad 5d - 1 = y^2, \qquad 13d - 1 = z^2.$$

Since x is clearly odd, $d = (x^2 + 1)/2$ is also odd because $4 \nmid x^2 + 1$. It follows that y and z are even, say $y = 2y_1$ and $z = 2z_1$. Hence $(z_1 - y_1)(z_1 + y_1) = (z^2 - y^2)/4 = 2d$. But in this case one of the factors $z_1 - y_1$, $z_1 + y_1$ is odd and the other one is even, which is impossible.

6. There are five such numbers:

$$\begin{aligned} 69300 &= 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 : & 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 &= 108 \text{ divisors;} \\ 50400 &= 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 : & 6 \cdot 3 \cdot 3 \cdot 2 &= 108 \text{ divisors;} \\ 60480 &= 2^6 \cdot 3^3 \cdot 5 \cdot 7 : & 7 \cdot 4 \cdot 2 \cdot 2 &= 112 \text{ divisors;} \\ 55440 &= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 : & 5 \cdot 3 \cdot 2 \cdot 2 \cdot 2 &= 120 \text{ divisors;} \\ 65520 &= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 : & 5 \cdot 3 \cdot 2 \cdot 2 \cdot 2 &= 120 \text{ divisors.} \end{aligned}$$

7. Let $P(x) = (x - x_0)(x - x_1) \cdots (x - x_n)(x - x_{n+1})$. Then

$$P'(x) = \sum_{j=0}^{n+1} \frac{P(x)}{x - x_j} \qquad \text{and} \qquad P''(x) = \sum_{j=0}^{n+1} \sum_{k \neq j} \frac{P(x)}{(x - x_j)(x - x_k)} .$$

Therefore

$$P''(x_i) = 2P'(x_i) \sum_{j \neq i} \frac{1}{(x_i - x_j)}$$

for $i = 0, 1, \dots, n + 1$, and the given condition implies $P''(x_i) = 0$ for $i = 1, 2, \dots, n$. Consequently,

$$x(x-1)P''(x) = (n+2)(n+1)P(x). \tag{1}$$

It is easy to observe that there is a unique monic polynomial of degree $n+2$ satisfying differential equation (1). On the other hand, the polynomial $Q(x) = (-1)^n P(1-x)$ also satisfies this equation, is monic, and $\deg Q = n+2$. Therefore $(-1)^n P(1-x) = P(x)$, and the result follows.

8. We shall solve the problem in the alternative formulation. Let $L_G(v)$ denote the length of the longest directed chain of edges in the given graph G that begins in a vertex v and is arranged decreasingly relative to the numbering. By the pigeonhole principle it suffices to show that $\sum_v L(v) \geq 2q$ in every such graph. We do this by induction on q.
For $q = 1$ the claim is obvious. We assume that it is true for $q-1$ and consider a graph G with q edges numbered $1, \dots, q$. Let the edge number q connect vertices u and w. Removing this edge, we get a graph G' with $q-1$ edges. We then have

$$L_G(u) \geq L_{G'}(w)+1,\ L_G(w) \geq L_{G'}(u)+1,\ L_G(v) \geq L_{G'}(v) \text{ for other } v.$$

Since $\sum L_{G'}(v) \geq 2(q-1)$ by inductive assumption, it follows that $\sum L_G(v) \geq 2(q-1)+2 = 2q$ as desired.

Second solution. Let us place a spider at each vertex of the graph. Let us now interchange the positions of the two spiders at the endpoints of each edge, listing the edges increasingly with respect to the numbering. This way we will move spiders exactly $2q$ times (two for each edge). Hence there is a spider that will be moved at least $2q/n$ times. All that remains is to notice that the path of each spider consists of edges numbered in increasing order.

Remark. A chain of the stated length having all vertices distinct does not necessarily exist. An example is $n = 4$, $q = 6$ with the numbering following the order ab, cd, ac, bd, ad, bc.

9. We shall use induction on the number n of points. The case $n = 1$ is trivial. Let us suppose that the statement is true for all $1, 2, \dots, n-1$, and that we are given a set T of n points.
If there exists a point $P \in T$ and a line l that is parallel to an axis and contains P and no other points of T, then by the inductive hypothesis we can color the set $T \setminus \{P\}$ and then use a suitable color for P. Let us now suppose that whenever a line parallel to an axis contains a point of T, it contains another point of T. It follows that for an arbitrary point $P_0 \in T$ we can choose points $P_1, P_2, \dots$ such that $P_k P_{k+1}$ is parallel to the x-axis for k even, and to the y-axis for k odd. We eventually come to a pair of integers (r, s) of the same parity, $0 \leq r < s$, such that lines $P_r P_{r+1}$ and $P_s P_{s+1}$ coincide. Hence the closed polygonal line $P_{r+1} P_{r+2} \dots P_s P_{r+1}$ is of even length. Thus we may color the points of this polygonal line alternately and then apply the inductive assumption for the rest of the set T. The induction is complete.

Second solution. Let $P_1, P_2, \ldots, P_k$ be the points lying on a line l parallel to an axis, going from left to right or from up to down. We draw segments joining P_1 with P_2, P_3 with P_4, and generally P_{2i-1} with P_{2i}. Having this done for every such line l, we obtain a set of segments forming certain polygonal lines. If one of these polygonal lines is closed, then it must have an even number of vertices. Thus, we can color the vertices on each of the polygonal lines alternately (a point not lying on any of the polygonal lines may be colored arbitrarily). The obtained coloring satisfies the conditions.

10. The set $X = \{1, \ldots, 1986\}$ splits into triads $T_1, \ldots, T_{662}$, where $T_j = \{3j-2, 3j-1, 3j\}$.
Let $\mathcal{F}$ be the family of all k-element subsets P such that $|P \cap T_j| = 1$ or 2 for some index j. If j_0 is the smallest such j_0, we define P' to be the k-element set obtained from P by replacing the elements of $P \cap T_{j_0}$ by the ones following cyclically inside T_{j_0}. Let $s(P)$ denote the remainder modulo 3 of the sum of elements of P. Then $s(P), s(P'), s(P'')$ are distinct, and $P''' = P$. Thus the operator $'$ gives us a bijective correspondence between the sets $X \in \mathcal{F}$ with $s(P) = 0$, those with $s(P) = 1$, and those with $s(P) = 2$.
If $3 \nmid k$ is not divisible by 3, then each k-element subset of X belongs to $\mathcal{F}$, and the game is fair. If $3 \mid k$, then k-element subsets not belonging to $\mathcal{F}$ are those that are unions of several triads. Since every such subset has the sum of elements divisible by 3, it follows that player A has the advantage.

11. Let X be a finite set in the plane and l_k a line containing exactly k points of X ($k = 1, \ldots, n$). Then l_n contains n points, l_{n-1} contains at least $n-2$ points not lying on l_n, l_{n-2} contains at least $n-4$ points not lying on l_n or l_{n-1}, etc. It follows that
$$|X| \geq g(n) = n + (n-2) + (n-4) + \cdots + \left(n - 2\left[\tfrac{n}{2}\right]\right).$$
Hence $f(n) \geq g(n) = \left[\frac{n+1}{2}\right]\left[\frac{n+2}{2}\right]$, where the last equality is easily proved by induction.
We claim that $f(n) = g(n)$. To prove this, we shall inductively construct a set X_n of cardinality $g(n)$ with the required property. For $n \leq 2$ a one-point and two-point set satisfy the requirements. Assume that X_n is a set of $g(n)$ points and that l_k is a line containing exactly k points of X_n, $k = 1, \ldots, n$. Consider any line l not parallel to any of the l_k's and not containing any point of X_n or any intersection point of the l_k. Let l intersect l_k in a point P_k, $k = 1, \ldots, n$, and let P_{n+1}, P_{n+2} be two points on l other than $P_1, \ldots, P_n$. We define $X_{n+2} = X_n \cup \{P_1, \ldots, P_{n+2}\}$. The set X_{n+2} consists of $g(n) + (n+2) = g(n+2)$ points. Since the lines $l, l_n, \ldots, l_2, l_1$ meet X_n in $n+2, n+1, \ldots, 3, 2$ points respectively (and there clearly exists a line containing only one point of X_{n+2}), this set also meets the demands.

12. We define $f(x_1,\dots,x_5) = \sum_{i=1}^{5}(x_{i+1} - x_{i-1})^2$ $(x_0 = x_5,\ x_6 = x_1)$. Assuming that $x_3 < 0$, according to the rules the lattice vector $X = (x_1, x_2, x_3, x_4, x_5)$ changes into $Y = (x_1, x_2 + x_3, -x_3, x_4 + x_3, x_5)$. Then

$$\begin{aligned} f(Y) - f(X) = {} & (x_2 + x_3 - x_5)^2 + (x_1 + x_3)^2 + (x_2 - x_4)^2 \\ & +(x_3 + x_5)^2 + (x_1 - x_3 - x_4)^2 - (x_2 - x_5)^2 \\ & -(x_3 - x_1)^2 - (x_4 - x_2)^2 - (x_5 - x_3)^2 - (x_1 - x_4)^2 \\ = {} & 2x_3(x_1 + x_2 + x_3 + x_4 + x_5) = 2x_3 S < 0. \end{aligned}$$

Thus f strictly decreases after each step, and since it takes only positive integer values, the number of steps must be finite.

Remark. One could inspect the behavior of $g(x) = \sum_{i=1}^{5}\sum_{j=1}^{5}|x_i + x_{i+1} + \cdots + x_{j-1}|$ instead. Then $g(Y) - g(X) = |S + x_3| - |S - x_3| > 0$.

13. Let us consider the infinite integer lattice and assume that having reached a point (x, n) or (n, y), the particle continues moving east and north following the rules of the game. The required probability p_k is equal to the probability of getting to one of the points $E_1(n, n+k)$, $E_2(n+k, n)$, but without passing through $(n, n+k-1)$ or $(n+k-1, n)$. Thus p is equal to the probability p_1 of getting to $E_1(n, n+k)$ via $D_1(n-1, n+k)$ plus the probability p_2 of getting to $E_2(n+k, n)$ via $D_2(n+k, n-1)$. Both p_1 and p_2 are easily seen to be equal to $\binom{2n+k-1}{n-1}2^{-2n-k}$, and therefore $p = \binom{2n+k-1}{n-1}2^{-2n-k+1}$.

14. We shall use the following simple fact.

Lemma. If $\widehat{k}$ is the image of a circle k under an inversion centered at a point Z, and O_1, O_2 are centers of k and $\widehat{k}$, then O_1, O_2, and Z are collinear.

Proof. The result follows immediately from the symmetry with respect to the line ZO_1.

Let I be the center of the inscribed circle i. Since $IX \cdot IA = IE^2$, the inversion with respect to i takes points A into X, and analogously B, C into Y, Z respectively. It follows from the lemma that the center of circle ABC, the center of circle XYZ, and point I are collinear.

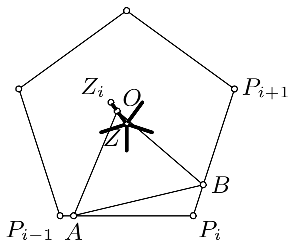
15. (a) This is the same problem as *SL82-14*.

(b) If S is the midpoint of AC, we have $B'S = AC\frac{\cos\angle D}{2\sin\angle D}$, $D'S = AC\frac{\cos\angle B}{2\sin\angle B}$, $B'D' = AC\left|\frac{\sin(\angle B + \angle D)}{2\sin\angle B\sin\angle D}\right|$. These formulas are true also if $\angle B > 90^\circ$ or $\angle D > 90^\circ$. We similarly obtain that $A''C'' = B'D'\left|\frac{\sin(\angle A' + \angle C')}{2\sin\angle A'\sin\angle C'}\right|$. Therefore

$$A''C'' = AC\frac{\sin^2(\angle A + \angle C)}{4\sin\angle A\sin\angle B\sin\angle C\sin\angle D}.$$

16. Let Z be the center of the polygon.

Suppose that at some moment we have $A \in P_{i-1}P_i$ and $B \in P_iP_{i+1}$, where P_{i-1}, P_i, P_{i+1} are adjacent vertices of the polygon. Since $\angle AOB = 180^\circ - \angle P_{i-1}P_iP_{i+1}$, the quadrilateral AP_iBO is cyclic. Hence $\angle AP_iO = \angle ABO = \angle AP_iZ$, which means that $O \in P_iZ$.



Moreover, from $OP_i = 2r \sin \angle P_iAO$, where r is the radius of circle AP_iBO, we obtain that $ZP_i \le OP_i \le ZP_i/\cos(\pi/n)$. Thus O traces a segment ZZ_i as A and B move along $P_{i-1}P_i$ and P_iP_{i+1} respectively, where Z_i is a point on the ray P_iZ with $P_iZ_i \cos(\pi/n) = P_iZ$. When A, B move along the whole circumference of the polygon, O traces an asterisk consisting of n segments of equal length emanating from Z and pointing away from the vertices.

17. We use complex numbers to represent the position of a point in the plane. For convenience, let $A_1, A_2, A_3, A_4, A_5, \dots$ be $A, B, C, A, B, \dots$ respectively, and let P_0 be the origin. After the kth step, the position of P_k will be $P_k = A_k + (P_{k-1} - A_k)u$, $k = 1, 2, 3, \dots$, where $u = e^{4\pi\imath/3}$. We easily obtain

$$P_k = (1-u)(A_k + uA_{k-1} + u^2A_{k-2} + \dots + u^{k-1}A_1).$$

The condition $P_0 \equiv P_{1986}$ is equivalent to $A_{1986} + uA_{1985} + \dots + u^{1984}A_2 + u^{1985}A_1 = 0$, which, having in mind that $A_1 = A_4 = A_7 = \cdots$, $A_2 = A_5 = A_8 = \cdots$, $A_3 = A_6 = A_9 = \cdots$, reduces to

$$662(A_3 + uA_2 + u^2A_1) = (1 + u^3 + \dots + u^{1983})(A_3 + uA_2 + u^2A_1) = 0.$$

It follows that $A_3 - A_1 = u(A_1 - A_2)$, and the assertion follows.

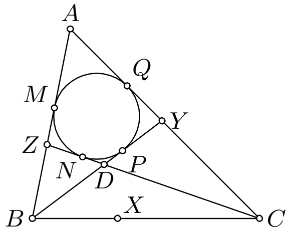
Second solution. Let f_P denote the rotation with center P through 120° clockwise. Let $f_1 = f_A$. Then $f_1(P_0) = P_1$. Let $B' = f_1(B)$, $C' = f_1(C)$, and $f_2 = f_{B'}$. Then $f_2(P_1) = P_2$ and $f_2(AB'C') = A'B'C''$. Finally, let $f_3 = f_{C''}$ and $f_3(A'B'C'') = A''B''C''$. Then $g = f_3f_2f_1$ is a translation sending P_0 to P_3 and C to C''. Now $P_{1986} = P_0$ implies that g^{662} is the identity, and thus $C = C''$.

Let K be such that ABK is equilateral and positively oriented. We observe that $f_2f_1(K) = K$; therefore the rotation f_2f_1 satisfies $f_2f_1(P) \ne P$ for $P \ne K$. Hence $f_2f_1(C) = C'' = C$ implies $K = C$.

18. We shall use the following criterion for a quadrangle to be circumscribable.

Lemma. The quadrangle $AYDZ$ is circumscribable if and only if $DB - DC = AB - AC$.

Proof. Suppose that $AYDZ$ is circumscribable and that the incircle is tangent to AZ, ZD, DY, YA at M, N, P, Q respectively. Then $DB - DC = PB - NC = MB - QC = AB - AC$. Conversely, assume that $DB - DC = AB - AC$ and let a tangent from D to the incircle of the triangle ACZ meet CZ and CA at $D' \neq Z$ and $Y' \neq A$ respectively. According to the first part we have $D'B - D'C = AB - AC$. It follows that $|D'B - DB| = |D'C - DC| = DD'$, implying that $D' \equiv D$.



Let us assume that $DZBX$ and $DXCY$ are circumscribable. Using the lemma we obtain $DC - DA = BC - BA$ and $DA - DB = CA - CB$. Adding these two inequalities yields $DC - DB = AC - AB$, and the statement follows from the lemma.

19. Let M and N be the midpoints of segments AB and CD, respectively. The given conditions imply that $\triangle ABD \cong \triangle BAC$ and $\triangle CDA \cong \triangle DCB$; hence $MC = MD$ and $NA = NB$. It follows that M and N both lie on the perpendicular bisectors of AB and CD, and consequently MN is the common perpendicular bisector of AB and CD. Points B and C are symmetric to A and D with respect to MN. Now if P is a point in space and P' the point symmetric to P with respect to MN, we have $BP = AP'$, $CP = DP'$, and thus $f(P) = AP + AP' + DP + DP'$. Let PP' intersect MN in Q. Then $AP + AP' \geq 2AQ$ and $DP + DP' \geq 2DQ$, from which it follows that $f(P) \geq 2(AQ + DQ) = f(Q)$. It remains to minimize $f(Q)$ with Q moving along the line MN.
Let us rotate point D around MN to a point D' that belongs to the plane AMN, on the side of MN opposite to A. Then $f(Q) = 2(AQ + D'Q) \geq AD'$, and equality occurs when Q is the intersection of AD' and MN. Thus $\min f(Q) = AD'$. We note that $4MD^2 = 2AD^2 + 2BD^2 - AB^2 = 2a^2 + 2b^2 - AB^2$ and $4MN^2 = 4MD^2 - CD^2 = 2a^2 + 2b^2 - AB^2 - CD^2$. Now, $AD'^2 = (AM + D'N)^2 + MN^2$, which together with $AM + D'N = (a+b)/2$ gives us

$$AD'^2 = \frac{a^2 + b^2 + AB \cdot CD}{2} = \frac{a^2 + b^2 + c^2}{2}.$$

We conclude that $\min f(Q) = \sqrt{(a^2 + b^2 + c^2)/2}$.

20. If the faces of the tetrahedron $ABCD$ are congruent triangles, we must have $AB = CD$, $AC = BD$, and $AD = BC$. Then the sum of angles at A is $\angle BAC + \angle CAD + \angle DAB = \angle BDC + \angle CBD + \angle DCB = 180°$.
We now assume that the sum of angles at each vertex is $180°$. Let us construct triangles BCD', CAD'', ABD''' in the plane ABC, exterior to $\triangle ABC$, such that $\triangle BCD' \cong \triangle BCD$, $\triangle CAD'' \cong \triangle CAD$, and $\triangle ABD''' \cong \triangle ABD$. Then by the assumption, $A \in D''D'''$, $B \in D'''D'$, and $C \in D'D''$. Since also $D''A = D'''A = DA$, etc., A, B, C are the mid-

points of segments $D''D''', D'''D', D'D''$ respectively. Thus the triangles ABC, BCD', CAD'', ABD''' are congruent, and the statement follows.

21. Since the sum of all edges of $ABCD$ is 3, the statement of the problem is an immediate consequence of the following statement:

Lemma. Let r be the inradius of a triangle with sides a, b, c. Then $a + b + c \geq 6\sqrt{3} \cdot r$, with equality if and only if the triangle is equilateral.

Proof. If S and p denotes the area and semiperimeter of the triangle, by Heron's formula and the AM–GM inequality we have

$$\begin{aligned} pr = S &= \sqrt{p(p-a)(p-b)(p-c)} \\ &\leq \sqrt{p\left(\frac{(p-a)+(p-b)+(p-c)}{3}\right)^3} = \sqrt{\frac{p^4}{27}} = \frac{p^2}{3\sqrt{3}}, \end{aligned}$$

i.e., $p \geq 3\sqrt{3} \cdot r$, which is equivalent to the claim.

4.28 Solutions to the Shortlisted Problems of IMO 1987

1. By (ii), $f(x) = 0$ has at least one solution, and there is the greatest among them, say x_0. Then by (v), for any x,

$$0 = f(x)f(x_0) = f(xf(x_0) + x_0 f(x) - x_0 x) = f(x_0(f(x) - x)). \qquad (1)$$

It follows that $x_0 \geq x_0(f(x) - x)$.
Suppose $x_0 > 0$. By (i) and (iii), since $f(x_0) - x_0 < 0 < f(0) - 0$, there is a number z between 0 and x_0 such that $f(z) = z$. By (1), $0 = f(x_0(f(z) - z)) = f(0) = 1$, a contradiction. Hence, $x_0 < 0$. Now the inequality $x_0 \geq x_0(f(x) - x)$ gives $f(x) - x \geq 1$ for all x; so, $f(1987) \geq 1988$. Therefore $f(1987) = 1988$.

2. Let d_i denote the number of cliques of which person i is a member. Clearly $d_i \geq 2$. We now distinguish two cases:
   (i) For some i, $d_i = 2$. Suppose that i is a member of two cliques, C_p and C_q. Then $|C_p| = |C_q| = n$, since for each couple other than i and his/her spouse, one member is in C_p and one in C_q. There are thus $(n-1)(n-2)$ pairs (r, s) of nonspouse persons distinct from i, where $r \in C_p$, $s \in C_q$. We observe that each such pair accounts for a different clique. Otherwise, we find two members of C_p or C_q who belong to one other clique. It follows that $k \geq 2 + (n-1)(n-2) \geq 2n$ for $n \geq 4$.
   (ii) For every i, $d_i \geq 3$. Suppose that $k < 2n$. For $i = 1, 2, \dots, 2n$ assign to person i an indeterminant x_i, and for $j = 1, 2, \dots, k$ set $y = \sum_{i \in C_j} x_i$. From linear algebra, we know that if $k < 2n$, then there exist $x_1, x_2, \dots, x_{2n}$, not all zero, such that $y_1 = y_2 = \cdots = y_k = 0$. On the other hand, suppose that $y_1 = y_2 = \cdots = y_k = 0$. Let M be the set of the couples and M' the set of all other pairs of persons. Then

$$\begin{aligned} 0 = \sum_{j=1}^{k} y_j^2 &= \sum_{i=1}^{2n} d_i x_i^2 + 2 \sum_{(i,j) \in M'} x_i x_j \\ &= \sum_{i=1}^{2n} (d_i - 2) x_i^2 + (x_1 + x_2 + \cdots + x_{2n})^2 + \sum_{(i,j) \in M} (x_i - x_j)^2 \\ &\geq \sum_{i=1}^{2n} x_i^2 > 0, \end{aligned}$$

   if not all $x_1, x_2, \dots, x_{2n}$ are zero, which is a contradiction. Hence $k \geq 2n$.

   Remark. The condition $n \geq 4$ is essential. For a party attended by 3 couples $\{(1,4), (2,5), (3,6)\}$, there is a collection of 4 cliques satisfying the conditions: $\{(1,2,3), (3,4,5), (5,6,1), (2,4,6)\}$.

3. The answer: yes. Set

$$p(k,m) = k + [1 + 2 + \cdots + (k+m)] = \frac{(k+m)^2 + 3k + m}{2}.$$

It is obviously of the desired type.

4. Setting $x_1 = \overrightarrow{AB}$, $x_2 = \overrightarrow{AD}$, $x_3 = \overrightarrow{AE}$, we have to prove that

$$\|x_1 + x_2\| + \|x_2 + x_3\| + \|x_3 + x_1\| \le \|x_1\| + \|x_2\| + \|x_3\| + \|x_1 + x_2 + x_3\|.$$

We have

$$\begin{aligned}
&(\|x_1\| + \|x_2\| + \|x_3\|)^2 - \|x_1 + x_2 + x_3\|^2 \\
&\quad = 2 \sum_{1 \le i < j \le 3} (\|x_i\|\|x_j\| - \langle x_i, x_j \rangle) = \sum_{1 \le i < j \le 3} \left[(\|x_i\| + \|x_j\|)^2 - \|x_i + x_j\|^2 \right] \\
&\quad = \sum_{1 \le i < j \le 3} (\|x_i\| + \|x_j\| + \|x_i + x_j\|)(\|x_i\| + \|x_j\| - \|x_i + x_j\|).
\end{aligned}$$

The following two inequalities are obvious:

$$\|x_i\| + \|x_j\| - \|x_i + x_j\| \ge 0, \tag{1}$$

$$\|x_i\| + \|x_j\| + \|x_i + x_j\| \le \|x_1\| + \|x_2\| + \|x_3\| + \|x_1 + x_2 + x_3\|. \tag{2}$$

It follows that

$$\begin{aligned}
&(\|x_1\| + \|x_2\| + \|x_3\|)^2 - \|x_1 + x_2 + x_3\|^2 \\
&\quad \le \left(\sum_{i=1}^{3} \|x_i\| + \left\| \sum_{i=1}^{3} x_i \right\| \right) \left(2 \sum_{i=1}^{3} \|x_i\| - \sum_{1 \le i < j \le 3} \|x_i + x_j\| \right),
\end{aligned}$$

and dividing by the positive number $\sum_{i=1}^{3} \|x_i\| + \left\| \sum_{i=1}^{3} x_i \right\|$ we obtain

$$\sum_{i=1}^{3} \|x_i\| - \left\| \sum_{i=1}^{3} x_i \right\| \le 2 \sum_{i=1}^{3} \|x_i\| - \sum_{1 \le i < j \le 3} \|x_i + x_j\|.$$

The inequality is proven. Let us analyze the cases of equality. If one of the vectors is null, then equality obviously holds. Suppose that $x_i \ne 0$, $i = 1, 2, 3$. For every i, j, at least one of (1) and (2) is equality. Equality in (1) holds if and only if x_i and x_j are collinear with the same direction, while in (2) it holds if and only if $-x_k$ and $x_1 + x_2 + x_3$ are collinear with the same direction. If not all the vectors are collinear, then there are at least two distinct pairs x_i, x_j, $i < j$, for which (2) is an equality, so at least two of x_i are collinear with $x_1 + x_2 + x_3$, but then so is the third; hence, the sum $x_1 + x_2 + x_3$ must be 0. Thus the cases of equality are (a) the vectors are collinear with the same direction; (b) the vectors are collinear, two of them have the same direction, say x_i, x_j, and $\|x_k\| \geq \|x_i\| + \|x_j\|$; (c) one of the vectors is 0; (d) their sum is 0.

Second solution. The following technique, although not quite elementary, is often used to effectively reduce geometric inequalities of first degree, like this one, to the one-dimensional case.

Let σ be a fixed sphere with center O. For an arbitrary segment d in space, and any line l, we denote by $\pi_l(d)$ the length of the projection of d onto l. Consider the integral of lengths of these projections on all possible directions of OP, with P moving on the sphere: $\int_\sigma \pi_{OP}(d)\, d\sigma$. It is clear that this value depends only on the length of d (because of symmetry); hence

$$\int_\sigma \pi_{OP} d\sigma = c \cdot |d| \qquad \text{for some constant } c \neq 0. \tag{1}$$

Notice that by the one-dimensional case, for any point $P \in \sigma$,

$$\pi_{OP}(x_1) + \pi_{OP}(x_2) + \pi_{OP}(x_3) + \pi_{OP}(x_1 + x_2 + x_3)$$
$$\geq \pi_{OP}(x_1 + x_2) + \pi_{OP}(x_1 + x_3) + \pi_{OP}(x_2 + x_3).$$

By integration on σ, using (1), we obtain

$$c(\|x_1\|+\|x_2\|+\|x_3\|+\|x_1+x_2+x_3\|) \geq c(\|x_1+x_2\|+\|x_1+x_3\|+\|x_2+x_3\|).$$

5. Assuming the notation $a = \overline{BC}$, $b = \overline{AC}$, $c = \overline{AB}$; $x = \overline{BL}$, $y = \overline{CM}$, $z = \overline{AN}$, from the Pythagorean theorem we obtain

$$(a-x)^2 + (b-y)^2 + (c-z)^2 = x^2 + y^2 + z^2$$
$$= \frac{x^2 + (a-x)^2 + y^2 + (b-y)^2 + z^2 + (c-z)^2}{2}.$$

Since $x^2+(a-x)^2 = a^2/2+(a-2x)^2/2 \geq a^2/2$ and similarly $y^2+(b-y)^2 \geq b^2/2$ and $z^2 + (c-z)^2 \geq c^2/2$, we get

$$x^2 + y^2 + z^2 \geq \frac{a^2 + b^2 + c^2}{4}.$$

Equality holds if and only if P is the circumcenter of the triangle ABC, i.e., when $x = a/2$, $y = b/2$, $z = c/2$.

6. Suppose w.l.o.g. that $a \geq b \geq c$. Then $1/(b+c) \geq 1/(a+c) \geq 1/(a+b)$. Chebyshev's inequality yields

$$\frac{a^n}{b+c} + \frac{b^n}{a+c} + \frac{c^n}{a+b} \geq \frac{1}{3}(a^n + b^n + c^n)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right). \tag{1}$$

By the Cauchy-Schwarz inequality we have

$$2(a+b+c)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right)\geq 9,$$

and the mean inequality yields $(a^n+b^n+c^n)/3 \geq [(a+b+c)/3]^n$. We obtain from (1) that

$$\begin{aligned}\frac{a^n}{b+c}+\frac{b^n}{a+c}+\frac{c^n}{a+b} &\geq \left(\frac{a+b+c}{3}\right)^n\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right)\\ &\geq \frac{3}{2}\left(\frac{a+b+c}{3}\right)^{n-1}=\left(\frac{2}{3}\right)^{n-2}S^{n-1}.\end{aligned}$$

7. For all real numbers v the following inequality holds:

$$\sum_{0\leq i<j\leq 4}(v_i-v_j)^2 \leq 5\sum_{i=0}^{4}(v_i-v)^2. \tag{1}$$

Indeed,

$$\begin{aligned}\sum_{0\leq i<j\leq 4}(v_i-v_j)^2 &= \sum_{0\leq i<j\leq 4}[(v_i-v)-(v_j-v)]^2\\ &= 5\sum_{i=0}^{4}(v_i-v)^2-\left(\sum_{i=0}^{4}(v_i-v)\right)^2 \leq 5\sum_{i=0}^{4}(v_i-v)^2.\end{aligned}$$

Let us first take v_i's, satisfying condition (1), so that w.l.o.g. $v_0\leq v_1\leq v_2\leq v_3\leq v_4\leq 1+v_0$. Defining $v_5=1+v_0$, we see that one of the differences $v_{j+1}-v_j$, $j=0,\dots,4$, is at most $1/5$. Take $v=(v_{j+1}+v_j)/2$, and then place the other three v_j's in the segment $[v-1/2, v+1/2]$. Now we have $|v-v_j|\leq 1/10$, $|v-v_{j+1}|\leq 1/10$, and $|v-v_k|\leq 1/2$, for any k different from $j, j+1$. The v_i's thus obtained have the required property. In fact, using the inequality (1), we obtain

$$\sum_{0\leq i<j\leq 4}(v_i-v_j)^2 \leq 5\left(2\left(\frac{1}{10}\right)^2+3\left(\frac{1}{2}\right)^2\right)=3.85<4.$$

Remark. The best possible estimate for the right-hand side is 2.

8. (a) Consider

$$a_i=ik+1,\quad i=1,2,\dots,m;\qquad b_j=jm+1,\quad j=1,2,\dots,k.$$

Assume that $mk \mid a_ib_j-a_sb_t=(ik+1)(jm+1)-(sk+1)(tm+1)=km(ij-st)+m(j-t)+k(i-s)$. Since m divides this sum, we get that $m \mid k(i-s)$, or, together with $\gcd(k,m)=1$, that $i=s$. Similarly $j=t$, which proves part (a).

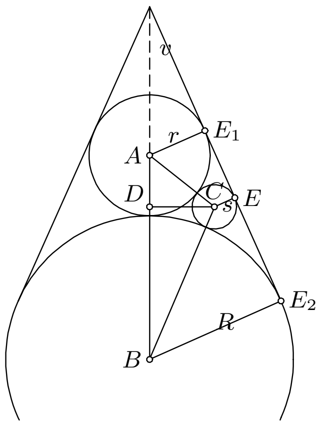
(b) Suppose the opposite, i.e., that all the residues are distinct. Then the residue 0 must also occur, say at a_1b_1: $mk \mid a_1b_1$; so, for some a' and b', $a' \mid a_1$, $b' \mid b_1$, and $a'b' = mk$. Assuming that for some $i, s \neq i$, $a' \mid a_i - a_s$, we obtain $mk = a'b' \mid a_ib_1 - a_sb_1$, a contradiction. This shows that $a' \geq m$ and similarly $b' \geq k$, and thus from $a'b' = mk$ we have $a' = m$, $b' = k$. We also get (1): all a_i's give distinct residues modulo $m = a'$, and all b_j's give distinct residues modulo $k = b'$.
Now let p be a common prime divisor of m and k. By $(*)$, exactly $\frac{p-1}{p}m$ of a_i's and exactly $\frac{p-1}{p}k$ of b_j's are not divisible by p. Therefore there are precisely $\frac{(p-1)^2}{p^2}mk$ products a_ib_j that are not divisible by p, although from the assumption that they all give distinct residues it follows that the number of such products is $\frac{p-1}{p}mk \neq \frac{(p-1)^2}{p^2}mk$. We have arrived at a contradiction, thus proving (b).

9. The answer is yes. Consider the curve

$$C = \{(x, y, z) \mid x = t, y = t^3, z = t^5, \quad t \in \mathbb{R}\}.$$

Any plane defined by an equation of the form $ax+by+cz+d = 0$ intersects the curve C at points (t, t^3, t^5) with t satisfying $ct^5 + bt^3 + at + d = 0$. This last equation has at least one but only finitely many solutions.

10. Denote by r, R (take w.l.o.g. $r < R$) the radii and by A, B the centers of the spheres S_1, S_2 respectively. Let s be the common radius of the spheres in the ring, C the center of one of them, say S, and D the foot of the perpendicular from C to AB. The centers of the spheres in the ring form a regular n-gon with center D, and thus $\sin(\pi/n) = s/CD$. Using Heron's formula on the triangle ABC, we obtain $(r+R)^2CD^2 = 4rRs(r + R + s)$, and hence



$$\sin^2 \frac{\pi}{n} = \frac{s^2}{CD^2} = \frac{(r + R)^2 s}{4(r + R + s)rR}. \tag{1}$$

Choosing the unit of length so that $r + R = 2$, for simplicity of writing, we write (1) as $1/\sin^2(\pi/n) = rR\,(1 + 2/s)$. Let now v be half the angle at the top of the cone. Then clearly $R - r = (R + r) \sin v = 2 \sin v$, giving us $R = 1 + \sin v, r = 1 - \sin v$. It follows that

$$\frac{1}{\sin^2 \frac{\pi}{n}} = \left(1 + \frac{2}{s}\right) \cos^2 v. \tag{2}$$

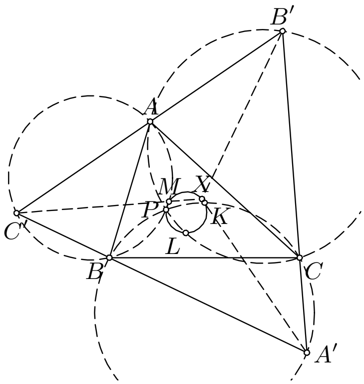
We need to express s as a function of R and r. Let E_1, E_2, E be collinear points of tangency of S_1, S_2, and S with the cone. Obviously, $E_1E_2 = E_1E + E_2E$, i.e., $2\sqrt{rs} + 2\sqrt{Rs} = 2\sqrt{Rr} = (R+r)\cos v = 2\cos v$. Hence,

$$\cos^2 v = s(\sqrt{R} + \sqrt{r})^2 = s(R + r + 2\sqrt{Rr}) = s(2 + 2\cos v).$$

Substituting this into (2), we obtain $2 + \cos v = 1/\sin(\pi/n)$. Therefore $1/3 < \sin(\pi/n) < 1/2$, and we conclude that the possible values for n are 7, 8, and 9.

11. Let A_1 be the set that contains 1, and let the minimal element of A_2 be less than that of A_3. We shall construct the partitions with required properties by allocating successively numbers to the subsets that always obey the rules. The number 1 must go to A_1; we show that for every subsequent number we have exactly two possibilities. Actually, while A_2 and A_3 are both empty, every successive number can enter either A_1 or A_2. Further, when A_2 is no longer empty, we use induction on the number to be placed, denote it by m: if m can enter A_i or A_j but not A_k, and it enters A_i, then $m+1$ can be placed in A_i or A_k, but not in A_j. The induction step is finished. This immediately gives us that the final answer is 2^{n-1}.

12. Here all angles will be oriented and measured counterclockwise. Note that $\measuredangle CA'B = \measuredangle AB'C = \measuredangle BC'A = \pi/3$. Let a', b', c' denote respectively the inner bisectors of angles A', B', C' in triangle $A'B'C'$. The lines a', b', c' meet at the centroid X of $A'B'C'$, and $\measuredangle(a', b') = \measuredangle(b', c') = \measuredangle(c', a') = 2\pi/3$. Now let K, L, M be the points such that $KB = KC$, $LC = LA$, $MA = MB$, and $\measuredangle BKC = \measuredangle CLA = \measuredangle AMB = 2\pi/3$, and let C_1, C_2, C_3 be the circles circumscribed about triangles



BKC, CLA, and AMB respectively. These circles are characterized by $C_1 = \{Z \mid \measuredangle BZC = 2\pi/3\}$, etc.; hence we deduce that they meet at a point P such that $\measuredangle BPC = \measuredangle CPA = \measuredangle APB = 2\pi/3$ (Torricelli's point). Points A', B', C' run over $C_1 \setminus \{P\}$, $C_2 \setminus \{P\}$, $C_3 \setminus \{P\}$ respectively. As for a', b', c', we see that $K \in a'$, $L \in b'$, $M \in c'$, and also that they can take all possible directions except KP, LP, MP respectively (if $K = P$, KP is assumed to be the corresponding tangent at K). Then, since $\measuredangle KXL = 2\pi/3$, X runs over the circle defined by $\{Z \mid \measuredangle KZL = 2\pi/3\}$, without P. But analogously, X runs over the circle $\{Z \mid \measuredangle LZM = 2\pi/3\}$, from which we can conclude that these two circles are the same, both equal to the circumcircle of KLM, and consequently also that triangle KLM is equilateral (which is, anyway, a well-known fact). Therefore, the locus of the points X is the circumcircle of KLM minus point P.

13. We claim that the points $P_i(i, i^2)$, $i = 1, 2, \dots, 1987$, satisfy the conditions. In fact:
   (i) $\overline{P_iP_j} = \sqrt{(i-j)^2 + (i^2 - j^2)^2} = |i-j|\sqrt{1+(i+j)^2}$.
   It is known that for each positive integer n, $\sqrt{n}$ is either an integer or an irrational number. Since $i + j < \sqrt{1+(i+j)^2} < i + j + 1$, $\sqrt{1+(i+j)^2}$ is not an integer, it is irrational, and so is $\overline{P_iP_j}$.
   (ii) The area A of the triangle $P_iP_jP_k$, for distinct i, j, k, is given by

$$\begin{aligned} A &= \left| \frac{i^2+j^2}{2}(i-j) + \frac{j^2+k^2}{2}(j-k) + \frac{k^2+i^2}{2}(k-i) \right| \\ &= \left| \frac{(i-j)(j-k)(k-i)}{2} \right| \in \mathbb{Q} \setminus \{0\}, \end{aligned}$$

   also showing that this triangle is nondegenerate.

14. Let x_n be the total number of counted words of length n, and y_n, z_n, u_n, z_n, y_n the numbers of counted words of length n starting with $0, 1, 2, 3, 4$, respectively (indeed, by symmetry, words starting with 0 are equally numbered as those starting with 4, etc.). We have the clear relations

$$\begin{aligned} &(1)\ y_n = z_{n-1}; && (2)\ z_n = y_{n-1} + u_{n-1}; \\ &(3)\ u_n = 2z_{n-1}; && (4)\ x_n = 2y_n + 2z_n + u_n. \end{aligned}$$

From (1), (2), and (3) we get $z_n = z_{n-2} + 2z_{n-2} = 3z_{n-2}$, with $z_1 = 1$, $z_2 = 2$, which gives

$$z_{2n} = 2 \cdot 3^{n-1}, \qquad z_{2n+1} = 3^n.$$

Then (1), (3), and (4) obviously imply

$$\begin{aligned} y_{2n} &= 3^{n-1}, & y_{2n+1} &= 2 \cdot 3^{n-1}; \\ u_{2n} &= 2 \cdot 3^{n-1}, & u_{2n+1} &= 4 \cdot 3^{n-1}; \\ x_{2n} &= 8 \cdot 3^{n-1}, & x_{2n+1} &= 14 \cdot 3^{n-1}; \end{aligned}$$

with the initial number $x_1 = 5$.

15. Since $x_1^2 + x_2^2 + \dots + x_n^2 = 1$, we get by the Cauchy-Schwarz inequality

$$|x_1| + |x_2| + \dots + |x_n| \le \sqrt{n(x_1^2 + x_2^2 + \dots + x_n^2)} = \sqrt{n}.$$

Hence all k^n sums of the form $e_1x_1 + e_2x_2 + \dots + e_nx_n$, with $e_i \in \{0, 1, 2, \dots, k-1\}$, must lie in some closed interval $\mathfrak{I}$ of length $(k-1)\sqrt{n}$. This interval can be covered with $k^n - 1$ closed subintervals of length $\frac{k-1}{k^n-1}\sqrt{n}$. By the pigeonhole principle there must be two of these sums lying in the same subinterval. Their difference, which is of the form $e_1x_1 + e_2x_2 + \cdots + e_nx_n$ where $e_i \in \{0, \pm 1, \dots, \pm(k-1)\}$, satisfies

$$|e_1x_1 + e_2x_2 + \cdots + e_nx_n| \le \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

16. We assume that $S = \{1, 2, \dots, n\}$, and use the obvious fact

$$\sum_{k=0}^{n} p_n(k) = n! \tag{0}$$

(a) To each permutation π of S we assign an n-vector $(e_1, e_2, \dots, e_n)$, where e_i is 1 if i is a fixed point of π, and 0 otherwise. Since exactly $p_n(k)$ of the assigned vectors contain exactly k "1"s, the considered sum $\sum_{k=0}^{n} kp_n(k)$ counts all the "1"s occurring in all the $n!$ assigned vectors. But for each i, $1 \le i \le n$, there are exactly $(n-1)!$ permutations that fix i; i.e., exactly $(n-1)!$ of the vectors have $e_i = 1$. Therefore the total number of "1"s is $n \cdot (n-1)! = n!$, implying

$$\sum_{k=0}^{n} kp_n(k) = n!. \tag{1}$$

(b) In this case, to each permutation π of S we assign a vector $(d_1, \dots, d_n)$ instead, with $d_i = k$ if i is a fixed point of π, and $d_i = 0$ otherwise, where k is the number of fixed points of π.
Let us count the sum Z of all components d_i for all the $n!$ permutations. There are $p_n(k)$ such vectors with exactly k components equal to k, and sums of components equal to k^2. Thus, $Z = \sum_{k=0}^{n} k^2 p_n(k)$. On the other hand, we may first calculate the sum of all components d_i for fixed i. In fact, the value $d_i = k > 0$ will occur exactly $p_{n-1}(k-1)$ times, so that the sum of the d_i's is $\sum_{k=1}^{n} kp_{n-1}(k-1) = \sum_{k=0}^{n-1}(k+1)p_{n-1}(k) = 2(n-1)!$. Summation over i yields

$$Z = \sum_{k=0}^{n} k^2 p_n(k) = 2n!. \tag{2}$$

From (0), (1), and (2), we conclude that

$$\sum_{k=0}^{n}(k-1)^2 p_n(k) = \sum_{k=0}^{n} k^2 p_n(k) - 2\sum_{k=0}^{n} kp_n(k) + \sum_{k=0}^{n} p_n(k) = n!.$$

Remark. Only the first part of this problem was given on the IMO.

17. The number of 4-colorings of the set M is equal to 4^{1987}. Let A be the number of arithmetic progressions in M with 10 terms. The number of colorings containing a monochromatic arithmetic progression with 10 terms is less than $4A \cdot 4^{1977}$. So, if $A < 4^9$, then there exist 4-colorings with the required property.

Now we estimate the value of A. If the first term of a 10-term progression is k and the difference is d, then $1 \le k \le 1978$ and $d \le \left[\frac{1987-k}{9}\right]$; hence

$$A = \sum_{k=1}^{1978} \left[\frac{1987-k}{9}\right] < \frac{1986 + 1985 + \cdots + 9}{9} = \frac{1995 \cdot 1978}{18} < 4^9.$$

18. Note first that the statement that some $a+x, a+y, a+x+y$ belong to a class C is equivalent to the following statement:
(1) There are positive integers $p, q \in C$ such that $p < q \le 2p$.
Indeed, given p, q, take simply $x = y = q - p$, $a = 2p - q$; conversely, if a, x, y ($x \le y$) exist such that $a+x, a+y, a+x+y \in C$, take $p = a+y$, $q = a+x+y$: clearly, $p < q \le 2p$.
We will show that $h(r) = 2r$. Let $\{1, 2, \dots, 2r\} = C_1 \cup C_2 \cup \dots \cup C_r$ be an arbitrary partition into r classes. By the pigeonhole principle, two among the $r+1$ numbers $r, r+1, \dots, 2r$ belong to the same class, say $i, j \in C_k$. If w.l.o.g. $i < j$, then obviously $i < j \le 2i$, and so by (1) this C_k has the required property.
On the other hand, we consider the partition

$$\{1, 2, \dots, 2r-t\} = \bigcup_{k=1}^{r-t} \{k, k+r\} \cup \{r-t+1\} \cup \dots \cup \{r\}$$

and prove that (1), and thus also the required property, does not hold. In fact, none of the classes in the partition contains p and q with $p < q \le 2p$, because $k + r > 2k$.

19. The facts given in the problem allow us to draw a triangular pyramid with angles $2\alpha, 2\beta, 2\gamma$ at the top and lateral edges of length $1/2$. At the base there is a triangle whose side lengths are exactly $\sin\alpha, \sin\beta, \sin\gamma$. The area of this triangle does not exceed the sum of areas of the lateral sides, which equals $(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)/8$.

20. Let y be the smallest nonnegative integer with $y \le p-2$ for which $f(y)$ is a composite number. Denote by q the smallest prime divisor of $f(y)$. We claim that $y < q$.
Suppose the contrary, that $y \ge q$. Let r be a positive integer such that $y \equiv r \pmod q$. Then $f(y) \equiv f(r) \equiv 0 \pmod q$, and since $q \le y \le p-2 \le f(r)$, we conclude that $q \mid f(r)$, which is a contradiction to the minimality of y.
Now, we will prove that $q > 2y$. Suppose the contrary, that $q \le 2y$. Since

$$f(y) - f(x) = (y-x)(y+x+1),$$

we observe that $f(y) - f(q-1-y) = (2y-q+1)q$, from which it follows that $f(q-1-y)$ is divisible by q. But by the assumptions, $q-1-y < y$, implying that $f(q-1-y)$ is prime and therefore equal to q. This is impossible, because

$$f(q-1-y) = (q-1-y)^2 + (q-1-y) + p > q + p - y - 1 \geq q.$$

Therefore $q \geq 2y+1$. Now, since $f(y)$, being composite, cannot be equal to q, and q is its smallest prime divisor, we obtain that $f(y) \geq q^2$. Consequently,

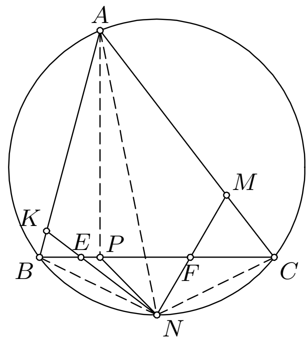
$$y^2 + y + p \geq q^2 \geq (2y+1)^2 = 4y^2 + 4y + 1 \Rightarrow 3(y^2+y) \leq p-1,$$

and from this we easily conclude that $y < \sqrt{p/3}$, which contradicts the condition of the problem. In this way, all the numbers

$$f(0), f(1), \ldots, f(p-2)$$

must be prime.

21. Let P be the second point of intersection of segment BC and the circle circumscribed about quadrilateral $AKLM$. Denote by E the intersection point of the lines KN and BC and by F the intersection point of the lines MN and BC. Then $\angle BCN = \angle BAN$ and $\angle MAL = \angle MPL$, as angles on the same arc. Since AL is a bisector, $\angle BCN = \angle BAL = \angle MAL = \angle MPL$, and consequently $PM \parallel NC$. Similarly we prove $KP \parallel BN$. Then the quadrilaterals $BKPN$ and $NPMC$ are trapezoids; hence



$$S_{BKE} = S_{NPE} \quad \text{and} \quad S_{NPF} = S_{CMF}.$$

Therefore $S_{ABC} = S_{AKNM}$.

22. Suppose that there exists such function f. Then we obtain

$$f(n+1987) = f(f(f(n))) = f(n) + 1987 \qquad \text{for all } n \in \mathbb{N},$$

and from here, by induction, $f(n+1987t) = f(n) + 1987t$ for all $n, t \in \mathbb{N}$. Further, for any $r \in \{0, 1, \ldots, 1986\}$, let $f(r) = 1987k + l$, $k, l \in \mathbb{N}$, $l \leq 1986$. We have

$$r + 1987 = f(f(r)) = f(l + 1987k) = f(l) + 1987k,$$

and consequently there are two possibilities:

(i) $k = 1 \Rightarrow f(r) = l + 1987$ and $f(l) = r$;

(ii) $k = 0 \Rightarrow f(r) = l$ and $f(l) = r + 1987$;

in both cases, $r \neq l$. In this way, the set $\{0, 1, \dots, 1986\}$ decomposes to pairs $\{a, b\}$ such that

$$f(a) = b \text{ and } f(b) = a + 1987, \quad \text{or} \quad f(b) = a \text{ and } f(a) = b + 1987.$$

But the set $\{0, 1, \dots, 1986\}$ has an odd number of elements, and cannot be decomposed into pairs. Contradiction.

23. If we prove the existence of $p, q \in \mathbb{N}$ such that the roots r, s of

$$f(x) = x^2 - kp \cdot x + kq = 0$$

are irrational real numbers with $0 < s < 1$ (and consequently $r > 1$), then we are done, because from $r + s, rs \equiv 0 \pmod{k}$ we get $r^m + s^m \equiv 0 \pmod{k}$, and $0 < s^m < 1$ yields the assertion.
To prove the existence of such natural numbers p and q, we can take them such that $f(0) > 0 > f(1)$, i.e.,

$$kq > 0 > k(q - p) + 1 \quad \Rightarrow \quad p > q > 0.$$

The irrationality of r can be obtained by taking $q = p - 1$, because the discriminant $D = (kp)^2 - 4kp + 4k$, for $(kp - 2)^2 < D < (kp - 1)^2$, is not a perfect square for $p \geq 2$.

4.29 Solutions to the Shortlisted Problems of IMO 1988

1. Assume that p and q are real and $b_0, b_1, b_2, \dots$ is a sequence such that $b_n = pb_{n-1} + qb_{n-2}$ for all $n > 1$. From the equalities $b_n = pb_{n-1} + qb_{n-2}$, $b_{n+1} = pb_n + qb_{n-1}$, $b_{n+2} = pb_{n+1} + qb_n$, eliminating b_{n+1} and b_{n-1} we obtain that $b_{n+2} = (p^2 + 2q)b_n - q^2 b_{n-2}$. So the sequence $b_0, b_2, b_4, \dots$ has the property

$$b_{2n} = Pb_{2n-2} + Qb_{2n-4}, \qquad P = p^2 + 2q, \;\; Q = -q^2. \tag{1}$$

We shall solve the problem by induction. The sequence a_n has $p = 2$, $q = 1$, and hence $P = 6$, $Q = -1$.
Let $k = 1$. Then $a_0 = 0$, $a_1 = 1$, and a_n is of the same parity as a_{n-2}; i.e., it is even if and only if n is even.
Let $k \geq 1$. We assume that for $n = 2^k m$, the numbers a_n are divisible by 2^k, but divisible by 2^{k+1} if and only if m is even. We assume also that the sequence $c_0, c_1, \dots$, with $c_m = a_{m \cdot 2^k}$, satisfies the condition $c_n = pc_{n-1} - c_{n-2}$, where $p \equiv 2 \pmod 4$ (for $k = 1$ it is true). We shall prove the same statement for $k+1$. According to (1), $c_{2n} = Pc_{2n-2} - c_{2n-4}$, where $P = p^2 - 2$. Obviously $P \equiv 2 \pmod 4$. Since $P = 4s + 2$ for some integer s, and $c_{2n} = 2^{k+1} d_{2n}$, $c_0 = 0$, $c_1 \equiv 2^k \pmod{2^{k+1}}$, and $c_2 = pc_1 \equiv 2^{k+1} \pmod{2^{k+2}}$, we have

$$c_{2n} = (4s + 2)2^{k+1} d_{2n-2} - c_{2n-4} \equiv c_{2n-4} \pmod{2^{k+2}},$$

i.e., $0 \equiv c_0 \equiv c_4 \equiv c_8 \equiv \cdots$ and $2^{k+1} \equiv c_2 \equiv c_6 \equiv \cdots \pmod{2^{k+2}}$, which proves the statement.

Second solution. The recursion is solved by

$$a_n = \frac{1}{2\sqrt{2}} \left((1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right) = \binom{n}{1} + 2\binom{n}{3} + 2^2 \binom{n}{5} + \cdots .$$

Let $n = 2^k m$ with m odd; then for $p > 0$ the summand

$$2^p \binom{n}{2p+1} = 2^{k+p} m \frac{(n-1)\dots(n-2p)}{(2p+1)!} = 2^{k+p} \frac{m}{2p+1} \binom{n-1}{2p}$$

is divisible by 2^{k+p}, because the denominator $2p+1$ is odd. Hence

$$a_n = n + \sum_{p>0} 2^p \binom{n}{2p+1} = 2^k m + 2^{k+1} N$$

for some integer N, so that a_n is exactly divisible by 2^k.

Third solution. It can be proven by induction that $a_{2n} = 2a_n(a_n + a_{n+1})$. The required result follows easily, again by induction on k.

2. For polynomials $f(x), g(x)$ with integer coefficients, we use the notation $f(x) \sim g(x)$ if all the coefficients of $f - g$ are even. Let $n = 2^s$. It is immediately shown by induction that $(x^2 + x + 1)^{2^s} \sim x^{2^{s+1}} + x^{2^s} + 1$, and the required number for $n = 2^s$ is 3.

Let $n = 2^s - 1$. If s is odd, then $n \equiv 1 \pmod 3$, while for s even, $n \equiv 0 \pmod 3$. Consider the polynomial

$$R_s(x) = \begin{cases} \begin{array}{l}(x+1)(x^{2n-1} + x^{2n-4} + \cdots + x^{n+3}) + x^{n+1} \\ +x^n + x^{n-1} + (x+1)(x^{n-4} + x^{n-7} + \cdots + 1),\end{array} & 2 \nmid s; \\ \begin{array}{l}(x+1)(x^{2n-1} + x^{2n-4} + \cdots + x^{n+2}) + x^n \\ +(x+1)(x^{n-3} + x^{n-6} + \cdots + 1),\end{array} & 2 \mid s. \end{cases}$$

It is easily checked that $(x^2+x+1)R_s(x) \sim x^{2^{s+1}} + x^{2^s} + 1 \sim (x^2+x+1)^{2^s}$, so that $R_s(x) \sim (x^2+x+1)^{2^s-1}$. In this case, the number of odd coefficients is $(2^{s+2} - (-1)^s)/3$.

Now we pass to the general case. Let the number n be represented in the binary system as

$$n = \underbrace{11\dots1}_{a_k}\underbrace{00\dots0}_{b_k}\underbrace{11\dots1}_{a_{k-1}}\underbrace{00\dots0}_{b_{k-1}}\dots\underbrace{11\dots1}_{a_1}\underbrace{00\dots0}_{b_1},$$

$b_i > 0$ $(i > 1)$, $b_1 \ge 0$, and $a_i > 0$. Then $n = \sum_{i=1}^k 2^{s_i}(2^{a_i} - 1)$, where $s_i = b_1 + a_1 + b_2 + a_2 + \cdots + b_i$, and hence

$$u_n(x) = (x^2+x+1)^n = \prod_{i=1}^k (x^2+x+1)^{2^{s_i}(2^{a_i}-1)} \sim \prod_{i=1}^k R_{a_i}(x^{2^{s_i}}).$$

Let $R_{a_i}(x^{2^{s_i}}) \sim x^{r_{i,1}} + \cdots + x^{r_{i,d_i}}$; clearly $r_{i,j}$ is divisible by 2^{s_i} and $r_{i,j} \le 2^{s_i+1}(2^{a_i} - 1) < 2^{s_{i+1}}$, so that for any j, $r_{i,j}$ can have nonzero binary digits only in some position t, $s_i \le t \le s_{i+1} - 1$. Therefore, in

$$\prod_{i=1}^k R_{a_i}(x^{2^{s_i}}) \sim \prod_{i=1}^k (x^{r_{i,1}} + \cdots + x^{r_{i,d_i}}) = \sum_{i=1}^k \sum_{p_i=1}^{d_i} x^{r_{1,p_1}+r_{2,p_2}+\cdots+r_{k,p_k}}$$

all the exponents $r_{1,p_1} + r_{2,p_2} + \cdots + r_{k,p_k}$ are different, so that the number of odd coefficients in $u_n(x)$ is

$$\prod_{i=1}^k d_i = \prod_{i=1}^k \frac{2^{a_i+2} - (-1)^{a_i}}{3}.$$

3. Let R be the circumradius, r the inradius, s the semiperimeter, Δ the area of ABC and Δ' the area of $A'B'C'$. The angles of triangle $A'B'C'$ are $A' = 90^\circ - A/2$, $B' = 90^\circ - B/2$, and $C' = 90^\circ - C/2$, and hence

$$\Delta = 2R^2 \sin A \sin B \sin C$$

$$\text{and } \Delta' = 2R^2 \sin A' \sin B' \sin C' = 2R^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Hence,

$$\frac{\Delta}{\Delta'} = \frac{\sin A \sin B \sin C}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{2r}{R},$$

where we have used that $r = AI\sin(A/2) = \cdots = 4R\sin(A/2)\cdot\sin(B/2)\cdot\sin(C/2)$. Euler's inequality $2r \le R$ shows that $\Delta \le \Delta'$.

Second solution. Let H be orthocenter of triangle ABC, and H_a, H_b, H_c points symmetric to H with respect to BC, CA, AB, respectively. Since $\angle BH_aC = \angle BHC = 180^\circ - \angle A$, points H_a, H_b, H_c lie on the circumcircle of ABC, and the area of the hexagon $AH_cBH_aCH_b$ is double the area of ABC. (1)

Let us apply the analogous result for the triangle $A'B'C'$. Since its orthocenter is the incenter I of ABC, and the point symmetric to I with respect to $B'C'$ is the point A, we find by (1) that the area of the hexagon $AC'BA'CB'$ is double the area of $A'B'C'$.

But it is clear that the area of ΔCH_aB is less than or equal to the area of $\Delta CA'B$ etc.; hence, the area of $AH_cBH_aCH_b$ does not exceed the area of $AC'BA'CB'$. The statement follows immediately.

4. Suppose that the numbers of any two neighboring squares differ by at most $n-1$. For $k = 1, 2, \dots, n^2 - n$, let A_k, B_k, and C_k denote, respectively, the sets of squares numbered by $1, 2, \dots, k$; of squares numbered by $k + n, k+n+1, \dots, n^2$; and of squares numbered by $k+1, \dots, k+n-1$. By the assumption, the squares from A_k and B_k have no edge in common; C_k has $n-1$ elements only. Consequently, for each k there exists a row and a column all belonging either to A_k, or to B_k.

   For $k = 1$, it must belong to B_k, while for $k = n^2 - n$ it belongs to A_k. Let k be the smallest index such that A_k contains a whole row and a whole column. Since B_{k-1} has that property too, it must have at least two squares in common with A_k, which is impossible.

5. Let $n = 2k$ and let $A = \{A_1, \dots, A_{2k+1}\}$ denote the family of sets with the desired properties. Since every element of their union B belongs to at least two sets of A, it follows that $A_j = \bigcup_{i\ne j} A_i \cap A_j$ holds for every $1 \le j \le 2k+1$. Since each intersection in the sum has at most one element and A_j has $2k$ elements, it follows that every element of A_j, i.e., in general of B, is a member of exactly two sets.

   We now prove that k is even, assuming that the marking described in the problem exists. We have already shown that for every two indices $1 \le j \le 2k+1$ and $i \ne j$ there exists a unique element contained in both A_i and A_j. On a $2k \times 2k$ matrix let us mark in the ith column and jth row for $i \ne j$ the number that was joined to the element of B in $A_i \cap A_j$. In the ith row and column let us mark the number of the element of B in $A_i \cap A_{2k+1}$. In each row from the conditions of the marking there must be an even number of zeros. Hence, the total number of zeros in the matrix is even. The matrix is symmetric with respect to its main diagonal; hence it has an even number of zeros outside its main diagonal. Hence, the number of zeros on the main diagonal must also be even and this number equals the number of elements in A_{2k+1} that are marked with 0, which is k. Hence k must be even.

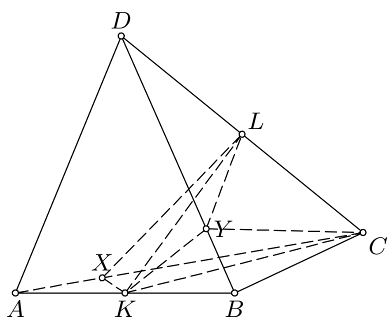
For even k we note that the dimensions of a $2k \times 2k$ matrix are divisible by 4. Tiling the entire matrix with the 4×4 submatrix

$$Q = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

we obtain a marking that indeed satisfies all the conditions of the problem; hence we have shown that the marking is possible if and only if k is even.

6. Let ω be the plane through AB, parallel to CD. Define the point transformation $f : X \mapsto X'$ in space as follows. If $X \in KL$, then $X' = X$; otherwise, let ω_X be the plane through X parallel to ω: then X' is the point symmetric to X with respect to the intersection point of KL with ω_X. Clearly, $f(A) = B$, $f(B) = A$, $f(C) = D$, $f(D) = C$; hence f maps the tetrahedron onto itself.
We shall show that f preserves volumes. Let $s : X \mapsto X''$ denote the symmetry with respect to KL, and g the transformation mapping X'' into X'; then $f = g \circ s$. If points $X_1'' = s(X_1)$ and $X_2'' = s(X_2)$ have the property that $X_1''X_2''$ is parallel to KL, then the segments $X_1''X_2''$ and $X_1'X_2'$ have the same length and lie on the same line. Then by Cavalieri's principle g preserves volume, and so does f.
Now, if α is any plane containing the line KL, the two parts of the tetrahedron on which it is partitioned by α are transformed into each other by f, and therefore have the same volumes.

Second solution. Suppose w.l.o.g. that the plane α through KL meets the interiors of edges AC and BD at X and Y. Let $\overrightarrow{AX} = \lambda\overrightarrow{AC}$ and $\overrightarrow{BY} = \mu\overrightarrow{BD}$, for $0 \le \lambda, \mu \le 1$. Then the vectors $\overrightarrow{KX} = \lambda\overrightarrow{AC} - \overrightarrow{AB}/2$, $\overrightarrow{KY} = \mu\overrightarrow{BD} + \overrightarrow{AB}/2$, $\overrightarrow{KL} = \overrightarrow{AC}/2 + \overrightarrow{BD}/2$ are coplanar; i.e., there exist real numbers a, b, c, not all zero, such that



$$\vec{0} = a\overrightarrow{KX} + b\overrightarrow{KY} + c\overrightarrow{KL} = (\lambda a + c/2)\overrightarrow{AC} + (\mu b + c/2)\overrightarrow{BD} + \frac{b-a}{2}\overrightarrow{AB}.$$

Since $\overrightarrow{AC}, \overrightarrow{BD}, \overrightarrow{AB}$ are linearly independent, we must have $a = b$ and $\lambda = \mu$. We need to prove that the volume of the polyhedron $KXLYBC$, which is one of the parts of the tetrahedron $ABCD$ partitioned by α, equals half of the volume V of $ABCD$. Indeed, we obtain

$$V_{KXLYBC} = V_{KXXLC} + V_{KBYLC} = \frac{1}{4}(1-\lambda)V + \frac{1}{4}(1+\mu)V = \frac{1}{2}V.$$

7. The algebraic equation $x^3 - 3x^2 + 1 = 0$ admits three real roots β, γ, a, with
$$-0.6 < \beta < -0.5, \qquad 0.6 < \gamma < 0.7, \qquad \sqrt{8} < a < 3.$$
Define, for all integers n,
$$u_n = \beta^n + \gamma^n + a^n.$$
It holds that $u_{n+3} = 3u_{n+2} - u_n$.
Obviously, $0 < \beta^n + \gamma^n < 1$ for all $n \geq 2$, and we see that $u_n - 1 = [a^n]$ for $n \geq 2$. It is now a question whether $u_{1788} - 1$ and $u_{1988} - 1$ are divisible by 17.
Working modulo 17, we get $u_0 \equiv 3$, $u_1 \equiv 3$, $u_2 \equiv 9$, $u_3 \equiv 7$, $u_4 \equiv 1, \dots, u_{16} = 3$, $u_{17} = 3$, $u_{18} = 9$. Thus, u_n is periodic modulo 17, with period 16. Since $1788 = 16 \cdot 111 + 12$, $1988 = 16 \cdot 124 + 4$, it follows that $u_{1788} \equiv u_{12} \equiv 1$ and $u_{1988} \equiv u_4 = 1$. So, $[a^{1788}]$ and $[a^{1988}]$ are divisible by 17.

Second solution. The polynomial $x^3 - 3x^2 + 1$ allows the factorization modulo 17 as $(x-4)(x-5)(x+6)$. Hence it is easily seen that $u_n \equiv 4^n + 5^n + (-6)^n$. Fermat's theorem gives us $4^n \equiv 5^n \equiv (-6)^n \equiv 1$ for $16 \mid n$, and the rest follows easily.

Remark. In fact, the roots of $x^3 - 3x^2 + 1 = 0$ are $\frac{1}{2\sin 10^\circ}$, $\frac{1}{2\sin 50^\circ}$, and $-\frac{1}{2\sin 70^\circ}$.

8. Consider first the case that the vectors are on the same line. Then if e is a unit vector, we can write $u_1 = x_1 e, \dots, u_n = x_n e$ for scalars x_i, $|x_i| \leq 1$, with zero sum. It is now easy to permute $x_1, x_2, \dots, x_n$ into $z_1, z_2, \dots z_n$ so that $|z_1| \leq 1, |z_1 + z_2| \leq 1, \dots, |z_1 + z_2 + \cdots + z_{n-1}| \leq 1$. Indeed, suppose w.l.o.g. that $z_1 = x_1 \geq 0$; then we choose $z_2, \dots, z_r$ from the x_i's to be negative, until we get to the first r with $x_1 + x_2 + \cdots + x_r \leq 0$; we continue successively choosing positive z_j's from the remaining x_i's until we get the first partial sum that is positive, and so on. It is easy to verify that $|z_1 + z_2 + \cdots + z_j| \leq 1$ for all $j = 1, 2, \dots, n$.
Now we pass to the general case. Let s be the longest vector that can be obtained by summing a subset of $u_1, \dots, u_m$, and assume w.l.o.g. that $s = u_1 + \cdots + u_p$. Further, let δ and δ' respectively be the lines through the origin O in the direction of s and perpendicular to s, and e, e' respectively the unit vectors on δ and δ'. Put $u_i = x_i e + y_i e'$, $i = 1, 2, \dots, m$. By the definition of δ and δ', we have $|x_i|, |y_i| \leq 1$; $x_1 + \cdots + x_m = y_1 + \cdots + y_m = 0$; $y_1 + \cdots + y_p = y_{p+1} + \cdots + y_m = 0$; we also have $x_{p+1}, \dots, x_m \leq 0$ (otherwise, if $x_i > 0$ for some i, then $|s + v_i| > |s|$), and similarly $x_1, \dots, x_p \geq 0$. Finally, suppose by the one-dimensional case that $y_1, \dots, y_p$ and $y_{p+1}, \dots, y_m$ are permuted in such a way that all the sums $y_1 + \cdots + y_i$ and $y_{p+1} + \cdots + y_{p+i}$ are ≤ 1 in absolute value.
We apply the construction of the one-dimensional case to $x_1, \dots, x_m$ taking, as described above, positive z_i's from $x_1, x_2, \dots, x_p$ and negative ones

from $x_{p+1}, \dots, x_m$, but so that the order is preserved; this way we get a permutation $x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_m}$. It is then clear that each sum $y_{\sigma_1} + y_{\sigma_2} + \dots + y_{\sigma_k}$ decomposes into the sum $(y_1 + y_2 + \dots + y_l) + (y_{p+1} + \dots + y_{p+n})$ (because of the preservation of order), and that each of these sums is less than or equal to 1 in absolute value. Thus each sum $u_{\sigma_1} + \dots + u_{\sigma_k}$ is composed of a vector of length at most 2 and an orthogonal vector of length at most 1, and so is itself of length at most $\sqrt{5}$.

9. Let us assume $\frac{a^2+b^2}{ab+1} = k \in \mathbb{N}$. We then have $a^2 - kab + b^2 = k$. Let us assume that k is not an integer square, which implies $k \geq 2$. Now we observe the minimal pair (a, b) such that $a^2 - kab + b^2 = k$ holds. We may assume w.l.o.g. that $a \geq b$. For $a = b$ we get $k = (2-k)a^2 \leq 0$; hence we must have $a > b$.
Let us observe the quadratic equation $x^2 - kbx + b^2 - k = 0$, which has solutions a and a_1. Since $a + a_1 = kb$, it follows that $a_1 \in \mathbb{Z}$. Since $a > kb$ implies $k > a + b^2 > kb$ and $a = kb$ implies $k = b^2$, it follows that $a < kb$ and thus $b^2 > k$. Since $aa_1 = b^2 - k > 0$ and $a > 0$, it follows that $a_1 \in \mathbb{N}$ and $a_1 = \frac{b^2-k}{a} < \frac{a^2-1}{a} < a$. We have thus found an integer pair (a_1, b) with $0 < a_1 < a$ that satisfies the original equation. This is a contradiction of the initial assumption that (a, b) is minimal. Hence k must be an integer square.

10. We claim that if the family $\{A_1, \dots, A_t\}$ separates the n-set N, then $2^t \geq n$. The proof goes by induction. The case $t = 1$ is clear, so suppose that the claim holds for $t - 1$. Since A_t does not separate elements of its own or its complement, it follows that $\{A_1, \dots, A_{t-1}\}$ is separating for both A_t and $N \smallsetminus A_t$, so that $|A_t|, |N \smallsetminus A_t| \leq 2^{t-1}$. Then $|N| \leq 2 \cdot 2^{t-1} = 2^t$, as claimed.
Also, if the set N with $N = 2^t$ is separated by $\{A_1, \dots, A_t\}$, then (precisely) one element of N is not covered. To show this, we again use induction. This is trivial for $t = 1$, so let $t \geq 1$. Since $A_1, \dots, A_{t-1}$ separate both A_t and $N \smallsetminus A_t$, $N \smallsetminus A_t$ must have exactly 2^{t-1} elements, and thus one of its elements is not covered by $A_1, \dots, A_{t-1}$, and neither is covered by A_t. We conclude that a separating and covering family of t subsets can exist only if $n \leq 2^t - 1$.
We now construct such subsets for the set N if $2^{t-1} \leq n \leq 2^t - 1$, $t \geq 1$. For $t = 1$, put $A_1 = \{1\}$. In the step from t to $t+1$, let $N = N' \cup N'' \cup \{y\}$, where $|N'|, |N''| \leq 2^{t-1}$; let $A'_1, \dots, A'_t$ be subsets covering and separating N' and $A''_1, \dots, A''_t$ such subsets for N''. Then the subsets $A_i = A'_i \cup A''_i$ $(i = 1, \dots, t)$ and $A_{t+1} = N'' \cup \{y\}$ obviously separate and cover N.
The answer: $t = [\log_2 n] + 1$.

Second solution. Suppose that the sets $A_1, \dots, A_t$ cover and separate N. Label each element $x \in N$ with a string of $(x_1 x_2 \dots x_t)$ of 0's and 1's, where x_i is 1 when $x \in A_i$, 0 otherwise. Since the A_i's separate, these strings are distinct; since they cover, the string $(00 \dots 0)$ does not occur.

Hence $n \le 2^t - 1$. Conversely, for $2^{t-1} \le n < 2^t$, represent the elements of N in base 2 as strings of 0's and 1's of length t. For $1 \le i \le t$, take A_i to be the set of numbers in N whose binary string has a 1 in the ith place. These sets clearly cover and separate.

11. The answer is 32. Write the combinations as triples $k = (x, y, z)$, $0 \le x, y, z \le 7$. Define the sets $K_1 = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$, $K_2 = \{(2,0,0),(0,2,0),(0,0,2),(2,2,2)\}$, $K_3 = \{(0,0,0),(4,4,4)\}$, and $K = \{k = k_1 + k_2 + k_3 \mid k_i \in K_i,\ i = 1,2,3\}$. There are 32 combinations in K. We shall prove that these combinations will open the safe in every case.
Let $t = (a, b, c)$ be the right combination. Set $k_3 = (0,0,0)$ if at least two of a, b, c are less than 4, and $k_3 = (4,4,4)$ otherwise. In either case, the difference $t - k_3$ contains two nonnegative elements not greater than 3. Choosing a suitable k_2 we can achieve that $t - k_3 - k_2$ contains two elements that are 0, 1. So, there exists k_1 such that $t - k_3 - k_2 - k_1 = t - k$ contains two zeros, for $k \in K$. This proves that 32 is sufficient.
Suppose that K is a set of at most 31 combinations. We say that $k \in K$ *covers* the combination k_1 if k and k_1 differ in at most one position. One of the eight sets $M_i = \{(i, y, z) \mid 0 \le y, z \le 7\}$, $i = 0, 1, \dots, 7$, contains at most three elements of K. Suppose w.l.o.g. that this is M_0. Further, among the eight sets $N_j = \{(0, j, z) \mid 0 \le z \le 7\}$, $j = 0, \dots, 7$, there are at least five, say w.l.o.g. $N_0, \dots, N_4$, not containing any of the combinations from K.
Of the 40 elements of the set $N = \{(0, y, z) \mid 0 \le y \le 4,\ 0 \le z \le 7\}$, at most $5 \cdot 3 = 15$ are covered by $K \cap M_0$, and at least 25 aren't. Consequently, the intersection of K with $L = \{(x, y, z) \mid 1 \le x \le 7, 0 \le y \le 4, 0 \le z \le 7\}$ contains at least 25 elements. So K has at most $31 - 25 = 6$ elements in the set $P = \{(x, y, z) \mid 0 \le x \le 7, 5 \le y \le 7, 0 \le z \le 7\}$. This implies that for some $j \in \{5, 6, 7\}$, say w.l.o.g. $j = 7$, K contains at most two elements in $Q_j = \{(x, y, z) \mid 0 \le x, z \le 7,\ y = j\}$; denote them by l_1, l_2. Of the 64 elements of Q_7, at most 30 are covered by l_1 and l_2. But then there remain 34 uncovered elements, which must be covered by different elements of $K \setminus Q_7$, having itself less at most 29 elements. Contradiction.

12. Let $E(XYZ)$ stand for the area of a triangle XYZ. We have

$$\frac{E_1}{E} = \frac{E(AMR)}{E(AMK)} \cdot \frac{E(AMK)}{E(ABK)} \cdot \frac{E(ABK)}{E(ABC)} = \frac{MR}{MK} \cdot \frac{AM}{AB} \cdot \frac{BK}{BC} \qquad \Rightarrow$$

$$\left(\frac{E_1}{E}\right)^{1/3} \le \frac{1}{3}\left(\frac{MR}{MK} + \frac{AM}{AB} + \frac{BK}{BC}\right).$$

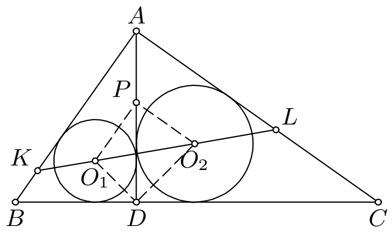
We similarly obtain

$$\left(\frac{E_2}{E}\right)^{1/3} \le \frac{1}{3}\left(\frac{KR}{MK} + \frac{BM}{AB} + \frac{CK}{BC}\right).$$

Therefore $(E_1/E)^{1/3} + (E_2/E)^{1/3} \le 1$, i.e., $\sqrt[3]{E_1} + \sqrt[3]{E_2} \le \sqrt[3]{E}$. Analogously, $\sqrt[3]{E_3} + \sqrt[3]{E_4} \le \sqrt[3]{E}$ and $\sqrt[3]{E_5} + \sqrt[3]{E_6} \le \sqrt[3]{E}$; hence

$$\begin{aligned} &8\sqrt[6]{E_1E_2E_3E_4E_5E_6} \\ &\quad = 2(\sqrt[3]{E_1}\sqrt[3]{E_2})^{1/2} \cdot 2(\sqrt[3]{E_3}\sqrt[3]{E_4})^{1/2} \cdot 2(\sqrt[3]{E_5}\sqrt[3]{E_6})^{1/2} \\ &\quad \le (\sqrt[3]{E_1} + \sqrt[3]{E_2}) \cdot (\sqrt[3]{E_3} + \sqrt[3]{E_4}) \cdot (\sqrt[3]{E_5} + \sqrt[3]{E_6}) \le E. \end{aligned}$$

13. Let $AB = c$, $AC = b$, $\angle CBA = \beta$, $BC = a$, and $AD = h$.
Let r_1 and r_2 be the inradii of ABD and ADC respectively and O_1 and O_2 the centers of the respective incircles. We obviously have $r_1/r_2 = c/b$. We also have $DO_1 = \sqrt{2}r_1$, $DO_2 = \sqrt{2}r_2$, and $\angle O_1DA = \angle O_2DA = 45°$. Hence $\angle O_1DO_2 = 90°$ and $DO_1/DO_2 = c/b$ from which it follows that $\triangle O_1DO_2 \sim \triangle BAC$.



We now define P as the intersection of the circumcircle of $\triangle O_1DO_2$ with DA. From the above similarity we have $\angle DPO_2 = \angle DO_1O_2 = \beta = \angle DAC$. It follows that $PO_2 \parallel AC$ and from $\angle O_1PO_2 = 90°$ it also follows that $PO_1 \parallel AB$. We also have $\angle PO_1O_2 = \angle PO_2O_1 = 45°$; hence $\angle LKA = \angle KLA = 45°$, and thus $AK = AL$. From $\angle O_1KA = \angle O_1DA = 45°$, $O_1A = O_1A$, and $\angle O_1KA = \angle O_1DA$ we have $\triangle O_1KA \cong \triangle O_1DA$ and hence $AL = AK = AD = h$. Thus

$$\frac{E}{E_1} = \frac{ah/2}{h^2/2} = \frac{a}{h} = \frac{a^2}{ah} = \frac{b^2+c^2}{bc} \ge 2\,.$$

Remark. It holds that for an arbitrary triangle ABC, $AK = AL$ if and only if $AB = AC$ or $\measuredangle BAC = 90°$.

14. Consider an array $[a_{ij}]$ of the given property and denote the sums of the rows and the columns by r_i and c_j respectively. Among the r_i's and c_j's, one element of $[-n, n]$ is missing, so that there are at least n nonnegative and n nonpositive sums. By permuting rows and columns we can obtain an array in which $r_1, \dots, r_k$ and $c_1, \dots, c_{n-k}$ are nonnegative. Clearly

$$\sum_{i=1}^{n} |r_i| + \sum_{j=1}^{n} |c_j| \ge \sum_{r=-n}^{n} |r| - n = n^2.$$

But on the other hand,

$$\sum_{i=1}^{n}|r_i|+\sum_{j=1}^{n}|c_j|=\sum_{i=1}^{k}r_i-\sum_{i=k+1}^{n}r_i+\sum_{j=1}^{n-k}c_j-\sum_{j=n-k+1}^{n}c_j=$$
$$=\sum_{i\le k}a_{ij}-\sum_{i>k}a_{ij}+\sum_{j\le n-k}a_{ij}-\sum_{j>n-k}a_{ij}=$$
$$=2\sum_{i=1}^{k}\sum_{j=1}^{n-k}a_{ij}-2\sum_{i=k+1}^{n}\sum_{j=n-k+1}^{n}a_{ij}\le 4k(n-k).$$

This yields $n^2 \le 4k(n-k)$, i.e., $(n-2k)^2 \le 0$, and thus n must be even. We proceed to show by induction that for all even n an array of the given type exists. For $n = 2$ the array in Fig. 1 is good. Let such an $n \times n$ array be given for some even $n \ge 2$, with $c_1 = n$, $c_2 = -n+1$, $c_3 = n-2, \dots, c_{n-1} = 2$, $c_n = -1$ and $r_1 = n-1$, $r_2 = -n+2, \dots, r_{n-1} = 1$, $r_n = 0$. Upon enlarging this array as indicated in Fig. 2, the positive sums are increased by 2, the nonpositive sums are decreased by 2, and the missing sums $-1, 0, 1, 2$ occur in the new rows and columns, so that the obtained array $(n+2) \times (n+2)$ is of the same type.

1	0
1	-1

Fig. 1

<table>
<tr><td colspan="4" rowspan="4">$n \times n$</td><td>1</td><td>1</td></tr>
<tr><td>-1</td><td>-1</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>-1</td><td>-1</td></tr>
<tr><td>1</td><td>-1</td><td>1</td><td>-1</td><td>1</td><td>0</td></tr>
<tr><td>1</td><td>-1</td><td>1</td><td>-1</td><td>1</td><td>-1</td></tr>
</table>

Fig. 2

15. Referring to the description of L_A, we have $\angle AMN = \angle AHN = 90^\circ - \angle HAC = \angle C$, and similarly $\angle ANM = \angle B$. Since the triangle ABC is acute-angled, the line L_A lies inside the angle A. Hence if $P = L_A \cap BC$ and $Q = L_B \cap AC$, we get $\angle BAP = 90^\circ - \angle C$; hence AP passes through the circumcenter O of ΔABC. Similarly we prove that L_B and L_C contains the circumcenter O also. It follows that L_A, L_B and L_C intersect at the point O.

Remark. Without identifying the point of intersection, one can prove the concurrence of the three lines using Ceva's theorem, in usual or trigonometric form.

16. Let $f(x) = \sum_{k=1}^{70} \frac{k}{x-k}$. For all integers $i = 1, \dots, 70$ we have that $f(x)$ tends to plus infinity as x tends downward to i, and $f(x)$ tends to minus infinity as x tends upward to i. As x tends to infinity, $f(x)$ tends to 0. Hence it follows that there exist $x_1, x_2, \dots, x_{70}$ such that $1 < x_1 < 2 < x_2 < 3 < \cdots < x_{69} < 70 < x_{70}$ and $f(x_i) = \frac{5}{4}$ for all $i = 1, \dots, 70$. Then the solution to the inequality is given by $S = \bigcup_{i=1}^{70}(i, x_i]$.
For numbers x for which $f(x)$ is well-defined, the equality $f(x) = \frac{5}{4}$ is equivalent to

$$p(x) = \prod_{j=1}^{70}(x-j) - \frac{4}{5}\sum_{k=1}^{70} k \prod_{\substack{j=1 \\ j\neq k}}^{70}(x-j) = 0.$$

The numbers $x_1, x_2, \dots, x_{70}$ are then the zeros of this polynomial. The sum $\sum_{i=1}^{70} x_i$ is then equal to minus the coefficient of x^{69} in p, which equals $\sum_{i=1}^{70}\left(i + \frac{4}{5}i\right)$. Finally,

$$|S| = \sum_{i=1}^{70}(x_i - i) = \frac{4}{5}\cdot\sum_{i=1}^{70} i = \frac{4}{5}\cdot\frac{70\cdot 71}{2} = 1988\ .$$

17. Let AC and AD meet BE in R, S, respectively. Then by the conditions of the problem,

$$\begin{aligned}
&\angle AEB = \angle EBD = \angle BDC = \angle DBC = \angle ADB = \angle EAD = \alpha,\\
&\angle ABE = \angle BEC = \angle ECD = \angle CED = \angle ACE = \angle BAC = \beta,\\
&\angle BCA = \angle CAD = \angle ADE = \gamma.
\end{aligned}$$

Since $\angle SAE = \angle SEA$, it follows that $AS = SE$, and analogously $BR = RA$. But $BSDC$ and $REDC$ are parallelograms; hence $BS = CD = RE$, giving us $BR = SE$ and $AR = AS$. Then also $AC = AD$, because $RS \parallel CD$. We deduce that $2\beta = \angle ACD = \angle ADC = 2\alpha$, i.e., $\alpha = \beta$.
It will be sufficient to show that $\alpha = \gamma$, since that will imply $\alpha = \beta = \gamma = 36^\circ$. We have that the sum of the interior angles of ACD is $4\alpha + \gamma = 180^\circ$. We have

$$\frac{\sin\gamma}{\sin\alpha} = \frac{AE}{DE} = \frac{AE}{CD} = \frac{AE}{RE} = \frac{\sin(2\alpha+\gamma)}{\sin(\alpha+\gamma)},$$

i.e., $\cos\alpha - \cos(\alpha+2\gamma) = 2\sin\gamma\sin(\alpha+\gamma) = 2\sin\alpha\sin(2\alpha+\gamma) = \cos(\alpha+\gamma) - \cos(3\alpha+\gamma)$. From $4\alpha+\gamma = 180^\circ$ we obtain $-\cos(3\alpha+\gamma) = \cos\alpha$. Hence

$$\cos(\alpha+\gamma) + \cos(\alpha+2\gamma) = 2\cos\frac{\gamma}{2}\cos\frac{2\alpha+3\gamma}{2} = 0,$$

so that $2\alpha + 3\gamma = 180^\circ$. It follows that $\alpha = \gamma$.

Second solution. We have $\angle BEC = \angle ECD = \angle DEC = \angle ECA = \angle CAB$, and hence the trapezoid $BAEC$ is cyclic; consequently, $AE = BC$. Similarly $AB = ED$, and $ABCD$ is cyclic as well. Thus $ABCDE$ is cyclic and has all sides equal; i.e., it is regular.

18. (i) Define $\angle APO = \phi$ and $S = AB^2 + AC^2 + BC^2$. We calculate $PA = 2r\cos\phi$ and $PB, PC = \sqrt{R^2 - r^2\cos^2\phi} \pm r\sin\phi$. We also have $AB^2 = PA^2 + PB^2$, $AC^2 = PA^2 + PC^2$ and $BC = BP + PC$. Combining all these we obtain

$$\begin{aligned}
S = AB^2 + AC^2 + BC^2 &= 2(PA^2 + PB^2 + PC^2 + PB\cdot PC)\\
&= 2(4r^2\cos^2\phi + 2(R^2 - r^2\cos^2\phi + r^2\sin^2\phi) + R^2 - r^2)\\
&= 6R^2 + 2r^2.
\end{aligned}$$

Hence it follows that S is constant; i.e., it does not depend on ϕ.

(ii) Let B_1 and C_1 respectively be points such that $APBB_1$ and $APCC_1$ are rectangles. It is evident that B_1 and C_1 lie on the larger circle and that $\overrightarrow{PU} = \frac{1}{2}\overrightarrow{PB_1}$ and $\overrightarrow{PV} = \frac{1}{2}\overrightarrow{PC_1}$. It is evident that we can arrange for an arbitrary point on the larger circle to be B_1 or C_1. Hence, the locus of U and V is equal to the circle obtained when the larger circle is shrunk by a factor of $1/2$ with respect to point P.

19. We will show that $f(n) = n$ for every n (thus also $f(1988) = 1988$).
Let $f(1) = r$ and $f(2) = s$. We obtain respectively the following equalities: $f(2r) = f(r+r) = 2$; $f(2s) = f(s+s) = 4$; $f(4) = f(2+2) = 4r$; $f(8) = f(4+4) = 4s$; $f(5r) = f(4r+r) = 5$; $f(r+s) = 3$; $f(8) = f(5+3) = 6r+s$. Then $4s = 6r + s$, which means that $s = 2r$.
Now we prove by induction that $f(nr) = n$ and $f(n) = nr$ for every $n \geq 4$. First we have that $f(5) = f(2+3) = 3r + s = 5r$, so that the statement is true for $n = 4$ and $n = 5$. Suppose that it holds for $n-1$ and n. Then $f(n+1) = f(n-1+2) = (n-1)r + 2r = (n+1)r$, and $f((n+1)r) = f((n-1)r + 2r) = (n-1) + 2 = n+1$. This completes the induction.
Since $4r \geq 4$, we have that $f(4r) = 4r^2$, and also $f(4r) = 4$. Then $r = 1$, and consequently $f(n) = n$ for every natural number n.

Second solution. $f(f(1)+n+m) = f(f(1)+f(f(n)+f(m))) = 1+f(n)+f(m)$, so $f(n) + f(m)$ is a function of $n + m$. Hence $f(n+1) + f(1) = f(n)+f(2)$ and $f(n+1)-f(n) = f(2)-f(1)$, implying that $f(n) = An+B$ for some constants A, B. It is easy to check that $A = 1$, $B = 0$ is the only possibility.

20. Suppose that $A_n = \{1, 2, \dots, n\}$ is partitioned into B_n and C_n, and that neither B_n nor C_n contains 3 distinct numbers one of which is equal to the product of the other two. If $n \geq 96$, then the divisors of 96 must be split up. Let w.l.o.g. $2 \in B_n$. There are four cases.

(i) $3 \in B_n, 4 \in B_n$. Then $6, 8, 12 \in C_n \Rightarrow 48, 96 \in B_n$. A contradiction for $96 = 2 \cdot 48$.

(ii) $3 \in B_n, 4 \in C_n$. Then $6 \in C_n$, $24 \in B_n$, $8, 12, 48 \in C_n$. A contradiction for $48 = 6 \cdot 8$.

(iii) $3 \in C_n, 4 \in B_n$. Then $8 \in C_n$, $24 \in B_n$, $6, 48 \in C_n$. A contradiction for $48 = 6 \cdot 8$.

(iv) $3 \in C_n, 4 \in C_n$. Then $12 \in B_n$, $6, 24 \in C_n$. A contradiction for $24 = 4 \cdot 6$.

If $n = 95$, there is a very large number of ways of partitioning A_n. For example, $B_n = \{1, p, p^2, p^3q^2, p^4q, p^2qr \mid p, q, r = \text{distinct primes}\}$, $C_n = \{p^3, p^4, p^5, p^6, pq, p^2q, p^3q, p^2q^2, pqr \mid p, q, r = \text{distinct primes}\}$. Then $B_{95} = \{1, 2, 3, 4, 5, 7, 9, 11, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47, 48, 49, 53, 59, 60, 61, 67, 71, 72, 73, 79, 80, 83, 84, 89, 90\}$.

21. Let X be the set of all ordered triples $a = (a_1, a_2, a_3)$ for $a_i \in \{0, 1, \dots, 7\}$. Write $a \prec b$ if $a_i \le b_i$ for $i = 1, 2, 3$ and $a \ne b$. Call a subset $Y \subset X$ *independent* if there are no $a, b \in Y$ with $a \prec b$. We shall prove that an independent set contains at most 48 elements.
For $j = 0, 1, \dots, 21$ let $X_j = \{(a_1, a_2, a_3) \in X \mid a_1 + a_2 + a_3 = j\}$. If $x \prec y$ and $x \in X_j$, $y \in X_{j+1}$ for some j, then we say that y is a *successor* of x, and x a *predecessor* of y.
Lemma. If A is an m-element subset of X_j and $j \le 10$, then there are at least m distinct successors of the elements of A.
Proof. For $k = 0, 1, 2, 3$ let $X_{j,k} = \{(a_1, a_2, a_3) \in X_j \mid \min(a_1, a_2, a_3, 7 - a_1, 7 - a_2, 7 - a_3) = k\}$. It is easy to verify that every element of $X_{j,k}$ has at least two successors in $X_{j+1,k}$ and every element of $X_{j+1,k}$ has at most two predecessors in $X_{j,k}$. Therefore the number of elements of $A \cap X_{j,k}$ is not greater than the number of their successors. Since X_j is a disjoint union of $X_{j,k}$, $k = 0, 1, 2, 3$, the lemma follows.
Similarly, elements of an m-element subset of X_j, $j \ge 11$, have at least m predecessors.
Let Y be an independent set, and let p, q be integers such that $p < 10 < q$. We can transform Y by replacing all the elements of $Y \cap X_p$ with their successors, and all the elements of $Y \cap X_q$ with their predecessors. After this transformation Y will still be independent, and by the lemma its size will not be reduced. Every independent set can be eventually transformed in this way into a subset of X_{10}, and X_{10} has exactly 48 elements.

22. Set $X = \sum_{i=1}^{p} x_i$ and w.l.o.g. assume that $X \ge 0$ (if $(x_1, \dots, x_p)$ is a solution, then $(-x_1, \dots, -x_p)$ is a solution too). Since $x^2 \ge x$ for all integers x, it follows that $\sum_{i=1}^{p} x_i^2 \ge X$.
If the last inequality is an equality, then all x_i's are 0 or 1; then, taking that there are a 1's, the equation becomes $4p + 1 = 4(a + 1) + \frac{4}{a-1}$, which forces $p = 6$ and $a = 5$.
Otherwise, we have $X + 1 \le \sum_{i=1}^{p} x_i^2 = \frac{4}{4p+1} X^2 + 1$, so $X \ge p + 1$. Also, by the Cauchy–Schwarz inequality, $X^2 \le p \sum_{i=1}^{p} x_i^2 = \frac{4p}{4p+1} X^2 + p$, so $X^2 \le 4p^2 + p$ and $X \le 2p$. Thus $1 \le X/p \le 2$. However,

$$\begin{aligned}\sum_{i=1}^{p} \left(x_i - \frac{X}{p}\right)^2 &= \sum x_i^2 - \frac{2X}{p} \sum x_i + \frac{X^2}{p} \\ &= \sum x_i^2 - p\frac{X^2}{p^2} = 1 - \frac{X^2}{p(4p+1)} < 1,\end{aligned}$$

and we deduce that $-1 < x_i - X/p < 1$ for all i. This finally gives $x_i \in \{1, 2\}$. Suppose there are b 2's. Then $3b + p = 4(b + p)^2/(4p + 1) + 1$, so $p = b + 1/(4b - 3)$, which leads to $p = 2$, $b = 1$.
Thus there are no solutions for any $p \notin \{2, 6\}$.

Remark. The condition $p = n(n+1)$, $n \ge 3$, was unnecessary in the official solution, too (its only role was to simplify showing that $X \ne p - 1$).

23. Denote by R the intersection point of lines AQ and BC. We know that $BR : RC = c : b$ and $AQ : QR = (b+c) : a$. By applying Stewart's theorem to ΔPBC and ΔPAR we obtain

$$\begin{aligned} a \cdot AP^2 + b \cdot BP^2 + c \cdot CP^2 &= aPA^2 + (b+c)PR^2 + (b+c)RB \cdot RC \\ &= (a+b+c)QP^2 + (b+c)RB \cdot RC + (a+b+c)QA \cdot QR. \end{aligned} \tag{1}$$

On the other hand, putting $P = Q$ into (1), we get that

$$a \cdot AQ^2 + b \cdot BQ^2 + c \cdot CQ^2 = (b+c)RB \cdot RC + (a+b+c)QA \cdot QR,$$

and the required statement follows.

Second solution. At vertices A, B, C place weights equal to a, b, c in some units respectively, so that Q is the center of gravity of the system. The left side of the equality to be proved is in fact the moment of inertia of the system about the axis through P and perpendicular to the plane ABC. On the other side, the right side expresses the same, due to the parallel axes theorem.

Alternative approach. Analytical geometry. The fact that all the variable segments appear squared usually implies that this is a good approach. Assign coordinates $A(x_a, y_a)$, $B(x_b, y_b)$, $C(x_c, y_c)$, and $P(x, y)$, use that $(a+b+c)\mathbf{Q} = a\mathbf{A} + b\mathbf{B} + c\mathbf{C}$, and calculate. Alternatively, differentiate $f(x, y) = a \cdot AP^2 + b \cdot BP^2 + c \cdot CP^2 - (a+b+c)QP^2$ and show that it is constant.

24. The first condition means in fact that $a_k - a_{k+1}$ is decreasing. In particular, if $a_k - a_{k+1} = -\delta < 0$, then $a_k - a_{k+m} = (a_k - a_{k+1}) + \cdots + (a_{k+m-1} - a_{k+m}) < -m\delta$, which implies that $a_{k+m} > a_k + m\delta$, and consequently $a_{k+m} > 1$ for large enough m, a contradiction. Thus $a_k - a_{k+1} \geq 0$ for all k.
Suppose that $a_k - a_{k+1} > 2/k^2$. Then for all $i < k$, $a_i - a_{i+1} > 2/k^2$, so that $a_i - a_{k+1} > 2(k+1-i)/k^2$, i.e., $a_i > 2(k+1-i)/k^2$, $i = 1, 2, \ldots, k$. But this implies $a_1 + a_2 + \cdots + a_k > 2/k^2 + 4/k^2 + \cdots + 2k/k^2 = k(k+1)/k^2$, which is impossible. Therefore $a_k - a_{k+1} \leq 2/k^2$ for all k.

25. Observe that $1001 = 7 \cdot 143$, i.e., $10^3 = -1 + 7a$, $a = 143$. Then by the binomial theorem, $10^{21} = (-1+7a)^7 = -1 + 7^2 b$ for some integer b, so that we also have $10^{21n} \equiv -1 \pmod{49}$ for any odd integer $n > 0$. Hence $N = \frac{9}{49}(10^{21n} + 1)$ is an integer of $21n$ digits, and $N(10^{21n} + 1) = \left(\frac{3}{7}(10^{21n} + 1)\right)^2$ is a double number that is a perfect square.

26. The overline in this problem will exclusively denote binary representation. We will show by induction that if $n = \overline{c_k c_{k-1} \ldots c_0} = \sum_{i=0}^{k} c_i 2^i$ is the binary representation of n ($c_i \in \{0, 1\}$), then $f(n) = \overline{c_0 c_1 \ldots c_k} = \sum_{i=0}^{k} c_i 2^{k-i}$ is the number whose binary representation is the palindrome of the binary representation of n. This evidently holds for $n \in \{1, 2, 3\}$.

Let us assume that the claim holds for all numbers up to $n-1$ and show it holds for $n = \overline{c_k c_{k-1} \dots c_0}$. We observe three cases:

(i) $c_0 = 0 \Rightarrow n = 2m \Rightarrow f(n) = f(m) = \overline{0c_1 \dots c_k} = \overline{c_0 c_1 \dots c_k}$.

(ii) $c_0 = 1,\ c_1 = 0 \Rightarrow n = 4m+1 \Rightarrow f(n) = 2f(2m+1) - f(m) = 2 \cdot \overline{1c_2 \dots c_k} - \overline{c_2 \dots c_k} = 2^k + 2 \cdot \overline{c_2 \dots c_k} - \overline{c_2 \dots c_k} = \overline{10c_2 \dots c_k} = \overline{c_0 c_1 \dots c_k}$.

(iii) $c_0 = 1,\ c_1 = 1 \Rightarrow n = 4m+3 \Rightarrow f(n) = 3f(2m+1) - 2f(m) = 3 \cdot \overline{1c_2 \dots c_k} - 2 \cdot \overline{c_2 \dots c_k} = 2^k + 2^{k-1} + 3 \cdot \overline{c_2 \dots c_k} - 2 \cdot \overline{c_2 \dots c_k} = \overline{11c_2 \dots c_k} = \overline{c_0 c_1 \dots c_k}$.

We thus have to find the number of palindromes in binary representation smaller than $1998 = \overline{11111000100}$. We note that for all $m \in \mathbb{N}$ the numbers of $2m$- and $(2m-1)$-digit binary palindromes are both equal to 2^{m-1}. We also note that $\overline{11111011111}$ and $\overline{11111111111}$ are the only 11-digit palindromes larger than 1998. Hence we count all palindromes of up to 11 digits and exclude the largest two. The number of $n \le 1998$ such that $f(n) = n$ is thus equal to $1+1+2+2+4+4+8+8+16+16+32-2 = 92$.

27. Consider a Cartesian system with the x-axis on the line BC and origin at the foot of the perpendicular from A to BC, so that A lies on the y-axis. Let A be $(0,\alpha)$, $B(-\beta,0)$, $C(\gamma,0)$, where $\alpha,\beta,\gamma > 0$ (because ABC is acute-angled). Then

$$\tan B = \frac{\alpha}{\beta}, \qquad \tan C = \frac{\alpha}{\gamma} \qquad \text{and} \qquad \tan A = -\tan(B+C) = \frac{\alpha(\beta+\gamma)}{\alpha^2 - \beta\gamma};$$

here $\tan A > 0$, so $\alpha^2 > \beta\gamma$. Let L have equation $x\cos\theta + y\sin\theta + p = 0$. Then

$$\begin{aligned}
&u^2 \tan A + v^2 \tan B + w^2 \tan C \\
&= \frac{\alpha(\beta+\gamma)}{\alpha^2-\beta\gamma}(\alpha\sin\theta + p)^2 + \frac{\alpha}{\beta}(-\beta\cos\theta + p)^2 + \frac{\alpha}{\gamma}(\gamma\cos\theta + p)^2 \\
&= (\alpha^2\sin^2\theta + 2\alpha p\sin\theta + p^2)\frac{\alpha(\beta+\gamma)}{\alpha^2-\beta\gamma} + \alpha(\beta+\gamma)\cos^2\theta + \frac{\alpha(\beta+\gamma)}{\beta\gamma}p^2 \\
&= \frac{\alpha(\beta+\gamma)}{\beta\gamma(\alpha^2-\beta\gamma)}(\alpha^2p^2 + 2\alpha p\beta\gamma\sin\theta + \alpha^2\beta\gamma\sin^2\theta + \beta\gamma(\alpha^2-\beta\gamma)\cos^2\theta) \\
&= \frac{\alpha(\beta+\gamma)}{\beta\gamma(\alpha^2-\beta\gamma)}\left[(\alpha p + \beta\gamma\sin\theta)^2 + \beta\gamma(\alpha^2-\beta\gamma)\right] \ge \alpha(\beta+\gamma) = 2\Delta,
\end{aligned}$$

with equality when $\alpha p + \beta\gamma\sin\theta = 0$, i.e., if and only if L passes through $(0, \beta\gamma/\alpha)$, which is the orthocenter of the triangle.

28. The sequence is uniquely determined by the conditions, and $a_1 = 2$, $a_2 = 7$, $a_3 = 25$, $a_4 = 89$, $a_5 = 317, \dots$; it satisfies $a_n = 3a_{n-1} + 2a_{n-2}$ for $n = 3,4,5$. We show that the sequence b_n given by $b_1 = 2$, $b_2 = 7$, $b_n = 3b_{n-1} + 2b_{n-2}$ has the same inequality property, i.e., that $b_n = a_n$:

$$b_{n+1}b_{n-1} - b_n^2 = (3b_n + 2b_{n-1})b_{n-1} - b_n(3b_{n-1} + 2b_{n-2}) = -2(b_nb_{n-2} - b_{n-1}^2)$$

for $n > 2$ gives that $b_{n+1}b_{n-1} - b_n^2 = (-2)^{n-2}$ for all $n \geq 2$. But then

$$\left| b_{n+1} - \frac{b_n^2}{b_{n-1}} \right| = \frac{2^{n-2}}{b_{n-1}} < \frac{1}{2},$$

since it is easily shown that $b_{n-1} > 2^{n-1}$ for all n. It is obvious that $a_n = b_n$ are odd for $n > 1$.

29. Let the first train start from Signal 1 at time 0, and let t_j be the time it takes for the jth train in the series to travel from one signal to the next. By induction on k, we show that Train k arrives at signal n at time $s_k + (n-2)m_k$, where $s_k = t_1 + \cdots + t_k$ and $m_k = \max_{j=1,\dots,k} t_j$.
For $k = 1$ the statement is clear. We now suppose that it is true for k trains and for every n, and add a $(k+1)$th train behind the others at Signal 1. There are two cases to consider:
(i) $t_{k+1} \geq m_k$, i.e., $m_{k+1} = t_{k+1}$. Then Train $k+1$ leaves Signal 1 when all the others reach Signal 2, which by the induction happens at time s_k. Since by the induction hypothesis Train k arrives at Signal $i+1$ at time $s_k+(i-1)m_k \leq s_k+(i-1)t_{k+1}$, Train $k+1$ is never forced to stop. The journey finishes at time $s_k + (n-1)t_{k+1} = s_{k+1} + (n-2)m_{k+1}$.
(ii) $t_{k+1} < m_k$, i.e., $m_{k+1} = m_k$. Train $k+1$ leaves Signal 1 at time s_k, and reaches Signal 2 at time $s_k + t_{k+1}$, but must wait there until all the other trains get to Signal 3, i.e., until time $s_k + m_k$ (by the induction hypothesis). So it reaches Signal 3 only at time $s_k + m_k + t_{k+1}$. Similarly, it gets to Signal 4 at time $s_k + 2m_k + t_{k+1}$, etc. Thus the entire schedule finishes at time $s_k + (n-2)m_k + t_{k+1} = s_{k+1} + (n-2)m_{k+1}$.

30. Let Δ_1, s_1, r' denote the area, semiperimeter, and inradius of triangle ABM, Δ_2, s_2, r' the same quantities for triangle MBC, and Δ, s, r those for $\triangle ABC$. Also, let P' and Q' be the points of tangency of the incircle of $\triangle ABM$ with the side AB and of the incircle of $\triangle MBC$ with the side BC, respectively, and let P, Q be the points of tangency of the incircle of $\triangle ABC$ with the sides AB, BC. We have $\Delta_1 = s_1 r'$, $\Delta_2 = s_2 r'$, $\Delta = sr$, so that $sr = (s_1 + s_2)r'$. Then

$$s_1 + s_2 = s + BM \qquad \Rightarrow \qquad \frac{r'}{r} = \frac{s}{s + BM}. \tag{1}$$

On the other hand, from similarity of triangles it follows that $AP'/AP = CQ'/CQ = r'/r$. By a well-known formula we find that $AP = s - BC$, $CQ = s - AB$, $AP' = s_1 - BM$, $CQ' = s_2 - BM$, and therefore deduce that

$$\frac{r'}{r} = \frac{s_1 - BM}{s - BC} = \frac{s_2 - BM}{s - AB} \Rightarrow \frac{r'}{r} = \frac{s_1 + s_2 - 2BM}{2s - AB - BC} = \frac{s - BM}{AC}. \tag{2}$$

It follows from (1) and (2) that $(s - BM)/AC = s/(s + BM)$, giving us $s^2 - BM^2 = s \cdot AC$. Finally,

$$BM^2 = s(s - AC) = s \cdot BP = s \cdot r \cot \frac{B}{2} = \Delta \cot \frac{B}{2}.$$

31. Denote the number of participants by $2n$, and assign to each seat one of the numbers $1, 2, \ldots, 2n$. Let the participant who was sitting at the seat k before the break move to seat $\pi(k)$. It suffices to prove that for every permutation π of the set $\{1, 2, \ldots, 2n\}$, there exist distinct i, j such that $\pi(i) - \pi(j) = \pm(i - j)$, the differences being calculated modulo $2n$.
If there are distinct i and j such that $\pi(i) - i = \pi(j) - j$ modulo $2n$, then we are done.
Suppose that all the differences $\pi(i) - i$ are distinct modulo $2n$. Then they take values $0, 1, \ldots, 2n - 1$ in some order, and consequently

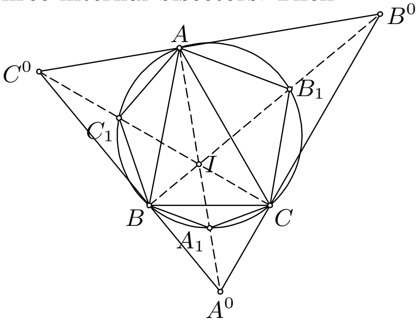
$$\sum_{i=1}^{2n} (\pi(i) - i) = 0 + 1 + \cdots + (2n - 1) \equiv n(2n - 1) \pmod{2n}.$$

On the other hand, $\sum_{i=1}^{2n}(\pi(i) - i) = \sum \pi(i) - \sum i = 0$, which is a contradiction because $n(2n - 1)$ is not divisible by $2n$.

Remark. For an odd number of participants, the statement is false. For example, the permutation $(a, 2a, \ldots, (2n+1)a)$ of $(1, 2, \ldots, 2n+1)$ modulo $2n+1$ does not satisfy the statement when $\gcd(a^2 - 1, 2n+1) = 1$. Check that such an a always exists.

4.30 Solutions to the Shortlisted Problems of IMO 1989

1. Let I denote the intersection of the three internal bisectors. Then $IA_1 = A_1A^0$. One way proving this is to realize that the circumcircle of ABC is the nine-point circle of $A^0B^0C^0$, hence it bisects IA^0, since I is the orthocenter of $A^0B^0C^0$. Another way is through noting that $IA_1 = A_1B$, which follows from $\angle A_1IB = \angle IBA_1 = (\angle A + \angle B)/2$, and $A_1B = A_1A^0$ which follows from $\angle A_1A^0B = \angle A_1BA^0 = 90° - \angle IBA_1$. Hence, we obtain $S_{IA_1B} = S_{A^0A_1B}$.



Repeating this argument for the six triangles that have a vertex at I and adding them up gives us $S_{A^0B^0C^0} = 2S_{AC_1BA_1CB_1}$. To prove $S_{AC_1BA_1CB_1} \geq 2S_{ABC}$, draw the three altitudes in triangle ABC intersecting in H. Let X, Y, and Z be the symmetric points of H with respect the sides BC, CA, and AB, respectively. Then X, Y, Z are points on the circumcircle of $\triangle ABC$ (because $\angle BXC = \angle BHC = 180° - \angle A$). Since A_1 is the midpoint of the arc BC, we have $S_{BA_1C} \geq S_{BXC}$. Hence

$$S_{AC_1BA_1CB_1} \geq S_{AZBXCY} = 2(S_{BHC} + S_{CHA} + S_{AHB}) = 2S_{ABC}.$$

2. Let the carpet have width x, length y. Suppose that the carpet $EFGH$ lies in a room $ABCD$, E being on AB, F on BC, G on CD, and H on DA. Then $\triangle AEH \equiv \triangle CGF \sim \triangle BFE \equiv \triangle DHG$. Let $\frac{y}{x} = k$, $AE = a$ and $AH = b$. In that case $BE = kb$ and $DH = ka$.
   Thus $a + kb = 50$, $ka + b = 55$, whence $a = \frac{55k-50}{k^2-1}$ and $b = \frac{50k-55}{k^2-1}$. Hence $x^2 = a^2 + b^2 = \frac{5525k^2-11000k+5525}{(k^2-1)^2}$, i.e.,

$$x^2(k^2-1)^2 = 5525k^2 - 11000k + 5525.$$

   Similarly, from the equations for the second storeroom, we get

$$x^2(k^2-1)^2 = 4469k^2 - 8360k + 4469.$$

   Combining the two equations, we get $5525k^2 - 11000k + 5525 = 4469k^2 - 8360k + 4469$, which implies $k = 2$ or $1/2$. Without loss of generality we have $y = 2x$ and $a + 2b = 50$, $2a + b = 55$; hence $a = 20$, $b = 15$, $x = \sqrt{15^2 + 20^2} = 25$, and $y = 50$. We have thus shown that the carpet is 25 feet by 50 feet.

3. Let the carpet have width x, length y. Let the length of the storerooms be q. Let $y/x = k$. Then, as in the previous problem, $(kq-50)^2 + (50k-q)^2 = (kq-38)^2 + (38k-q)^2$, i.e.,

$$kq = 22(k^2 + 1). \tag{1}$$

Also, as before, $x^2 = \left(\frac{kq-50}{k^2-1}\right)^2 + \left(\frac{50k-q}{k^2-1}\right)^2$, i.e.,

$$x^2(q^2 - 1)^2 = (k^2 + 1)(q^2 - 1900), \tag{2}$$

which, together with (1), yields

$$x^2k^2(k^2 - 1)^2 = (k^2 + 1)(484k^4 - 932k^2 + 484).$$

Since k is rational, let $k = c/d$, where c and d are integers with $\gcd(c, d) = 1$. Then we obtain

$$x^2c^2(c^2 - d^2)^2 = c^2(484c^4 - 448c^2d^2 - 448d^4) + 484d^6.$$

We thus have $c^2 \mid 484d^6$, but since $(c, d) = 1$, we have $c^2 \mid 484 \Rightarrow c \mid 22$. Analogously, $d \mid 22$; thus $k = 1, 2, 11, 22, \frac{1}{2}, \frac{1}{11}, \frac{1}{22}, \frac{2}{11}, \frac{11}{2}$. Since reciprocals lead to the same solution, we need only consider $k \in \left\{1, 2, 11, 22, \frac{11}{2}\right\}$, yielding $q = 44, 55, 244, 485, 125$, respectively. We can test these values by substituting them into (2). Only $k = 2$ gives us an integer solution, namely $x = 25, y = 50$.

4. First we note that for every integer $k > 0$ and prime number p, p^k doesn't divide $k!$. This follows from the fact that the highest exponent r of p for which $p^r|k!$ is

$$r = \left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \cdots < \frac{k}{p} + \frac{k}{p^2} + \cdots = \frac{k}{p-1} < k.$$

Now suppose that α is a rational root of the given equation. Then

$$\alpha^n + \frac{n!}{(n-1)!}\alpha^{n-1} + \cdots + \frac{n!}{2!}\alpha^2 + \frac{n!}{1!}\alpha + n! = 0, \tag{1}$$

from which we can conclude that α must be an integer, not equal to ± 1. Let p be a prime divisor of n and let r be the highest exponent of p for which $p^r|n!$. Then $p \mid \alpha$. Since $p^k|\alpha^k$ and $p^k \nmid k!$, we obtain that $p^{r+1} \mid n!\alpha^k/k!$ for $k = 1, 2, \dots, n$. But then it follows from (1) that $p^{r+1} \mid n!$, a contradiction.

5. According to the Cauchy–Schwarz inequality,

$$\left(\sum_{i=1}^{n} a_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} 1^2\right) = n\left(\sum_{i=1}^{n} a_i^2\right).$$

Since $r_1 + \cdots + r_n = -n$, applying this inequality we obtain $r_1^2 + \ldots + r_n^2 \ge n$, and applying it three more times, we obtain

$$r_1^{16} + \cdots + r_n^{16} \ge n,$$

with equality if and only if $r_1 = r_2 = \ldots = r_n = -1$ and $p(x) = (x + 1)^n$.

6. Let us denote the measures of the inner angles of the triangle ABC by α, β, γ. Then $P = r^2(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)/2$. Since the inner angles of the triangle $A'B'C'$ are $(\beta+\gamma)/2, (\gamma+\alpha)/2, (\alpha+\beta)/2$, we also have $Q = r^2[\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta)]/2$. Applying the AM–GM mean inequality, we now obtain

$$\begin{aligned} 16Q^3 &= \frac{16}{8} r^6 (\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta))^3 \\ &\geq 54 r^6 \sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta) \\ &= 27r^6[\cos(\alpha-\beta) - \cos(\alpha+\beta+2\gamma)]\sin(\alpha+\beta) \\ &= 27r^6[\cos(\alpha-\beta) + \cos\gamma]\sin(\alpha+\beta) \\ &= \frac{27}{2} r^6 [\sin(\alpha+\beta+\gamma) + \sin(\alpha+\beta-\gamma) + \sin 2\alpha + \sin 2\beta] \\ &= \frac{27}{2} r^6 [\sin(2\gamma) + \sin 2\alpha + \sin 2\beta] = 27r^4 P. \end{aligned}$$

This completes the proof.

7. Assume that P_1 and P_2 are points inside E, and that the line P_1P_2 intersects the perimeter of E at Q_1 and Q_2. If we prove the statement for Q_1 and Q_2, we are done, since these arcs can be mapped homothetically to join P_1 and P_2.
Let V_1, V_2 be two vertices of E. Then applying two homotheties to the inscribed circle of E one can find two arcs (one of them may be a side of E) joining these two points, both tangent to the sides of E that meet at V_1 and V_2. If A is any point of the side V_2V_3, two homotheties with center V_1 take the arcs joining V_1 to V_2 and V_3 into arcs joining V_1 to A; their angle of incidence at A remains $(1-2/n)\pi$.
Next, for two arbitrary points Q_1 and Q_2 on two different sides V_1V_2 and V_3V_4, we join V_1 and V_2 to Q_2 with pairs of arcs that meet at Q_2 and have an angle of incidence $(1-2/n)\pi$. The two arcs that meet the line Q_1Q_2 again outside E meet at Q_2 at an angle greater than or equal to $(1-2/n)\pi$. Two homotheties with center Q_2 carry these arcs to ones meeting also at Q_1 with the same angle of incidence.

8. Let A, B, C, D denote the vertices of R. We consider the set $\mathcal{S}$ of all points E of the plane that are vertices of at least one rectangle, and its subset $\mathcal{S}'$ consisting of those points in $\mathcal{S}$ that have both coordinates integral in the orthonormal coordinate system with point A as the origin and lines AB, AD as axes.
First, to each $E \in \mathcal{S}$ we can assign an integer n_E as the number of rectangles R_i with one vertex at E. It is easy to check that $n_E = 1$ if E is one of the vertices A, B, C, D; in all other cases n_E is either 2 or 4.
Furthermore, for each rectangle R_i we define $f(R_i)$ as the number of vertices of R_i that belong to $\mathcal{S}'$. Since every R_i has at least one side of integer length, $f(R_i)$ can take only values 0, 2, or 4. Therefore we have

$$\sum_{i=1}^{n} f(R_i) \equiv 0 \pmod 2.$$

On the other hand, $\sum_{i=1}^{n} f(R_i)$ is equal to $\sum_{E\in\mathcal{S}'} n_E$, implying that

$$\sum_{E\in\mathcal{S}'} n_E \equiv 0 \pmod 2.$$

However, since $n_A = 1$, at least one other n_E, where $E \in \mathcal{S}'$, must be odd, and that can happen only for E being B, C, or D. We conclude that at least one of the sides of R has integral length.

Second solution. Consider the coordinate system introduced above. If D is a rectangle whose sides are parallel to the axes of the system, it is easy to prove that

$$\int_D \sin 2\pi x \sin 2\pi y \; dx\, dy = 0$$

if and only if at least one side of D has integral length. This holds for all R_i's, so that adding up these equalities we get $\int_R \sin 2\pi x \sin 2\pi y \; dx\, dy = 0$. Thus, R also has a side of integral length.

9. From $a_{n+1} + b_{n+1}\sqrt[3]{2} + c_{n+1}\sqrt[3]{4} = (a_n + b_n\sqrt[3]{2} + c_n\sqrt[3]{4})(1 + 4\sqrt[3]{2} - 4\sqrt[3]{4})$ we obtain $a_{n+1} = a_n - 8b_n + 8c_n$. Since $a_0 = 1$, a_n is odd for all n.
For an integer $k > 0$, we can write $k = 2^l k'$, k' being odd and l a nonnegative integer. Let us set $v(k) = l$, and define $\beta_n = v(b_n), \gamma_n = v(c_n)$. We prove the following lemmas:
Lemma 1. For every integer $p \geq 0$, b_{2^p} and c_{2^p} are nonzero, and $\beta_{2^p} = \gamma_{2^p} = p + 2$.
Proof. By induction on p. For $p = 0$, $b_1 = 4$ and $c_1 = -4$, so the assertion is true. Suppose that it holds for p. Then

$$(1+4\sqrt[3]{2}-4\sqrt[3]{4})^{2^{p+1}} = (a+2^{p+2}(b'\sqrt[3]{2}+c'\sqrt[3]{4}))^2 \text{ with } a, b', \text{ and } c' \text{ odd.}$$

Then we easily obtain that $(1 + 4\sqrt[3]{2} - 4\sqrt[3]{4})^{2^{p+1}} = A + 2^{p+3}(B\sqrt[3]{2} + C\sqrt[3]{4})$, where $A, B = ab' + 2^{p+1}E, C = ac' + 2^{p+1}F$ are odd. Therefore Lemma 1 holds for $p + 1$.
Lemma 2. Suppose that for integers $n, m \geq 0$, $\beta_n = \gamma_n = \lambda > \beta_m = \gamma_m = \mu$. Then b_{n+m}, c_{n+m} are nonzero and $\beta_{n+m} = \gamma_{n+m} = \mu$.
Proof. Calculating $(a' + 2^\lambda(b'\sqrt[3]{2} + c'\sqrt[3]{4}))(a'' + 2^\mu(b''\sqrt[3]{2} + c''\sqrt[3]{4}))$, with $a', b', c', a'', b'', c''$ odd, we easily obtain the product $A + 2^\mu(B\sqrt[3]{2} + C\sqrt[3]{4})$, where $A, B = a'b'' + 2^{\lambda-\mu}E$, and $C = a'c'' + 2^{\lambda-\mu}F$ are odd, which proves Lemma 2.
Since every integer $n > 0$ can be written as $n = 2^{p_r} + \cdots + 2^{p_1}$, with $0 \leq p_1 < \cdots < p_r$, from Lemmas 1 and 2 it follows that c_n is nonzero, and that $\gamma_n = p_1 + 2$.

Remark. b_{1989} and c_{1989} are divisible by 4, but not by 8.

10. Plugging in $wz+a$ instead of z into the functional equation, we obtain
$$f(wz+a)+f(w^2z+wa+a)=g(wz+a). \tag{1}$$
By repeating this process, this time in (1), we get
$$f(w^2z+wa+a)+f(z)=g(w^2z+wa+a). \tag{2}$$
Solving the system of linear equations (1), (2) and the original functional equation, we easily get
$$f(z)=\frac{g(z)+g(w^2z+wa+a)-g(wz+a)}{2}.$$
This function thus uniquely satisfies the original functional equation.

11. Call a binary sequence S of length n *repeating* if for some $d \mid n$, $d>1$, S can be split into d identical blocks. Let x_n be the number of nonrepeating binary sequences of length n. The total number of binary sequences of length n is obviously 2^n. Any sequence of length n can be produced by repeating its unique longest nonrepeating initial block according to need. Hence, we obtain the recursion relation $\sum_{d|n} x_d = 2^n$. This, along with $x_1 = 2$, gives us $a_n = x_n$ for all n.
We now have that the sequences counted by x_n can be grouped into groups of n, the sequences in the same group being cyclic shifts of each other. Hence, $n \mid x_n = a_n$.

12. Assume that each car starts with a unique ranking number. Suppose that while turning back at a meeting point two cars always exchanged their ranking numbers. We can observe that ranking numbers move at a constant speed and direction. One hour later, after several exchanges, each starting point will be occupied by a car of the same ranking number and proceeding in the same direction as the one that started from there one hour ago.
We now give the cars back their original ranking numbers. Since the sequence of the cars along the track cannot be changed, the only possibility is that the original situation has been rotated, maybe onto itself. Hence for some $d \mid n$, after d hours each car will be at its starting position and orientation.

13. Let us construct the circles σ_1 with center A and radius $R_1 = AD$, σ_2 with center B and radius $R_2 = BC$, and σ_3 with center P and radius x. The points C and D lie on σ_2 and σ_1 respectively, and CD is tangent to σ_3. From this it is plain that the greatest value of x occurs when CD is also tangent to σ_1 and σ_2. We shall show that in this case the required inequality is really an equality, i.e., that $\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$. Then the inequality will immediately follow.
Denote the point of tangency of CD with σ_3 by M. By the Pythagorean theorem we have $CD = \sqrt{(R_1+R_2)^2-(R_1-R_2)^2} = 2\sqrt{R_1R_2}$. On the

other hand, $CD = CM + MD = 2\sqrt{R_2 x} + 2\sqrt{R_1 x}$. Hence, we obtain $\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{R_1}} + \frac{1}{\sqrt{R_2}}$.

14. *Lemma 1.* In a quadrilateral $ABCD$ circumscribed about a circle, with points of tangency P, Q, R, S on DA, AB, BC, CD respectively, the lines AC, BD, PR, QS concur.

Proof. Follows immediately, for example, from Brianchon's theorem.

Lemma 2. Let a variable chord XY of a circle $C(I, r)$ subtend a right angle at a fixed point Z within the circle. Then the locus of the midpoint P of XY is a circle whose center is at the midpoint M of IZ and whose radius is $\sqrt{r^2/2 - IZ^2/4}$.

Proof. From $\angle XZY = 90^\circ$ follows $\overrightarrow{ZX} \cdot \overrightarrow{ZY} = (\overrightarrow{IX} - \overrightarrow{IZ}) \cdot (\overrightarrow{IY} - \overrightarrow{IZ}) = 0$. Therefore,

$$\begin{aligned}\overrightarrow{MP}^2 &= (\overrightarrow{MI} + \overrightarrow{IP})^2 = \frac{1}{4}(-\overrightarrow{IZ} + \overrightarrow{IX} + \overrightarrow{IY})^2 \\ &= \frac{1}{4}(IX^2 + IY^2 - IZ^2 + 2(\overrightarrow{IX} - \overrightarrow{IZ}) \cdot (\overrightarrow{IY} - \overrightarrow{IZ})) \\ &= \frac{1}{2}r^2 - \frac{1}{4}IZ^2.\end{aligned}$$

Lemma 3. Using notation as in Lemma 1, if $ABCD$ is cyclic, PR is perpendicular to QS.

Proof. Consider the inversion in $C(I, r)$, mapping A to A' etc. (P, Q, R, S are fixed). As is easily seen, A', B', C', D' will lie at the midpoints of PQ, QR, RS, SP, respectively. $A'B'C'D'$ is a parallelogram, but also cyclic, since inversion preserves circles; thus it must be a rectangle, and so $PR \perp QS$.

Now we return to the main result. Let I and O be the incenter and circumcenter, Z the intersection of the diagonals, and $P, Q, R, S, A', B', C', D'$ points as defined in Lemmas 1 and 3. From Lemma 3, the chords PQ, QR, RS, SP subtend 90° at Z. Therefore by Lemma 2 the points A', B', C', D' lie on a circle whose center is the midpoint Y of IZ. Since this circle is the image of the circle $ABCD$ under the considered inversion (centered at I), it follows that I, O, Y are collinear, and hence so are I, O, Z.

Remark. This is the famous Newton's theorem for bicentric quadrilaterals.

15. By Cauchy's inequality, $44 < \sqrt{1989} < a + b + c + d \le \sqrt{2 \cdot 1989} < 90$. Since $m^2 = a + b + c + d$ is of the same parity as $a^2 + b^2 + c^2 + d^2 = 1989$, m^2 is either 49 or 81. Let $d = \max\{a, b, c, d\}$.

Suppose that $m^2 = 49$. Then $(49 - d)^2 = (a + b + c)^2 > a^2 + b^2 + c^2 = 1989 - d^2$, and so $d^2 - 49d + 206 > 0$. This inequality does not hold for $5 \le d \le 44$. Since $d \ge \sqrt{1989/4} > 22$, d must be at least 45, which is impossible because $45^2 > 1989$. Thus we must have $m^2 = 81$ and $m = 9$. Now, $4d > 81$ implies $d \ge 21$. On the other hand, $d < \sqrt{1989}$, and hence

$d = 25$ or $d = 36$. Suppose that $d = 25$ and put $a = 25 - p$, $b = 25 - q$, $c = 25 - r$ with $p, q, r \geq 0$. From $a+b+c = 56$ it follows that $p+q+r = 19$, which, together with $(25-p)^2 + (25-q)^2 + (25-r)^2 = 1364$, gives us $p^2 + q^2 + r^2 = 439 > 361 = (p+q+r)^2$, a contradiction. Therefore $d = 36$ and $n = 6$.

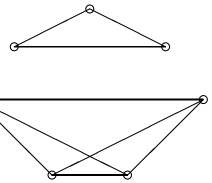
Remark. A little more calculation yields the unique solution $a = 12$, $b = 15$, $c = 18$, $d = 36$.

16. Define $S_k = \sum_{i=0}^{k} a_i$ $(k = 0, 1, \ldots, n)$ and $S_{-1} = 0$. We note that $S_{n-1} = S_n$. Hence

$$\begin{aligned} S_n &= \sum_{k=0}^{n-1} a_k = nc + \sum_{k=0}^{n-1} \sum_{i=k}^{n-1} a_{i-k}(a_i + a_{i+1}) \\ &= nc + \sum_{i=0}^{n-1} \sum_{k=0}^{i} a_{i-k}(a_i + a_{i+1}) = nc + \sum_{i=0}^{n-1} (a_i + a_{i+1}) \sum_{k=0}^{i} a_{i-k} \\ &= nc + \sum_{i=0}^{n-1} (S_{i+1} - S_{i-1}) S_i = nc + S_n^2, \end{aligned}$$

i.e., $S_n^2 - S_n + nc = 0$. Since S_n is real, the discriminant of the quadratic equation must be positive, and hence $c \leq \frac{1}{4n}$.

17. A figure consisting of 9 lines is shown below.



Now we show that 8 lines are not sufficient. Assume the opposite. By the pigeonhole principle, there is a vertex, say A, that is joined to at most 2 other vertices. Let B, C, D, E denote the vertices to which A is not joined, and F, G the other two vertices. Then any two vertices of B, C, D, E must be mutually joined for an edge to exist within the triangle these two points form with A. This accounts for 6 segments. Since only two segments remain, among A, F, and G at least two are not joined. Taking these two and one of B, C, D, E that is not joined to any of them (it obviously exists), we get a triple of points, no two of which are joined; a contradiction.

Second solution. Since (a) is equivalent to the fact that no three points make a "blank triangle," by Turan's theorem the number of "blank edges" cannot exceed $[7^2/4] = 12$, leaving at least $7 \cdot 6/2 - 12 = 9$ segments. For general n, the answer is $\left[(n-1)/2\right]^2$.

18. Consider the triangle MA_iM_i. Obviously, the point M_i is the image of A_i under the composition C of rotation $R_M^{\alpha/2-90^\circ}$ and homothety $H_M^{2\sin(\alpha/2)}$. Therefore, the polygon $M_1M_2\dots M_n$ is obtained as the image of $A_1A_2\dots A_n$ under the rotational homothety C with coefficient $2\sin(\alpha/2)$. Therefore $S_{M_1M_2\dots M_n} = 4\sin^2(\alpha/2)\cdot S$.

19. Let us color the board in a chessboard fashion. Denote by S_b and S_w respectively the sum of numbers in the black and in the white squares. It is clear that every allowed move leaves the difference $S_b - S_w$ unchanged. Therefore a necessary condition for annulling all the numbers is $S_b = S_w$. We now show it is sufficient. Assuming $S_b = S_w$ let us observe a triple of (different) cells a, b, c with respective values x_a, x_b, x_c where a and c are both adjacent to b. We first prove that we can reduce x_a to be 0 if $x_a > 0$. If $x_a \le x_b$, we subtract x_a from both a and b. If $x_a > x_b$, we add $x_a - x_b$ to b and c and proceed as in the previous case. Applying the reduction in sequence, along the entire board, we reduce all cells except two neighboring cells to be 0. Since $S_b = S_w$ is invariant, the two cells must have equal values and we can thus reduce them both to 0.

20. Suppose $k \ge 1/2 + \sqrt{2n}$. Consider a point P in S. There are at least k points in S having all the same distance to P, so there are at least $\binom{k}{2}$ pairs of points A, B with $AP = BP$. Since this is true for every point $P \in S$, there are at least $n\binom{k}{2}$ triples of points (A, B, P) for which $AP = BP$ holds. However,

$$\begin{aligned} n\binom{k}{2} &= n\frac{k(k-1)}{2} \ge \frac{n}{2}\left(\sqrt{2n}+\frac{1}{2}\right)\left(\sqrt{2n}-\frac{1}{2}\right) \\ &= \frac{n}{2}\left(2n-\frac{1}{4}\right) > n(n-1) = 2\binom{n}{2}. \end{aligned}$$

Since $\binom{n}{2}$ is the number of all possible pairs (A, B) with $A, B \in S$, there must exist a pair of points A, B with more than two points P_i such that $AP_i = BP_i$. These points P_i are collinear (they lie on the perpendicular bisector of AB), contradicting condition (1).

21. In order to obtain a triangle as the intersection we must have three points P, Q, R on three sides of the tetrahedron passing through one vertex, say T. It is clear that we may suppose w.l.o.g. that P is a vertex, and Q and R lie on the edges TP_1 and TP_2 (P_1, P_2 are vertices) or on their extensions respectively. Suppose that $\overrightarrow{TQ} = \lambda\overrightarrow{TP_1}$ and $\overrightarrow{TR} = \mu\overrightarrow{TP_2}$, where $\lambda, \mu > 0$. Then

$$\cos\angle QPR = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\overline{PQ}\cdot\overline{PR}} = \frac{(\lambda-1)(\mu-1)+1}{2\sqrt{\lambda^2-\lambda+1}\sqrt{\mu^2-\mu+1}}.$$

In order to obtain an obtuse angle (with $\cos < 0$) we must choose $\mu < 1$ and $\lambda > \frac{2-\mu}{1-\mu} > 1$. Since $\sqrt{\lambda^2-\lambda+1} > \lambda - 1$ and $\sqrt{\mu^2-\mu+1} > 1-\mu$, we get that for $(\lambda-1)(\mu-1)+1 < 0$,

$$\cos\angle QPR > \frac{1-(1-\mu)(\lambda-1)}{2(1-\mu)(\lambda-1)} > -\frac{1}{2}; \qquad \text{hence } \angle QPR < 120^\circ.$$

Remark. After obtaining the formula for $\cos\angle QPR$, the official solution was as follows: For fixed $\mu_0 < 1$ and $\lambda > 1$, $\cos\angle QPR$ is a decreasing function of λ: indeed,

$$\frac{\partial \cos\angle QPR}{\partial\lambda} = \frac{\mu-(3-\mu)\lambda}{4(\lambda^2-\lambda+1)^{3/2}(\mu^2-\mu+1)^{1/2}} < 0.$$

Similarly, for a fixed, sufficiently large λ_0, $\cos\angle QPR$ is decreasing for μ decreasing to 0. Since $\lim_{\lambda\to 0,\mu\to 0+}\cos\angle QPR = -1/2$, we conclude that $\angle QPR < 120^\circ$.

22. The statement remains valid if 17 is replaced by any divisor k of $1989 = 3^2\cdot 13\cdot 17$, $1 < k < 1989$, so let k be one such divisor. The set $\{1,2,\dots,1989\}$ can be partitioned as $\{1,2,\dots,3k\}\cup\bigcup_{j=1}^{L}\{(2j+1)k+1,(2j+1)k+2,\dots,(2j+1)k+2k\} = X\cup Y_1\cup\cdots\cup Y_L$, where $L = (1989-3k)/2k$. The required statement will be an obvious consequence of the following two claims.

   Claim 1. The set $X = \{1,2,\dots,3k\}$ can be partitioned into k disjoint subsets, each having 3 elements and the same sum.

   Proof. Since k is odd, let $t = k-1/2$ and $X = \{1,2,\dots,6t+3\}$. For $l = 1,2,\dots,t$, define

$$\begin{aligned} X_{2l-1} &= \{l, 3t+1+l, 6t+5-2l\},\\ X_{2l} &= \{t+1+l, 2t+1+l, 6t+4-2l\}\\ X_{2t+1} &= X_k = \{t+1, 4t+2, 4t+3\}. \end{aligned}$$

   It is easily seen that these three subsets are disjoint and that the sum of elements in each set is $9t+6$.

   Claim 2. Each $Y_j = \{(2j+1)k+1,\dots,(2j+1)k+2k\}$ can be partitioned into k disjoint subsets, each having 2 elements and the same sum.

   Proof. The obvious partitioning works:

$$Y_j = \{(2j+1)k+1,(2j+1)k+2k\}\cup\cdots\cup\{(2j+1)k+k,(2j+1)k+(k+1)\}.$$

23. Two numbers $x, y \in \{1,\dots,2n\}$ will be called *twins* if $|x-y| = n$. Then the set $\{1,\dots,2n\}$ splits into n pairs of twins. A permutation $(x_1,\dots,x_{2n})$ of this set is said to be of type T_k if $|x_i - x_{i+1}| = n$ holds for exactly k indices i (thus a permutation of type T_0 contains no pairs of neighboring twins). Denote by $F_k(n)$ the number of T_k-type permutations of $\{1,\dots,2n\}$.

   Let $(x_1,\dots,x_{2n})$ be a permutation of type T_0. Removing x_{2n} and its twin, we obtain a permutation of $2n-2$ elements consisting of $n-1$ pairs of twins. This new permutation is of one of the following types:

(i) type T_0: x_{2n} can take $2n$ values, and its twin can take any of $2n-2$ positions;
(ii) type T_1: x_{2n} can take any one of $2n$ values, but its twin must be placed to separate the unique pair of neighboring twins in the new permutation.

The recurrence formula follows:

$$F_0(n) = 2n[(2n-2)F_0(n-1) + F_1(n-1)]. \tag{1}$$

Now let $(x_1, \dots, x_{2n})$ be a permutation of type T_1, and let (x_j, x_{j+1}) be the unique neighboring twin pair. Similarly, on removing this pair we get a permutation of $2n-2$ elements, either of type T_0 or of type T_1. The pair (x_j, x_{j+1}) is chosen out of n twin pairs and can be arranged in two ways. Also, in the first case it can be placed anywhere ($2n-1$ possible positions), but in the second case it must be placed to separate the unique pair of neighboring twins. Hence,

$$F_1(n) = 2n[(2n-1)F_0(n-1) + F_1(n-1)] = F_0(n) + 2nF_0(n-1). \tag{2}$$

This implies that $F_0(n) < F_1(n)$. Therefore the permutations with at least one neighboring twin pair are more numerous than those with no such pairs.

Remark 1. As in the official solution, formulas (1) and (2) together give for F_0 the recurrence

$$F_0(n) = 2n[(2n-1)F_0(n-1) + (2n-2)F_0(n-2)].$$

For the ratio $p_n = F_0(n)/(2n)!$, simple algebraic manipulation yields $p_n = p_{n-1} + \frac{p_{n-2}}{(2n-3)(2n-1)}$. Since $p_1 = 0$, we get

$$p_n < p_{n-1} + \frac{1}{(2n-3)(2n-1)} = p_{n-1} + \frac{1}{2(2n-3)} - \frac{1}{2(2n-1)} < \cdots < \frac{1}{2}.$$

Remark 2. Using the inclusion–exclusion principle, the following formula can be obtained:

$$F_0(n) = 2^0\binom{n}{0}(2n)! - 2^1\binom{n}{1}(2n-1)! + 2^2\binom{n}{2}(2n-2)! - \cdots$$
$$\cdots + (-1)^{n-1}2^n\binom{n}{n}n!.$$

One consequence is that in fact, $\lim_{n\to\infty} p_n = 1/e$.

Second solution. Let $f : T_0 \to T_1$ be the mapping defined as follows: if $(x_1, x_2, \dots, x_{2n}) \in T_0$ and x_k, $k > 2$, is the twin of x_1, then

$$f(x_1, x_2, \dots, x_{2n}) = (x_2, \dots, x_{k-1}, x_1, x_k, \dots, x_{2n}).$$

The mapping f is injective, but not surjective. Thus $F_0(n) < F_1(n)$.

24. Instead of Euclidean distance, we will use the angles $\angle A_iOA_j$, O denoting the center of the sphere. Let $\{A_1, \dots, A_5\}$ be any set for which $\min_{i \neq j} \angle A_iOA_j \geq \pi/2$ (such a set exists: take for example five vertices of an octagon). We claim that two of the A_i's must be antipodes, thus implying that $\min_{i \neq j} \angle A_iOA_j$ is exactly equal to $\pi/2$, and consequently that $\min_{i \neq j} A_iA_j = \sqrt{2}$.
Suppose no two of the five points are antipodes. Visualize A_5 as the south pole. Then $A_1, \dots, A_4$ lie in the northern hemisphere, including the equator (but excluding the north pole). No two of $A_1, \dots, A_4$ can lie in the interior of a quarter of this hemisphere, which means that any two of them differ in longitude by at least $\pi/2$. Hence, they are situated on four meridians that partition the sphere into quarters. Finally, if one of them does not lie on the equator, its two neighbors must. Hence, in any case there will exist an antipodal pair, giving us a contradiction.

25. We may assume w.l.o.g. that $a > 0$ (because $a, b < 0$ is impossible, and $a, b \neq 0$ from the condition of the problem). Let $(x_0, y_0, z_0, w_0) \neq (0, 0, 0, 0)$ be a solution of $x^2 - ay^2 - bz^2 + abw^2$. Then

$$x_0^2 - ay_0^2 = b(z_0^2 - aw_0^2).$$

Multiplying both sides by $(z_0^2 - aw_0^2)$, we get

$$\begin{aligned} &(x_0^2 - ay_0^2)(z_0^2 - aw_0^2) - b(z_0^2 - aw_0^2)^2 = 0 \\ \Leftrightarrow\ &(x_0z_0 - ay_0w_0)^2 - a(y_0z_0 - x_0w_0)^2 - b(z_0^2 - aw_0^2)^2 = 0. \end{aligned}$$

Hence, for $x_1 = x_0z_0 - ay_0w_0,\quad y_1 = y_0z_0 - x_0w_0,\quad z_1 = z_0^2 - aw_0^2$, we have

$$x_1^2 - ay_1^2 - bz_1^2 = 0.$$

If (x_1, y_1, z_1) is the trivial solution, then $z_1 = 0$ implies $z_0 = w_0 = 0$ and similarly $x_0 = y_0 = 0$ because a is not a perfect square. This contradicts the initial assumption.

26. By the Cauchy–Schwarz inequality,

$$\left(\sum_{i=1}^{n} x_i \right)^2 \leq n \sum_{i=1}^{n} x_i^2.$$

Since $\sum_{i=1}^{n} x_i = a - x_0$ and $\sum_{i=1}^{n} x_i^2 = b - x_0^2$, we have $(a - x_0)^2 \leq n(b - x_0^2)$, i.e.,

$$(n + 1)x_0^2 - 2ax_0 + (a^2 - nb) \leq 0.$$

The discriminant of this quadratic is $D = 4n(n + 1)\left[b - a^2/(n + 1)\right]$, so we conclude that

(i) if $a^2 > (n + 1)b$, then such an x_0 does not exist;
(ii) if $a^2 = (n + 1)b$, then $x_0 = a/n + 1$; and

(iii) if $a^2 < (n+1)b$, then $\frac{a-\sqrt{D}/2}{n+1} \le x_0 \le \frac{a+\sqrt{D}/2}{n+1}$.
It is easy to see that these conditions for x_0 are also sufficient.

27. Let n be the required exponent, and suppose $n = 2^k q$, where q is an odd integer. Then we have

$$m^n - 1 = (m^{2^k} - 1)[(m^{2^k(q-1)} + \cdots + m^{2^k} + 1] = (m^{2^k} - 1)A,$$

where A is odd. Therefore $m^n - 1$ and $m^{2^k} - 1$ are divisible by the same power of 2, and so $n = 2^k$.
Next, we observe that

$$\begin{aligned} m^{2^k} - 1 &= (m^{2^{k-1}} - 1)(m^{2^{k-1}} + 1) = \dots \\ &= (m^2 - 1)(m^2 + 1)(m^4 + 1)\cdots(m^{2^{k-1}} + 1). \end{aligned}$$

Let s be the maximal positive integer for which $m \equiv \pm 1 \pmod{2^s}$. Then $m^2 - 1$ is divisible by 2^{s+1} and not divisible by 2^{s+2}. All the numbers $m^2 + 1, m^4 + 1, \dots, m^{2^{k-1}} + 1$ are divisible by 2 and not by 4. Hence $m^{2^k} - 1$ is divisible by 2^{s+k} and not by 2^{s+k+1}.
It follows from the above consideration that the smallest exponent n equals 2^{1989-s} if $s \le 1989$, and $n = 1$ if $s > 1989$.

28. Assume w.l.o.g. that the rays OA_1, OA_2, OA_3, OA_4 are arranged clockwise. Setting $OA_1 = a$, $OA_2 = b$, $OA_3 = c$, $OA_4 = d$, and $\angle A_1OA_2 = x$, $\angle A_2OA_3 = y$, $\angle A_3OA_4 = z$, we have

$$\begin{aligned} S_1 &= \sigma(OA_1A_2) = \frac{1}{2}ab|\sin x|,\ S_2 = \sigma(OA_1A_3) = \frac{1}{2}ac|\sin(x+y)|, \\ S_3 &= \sigma(OA_1A_4) = \frac{1}{2}ad|\sin(x+y+z)|,\ S_4 = \sigma(OA_2A_3) = \frac{1}{2}bc|\sin y|, \\ S_5 &= \sigma(OA_2A_4) = \frac{1}{2}bd|\sin(y+z)|,\ S_6 = \sigma(OA_3A_4) = \frac{1}{2}cd|\sin z|. \end{aligned}$$

Since $\sin(x+y+z)\sin y + \sin x \sin z = \sin(x+y)\sin(y+z)$, it follows that there exists a choice of $k, l \in \{0, 1\}$ such that

$$S_1S_6 + (-1)^k S_2S_5 + (-1)^l S_3S_4 = 0.$$

For example (w.l.o.g.), if $S_3S_4 = S_1S_6 + S_2S_5$, we have

$$\left(\max_{1\le i\le 6} S_i\right)^2 \ge S_3S_4 = S_1S_6 + S_2S_5 \ge 1 + 1 = 2,$$

i.e., $\max_{1\le i\le 6} S_i \ge \sqrt{2}$ as claimed.

29. Let P_i, sitting at the place A, and P_j sitting at B, be two birds that can see each other. Let k and l respectively be the number of birds visible from B but not from A, and the number of those visible from A but not from

B. Assume that $k \geq l$. Then if all birds from B fly to A, each of them will see l new birds, but won't see k birds anymore. Hence the total number of mutually visible pairs does not increase, while the number of distinct positions occupied by at least one bird decreases by one. Repeating this operation as many times as possible one can arrive at a situation in which two birds see each other if and only if they are in the same position. The number of such distinct positions is at most 35, while the total number of mutually visible pairs is not greater than at the beginning. Thus the problem is equivalent to the following one:

(1) If $x_i \geq 0$ are integers with $\sum_{j=1}^{35} x_j = 155$, find the least possible value of $\sum_{j=1}^{35} (x_j^2 - x_j)/2$.

If $x_j \geq x_i + 2$ for some i, j, then the sum of $(x_j^2 - x_j)/2$ decreases (for $x_j - x_i - 2$) if x_i, x_j are replaced with $x_i + 1, x_j - 1$. Consequently, our sum attains its minimum when the x_i's differ from each other by at most 1. In this case, all the x_i's are equal to either $[155/35] = 4$ or $[155/35] + 1 = 5$, where $155 = 20 \cdot 4 + 15 \cdot 5$. It follows that the (minimum possible) number of mutually visible pairs is $20 \cdot \frac{4 \cdot 3}{2} + 15 \cdot \frac{5 \cdot 4}{2} = 270$.

Second solution for (1). Considering the graph consisting of birds as vertices and pairs of mutually nonvisible birds as edges, we see that there is no complete 36-subgraph. Turan's theorem gives the answer immediately. (See problem (SL89-17).)

30. For all n such N exists. For a given n choose $N = (n+1)!^2 + 1$. Then $1 + j$ is a proper factor of $N + j$ for $1 \leq j \leq n$. So if $N + j = p^m$ is a power of a prime p, then $1 + j = p^r$ for some integer r, $1 \leq r < m$. But then p^{r+1} divides both $(n+1)!^2 = N - 1$ and $p^m = N + j$, implying that $p^{r+1} \mid 1 + j$, which is impossible. Thus none of $N+1, N+2, \dots, N+n$ is a power of a prime.

    Second solution. Let $p_1, p_2, \dots, p_{2n}$ be distinct primes. By the Chinese remainder theorem, there exists a natural number N such that $p_1 p_2 \mid N + 1$, $p_3 p_4 \mid N + 2$, $\dots$, $p_{2n-1} p_{2n} \mid N + n$, and then obviously none of the numbers $N+1, \dots, N+n$ can be a power of a prime.

31. Let us denote by N_{pqr} the number of solutions for which $a_p/x_p \geq a_q/x_q \geq a_r/x_r$, where (p, q, r) is one of six permutations of $(1, 2, 3)$. It is clearly enough to prove that $N_{pqr} + N_{qpr} \leq 2a_1a_2(3 + \ln(2a_1))$.
    First, from

$$\frac{3a_p}{x_p} \geq \frac{a_p}{x_p} + \frac{a_q}{x_q} + \frac{a_r}{x_r} = 1 \qquad \text{and} \qquad \frac{a_p}{x_p} < 1$$

    we get $a_p + 1 \leq x_p \leq 3a_p$. Similarly, for fixed x_p we have

$$\frac{2a_q}{x_q} \geq \frac{a_q}{x_q} + \frac{a_r}{x_r} = 1 - \frac{a_p}{x_p} \qquad \text{and} \qquad \frac{a_q}{x_q} \leq \min\left(\frac{a_p}{x_p}, 1 - \frac{a_p}{x_p}\right),$$

which gives $\max\{a_q \cdot x_p/a_p, a_q \cdot x_p/(x_p - a_p)\} \le x_q \le 2a_q \cdot x_p/(x_p - a_p)$, i.e., if $a_p + 1 \le x_p \le 2a_p$ there are at most $a_q \cdot x_p/(x_p - a_p) + 1/2$ possible values for x_q (because there are $[2x] - [x] = [x + 1/2]$ integers between x and $2x$), and if $2a_p + 1 \le x_p \le 3a_p$, at most $2a_q \cdot x_p/(x_p - a_p) - a_q \cdot x_p/a_p + 1$ possible values. Given x_p and x_q, x_r is uniquely determined. Hence

$$\begin{aligned}
N_{pqr} &\le \sum_{x_p=a_p+1}^{2a_p} \left(\frac{a_q \cdot x_p}{x_p - a_p} + \frac{1}{2}\right) + \sum_{x_p=2a_p+1}^{3a_p} \left(\frac{2a_q \cdot x_p}{x_p - a_p} - \frac{a_q \cdot x_p}{a_p} + 1\right) \\
&= \frac{3a_p}{2} + a_q \sum_{k=1}^{a_p} \left[\frac{k + a_p}{k} + \left(\frac{2(k + 2a_p)}{k + a_p} - \frac{k + 2a_p}{a_p}\right)\right] \\
&= \frac{3a_p}{2} + a_q \sum_{k=1}^{a_p} \left[1 - \frac{k}{a_p} + a_p \left(\frac{1}{k} + \frac{2}{k + a_p}\right)\right] \\
&= \frac{3a_p}{2} - \frac{a_q}{2} + a_p a_q \left(\frac{1}{2} + \sum_{k=1}^{a_p} \left(\frac{1}{k} + \frac{2}{k + a_p}\right)\right) \\
&\le a_p a_q \left(\frac{3}{2a_q} - \frac{1}{2a_p} + \ln(2a_p) + \frac{5}{2} - \ln 2\right),
\end{aligned}$$

where we have used $\sum_{k=1}^{n} (1/k + 2/(k + n)) \le \ln(2n) + 2 - \ln 2$ (this can be proved by induction). Hence,

$$N_{pqr} + N_{qpr} \le 2a_p a_q (1 + 0.5 + \ln(2a_p) + 2 - \ln 2) < 2a_1 a_2 (2.81 + \ln(2a_1)).$$

Remark. The official solution was somewhat simpler, but used that the interval $(x, 2x]$, for real x, cannot contain more than x integers, which is false in general. Thus it could give only a weaker estimate $N \le 6a_1 a_2 (9/2 - \ln 2 + \ln(2a_1))$.

32. Let CC' be an altitude, and R the circumradius. Then, since $AH = R$, we have $AC' = |R \sin B|$ and hence (1) $CC' = |R \sin B \tan A|$. On the other hand, $CC' = |BC \sin B| = 2|R \sin A \sin B|$, which together with (1) yields $2|\sin A| = |\tan A| \Rightarrow |\cos A| = 1/2$. Hence, $\angle A$ is 60°. (Without the condition that the triangle is acute, $\angle A$ could also be 120°.)

Second Solution. For a point X, let $\overline{X}$ denote the vector OX. Then $|\overline{A}| = |\overline{B}| = |\overline{C}| = R$ and $\overline{H} = \overline{A} + \overline{B} + \overline{C}$, and moreover,

$$R^2 = (\overline{H} - \overline{A})^2 = (\overline{B} + \overline{C})^2 = 2\overline{B}^2 + 2\overline{C}^2 - (\overline{B} - \overline{C})^2 = 4R^2 - BC^2.$$

It follows that $\sin A = \frac{BC}{2R} = \sqrt{3}/2$, i.e., that $\angle A = 60^\circ$.

Third Solution. Let A_1 be the midpoint of BC. It is well known that $AH = 2OA_1$, and since $AH = AO = BO$, it means that in the right-angled triangle BOA_1 the relation $BO = 2OA_1$ holds. Thus $\angle BOA_1 = \angle A = 60^\circ$.

4.31 Solutions to the Shortlisted Problems of IMO 1990

1. Let N be a number that can be written as a sum of 1990 consecutive integers and as a sum of consecutive positive integers in exactly 1990 ways. The former requirement gives us $N = m + (m+1) + \cdots + (m+1989) = 995(2m+1989)$ for some m. Thus $2 \nmid N$, $5 \mid N$, and $199 \mid N$. The latter requirement tells us that there are exactly 1990 ways to express N as $n + (n+1) + \cdots + (n+k)$, or equivalently, express $2N$ as $(k+1)(2n+k)$. Since N is odd, it follows that one of the factors $k+1$ and $2n+k$ is odd and the other is divisible by 2, but not by 4. Evidently $k+1 < 2n+k$. On the other hand, every factorization $2N = ab$, $1 < a < b$, corresponds to a single pair (n,k), where $n = \frac{b-a+1}{2}$ (which is an integer) and $k = a-1$. The number of such factorizations is equal to $d(2N)/2 - 1$ because $a = b$ is impossible (here $d(x)$ denotes the number of positive divisors of an $x \in \mathbb{N}$). Hence we must have $d(2N) = 2 \cdot 1991 = 3982$. Now let $2N = 2 \cdot 5^{e_1} \cdot 199^{e_2} \cdot p_3^{e_3} \cdots p_r^{e_r}$ be a factorization of $2N$ into prime numbers, where $p_3, \dots, p_r$ are distinct primes other than 2, 5, and 199 and $e_1, \cdots, e_r$ are positive integers. Then $d(2N) = 2(e_1+1)(e_2+1)\cdots(e_r+1)$, from which we deduce $(e_1+1)(e_2+1)\cdots(e_r+1) = 1991 = 11 \cdot 181$. We thus get $\{e_1, e_2\} = \{10, 180\}$ and $e_3 = \cdots = e_r = 0$. Hence $N = 5^{10} \cdot 199^{180}$ and $N = 5^{180} \cdot 199^{10}$ are the only possible solutions. These numbers indeed satisfy the desired properties.

2. We will call a cycle with m committees and n countries an (m,n) cycle. We will number the delegates from each country with numbers $1,2,3$ and denote committees by arrays of these integers (of length n) defining which of the delegates from each country is in the committee. We will first devise methods of constructing larger cycles out of smaller cycles.
Let $A_1, \dots, A_m$ be an (m,n) cycle, where m is odd. Then the following is a $(2m, n+1)$ cycle:
$$(A_1,1), (A_2,2), \dots, (A_m,1), (A_1,2), (A_2,1), \dots, (A_m,2).$$
Also, let $A_1, \dots, A_m$ be an (m,n) cycle and $k \le m$ an even integer. Then the cycle
$$\begin{gathered}(A_1,3), (A_2,1), (A_3,2), \dots, (A_{k-2},1), (A_{k-1},2),\\ (A_k,3), (A_{k-1},1), (A_{k-2},2), \dots, (A_2,2)\end{gathered}$$
is a $(2(k-1), n+1)$ cycle.
Starting from the ((1),(2),(3)) cycle with parameters $(3,1)$ we can sequentially construct larger cycles using the shown methods. The obtained cycles have parameters as follows:
$$(6,2), (10,3), \dots, (2^k+2, k), \dots, (1026, 10), (1990, 11).$$
Thus there exists a cycle of 1990 committees with 11 countries.

3. A segment connecting two points which divides the given circle into two arcs one of which contains exactly n points in its interior we will call a *good* segment. Good segments determine one or more closed polygonal lines that we will call *stars*. Let us compute the number of stars. Note first that $\gcd(n+1, 2n-1) = \gcd(n+1, 3)$.
   (i) Suppose that $3 \nmid n+1$. Then the good segments form a single star. Among any n points, two will be adjacent vertices of the star. On the other hand, we can select $n-1$ alternate points going along the star, and in this case no two points lie on a good segment. Hence $N = n$.
   (ii) If $3 \mid n+1$, we obtain three stars of $\left[\frac{2n-1}{3}\right]$ vertices. If more than $\left[\frac{2n-1}{6}\right] = \frac{n-2}{3}$ points are chosen on any of the stars, then two of them will be connected with a good segment. On the other hand, we can select $\frac{n-2}{3}$ alternate points on each star, which adds up to $n-2$ points in total, no two of which lie on a good segment. Hence $N = n-1$.

   To sum up, $N = n$ for $3 \nmid 2n-1$ and $N = n-1$ for $3 \mid 2n-1$.

4. Assuming that A_1 is not such a set A_i, it follows that for every m there exist m consecutive numbers not in A_1. It follows that $A_2 \cup A_3 \cup \cdots \cup A_r$ contains arbitrarily long sequences of numbers. Inductively, let us assume that $A_j \cup A_{j+1} \cup \cdots \cup A_r$ contains arbitrarily long sequences of consecutive numbers and none of $A_1, A_2, \dots, A_{j-1}$ is the desired set A_i. Let us assume that A_j is also not A_i. Hence for each m there exists $k(m)$ such that among $k(m)$ elements of A_j there exist two consecutive elements that differ by at least m. Let us consider $m \cdot k(m)$ consecutive numbers in $A_j \cup \cdots \cup A_r$, which exist by the induction hypothesis. Then either A_j contains fewer than $k(m)$ of these integers, in which case $A_{j+1} \cup \cdots \cup A_r$ contains m consecutive integers by the pigeonhole principle or A_j contains $k(m)$ integers among which there exists a gap of length m of consecutive integers that belong to $A_{j+1} \cup \cdots \cup A_r$. Hence we have proven that $A_{j+1} \cup \cdots \cup A_r$ contains sequences of integers of arbitrary length. By induction, assuming that $A_1, A_2, \dots, A_{r-1}$ do not satisfy the conditions to be the set A_i, it follows that A_r contains sequences of consecutive integers of arbitrary length and hence satisfies the conditions necessary for it to be the set A_i.

5. Let O be the circumcenter of ABC, E the midpoint of OH, and R and r the radii of the circumcircle and incircle respectively. We use the following facts from elementary geometry: $\overrightarrow{OH} = 3\overrightarrow{OG}$, $OK^2 = R^2 - 2Rr$, and $KE = \frac{R}{2} - r$. Hence $\overrightarrow{KH} = 2\overrightarrow{KE} - \overrightarrow{KO}$ and $\overrightarrow{KG} = \frac{2\overrightarrow{KE} + \overrightarrow{KO}}{3}$. We then obtain
$$\overrightarrow{KH} \cdot \overrightarrow{KG} = \frac{1}{3}(4KE^2 - KO^2) = -\frac{2}{3}r(R-2r) < 0 .$$
   Hence $\cos\angle GKH < 0 \Rightarrow \angle GKH > 90^\circ$.

6. Let W denote the set of all n_0 for which player A has a winning strategy, L the set of all n_0 for which player B has a winning strategy, and T the set of all n_0 for which a tie is ensured.

Lemma. Assume $\{m, m+1, \dots 1990\} \subseteq W$ and that there exists $s \le 1990$ such that $s/p^r \ge m$, where p^r is the largest degree of a prime that divides s. Then all integers x such that $\sqrt{s} \le x < m$ also belong in W.

Proof. Starting from x, player A can choose s, and by definition of s, player B cannot choose a number smaller than m. This ensures player A the victory.

We now have trivially that since $45^2 = 2025 > 1990$, it follows that for $n_0 \in \{45, \dots, 1990\}$ player A can choose 1990 in the first move. Hence $\{45, \dots, 1990\} \subseteq W$. Using $m = 45$ and selecting $s = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7$ we apply the lemma to get that all integers x such that $\sqrt{420} < 21 \le x \le 1990$ are in W. Again, using $m = 21$ and selecting $s = 168 = 2^3 \cdot 3 \cdot 7$ we apply the lemma to get that all integers x such that $\sqrt{168} < 13 \le x \le 1990$ are in W. Selecting $s = 105$ we obtain the new value for m at $m = 11$. Selecting $s = 60$ we obtain $m = 8$. Thus $\{8, \dots, 1990\} \subseteq W$.

For $n_0 > 1990$ there exists $r \in N$ such that $2^r \cdot 3^2 < n_0 \le 2^{r+1} \cdot 3^2 < n_0^2$. Player A can take $n_1 = 2^{r+1} \cdot 3^2$. The number player B selects has to satisfy $8 \le n_2 < n_0$. After finitely many steps he will select $8 \le n_{2r} \le 1990$, and A will have a winning strategy. Hence all $m \ge 8$ belong to W.

Now let us consider the case $n_0 \le 5$. Since the smallest number divisible by three different primes is 30 and $n_0^2 \le 5^2 = 25 < 30$, it follows that n_1 is of the form $n_1 = p^r$ or $n_1 = p^r \cdot q^s$, where p and q are two different primes. In the first case player B can choose 1 and win, while in the second case he can select the smaller of p^r, q^s, which is also smaller than $\sqrt{n_1} \le n_0$. Thus player B can eventually reach $n_{2k} = 1$. Thus $\{2, 3, 4, 5\} \subseteq L$.

Finally, for $n_0 = 6$ or $n_0 = 7$ player A must select a number divisible by at least three primes, which must be $30 = 2 \cdot 3 \cdot 5$ or $42 = 2 \cdot 3 \cdot 7$; otherwise, B can select a degree of a prime smaller than n_0, yielding $n_2 < 6$ and victory for B. Player B must select a number smaller than 8. Hence, he has to select 6 in both cases. Afterwards, to avoid losing the game, player A will always choose 30 and player B always 6. In this case we would have a tie. Hence $T \subseteq \{6, 7\}$.

Considering that we have accounted for all integers $n_0 > 1$, the final solution is $L = \{2, 3, 4, 5\}$, $T = \{6, 7\}$, and $W = \{x \in \mathbb{N} \mid x \ge 8\}$.

7. Let $f(n) = g(n)2^{n^2}$ for all n. The recursion then transforms into $g(n+2) - 2g(n+1) + g(n) = n \cdot 16^{-n-1}$ for $n \in \mathbb{N}_0$. By summing this equation from 0 to $n-1$, we get

$$g(n+1) - g(n) = \frac{1}{15^2} \cdot (1 - (15n+1)16^{-n}).$$

By summing up again from 0 to $n-1$ we get $g(n) = \frac{1}{15^3} \cdot (15n - 32 + (15n+2)16^{-n+1})$. Hence

$$f(n) = \frac{1}{15^3} \cdot (15n + 2 + (15n - 32)16^{n-1}) \cdot 2^{(n-2)^2}.$$

Now let us look at the values of $f(n)$ modulo 13:

$$f(n) \equiv 15n + 2 + (15n - 32)16^{n-1} \equiv 2n + 2 + (2n - 6)3^{n-1}.$$

We have $3^3 \equiv 1 \pmod{13}$. Plugging in $n \equiv 1 \pmod{13}$ and $n \equiv 1 \pmod 3$ for $n = 1990$ gives us $f(1990) \equiv 0 \pmod{13}$. We similarly calculate $f(1989) \equiv 0$ and $f(1991) \equiv 0 \pmod{13}$.

8. Since $2^{1990} < 8^{700} < 10^{700}$, we have $f_1(2^{1990}) < (9\cdot700)^2 < 4\cdot10^7$. We then have $f_2(2^{1990}) < (3+9\cdot7)^2 < 4900$ and finally $f_3(2^{1990}) < (3+9\cdot3)^2 = 30^2$. It is easily shown that $f_k(n) \equiv f_{k-1}(n)^2 \pmod 9$. Since $2^6 \equiv 1 \pmod 9$, we have $2^{1990} \equiv 2^4 \equiv 7$ (all congruences in this problem will be mod 9). It follows that $f_1(2^{1990}) \equiv 7^2 \equiv 4$ and $f_2(2^{1990}) \equiv 4^2 \equiv 7$. Indeed, it follows that $f_{2k}(2^{1990}) \equiv 7$ and $f_{2k+1}(2^{1990}) \equiv 4$ for all integer $k > 0$. Thus $f_3(2^{1990}) = r^2$ where $r < 30$ is an integer and $r \equiv f_2(2^{1990}) \equiv 7$. It follows that $r \in \{7, 16, 25\}$ and hence $f_3(2^{1990}) \in \{49, 256, 625\}$. It follows that $f_4(2^{1990}) = 169$, $f_5(2^{1990}) = 256$, and inductively $f_{2k}(2^{1990}) = 169$ and $f_{2k+1}(2^{1990}) = 256$ for all integer $k > 1$. Hence $f_{1991}(2^{1990}) = 256$.

9. Let a, b, c be the lengths of the sides of $\triangle ABC$, $s = \frac{a+b+c}{2}$, r the inradius of the triangle, and c_1 and b_1 the lengths of AB_2 and AC_2 respectively. As usual we will denote by $S(XYZ)$ the area of $\triangle XYZ$. We have

$$S(AC_1B_2) = \frac{AC_1 \cdot AB_2}{AC \cdot AB} S(ABC) = \frac{c_1 rs}{2b},$$

$$S(AKB_2) = \frac{c_1 r}{2}, \quad S(AC_1K) = \frac{cr}{4}.$$

From $S(AC_1B_2) = S(AKB_2) + S(AC_1K)$ we get $\frac{c_1rs}{2b} = \frac{c_1r}{2} + \frac{cr}{4}$; therefore $(a - b + c)c_1 = bc$. By looking at the area of $\triangle AB_1C_2$ we similarly obtain $(a + b - c)b_1 = bc$. From these two equations and from $S(ABC) = S(AB_2C_2)$, from which we have $b_1c_1 = bc$, we obtain

$$a^2 - (b - c)^2 = bc \Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \cos(\angle BAC) = \frac{1}{2} \Rightarrow \angle BAC = 60^\circ.$$

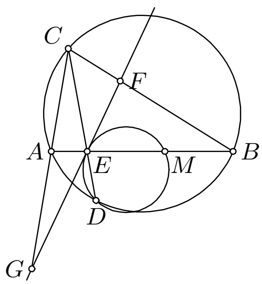
10. Let r be the radius of the base and h the height of the cone. We may assume w.l.o.g. that $r = 1$. Let A be the top of the cone, BC the diameter of the circumference of the base such that the plane touches the circumference at B, O the center of the base, and H the midpoint of OA (also belonging to the plane). Let BH cut the sheet of the cone at D. By applying Menelaus's theorem to $\triangle AOC$ and $\triangle BHO$, we conclude that $\frac{AD}{DC} = \frac{CB}{BO} \cdot \frac{OH}{HA} = \frac{1}{2}$ and $\frac{HD}{DB} = \frac{HA}{AO} \cdot \frac{OC}{CB} = \frac{1}{4}$.
The plane cuts the cone in an ellipse whose major axis is BD. Let E be the center of this ellipse and FG its minor axis. We have $\frac{BE}{ED} = \frac{1}{2}$. Let E', F', G' be radial projections of E, F, G from A onto the base of the cone. Then E sits on BC. Let $h(X)$ denote the height of a point X with respect to the base of the cone. We have $h(E) = h(D)/2 = h/3$.

Hence $EF = 2E'F'/3$. Applying Menelaus's theorem to $\triangle BHO$ we get $\frac{OE'}{E'B} = \frac{BE}{EH} \cdot \frac{HA}{AO} = 1$. Hence $EF = \frac{2}{3}\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}$.
Let d denote the distance from A to the plane. Let V_1 and V denote the volume of the cone above the plane (on the same side of the plane as A) and the total volume of the cone. We have

$$\frac{V_1}{V} = \frac{BE \cdot EF \cdot d}{h} = \frac{(2BH/3)(1/\sqrt{3})(2S_{AHB}/BH)}{h}$$
$$= \frac{(2/3)(1/\sqrt{3})(h/2)}{h} = \frac{1}{3\sqrt{3}}.$$

Since this ratio is smaller than $1/2$, we have indeed selected the correct volume for our ratio.

11. Assume $\mathcal{B}(A,E,M,B)$. Since A,B,C,D lie on a circle, we have $\angle GCE = \angle MBD$ and $\angle MAD = \angle FCE$. Since FD is tangent to the circle around $\triangle EMD$ at E, we have $\angle MDE = \angle FEB = \angle AEG$. Consequently, $\angle CEF = 180^\circ - \angle CEA - \angle FEB = 180^\circ - \angle MED - \angle MDE = \angle EMD$ and $\angle CEG = 180^\circ - \angle CEF = 180^\circ - \angle EMD = \angle DMB$. It follows that $\triangle CEF \sim \triangle AMD$ and $\triangle CEG \sim \triangle BMD$. From the first similarity we obtain $CE \cdot MD = AM \cdot EF$, and from the second we obtain $CE \cdot MD = BM \cdot EG$. Hence



$$AM \cdot EF = BM \cdot EG \Longrightarrow$$
$$\frac{GE}{EF} = \frac{AM}{BM} = \frac{\lambda}{1-\lambda}.$$

If $\mathcal{B}(A,M,E,B)$, interchanging the roles of A and B we similarly obtain $\frac{GE}{EF} = \frac{\lambda}{1-\lambda}$.

12. Let $d(X,l)$ denote the distance of a point X from a line l. Using the elementary facts that $AF : FC = c : a$ and $BD : DC = c : b$, we obtain $d(F,L) = \frac{a}{a+c}h_c$ and $d(D,L) = \frac{b}{b+c}h_c$, where h_a is the altitude of $\triangle ABC$ from A. We also have $\angle FGC = \beta/2$, $\angle DEC = \alpha/2$. It follows that

$$DE = \frac{d(D,L)}{\sin(\alpha/2)} \quad \text{and} \quad FG = \frac{d(F,L)}{\sin(\beta/2)}. \tag{1}$$

Now suppose that $a > b$. Since the function $f(x) = \frac{x}{x+c}$ is strictly increasing, we deduce $d(F,L) > d(D,L)$. Furthermore, $\sin(\alpha/2) > \sin(\beta/2)$, so we get from (1) that $FG > DE$.
Similarly, $a < b$ implies $FG < DE$. Hence we must have $a = b$, i.e., $AC = BC$.

13. We will call the ground the "zeroth" rung. We will prove that the minimum n is $n = a+b-(a,b)$. It is plain that if $(a,b) = k > 1$, the scientist can climb

only onto the rungs divisible by k and we can just observe these rungs to obtain the situation equivalent to $a' = a/k$, $b' = b/k$, and $n' = a' + b' - 1$. Thus let us assume that $(a, b) = 1$ and show that $n = a + b - 1$.

We obviously have $n > a$. Consider $n = a + b - k$, $k \geq 1$, and let us assume without loss of generality that $a > b$ (otherwise, we can reverse the problem starting from the top rung in our round trip). Then we can uniquely define the numbers r_i, $0 \leq r_i < b$, by $r_i \equiv ia \pmod{b}$. We now describe the only possible sequence of moves. From a position $0 \leq p \leq b-k$ we can move only a rungs upward and for $p > b - 1$ we can move only b rungs downward. If we end up at $b - k < p \leq b - 1$, we are stuck. Hence, given that we are at r_i, if $r_i \leq b - k$, we can move to $a + r_i$, and when we descend as far as we can go we will end up at $r_{i+1} \equiv a + r_i \pmod{b}$.

If the mathematician climbs to the highest rung and then comes back to $r_i = 0$, then we deduce $b \mid ia$, so $i \geq b$. But since $(a, b) = 1$, there exists $0 < j < b$ such that $r_j \equiv ja \equiv b - 1 \pmod{b}$. Thus the mathematician has visited the position $b - 1$. For him not to get stuck we must have $k \leq 1$ and $n \geq a + b - 1$. For $n = a + b - 1$ by induction he can come to any position r_i, $i \geq 0$, so he eventually comes to $r_j = b - 1$, climbs to the highest rung, and then continues until he gets to $r_b = 0$. Hence the answer to the problem is $n = a + b - 1$.

14. Let V be the set of all midpoints of bad sides, and E the set of segments connecting two points in V that belong to the same triangle. Each edge in E is parallel to exactly one good side and thus is parallel to the coordinate grid and has half-integer coordinates. Thus, the edges of E are a subset of the grid formed by joining the centers of the squares in the original grid to each other. Let G be a graph whose set of vertices is V and set of edges is E. The degree of each vertex X, denoted by $d(X)$, is 0, 1, or 2. We observe the following cases:
   (i) $d(X) = 0$ for some X. Then both triangles containing X have two good sides.
   (ii) $d(X) = 1$ for some X. Since $\sum_{X \in V} d(X) = 2|E|$ is even, it follows that at least another vertex Y has the degree 1. Hence both X and Y belong to triangles having two good sides.
   (iii) $d(X) = 2$ for all $X \in V$. We will show that this case cannot occur. We prove first that centers of all the squares of the $m \times n$ board belong to $V \cup E$. A bad side contains no points with half-integer coordinates in its interior other than its midpoint. Therefore either a point X is in V, or it lies on the segment connecting the midpoints of the two bad sides. Evidently, the graph G can be partitioned into disjoint cycles. Each center of a square is passed exactly once in exactly one cycle. Let us color the board black and white in a standard chessboard fashion. Each cycle passes through centers that must alternate in color, and hence it contains an equal number of black and white centers. Consequently,

the numbers of black and white squares on the entire board must be equal, contradicting the condition that m and n are odd.

Our proof is thus completed.

15. Let $S(Z)$ denote the sum of all the elements of a set Z. We have $S(X) = (k+1)\cdot 1990 + \frac{k(k+1)}{2}$. To partition the set into two parts with equal sums, $S(X)$ must be even and hence $\frac{k(k+1)}{2}$ must be even. Hence k is of the form $4r$ or $4r+3$, where r is an integer.

For $k = 4r+3$ we can partition X into consecutive fourtuplets $\{1990 + 4l, 1990 + 4l + 1, 1990 + 4l + 2, 1990 + 4l + 3\}$ for $0 \le l \le r$ and put $1990 + 4l, 1990 + 4l + 3 \in A$ and $1990 + 4l + 1, 1990 + 4l + 2 \in B$ for all l. This would give us $S(A) = S(B) = (3980 + 4r + 3)(r + 1)$.

For $k = 4r$ the numbers of elements in A and B must differ. Let us assume without loss of generality $|A| < |B|$. Then $S(A) \le (1990+2r+1)+(1990+2r+2)+\cdots+(1990+4r)$ and $S(B) \ge 1990+1991+\cdots+(1990+2r)$. Plugging these inequalities into the condition $S(A) = S(B)$ gives us $r \ge 23$ and consequently $k \ge 92$. We note that $B = \{1990, 1991, \dots, 2034, 2052, 2082\}$ and $A = \{2035, 2036, \dots, 2051, 2053, \dots, 2081\}$ is a partition for $k = 92$ that satisfies $S(A) = S(B)$. To construct a partition out of higher $k = 4r$ we use the $k = 92$ partition for the first 93 elements and construct for the remaining elements as was done for $k = 4r + 3$.

Hence we can construct a partition exactly for the integers k of the form $k = 4r + 3$, $r \ge 0$, and $k = 4r$, $r \ge 23$.

16. Let $A_0A_1\dots A_{1989}$ be the desired 1990-gon. We also define $A_{1990} = A_0$. Let O be an arbitrary point. For $1 \le i \le 1990$ let B_i be a point such that $\overrightarrow{OB_i} = \overrightarrow{A_{i-1}A_i}$. We define $B_0 = B_{1990}$. The points B_i must satisfy the following properties: $\angle B_iOB_{i+1} = \frac{2\pi}{1990}$, $0 \le i \le 1989$, lengths of OB_i are a permutation of $1^2, 2^2, \dots, 1989^2, 1990^2$, and $\sum_{i=0}^{1989} \overrightarrow{OB_i} = \overrightarrow{0}$. Conversely, any such set of points B_i corresponds to a desired 1990-gon. Hence, our goal is to construct vectors $\overrightarrow{OB_i}$ satisfying all the stated properties.

Let us group vectors of lengths $(2n-1)^2$ and $(2n)^2$ into pairs and put them diametrically opposite each other. The length of the resulting vectors is $4n - 1$. The problem thus reduces to arranging vectors of lengths $3, 7, 11, \dots, 3979$ at mutual angles of $\frac{2\pi}{995}$ such that their sum is $\overrightarrow{0}$. We partition the 995 directions into 199 sets of five directions at mutual angles $\frac{2\pi}{5}$. The directions when intersected with a unit circle form a regular pentagon. We group the set of lengths of vectors $3, 7, \dots, 3979$ into 199 sets of five consecutive elements of the set. We place each group of lengths on directions belonging to the same group of directions, thus constructing five vectors. We use that $\overrightarrow{OC_1} + \cdots + \overrightarrow{OC_n} = 0$ where O is the center of a regular n-gon $C_1 \dots C_n$. In other words, vectors of equal lengths along directions that form a regular n-gon cancel each other out. Such are the groups of five directions. Hence, we can assume for each group of five lengths for its lengths to be $\{0, 4, 8, 12, 16\}$. We place these five lengths in a random fashion on a single group of directions. We then rotate the configuration clockwise by $\frac{2\pi}{199}$ to cover other groups of directions and repeat until all groups of directions are exhausted. It follows that all vectors of each of the lengths $\{0, 4, 8, 12, 16\}$ will form a regular 199-gon and will thus cancel each other out.

We have thus constructed a way of obtaining points B_i and have hence shown the existence of the 1990-gon satisfying (i) and (ii).

17. Let us set a coordinate system denoting the vertices of the block. The vertices of the unit cubes of the block can be described as $\{(x, y, z) \mid 0 \le x \le p,\ 0 \le y \le q,\ 0 \le z \le r\}$, and we restrict our attention to only these points. Suppose the point A is fixed at (a, b, c). Then for every other necklace point (x, y, z) numbers $x - a$, $y - b$, and $z - c$ must be of equal parity. Conversely, every point (x, y, z) such that $x - a$, $y - b$, and $z - c$ are of the same parity has to be a necklace point. Consider the graph G whose vertices are all such points and edges are all diagonals of the unit cubes through these points. In part (a) we are looking for an open or closed Euler path, while in part (b) we are looking for a closed Euler path.

    Necklace points in the interior of the (p, q, r) box have degree 8, points on the surface have degree 4, points on the edge have degree 2, and points on the corner have degree 1. A closed Euler path can be formed if and only if all vertices are of an even degree, while an open Euler path can be formed if and only if exactly two vertices have an odd degree. Hence the problem in part (a) amounts to being able to choose a point A such that 0 or 2 corner vertices are necklace vertices, whereas in part (b) no corner points can be necklace vertices. We distinguish two cases.

    (i) At least two of p, q, r, say p, q, are even. We can choose $a = 1,\ b = c = 0$. In this case none of the corners is a necklace point. Hence a closed Euler path exists.

    (ii) At most one of p, q, r is even. However one chooses A, exactly two necklace points are at the corners. Hence, an open Euler path exists, but it is impossible to form a closed path.

    Hence, in part (a), a box can be made of all (p, q, r) and in part (b) only those (p, q, r) where at least two of the numbers are even.

18. Clearly, it suffices to consider the case $(a, b) = 1$. Let S be the set of integers such that $M - b \le x \le M + a - 1$. Then $f(S) \subseteq S$ and $0 \in S$. Consequently, $f^k(0) \in S$. Let us assume for $k > 0$ that $f^k(0) = 0$. Since $f(m) = m + a$ or $f(m) = m - b$, it follows that k can be written as $k = r + s$, where $ra - sb = 0$. Since a and b are relatively prime, it follows that $k \ge a + b$.

    Let us now prove that $f^{a+b}(0) = 0$. In this case $a + b = r + s$ and hence $f^{a+b}(0) = (a + b - s)a - sb = (a + b)(a - s)$. Since $a + b \mid f^{a+b}(0)$ and $f^{a+b}(0) \in S$, it follows that $f^{a+b}(0) = 0$. Thus for $(a, b) = 1$ it follows that $k = a + b$. For other a and b we have $k = \frac{a+b}{(a,b)}$.

19. Let d_1, d_2, d_3, d_4 be the distances of the point P to the tetrahedron. Let d be the height of the regular tetrahedron. Let $x_i = d_i/d$. Clearly, $x_1 + x_2 + x_3 + x_4 = 1$, and given this condition, the parameters vary freely as we vary P within the tetrahedron. The four tetrahedra have volumes x_1^3, x_2^3, x_3^3, and x_4^3, and the four parallelepipeds have volumes of $6x_2x_3x_4$, $6x_1x_3x_4$, $6x_1x_2x_4$, and $6x_1x_2x_3$. Hence, using $x_1 + x_2 + x_3 + x_4 = 1$ and setting $g(x) = x^2(1-x)$, we directly verify that

$$\begin{aligned} f(P) = f(x_1, x_2, x_3, x_4) &= 1 - \sum_{i=1}^{4} x_i^3 - 6 \sum_{1 \le i < j < k \le 4} x_i x_j x_k \\ &= 3(g(x_1) + g(x_2) + g(x_3) + g(x_4)) \,. \end{aligned}$$

We note that $g(0) = 0$ and $g(1) = 0$. Hence, as x_1 tends to 1 and other variables tend to 0, $f(x_1, x_2, x_3, x_4) = 0$. Thus $f(P)$ is sharply bounded downwards at 0.

We now find an upper bound. We note that

$$\begin{aligned} g(x_i + x_j) &= (x_i + x_j)^2 (1 - x_1 - x_2) \\ &= g(x_i) + g(x_j) + 2x_i x_j \left(1 - \frac{3}{2}(x_i + x_j)\right); \end{aligned}$$

thus for $x_i + x_j \le 2/3$ and $x_i, x_j > 0$ we have $g(x_i + x_j) + g(0) \ge g(x_i) + g(x_j)$. Equality holds only when $x_i + x_j = 2/3$.

Assuming without loss of generality $x_1 \ge x_2 \ge x_3 \ge x_4$, we have $g(x_1) + g(x_2) + g(x_3) + g(x_4) < g(x_1) + g(x_2) + g(x_3 + x_4)$. Assuming $y_1 + y_2 + y_3 = 1$ and $y_1 \ge y_2 \ge y_3$, we have $g(y_1) + g(y_2) + g(y_3) \le g(y_1) + g(y_2 + y_3)$. Hence $g(x_1) + g(x_2) + g(x_3) + g(x_4) < g(x) + g(1-x)$ for some x. We also have $g(x) + g(1-x) = x(1-x) \le 1/4$. Hence $f(P) \le 3/4$. Equality holds for $x_1 = x_2 = 1/2$, $x_3 = x_4 = 0$ (corresponding to the midpoint of an edge), and as the variables converge to these values, $f(P)$ converges to $3/4$. Hence the bounds for $f(P)$ are

$$0 < f(P) < \frac{3}{4}.$$

20. Let n be the unique integer such that $2^{n-1} \le k < 2^n$. Let $S(n)$ be the set of numbers less than 10^n that are written with only the digits $\{0, 1\}$ in the decimal system. Evidently $|S(n)| = 2^n > k$ and hence there exist two numbers $x, y \in S(n)$ such that $k \mid x - y$.

Let us show that $w = |x - y|$ is the desired number. By definition $k \mid w$. We also have

$$w < 1.2 \cdot 10^{n-1} \le 1.2 \cdot (2^3\sqrt{2})^{n-1} \le 1.2 \cdot k^3\sqrt{k} \le k^4.$$

Finally, since $x, y \in S(n)$, it follows that $w = |x - y|$ can be written using only the digits $\{0, 1, 8, 9\}$. This completes the proof.

21. We must solve the congruence $(1+2^p+2^{n-p})N \equiv 1 \pmod{2^n}$. Since $(1+2^p+2^{n-p})$ and 2^n are coprime, there clearly exists a unique N satisfying this equation and $0 < N < 2^n$.

Let us assume $n = mp$. Then we have $(1+2^p)\left(\sum_{j=0}^{m-1}(-1)^j 2^{jp}\right) \equiv 1 \pmod{2^n}$ and $(1+2^{n-p})(1-2^{n-p}) \equiv 1 \pmod{2^n}$. By multiplying the two congruences we obtain

$$(1+2^p)(1+2^{n-p})(1-2^{n-p})\left(\sum_{j=0}^{m-1}(-1)^j 2^{jp}\right) \equiv 1 \pmod{2^n}\,.$$

Since $(1+2^p)(1+2^{n-p}) \equiv (1+2^p+2^{n-p}) \pmod{2^n}$, it follows that $N \equiv (1-2^{n-p})\left(\sum_{j=0}^{m-1}(-1)^j 2^{jp}\right) \pmod{2^n}$. The integer $N = \sum_{j=0}^{m-1}(-1)^j 2^{jp} - 2^{n-p} + 2^n$ satisfies the congruence and $0 < N \le 2^n$. Using that for $a > b$ we have in binary representation

$$2^a - 2^b = \underbrace{11\dots11}_{a-b \text{ times}}\underbrace{00\dots00}_{b \text{ times}},$$

the binary representation of N is calculated as follows:

$$N = \begin{cases} \underbrace{11\dots11}_{p \text{ times}}\underbrace{11\dots11}_{p \text{ times}}\underbrace{00\dots00}_{p \text{ times}}\dots\underbrace{11\dots11}_{p \text{ times}}\underbrace{00\dots00}_{p-1 \text{ times}}1, & 2 \nmid \frac{n}{p}, \\ \underbrace{11\dots11}_{p-1 \text{ times}}\underbrace{00\dots00}_{p+1 \text{ times}}\underbrace{11\dots11}_{p \text{ times}}\underbrace{00\dots00}_{p \text{ times}}\dots\underbrace{11\dots11}_{p \text{ times}}\underbrace{00\dots00}_{p-1 \text{ times}}1, & 2 \mid \frac{n}{p}. \end{cases}$$

22. We can assume without loss of generality that each connection is serviced by only one airline and the problem reduces to finding two disjoint monochromatic cycles of the same color and of odd length on a complete graph of 10 points colored by two colors. We use the following two standard lemmas:

Lemma 1. Given a complete graph on six points whose edges are colored with two colors there exists a monochromatic triangle.

Proof. Let us denote the vertices by $c_1, c_2, c_3, c_4, c_5, c_6$. By the pigeonhole principle at least three vertices out of c_1, say c_2, c_3, c_4, are of the same color, let us call it red. Assuming that at least one of the edges connecting points c_2, c_3, c_4 is red, the connected points along with c_1 form a red triangle. Otherwise, edges connecting c_2, c_3, c_4 are all of the opposite color, let us call it blue, and hence in all cases we have a monochromatic triangle.

Lemma 2. Given a complete graph on five points whose edges are colored two colors there exists a monochromatic triangle or a monochromatic cycle of length five.

Proof. Let us denote the vertices by c_1, c_2, c_3, c_4, c_5. Assume that out of a point c_i three vertices are of the same color. We can then proceed as in Lemma 1 to obtain a monochromatic triangle. Otherwise, each point is connected to other points with exactly two red and two blue vertices. Hence, we obtain monochromatic cycles starting from a single point and moving along the edges of the same color. Since each cycle must be of length at least three (i.e., we cannot have more than one cycle of one color), it follows that for both red and blue we must have one cycle of length five of that color.

We now apply the lemmas. Let us denote the vertices by $c_1, c_2, \dots, c_{10}$. We apply Lemma 1 to vertices $c_1, \dots, c_6$ to obtain a monochromatic triangle. Out of the seven remaining vertices we select 6 and again apply Lemma 1 to obtain another monochromatic triangle. If they are of the same color, we are done. Otherwise, out of the nine edges connecting the two triangles of opposite color at least 5 are of the same color, we can assume blue w.l.o.g., and hence a vertex of a red triangle must contain at least two blue edges whose endpoints are connected with a blue edge. Hence there exist two triangles of different colors joined at a vertex. These take up five points. Applying Lemma 2 on the five remaining points, we obtain a monochromatic cycle of odd length that is of the same color as one of the two joined triangles and disjoint from both of them.

23. Let us assume $n > 1$. Obviously n is odd. Let $p \geq 3$ be the smallest prime divisor of n. In this case $(p-1, n) = 1$. Since $2^n + 1 \mid 2^{2n} - 1$, we have that $p \mid 2^{2n} - 1$. Thus it follows from Fermat's little theorem and elementary number theory that $p \mid (2^{2n} - 1, 2^{p-1} - 1) = 2^{(2n,p-1)} - 1$. Since $(2n, p-1) \leq 2$, it follows that $p \mid 3$ and hence $p = 3$.

Let us assume now that n is of the form $n = 3^k d$, where $2, 3 \nmid d$. We first prove that $k = 1$.

Lemma. If $2^m - 1$ is divisible by 3^r, then m is divisible by 3^{r-1}.

Proof. This is the lemma from (SL97-14) with $p = 3$, $a = 2^2$, $k = m$, $\alpha = 1$, and $\beta = r$.

Since 3^{2k} divides $n^2 \mid 2^{2n} - 1$, we can apply the lemma to $m = 2n$ and $r = 2k$ to conclude that $3^{2k-1} \mid n = 3^k d$. Hence $k = 1$.

Finally, let us assume $d > 1$ and let q be the smallest prime factor of d. Obviously q is odd, $q \geq 5$, and $(n, q-1) \in \{1, 3\}$. We then have $q \mid 2^{2n} - 1$ and $q \mid 2^{q-1} - 1$. Consequently, $q \mid 2^{(2n,q-1)} - 1 = 2^{2(n,q-1)} - 1$, which divides $2^6 - 1 = 63 = 3^2 \cdot 7$, so we must have $q = 7$. However, in that case we obtain $7 \mid n \mid 2^n + 1$, which is a contradiction, since powers of two can only be congruent to 1,2 and 4 modulo 7. It thus follows that $d = 1$ and $n = 3$. Hence $n > 1 \Rightarrow n = 3$.

It is easily verified that $n = 1$ and $n = 3$ are indeed solutions. Hence these are the only solutions.

24. Let us denote $A = b + c + d$, $B = a + c + d$, $C = a + b + d$, $D = a + b + c$. Since $ab + bc + cd + da = 1$ the numbers A, B, C, D are all positive. By trivially applying the AM-GM inequality we have:

$$a^2 + b^2 + c^2 + d^2 \geq ab + bc + cd + da = 1 \ .$$

We will prove the inequality assuming only that A, B, C, D are positive and $a^2 + b^2 + c^2 + d^2 \geq 1$. In this case we may assume without loss of generality that $a \geq b \geq c \geq d \geq 0$. Hence $a^3 \geq b^3 \geq c^3 \geq d^3 \geq 0$ and $\frac{1}{A} \geq \frac{1}{B} \geq \frac{1}{C} \geq \frac{1}{D} > 0$. Using the Chebyshev and Cauchy inequalities we obtain:

$$\begin{aligned}
&\frac{a^3}{A} + \frac{b^3}{B} + \frac{c^3}{C} + \frac{d^3}{D} \\
&\quad \geq \frac{1}{4}(a^3 + b^3 + c^3 + d^3)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}\right) \\
&\quad \geq \frac{1}{16}(a^2 + b^2 + c^2 + d^2)(a + b + c + d)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}\right) \\
&\quad = \frac{1}{48}(a^2 + b^2 + c^2 + d^2)(A + B + C + D)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}\right) \geq \frac{1}{3}\,.
\end{aligned}$$

This completes the proof.

25. Plugging in $x = 1$ we get $f(f(y)) = f(1)/y$ and hence $f(y_1) = f(y_2)$ implies $y_1 = y_2$ i.e. that the function is bijective. Plugging in $y = 1$ gives us $f(xf(1)) = f(x) \Rightarrow xf(1) = x \Rightarrow f(1) = 1$. Hence $f(f(y)) = 1/y$. Plugging in $y = f(z)$ implies $1/f(z) = f(1/z)$. Finally setting $y = f(1/t)$ into the original equation gives us $f(xt) = f(x)/f(1/t) = f(x)f(t)$.
Conversely, any functional equation on $\mathbb{Q}^+$ satisfying (i) $f(xt) = f(x)f(t)$ and (ii) $f(f(x)) = \frac{1}{x}$ for all $x, t \in \mathbb{Q}^+$ also satisfies the original functional equation: $f(xf(y)) = f(x)f(f(y)) = \frac{f(x)}{y}$. Hence it suffices to find a function satisfying (i) and (ii).
We note that all elements $q \in \mathbb{Q}^+$ are of the form $q = \prod_{i=1}^n p_i^{a_i}$ where p_i are prime and $a_i \in \mathbb{Z}$. The criterion (a) implies $f(q) = f(\prod_{i=1}^n p_i^{a_i}) = \prod_{i=1}^n f(p_i)^{a_i}$. Thus it is sufficient to define the function on all primes. For the function to satisfy (b) it is necessary and sufficient for it to satisfy $f(f(p)) = \frac{1}{p}$ for all primes p. Let q_i denote the i-th smallest prime. We define our function f as follows:

$$f(q_{2k-1}) = q_{2k}, \quad f(q_{2k}) = \frac{1}{q_{2k-1}}, \quad k \in \mathbb{N}\,.$$

Such a function clearly satisfies (b) and along with the additional condition $f(xt) = f(x)f(t)$ it is well defined for all elements of $\mathbb{Q}^+$ and it satisfies the original functional equation.

26. We note that $|P(x)/x| \to \infty$. Hence, there exists an integer number M such that $M > |q_1|$ and $|P(x)| \leq |x| \Rightarrow |x| < M$. It follows that $|q_i| < M$ for all $i \in \mathbb{N}$ because assuming $|q_i| \geq M$ for some i we get $|q_{i-1}| = |P(q_i)| > |q_i| \geq M$ and this ultimately contradicts $|q_1| < M$.
Let us define $q_1 = \frac{r}{s}$ and $P(x) = \frac{ax^3+bx^2+cx+d}{e}$ where r, s, a, b, c, d, e are all integers. For $N = sa$ we shall prove by induction that Nq_i is an integer for all $i \in \mathbb{N}$. By definition $N \neq 0$.

For $i = 1$ this obviously holds. Assume it holds for some $i \in \mathbb{N}$. Then using $q_i = P(q_{i+1})$ we have that Nq_{i+1} is a zero of the polynomial

$$Q(x) = \frac{e}{a}N^3\left(P\left(\frac{x}{N}\right) - q_i\right)$$
$$= x^3 + (sb)x^2 + (s^2ac)x + (s^3a^2d - s^2ae(Nq_i)) .$$

Since $Q(x)$ is a monic polynomial with integer coefficients (a conclusion for which we must assume the induction hypothesis) and Nq_{i+1} is rational it follows by the rational root theorem that Nq_{i+1} is an integer.

It follows that all q_i are multiples of $1/N$. Since $-M < q_i < M$ we conclude that q_i can take less than $T = 2M|N|$ distinct values. Therefore for each j there are m_j and $m_j + k_j$ ($k_j > 0$) both belonging to the set $\{jT+1, jT+2, \ldots, jT+T\}$ such that $q_{m_j} = q_{m_j+k_j}$. Since $k_j < T$ for all k_j it follows that there exists a positive integer k which appears an infinite number of times in the sequence k_j, i.e. there exist infinitely many integers m such that $q_m = q_{m+k}$. Moreover, $q_m = q_{m+k}$ clearly implies $q_n = q_{n+k}$ for all $n \le m$. Hence $q_n = q_{n+k}$ holds for all n.

27. Let us denote by $A_n(k)$ the n-digit number which consists of $n-1$ ones and one digit seven in the $k+1$-th rightmost position ($0 \le k < n$). Then $A_n(k) = (10^n + 54 \cdot 10^k - 1)/9$.

We note that if $3 \mid n$ we have that $3 \mid A_n(k)$ for all k. Hence n cannot be divisible by 3.

Now let $3 \nmid n$. We claim that for each such $n \ge 5$, there exists $k < n$ for which $7 \mid A_n(k)$. We see that $A_n(k)$ is divisible by 7 if and only if $10^n - 1 \equiv 2 \cdot 10^k \pmod 7$. There are several cases.

$n \equiv 1 \pmod 6$. Then $10^n - 1 \equiv 2 \equiv 2 \cdot 10^0$, so $7 \mid A_n(0)$.
$n \equiv 2 \pmod 6$. Then $10^n - 1 \equiv 1 \equiv 2 \cdot 10^4$, so $7 \mid A_n(4)$.
$n \equiv 4 \pmod 6$. Then $10^n - 1 \equiv 3 \equiv 2 \cdot 10^5$, so $7 \mid A_n(5)$.
$n \equiv 5 \pmod 6$. Then $10^n - 1 \equiv 4 \equiv 2 \cdot 10^2$, so $7 \mid A_n(2)$.

The remaining cases are $n = 1, 2, 4$. For $n = 4$ the number $1711 = 29 \cdot 59$ is composite, while it is easily checked that $n = 1$ and $n = 2$ are solutions. Hence the answer is $n = 1, 2$.

28. Let us first prove the following lemma.

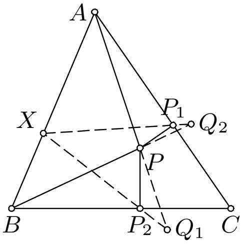
Lemma. Let $(b'/a', d'/c')$ and $(b''/a'', d''/c'')$ be two points with rational coordinates where the fractions given are irreducible. If both a' and c' are odd and the distance between the two points is 1 then it follows that a'' and c'' are odd, and that $b' + d'$ and $b'' + d''$ are of a different parity.

Proof. Let b/a and d/c be irreducible fractions such that $b'/a' - b''/a'' = b/a$ and $d'/c' - d''/c'' = d/c$. Then it follows that $b^2/a^2 + d^2/c^2 = 1 \Rightarrow b^2c^2 + a^2d^2 = a^2c^2$. Since $(a, b) = 1$ and $(c, d) = 1$ it follows that $a \mid c$, $c \mid a$ and hence $a = c$. Consequently $b^2 + d^2 = a^2$. Since a is mutually co-prime to b and d it follows that a and $b+d$ are odd. From $b''/a'' = b/a + b'/a'$ we get that $a'' \mid aa'$, so a'' is odd. Similarly, c'' is odd as well. Now it follows that $b'' \equiv b + b'$ and similarly $d'' \equiv d + d'$ (mod 2). Hence $b'' + d'' \equiv b' + d' + b + d \equiv b' + d' + 1 \pmod 2$, from which it follows that $b' + d'$ and $b'' + d''$ are of a different parity.

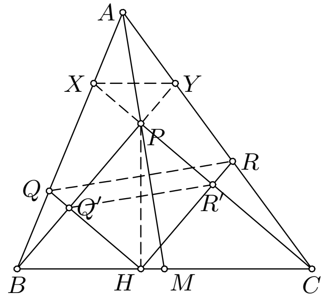
Without loss of generality we start from the origin of the coordinate system $(0/1, 0/1)$. Initially $b + d = 0$ and after moving to each subsequent point along the broken line $b + d$ changes parity by the lemma. Hence it will not be possible to return to the origin after an odd number of steps since $b + d$ will be odd.

4.32 Solutions to the Shortlisted Problems of IMO 1991

1. All the angles $\angle PP_1C$, $\angle PP_2C$, $\angle PQ_1C$, $\angle PQ_2C$ are right, hence P_1, P_2, Q_1, Q_2 lie on the circle with diameter PC. The result now follows immediately from Pascal's theorem applied to the hexagon $P_1PP_2Q_1CQ_2$. It tells us that the points of intersection of the three pairs of lines P_1C, PQ_1 (intersection A), P_1Q_2, P_2Q_1 (intersection X) and PQ_2, P_2C (intersection B) are collinear.



2. Let HQ meet PB at Q' and HR meet PC at R'. From $MP = MB = MC$ we have $\angle BPC = 90^o$. So $PR'HQ'$ is a rectangle. Since PH is perpendicular to BC, it follows that the circle with diameter PH, through P, R', H, Q', is tangent to BC. It is now sufficient to show that QR is parallel to $Q'R'$. Let CP meet AB at X, and BP meet AC at Y. Since P is on the median, it follows (for example, by Ceva's theorem) that $AX/XB = AY/YC$, i.e. that XY is parallel to BC. Consequently, $PY/BP = PX/CP$. Since HQ is parallel to CX, we have $QQ'/HQ' = PX/CP$ and similarly $RR'/HR' = PY/BP$. It follows that $QQ'/HQ' = RR'/HR'$, hence QR is parallel to $Q'R'$ as required.



Second solution. It suffices to show that $\angle RHC = \angle RQH$, or equivalently $RH : QH = PC : PB$. We assume $PC : PB = 1 : x$. Let $X \in AB$ and $Y \in AC$ be points such that $MX \perp PB$ and $MY \perp PC$. Since MX bisects $\angle AMB$ and MY bisects AMC, we deduce

$$AX : XB = AM : MB = AY : YC \Rightarrow XY \parallel BC \Rightarrow$$
$$\Rightarrow \triangle XYM \sim \triangle CBP \Rightarrow XM : MY = 1 : x.$$

Now from $CH : HB = 1 : x^2$ we obtain $RH : MY = CH : CM = 1 : \frac{1+x^2}{2}$ and $QH : MX = BH : BM = x^2 : \frac{1+x^2}{2}$. Therefore

$$RH : QH = \frac{2}{1+x^2} MY : \frac{2x^2}{1+x^2} MX = 1 : x.$$

3. Consider the problem with the unit circle on the complex plane. For convenience, we use the same letter for a point in the plane and its corresponding complex number.
Lemma 1. Line $l(S, PQR)$ contains the point $Z = \frac{P+Q+R+S}{2}$.

Proof. Suppose P', Q', R' are the feet of perpendiculars from S to QR, RP, PQ respectively. It suffices to show that P', Q', R', Z are on the same line. Let us first represent P' by Q, R, S. Since $P' \in QR$, we have $\frac{P'-Q}{R-Q} = \overline{\left(\frac{P'-Q}{R-Q}\right)}$, that is,

$$(P'-Q)(\overline{R}-\overline{Q}) = (\overline{P'}-\overline{Q})(R-Q). \tag{1}$$

On the other hand, since $SP' \perp QR$, the ratio $\frac{P'-S}{R-Q}$ is purely imaginary. Thus

$$(P'-S)(\overline{R}-\overline{Q}) = -(\overline{P'}-\overline{S})(R-Q). \tag{2}$$

Eliminating $\overline{P'}$ from (1) and (2) and using the fact that $\overline{X} = X^{-1}$ for X on the unit circle, we obtain $P' = (Q+R+S-QR/S)/2$ and analogously $Q' = (P+R+S-PR/S)/2$ and $R' = (P+Q+S-PQ/S)/2$. Hence $Z-P' = (P+QR/S)/2$, $Z-Q' = (Q+PR/S)/2$ and $Z-R' = (R+PQ/S)/2$. Setting $P=p^2$, $Q=q^2$, $R=r^2$, $S=s^2$ we obtain $Z-P' = \frac{pqr}{2s}\left(\frac{ps}{qr}+\frac{qr}{ps}\right)$, $Z-Q' = \frac{pqr}{2s}\left(\frac{qs}{pr}+\frac{pr}{qs}\right)$ and $Z-P' = \frac{pqr}{2s}\left(\frac{rs}{pq}+\frac{pq}{rs}\right)$.

Since $x+x^{-1} = 2\mathrm{Re}\, x$ is real for all x on the unit circle, it follows that the ratio of every pair of these differences is real, which means that Z, P', Q', R' belong to the same line.

Lemma 2. If P, Q, R, S are four different points on a circle, then the lines $l(P, QRS)$, $l(Q, RSP)$, $l(R, SPQ)$, $l(S, PQR)$ intersect at one point.

Proof. By Lemma 1, they all pass through $\frac{P+Q+R+S}{2}$.

Now we can find the needed conditions for $A, B, \dots, F$. In fact, the lines $l(A, BDF)$, $l(D, ABF)$ meet at $Z_1 = \frac{A+B+D+F}{2}$, and $l(B, ACE)$, $l(E, ABC)$ meet at $Z_2 = \frac{A+B+C+E}{2}$. Hence, $Z_1 \equiv Z_2$ if and only if $D-C = E-F \Leftrightarrow CDEF$ is a rectangle.

Remark. The line $l(S, PQR)$ is widely known as Simson line; the proof that the feet of perpendiculars are collinear is straightforward. The key claim, Lemma 1, is a known property of Simson lines, and can be shown elementarily:

* $l(S, PQR)$ passes through the midpoint X of HS, where H is the orthocenter of PQR.

4. Assume the contrary, that $\angle MAB$, $\angle MBC$, $\angle MCA$ are all greater than 30°. By the sine Ceva theorem, it holds that

$$\begin{aligned}&\sin\angle MAC \sin\angle MBA \sin\angle MCB \\ &= \sin\angle MAB \sin\angle MBC \sin\angle MCA > \sin^3 30^\circ = \frac{1}{8}\,.\end{aligned} \tag{$*$}$$

On the other hand, since $\angle MAC + \angle MBA + \angle MCB < 180^\circ - 3\cdot 30^\circ = 90^\circ$, Jensen's inequality applied on the concave function $\ln\sin x$ $(x \in [0,\pi])$ gives us $\sin\angle MAC \sin\angle MBA \sin\angle MCB < \sin^3 30^\circ$, contradicting $(*)$.

Second solution. Denote the intersections of PA, PB, PC with BC, CA, AB by A_1, B_1, C_1, respectively. Suppose that each of the angles $\angle PAB$, $\angle PBC, \angle PCA$ is greater than 30^o and denote $PA = 2x$, $PB = 2y$, $PC = 2z$. Then $PC_1 > x$, $PA_1 > y$, $PB_1 > z$. On the other hand, we know that

$$\frac{PC_1}{PC + PC_1} + \frac{PA_1}{PA + PA_1} + \frac{PB_1}{PB + PB_1} = \frac{S_{ABP}}{S_{ABC}} + \frac{S_{PBC}}{S_{ABC}} + \frac{S_{APC}}{S_{ABC}} = 1.$$

Since the function $\frac{t}{p+t}$ is increasing, we obtain $\frac{x}{2z+x} + \frac{y}{2x+y} + \frac{z}{2y+z} < 1$. But on the contrary, Cauchy-Schwartz inequality (or alternatively Jensen's inequality) yields

$$\frac{x}{2z+x} + \frac{y}{2x+y} + \frac{z}{2y+z} \geq \frac{(x+y+z)^2}{x(2z+x)+y(2x+y)+z(2y+z)} = 1.$$

5. Let P_1 be the point on the side BC such that $\angle BFP_1 = \beta/2$. Then $\angle BP_1F = 180^o - 3\beta/2$, and the sine law gives us $\frac{BF}{BP_1} = \frac{\sin(3\beta/2)}{\sin(\beta/2)} = 3 - 4\sin^2(\beta/2) = 1 + 2\cos\beta$.
   Now we calculate $\frac{BF}{BP}$. We have $\angle BIF = 120^o - \beta/2$, $\angle BFI = 60^o$ and $\angle BIC = 120^o$, $\angle BCI = \gamma/2 = 60^o - \beta/2$. By the sine law,

$$BF = BI\frac{\sin(120^o - \beta/2)}{\sin 60^o}, \qquad BP = \frac{1}{3}BC = BI\frac{\sin 120^o}{3\sin(60^o - \beta/2)}.$$

   It follows that $\frac{BF}{BP} = \frac{3\sin(60^o-\beta/2)\sin(60^o+\beta/2)}{\sin^2 60^o} = 4\sin(60^o - \beta/2)\sin(60^o + \beta/2) = 2(\cos\beta - \cos 120^o) = 2\cos\beta + 1 = \frac{BF}{BP_1}$. Therefore $P \equiv P_1$.

6. Let a, b, c be sides of the triangle. Let A_1 be the intersection of line AI with BC. By the known fact, $BA_1 : A_1C = c : b$ and $AI : IA_1 = AB : BA_1$, hence $BA_1 = \frac{ac}{b+c}$ and $\frac{AI}{IA_1} = \frac{AB}{BA_1} = \frac{b+c}{a}$. Consequently $\frac{AI}{l_A} = \frac{b+c}{a+b+c}$.
   Put $a = n+p$, $b = p+m$, $c = m+n$: it is obvious that m, n, p are positive. Our inequality becomes

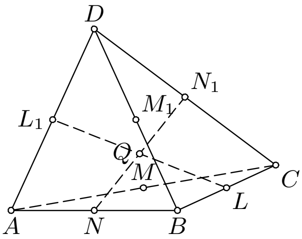
$$2 < \frac{(2m+n+p)(m+2n+p)(m+n+2p)}{(m+n+p)^3} \leq \frac{64}{27}.$$

   The right side inequality immediately follows from the inequality between arithmetic and geometric means applied on $2m+n+p$, $m+2n+p$ and $m+n+2p$. For the left side inequality, denote by $T = m+n+p$. Then we can write $(2m+n+p)(m+2n+p)(m+n+2p) = (T+m)(T+n)(T+p)$ and

$$(T+m)(T+n)(T+p) = T^3 + (m+n+p)T^2 + (mn+np+pn)T + mnp > 2T^3.$$

   Remark. The inequalities cannot be improved. In fact, $\frac{AI\cdot BI\cdot CI}{l_A l_B l_C}$ is equal to $8/27$ for $a = b = c$, while it can be arbitrarily close to $1/4$ if $a = b$ and c is sufficiently small.

7. The given equations imply $AB = CD$, $AC = BD$, $AD = BC$. Let L_1, M_1, N_1 be the midpoints of AD, BD, CD respectively. Then the above



equalities yield

$$\begin{gathered} L_1M_1 = AB/2 = LM, \\ L_1M_1 \parallel AB \parallel LM; \\ L_1M = CD/2 = LM_1, \\ L_1M \parallel CD \parallel LM_1. \end{gathered}$$

Thus L, M, L_1, M_1 are coplanar and LML_1M_1 is a rhombus as well as MNM_1N_1 and LNL_1N_1. Then the segments LL_1, MM_1, NN_1 have the common midpoint Q and $QL \perp QM$, $QL \perp QN$, $QM \perp QN$. We also infer that the line NN_1 is perpendicular to the plane LML_1M_1 and hence to the line AB. Thus $QA = QB$, and similarly, $QB = QC = QD$, hence Q is just the center O, and $\angle LOM = \angle MON = \angle NOL = 90°$.

8. Let $P_1(x_1, y_1), P_2(x_2, y_2), \dots, P_n(x_n, y_n)$ be the n points of S in the coordinate plane. We may assume $x_1 < x_2 < \cdots < x_n$ (choosing adequate axes and renumbering the points if necessary). Define d to be half the minimum distance of P_i from the line P_jP_k, where i, j, k go through all possible combinations of mutually distinct indices.
First we define a set T containing $2n - 4$ points:

$$T = \{(x_i, y_i - d), (x_i, y_i + d) \mid i = 2, 3, \dots, n-1\}.$$

Consider any triangle $P_kP_lP_m$, where $k < l < m$. Its interior contains at least one of the two points $(x_l, y_l \pm d)$, so T is a set of $2n-4$ points with the required property. However, at least one of the points of T is useless. The convex hull of S is a polygon with at least three points in S as vertices. Let P_j be a vertex of that hull distinct from P_1 and P_n. Clearly one of the points $(x_j, y_j \pm d)$ lies outside the convex hull, and thus can be left out. The remaining set of $2n - 5$ points satisfies the conditions.

9. Let A_1, A_2 be two points of E which are joined. In $E \setminus \{A_1, A_2\}$, there are at most 397 points to which A_1 is not joined, and at most as much to which A_2 is not joined. Consequently, there exists a point A_3 which is joined to both A_1 and A_2. There are at most $3 \cdot 397 = 1191$ points of $E \setminus \{A_1, A_2, A_3\}$ to which at least one of A_1, A_2, A_3 is not joined, hence it is possible to choose a point A_4 joined to A_1, A_2, A_3. Similarly, there exists a point A_5 which is joined to all A_1, A_2, A_3, A_4. Finally, among the remaining 1986 points, there are at most $5 \cdot 397 = 1985$ which are not joined to one of the points $A_1, \dots, A_5$. Thus there is at least one point A_6 joined to all $A_1, \dots, A_5$. It is clear that $A_1, \dots, A_6$ are pairwise joined.

Solution of the alternative version. Let be given 1991 points instead. Number the points from 1 to 1991, and join i and j if and only if $i - j$ is not a multiple of 5. Then each i is joined to 1592 or 1593 other points, and obviously among any six points there are two which are not joined.

10. We start at some vertex v_0 and walk along distinct edges of the graph, numbering them $1, 2, \dots$ in the order of appearance, until this is no longer possible without reusing an edge. If there are still edges which are not numbered, one of them has a vertex which has already been visited (else G would not be connected). Starting from this vertex, we continue to walk along unused edges resuming the numbering, until we eventually get stuck. Repeating this procedure as long as possible, we shall number all the edges.
Let v be a vertex which is incident with $e \geq 2$ edges. If $v = v_0$, then it is on the edge 1, so the gcd at v is 1. If $v \neq v_0$, suppose that it was reached for the first time by the edge r. At that time there was at least one unused edge incident with v (as $e \geq 2$), hence one of them was labelled by $r+1$. The gcd at v again equals $\gcd(r, r+1) = 1$.

11. To start with, observe that $\frac{1}{n-m}\binom{n-m}{m} = \frac{1}{n}\left[\binom{n-m}{m} + \binom{n-m-1}{m-1}\right]$.
For $n = 1, 2, \dots$ set $S_n = \sum_{m=0}^{[n/2]}(-1)^m\binom{n-m}{m}$. Using the identity $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$ we obtain the following relation for S_n:

$$\begin{aligned} S_{n+1} &= \sum_m (-1)^m \binom{n-m+1}{m} \\ &= \sum_m (-1)^m \binom{n-m}{m} + \sum_m (-1)^m \binom{n-m}{m-1} = S_n - S_{n-1}. \end{aligned}$$

Since the initial members of the sequence S_n are $1, 1, 0, -1, -1, 0, 1, 1, \dots$, we thus find that S_n is periodic with period 6.
Now the sum from the problem reduces to

$$\begin{aligned} &\frac{1}{1991}\binom{1991}{0} - \frac{1}{1991}\left[\binom{1990}{1} + \binom{1989}{0}\right] + \cdots - \frac{1}{1991}\left[\binom{996}{995} + \binom{995}{994}\right] \\ &= \frac{1}{1991}(S_{1991} - S_{1989}) = \frac{1}{1991}(0 - (-1)) = \frac{1}{1991}. \end{aligned}$$

12. Let A_m be the set of those elements of S which are divisible by m. By the inclusion-exclusion principle, the number of elements divisible by 2, 3, 5 or 7 equals

$$\begin{aligned} &|A_2 \cup A_3 \cup A_5 \cup A_7| \\ &= |A_2| + |A_3| + |A_5| + |A_7| - |A_6| - |A_{10}| - |A_{14}| - |A_{15}| \\ &\quad - |A_{21}| - |A_{35}| + |A_{30}| + |A_{42}| + |A_{70}| + |A_{105}| - |A_{210}| \\ &= 140 + 93 + 56 + 40 - 46 - 28 - 20 - 18 \\ &\quad - 13 - 8 + 9 + 6 + 4 + 2 - 1 = 216. \end{aligned}$$

Among any five elements of the set $A_2 \cup A_3 \cup A_5 \cup A_7$, one of the sets A_2, A_3, A_5, A_7 contains at least two, and those two are not relatively prime. Therefore $n > 216$.

We claim that the answer is $n = 217$. First notice that the set $A_2 \cup A_3 \cup A_5 \cup A_7$ consists of four prime $(2, 3, 5, 7)$ and 212 composite numbers. The set $S \setminus A$ contains exactly 8 composite numbers: namely, $11^2, 11 \cdot 13, 11 \cdot 17, 11 \cdot 19, 11 \cdot 23, 13^2, 13 \cdot 17, 13 \cdot 19$. Thus S consists of the unity, 220 composite numbers and 59 primes.

Let A be a 217-element subset of S, and suppose that there are no five pairwise relatively prime numbers in A. Then A can contain at most 4 primes (or unity and three primes) and at least 213 composite numbers. Hence the set $S \setminus A$ contains at most 7 composite numbers. Consequently, at least one of the following 8 five-element sets is disjoint with $S \setminus A$, and is thus entirely contained in A:

$$\begin{array}{ll}
\{2\cdot 23, 3\cdot 19, 5\cdot 17, 7\cdot 13, 11\cdot 11\}, & \{2\cdot 29, 3\cdot 23, 5\cdot 19, 7\cdot 17, 11\cdot 13\},\\
\{2\cdot 31, 3\cdot 29, 5\cdot 23, 7\cdot 19, 11\cdot 17\}, & \{2\cdot 37, 3\cdot 31, 5\cdot 29, 7\cdot 23, 11\cdot 19\},\\
\{2\cdot 41, 3\cdot 37, 5\cdot 31, 7\cdot 29, 11\cdot 23\}, & \{2\cdot 43, 3\cdot 41, 5\cdot 37, 7\cdot 31, 13\cdot 17\},\\
\{2\cdot 47, 3\cdot 43, 5\cdot 41, 7\cdot 37, 13\cdot 19\}, & \{2\cdot 2, 3\cdot 3, 5\cdot 5, 7\cdot 7, 13\cdot 13\}.
\end{array}$$

As each of these sets consists of five numbers relatively prime in pairs, the claim is proved.

13. Call a sequence $e_1, \dots, e_n$ *good* if $e_1a_1 + \cdots + e_na_n$ is divisible by n. Among the sums $s_0 = 0$, $s_1 = a_1$, $s_2 = a_1 + a_2$, $\dots$, $s_n = a_1 + \cdots + a_n$, two give the same remainder modulo n, and their difference corresponds to a good sequence. To show that, permuting the a_i's, we can find $n - 1$ different sequences, we use the following

Lemma. Let A be a $k \times n$ $(k \le n-2)$ matrix of zeros and ones, whose every row contains at least one 0 and at least two 1's. Then it is possible to permute columns of A is such a way that in any row 1's do not form a block.

Proof. We will use the induction on k. The case $k = 1$ and arbitrary $n \ge 3$ is trivial. Suppose that $k \ge 2$ and that for $k - 1$ and any $n \ge k + 1$ the lemma is true. Consider a $k \times n$ matrix A, $n \ge k + 2$. We mark an element a_{ij} if either it is the only zero in the i-th row, or one of the 1's in the row if it contains exactly two 1's. Since $n \ge 4$, every row contains at most two marked elements, which adds up to at most $2k < 2n$ marked elements in total. It follows that there is a column with at most one marked element. Assume w.l.o.g. that it is the first column and that a_{1j} isn't marked for $j > 1$. The matrix B, obtained by omitting the first row and first column from A, satisfies the conditions of the lemma. Therefore, we can permute columns of B and get the required form. Considered as a permutation of column of A, this permutation may leave a block of 1's only in the first row of A. In the case that it is so, if $a_{11} = 1$ we put the first column in the last place, otherwise we put it between any two columns having 1's in the first row. The obtained matrix has the required property.

Suppose now that we have got k different nontrivial good sequences $e_1^i, \dots, e_n^i$, $i = 1, \dots, k$, and that $k \le n - 2$. The matrix $A = (e_j^i)$

fulfils the conditions of Lemma, hence there is a permutation σ from Lemma. Now among the sums $s_0 = 0$, $s_1 = a_{\sigma(1)}$, $s_2 = a_{\sigma(1)} + a_{\sigma(2)}$, $\dots$, $s_n = a_{\sigma(1)} + \cdots + a_{\sigma(n)}$, two give the same remainder modulo n. Let $s_p \equiv s_q \pmod{n}$, $p < q$. Then $n \mid s_q - s_p = a_{\sigma(p+1)} + \cdots + a_{\sigma(q)}$, and this yields a good sequence $e_1, \dots, e_n$ with $e_{\sigma(p+1)} = \cdots = e_{\sigma(q)} = 1$ and other e's equal to zero. Since from the construction we see that none of the sequences $e_{\sigma(j)^i}$ has all 1's in a block, in this way we have got a new nontrivial good sequence, and we can continue this procedure until there are $n-1$ sequences. Together with the trivial $0, \dots, 0$ sequence, we have found n good sequences.

14. Suppose that $f(x_0), f(x_0+1), \dots, f(x_0+2p-2)$ are squares. If $p \mid a$ and $p \nmid b$, then $f(x) \equiv bx+c \pmod{p}$ for $x = x_0, \dots, x_0+p-1$ form a complete system of residues modulo p. However, a square is always congruent to exactly one of the $\frac{p+1}{2}$ numbers $0, 1^2, 2^2, \dots, (\frac{p-1}{2})^2$ and thus cannot give every residue modulo p. Also, if $p \mid a$ and $p \mid b$, then $p \mid b^2 - 4ac$.
We now assume $p \nmid a$. The following identities hold for any quadric polynomial:
$$4a \cdot f(x) = (2ax+b)^2 - (b^2 - 4ac) \tag{1}$$
and
$$f(x+p) - f(x) = p(2ax+b) + p^2 a. \tag{2}$$
Suppose that there is an y, $x_0 \le y \le x_0+p-2$, for which $f(y)$ is divisible by p. Then both $f(y)$ and $f(y+p)$ are squares divisible by p, and therefore both are divisible by p^2. But relation (2) implies that $p \mid 2ay+b$, and hence by (1) $b^2 - 4ac$ is divisible by p as well.
Therefore it suffices to show that such an y exists, and for that aim we prove that there are two such y in $[x_0, x_0+p-1]$. Assume the opposite. Since for $x = x_0, x_0+1, \dots, x_0+p-1$ $f(x)$ is congruent modulo p to one of the $\frac{p-1}{2}$ numbers $1^2, 2^2, \dots, \left(\frac{p-1}{2}\right)^2$, it follows by the pigeon-hole principle that for some mutually distinct $u, v, w \in \{x_0, \dots, x_0+p-1\}$ we have $f(u) \equiv f(v) \equiv f(w) \pmod{p}$. Consequently the difference $f(u) - f(v) = (u-v)(a(u+v)+b)$ is divisible by p, but it is clear that $p \nmid u - v$, hence $a(u+v) \equiv -b \pmod{p}$. Similarly $a(u+w) \equiv -b \pmod{p}$, which together with the previous congruence yields $p \mid a(v-w) \Rightarrow p \mid v - w$ which is clearly impossible. It follows that $p \mid f(y_1)$ for at least one y_1, $x_0 \le y_1 < x_0 + p$.
If y_2, $x_0 \le y_2 < x_0 + p$ is such that $a(y_1+y_2) + b \equiv 0 \pmod{p}$, we have $p \mid f(y_1) - f(y_2) \Rightarrow p \mid f(y_2)$. If $y_1 = y_2$, then by (1) $p \mid b^2 - 4ac$. Otherwise, among y_1, y_2 one belongs to $[x_0, x_0+p-2]$ as required.

Second solution. Using Legendre's symbols $\left(\frac{a}{p}\right)$ for quadratic residues we can prove a stronger statement for $p \ge 5$. It can be shown that
$$\sum_{x=0}^{p-1} \left(\frac{ax^2+bx+c}{p}\right) = -\left(\frac{a}{p}\right) \qquad \text{if} \quad p \nmid b^2 - 4ac,$$

hence for at most $\frac{p+3}{2}$ values of x between x_0 and $x_0 + p - 1$ inclusive, $ax^2 + bx + c$ is a quadratic residue or 0 modulo p. Therefore, if $p \geq 5$ and $f(x)$ is a square for $\frac{p+5}{2}$ consecutive values, then $p \mid b^2 - 4ac$.

15. Assume that the sequence has the period T. We can find integers $k > m > 0$, as large as we like, such that $10^k \equiv 10^m \pmod{T}$, using for example Euler's theorem. It is obvious that $a_{10^k-1} = a_{10^k}$ and hence, taking k sufficiently large and using the periodicity, we see that

$$a_{2\cdot 10^k-10^m-1} = a_{10^k-1} = a_{10^k} = a_{2\cdot 10^k-10^m}.$$

Since $(2 \cdot 10^k - 10^m)! = (2 \cdot 10^k - 10^m)(2 \cdot 10^k - 10^m - 1)!$ and the last nonzero digit of $2 \cdot 10^k - 10^m$ is nine, we must have $a_{2\cdot 10^k-10^m-1} = 5$ (if s is a digit, the last digit of $9s$ is s only if $s = 5$). But this means that 5 divides $n!$ with a greater power than 2 does, which is impossible. Indeed, if the exponents of these powers are α_2, α_5 respectively, then $\alpha_5 = [n/5] + [n/5^2] + \cdots \leq \alpha_2 = [n/2] + [n/2^2] + \cdots$.

16. Let p be the least prime number that does not divide n: thus $a_1 = 1$ and $a_2 = p$. Since $a_2 - a_1 = a_3 - a_2 = \cdots = r$, the a_i's are $1, p, 2p-1, 3p-2, \dots$. We have the following cases:

$p = 2$. Then $r = 1$ and the numbers $1, 2, 3, \dots, n-1$ are relatively prime to n, hence n is a prime.

$p = 3$. Then $r = 2$, so every odd number less than n is relatively prime to n, from which we deduce that n has no odd divisors. Therefore $n = 2^k$ for some $k \in \mathbb{N}$.

$p > 3$. Then $r = p - 1$ and $a_{k+1} = a_1 + k(p-1) = 1 + k(p-1)$. Since $n - 1$ also must belong to the progression, we have $p - 1 \mid n - 2$. Let q be any prime divisor of $p - 1$. Then also $q \mid n - 2$. On the other hand, since $q < p$, it must divide n too, therefore $q \mid 2$, i.e. $q = 2$. This means that $p - 1$ has no prime divisors other than 2 and thus $p = 2^l + 1$ for some $l \geq 2$. But in order for p to be prime, l must be even (because $3 \mid 2^l + 1$ for l odd). Now we recall that $2p - 1$ is also relatively prime to n; but $2p - 1 = 2^{l+1} + 1$ is divisible by 3, which is a contradiction because $3 \mid n$.

17. Taking the equation $3^x + 4^y = 5^z$ $(x, y, z > 0)$ modulo 3, we get that $5^z \equiv 1 \pmod 3$, hence z is even, say $z = 2z_1$. The equation then becomes $3^x = 5^{2z_1} - 4^y = (5^{z_1} - 2^y)(5^{z_1} + 2^y)$. Each factor $5^{z_1} - 2^y$ and $5^{z_1} + 2^y$ is a power of 3, for which the only possibility is $5^{z_1} + 2^y = 3^x$ and $5^{z_1} - 2^y = 1$. Again modulo 3 these equations reduce to $(-1)^{z_1} + (-1)^y = 0$ and $(-1)^{z_1} - (-1)^y = 1$, implying that z_1 is odd and y is even. Particularly, $y \geq 2$. Reducing the equation $5^{z_1} + 2^y = 3^x$ modulo 4 we get that $3^x \equiv 1$, hence x is even. Now if $y > 2$, modulo 8 this equation yields $5 \equiv 5^{z_1} \equiv 3^x \equiv 1$, a contradiction. Hence $y = 2$, $z_1 = 1$. The only solution of the original equation is $x = y = z = 2$.

18. For integers $a > 0$, $n > 0$ and $\alpha \geq 0$, we shall write $a^\alpha \parallel n$ when $a^\alpha \mid n$ and $a^{\alpha+1} \nmid n$.
Lemma. For every odd number $a \geq 3$ and an integer $n \geq 0$ it holds that

$$a^{n+1} \parallel (a+1)^{a^n} - 1 \quad \text{and} \quad a^{n+1} \parallel (a-1)^{a^n} + 1.$$

Proof. We shall prove the first relation by induction (the second is analogous). For $n = 0$ the statement is obvious. Suppose that it holds for some n, i.e. that $(1+a)^{a^n} = 1 + Na^{n+1}$, $a \nmid N$. Then

$$(1+a)^{a^{n+1}} = (1+Na^{n+1})^a = 1 + a \cdot Na^{n+1} + \binom{a}{2} N^2 a^{2n+2} + Ma^{3n+3}$$

for some integer M. Since $\binom{a}{2}$ is divisible by a for a odd, we deduce that the part of the above sum behind $1 + a \cdot Na^{n+1}$ is divisible by a^{n+3}. Hence $(1+a)^{a^{n+1}} = 1 + N'a^{n+2}$, where $a \nmid N'$.

It follows immediately from Lemma that

$$1991^{1993} \parallel 1990^{1991^{1992}} + 1 \quad \text{and} \quad 1991^{1991} \parallel 1992^{1991^{1990}} - 1.$$

Adding these two relations we obtain immediately that $k = 1991$ is the desired value.

19. Set $x = \cos(\pi a)$. The given equation is equivalent to $4x^3 + 4x^2 - 3x - 2 = 0$, which factorizes as $(2x+1)(2x^2 + x - 2) = 0$.
The case $2x + 1 = 0$ yields $\cos(\pi a) = -1/2$ and $a = 2/3$. It remains to show that if x satisfies $2x^2 + x - 2 = 0$ then a is not rational. The polynomial equation $2x^2 + x - 2 = 0$ has two real roots, $x_{1,2} = \frac{-1 \pm \sqrt{17}}{4}$, and since $|x| \leq 1$ we must have $x = \cos \pi a = \frac{-1+\sqrt{17}}{4}$.
We now prove by induction that, for every integer $n \geq 0$, $\cos(2^n \pi a) = \frac{a_n + b_n\sqrt{17}}{4}$ for some odd integers a_n, b_n. The case $n = 0$ is trivial. Also, if $\cos(2^n \pi a) = \frac{a_n + b_n\sqrt{17}}{4}$, then

$$\begin{aligned} \cos(2^{n+1}\pi a) &= 2\cos^2(2^n \pi a) - 1 \\ &= \frac{1}{4}\left(\frac{a_n^2 + 17b_n^2 - 8}{2} + a_n b_n \sqrt{17}\right) = \frac{a_{n+1} + b_{n+1}\sqrt{17}}{4}. \end{aligned}$$

By the inductive step that a_n, b_n are odd, it is obvious that a_{n+1}, b_{n+1} are also odd. This proves the claim.
Note also that, since $a_{n+1} = \frac{1}{2}(a_n^2 + 17b_n^2 - 8) > a_n$, the sequence $\{a_n\}$ is strictly increasing. Hence the set of values of $\cos(2^n \pi a)$, $n = 0, 1, 2, \ldots$, is infinite (because $\sqrt{17}$ is irrational). However, if a were rational, then the set of values of $\cos m\pi a$, $m = 1, 2, \ldots$, would be finite, a contradiction. Therefore the only possible value for a is $2/3$.

20. We prove the result with 1991 replaced by any positive integer k. For natural numbers p, q, let $\epsilon = (\alpha p - [\alpha p])(\alpha q - [\alpha q])$. Then $0 < \epsilon < 1$ and

$$\epsilon = \alpha^2 pq - \alpha(p[\alpha q] + q[\alpha p]) + [\alpha p][\alpha q].$$

Multiplying this equality by $\alpha - k$ and using $\alpha^2 = k\alpha + 1$, i.e. $\alpha(\alpha - k) = 1$, we get

$$(\alpha - k)\epsilon = \alpha(pq + [\alpha p][\alpha q]) - (p[\alpha q] + q[\alpha p] + k[\alpha p][\alpha q]).$$

Since $0 < (\alpha - k)\epsilon < 1$, we have $[\alpha(p*q)] = p[\alpha q] + q[\alpha p] + k[\alpha p][\alpha q]$. Now

$$\begin{aligned}(p*q)*r &= (p*q)r + [\alpha(p*q)][\alpha r] = \\ &= pqr + [\alpha p][\alpha q]r + [\alpha q][\alpha r]p + [\alpha r][\alpha p]q + k[\alpha p][\alpha q][\alpha r].\end{aligned}$$

Since the last expression is symmetric, the same formula is obtained for $p*(q*r)$.

21. The polynomial $g(x)$ factorizes as $g(x) = f(x)^2 - 9 = (f(x)-3)(f(x)+3)$. If one of the equations $f(x) + 3 = 0$ and $f(x) - 3 = 0$ has no integer solutions, then the number of integer solutions of $g(x) = 0$ clearly does not exceed 1991.

    Suppose now that both $f(x) + 3 = 0$ and $f(x) - 3 = 0$ have integer solutions. Let $x_1, \dots, x_k$ be distinct integer solutions of the former, and $x_{k+1}, \dots, x_{k+l}$ be distinct integer solutions of the latter equation. There exist monic polynomials $p(x), q(x)$ with integer coefficients such that $f(x) + 3 = (x - x_1)(x - x_2)\dots(x - x_k)p(x)$ and $f(x) - 3 = (x - x_{k+1})(x - x_{k+2})\dots(x - x_{k+l})q(x)$. Thus we obtain

$$(x-x_1)(x-x_2)\dots(x-x_k)p(x)-(x-x_{k+1})(x-x_{k+2})\dots(x-x_{k+l})q(x) = 6.$$

    Putting $x = x_{k+1}$ we get $(x_{k+1} - x_1)(x_{k+1} - x_2)\cdots(x_{k+1} - x_k) \mid 6$, and since the product of more than four distinct integers cannot divide 6, this implies $k \le 4$. Similarly $l \le 4$; hence $g(x) = 0$ has at most 8 distinct integer solutions.

    Remark. The proposer provided a solution for the upper bound of 1995 roots which was essentially the same as that of (IMO74-6).

22. Suppose w.l.o.g. that the center of the square is at the origin $O(0,0)$. We denote the curve $y = f(x) = x^3 + ax^2 + bx + c$ by γ and the vertices of the square by A, B, C, D in this order.

    At first, the symmetry with respect to the point O maps γ into the curve $\overline{\gamma}$ ($y = f(-x) = x^3 - ax^2 + bx - c$). Obviously $\overline{\gamma}$ also passes through A, B, C, D, and thus has four different intersection points with γ. Then $2ax^2 + 2c$ has at least four distinct solution, which implies $a = c = 0$. Particularly, γ passes through O and intersects all quadrants, and hence $b < 0$.

    Further, the curve γ', obtained by rotation of γ around O for 90°, has an equation $-x = f(y)$ and also contains the points A, B, C, D and O. The intersection points (x, y) of $\gamma \cap \gamma'$ are determined by $-x = f(f(x))$, and hence they are roots of a polynomial $p(x) = f(f(x)) + x$ of 9-th degree.

But the number of times that one cubic actually crosses the other in each quadrant is in the general case even (draw the picture!), and since $ABCD$ is the only square lying on $\gamma \cap \gamma'$, the intersection points A, B, C, D must be double. It follows that

$$p(x) = x[(x-r)(x+r)(x-s)(x+s)]^2, \tag{1}$$

where r, s are the x-coordinates of A and B. On the other hand, $p(x)$ is defined by $(x^3+bx)^3+b(x^3+bx)+x$, and therefore equating of coefficients with (1) yields

$$\begin{aligned} 3b &= -2(r^2+s^2), & 3b^2 &= (r^2+s^2)^2 + 2r^2s^2, \\ b(b^2+1) &= -2r^2s^2(r^2+s^2), & b^2+1 &= r^4s^4. \end{aligned}$$

Straightforward solving this system of equations gives $b = -\sqrt{8}$ and $r^2 + s^2 = \sqrt{18}$.

The line segment from O to (r, s) is half a diagonal of the square, and thus a side of the square has length $a = \sqrt{2(r^2+s^2)} = \sqrt[4]{72}$.

23. From (i), replacing m by $f(f(m))$, we get

$$\begin{aligned} & f(\, f(f(m)) + f(f(n))\,) = -f(f(\, f(f(m)) + 1)) - n; \\ \text{analogously} \quad & f(\, f(f(n)) + f(f(m))\,) = -f(f(\, f(f(n)) + 1)) - m. \end{aligned}$$

From these relations we get $f(f(f(f(m))+1)) - f(f(f(f(n))+1)) = m-n$. Again from (i),

$$\begin{aligned} & f(f(\, f(f(m)) + 1)) = f(-m - f(f(2))\,) \\ \text{and} \quad & f(f(\, f(f(n)) + 1)) = f(-n - f(f(2))\,). \end{aligned}$$

Setting $f(f(2)) = k$ we obtain $f(-m-k) - f(-n-k) = m-n$ for all integers m, n. This implies $f(m) = f(0) - m$. Then also $f(f(m)) = m$, and using this in (i) we finally get

$$f(n) = -n - 1 \qquad \text{for all integers } n.$$

Particularly $f(1991) = -1992$.

From (ii) we obtain $g(n) = g(-n-1)$ for all integers n. Since g is a polynomial, it must also satisfy $g(x) = g(-x-1)$ for all real x. Let us now express g as a polynomial on $x + 1/2$: $g(x) = h(x+1/2)$. Then h satisfies $h(x+1/2) = h(-x-1/2)$, i.e. $h(y) = h(-y)$, hence it is a polynomial in y^2; thus g is a polynomial in $(x+1/2)^2 = x^2+x+1/4$. Hence $g(n) = p(n^2+n)$ (for some polynomial p) is the most general form of g.

24. Let $y_k = a_k - a_{k+1} + a_{k+2} - \cdots + a_{k+n-1}$ for $k = 1, 2, \ldots, n$, where we define $x_{i+n} = x_i$ for $1 \le i \le n$. We then have $y_1 + y_2 = 2a_1$, $y_2 + y_3 = 2a_2, \ldots, y_n + y_1 = 2a_n$.

(i) Let $n = 4k-1$ for some integer $k > 0$. Then for each $i = 1, 2, \dots, n$ we have that $y_i = (a_i + a_{i+1} + \cdots + a_{i-1}) - 2(a_{i+1} + a_{i+3} + \cdots + a_{i-2}) = 1 + 2 + \cdots + (4k-1) - 2(a_{i+1} + a_{i+3} + \cdots + a_{i-2})$ is even. Suppose now that $a_1, \dots, a_n$ is a good permutation. Then each y_i is positive and even, so $y_i \geq 2$. But for some $t \in \{1, \dots, n\}$ we must have $a_t = 1$, and thus $y_t + y_{t+1} = 2a_t = 2$ which is impossible. Hence the numbers $n = 4k-1$ are not good.

(ii) Let $n = 4k+1$ for some integer $k > 0$. Then $2, 4, \dots, 4k, 4k+1, 4k-1, \dots, 3, 1$ is a permutation with the desired property. Indeed, in this case $y_1 = y_{4k+1} = 1$, $y_2 = y_{4k} = 3, \dots, y_{2k} = y_{2k+2} = 4k-1$, $y_{2k+1} = 4k+1$.

Therefore all nice numbers are given by $4k+1$, $k \in \mathbb{N}$.

25. Since replacing x_1 by 1 can only reduce the set of indices i for which the desired inequality holds, we may assume $x_1 = 1$. Similarly we may assume $x_n = 0$. Now we can let i be the largest index such that $x_i > 1/2$. Then $x_{i+1} \leq 1/2$, hence

$$x_i(1 - x_{i+1}) \geq \frac{1}{4} = \frac{1}{4} x_1 (1 - x_n).$$

26. Without loss of generality we can assume $b_1 \geq b_2 \geq \cdots \geq b_n$. We denote by A_i the product $a_1 a_2 \dots a_{i-1} a_{i+1} \dots a_n$. If for some $i < j$ holds $A_i < A_j$, then $b_i A_i + b_j A_j \leq b_i A_j + b_j A_i$ (or equivalently $(b_i - b_j)(A_i - A_j) \leq 0$). Therefore the sum $\sum_{i=1}^n b_i A_i$ does not decrease when we rearrange the numbers $a_1, \dots, a_n$ so that $A_1 \geq \cdots \geq A_n$, and consequently $a_1 \leq \cdots \leq a_n$. Further, for fixed a_i's and $\sum b_i = 1$, the sum $\sum_{i=1}^n b_i A_i$ is maximal when b_1 takes the largest possible value, i.e. $b_1 = p$, b_2 takes the remaining largest possible value $b_2 = 1 - p$, whereas $b_3 = \cdots = b_n = 0$. In this case

$$\begin{aligned}\sum_{i=1}^n b_i A_i &= pA_1 + (1-p)A_2 = a_3 \dots a_n (pa_2 + (1-p)a_1) \\ &\leq p(a_1 + a_2) a_3 \dots a_n \leq \frac{p}{(n-1)^{n-1}} ,\end{aligned}$$

using the inequality between the geometric and arithmetic means for $a_3, \dots, a_n, a_1 + a_2$.

27. Write $F(x_1, \dots, x_n) = \sum_{i<j} x_i x_j (x_i + x_j)$. Choose an n-tuple $(x_1, \dots, x_n)$, $\sum_{i=1}^n x_i = 1$, $x_i \geq 0$ with at least three nonzero components, and assume w.l.o.g. that $x_1 \geq \cdots \geq x_{k-1} \geq x_k \geq x_{k+1} = \cdots = x_n = 0$. We claim that replacing x_{k-1}, x_k with $x_{k-1} + x_k, 0$ the value of F increases. Write for brevity $x_{k-1} = a$, $x_k = b$. Then

$$F(\dots, a+b, 0, 0, \dots) - F(\dots, a, b, 0, \dots)$$

$$= \sum_{i=1}^{k-2} x_i (a+b)(x_i + a + b) - \sum_{i=1}^{k-2} [x_i a (x_i + a) + x_i b (x_i + b)] - ab(a+b)$$

$$= ab\left(2\sum_{i=1}^{k-2} x_i - a - b\right) = ab(2 - 3(a+b)) > 0,$$

because $x_{k-1} + x_k \le \frac{2}{3}(x_1 + x_{k-1} + x_{k-2}) \le \frac{2}{3}$. Repeating this procedure we can reduce the number of nonzero x_i's to two, increasing the value of F in each step. It remains to maximize F over n-tuples $(x_1, x_2, 0, \dots, 0)$ with $x_1, x_2 \ge 0$, $x_1 + x_2 = 1$: in this case F equals x_1x_2 and attains its maximum value $\frac{1}{4}$ when $x_1 = x_2 = \frac{1}{2}$, $x_3 = \dots, x_n = 0$.

28. Let $x_n = c(n\sqrt{2} - [n\sqrt{2}])$ for some constant $c > 0$. For $i > j$, putting $p = [i\sqrt{2}] - [j\sqrt{2}]$, we have

$$|x_i - x_j| = c|(i-j)\sqrt{2} - p| = \frac{|2(i-j)^2 - p^2|c}{(i-j)\sqrt{2}+p} \ge \frac{c}{(i-j)\sqrt{2}+p} \ge \frac{c}{4(i-j)},$$

because $p < (i-j)\sqrt{2} + 1$. Taking $c = 4$, we obtain that for any $i > j$, $(i-j)|x_i - x_j| \ge 1$. Of course, this implies $(i-j)^a|x_i - x_j| \ge 1$ for any $a > 1$.

Remark. The constant 4 can be replaced with $3/2 + \sqrt{2}$.

Second solution. Another example of a sequence $\{x_n\}$ is constructed in the following way: $x_1 = 0$, $x_2 = 1$, $x_3 = 2$ and $x_{3^k i+m} = x_m + \frac{i}{3^k}$ for $i = 1, 2$ and $1 \le m \le 3^k$. It is easily shown that $|i-j| \cdot |x_i - x_j| \ge 1/3$ for any $i \ne j$.

Third solution. If $n = b_0 + 2b_1 + \cdots + 2^k b_k$, $b_i \in \{0, 1\}$, then one can set x_n to be $= b_0 + 2^{-a}b_1 + \cdots + 2^{-ka}b_k$. In this case it holds that $|i-j|^a|x_i - x_j| \ge \frac{2^a-2}{2^a-1}$.

29. One easily observes that the following sets are super-invariant: one-point set, its complement, closed and open half-lines or their complements, and the whole real line. To show that these are the only possibilities, we first observe that S is super-invariant if and only if for each $a > 0$ there is a b such that $x \in S \Leftrightarrow ax + b \in S$.
   (i) Suppose that for some a there are two such b's: b_1 and b_2. Then $x \in S \Leftrightarrow ax + b_1 \in S$ and $x \in S \Leftrightarrow ax + b_2 \in S$, which implies that S is periodic: $y \in S \Leftrightarrow y + \frac{b_1 - b_2}{a} \in S$. Since S is identical to a translate of any stretching of S, all positive numbers are periods of S. Therefore $S \equiv \mathbb{R}$.
   (ii) Assume that, for each a, $b = f(a)$ is unique. Then for any a_1 and a_2,

$$\begin{aligned} x \in S &\Leftrightarrow a_1x + f(a_1) \in S \Leftrightarrow a_1a_2x + a_2f(a_1) + f(a_2) \in S \\ &\Leftrightarrow a_2x + f(a_2) \in S \Leftrightarrow a_1a_2x + a_1f(a_2) + f(a_1) \in S. \end{aligned}$$

   As above it follows that $a_1f(a_2) + f(a_1) = a_2f(a_1) + f(a_2)$, or equivalently $f(a_1)(a_2-1) = f(a_2)(a_1-1)$. Hence (for some c), $f(a) = c(a-1)$ for all a. Now $x \in S \Leftrightarrow ax + c(a-1) \in S$ actually means that $y - c \in S \Leftrightarrow ay - c \in S$ for all a. Then it is easy to conclude that $\{y - c \mid y \in S\}$ is either a half-line or the whole line, and so is S.

30. Let a and b be the integers written by A and B respectively, and let $x < y$ be the two integers written by the referee. Suppose that none of A and B ever answers "yes".

Initially, regardless of a, A knows that $0 \leq b \leq y$ and answers "no". In the second step, B knows that A obtained $0 \leq b \leq y$, but if a were greater than x, A would know that $a + b = y$ and would thus answer "yes". So B concludes $0 \leq a \leq x$ but answers "no". The process continues.

Suppose that, in the n-th step, A knows that B obtained $r_{n-1} \leq a \leq s_{n-1}$. If $b > x - r_{n-1}$, B would know that $a + b > x$ and hence $a + b = y$, while if $b < y - s_{n-1}$, B would know that $a + b < y$, i.e. $a + b = x$: in both cases he would be able to guess a. However, B answered "no", from which A concludes $y - s_{n-1} \leq b \leq x - r_{n-1}$. Put $r_n = y - s_{n-1}$ and $s_n = x - r_{n-1}$. Similarly, in the next step B knows that A obtained $r_n \leq b \leq s_n$ and, since A answered "no", concludes $y - s_n \leq a \leq x - r_n$. Put $r_{n+1} = y - s_n$ and $s_{n+1} = x - r_n$.

Notice that in both cases $s_{i+1} - r_{i+1} = s_i - r_i - (y - x)$. Since $y - x > 0$, there exists an m for which $s_m - r_m < 0$, a contradiction.

4.33 Solutions to the Shortlisted Problems of IMO 1992

1. Assume that a pair (x, y) with $x < y$ satisfies the required conditions. We claim that the pair (y, x_1) also satisfies the conditions, where $x_1 = \frac{y^2+m}{x}$ (note that $x_1 > y$ is a positive integer). This will imply the desired result, since starting from the pair $(1, 1)$ we can obtain arbitrarily many solutions. First, we show that $\gcd(x_1, y) = 1$. Suppose to the contrary that $\gcd(x_1, y) = d > 1$. Then $d \mid x_1 \mid y^2+m \Rightarrow d \mid m$, which implies $d \mid y \mid x^2+m \Rightarrow d \mid x$. But this last is impossible, since $\gcd(x, y) = 1$. Thus it remains to show that $x_1 \mid y^2 + m$ and $y \mid x_1^2 + m$. The former relation is obvious. Since $\gcd(x, y) = 1$, the latter is equivalent to $y \mid (xx_1)^2 + mx^2 = y^4 + 2my^2 + m^2 + mx^2$, which is true because $y \mid m(m+x^2)$ by the assumption. Hence (y, x_1) indeed satisfies all the required conditions.

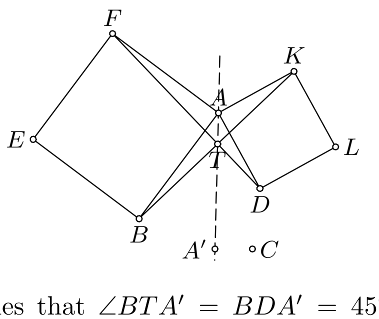
   Remark. The original problem asked to prove the existence of a pair (x, y) of positive integers satisfying the given conditions such that $x+y \le m+1$. The problem in this formulation is trivial, since the pair $x = y = 1$ satisfies the conditions. Moreover, this is sometimes the only solution with $x + y \le m + 1$. For example, for $m = 3$ the least nontrivial solution is $(x_0, y_0) = (1, 4)$.

2. Let us define x_n inductively as $x_n = f(x_{n-1})$, where $x_0 \ge 0$ is a fixed real number. It follows from the given equation in f that $x_{n+2} = -ax_{n+1} + b(a + b)x_n$. The general solution to this equation is of the form

$$x_n = \lambda_1 b^n + \lambda_2(-a - b)^n,$$

   where $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfy $x_0 = \lambda_1 + \lambda_2$ and $x_1 = \lambda_1 b - \lambda_2(a + b)$. In order to have $x_n \ge 0$ for all n we must have $\lambda_2 = 0$. Hence $x_0 = \lambda_1$ and $f(x_0) = x_1 = \lambda_1 b = bx_0$. Since x_0 was arbitrary, we conclude that $f(x) = bx$ is the only possible solution of the functional equation. It is easily verified that this is indeed a solution.

3. Consider two squares $AB'CD'$ and $A'BC'D$. Since $AC \perp BD$, these two squares are homothetic, which implies that the lines AA', BB', CC', DD' are concurrent at a certain point O.

   Since the rotation about A by 90° takes ΔABK into ΔAFD, it follows that $BK \perp DF$. Denote by T the intersection of BK and DF. The rotation about some point X by 90° maps BK into DF if and only if TX bisects an angle between BK and DF. Therefore $\angle FTA = \angle ATK = 45^\circ$. Moreover, the quadrilateral $BA'DT$ is cyclic, which implies that $\angle BTA' = BDA' = 45^\circ$ and consequently that the points A, T, A' are collinear. It follows that the point O lies on a bisector of $\angle BTD$ and therefore the rotation $\mathcal{R}$ about O by 90° takes BK into DF. Analogously, $\mathcal{R}$ maps the lines CE, DG, AI into AH, BJ, CL. Hence the quadrilateral $P_1Q_1R_1S_1$ is the image of the quadrilateral $P_2Q_2R_2S_2$, and the result follows.

4. There are 36 possible edges in total. If not more than 3 edges are left undrawn, then we can choose 6 of the given 9 points no two of which are connected by an undrawn edge. These 6 points together with the edges between them form a two-colored complete graph, and thus by a well-known result there exists at least one monochromatic triangle. It follows that $n \leq 33$.

In order to show that $n = 33$, we shall give an example of a graph with 32 edges that does not contain a monochromatic triangle. Let us start with a complete graph C_5 with 5 vertices. Its edges can be colored in two colors so that there is no monochromatic triangle (Fig. 1). Furthermore, given a graph $\mathcal{H}$ with k vertices without monochromatic triangles, we can add to it a new vertex, join it to all vertices of $\mathcal{H}$ except A, and color each edge BX in the same way as AX. The obtained graph obviously contains no monochromatic triangles. Applying this construction four times to the graph C_5 we get an example like that of Fig. 2.

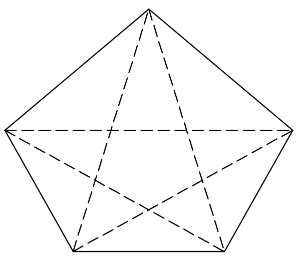


Fig. 1

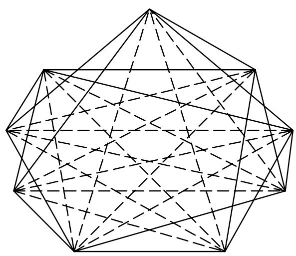


Fig. 2

Second solution. For simplicity, we call the colors red and blue.

Let $r(k, l)$ be the least positive integer r such that each complete r-graph whose edges are colored in red and blue contains either a complete red k-graph or a complete blue l-graph. Also, let $t(n, k)$ be the greatest possible number of edges in a graph with n vertices that does not contain a complete k-graph. These numbers exist by the theorems of Ramsey and Turán.

Let us assume that $r(k, l) < n$. Every graph with n vertices and $t(n, r(k, l))$ $+1$ edges contains a complete subgraph with $r(k, l)$ vertices, and this subgraph contains either a red complete k-graph or a blue complete l-graph.

We claim that $t(n, r(k, l)) + 1$ is the smallest number of edges with the above property. By the definition of $r(k, l)$ there exists a coloring of the complete graph H with $r(k, l) - 1$ vertices in two colors such that no red complete k-graph or blue complete l-graph exists. Let c_{ij} be the color in

which the edge (i,j) of H is colored, $1 \le i < j \le r(k,l)-1$. Consider a complete $r(k,l)-1$-partite graph G with n vertices and exactly $t(n, r(k,l))$ edges and denote its partitions by P_i, $i = 1, \dots, r(k,l)-1$. If we color each edge of H between P_i and P_j $(j < i)$ in the color c_{ij}, we obviously obtain a graph with n vertices and $t(n, r(k,l))$ edges in two colors that contains neither a red complete k-graph nor a blue complete l-graph.
Therefore the answer to our problem is $t(9, r(3,3)) + 1 = t(9,6) + 1 = 33$.

5. Denote by K, L, M, and N the midpoints of the sides AB, BC, CD, and DA, respectively. The quadrilateral $KLMN$ is a rhombus. We shall prove that $O_1O_3 \parallel KM$. Similarly, $O_2O_4 \parallel LN$, and the desired result follows immediately.
We have $\overrightarrow{O_1O_3} = \overrightarrow{KM} + \left(\overrightarrow{O_1K} + \overrightarrow{MO_3}\right)$. Assume that $ABCD$ is positively oriented. A rotational homothety $\mathcal{R}$ with angle $-90°$ and coefficient $1/\sqrt{3}$ takes the vectors $\overrightarrow{BK}$ and $\overrightarrow{CM}$ into $\overrightarrow{O_1K}$ and $\overrightarrow{MO_3}$ respectively. Therefore
$$\begin{aligned}\overrightarrow{O_1O_3} &= \overrightarrow{KM} + (\overrightarrow{O_1K} + \overrightarrow{MO_3}) = \overrightarrow{KM} + \mathcal{R}(\overrightarrow{BK} + \overrightarrow{CM}) \\ &= \overrightarrow{KM} + \frac{1}{2}\mathcal{R}(\overrightarrow{BA} + \overrightarrow{CD}) = \overrightarrow{KM} + \mathcal{R}(\overrightarrow{LN}).\end{aligned}$$
Since $LN \perp KM$, it follows that $\mathcal{R}(LN)$ is parallel to KM and so is O_1O_3.

6. It is easy to see that f is injective and surjective. From $f(x^2 + f(y)) = f((-x)^2 + f(y))$ it follows that $f(x)^2 = (f(-x))^2$, which implies $f(-x) = -f(x)$ because f is injective. Furthermore, there exists $z \in \mathbb{R}$ such that $f(z) = 0$. From $f(-z) = -f(z) = 0$ we deduce that $z = 0$. Now we have $f(x^2) = f(x^2 + f(0)) = 0 + (f(x))^2 = f(x)^2$, and consequently $f(x) = f(\sqrt{x})^2 > 0$ for all $x > 0$. It also follows that $f(x) < 0$ for $x < 0$. In other words, f preserves sign.
Now setting $x > 0$ and $y = -f(x)$ in the given functional equation we obtain
$$f(x - f(x)) = f(\sqrt{x}^2 + f(-x)) = -x + f(\sqrt{x})^2 = -(x - f(x)).$$
But since f preserves sign, this implies that $f(x) = x$ for $x > 0$. Moreover, since $f(-x) = -f(x)$, it follows that $f(x) = x$ for all x. It is easily verified that this is indeed a solution.

7. Let G_1, G_2 touch the chord BC at P, Q and touch the circle G at R, S respectively. Let D be the midpoint of the complementary arc BC of G. The homothety centered at R mapping G_1 onto G also maps the line BC onto a tangent of G parallel to BC. It follows that this line touches G at point D, which is therefore the image of P under the homothety. Hence R, P, and D are collinear. Since $\angle DBP = \angle DCB = \angle DRB$, it follows that $\triangle DBP \sim \triangle DRB$ and consequently that $DP \cdot DR = DB^2$. Similarly, points S, Q, D are collinear and satisfy $DQ \cdot DS = DB^2 = DP \cdot DR$.

Hence D lies on the radical axis of the circles G_1 and G_2, i.e., on their common tangent AW, which also implies that AW bisects the angle BAD. Furthermore, since $DB = DC = DW = \sqrt{DP \cdot DR}$, it follows from the lemma of (SL99-14) that W is the incenter of $\triangle ABC$.

Remark. According to the third solution of (SL93-3), both PW and QW contain the incenter of $\triangle ABC$, and the result is immediate. The problem can also be solved by inversion centered at W.

8. For simplicity, we shall write n instead of 1992.

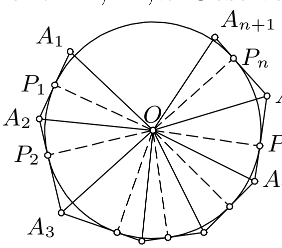
Lemma. There exists a tangent n-gon $A_1A_2 \dots A_n$ with sides $A_1A_2 = a_1$, $A_2A_3 = a_2, \dots, A_nA_1 = a_n$ if and only if the system

$$x_1 + x_2 = a_1,\ x_2 + x_3 = a_2,\ ,\dots,\ x_n + x_1 = a_n \tag{1}$$

has a solution $(x_1, \dots, x_n)$ in positive reals.

Proof. Suppose that such an n-gon $A_1A_2 \dots A_n$ exists. Let the side A_iA_{i+1} touch the inscribed circle at point P_i (where $A_{n+1} = A_1$). Then $x_1 = A_1P_n = A_1P_1$, $x_2 = A_2P_1 = A_2P_2, \dots, x_n = A_nP_{n-1} = A_nP_n$ is clearly a positive solution of (1).

Now suppose that the system (1) has a positive real solution $(x_1, \dots, x_n)$. Let us draw a polygonal line $A_1A_2 \dots A_{n+1}$ touching a circle of radius r at points $P_1, P_2, \dots, P_n$ respectively such that $A_1P_1 = A_{n+1}P_n = x_1$ and $A_iP_i = A_iP_{i-1} = x_i$ for $i = 2, \dots, n$. Observe that $OA_1 = OA_{n+1} = \sqrt{x_1^2 + r^2}$ and the function $f(r) = \angle A_1OA_2 + \angle A_2OA_3 + \cdots + \angle A_nOA_{n+1} = 2(\arctan \frac{x_1}{r} + \cdots + \arctan \frac{x_n}{r})$ is continuous. Thus $A_1A_2 \dots A_{n+1}$ is a closed simple polygonal line if and only if $f(r) = 360°$. But such an r exists, since $f(r) \to 0$



when $r \to \infty$ and $f(r) \to \infty$ when $r \to 0$. This proves the second direction of the lemma.

For $n = 4k$, the system (1) is solvable in positive reals if $a_i = i$ for $i \equiv 1, 2 \pmod 4$, $a_i = i+1$ for $i \equiv 3$ and $a_i = i-1$ for $i \equiv 0 \pmod 4$. Indeed, one solution is given by $x_i = 1/2$ for $i \equiv 1$, $x_i = 3/2$ for $i \equiv 3$ and $x_i = i - 3/2$ for $i \equiv 0, 2 \pmod 4$.

Remark. For $n = 4k + 2$ there is no such n-gon. In fact, solvability of the system (1) implies $a_1 + a_3 + \cdots = a_2 + a_4 + \cdots$, while in the case $n = 4k + 2$ the sum $a_1 + a_2 + \cdots + a_n$ is odd.

9. Since the equation $x^3 - x - c = 0$ has only one real root for every $c > 2/(3\sqrt{3})$, α is the unique real root of $x^3 - x - 33^{1992} = 0$. Hence $f^n(\alpha) = f(\alpha) = \alpha$.

Remark. Consider any irreducible polynomial $g(x)$ in the place of $x^3 - x - 33^{1992}$. The problem amounts to proving that if α and $f(\alpha)$ are roots of g, then any $f^{(n)}(\alpha)$ is also a root of g. In fact, since $g(f(x))$ vanishes at $x=\alpha$, it must be divisible by the minimal polynomial of α, that is, $g(x)$. It follows by induction that $g(f^{(n)}(x))$ is divisible by $g(x)$ for all $n \in \mathbb{N}$, and hence $g(f^{(n)}(\alpha)) = 0$.

10. Let us set $S(x) = \{(y,z) \mid (x,y,z) \in V\}$, $S_y(x) = \{z \mid (x,z) \in S_y\}$ and $S_z(x) = \{y \mid (x,y) \in S_z\}$. Clearly $S(x) \subset S_x$ and $S(x) \subset S_y(x) \times S_z(x)$. It follows that

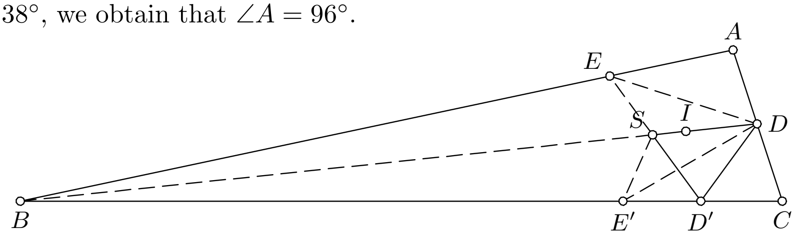
$$\begin{aligned} |V| &= \sum_x |S(x)| \le \sum_x \sqrt{|S_x||S_y(x)||S_z(x)|} \\ &= \sqrt{|S_x|} \sum_x \sqrt{|S_y(x)||S_z(x)|}. \end{aligned} \tag{1}$$

Using the Cauchy–Schwarz inequality we also get

$$\sum_x \sqrt{|S_y(x)||S_z(x)|} \le \sqrt{\sum_x |S_y(x)|}\sqrt{\sum_x |S_z(x)|} = \sqrt{|S_y||S_z|}. \tag{2}$$

Now (1) and (2) together yield $|V| \le \sqrt{|S_x||S_y||S_z|}$.

11. Let I be the incenter of $\triangle ABC$. Since $90^\circ + \alpha/2 = \angle BIC = \angle DIE = 138^\circ$, we obtain that $\angle A = 96^\circ$.



Let D' and E' be the points symmetric to D and E with respect to CE and BD respectively, and let S be the intersection point of ED' and BD. Then $\angle BDE' = 24^\circ$ and $\angle D'DE' = \angle D'DE - \angle E'DE = 24^\circ$,which means that DE' bisects the angle SDD'. Moreover, $\angle E'SB = \angle ESB = \angle EDS + \angle DES = 60^\circ$ and hence SE' bisects the angle $D'SB$. It follows that E' is the excenter of $\triangle D'DS$ and consequently $\angle D'DC = \angle DD'C = \angle SD'E' = (180^\circ - 72^\circ)/2 = 54^\circ$. Finally, $\angle C = 180^\circ - 2 \cdot 54^\circ = 72^\circ$ and $\angle B = 12^\circ$.

12. Let us set $\deg f = n$ and $\deg g = m$. We shall prove the result by induction on n. If $n < m$, then $\deg_x[f(x) - f(y)] < \deg_x[g(x) - g(y)]$, which implies that $f(x) - f(y) = 0$, i.e., that f is constant. The statement trivially holds. Assume now that $n \ge m$. Transition to $f_1(x) = f(x) - f(0)$ and $g_1(x) = g(x) - g(0)$ allows us to suppose that $f(0) = g(0) = 0$. Then the given condition for $y = 0$ gives us $f(x) = f_1(x)g(x)$, where $f_1(x) = a(x,0)$ and $\deg f_1 = n - m$. We now have

$$\begin{aligned} a(x,y)(g(x) - g(y)) &= f(x) - f(y) = f_1(x)g(x) - f_1(y)g(y) \\ &= [f_1(x) - f_1(y)]g(x) + f_1(y)[g(x) - g(y)]. \end{aligned}$$

Since $g(x)$ is relatively prime to $g(x)-g(y)$, it follows that $f_1(x)-f_1(y) = b(x,y)(g(x)-g(y))$ for some polynomial $b(x,y)$. By the induction hypothesis there exists a polynomial h_1 such that $f_1(x) = h_1(g(x))$ and consequently $f(x) = g(x) \cdot h_1(g(x)) = h(g(x))$ for $h(t) = th_1(t)$. Thus the induction is complete.

13. Let us define

$$\begin{aligned} F(p,q,r) &= \frac{(pqr-1)}{(p-1)(q-1)(r-1)} \\ &= 1 + \frac{1}{p-1} + \frac{1}{q-1} + \frac{1}{r-1} \\ &\quad + \frac{1}{(p-1)(q-1)} + \frac{1}{(q-1)(r-1)} + \frac{1}{(r-1)(p-1)}. \end{aligned}$$

Obviously F is a decreasing function of p,q,r. Suppose that $1 < p < q < r$ are integers for which $F(p,q,r)$ is an integer. Observe that p,q,r are either all even or all odd. Indeed, if for example p is odd and q is even, then $pqr-1$ is odd while $(p-1)(q-1)(r-1)$ is even, which is impossible. Also, if p,q,r are even then $F(p,q,r)$ is odd.

If $p \geq 4$, then $1 < F(p,q,r) \leq F(4,6,8) = 191/105 < 2$, which is impossible. Hence $p \leq 3$.

Let $p = 2$. Then q,r are even and $1 < F(2,q,r) \leq F(2,4,6) = 47/15 < 4$. Therefore $F(2,q,r) = 3$. This equality reduces to $(q-3)(r-3) = 5$, with the unique solution $q = 4$, $r = 8$.

Let $p = 3$. Then q,r are odd and $1 < F(3,q,r) \leq F(3,5,7) = 104/48 < 3$. Therefore $F(3,q,r) = 2$. This equality reduces to $(q-4)(r-4) = 11$, which leads to $q = 5$, $r = 15$.

Hence the only solutions (p,q,r) of the problem are $(2,4,8)$ and $(3,5,15)$.

14. We see that $x_1 = 2^0$. Suppose that for some $m,r \in \mathbb{N}$ we have $x_m = 2^r$. Then inductively $x_{m+i} = 2^{r-i}(2i+1)$ for $i = 1,2,\dots,r$ and $x_{m+r+1} = 2^{r+1}$. Since every natural number can be uniquely represented as the product of an odd number and a power of two, we conclude that every natural number occurs in our sequence exactly once.

Moreover, it follows that $2k-1 = x_{k(k+1)/2}$. Thus $x_n = 1992 = 2^3 \cdot 249$ implies that $x_{n+3} = 255 = 2\cdot 128 - 1 = x_{128\cdot 129/2} = x_{8256}$. Hence $n = 8253$.

15. The result follows from the following lemma by taking $n = \frac{1992\cdot 1993}{2}$ and $M = \{d, 2d, \dots, 1992d\}$.

Lemma. For every $n \in \mathbb{N}$ there exists a natural number d such that all the numbers $d, 2d, \dots, nd$ are of the form m^k ($m,k \in \mathbb{N}$, $k \geq 2$).

Proof. Let $p_1, p_2, \dots, p_n$ be distinct prime numbers. We shall find d in the form $d = 2^{\alpha_2}3^{\alpha_3}\cdots n^{\alpha_n}$, where $\alpha_i \geq 0$ are integers such that kd is a perfect p_kth power. It is sufficient to find α_i, $i = 2,3,\dots,n$, such that $\alpha_i \equiv 0 \pmod{p_j}$ if $i \neq j$ and $\alpha_i \equiv -1 \pmod{p_j}$ if $i = j$. But the existence of such α_i's is an immediate consequence of the Chinese remainder theorem.

16. Observe that $x^4+x^3+x^2+x+1=(x^2+3x+1)^2-5x(x+1)^2$. Thus for $x=5^{25}$ we have

$$\begin{aligned} N &= x^4+x^3+x^2+x+1 \\ &= (x^2+3x+1-5^{13}(x+1))(x^2+3x+1+5^{13}(x+1)) = A\cdot B. \end{aligned}$$

Clearly, both A and B are positive integers greater than 1.

17. (a) Let $n=\sum_{i=1}^{k} 2^{a_i}$, so that $\alpha(n)=k$. Then

$$n^2=\sum_i 2^{2a_i}+\sum_{i<j} 2^{a_i+a_j+1}$$

has at most $k+\binom{k}{2}=\frac{k(k+1)}{2}$ binary ones.

(b) The above inequality is an equality for all numbers $n_k=2^k$.

(c) Put $n_m=2^{2^m-1}-\sum_{j=1}^{m} 2^{2^m-2^j}$, where $m>1$. It is easy to see that $\alpha(n_m)=2^m-m$. On the other hand, squaring and simplifying yields $n_m^2=1+\sum_{i<j} 2^{2^{m+1}+1-2^i-2^j}$. Hence $\alpha(n_m^2)=1+\frac{m(m+1)}{2}$ and thus

$$\frac{\alpha(n_m^2)}{\alpha(n_m)}=\frac{2+m(m+1)}{2(2^m-m)}\to 0 \quad \text{as } m\to\infty.$$

Solution to the alternative parts.

(1) Let $n=\sum_{i=1}^{n} 2^{2^i}$. Then $n^2=\sum_{i=1}^{n} 2^{2^{i+1}}+\sum_{i<j} 2^{2^i+2^j+1}$ has exactly $\frac{k(k+1)}{2}$ binary ones, and therefore $\frac{\alpha(n^2)}{\alpha(n)}=\frac{2k}{k(k+1)}\to\infty$.

(2) Consider the sequence n_i constructed in part (c). Let $\theta>1$ be a constant to be chosen later, and let $N_i=2^{m_i}n_i-1$ where $m_i>\alpha(n_i)$ is such that $m_i/\alpha(n_i)\to\theta$ as $i\to\infty$. Then $\alpha(N_i)=\alpha(n_i)+m_i-1$, whereas $N_i^2=2^{2m_i}n_i^2-2^{m_i+1}n_i+1$ and $\alpha(N_i^2)=\alpha(n_i^2)-\alpha(n_i)+m_i$. It follows that

$$\lim_{i\to\infty}\frac{\alpha(N_i^2)}{\alpha(N_i)}=\lim_{i\to\infty}\frac{\alpha(n_i^2)+(\theta-1)\alpha(n_i)}{(1+\theta)\alpha(n_i)}=\frac{\theta-1}{\theta+1},$$

which is equal to $\gamma\in[0,1]$ for $\theta=\frac{1+\gamma}{1-\gamma}$ (for $\gamma=1$ we set $m_i/\alpha(n_i)\to\infty$).

(3) Let be given a sequence $(n_i)_{i=1}^{\infty}$ with $\alpha(n_i^2)/\alpha(n_i)\to\gamma$. Taking $m_i>\alpha(n_i)$ and $N_i=2^{m_i}n_i+1$ we easily find that $\alpha(N_i)=\alpha(n_i)+1$ and $\alpha(N_i^2)=\alpha(n_i^2)+\alpha(n_i)+1$. Hence $\alpha(N_i^2)/\alpha(N_i)=\gamma+1$. Continuing this procedure we can construct a sequence t_i such that $\alpha(t_i^2)/\alpha(t_i)=\gamma+k$ for an arbitrary $k\in\mathbb{N}$.

18. Let us define inductively $f^1(x)=f(x)=\frac{1}{x+1}$ and $f^n(x)=f(f^{n-1}(x))$, and let $g_n(x)=x+f(x)+f^2(x)+\cdots+f^n(x)$. We shall prove first the following statement.

Lemma. The function $g_n(x)$ is strictly increasing on $[0,1]$, and $g_{n-1}(1) = F_1/F_2 + F_2/F_3 + \cdots + F_n/F_{n+1}$.

Proof. Since $f(x) - f(y) = \frac{y-x}{(1+x)(1+y)}$ is smaller in absolute value than $x - y$, it follows that $x > y$ implies $f^{2k}(x) > f^{2k}(y)$ and $f^{2k+1}(x) < f^{2k+1}(y)$, and moreover that for every integer $k \geq 0$,

$$[f^{2k}(x) - f^{2k}(y)] + [f^{2k+1}(x) - f^{2k+1}(y)] > 0.$$

Hence if $x > y$, we have $g_n(x) - g_n(y) = (x-y) + [f(x) - f(y)] + \cdots + [f^n(x) - f^n(y)] > 0$, which yields the first part of the lemma.

The second part follows by simple induction, since $f^k(1) = F_{k+1}/F_{k+2}$.

If some $x_i = 0$ and consequently $x_j = 0$ for all $j \geq i$, then the problem reduces to the problem with $i-1$ instead of n. Thus we may assume that all $x_1, \dots, x_n$ are different from 0. If we write $a_i = [1/x_i]$, then $x_i = \frac{1}{a_i + x_{i+1}}$. Thus we can regard x_i as functions of x_n depending on $a_1, \dots, a_{n-1}$.

Suppose that $x_n, a_{n-1}, \dots, a_3, a_2$ are fixed. Then $x_2, x_3, \dots, x_n$ are all fixed, and $x_1 = \frac{1}{a_1 + x_2}$ is maximal when $a_1 = 1$. Hence the sum $S = x_1 + x_2 + \cdots + x_n$ is maximized for $a_1 = 1$.

We shall show by induction on i that S is maximized for $a_1 = a_2 = \cdots = a_i = 1$. In fact, assuming that the statement holds for $i-1$ and thus $a_1 = \cdots = a_{i-1} = 1$, having $x_n, a_{n-1}, \dots, a_{i+1}$ fixed we have that $x_n, \dots, x_{i+1}$ are also fixed, and that $x_{i-1} = f(x_i), \dots, x_1 = f^{i-1}(x_i)$. Hence by the lemma, $S = g_{i-1}(x_i) + x_{i+1} + \cdots + x_n$ is maximal when $x_i = \frac{1}{a_i + x_{i+1}}$ is maximal, that is, for $a_i = 1$. Thus the induction is complete.

It follows that $x_1 + \cdots + x_n$ is maximal when $a_1 = \cdots = a_{n-1} = 1$, so that $x_1 + \cdots + x_n = g_{n-1}(x_1)$. By the lemma, the latter does not exceed $g_{n-1}(1)$. This completes the proof.

Remark. The upper bound is the best possible, because it is approached by taking x_n close to 1 and inductively (in reverse) defining $x_{i-1} = \frac{1}{1+x_i} = \frac{1}{a_i + x_i}$.

19. Observe that $f(x) = (x^4 + 2x^2 + 3)^2 - 8(x^2 - 1)^2 = [x^4 + 2(1 - \sqrt{2})x^2 + 3 + 2\sqrt{2}][x^4 + 2(1 + \sqrt{2})x^2 + 3 - 2\sqrt{2}]$. Now it is easy to find that the roots of f are

$$x_{1,2,3,4} = \pm i\left(i\sqrt[4]{2} \pm 1\right) \quad \text{and} \quad x_{5,6,7,8} = \pm i\left(\sqrt[4]{2} \pm 1\right).$$

In other words, $x_k = \alpha_i + \beta_j$, where $\alpha_i^2 = -1$ and $\beta_j^4 = 2$.

We claim that any root of f can be obtained from any other using rational functions. In fact, we have

$$\begin{aligned}
x^3 &= -\alpha_i - 3\beta_j + 3\alpha_i\beta_j^2 + \beta_j^3, \\
x^5 &= 11\alpha_i + 7\beta_j - 10\alpha_i\beta_j^2 - 10\beta_j^3 \\
x^7 &= -71\alpha_i - 49\beta_j + 35\alpha_i\beta_j^2 + 37\beta_j^3,
\end{aligned}$$

from which we easily obtain that

$$\alpha_i = 24^{-1}(127x+5x^3+19x^5+5x^7), \qquad \beta_j = 24^{-1}(151x+5x^3+19x^5+5x^7).$$

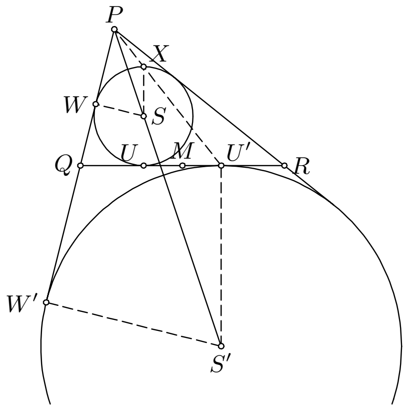
Since all other values of α and β can be obtained as rational functions of α_i and β_j, it follows that all the roots x_l are rational functions of a particular root x_k.

We now note that if x_1 is an integer such that $f(x_1)$ is divisible by p, then $p > 3$ and $x_1 \in \mathbb{Z}_p$ is a root of the polynomial f. By the previous consideration, all remaining roots $x_2, \dots, x_8$ of f over the field $\mathbb{Z}_p$ are rational functions of x_1, since 24 is invertible in Z_p. Then $f(x)$ factors as

$$f(x) = (x - x_1)(x - x_2)\cdots(x - x_8),$$

and the result follows.

20. Denote by U the point of tangency of the circle C and the line l. Let X and U' be the points symmetric to U with respect to S and M respectively; these points do not depend on the choice of P. Also, let C' be the excircle of $\triangle PQR$ corresponding to P, S' the center of C', and W, W' the points of tangency of C and C' with the line PQ respectively. Obviously, $\triangle WSP \sim \triangle W'S'P$. Since $SX \parallel S'U'$ and $SX : S'U' = SW : S'W' = SP : S'P$, we deduce that $\Delta SXP \sim \Delta S'U'P$, and consequently that P lies on the line XU'. On the other hand, it is easy to show that each point P of the ray $U'X$ over X satisfies the required condition. Thus the desired locus is the extension of $U'X$ over X.



21. (a) Representing n^2 as a sum of $n^2 - 13$ squares is equivalent to representing 13 as a sum of numbers of the form $x^2 - 1$, $x \in \mathbb{N}$, such as $0, 3, 8, 15, \dots$. But it is easy to check that this is impossible, and hence $s(n) \le n^2 - 14$.

(b) Let us prove that $s(13) = 13^2 - 14 = 155$. Observe that

$$\begin{aligned} 13^2 &= 8^2 + 8^2 + 4^2 + 4^2 + 3^2 \\ &= 8^2 + 8^2 + 4^2 + 4^2 + 2^2 + 2^2 + 1^2 \\ &= 8^2 + 8^2 + 4^2 + 3^2 + 3^2 + 2^2 + 1^2 + 1^2 + 1^2. \end{aligned}$$

Given any representation of n^2 as a sum of m squares one of which is even, we can construct a representation as a sum of $m + 3$ squares by dividing the odd square into four equal squares. Thus the first equality enables us to construct representations with $5, 8, 11, \dots, 155$ squares, the second to construct ones with $7, 10, 13, \dots, 154$ squares, and the third with $9, 12, \dots, 153$ squares. It remains only to represent 13^2 as a sum of $k = 2, 3, 4, 6$ squares. This can be done as follows:

$$\begin{aligned}13^2 &= 12^2 + 5^2 = 12^2 + 4^2 + 3^2 \\ &= 11^2 + 4^2 + 4^2 + 4^2 = 12^2 + 3^2 + 2^2 + 2^2 + 2^2 + 2^2.\end{aligned}$$

(c) We shall prove that whenever $s(n) = n^2 - 14$ for some $n \geq 13$, it also holds that $s(2n) = (2n)^2 - 14$. This will imply that $s(n) = n^2 - 14$ for any $n = 2^t \cdot 13$.
If $n^2 = x_1^2 + \cdots + x_r^2$, then we have $(2n)^2 = (2x_1)^2 + \cdots + (2x_r)^2$. Replacing $(2x_i)^2$ with $x_i^2 + x_i^2 + x_i^2 + x_i^2$ as long as it is possible we can obtain representations of $(2n)^2$ consisting of $r, r+3, \dots, 4r$ squares. This gives representations of $(2n)^2$ into k squares for any $k \leq 4n^2 - 62$. Further, we observe that each number $m \geq 14$ can be written as a sum of $k \geq m$ numbers of the form $x^2 - 1$, $x \in \mathbb{N}$, which is easy to verify. Therefore if $k \leq 4n^2 - 14$, it follows that $4n^2 - k$ is a sum of k numbers of the form $x^2 - 1$ (since $k \geq 4n^2 - k \geq 14$), and consequently $4n^2$ is a sum of k squares.

Remark. One can find exactly the value of $f(n)$ for each n:

$$f(n) = \begin{cases} 1, & \text{if } n \text{ has a prime divisor congruent to 3 mod 4;} \\ 2, & \text{if } n \text{ is of the form } 5 \cdot 2^k,\ k \text{ a positive integer;} \\ n^2 - 14, & \text{otherwise.} \end{cases}$$

4.34 Solutions to the Shortlisted Problems of IMO 1993

1. First we notice that for a rational point O (i.e., with rational coordinates), there exist 1993 rational points in each quadrant of the unit circle centered at O. In fact, it suffices to take

$$X = \left\{ O + \left(\pm\frac{t^2-1}{t^2+1}, \pm\frac{2t}{t^2+1} \right) \middle| \, t = 1, 2, \dots, 1993 \right\}.$$

Now consider the set $A = \{(i/q, j/q) \mid i, j = 0, 1, \dots, 2q\}$, where $q = \prod_{i=1}^{1993}(t^2+1)$. We claim that A gives a solution for the problem. Indeed, for any $P \in A$ there is a quarter of the unit circle centered at P that is contained in the square $[0,2] \times [0,2]$. As explained above, there are 1993 rational points on this quarter circle, and by definition of q they all belong to A.

Remark. Substantially the same problem was proposed by Bulgaria for IMO 71: see (SL71-2), where we give another possible construction of a set A.

2. It is well known that $r \le \frac{1}{2}R$. Therefore $\frac{1}{3}(1+r)^2 \le \frac{1}{3}\left(1+\frac{1}{2}\right)^2 = \frac{3}{4}$. It remains only to show that $p \le \frac{1}{4}$. We note that p does not exceed one half of the circumradius of $\triangle A'B'C'$. However, by the theorem on the nine-point circle, this circumradius is equal to $\frac{1}{2}R$, and the conclusion follows.

Second solution. By a well-known relation we have $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$ ($= 1 + r$ when $R = 1$). Next, recalling that the incenter of $\triangle A'B'C'$ is at the orthocenter of $\triangle ABC$, we easily obtain $p = 2\cos A \cos B \cos C$. Cosines of angles of a triangle satisfy the identity $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$ (the proof is straightforward: see (SL81-11)). Thus

$$\begin{aligned} p + \frac{1}{3}(1+r)^2 &= 2\cos A \cos B \cos C + \frac{1}{3}(\cos A + \cos B + \cos C)^2 \\ &\le 2\cos A \cos B \cos C + \cos^2 A + \cos^2 B + \cos^2 C = 1. \end{aligned}$$

3. Let O_1 and ρ be the center and radius of k_c. It is clear that C, I, O_1 are collinear and $CI/CO_1 = r/\rho$. By Stewart's theorem applied to $\triangle OCO_1$,

$$OI^2 = \frac{r}{\rho} OO_1^2 + \left(1 - \frac{r}{\rho}\right) OC^2 - CI \cdot IO_1. \qquad (1)$$

Since $OO_1 = R - \rho$, $OC = R$ and by Euler's formula $OI^2 = R^2 - 2Rr$, substituting these values in (1) gives $CI \cdot IO_1 = r\rho$, or equivalently $CO_1 \cdot IO_1 = \rho^2 = DO_1^2$. Hence the triangles CO_1D and DO_1I are similar, implying $\angle DIO_1 = 90^\circ$. Since $CD = CE$ and the line CO_1 bisects the segment DE, it follows that I is the midpoint of DE.

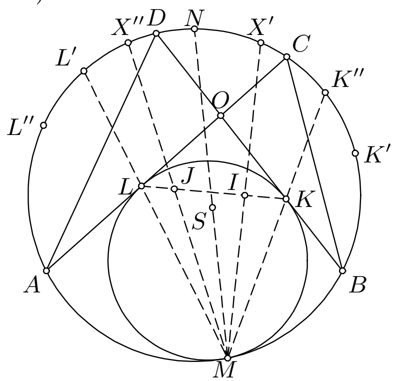
Second solution. Under the inversion with center C and power ab, k_c is transformed into the excircle of $\widehat{A}\widehat{B}C$ corresponding to C. Thus $CD =$

$\frac{ab}{s}$, where s is the common semiperimeter of $\triangle ABC$ and $\triangle \widehat{A}\widehat{B}C$, and consequently the distance from D to BC is $\frac{ab}{s}\sin C = \frac{2S_{ABC}}{s} = 2r$. The statement follows immediately.

Third solution. We shall prove a stronger statement: Let $ABCD$ be a convex quadrilateral inscribed in a circle k, and k' the circle that is tangent to segments BO, AO at K, L respectively (where $O = BD \cap AC$), and internally to k at M. Then KL contains the incenters I, J of $\triangle ABC$ and $\triangle ABD$.

Let K', K'', L', L'', N denote the midpoints of arcs BC, BD, AC, AD, AB that don't contain M; X', X'' the points on k defined by $X'N = NX'' = K'K'' = L'L''$ (as oriented arcs); and set $S = AK' \cap BL''$, $\overline{M} = NS \cap k$, $\overline{K} = K''M \cap BO$, $\overline{L} = L'M \cap AO$.

It is clear that $I = AK' \cap BL'$, $J = AK'' \cap BL''$. Furthermore, $X'\overline{M}$ contains I (to see this, use the fact that for A, B, C, D, E, F on k, lines AD, BE, CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$, and then express $A\overline{M}/\overline{M}B$ by applying this rule to $AMBK'NL''$ and show that $AK', \overline{M}X', BL'$ are concurrent).



Analogously, $X''\overline{M}$ contains J. Now the points $B, \overline{K}, I, S, \overline{M}$ lie on a circle ($\angle B\overline{K}\overline{M} = \angle BI\overline{M} = \angle BS\overline{M}$), and points $A, \overline{L}, J, S, \overline{M}$ do so as well. Lines $I\overline{K}, J\overline{L}$ are parallel to $K''L'$ (because $\angle\overline{M}\overline{K}I = \angle\overline{M}BI = \angle\overline{M}K''L'$). On the other hand, the quadrilateral $ABIJ$ is cyclic, and simple calculation with angles shows that IJ is also parallel to $K''L'$. Hence $\overline{K}, I, J, \overline{L}$ are collinear.

Finally, $\overline{K} \equiv K$, $\overline{L} \equiv L$, and $\overline{M} \equiv M$ because the homothety centered at M that maps k' to k sends K to K'' and L to L' (thus M, K, K'', as well as M, L, L', must be collinear). As is seen now, the deciphered picture yields many other interesting properties. Thus, for example, N, S, M are collinear, i.e., $\angle AMS = \angle BMS$.

Fourth solution. We give an alternative proof of the more general statement in the third solution. Let W be the foot of the perpendicular from B to AC. We define $q = CW$, $h = BW$, $t = OL = OK$, $x = AL$, $\theta = \measuredangle WBO$ (θ is negative if $\mathcal{B}(O, W, A)$, $\theta = 0$ if $W = O$), and as usual, $a = BC$, $b = AC$, $c = AB$. Let $\alpha = \measuredangle KLC$ and $\beta = \measuredangle ILC$ (both angles must be acute). Our goal is to prove $\alpha = \beta$. We note that $90^\circ - \theta = 2\alpha$. One easily gets

$$\tan\alpha = \frac{\cos\theta}{1 + \sin\theta}\,, \quad \tan\beta = \frac{\frac{2S_{ABC}}{a+b+c}}{\frac{b+c-a}{2} - x}\,. \tag{1}$$

Applying Casey's theorem to A, B, C, k', we get $AC \cdot BK + AL \cdot BC = AB \cdot CL$, i.e., $b\left(\frac{h}{\cos\theta} - t\right) + xa = c(b-x)$. Using that $t = b - x - q - h\tan\theta$ we get

$$x = \frac{b(b+c-q) - bh\left(\frac{1}{\cos\theta} + \tan\theta\right)}{a+b+c}. \qquad (2)$$

Plugging (2) into the second equation of (1) and using $bh = 2S_{ABC}$ and $c^2 = b^2 + a^2 - 2bq$, we obtain $\tan\alpha = \tan\beta$, i.e., $\alpha = \beta$, which completes our proof.

4. Let h be the altitude from A and $\varphi = \angle BAD$. We have $BM = \frac{1}{2}(BD + AB - AD)$ and $MD = \frac{1}{2}(BD - AB + AD)$, so

$$\begin{aligned}\frac{1}{MB} + \frac{1}{MD} &= \frac{BD}{MB \cdot MD} = \frac{4BD}{BD^2 - AB^2 - AD^2 + 2AB \cdot AD} \\ &= \frac{4BD}{2AB \cdot AD(1-\cos\varphi)} = \frac{2BD\sin\varphi}{2S_{ABD}(1-\cos\varphi)} \\ &= \frac{2BD\sin\varphi}{BD \cdot h(1-\cos\varphi)} = \frac{2}{h\tan\frac{\varphi}{2}}.\end{aligned}$$

It follows that $\frac{1}{MB} + \frac{1}{MD}$ depends only on h and φ. Specially, $\frac{1}{NC} + \frac{1}{NE} = \frac{2}{h\tan(\varphi/2)}$ as well.

5. For $n = 1$ the game is trivially over. If $n = 2$, it can end, for example, in the following way:

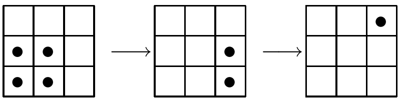


Fig. 1

The sequence of moves shown in Fig. 2 enables us to remove three pieces placed in a 1×3 rectangle, using one more piece and one more free cell. In that way, for any $n \geq 4$ we can reduce an $(n+3) \times (n+3)$ square to an $n \times n$ square (Fig. 3). Therefore the game can end for every n that is not divisible by 3.

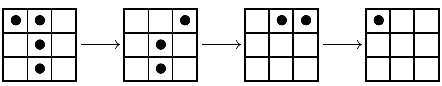


Fig. 2

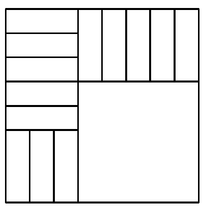


Fig. 3

Suppose now that one can play the game on a $3k \times 3k$ square so that at the end only one piece remains. Denote the cells by (i, j), $i, j \in \{1, \ldots, 3k\}$, and let S_0, S_1, S_2 denote the numbers of pieces on those squares (i, j) for which $i+j$ gives remainder 0, 1, 2 respectively upon division by 3. Initially $S_0 = S_1 = S_2 = 3k^2$. After each move, two of S_0, S_1, S_2 diminish and one increases by one. Thus each move reverses the parity of the S_i's, so that S_0, S_1, S_2 are always of the same parity. But in the final position one of the S_i's must be equal to 1 and the other two must be 0, which is impossible.

6. Notice that for $\alpha = \frac{1+\sqrt{5}}{2}$, $\alpha^2 n = \alpha n + n$ for all $n \in \mathbb{N}$. We shall show that $f(n) = \left[\alpha n + \frac{1}{2}\right]$ (the closest integer to αn) satisfies the requirements. Observe that f is strictly increasing and $f(1) = 2$. By the definition of f, $|f(n) - \alpha n| \le \frac{1}{2}$ and $f(f(n)) - f(n) - n$ is an integer. On the other hand,

$$\begin{aligned}|f(f(n)) - f(n) - n| &= |f(f(n)) - f(n) - \alpha^2 n + \alpha n| \\ &= |f(f(n)) - \alpha f(n) + \alpha f(n) - \alpha^2 n - f(n) + \alpha n| \\ &= |(\alpha - 1)(f(n) - \alpha n) + (f(f(n)) - \alpha f(n))| \\ &\le (\alpha - 1)|f(n) - \alpha n| + |f(f(n)) - \alpha f(n)| \\ &\le \frac{1}{2}(\alpha - 1) + \frac{1}{2} = \frac{1}{2}\alpha < 1,\end{aligned}$$

which implies that $f(f(n)) - f(n) - n = 0$.

7. Multiplying by a and c the equation

$$ax^2 + 2bxy + cy^2 = P^k n, \tag{1}$$

gives $(ax + by)^2 + Py^2 = aP^k n$ and $(bx + cy)^2 + Px^2 = cP^k n$.
It follows immediately that $M(n)$ is finite; moreover, $(ax+by)^2$ and $(bx+cy)^2$ are divisible by P, and consequently $ax + by$, $bx + cy$ are divisible by P because P is not divisible by a square greater than 1. Thus there exist integers X, Y such that $bx + cy = PX$, $ax + by = -PY$. Then $x = -bX - cY$ and $y = aX + bY$. Introducing these values into (1) and simplifying the expression obtained we get

$$aX^2 + 2bXY + cY^2 = P^{k-1} n. \tag{2}$$

Hence $(x, y) \mapsto (X, Y)$ is a bijective correspondence between integral solutions of (1) and (2), so that $M(P^k n) = M(P^{k-1} n) = \cdots = M(n)$.

8. Suppose that $f(n) = 1$ for some $n > 0$. Then $f(n + 1) = n + 2$, $f(n + 2) = 2n + 4$, $f(n + 3) = n + 1$, $f(n + 4) = 2n + 5$, $f(n + 5) = n$, and so by induction $f(n + 2k) = 2n + 3 + k$, $f(n + 2k - 1) = n + 3 - k$ for $k = 1, 2, \dots, n + 2$. Particularly, $n' = 3n + 3$ is the smallest value greater than n for which $f(n') = 1$. It follows that all numbers n with $f(n) = 1$ are given by $n = b_i$, where $b_0 = 1$, $b_n = 3b_{n-1} + 3$. Furthermore, $b_n = 3 + 3b_{n-1} = 3 + 3^2 + 3^2 b_{n-2} = \cdots = 3 + 3^2 + \cdots + 3^n + 3^n = \frac{1}{2}(5 \cdot 3^n - 3)$.
It is seen from above that if $n \le b_i$, then $f(n) \le f(b_i - 1) = b_i + 1$. Hence if $f(n) = 1993$, then $n \ge b_i \ge 1992$ for some i. The smallest such b_i is

$b_7 = 5466$, and $f(b_i + 2k - 1) = b_i + 3 - k = 1993$ implies $k = 3476$. Thus the least integer in S is $n_1 = 5466 + 2 \cdot 3476 - 1 = 12417$.
All the elements of S are given by $n_i = b_{i+6} + 2k - 1$, where $b_{i+6} + 3 - k = 1993$, i.e., $k = b_{i+6} - 1990$. Therefore $n_i = 3b_{i+6} - 3981 = \frac{1}{2}(5 \cdot 3^{i+7} - 7971)$. Clearly S is infinite and $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 3$.

9. We shall first complete the "multiplication table" for the sets A, B, C. It is clear that this multiplication is commutative and associative, so that we have the following relations:

$$\begin{aligned}&AC = (AB)B = BB = C;\\&A^2 = AA = (AB)C = BC = A;\\&C^2 = CC = B(BC) = BA = B.\end{aligned}$$

(a) Now put 1 in A and distribute the primes arbitrarily in A, B, C. This distribution uniquely determines the partition of $\mathbb{Q}^+$ with the stated property. Indeed, if an arbitrary rational number

$$x = p_1^{\alpha_1} \cdots p_k^{\alpha_k} q_1^{\beta_1} \cdots q_l^{\beta_l} r_1^{\gamma_1} \cdots r_m^{\gamma_m}$$

is given, where $p_i \in A$, $q_i \in B$, $r_i \in C$ are primes, it is easy to see that x belongs to A, B, or C according as $\beta_1 + \cdots + \beta_l + 2\gamma_1 + \cdots + 2\gamma_m$ is congruent to 0, 1, or 2 (mod 3).

(b) In every such partition, cubes all belong to A. In fact, $A^3 = A^2A = AA = A$, $B^3 = B^2B = CB = A$, $C^3 = C^2C = BC = A$.

(c) By (b) we have $1, 8, 27 \in A$. Then $2 \notin A$, and since the problem is symmetric with respect to B, C, we can assume $2 \in B$ and consequently $4 \in C$. Also $7 \notin A$, and also $7 \notin B$ (otherwise, $28 = 4 \cdot 7 \in A$ and $27 \in A$), so $7 \in C$, $14 \in A$, $28 \in B$. Further, we see that $3 \notin A$ (since otherwise $9 \in A$ and $8 \in A$). Put 3 in C. Then $5 \notin B$ (otherwise $15 \in A$ and $14 \in A$), so let $5 \in C$ too. Consequently $6, 10 \in A$. Also $13 \notin A$, and $13 \notin C$ because $26 \notin A$, so $13 \in B$. Now it is easy to distribute the remaining primes $11, 17, 19, 23, 29, 31$: one possibility is

$$\begin{aligned}A &= \{1, 6, 8, 10, 14, 19, 23, 27, 29, 31, 33, \dots\},\\C &= \{3, 4, 5, 7, 18, 22, 24, 26, 30, 32, 34, \dots\},\\B &= \{2, 9, 11, 12, 13, 15, 16, 17, 20, 21, 25, 28, 35, \dots\}.\end{aligned}$$

Remark. It can be proved that $\min\{n \in \mathbb{N} \mid n \in A, n+1 \in A\} \le 77$.

10. (a) Let $n = p$ be a prime and let $p \mid a^p - 1$. By Fermat's theorem $p \mid a^{p-1} - 1$, so that $p \mid a^{\gcd(p,p-1)} - 1 = a - 1$, i.e., $a \equiv 1 \pmod p$. Since then $a^i \equiv 1 \pmod p$, we obtain $p \mid a^{p-1} + \cdots + a + 1$ and hence $p^2 \mid a^p - 1 = (a-1)(a^{p-1} + \cdots + a + 1)$.

(b) Let $n = p_1 \cdots p_k$ be a product of distinct primes and let $n \mid a^n - 1$. Then from $p_i \mid a^n - 1 = (a^{(n/p_i)})^{p_i} - 1$ and part (a) we conclude that $p_i^2 \mid a^n - 1$. Since this is true for all indices i, we also have $n^2 \mid a^n - 1$; hence n has the property P.

11. Due to the extended Eisenstein criterion, f must have an irreducible factor of degree not less than $n-1$. Since f has no integral zeros, it must be irreducible.

Second solution. The proposer's solution was as follows. Suppose that $f(x) = g(x)h(x)$, where g, h are nonconstant polynomials with integer coefficients. Since $|f(0)| = 3$, either $|g(0)| = 1$ or $|h(0)| = 1$. We may assume $|g(0)| = 1$ and that $g(x) = (x-\alpha_1)\cdots(x-\alpha_k)$. Then $|\alpha_1 \cdots \alpha_k| = 1$. Since $\alpha_i^{n-1}(\alpha_i + 5) = -3$, taking the product over $i = 1, 2, \dots, k$ yields $|(\alpha_1 + 5)\cdots(\alpha_k + 5)| = |g(-5)| = 3^k$. But $f(-5) = g(-5)h(-5) = 3$, so the only possibility is $\deg g = k = 1$. This is impossible, because f has no integral zeros.

Remark. Generalizing this solution, it can be shown that if a, m, n are positive integers and $p < a-1$ is a prime, then $F(x) = x^m(x-a)^n + p$ is irreducible. The details are left to the reader.

12. Let $x_1 < x_2 < \cdots < x_n$ be the elements of S. We use induction on n. The result is trivial for $k = 1$ or $n = k$, so assume that it is true for $n-1$ numbers. Then there exist $m = (k-1)(n-k)+1$ distinct sums of $k-1$ numbers among $x_2, \dots, x_n$; call these sums S_i, $S_1 < S_2 < \cdots < S_m$. Then $x_1 + S_1, x_1 + S_2, \dots, x_1 + S_m$ are distinct sums of k of the numbers $x_1, x_2, \dots, x_n$. However, the biggest of these sums is

$$x_1 + S_m \le x_1 + x_{n-k+2} + x_{n-k+3} + \cdots + x_n;$$

hence we can find $n-k$ sums that are greater and thus not included here: $x_2+x_{n-k+2}+\cdots+x_n, x_3+x_{n-k+2}+\cdots+x_n, \dots, x_{n-k+1}+x_{n-k+2}+\cdots+x_n$. This counts for $k(n-k)+1$ sums in total.

Remark. Equality occurs if S is an arithmetic progression.

13. For an odd integer $N > 1$, let $S_N = \{(m,n) \in S \mid m+n = N\}$. If $f(m,n) = (m_1, n_1)$, then $m_1 + n_1 = m + n$ with m_1 odd and $m_1 \le \frac{n}{2} < \frac{N}{2} < n_1$, so f maps S_N to S_N. Also f is bijective, since if $f(m,n) = (m_1, n_1)$, then n is uniquely determined as the even number of the form $2^k m_1$ that belongs to the interval $[\frac{N+1}{2}, N]$, and this also determines m.
Note that S_N has at most $\left[\frac{N+1}{4}\right]$ elements, with equality if and only if N is prime. Thus if $(m,n) \in S_N$, there exist s, r with $1 \le s < r \le \left[\frac{N+5}{4}\right]$ such that $f^s(m,n) = f^r(m,n)$. Consequently $f^t(m,n) = (m,n)$, where $t = r-s$, $0 < t \le \left[\frac{N+1}{4}\right] = \left[\frac{m+n+1}{4}\right]$.
Suppose that $(m,n) \in S_N$ and t is the least positive integer with $f^t(m,n) = (m,n)$. We write $(m,n) = (m_0, n_0)$ and $f^i(m,n) = (m_i, n_i)$ for $i = 1, \dots, t$. Then there exist positive integers a_i such that $2^{a_i} m_i = n_{i-1}$, $i = 1, \dots, t$. Since $m_t = m_0$, multiplying these equalities gives

$$\begin{aligned} 2^{a_1+a_2+\cdots+a_t} m_0 m_1 \cdots m_{t-1} &= n_0 n_1 \cdots n_{t-1} \\ &\equiv (-1)^t m_0 m_1 \cdots m_{t-1} \pmod{N}. \end{aligned} \tag{1}$$

It follows that $N \mid 2^k \pm 1$ and consequently $N \mid 2^{2k} - 1$, where $k = a_1 + \cdots + a_t$. On the other hand, it also follows that $2^k \mid n_0 n_1 \cdots n_{t-1} \mid (N-1)(N-3)\cdots(N-2[N/4])$. But since

$$\frac{(N-1)(N-3)\cdots\left(N-2\left[\frac{N}{4}\right]\right)}{1\cdot 3\cdots\left(2\left[\frac{N-2}{4}\right]+1\right)} = \frac{2\cdot 4\cdots(N-1)}{1\cdot 2\cdots\frac{N-1}{2}} = 2^{\frac{N-1}{2}},$$

we conclude that $0 < k \le \frac{N-1}{2}$, where equality holds if and only if $\{n_1, \dots, n_t\}$ is the set of all even integers from $\frac{N+1}{2}$ to $N-1$, and consequently $t = \frac{N+1}{4}$.
Now if $N \nmid 2^h - 1$ for $1 \le h < N-1$, we must have $2k = N-1$. Therefore $t = \frac{N+1}{4}$.

14. Consider any point T inside the triangle ABC or on its boundary. Since

$$\begin{aligned} 2S &= 2(S_{AETF} + S_{BFTD} + S_{CDTE}) \\ &\le AT\cdot EF + BT\cdot FD + CT\cdot DE = (AT+BT+CT)DE, \end{aligned}$$

it suffices to find a point T such that

$$(AT+BT+CT)^2 \ge \frac{a^2+b^2+c^2+4S\sqrt{3}}{2}.$$

We distinguish two cases:

(i) If all angles of $\triangle ABC$ are less than 120°, then the sum $AT+BT+CT$ attains its minimum when T is the Torricelli point, i.e., such that $\angle ATB = \angle BTC = \angle CTA = 120^\circ$. In this case, by the cosine theorem we get

$$\begin{aligned} AT^2 + AT\cdot BT + BT^2 &= c^2, \\ BT^2 + BT\cdot CT + CT^2 &= a^2, \\ CT^2 + CT\cdot AT + AT^2 &= b^2, \\ 3(AT\cdot BT + BT\cdot CT + CT\cdot AT) &= 4\sqrt{3}(S_{ATB} + S_{BTC} + S_{CTA}) \\ &= 4\sqrt{3}S. \end{aligned}$$

Adding these four equalities, we obtain $2(AT+BT+CT)^2 = a^2 + b^2 + c^2 + 4\sqrt{3}S$.

(ii) Let $\angle ACB \ge 120^\circ$. We claim that $T = C$ satisfies the requirements. Indeed, $a^2 + b^2 + c^2 + 4\sqrt{3}S = a^2 + b^2 + (a^2 + b^2 - 2ab\cos\angle C) + 2\sqrt{3}ab\sin\angle C = 2(a^2+b^2) + 2ab(\sqrt{3}\sin\angle C - \cos\angle C) = 2(a^2+b^2) + 4ab\sin(\angle C - 30^\circ) \le 2(a+b)^2$, which proves the desired inequality.

15. Denote by $d(PQR)$ the diameter of a triangle PQR. It is clear that $d(PQR)\cdot m(PQR) = 2S_{PQR}$. So if the point X lies inside the triangle ABC or on its boundary, we have $d(ABX), d(BCX), d(CAX) \le d(ABC)$, which implies

$$\begin{aligned} m(ABX)+m(BCX)+m(CAX) &= \frac{2S_{ABX}}{d(ABX)}+\frac{2S_{BCX}}{d(BCX)}+\frac{2S_{CAX}}{d(CAX)} \\ &\geq \frac{2S_{ABX}+2S_{BCX}+2S_{CAX}}{d(ABC)} \\ &= \frac{2S_{ABC}}{d(ABC)} = m(ABC). \end{aligned}$$

If X is outside $\triangle ABC$ but inside the angle BAC, consider the point Y of intersection of AX and BC. Then $m(ABX)+m(BCX)+m(CAX) \geq m(ABY)+m(BCY)+m(CAY) \geq m(ABC)$. Also, if X is inside the opposite angle of $\angle BAC$ (i.e., $\angle DAE$, where $\mathcal{B}(D,A,B)$ and $\mathcal{B}(E,A,C)$), then $m(ABX)+m(BCX)+m(CAX) \geq m(BCX) \geq m(ABC)$. Since these are substantially all possible different positions of point X, we have finished the proof.

16. Let $S_n = \{A = (a_1,\dots,a_n) \mid 0 \leq a_i < i\}$. For $A = (a_1,\dots,a_n)$, let $A' = (a_1,\dots,a_{n-1})$, so that we can write $A = (A', a_n)$. The proof of the statement from the problem will be given by induction on n. For $n = 2$ there are two possibilities for A_0, so one directly checks that $A_2 = A_0$. Now assume that $n \geq 3$ and that $A_0 = (A_0', a_{0n}) \in S_n$. It is clear that then any A_i is in S_n too. By the induction hypothesis there exists $k \in \mathbb{N}$ such that $A_k' = A_{k+2}' = A_{k+4}' = \cdots$ and $A_{k+1}' = A_{k+3}' = \cdots$. Observe that if we increase (decrease) a_{kn}, $a_{k+1,n}$ will decrease (respectively increase), and this will also increase (respectively decrease) $a_{k+2,n}$. Hence $a_{kn}, a_{k+2,n}, a_{k+4,n}, \dots$ is monotonically increasing or decreasing, and since it is bounded (by 0 and $n-1$), it follows that we will eventually have $a_{k+2i,n} = a_{k+2i+2,n} = \cdots$. Consequently $A_{k+2i} = A_{k+2i+2}$.

17. We introduce the rotation operation *Rot* to the left by one, so that $\mathit{Step}_j = \mathit{Rot}^{-j} \circ \mathit{Step}_0 \circ \mathit{Rot}^j$. Now writing $\mathit{Step}^* = \mathit{Rot} \circ \mathit{Step}_0$, the problem is transformed into the question whether there is an $M(n)$ such that all lamps are *ON* again after $M(n)$ successive applications of Step^*.
We operate in the field $\mathbb{Z}_2$, representing *OFF* by 0 and *ON* by 1. So if the status of L_j at some moment is given by $v_j \in \mathbb{Z}_2$, the effect of Step_j is that v_j is replaced by $v_j + v_{j-1}$. With the n-tuple $v_0,\dots,v_{n-1}$ we associate the polynomial

$$P(x) = v_{n-1}x^{n-1} + v_0x^{n-2} + v_1x^{n-3} + \cdots + v_{n-2}.$$

By means of Step^*, this polynomial is transformed into the polynomial $Q(x)$ over $\mathbb{Z}$ of degree less than n that satisfies $Q(x) \equiv xP(x)$ (mod $x^n + x^{n-1} + 1$). From now on, the sign $\equiv$ always stands for congruence with this modulus.

(i) It suffices to show the existence of $M(n)$ with $x^{M(n)} \equiv 1$. Because the number of residue classes is finite, there are r, q, $r < q$ such that $x^q \equiv x^r$, i.e., $x^r(x^{q-r} - 1) = 0$. One can take $M(n) = q - r$. (Or simply note that there are only finitely many possible configurations;

since each operation is bijective, the configuration that reappears first must be $ON, ON, \dots, ON$.)

(ii) We shall prove that if $n = 2^k$, then $x^{n^2-1} \equiv 1$. We have $x^{n^2} \equiv (x^{n-1}+1)^n \equiv x^{n^2-n}+1$, because all binomial coefficients of order $n = 2^k$ are even, apart from the first one and the last one. Since also $x^{n^2} \equiv x^{n^2-1} + x^{n^2-n}$, this is what we wanted.

(iii) Now if $n = 2^k+1$, we prove that $x^{n^2-n+1} \equiv 1$. We have $x^{n^2-1} \equiv (x^{n+1})^{n-1} \equiv (x+x^n)^{n-1} \equiv x^{n-1} + x^{n^2-n}$ (again by evenness of binomial coefficients of order $n-1 = 2^k$). Together with $x^{n^2} \equiv x^{n^2-1} + x^{n^2-n}$, this leads to $x^{n^2} \equiv x^{n-1}$.

18. Let B_n be the set of sequences with the stated property ($S_n = |B_n|$). We shall prove by induction on n that $S_n \ge \frac{3}{2} S_{n-1}$ for every n.
Suppose that for every $i \le n$, $S_i \ge \frac{3}{2} S_{i-1}$, and consequently $S_i \le \left(\frac{2}{3}\right)^{n-i} S_n$. Let us consider the $2S_n$ sequences obtained by putting 0 or 1 at the end of any sequence from B_n. If some sequence among them does not belong to B_{n+1}, then for some $k \ge 1$ it can be obtained by extending some sequence from B_{n+1-6k} by a sequence of k terms repeated six times. The number of such sequences is $2^k S_{n+1-6k}$. Hence the number of sequences not satisfying our condition is not greater than

$$\sum_{k \ge 1} 2^k S_{n+1-6k} \le \sum_{k \ge 1} 2^k \left(\frac{2}{3}\right)^{6k-1} S_n = \frac{3}{2} S_n \frac{2(2/3)^6}{1-2(2/3)^6} = \frac{192}{601} S_n < \frac{1}{2} S_n.$$

Therefore S_{n+1} is not smaller than $2S_n - \frac{1}{2} S_n = \frac{3}{2} S_n$. Thus we have $S_n \ge \left(\frac{3}{2}\right)^n$.

19. Let s be the minimum number of nonzero digits that can appear in the b-adic representation of any number divisible by $b^n - 1$. Among all numbers divisible by $b^n - 1$ and having s nonzero digits in base b, we choose the number A with the minimum sum of digits. Let $A = a_1 b^{n_1} + \dots + a_s b^{n_s}$, where $0 < a_i \le b-1$ and $n_1 > n_2 > \dots > n_s$.
First, suppose that $n_i \equiv n_j \pmod{n}$, $i \ne j$. Consider the number

$$B = A - a_i b^{n_i} - a_j b^{n_j} + (a_i + a_j) b^{n_j + kn},$$

with k chosen large enough so that $n_j + kn > n_1$: this number is divisible by $b^n - 1$ as well. But if $a_i + a_j < b$, then B has $s-1$ digits in base b, which is impossible; on the other hand, $a_i + a_j \ge b$ is also impossible, for otherwise B would have sum of digits less for $b-1$ than that of A (because B would have digits 1 and $a_i + a_j - b$ in the positions $n_j + kn + 1$, $n_j + kn$). Therefore $n_i \not\equiv n_j$ if $i \ne j$.
Let $n_i \equiv r_i$, where $r_i \in \{0, 1, \dots, n-1\}$ are distinct. The number $C = a_1 b^{r_1} + \dots + a_s b^{r_s}$ also has s digits and is divisible by $b^n - 1$. But since $C < b^n$, the only possibility is $C = b^n - 1$ which has exactly n digits in base b. It follows that $s = n$.

20. For every real x we shall denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the greatest integer less than or equal to x and the smallest integer greater than or equal to x respectively. The condition $c_i + nk_i \in [1-n, n]$ is equivalent to $k_i \in I_i = \left[\frac{1-c_i}{n} - 1, 1 - \frac{c_i}{n}\right]$. For every c_i, this interval contains two integers (not necessarily distinct), namely $p_i = \left\lceil \frac{1-c_i}{n} - 1 \right\rceil \le q_i = \left\lfloor 1 - \frac{c_i}{n} \right\rfloor$. In order to show that there exist integers $k_i \in I_i$ with $\sum_{i=1}^n k_i = 0$, it is sufficient to show that $\sum_{i=1}^n p_i \le 0 \le \sum_{i=1}^n q_i$.
Since $p_i < \frac{1-c_i}{n}$, we have

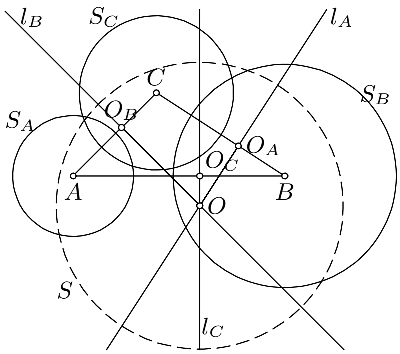
$$\sum_{i=1}^n p_i < 1 - \sum_{i=1}^n \frac{c_i}{n} \le 1,$$

and consequently $\sum_{i=1}^n p_i \le 0$ because the p_i's are integers. On the other hand, $q_i > -\frac{c_i}{n}$ implies

$$\sum_{i=1}^n q_i > -\sum_{i=1}^n \frac{c_i}{n} \ge -1,$$

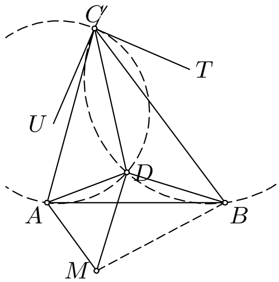
which leads to $\sum_{i=1}^n q_i \ge 0$. The proof is complete.

21. Assume that S is a circle with center O that cuts S_i diametrically in points P_i, Q_i, $i \in \{A, B, C\}$, and denote by r_i, r the radii of S_i and S respectively. Since OA is perpendicular to $P_A Q_A$, it follows by Pythagoras's theorem that $OA^2 + AP_A^2 = OP_A^2$, i.e., $r_A^2 + OA^2 = r^2$. Analogously $r_B^2 + OB^2 = r^2$ and $r_C^2 + OC^2 = r^2$. Thus if O_A, O_B, O_C are the feet of perpendiculars from O to BC, CA, AB respectively, then $O_C A^2 - O_C B^2 = r_B^2 - r_A^2$. Since the left-hand side is a monotonic function of $O_C \in AB$, the point O_C is uniquely determined by the imposed conditions. The same holds for O_A and O_B.



If A, B, C are not collinear, then the positions of O_A, O_B, O_C uniquely determine the point O, and therefore the circle S also. On the other hand, if A, B, C are collinear, all one can deduce is that O lies on the lines l_A, l_B, l_C through O_A, O_B, O_C, perpendicular to BC, CA, AB respectively. By this, l_A, l_B, l_C are parallel, so O can be either anywhere on the line if these lines coincide, or nowhere if they don't coincide. So if there exists more than one circle S, A, B, C lie on a line and the foot O' of the perpendicular from O to the line ABC is fixed. If X, Y are the intersection points of S and the line ABC, then $r^2 = OX^2 = OA^2 + r_A^2$ and consequently $O'X^2 = O'A^2 + r_A^2$, which implies that X, Y are fixed.

22. Let M be the point inside $\angle ADB$ that satisfies $DM = DB$ and $DM \perp$



DB. Then $\angle ADM = \angle ACB$ and $AD/DM = AC/CB$. It follows that the triangles ADM, ACB are similar; hence $\angle CAD = \angle BAM$ (because $\angle CAB = \angle DAM$) and $AB/AM = AC/AD$. Consequently the triangles CAD, BAM are similar and therefore $\frac{AC}{AB} = \frac{CD}{BM} = \frac{CD}{\sqrt{2}BD}$. Hence $\frac{AB \cdot CD}{AC \cdot BD} = \sqrt{2}$.

Let CT, CU be the tangents at C to the circles ACD, BCD respectively. Then (in oriented angles) $\angle TCU = \angle TCD + \angle DCU = \angle CAD + \angle CBD = 90°$, as required.

Second solution to the first part. Denote by E, F, G the feet of the perpendiculars from D to BC, CA, AB. Consider the pedal triangle EFG. Since $FG = AD \sin \angle A$, from the sine theorem we have $FG : GE : EF = (CD \cdot AB) : (BD \cdot AC) : (AD \cdot BC)$. Thus $EG = FG$. On the other hand, $\angle EGF = \angle EGD + \angle DGF = \angle CBD + \angle CAD = 90°$ implies that $EF : EG = \sqrt{2} : 1$; hence the required ratio is $\sqrt{2}$.

Third solution to the first part. Under inversion centered at C and with power $r^2 = CA \cdot CB$, the triangle DAB maps into a right-angled isosceles triangle $D^*A^*B^*$, where

$$D^*A^* = \frac{AD \cdot BC}{CD}, \ D^*B^* = \frac{AC \cdot BD}{CD}, \ A^*B^* = \frac{AB \cdot CD}{CD}.$$

Thus $D^*B^* : A^*B^* = \sqrt{2}$, and this is the required ratio.

23. Let the given numbers be $a_1, \dots, a_n$. Put $s = a_1 + \cdots + a_n$ and $m = \operatorname{lcm}(a_1, \dots, a_n)$ and write $m = 2^k r$ with $k \geq 0$ and r odd. Let the binary expansion of r be $r = 2^{k_0} + 2^{k_1} + \cdots + 2^{k_t}$, with $0 = k_0 < \cdots < k_t$. Adjoin to the set $\{a_1, \dots, a_n\}$ the numbers $2^{k_i}s$, $i = 1, 2, \dots, t$. The sum of the enlarged set is rs. Finally, adjoin $rs, 2rs, 2^2rs, \dots, 2^{l-1}rs$ for $l = \max\{k, k_t\}$. The resulting set has sum $2^l rs$, which is divisible by m and so by each of a_j, and also by the $2^i s$ above and by $rs, 2rs, \dots, 2^{l-1}rs$. Therefore this is a DS-set.

Second solution. We show by induction that there is a DS-set containing 1 and n. For $n = 2, 3$, take $\{1, 2, 3\}$. Assume that $\{1, n, b_1, \dots, b_k\}$ is a DS-set. Then $\{1, n+1, n, 2(n+1)n, 2(n+1)b_1, \dots, 2(n+1)b_k\}$ is a DS-set too.
For given $a_1, \dots, a_n$ let m be a sufficiently large common multiple of the a_i's such that $u = m - (a_1 + \cdots + a_n) \neq a_i$ for all i. There exist $b_1, \dots, b_k$ such that $\{1, u, b_1, \dots, b_k\}$ is a DS-set. It is clear that $\{a_1, \dots, a_n, u, mu, mb_1, \dots, mb_k\}$ is a DS-set containing $a_1, \dots, a_n$.

24. By the Cauchy–Schwarz inequality, if $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ are positive numbers, then

$$\left(\sum_{i=1}^{n}\frac{x_i}{y_i}\right)\left(\sum_{i=1}^{n}x_iy_i\right)\ge\left(\sum_{i=1}^{n}x_i\right)^2.$$

Applying this to the numbers a,b,c,d and $b+2c+3d, c+2d+3a, d+2a+3b, a+2b+3c$ (here $n=4$), we obtain

$$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c}$$
$$\ge\frac{(a+b+c+d)^2}{4(ab+ac+ad+bc+bd+cd)}\ge\frac{2}{3}.$$

The last inequality follows, for example, from $(a-b)^2+(a-c)^2+\cdots+(c-d)^2\ge 0$. Equality holds if and only if $a=b=c=d$.

Second solution. Putting $A=b+2c+3d$, $B=c+2d+3a$, $C=d+2a+3b$, $D=a+2b+3c$, our inequality transforms into

$$\frac{-5A+7B+C+D}{24A}+\frac{-5B+7C+D+A}{24B}$$
$$+\frac{-5C+7D+A+B}{24C}+\frac{-5D+7A+B+C}{24D}\ge\frac{2}{3}.$$

This follows from the arithmetic-geometric mean inequality, since $\frac{B}{A}+\frac{C}{B}+\frac{D}{C}+\frac{A}{D}\ge 4$, etc.

25. We need only consider the case $a>1$ (since the case $a<-1$ is reduced to $a>1$ by taking $a'=-a$, $x_i'=-x_i$). Since the left sides of the equations are nonnegative, we have $x_i\ge-\frac{1}{a}>-1$, $i=1,\dots,1000$. Suppose w.l.o.g. that $x_1=\max\{x_i\}$. In particular, $x_1\ge x_2,x_3$. If $x_1\ge 0$, then we deduce that $x_{1000}^2\ge 1\Rightarrow x_{1000}\ge 1$; further, from this we deduce that $x_{999}>1$ etc., so either $x_i>1$ for all i or $x_i<0$ for all i.
   (i) $x_i>1$ for every i. Then $x_1\ge x_2$ implies $x_1^2\ge x_2^2$, so $x_2\ge x_3$. Thus $x_1\ge x_2\ge\cdots\ge x_{1000}\ge x_1$, and consequently $x_1=\cdots=x_{1000}$. In this case the only solution is $x_i=\frac{1}{2}(a+\sqrt{a^2+4})$ for all i.
   (ii) $x_i<0$ for every i. Then $x_1\ge x_3$ implies $x_1^2\le x_3^2\Rightarrow x_2\le x_4$. Similarly, this leads to $x_3\ge x_5$, etc. Hence $x_1\ge x_3\ge x_5\ge\cdots\ge x_{999}\ge x_1$ and $x_2\le x_4\le\cdots\le x_2$, so we deduce that $x_1=x_3=\cdots$ and $x_2=x_4=\cdots$. Therefore the system is reduced to $x_1^2=ax_2+1$, $x_2^2=ax_1+1$. Subtracting these equations, one obtains $(x_1-x_2)(x_1+x_2+a)=0$. There are two possibilities:
      (1) If $x_1=x_2$, then $x_1=x_2=\cdots=\frac{1}{2}(a-\sqrt{a^2+4})$.
      (2) $x_1+x_2+a=0$ is equivalent to $x_1^2+ax_1+(a^2-1)=0$. The discriminant of the last equation is $4-3a^2$. Therefore if $a>\frac{2}{\sqrt{3}}$, this case yields no solutions, while if $a\le\frac{2}{\sqrt{3}}$, we obtain $x_1=\frac{1}{2}(-a-\sqrt{4-3a^2})$, $x_2=\frac{1}{2}(-a+\sqrt{4-3a^2})$, or vice versa.

26. Set

$$f(a,b,c,d) = abc + bcd + cda + dab - \frac{176}{27}abcd$$
$$= ab(c+d) + cd\left(a + b - \frac{176}{27}ab\right).$$

If $a + b - \frac{176}{a}b \leq 0$, by the arithmetic-geometric inequality we have $f(a,b,c,d) \leq ab(c+d) \leq \frac{1}{27}$.
On the other hand, if $a + b - \frac{176}{a}b > 0$, the value of f increases if c, d are replaced by $\frac{c+d}{2}, \frac{c+d}{2}$. Consider now the following fourtuplets:

$$P_0(a,b,c,d),\ P_1\left(a, b, \frac{c+d}{2}, \frac{c+d}{2}\right),\ P_2\left(\frac{a+b}{2}, \frac{a+b}{2}, \frac{c+d}{2}, \frac{c+d}{2}\right),$$
$$P_3\left(\frac{1}{4}, \frac{a+b}{2}, \frac{c+d}{2}, \frac{1}{4}\right),\ P_4\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

From the above considerations we deduce that for $i = 0, 1, 2, 3$ either $f(P_i) \leq f(P_{i+1})$, or directly $f(P_i) \leq 1/27$. Since $f(P_4) = 1/27$, in every case we are led to

$$f(a,b,c,d) = f(P_0) \leq \frac{1}{27}.$$

Equality occurs only in the cases $(0, 1/3, 1/3, 1/3)$ (with permutations) and $(1/4, 1/4, 1/4, 1/4)$.

Remark. Lagrange multipliers also work. On the boundary of the set one of the numbers a, b, c, d is 0, and the inequality immediately follows, while for an extremum point in the interior, among a, b, c, d there are at most two distinct values, in which case one easily verifies the inequality.

4.35 Solutions to the Shortlisted Problems of IMO 1994

1. Obviously $a_0 > a_1 > a_2 > \cdots$. Since $a_k - a_{k+1} = 1 - \frac{1}{a_k+1}$, we have $a_n = a_0 + (a_1 - a_0) + \cdots + (a_n - a_{n-1}) = 1994 - n + \frac{1}{a_0+1} + \cdots + \frac{1}{a_{n-1}+1} > 1994 - n$. Also, for $1 \le n \le 998$,
$$\frac{1}{a_0+1} + \cdots + \frac{1}{a_{n-1}+1} < \frac{n}{a_{n-1}+1} < \frac{998}{a_{997}+1} < 1$$
because as above, $a_{997} > 997$. Hence $\lfloor a_n \rfloor = 1994 - n$.

2. We may assume that $a_1 > a_2 > \cdots > a_m$. We claim that for $i = 1, \dots, m$, $a_i + a_{m+1-i} \ge n+1$. Indeed, otherwise $a_i + a_{m+1-i}, \dots, a_i + a_{m-1}$, $a_i + a_m$ are i different elements of A greater than a_i, which is impossible. Now by adding for $i = 1, \dots, m$ we obtain $2(a_1 + \cdots + a_m) \ge m(n+1)$, and the result follows.

3. The last condition implies that $f(x) = x$ has at most one solution in $(-1, 0)$ and at most one solution in $(0, \infty)$. Suppose that for $u \in (-1, 0)$, $f(u) = u$. Then putting $x = y = u$ in the given functional equation yields $f(u^2 + 2u) = u^2 + 2u$. Since $u \in (-1, 0) \Rightarrow u^2 + 2u \in (-1, 0)$, we deduce that $u^2 + 2u = u$, i.e., $u = -1$ or $u = 0$, which is impossible. Similarly, if $f(v) = v$ for $v \in (0, \infty)$, we are led to the same contradiction.
However, for all $x \in S$, $f(x + (1+x)f(x)) = x + (1+x)f(x)$, so we must have $x + (1+x)f(x) = 0$. Therefore $f(x) = -\frac{x}{1+x}$ for all $x \in S$. It is directly verified that this function satisfies all the conditions.

4. Suppose that $\alpha = \beta$. The given functional equation for $x = y$ yields $f(x/2) = x^{-\alpha} f(x)^2/2$; hence the functional equation can be written as
$$f(x)f(y) = \frac{1}{2} x^\alpha y^{-\alpha} f(y)^2 + \frac{1}{2} y^\alpha x^{-\alpha} f(x)^2,$$
i.e.,
$$\left((x/y)^{\alpha/2} f(y) - (y/x)^{\alpha/2} f(x) \right)^2 = 0.$$
Hence $f(x)/x^\alpha = f(y)/y^\alpha$ for all $x, y \in \mathbb{R}^+$, so $f(x) = \lambda x^\alpha$ for some λ. Substituting into the functional equation we obtain that $\lambda = 2^{1-\alpha}$ or $\lambda = 0$. Thus either $f(x) \equiv 2^{1-\alpha} x^\alpha$ or $f(x) \equiv 0$.
Now let $\alpha \ne \beta$. Interchanging x with y in the given equation and subtracting these equalities from each other, we get $(x^\alpha - x^\beta) f(y/2) = (y^\alpha - y^\beta) f(x/2)$, so for some constant $\lambda \ge 0$ and all $x \ne 1$, $f(x/2) = \lambda(x^\alpha - x^\beta)$. Substituting this into the given equation, we obtain that only $\lambda = 0$ is possible, i.e., $f(x) \equiv 0$.

5. If $f^{(n)}(x) = \frac{p_n(x)}{q_n(x)}$ for some positive integer n and polynomials p_n, q_n, then
$$f^{(n+1)}(x) = f\left(\frac{p_n(x)}{q_n(x)} \right) = \frac{p_n(x)^2 + q_n(x)^2}{2 p_n(x) q_n(x)}.$$

Note that $f^{(0)}(x) = x/1$. Thus $f^{(n)}(x) = \frac{p_n(x)}{q_n(x)}$, where the sequence of polynomials p_n, q_n is defined recursively by

$$p_0(x) = x, \quad q_0(x) = 1, \quad \text{and}$$

$$p_{n+1}(x) = p_n(x)^2 + q_n(x)^2, \quad q_{n+1}(x) = 2p_n(x)q_n(x).$$

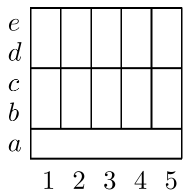
Furthermore, $p_0(x) \pm q_0(x) = x \pm 1$ and $p_{n+1}(x) \pm q_{n+1}(x) = p_n(x)^2 + q_n(x)^2 \pm 2p_n(x)q_n(x) = (p_n(x) \pm q_n(x))^2$, so $p_n(x) \pm q_n(x) = (x \pm 1)^{2^n}$ for all n. Hence

$$p_n(x) = \frac{(x+1)^{2^n} + (x-1)^{2^n}}{2} \quad \text{and} \quad q_n(x) = \frac{(x+1)^{2^n} - (x-1)^{2^n}}{2}.$$

Finally,

$$\frac{f^{(n)}(x)}{f^{(n+1)}(x)} = \frac{p_n(x)q_{n+1}(x)}{q_n(x)p_{n+1}(x)} = \frac{2p_n(x)^2}{p_{n+1}(x)} = \frac{((x+1)^{2^n} + (x-1)^{2^n})^2}{(x+1)^{2^{n+1}} + (x-1)^{2^{n+1}}}$$
$$= 1 + \frac{2\left(\frac{x+1}{x-1}\right)^{2^n}}{1 + \left(\frac{x+1}{x-1}\right)^{2^{n+1}}} = 1 + \frac{1}{f\left(\left(\frac{x+1}{x-1}\right)^{2^n}\right)}.$$

6. Call the first and second player M and N respectively. N can keep $A \le 6$. Indeed, let 10 dominoes be placed as shown in the picture, and whenever M marks a 1 in a cell of some domino, let N mark 0 in the other cell of that domino if it is still empty. Since any 3×3 square contains at least three complete dominoes, there are at least three 0's inside. Hence $A \le 6$.



We now show that M can make $A = 6$. Let him start by marking 1 in $c3$. By symmetry, we may assume that N's response is made in row 4 or 5. Then M marks 1 in $c2$. If N puts 0 in $c1$, then M can always mark two 1's in $b \times \{1, 2, 3\}$ as well as three 1's in $\{a, d\} \times \{1, 2, 3\}$. Thus either $\{a, b, c\} \times \{1, 2, 3\}$ or $\{b, c, d\} \times \{1, 2, 3\}$ will contain six 1's. However, if N does not play his second move in $c1$, then M plays there, and thus he can easily achieve to have six 1's either in $\{a, b, c\} \times \{1, 2, 3\}$ or $\{c, d, e\} \times \{1, 2, 3\}$.

7. Let $a_1, a_2, \dots, a_m$ be the ages of the male citizens ($m \ge 1$). We claim that the age of each female citizen can be expressed in the form $c_1a_1 + \cdots + c_ma_m$ for some constants $c_i \ge 0$, and we will prove this by induction on the number n of female citizens.
The claim is clear if $n = 1$. Suppose it holds for n and consider the case of $n + 1$ female citizens. Choose any of them, say A of age x who knows k

citizens (at least one male). By the induction hypothesis, the age of each of the other n females is expressible as $c_1a_1 + \cdots + c_ma_m + c_0x$, where $c_i \geq 0$ and $c_0 + c_1 + \cdots + c_m = 1$. Consequently, the sum of ages of the k citizens who know A is $kx = b_1a_1 + \cdots + b_ma_m + b_0x$ for some constants $b_i \geq 0$ with sum k. But A knows at least one male citizen (who does not contribute to the coefficient of x), so $b_0 \leq k-1$. Hence $x = \frac{b_1a_1+\cdots+b_ma_m}{k-b_0}$, and the claim follows.

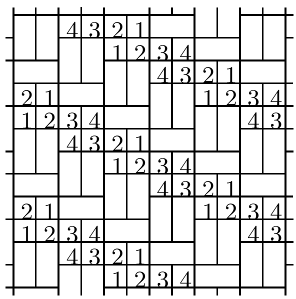
8. (a) Let a, b, c, $a \leq b \leq c$ be the amounts of money in dollars in Peter's first, second, and third account, respectively. If $a = 0$, then we are done, so suppose that $a > 0$. Let Peter make transfers of money into the first account as follows. Write $b = aq + r$ with $0 \leq r < a$ and let $q = m_0 + 2m_1 + \cdots + 2^k m_k$ be the binary representation of q ($m_i \in \{0, 1\}$, $m_k = 1$). In the ith transfer, $i = 1, 2, \ldots, k+1$, if $m_i = 1$ he transfers money from the second account, while if $m_i = 0$ he does so from the third. In this way he has transferred exactly $(m_0+2m_1+\cdots+2^k m_k)a$ dollars from the second account, thus leaving r dollars in it, $r < a$. Repeating this procedure, Peter can diminish the amount of money in the smallest account to zero, as required.
   (b) If Peter has an odd number of dollars, he clearly cannot transfer his money into one account.

9. (a) For $i = 1, \ldots, n$, let d_i be 0 if the card i is in the ith position, and 1 otherwise. Define $b = d_1 + 2d_2 + 2^2d_3 + \cdots + 2^{n-1}d_n$, so that $0 \leq b \leq 2^n - 1$, and $b = 0$ if and only if the game is over. After each move some digit d_l changes from 1 to 0 while $d_{l+1}, d_{l+2}, \ldots$ remain unchanged. Hence b decreases after each move, and consequently the game ends after at most $2^n - 1$ moves.
   (b) Suppose the game lasts exactly $2^n - 1$ moves. Then each move decreases b for exactly one, so playing the game in reverse (starting from the final configuration), every move is uniquely determined. It follows that if the configuration that allows a game lasting $2^n - 1$ moves exists, it must be unique.
   Consider the initial configuration $0, n, n-1, \ldots, 2, 1$. We prove by induction that the game will last exactly $2^n - 1$ moves, and that the card 0 will get to the 0th position only in the last move. This is trivial for $n = 1$, so suppose that the claim is true for some $n = m - 1 \geq 1$ and consider the case $n = m$. Obviously the card 0 does not move until the card m gets to the 0-th position. But if we ignore the card 0 and consider the card m to be the card 0, the induction hypothesis gives that the card m will move to the 0th position only after $2^{m-1} - 1$ moves. After these $2^{m-1} - 1$ moves, we come to the configuration $0, m-1, \ldots, 2, 1, m$. The next move yields $m, 0, m-1, \ldots, 2, 1$, so by the induction hypothesis again we need $2^{m-1} - 1$ moves more to finish the game.

10. (a) The case $n > 1994$ is trivial. Suppose that $n = 1994$. Label the girls G_1 to G_{1994}, and let G_1 initially hold all the cards. At any moment give to each card the value i, $i = 1, \dots, 1994$, if G_i holds it. Define the characteristic C of a position as the sum of all these values. Initially $C = 1994$. In each move, if G_i passes cards to G_{i-1} and G_{i+1} (where $G_0 = G_{1994}$ and $G_{1995} = G_1$), C changes for ± 1994 or does not change, so that it remains divisible by 1994. But if the game ends, the characteristic of the final position will be $C = 1 + 2 + \cdots + 1994 = 997 \cdot 1995$, which is not divisible by 1994.

    (b) Whenever a card is passed from one girl to another for the first time, let the girls sign their names on it. Thereafter, if one of them passes a card to her neighbor, we shall assume that the passed card is exactly the one signed by both of them. Thus each signed card is stuck between two neighboring girls, so if $n < 1994$, there are two neighbors who never exchange cards. Consequently, there is a girl G who played only a finite number of times. If her neighbor plays infinitely often, then after her last move, G will continue to accumulate cards indefinitely, which is impossible. Hence every girl plays finitely many times.

11. Tile the table with dominoes and numbers as shown in the picture. The second player will not lose if whenever the first player plays in a cell of a domino, he plays in the other cell of the domino, and whenever the first player plays on a number, he plays on the same number that is diagonally adjacent.



12. Define S_n recursively as follows: Let $S_2 = \{(0,0),(1,1)\}$ and $S_{n+1} = S_n \cup T_n$, where $T_n = \{(x + 2^{n-1}, y + M_n) \mid (x,y) \in S_n\}$, with M_n chosen large enough so that the entire set T_n lies above every line passing through two points of S_n. By definition, S_n has exactly 2^{n-1} points and contains no three collinear points. We claim that no $2n$ points of this set are the vertices of a convex $2n$-gon.

    Consider an arbitrary convex polygon $\mathcal{P}$ with vertices in S_n. Join by a diagonal d the two vertices of $\mathcal{P}$ having the smallest and greatest x-coordinates. This diagonal divides $\mathcal{P}$ into two convex polygons $\mathcal{P}_1, \mathcal{P}_2$, the former lying above d. We shall show by induction that both $\mathcal{P}_1, \mathcal{P}_2$ have at most n vertices. Assume to the contrary that $\mathcal{P}_1$ has at least $n+1$ vertices $A_1(x_1, y_1), \dots, A_{n+1}(x_{n+1}, y_{n+1})$ in S_n, with $x_1 < \cdots < x_{n+1}$. It follows that $\frac{y_2 - y_1}{x_2 - x_1} > \cdots > \frac{y_{n+1} - y_n}{x_{n+1} - x_n}$. By the induction hypothesis, not more than $n - 1$ of these vertices belong to S_{n-1} or T_{n-1}, so let $A_{k-1}, A_k \in S_{n-1}$, $A_{k+1} \in T_{n-1}$. But by the construction of T_{n-1}, $\frac{y_{k+1} - y_k}{x_{k+1} - x_k} > \frac{y_k - y_{k-1}}{x_k - x_{k-1}}$, which gives a contradiction. Similarly, $\mathcal{P}_2$ has no more than n vertices, and therefore $\mathcal{P}$ itself has at most $2n-2$ vertices.

13. Extend AD and BC to meet at P, and let Q be the foot of the perpendicular from P to AB. Denote by O the center of Γ. Since $\triangle PAQ \sim \triangle OAD$ and $\triangle PBQ \sim \triangle OBC$, we obtain $\frac{AQ}{AD} = \frac{PQ}{OD} = \frac{PQ}{OC} = \frac{BQ}{BC}$. Therefore $\frac{AQ}{QB} \cdot \frac{BC}{CP} \cdot \frac{PD}{DA} = 1$, so by the converse Ceva theorem, AC, BD, and PQ are concurrent. It follows that $Q \equiv F$. Finally, since the points O, C, P, D, F are concyclic, we have $\angle DFP = \angle DOP = \angle POC = \angle PFC$.

14. Although it does not seem to have been noticed at the jury, the statement of the problem is *false*. For $A(0,0), B(0,4), C(1,4), D(7,0)$, we have $M(4,2)$, $P(2,1)$, $Q(2,3)$ and $N(9/2, 1/2) \notin \triangle ABM$.
The official solution, if it can be called so, actually shows that N lies inside $ABCD$ and goes as follows: The case $AD = BC$ is trivial, so let $AD > BC$. Let L be the midpoint of AB. Complete the parallelograms $ADMX$ and $BCMY$. Now $N = DX \cap CY$, so let CY and DX intersect AB at K and H respectively. From $LX = LY$ and

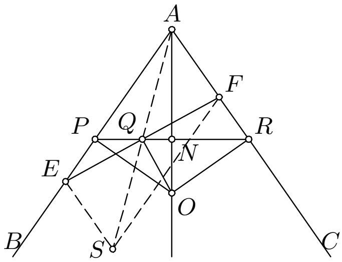
$$\frac{HL}{LX} = \frac{HA}{AD} < \frac{LA}{AD} < \frac{KB}{AD} < \frac{KB}{BC} = \frac{KL}{LY}$$

we get $HL < KL$, and the statement follows.

15. We shall prove that AD is a common tangent of ω and ω_2. Denote by K, L the points of tangency of ω with l_1 and l_2 respectively. Let r, r_1, r_2 be the radii of $\omega, \omega_1, \omega_2$ respectively, and set $KA = x$, $LB = y$. It will be enough if we show that $xy = 2r^2$, since this will imply that $\triangle KLB$ and $\triangle AKO$ are similar, where O is the center of ω, and consequently that $OA \perp KD$ (because $D \in KB$). Now if O_1 is the center of ω_1, we have $x^2 = KA^2 = OO_1^2 - (KO - AO_1)^2 = (r + r_1)^2 - (r - r_1)^2 = 4rr_1$ and analogously $y^2 = 4rr_2$. But we also have $(r_1 + r_2)^2 = O_1O_2^2 = (x-y)^2 + (2r - r_1 - r_2)^2$, so $x^2 - 2xy + y^2 = 4r(r_1 + r_2 - r)$, from which we obtain $xy = 2r^2$ as claimed. Hence AD is tangent to both ω, ω_2, and similarly BC is tangent to ω, ω_1.
It follows that Q lies on the radical axes of pairs of circles (ω, ω_1) and (ω, ω_2). Therefore Q also lies on the radical axis of (ω_1, ω_2), i.e., on the common tangent at E of ω_1 and ω_2. Hence $QC = QD = QE$.

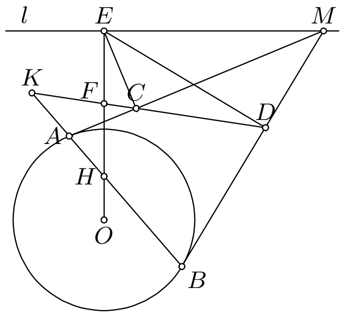
Second solution. An inversion with center at D maps ω and ω_2 to parallel lines, ω_1 and l_2 to disjoint equal circles touching ω, ω_2, and l_1 to a circle externally tangent to ω_1, l_2, and to ω. It is easy to see that the obtained picture is symmetric (with respect to a diameter of l_1), and that line AD is parallel to the lines ω and ω_2. Going back to the initial picture, this means that AD is a common tangent of ω and ω_2. The end is like that in the first solution.

16. First, assume that $\angle OQE = 90°$. Extend PN to meet AC at R. Then $OEPQ$ and $ORFQ$ are cyclic quadrilaterals; hence we have $\angle OEQ = \angle OPQ = \angle ORQ = \angle OFQ$. It follows that $\triangle OEQ \cong \triangle OFQ$ and $QE = QF$.



Now suppose $QE = QF$. Let S be the point symmetric to A with respect to Q, so that the quadrilateral $AESF$ is a parallelogram. Draw the line $E'F'$ through Q so that $\angle OQE' = 90°$ and $E' \in AB$, $F' \in AC$. By the first part $QE' = QF'$; hence $AE'SF'$ is also a parallelogram. It follows that $E \equiv E'$, $F \equiv F'$, and $\angle OQE = 90°$.

17. We first prove that AB cuts OE in a fixed point H. Note that $\angle OAH = \angle OMA = \angle OEA$ (because O, A, E, M lie on a circle); hence $\triangle OAH \sim \triangle OEA$. This implies $OH \cdot OE = OA^2$, i.e., H is fixed.



Let the lines AB and CD meet at K. Since $EAOBM$ and $ECDM$ are cyclic, we have $\angle EAK = \angle EMB = \angle ECK$, so $ECAK$ is cyclic. Therefore $\angle EKA = 90°$, hence $EKBD$ is also cyclic and $EK \parallel OM$. Then $\angle EKF = \angle EBD = \angle EOM = \angle OEK$, from which we deduce that $KF = FE$. However, since $\angle EKH = 90°$, the point F is the midpoint of EH; hence it is fixed.

18. Since for each of the subsets $\{1,4,9\}$, $\{2,6,12\}$, $\{3,5,15\}$ and $\{7,8,14\}$ the product of its elements is a square and these subsets are disjoint, we have $|M| \le 11$. Suppose that $|M| = 11$. Then $10 \in M$ and none of the disjoint subsets $\{1,4,9\}, \{2,5\}, \{6,15\}, \{7,8,14\}$ is a subset of M. Consequently $\{3,12\} \subset M$, so none of $\{1\}, \{4\}, \{9\}, \{2,6\}, \{5,15\}$, and $\{7,8,14\}$ is a subset of M: thus $|M| \le 9$, a contradiction. It follows that $|M| \le 10$, and this number is attained in the case $M = \{1,4,5,6,7,10,11,12,13,14\}$.

19. Since $mn-1$ and m^3 are relatively prime, $mn-1$ divides n^3+1 if and only if it divides $m^3(n^3+1) = (m^3n^3-1)+m^3+1$. Thus

$$\frac{n^3+1}{mn-1} \in \mathbb{Z} \Leftrightarrow \frac{m^3+1}{mn-1} \in \mathbb{Z};$$

hence we may assume that $m \ge n$. If $m = n$, then $\frac{n^3+1}{n^2-1} = n + \frac{1}{n-1}$ is an integer, so $m = n = 2$. If $n = 1$, then $\frac{2}{m-1} \in \mathbb{Z}$, which happens only when $m = 2$ or $m = 3$. Now suppose $m > n \ge 2$. Since $m^3+1 \equiv 1$ and

$mn-1 \equiv -1 \pmod{n}$, we deduce $\frac{n^3+1}{mn-1} = kn-1$ for some integer $k > 0$. On the other hand, $kn-1 < \frac{n^3+1}{n^2-1} = n+\frac{1}{n-1} \le 2n-1$ gives that $k = 1$, and therefore $n^3+1 = (mn-1)(n-1)$. This yields $m = \frac{n^2+1}{n-1} = n+1+\frac{2}{n-1} \in \mathbb{N}$, so $n \in \{2,3\}$ and $m = 5$. The solutions with $m < n$ are obtained by symmetry.

There are 9 solutions in total: $(1,2)$, $(1,3)$, $(2,1)$, $(3,1)$, $(2,2)$, $(2,5)$, $(3,5)$, $(5,2)$, $(5,3)$.

20. Let A be the set of all numbers of the form $p_1p_2\dots p_{p_1}$, where $p_1 < p_2 < \dots < p_{p_1}$ are primes. In other words, $A = \{2\cdot 3, 2\cdot 5, \dots\} \cup \{3\cdot 5\cdot 7, 3\cdot 5\cdot 11, \dots\} \cup \{5\cdot 7\cdot 11\cdot 13\cdot 17, \dots\} \cup \cdots$.
This set satisfies the requirements of the problem. Indeed, for any infinite set of primes $P = \{q_1, q_2, \dots\}$ (where $q_1 < q_2 < \cdots$) we have
$$m = q_1q_2\cdots q_{q_1} \in A \quad \text{and} \quad n = q_2q_3\cdots q_{q_1+1} \notin A.$$

21. Note first that $y_n = 2^k$ $(k \ge 2)$ and $z_k \equiv 1 \pmod 4$ for all n, so if x_n is odd, x_{n+1} will be even. Further, it is shown by induction on n that $y_n > z_n$ when x_{n-1} is even and $2y_n > z_n > y_n$ when x_{n-1} is odd. In fact, $n = 1$ is the trivial case, while if it holds for $n \ge 1$, then $y_{n+1} = 2y_n > z_n = z_{n+1}$ if x_n is even, and $2y_{n+1} = 2y_n > y_n + z_n = z_{n+1}$ if x_n is odd (since then x_{n-1} is even).
If $x_1 = 0$, then $x_0 = 3$ is good. Suppose $x_n = 0$ for some $n \ge 2$. Then x_{n-1} is odd and x_{n-2} is even, so that $y_{n-1} > z_{n-1}$. We claim that a pair (y_{n-1}, z_{n-1}), where $2^k = y_{n-1} > z_{n-1} > 0$ and $z_{n-1} \equiv 1 \pmod 4$, uniquely determines $x_0 = f(y_{n-1}, z_{n-1})$. We see that $x_{n-1} = \frac{1}{2}y_{n-1} + z_{n-1}$, and define (x_k, y_k, z_k) backwards as follows, until we get $(y_k, z_k) = (4,1)$. If $y_k > z_k$, then x_{k-1} must have been even, so we define $(x_{k-1}, y_{k-1}, z_{k-1}) = (2x_k, y_k/2, z_k)$; otherwise x_{k-1} must have been odd, so we put $(x_{k-1}, y_{k-1}, z_{k-1}) = (x_k - y_k/2 + z_k, y_k, z_k - y_k)$. We eventually arrive at $(y_0, z_0) = (4,1)$ and a good integer $x_0 = f(y_{n-1}, z_{n-1})$, as claimed. Thus for example $(y_{n-1}, z_{n-1}) = (64, 61)$ implies $x_{n-1} = 93$, $(x_{n-2}, y_{n-2}, z_{n-2}) = (186, 32, 61)$ etc., and $x_0 = 1953$, while in the case of $(y_{n-1}, z_{n-1}) = (128, 1)$ we get $x_0 = 2080$.
Note that $y' > y \Rightarrow f(y', z') > f(y, z)$ and $z' > z \Rightarrow f(y, z') > f(y, z)$. Therefore there are no y, z for which $1953 < f(y,z) < 2080$. Hence all good integers less than or equal to 1994 are given as $f(y,z)$, $y = 2^k \le 64$ and $0 < z \equiv 1 \pmod 4$, and the number of such (y,z) equals $1+2+4+8+16 = 31$. So the answer is 31.

22. (a) Denote by $b(n)$ the number of 1's in the binary representation of n. Since $b(2k+2) = b(k+1)$ and $b(2k+1) = b(k)+1$, we deduce that
$$f(k+1) = \begin{cases} f(k)+1, & \text{if } b(k) = 2; \\ f(k), & \text{otherwise.} \end{cases} \tag{1}$$
The set of k's with $b(k) = 2$ is infinite, so it follows that $f(k)$ is unbounded. Hence f takes all natural values.

(b) Since f is increasing, k is a unique solution of $f(k) = m$ if and only if $f(k-1) < f(k) < f(k+1)$. By (1), this inequality is equivalent to $b(k-1) = b(k) = 2$. It is easy to see that then $k-1$ must be of the form $2^t + 1$ for some t. In this case, $\{k+1, \dots, 2k\}$ contains the number $2^{t+1} + 3 = 10\dots011_2$ and $\frac{t(t-1)}{2}$ binary $(t+1)$-digit numbers with three 1's, so $m = f(k) = \frac{t(t-1)}{2} + 1$.

23. (a) Let p be a prime divisor of x_i, $i > 1$, and let $x_j \equiv u_j \pmod{p}$ where $0 \le u_j \le p-1$ (particularly $u_i \equiv 0$). Then $u_{j+1} \equiv u_j u_{j-1} + 1 \pmod{p}$. The number of possible pairs (u_j, u_{j+1}) is finite, so u_j is eventually periodic. We claim that for some $d_p > 0$, $u_{i+d_p} = 0$. Indeed, suppose the contrary and let $(u_m, u_{m+1}, \dots, u_{m+d-1})$ be the first period for $m \ge i$. Then $m \ne i$. By the assumption $u_{m-1} \not\equiv u_{m+d-1}$, but $u_{m-1}u_m \equiv u_{m+1} - 1 \equiv u_{m+d+1} - 1 \equiv u_{m+d-1}u_{m+d} \equiv u_{m+d-1}u_m \pmod{p}$, which is impossible if $p \nmid u_m$. Hence there is a d_p with $u_i = u_{i+d_p} = 0$ and moreover $u_{i+1} = u_{i+d_p+1} = 1$, so the sequence u_j is periodic with period d_p starting from u_i. Let m be the least common multiple of all d_p's, where p goes through all prime divisors of x_i. Then the same primes divide every x_{i+km}, $k = 1, 2, \dots$, so for large enough k and $j = i + km$, $x_i^i \mid x_j^j$.

(b) If $i = 1$, we cannot deduce that $x_{i+1} \equiv 1 \pmod{p}$. The following example shows that the statement from (a) need not be true in this case. Take $x_1 = 22$ and $x_2 = 9$. Then x_n is even if and only if $n \equiv 1 \pmod{3}$, but modulo 11 the sequence $\{x_n\}$ is $0, 9, 1, 10, 0, 1, 1, 2, 3, 7, 0, \dots$, so $11 \mid x_n$ $(n > 1)$ if and only if $n \equiv 5 \pmod{6}$. Thus for no $n > 1$ can we have $22 \mid x_n$.

24. A multiple of 10 does not divide any wobbly number. Also, if $25 \mid n$, then every multiple of n ends with 25, 50, 75, or 00; hence it is not wobbly. We now show that every other number n divides some wobbly number.

(i) Let n be odd and not divisible by 5. For any $k \ge 1$ there exists l such that $(10^k - 1)n$ divides $10^l - 1$, and thus also divides $10^{kl} - 1$. Consequently, $v_k = \frac{10^{kl}-1}{10^k-1}$ is divisible by n, and it is wobbly when $k = 2$ (indeed, $v_2 = 101\dots01$).

If n is divisible by 5, one can simply take $5v_2$ instead.

(ii) Let n be a power of 2. We prove by induction on m that 2^{2m+1} has a wobbly multiple w_m with exactly m nonzero digits. For $m = 1$, take $w_1 = 8$. Suppose that for some $m \ge 1$ there is a wobbly $w_m = 2^{2m+1}d_m$. Then the numbers $a \cdot 10^{2m} + w_m$ are wobbly and divisible by 2^{2m+1} when $a \in \{2, 4, 6, 8\}$. Moreover, one of these numbers is divisible by 2^{2m+3}. Indeed, it suffices to choose a such that $\frac{a}{2} + d_m$ is divisible by 4. This proves the induction step.

(iii) Let $n = 2^m r$, where $m \ge 1$ and r is odd, $5 \nmid r$. Then $v_{2m}w_m$ is wobbly and divisible by both 2^m and r (using notation from (i), $r \mid v_{2m}$).

4.36 Solutions to the Shortlisted Problems of IMO 1995

1. Let $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$. Then $xyz = 1$ and

$$S = \frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}.$$

We must prove that $S \geq \frac{3}{2}$. From the Cauchy–Schwarz inequality,

$$[(y+z)+(z+x)+(x+y)] \cdot S \geq (x+y+z)^2 \quad \Rightarrow \quad S \geq \frac{x+y+z}{2}\,.$$

It follows from the A-G mean inequality that $\frac{x+y+z}{2} \geq \frac{3}{2}\sqrt[3]{xyz} = \frac{3}{2}$; hence the proof is complete. Equality holds if and only if $x = y = z = 1$, i.e., $a = b = c = 1$.

Remark. After reducing the problem to $\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}$, we can solve the problem using Jensen's inequality applied to the function $g(u,v) = u^2/v$. The problem can also be solved using Muirhead's inequality.

2. We may assume $c \geq 0$ (otherwise, we may simply put $-y_i$ in the place of y_i). Also, we may assume $a \geq b$. If $b \geq c$, it is enough to take $n = a+b-c$, $x_1 = \cdots = x_a = 1$, $y_1 = \cdots = y_c = y_{a+1} = \cdots = y_{a+b-c} = 1$, and the other x_i's and y_i's equal to 0, so we need only consider the case $a > c > b$. We proceed to prove the statement of the problem by induction on $a+b$. The case $a+b = 1$ is trivial. Assume that the statement is true when $a+b \leq N$, and let $a+b = N+1$. The triple $(a+b-2c, b, c-b)$ satisfies the condition (since $(a+b-2c)b - (c-b)^2 = ab - c^2$), so by the induction hypothesis there are n-tuples $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ with the wanted property. It is easy to verify that $(x_i + y_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ give a solution for (a,b,c).

3. Write $A_i = \frac{a_i^2 + a_{i+1}^2 - a_{i+2}^2}{a_i + a_{i+1} - a_{i+2}} = a_i + a_{i+1} + a_{i+2} - \frac{2a_i a_{i+1}}{a_i + a_{i+1} - a_{i+2}}$. Since $2a_i a_{i+1} \geq 4(a_i + a_{i+1} - 2)$ (which is equivalent to $(a_i - 2)(a_{i+1} - 2) \geq 0$), it follows that $A_i \leq a_i + a_{i+1} + a_{i+2} - 4\left(1 + \frac{a_{i+2}-2}{a_i + a_{i+1} - a_{i+2}}\right) \leq a_i + a_{i+1} + a_{i+2} - 4\left(1 + \frac{a_{i+2}-2}{4}\right)$, because $1 \leq a_i + a_{i+1} - a_{i+2} \leq 4$. Therefore $A_i \leq a_i + a_{i+1} - 2$, so $\sum_{i=1}^n A_i \leq 2s - 2n$ as required.

4. The second equation is equivalent to $\frac{a^2}{yz} + \frac{b^2}{zx} + \frac{c^2}{xy} + \frac{abc}{xyz} = 4$. Let $x_1 = \frac{a}{\sqrt{yz}}$, $y_1 = \frac{b}{\sqrt{zx}}$, $z_1 = \frac{c}{\sqrt{xy}}$. Then $x_1^2 + y_1^2 + z_1^2 + x_1 y_1 z_1 = 4$, where $0 < x_1, y_1, z_1 < 2$. Regarding this as a quadratic equation in z_1, the discriminant $(4 - x_1^2)(4 - y_1^2)$ suggests that we let $x_1 = 2\sin u$, $y_1 = 2\sin v$, $0 < u, v < \pi/2$. Then it is directly shown that z_1 will be exactly $2\cos(u+v)$ as the only positive solution of the quadratic equation.
Thus $a = 2\sqrt{yz}\sin u$, $b = 2\sqrt{xz}\sin v$, $c = 2\sqrt{xy}(\cos u \cos v - \sin u \sin v)$, so from $x + y + z - a - b - c = 0$ we obtain

$$(\sqrt{x}\cos v - \sqrt{y}\cos u)^2 + (\sqrt{x}\sin v + \sqrt{y}\sin u - \sqrt{z})^2 = 0,$$

which implies

$$\sqrt{z}=\sqrt{x}\sin v+\sqrt{y}\sin u=\frac{1}{2}(y_1\sqrt{x}+x_1\sqrt{y})=\frac{1}{2}\left(\frac{b}{\sqrt{zx}}\sqrt{x}+\frac{a}{\sqrt{yz}}\sqrt{y}\right).$$

Therefore $z=\frac{a+b}{2}$. Similarly, $x=\frac{b+c}{2}$ and $y=\frac{c+a}{2}$. It is clear that the triple $(x,y,z)=\left(\frac{b+c}{2},\frac{c+a}{2},\frac{a+b}{2}\right)$ is indeed a (unique) solution of the given system of equations.

Second solution. Put $x=\frac{b+c}{2}-u$, $y=\frac{c+a}{2}-v$, $z=\frac{a+b}{2}-w$, where $u\le\frac{b+c}{2}$, $v\le\frac{c+a}{2}$, $w\le\frac{a+b}{2}$ and $u+v+w=0$. The equality $abc+a^2x+b^2y+c^2z=4xyz$ becomes $2(au^2+bv^2+cw^2+2uvw)=0$. Now $uvw>0$ is clearly impossible. On the other hand, if $uvw\le 0$, then two of u,v,w are nonnegative, say $u,v\ge 0$. Taking into account $w=-u-v$, the above equality reduces to $2[(a+c-2v)u^2+(b+c-2u)v^2+2cuv]=0$, so $u=v=0$.

Third solution. The fact that we are given two equations and three variables suggests that this is essentially a problem on inequalities. Setting $f(x,y,z)=4xyz-a^2x-b^2y-c^2z$, we should show that $\max f(x,y,z)=abc$, for $0<x,y,z$, $x+y+z=a+b+c$, and find when this value is attained. Thus we apply Lagrange multipliers to $F(x,y,z)=f(x,y,z)-\lambda(x+y+z-a-b-c)$, and obtain that f takes a maximum at (x,y,z) such that $4yz-a^2=4zx-b^2=4xy-c^2=\lambda$ and $x+y+z=a+b+c$. The only solution of this system is $(x,y,z)=\left(\frac{b+c}{2},\frac{c+a}{2},\frac{a+b}{2}\right)$.

5. Suppose that a function f satisfies the condition, and let c be the least upper bound of $\{f(x)\mid x\in\mathbb{R}\}$. We have $c\ge 2$, since $f(2)=f(1+1/1^2)=f(1)+f(1)^2=2$. Also, since c is the least upper bound, for each $k=1,2,\dots$ there is an $x_k\in\mathbb{R}$ such that $f(x_k)\ge c-1/k$. Then

$$c\ge f\left(x_k+\frac{1}{x_k^2}\right)\ge c-\frac{1}{k}+f\left(\frac{1}{x_k}\right)^2 \implies f\left(\frac{1}{x_k}\right)\ge-\frac{1}{\sqrt{k}}.$$

On the other hand,

$$c\ge f\left(\frac{1}{x_k}+x_k^2\right)=f\left(\frac{1}{x_k}\right)+f(x_k)^2\ge-\frac{1}{\sqrt{k}}+\left(c-\frac{1}{k}\right)^2.$$

It follows that

$$\frac{1}{\sqrt{k}}-\frac{1}{k^2}\ge c\left(c-1-\frac{2}{k}\right),$$

which cannot hold for k sufficiently large.

Second solution. Assume that f exists and let n be the least integer such that $f(x)\le\frac{n}{4}$ for all x. Since $f(2)=2$, we have $n\ge 8$. Let $f(x)>\frac{n-1}{4}$. Then $f(1/x)=f(x+1/x^2)-f(x)<1/4$, so $f(1/x)>-1/2$. On the other hand, this implies $\left(\frac{n-1}{4}\right)^2<f(x)^2=f(1/x+x^2)-f(1/x)<\frac{n}{4}+\frac{1}{2}$, which is impossible when $n\ge 8$.

6. Let $y_i = x_{i+1} + \cdots + x_n$, $Y = \sum_{j=2}^{n}(j-1)x_j$, and $z_i = \frac{n(n-1)}{2}y_i - (n-i)Y$. Then $\frac{n(n-1)}{2}\sum_{i<j} x_i x_j - \left(\sum_{i=1}^{n-1}(n-i)x_i\right) Y = \frac{n(n-1)}{2}\sum_{i=1}^{n-1} x_i y_i - \sum_{i=1}^{n-1}(n-i)x_i Y = \sum_{i=1}^{n-1} x_i z_i$, so it remains to show that $\sum_{i=1}^{n-1} x_i z_i > 0$. Since $\sum_{i=1}^{n-1} y_i = Y$ and $\sum_{i=1}^{n-1}(n-i) = \frac{n(n-1)}{2}$, we have $\sum z_i = 0$. Note that $Y < \sum_{j=2}^{n}(j-1)x_n = \frac{n(n-1)}{2}x_n$, and consequently $z_{n-1} = \frac{n(n-1)}{2}x_n - Y > 0$. Furthermore, we have

$$\frac{z_{i+1}}{n-i-1} - \frac{z_i}{n-i} = \frac{n(n-1)}{2}\left(\frac{y_{i+1}}{n-i-1} - \frac{y_i}{n-i}\right) > 0,$$

which means that $\frac{z_1}{n-1} < \frac{z_2}{n-2} < \cdots < \frac{z_{n-1}}{1}$. Therefore there is a k for which $z_1, \dots, z_k \le 0$ and $z_{k+1}, \dots, z_{n-1} > 0$. But then $z_i(x_i - x_k) \ge 0$, i.e., $x_i z_i \ge x_k z_i$ for all i, so $\sum_{i=1}^{n-1} x_i z_i > \sum_{i=1}^{n-1} x_k z_i = 0$ as required.

Second solution. Set $X = \sum_{j=1}^{n-1}(n-j)x_j$ and $Y = \sum_{j=2}^{n}(j-1)x_j$. Since $4XY = (X+Y)^2 - (X-Y)^2$, the RHS of the inequality becomes

$$XY = \frac{1}{4}\left[(n-1)^2\left(\sum_{i=1}^{n} x_i\right)^2 - \left(\sum_{i=1}^{n}(2i-1-n)x_i\right)^2\right].$$

The LHS equals $\frac{1}{4}\left((n-1)^2\left(\sum_{i=1}^{n} x_i\right)^2 - (n-1)\sum_{i<j}(x_j - x_i)^2\right)$. Since $\sum_{i=1}^{n}(2i-1-n)x_i = \sum_{i<j}(x_j - x_i)$ also holds, we must prove that

$$\left(\sum_{i<j}(x_j - x_i)\right)^2 > (n-1)\sum_{i<j}(x_j - x_i)^2. \tag{1}$$

Putting $x_{i+1} - x_i = d_i > 0$ (so, $x_j - x_i = d_i + d_{i+1} + \cdots + d_{j-1}$) and expanding the obtained expressions, we reduce this inequality to $\sum_k k^2(n-k)^2 d_k^2 + 2\sum_{k<l} kl(n-k)(n-l)d_k d_l > \sum_k (n-1)k(n-k)d_k^2 + 2\sum_{k<l}(n-1)k(n-l)d_k d_l$, which is verified immediately by comparing coefficients.

Remark. An inequality significantly stronger than (1) in the second solution has appeared later, as *IMO 03-5*.

7. The result is trivial if O coincides with X or Y, so let us assume it does not. From $OB \cdot ON = OC \cdot OM = OX \cdot OY$ we deduce that $BCMN$ is a cyclic quadrilateral. Further, if O lies between X and Y, then $\angle MAD + \angle MND = \angle MAD + \angle MNB + \angle BND = \angle MAD + \angle MCA + \angle AMC = 180^\circ$. Similarly, we also have $\angle MAD + \angle MND = 180^\circ$ if O is not on the segment XY. Therefore $ADNM$ is cyclic. Now let AM and DN intersect at Z and let the line ZX intersect the two circles at Y_1 and Y_2. Then $ZX \cdot ZY_1 = ZM \cdot ZA = ZN \cdot ZD = ZX \cdot ZY_2$. Hence $Y_1 = Y_2 = Y$, implying that Z lies on XY.

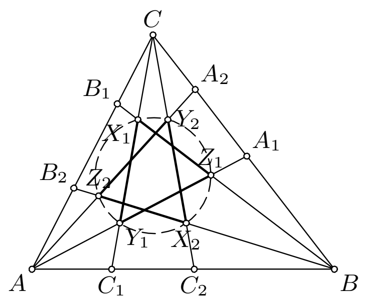
Second solution. Let Z_1, Z_2 be the points in which AM, DN respectively meet XY, and $P = BC \cap XY$. Then, from $\triangle OPC \sim \triangle APZ_1$, we have $PZ_1 = \frac{PA\cdot PC}{PO} = \frac{PX^2}{PO}$ and analogously $PZ_2 = \frac{PX^2}{PO}$. Hence, we conclude that $Z_1 \equiv Z_2$.

8. Let A', B', C' be the points symmetric to A, B, C with respect to the midpoints of BC, CA, AB respectively. From the condition on X we have $XB^2 - XC^2 = AC^2 - AB^2 = A'B^2 - A'C^2$, and hence X must lie on the line through A' perpendicular to BC. Similarly, X lies on the line through B' perpendicular to CA. It follows that there is a unique position for X, namely the orthocenter of $\triangle A'B'C'$. It easily follows that this point X satisfies the original equations.

9. If EF is parallel to BC, $\triangle ABC$ must be isosceles and E, Y are symmetric to F, Z with respect to AD, so the result follows. Now suppose that EF meets BC at P. By Menelaus's theorem, $\frac{BP}{CP} = \frac{BF}{FA} \cdot \frac{AE}{EC} = \frac{BD}{DC}$ (since $BD = BF$, $CD = CE$, $AE = AF$). It follows that the point P depends only on D and not on A. In particular, the same point is obtained as the intersection of ZY with BC. Therefore $PE \cdot PF = PD^2 = PY \cdot PZ$, from which it follows that $EFZY$ is a cyclic quadrilateral.

   Second solution. Since $CD = CY = CE$ and $BD = BZ = BF$, all angles of $EFZY$ can be calculated in terms of angles of ABC and $YZBC$. In fact, $\angle FEY = \frac{1}{2}(\angle A + \angle C + \angle BCY)$ and $\angle FZY = \frac{1}{2}(180° + \angle B + \angle BCY)$, which gives us $\angle FEY + \angle FZY = 180°$.

10. Let the two triangles be $X_1Y_1Z_1$, $X_2Y_2Z_2$, with $X_1 = BB_1 \cap CC_1$, $Y_1 = CC_1 \cap AA_1$, $Z_1 = AA_1 \cap BB_1$, $X_2 = BB_2 \cap CC_2$, $Y_2 = CC_2 \cap AA_2$, $Z_2 = AA_2 \cap BB_2$. First, we observe that $\angle ABB_2 = \angle ACC_1$ and $\angle ABB_1 = \angle ACC_2$. Consequently $\angle BZ_1A_1 = \angle BAA_1 + \angle ABB_1 = \angle BCC_2 + \angle C_2CA = \angle C$ and similarly $\angle AZ_2B_2 = \angle C$, $\angle AY_1C_1 = \angle CY_2A_2 = \angle B$. Also, $\triangle ABB_2 \sim \triangle ACC_1$; hence $AC_1/AC = AB_2/AB$.



From the sine formula, we obtain

$$\frac{AZ_1}{\sin\angle ABZ_1} = \frac{AB}{\sin\angle AZ_1B} = \frac{AB}{\sin\angle C} = \frac{AC}{\sin\angle B} = \frac{AC}{\sin\angle AY_2C} = \frac{AY_2}{\sin\angle ACY_2} \implies AZ_1 = AY_2.$$

Analogously, $BX_1 = BZ_2$ and $CY_1 = CX_2$. Furthermore, again from the sine formula,

$$\frac{AY_1}{\sin\angle AC_1Y_1} = \frac{AC_1}{\sin\angle AY_1C_1} = \frac{AC_1}{AC}\frac{AC}{\sin\angle B}$$
$$= \frac{AB_2}{AB}\frac{AB}{\sin\angle C} = \frac{AB_2}{\sin\angle AZ_2B_2} = \frac{AZ_2}{\sin\angle AB_2Z_2}.$$

Hence, $AY_1 = AZ_2$ and, analogously, $BZ_1 = BX_2$ and $CX_1 = CY_2$. We deduce that $Y_1Z_2 \parallel BC$ and $Z_2X_1 \parallel AC$, which gives us $\angle Y_1Z_2X_1 = 180^\circ - \angle C = 180^\circ - \angle Y_1Z_1X_1$. It follows that Z_2 lies on the circle circumscribed about $\triangle X_1Y_1Z_1$. Similarly, so do X_2 and Y_2.

Second solution. Let H be the orthocenter of $\triangle ABC$. Triangles AHB, BHC, CHA, ABC have the same circumradius R. Additionally,

$$\angle HAA_i = \angle HBB_i = \angle HCC_i = \theta \quad (i = 1, 2).$$

Since $\angle HBX_1 = \angle HCX_1 = \theta$, BCX_1H is concyclic and therefore $HX_1 = 2R\sin\theta$. The same holds for $HY_1, HZ_1, HX_2, HY_2, HZ_2$. Hence X_i, Y_i, Z_i $(i = 1, 2)$ lie on a circle centered at H.

11. Triangles BCD and EFA are equilateral, and hence BE is an axis of symmetry of $ABDE$. Let C', F' respectively be the points symmetric to C, F with respect to BE. The points G and H lie on the circumcircles of ABC' and DEF' respectively (because, for instance, $\angle AGB = 120^\circ = 180^\circ - \angle AC'B$); hence from Ptolemy's theorem we have $AG + GB = C'G$ and $DH + HE = HF'$. Therefore

$$AG + GB + GH + DH + HE = C'G + GH + HF' \geq C'F' = CF,$$

with equality if and only if G and H both lie on $C'F'$.

Remark. Since by Ptolemy's inequality $AG + GB \geq C'G$ and $DH + HE \geq HF'$, the result holds without the condition $\angle AGB = \angle DHE = 120^\circ$.

12. Let O be the circumcenter and R the circumradius of $A_1A_2A_3A_4$. We have $OA_i^2 = (\overrightarrow{OG} + (\overrightarrow{OA_i} - \overrightarrow{OG}))^2 = OG^2 + GA_i^2 + 2\overrightarrow{OG}\cdot\overrightarrow{GA_i}$. Summing up these equalities for $i = 1, 2, 3, 4$ and using that $\sum_{i=1}^4 \overrightarrow{GA_i} = \overrightarrow{0}$, we obtain

$$\sum_{i=1}^4 OA_i^2 = 4OG^2 + \sum_{i=1}^4 GA_i^2 \iff \sum_{i=1}^4 GA_i^2 = 4(R^2 - OG^2). \quad (1)$$

Now we have that the potential of G with respect to the sphere equals $GA_i \cdot GA_i' = R^2 - OG^2$. Plugging in these expressions for GA_i', we reduce the inequalities we must prove to

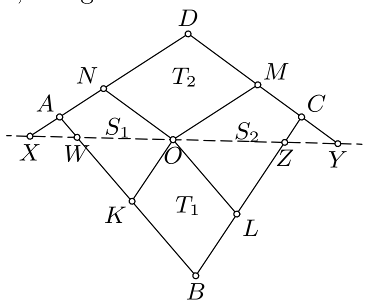
$$GA_1 \cdot GA_2 \cdot GA_3 \cdot GA_4 \leq (R^2 - OG^2)^2 \quad (2)$$

and $$(R^2 - OG^2)\sum_{i=1}^4 \frac{1}{GA_i} \geq \sum_{i=1}^4 GA_i. \quad (3)$$

Inequality (2) immediately follows from (1) and the quadratic-geometric mean inequality for GA_i. Since from the Cauchy–Schwarz inequality we have $\sum_{i=1}^4 GA_i^4 \geq \frac{1}{4}\left(\sum_{i=1}^4 GA_i\right)^2$ and $\left(\sum_{i=1}^4 GA_i\right)\left(\sum_{i=1}^4 \frac{1}{GA_i}\right) \geq 16$, inequality (3) follows from (1) and from

$$\left(\sum_{i=1}^{4} GA_i^2\right)\left(\sum_{i=1}^{4}\frac{1}{GA_i}\right) \geq \frac{1}{4}\left(\sum_{i=1}^{4} GA_i\right)^2\left(\sum_{i=1}^{4}\frac{1}{GA_i}\right) \geq 4\sum_{i=1}^{4} GA_i.$$

13. If O lies on AC, then $ABCD$, $AKON$, and $OLCM$ are similar; hence $AC = AO + OC$ implies $\sqrt{S} = \sqrt{S_1} + \sqrt{S_2}$. Assume that O does not lie on AC and that w.l.o.g. it lies inside triangle ADC. Let us denote by T_1, T_2 the areas of parallelograms $KBLO, NOMD$ respectively. Consider a line through O that intersects AD, DC, CB, BA respectively at X, Y, Z, W so that $OW/OX = OZ/OY$ (such a line exists by a continuity argument: the left side is smaller when $W = X = A$, but greater when $Y = Z = C$).



The desired inequality is equivalent to $T_1 + T_2 \geq 2\sqrt{S_1 S_2}$. Since triangles WKO, OLZ, WBZ are similar and $WO + OZ = WZ$, we have $\sqrt{S_{WKO}} + \sqrt{S_{OLZ}} = \sqrt{S_{WBZ}} = \sqrt{S_{WKO} + S_{OLZ} + T_1}$, which implies $T_1 = 2\sqrt{S_{WKO}S_{OLZ}}$. Similarly, $T_2 = 2\sqrt{S_{XNO}S_{OMY}}$. Since $OW/OZ = OX/OY$, we have $S_{WKO}/S_{XNO} = S_{OLZ}/S_{OMY}$. Therefore we obtain

$$\begin{aligned} T_1 + T_2 &= 2\sqrt{S_{WKO}S_{OLZ}} + 2\sqrt{S_{XNO}S_{OMY}} \\ &= 2\sqrt{(S_{WKO} + S_{XNO})(S_{OLZ} + S_{OMY})} \geq 2\sqrt{S_1 S_2}. \end{aligned}$$

Second solution. By an affine transformation of the plane one can transform any nondegenerate quadrilateral into a cyclic one, thereby preserving parallelness and ratios of areas. Thus we may assume w.l.o.g. that $ABCD$ is cyclic.

By a well-known formula, the area of a cyclic quadrilateral with sides a, b, c, d and semiperimeter p is given by

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}\,.$$

Let us set $AK = a_1$, $KB = b_1$, $BL = a_2$, $LC = b_2$, $CM = a_3$, $MD = b_3$, $DN = a_4$, $NA = b_4$. Then the sides of quadrilateral $AKON$ are a_i, the sides of $CLOM$ are b_i, and the sides of $ABCD$ are $a_i + b_i$ $(i = 1, 2, 3, 4)$. If p and q are the semiperimeters of $AKON$ and $CLOM$, and $x_i = p - a_i$, $y_i = q - b_i$, then we have $S_1 = \sqrt{x_1x_2x_3x_4}$, $S_2 = \sqrt{y_1y_2y_3y_4}$, and $S = \sqrt{(x_1+y_1)(x_2+y_2)(x_3+y_3)(x_4+y_4)}\,$. Thus we need to show that

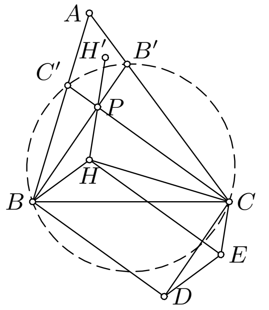
$$\sqrt[4]{x_1x_2x_3x_4} + \sqrt[4]{y_1y_2y_3y_4} \le \sqrt[4]{(x_1+y_1)(x_2+y_2)(x_3+y_3)(x_4+y_4)}\,.$$

By setting $y_i = t_ix_i$ we reduce this inequality to $1 + \sqrt[4]{t_1t_2t_3t_4} \le \sqrt[4]{(1+t_1)(1+t_2)(1+t_3)(1+t_4)}$. One way to prove the last inequality is to apply the simple inequality

$$1 + \sqrt{uv} \le \sqrt{(1+u)(1+v)}$$

to $\sqrt{t_1t_2}$, $\sqrt{t_3t_4}$ and then to t_1, t_2 and t_3, t_4.

14. Let BB' cut CC' at P. Since $\angle B'BC' = \angle B'CC'$, it follows that $\angle PBH = \angle PCH$. Let D and E be points such that $BPCD$ and $HPCE$ are parallelograms (consequently, so is $BHED$). Triangles BAC and $C'AB'$ are similar, from which we deduce that $\triangle B'H'C'$ and $\triangle BHC$ are similar, as well as $\triangle B'PC'$ and $\triangle BDC$. Hence $B'PC'H'$ and $BDCH$ are similar, from which we obtain $\angle H'PB' = \angle HDB$. Now $\angle CDE = \angle PBH = \angle PCH = \angle CHE$ implies that $HCED$ is a cyclic quadrilateral. Therefore $\angle BPH = \angle DCE = \angle DHE = \angle HDB = \angle H'PB'$; hence HH' also passes through P.



Second solution. Observe that $\triangle HBC \sim \triangle H'B'C'$, $\angle PBH = \angle PCH$ and $\angle PB'H' = \angle PC'H'$.
By Ceva's theorem in trigonometric form applied to $\triangle BPC$ and the point H, we have $\frac{\sin\angle BPH}{\sin\angle HPC} = \frac{\sin\angle HBP}{\sin\angle HBC} \cdot \frac{\sin\angle HCB}{\sin\angle HCP} = \frac{\sin\angle HCB}{\sin\angle HBC}$. Similarly, Ceva's theorem for $\triangle B'PC'$ and point H' yields $\frac{\sin\angle B'PH'}{\sin\angle H'PC'} = \frac{\sin\angle H'C'B'}{\sin\angle H'B'C'}$. Thus it follows that

$$\frac{\sin\angle B'PH'}{\sin\angle H'PC'} = \frac{\sin\angle BPH}{\sin\angle HPC},$$

which finally implies that $\angle BPH = \angle B'PH'$.

15. We show by induction on k that there exists a positive integer a_k for which $a_k^2 \equiv -7 \pmod{2^k}$. The statement of the problem follows, since every $a_k + r2^k$ $(r = 0, 1, \dots)$ also satisfies this condition.
Note that for $k = 1, 2, 3$ one can take $a_k = 1$. Now suppose that $a_k^2 \equiv -7 \pmod{2^k}$ for some $k > 3$. Then either $a_k^2 \equiv -7 \pmod{2^{k+1}}$ or $a_k^2 \equiv 2^k - 7 \pmod{2^{k+1}}$. In the former case, take $a_{k+1} = a_k$. In the latter case, set $a_{k+1} = a_k + 2^{k-1}$. Then $a_{k+1}^2 = a_k^2 + 2^k a_k + 2^{2k-2} \equiv a_k^2 + 2^k \equiv -7 \pmod{2^{k+1}}$ because a_k is odd.

16. If A is odd, then every number in M_1 is of the form $x(x+A) + B \equiv B \pmod 2$, while numbers in M_2 are congruent to C modulo 2. Thus it is enough to take $C \equiv B + 1 \pmod 2$.

If A is even, then all numbers in M_1 have the form $\left(X + \frac{A}{2}\right)^2 + B - \frac{A^2}{4}$ and are congruent to $B - \frac{A^2}{4}$ or $B - \frac{A^2}{4} + 1$ modulo 4, while numbers in M_2 are congruent to C modulo 4. So one can choose any $C \equiv B - \frac{A^2}{4} + 2 \pmod 4$.

17. For $n = 4$, the vertices of a unit square $A_1A_2A_3A_4$ and $p_1 = p_2 = p_3 = p_4 = \frac{1}{6}$ satisfy the conditions. We claim that there are no solutions for $n = 5$ (and thus for any $n \geq 5$).
Suppose to the contrary that points A_i and p_i, $i = 1, \dots, 5$, satisfy the conditions. Denote the area of $\triangle A_iA_jA_k$ by $S_{ijk} = p_i + p_j + p_k$, $1 \leq i < j < k \leq 5$. Observe that all the p_i's must be distinct. Indeed, if $p_4 = p_5$, then $S_{124} = S_{125}$ and $S_{234} = S_{235}$, which implies that A_4A_5 is parallel to A_1A_2 and A_2A_3, so A_1, A_2, A_3 are collinear, which is impossible. Also note that if $A_iA_jA_kA_l$ is convex, then $S_{ijk} + S_{ikl} = S_{ijl} + S_{jkl}$ gives $p_i + p_k = p_j + p_l$. Now consider the convex hull of A_1, A_2, A_3, A_4, A_5. There are three cases.
    (i) The convex hull is the pentagon $A_1A_2A_3A_4A_5$. Then $A_1A_2A_3A_4$ and $A_1A_2A_3A_5$ are convex, so we have $p_1 + p_3 = p_2 + p_4$ and $p_1 + p_3 = p_2 + p_5$. Hence $p_4 = p_5$, a contradiction.
    (ii) The convex hull is w.l.o.g. the quadrilateral $A_1A_2A_3A_4$. Assume that A_5 lies within $A_1A_3A_4$. Then $A_1A_2A_3A_5$ is also convex, so as in (1) we get $p_4 = p_5$.
    (iii) The convex hull is w.l.o.g. the triangle $A_1A_2A_3$. Since $S_{124} + S_{134} + S_{234} = S_{125} + S_{135} + S_{235}$, we conclude that again $p_4 = p_5$.

18. Let $x = za$ and $y = zb$, where a and b are relatively prime. The given Diophantine equation becomes $a + zb^2 + z^2 = z^2ab$, so $a = zc$ for some $c \in \mathbb{Z}$. We obtain $c + b^2 + z = z^2cb$, or $c = \frac{b^2+z}{z^2b-1}$.
    (i) If $z = 1$, then $c = \frac{b^2+1}{b-1} = b + 1 + \frac{2}{b-1}$, so $b = 2$ or $b = 3$. These values yield two solutions: $(x, y) = (5, 2)$ and $(x, y) = (5, 3)$.
    (ii) If $z = 2$, then $16c = \frac{16b^2+32}{4b-1} = 4b + 1 + \frac{33}{4b-1}$, so $b = 1$ or $b = 3$. In this case $(x, y) = (4, 2)$ or $(x, y) = (4, 6)$.
    (iii) Let $z \geq 3$. First, we see that $z^2c = \frac{z^2b^2+z^3}{z^2b-1} = b + \frac{b+z^3}{z^2b-1}$. Thus $\frac{b+z^3}{z^2b-1}$ must be a positive integer, so $b + z^3 \geq z^2b - 1$, which implies $b \leq \frac{z^2-z+1}{z-1}$. It follows that $b \leq z$. But then $b^2 + z \leq z^2 + b < z^2b - 1$, with the last inequality because $(z^2-1)(b-1) > 2$. Therefore $c = \frac{b^2+z}{z^2b-1} < 1$, a contradiction.
The only solutions for (x, y) are $(4, 2), (4, 6), (5, 2), (5, 3)$.

19. For each two people let n be the number of people exchanging greetings with both of them. To determine n in terms of k, we shall count in two ways the number of triples (A, B, C) of people such that A exchanged greetings with both B and C, but B and C mutually did not.
There are $12k$ possibilities for A, and for each A there are $(3k + 6)$ possibilities for B. Since there are n people who exchanged greetings with both

A and B, there are $3k+5-n$ who did so with A but not with B. Thus the number of triples (A,B,C) is $12k(3k+6)(3k+5-n)$. On the other hand, there are $12k$ possible choices of B, and $12k-1-(3k+6)=9k-7$ possible choices of C; for every B,C, A can be chosen in n ways, so the number of considered triples equals $12kn(9k-7)$.
Hence $(3k+6)(3k+5-n)=n(9k-7)$, i.e., $n=\frac{3(k+2)(3k+5)}{12k-1}$. This gives us that $\frac{4n}{3}=\frac{12k^2+44k+40}{12k-1}=k+4-\frac{3k-44}{12k-1}$ is an integer too. It is directly verified that only $k=3$ gives an integer value for n, namely $n=6$.

Remark. The solution is complete under the assumption that such a k exists. We give an example of such a party with 36 persons, $k=3$. Let the people sit in a 6×6 array $[P_{ij}]_{i,j=1}^{6}$, and suppose that two persons P_{ij},P_{kl} exchanged greetings if and only if $i=k$ or $j=l$ or $i-j\equiv k-l$ (mod 6). Thus each person exchanged greetings with exactly 15 others, and it is easily verified that this party satisfies the conditions.

20. We shall consider the set $M=\{0,1,\dots,2p-1\}$ instead. Let $M_1=\{0,1,\dots,p-1\}$ and $M_2=\{p,p+1,\dots,2p-1\}$. We shall denote by $|A|$ and $\sigma(A)$ the number of elements and the sum of elements of the set A; also, let C_p be the family of all p-element subsets of M. Define the mapping $T: C_p\to C_p$ as $T(A)=\{x+1\mid x\in A\cap M_1\}\cup\{A\cap M_2\}$, the addition being modulo p. There are exactly two fixed points of T: these are M_1 and M_2. Now if A is any subset from C_p distinct from M_1,M_2, and $k=|A\cap M_1|$ with $1\le k\le p-1$, then for $i=0,1,\dots,p-1$, $\sigma(T^i(A))=\sigma(A)+ik \pmod p$. Hence subsets $A,T(A),\dots,T^{p-1}(A)$ are distinct, and exactly one of them has sum of elements divisible by p. Since $\sigma(M_1),\sigma(M_2)$ are divisible by p and $C_p\setminus\{M_1,M_2\}$ decomposes into families of the form $\{A,T(A),\dots,T^{p-1}(A)\}$, we conclude that the required number is $\frac{1}{p}(|C_p|-2)+2=\frac{1}{p}\left(\binom{2p}{p}-2\right)+2$.

Second solution. Let C_k be the family of all k-element subsets of $\{1,2,\dots,2p\}$. Denote by M_k ($k=1,2,\dots,p$) the family of p-element multisets with k distinct elements from $\{1,2,\dots,2p\}$, exactly one of which appears more than once, that have sum of elements divisible by p. It is clear that every subset from C_k, $k<p$, can be complemented to a multiset from $M_k\cup M_{k+1}$ in exactly two ways, since the equation $(p-k)a\equiv 0 \pmod p$ has exactly two solutions in $\{1,2,\dots,2p\}$. On the other hand, every multiset from M_k can be obtained by completing exactly one subset from C_k. Additionally, a multiset from M_k can be obtained from exactly one subset from C_{k-1} if $k<p$, and from exactly p subsets from C_{k-1} if $k=p$. Therefore $|M_k|+|M_{k+1}|=2|C_k|=2\binom{2p}{k}$ for $k=1,2,\dots,p-2$, and $|M_{p-1}|+p|M_p|=2|C_{p-1}|=2\binom{2p}{p-1}$. Since $M_1=2p$, it is not difficult to show using recursion that $|M_p|=\frac{1}{p}\left(\binom{2p}{p}-2\right)+2$.

Third solution. Let $\omega = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$. We have $\prod_{i=1}^{2p}(x-\omega^i) = (x^p-1)^2 = x^{2p} - 2x^p + 1$; hence comparing the coefficients at x^p, we obtain $\sum \omega^{i_1+\cdots+i_p} = \sum_{i=0}^{p-1} a_i\omega^i = 2$, where the first sum runs over all p-subsets $\{i_1,\dots,i_p\}$ of $\{1,\dots,2p\}$, and a_i is the number of such subsets for which $i_1+\cdots+i_p \equiv i \pmod p$. Setting $q(x) = -2 + \sum_{i=0}^{p-1} a_i x^i$, we obtain $q(\omega^j) = 0$ for $j = 1,2,\dots,p-1$. Hence $1+x+\cdots+x^{p-1} \mid q(x)$, and since $\deg q = p-1$, we have $q(x) = -2+\sum_{i=0}^{p-1} a_i x^i = c(1+x+\cdots+x^{p-1})$ for some constant c. Thus $a_0 - 2 = a_1 = \cdots = a_{p-1}$, which together with $a_0+\cdots+a_{p-1} = \binom{2p}{p}$ yields $a_0 = \frac{1}{p}\left(\binom{2p}{p} - 2\right) + 2$.

21. We shall show that there is no such n. Certainly, $n=2$ does not work, so suppose $n \ge 3$. Let a,b be distinct elements of A_1, and c any integer greater than $-a$ and $-b$. We claim that $a+c, b+c$ belong to the same subsets. Suppose to the contrary that $a+c \in A_1$ and $b+c \in A_2$, and take arbitrary elements $x_i \in A_i$, $i = 3,\dots,n$. The number $b+x_3+\cdots+x_n$ is in A_2, so that $s = (a+c)+(b+x_3+\cdots+x_n)+x_4+\cdots+x_n$ must be in A_3. On the other hand, $a+x_3+\cdots+x_n \in A_2$, so $s = (a+x_3+\cdots+x_n) + (b+c)+x_4+\cdots+x_n$ is in A_1, a contradiction. Similarly, if $a+c \in A_2$ and $b+c \in A_3$, then $s = a+(b+c)+x_4+\cdots+x_n$ belongs to A_2, but also $s = b+(a+c)+x_4+\cdots+x_n \in A_3$, which is impossible.
For $i = 1,\dots,n$ choose $x_i \in A_i$; set $s = x_1+\cdots+x_n$ and $y_i = s - x_i$. Then $y_i \in A_i$. By what has been proved above, $2x_i = x_i + x_i$ belongs to the same subset as $x_i + y_i = s$ does. It follows that all numbers $2x_i$, $i = 1,\dots,n$, are in the same subset. Since we can arbitrarily take x_i from each set A_i, it follows that all even numbers belong to the same set, say A_1. Similarly, $2x_i + 1 = (x_i+1)+x_i$ is in the subset to which $(x_i+1)+y_i = s+1$ belongs for all $i = 1,\dots,n$; hence all odd numbers greater than 1 are in the same subset, say A_2. By the above considerations, $3-2 = 1 \in A_2$ also. But then nothing remains in $A_3,\dots,A_n$, a contradiction.

22. Let $u = \sqrt{2p} - \sqrt{x} - \sqrt{y}$ and $v = u(2\sqrt{2p} - u) = 2p - (\sqrt{2p}-u)^2 = 2p - x - y - \sqrt{4xy}$ for $x,y \in \mathbb{N}$, $x \le y$. Obviously $u \ge 0$ if and only if $v \ge 0$, and u, v attain minimum positive values simultaneously. Note that $v \ne 0$. Otherwise $u = 0$ too, so $y = (\sqrt{2p} - \sqrt{x})^2 = 2p - x - 2\sqrt{2px}$, which implies that $2px$ is a square, and consequently x is divisible by $2p$, which is impossible.
Now let z be the smallest integer greater than $\sqrt{4xy}$. We have $z^2 - 1 \ge 4xy$, $z \le 2p - x - y$, and $z \le p$ because $\sqrt{4xy} \le (\sqrt{x}+\sqrt{y})^2 < 2p$. It follows that

$$v = 2p - x - y - \sqrt{4xy} \ge z - \sqrt{z^2-1} = \frac{1}{z+\sqrt{z^2-1}} \ge \frac{1}{p+\sqrt{p^2-1}}.$$

Equality holds if and only if $z = x+y = p$ and $4xy = p^2 - 1$, which is satisfied only when $x = \frac{p-1}{2}$ and $y = \frac{p+1}{2}$. Hence for these values of x, y, both u and v attain positive minima.

23. By putting $F(1) = 0$ and $F(361) = 1$, condition (c) becomes $F(F(n^{163})) = F(F(n))$ for $n \geq 2$. For $n = 2, 3, \dots, 360$ let $F(n) = n$, and inductively define $F(n)$ for $n \geq 362$ as follows:

$$F(n) = \begin{cases} F(m), & \text{if } n = m^{163},\ m \in \mathbb{N}; \\ \text{the least number not in } \{F(k) \mid k < n\}, & \text{otherwise.} \end{cases}$$

Obviously, (a) each nonnegative integer appears in the sequence because there are infinitely many numbers not of the form m^{163}, and (b) each positive integer appears infinitely often because $F(m^{163}) = F(m)$. Since $F(n^{163}) = F(n)$, (c) also holds.

Second solution. Another example of such a sequence is as follows: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, is the factorization of n into primes, we put $F(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ and $F(1) = 0$. Conditions (a) and (b) are evidently satisfied for this F, while (c) follows from $F(F(n^{163})) = F(163F(n)) = F(F(n))+1$ (because 163 is a prime) and $F(F(361)) = F(F(19^2)) = F(2) = 1$.

24. The given condition is equivalent to $(2x_i - x_{i-1})(x_i x_{i-1} - 1) = 0$, so either $x_i = \frac{1}{2}x_{i-1}$ or $x_i = \frac{1}{x_{i-1}}$. We shall show by induction on n that for any $n \geq 0$, $x_n = 2^{k_n} x_0^{e_n}$ for some integer k_n, where $|k_n| \leq n$ and $e_n = (-1)^{n-k_n}$. Indeed, this is true for $n = 0$. If it holds for some n, then $x_{n+1} = \frac{1}{2}x_n = 2^{k_n-1}x_0^{e_n}$ (hence $k_{n+1} = k_n - 1$ and $e_{n+1} = e_n$) or $x_{n+1} = \frac{1}{x_n} = 2^{-k_n}x_0^{-e_n}$ (hence $k_{n+1} = -k_n$ and $e_{n+1} = -e_n$).
Thus $x_0 = x_{1995} = 2^{k_{1995}} x_0^{e_{1995}}$. Note that $e_{1995} = 1$ is impossible, since in that case k_{1995} would be odd, although it should equal 0. Therefore $e^{1995} = -1$, which gives $x_0^2 = 2^{k_{1995}} \leq 2^{1994}$, so the maximal value that x_0 can have is 2^{997}. This value is attained in the case $x_i = 2^{997-i}$ for $i = 0, \dots, 997$ and $x_i = 2^{i-998}$ for $i = 998, \dots, 1995$.

Second solution. First we show that there is an n, $0 \leq n \leq 1995$, such that $x_n = 1$. Suppose the contrary. Then each of x_n belongs to one of the intervals $I_{-i-1} = [2^{-i-1}, 2^{-i})$ or $I_i = (2^i, 2^{i+1}]$, where $i = 0, 1, 2, \dots$. Let $x_n \in I_{i_n}$. Note that by the formula for x_n, i_n and i_{n-1} are of different parity. Hence i_0 and i_{1995} are also of different parity, contradicting $x_0 = x_{1995}$.
It follows that for some n, $x_n = 1$. Now if $n \leq 997$, then $x_0 \leq 2^{997}$, while if $n \geq 998$, we also have $x_0 = x_{1995} \leq 2^{997}$.

25. By the definition of $q(x)$, it divides x for all integers $x > 0$, so $f(x) = xp(x)/q(x)$ is a positive integer too. Let $\{p_0, p_1, p_2, \dots\}$ be all prime numbers in increasing order. Since it easily follows by induction that all x_n's are square-free, we can assign to each of them a unique code according to which primes divide it: if p_m is the largest prime dividing x_n, the code corresponding to x_n will be $\dots 0 s_m s_{m-1} \dots s_0$, with $s_i = 1$ if $p_i \mid x_n$ and $s_i = 0$ otherwise. Let us investigate how f acts on these codes. If the code of x_n ends with 0, then x_n is odd, so the code of $f(x_n) = x_{n+1}$ is obtained from that of x_n by replacing $s_0 = 0$ by $s_0 = 1$. Furthermore, if the code of

x_n ends with $011\dots1$, then the code of x_{n+1} ends with $100\dots0$ instead. Thus if we consider the codes as binary numbers, f acts on them as an addition of 1. Hence the code of x_n is the binary representation of n and thus x_n uniquely determines n.
Specifically, if $x_n = 1995 = 3\cdot 5\cdot 7\cdot 19$, then its code is 10001110 and corresponds to $n = 142$.

26. For $n = 1$ the result is trivial, since $x_1 = 1$. Suppose now that $n \ge 2$ and let $f_n(x) = x^n - \sum_{i=0}^{n-1} x^i$. Note that x_n is the unique positive real root of f_n, because $\frac{f_n(x)}{x^{n-1}} = x - 1 - \frac{1}{x} - \cdots - \frac{1}{x^{n-1}}$ is strictly increasing on $\mathbb{R}^+$. Consider $g_n(x) = (x-1)f_n(x) = (x-2)x^n + 1$. Obviously $g_n(x)$ has no positive roots other than 1 and $x_n > 1$. Observe that $\left(1 - \frac{1}{2^n}\right)^n > 1 - \frac{n}{2^n} \ge \frac{1}{2}$ for $n \ge 2$ (by Bernoulli's inequality). Since then

$$g_n\left(2 - \frac{1}{2^n}\right) = -\frac{1}{2^n}\left(2 - \frac{1}{2^n}\right)^n + 1 = 1 - \left(1 - \frac{1}{2^{n+1}}\right)^n > 0,$$

and

$$g_n\left(2 - \frac{1}{2^{n-1}}\right) = -\frac{1}{2^{n-1}}\left(2 - \frac{1}{2^{n-1}}\right)^n + 1 = 1 - 2\left(1 - \frac{1}{2^n}\right)^n < 0,$$

we conclude that x_n is between $2 - \frac{1}{2^{n-1}}$ and $2 - \frac{1}{2^n}$, as required.
Remark. Moreover, $\lim_{n\to\infty} 2^n(2 - x_n) = 1$.

27. Computing the first few values of $f(n)$, we observe the following pattern:

$$\begin{array}{rl} f(4k) = k,\ k \ge 3, & f(8) = 3; \\ f(4k+1) = 1,\ k \ge 4, & f(5) = f(13) = 2; \\ f(4k+2) = k-3,\ k \ge 7, & f(2) = 1, f(6) = f(10) = 2, \\ & f(14) = f(18) = 3, f(26) = 4; \\ f(4k+3) = 2. & \end{array}$$

We shall prove these statements simultaneously by induction on n, having verified them for $k \le 7$.
(i) Let $n = 4k$. Since $f(3) = f(7) = \cdots = f(4k-1) = 2$, we have $f(4k) \ge k$. But $f(n) \le \max_{m<n} f(m) + 1 \le (k-1)+1$, so $f(4k) = k$.
(ii) Let $n = 4k+1$, $k \ge 7$. Since $f(4k) = k$ and $f(m) < k$ for $m < 4k$, we deduce that $f(4k+1) = 1$.
(iii) Let $n = 4k+2$, $k \ge 7$. Since $f(17) = f(21) = \cdots = f(4k+1) = 1$, we obtain $f(4k+2) \ge k-3$. On the other hand, if $f(4k+1) = f(4k+1-d) = 1$, then $d \ge 8$, and $4k+1-8(k-3) < 0$. So $f(4k+2) = k-3$.
(iv) Let $n = 4k+3$, $k \ge 7$. We have $f(4k+2) = k-3$ and $f(m) = k-3$ for exactly one $m < 4k+2$ (namely for $m = 4k-12$); hence $f(4k+3) = 2$.
Therefore, for example, $f(4n+8) = n+2$ for all n; hence we can take $a = 4$ and $b = 8$.

28. Let $F(x) = f(x) - 95$ for $x \geq 1$. Writing k for $m + 95$, the given condition becomes

$$F(k + F(n)) = F(k) + n, \qquad k \geq 96, n \geq 1. \tag{1}$$

Thus for $x, z \geq 96$ and an arbitrary y we have $F(x + y) + z = F(x + y + F(z)) = F(x + F(F(y) + z)) = F(x) + F(y) + z$, and consequently $F(x + y) = F(x) + F(y)$ whenever $x \geq 96$. Moreover, since then $F(x + y) + F(96) = F(x + y + 96) = F(x) + F(y + 96) = F(x) + F(y) + F(96)$ for any x, y, we obtain

$$F(x + y) = F(x) + F(y), \qquad x, y \in \mathbb{N}. \tag{2}$$

It follows by induction that $F(n) = nc$ for all n, where $F(1) = c$. Equation (1) becomes $ck + c^2 n = ck + n$, and yields $c = 1$. Hence $F(n) = n$ and $f(n) = n + 95$ for all n.

Finally, $\sum_{k=1}^{19} f(k) = 96 + 97 + \cdots + 114 = 1995$.

Second solution. First we show that $f(n) > 95$ for all n. If to the contrary $f(n) \leq 95$, we have $f(m) = n + f(m + 95 - f(n))$, so by induction $f(m) = kn + f(m + k(95 - f(n))) \geq kn$ for all k, which is impossible. Now for $m > 95$ we have $f(m + f(n) - 95) = n + f(m)$, and again by induction $f(m + k(f(n) - 95)) = kn + f(m)$ for all m, n, k. It follows that with n fixed,

$$(\forall m) \lim_{k \to \infty} \frac{f(m + k(f(n) - 95))}{m + k(f(n) - 95)} = \frac{n}{f(n) - 95};$$

hence

$$\lim_{s \to \infty} \frac{f(s)}{s} = \frac{n}{f(n) - 95}.$$

Hence $\frac{n}{f(n)-95}$ does not depend on n, i.e., $f(n) \equiv cn + 95$ for some constant c. It is easily checked that only $c = 1$ is possible.

4.37 Solutions to the Shortlisted Problems of IMO 1996

1. We have $a^5+b^5-a^2b^2(a+b)=(a^3-b^3)(a^2-b^2)\ge 0$, i.e. $a^5+b^5\ge a^2b^2(a+b)$. Hence

$$\frac{ab}{a^5+b^5+ab}\le\frac{ab}{a^2b^2(a+b)+ab}=\frac{abc^2}{a^2b^2c^2(a+b)+abc^2}=\frac{c}{a+b+c}.$$

Now, the left side of the inequality to be proved does not exceed $\frac{c}{a+b+c}+\frac{a}{a+b+c}+\frac{b}{a+b+c}=1$. Equality holds if and only if $a=b=c$.

2. Clearly $a_1>0$, and if $p\ne a_1$, we must have $a_n<0$, $|a_n|>|a_1|$, and $p=-a_n$. But then for sufficiently large odd k, $-a_n^k=|a_n|^k>(n-1)|a_1|^k$, so that $a_1^k+\cdots+a_n^k\le(n-1)|a_1|^k-|a_n|^k<0$, a contradiction. Hence $p=a_1$.
Now let $x>a_1$. From $a_1+\cdots+a_n\ge 0$ we deduce $\sum_{j=2}^n(x-a_j)\le(n-1)\left(x+\frac{a_1}{n-1}\right)$, so by the AM–GM inequality,

$$(x-a_2)\cdots(x-a_n)\le\left(x+\frac{a_1}{n-1}\right)^{n-1}\le x^{n-1}+x^{n-2}a_1+\cdots+a_1^{n-1}. \quad (1)$$

The last inequality holds because $\binom{n-1}{r}\le(n-1)^r$ for all $r\ge 0$. Multiplying (1) by $(x-a_1)$ yields the desired inequality.

3. Since $a_1>2$, it can be written as $a_1=b+b^{-1}$ for some $b>0$. Furthermore, $a_1^2-2=b^2+b^{-2}$ and hence $a_2=(b^2+b^{-2})(b+b^{-1})$. We prove that

$$a_n=\left(b+b^{-1}\right)\left(b^2+b^{-2}\right)\left(b^4+b^{-4}\right)\cdots\left(b^{2^{n-1}}+b^{-2^{n-1}}\right)$$

by induction. Indeed, $\frac{a_{n+1}}{a_n}=\left(\frac{a_n}{a_{n-1}}\right)^2-2=\left(b^{2^{n-1}}+b^{-2^{n-1}}\right)^2-2=b^{2^n}+b^{-2^n}$.
Now we have

$$\sum_{i=1}^n\frac{1}{a_i}=1+\frac{b}{b^2+1}+\frac{b^3}{(b^2+1)(b^4+1)}+\cdots \\ \cdots+\frac{b^{2^n-1}}{(b^2+1)(b^4+1)\dots(b^{2^n}+1)}. \quad (1)$$

Note that $\frac12(a+2-\sqrt{a^2-4})=1+\frac1b$; hence we must prove that the right side in (1) is less than $\frac1b$. This follows from the fact that

$$\frac{b^{2^k}}{(b^2+1)(b^4+1)\cdots(b^{2^k}+1)} \\ =\frac{1}{(b^2+1)(b^4+1)\cdots(b^{2^{k-1}}+1)}-\frac{1}{(b^2+1)(b^4+1)\cdots(b^{2^k}+1)};$$

hence the right side in (1) equals $\frac1b\left(1-\frac{1}{(b^2+1)(b^4+1)\dots(b^{2^n}+1)}\right)$, and this is clearly less than $1/b$.

4. Consider the function

$$f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}.$$

Since f is strictly decreasing from $+\infty$ to 0 on the interval $(0, +\infty)$, there exists exactly one $R > 0$ for which $f(R) = 1$. This R is also the only positive real root of the given polynomial.
Since $\ln x$ is a concave function on $(0, +\infty)$, Jensen's inequality gives us

$$\sum_{j=1}^{n} \frac{a_j}{A} \left(\ln \frac{A}{R^j} \right) \le \ln \left(\sum_{j=1}^{n} \frac{a_j}{A} \cdot \frac{A}{R^j} \right) = \ln f(R) = 0.$$

Therefore $\sum_{j=1}^{n} a_j (\ln A - j \ln R) \le 0$, which is equivalent to $A \ln A \le B \ln R$, i.e., $A^A \le R^B$.

5. Considering the polynomials $\pm P(\pm x)$ we may assume w.l.o.g. that $a, b \ge 0$. We have four cases:
(1) $c \ge 0, d \ge 0$. Then $|a| + |b| + |c| + |d| = a + b + c + d = P(1) \le 1$.
(2) $c \ge 0, d < 0$. Then $|a|+|b|+|c|+|d| = a+b+c-d = P(1)-2P(0) \le 3$.
(3) $c < 0, d \ge 0$. Then

$$\begin{aligned} |a| + |b| + |c| + |d| &= a + b - c + d \\ &= \frac{4}{3}P(1) - \frac{1}{3}P(-1) - \frac{8}{3}P(1/2) + \frac{8}{3}P(-1/2) \le 7. \end{aligned}$$

(4) $c < 0, d < 0$. Then

$$\begin{aligned} |a| + |b| + |c| + |d| &= a + b - c - d \\ &= \frac{5}{3}P(1) - 4P(1/2) + \frac{4}{3}P(-1/2) \le 7. \end{aligned}$$

Remark. It can be shown that the maximum of 7 is attained only for $P(x) = \pm(4x^3 - 3x)$.

6. Let $f(x), g(x)$ be polynomials with integer coefficients such that

$$f(x)(x+1)^n + g(x)(x^n + 1) = k_0. \qquad (*)$$

Write $n = 2^r m$ for m odd and note that $x^n + 1 = (x^{2^r} + 1)B(x)$, where $B(x) = x^{2^r(m-1)} - x^{2^r(m-2)} + \cdots - x^{2^r} + 1$. Moreover, $B(-1) = 1$; hence $B(x) - 1 = (x+1)c(x)$ and thus

$$R(x)B(x) + 1 = (B(x) - 1)^n = (x+1)^n c(x)^n \qquad (1)$$

for some polynomials $c(x)$ and $R(x)$.
The zeros of the polynomial $x^{2^r} + 1$ are ω_j, with $\omega_1 = \cos \frac{\pi}{2^r} + i \sin \frac{\pi}{2^r}$, and $\omega_j = \omega^{2j-1}$ for $1 \le j \le 2^r$. We have

$$(\omega_1 + 1)(\omega_2 + 1) \cdots (\omega_{2^{r+1}} + 1) = 2. \tag{2}$$

From $(*)$ we also get $f(\omega_j)(\omega_j + 1)^n = k_0$ for $j = 1, 2, \ldots, 2^r$. Since $A = f(\omega_1)f(\omega_2)\cdots f(\omega_{2^r})$ is a symmetric polynomial in $\omega_1, \ldots, \omega_{2^r}$ with integer coefficients, A is an integer. Consequently, taking the product over $j = 1, 2, \ldots, 2^r$ and using (2) we deduce that $2^n A = k_0^{2^r}$ is divisible by $2^n = 2^{2^r m}$. Hence $2^m \mid k_0$.
Furthermore, since $\omega_j + 1 = (\omega_1 + 1)p_j(\omega_1)$ for some polynomial p_j with integer coefficients, (2) gives $(\omega_1 + 1)^{2^r} p(\omega_1) = 2$, where $p(x) = p_2(x) \cdots p_{2^r}(x)$ has integer coefficients. But then the polynomial $(x + 1)^{2^r} p(x) - 2$ has a zero $x = \omega_1$, so it is divisible by its minimal polynomial $x^{2^r} + 1$. Therefore

$$(x + 1)^{2^r} p(x) = 2 + (x^{2^r} + 1)q(x) \tag{3}$$

for some polynomial $q(x)$. Raising (3) to the mth power we get $(x + 1)^n p(x)^n = 2^m + (x^{2^r} + 1)Q(x)$ for some polynomial $Q(x)$ with integer coefficients. Now using (1) we obtain

$$\begin{aligned}(x + 1)^n c(x)^n (x^{2^r} + 1)Q(x) = {} & (x^{2^r} + 1)Q(x) + (x^{2^r} + 1)Q(x)B(x)R(x) \\ = {} & (x + 1)^n p(x)^n - 2^m + (x^n + 1)Q(X)R(x).\end{aligned}$$

Therefore $(x+1)^n f(x) + (x^n + 1)g(x) = 2^m$ for some polynomials $f(x), g(x)$ with integer coefficients, and $k_0 = 2^m$.

7. We are given that $f(x+a+b) - f(x+a) = f(x+b) - f(x)$, where $a = 1/6$ and $b = 1/7$. Summing up these equations for $x, x+b, \ldots, x+6b$ we obtain $f(x+a+1) - f(x+a) = f(x+1) - f(x)$. Summing up the new equations for $x, x + a, \ldots, x + 5a$ we obtain that

$$f(x + 2) - f(x + 1) = f(x + 1) - f(x).$$

It follows by induction that $f(x + n) - f(x) = n[f(x + 1) - f(x)]$. If $f(x + 1) \neq f(x)$, then $f(x + n) - f(x)$ will exceed in absolute value an arbitrarily large number for a sufficiently large n, contradicting the assumption that f is bounded. Hence $f(x + 1) = f(x)$ for all x.

8. Putting $m = n = 0$ we obtain $f(0) = 0$ and consequently $f(f(n)) = f(n)$ for all n. Thus the given functional equation is equivalent to

$$f(m + f(n)) = f(m) + f(n), \qquad f(0) = 0\,.$$

Clearly one solution is $(\forall x)\ f(x) = 0$. Suppose f is not the zero function. We observe that f has nonzero fixed points (for example, any $f(n)$ is a fixed point). Let a be the smallest nonzero fixed point of f. By induction, each ka ($k \in \mathbb{N}$) is a fixed point too. We claim that all fixed points of f are of this form. Indeed, suppose that $b = ka + i$ is a fixed point, where $i < a$. Then

$$b = f(b) = f(ka+i) = f(i+f(ka)) = f(i) + f(ka) = f(i) + ka;$$

hence $f(i) = i$. Hence $i = 0$.
Since the set of values of f is a set of its fixed points, it follows that for $i = 0, 1, \dots, a-1$, $f(i) = an_i$ for some integers $n_i \geq 0$ with $n_0 = 0$.
Let $n = ka+i$ be any positive integer, $0 \leq i < a$. As before, the functional equation gives us

$$f(n) = f(ka+i) = f(i) + ka = (n_i + k)a.$$

Besides the zero function, this is the general solution of the given functional equation. To verify this, we plug in $m = ka+i$, $n = la+j$ and obtain

$$\begin{aligned} f(m+f(n)) &= f(ka+i+f(la+j)) = f((k+l+n_j)a+i) \\ &= (k+l+n_j+n_i)a = f(m)+f(n). \end{aligned}$$

9. From the definition of $a(n)$ we obtain

$$a(n) - a([n/2]) = \begin{cases} 1 \text{ if } n \equiv 0 \text{ or } n \equiv 3 \pmod 4; \\ -1 \text{ if } n \equiv 1 \text{ or } n \equiv 2 \pmod 4). \end{cases}$$

Let $n = \overline{b_k b_{k-1} \dots b_1 b_0}$ be the binary representation of n, where we assume $b_k = 1$. If we define $p(n)$ and $q(n)$ to be the number of indices $i = 0, 1, \dots, k-1$ with $b_i = b_{i+1}$ and the number of $i = 0, 1, \dots, k-1$ with $b_i \neq b_{i+1}$ respectively, we get

$$a(n) = p(n) - q(n). \tag{1}$$

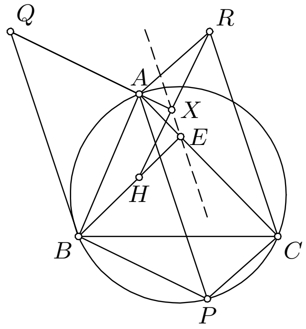
(a) The maximum value of $a(n)$ for $n \leq 1996$ is 9 when $p(n) = 9$ and $q(n) = 0$, i.e., in the case $n = \overline{1111111111}_2 = 1023$.
The minimum value is -10 and is attained when $p(n) = 0$ and $q(n) = 10$, i.e., only for $n = \overline{10101010101}_2 = 1365$.

(b) From (1) we have that $a(n) = 0$ is equivalent to $p(n) = q(n) = k/2$. Hence k must be even, and the $k/2$ indices i for which $b_i = b_{i+1}$ can be chosen in exactly $\binom{k}{k/2}$ ways. Thus the number of positive integers $n < 2^{11} = 2048$ with $a(n) = 0$ is equal to

$$\binom{0}{0} + \binom{2}{1} + \binom{4}{2} + \binom{6}{3} + \binom{8}{4} + \binom{10}{5} = 351.$$

But five of these numbers exceed 1996: these are $2002 = \overline{11111010010}_2$, $2004 = \overline{11111010100}_2$, $2006 = \overline{11111010110}_2$, $2010 = \overline{11111011010}_2$, $2026 = \overline{11111101010}_2$. Therefore there are 346 numbers $n \leq 1996$ for which $a(n) = 0$.

10. We first show that H is the common orthocenter of the triangles ABC and AQR.



Let G, G', H' be respectively the centroid of $\triangle ABC$, the centroid of $\triangle PBC$, and the orthocenter of $\triangle PBC$. Since the triangles ABC and PBC have a common circumcenter, from the properties of the Euler line we get $\overrightarrow{HH'} = 3\overrightarrow{GG'} = \overrightarrow{AP}$. But $\triangle AQR$ is exactly the image of $\triangle PBC$ under translation by $\overrightarrow{AP}$; hence the orthocenter of AQR coincides with H. (*Remark:* This can be shown by noting that $AHBQ$ is cyclic.)

Now we have that $RH \perp AQ$; hence $\angle AXH = 90^\circ = \angle AEH$. It follows that $AXEH$ is cyclic; hence

$$\angle EXQ = 180^\circ - \angle AHE = 180^\circ - \angle BCA = 180^\circ - \angle BPA = \angle PAQ$$

(as oriented angles). Hence $EX \parallel AP$.

11. Let X, Y, Z respectively be the feet of the perpendiculars from P to BC, CA, AB. Examining the cyclic quadrilaterals $AZPY$, $BXPZ$, $CYPX$, one can easily see that $\angle XZY = \angle APB - \angle C$ and $XY = PC \sin\angle C$. The first relation gives that XYZ is isosceles with $XY = XZ$, so from the second relation $PB \sin\angle B = PC \sin\angle C$. Hence $AB/PB = AC/PC$. This implies that the bisectors BD and CD of $\angle ABP$ and $\angle ACP$ divide the segment AP in equal ratios; i.e., they concur with AP.

Second solution. Take that X, Y, Z are the points of intersection of AP, BP, CP with the circumscribed circle of ABC instead. We similarly obtain $XY = XZ$. If we write $AP \cdot PX = BP \cdot PY = CP \cdot PZ = k$, from the similarity of $\triangle APC$ and $\triangle ZPX$ we get

$$\frac{AC}{XZ} = \frac{AP}{PZ} = \frac{AP \cdot CP}{k},$$

i.e., $XZ = \frac{k \cdot AC \cdot BP}{AP \cdot BP \cdot CP}$. It follows again that $AC/AB = PC/PB$.

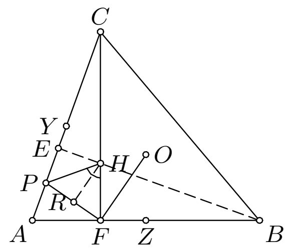
Third solution. Apply an inversion with center at A and radius r, and denote by $\overline{Q}$ the image of any point Q. Then the given condition becomes $\angle \overline{BCP} = \angle \overline{CBP}$, i.e., $\overline{BP} = \overline{PC}$. But

$$\overline{PB} = \frac{r^2}{AP \cdot AB} PB,$$

so $AC/AB = PC/PB$.

Remark. Moreover, it follows that the locus of P is an arc of the circle of Apollonius through C.

12. It is easy to see that P lies on the segment AC. Let E be the foot of the altitude BH and Y, Z the midpoints of AC, AB respectively. Draw the perpendicular HR to FP ($R \in FP$). Since Y is the circumcenter of $\triangle FCA$, we have $\angle FYA = 180^\circ - 2\angle A$. Also, $OFPY$ is cyclic; hence $\angle OPF = \angle OYF = 2\angle A - 90^\circ$. Next, $\triangle OZF$ and $\triangle HRF$ are similar, so $OZ/OF = HR/HF$. This leads to $HR \cdot OF = HF \cdot OZ = \frac{1}{2}HF \cdot HC = \frac{1}{2}HE \cdot HB = HE \cdot OY \implies HR/HE = OY/OF$. Moreover, $\angle EHR = \angle FOY$; hence the triangles EHR and FOY are similar. Consequently $\angle HPC = \angle HRE = \angle OYF = 2\angle A - 90^\circ$, and finally, $\angle FHP = \angle HPC + \angle HCP = \angle A$.



Second solution. As before, $\angle HFY = 90^\circ - \angle A$, so it suffices to show that $HP \perp FY$. The points O, F, P, Y lie on a circle, say Ω_1 with center at the midpoint Q of OP. Furthermore, the points F, Y lie on the nine-point circle Ω of $\triangle ABC$ with center at the midpoint N of OH. The segment FY is the common chord of Ω_1 and Ω, from which we deduce that $NQ \perp FY$. However, $NQ \parallel HP$, and the result follows.

Third solution. Let H' be the point symmetric to H with respect to AB. Then H' lies on the circumcircle of ABC. Let the line FP meet the circumcircle at U, V and meet $H'B$ at P'. Since $OF \perp UV$, F is the midpoint of UV. By the butterfly theorem, F is also the midpoint of PP'. Therefore $\triangle H'FP' \cong FHP$; hence $\angle FHP = \angle FH'B = \angle A$.

Remark. It is possible to solve the problem using trigonometry. For example, $\frac{FZ}{ZO} = \frac{FK}{KP} = \frac{\sin(A-B)}{\cos C}$, where K is on CF with $PK \perp CF$. Then $\frac{CF}{KP} = \frac{\sin(A-B)}{\cos C} + \tan A$, from which one obtains formulas for KP and KH. Finally, we can calculate $\tan \angle FHP = \frac{KP}{KH} = \cdots = \tan A$.

Second remark. Here is what happens when $BC \le CA$. If $\angle A > 45^\circ$, then $\angle FHP = \angle A$. If $\angle A = 45^\circ$, the point P escapes to infinity. If $\angle A < 45^\circ$, the point P appears on the extension of AC over C, and $\angle FHP = 180^\circ - \angle A$.

13. By the law of cosines applied to $\triangle CA_1B_1$, we obtain

$$A_1B_1^2 = A_1C^2 + B_1C^2 - A_1C \cdot B_1C \ge A_1C \cdot B_1C.$$

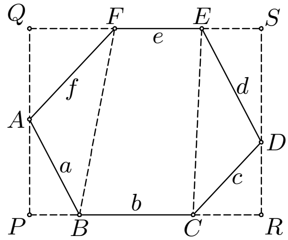
Analogously, $B_1C_1^2 \ge B_1A \cdot C_1A$ and $C_1A_1^2 \ge C_1B \cdot A_1B$, so that multiplying these inequalities yields

$$A_1B_1^2 \cdot B_1C_1^2 \cdot C_1A_1^2 \ge A_1B \cdot A_1C \cdot B_1A \cdot B_1C \cdot C_1A \cdot C_1B. \qquad (1)$$

Now, the lines AA_1, BB_1, CC_1 concur, so by Ceva's theorem, $A_1B \cdot B_1C \cdot C_1A = AB_1 \cdot BC_1 \cdot CA_1$, which together with (1) gives the desired inequality. Equality holds if and only if $CA_1 = CB_1$, etc.

14. Let a, b, c, d, e, and f denote the lengths of the sides AB, BC, CD, DE, EF, and FA respectively.

Note that $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$. Draw the lines PQ and RS through A and D perpendicular to BC and EF respectively ($P, R \in BC$, $Q, S \in EF$). Then $BF \geq PQ = RS$. Therefore $2BF \geq PQ + RS$, or



$$\begin{aligned} &2BF \geq (a\sin B + f\sin C) + (c\sin C + d\sin B), \\ \text{and similarly,}\quad &2BD \geq (c\sin A + b\sin B) + (e\sin B + f\sin A), \\ &2DF \geq (e\sin C + d\sin A) + (a\sin A + b\sin C). \end{aligned} \tag{1}$$

Next, we have the following formulas for the considered circumradii:

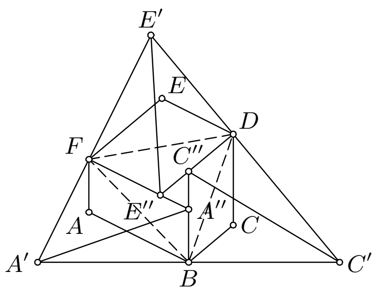
$$R_A = \frac{BF}{2\sin A}, \qquad R_C = \frac{BD}{2\sin C}, \qquad R_E = \frac{DF}{2\sin E}.$$

It follows from (1) that

$$\begin{aligned} R_A + R_C + R_E &\geq \frac{1}{4}a\left(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B}\right) + \frac{1}{4}b\left(\frac{\sin C}{\sin B} + \frac{\sin B}{\sin C}\right) + \cdots \\ &\geq \frac{1}{2}(a + b + \cdots) = \frac{P}{2}, \end{aligned}$$

with equality if and only if $\angle A = \angle B = \angle C = 120^\circ$ and $FB \perp BC$ etc., i.e., if and only if the hexagon is regular.

Second solution. Let us construct points A'', C'', E'' such that $ABA''F$, $CDC''B$, and $EFE''D$ are parallelograms. It follows that A'', C'', B are collinear and also C'', E'', B and E'', A'', F. Furthermore, let A' be the intersection of the perpendiculars through F and B to FA'' and BA'', respectively, and let C' and E' be analogously defined. Since $A'FA''B$ is cyclic with the diameter being $A'A''$ and since $\triangle FA''B \cong \triangle BAF$, it follows that $2R_A = A'A'' = x$.



Similarly, $2R_C = C'C'' = y$ and $2R_E = E'E'' = z$. We also have $AB = FA'' = y_a$, $AF = A''B = z_a$, $CD = C''B = z_c$, $CB = C''D = x_c$, $EF = E''D = x_e$, and $ED = E''F = y_e$. The original inequality we must prove now becomes

$$x + y + z \geq y_a + z_a + z_c + x_c + x_e + y_e\,. \tag{1}$$

We now follow and generalize the standard proof of the Erdős–Mordell inequality (for the triangle $A'C'E'$), which is what (1) is equivalent to when $A'' = C'' = E''$.
We set $C'E' = a$, $A'E' = c$ and $A'C' = e$. Let A_1 be the point symmetric to A'' with respect to the bisector of $\angle E'A'C'$. Let F_1 and B_1 be the feet of the perpendiculars from A_1 to $A'C'$ and $A'E'$, respectively. In that case, $A_1F_1 = A''F = y_a$ and $A_1B_1 = A''B = z_a$. We have

$$\begin{aligned} ax = A'A_1 \cdot E'C' \geq 2S_{A'E'A_1C'} &= 2S_{A'E'A_1} + 2S_{A'C'A_1} \\ &= cz_a + ey_a \ . \end{aligned}$$

Similarly, $cy \geq ex_c + az_c$ and $ez \geq ay_e + cx_e$. Thus

$$\begin{aligned} x + y + z &\geq \frac{c}{a}z_a + \frac{a}{c}z_c + \frac{e}{c}x_c + \frac{c}{e}x_e + \frac{a}{e}y_e + \frac{e}{a}y_a \\ &= \left(\frac{c}{a} + \frac{a}{c}\right)\left(\frac{z_a + z_c}{2}\right) + \left(\frac{c}{a} - \frac{a}{c}\right)\left(\frac{z_a - z_c}{2}\right) + \cdots \ . \end{aligned} \tag{2}$$

Let us set $a_1 = \frac{x_c - x_e}{2}$, $c_1 = \frac{y_e - y_a}{2}$, $e_1 = \frac{z_a - z_c}{2}$. We note that $\triangle A''C''E'' \sim \triangle A'C'E'$ and hence $a_1/a = c_1/c = e_1/e = k$. Thus $\left(\frac{c}{a} - \frac{a}{c}\right)e_1 + \left(\frac{e}{c} - \frac{c}{e}\right)a_1 + \left(\frac{a}{e} - \frac{e}{a}\right)c_1 = k\left(\frac{ce}{a} - \frac{ae}{c} + \frac{ea}{c} - \frac{ca}{e} + \frac{ac}{e} - \frac{ec}{a}\right) = 0$. Equation (2) reduces to

$$\begin{aligned} x + y + z \geq &\left(\frac{c}{a} + \frac{a}{c}\right)\left(\frac{z_a + z_c}{2}\right) + \left(\frac{e}{c} + \frac{c}{e}\right)\left(\frac{x_e + x_c}{2}\right) \\ &+ \left(\frac{a}{e} + \frac{e}{a}\right)\left(\frac{y_a + y_e}{2}\right). \end{aligned}$$

Using $c/a + a/c$, $e/c + c/e$, $a/e + e/a \geq 2$ we finally get $x + y + z \geq y_a + z_a + z_c + x_c + x_e + y_e$.
Equality holds if and only if $a = c = e$ and $A'' = C'' = E'' =$ center of $\triangle A'C'E'$, i.e., if and only if $ABCDEF$ is regular.

Remark. From the second proof it is evident that the Erdős–Mordell inequality is a special case of the problem. if P_a, P_b, P_c are the feet of the perpendiculars from a point P inside $\triangle ABC$ to the sides BC, CA, AB, and $P_aPP_bP'_c, P_bPP_cP'_a, P_cPP_aP'_b$ parallelograms, we can apply the problem to the hexagon $P_aP'_cP_bP'_aP_cP'_b$ to prove the Erdős–Mordell inequality for $\triangle ABC$ and point P.

15. Denote by $ABCD$ and $EFGH$ the two rectangles, where $AB = a$, $BC = b$, $EF = c$, and $FG = d$. Obviously, the first rectangle can be placed within the second one with the angle α between AB and EF if and only if

$$a\cos\alpha + b\sin\alpha \leq c, \qquad a\sin\alpha + b\cos\alpha \leq d. \tag{1}$$

Hence $ABCD$ can be placed within $EFGH$ if and only if there is an $\alpha \in [0, \pi/2]$ for which (1) holds.

The lines $l_1(ax+by=c)$ and $l_2(bx+ay=d)$ and the axes x and y bound a region $\mathcal{R}$. By (1), the desired placement of the rectangles is possible if and only if $\mathcal{R}$ contains some point $(\cos\alpha, \sin\alpha)$ of the unit circle centered at the origin $(0,0)$. This in turn holds if and only if the intersection point L of l_1 and l_2 lies outside the unit circle. It is easily computed that L has coordinates $\left(\frac{bd-ac}{b^2-a^2}, \frac{bc-ad}{b^2-a^2}\right)$. Now L being outside the unit circle is exactly equivalent to the inequality we want to prove.

Remark. If equality holds, there is exactly one way of placing. This happens, for example, when $(a,b)=(5,20)$ and $(c,d)=(16,19)$.

Second remark. This problem is essentially very similar to (SL89-2).

16. Let A_1 be the point of intersection of OA' and BC; similarly define B_1 and C_1. From the similarity of triangles OBA_1 and $OA'B$ we obtain $OA_1 \cdot OA' = R^2$. Now it is enough to show that $8OA_1 \cdot OB' \cdot OC' \le R^3$. Thus we must prove that

$$\lambda\mu\nu \le \frac{1}{8}, \qquad \text{where} \qquad \frac{OA_1}{OA} = \lambda, \quad \frac{OB_1}{OB} = \mu, \quad \frac{OC_1}{OC} = \nu. \tag{1}$$

On the other hand, we have

$$\frac{\lambda}{1+\lambda} + \frac{\mu}{1+\mu} + \frac{\nu}{1+\nu} = \frac{S_{OBC}}{S_{ABC}} + \frac{S_{AOC}}{S_{ABC}} + \frac{S_{ABO}}{S_{ABC}} = 1.$$

Simplifying this relation, we get

$$1 = \lambda\mu + \mu\nu + \nu\lambda + 2\lambda\mu\nu \ge 3(\lambda\mu\nu)^{2/3} + 2\lambda\mu\nu,$$

which cannot hold if $\lambda\mu\nu > \frac{1}{8}$. Hence $\lambda\mu\nu \le \frac{1}{8}$, with equality if and only if $\lambda = \mu = \nu = \frac{1}{2}$. This implies that O is the centroid of ABC, and consequently, that the triangle is equilateral.

Second solution. In the official solution, the inequality to be proved is transformed into

$$\cos(A-B)\cos(B-C)\cos(C-A) \ge 8\cos A\cos B\cos C.$$

Since $\frac{\cos(B-C)}{\cos A} = -\frac{\cos(B-C)}{\cos(B+C)} = \frac{\tan B\tan C+1}{\tan B\tan C-1}$, the last inequality becomes $(xy+1)(yz+1)(zx+1) \ge 8(xy-1)(yz-1)(zx-1)$, where we write x, y, z for $\tan A, \tan B, \tan C$. Using the relation $x+y+z = xyz$, we can reduce this inequality to

$$(2x+y+z)(x+2y+z)(x+y+2z) \ge 8(x+y)(y+z)(z+x).$$

This follows from the AM–GM inequality: $2x+y+z = (x+y)+(x+z) \ge 2\sqrt{(x+y)(x+z)}$, etc.

17. Let the diagonals AC and BD meet in X. Either $\angle AXB$ or $\angle AXD$ is geater than or equal to $90°$, so we assume w.l.o.g. that $\angle AXB \geq 90°$. Let $\alpha, \beta, \alpha', \beta'$ denote $\angle CAB$, $\angle ABD$, $\angle BDC$, $\angle DCA$. These angles are all acute and satisfy $\alpha + \beta = \alpha' + \beta'$. Furthermore,

$$R_A = \frac{AD}{2\sin\beta}, \qquad R_B = \frac{BC}{2\sin\alpha}, \qquad R_C = \frac{BC}{2\sin\alpha'}, \qquad R_D = \frac{AD}{2\sin\beta'}.$$

Let $\angle B + \angle D = 180°$. Then A, B, C, D are concyclic and trivially $R_A + R_C = R_B + R_D$.

Let $\angle B + \angle D > 180°$. Then D lies within the circumcircle of ABC, which implies that $\beta > \beta'$. Similarly $\alpha < \alpha'$, so we obtain $R_A < R_D$ and $R_C < R_B$. Thus $R_A + R_C < R_B + R_D$.

Let $\angle B + \angle D < 180°$. As in the previous case, we deduce that $R_A > R_D$ and $R_C > R_B$, so $R_A + R_C > R_B + R_D$.

18. We first prove the result in the simplest case. Given a 2-gon ABA and a point O, let a, b, c, h denote OA, OB, AB, and the distance of O from AB. Then $D = a + b$, $P = 2c$, and $H = 2h$, so we should show that

$$(a+b)^2 \geq 4h^2 + c^2. \tag{1}$$

Indeed, let l be the line through O parallel to AB, and D the point symmetric to B with respect to l. Then $(a+b)^2 = (OA+OB)^2 = (OA+OD)^2 \geq AD^2 = c^2 + 4h^2$.

Now we pass to the general case. Let $A_1A_2\dots A_n$ be the polygon $\mathcal{F}$ and denote by d_i, p_i, and h_i respectively OA_i, A_iA_{i+1}, and the distance of O from A_iA_{i+1} (where $A_{n+1} = A_1$). By the case proved above, we have for each i, $d_i + d_{i+1} \geq \sqrt{4h_i^2 + p_i^2}$. Summing these inequalities for $i = 1, \dots, n$ and squaring, we obtain

$$4D^2 \geq \left(\sum_{i=1}^n \sqrt{4h_i^2 + p_i^2}\right)^2.$$

It remains only to prove that $\sum_{i=1}^n \sqrt{4h_i^2 + p_i^2} \geq \sqrt{\sum_{i=1}^n (4h_i^2 + p_i^2)} = \sqrt{4H^2 + D^2}$. But this follows immediately from the Minkowski inequality. Equality holds if and only if it holds in (1) and in the Minkowski inequality, i.e., if and only if $d_1 = \dots = d_n$ and $h_1/p_1 = \dots = h_n/p_n$. This means that $\mathcal{F}$ is inscribed in a circle with center at O and $p_1 = \dots = p_n$, so $\mathcal{F}$ is a regular polygon and O its center.

19. It is easy to check that after 4 steps we will have all a, b, c, d even. Thus $|ab-cd|, |ac-bd|, |ad-bc|$ remain divisible by 4, and clearly are not prime. The answer is *no*.

Second solution. After one step we have $a+b+c+d = 0$. Then $ac - bd = ac + b(a+b+c) = (a+b)(b+c)$ etc., so

$$|ab - cd| \cdot |ac - bd| \cdot |ad - bc| = (a+b)^2(a+c)^2(b+c)^2.$$

However, the product of three primes cannot be a square, hence the answer is *no*.

20. Let $15a + 16b = x^2$ and $16a - 15b = y^2$, where $x, y \in \mathbb{N}$. Then we obtain

$$x^4 + y^4 = (15a+16b)^2 + (16a-15b)^2 = (15^2+16^2)(a^2+b^2) = 481(a^2+b^2).$$

In particular, $481 = 13 \cdot 37 \mid x^4 + y^4$. We have the following lemma.

Lemma. Suppose that $p \mid x^4 + y^4$, where $x, y \in \mathbb{Z}$ and p is an odd prime, where $p \not\equiv 1 \pmod 8$. Then $p \mid x$ and $p \mid y$.

Proof. Since $p \mid x^8 - y^8$ and by Fermat's theorem $p \mid x^{p-1} - y^{p-1}$, we deduce that $p \mid x^d - y^d$, where $d = (p-1, 8)$. But $d \neq 8$, so $d \mid 4$. Thus $p \mid x^4 - y^4$, which implies that $p \mid 2y^4$, i.e., $p \mid y$ and $p \mid x$.

In particular, we can conclude that $13 \mid x, y$ and $37 \mid x, y$. Hence x and y are divisible by 481. Thus each of them is at least 481.

On the other hand, $x = y = 481$ is possible. It is sufficient to take $a = 31 \cdot 481$ and $b = 481$.

Second solution. Note that $15x^2 + 16y^2 = 481a^2$. It can be directly verified that the divisibility of $15x^2 + 16y^2$ by 13 and by 37 implies that both x and y are divisible by both primes. Thus $481 \mid x, y$.

21. (a) It clearly suffices to show that for every integer c there exists a quadratic sequence with $a_0 = 0$ and $a_n = c$, i.e., that c can be expressed as $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$. Since

$$(n+1)^2 - (n+2)^2 - (n+3)^2 + (n+4)^2 = 4,$$

we observe that if our claim is true for c, then it is also true for $c \pm 4$. Thus it remains only to prove the claim for $c = 0, 1, 2, 3$. But one immediately finds $1 = 1^2$, $2 = -1^2 - 2^2 - 3^2 + 4^2$, and $3 = -1^2 + 2^2$, while the case $c = 0$ is trivial.

(b) We have $a_0 = 0$ and $a_n = 1996$. Since $a_n \leq 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$, we get $a_{17} \leq 1785$, so $n \geq 18$. On the other hand, a_{18} is of the same parity as $1^2 + 2^2 + \cdots + 18^2 = 2109$, so it cannot be equal to 1996. Therefore we must have $n \geq 19$. To construct a required sequence with $n = 19$, we note that $1^2 + 2^2 + \cdots + 19^2 = 2470 = 1996 + 2 \cdot 237$; hence it is enough to write 237 as a sum of distinct squares. Since $237 = 14^2 + 5^2 + 4^2$, we finally obtain

$$1996 = 1^2 + 2^2 + 3^2 - 4^2 - 5^2 + 6^2 + \cdots + 13^2 - 14^2 + 15^2 + \cdots + 19^2.$$

22. Let $a, b \in \mathbb{N}$ satisfy the given equation. It is not possible that $a = b$ (since it leads to $a^2 + 2 = 2a$), so we assume w.l.o.g. that $a > b$. Next, for $a > b = 1$ the equation becomes $a^2 = 2a$, and one obtains a solution $(a, b) = (2, 1)$.

Let $b > 1$. If $\left[\frac{a^2}{b}\right] = \alpha$ and $\left[\frac{b^2}{a}\right] = \beta$, then we trivially have $ab \geq \alpha\beta$. Since also $\frac{a^2+b^2}{ab} \geq 2$, we obtain $\alpha + \beta \geq \alpha\beta + 2$, or equivalently

$(\alpha-1)(\beta-1) \le -1$. But $\alpha \ge 1$, and therefore $\beta = 0$. It follows that $a > b^2$, i.e., $a = b^2 + c$ for some $c > 0$. Now the given equation becomes $b^3 + 2bc + \left[\frac{c^2}{b}\right] = \left[\frac{b^4+2b^2c+b^2+c^2}{b^3+bc}\right] + b^3 + bc$, which reduces to

$$(c-1)b + \left[\frac{c^2}{b}\right] = \left[\frac{b^2(c+1)+c^2}{b^3+bc}\right]. \tag{1}$$

If $c = 1$, then (1) always holds, since both sides are 0. We obtain a family of solutions $(a,b) = (n, n^2+1)$ or $(a,b) = (n^2+1, n)$. Note that the solution $(1,2)$ found earlier is obtained for $n = 1$.
If $c > 1$, then (1) implies that $\frac{b^2(c+1)+c^2}{b^3+bc} \ge (c-1)b$. This simplifies to

$$c^2(b^2-1) + b^2(c(b^2-2) - (b^2+1)) \le 0. \tag{2}$$

Since $c \ge 2$ and $b^2 - 2 \ge 0$, the only possibility is $b = 2$. But then (2) becomes $3c^2 + 8c - 20 \le 0$, which does not hold for $c \ge 2$.
Hence the only solutions are (n, n^2+1) and (n^2+1, n), $n \in \mathbb{N}$.

23. We first observe that the given functional equation is equivalent to

$$4f\left(\frac{(3m+1)(3n+1)-1}{3}\right) + 1 = (4f(m)+1)(4f(n)+1).$$

This gives us the idea of introducing a function $g : 3\mathbb{N}_0 + 1 \to 4\mathbb{N}_0 + 1$ defined as $g(x) = 4f\left(\frac{x-1}{3}\right) + 1$. By the above equality, g will be multiplicative, i.e.,

$$g(xy) = g(x)g(y) \qquad \text{for all } x, y \in 3\mathbb{N}_0 + 1.$$

Conversely, any multiplicative bijection g from $3\mathbb{N}_0 + 1$ onto $4\mathbb{N}_0 + 1$ gives us a function f with the required property: $f(x) = \frac{g(3x+1)-1}{4}$.
It remains to give an example of such a function g. Let P_1, P_2, Q_1, Q_2 be the sets of primes of the forms $3k+1$, $3k+2$, $4k+1$, and $4k+3$, respectively. It is well known that these sets are infinite. Take any bijection h from $P_1 \cup P_2$ onto $Q_1 \cup Q_2$ that maps P_1 bijectively onto Q_1 and P_2 bijectively onto Q_2. Now define g as follows: $g(1) = 1$, and for $n = p_1p_2\cdots p_m$ (p_i's need not be different) define $g(n) = h(p_1)h(p_2)\cdots h(p_m)$. Note that g is well-defined. Indeed, among the p_i's an even number are of the form $3k+2$, and consequently an even number of $h(p_i)$s are of the form $4k+3$. Hence the product of the $h(p_i)$'s is of the form $4k+1$. Also, it is obvious that g is multiplicative. Thus, the defined g satisfies all the required properties.

24. We shall work on the array of lattice points defined by $\mathcal{A} = \{(x,y) \in \mathbb{Z}^2 \mid 0 \le x \le 19,\ 0 \le y \le 11\}$. Our task is to move from $(0,0)$ to $(19,0)$ via the points of $\mathcal{A}$ so that each move has the form $(x,y) \to (x+a, y+b)$, where $a, b \in \mathbb{Z}$ and $a^2 + b^2 = r$.

(a) If r is even, then $a+b$ is even whenever $a^2+b^2=r$ $(a,b\in\mathbb{Z})$. Thus the parity of $x+y$ does not change after each move, so we cannot reach $(19,0)$ from $(0,0)$.
If $3\mid r$, then both a and b are divisible by 3, so if a point (x,y) can be reached from $(0,0)$, we must have $3\mid x$. Since $3\nmid 19$, we cannot get to $(19,0)$.

(b) We have $r=73=8^2+3^2$, so each move is either $(x,y)\to(x\pm8,y\pm3)$ or $(x,y)\to(x\pm3,y\pm8)$. One possible solution is shown in Fig. 1.

(c) We have $97=9^2+4^2$. Let us partition $\mathcal{A}$ as $\mathcal{B}\cup\mathcal{C}$, where $\mathcal{B}=\{(x,y)\in\mathcal{A}\mid 4\le y\le 7\}$. It is easily seen that moves of the type $(x,y)\to(x\pm9,y\pm4)$ always take us from the set $\mathcal{B}$ to $\mathcal{C}$ and vice versa, while the moves $(x,y)\to(x\pm4,y\pm9)$ always take us from $\mathcal{C}$ to $\mathcal{C}$. Furthermore, each move of the type $(x,y)\to(x\pm9,y\pm4)$ changes the parity of x, so to get from $(0,0)$ to $(19,0)$ we must have an odd number of such moves. On the other hand, with an odd number of such moves, starting from $\mathcal{C}$ we can end up only in $\mathcal{B}$, although the point $(19,0)$ is not in $\mathcal{B}$. Hence, the answer is *no*.

Remark. Part (c) can also be solved by examining all cells that can be reached from $(0,0)$. All these cells are marked in Fig. 2.

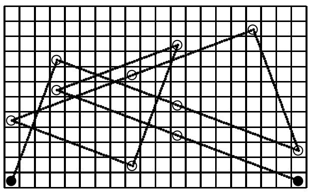


Fig. 1

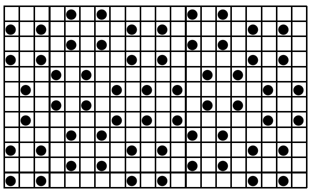


Fig. 2

25. Let the vertices in the bottom row be assigned an arbitrary coloring, and suppose that some two adjacent vertices receive the same color. The number of such colorings equals 2^n-2. It is easy to see that then the colors of the remaining vertices get fixed uniquely in order to satisfy the requirement. So in this case there are 2^n-2 possible colorings.
Next, suppose that the vertices in the bottom row are colored alternately red and blue. There are two such colorings. In this case, the same must hold for every row, and thus we get 2^n possible colorings.
It follows that the total number of considered colorings is $(2^n-2)+2^n=2^{n+1}-2$.

26. Denote the required maximum size by $M_k(m,n)$. If $m<\frac{n(n+1)}{2}$, then trivially $M=k$, so from now on we assume that $m\ge\frac{n(n+1)}{2}$.
First we give a lower bound for M. Let $r=r_k(m,n)$ be the largest integer such that $r+(r+1)+\cdots+(r+n-1)\le m$. This is equivalent to $nr\le m-\frac{n(n-1)}{2}\le n(r+1)$, so $r=\left[\frac{m}{n}-\frac{n-1}{2}\right]$. Clearly no n elements from $\{r+1,r+2,\dots,k\}$ add up to m, so

$$M \geq k - r_k(m,n) = k - \left[\frac{m}{n} - \frac{n-1}{2}\right]. \tag{1}$$

We claim that M is actually equal to $k - r_k(m,n)$. To show this, we shall prove by induction on n that if no n elements of a set $S \subseteq \{1,2,\dots,k\}$ add up to m, then $|S| \leq k - r_k(m,n)$.

For $n = 2$ the claim is true, because then for each $i = 1,\dots,r_k(m,2) = \left[\frac{m-1}{2}\right]$ at least one of i and $m-i$ must be excluded from S. Now let us assume that $n > 2$ and that the result holds for $n-1$. Suppose that $S \subseteq \{1,2,\dots,k\}$ does not contain n distinct elements with the sum m, and let x be the smallest element of S. We may assume that $x \leq r_k(m,n)$, because otherwise the statement is clear. Consider the set $S' = \{y - x \mid y \in S,\ y \neq x\}$. Then S' is a subset of $\{1,2,\dots,k-x\}$ no $n-1$ elements of which have the sum $m - nx$. Also, it is easily checked that $n - 1 \leq m - nx - 1 \leq k - x$, so we may apply the induction hypothesis, which yields that

$$|S| \leq 1 + k - x - r_k(m - nx, n-1) = k - \left[\frac{m-x}{n-1} - \frac{n}{2}\right]. \tag{2}$$

On the other hand, $\left(\frac{m-x}{n-1} - \frac{n}{2}\right) - r_k(m,n) = \frac{m - nx - \frac{n(n-1)}{2}}{n(n-1)} \geq 0$ because $x \leq r_k(m,n)$; hence (2) implies $|S| \leq k - r_k(m,n)$ as claimed.

27. Suppose that such sets of points $\mathcal{A},\mathcal{B}$ exist.
First, we observe that there exist five points A,B,C,D,E in $\mathcal{A}$ such that their convex hull does not contain any other point of $\mathcal{A}$. Indeed, take any point $A \in \mathcal{A}$. Since any two points of $\mathcal{A}$ are at distance at least 1, the number of points $X \in \mathcal{A}$ with $XA \leq r$ is finite for every $r > 0$. Thus it is enough to choose four points B,C,D,E of $\mathcal{A}$ that are closest to A. Now consider the convex hull $\mathcal{C}$ of A,B,C,D,E.
Suppose that $\mathcal{C}$ is a pentagon, say $ABCDE$. Then each of the disjoint triangles ABC, ACD, ADE contains a point of $\mathcal{B}$. Denote these points by P,Q,R. Then $\triangle PQR$ contains some point $F \in \mathcal{A}$, so F is inside $ABCDE$, a contradiction.
Suppose that $\mathcal{C}$ is a quadrilateral, say $ABCD$, with E lying within $ABCD$. Then the triangles ABE, BCE, CDE, DAE contain some points P,Q,R,S of $\mathcal{B}$ that form two disjoint triangles. It follows that there are two points of $\mathcal{A}$ inside $ABCD$, which is a contradiction.
Finally, suppose that $\mathcal{C}$ is a triangle with two points of $\mathcal{A}$ inside. Then $\mathcal{C}$ is the union of five disjoint triangles with vertices in $\mathcal{A}$, so there are at least five points of $\mathcal{B}$ inside $\mathcal{C}$. These five points make at least three disjoint triangles containing three points of $\mathcal{A}$. This is again a contradiction.
It follows that no such sets $\mathcal{A},\mathcal{B}$ exist.

28. Note that w.l.o.g., we can assume that p and q are coprime. Indeed, otherwise it suffices to consider the problem in which all x_i's and p,q are divided by $\gcd(p,q)$.

Let k, l be the number of indices i with $x_{i+1} - x_i = p$ and the number of those i with $x_{i+1} - x_i = -q$ $(0 \le i < n)$. From $x_0 = x_n = 0$ we get $kp = lq$, so for some integer $t > 1$, $k = qt$, $l = pt$, and $n = (p+q)t$.

Consider the sequence $y_i = x_{i+p+q} - x_i$, $i = 0, \dots, n-p-q$. We claim that at least one of the y_i's equals zero. We begin by noting that each y_i is of the form $up - vq$, where $u + v = p + q$; therefore $y_i = (u+v)p - v(p+q) = (p-v)(p+q)$ is always divisible by $p+q$. Moreover, $y_{i+1} - y_i = (x_{i+p+q+1} - x_{i+p+q}) - (x_{i+1} - x_i)$ is 0 or $\pm(p+q)$. We conclude that if no y_i is 0 then all y_i's are of the same sign. But this is in contradiction with the relation $y_0 + y_{p+q} + \cdots + y_{n-p-q} = x_n - x_0 = 0$. Consequently some y_i is zero, as claimed.

Second solution. As before we assume $(p, q) = 1$. Let us define a sequence of points $A_i(y_i, z_i)$ $(i = 0, 1, \dots, n)$ in $\mathbb{N}_0^2$ inductively as follows. Set $A_0 = (0, 0)$ and define (y_{i+1}, z_{i+1}) as $(y_i, z_i + 1)$ if $x_{i+1} = x_i + p$ and $(y_i + 1, z_i)$ otherwise. The points A_i form a trajectory L in $\mathbb{N}_0^2$ continuously moving upwards and rightwards by steps of length 1. Clearly, $x_i = pz_i - qy_i$ for all i. Since $x_n = 0$, it follows that $(z_n, y_n) = (kq, kp)$, $k \in \mathbb{N}$. Since $y_n + z_n = n > p + q$, it follows that $k > 1$. We observe that $x_i = x_j$ if and only if $A_iA_j \parallel A_0A_n$. We shall show that such i, j with $i < j$ and $(i, j) \neq (0, n)$ must exist.

If L meets A_0A_n in an interior point, then our statement trivially holds. From now on we assume the opposite. Let P_{ij} be the rectangle with sides parallel to the coordinate axes and with vertices at (ip, jq) and $((i+1)p, (j+1)q)$. Let L_{ij} be the part of the trajectory L lying inside P_{ij}. We may assume w.l.o.g. that the endpoints of L_{00} lie on the vertical sides of P_{00}. Then there obviously exists $d \in \{1, \dots, k-1\}$ such that the endpoints of L_{dd} lie on the horizontal sides of P_{dd}. Consider the translate L'_{dd} of L_{dd} for the vector $-d(p, q)$. The endpoints of L'_{dd} lie on the vertical sides of P_{00}. Hence L_{00} and L'_{dd} have some point $X \neq A_0$ in common. The translate Y of point X for the vector $d(p, q)$ belongs to L and satisfies $XY \parallel A_0A_n$.

29. Let the squares be indexed serially by the integers: $\dots, -1, 0, 1, 2, \dots$. When a bean is moved from i to $i+1$ or from $i+1$ to i for the first time, we may assign the index i to it. Thereafter, whenever some bean is moved in the opposite direction, we shall assume that it is exactly the one marked by i, and so on. Thus, each pair of neighboring squares has a bean stuck between it, and since the number of beans is finite, there are only finitely pairs of neighboring squares, and thus finitely many squares on which moves are made. Thus we may assume w.l.o.g. that all moves occur between 0 and $l \in \mathbb{N}$ and that all beans exist at all times within $[0, l]$.

Defining b_i to be the number of beans in the ith cell $(i \in \mathbb{Z})$ and b the total number of beans, we define the semi-invariant $S = \sum_{i \in \mathbb{Z}} i^2 b_i$. Since all moves occur above 0, the semi-invariant S increases by 2 with each move, and since we always have $S < b \cdot l^2$, it follows that the number of moves must be finite.

We now prove the uniqueness of the final configuration and the number of moves for some initial configuration $\{b_i\}$. Let $x_i \geq 0$ be the number of moves made in the ith cell ($i \in \mathbb{Z}$) during the game. Since the game is finite, only finitely many of x_i's are nonzero. Also, the number of beans in cell i, denoted as e_i, at the end is

$$(\forall i \in \mathbb{Z})\ e_i = b_i + x_{i-1} + x_{i+1} - 2x_i \in \{0, 1\}\,. \tag{1}$$

Thus it is enough to show that given $b_i \geq 0$, the sequence $\{x_i\}_{i\in\mathbb{Z}}$ of nonnegative integers satisfying (1) is unique.

Suppose the assertion is false, i.e., that there exists at least one sequence $b_i \geq 0$ for which there exist distinct sequences $\{x_i\}$ and $\{x_i'\}$ satisfying (1). We may choose such a $\{b_i\}$ for which $\min\{\sum_{i\in\mathbb{Z}} x_i, \sum_{i\in\mathbb{Z}} x_i'\}$ is minimal (since $\sum_{i\in\mathbb{Z}} x_i$ is always finite). We choose any index j such that $b_j > 1$. Such an index j exists, since otherwise the game is over. Then one must make at least one move in the jth cell, which implies that $x_j, x_j' \geq 1$. However, then the sequences $\{x_i\}$ and $\{x_i'\}$ with x_j and x_j' decreased by 1 also satisfy (1) for a sequence $\{b_i\}$ where b_{j-1}, b_j, b_{j+1} is replaced with $b_{j-1}+1, b_j-2, b_{j+1}+1$. This contradicts the assumption of minimal $\min\{\sum_{i\in\mathbb{Z}} x_i, \sum_{i\in\mathbb{Z}} x_i'\}$ for the initial $\{b_i\}$.

30. For convenience, we shall write $f^2, fg, \dots$ for the functions $f\circ f, f\circ g, \dots$. We need two lemmas.

Lemma 1. If $f(x) \in S$ and $g(x) \in T$, then $x \in S \cap T$.

Proof. The given condition means that $f^3(x) = g^2 f(x)$ and $gfg(x) = fg^2(x)$. Since $x \in S \cup T = U$, we have two cases:

$x \in S$. Then $f^2(x) = g^2(x)$, which also implies $f^3(x) = fg^2(x)$. Therefore $gfg(x) = fg^2(x) = f^3(x) = g^2 f(x)$, and since g is a bijection, we obtain $fg(x) = gf(x)$, i.e., $x \in T$.

$x \in T$. Then $fg(x) = gf(x)$, so $g^2 f(x) = gfg(x)$. It follows that $f^3(x) = g^2 f(x) = gfg(x) = fg^2(x)$, and since f is a bijection, we obtain $x \in S$.

Hence $x \in S\cap T$ in both cases. Similarly, $f(x) \in T$ and $g(x) \in S$ again imply $x \in S \cap T$.

Lemma 2. $f(S\cap T) = g(S\cap T) = S\cap T$.

Proof. By symmetry, it is enough to prove $f(S\cap T) = S\cap T$, or in other words that $f^{-1}(S\cap T) = S\cap T$. Since $S\cap T$ is finite, this is equivalent to $f(S\cap T) \subseteq S\cap T$.

Let $f(x) \in S\cap T$. Then if $g(x) \in S$ (since $f(x) \in T$), Lemma 1 gives $x \in S\cap T$; similarly, if $g(x) \in T$, then by Lemma 1, $x \in S\cap T$.

Now we return to the problem. Assume that $f(x) \in S$. If $g(x) \notin S$, then $g(x) \in T$, so from Lemma 1 we deduce that $x \in S\cap T$. Then Lemma 2 claims that $g(x) \in S\cap T$ too, a contradiction. Analogously, from $g(x) \in S$ we are led to $f(x) \in S$. This finishes the proof.

4.38 Solutions to the Shortlisted Problems of IMO 1997

1. Let ABC be the given triangle, with $\angle B = 90^\circ$ and $AB = m$, $BC = n$. For an arbitrary polygon $\mathcal{P}$ we denote by $w(\mathcal{P})$ and $b(\mathcal{P})$ respectively the total areas of the white and black parts of $\mathcal{P}$.
   (a) Let D be the fourth vertex of the rectangle $ABCD$. When m and n are of the same parity, the coloring of the rectangle $ABCD$ is centrally symmetric with respect to the midpoint of AC. It follows that $w(ABC) = \frac{1}{2}w(ABCD)$ and $b(ABC) = \frac{1}{2}b(ABCD)$; thus $f(m,n) = \frac{1}{2}|w(ABCD) - b(ABCD)|$. Hence $f(m,n)$ equals $\frac{1}{2}$ if m and n are both odd, and 0 otherwise.
   (b) The result when m,n are of the same parity follows from (a). Suppose that $m > n$, where m and n are of different parity. Choose a point E on AB such that $AE = 1$. Since by (a) $|w(EBC) - b(EBC)| = f(m-1,n) \le \frac{1}{2}$, we have $f(m,n) \le \frac{1}{2} + |w(EAC) - b(EAC)| \le \frac{1}{2} + S(EAC) = \frac{1}{2} + \frac{n-1}{2} = \frac{n}{2}$. Therefore $f(m,n) \le \frac{1}{2}\min(m,n)$.
   (c) Let us calculate $f(m,n)$ for $m = 2k+1$, $n = 2k$, $k \in \mathbb{N}$. With E defined as in (b), we have $BE = BC = 2k$. If the square at B is w.l.o.g. white, CE passes only through black squares. The white part of $\triangle EAC$ then consists of $2k$ similar triangles with areas $\frac{1}{2}\frac{i}{2k}\frac{i}{2k+1} = \frac{i^2}{4k(2k+1)}$, where $i = 1, 2, \dots, 2k$. The total white area of EAC is
   $$\frac{1}{4k(2k+1)}(1^2 + 2^2 + \cdots + (2k)^2) = \frac{4k+1}{12}.$$
   Therefore the black area is $(8k-1)/12$, and $f(2k+1,2k) = (2k-1)/6$, which is not bounded.

2. For any sequence $X = (x_1, x_2, \dots, x_n)$ let us define
   $$\overline{X} = (1, 2, \dots, x_1, 1, 2, \dots, x_2, \dots, 1, 2, \dots, x_n).$$
   Also, for any two sequences A, B we denote their concatenation by AB. It clearly holds that $\overline{AB} = \overline{A}\,\overline{B}$. The sequences $R_1, R_2, \dots$ are given by $R_1 = (1)$ and $R_n = \overline{R_{n-1}}(n)$ for $n > 1$.
   We consider the family of sequences Q_{ni} for $n, i \in \mathbb{N}$, $i \le n$, defined as follows:
   $$Q_{n1} = (1), \quad Q_{nn} = (n), \quad \text{and} \quad Q_{ni} = Q_{n-1,i-1}Q_{n-1,i} \quad \text{if } 1 < i < n.$$
   These sequences form a Pascal-like triangle, as shown in the picture below:
   $$\begin{array}{lccccccccc}
Q_{1i}: & & & & & 1 & & & & \\
Q_{2i}: & & & & 1 & & 2 & & & \\
Q_{3i}: & & & 1 & & 12 & & 3 & & \\
Q_{4i}: & & 1 & & 112 & & 123 & & 4 & \\
Q_{5i}: & 1 & & 1112 & & 112123 & & 1234 & & 5
\end{array}$$

We claim that R_n is in fact exactly $Q_{n1}Q_{n2}\dots Q_{nn}$. Before proving this, we observe that $Q_{ni} = \overline{Q_{n-1,i}}$. This follows by induction, because $Q_{ni} = Q_{n-1,i-1}Q_{n-1,i} = \overline{Q_{n-2,i-1}}\ \overline{Q_{n-2,i}} = \overline{Q_{n-1,i}}$ for $n \geq 3$, $i \geq 2$ (the cases $i = 1$ and $n = 1, 2$ are trivial). Now $R_1 = Q_{11}$ and

$$R_n = \overline{R_{n-1}}(n) = \overline{Q_{n-1,1}\dots Q_{n-1,n-1}}(n) = Q_{n,1}\dots Q_{n,n-1}Q_{n,n}$$

for $n \geq 2$, which justifies our claim by induction.
Now we know enough about the sequence R_n to return to the question of the problem. We use induction on n once again. The result is obvious for $n = 1$ and $n = 2$. Given any $n \geq 3$, consider the kth elements of R_n from the left, say u, and from the right, say v. Assume that u is a member of Q_{nj}, and consequently that v is a member of $Q_{n,n+1-j}$. Then u and v come from symmetric positions of R_{n-1} (either from $Q_{n-1,j}, Q_{n-1,n-j}$, or from $Q_{n-1,j-1}, Q_{n-1,n+1-j}$), and by the inductive hypothesis exactly one of them is 1.

3. (a) For $n = 4$, consider a convex quadrilateral $ABCD$ in which $AB = BC = AC = BD$ and $AD = DC$, and take the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$. For $n = 5$, take the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EA}$ for any regular pentagon $ABCDE$.
   (b) Let us draw the vectors of V as originated from the same point O. Consider any maximal subset $B \subset V$, and denote by u the sum of all vectors from B. If l is the line through O perpendicular to u, then B contains exactly those vectors from V that lie on the same side of l as u does, and no others. Indeed, if any $v \notin B$ lies on the same side of l, then $|u + v| \geq |u|$; similarly, if some $v \in B$ lies on the other side of l, then $|u - v| \geq |u|$.
   Therefore every maximal subset is determined by some line l as the set of vectors lying on the same side of l. It is obvious that in this way we get at most $2n$ sets.

4. (a) Suppose that an $n \times n$ coveralls matrix A exists for some $n > 1$. Let $x \in \{1, 2, \dots, 2n - 1\}$ be a fixed number that does not appear on the fixed diagonal of A. Such an element must exist, since the diagonal can contain at most n different numbers. Let us call the union of the ith row and the ith column the *ith cross*. There are n crosses, and each of them contains exactly one x. On the other hand, each entry x of A is contained in exactly two crosses. Hence n must be even. However, 1997 is an odd number; hence no coveralls matrix exists for $n = 1997$.
   (b) For $n = 2$, $A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ is a coveralls matrix. For $n = 4$, one such matrix is, for example,

$$A_4 = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 1 & 7 & 5 \\ 4 & 6 & 1 & 2 \\ 7 & 4 & 3 & 1 \end{bmatrix}.$$

This construction can be generalized. Suppose that we are given an $n \times n$ coveralls matrix A_n. Let B_n be the matrix obtained from A_n by adding $2n$ to each entry, and C_n the matrix obtained from B_n by replacing each diagonal entry (equal to $2n+1$ by induction) with $2n$. Then the matrix

$$A_{2n} = \begin{bmatrix} A_n & B_n \\ C_n & A_n \end{bmatrix}$$

is coveralls. To show this, suppose that $i \le n$ (the case $i > n$ is similar). The ith cross is composed of the ith cross of A_n, the ith row of B_n, and the ith column of C_n. The ith cross of A_i covers $1, 2, \dots, 2n-1$. The ith row of B_n covers all numbers of the form $2n+j$, where j is covered by the ith row of A_n (including $j = 1$). Similarly, the ith column of C_n covers $2n$ and all numbers of the form $2n+k$, where $k > 1$ is covered by the ith column of A_n. Thus we see that all numbers are accounted for in the ith cross of A_{2n}, and hence A_{2n} is a desired coveralls matrix. It follows that we can find a coveralls matrix whenever n is a power of 2.

Second solution for part b. We construct a coveralls matrix explicitly for $n = 2^k$. We consider the coordinates/cells of the matrix elements modulo n throughout the solution. We define the i-diagonal ($0 \le i < n$) to be the set of cells of the form $(j, j+i)$, for all j. We note that each cross contains exactly one cell from the 0-diagonal (the main diagonal) and two cells from each i-diagonal. For two cells within an i diagonal, x and y, we define x and y to be related if there exists a cross containing both x and y. Evidently, for every cell x not on the 0-diagonal there are exactly two other cells related to it. The relation thus breaks up each i-diagonal ($i > 0$) into cycles of length larger than 1. Due to the diagonal translational symmetry (modulo n), all the cycles within a given i-diagonal must be of equal length and thus of an even length, since $n = 2^k$.
The construction of a coveralls matrix is now obvious. We select a number, say 1, to place on all the cells of the 0-diagonal. We pair up the remaining numbers and assign each pair to an i-diagonal, say $(2i, 2i+1)$. Going along each cycle within the i-diagonal we alternately assign values of $2i$ and $2i+1$. Since the cycle has an even length, a cell will be related only to a cell of a different number, and hence each cross will contain both $2i$ and $2i+1$.

5. We shall prove first the 2-dimensional analogue:
Lemma. Given an equilateral triangle ABC and two points M, N on the sides AB and AC respectively, there exists a triangle with sides CM, BN, MN.
Proof. Consider a regular tetrahedron $ABCD$. Since $CM = DM$ and $BN = DN$, one such triangle is DMN.

Now, to solve the problem for a regular tetrahedron $ABCD$, we consider a 4-dimensional polytope $ABCDE$ whose faces $ABCD$, $ABCE$, $ABDE$, $ACDE$, $BCDE$ are regular tetrahedra. We don't know what it looks like, but it yields a desired triangle: for $M \in ABC$ and $N \in ADC$, we have $DM = EM$ and $BN = EN$; hence the desired triangle is EMN.

Remark. A solution that avoids embedding in $\mathbb{R}^4$ is possible, but no longer so short.

6. (a) One solution is

$$x = 2^{n^2}3^{n+1}, \qquad y = 2^{n^2-n}3^n, \qquad z = 2^{n^2-2n+2}3^{n-1}.$$

(b) Suppose w.l.o.g. that $\gcd(c, a) = 1$. We look for a solution of the form

$$x = p^m, \qquad y = p^n, \qquad z = qp^r, \qquad\qquad p, q, m, n, r \in \mathbb{N}.$$

Then $x^a + y^b = p^{ma} + p^{nb}$ and $z^c = q^c p^{rc}$, and we see that it is enough to assume $ma - 1 = nb = rc$ (there are infinitely many such triples (m, n, r)) and $q^c = p + 1$.

7. Let us set $AC = a$, $CE = b$, $EA = c$. Applying Ptolemy's inequality for the quadrilateral $ACEF$ we get

$$AC \cdot EF + CE \cdot AF \geq AE \cdot CF.$$

Since $EF = AF$, this implies $\frac{FA}{FC} \geq \frac{c}{a+b}$. Similarly $\frac{BC}{BE} \geq \frac{a}{b+c}$ and $\frac{DE}{DA} \geq \frac{b}{c+a}$. Now,

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

Hence it is enough to prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}. \tag{1}$$

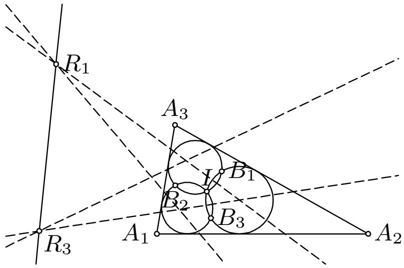
If we now substitute $x = b+c$, $y = c+a$, $z = a+b$ and $S = a+b+c$ the inequality (1) becomes equivalent to $S(1/x + 1/y + 1/y) - 3 \geq 3/2$ which follows immediately form $1/x + 1/y + 1/z \geq 9/(x+y+z) = 9/(2S)$.
Equality occurs if it holds in Ptolemy's inequalities and also $a = b = c$. The former happens if and only if the hexagon is cyclic. Hence the only case of equality is when $ABCDEF$ is regular.

8. (a) Denote by b and c the perpendicular bisectors of AB and AC respectively. If w.l.o.g. b and AD do not intersect (are parallel), then $\angle BCD = \angle BAD = 90°$, a contradiction. Hence V, W are well-defined. Now, $\angle DWB = 2\angle DAB$ and $\angle DVC = 2\angle DAC$ as oriented angles, and therefore $\angle(WB, VC) = 2(\angle DVC - \angle DWB) = 2\angle BAC = 2\angle BCD$ is not equal to 0. Consequently CV and BW meet at some T with $\angle BTC = 2\angle BAC$.

(b) Let B' be the second point of intersection of BW with Γ. Clearly $AD = BB'$. But we also have $\angle BTC = 2\angle BAC = 2\angle BB'C$, which implies that $CT = TB'$. It follows that $AD = BB' = |BT \pm TB'| = |BT \pm CT|$.

Remark. This problem is also solved easily using trigonometry.

9. For $i = 1, 2, 3$ (all indices in this problem will be modulo 3) we denote by O_i the center of C_i and by M_i the midpoint of the arc $A_{i+1}A_{i+2}$ that does not contain A_i. First we have that $O_{i+1}O_{i+2}$ is the perpendicular bisector of IB_i, and thus it contains the circumcenter R_i of A_iB_iI. Additionally, it is easy to show that $T_{i+1}A_i = T_{i+1}I$ and $T_{i+2}A_i = T_{i+2}I$, which implies that R_i lies on the line $T_{i+1}T_{i+2}$. Therefore $R_i = O_{i+1}O_{i+2} \cap T_{i+1}T_{i+2}$.



Now, the lines T_1O_1, T_2O_2, T_3O_3 are concurrent at I. By Desargues's theorem, the points of intersection of $O_{i+1}O_{i+2}$ and $T_{i+1}T_{i+2}$, i.e., the R_i's, lie on a line for $i = 1, 2, 3$.

Second solution. The centers of three circles passing through the same point I and not touching each other are collinear if and only if they have another common point. Hence it is enough to show that the circles A_iB_iI have a common point other than I.

Now apply inversion at center I and with an arbitrary power. We shall denote by X' the image of X under this inversion. In our case, the image of the circle C_i is the line $B'_{i+1}B'_{i+2}$ while the image of the line $A_{i+1}A_{i+2}$ is the circle $IA'_{i+1}A'_{i+2}$ that is tangent to $B'_iB'_{i+2}$, and $B'_iB'_{i+2}$. These three circles have equal radii, so their centers P_1, P_2, P_3 form a triangle also homothetic to $\triangle B'_1B'_2B'_3$. Consequently, points A'_1, A'_2, A'_3, that are the reflections of I across the sides of $P_1P_2P_3$, are vertices of a triangle also homothetic to $B'_1B'_2B'_3$. It follows that $A'_1B'_1, A'_2B'_2, A'_3B'_3$ are concurrent at some point J', i.e., that the circles A_iB_iI all pass through J.

10. Suppose that $k \geq 4$. Consider any polynomial $F(x)$ with integer coefficients such that $0 \leq F(x) \leq k$ for $x = 0, 1, \dots, k+1$. Since $F(k+1) - F(0)$ is divisible by $k + 1$, we must have $F(k + 1) = F(0)$. Hence

$$F(x) - F(0) = x(x - k - 1)Q(x)$$

for some polynomial $Q(x)$ with integer coefficients. In particular, $F(x) - F(0)$ is divisible by $x(k + 1 - x) > k + 1$ for every $x = 2, 3, \dots, k - 1$, so $F(x) = F(0)$ must hold for any $x = 2, 3, \dots, k - 1$. It follows that

$$F(x) - F(0) = x(x - 2)(x - 3) \cdots (x - k + 1)(x - k - 1)R(x)$$

for some polynomial $R(x)$ with integer coefficients. Thus $k \geq |F(1) - F(0)| = k(k-2)!|R(1)|$, although $k(k-2)! > k$ for $k \geq 4$. In this case we have $F(1) = F(0)$ and similarly $F(k) = F(0)$. Hence, the statement is true for $k \geq 4$.
It is easy to find counterexamples for $k \leq 3$. These are, for example,

$$F(x) = \begin{cases} x(2-x) & \text{for } k = 1, \\ x(3-x) & \text{for } k = 2, \\ x(2-x)^2(4-x) & \text{for } k = 3. \end{cases}$$

11. All real roots of $P(x)$ (if any) are negative: say $-a_1, -a_2, \dots, -a_k$. Then $P(x)$ can be factored as

$$P(x) = C(x+a_1)\cdots(x+a_k)(x^2 - b_1x + c_1)\cdots(x^2 - b_mx + c_m), \quad (1)$$

where $x^2 - b_ix + c_i$ are quadratic polynomials without real roots.
Since the product of polynomials with positive coefficients is again a polynomial with positive coefficients, it will be sufficient to prove the result for each of the factors in (1). The case of $x + a_j$ is trivial. It remains only to prove the claim for every polynomial $x^2 - bx + c$ with $b^2 < 4c$.
From the binomial formula, we have for any $n \in \mathbb{N}$,

$$(1+x)^n(x^2 - bx + c) = \sum_{i=0}^{n+2} \left[\binom{n}{i-2} - b\binom{n}{i-1} + c\binom{n}{i} \right] x^i = \sum_{i=0}^{n+2} C_i x^i,$$

where

$$C_i = \frac{n!\left((b+c+1)i^2 - ((b+2c)n + (2b+3c+1))i + c(n^2+3n+2)\right)x^i}{i!(n-i+2)!}.$$

The coefficients C_i of x^i appear in the form of a quadratic polynomial in i depending on n. We claim that for large enough n this polynomial has negative discriminant, and is thus positive for every i. Indeed, this discriminant equals $D = ((b+2c)n + (2b+3c+1))^2 - 4(b+c+1)c(n^2 + 3n+2) = (b^2 - 4c)n^2 - 2Un + V$, where $U = 2b^2 + bc + b - 4c$ and $V = (2b+c+1)^2 - 4c$, and since $b^2 - 4c < 0$, for large n it clearly holds that $D < 0$.

12. *Lemma.* For any polynomial P of degree at most n, the following equality holds:

$$\sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} P(i) = 0.$$

Proof. See (SL81-13).
Suppose to the contrary that the degree of f is at most $p-2$. Then it follows from the lemma that

$$0 = \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} f(i) \equiv \sum_{i=0}^{p-1} f(i) \pmod{p},$$

since $\binom{p-1}{i} = \frac{(p-1)(p-2)\cdots(p-i)}{i!} \equiv (-1)^i \pmod p$. But this is clearly impossible if $f(i)$ equals 0 or 1 modulo p and $f(0) = 0$, $f(1) = 1$.

Remark. In proving the essential relation $\sum_{i=0}^{p-1} f(i) \equiv 0 \pmod p$, it is clearly enough to show that $S_k = 1^k + 2^k + \cdots + (p-1)^k$ is divisible by p for every $k \le p-2$. This can be shown in two other ways.

(1) By induction. Assume that $S_0 \equiv \cdots \equiv S_{k-1} \pmod p$. By the binomial formula we have

$$0 \equiv \sum_{n=0}^{p-1} [(n+1)^{k+1} - n^{k+1}] \equiv (k+1)S_k + \sum_{i=0}^{k-1} \binom{k+1}{i} S_i \pmod p,$$

and the inductive step follows.

(2) Using the primitive root g modulo p. Then

$$S_k \equiv 1 + g^k + \cdots + g^{k(p-2)} = \frac{g^{k(p-1)} - 1}{g^k - 1} \equiv 0 \pmod p.$$

13. Denote $A(r)$ and $B(r)$ by $A(n,r)$ and $B(n,r)$ respectively.
The numbers $A(n,r)$ can be found directly: one can choose r girls and r boys in $\binom{n}{r}^2$ ways, and pair them in $r!$ ways. Hence

$$A(n,r) = \binom{n}{r}^2 \cdot r! = \frac{n!^2}{(n-r)!^2 r!}.$$

Now we establish a recurrence relation between the $B(n,r)$'s. Let $n \ge 2$ and $2 \le r \le n$. There are two cases for a desired selection of r pairs of girls and boys:

(i) One of the girls dancing is g_n. Then the other $r-1$ girls can choose their partners in $B(n-1, r-1)$ ways and g_n can choose any of the remaining $2n - r$ boys. Thus, the total number of choices in this case is $(2n-r)B(n-1, r-1)$.

(ii) g_n is not dancing. Then there are exactly $B(n-1, r)$ possible choices.

Therefore, for every $n \ge 2$ it holds that

$$B(n,r) = (2n-r)B(n-1, r-1) + B(n-1, r) \qquad \text{for } r = 2, \dots, n.$$

Here we assume that $B(n,r) = 0$ for $r > n$, while $B(n,1) = 1 + 3 + \cdots + (2n-1) = n^2$.
It is directly verified that the numbers $A(n,r)$ satisfy the same initial conditions and recurrence relations, from which it follows that $A(n,r) = B(n,r)$ for all n and $r \le n$.

14. We use the following nonstandard notation: (1°) for $x, y \in \mathbb{N}$, $x \sim y$ means that x and y have the same prime divisors; (2°) for a prime p and integers $r \ge 0$ and $x > 0$, $p^r \parallel x$ means that x is divisible by p^r, but not by p^{r+1}.
First, $b^m - 1 \sim b^n - 1$ is obviously equivalent to $b^m - 1 \sim \gcd(b^m - 1, b^n - 1) = b^d - 1$, where $d = \gcd(m,n)$. Setting $b^d = a$ and $m = kd$, we reduce

the condition of the problem to $a^k - 1 \sim a - 1$. We are going to show that this implies that $a+1$ is a power of 2. This will imply that d is odd (for even d, $a+1 = b^d + 1$ cannot be divisible by 4), and consequently $b+1$, as a divisor of $a+1$, is also a power of 2. But before that, we need the following important lemma (Theorem 2.126).

Lemma. Let a, k be positive integers and p an odd prime. If $\alpha \geq 1$ and $\beta \geq 0$ are such that $p^\alpha \parallel a - 1$ and $p^\beta \parallel k$, then $p^{\alpha+\beta} \parallel a^k - 1$.

Proof. We use induction on β. If $\beta = 0$, then $\frac{a^k-1}{a-1} = a^{k-1} + \cdots + a + 1 \equiv k \pmod p$ (because $a \equiv 1$), and it is not divisible by p.

Suppose that the lemma is true for some $\beta \geq 0$, and let $k = p^{\beta+1}t$ where $p \nmid t$. By the induction hypothesis, $a^{k/p} = a^{p^\beta t} = mp^{\alpha+\beta} + 1$ for some m not divisible by p. Furthermore,

$$a^k - 1 = (mp^{\alpha+\beta}+1)^p - 1 = (mp^{\alpha+\beta})^p + \cdots + \binom{p}{2}(mp^{\alpha+\beta})^2 + mp^{\alpha+\beta+1}.$$

Since $p \mid \binom{p}{2} = \frac{p(p-1)}{2}$, all summands except for the last one are divisible by $p^{\alpha+\beta+2}$. Hence $p^{\alpha+\beta+1} \parallel a^k - 1$, completing the induction.

Now let $a^k - 1 \sim a - 1$ for some $a, k > 1$. Suppose that p is an odd prime divisor of k, with $p^\beta \parallel k$. Then putting $X = a^{p^\beta - 1} + \cdots + a + 1$ we also have $(a-1)X = a^{p^\beta} - 1 \sim a - 1$; hence each prime divisor q of X must also divide $a - 1$. But then $a^i \equiv 1 \pmod q$ for each $i \in \mathbb{N}_0$, which gives us $X \equiv p^\beta \pmod q$. Therefore $q \mid p^\beta$, i.e., $q = p$; hence X is a power of p. On the other hand, since $p \mid a - 1$, we put $p^\alpha \parallel a - 1$. From the lemma we obtain $p^{\alpha+\beta} \parallel a^{p^\beta} - 1$, and deduce that $p^\beta \parallel X$. But X has no prime divisors other than p, so we must have $X = p^\beta$. This is clearly impossible, because $X > p^\beta$ for $a > 1$. Thus our assumption that k has an odd prime divisor leads to a contradiction: in other words, k must be a power of 2. Now $a^k - 1 \sim a - 1$ implies $a - 1 \sim a^2 - 1 = (a-1)(a+1)$, and thus every prime divisor q of $a+1$ must also divide $a - 1$. Consequently $q = 2$, so it follows that $a+1$ is a power of 2. As we explained above, this gives that $b+1$ is also a power of 2.

Remark. In fact, one can continue and show that k must be equal to 2. It is not possible for $a^4 - 1 \sim a^2 - 1$ to hold. Similarly, we must have $d = 1$. Therefore all possible triples (b, m, n) with $m > n$ are $(2^s - 1, 2, 1)$.

15. Let $a + bt$, $t = 0, 1, 2, \ldots$, be a given arithmetic progression that contains a square and a cube ($a, b > 0$). We use induction on the progression step b to prove that the progression contains a sixth power.
   (i) $b = 1$: this case is trivial.
   (ii) $b = p^m$ for some prime p and $m > 0$. The case $p^m \mid a$ trivially reduces to the previous case, so let us have $p^m \nmid a$.
   Suppose that $\gcd(a, p) = 1$. If x, y are integers such that $x^2 \equiv y^3 \equiv a$ (here all the congruences will be mod p^m), then $x^6 \equiv a^3$ and $y^6 \equiv a^2$. Consider an integer y_1 such that $yy_1 \equiv 1$. It satisfies $a^2(xy_1)^6 \equiv$

$x^6y^6y_1^6 \equiv x^6 \equiv a^3$, and consequently $(xy_1)^6 \equiv a$. Hence a sixth power exists in the progression.

If $\gcd(a,p) > 1$, we can write $a = p^k c$, where $k < m$ and $p \nmid c$. Since the arithmetic progression $x_t = a + bt = p^k(c + p^{m-k}t)$ contains a square, k must be even; similarly, it contains a cube, so $3 \mid k$. It follows that $6 \mid k$. The progression $c + p^{m-k}t$ thus also contains a square and a cube; hence by the previous case it contains a sixth power and thus x_t does also.

(iii) b is not a power of a prime, and thus can be expressed as $b = b_1b_2$, where $b_1, b_2 > 1$ and $\gcd(b_1, b_2) = 1$. It is given that progressions $a + b_1t$ and $a + b_2t$ both contain a square and a cube, and therefore by the inductive hypothesis they both contain sixth powers: say z_1^6 and z_2^6, respectively. By the Chinese remainder theorem, there exists $z \in \mathbb{N}$ such that $z \equiv z_1 \pmod{b_1}$ and $z \equiv z_2 \pmod{b_2}$. But then z^6 belongs to both of the progressions $a + b_1t$ and $a + b_2t$. Hence z^6 is a member of the progression $a + bt$.

16. Let $d_a(X), d_b(X), d_c(X)$ denote the distances of a point X interior to $\triangle ABC$ from the lines BC, CA, AB respectively. We claim that $X \in PQ$ if and only if $d_a(X) + d_b(X) = d_c(X)$. Indeed, if $X \in PQ$ and $PX = kPQ$ then $d_a(X) = kd_a(Q)$, $d_b(X) = (1-k)d_b(P)$, and $d_c(X) = (1-k)d_c(P) + kd_c(Q)$, and simple substitution yields $d_a(X) + d_b(X) = d_c(X)$. The converse follows easily. In particular, $O \in PQ$ if and only if $d_a(O) + d_b(O) = d_c(O)$, i.e., $\cos\alpha + \cos\beta = \cos\gamma$.

We shall now show that $I \in DE$ if and only if $AE + BD = DE$. Let K be the point on the segment DE such that $AE = EK$. Then $\angle EKA = \frac{1}{2}\angle DEC = \frac{1}{2}\angle CBA = \angle IBA$; hence the points A, B, I, K are concyclic. The point I lies on DE if and only if $\angle BKD = \angle BAI = \frac{1}{2}\angle BAC = \frac{1}{2}\angle CDE = \angle DBK$, which is equivalent to $KD = BD$, i.e., to $AE + BD = DE$. But since $AE = AB\cos\alpha$, $BD = AB\cos\beta$, and $DE = AB\cos\gamma$, we have that $I \in DE \Leftrightarrow \cos\alpha + \cos\beta = \cos\gamma$. The conditions for $O \in PQ$ and $I \in DE$ are thus equivalent.

Second solution. We know that three points X, Y, Z are collinear if and only if for some $\lambda, \mu \in \mathbb{R}$ with sum 1, we have $\lambda\overrightarrow{CX} + \mu\overrightarrow{CY} = \overrightarrow{CZ}$. Specially, if $\overrightarrow{CX} = p\overrightarrow{CA}$ and $\overrightarrow{CY} = q\overrightarrow{CB}$ for some p, q, and $\overrightarrow{CZ} = k\overrightarrow{CA} + l\overrightarrow{CB}$, then Z lies on XY if and only if $kq + lp = pq$.

Using known relations in a triangle we directly obtain

$$\overrightarrow{CP} = \frac{\sin\beta}{\sin\beta + \sin\gamma}\overrightarrow{CB}, \qquad \overrightarrow{CQ} = \frac{\sin\alpha}{\sin\alpha + \sin\gamma}\overrightarrow{CA},$$
$$\overrightarrow{CO} = \frac{\sin 2\alpha \cdot \overrightarrow{CA} + \sin 2\beta \cdot \overrightarrow{CB}}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}; \qquad \overrightarrow{CD} = \frac{\tan\beta}{\tan\beta + \tan\gamma}\overrightarrow{CB},$$
$$\overrightarrow{CE} = \frac{\tan\beta}{\tan\beta + \tan\gamma}\overrightarrow{CA}, \qquad \overrightarrow{CI} = \frac{\sin\alpha \cdot \overrightarrow{CA} + \sin\beta \cdot \overrightarrow{CB}}{\sin\alpha + \sin\beta + \sin\gamma}.$$

Now by the above considerations we get that the conditions (1) *P, Q, O are collinear* and (2) *D, E, I are collinear* are both equivalent to $\cos\alpha+\cos\beta=\cos\gamma$.

17. We note first that x and y must be powers of the same positive integer. Indeed, if $x=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ and $y=p_1^{\beta_1}\cdots p_k^{\beta_k}$ (some of α_i and β_i may be 0, but not both for the same index i), then $x^{y^2}=y^x$ implies $\frac{\alpha_i}{\beta_i}=\frac{x}{y^2}=\frac{p}{q}$ for some $p,q>0$ with $\gcd(p,q)=1$, so for $a=p_1^{\alpha_1/p}\cdots p_k^{\alpha_k/p}$ we can take $x=a^p$ and $y=a^q$.
If $a=1$, then $(x,y)=(1,1)$ is the trivial solution. Let $a>1$. The given equation becomes $a^{pa^{2q}}=a^{qa^p}$, which reduces to $pa^{2q}=qa^p$. Hence $p\neq q$, so we distinguish two cases:
   (i) $p>q$. Then from $a^{2q}<a^p$ we deduce $p>2q$. We can rewrite the equation as $p=a^{p-2q}q$, and putting $p=2q+d$, $d>0$, we obtain $d=q(a^d-2)$. By induction, $2^d-2>d$ for each $d>2$, so we must have $d\le 2$. For $d=1$ we get $q=1$ and $a=p=3$, and therefore $(x,y)=(27,3)$, which is indeed a solution. For $d=2$ we get $q=1$, $a=2$, and $p=4$, so $(x,y)=(16,2)$, which is another solution.
   (ii) $p<q$. As above, we get $q/p=a^{2q-p}$, and setting $d=2q-p>0$, this is transformed to $a^d=a^{(2a^d-1)p}$, or equivalently to $d=(2a^d-1)p$. However, this equality cannot hold, because $2a^d-1>d$ for each $a\ge 2$, $d\ge 1$.

   The only solutions are thus $(1,1)$, $(16,2)$, and $(27,3)$.

18. By symmetry, assume that $AB>AC$. The point D lies between M and P as well as between Q and R, and if we show that $DM\cdot DP=DQ\cdot DR$, it will imply that M,P,Q,R lie on a circle.
Since the triangles ABC, AEF, AQR are similar, the points B,C,Q,R lie on a circle. Hence $DB\cdot DC=DQ\cdot DR$, and it remains to prove that
$$DB\cdot DC=DM\cdot DP.$$
However, the points B,C,E,F are concyclic, but so are the points E,F,D,M (they lie on the nine-point circle), and we obtain $PB\cdot PC=PE\cdot PF=PD\cdot PM$. Set $PB=x$ and $PC=y$. We have $PM=\frac{x+y}{2}$ and hence $PD=\frac{2xy}{x+y}$. It follows that $DB=PB-PD=\frac{x(x-y)}{x+y}$, $DC=\frac{y(x-y)}{x+y}$, and $DM=\frac{(x-y)^2}{2(x+y)}$, from which we immediately obtain $DB\cdot DC=DM\cdot DP=\frac{xy(x-y)^2}{(x+y)^2}$, as needed.

19. Using that $a_{n+1}=0$ we can transform the desired inequality into
$$\sqrt{a_1+a_2+\cdots+a_{n+1}}\le\sqrt{1}\sqrt{a_1}+(\sqrt{2}-\sqrt{1})\sqrt{a_2}+\cdots+(\sqrt{n+1}-\sqrt{n})\sqrt{a_{n+1}}. \qquad (1)$$
We shall prove by induction on n that (1) holds for any $a_1\ge a_2\ge\cdots\ge a_{n+1}\ge 0$, i.e., not only when $a_{n+1}=0$. For $n=0$ the inequality is

obvious. For the inductive step from $n-1$ to n, where $n \geq 1$, we need to prove the inequality

$$\sqrt{a_1 + \cdots + a_{n+1}} - \sqrt{a_1 + \cdots + a_n} \leq (\sqrt{n+1} - \sqrt{n})\sqrt{a_{n+1}}. \qquad (2)$$

Putting $S = a_1 + a_2 + \cdots + a_n$, this simplifies to $\sqrt{S + a_{n+1}} - \sqrt{S} \leq \sqrt{na_{n+1} + a_{n+1}} - \sqrt{na_{n+1}}$. For $a_{n+1} = 0$ the inequality is obvious. For $a_{n+1} > 0$ we have that the function $f(x) = \sqrt{x + a_{n+1}} - \sqrt{x} = \frac{a_{n+1}}{\sqrt{x+a_{n+1}}+\sqrt{x}}$ is strictly decreasing on $\mathbb{R}^+$; hence (2) will follow if we show that $S \geq na_{n+1}$. However, this last is true because $a_1, \ldots, a_n \geq a_{n+1}$. Equality holds if and only if $a_1 = a_2 = \cdots = a_k$ and $a_{k+1} = \cdots = a_{n+1} = 0$ for some k.

Second solution. Setting $b_k = \sqrt{a_k} - \sqrt{a_{k+1}}$ for $k = 1, \ldots, n$ we have $a_i = (b_i + \cdots + b_n)^2$, so the desired inequality after squaring becomes

$$\sum_{k=1}^{n} kb_k^2 + 2 \sum_{1 \leq k < l \leq n} kb_k b_l \leq \sum_{k=1}^{n} k\, b_k^2 + 2 \sum_{1 \leq k < l \leq n} \sqrt{kl}\, b_k b_l,$$

which clearly holds.

20. To avoid dividing into cases regarding the position of the point X, we use oriented angles.
Let R be the foot of the perpendicular from X to BC. It is well known that the points P, Q, R lie on the corresponding Simson line. This line is a tangent to γ (i.e., the circle XDR) if and only if $\angle PRD = \angle RXD$. We have

$$\begin{aligned}\angle PRD &= \angle PXB = 90^\circ - \angle XBA = 90^\circ - \angle XBC + \angle ABC \\ &= 90^\circ - \angle DAC + \angle ABC\end{aligned}$$

and

$$\angle RXD = 90^\circ - \angle ADB = 90^\circ + \angle BCA - \angle DAC;$$

hence $\angle PRD = \angle RXD$ if and only if $\angle ABC = \angle BCA$, i.e, $AB = AC$.

21. For any permutation $\pi = (y_1, y_2, \ldots, y_n)$ of $(x_1, x_2, \ldots, x_n)$, denote by $S(\pi)$ the sum $y_1 + 2y_2 + \cdots + ny_n$. Suppose, contrary to the claim, that $|S(\pi)| > \frac{n+1}{2}$ for any π.
Further, we note that if π' is obtained from π by interchanging two neighboring elements, say y_k and y_{k+1}, then $S(\pi)$ and $S(\pi')$ differ by $|y_k + y_{k+1}| \leq n+1$, and consequently they must be of the same sign.
Now consider the identity permutation $\pi_0 = (x_1, \ldots, x_n)$ and the reverse permutation $\overline{\pi_0} = (x_n, \ldots, x_1)$. There is a sequence of permutations $\pi_0, \pi_1, \ldots, \pi_m = \overline{\pi_0}$ such that for each i, π_{i+1} is obtained from π_i by interchanging two neighboring elements. Indeed, by successive interchanges we can put x_n in the first place, then x_{n-1} in the second place, etc. Hence all $S(\pi_0), \ldots, S(\pi_m)$ are of the same sign. However, since $|S(\pi_0) + S(\pi_m)| = (n+1)|x_1 + \cdots + x_n| = n+1$, this implies that one of

$S(\pi_0)$ and $S(\overline{\pi_0})$ is smaller than $\frac{n+1}{2}$ in absolute value, contradicting the initial assumption.

22. (a) Suppose that f and g are such functions. From $g(f(x)) = x^3$ we have $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. In particular, $f(-1), f(0)$, and $f(1)$ are three distinct numbers. However, since $f(x)^2 = f(g(f(x))) = f(x^3)$, each of the numbers $f(-1), f(0), f(1)$ is equal to its square, and so must be either 0 or 1. This contradiction shows that no such f, g exist.

(b) The answer is *yes*. We begin with constructing functions $F, G : (1, \infty) \to (1, \infty)$ with the property $F(G(x)) = x^2$ and $G(F(x)) = x^4$ for $x > 1$. Define the functions φ, ψ by $F(2^{2^t}) = 2^{2^{\varphi(t)}}$ and $G(2^{2^t}) = 2^{2^{\psi(t)}}$. These functions determine F and G on the entire interval $(1, \infty)$, and satisfy $\varphi(\psi(t)) = t + 1$ and $\psi(\varphi(t)) = t + 2$. It is easy to find examples of φ and ψ: for example, $\varphi(t) = \frac{1}{2}t + 1$, $\psi(t) = 2t$. Thus we also arrive at an example for F, G:

$$F(x) = 2^{2^{\frac{1}{2}\log_2 \log_2 x + 1}} = 2^{2\sqrt{\log_2 x}}, \qquad G(x) = 2^{2^{2 \log_2 \log_2 x}} = 2^{\log_2^2 x}.$$

It remains only to extend these functions to the whole of $\mathbb{R}$. This can be done as follows:

$$\widetilde{f}(x) = \begin{cases} F(x) & \text{for } x > 1, \\ 1/F(1/x) & \text{for } 0 < x < 1, \\ x & \text{for } x \in \{0, 1\}; \end{cases} \quad \widetilde{g}(x) = \begin{cases} G(x) & \text{for } x > 1, \\ 1/G(1/x) & \text{for } 0 < x < 1, \\ x & \text{for } x \in \{0, 1\}; \end{cases}$$

$$\text{and then} \qquad f(x) = \widetilde{f}(|x|), \quad g(x) = \widetilde{g}(|x|) \qquad \text{for } x \in \mathbb{R}.$$

It is directly verified that these functions have the required property.

23. Let K, L, M, and N be the projections of O onto the lines AB, BC, CD, and DA, and let α_1, α_2, α_3, α_4, β_1, β_2, β_3, β_4 denote the angles OAB, OBC, OCD, ODA, OAD, OBA, OCB, ODC, respectively.

We start with the following observation: Since NK is a chord of the circle with diameter OA, we have $OA \sin \angle A = NK = ON \cos \alpha_1 + OK \cos \beta_1$ (because $\angle ONK = \alpha_1$ and $\angle OKN = \beta_1$). Analogous equalities also hold: $OB \sin \angle B = KL = OK \cos \alpha_2 + OL \cos \beta_2$, $OC \sin \angle C = LM = OL \cos \alpha_3 + OM \cos \beta_3$ and $OD \sin \angle D = MN = OM \cos \alpha_4 + ON \cos \beta_4$. Now the condition in the problem can be restated as $NK + LM = KL + MN$ (i.e., $KLMN$ is circumscribed), i.e.,

$$\begin{aligned} OK(\cos \beta_1 - \cos \alpha_2) + OL(\cos \alpha_3 - \cos \beta_2) \\ +OM(\cos \beta_3 - \cos \alpha_4) + ON(\cos \alpha_1 - \cos \beta_4) = 0. \end{aligned} \tag{1}$$

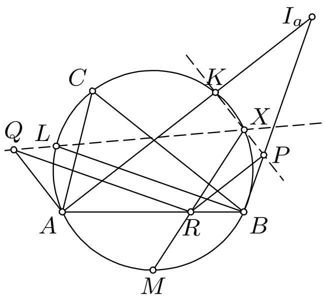
To prove that $ABCD$ is cyclic, it suffices to show that $\alpha_1 = \beta_4$. Assume the contrary, and let w.l.o.g. $\alpha_1 > \beta_4$. Then point A lies inside the circle BCD, which is further equivalent to $\beta_1 > \alpha_2$. On the other hand, from $\alpha_1 + \beta_2 = \alpha_3 + \beta_4$ we deduce $\alpha_3 > \beta_2$, and similarly $\beta_3 > \alpha_4$. Therefore, since the cosine is strictly decreasing on $(0, \pi)$, the left side of (1) is strictly negative, yielding a contradiction.

24. There is a bijective correspondence between representations in the given form of $2k$ and $2k+1$ for $k = 0, 1, \dots$, since adding 1 to every representation of $2k$, we obtain a representation of $2k+1$, and conversely, every representation of $2k+1$ contains at least one 1, which can be removed. Hence, $f(2k+1) = f(2k)$.
Consider all representations of $2k$. The number of those that contain at least one 1 equals $f(2k-1) = f(2k-2)$, while the number of those not containing a 1 equals $f(k)$ (the correspondence is given by division of summands by 2). Therefore
$$f(2k) = f(2k-2) + f(k). \tag{1}$$
Summing these equalities over $k = 1, \dots, n$, we obtain
$$f(2n) = f(0) + f(1) + \cdots + f(n). \tag{2}$$
We first prove the right-hand inequality. Since f is increasing, and $f(0) + f(1) = f(2)$, (2) yields $f(2n) \le nf(n)$ for $n \ge 2$. Now $f(2^3) = f(0) + \cdots + f(4) = 10 < 2^{3^2/2}$, and one can easily conclude by induction that $f(2^{n+1}) \le 2^n f(2^n) < 2^n \cdot 2^{n^2/2} < 2^{(n+1)^2/2}$ for each $n \ge 3$.
We now derive the lower estimate. It follows from (1) that $f(x+2) - f(x)$ is increasing. Consequently, for each m and $k < m$ we have $f(2m+2k) - f(2m) \ge f(2m+2k-2) - f(2m-2) \ge \cdots \ge f(2m) - f(2m-2k)$, so $f(2m+2k) + f(2m-2k) \ge 2f(2m)$. Adding all these inequalities for $k = 1, 2, \dots, m$, we obtain $f(0) + f(2) + \cdots + f(4m) \ge (2m+1)f(2m)$. But since $f(2) = f(3)$, $f(4) = f(5)$ etc., we also have $f(1) + f(3) + \cdots + f(4m-1) > (2m-1)f(2m)$, which together with the above inequality gives
$$f(8m) = f(0) + f(1) + \cdots + f(4m) > 4mf(2m). \tag{3}$$
Finally, we have that the inequality $f(2^n) > 2^{n^2/4}$ holds for $n = 2$ and $n = 3$, while for larger n we have by induction $f(2^n) > 2^{n-1} f(2^{n-2}) > 2^{n-1+(n-2)^2/4} = 2^{n^2/4}$. This completes the proof.

Remark. Despite the fact that the lower estimate is more difficult, it is much weaker than the upper estimate. It can be shown that $f(2^n)$ eventually (for large n) exceeds 2^{cn^2} for any $c < \frac{1}{2}$.

25. Let MR meet the circumcircle of triangle ABC again at a point X. We claim that X is the common point of the lines KP, LQ, MR. By symmetry, it will be enough to show that X lies on KP. It is easy to see that X and P lie on the same side of AB as K. Let $I_a = AK \cap BP$ be the excenter of $\triangle ABC$ corresponding to A. It is easy to calculate that $\angle AI_aB = \gamma/2$, from which we get $\angle RPB = \angle AI_aB = \angle MCB = \angle RXB$. Therefore R, B, P, X are concyclic. Now if P and K are on distinct sides of BX (the

other case is similar), we have $\angle RXP = 180^\circ - \angle RBP = 90^\circ - \beta/2 = \angle MAK = 180^\circ - \angle RXK$, from which it follows that K, X, P are collinear, as claimed.



Remark. It is not essential for the statement of the problem that R be an internal point of AB. Work with cases can be avoided using oriented angles.

26. Let us first examine the case that all the inequalities in the problem are actually equalities. Then $a_{n-2} = a_{n-1} + a_n$, $a_{n-3} = 2a_{n-1} + a_n, \dots, a_0 = F_n a_{n-1} + F_{n-1} a_n = 1$, where F_n is the nth Fibonacci number. Then it is easy to see (from $F_1 + F_2 + \cdots + F_k = F_{k+2}$) that $a_0 + \cdots + a_n = (F_{n+2} - 1)a_{n-1} + F_{n+1} a_n = \frac{F_{n+2}-1}{F_n} + \left(F_{n+1} - \frac{F_{n-1}(F_{n+2}-1)}{F_n}\right) a_n$. Since $\frac{F_{n-1}(F_{n+2}-1)}{F_n} \le F_{n+1}$, it follows that $a_0 + a_1 + \cdots + a_n \ge \frac{F_{n+2}-1}{F_n}$, with equality holding if and only if $a_n = 0$ and $a_{n-1} = \frac{1}{F_n}$.
We denote by M_n the required minimum in the general case. We shall prove by induction that $M_n = \frac{F_{n+2}-1}{F_n}$. For $M_1 = 1$ and $M_2 = 2$ it is easy to show that the formula holds; hence the inductive basis is true. Suppose that $n > 2$. The sequences $1, \frac{a_2}{a_1}, \dots, \frac{a_n}{a_1}$ and $1, \frac{a_3}{a_2}, \dots, \frac{a_n}{a_2}$ also satisfy the conditions of the problem. Hence we have

$$a_0 + \cdots + a_n = a_0 + a_1 \left(1 + \frac{a_2}{a_1} + \cdots + \frac{a_n}{a_1}\right) \ge 1 + a_1 M_{n-1}$$

and

$$a_0 + \cdots + a_n = a_0 + a_1 + a_2 \left(1 + \frac{a_3}{a_2} + \cdots + \frac{a_n}{a_2}\right) \ge 1 + a_1 + a_2 M_{n-2}.$$

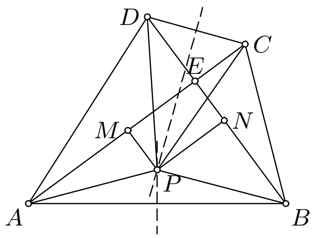
Multiplying the first inequality by $M_{n-2} - 1$ and the second one by M_{n-1}, adding the inequalities and using that $a_1 + a_2 \ge 1$, we obtain $(M_{n-1} + M_{n-2} + 1)(a_0 + \cdots + a_n) \ge M_{n-1}M_{n-2} + M_{n-1} + M_{n-2} + 1$, so

$$M_n \ge \frac{M_{n-1}M_{n-2} + M_{n-1} + M_{n-2} + 1}{M_{n-1} + M_{n-2} + 1}.$$

Since $M_{n-1} = \frac{F_{n+1}-1}{F_{n-1}}$ and $M_{n-2} = \frac{F_n - 1}{F_{n-2}}$, the above inequality easily yields $M_n \ge \frac{F_{n+2}-1}{F_n}$. However, we have shown above that equality can occur; hence $\frac{F_{n+2}-1}{F_n}$ is indeed the required minimum.

4.39 Solutions to the Shortlisted Problems of IMO 1998

1. We begin with the following observation: Suppose that P lies in $\triangle AEB$, where E is the intersection of AC and BD (the other cases are similar). Let M, N be the feet of the perpendiculars from P to AC and BD respectively. We have $S_{ABP} = S_{ABE} - S_{AEP} - S_{BEP} = \frac{1}{2}(AE \cdot BE - AE \cdot EN - BE \cdot EM) = \frac{1}{2}(AM \cdot BN - EM \cdot EN)$. Similarly, $S_{CDP} = \frac{1}{2}(CM \cdot DN - EM \cdot EN)$. Therefore, we obtain
$$S_{ABP} - S_{CDP} = \frac{AM \cdot BN - CM \cdot DN}{2}. \tag{1}$$



Now suppose that $ABCD$ is cyclic. Then P is the circumcenter of $ABCD$; hence M and N are the midpoints of AC and BD. Hence $AM = CM$ and $BN = DN$; thus (1) gives us $S_{ABP} = S_{CDP}$.
On the other hand, suppose that $ABCD$ is not cyclic and let w.l.o.g. $PA = PB > PC = PD$. Then we must have $AM > CM$ and $BN > DN$, and consequently by (1), $S_{ABP} > S_{CDP}$. This proves the other implication.

Second solution. Let F and G denote the midpoints of AB and CD, and assume that P is on the same side of FG as B and C. Since $PF \perp AB$, $PG \perp CD$, and $\angle FEB = \angle ABE$, $\angle GEC = \angle DCE$, a direct computation yields $\angle FPG = \angle FEG = 90^\circ + \angle ABE + \angle DCE$.
Taking into account that $S_{ABP} = \frac{1}{2}AB \cdot FP = FE \cdot FP$, we note that $S_{ABP} = S_{CDP}$ is equivalent to $FE \cdot FP = GE \cdot GP$, i.e., to $FE/EG = GP/PF$. But this last is equivalent to triangles EFG and PGF being similar, which holds if and only if $EFPG$ is a parallelogram. This last is equivalent to $\angle EFP = \angle EGP$, or $2\angle ABE = 2\angle DCE$. Thus $S_{ABP} = S_{CDP}$ is equivalent to $ABCD$ being cyclic.

Remark. The problems also allows an analytic solution, for example putting the x and y axes along the diagonals AC and BD.

2. If AD and BC are parallel, then $ABCD$ is an isosceles trapezoid with $AB = CD$, so P is the midpoint of EF. Let M and N be the midpoints of AB and CD. Then $MN \parallel BC$, and the distance $d(E, MN)$ equals the distance $d(F, MN)$ because B and D are the same distance from MN and $EM/BM = FN/DN$. It follows that the midpoint P of EF lies on MN, and consequently $S_{APD} : S_{BPC} = AD : BC$.
If AD and BC are not parallel, then they meet at some point Q. It is plain that $\triangle QAB \sim \triangle QCD$, and since $AE/AB = CF/CD$, we also deduce that $\triangle QAE \sim \triangle QCF$. Therefore $\angle AQE = \angle CQF$. Further, from these similarities we obtain $QE/QF = QA/QC = AB/CD = PE/PF$,

which in turn means that QP is the internal bisector of $\angle EQF$. But since $\angle AQE = \angle CQF$, this is also the internal bisector of $\angle AQB$. Hence P is at equal distances from AD and BC, so again $S_{APD} : S_{BPC} = AD : BC$.

Remark. The part $AB \parallel CD$ could also be regarded as a limiting case of the other part.

Second solution. Denote $\lambda = \frac{AE}{AB}$, $AB = a$, $BC = b$, $CD = c$, $DA = d$, $\angle DAB = \alpha$, $\angle ABC = \beta$. Since $d(P, AD) = \frac{c \cdot d(E,AD) + a \cdot d(F,AD)}{a+c}$, we have $S_{APD} = \frac{cS_{EAD} + aS_{FAD}}{a+c} = \frac{\lambda c S_{ABD} + (1-\lambda) a S_{ACD}}{a+c}$. Since $S_{ABD} = \frac{1}{2} ad \sin\alpha$ and $S_{ACD} = \frac{1}{2} cd \sin\beta$, we are led to $S_{APD} = \frac{acd}{a+c}[\lambda \sin\alpha + (1-\lambda)\sin\beta]$, and analogously $S_{BPC} = \frac{abc}{a+c}[\lambda \sin\alpha + (1-\lambda)\sin\beta]$. Thus we obtain $S_{APD} : S_{BPC} = d : b$.

3. *Lemma.* If U, W, V are three points on a line l in this order, and X a point in the plane with $XW \perp UV$, then $\angle UXV < 90^\circ$ if and only if $XW^2 > UW \cdot VW$.

   Proof. Let $XW^2 > UW \cdot VW$, and let X_0 be a point on the segment XW such that $X_0W^2 \geq UW \cdot VW$. Then $X_0W/UW = VW/X_0W$, so that triangles X_0WU and VWX_0 are similar. Thus $\angle UX_0V = \angle UX_0W + \angle WUX_0 = 90^\circ$, which immediately implies that $\angle UXV < 90^\circ$.
   Similarly, if $XW^2 \leq UW \cdot VW$, then $\angle UXV \geq 90^\circ$.

   Since $BI \perp RS$, it will be enough by the lemma to show that $BI^2 > BR \cdot BS$. Note that $\triangle BKR \sim \triangle BSL$: in fact, we have $\angle KBR = \angle SBL = 90^\circ - \beta/2$ and $\angle BKR = \angle AKM = \angle KLM = \angle BSL = 90^\circ - \alpha/2$. In particular, we obtain $BR/BK = BL/BS = BK/BS$, so that $BR \cdot BS = BK^2 < BI^2$.

   Second solution. Let E, F be the midpoints of KM and LM respectively. The quadrilaterals $RBIE$ and $SBIF$ are inscribed in the circles with diameters IR and IS. Now we have $\angle RIS = \angle RMS + \angle IRM + \angle ISM = 90^\circ - \beta/2 + \angle IBE + \angle IBF = 90^\circ - \beta/2 + \angle EBF$.
   On the other hand, BE and BF are medians in $\triangle BKM$ and $\triangle BLM$ in which $BM > BK$ and $BM > BL$. We conclude that $\angle MBE < \frac{1}{2}\angle MBK$ and $\angle MBF < \frac{1}{2}\angle MBL$. Adding these two inequalities gives $\angle EBF < \beta/2$. Therefore $\angle RIS < 90^\circ$.
   Remark. It can be shown (using vectors) that the statement remains true for an arbitrary line t passing through B.

4. Let K be the point on the ray BN with $\angle BCK = \angle BMA$. Since $\angle KBC = \angle ABM$, we get $\triangle BCK \sim \triangle BMA$. It follows that $BC/BM = BK/BA$, which implies that also $\triangle BAK \sim \triangle BMC$. The quadrilateral $ANCK$ is cyclic, because $\angle BKC = \angle BAM = \angle NAC$. Then by Ptolemy's theorem we obtain

$$AC \cdot BK = AC \cdot BN + AN \cdot CK + CN \cdot AK. \tag{1}$$

   On the other hand, from the similarities noted above we get

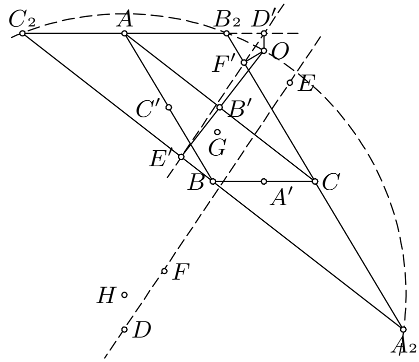
$$CK = \frac{BC \cdot AM}{BM}, \; AK = \frac{AB \cdot CM}{BM} \; \text{ and } BK = \frac{AB \cdot BC}{BM}.$$

After substitution of these values, the equality (1) becomes

$$\frac{AB \cdot BC \cdot AC}{BM} = AC \cdot BN + \frac{BC \cdot AM \cdot AN}{BM} + \frac{AB \cdot CM \cdot CN}{BM},$$

which is exactly the equality we must prove multiplied by $\frac{AB \cdot BC \cdot CA}{BM}$.

5. Let G be the centroid of $\triangle ABC$ and $\mathcal{H}$ the homothety with center G and ratio $-\frac{1}{2}$. It is well-known that $\mathcal{H}$ maps H into O. For every other point X, let us denote by X' its image under $\mathcal{H}$. Also, let $A_2B_2C_2$ be the triangle in which A, B, C are the midpoints of B_2C_2, C_2A_2, and A_2B_2, respectively.



It is clear that A', B', C' are the midpoints of sides BC, CA, AB respectively. Furthermore, D' is the reflection of A' across $B'C'$. Thus D' must lie on B_2C_2 and $A'D' \perp B_2C_2$. However, it also holds that $OA' \perp B_2C_2$, so we conclude that O, D', A' are collinear and D' is the projection of O on B_2C_2. Analogously, E', F' are the projections of O on C_2A_2 and A_2B_2.
Now we apply Simson's theorem. It claims that D', E', F' are collinear (which is equivalent to D, E, F being collinear) if and only if O lies on the circumcircle of $A_2B_2C_2$. However, this circumcircle is centered at H with radius $2R$, so the last condition is equivalent to $HO = 2R$.

6. Let P be the point such that $\triangle CDP$ and $\triangle CBA$ are similar and equally oriented. Since then $\angle DCP = \angle BCA$ and $\frac{BC}{CA} = \frac{DC}{CP}$, it follows that $\angle ACP = \angle BCD$ and $\frac{AC}{CP} = \frac{BC}{CD}$, so $\triangle ACP \sim \triangle BCD$. In particular, $\frac{BC}{CA} = \frac{DB}{PA}$.
Furthermore, by the conditions of the problem we have $\angle EDP = 360^\circ - \angle B - \angle D = \angle F$ and $\frac{PD}{DE} = \frac{PD}{CD} \cdot \frac{CD}{DE} = \frac{AB}{BC} \cdot \frac{CD}{DE} = \frac{AF}{FE}$. Therefore $\triangle EDP \sim \triangle EFA$ as well, so that similarly as above we conclude that $\triangle AEP \sim \triangle FED$ and consequently $\frac{AE}{EF} = \frac{PA}{FD}$.
Finally, $\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = \frac{DB}{PA} \cdot \frac{PA}{FD} \cdot \frac{FD}{DB} = 1$.
Second solution. Let a, b, c, d, e, f be the complex coordinates of A, B, C, D, E, F, respectively. The condition of the problem implies that $\frac{a-b}{b-c} \cdot \frac{c-d}{d-e} \cdot \frac{e-f}{f-a} = -1$.
On the other hand, since $(a-b)(c-d)(e-f) + (b-c)(d-e)(f-a) = (b-c)(a-e)(f-d) + (c-a)(e-f)(d-b)$ holds identically, we immediately deduce that $\frac{b-c}{c-a} \cdot \frac{a-e}{e-f} \cdot \frac{f-d}{d-b} = -1$. Taking absolute values gives $\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1$.

7. We shall use the following result.

Lemma. In a triangle ABC with $BC = a$, $CA = b$, and $AB = c$,

i. $\angle C = 2\angle B$ if and only if $c^2 = b^2 + ab$;

ii. $\angle C + 180^\circ = 2\angle B$ if and only if $c^2 = b^2 - ab$.

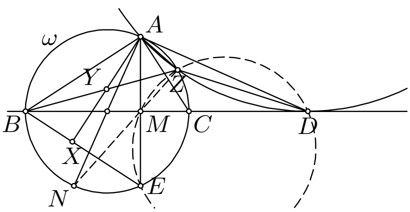
Proof.

i. Take a point D on the extension of BC over C such that $CD = b$. The condition $\angle C = 2\angle B$ is equivalent to $\angle ADC = \frac{1}{2}\angle C = \angle B$, and thus to $AD = AB = c$. This is further equivalent to triangles CAD and ABD being similar, so $CA/AD = AB/BD$, i.e., $c^2 = b(a+b)$.

ii. Take a point E on the ray CB such that $CE = b$. As above, $\angle C + 180^\circ = 2\angle B$ if and only if $\triangle CAE \sim \triangle ABE$, which is equivalent to $EB/BA = EA/AC$, or $c^2 = b(b-a)$.

Let F, G be points on the ray CB such that $CF = \frac{1}{3}a$ and $CG = \frac{4}{3}a$. Set $BC = a$, $CA = b$, $AB = c$, $EC = b_1$, and $EB = c_1$. By the lemma it follows that $c^2 = b^2 + ab$. Also $b_1 = AG$ and $c_1 = AF$, so Stewart's theorem gives us $c_1^2 = \frac{2}{3}b^2 + \frac{1}{3}c^2 - \frac{2}{9}a^2 = b^2 + \frac{1}{3}ab - \frac{2}{9}a^2$ and $b_1^2 = -\frac{1}{3}b^2 + \frac{4}{3}c^2 + \frac{4}{9}a^2 = b^2 + \frac{4}{3}ab + \frac{4}{9}a^2$. It follows that $b_1 = \frac{2}{3}a + b$ and $c_1^2 = b_1^2 - \left(ab + \frac{2}{3}a^2\right) = b_1^2 - ab_1$. The statement of the problem follows immediately by the lemma.

8. Let M be the point of intersection of AE and BC, and let N be the point on ω diametrically opposite A.



Since $\angle B < \angle C$, points N and B are on the same side of AE. Furthermore, $\angle NAE = \angle BAX = 90^\circ - \angle ABE$; hence the triangles NAE and BAX are similar. Consequently, $\triangle BAY$ and $\triangle NAM$ are also similar, since M is the midpoint of AE. Thus $\angle ANZ = \angle ABZ = \angle ABY = \angle ANM$, implying that N, M, Z are collinear. Now we have $\angle ZMD = 90^\circ - \angle ZMA = \angle EAZ = \angle ZED$ (the last equality because ED is tangent to ω); hence $ZMED$ is a cyclic quadrilateral. It follows that $\angle ZDM = \angle ZEA = \angle ZAD$, which is enough to conclude that MD is tangent to the circumcircle of AZD.

Remark. The statement remains valid if $\angle B \geq \angle C$.

9. Set $a_{n+1} = 1 - (a_1 + \cdots + a_n)$. Then $a_{n+1} > 0$, and the desired inequality becomes

$$\frac{a_1 a_2 \cdots a_{n+1}}{(1-a_1)(1-a_2)\cdots(1-a_{n+1})} \leq \frac{1}{n^{n+1}}.$$

To prove it, we observe that

$$1 - a_i = a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_{n+1} \geq n\sqrt[n]{a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1}}.$$

Multiplying these inequalities for $i = 1, 2, \ldots, n+1$, we get exactly the inequality we need.

10. We shall first prove the inequality for n of the form 2^k, $k = 0, 1, 2, \dots$. The case $k = 0$ is clear. For $k = 1$, we have

$$\frac{1}{r_1+1} + \frac{1}{r_2+1} - \frac{2}{\sqrt{r_1 r_2}+1} = \frac{(\sqrt{r_1 r_2}-1)(\sqrt{r_1}-\sqrt{r_2})^2}{(r_1+1)(r_2+1)(\sqrt{r_1 r_2}+1)} \geq 0.$$

For the inductive step it suffices to show that the claim for k and 2 implies that for $k+1$. Indeed,

$$\begin{aligned}\sum_{i=1}^{2^{k+1}} \frac{1}{r_i+1} &\geq \frac{2^k}{\sqrt[2^k]{r_1 r_2 \cdots r_{2^k}}+1} + \frac{2^k}{\sqrt[2^k]{r_{2^k+1} r_{2^k+2} \cdots r_{2^{k+1}}}+1} \\ &\geq \frac{2^{k+1}}{\sqrt[2^{k+1}]{r_1 r_2 \cdots r_{2^{k+1}}}+1},\end{aligned} \tag{1}$$

and the induction is complete.

We now show that if the statement holds for 2^k, then it holds for every $n < 2^k$ as well. Put $r_{n+1} = r_{n+2} = \cdots = r_{2^k} = \sqrt[n]{r_1 r_2 \dots r_n}$. Then (1) becomes

$$\frac{1}{r_1+1} + \cdots + \frac{1}{r_n+1} + \frac{2^k-n}{\sqrt[n]{r_1 \cdots r_n}+1} \geq \frac{2^k}{\sqrt[n]{r_1 \cdots r_n}+1}.$$

This proves the claim.

Second solution. Define $r_i = e^{x_i}$, where $x_i > 0$. The function $f(x) = \frac{1}{1+e^x}$ is convex for $x > 0$: indeed, $f''(x) = \frac{e^x(e^x-1)}{(e^x+1)^3} > 0$. Thus by Jensen's inequality applied to $f(x_1), \dots, f(x_n)$, we get $\frac{1}{r_1+1} + \cdots + \frac{1}{r_n+1} \geq \frac{n}{\sqrt[n]{r_1 \cdots r_n}+1}$.

11. The given inequality is equivalent to $x^3(x+1) + y^3(y+1) + z^3(z+1) \geq \frac{3}{4}(x+1)(y+1)(z+1)$. By the A-G mean inequality, it will be enough to prove a stronger inequality:

$$x^4 + x^3 + y^4 + y^3 + z^4 + z^3 \geq \frac{1}{4}[(x+1)^3 + (y+1)^3 + (z+1)^3]. \tag{1}$$

If we set $S_k = x^k + y^k + z^k$, (1) takes the form $S_4 + S_3 \geq \frac{1}{4}S_3 + \frac{3}{4}S_2 + \frac{3}{4}S_1 + \frac{3}{4}$. Note that by the A-G mean inequality, $S_1 = x + y + z \geq 3$. Thus it suffices to prove the following:

If $S_1 \geq 3$ and $m > n$ are positive integers, then $S_m \geq S_n$.

This can be shown in many ways. For example, by Hölder's inequality,

$$(x^m + y^m + z^m)^{n/m}(1+1+1)^{(m-n)/m} \geq x^n + y^n + z^n.$$

(Another way is using the Chebyshev inequality: if $x \geq y \geq z$ then $x^{k-1} \geq y^{k-1} \geq z^{k-1}$; hence $S_k = x \cdot x^{k-1} + y \cdot y^{k-1} + z \cdot z^{k-1} \geq \frac{1}{3}S_1 S_{k-1}$, and the claim follows by induction.)

Second solution. Assume that $x \geq y \geq z$. Then also $\frac{1}{(y+1)(z+1)} \geq \frac{1}{(x+1)(z+1)} \geq \frac{1}{(x+1)(y+1)}$. Hence Chebyshev's inequality gives that

$$\begin{aligned}&\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+x)(1+z)} + \frac{z^3}{(1+x)(1+y)} \\ &\geq \frac{1}{3}\frac{(x^3+y^3+z^3)\cdot(3+x+y+z)}{(1+x)(1+y)(1+z)}.\end{aligned}$$

Now if we put $x+y+z=3S$, we have $x^3+y^3+z^3 \geq 3S$ and $(1+x)(1+y)(1+z) \leq (1+a)^3$ by the A-G mean inequality. Thus the needed inequality reduces to $\frac{6S^3}{(1+S)^3} \geq \frac{3}{4}$, which is obviously true because $S \geq 1$.

Remark. Both these solutions use only that $x+y+z \geq 3$.

12. The assertion is clear for $n=0$. We shall prove the general case by induction on n. Suppose that $c(m,i) = c(m, m-i)$ for all i and $m \leq n$. Then by the induction hypothesis and the recurrence formula we have $c(n+1,k) = 2^k c(n,k) + c(n,k-1)$ and $c(n+1, n+1-k) = 2^{n+1-k}c(n, n+1-k) + c(n, n-k) = 2^{n+1-k}c(n,k-1) + c(n,k)$. Thus it remains only to show that

$$(2^k - 1)c(n,k) = (2^{n+1-k}-1)c(n,k-1).$$

We prove this also by induction on n. By the induction hypothesis,

$$c(n-1,k) = \frac{2^{n-k}-1}{2^k-1}c(n-1,k-1)$$

and

$$c(n-1,k-2) = \frac{2^{k-1}-1}{2^{n+1-k}-1}c(n-1,k-1).$$

Using these formulas and the recurrence formula we obtain $(2^k-1)c(n,k) - (2^{n+1-k}-1)c(n,k-1) = (2^{2k}-2^k)c(n-1,k) - (2^n - 3\cdot 2^{k-1}+1)c(n-1,k-1) - (2^{n+1-k}-1)c(n-1,k-2) = (2^n - 2^k)c(n-1,k-1) - (2^n - 3\cdot 2^{k-1}+1)c(n-1,k-1) - (2^{k-1}-1)c(n-1,k-1) = 0$. This completes the proof.

Second solution. The given recurrence formula resembles that of binomial coefficients, so it is natural to search for an explicit formula of the form $c(n,k) = \frac{F(n)}{F(k)F(n-k)}$, where $F(m) = f(1)f(2)\cdots f(m)$ (with $F(0)=1$) and f is a certain function from the natural numbers to the real numbers. If there is such an f, then $c(n,k) = c(n,n-k)$ follows immediately.
After substitution of the above relation, the recurrence equivalently reduces to $f(n+1) = 2^k f(n-k+1) + f(k)$. It is easy to see that $f(m) = 2^m - 1$ satisfies this relation.

Remark. If we introduce the polynomial $P_n(x) = \sum_{k=0}^{n} c(n,k)x^k$, the recurrence relation gives $P_0(x) = 1$ and $P_{n+1}(x) = xP_n(x) + P_n(2x)$. As a consequence of the problem, all polynomials in this sequence are symmetric, i.e., $P_n(x) = x^n P_n(x^{-1})$.

13. Denote by $\mathcal{F}$ the set of functions considered. Let $f \in \mathcal{F}$, and let $f(1) = a$. Putting $n = 1$ and $m = 1$ we obtain $f(f(z)) = a^2 z$ and $f(az^2) = f(z)^2$ for all $z \in \mathbb{N}$. These equations, together with the original one, imply $f(x)^2 f(y)^2 = f(x)^2 f(ay^2) = f(x^2 f(f(ay^2))) = f(x^2 a^3 y^2) = f(a(axy)^2) = f(axy)^2$, or $f(axy) = f(x)f(y)$ for all $x, y \in \mathbb{N}$. Thus $f(ax) = af(x)$, and we conclude that

$$af(xy) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{N}. \tag{1}$$

We now prove that $f(x)$ is divisible by a for each $x \in \mathbb{N}$. In fact, we inductively get that $f(x)^k = a^{k-1} f(x^k)$ is divisible by a^{k-1} for every k. If p^α and p^β are the exact powers of a prime p that divide $f(x)$ and a respectively, we deduce that $k\alpha \geq (k-1)\beta$ for all k, so we must have $\alpha \geq \beta$ for any p. Therefore $a \mid f(x)$.

Now we consider the function on natural numbers $g(x) = f(x)/a$. The above relations imply

$$g(1) = 1, \quad g(xy) = g(x)g(y), \quad g(g(x)) = x \quad \text{for all } x, y \in \mathbb{N}. \tag{2}$$

Since $g \in \mathcal{F}$ and $g(x) \leq f(x)$ for all x, we may restrict attention to the functions g only.

Clearly g is bijective. We observe that g maps a prime to a prime. Assume to the contrary that $g(p) = uv$, $u, v > 1$. Then $g(uv) = p$, so either $g(u) = 1$ and $g(v) = 1$. Thus either $g(1) = u$ or $g(1) = v$, which is impossible.

We return to the problem of determining the least possible value of $g(1998)$. Since $g(1998) = g(2 \cdot 3^3 \cdot 37) = g(2) \cdot g(3)^3 \cdot g(37)$, and $g(2)$, $g(3)$, $g(37)$ are distinct primes, $g(1998)$ is not smaller than $2^3 \cdot 3 \cdot 5 = 120$. On the other hand, the value of 120 is attained for any function g satisfying (2) and $g(2) = 3$, $g(3) = 2$, $g(5) = 37$, $g(37) = 5$. Hence the answer is 120.

14. If $x^2 y + x + y$ is divisible by $xy^2 + y + 7$, then so is the number $y(x^2 y + x + y) - x(xy^2 + y + 7) = y^2 - 7x$.

If $y^2 - 7x \geq 0$, then since $y^2 - 7x < xy^2 + y + 7$, it follows that $y^2 - 7x = 0$. Hence $(x, y) = (7t^2, 7t)$ for some $t \in \mathbb{N}$. It is easy to check that these pairs really are solutions.

If $y^2 - 7x < 0$, then $7x - y^2 > 0$ is divisible by $xy^2 + y + 7$. But then $xy^2 + y + 7 \leq 7x - y^2 < 7x$, from which we obtain $y \leq 2$. For $y = 1$, we are led to $x + 8 \mid 7x - 1$, and hence $x + 8 \mid 7(x + 8) - (7x - 1) = 57$. Thus the only possibilities are $x = 11$ and $x = 49$, and the obtained pairs $(11, 1), (49, 1)$ are indeed solutions. For $y = 2$, we have $4x + 9 \mid 7x - 4$, so that $7(4x + 9) - 4(7x - 4) = 79$ is divisible by $4x + 9$. We do not get any new solutions in this case.

Therefore all required pairs (x, y) are $(7t^2, 7t)$ $(t \in \mathbb{N})$, $(11, 1)$, and $(49, 1)$.

15. The condition is obviously satisfied if $a = 0$ or $b = 0$ or $a = b$ or a, b are both integers. We claim that these are the only solutions.

Suppose that a, b belong to none of the above categories. The quotient $a/b = \lfloor a \rfloor / \lfloor b \rfloor$ is a nonzero rational number: let $a/b = p/q$, where p and q are coprime nonzero integers.

Suppose that $p \notin \{-1, 1\}$. Then p divides $\lfloor an \rfloor$ for all n, so in particular p divides $\lfloor a \rfloor$ and thus $a = kp + \varepsilon$ for some $k \in \mathbb{N}$ and $0 \le \varepsilon < 1$. Note that $\varepsilon \neq 0$, since otherwise $b = kq$ would also be an integer. It follows that there exists an $n \in \mathbb{N}$ such that $1 \le n\varepsilon < 2$. But then $\lfloor na \rfloor = \lfloor knp + n\varepsilon \rfloor = knp + 1$ is not divisible by p, a contradiction. Similarly, $q \notin \{-1, 1\}$ is not possible. Therefore we must have $p, q = \pm 1$, and since $a \neq b$, the only possibility is $b = -a$. However, this leads to $\lfloor -a \rfloor = -\lfloor a \rfloor$, which is not valid if a is not an integer.

16. Let S be a set of integers such that for no four distinct elements $a, b, c, d \in S$, it holds that $20 \mid a + b - c - d$. It is easily seen that there cannot exist distinct elements a, b, c, d with $a \equiv b$ and $c \equiv d \pmod{20}$. Consequently, if the elements of S give k different residues modulo 20, then S itself has at most $k + 2$ elements.

    Next, consider these k elements of S with different residues modulo 20. They give $\frac{k(k-1)}{2}$ different sums of two elements. For $k \ge 7$ there are at least 21 such sums, and two of them, say $a + b$ and $c + d$, are equal modulo 20; it is easy to see that a, b, c, d are discinct. It follows that k cannot exceed 6, and consequently S has at most 8 elements.

    An example of a set S with 8 elements is $\{0, 20, 40, 1, 2, 4, 7, 12\}$. Hence the answer is $n = 9$.

17. Initially, we determine that the first few values for a_n are 1, 3, 4, 7, 10, 12, 13, 16, 19, 21, 22, 25. Since these are exactly the numbers of the forms $3k + 1$ and $9k + 3$, we conjecture that this is the general pattern. In fact, it is easy to see that the equation $x + y = 3z$ has no solution in the set $K = \{3k + 1, 9k + 3 \mid k \in \mathbb{N}\}$. We shall prove that the sequence $\{a_n\}$ is actually this set ordered increasingly.

    Suppose $a_n > 25$ is the first member of the sequence not belonging to K. We have several cases:

    (i) $a_n = 3r + 2$, $r \in \mathbb{N}$. By the assumption, one of $r + 1, r + 2, r + 3$ is of the form $3k + 1$ (and smaller than a_n), and therefore is a member a_i of the sequence. Then $3a_i$ equals $a_n + 1$, $a_n + 4$, or $a_n + 7$, which is a contradiction because $1, 4, 7$ are in the sequence.

    (ii) $a_n = 9r$, $r \in \mathbb{N}$. Then $a_n + a_2 = 3(3r + 1)$, although $3r + 1$ is in the sequence, a contradiction.

    (iii) $a_n = 9r + 6$, $r \in \mathbb{N}$. Then one of the numbers $3r + 3$, $3r + 6$, $3r + 9$ is a member a_j of the sequence, and thus $3a_j$ is equal to $a_n + 3$, $a_n + 12$, or $a_n + 21$, where $3, 12, 21$ are members of the sequence, again a contradiction.

    Once we have revealed the structure of the sequence, it is easy to compute a_{1998}. We have $1998 = 4 \cdot 499 + 2$, which implies $a_{1998} = 9 \cdot 499 + a_2 = 4494$.

18. We claim that, if $2^n - 1$ divides $m^2 + 9$ for some $m \in \mathbb{N}$, then n must be a power of 2. Suppose otherwise that n has an odd divisor $d > 1$. Then $2^d - 1 \mid 2^n - 1$ is also a divisor of $m^2 + 9 = m^2 + 3^2$. However, $2^d - 1$ has some prime divisor p of the form $4k - 1$, and by a well-known fact, p divides both m and 3. Hence $p = 3$ divides $2^d - 1$, which is impossible, because for d odd, $2^d \equiv 2 \pmod 3$. Hence $n = 2^r$ for some $r \in \mathbb{N}$.
Now let $n = 2^r$. We prove the existence of m by induction on r. The case $r = 1$ is trivial. Now for any $r > 1$ note that $2^{2^r} - 1 = (2^{2^{r-1}} - 1)(2^{2^{r-1}} + 1)$. The induction hypothesis claims that there exists an m_1 such that $2^{2^{r-1}} - 1 \mid m_1^2 + 9$. We also observe that $2^{2^{r-1}} + 1 \mid m_2^2 + 9$ for simple $m_2 = 3 \cdot 2^{2^{r-2}}$. By the Chinese remainder theorem, there is an $m \in \mathbb{N}$ that satisfies $m \equiv m_1 \pmod{2^{2^{r-1}} - 1}$ and $m \equiv m_2 \pmod{2^{2^{r-1}} + 1}$. It is easy to see that this $m^2 + 9$ will be divisible by both $2^{2^{r-1}} - 1$ and $2^{2^{r-1}} + 1$, i.e., that $2^{2^r} - 1 \mid m^2 + 9$. This completes the induction.

19. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where p_i are distinct primes and α_i natural numbers, we have $\tau(n) = (\alpha_1 + 1) \cdots (\alpha_r + 1)$ and $\tau(n^2) = (2\alpha_1 + 1) \dots (2\alpha_r + 1)$. Putting $k_i = \alpha_i + 1$, the problem reduces to determining all natural values of m that can be represented as

$$m = \frac{2k_1 - 1}{k_1} \cdot \frac{2k_2 - 1}{k_2} \cdots \frac{2k_r - 1}{k_r}. \tag{1}$$

Since the numerator $\tau(n^2)$ is odd, m must be odd too. We claim that every odd m has a representation of the form (1). The proof will be done by induction.
This is clear for $m = 1$. Now for every $m = 2k - 1$ with k odd the result follows easily, since $m = \frac{2k-1}{k} \cdot k$, and k can be written as (1). We cannot do the same if k is even; however, in the case $m = 4k - 1$ with k odd, we can write it as $m = \frac{12k-3}{6k-1} \cdot \frac{6k-1}{3k} \cdot k$, and this works.
In general, suppose that $m = 2^t k - 1$, with k odd. Following the same pattern, we can write m as

$$m = \frac{2^t(2^t - 1)k - (2^t - 1)}{2^{t-1}(2^t - 1)k - (2^{t-1} - 1)} \cdots \frac{4(2^t - 1)k - 3}{2(2^t - 1)k - 1} \cdot \frac{2(2^t - 1)k - 1}{(2^t - 1)k} \cdot k.$$

The induction is finished. Hence m can be represented as $\frac{\tau(n^2)}{\tau(n)}$ if and only if it is odd.

20. We first consider the special case $n = 3^r$. Then the simplest choice $\frac{10^n - 1}{9} = 11\dots1$ (n digits) works. This can be shown by induction: it is true for $r = 1$, while the inductive step follows from $10^{3^r} - 1 = (10^{3^{r-1}} - 1)(10^{2 \cdot 3^{r-1}} + 10^{3^{r-1}} + 1)$, because the second factor is divisible by 3.
In the general case, let $k \ge n/2$ be a positive integer and $a_1, \dots, a_{n-k}$ be nonzero digits. We have

$$\begin{aligned} A &= (10^k - 1)\overline{a_1 a_2 \dots a_{n-k}} \\ &= \overline{a_1 a_2 \dots a_{n-k-1} a'_{n-k} \underbrace{99\dots99}_{2k-n} b_1 b_2 \dots b_{n-k-1} b'_{n-k}}, \end{aligned}$$

where $a'_{n-k} = a_{n-k} - 1$, $b_i = 9 - a_i$, and $b'_{n-k} = 9 - a'_{n-k}$. The sum of digits of A equals $9k$ independently of the choice of digits $a_1, \dots, a_{n-k}$. Thus we need only choose $k \ge \frac{n}{2}$ and digits $a_1, \dots, a_{n-k-1} \notin \{0, 9\}$ and $a_{n-k} \in \{0, 1\}$ in order for the conditions to be fulfilled. Let us choose

$$k = \begin{cases} 3^r, & \text{if } 3^r < n \le 2 \cdot 3^r \text{ for some } r \in \mathbb{Z}, \\ 2 \cdot 3^r, & \text{if } 2 \cdot 3^r < n \le 3^{r+1} \text{ for some } r \in \mathbb{Z}; \end{cases}$$

and $\overline{a_1 a_2 \dots a_{n-k}} = \overline{22\dots2}$. The number

$$A = \overline{\underbrace{22\dots2}_{n-k-1} 1 \underbrace{99\dots99}_{2k-n} \underbrace{77\dots7}_{n-k-1} 8}$$

thus obtained is divisible by $2 \cdot (10^k - 1)$, which is, as explained above, divisible by $18 \cdot 3^r$. Finally, the sum of digits of A is either $9 \cdot 3^r$ or $18 \cdot 3^r$; thus A has the desired properties.

21. Such a sequence is obviously strictly increasing. We note that it must be unique. Indeed, given $a_0, a_1, \dots, a_{n-1}$, then a_n is the least positive integer not of the form $a_i + 2a_j + 4a_k$, $i, j, k < n$.
We easily get that the first few a_n's are $0, 1, 8, 9, 64, 65, 72, 73, \dots$. Let $\{c_n\}$ be the increasing sequence of all positive integers that consist of zeros and ones in base 8, i.e., those of the form $t_0 + 2^3 t_1 + \dots + 2^{3q} t_q$ where $t_i \in \{0, 1\}$. We claim that $a_n = c_n$. To prove this, it is enough to show that each $m \in \mathbb{N}$ can be uniquely written as $c_i + 2c_j + 4c_k$. If $m = t_0 + 2t_1 + \dots + 2^r t_r$ $(t_i \in \{0, 1\})$, then $m = c_i + 2c_j + 2^2 c_k$ is obviously possible if and only if $c_i = t_0 + 2^3 t_3 + 2^6 t_6 + \cdots$, $c_j = t_1 + 2^3 t_4 + \dots$, and $c_k = t_2 + 2^3 t_5 + \cdots$.
Hence for $n = s_0 + 2s_1 + \dots + 2^r s_r$ we have $a_n = s_0 + 8s_1 + \dots + 8^r s_r$. In particular, $1998 = 2 + 2^2 + 2^3 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10}$, so $a_{1998} = 8 + 8^2 + 8^3 + 8^6 + 8^7 + 8^8 + 8^9 + 8^{10} = 1227096648$.
Second solution. Define $f(x) = x^{a_0} + x^{a_1} + \cdots$. Then the assumed property of $\{a_n\}$ gives

$$f(x)f(x^2)f(x^4) = \sum_{i,j,k} x^{a_i + 2a_j + 4a_k} = \sum_n x^n = \frac{1}{1-x}.$$

We also get as a consequence $f(x^2)f(x^4)f(x^8) = \frac{1}{1-x^2}$, which gives $f(x) = (1 + x)f(x^8)$. Continuing this, we obtain

$$f(x) = (1 + x)(1 + x^8)(1 + x^{8^2}) \cdots .$$

Hence the a_n's are integers that have only 0's and 1's in base 8.

22. We can obviously change each x into $\lfloor x \rfloor$ or $\lceil x \rceil$ so that the column sums remain unchanged. However, this does not necessarily match the row sums as well, so let us consider the sum S of the absolute values of the changes in the row sums. It is easily seen that S is even, and we want it to be 0.
A row may have a higher or lower sum than desired. Let us mark a cell by $-$ if its entry x was changed to $\lfloor x \rfloor$, and by $+$ if it was changed to $\lceil x \rceil$ instead. We call a row R_2 *accessible* from a row R_1 if there is a column C such that $C \cap R_1$ is marked $+$ and $C \cap R_2$ is marked $-$. Note that a column containing a $+$ must contain a $-$ as well, because column sums are unchanged. Hence from each row with a higher sum we can access another row.
Assume that the row sum in R_1 is higher. If $R_1, R_2, \ldots, R_k$ is a sequence of rows such that R_{i+1} is accessible from R_i via some column C_i and such that the row sum in R_k is lower, then by changing the signs in $C_i \cap R_i$ and $C_i \cap R_{i+1}$ $(i = 1, 2, \ldots, k-1)$ we decrease S by 2, leaving column sums unchanged. We claim that such a sequence of rows always exists.
Let $\mathcal{R}$ be the union of all rows that are accessible from R_1, directly or indirectly; let $\overline{\mathcal{R}}$ be the union of the remaining rows. We show that for any column C, the sum in $\mathcal{R} \cap C$ is not higher. If $\mathcal{R} \cap C$ contains no $+$'s, then this is clear. If $\mathcal{R} \cap C$ contains a $+$, since the rows of $\overline{\mathcal{R}}$ are not accessible, the set $\overline{\mathcal{R}} \cap C$ contains no $-$'s. It follows that the sum in $\overline{\mathcal{R}} \cap C$ is not lower, and since column sums are unchanged, we again come to the same conclusion. Thus the total sum in $\mathcal{R}$ is not higher. Therefore, there is a row in $\mathcal{R}$ with too low a sum, justifying our claim.

23. (a) If n is even, then every odd integer is unattainable. Assume that $n \geq 9$ is odd. Let a be obtained by addition from some b, and b from c by multiplication. Then a is $2c+2$, $2c+n$, $nc+2$, or $nc+n$, and is in every case congruent to $2c+2$ modulo $n-2$. In particular, if $a \equiv -2 \pmod{n-2}$, then also $b \equiv -4$ and $c \equiv -2 \pmod{n-2}$.
Now consider any $a = kn(n-2) - 2$, where k is odd. If it is attainable, but not divisible by 2 or n, it must have been obtained by addition. Thus all predecessors of a are congruent to either -2 or $-4 \pmod{n-2}$, and none of them equals 1, a contradiction.
(b) Call an attainable number *addy* if the last operation is addition, and *multy* if the last operation is multiplication. We prove the following claims by simultaneous induction on k:
(1) $n = 6k$ is both addy and multy;
(2) $n = 6k+1$ is addy for $k \geq 2$;
(3) $n = 6k+2$ is addy for $k \geq 1$;
(4) $n = 6k+3$ is addy;
(5) $n = 6k+4$ is multy for $k \geq 1$;
(6) $n = 6k+5$ is addy.
The cases $k \leq 1$ are easily verified. For $k \geq 2$, suppose all six statements hold up to $k-1$.

Since $6k-3$ is addy, $6k$ is multy.
Next, $6k-2$ is multy, so both $6k=(6k-2)+2$ and $6k+1=(6k-2)+3$ are addy.
Since $6k$ is multy, both $6k+2$ and $6k+3$ are addy.
Number $6k+4=2\cdot(3k+2)$ is multy, because $3k+2$ is addy (being either $6l+2$ or $6l+5$).
Finally, we have $6k+5=3\cdot(2k+1)+2$. Since $2k+1$ is $6l+1$, $6l+3$, or $6l+5$, it is addy except for 7. Hence $6k+5$ is addy except possibly for 23. But $23=((1\cdot 2+2)\cdot 2+2)\cdot 2+3$ is also addy.
This completes the induction. Now 1 is given and $2=1\cdot 2$, $4=1+3$. It is easily checked that 7 is not attainable, and hence it is the only unattainable number.

24. Let $f(n)$ be the minimum number of moves needed to monotonize *any* permutation of n distinct numbers. Let us be given a permutation π of $\{1,2,\dots,n\}$, and let k be the first element of π. In $f(n-1)$ moves, we can transform π to either $(k,1,2,\dots,k-1,k+1,\dots,n)$ or $(k,n,n-1,\dots,k+1,k-1,\dots,1)$. Now the former can be changed to $(k,k-1,\dots,2,1,k+1,\dots,n)$, which is then monotonized in the next move. Similarly, the latter also can be monotonized in two moves. It follows that $f(n)\le f(n-1)+2$. Thus we shall be done if we show that $f(5)\le 4$.
First we note that $f(3)=1$. Consider a permutation of $\{1,2,3,4\}$. If either 1 or 4 is the first or the last element, we need one move to monotonize the other three elements, and at most one more to monotonize the whole permutation. Of the remaining four permutations, $(2,1,4,3)$ and $(3,4,1,2)$ can also be monotonized in two moves. The permutations $(2,4,1,3)$ and $(3,1,4,2)$ require 3 moves, but by this we can choose whether to change them into $(1,2,3,4)$ or $(4,3,2,1)$.
We now consider a permutation of $\{1,2,3,4,5\}$. If either 1 or 5 is in the first or last position, we can monotonize the rest in 3 moves, but in such a way that the whole permutation can be monotonized in the next move. If this is not the case, then either 1 or 5 is in the second or fourth position. Then we simply switch it to the outside in one move and continue as in the former case. Hence $f(5)=4$, as desired.

25. We use induction on n. For $n=3$, we have a single two-element subset $\{i,j\}$ that is split by (i,k,j) (where k is the third element of U). Assume that the result holds for some $n\ge 3$, and consider a family $\mathcal{F}$ of $n-1$ proper subsets of $U=\{1,2,\dots,n+1\}$, each with at least 2 elements.
To continue the induction, we need an element $a\in U$ that is contained in all n-element subsets of $\mathcal{F}$, but in at most one of the two-element subsets. We claim that such an a exists. Let $\mathcal{F}$ contain k n-element subsets and m 2-element subsets ($k+m\le n-1$). The intersection of the n-element subsets contains exactly $n+1-k\ge m+2$ elements. On the other hand, at most m elements belong to more than one 2-element subset, which justifies our claim.

Now let A be the 2-element subset that contains a, if it exists; otherwise, let A be any subset from $\mathcal{F}$ containing a. Excluding a from all the subsets from $\mathcal{F} \setminus \{A\}$, we get at most $n-2$ subsets of $U \setminus \{a\}$ with at least 2 and at most $n-1$ elements. By the inductive hypothesis, we can arrange $U \setminus \{a\}$ so that we split all the subsets of $\mathcal{F}$ except A. It remains to place a, and we shall make a desired arrangement if we put it anywhere away from A.

26. Put $n = 2r+1$. Since each of the $\binom{n}{2}$ pairs of judges agrees on at most two candidates, the total number of agreements is at most $k\binom{n}{2}$. On the other hand, if the ith candidate is passed by x_i judges and failed by $n - x_i$ judges, then the number of agreements on this candidate equals

$$\binom{x_i}{2} + \binom{n-x_i}{2} = \frac{x_i^2 + (n-x_i)^2 - n}{2} \geq \frac{r^2 + (n-r)^2 - n}{2} = \frac{(n-1)^2}{4}.$$

Therefore the total number of agreements is at least $\frac{m(n-1)^2}{4}$, which implies that

$$k\binom{n}{2} \geq \frac{m(n-1)^2}{4}, \qquad \text{hence} \qquad \frac{k}{m} \geq \frac{n-1}{2n}.$$

Remark. The obtained inequality is sharp. Indeed, if $m = \binom{2r+1}{r}$ and each candidate is passed by a different subset of r judges, we get equality. A similar example shows that the result is not valid for even n. In that case the weaker estimate $\frac{k}{m} \geq \frac{n-2}{2n-2}$ holds.

27. Since this is essentially a graph problem, we call the points and segments vertices and edges of the graph. We first prove that the task is impossible if $k \leq 4$.

Cases $k \leq 2$ are trivial. If $k = 3$, then among the edges from a vertex A there are two of the same color, say AB and AC, so we don't have all the three colors among the edges joining A, B, C.

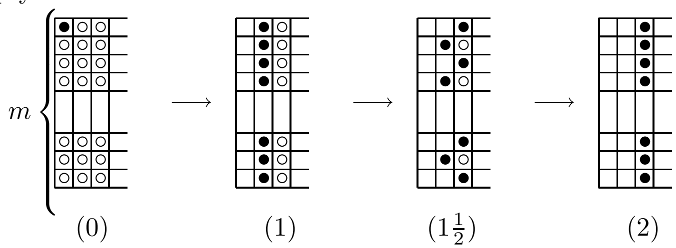
Now let $k = 4$, and assume that there is a desired coloring. Consider the edges incident with a vertex A. At least three of them have the same color, say blue. Suppose that four of them, AB, AC, AD, AE, are blue. There is a blue edge, say BC, among the ones joining B, C, D, E. Then four of the edges joining A, B, C, D are blue, and we cannot complete the coloring. So, exactly three edges from A are blue: AB, AC, AD. Also, of the edges connecting any three of the 6 vertices other than A, B, C, D, one is blue (because the edges joining them with A are not so). By a classical result, there is a blue triangle EFG with vertices among these six. Now one of EB, EC, ED must be blue as well, because none of BC, BD, CD is. Let it be EB. Then four of the edges joining B, E, F, G are blue, which is impossible.

For $k = 5$ the task is possible. Label the vertices $0, 1, \dots, 9$. For each color, we divide the vertices into four groups and paint in this color every edge joining two from the same group, as shown below. Then among any 5 vertices, 2 must belong to the same group, and the edge connecting them has the considered color.

yellow:	01 12 20	36 69 93	57	48
red:	23 34 42	58 81 15	79	60
blue:	45 56 64	70 03 37	91	82
green:	67 78 86	92 25 59	13	04
orange:	89 90 08	14 47 71	35	26.

A desired coloring can be made for $k \geq 6$ as well. Paint the edge ij in the $(i+j)$th color for $i < j \leq 8$, and in the $2i$th color if $j = 9$ (the addition being modulo 9). We can ignore the edges painted with the extra colors. Then the edges of one color appear as five disjoint segments, so that any complete k-graph for $k \geq 5$ contains one of them.

28. Let A be the number of markers with white side up, and B the number of pairs of markers whose squares share a side.
We claim that $A + B$ does not change its parity as the game progresses. Suppose that in some move we remove a marker that has exactly k neighbors, among them r with white side up ($0 \leq r \leq k \leq 4$). Of course, this marker has its black side up. When it is removed, the r white markers get black side up, while the $k - r$ black ones become white. Thus A changes by $k - 2r$. As for B, it decreases by k. It follows that A decreases by $2r$ and preserves its parity, as claimed.
Initially, $A = mn - 1$ and $B = m(n-1) + n(m-1)$; hence $A + B$ equals $3mn - m - n - 1$. If we succeed in removing all the markers, we end up with $A + B = 0$. Hence $3mn - m - n - 1 = (m-1)(n-1) + 2(mn-1)$ must be even, or equivalently at least one of m and n is odd.
On the other hand, the game can be finished successfully if m or n is odd. Assume that m is odd. As shown in the picture, we can arrive at the position (1) in m moves; with $\frac{m+1}{2}$ moves we reduce it to the position $(1\frac{1}{2})$, and with the next $\frac{m-1}{2}$ moves to the position (2). We continue until we empty all the columns.



4.40 Solutions to the Shortlisted Problems of IMO 1999

1. Obviously $(1, p)$ (where p is an arbitrary prime) and $(2, 2)$ are solutions and the only solutions to the problem for $x < 3$ or $p < 3$.
Let us now assume $x, p \geq 3$. Since p is odd, $(p-1)^x+1$ is odd, and hence x is odd. Let q be the largest prime divisor of x, which also must be odd. We have $q \mid x \mid x^{p-1} \mid (p-1)^x+1 \Rightarrow (p-1)^x \equiv -1 \pmod q$. Also from Fermat's little theorem $(p-1)^{q-1} \equiv 1 \pmod q$. Since $q-1$ and x are coprime, there exist integers α, β such that $x\alpha = (q-1)\beta + 1$. We also note that α must be odd. We now have $p - 1 \equiv (p-1)^{(q-1)\beta+1} \equiv (p-1)^{x\alpha} \equiv -1 \pmod q$ and hence $q \mid p \Rightarrow q = p$. Since x is odd, $p \mid x$, and $x \leq 2p$, it follows $x = p$ for all $x, p \geq 3$. Thus
$$p^{p-1} \mid (p-1)^x + 1 = p^2 \cdot \left(p^{p-2} - \binom{p}{1} p^{p-1} + \cdots - \binom{p}{p-2} + 1\right).$$
Since the expression in parenthesis is not divisible by p, it follows that $p^{p-1} | p^2$ and hence $p \leq 3$. One can easily verify that $(3, 3)$ is a valid solution.
We have shown that the only solutions are $(1, p)$, $(2, 2)$, and $(3, 3)$, where p is an arbitrary prime.

2. We first prove that every rational number in the interval $(1, 2)$ can be represented in the form $\frac{a^3+b^3}{a^3+d^3}$. Taking b, d such that $b \neq d$ and $a = b + d$, we get $a^2 - ab + b^2 = a^2 - ad + d^2$ and
$$\frac{a^3 + b^3}{a^3 + d^3} = \frac{(a + b)(a^2 - ab + b^2)}{(a + d)(a^2 - ad + d^2)} = \frac{a + b}{a + d}.$$
For a given rational number $1 < m/n < 2$ we can select $a = m + n$ and $b = 2m - n$ such that along with $d = a - b$ we have $\frac{a+b}{a+d} = \frac{m}{n}$. This completes the proof of the first statement.
For m/n outside of the interval we can easily select a rational number p/q such that $\sqrt[3]{\frac{n}{m}} < \frac{p}{q} < \sqrt[3]{\frac{2n}{m}}$. In other words $1 < \frac{p^3 m}{q^3 n} < 2$. We now proceed to obtain a, b and d for $\frac{p^3 m}{q^3 n}$ as before, and we finally have
$$\frac{p^3 m}{q^3 n} = \frac{a^3 + b^3}{a^3 + d^3} \Rightarrow \frac{m}{n} = \frac{(aq)^3 + (bq)^3}{(ap)^3 + (dp)^3}.$$
Thus we have shown that all positive rational numbers can be expressed in the form $\frac{a^3+b^3}{c^3+d^3}$.

3. We first prove the following lemma.
Lemma. For $d, c \in \mathbb{N}$ and $d^2 \mid c^2 + 1$ there exists $b \in \mathbb{N}$ such that $d^2(d^2 + 1) \mid b^2 + 1$.
Proof. It is enough to set $b = c + d^2c - d^3 = c + d^2(c - d)$.

Using the lemma it suffices to find increasing sequences d_n and c_n such that $c_n - d_n$ is an increasing sequence and $d_n^2 \mid c_n^2 + 1$. We then obtain the desired sequences a_n and b_n from $a_n = d_n^2$ and $b_n = c_n + d_n^2(c_n - d_n)$. It is easy to check that $d_n = 2^{2n} + 1$ and $c_n = 2^{nd_n}$ satisfy the required conditions. Hence we have demonstrated the existence of increasing sequences a_n and b_n such that $a_n(a_n + 1) \mid b_n^2 + 1$.

Remark. There are many solutions to this problem. For example, it is sufficient to prove that the Pell-type equation $5a_n(a_n + 1) = b_n^2 + 1$ has an infinity of solutions in positive integers. Alternatively, one can show that $a_n(a_n + 1)$ can be represented as a sum of two coprime squares for infinitely many a_n, which implies the existence of b_n.

4. (a) The fundamental period of p is the smallest integer $d(p)$ such that $p \mid 10^{d(p)} - 1$.
Let s be an arbitrary prime and set $N_s = 10^{2s} + 10^s + 1$. In that case $N_s \equiv 3 \pmod 9$. Let $p_s \neq 37$ be a prime dividing $N_s/3$. Clearly $p_s \neq 3$. We claim that such a prime exists and that $3 \mid d(p_s)$. The prime p_s exists, since otherwise N_s could be written in the form $N_s = 3 \cdot 37^k \equiv 3 \pmod 4$, while on the other hand for $s > 1$ we have $N_s \equiv 1 \pmod 4$. Now we prove $3 \mid d(p_s)$. We have $p_s \mid N_s \mid 10^{3s} - 1$ and hence $d(p_s) \mid 3s$. We cannot have $d(p_s) \mid s$, for otherwise $p_s \mid 10^s - 1 \Rightarrow p_s \mid (10^{2s} + 10^s + 1, 10^s - 1) = 3$; and we cannot have $d(p_s) \mid 3$, for otherwise $p_s \mid 10^3 - 1 = 999 = 3^3 \cdot 37$, both of which contradict $p_s \neq 3, 37$. It follows that $d(p_s) = 3s$. Hence for every prime s there exists a prime p_s such that $d(p_s) = 3s$. It follows that the cardinality of S is infinite.
(b) Let $r = r(s)$ be the fundamental period of $p \in S$. Then $p \mid 10^{3r} - 1$, $p \nmid 10^r - 1 \Rightarrow p \mid 10^{2r} + 10^r + 1$. Let $x_j = \frac{10^{j-1}}{p}$ and $y_j = \{x_j\} = 0.a_j a_{j+1} a_{j+2} \dots$. Then $a_j < 10y_j$, and hence

$$f(k,p) = a_k + a_{k+r} + a_{k+2r} < 10(y_k + y_{k+r} + y_{k+2r}) .$$

We note that $x_k + x_{k+s(p)} + x_{k+2s(p)} = \frac{10^{k-1}N_p}{p}$ is an integer, from which it follows that $y_k + y_{k+s(p)} + y_{k+2s(p)} \in \mathbb{N}$. Hence $y_k + y_{k+s(p)} + y_{k+2s(p)} \leq 2$. It follows that $f(k,p) < 20$. We note that $f(2,7) = 4 + 8 + 7 = 19$. Hence 19 is the greatest possible value of $f(k,p)$.

5. Since one can arbitrarily add zeros at the end of m, which increases divisibility by 2 and 5 to an arbitrary exponent, it suffices to assume $2, 5 \nmid n$. If $(n, 10) = 1$, there exists an integer $w \geq 2$ such that $10^w \equiv 1 \pmod n$. We also note that $10^{iw} \equiv 1 \pmod n$ and $10^{jw+1} \equiv 10 \pmod n$ for all integers i and j. Let us assume that m is of the form $m = \sum_{i=1}^{u} 10^{iw} + \sum_{j=1}^{v} 10^{jw+1}$ for integers $u, v \geq 0$ (where if u or v is 0, the corresponding sum is 0). Obviously, the sum of the digits of m is equal to $u + v$, and also $m \equiv u + 10v \pmod n$. Hence our problem reduces to finding integers $u, v \geq 0$ such that $u + v = k$ and $n \mid u + 10v = k + 9v$. Since $(n, 9) = 1$, it follows that there exists some v_0 such that $0 \leq v_0 < n \leq k$ and $9v_0 \equiv$

$-k \pmod{n} \Rightarrow n \mid k + 9v_0$. Taking this v_0 and setting $u_0 = k - v_0$ we obtain the desired parameters for defining m.

6. Let N be the smallest integer greater than M. We take the difference of the numbers in the progression to be of the form $10^m + 1$, $m \in \mathbb{N}$. Hence we can take $a_n = a_0 + n(10^m + 1) = \overline{b_s b_{s-1} \dots b_0}$ where a_0 is the initial term in the progression and $\overline{b_s b_{s-1} \dots b_0}$ is the decimal representation of a_n. Since $2m$ is the smallest integer x such that $10^x \equiv 1 \pmod{10^m + 1}$, it follows that $10^k \equiv 10^l \pmod{10^m + 1} \Leftrightarrow k \equiv l \pmod{2m}$. Hence

$$a_0 \equiv a_n = \overline{b_s b_{s-1} \dots b_0} \equiv \sum_{i=0}^{2m-1} c_i 10^i \pmod{10^m + 1},$$

where $c_i = b_i + b_{2m+i} + b_{4m+i} + \cdots \geq 0$ for $i = 0, 1, \dots, 2m - 1$ (these c_i also depend on n). We note that $\sum_{i=0}^{2m-1} c_i 10^i$ is invariant modulo $10^m + 1$ for all n and that $\sum_{i=0}^{2m-1} c_i = \sum_{j=0}^{s} b_j$ for a given n. Hence we must choose a_0 and m such that a_0 is not congruent to any number of the form $\sum_{i=0}^{2m-1} c_i 10^i$, where $c_0 + c_1 + \cdots + c_{2m-1} \leq N$ $(c_0, c_1, \dots, c_{2m-1} \geq 0)$. The number of ways to select the nonnegative integers $c_0, c_1, \dots, c_{2m-1}$ such that $c_0 + c_1 + \cdots + c_{2m-1} \leq N$ is equal to the number of strictly increasing sequences $0 \leq c_0 < c_0 + c_1 + 1 < c_0 + c_1 + c_2 + 2 + \cdots < c_0 + c_1 + \cdots + c_{2m-1} + 2m - 1 \leq N + 2m - 1$, which is equal to the number of $2m$-element subsets of $\{0, 1, 2, \dots, N + 2m - 1\}$, which is $\binom{N+2m}{N}$. For sufficiently large m we have $\binom{N+2m}{N} < 10^m$, and hence in this case one can select a_0 such that a_0 is not congruent to $\sum_{i=0}^{2m-1} c_i 10^i$ modulo $10^m + 1$ for any set of integers $c_0, c_1, \dots, c_{2m-1}$ such that $c_0 + c_1 + \cdots + c_{2m-1} \leq N$. Thus we have found the desired arithmetic progression.

7. We use the following simple lemma.

Lemma. Suppose that M is the interior point of a convex quadrilateral $ABCD$. Then it follows that $MA + MB < AD + DC + CB$.

Proof. We repeatedly make use of the triangle inequality. The line AM, in addition to A, intersects the quadrilateral in a second point N. In that case $AM + MB < AN + NB < AD + DC + CB$.

We now apply this lemma in the following way. Let D, E, and F be median points of BC, AC, and AB. Any point M in the interior of $\triangle ABC$ is contained in at least two of the three convex quadrilaterals $ABDE$, $BCEF$, and $CAFD$. Let us assume without loss of generality that M is in the interior of $BCEF$ and $CAFD$. In that case we apply the lemma to obtain $AM + CM < AF + FD + DC$ and $BM + CM < CE + EF + FB$ to obtain

$$\begin{aligned} CM + AM + BM + CM &< AF + FD + DC + CE + EF + FB \\ &= AB + AC + BC \end{aligned}$$

from which the required conclusion immediately follows.

8. Let A, B, C, and D be inverses of four of the five points, with the fifth point being the pole of the inversion. A separator through the pole transforms into a line containing two of the remaining four points such that the remaining two points are on opposite sides of the line. A separator not containing the pole transforms into a circle through three of the points with the fourth point in its interior. Let K be the convex hull of A, B, C, and D. We observe two cases:
   (i) K is a quadrilateral, for example $ABCD$. In that case the four separators are the two diagonals and two circles ABC and ADC if $\angle A + \angle C < 180°$, or BAD and BCD otherwise. The remaining six viable circles and lines are clearly not separators.
   (ii) K is a triangle, for example ABC with D in its interior. In that case the separators are lines DA, DB, DC and the circle ABC. No other lines and circles qualify.

   We have thus shown that any set of five points satisfying the stated conditions will have exactly four separators.

9. Let r_{PQ} denote a reflection about the planar bisector of PQ with $P, Q \in S$. Let G be the centroid of S. From $r_{PQ}(S) = S$ it follows that $r_{PQ}(G) = G$. Hence G belongs to the perpendicular bisector of PQ and thus $GP = GQ$. Consequently the whole of S lies on a sphere Σ centered at G. We note the following two cases:
   (a) S is a subset of a plane π. In this case S is included in a circle k, G being its center. Hence its n points form a convex polygon $A_1A_2\dots A_n$. When applying $r_{A_iA_{i+2}}$ for some $0 < i < n-1$ the point A_{i+1} transforms into some point of S lying on the same side of A_iA_{i+1}, which has to be A_{i+1} itself. It thus follows that $A_iA_{i+1} = A_{i+1}A_{i+2}$ for all $0 < i < n-1$ and hence $A_1A_2\dots A_n$ is a regular n-gon.
   (b) The points in S are not coplanar. It follows that S is a polyhedron P inscribed in a sphere Σ centered at G. By applying the previous case to the faces of the polyhedron, it follows that all faces are regular n-gons.

   Let us take an arbitrary vertex V and let VV_1, VV_2 and VV_3 be three consecutive edges stemming from V (V, V_1, V_2, and V_3 defining two adjacent faces of P). We now look at $r_{V_1V_3}$. Since this transformation leaves the half-planes $[V_1V_3, V_2$ and $[V_1V_3, V$ invariant and since V_2 and V are the only points of P on the respective half-planes, it follows that $r_{V_1V_3}$ leaves V and V_2 invariant. This transform also swaps V_1 and V_3. Hence, the face determined by VV_1V_2 is transformed by $r_{V_1V_3}$ into the face VV_3V_2, and thus the two faces sharing VV_2 are congruent. We conclude that all faces are congruent and similarly that vertices are endpoints of the same number of edges; hence P is a regular polyhedron.

   Finally, we have to rule out S being vertices of a cube, a dodecahedron, or an icosahedron. In all of these cases if we select two diametrically opposite points P and Q, then $S\backslash\{P,Q\}$ is not symmetric with respect to the bisector of PQ, which prevents r_{PQ} from being an invariant transformation of S.

It thus follows that the only viable finite completely symmetric sets are vertices of regular n-gons, the tetrahedron, and the octahedron. It is not explicitly asked for, but it is easy to verify that all of these are indeed completely symmetric.

Remark. On the IMO, a simpler version of this problem was adopted, adding the condition that S belongs to a plane and thus eliminating the need for the second case altogether.

10. We use the following lemma.

Lemma. Let ABC be a triangle and $X \in AB$ such that $\overrightarrow{AX} : \overrightarrow{XB} = m : n$. Then $(m+n)\cot\angle CXB = n\cot A - m\cot B$ and $m\cot\angle ACX = (n+m)\cot C + n\cot A$.

Proof. Let CD be the altitude from C and h its length. Then using oriented segments we have $AX = AD + DX = h\cot A - h\cot\angle CXB$ and $BX = BD + DX = h\cot B + h\cot\angle CXB$. The first formula in the lemma now follows from $n\cdot AX = m\cdot BX$. The second formula immediately follows from the first part applied to the triangle ACX and the point $X' \in AC$ such that $XX' \parallel BC$.

Let us set $\cot A = x$, $\cot B = y$, and $\cot C = z$. Applying the second formula in the lemma to $\triangle ABC$ and the point X, we obtain $4\cot\angle ACX = 9z+5x$. Applying the first formula in the lemma to $\triangle CXZ$ and the point Y and using $\angle XYZ = 45^\circ$ and $\cot\angle CXZ = -y$, we obtain $3\cot\angle XYZ = \cot\angle ACX - 2\cot\angle CXZ = \frac{9z+5x}{4} + 2y \Rightarrow 5x + 8y + 9z = 12$.
We now use the well-known relation for cotangents of a triangle $xy+yz+xz = 1$ to get $9 = 9(x+y)z + 9xy = (x+y)(12-5x-8z) + 9xy = 9 \Rightarrow (4y+x-3)^2 + 9(x-1)^2 = 0 \Rightarrow x = 1,\ y = \frac{1}{2},\ z = \frac{1}{3}$. It follows that x, y, and z have fixed values, and hence all triangles T in Σ are similar, with their smallest angle A having cotangent 1 and thus being equal to $\angle A = 45^\circ$.

11. Let $\Omega(I,r)$ be the incircle of $\triangle ABC$. Let D, E, and F denote the points where Ω touches BC, AC, and AB, respectively. Let P, Q, and R denote the midpoints of EF, DF, and DE respectively. We prove that Ω_a passes through Q and R. Since $\triangle IQD \sim \triangle IDB$ and $\triangle IRD \sim \triangle IDC$, we obtain $IQ\cdot IB = IR\cdot IC = r^2$. We conclude that B, C, Q, and R lie on a single circle Γ_a. Moreover, since the power of I with respect to Γ_a is r^2, it follows for a tangent IX from I to Γ_a that X lies on Ω and hence Ω is perpendicular to Γ_a. From the uniqueness of Ω_a it follows that $\Omega_a = \Gamma_a$. Thus Ω_a contains Q and R. Similarly Ω_b contains P and R and Ω_c contains P and Q. Hence, $A' = P$, $B' = Q$ and $C' = R$. Therefore the radius of the circumcircle of $\triangle A'B'C'$ is half the radius of Ω.

12. We first introduce the following lemmas.

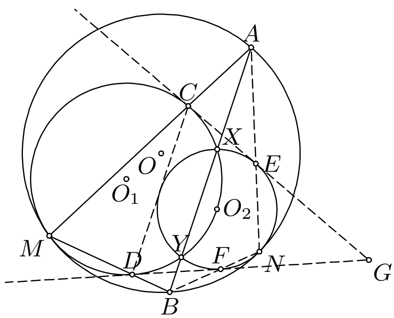
Lemma 1. Let ABC be a triangle, I its inenter and I_a the center of the excircle touching BC. Let A' be the center of the arc $\widehat{BC}$ of the circumcircle not containing A. Then $A'B = A'C = A'I = A'I_a$.

Proof. The result follows from a straightforward calculation of the relevant angles.

Lemma 2. Let two circles k_1 and k_2 meet each other at points X and Y and touch a circle k internally in points M and N, respectively. Let A be one of the intersections of the line XY with k. Let AM and AN intersect k_1 and k_2 respectively at C and E. Then CE is a common tangent of k_1 and k_2.

Proof. Since $AC \cdot AM = AX \cdot AY = AE \cdot AN$, the points M, N, E, C lie on a circle. Let MN meet k_1 again at Z. If M' is any point on the common tangent at M, then $\angle MCZ = \angle M'MZ = \angle M'MN = \angle MAN$ (as oriented angles), implying that $CZ \parallel AN$. It follows that $\angle ACE = \angle ANM = \angle CZM$. Hence CE is tangent to k_1 and analogously to k_2.

In the main problem, let us define E and F respectively as intersections of NA and NB with Ω_2. Then applying Lemma 2 we get that CE and DF are the common tangents of Ω_1 and Ω_2.



If the circles have the same radii, the result trivially holds. Otherwise, let G be the intersection of CE and DF. Let O_1 and O_2 be the centers of Ω_1 and Ω_2. Since $O_1D = O_1C$ and $\angle O_1DG = \angle O_1CG = 90°$, it follows that O_1 is the midpoint of the shorter arc of the circumcircle of $\triangle CDG$. The center O_2 is located on the bisector of $\angle CGD$, since Ω_2 touches both GC and GD.

However, it also sits on Ω_1, and using Lemma 1 we obtain that O_2 is either at the incenter or at the excenter of $\triangle CDG$ opposite G. Hence, Ω_2 is either the incircle or the excircle of CDG and thus in both cases touches CD.

Second solution. Let O be the center of Γ, and r, r_1, r_2 the radii of $\Gamma, \Gamma_1, \Gamma_2$. It suffices to show that the distance $d(O_2, CD)$ is equal to r_2.

The homothety with center M and ratio r/r_1 takes Γ_1, C, D into Γ, A, B, respectively; hence $CD \parallel AB$ and $d(C, AB) = \frac{r-r_1}{r} d(M, AB)$. Let O_1O_2 meet XY at R. Then $d(O_2, CD) = O_2R + \frac{r-r_1}{r} d(M, AB)$, i.e.,

$$d(O_2, CD) = O_2R + \frac{r-r_1}{r}[O_1O_2 - O_2R + r_1 \cos\angle OO_1O_2], \qquad (1)$$

since O, O_1, and M are collinear. We have $O_1X = O_1O_2 = r_1$, $OO_1 = r - r_1$, $OO_2 = r - r_2$, and $O_2X = r_2$. Using the cosine law in the triangles OO_1O_2 and XO_1O_2, we obtain that $\cos\angle OO_1O_2 = \frac{2r_1^2 - 2rr_1 + 2rr_2 - r_2^2}{2r_1(r-r_1)}$ and $O_2R = \frac{r_2^2}{2r_1}$. Substituting these values in (1) we get $d(O_2, CD) = r_2$.

13. Let us construct a convex quadrilateral $PQRS$ and an interior point T such that $\triangle PTQ \cong \triangle AMB$, $\triangle QTR \sim \triangle AMD$, and $\triangle PTS \sim \triangle CMD$. We then have $TS = \frac{MD \cdot PT}{MC} = MD$ and $\frac{TR}{TS} = \frac{TR \cdot TQ \cdot TP}{TQ \cdot TP \cdot TS} = \frac{MD \cdot MB \cdot MC}{MA \cdot MA \cdot MD} = \frac{MB}{MC}$ (using $MA = MC$). We also have $\angle STR = \angle BMC$ and therefore $\triangle RTS \sim \triangle BMC$. Now the relations between angles become

$$\angle TPS + \angle TQR = \angle PTQ \quad \text{and} \quad \angle TPQ + \angle TSR = \angle PTS,$$

implying that $PQ \parallel RS$ and $QR \parallel PS$. Hence $PQRS$ is a parallelogram and hence $AB = PQ = RS$ and $QR = PS$. It follows that $\frac{BC}{MC} = \frac{RS}{TS} = \frac{AB}{MD} \Rightarrow AB \cdot CM = BC \cdot MD$ and $\frac{AD \cdot BM}{AM} = \frac{AD \cdot QT}{AM} = QR = PS = \frac{CD \cdot TS}{MD} = CD \Rightarrow BM \cdot AD = MA \cdot CD$.

14. We first introduce the same lemma as in problem 12 and state it here without proof.

Lemma. Let ABC be a triangle and I the center of its incircle. Let M be the center of the arc $\widehat{BC}$ of the circumcircle not containing A. Then $MB = MC = MI$.

Let the circle XO_1O_2 intersect the circle Ω again at point T. Let M and N be respectively the midpoints of arcs $\widehat{BC}$ and $\widehat{AC}$, and let P be the intersection of Ω and the line through C parallel to MN. Then the lemma gives $MP = NC = NI = NO_1$ and $NP = MC = MI = MO_2$. Since O_1 and O_2 lie on XN and XM respectively, we have $\angle NTM = \angle NXM = \angle O_1XO_2 = \angle O_1TO_2$ and hence $\angle NTO_1 = \angle MTO_2$. Moreover, $\angle TNO_1 = \angle TNX = \angle TMO_2$, from which it follows that $\triangle O_1NT \sim \triangle O_2MT$. Thus $\frac{NT}{MP} = \frac{NT}{NO_1} = \frac{MT}{MO_2} = \frac{MT}{NP} \Rightarrow MP \cdot MT = NP \cdot NT \Rightarrow S_{MPT} = S_{NPT}$. It follows that TP bisects the segment MN, and hence it passes through I. We conclude that T belongs to the line PI and does not depend on X.

Remark. An alternative approach is to apply an inversion at point C. Points O_1 and O_2 become excenters of $\triangle AXC$ and $\triangle BXC$, and T becomes the projection of I_c onto AB.

15. For all $x_i = 0$ any C will do, so we may assume the contrary. Since the equation is symmetric and homogeneous, we may assume $\sum_i x_i = 1$. The equation now becomes $F(x_1, x_2, \ldots, x_n) = \sum_{i<j} x_i x_j (x_i^2 + x_j^2) = \sum_i x_i^2 \sum_{j \neq i} x_j = \sum_i x_i^3 (1 - x_i) = \sum_i f(x_i) \leq C$, where we define $f(x) = x^3 - x^4$. We note that for $x, y \geq 0$ and $x + y \leq 2/3$,

$$f(x+y) + f(0) - f(x) - f(y) = 3xy(x+y)\left(\frac{2}{3} - x - y\right) \geq 0 \,. \tag{1}$$

We note that if at least three elements of $\{x_1, x_2, \ldots, x_n\}$ are nonzero the condition of (1) always holds for the two smallest ones. Hence, applying (1) repeatedly, we obtain $F(x_1, x_2, \ldots, x_n) \leq F(a, 1-a, 0, \ldots, 0) = \frac{1}{2}(2a(1-a))(1 - 2a(1-a)) \leq \frac{1}{8} = F\left(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right)$. Thus we have $C = \frac{1}{8}$ (for all

n), and equality holds only when two x_i are equal and the remaining ones are 0.

Second solution. Let $M = x_1^2 + x_2^2 + \cdots + x_n^2$. Using $ab \le (a+2b)^2/8$ we have

$$\sum_{1\le i<j\le n} x_i x_j (x_i^2 + x_j^2) \le M \sum_{i<j} x_i x_j \\ \le \frac{1}{8}\left(M + 2\sum_{i<j} x_i x_j\right)^2 = \frac{1}{8}\left(\sum_{i=1}^{n} x_i\right)^4.$$

Equality holds if and only if $M = 2\sum_{i<j} x_i x_j$ and $x_i x_j (x_i^2 + x_j^2) = M x_i x_j$ for all $i < j$, which holds if and only if $n-2$ of the x_i are zero and the remaining two are equal.

Remark. Problems (SL90-26) and (SL91-27) are very similar.

16. Let $C(A)$ denote the characteristic of an arrangement A. We shall prove that $\max C(A) = \frac{n+1}{n}$.
Let us prove first $C(A) \le \frac{n+1}{n}$ for all A. Among elements $\{n^2-n, n^2-n+1, \dots, n^2\}$, by the pigeonhole principle, in at least one row and at least one column there exist two elements, and hence one pair in the same row or column that is not (n^2-n, n^2). Hence

$$C(A) \le \max\left\{\frac{n^2}{n^2-n+1}, \frac{n^2-1}{n^2-n}\right\} = \frac{n^2-1}{n^2-n} = \frac{n+1}{n}\ .$$

We now consider the following arrangement:

$$a_{ij} = \begin{cases} i + n(j-i-1) & \text{if } i < j, \\ i + n(n-i+j-1) & \text{if } i \ge j. \end{cases}$$

We claim that $C(a) = \frac{n+1}{n}$. Indeed, in this arrangement no two numbers in the same row or column differ by less than $n-1$, and in addition, n^2 and n^2-n+1 are in different rows and columns, and hence

$$C(A) \ge \frac{n^2-1}{n^2-n} = \frac{n+1}{n}.$$

17. A game is determined by the ordering $t_1, \dots, t_N$ of the $N = \binom{n}{2}$ transpositions (i,j) of the set $\{1,2,\dots,n\}$. The game is nice if the permutation $P = t_N t_{N-1} \dots t_1$ has no fixed point, and tiresome if P is the identity (denoted by I). Recall that every permutation can be written as a composition of disjoint cycles.
We claim that there exists a nice game if and only if $n \ne 3$.
For $n = 2$, $P_2 = t_1 = (1,2)$ is obviously nice. For $n = 3$ each game has the form $P = (b,c)(a,c)(a,b) = (a,c)$ for an appropriate notation of the players, which cannot be nice. Now for $n \ge 4$ we define

$P_n = (1,2)(1,3)(2,3)\cdots(1,n)(2,n)\cdots(n-1,n)$. We obtain inductively that $P_n = P_{n-1}(1,n,n-1,\dots,2) = (1,n)(2,n-1)\cdots(i,n+1-i)\cdots$ is nice for all even n.

Also, if $n = 2k+1$ is odd, then $Q_n = P_{n-1}(1,n)(2,n)\cdots(k,n)(n-1,n)(n-2,n)\cdots(k+1,n)$ maps i to $n+1-i$ for $i \le k$, to $n-1-i$ for $k+1 \le i \le 2k-1$, and to $3k+1-i$ if $i \in \{2k, 2k+1\}$. Hence Q_n is nice. This justifies our claim.

Now we prove that a tiresome game exists if and only if $n \equiv 0, 1 \pmod 4$. Evidently every transposition changes the sign of the permutation. Thus the sign of P is $(-1)^{\binom{n}{2}}$ and for P to be the identity we must have $2 \mid \binom{n}{2} \Rightarrow n \equiv 0, 1 (\mathrm{mod}\ 4)$.

Let us now construct tiresome games for the allowed n. For $n = 4k$ we divide the girls into groups of 4. In each group we perform the following game: $(3,4)(1,3)(2,4)(2,3)(1,4)(1,2) = I$. On the other hand, among two different groups (call them $\{1,2,3,4\}$ and $\{5,6,7,8\}$) we perform

$$(4,7)(3,7)(4,6)(1,6)(2,8)(3,8)(2,7)(2,6)$$
$$(4,5)(4,8)(1,7)(1,8)(3,5)(3,6)(2,5)(1,5) = I.$$

For $n = 4k+1$ we divide into groups of four as before, with one girl remaining. Every time a group (denoted $\{1,2,3,4\}$) is to play a game the remaining girl (denoted 5) joins in, and they play

$$(3,5)(3,4)(4,5)(1,3)(2,4)(2,3)(1,4)(1,5)(1,2)(2,5) = I.$$

This completes the proof.

18. Define $f(x,y) = x^2 - xy + y^2$. Let us assume that three such sets A, B, and C do exist and that w.l.o.g. 1, b, and c $(c > b)$ are respectively their smallest elements.

Lemma 1. Numbers x, y, and $x+y$ cannot belong to three different sets.

Proof. The number $f(x, x+y) = f(y, x+y)$ must belong to both the set containing y and the set containing x, a contradiction.

Lemma 2. The subset C contains a multiple of b. Moreover, if kb is the smallest such multiple, then $(k-1)b \in B$ and $(k-1)b+1, kb+1 \in A$.

Proof. Let r be the residue of c modulo b. If $r = 0$, the first statement automatically holds. Let $0 < r < b$. In that case $r \in A$, and $c - r$ is then not in B according to Lemma 1. Hence $c - r \in A$ and since $b \mid c - r$, it follows that $b \mid f(c-r, b) \in C$, thus proving the first statement. It follows immediately from Lemma 1 that $(k-1)b \in B$. Now by Lemma 1, $(k-1)b + 1 = kb - (b-1)$ must be in A; similarly, $kb + 1 = [(k-1)b + 1] + b \in A$ as well.

Let us show by induction that $(nk-1)b + 1, nkb + 1 \in A$ for all integers n. The inductive basis has been shown in Lemma 2. Assuming that $[(n-1)k - 1]b + 1 \in A$ and $(n-1)kb + 1 \in A$, we get that $(nk-1)b + 1 = ((n-1)kb + 1) + (k-1)b = [((n-1)k-1)b + 1] + kb$ belongs to A and

$nkb+1 = ((nk-1)b+1)+b = ((n-1)kb+1)+kb \Rightarrow nkb+1 \in A$. This finishes the inductive step. In particular, $f(kb, kb+1) = (kb+1)kb+1 \in A$. However, since $kb \in C$, $kb+1 \in A$, it follows that $f(kb, kb+1) \in B$, which is a contradiction.

19. Let $A = \{f(x) \mid x \in \mathbb{R}\}$ and $f(0) = c$. Plugging in $x = y = 0$ we get $f(-c) = f(c) + c - 1$, hence $c \neq 0$. If $x \in A$, then taking $x = f(y)$ in the original functional equation we get $f(x) = \frac{c+1}{2} - \frac{x^2}{2}$ for all $x \in A$.
We now show that $A - A = \{x_1 - x_2 \mid x_1, x_2 \in A\} = \mathbb{R}$. Indeed, plugging in $y = 0$ into the original equation gives us $f(x-c) - f(x) = cx + f(c) - 1$, an expression that evidently spans all the real numbers. Thus, each x can be represented as $x = x_1 - x_2$, where $x_1, x_2 \in A$. Plugging $x = x_1$ and $f(y) = x_2$ into the original equation gives us

$$f(x) = f(x_1 - x_2) = f(x_1) + x_1x_2 + f(x_2) - 1 = c - \frac{x_1^2 + x_2^2}{2} + x_1x_2 = c - \frac{x^2}{2}.$$

Hence we must have $c = \frac{c+1}{2}$, which gives us $c = 1$. Thus $f(x) = 1 - \frac{x^2}{2}$ for all $x \in \mathbb{R}$. It is easily checked that this function satisfies the original functional equation.

20. We first introduce some useful notation. An arrangement around the circle will be denoted by $x = \{x_1, x_2, \dots, x_n\}$, where the elements are arranged clockwise and x_1 is fixed to be the smallest number. We will call an arrangement *balanced* if $x_1 \le x_n \le x_2 \le x_{n-1} \le x_3 \le x_{n-2} \le \cdots$ (the string of inequalities continues until all the elements are accounted for). We will denote the permutation of $x = \{x_1, x_2, \dots, x_n\}$ in ascending order by $x' = \{x_1', x_2', \dots, x_n'\}$. We will let $f_i(x) = \{f_i(x)_1, f_i(x)_2, \dots, f_i(x)_{n-1}\}$ denote the arrangement after one iteration of the algorithm where x_i was the deleted element.
Lemma 1. If an arrangement x is balanced, then $f_1(x)$ is also balanced.
Proof. In one iteration we have $\{x_1, \dots, x_n\} \to \{x_n + x_2, x_2 + x_3, \dots, x_{n-1} + x_n\}$. Since $x_n \le x_2 \le x_{n-1} \le x_3 \le x_{n-2} \le \cdots$, it follows that $x_n + x_2 \le x_n + x_{n-1} \le x_2 + x_3 \le x_{n-1} + x_{n-2} \le \cdots$, which means that $f_1(x)$ is balanced.
We will first show by induction that $S_{\max}$ can be reached by using the balanced initial arrangement $\{a_1, a_3, a_5, \dots, a_6, a_4, a_2\}$ and repeatedly deleting the smallest member. For $n = 3$ we have $S_3 = a_2 + a_3$, in accordance with the formula. Assuming that the formula holds for a given n, we note that for an arrangement $x = \{a_1, a_3, a_5, \dots, a_6, a_4, a_2\}$ the arrangement $f_1(x)$ is also balanced. We now apply the induction hypothesis and use that $\binom{n-2}{i} + \binom{n-2}{i-1} = \binom{n-1}{i}$:

$$\begin{aligned} S(x) &= S(f_1(x)) \\ &= \sum_{k=2}^{n-1} \binom{n-2}{[k/2]-1}(a_k + a_{k+2}) + \binom{n-2}{[n/2]-1}(a_n + a_{n+1}) = S_{\max}. \end{aligned}$$

We now prove that every other arrangement yields a smaller value. We shall write $\{x_1, \dots, x_n\} \le \{y_1, \dots, y_n\}$ whenever $x'_n + x'_{n-1} + \cdots + x'_i \le y'_n + y'_{n-1} + \cdots + y'_i$ holds for all $1 \le i \le n$.

Lemma 2. Let x be an arbitrary arrangement and y a balanced arrangement, both of n elements, such that $x \le y$. Then it follows that $f_i(x) \le f_1(y)$, for all i.

Proof. For any $1 \le j \le n-1$ there exists k_j such that $f_i(x)_j = x_{k_j} + x_{k_j+1}$ (assuming $k_j + 1 = 1$ if $k_j = n-1$). Then we have

$$\begin{aligned} f_i(x)_{n-1} + \cdots + f_i(x)_{n-j} &= (x_{k_1} + x_{k_1+1}) + \cdots + (x_{k_j} + x_{k_j+1}) \\ &\le 2x'_n + \cdots + 2x'_{n-i+1} + x'_{n-i} + x'_{n-i-1} \\ &= f_1(y)_{n-1} + \cdots + f_1(y)_{n-j} \end{aligned}$$

for all j, and hence $f_i(x) \le f_1(y)$.

An immediate consequence of Lemma 2 is $f^{n-2}(x) \le f_1^{n-2}(y)$, implying $S = f^{n-2}(x)_1 + f^{n-2}(x)_2 \le f_1^{n-2}(y)_1 + f_1^{n-2}(y)_2 = S_{\max}(y)$. Thus the proof is finished.

21. Let us call $f(n,s)$ the number of paths from $(0,0)$ to (n,n) that contain exactly s steps. Evidently, for all n we have $f(n,1) = f(2,2) = 1$, in accordance with the formula. Let us thus assume inductively for a given $n > 2$ that for all s we have $f(n,s) = \frac{1}{s}\binom{n-1}{s-1}\binom{n}{s-1}$. We shall prove that the given formula holds also for all $f(n+1,s)$, where $s \ge 2$.

We say that an $(n+1,s)$- or $(n+1,s+1)$-path is *related* to a given (n,s)-path if it is obtained from the given path by inserting a step EN between two moves or at the beginning or the end of the path. We note that by inserting the step between two moves that form a step one obtains an $(n+1,s)$-path; in all other cases one obtains an $(n+1,s+1)$-path. For each (n,s)-path there are exactly $2n+1-s$ related $(n+1,s+1)$-paths, and for each $(n,s+1)$-path there are $s+1$ related $(n+1,s+1)$-paths. Also, each $(n+1,s+1)$-path is related to exactly $s+1$ different (n,s)- or $(n,s+1)$-paths. Thus:

$$\begin{aligned} (s+1)f(n+1,s+1) &= (2n+1-s)f(n,s) + (s+1)f(n,s+1) \\ &= \frac{2n+1-s}{s}\binom{n-1}{s-1}\binom{n}{s-1} + \binom{n-1}{s}\binom{n}{s} \\ &= \binom{n}{s}\binom{n+1}{s}, \end{aligned}$$

i.e., $f(n+1,s+1) = \frac{1}{s+1}\binom{n}{s}\binom{n+1}{s}$. This completes the proof.

22. (a) Color the first, third, and fifth row red, and the remaining squares white. There in total n pieces and $3n$ red squares. Since each piece can cover at most three red squares, it follows that each piece colors exactly three red squares. Then it follows that the two white squares it covers must be on the same row; otherwise, the piece has to cover

at least three. Hence, each white row can be partitioned into pairs of squares belonging to the same piece. Thus it follows that the number of white squares in a row, which is n, must be even.

(b) Let a_k denote the number of different tilings of a $5 \times 2k$ rectangle. Let b_k be the number of tilings that cannot be partitioned into two smaller tilings along a vertical line (without cutting any pieces). It is easy to see that $a_1 = b_1 = 2$, $b_2 = 2$, $a_2 = 6 = 2 \cdot 3$, $b_3 = 4$, and subsequently, by induction, $b_{3k} \geq 4$, $b_{3k+1} \geq 2$, and $b_{3k+2} \geq 2$. We also have $a_k = b_k + \sum_{i=1}^{k-1} b_i a_{k-i}$. For $k \geq 3$ we now have inductively

$$a_k > 2 + \sum_{i=1}^{k-1} 2a_{k-i} \geq 2 \cdot 3^{k-1} + 2a_{k-1} \geq 2 \cdot 3^k \ .$$

23. Let $r(m)$ denote the rest period before the mth catch, $t(m)$ the number of minutes before the mth catch, and $f(n)$ as the number of flies caught in n minutes. We have $r(1) = 1$, $r(2m) = r(m)$, and $r(2m+1) = f(m) + 1$. We then have by induction that $r(m)$ is the number of ones in the binary representation of m. We also have $t(m) = \sum_{i=1}^{m} r(i)$ and $f(t(m)) = m$. From the recursive relations for r we easily derive $t(2m+1) = 2t(m)+m+1$ and consequently $t(2m) = 2t(m) + m - r(m)$. We then have, by induction on p, $t(2^p m) = 2^p t(m) + p \cdot m \cdot 2^{p-1} - (2^p - 1)r(m)$.

(a) We must find the smallest number m such that $r(m+1) = 9$. The smallest number with nine binary digits is $\overline{111111111}_2 = 511$; hence the required m is 510.

(b) We must calculate $t(98)$. Using the recursive formulas we have $t(98) = 2t(49) + 49 - r(49)$, $t(49) = 2t(24) + 25$, and $t(24) = 8t(3) + 36 - 7r(3)$. Since we have $t(3) = 4$, $r(3) = 2$ and $r(49) = r(\overline{110001}_2) = 3$, it follows $t(24) = 54 \Rightarrow t(49) = 133 \Rightarrow t(98) = 312$.

(c) We must find m_c such that $t(m_c) \leq 1999 < t(m_c + 1)$. One can estimate where this occurs using the formula $t(2^p(2^q - 1)) = (p + q)2^{p+q-1} - p\, 2^{p-1} - q\, 2^p + q$, provable from the recursive relations. It suffices to note that $t(462) = 1993$ and $t(463) = 2000$; hence $m_c = 462$.

24. Let $S = \{0, 1, \dots, N^2 - 1\}$ be the group of residues (with respect to addition modulo N^2) and A an n-element subset. We will use $|X|$ to denote the number of elements of a subset X of S, and $\overline{X}$ to refer to the complement of X in S. For $i \in S$ we also define $A_i = \{a + i \mid a \in A\}$. Our task is to select $0 \leq i_1 < \cdots < i_N \leq N^2 - 1$ such that $\left|\bigcup_{j=1}^{N} A_{i_j}\right| \geq \frac{1}{2}|S|$. Each $x \in S$ appears in exactly N sets A_i. We have

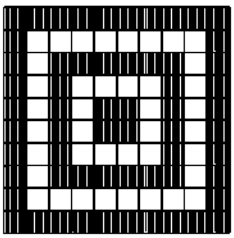
$$\sum_{i_1<\cdots<i_N}\left|\bigcap_{j=1}^{N}\overline{A}_{i_j}\right| = \sum_{i_1<\cdots<i_N} |\{x\in S \mid x\notin A_{i_1},\ldots,A_{i_N}\}|$$
$$= \sum_{x\in S} |\{i_1<\cdots<i_N | x\notin A_{i_1},\ldots,A_{i_N}\}|$$
$$= \sum_{x\in S}\binom{N^2-N}{N} = \binom{N^2-N}{N}|S|\,.$$

Hence

$$\sum_{i_1<\cdots<i_N}\left|\bigcup_{j=1}^{N} A_{i_j}\right| = \sum_{i_1<\cdots<i_N}\left(|S| - \left|\bigcap_{j=1}^{N}\overline{A}_{i_j}\right|\right)$$
$$= \left(\binom{N^2}{N} - \binom{N^2-N}{N}\right)|S|.$$

Thus, by the pigeonhole principle, one can choose $i_1<\cdots<i_N$ such that $\left|\bigcup_{j=1}^{N} A_{i_j}\right| \geq \left(1-\binom{N^2-N}{N}/\binom{N^2}{N}\right)|S|$. Since $\binom{N^2}{N}/\binom{N^2-N}{N} \geq \left(\frac{N^2}{N^2-N}\right)^N$ $=\left(1+\frac{1}{N-1}\right)^N > e > 2$, it follows that $\left|\bigcup_{j=1}^{N} A_{i_j}\right| \geq \frac{1}{2}|S|$; hence the chosen $i_1<\cdots<i_N$ are indeed the elements of B that satisfy the conditions of the problem.

25. Let $n=2k$. Color the cells neighboring the edge of the board black. Then color the cells neighboring the black cells white. Then in alternation color the still uncolored cells neighboring the white or black cells on the boundary the opposite color and repeat until all cells are colored.



We call the cells colored the same color in each such iteration a "frame." In the color scheme described, each cell (white or black) neighbors exactly two black cells. The number of black cells is $2k(k+1)$, and hence we need to mark at least $k(k+1)$ cells.
On the other hand, going along each black-colored frame, we can alternately mark two consecutive cells and then not mark two consecutive cells. Every cell on the black frame will have one marked neighbor. One can arrange these sequences on two consecutive black frames such that each cell in the white frame in between has exactly one neighbor. Hence, starting from a sequence on the largest frame we obtain a marking that contains exactly half of all the black cells, i.e., $k(k+1)$ and neighbors every cell. It follows that the desired minimal number of markings is $k(k+1)$.

Remark. For $n=4k-1$ and $n=4k+1$ one can perform similar markings to obtain minimal numbers $4k^2-1$ and $(2k+1)^2$, respectively.

26. We denote colors by capital initial letters. Let us suppose that there exists a coloring $f : \mathbb{Z} \to \{R, G, B, Y\}$ such that for any $a \in \mathbb{Z}$ we have $f\{a, a+x, a+y, a+x+y\} = \{R, G, B, Y\}$. We now define a coloring of an integer lattice $g : \mathbb{Z} \times \mathbb{Z} \to \{R, G, B, Y\}$ by the rule $g(i, j) = f(xi + yj)$. It follows that every unit square in g must have its vertices colored by four different colors.
If there is a row or column with period 2, then applying the condition to adjacent unit squares, we get (by induction) that all rows or columns, respectively, have period 2.
On the other hand, taking a row to be not of period 2, i.e., containing a sequence of three distinct colors, for example GRY, we get that the next row must contain in these columns YBG, and the following GRY, and so on. It would follow that a column in this case must have period 2. A similar conclusion holds if we start with an aperiodic column. Hence either all rows or all columns must have period 2.
Let us assume w.l.o.g. that all rows have a period of 2. Assuming w.l.o.g. $\{g(0,0), g(1,0)\} = \{G, B\}$, we get that the even rows are painted with $\{G, B\}$ and odd with $\{Y, R\}$. Since x is odd, it follows that $g(y, 0)$ and $g(0, x)$ are of different color. However, since $g(y, 0) = f(xy) = g(0, x)$, this is a contradiction. Hence the statement of the problem holds.

27. Denote $A = \{0, 1, 2\}$ and $B = \{0, 1, 3\}$. Let $f_T(x) = \sum_{a \in T} x^a$. Then define $F_T(x) = f_T(x) f_T(x^2) \cdots f_T(x^{p-1})$. We can write $F_T(x) = \sum_{i=0}^{p(p-1)} a_i x^i$, where a_i is the number of ways to select an array $\{x_1, \dots, x_{p-1}\}$ where $x_i \in T$ for all i and $x_1 + 2x_2 + \cdots + (p-1)x_{p-1} = i$. Let $w = \cos(2\pi/p) + i \sin(2\pi/p)$, a pth root of unity. Noting that

$$1 + w^j + w^{2j} + \cdots + w^{(p-1)j} = \begin{cases} p, & p \mid j, \\ 0, & p \nmid j, \end{cases}$$

it follows that $F_T(1) + F_T(w) + \cdots + F_T(w^{p-1}) = pE(T)$.
Since $|A| = |B| = 3$, it follows that $F_A(1) = F_B(1) = 3^{p-1}$. We also have for $p \nmid i, j$ that $F_T(w^i) = F_T(w)$. Finally, we have

$$F_A(w) = \prod_{i=1}^{p-1} (1 + w^i + w^{2i}) = \prod_{i=1}^{p-1} \frac{1 - w^{3i}}{1 - w^i} = 1.$$

Hence, combining these results, we obtain

$$E(A) = \frac{3^{p-1} + p - 1}{p} \quad \text{and} \quad E(B) = \frac{3^{p-1} + (p-1) F_B(w)}{p}.$$

It remains to demonstrate that $F_B(w) \geq 1$ for all p and that equality holds only for $p = 5$. Since $E(B)$ is an integer, it follows that $F_B(w)$ is an integer and $F_B(w) \equiv 1 \pmod{p}$. Since $f_B(w^{p-i}) = \overline{f_B(w^i)}$, it follows that $F_B(w) = |f_B(w)|^2 \left|f_B(w^2)\right|^2 \cdots \left|f_B\left(w^{(p-1)/2}\right)\right|^2 > 0$. Hence $F_B(w) \geq 1$.

It remains to show that $F_B(w) = 1$ if and only if $p = 5$. We have the formula $(x-w)(x-w^2)\cdots(x-w^{p-1}) = x^{p-1}+x^{p-2}+\cdots+x+1 = \frac{x^p-1}{x-1}$. Let $f_B(x) = x^3 + x + 1 = (x-\lambda)(x-\mu)(x-\nu)$, where λ, μ, and ν are the three zeros of the polynomial $f_B(x)$. It follows that

$$F_B(w) = \left(\frac{\lambda^p - 1}{\lambda - 1}\right)\left(\frac{\mu^p - 1}{\mu - 1}\right)\left(\frac{\nu^p - 1}{\nu - 1}\right) = -\frac{1}{3}(\lambda^p - 1)(\mu^p - 1)(\nu^p - 1),$$

since $(\lambda - 1)(\mu - 1)(\nu - 1) = -f_B(1) = -3$. We also have $\lambda + \mu + \nu = 0$, $\lambda\mu\nu = -1$, $\lambda\mu + \lambda\nu + \mu\nu = 1$, and $\lambda^2 + \mu^2 + \nu^2 = (\lambda + \mu + \nu)^2 - 2(\lambda\mu + \lambda\nu + \mu\nu) = -2$. By induction (using that $(\lambda^r + \mu^r + \nu^r) + (\lambda^{r-2} + \mu^{r-2} + \nu^{r-2}) + (\lambda^{r-3} + \mu^{r-3} + \nu^{r-3}) = 0$), it follows that $\lambda^r + \mu^r + \nu^r$ is an integer for all $r \in \mathbb{N}$.

Let us assume $F_B(x) = 1$. It follows that $(\lambda^p - 1)(\mu^p - 1)(\nu^p - 1) = -3$. Hence $\lambda^p,\ \mu^p,\ \nu^p$ are roots of the polynomial $p(x) = x^3 - qx^2 + (1+q)x + 1$, where $q = \lambda^p + \mu^p + \nu^p$. Since $f_B(x)$ is an increasing function in real numbers, it follows that it has only one real root (w.l.o.g.) λ, the other two roots being complex conjugates. From $f_B(-1) < 0 < f_B(-1/2)$ it follows that $-1 < \lambda < -1/2$. It also follows that λ^p is the x coordinate of the intersection of functions $y = x^3 + x + 1$ and $y = q(x^2 - x)$. Since $\lambda < \lambda^p < 0$, it follows that $q > 0$; otherwise, $q(x^2 - x)$ intersects $x^3 + x + 1$ at a value smaller than λ. Additionally, as p increases, λ^p approaches 0, and hence q must increase.

For $p = 5$ we have $1 + w + w^3 = -w^2(1 + w^2)$ and hence $G(w) = \prod_{i=1}^{p-1}(1 + w^{2j}) = 1$. For a zero of $f_B(x)$ we have $x^5 = -x^3 - x^2 = -x^2 + x + 1$ and hence $q = \lambda^5 + \mu^5 + \nu^5 = -(\lambda^2 + \mu^2 + \nu^2) + (\lambda + \mu + \nu) + 3 = 5$.

For $p > 5$ we also have $q \geq 6$. Assuming again $F_B(x) = 1$ and defining $p(x)$ as before, we have $p(-1) < 0$, $p(0) > 0$, $p(2) < 0$, and $p(x) > 0$ for a sufficiently large $x > 2$. It follows that $p(x)$ must have three distinct real roots. However, since $\mu^p, \nu^p \in \mathbb{R} \Rightarrow \nu^p = \overline{\mu^p} = \mu^p$, it follows that $p(x)$ has at most two real roots, which is a contradiction. Hence, it follows that $F_B(x) > 1$ for $p > 5$ and thus $E(A) \leq E(B)$, where equality holds only for $p = 5$.

4.41 Solutions to the Shortlisted Problems of IMO 2000

1. In order for the trick to work, whenever $x + y = z + t$ and the cards x, y are placed in different boxes, either z, t are in these boxes as well or they are both in the remaining box.

   Case 1. The cards $i, i + 1, i + 2$ are in different boxes for some i. Since $i + (i + 3) = (i + 1) + (i + 2)$, the cards i and $i + 3$ must be in the same box; moreover, $i - 1$ must be in the same box as $i + 2$, etc. Hence the cards $1, 4, 7, \dots, 100$ are placed in one box, the cards $2, 5, \dots, 98$ are in the second, while $3, 6, \dots, 99$ are in the third box. The number of different arrangements of the cards is 6 in this case.

   Case 2. No three successive cards are all placed in different boxes. Suppose that 1 is in the blue box, and denote by w and r the smallest numbers on cards lying in the white and red boxes; assume w.l.o.g. that $w < r$. The card $w + 1$ is obviously not red, from which it follows that $r > w + 1$. Now suppose that $r < 100$. Since $w + r = (w - 1) + (r + 1)$, $r + 1$ must be in the blue box. But then $(r + 1) + w = r + (w + 1)$ implies that $w + 1$ must be red, which is a contradiction. Hence the red box contains only the card 100. Since $99 + w = 100 + (w - 1)$, we deduce that the card 99 is in the white box. Moreover, if any of the cards k, $2 \le k \le 99$, were in the blue box, then since $k + 99 = (k - 1) + 100$, the card $k - 1$ should be in the red box, which is impossible. Hence the blue box contains only the card 1, whereas the cards $2, 3, \dots, 99$ are all in the white box.

   In general, one box contains 1, another box only 100, while the remaining contains all the other cards. There are exactly 6 such arrangements, and the trick works in each of them.

   Therefore the answer is 12.

2. Since the volume of each brick is 12, the side of any such cube must be divisible by 6.

   Suppose that a cube of side $n = 6k$ can be built using $\frac{n^3}{12} = 18k^3$ bricks. Set a coordinate system in which the cube is given as $[0, n] \times [0, n] \times [0, n]$ and color in black each unit cube $[2p, 2p + 1] \times [2q, 2q + 1] \times [2r, 2r + 1]$. There are exactly $\frac{n^3}{9} = 27k^3$ black cubes. Each brick covers either one or three black cubes, which is in any case an odd number. It follows that the total number of black cubes must be even, which implies that k is even. Hence $12 \mid n$.

   On the other hand, two bricks can be fitted together to give a $2 \times 3 \times 4$ box. Using such boxes one can easily build a cube of side 12, and consequently any cube of side divisible by 12.

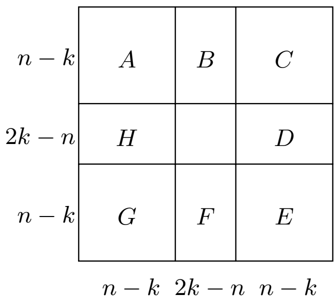
3. Clearly $m(S)$ is the number of pairs of point and triangle $(P_t, P_iP_jP_k)$ such that P_t lies inside the circle $P_iP_jP_k$. Consider any four-element set $S_{ijkl} = \{P_i, P_j, P_k, P_l\}$. If the convex hull of S_{ijkl} is the triangle $P_iP_jP_k$, then we have $a_i = a_j = a_k = 0$, $a_l = 1$. Suppose that the convex hull is

the quadrilateral $P_iP_jP_kP_l$. Since this quadrilateral is not cyclic, we may suppose that $\angle P_i + \angle P_k < 180° < \angle P_j + \angle P_l$. In this case $a_i = a_k = 0$ and $a_j = a_l = 1$. Therefore $m(S_{ijkl})$ is 2 if P_i, P_j, P_k, P_l are vertices of a convex quadrilateral, and 1 otherwise.

There are $\binom{n}{4}$ four-element subsets S_{ijkl}. If $a(S)$ is the number of such subsets whose points determine a convex quadrilateral, we have $m(S) = 2a(S) + \left(\binom{n}{4} - a(S)\right) = \binom{n}{4} + a(S) \le 2\binom{n}{4}$. Equality holds if and only if every four distinct points of S determine a convex quadrilateral, i.e. if and only if the points of S determine a convex polygon. Hence $f(n) = 2\binom{n}{4}$ has the desired property.

4. By a *good placement* of pawns we mean the placement in which there is no block of k adjacent unoccupied squares in a row or column.

   We can make a good placement as follows: Label the rows and columns with $0, 1, \dots, n-1$ and place a pawn on a square (i, j) if and only if k divides $i + j + 1$. This is obviously a good placement in which the pawns are placed on three lines with k, $2n-2k$, and $2n-3k$ squares, which adds up to $4n - 4k$ pawns in total.

   Now we shall prove that a good placement must contain at least $4n - 4k$ pawns. Suppose we have a good placement of m pawns. Partition the board into nine rectangular regions as shown in the picture. Let $a, b, \dots, h$ be the numbers of pawns in the rectangles $A, B, \dots, H$ respectively. Note that each row that passes through A, B, and C either contains a pawn inside B, or contains a pawn in both A and C. It follows that $a + c + 2b \ge 2(n-k)$. We similarly obtain that $c + e + 2d$, $e + g + 2f$, and $g + a + 2h$ are all at least $2(n-k)$. Adding and dividing by 2 yields $a + b + \cdots + h \ge 4(n-k)$, which proves the statement.

5. We say that a vertex of a nice region is convex if the angle of the region at that vertex equals $90°$; otherwise (if the angle is $270°$), we say that a vertex is concave.

   For a simple broken line C contained in the boundary of a nice region R we call the pair (R, C) a *boundary pair*. Such a pair is called *outer* if the region R is inside the broken line C, and *inner* otherwise. Let $\mathcal{B}_i$, $\mathcal{B}_o$ be the sets of inner and outer boundary pairs of nice regions respectively, and let $\mathcal{B} = \mathcal{B}_i \cup \mathcal{B}_o$. For a boundary pair $b = (R, C)$ denote by c_b and v_b respectively the number of convex and concave vertices of R that belong to C. We have the following facts:

   (1) Each vertex of a rectangle corresponds to one concave angle of a nice region and vice versa. This correspondence is bijective, so $\sum_{b \in \mathcal{B}} v_b = 4n$.

(2) For a boundary pair $b = (R, C)$ the sum of angles of R that are on C equals $(c_b + v_b - 2)180^\circ$ if b is outer, and $(c_b + v_b + 2)180^\circ$ if b is inner. On the other hand the sum of angles is obviously equal to $c_b \cdot 90^\circ + v_b \cdot 270^\circ$. It immediately follows that $c_b - v_b = \begin{cases} 4 \text{ if } b \in \mathcal{B}_o, \\ -4 \text{ if } b \in \mathcal{B}_i. \end{cases}$

(3) Since every vertex of a rectangle appears in exactly two boundary pairs and each boundary pair contains at least one vertex of a rectangle, the number K of boundary pairs is less than or equal to $8n$.

(4) The set $\mathcal{B}_i$ is nonempty, because every boundary of the infinite region is inner.

Consequently, the sum of the numbers of the vertices of all nice regions is equal to

$$\sum_{b \in \mathcal{B}} (c_b + v_b) = \sum_{b \in \mathcal{B}} (2v_b + (c_b - v_b)) \leq 2 \cdot 4n + 4(K-1) - 4 \leq 40n - 8.$$

6. Every integer z has a unique representation $z = px + qy$, where $x, y \in \mathbb{Z}$, $0 \leq x \leq q-1$. Consider the region T in the xy-plane defined by the last inequality and $px + qy \geq 0$. There is a bijective correspondence between lattice points of this region and nonnegative integers given by $(x, y) \mapsto z = px + qy$. Let us mark all lattice points of T whose corresponding integers belong to S and color in black the unit squares whose left-bottom vertices are at marked points. Due to the condition for S, this coloring has the property that all points lying on the right or above a colored point are colored as well. In particular, since the point $(0,0)$ is colored, all points above or on the line $y = 0$ are colored. What we need is the number of such colorings of T.

The border of the colored subregion C of T determines a path from $(0,0)$ to $(q, -p)$ consisting of consecutive unit moves either to the right or downwards. There are $\binom{p+q}{p}$ such paths in total. We must find the number of such paths not going below the line $l : px + qy = 0$.

Consider any path $\gamma = A_0 A_1 \dots A_{p+q}$ from $A_0 = (0,0)$ to $A_{p+q} = (q, -p)$. We shall see the path γ as a sequence $G_1 G_2 \dots G_{p+q}$ of moves to the right (R) or downwards (D) with exactly p D's and q R's.

Two paths are said to be *equivalent* if one is obtained from the other by a circular shift of the corresponding sequence $G_1 G_2 \dots G_{p+q}$. We note that all the $p+q$ circular shifts of a path are distinct. Indeed, $G_1 \dots G_{p+q} \equiv G_{i+1} \dots G_{i+p+q}$ would imply $G_1 = G_{i+1} = G_{2i+1} = \cdots$ (where $G_{j+p+q} = G_j$), so $G_1 = \cdots = G_{p+q}$, which is impossible. Hence each equivalence class contains exactly $p+q$ paths.

Let l_i, $0 \leq i < p+q$, be the line through A_i that is parallel to the line l. Since $\gcd(p, q) = 1$, all these lines are distinct.

Let l_m be the unique lowest line among the l_i's. Then the path $G_{m+1} G_{m+2} \dots G_{m+p+q}$ is above the line l. Every other cyclic shift gives rise to a path having at least one vertex below the line l. Thus each equivalence class contains exactly one path above the line l, so the number of such paths is equal to $\frac{1}{p+q}\binom{p+q}{p}$. Therefore the answer is $\frac{1}{p+q}\binom{p+q}{p}$.

7. Elementary computation gives $\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)=ab-a+\frac{a}{c}-b+1-\frac{1}{c}+1-\frac{1}{b}+\frac{1}{bc}$. Using $ab=\frac{1}{c}$ and $\frac{1}{bc}=a$ we obtain

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)=\frac{a}{c}-b-\frac{1}{b}+2\leq\frac{a}{c},$$

since $b+\frac{1}{b}\geq 2$. Similarly we obtain

$$\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq\frac{b}{a}\quad\text{and}\quad\left(c-1+\frac{1}{a}\right)\left(a-1+\frac{1}{b}\right)\leq\frac{c}{b}.$$

The desired inequality follows from the previous three inequalities. Equality holds if and only if $a=b=c=1$.

8. We note that $\{ta\}$ lies in $\left(\frac{1}{3},\frac{2}{3}\right]$ if and only if there is an integer k such that $k+\frac{1}{3}<ta\leq k+\frac{2}{3}$, i.e., if and only if $t\in I_k=\left(\frac{k+1/3}{a},\frac{k+2/3}{a}\right]$ for some k. Similarly, t should belong to the sets $J_m=\left(\frac{m+1/3}{b},\frac{m+2/3}{b}\right]$ and $K_n=\left(\frac{n+1/3}{c},\frac{n+2/3}{c}\right]$ for some m,n. We have to show that $I_k\cap J_m\cap K_n$ is nonempty for some integers k,m,n.
The intervals K_n are separated by a distance $\frac{2}{3c}$, and since $\frac{2}{3c}<\frac{1}{3b}$, each of the intervals J_m intersects at least one of the K_n's. Hence it is enough to prove that $J_m\subset I_k$ for some k,m.
Let u_m and v_m be the left and right endpoints of J_m. Since $av_m=au_m+\frac{a}{3b}<au_m+\frac{1}{6}$, it will suffice to show that there is an integer m such that the fractional part of au_m lies in $\left[\frac{1}{3},\frac{1}{2}\right]$.
Let $a=d\alpha$, $b=d\beta$, $\gcd(\alpha,\beta)=1$. Setting $m=d\mu$ we obtain that $au_m=a\frac{m+1/3}{b}=\frac{\alpha m}{d\beta}+\frac{\alpha}{3\beta}=\frac{\alpha\mu}{\beta}+\frac{\alpha}{3\beta}$. Since $\alpha\mu$ gives all possible residues modulo β, every term of the arithmetic progression $\frac{j}{\beta}+\frac{\alpha}{3\beta}$ $(j\in\mathbb{Z})$ has its fractional part equal to the fractional part of some au_m. Now for $\beta\geq 6$ the progression step is $\frac{1}{\beta}\leq\frac{1}{6}$, so at least one of the au_m has its fractional part in $[1/3,1/2]$. If otherwise $\beta\leq 5$, the only irreducible fractions $\frac{\alpha}{\beta}$ that satisfy $2\alpha<\beta$ are $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{2}{5}$; hence one can take m to be $1,1,2,3$ respectively. This justifies our claim.

9. Let us first solve the problem under the assumption that $g(\alpha)=0$ for some α.
Setting $y=\alpha$ in the given equation yields $g(x)=(\alpha+1)f(x)-xf(\alpha)$. Then the given equation becomes $f(x+g(y))=(\alpha+1-y)f(x)+(f(y)-f(\alpha))x$, so setting $y=\alpha+1$ we get $f(x+n)=mx$, where $n=g(\alpha+1)$ and $m=f(\alpha+1)-f(\alpha)$. Hence f is a linear function, and consequently g is also linear. If we now substitute $f(x)=ax+b$ and $g(x)=cx+d$ in the given equation and compare the coefficients, we easily find that

$$f(x) = \frac{cx - c^2}{1 + c} \quad \text{and} \quad g(x) = cx - c^2, \qquad c \in \mathbb{R} \setminus \{-1\}.$$

Now we prove the existence of α such that $g(\alpha) = 0$. If $f(0) = 0$ then putting $y = 0$ in the given equation we obtain $f(x + g(0)) = g(x)$, so we can take $\alpha = -g(0)$.
Now assume that $f(0) = b \neq 0$. By replacing x by $g(x)$ in the given equation we obtain $f(g(x) + g(y)) = g(x)f(y) - yf(g(x)) + g(g(x))$ and, analogously, $f(g(x) + g(y)) = g(y)f(x) - xf(g(y)) + g(g(y))$. The given functional equation for $x = 0$ gives $f(g(y)) = a - by$, where $a = g(0)$. In particular, g is injective and f is surjective, so there exists $c \in \mathbb{R}$ such that $f(c) = 0$. Now the above two relations yield

$$g(x)f(y) - ay + g(g(x)) = g(y)f(x) - ax + g(g(y)). \tag{1}$$

Plugging $y = c$ in (1) we get $g(g(x)) = g(c)f(x) - ax + g(g(c)) + ac = kf(x) - ax + d$. Now (1) becomes $g(x)f(y) + kf(x) = g(y)f(x) + kf(y)$. For $y = 0$ we have $g(x)b + kf(x) = af(x) + kb$, whence

$$g(x) = \frac{a - k}{b} f(x) + k.$$

Note that $g(0) = a \neq k = g(c)$, since g is injective. From the surjectivity of f it follows that g is surjective as well, so it takes the value 0.

10. Clearly $F(0) = 0$ by (i). Moreover, it follows by induction from (i) that $F(2^n) = f_{n+1}$ where f_n denotes the nth Fibonacci's number. In general, if $n = \epsilon_k 2^k + \epsilon_{k-1}2^{k-1} + \cdots + \epsilon_1 \cdot 2 + \epsilon_0$ (where $\epsilon_i \in \{0, 1\}$), it is straightforward to verify that

$$F(n) = \epsilon_k f_{k+1} + \epsilon_{k-1} f_k + \cdots + \epsilon_1 f_2 + \epsilon_0 f_1. \tag{1}$$

We observe that if the binary representation of n contains no two adjacent ones, then $F(3n) = F(4n)$. Indeed, if $n = \epsilon_{k_r} 2^{k_r} + \cdots + \epsilon_{k_0} 2^{k_0}$, where $k_{i+1} - k_i \geq 2$ for all i, then $3n = \epsilon_{k_r}(2^{k_r+1} + 2^{k_r}) + \cdots + \epsilon_{k_0}(2^{k_0+1} + 2^{k_0})$. According to this, in computing $F(3n)$ each f_{i+1} in (1) is replaced by $f_{i+1} + f_{i+2} = f_{i+3}$, leading to the value of $F(4n)$.
We shall prove the converse: $F(3n) \leq F(4n)$ holds for all $n \geq 0$, with equality if and only if the binary representation of n contains no two adjacent ones.
We prove by induction on $m \geq 1$ that this holds for all n satisfying $0 \leq n < 2^m$. The verification for the early values of m is direct. Assume it is true for a certain m and let $2^m \leq n \leq 2^{n+1}$. If $n = 2^m + p$, $0 \leq p < 2^m$, then (1) implies $F(4n) = F(2^{m+2} + 4p) = f_{m+3} + F(4p)$. Now we distinguish three cases:

(i) If $3p < 2^m$, then the binary representation of $3p$ does not carry into that of $3 \cdot 2^m$. Then it follows from (1) and the induction hypothesis that

$$F(3n) = F(3 \cdot 2^m) + F(3p) = f_{m+3} + F(3p) \le f_{m+3} + F(4p) = F(4n).$$

Equality holds if and only if $F(3p) = F(4p)$, i.e. p has no two adjacent binary ones.

(ii) If $2^m \le 3p < 2^{m+1}$, then the binary representation of $3p$ carries 1 into that of $3 \cdot 2^m$. Thus $F(3n) = f_{m+3} + (F(3p) - f_{m+1}) = f_{m+2} + F(3p) < f_{m+3} + F(4p) = F(4n)$.

(iii) If $2^{m+1} \le p < 3 \cdot 2^m$, then the binary representation of $3p$ caries 10 into that of $3 \cdot 2^m$, which implies

$$F(3n) = f_{m+3} + f_{m+1} + (F(3p) - f_{m+2}) = 2f_{m+1} + F(3p) < F(4n).$$

It remains to compute the number of integers in $[0, 2^m)$ with no two adjacent binary 1's. Denote their number by u_m. Among them there are u_{m-1} less than 2^{m-1} and u_{m-2} in the segment $[2^{m-1}, 2^m)$. Hence $u_m = u_{m-1} + u_{m-2}$ for $m \ge 3$. Since $u_1 = 2 = f_3$, $u_2 = 3 = f_4$, we conclude that $u_m = f_{m+2} = F(2^{m+1})$.

11. We claim that for $\lambda \ge \frac{1}{n-1}$ we can take all fleas as far to the right as we want. In every turn we choose the leftmost flea and let it jump over the rightmost one. Let d and δ denote the maximal and the minimal distances between two fleas at some moment. Clearly, $d \ge (n-1)\delta$. After the leftmost flea jumps over the rightmost one, the minimal distance does not decrease, because $\lambda d \ge \delta$. However, the position of the leftmost flea moved to the right by at least δ, and consequently we can move the fleas arbitrarily far to the right after a finite number of moves.
Suppose now that $\lambda < \frac{1}{n-1}$. Under this assumption we shall prove that there is a number M that cannot be reached by any flea. Let us assign to each flea the coordinate on the real axis in which it is settled. Denote by s_k the sum of all the numbers in the kth step, and by w_k the coordinate of the rightmost flea. Clearly, $s_k \le nw_k$. We claim that the sequence w_k is bounded.
In the $(k+1)$th move let a flea A jump over B, landing at C, and let a, b, c be their respective coordinates. We have $s_{k+1} - s_k = c - a$. Then by the given rule, $\lambda(b-a) = c - b = s_{k+1} - s_k + a - b$, which implies $s_{k+1} - s_k = (1+\lambda)(b-a) = \frac{1+\lambda}{\lambda}(c-b)$. Hence $s_{k+1} - s_k \ge \frac{1+\lambda}{\lambda}(w_{k+1} - w_k)$. Summing up these inequalities for $k = 0, \dots, n-1$ yields $s_n - s_0 \ge \frac{1+\lambda}{\lambda}(w_n - w_0)$. Now using $s_n \le nw_n$ we conclude that

$$\left(\frac{1+\lambda}{\lambda} - n\right) w_n \le \frac{1+\lambda}{\lambda} w_0 - s_0.$$

Since $\frac{1+\lambda}{\lambda} - n > 0$, this proves the result.

12. Since $D(A) = D(B)$, we can define $f(i) > g(i) \ge 0$ that satisfy $b_i - b_{i-1} = a_{f(i)} - a_{g(i)}$ for all i.
The number $b_{i+1} - b_{i-1} \in D(B) = D(A)$ can be written in the form $a_u - a_v$, $u > v \ge 0$. Then $b_{i+1} - b_{i-1} = b_{i+1} - b_i + b_i - b_{i-1}$ implies

$a_{f(i+1)} + a_{f(i)} + a_v = a_{g(i+1)} + a_{g(i)} + a_u$, so the B_3 property of A implies that $(f(i+1), f(i), v)$ and $(g(i+1), g(i), u)$ coincide up to a permutation. It follows that either $f(i+1) = g(i)$ or $f(i) = g(i+1)$. Hence if we define $R = \{i \in \mathbb{N}_0 \mid f(i+1) = g(i)\}$ and $S = \{i \in \mathbb{N}_0 \mid f(i) = g(i+1)\}$ it holds that $R \cup S = \mathbb{N}_0$.

Lemma. If $i \in R$, then also $i+1 \in R$.

Proof. Suppose to the contrary that $i \in R$ and $i+1 \in S$, i.e., $g(i) = f(i+1) = g(i+2)$. There are integers x and y such that $b_{i+2} - b_{i-1} = a_x - a_y$. Then $a_x - a_y = a_{f(i+2)} - a_{g(i+2)} + a_{f(i+1)} - a_{g(i+1)} + a_{f(i)} - a_{g(i)} = a_{f(i+2)} + a_{f(i)} - a_{g(i+1)} - a_{g(i)}$, so by the B_3 property $(x, g(i+1), g(i))$ and $(y, f(i+2), f(i))$ coincide up to a permutation. But this is impossible, since $f(i+2), f(i) > g(i+2) = g(i) = f(i+1) > g(i+1)$. This proves the lemma.

Therefore if $i \in R \neq \emptyset$, then it follows that every $j > i$ belongs to R. Consequently $g(i) = f(i+1) > g(i+1) = f(i+2) > g(i+2) = f(i+3) > \cdots$ is an infinite decreasing sequence of nonnegative integers, which is impossible. Hence $S = \mathbb{N}_0$, i.e.,

$$b_{i+1} - b_i = a_{f(i+1)} - a_{f(i)} \qquad \text{for all } i \in \mathbb{N}_0.$$

Thus $f(0) = g(1) < f(1) < f(2) < \cdots$, implying $f(i) \geq i$. On the other hand, for any i there exist j, k such that $a_{f(i)} - a_i = b_j - b_k = a_{f(j)} - a_{f(k)}$, so by the B_3 property $i \in \{f(i), f(k)\}$ is a value of f. Hence we must have $f(i) = i$ for all i, which finally gives $A = B$.

13. One can easily find n-independent polynomials for $n = 0, 1$. For example, $P_0(x) = 2000x^{2000} + \cdots + 2x^2 + x + 0$ is 0-independent (for $Q \in M(P_0)$ it suffices to exchange the coefficient 0 of Q with the last term), and $P_1(x) = 2000x^{2000} + \cdots + 2x^2 + x - (1 + 2 + \cdots + 2000)$ is 1-independent (since any $Q \in M(P_1)$ vanishes at $x = 1$). Let us show that no n-independent polynomials exist for $n \notin \{0, 1\}$.

Consider separately the case $n = -1$. For any set T we denote by $S(T)$ the sum of elements of T. Suppose that $P(x) = a_{2000}x^{2000} + \cdots + a_1 x + a_0$ is -1-independent. Since $P(-1) = (a_0 + a_2 + \cdots + a_{2000}) - (a_1 + a_3 + \cdots + a_{1999})$, this means that for any subset E of the set $C = \{a_0, a_1, \ldots, a_{2000}\}$ having 1000 or 1001 elements there exist elements $e \in E$ and $f \in C \setminus E$ such that $S(E \cup \{f\} \setminus \{e\}) = \frac{1}{2}S(C)$, or equivalently that $S(E) - \frac{1}{2}S(C) = e - f$. We may assume w.l.o.g. that $a_0 < a_1 < \cdots < a_{2000}$.

Suppose that E is a 1000-element subset of C containing b_0, b_1 but not b_{1999}, b_{2000}. By the -1-independence of P there exist $e \in E$ and $f \in C \setminus E$ such that $S(E) - \frac{1}{2}S(C) = e - f$. The same must hold for the set $E' = E \cup \{b_{1999}, b_{2000}\} \setminus \{b_0, b_1\}$, so for some $e' \in E'$ and $f' \in C \setminus E'$ we have $S(E') - \frac{1}{2}S(C) = e' - f'$. It follows that $b_{1999} + b_{2000} - b_0 - b_1 = S(E') - S(E) = e + e' - f - f'$. Therefore the transposition $e \leftrightarrow f$ must involve at least one of the elements $b_0, b_1, b_{1999}, b_{2000}$.

There are 7994 possible transpositions involving one of these four elements. On the other hand, by (SL93-12) the subsets E of C containing b_0, b_1 but not b_{1999}, b_{2000} give at least $998\cdot999+1$ distinct sums of elements, far exceeding 7994. This is a contradiction.

For the case $|n| \geq 2$ we need the following lemma.

Lemma. Let $n \geq 2$ be a natural number and $P(x) = a_m x^m + \cdots + a_1 x + a_0$ a polynomial with distinct coefficients. Then the set $\{Q(n) \mid Q \in M(P)\}$ contains at least 2^m elements.

Proof. We shall use induction on m. The statement is easily verified for $m = 1$. Assume w.l.o.g. that $a_m < \cdots < a_1 < a_0$. Consider two polynomials Q_k and Q_{k+1} of the form

$$Q_k(x) = a_m x^m + \cdots + a_k x^k + a_0 x^{k-1} + b_{k-1} x^{k-2} + \cdots + b_1,$$
$$Q_{k+1}(x) = a_m x^m + \cdots + a_{k+1} x^{k+1} + a_0 x^k + c_k x^{k-1} + \cdots + c_1,$$

where $(b_{k-1}, \dots, b_1)$ and $(c_k, \dots, c_1)$ are permutations of the sets $\{a_{k-1}, \dots, a_1\}$ and $\{a_k, \dots, a_1\}$ respectively. We claim that $Q_{k+1}(n) \geq Q_k(n)$. Indeed, since $a_0 - c_k \leq a_0 - a_k$ and $b_j - c_j < a_0 - a_k$ for $1 \leq j \leq n-1$, we have $Q_{k+1}(n) - Q_k(x) = (a_0 - a_k)n^k - (a_0 - c_k)n^{k-1} - (b_{k-1} - c_{k-1})n^{k-2} - \cdots - (b_1 - c_1) \geq (a_0 - a_k)(n^k - n^{k-1} - \cdots - n - 1) > 0$. Furthermore, by the induction hypothesis the polynomials of the form $Q_k(x)$ take at least 2^{k-2} values at $x = n$. Hence the total number of values of $Q(n)$ for $Q \in M(P)$ is at least $1 + 1 + 2 + 2^2 + \cdots + 2^{m-1} = 2^m$.

Now we return to the main result. Suppose that $P(x) = a_{2000}x^{2000} + a_{1999}x^{1999} + a_0$ is an n-independent polynomial. Since $P_2(x) = a_{2000}x^{2000} + a_{1998}x^{1998} + \cdots + a_2 x^2 + a_0$ is a polynomial in $t = x^2$ of degree 1000, by the lemma it takes at least 2^{1000} distinct values at $x = n$. Hence $\{Q(n) \mid Q \in M(P)\}$ contains at least 2^{1000} elements. On the other hand, interchanging the coefficients b_i and b_j in a polynomial $Q(x) = b_{2000}x^{2000} + \cdots + b_0$ modifies the value of Q at $x = n$ by $(b_i - b_j)(n^i - n^j) = (a_k - a_l)(n^i - n^j)$ for some k, l. Hence there are fewer than 2001^4 possible modifications of the value at n. Since $2001^4 < 2^{1000}$, we have arrived at a contradiction.

14. The given condition is obviously equivalent to $a^2 \equiv 1 \pmod{n}$ for all integers a coprime to n. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n onto primes. Since by the Chinese remainder theorem the numbers coprime to n can give any remainder modulo $p_i^{\alpha_i}$ except 0, our condition is equivalent to $a^2 \equiv 1 \pmod{p_i^{\alpha_i}}$ for all i and integers a coprime to p_i.
Now if $p_i \geq 3$, we have $2^2 \equiv 1 \pmod{p_i^{\alpha_i}}$, so $p_i = 3$ and $\alpha_i = 2$. If $p_j = 2$, then $3^2 \equiv 1 \pmod{2^{\alpha_j}}$ implies $\alpha_j \leq 3$. Hence n is a divisor of $2^3 \cdot 3 = 24$. Conversely, each $n \mid 24$ has the desired property.

15. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n onto primes ($p_1 < p_2 < \cdots < p_k$). Since $4n$ is a perfect cube, we deduce that $p_1 = 2$ and $\alpha_1 = 3\beta_1 + 1$, $\alpha_2 = 3\beta_2, \dots, \alpha_k = 3\beta_k$ for some integers $\beta_i \geq 0$. Using $d(n) = (\alpha_1 + 1)\cdot(\alpha_2 + 1)\cdots(\alpha_k + 1)$ we can rewrite the equation $d(n)^3 = 4n$ as

$$(3\beta_1+2)\cdot(3\beta_2+1)\cdots(3\beta_k+1)=2^{\beta_1+1}p_2^{\beta_2}\cdots p_k^{\beta_k}.$$

Since $d(n)$ is not divisible by 3, it follows that $p_i \geq 5$ for $i \geq 2$. Thus the above equation is equivalent to

$$\frac{3\beta_1+2}{2^{\beta_1+1}}=\frac{p_2^{\beta_2}}{3\beta_2+1}\cdots\frac{p_k^{\beta_k}}{3\beta_k+1}. \tag{1}$$

For $i \geq 2$ we have $p_i^{\beta_i} \geq (1+4)^{\beta_i} \geq 1+4\beta_i$; hence (1) implies that $\frac{3\beta_1+2}{2^{\beta_1+1}} \geq 1$, which leads to $\beta_1 \leq 2$.
For $\beta_1=0$ or $\beta_1=2$ we have that $\frac{3\beta_1+2}{2^{\beta_1+1}}=1$, and therefore $\beta_2=\cdots=\beta_k=0$. This yields the solutions $n=2$ and $n=2^7=128$.
For $\beta_1=1$ the left-hand side of (1) equals $\frac{5}{4}$. On the other hand, if $p_i>5$ or $\beta_i>1$, then $\frac{p_i^{\beta_i}}{3\beta_i+1}>\frac{5}{4}$, which is impossible. We conclude that $p_2=5$ and $k=2$, so $n=2000$.
Hence the solutions for n are $2, 128$, and 2000.

16. More generally, we will prove by induction on k that for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ that has exactly k distinct prime divisors such that $n_k \mid 2^{n_k}+1$ and $3 \mid n_k$.
For $k=1$, $n_1=3$ satisfies the given conditions. Now assume that $k \geq 1$ and $n_k=3^\alpha m$ where $3 \nmid m$, so that m has exactly $k-1$ prime divisors. Then the number $3n_k=3^{\alpha+1}m$ has exactly k prime divisors and $2^{3n_k}+1=(2^{n_k}+1)(2^{2n_k}-2^{n_k}+1)$ is divisible by $3n_k$, since $3 \mid 2^{2n_k}-2^{n_k}+1$. We shall find a prime p not dividing n_k such that $n_{k+1}=3pn_k$. It is enough to find p such that $p \mid 2^{3n_k}+1$ and $p \nmid 2^{n_k}+1$.
Moreover, we shall show that for every integer $a>2$ there exists a prime number p that divides $a^3+1=(a+1)(a^2-a+1)$ but not $a+1$. To prove this we observe that $\gcd(a^2-a+1, a+1)=\gcd(3, a+1)$. Now if $3 \nmid a+1$, we can simply take $p=3$; otherwise, if $a=3b-1$, then $a^2-a+1=9b^2-9b+3$ is not divisible by 3^2; hence we can take for p any prime divisor of $\frac{a^2-a+1}{3}$.

17. Trivially all triples $(a,1,n)$ and $(1,m,n)$ are solutions. Assume now that $a>1$ and $m>1$.
If m is even, then $a^m+1 \equiv (-1)^m+1 \equiv 2 \pmod{a+1}$, which implies that $a^m+1=2^t$. In particular, a is odd. But this is impossible, since $2<a^m+1=(a^{m/2})^2+1 \equiv 2 \pmod 4$. Hence m is odd.
Let p be an arbitrary prime divisor of m and $m=pm_1$. Then $a^m+1 \mid (a+1)^n \mid (a^{m_1}+1)^n$, so $b^p+1 \mid (b+1)^n$ for $b=a^{m_1}$. It follows that

$$P=\frac{b^p+1}{b+1}=b^{p-1}-b^{p-2}+\cdots+1 \mid (b+1)^n.$$

Since $P \equiv p \pmod{b+1}$, we deduce that P has no prime divisors other than p; hence P is a power of p and $p \mid b+1$. Let $b=kp-1$, $k \in \mathbb{N}$. Then by

the binomial formula we have $b^i = (kp-1)^i \equiv (-1)^{i+1}(ikp-1) \pmod{p^2}$, and therefore $P \equiv -kp((p-1)+(p-2)+\cdots+1)+p \equiv p \pmod{p^2}$. We conclude that $P \le p$. But we also have $P \ge b^{p-1} - b^{p-2} \ge b^{p-2} > p$ for $p > 3$, so we must have $P = p = 3$ and $b = 2$. Since $b = a^{m_1}$, we obtain $a = 2$ and $m = 3$. The triple $(2,3,n)$ is indeed a solution if $n \ge 2$.
Hence the set of solutions is $\{(a,1,n),(1,m,n) \mid a,m,n \in \mathbb{N}\} \cup \{(2,3,n) \mid n \ge 2\}$.

Remark. This problem is very similar to (SL97-14).

18. It is known that the area of the triangle is $S = pr = p^2/n$ and $S = \sqrt{p(p-a)(p-b)(p-c)}$. It follows that $p^3 = n^2(p-a)(p-b)(p-c)$, which by putting $x = p-a$, $y = p-b$, and $z = p-c$ transforms into

$$(x+y+z)^3 = n^2xyz. \tag{1}$$

We will be done if we show that (1) has a solution in positive integers for infinitely many natural numbers n. Let us assume that $z = k(x+y)$ for an integer $k > 0$. Then (1) becomes $(k+1)^3(x+y)^2 = kn^2xy$. Further, by setting $n = 3(k+1)$ this equation reduces to

$$(k+1)(x+y)^2 = 9kxy. \tag{2}$$

Set $t = x/y$. Then (2) has solutions in positive integers if and only if $(k+1)(t+1)^2 = 9kt$ has a rational solution, i.e., if and only if its discriminant $D = k(5k-4)$ is a perfect square. Setting $k = u^2$, we are led to show that $5u^2 - 4 = v^2$ has infinitely many integer solutions. But this is a classic Pell-type equation, whose solution is every Fibonacci number $u = F_{2i+1}$. This completes the proof.

19. Suppose that a natural number N satisfies $N = a_1^2 + \cdots + a_k^2$, $2N = b_1^2 + \cdots + b_l^2$, where a_i, b_j are natural numbers such that none of the ratios $a_i/a_j, b_i/b_j, a_i/b_j, b_j/a_i$ is a power of 2.
We claim that every natural number $n > \sum_{i=0}^{4N-2}(2iN+1)^2$ can be represented as a sum of distinct squares. Suppose $n = 4qN + r$, $0 \le r < 4N$. Then

$$n = 4Ns + \sum_{i=0}^{r-1}(2iN+1)^2$$

for some positive integer s, so it is enough to show that $4Ns$ is a sum of distinct even squares. Let $s = \sum_{c=1}^{C} 2^{2u_c} + \sum_{d=1}^{D} 2^{2v_d+1}$ be the binary expansion of s. Then

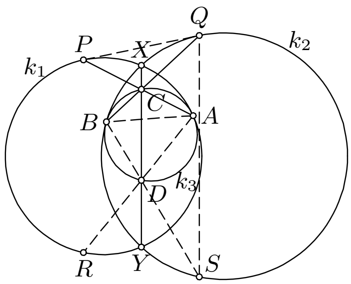
$$4Ns = \sum_{c=1}^{C}\sum_{i=1}^{k}(2^{u_c+1}a_i)^2 + \sum_{d=1}^{D}\sum_{j=1}^{l}(2^{u_d+1}b_j)^2,$$

where all the summands are distinct by the condition on a_i, b_j.

It remains to choose an appropriate N: for example $N = 29$, because $29 = 5^2 + 2^2$ and $58 = 7^2 + 3^2$.

Second solution. It can be directly checked that every odd integer $67 < n \le 211$ can be represented as a sum of distinct squares. For any $n > 211$ we can choose an integer m such that $m^2 > \frac{n}{2}$ and $n - m^2$ is odd and greater than 67, and therefore by the induction hypothesis can be written as a sum of distinct squares. Hence n is also a sum of distinct squares.

20. Denote by k_1, k_2 the given circles and by k_3 the circle through A, B, C, D. We shall consider the case that k_3 is inside k_1 and k_2, since the other case is analogous.

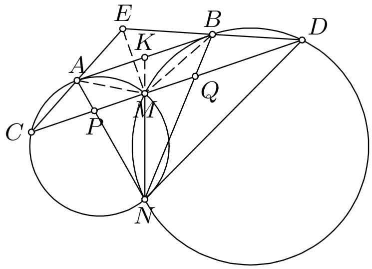


Let AC and AD meet k_1 at points P and R, and BC and BD meet k_2 at Q and S respectively. We claim that PQ and RS are the common tangents to k_1 and k_2, and therefore P, Q, R, S are the desired points. The circles k_1 and k_3 are tangent to each other, so we have $DC \parallel RP$. Since

$$AC \cdot CP = XC \cdot CY = BC \cdot CQ,$$

the quadrilateral $ABQP$ is cyclic, implying that $\angle APQ = \angle ABQ = \angle ADC = \angle ARP$. It follows that PQ is tangent to k_1. Similarly, PQ is tangent to k_2.

21. Let K be the intersection point of the lines MN and AB.

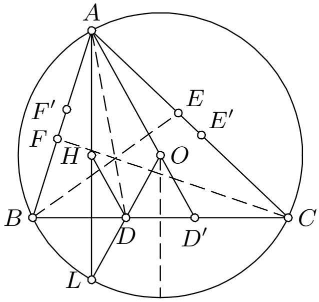


Since $KA^2 = KM \cdot KN = KB^2$, it follows that K is the midpoint of the segment AB, and consequently M is the midpoint of AB. Thus it will be enough to show that $EM \perp PQ$, or equivalently that $EM \perp AB$. However, since AB is tangent to the circle G_1 we have $\angle BAM = \angle ACM = \angle EAB$, and similarly $\angle ABM = \angle EBA$. This implies that the triangles EAB and MAB are congruent. Hence E and M are symmetric with respect to AB; hence $EM \perp AB$.

Remark. The proposer has suggested an alternative version of the problem: to prove that EN bisects the angle CND. This can be proved by noting that $EANB$ is cyclic.

22. Let L be the point symmetric to H with respect to BC. It is well known that L lies on the circumcircle k of $\triangle ABC$. Let D be the intersection point of OL and BC. We similarly define E and F. Then

$$OD + DH = OD + DL = OL = OE + EH = OF + FH.$$



We shall prove that AD, BE, and CF are concurrent. Let line AO meet BC at D'. It is easy to see that $\angle OD'D = \angle ODD'$; hence the perpendicular bisector of BC bisects DD' as well. Hence $BD = CD'$. If we define E' and F' analogously, we have $CE = AE'$ and $AF = BF'$. Since the lines AD', BE', CF' meet at O, it follows that $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = 1$. This proves our claim by Ceva's theorem.

23. First, suppose that there are numbers (b_i, c_i) assigned to the vertices of the polygon such that

$$A_iA_j = b_jc_i - b_ic_j \quad \text{ for all } i, j \text{ with } 1 \le i \le j \le n. \tag{1}$$

In order to show that the polygon is cyclic, it is enough to prove that A_1, A_2, A_3, A_i lie on a circle for each i, $4 \le i \le n$, or equivalently, by Ptolemy's theorem, that $A_1A_2 \cdot A_3A_i + A_2A_3 \cdot A_iA_1 = A_1A_3 \cdot A_2A_i$. But this is straightforward with regard to (1).
Now suppose that $A_1A_2 \dots A_n$ is a cyclic quadrilateral. By Ptolemy's theorem we have $A_iA_j = A_2A_j \cdot \frac{A_1A_i}{A_1A_2} - A_2A_i \cdot \frac{A_1A_j}{A_1A_2}$ for all i, j. This suggests taking $b_1 = -A_1A_2$, $b_i = A_2A_i$ for $i \ge 2$ and $c_i = \frac{A_1A_i}{A_1A_2}$ for all i. Indeed, using Ptolemy's theorem, one easily verifies (1).

24. Since $\angle ABT = 180^\circ - \gamma$ and $\angle ACT = 180^\circ - \beta$, the law of sines gives $\frac{BP}{PC} = \frac{S_{ABT}}{S_{ACT}} = \frac{AB \cdot BT \cdot \sin\gamma}{AB \cdot BT \cdot \sin\beta} = \frac{AB \sin\gamma}{AC \sin\beta} = \frac{c^2}{b^2}$, which implies $BP = \frac{c^2 a}{b^2 + c^2}$. Denote by M and N the feet of perpendiculars from P and Q on AB. We have $\cot\angle ABQ = \frac{BN}{NQ} = \frac{2BN}{PM} = \frac{BA+BM}{BP\sin\beta} = \frac{c+BP\cos\beta}{BP\sin\beta} = \frac{b^2+c^2+ac\cos\beta}{ca\sin\beta} = \frac{2(b^2+c^2)+a^2+c^2-b^2}{2ca\sin\beta} = \frac{a^2+b^2+3c^2}{4S_{ABC}} = 2\cot\alpha + 2\cot\beta + \cot\gamma$. Similarly, $\cot\angle BAS = 2\cot\alpha + 2\cot\beta + \cot\gamma$; hence $\angle ABQ = \angle BAS$.
Now put $p = \cot\alpha$ and $q = \cot\beta$. Since $p+q \ge 0$, the A-G mean inequality gives us $\cot\angle ABQ = 2p + 2q + \frac{1-pq}{p+q} \ge 2p + 2q + \frac{1-(p+q)^2/4}{p+q} = \frac{7}{4}(p+q) + \frac{1}{p+q} \ge 2\sqrt{\frac{7}{4}} = \sqrt{7}$. Hence $\angle ABQ \le \arctan\frac{1}{\sqrt{7}}$. Equality holds if and only if $\cot\alpha = \cot\beta = \frac{1}{\sqrt{7}}$, i.e., when $a : b : c = 1 : 1 : \frac{1}{\sqrt{2}}$.

25. By the condition of the problem, $\triangle ADX$ and $\triangle BCX$ are similar. Then there exist points Y' and Z' on the perpendicular bisector of AB such that $\triangle AY'Z'$ is similar and oriented the same as $\triangle ADX$, and $\triangle BY'Z'$ is (being congruent to $\triangle AY'Z'$) similar and oriented the same as $\triangle BCX$. Since then $AD/AY' = AX/AZ'$ and $\angle DAY' = \angle XAZ'$, $\triangle ADY'$ and $\triangle AXZ'$ are also similar, implying $\frac{AD}{AX} = \frac{DY'}{XZ'}$. Analogously, $\frac{BC}{BX} = \frac{CY'}{XZ'}$. It follows from $\frac{AD}{AX} = \frac{BC}{BX}$ that $CY' = DY'$, which means that Y' lies on the perpendicular bisector of CD. Hence $Y' \equiv Y$.

Now $\angle AYB = 2\angle AYZ' = 2\angle ADX$, as desired.

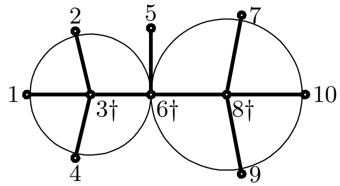
26. The problem can be reformulated in the following way: *Given a set S of ten points in the plane such that the distances between them are all distinct, for each point $P \in S$ we mark the point $Q \in S \setminus \{P\}$ nearest to P. Find the least possible number of marked points.*
Observe that each point $A \in S$ is the nearest to at most five other points. Indeed, for any six points $P_1, \dots, P_6$ one of the angles P_iAP_j is at most $60°$, in which case P_iP_j is smaller than one of the distances AP_i, AP_j. It follows that at least two points are marked.
Now suppose that exactly two points, say A and B, are marked. Then AB is the minimal distance of the points from S, so by the previous observation the rest of the set S splits into two subsets of four points according to whether the nearest point is A or B. Let these subsets be $\{A_1, A_2, A_3, A_4\}$ and $\{B_1, B_2, B_3, B_4\}$ respectively. Assume that the points are labelled so that the angles A_iAA_{i+1} are successively adjacent as well as the angles B_iBB_{i+1}, and that A_1, B_1 lie on one side of AB, and A_4, B_4 lie on the other side. Since all the angles A_iAA_{i+1} and B_iBB_{i+1} are greater than $60°$, it follows that

$$\angle A_1AB + \angle BAA_4 + \angle B_1BA + \angle ABB_4 < 360°.$$

Therefore $\angle A_1AB + \angle B_1BA < 180°$ or $\angle A_4AB + \angle B_4BA < 180°$. Without loss of generality, let us assume the first inequality.
On the other hand, note that the quadrilateral ABB_1A_1 is convex because A_1 and B_1 are on different sides of the perpendicular bisector of AB. From $A_1B_1 > A_1A$ and $BB_1 > AB$ we obtain $\angle A_1AB_1 > \angle A_1B_1A$ and $\angle BAB_1 > \angle AB_1B$. Adding these relations yields $\angle A_1AB > \angle A_1B_1B$. Similarly, $\angle B_1BA > \angle B_1A_1A$. Adding these two inequalities, we get

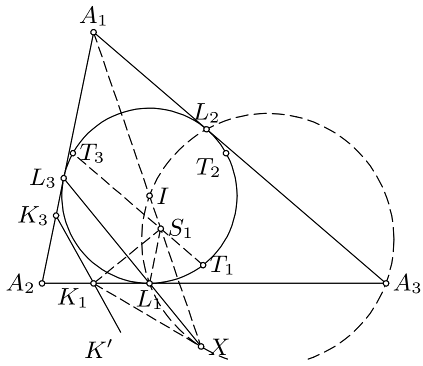
$$180° > \angle A_1AB + \angle B_1BA > \angle A_1B_1B + \angle B_1A_1A;$$

hence the sum of the angles of the quadrilateral ABB_1A_1 is less than $360°$, which is a contradiction. Thus at least 3 points are marked.
An example of a configuration in which exactly 3 gangsters are killed is shown below.



27. Denote by $\alpha_1, \alpha_2, \alpha_3$ the angles of $\triangle A_1A_2A_3$ at vertices A_1, A_2, A_3 respectively. Let T_1, T_2, T_3 be the points symmetric to L_1, L_2, L_3 with respect to A_1I, A_2I, and A_3I respectively. We claim that $T_1T_2T_3$ is the desired triangle.

Denote by S_1 and R_1 the points symmetric to K_1 and K_3 with respect to L_1L_3. It is enough to show that T_1 and T_3 lie on the line R_1S_1. To prove this, we shall prove that $\angle K_1S_1T_1 = \angle K'K_1S_1$ for a point K' on the line K_1K_3 such that K_3 and K' lie on different sides of K_1. We show first that $S_1 \in A_1I$. Let X be the point of intersection of lines A_1I and L_1L_3. We see from the triangle A_1L_3X that $\angle L_1XI = \alpha_3/2 = \angle L_1A_3I$, which implies that L_1XA_3I is cyclic.



We now have $\angle A_1XA_3 = 90^\circ = \angle A_1K_1A_3$; hence $A_1K_1XA_3$ is also cyclic. It follows that $\angle K_1XI = \angle K_1A_3A_1 = \alpha_3 = 2\angle L_1XI$; hence X_1L_1 bisects the angle K_1X_1I. Hence $S_1 \in XI$ as claimed. Now we have $\angle K_1S_1T_1 = \angle K_1S_1L_1 + 2\angle L_1S_1X = \angle S_1K_1L_1 + 2\angle L_1K_1X$. It remains to prove that K_1X bisects $\angle A_3K_1K'$. From the cyclic quadrilateral $A_1K_1XA_3$ we see that $\angle XK_1A_3 = \alpha_1/2$. Since $A_1K_3K_1A_3$ is cyclic, we also have $\angle K'K_1A_3 = \alpha_1 = 2\angle XK_1A_3$, which proves the claim.

4.42 Solutions to the Shortlisted Problems of IMO 2001

1. First, let us show that such a function is at most unique. Suppose that f_1 and f_2 are two such functions, and consider $g = f_1 - f_2$. Then g is zero on the boundary and satisfies

$$g(p, q, r) = \frac{1}{6}[g(p+1, q-1, r) + \cdots + g(p, q-1, r+1)],$$

i.e., $g(p, q, r)$ is equal to the average of the values of g at six points $(p + 1, q - 1, r), \dots$ that lie in the plane π given by $x + y + z = p + q + r$. Suppose that (p, q, r) is the point at which g attains its maximum in absolute value on $\pi \cap T$. The averaging property of g implies that the values of g at $(p+1, q-1, r)$ etc. are all equal to $g(p, q, r)$. Repeating this argument we obtain that g is constant on the whole of $\pi \cap T$, and hence it equals 0 everywhere. Therefore $f_1 \equiv f_2$.

It remains to guess f. It is natural to try $\overline{f}(p, q, r) = pqr$ first: it satisfies $\overline{f}(p, q, r) = \frac{1}{6}[\overline{f}(p+1, q-1, r) + \cdots + \overline{f}(p, q-1, r+1)] + \frac{p+q+r}{3}$. Thus we simply take

$$f(p, q, r) = \frac{3}{p+q+r}\overline{f}(p, q, r) = \frac{3pqr}{p+q+r}$$

and directly check that it satisfies the required property. Hence this is the unique solution.

2. It follows from Bernoulli's inequality that for each $n \in \mathbb{N}$, $\left(1 + \frac{1}{n}\right)^n \geq 2$, or $\sqrt[n]{2} \leq 1 + \frac{1}{n}$. Consequently, it will be enough to show that $1 + a_n > \left(1 + \frac{1}{n}\right) a_{n-1}$. Assume the opposite. Then there exists N such that for each $n \geq N$,

$$1 + a_n \leq \left(1 + \frac{1}{n}\right) a_{n-1}, \quad \text{i.e.,} \quad \frac{1}{n+1} + \frac{a_n}{n+1} \leq \frac{a_{n-1}}{n}.$$

Summing for $n = N, \dots, m$ yields $\frac{a_m}{m+1} \leq \frac{a_{N-1}}{N} - \left(\frac{1}{N+1} + \cdots + \frac{1}{m+1}\right)$. However, it is well known that the sum $\frac{1}{N+1} + \cdots + \frac{1}{m+1}$ can be arbitrarily large for m large enough, so that $\frac{a_m}{m+1}$ is eventually negative. This contradiction yields the result.

Second solution. Suppose that $1 + a_n \leq \sqrt[n]{2} a_{n-1}$ for all $n \geq N$. Set $b_n = 2^{-(1+1/2+\cdots+1/n)}$ and multiply both sides of the above inequality to obtain $b_n + b_n a_n \leq b_{n-1} a_{n-1}$. Thus

$$b_N a_N > b_N a_N - b_n a_n \geq b_N + b_{N+1} + \cdots + b_n.$$

However, it can be shown that $\sum_{n>N} b_N$ diverges: in fact, since $1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln n$, we have $b_n > 2^{-1-\ln n} = \frac{1}{2} n^{-\ln 2} > \frac{1}{2n}$, and we already know that $\sum_{n>N} \frac{1}{2n}$ diverges.

Remark. As can be seen from both solutions, the value 2 in the problem can be increased to e.

3. By the arithmetic–quadratic mean inequality, it suffices to prove that

$$\frac{x_1^2}{(1+x_1^2)^2}+\frac{x_2^2}{(1+x_1^2+x_2^2)^2}+\cdots+\frac{x_n^2}{(1+x_1^2+\cdots+x_n^2)^2}<1.$$

Observe that for $k\geq 2$ the following holds:

$$\begin{aligned}\frac{x_k^2}{(1+x_1^2+\cdots+x_k^2)^2} &\leq \frac{x_k^2}{(1+\cdots+x_{k-1}^2)(1+\cdots+x_k^2)}\\ &= \frac{1}{1+x_1^2+\cdots+x_{k-1}^2}-\frac{1}{1+x_1^2+\cdots+x_k^2}.\end{aligned}$$

For $k=1$ we have $\frac{x_1^2}{(1+x_1)^2}\leq 1-\frac{1}{1+x_1^2}$. Summing these inequalities, we obtain

$$\frac{x_1^2}{(1+x_1^2)^2}+\cdots+\frac{x_n^2}{(1+x_1^2+\cdots+x_n^2)^2}\leq 1-\frac{1}{1+x_1^2+\cdots+x_n^2}<1.$$

Second solution. Let $a_n(k)=\sup\left(\frac{x_1}{k^2+x_1^2}+\cdots+\frac{x_n}{k^2+x_1^2+\cdots+x_n^2}\right)$ and $a_n=a_n(1)$. We must show that $a_n<\sqrt{n}$. Replacing x_i by kx_i shows that $a_n(k)=a_n/k$. Hence

$$a_n=\sup_{x_1}\left(\frac{x_1}{1+x_1^2}+\frac{a_{n-1}}{\sqrt{1+x_1^2}}\right)=\sup_{\theta}(\sin\theta\cos\theta+a_{n-1}\cos\theta), \qquad (1)$$

where $\tan\theta=x_1$. The above supremum can be computed explicitly:

$$a_n=\frac{1}{8\sqrt{2}}\left(3a_{n-1}+\sqrt{a_{n-1}^2+8}\right)\sqrt{4-a_{n-1}^2+a_{n-1}\sqrt{a_{n-1}^2+8}}.$$

However, the required inequality is weaker and can be proved more easily: if $a_{n-1}<\sqrt{n-1}$, then by (1) $a_n<\sin\theta+\sqrt{n-1}\cos\theta=\sqrt{n}\sin(\theta+\alpha)\leq\sqrt{n}$, for $\alpha\in(0,\pi/2)$ with $\tan\alpha=\sqrt{n}$.

4. Let $(*)$ denote the given functional equation. Substituting $y=1$ we get $f(x)^2=xf(x)f(1)$. If $f(1)=0$, then $f(x)=0$ for all x, which is the trivial solution. Suppose $f(1)=C\neq 0$. Let $G=\{y\in\mathbb{R}\mid f(y)\neq 0\}$. Then

$$f(x)=\begin{cases}Cx & \text{if } x\in G,\\ 0 & \text{otherwise.}\end{cases} \qquad (1)$$

We must determine the structure of G so that the function defined by (1) satisfies $(*)$.
(1) Clearly $1\in G$, because $f(1)\neq 0$.
(2) If $x\in G$, $y\notin G$, then by $(*)$ it holds $f(xy)f(x)=0$, so $xy\notin G$.
(3) If $x,y\in G$, then $x/y\in G$ (otherwise by 2°, $y(x/y)=x\notin G$).

(4) If $x, y \in G$, then by 2° we have $x^{-1} \in G$, so $xy = y/x^{-1} \in G$. Hence G is a set that contains 1, does not contain 0, and is closed under multiplication and division. Conversely, it is easy to verify that every such G in (1) gives a function satisfying $(*)$.

5. Let $a_1, a_2, \dots, a_n$ satisfy the conditions of the problem. Then $a_k > a_{k-1}$, and hence $a_k \ge 2$ for $k = 1, \dots, n$. The inequality $(a_{k+1} - 1)a_{k-1} \ge a_k^2(a_k - 1)$ can be rewritten as

$$\frac{a_{k-1}}{a_k} + \frac{a_k}{a_{k+1} - 1} \le \frac{a_{k-1}}{a_k - 1}.$$

Summing these inequalities for $k = i+1, \dots, n-1$ and using the obvious inequality $\frac{a_{n-1}}{a_n} < \frac{a_{n-1}}{a_n - 1}$, we obtain $\frac{a_i}{a_{i+1}} + \dots + \frac{a_{n-1}}{a_n} < \frac{a_i}{a_{i+1}-1}$. Therefore

$$\frac{a_i}{a_{i+1}} \le \frac{99}{100} - \frac{a_0}{a_1} - \dots - \frac{a_{i-1}}{a_i} < \frac{a_i}{a_{i+1} - 1} \qquad \text{for } i = 1, 2, \dots, n-1. \quad (1)$$

Consequently, given $a_0, a_1, \dots, a_i$, there is at most one possibility for a_{i+1}. In our case, (1) yields $a_1 = 2$, $a_2 = 5$, $a_3 = 56$, $a_4 = 280^2 = 78400$. These values satisfy the conditions of the problem, so that this is a unique solution.

6. We shall determine a constant $k > 0$ such that

$$\frac{a}{\sqrt{a^2 + 8bc}} \ge \frac{a^k}{a^k + b^k + c^k} \qquad \text{for all } a, b, c > 0. \qquad (1)$$

This inequality is equivalent to $(a^k + b^k + c^k)^2 \ge a^{2k-2}(a^2 + 8bc)$, which further reduces to

$$(a^k + b^k + c^k)^2 - a^{2k} \ge 8a^{2k-2}bc.$$

On the other hand, the AM–GM inequality yields

$$(a^k + b^k + c^k)^2 - a^{2k} = (b^k + c^k)(2a^k + b^k + c^k) \ge 8a^{k/2}b^{3k/4}c^{3k/4},$$

and therefore $k = 4/3$ is a good choice. Now we have

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}}$$
$$\ge \frac{a^{4/3}}{a^{4/3} + b^{4/3} + c^{4/3}} + \frac{b^{4/3}}{a^{4/3} + b^{4/3} + c^{4/3}} + \frac{c^{4/3}}{a^{4/3} + b^{4/3} + c^{4/3}} = 1.$$

Second solution. The numbers $x = \frac{a}{\sqrt{a^2+8bc}}$, $y = \frac{b}{\sqrt{b^2+8ca}}$ and $z = \frac{c}{\sqrt{c^2+8ab}}$ satisfy

$$f(x, y, z) = \left(\frac{1}{x^2} - 1\right)\left(\frac{1}{y^2} - 1\right)\left(\frac{1}{z^2} - 1\right) = 8^3.$$

Our task is to prove $x+y+z \geq 1$.
Since f is decreasing on each of the variables x, y, z, this is the same as proving that $x, y, z > 0$, $x+y+z=1$ implies $f(x,y,z) \geq 8^3$. However, since $\frac{1}{x^2} - 1 = \frac{(x+y+z)^2 - x^2}{x^2} = \frac{(2x+y+z)(y+z)}{x^2}$, the inequality $f(x,y,z) \geq 8^3$ becomes

$$\frac{(2x+y+z)(x+2y+z)(x+y+2z)(y+z)(z+x)(x+y)}{x^2y^2z^2} \geq 8^3,$$

which follows immediately by the AM–GM inequality.

Third solution. We shall prove a more general fact: the inequality $\frac{a}{\sqrt{a^2+kbc}} + \frac{b}{\sqrt{b^2+kca}} + \frac{c}{\sqrt{c^2+kab}} \geq \frac{3}{\sqrt{1+k}}$ is true for all $a, b, c > 0$ if and only if $k \geq 8$.
Firstly suppose that $k \geq 8$. Setting $x = bc/a^2$, $y = ca/b^2$, $z = ab/c^2$, we reduce the desired inequality to

$$F(x,y,z) = f(x) + f(y) + f(z) \geq \frac{3}{\sqrt{1+k}}, \quad \text{where } f(t) = \frac{1}{\sqrt{1+kt}}, \tag{2}$$

for $x, y, z > 0$ such that $xyz = 1$. We shall prove (2) using the method of Lagrange multipliers.
The boundary of the set $D = \{(x,y,z) \in \mathbb{R}_+^3 \mid xyz = 1\}$ consists of points (x,y,z) with one of x, y, z being 0 and another one being $+\infty$. If w.l.o.g. $x = 0$, then $F(x,y,z) \geq f(x) = 1 \geq 3/\sqrt{1+k}$.
Suppose now that (x,y,z) is a point of local minimum of F on D. There exists $\lambda \in \mathbb{R}$ such that (x,y,z) is stationary point of the function $F(x,y,z) + \lambda xyz$. Then (x,y,z,λ) is a solution to the system $f'(x) + \lambda yz = f'(y) + \lambda xz = f'(z) + \lambda xy = 0$, $xyz = 1$. Eliminating λ gives us

$$xf'(x) = yf'(y) = zf'(z), \quad xyz = 1. \tag{3}$$

The function $tf'(t) = \frac{-kt}{2(1+kt)^{3/2}}$ decreases on the interval $(0, 2/k]$ and increases on $[2/k, +\infty)$ because $(tf'(t))' = \frac{k(kt-2)}{4(1+kt)^{5/2}}$. It follows that two of the numbers x, y, z are equal. If $x = y = z$, then $(1,1,1)$ is the only solution to (3). Suppose that $x = y \neq z$. Since $(yf'(y))^2 - (zf'(z))^2 = \frac{k^2(z-y)(k^3y^2z^2 - 3kyz - y - z)}{4(1+ky)^3(1+kz)^3}$, (3) gives us $y^2z = 1$ and $k^3y^2z^2 - 3kyz - y - z = 0$. Eliminating z we obtain an equation in y, $k^3/y^2 - 3k/y - y - 1/y^2 = 0$, whose only real solution is $y = k-1$. Thus $(k-1, k-1, 1/(k-1)^2)$ and the cyclic permutations are the only solutions to (3) with x, y, z being not all equal. Since $F(k-1, k-1, 1/(k-1)^2) = (k+1)/\sqrt{k^2-k+1} > F(1,1,1) = 1$, the inequality (2) follows.
For $0 < k < 8$ we have that $\frac{a}{\sqrt{a^2+kbc}} + \frac{b}{\sqrt{b^2+kca}} + \frac{c}{\sqrt{c^2+kab}} > \frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$. If we fix c and let a, b tend to 0, the first two summands will tend to 0 while the third will tend to 1. Hence the inequality cannot be improved.

7. It is evident that arranging of A in increasing order does not diminish m. Thus we can assume that A is nondecreasing. Assume w.l.o.g. that $a_1 = 1$, and let b_i be the number of elements of A that are equal to i $(1 \le i \le n = a_{2001})$. Then we have $b_1 + b_2 + \cdots + b_n = 2001$ and

$$m = b_1b_2b_3 + b_2b_3b_4 + \cdots + b_{n-2}b_{n-1}b_n. \tag{1}$$

Now if b_i, b_j $(i < j)$ are two largest b's, we deduce from (1) and the AM–GM inequality that $m \le b_ib_j(b_1+\cdots+b_{i-1}+b_{i+1}+\cdots+b_{j-1}+b_{j+1}+b_n) \le \left(\frac{2001}{3}\right)^3 = 667^3$ $(b_1b_2b_3 \le b_1b_ib_j$, etc.). The value 667^3 is attained for $b_1 = b_2 = b_3 = 667$ (i.e., $a_1 = \cdots = a_{667} = 1$, $a_{668} = \cdots = a_{1334} = 2$, $a_{1335} = \cdots = a_{2001} = 3$). Hence the maximum of m is 667^3.

8. Suppose to the contrary that all the $S(a)$'s are different modulo $n!$. Then the sum of $S(a)$'s over all permutations a satisfies $\sum_a S(a) \equiv 0+1+\cdots+(n!-1) = \frac{(n!-1)n!}{2} \equiv \frac{n!}{2} \pmod{n!}$. On the other hand, the coefficient of c_i in $\sum_a S(a)$ is equal to $(n-1)!(1+2+\cdots+n) = \frac{n+1}{2}n!$ for all i, from which we obtain

$$\sum_a S(a) \equiv \frac{n+1}{2}(c_1 + \cdots + c_n)n! \equiv 0 \pmod{n!}$$

for odd n. This is a contradiction.

9. Consider one such party. The result is trivially true if there is only one 3-clique, so suppose there exist at least two 3-cliques C_1 and C_2. We distinguish two cases:
   (i) $C_1 = \{a, b, c\}$ and $C_2 = \{a, d, e\}$ for some distinct people a, b, c, d, e. If the departure of a destroys all 3-cliques, then we are done. Otherwise, there is a third 3-clique C_3, which has a person in common with each of C_1, C_2 and does not include a: say, $C_3 = \{b, d, f\}$ for some f. We thus obtain another 3-clique $C_4 = \{a, b, d\}$, which has two persons in common with C_3, and the case (ii) is applied.
   (ii) $C_1 = \{a, b, c\}$ and $C_2 = \{a, b, d\}$ for distinct people a, b, c, d. If the departure of a, b leaves no 3-clique, then we are done. Otherwise, for some e there is a clique $\{c, d, e\}$.
   We claim that then the departure of c, d breaks all 3-cliques. Suppose the opposite, that a 3-clique C remains. Since C shares a person with each of the 3-cliques $\{c, d, a\}, \{c, d, b\}, \{c, d, e\}$, it must be $C = \{a, b, e\}$. However, then $\{a, b, c, d, e\}$ is a 5-clique, which is assumed to be impossible.

10. For convenience let us write $a = 1776$, $b = 2001$, $0 < a < b$. There are two types of historic sets:

$$(1)\ \{x, x+a, x+a+b\} \qquad \text{and} \qquad (2)\ \{x, x+b, x+a+b\}.$$

We construct a sequence of historic sets $H_1, H_2, H_3, \ldots$ inductively as follows:

(i) $H_1 = \{0, a, a+b\}$, and

(ii) Let y_n be the least nonnegative integer not occurring in $U_n = H_1 \cap \cdots \cap H_n$. We take H_{n+1} to be $\{y_n, y_n + a, y_n + a + b\}$ if $y_n + a \notin U_n$, and $\{y_n, y_n + b, y_n + a + b\}$ otherwise.

It remains to show that this construction never fails. Suppose that it failed at the construction of H_{n+1}. The element $y_n + a + b$ is not contained in U_n, since by the construction the smallest elements of $H_1, \dots, H_n$ are all less than y_n. Hence the reason for the failure must be the fact that both $y_n + a$ and $y_n + b$ are covered by U_n. Further, $y_n + b$ must have been the largest element of its set H_k, so the smallest element of H_k equals $y_n - a$. But since y_n is not covered, we conclude that H_k is of type (2). This is a contradiction, because y_n was free, so by the algorithm we had to choose for H_k the set of type (1) (that is, $\{y_n - a, y_n, y_n + b\}$) first.

11. Let $(x_0, x_1, \dots, x_n)$ be any such sequence: its terms are clearly nonnegative integers. Also, $x_0 = 0$ yields a contradiction, so $x_0 > 0$. Let m be the number of positive terms among $x_1, \dots, x_n$. Since x_i counts the terms equal to i, the sum $x_1 + \cdots + x_n$ counts the total number of positive terms in the sequence, which is known to be $m+1$. Therefore among $x_1, \dots, x_n$ exactly $m-1$ terms are equal to 1, one is equal to 2, and the others are 0. Only x_0 can exceed 2, and consequently at most one of $x_3, x_4, \dots$ can be positive. It follows that $m \le 3$.

(i) $m = 1$: Then $x_2 = 2$ (since $x_1 = 2$ is impossible), so $x_0 = 2$. The resulting sequence is $(2, 0, 2, 0)$.

(ii) $m = 2$: Either $x_1 = 2$ or $x_2 = 2$. These cases yield $(1, 2, 1, 0)$ and $(2, 1, 2, 0, 0)$ respectively.

(iii) $m = 3$: This means that $x_k > 0$ for some $k > 2$. Hence $x_0 = k$ and $x_k = 1$. Further, $x_1 = 1$ is impossible, so $x_1 = 2$ and $x_2 = 1$; there are no more positive terms in the sequence. The resulting sequence is $(p, 2, 1, \underbrace{0, \dots, 0}_{p-3}, 1, 0, 0, 0)$.

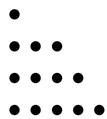
12. For each balanced sequence $a = (a_1, a_2, \dots, a_{2n})$ denote by $f(a)$ the sum of j's for which $a_j = 1$ (for example, $f(100101) = 1 + 4 + 6 = 11$). Partition the $\binom{2n}{n}$ balanced sequences into $n+1$ classes according to the residue of f modulo $n+1$. Now take S to be a class of minimum size: obviously $|S| \le \frac{1}{n+1}\binom{2n}{n}$. We claim that every balanced sequence a is either a member of S or a neighbor of a member of S. We consider two cases.

(i) Let a_1 be 1. It is easy to see that moving this 1 just to the right of the kth 0, we obtain a neighboring balanced sequence b with $f(b) = f(a) + k$. Thus if $a \notin S$, taking a suitable $k \in \{1, 2, \dots, n\}$ we can achieve that $b \in S$.

(ii) Let a_1 be 0. Taking this 0 just to the right of the kth 1 gives a neighbor b with $f(b) = f(a) - k$, and the conclusion is similar to that of (i).

This justifies our claim.

13. At any moment, let p_i be the number of pebbles in the ith column, $i = 1, 2, \dots$. The final configuration has obvious properties $p_1 \geq p_2 \geq \cdots$ and $p_{i+1} \in \{p_i, p_i - 1\}$. We claim that $p_{i+1} = p_i > 0$ is possible for at most one i.
Assume the opposite. Then the final configuration has the property that for some r and $s > r$ we have $p_{r+1} = p_r$, $p_{s+1} = p_s > 0$ and $p_{r+k} = p_{r+1} - k + 1$ for all $k = 1, \dots, s - r$. Consider the earliest configuration, say C, with this property. What was the last move before C? The only possibilities are moving a pebble either from the rth or from the sth column; however, in both cases the configuration preceding this last move had the same property, contradicting the assumption that C is the earliest.
Therefore the final configuration looks as follows: $p_1 = a \in \mathbb{N}$, and for some r, p_i equals $a - (i-1)$ if $i \leq r$, and $a - (i-2)$ otherwise. It is easy to determine a, r: since $n = p_1 + p_2 + \cdots = \frac{(a+1)(a+2)}{2} - r$, we get $\frac{a(a+1)}{2} \leq n < \frac{(a+1)(a+2)}{2}$, from which we uniquely find a and then r as well.

The final configuration for $n = 13$: 

14. We say that a problem is *difficult for boys* if at most two boys solved it, and *difficult for girls* if at most two girls solved it.
Let us estimate the number of pairs boy-girl both of whom solved some problem difficult for boys. Consider any girl. By the condition (ii), among the six problems she solved, at least one was solved by at least 3 boys, and hence at most 5 were difficult for boys. Since each of these problems was solved by at most 2 boys and there are 21 girls, the considered number of pairs does not exceed $5 \cdot 2 \cdot 21 = 210$.
Similarly, there are at most 210 pairs boy-girl both of whom solved some problem difficult for girls. On the other hand, there are $21^2 > 2 \cdot 210$ pairs boy-girl, and each of them solved one problem in common. Thus some problems were difficult neither for girls nor for boys, as claimed.

Remark. The statement can be generalized: if $2(m-1)(n-1)+1$ boys and as many girls participated, and nobody solved more than m problems, then some problem was solved by at least n boys and n girls.

15. Let $MNPQ$ be the square inscribed in $\triangle ABC$ with $M \in AB$, $N \in AC$, $P, Q \in BC$, and let AA_1 meet MN, PQ at K, X respectively. Put $MK = PX = m$, $NK = QX = n$, and $MN = d$. Then

$$\frac{BX}{XC} = \frac{m}{n} = \frac{BX+m}{XC+n} = \frac{BP}{CQ} = \frac{d\cot\beta + d}{d\cot\gamma + d} = \frac{\cot\beta + 1}{\cot\gamma + 1}.$$

Similarly, if BB_1 and CC_1 meet AC and BC at Y, Z respectively then $\frac{CY}{YA} = \frac{\cot\gamma+1}{\cot\alpha+1}$ and $\frac{AZ}{ZB} = \frac{\cot\alpha+1}{\cot\beta+1}$. Therefore $\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = 1$, so by Ceva's theorem, AX, BY, CZ have a common point.

Second solution. Let A_2 be the center of the square constructed over BC outside $\triangle ABC$. Since this square and the inscribed square corresponding to the side BC are homothetic, A, A_1, and A_2 are collinear. Points B_2, C_2 are analogously defined. Denote the angles $BAA_2, A_2AC, CBB_2, B_2BA, ACC_2, C_2CB$ by $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$. By the law of sines we have

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{\sin(\beta+45^\circ)}{\sin(\gamma+45^\circ)}, \quad \frac{\sin\beta_1}{\sin\beta_2} = \frac{\sin(\gamma+45^\circ)}{\sin(\alpha+45^\circ)}, \quad \frac{\sin\gamma_1}{\sin\gamma_2} = \frac{\sin(\alpha+45^\circ)}{\sin(\beta+45^\circ)}.$$

Since the product of these ratios is 1, by the trigonometric Ceva's theorem AA_2, BB_2, CC_2 are concurrent.

16. Since $\angle OCP = 90^\circ - \angle A$, we are led to showing that $\angle OCP > \angle COP$, i.e., $OP > CP$. By the triangle inequality it suffices to prove $CP < \frac{1}{2}CO$. Let $CO = R$. The law of sines yields $CP = AC\cos\gamma = 2R\sin\beta\cos\gamma < 2R\sin\beta\cos(\beta+30^\circ)$. Finally, we have

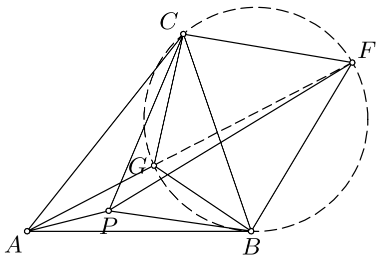
$$2\sin\beta\cos(\beta+30^\circ) = \sin(2\beta+30^\circ) - \sin 30^\circ \le \frac{1}{2},$$

which completes the proof.

17. Let us investigate a more general problem, in which G is any point of the plane such that AG, BG, CG are sides of a triangle.
Let F be the point in the plane such that $BC : CF : FB = AG : BG : CG$ and F, A lie on different sides of BC. Then by Ptolemy's inequality, on $BPCF$ we have $AG \cdot AP + BG \cdot BP + CG \cdot CP = AG \cdot AP + \frac{AG}{BC}(CF \cdot BP + BF \cdot CP) \ge AG \cdot AP + \frac{AG}{BC} BC \cdot PF$. Hence

$$AG \cdot AP + BG \cdot BP + CG \cdot CP \ge AG \cdot AF, \tag{1}$$

where equality holds if and only if P lies on the segment AF and on the circle BCF. Now we return to the case of G the centroid of $\triangle ABC$.
We claim that F is then the point $\widehat{G}$ in which the line AG meets again the circumcircle of $\triangle BGC$. Indeed, if M is the midpoint of AB, by the law of sines we have $\frac{BC}{C\widehat{G}} = \frac{\sin\angle B\widehat{G}C}{\sin\angle CB\widehat{G}} = \frac{\sin\angle BGM}{\sin\angle AGM} = \frac{AG}{BG}$, and similarly $\frac{BC}{B\widehat{G}} = \frac{AG}{CG}$. Thus (1) implies



$$AG \cdot AP + BG \cdot BP + CG \cdot CP \ge AG \cdot A\widehat{G}.$$

It is easily seen from the above considerations that equality holds if and only if $P \equiv G$, and then the (minimum) value of $AG \cdot AP + BG \cdot BP + CG \cdot CP$ equals

$$AG^2 + BG^2 + CG^2 = \frac{a^2 + b^2 + c^2}{3}.$$

Second solution. Notice that $AG \cdot AP \geq \overrightarrow{AG} \cdot \overrightarrow{AP} = \overrightarrow{AG} \cdot (\overrightarrow{AG} + \overrightarrow{PG})$. Summing this inequality with analogous inequalities for $BG \cdot BP$ and $CG \cdot CP$ gives us $AG \cdot AP + BG \cdot BP + CG \cdot CP \geq AG^2 + BG^2 + CG^2 + (\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG}) \cdot \overrightarrow{PG} = AG^2 + BG^2 + CG^2 = \frac{a^2+b^2+c^2}{3}$. Equality holds if and only if $P \equiv Q$.

18. Let $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ denote the angles $\angle MAB$, $\angle MBC$, $\angle MCA$, $\angle MAC$, $\angle MBA$, $\angle MCB$ respectively. Then $\frac{MB' \cdot MC'}{MA^2} = \sin\alpha_1 \sin\alpha_2$, $\frac{MC' \cdot MA'}{MB^2} = \sin\beta_1 \sin\beta_2$, $\frac{MA' \cdot MB'}{MC^2} = \sin\gamma_1 \sin\gamma_2$; hence

$$p(M)^2 = \sin\alpha_1 \sin\alpha_2 \sin\beta_1 \sin\beta_2 \sin\gamma_1 \sin\gamma_2.$$

Since

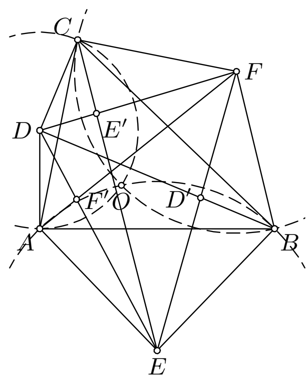
$$\sin\alpha_1 \sin\alpha_2 = \frac{1}{2}(\cos(\alpha_1 - \alpha_2) - \cos(\alpha_1 + \alpha_2) \leq \frac{1}{2}(1 - \cos\alpha) = \sin^2\frac{\alpha}{2},$$

we conclude that

$$p(M) \leq \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2}.$$

Equality occurs when $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, and $\gamma_1 = \gamma_2$, that is, when M is the incenter of $\triangle ABC$.
It is well known that $\mu(ABC) = \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2}$ is maximal when $\triangle ABC$ is equilateral (it follows, for example, from Jensen's inequality applied to $\ln \sin x$). Hence $\max \mu(ABC) = \frac{1}{8}$.

19. It is easy to see that the hexagon $AEBFCD$ is convex and $\angle AEB + \angle BFC + \angle CDA = 360°$. Using this relation we obtain that the circles $\omega_1, \omega_2, \omega_3$ with centers at D, E, F and radii DA, EB, FC respectively all pass through a common point O. Indeed, if $\omega_1 \cap \omega_2 = \{O\}$, then $\angle AOB = 180° - \angle AEB/2$ and $\angle BOC = 180° - \angle BFC/2$; hence $\angle COA = 180° - \angle CDA/2$ as well, i.e., $O \in \omega_3$. The point O is the reflection of A with respect to DE. Similarly, it is also the reflection of B with respect to EF, and that of C with respect to FD. Hence



$$\frac{DB}{DD'} = 1 + \frac{D'B}{DD'} = 1 + \frac{S_{EBF}}{S_{EDF}} = 1 + \frac{S_{OEF}}{S_{DEF}}.$$

Analogously $\frac{EC}{EE'} = 1 + \frac{S_{ODF}}{S_{DEF}}$ and $\frac{FA}{FF'} = 1 + \frac{S_{ODE}}{S_{DEF}}$. Adding these relations gives us

$$\frac{DB}{DD'} + \frac{EC}{EE'} + \frac{FA}{FF'} = 3 + \frac{S_{OEF} + S_{ODF} + S_{ODE}}{S_{DEF}} = 4.$$

20. By Ceva's theorem, we can choose real numbers x, y, z such that

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}} = \frac{z}{y}, \quad \frac{\overrightarrow{CE}}{\overrightarrow{EA}} = \frac{x}{z}, \quad \text{and} \quad \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = \frac{y}{x}.$$

The point P lies outside the triangle ABC if and only if x, y, z are not all of the same sign. In what follows, S_X will denote the signed area of a figure X.

Let us assume that the area S_{ABC} of $\triangle ABC$ is 1. Since $S_{PBC} : S_{PCA} : S_{PAB} = x : y : z$ and $S_{PBD} : S_{PDC} = z : y$, it follows that $S_{PBD} = \frac{z}{y+z}\frac{x}{x+y+z}$. Hence $S_{PBD} = \frac{1}{y(y+z)}\frac{xyz}{x+y+z}$, $S_{PCE} = \frac{1}{z(z+x)}\frac{xyz}{x+y+z}$, $S_{PAF} = \frac{1}{x(x+y)}\frac{xyz}{x+y+z}$. By the condition of the problem we have $|S_{PBD}| = |S_{PCE}| = |S_{PAF}|$, or

$$|x(x+y)| = |y(y+z)| = |z(z+x)|.$$

Obviously x, y, z are nonzero, so that we can put w.l.o.g. $z = 1$. At least two of the numbers $x(x+y), y(y+1), 1(1+x)$ are equal, so we can assume that $x(x+y) = y(y+1)$. We distinguish two cases:

(i) $x(x+y) = y(y+1) = 1+x$. Then $x = y^2 + y - 1$, from which we obtain $(y^2+y-1)(y^2+2y-1) = y(y+1)$. Simplification gives $y^4 + 3y^3 - y^2 - 4y + 1 = 0$, or

$$(y-1)(y^3 + 4y^2 + 3y - 1) = 0.$$

If $y = 1$, then also $z = x = 1$, so P is the centroid of $\triangle ABC$, which is not an exterior point. Hence $y^3 + 4y^2 + 3y - 1 = 0$. Now the signed area of each of the triangles PBD, PCE, PAF equals

$$\begin{aligned} S_{PAF} &= \frac{yz}{(x+y)(x+y+z)} \\ &= \frac{y}{(y^2+2y-1)(y^2+2y)} = \frac{1}{y^3+4y^2+3y-2} = -1. \end{aligned}$$

It is easy to check that not both of x, y are positive, implying that P is indeed outside $\triangle ABC$. This is the desired result.

(ii) $x(x+y) = y(y+1) = -1-x$. In this case we are led to

$$f(y) = y^4 + 3y^3 + y^2 - 2y + 1 = 0.$$

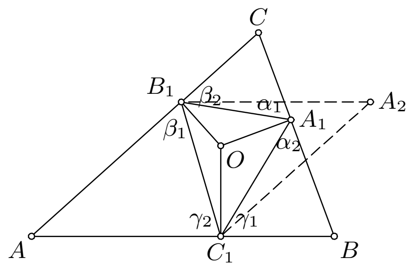
We claim that this equation has no real solutions. In fact, assume that y_0 is a real root of $f(y)$. We must have $y_0 < 0$, and hence $u = -y_0 > 0$ satisfies $3u^3 - u^4 = (u+1)^2$. On the other hand,

$$\begin{aligned}3u^3 - u^4 = u^3(3-u) &= 4u\left(\frac{u}{2}\right)\left(\frac{u}{2}\right)(3-u)\\ &\le 4u\left(\frac{u/2+u/2+3-u}{3}\right)^3 = 4u\\ &\le (u+1)^2,\end{aligned}$$

where at least one of the inequalities is strict, a contradiction.

Remark. The official solution was incomplete, missing the case (ii).

21. We denote by $p(XYZ)$ the perimeter of a triangle XYZ.
If O is the circumcenter of $\triangle ABC$, then A_1, B_1, C_1 are the midpoints of the corresponding sides of the triangle, and hence $p(A_1B_1C_1) = p(AB_1C_1) = p(A_1BC_1) = p(A_1B_1C)$.
Conversely, suppose that $p(A_1B_1C_1) \ge p(AB_1C_1), p(A_1BC_1), p(A_1B_1C)$. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$, γ_2 denote $\angle B_1A_1C$, $\angle C_1A_1B$, $\angle C_1B_1A$, $\angle A_1B_1C$, $\angle A_1C_1B$, $\angle B_1C_1A$.



Suppose that $\gamma_1, \beta_2 \ge \alpha$. If A_2 is the fourth vertex of the parallelogram $B_1AC_1A_2$, then these conditions imply that A_1 is in the interior or on the border of $\triangle B_1C_1A_2$, and therefore $p(A_1B_1C_1) \le p(A_2B_1C_1) = p(AB_1C_1)$. Moreover, if one of the inequalities $\gamma_1 \ge \alpha$, $\beta_2 \ge \alpha$ is strict, then $p(A_1B_1C_1)$ is strictly less than $p(AB_1C_1)$, contrary to the assumption. Hence

$$\begin{aligned}\beta_2 \ge \alpha &\Longrightarrow \gamma_1 \le \alpha,\\ \gamma_2 \ge \beta &\Longrightarrow \alpha_1 \le \beta,\\ \alpha_2 \ge \gamma &\Longrightarrow \beta_1 \le \gamma,\end{aligned} \tag{1}$$

the last two inequalities being obtained analogously to the first one. Because of the symmetry, there is no loss of generality in assuming that $\gamma_1 \le \alpha$. Then since $\gamma_1 + \alpha_2 = 180^\circ - \beta = \alpha + \gamma$, it follows that $\alpha_2 \ge \gamma$. From (1) we deduce $\beta_1 \le \gamma$, which further implies $\gamma_2 \ge \beta$. Similarly, this leads to $\alpha_1 \le \beta$ and $\beta_2 \ge \alpha$. To sum up,

$$\gamma_1 \le \alpha \le \beta_2, \qquad \alpha_1 \le \beta \le \gamma_2, \qquad \beta_1 \le \gamma \le \alpha_2.$$

Since OA_1BC_1 and OB_1CA_1 are cyclic, we have $\angle A_1OB = \gamma_1$ and $\angle A_1OC = \beta_2$. Hence $BO : CO = \cos\beta_2 : \cos\gamma_1$, hence $BO \le CO$. Analogously, $CO \le AO$ and $AO \le BO$. Therefore $AO = BO = CO$, i.e., O is the circumcenter of ABC.

22. Let S and T respectively be the points on the extensions of AB and AQ over B and Q such that $BS = BP$ and $QT = QB$. It is given that $AS = AB + BP = AQ + QB = AT$. Since $\angle PAS = \angle PAT$, the triangles APS

and APT are congruent, from which we deduce that $\angle ATP = \angle ASP = \beta/2 = \angle QBP$. Hence $\angle QTP = \angle QBP$.

If P does not lie on BT, then the last equality implies that $\triangle QBP$ and $\triangle QTP$ are congruent, so P lies on the internal bisector of $\angle BQT$. But P also lies on the internal bisector of $\angle QAB$; consequently, P is an excenter of $\triangle QAB$, thus lying on the internal bisector of $\angle QBS$ as well. It follows that $\angle PBQ = \beta/2 = \angle PBS = 180^\circ - \beta$, so $\beta = 120^\circ$, which is impossible. Therefore $P \in BT$, which means that $T \equiv C$. Now from $QC = QB$ we conclude that $120^\circ - \beta = \gamma = \beta/2$, i.e., $\beta = 80^\circ$ and $\gamma = 40^\circ$.

23. For each positive integer x, define $\alpha(x) = x/10^r$ if r is the positive integer satisfying $10^r \le x < 10^{r+1}$. Observe that if $\alpha(x)\alpha(y) < 10$ for some $x, y \in \mathbb{N}$, then $\alpha(xy) = \alpha(x)\alpha(y)$. If, as usual, $[t]$ means the integer part of t, then $[\alpha(x)]$ is actually the leftmost digit of x.

    Now suppose that n is a positive integer such that $k \le \alpha((n+k)!) < k+1$ for $k = 1, 2, \dots, 9$. We have

$$1 < \alpha(n+k) = \frac{\alpha((n+k)!)}{\alpha((n+k-1)!)} < \frac{k+1}{k-1} \le 3 \qquad \text{for } 2 \le k \le 9,$$

    from which we obtain $\alpha(n+k+1) > \alpha(n+k)$ (the opposite can hold only if $\alpha(n+k) \ge 9$). Therefore

$$1 < \alpha(n+2) < \cdots < \alpha(n+9) \le \frac{5}{4}.$$

    On the other hand, this implies that $\alpha((n+4)!) = \alpha((n+1)!)\alpha(n+2)\alpha(n+3)\alpha(n+4) < (5/4)^3\alpha((n+1)!) < 4$, contradicting the assumption that the leftmost digit of $(n+4)!$ is 4.

24. We shall find the general solution to the system. Squaring both sides of the first equation and subtracting twice the second equation we obtain $(x-y)^2 = z^2 + u^2$. Thus $(z, u, x-y)$ is a Pythagorean triple. Then it is well known that there are positive integers t, a, b such that $z = t(a^2 - b^2)$, $u = 2tab$ (or vice versa), and $x - y = t(a^2 + b^2)$. Using that $x + y = z + u$ we come to the general solution:

$$x = t(a^2 + ab), \qquad y = t(ab - b^2); \qquad\qquad z = t(a^2 - b^2), \qquad u = 2tab.$$

    Putting $a/b = k$ we obtain

$$\frac{x}{y} = \frac{k^2 + k}{k - 1} = 3 + (k-1) + \frac{2}{k-1} \ge 3 + 2\sqrt{2},$$

    with equality for $k - 1 = \sqrt{2}$. On the other hand, k can be arbitrarily close to $1 + \sqrt{2}$, and so x/y can be arbitrarily close to $3 + 2\sqrt{2}$. Hence $m = 3 + 2\sqrt{2}$.

    Remark. There are several other techniques for solving the given system. The exact lower bound of m itself can be obtained as follows: by the system $\left(\frac{x}{y}\right)^2 - 6\frac{x}{y} + 1 = \left(\frac{z-u}{y}\right)^2 \ge 0$, so $x/y \ge 3 + 2\sqrt{2}$.

25. Define $b_n = |a_{n+1} - a_n|$ for $n \geq 1$. From the equalities $a_{n+1} = b_{n-1} + b_{n-2}$, from $a_n = b_{n-2} + b_{n-3}$ we obtain $b_n = |b_{n-1} - b_{n-3}|$. From this relation we deduce that $b_m \leq \max(b_n, b_{n+1}, b_{n+2})$ for all $m \geq n$, and consequently b_n is bounded.

Lemma. If $\max(b_n, b_{n+1}, b_{n+2}) = M \geq 2$, then $\max(b_{n+6}, b_{n+7}, b_{n+8}) \leq M - 1$.

Proof. Assume the opposite. Suppose that $b_j = M$, $j \in \{n, n+1, n+2\}$, and let $b_{j+1} = x$ and $b_{j+2} = y$. Thus $b_{j+3} = M - y$. If $x, y, M - y$ are all less than M, then the contradiction is immediate. The remaining cases are these:

(i) $x = M$. Then the sequence has the form $M, M, y, M - y, y, \dots$, and since $\max(y, M - y, y) = M$, we must have $y = 0$ or $y = M$.

(ii) $y = M$. Then the sequence has the form $M, x, M, 0, x, M - x, \dots$, and since $\max(0, x, M - x) = M$, we must have $x = 0$ or $x = M$.

(iii) $y = 0$. Then the sequence is $M, x, 0, M, M - x, M - x, x, \dots$, and since $\max(M - x, x, x) = M$, we have $x = 0$ or $x = M$.

In every case M divides both x and y. From the recurrence formula M also divides b_i for every $i < j$. However, $b_2 = 12^{12} - 11^{11}$ and $b_4 = 11^{11}$ are relatively prime, a contradiction.

From $\max(b_1, b_2, b_3) \leq 13^{13}$ and the lemma we deduce inductively that $b_n \leq 1$ for all $n \geq 6 \cdot 13^{13} - 5$. Hence $a_n = b_{n-2} + b_{n-3}$ takes only the values 0, 1, 2 for $n \geq 6 \cdot 13^{13} - 2$. In particular, $a_{14^{14}}$ is 0, 1, or 2. On the other hand, the sequence a_n modulo 2 is as follows: $1, 0, 1, 0, 0, 1, 1; 1, 0, 1, 0, \dots$; and therefore it is periodic with period 7. Finally, $14^{14} \equiv 0$ modulo 7, from which we obtain $a_{14^{14}} \equiv a_7 \equiv 1 \pmod 2$. Therefore $a_{14^{14}} = 1$.

26. Let C be the set of those $a \in \{1, 2, \dots, p-1\}$ for which $a^{p-1} \equiv 1 \pmod{p^2}$. At first, we observe that $a, p - a$ do not both belong to C, regardless of the value of a. Indeed, by the binomial formula,

$$(p-a)^{p-1} - a^{p-1} \equiv -(p-1)p\, a^{p-2} \not\equiv 0 \pmod{p^2}.$$

As a consequence we deduce that $|C| \leq \frac{p-1}{2}$. Further, we observe that $p - k \in C \Leftrightarrow k \equiv k(p-k)^{p-1} \pmod{p^2}$, i.e.,

$$p - k \in C \Leftrightarrow k \equiv k(k^{p-1} - (p-1)p\, k^{p-2}) \equiv k^p + p \pmod{p^2}. \qquad (1)$$

Now assume the contrary to the claim, that for every $a = 1, \dots, p-2$ one of $a, a+1$ is in C. In this case it is not possible that $a, a+1$ are both in C, for then $p-a, p-a-1 \notin C$. Thus, since $1 \in C$, we inductively obtain that $2, 4, \dots, p-1 \notin C$ and $1, 3, 5, \dots, p-2 \in C$. In particular, $p-2, p-4 \in C$, which is by (1) equivalent to $2 \equiv 2^p + p$ and $4 \equiv 4^p + p \pmod{p^2}$. However, squaring the former equality and subtracting the latter, we obtain $2^{p+1}p \equiv p \pmod{p^2}$, or $4 \equiv 1 \pmod p$, which is a contradiction unless $p = 3$. This finishes the proof.

27. The given equality is equivalent to $a^2 - ac + c^2 = b^2 + bd + d^2$. Hence $(ab + cd)(ad + bc) = ac(b^2 + bd + d^2) + bd(a^2 - ac + c^2)$, or equivalently,

$$(ab + cd)(ad + bc) = (ac + bd)(a^2 - ac + c^2). \tag{1}$$

Now suppose that $ab + cd$ is prime. It follows from $a > b > c > d$ that

$$ab + cd > ac + bd > ad + bc; \tag{2}$$

hence $ac + bd$ is relatively prime with $ab + cd$. But then (1) implies that $ac + bd$ divides $ad + bc$, which is impossible by (2).

Remark. Alternatively, (1) could be obtained by applying the law of cosines and Ptolemy's theorem on a quadrilateral $XYZT$ with $XY = a$, $YZ = c$, $ZT = b$, $TX = d$ and $\angle Y = 60°$, $\angle T = 120°$.

28. Yes. The desired result is an immediate consequence of the following fact applied on $p = 101$.

Lemma. For any odd prime number p, there exist p nonnegative integers less than $2p^2$ with all pairwise sums mutually distinct.

Proof. We claim that the numbers $a_n = 2np + (n^2)$ have the desired property, where (x) denotes the remainder of x upon division by p. Suppose that $a_k + a_l = a_m + a_n$. By the construction of a_i, we have $2p(k+l) \leq a_k + a_l < 2p(k+l+1)$. Hence we must have $k+l = m+n$, and therefore also $(k^2) + (l^2) = (m^2) + (n^2)$. Thus

$$k + l \equiv m + n \quad \text{and} \quad k^2 + l^2 \equiv m^2 + n^2 \pmod{p}.$$

But then it holds that $(k-l)^2 = 2(k^2+l^2) - (k+l)^2 \equiv (m-n)^2 \pmod{p}$, so $k - l \equiv \pm(m - n)$, which leads to $(k, l) = (m, n)$. This proves the lemma.

4.43 Solutions to the Shortlisted Problems of IMO 2002

1. Consider the given equation modulo 9. Since each cube is congruent to either -1, 0 or 1, whereas $2002^{2002} \equiv 4^{2002} = 4 \cdot 64^{667} \equiv 4 \pmod 9$, we conclude that $t \geq 4$.
   On the other hand, $2002^{2002} = 2002 \cdot (2002^{667})^3 = (10^3 + 10^3 + 1^3 + 1^3)(2002^{667})^3$ is a solution with $t = 4$. Hence the answer is 4.

2. Set $S = d_1d_2 + \cdots + d_{k-1}d_k$. Since $d_i/n = 1/d_{k+1-i}$, we have $\frac{S}{n^2} = \frac{1}{d_kd_{k-1}} + \cdots + \frac{1}{d_2d_1}$. Hence
$$\frac{1}{d_2d_1} \leq \frac{S}{n^2} \leq \left(\frac{1}{d_{k-1}} - \frac{1}{d_k}\right) + \cdots + \left(\frac{1}{d_1} - \frac{1}{d_2}\right) = 1 - \frac{1}{d_k} < 1,$$
   or (since $d_1 = 1$) $1 < \frac{n^2}{S} \leq d_2$. This shows that $S < n^2$.
   Also, if S is a divisor of n^2, then n^2/S is a nontrivial divisor of n^2 not exceeding d_2. But d_2 is obviously the least prime divisor of n (and also of n^2), so we must have $n^2/S = d_2$, which holds if and only if n is prime.

3. We observe that if a, b are coprime odd numbers, then $\gcd(2^a+1, 2^b+1) = 3$. In fact, this g.c.d. divides $\gcd(2^{2a}-1, 2^{2b}-1) = 2^{\gcd(2a,2b)}-1 = 2^2-1 = 3$, while 3 obviously divides both $2^a + 1$ and $2^b + 1$. In particular, if $3 \nmid b$, then $3^2 \nmid 2^b+1$, so 2^a+1 and $(2^b+1)/3$ are coprime; consequently $2^{ab}+1$ (being divisible by $2^a + 1$, $2^b + 1$) is divisible by $\frac{(2^a+1)(2^b+1)}{3}$.
   Now we prove the desired result by induction on n. For $n = 1$, $2^{p_1} + 1$ is divisible by 3 and exceeds 3^2, so it has at least 4 divisors. Assume that $2^a+1 = 2^{p_1\cdots p_{n-1}}+1$ has at least 4^{n-1} divisors and consider $N = 2^{ab}+1 = 2^{p_1\cdots p_n} + 1$ (where $b = p_n$). As above, $2^a + 1$ and $\frac{2^b+1}{3}$ are coprime, and thus $Q = (2^a + 1)(2^b + 1)/3$ has at least $2 \cdot 4^{n-1}$ divisors. Moreover, N is divisible by Q and is greater than Q^2 (indeed, $N > 2^{ab} > 2^{2a}2^{2b} > Q^2$ if $a, b \geq 5$). Then N has at least twice as many divisors as Q (because for every $d \mid Q$ both d and N/d are divisors of N), which counts up to 4^n divisors, as required.

   Remark. With some knowledge of cyclotomic polynomials, one can show that $2^{p_1\cdots p_n} + 1$ has at least $2^{2^{n-1}}$ divisors, far exceeding 4^n.

4. For $a = b = c = 1$ we obtain $m = 12$. We claim that the given equation has infinitely many solutions in positive integers a, b, c for this value of m. After multiplication by $abc(a+b+c)$ the equation $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{abc}-\frac{12}{a+b+c} = 0$ becomes
$$a^2(b + c) + b^2(c + a) + c^2(a + b) + a + b + c - 9abc = 0. \qquad (1)$$
   We must show that this equation has infinitely many solutions in positive integers. Suppose that (a, b, c) is one such solution with $a < b < c$. Regarding (1) as a quadratic equation in a, we see by Vieta's formula that $\left(b, c, \frac{bc+1}{a}\right)$ also satisfies (1).

Define $(a_n)_{n=0}^{\infty}$ by $a_0 = a_1 = a_2 = 1$ and $a_{n+1} = \frac{a_n a_{n-1}+1}{a_{n-2}}$ for each $n > 1$.

We show that all a_n's are integers. This procedure is fairly standard. The above relation for n and $n-1$ gives $a_{n+1}a_{n-2} = a_n a_{n-1} + 1$ and $a_{n-1}a_{n-2} + 1 = a_n a_{n-3}$, so that adding yields $a_{n-2}(a_{n-1} + a_{n+1}) = a_n(a_{n-1} + a_{n-3})$. Therefore $\frac{a_{n+1}+a_{n-1}}{a_n} = \frac{a_{n-1}+a_{n-3}}{a_{n-2}} = \cdots$, from which it follows that

$$\frac{a_{n+1} + a_{n-1}}{a_n} = \begin{cases} \frac{a_2+a_0}{a_1} = 2 \text{ for } n \text{ odd;} \\ \frac{a_3+a_1}{a_2} = 3 \text{ for } n \text{ even.} \end{cases}$$

It is now an immediate consequence that every a_n is integral. Also, the above consideration implies that (a_{n-1}, a_n, a_{n+1}) is a solution of (1) for each $n \geq 1$. Since a_n is strictly increasing, this gives an infinity of solutions in integers.

Remark. There are infinitely many values of $m \in \mathbb{N}$ for which the given equation has at least one solution in integers, and each of those values admits an infinity of solutions.

5. Consider all possible sums $c_1a_1 + c_2a_2 + \cdots + c_na_n$, where each c_i is an integer with $0 \leq c_i < m$. There are m^n such sums, and if any two of them give the same remainder modulo m^n, say $\sum c_ia_i \equiv \sum d_ia_i \pmod{m^n}$, then $\sum(c_i - d_i)a_i$ is divisible by m^n, and since $|c_i - d_i| < m$, we are done. We claim that two such sums must exist.
Suppose to the contrary that the sums $\sum_i c_ia_i$ $(0 \leq c_i < m)$ give all the different remainders modulo m^n. Consider the polynomial

$$P(x) = \sum x^{c_1a_1+\cdots+c_na_n},$$

where the sum is taken over all $(c_1, \ldots, c_n)$ with $0 \leq c_i < m$. If ξ is a primitive m^nth root of unity, then by the assumption we have

$$P(\xi) = 1 + \xi + \cdots + \xi^{m^n-1} = 0.$$

On the other hand, $P(x)$ can be factored as

$$P(x) = \prod_{i=1}^{n}(1 + x^{a_i} + \cdots + x^{(m-1)a_i}) = \prod_{i=1}^{n} \frac{1 - x^{ma_i}}{1 - x^{a_i}},$$

so that none of its factors is zero at $x = \xi$ because ma_i is not divisible by m^n. This is obviously a contradiction.

Remark. The example $a_i = m^{i-1}$ for $i = 1, \ldots, n$ shows that the condition that no a_i is a multiple of m^{n-1} cannot be removed.

6. Suppose that (m, n) is such a pair. Assume that division of the polynomial $F(x) = x^m + x - 1$ by $G(x) = x^n + x^2 - 1$ gives the quotient $Q(x)$ and remainder $R(x)$. Since $\deg R(x) < \deg G(x)$, for x large enough $|R(x)| < |G(x)|$; however, $R(x)$ is divisible by $G(x)$ for infinitely many integers x, so

it is equal to zero infinitely often. Hence $R \equiv 0$, and thus $F(x)$ is exactly divisible by $G(x)$.

The polynomial $G(x)$ has a root α in the interval $(0,1)$, because $G(0) = -1$ and $G(1) = 1$. Then also $F(\alpha) = 0$, so that

$$\alpha^m + \alpha = \alpha^n + \alpha^2 = 1.$$

If $m \geq 2n$, then $1 - \alpha = \alpha^m \leq (\alpha^n)^2 = (1-\alpha^2)^2$, which is equivalent to $\alpha(\alpha-1)(\alpha^2+\alpha-1) \geq 0$. But this last is not possible, because $\alpha^2+\alpha-1 > \alpha^m + \alpha - 1 = 0$; hence $m < 2n$.

Now we have $F(x)/G(x) = x^{m-n} - (x^{m-n+2} - x^{m-n} - x + 1)/G(x)$, so $H(x) = x^{m-n+2} - x^{m-n} - x + 1$ is also divisible by $G(x)$; but $\deg H(x) = m - n + 2 \leq n + 1 = \deg G(x) + 1$, from which we deduce that either $H(x) = G(x)$ or $H(x) = (x-a)G(x)$ for some $a \in \mathbb{Z}$. The former case is impossible. In the latter case we must have $m = 2n - 1$, and thus $H(x) = x^{n+1} - x^{n-1} - x + 1$; on the other hand, putting $x = 1$ gives $a = 1$, so $H(x) = (x-1)(x^n + x^2 - 1) = x^{n+1} - x^n + x^3 - x^2 - x + 1$. This is possible only if $n = 3$ and $m = 5$.

Remark. It is an old (though difficult) result that the polynomial $x^n \pm x^k \pm 1$ is either irreducible or equals $x^2 \pm x + 1$ times an irreducible factor.

7. To avoid working with cases, we use oriented angles modulo $180°$. Let K be the circumcenter of $\triangle BCD$, and X any point on the common tangent to the circles at D. Since the tangents at the ends of a chord are equally inclined to the chord, we have $\angle BAC = \angle ABD + \angle BDC + \angle DCA = \angle BDX + \angle BDC + \angle XDC = 2\angle BDC = \angle BKC$. It follows that B, C, A, K are concyclic, as required.

8. Construct equilateral triangles ACP and ABQ outside the triangle ABC. Since $\angle APC + \angle AFC = 60° + 120° = 180°$, the points A, C, F, P lie on a circle; hence $\angle AFP = \angle ACP = 60° = \angle AFD$, so D lies on the segment FP; similarly, E lies on FQ. Further, note that

$$\frac{FP}{FD} = 1 + \frac{DP}{FD} = 1 + \frac{S_{APC}}{S_{AFC}} \geq 4$$

with equality if F is the midpoint of the smaller arc AC: hence $FD \leq \frac{1}{4}FP$ and $FE \leq \frac{1}{4}FQ$. Then by the law of cosines,

$$\begin{aligned} DE &= \sqrt{FD^2 + FE^2 + FD \cdot FE} \\ &\leq \frac{1}{4}\sqrt{FP^2 + FQ^2 + FP \cdot FQ} = \frac{1}{4}PQ \leq AP + AQ = AB + AC. \end{aligned}$$

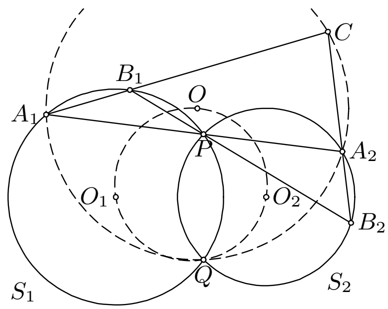
Equality holds if and only if $\triangle ABC$ is equilateral.

9. Since $\angle BCA = \frac{1}{2}\angle BOA = \angle BOD$, the lines CA and OD are parallel, so that $ODAI$ is a parallelogram. It follows that $AI = OD = OE = AE = AF$. Hence

$$\angle IFE = \angle IFA - \angle EFA = \angle AIF - \angle ECA = \angle AIF - \angle ACF = \angle CFI.$$

Also, from $AE = AF$ we get that CI bisects $\angle ECF$. Therefore I is the incenter of $\triangle CEF$.

10. Let O be the circumcenter of A_1A_2C, and O_1, O_2 the centers of S_1, S_2 respectively.
First, from $\angle A_1QA_2 = 180° - \angle PA_1Q - \angle QA_2P = \frac{1}{2}(360° - \angle PO_1Q - \angle QO_2P) = \angle O_1QO_2$ we obtain $\angle A_1QA_2 = \angle B_1QB_2 = \angle O_1QO_2$.
Therefore $\angle A_1QA_2 = \angle B_1QP + \angle PQB_2 = \angle CA_1P + \angle CA_2P = 180° - \angle A_1CA_2$, from which we conclude that Q lies on the circumcircle of $\triangle A_1A_2C$. Hence $OA_1 = OQ$. However, we also have $O_1A_2 = O_1Q$. Consequently, O, O_1 both lie on the perpendicular bisector of A_1Q, so $OO_1 \perp A_1Q$. Similarly, $OO_2 \perp A_2Q$, leading to $\angle O_2OO_1 = 180° - \angle A_1QA_2 = 180° - \angle O_1QO_2$. Hence, O lies on the circle through O_1, O_2, Q, which is fixed.



11. When S is the set of vertices of a regular pentagon, then it is easily verified that $\frac{M(S)}{m(S)} = \frac{1+\sqrt{5}}{2} = \alpha$. We claim that this is the best possible. Let A, B, C, D, E be five arbitrary points, and assume that $\triangle ABC$ has the area $M(S)$. We claim that some triangle has area less than or equal to $M(S)/\alpha$.
Construct a larger triangle $A'B'C'$ with $C \in A'B' \parallel AB$, $A \in B'C' \parallel BC$, $B \in C'A' \parallel CA$. The point D, as well as E, must lie on the same side of $B'C'$ as BC, for otherwise $\triangle DBC$ would have greater area than $\triangle ABC$. A similar result holds for the other edges, and therefore D, E lie inside the triangle $A'B'C'$ or on its boundary. Moreover, at least one of the triangles $A'BC, AB'C, ABC'$, say ABC', contains neither D nor E. Hence we can assume that D, E are contained inside the quadrilateral $A'B'AB$.
An affine linear transformation does not change the ratios between areas. Thus if we apply such an affine transformation mapping A, B, C into the vertices $ABMCN$ of a regular pentagon, we won't change $M(S)/m(S)$. If now D or E lies inside $ABMCN$, then we are done. Suppose that both D and E are inside the triangles CMA', CNB'. Then $CD, CE \le CM$ (because $CM = CN = CA' = CB'$) and $\angle DCE$ is either less than or equal to $36°$ or greater than or equal to $108°$, from which we obtain that the area of $\triangle CDE$ cannot exceed the area of $\triangle CMN = M(S)/\alpha$. This completes the proof.

12. Let $l(MN)$ denote the length of the shorter arc MN of a given circle.

Lemma. Let PR, QS be two chords of a circle k of radius r that meet each other at a point X, and let $\angle PXQ = \angle RXS = 2\alpha$. Then $l(PQ) + l(RS) = 4\alpha r$.

Proof. Let O be the center of the circle. Then $l(PQ) + l(RS) = \angle POQ \cdot r + \angle ROS \cdot r = 2(\angle QSP + \angle RPS)r = 2\angle QXP \cdot r = 4\alpha r$.

Consider a circle k, sufficiently large, whose interior contains all the given circles. For any two circles C_i, C_j, let their exterior common tangents PR, QS $(P, Q, R, S \in k)$ form an angle 2α. Then $O_iO_j = \frac{2}{\sin\alpha}$, so $\alpha > \sin\alpha = \frac{2}{O_iO_j}$. By the lemma we have $l(PQ) + l(RS) = 4\alpha r \geq \frac{8r}{O_iO_j}$, and hence

$$\frac{1}{O_iO_j} \leq \frac{l(PQ) + l(RS)}{8r}. \tag{1}$$

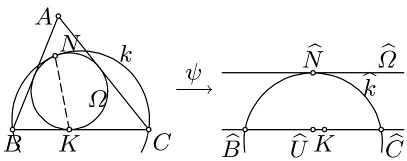
Now sum all these inequalities for $i < j$. The result will follow if we show that every point of the circle k belongs to at most $n-1$ arcs such as PQ, RS. Indeed, that will imply that the sum of all the arcs is at most $2(n-1)\pi r$, hence from (1) we conclude that $\sum \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}$.

Consider an arbitrary point T of k. We prove by induction (the basis $n = 1$ is trivial) that the number of pairs of circles that are simultaneously intercepted by a ray from T is at most $n-1$. Let Tu be a ray touching k at T. If we let this ray rotate around T, it will at some moment intercept a pair of circles for the first time, say C_1, C_2. At some further moment the interception with one of these circles, say C_1, is lost and never established again. Thus the pair (C_1, C_2) is the only pair containing C_1 that is intercepted by some ray from T. On the other hand, by the inductive hypothesis the number of such pairs not containing C_1 does not exceed $n-2$, justifying our claim.

13. Let k be the circle through B, C that is tangent to the circle Ω at point N'. We must prove that K, M, N' are collinear. Since the statement is trivial for $AB = AC$, we may assume that $AC > AB$. As usual, $R, r, \alpha, \beta, \gamma$ denote the circumradius and the inradius and the angles of $\triangle ABC$, respectively.

We have $\tan\angle BKM = DM/DK$. Straightforward calculation gives $DM = \frac{1}{2}AD = R\sin\beta\sin\gamma$ and $DK = \frac{DC-DB}{2} - \frac{KC-KB}{2} = R\sin(\beta-\gamma) - R(\sin\beta - \sin\gamma) = 4R\sin\frac{\beta-\gamma}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$, so we obtain

$$\tan\angle BKM = \frac{\sin\beta\sin\gamma}{4\sin\frac{\beta-\gamma}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} = \frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\beta-\gamma}{2}}.$$



To calculate the angle BKN', we apply the inversion ψ with center at K and power $BK \cdot CK$. For each object X, we denote by $\widehat{X}$ its image under ψ. The incircle Ω maps to a line $\widehat{\Omega}$ parallel to $\widehat{B}\widehat{C}$, at distance $\frac{BK \cdot CK}{2r}$ from $\widehat{B}\widehat{C}$. Thus the point $\widehat{N'}$ is the projection of the midpoint $\widehat{U}$ of $\widehat{B}\widehat{C}$ onto $\widehat{\Omega}$. Hence

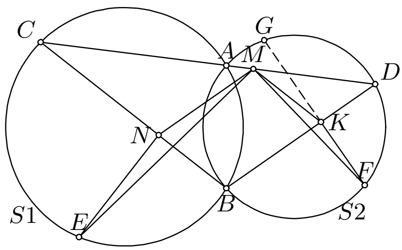
$$\tan \angle BKN' = \tan \angle \widehat{B}K\widehat{N'} = \frac{\widehat{U}\widehat{N'}}{\widehat{U}K} = \frac{BK \cdot CK}{r(CK - BK)}.$$

Again, one easily checks that $KB \cdot KC = bc \sin^2 \frac{\alpha}{2}$ and $r = 4R \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2}$, which implies

$$\begin{aligned} \tan \angle BKN' &= \frac{bc \sin^2 \frac{\alpha}{2}}{r(b-c)} \\ &= \frac{4R^2 \sin \beta \sin \gamma \sin^2 \frac{\alpha}{2}}{4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \cdot 2R(\sin \beta - \sin \gamma)} = \frac{\cos \frac{\beta}{2} \cos \frac{\gamma}{2}}{\sin \frac{\beta - \gamma}{2}}. \end{aligned}$$

Hence $\angle BKM = \angle BKN'$, which implies that K, M, N' are indeed collinear; thus $N' \equiv N$.

14. Let G be the other point of intersection of the line FK with the arc BAD.



Since $BN/NC = DK/KB$ and $\angle CEB = \angle BGD$ the triangles CEB and BGD are similar. Thus $BN/NE = DK/KG = FK/KB$. From $BN = MK$ and $BK = MN$ it follows that $MN/NE = FK/KM$. But we also have that $\angle MNE = 90° + \angle MNB = 90° + \angle MKB = \angle FKM$, and hence $\triangle MNE \sim \triangle FKM$.

Now $\angle EMF = \angle NMK - \angle NME - \angle KMF = \angle NMK - \angle NME - \angle NEM = \angle NMK - 90° + \angle BNM = 90°$ as claimed.

15. We observe first that f is surjective. Indeed, setting $y = -f(x)$ gives $f(f(-f(x)) - x) = f(0) - 2x$, where the right-hand expression can take any real value.

In particular, there exists x_0 for which $f(x_0) = 0$. Now setting $x = x_0$ in the functional equation yields $f(y) = 2x_0 + f(f(y) - x_0)$, so we obtain

$$f(z) = z - x_0 \quad \text{for } z = f(y) - x_0.$$

Since f is surjective, z takes all real values. Hence for all z, $f(z) = z + c$ for some constant c, and this is indeed a solution.

16. For $n \geq 2$, let $(k_1, k_2, \ldots, k_n)$ be the permutation of $\{1, 2, ..., n\}$ with $a_{k_1} \leq a_{k_2} \leq \cdots \leq a_{k_n}$. Then from the condition of the problem, using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}c &\ge a_{k_n} - a_{k_1} = |a_{k_n} - a_{k_{n-1}}| + \cdots + |a_{k_3} - a_{k_2}| + |a_{k_2} - a_{k_1}| \\ &\ge \frac{1}{k_1+k_2} + \frac{1}{k_2+k_3} + \cdots + \frac{1}{k_{n-1}+k_n} \\ &\ge \frac{(n-1)^2}{(k_1+k_2)+(k_2+k_3)+\cdots+(k_{n-1}+k_n)} \\ &= \frac{(n-1)^2}{2(k_1+k_2+\cdots+k_n)-k_1-k_n} \ge \frac{(n-1)^2}{n^2+n-3} \ge \frac{n-1}{n+2}.\end{aligned}$$

Therefore $c \ge 1 - \frac{3}{n+2}$ for every positive integer n. But if $c < 1$, this inequality is obviously false for all $n > \frac{3}{1-c} - 2$. We conclude that $c \ge 1$.

Remark. The least value of c is not greater than $2\ln 2$. An example of a sequence $\{a_n\}$ with $0 \le a_n \le 2\ln 2$ can be constructed inductively as follows: Given $a_1, a_2, \dots, a_{n-1}$, then a_n can be any number from $[0, 2\ln 2]$ that does not belong to any of the intervals $\left(a_i - \frac{1}{i+n}, a_i + \frac{1}{i+n}\right)$ $(i = 1, 2, \dots, n-1)$, and the total length of these intervals is always less than or equal to

$$\frac{2}{n+1} + \frac{2}{n+2} + \cdots + \frac{2}{2n-1} < 2\ln 2.$$

17. Let x, y be distinct integers satisfying $xP(x) = yP(y)$; this is equivalent to $a(x^4 - y^4) + b(x^3 - y^3) + c(x^2 - y^2) + d(x - y) = 0$. Dividing by $x - y$ we obtain

$$a(x^3 + x^2y + xy^2 + y^3) + b(x^2 + xy + y^2) + c(x + y) + d = 0.$$

Putting $x + y = p$, $x^2 + y^2 = q$ leads to $x^2 + xy + y^2 = \frac{p^2+q}{2}$, so the above equality becomes

$$apq + \frac{b}{2}(p^2 + q) + cp + d = 0, \qquad \text{i.e.} \qquad (2ap + b)q = -(bp^2 + 2cp + 2d).$$

Since $q \ge p^2/2$, it follows that $p^2|2ap + b| \le 2|bp^2 + 2cp + 2d|$, which is possible only for finitely many values of p, although there are infinitely many pairs (x, y) with $xP(x) = yP(y)$. Hence there exists p such that $xP(x) = (p - x)P(p - x)$ for infinitely many x, and therefore for all x.
If $p \ne 0$, then p is a root of $P(x)$. If $p = 0$, the above relation gives $P(x) = -P(-x)$. This forces $b = d = 0$, so $P(x) = x(ax^2 + c)$. Thus 0 is a root of $P(x)$.

18. Putting $x = z = 0$ and $t = y$ into the given equation gives $4f(0)f(y) = 2f(0)$ for all y. If $f(0) \ne 0$, then we deduce $f(y) = \frac{1}{2}$, i.e., f is identically equal to $\frac{1}{2}$.
Now we suppose that $f(0) = 0$. Setting $z = t = 0$ we obtain

$$f(xy) = f(x)f(y) \qquad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Thus if $f(y) = 0$ for some $y \ne 0$, then f is identically zero. So, assume $f(y) \ne 0$ whenever $y \ne 0$.

Next, we observe that f is strictly increasing on the set of positive reals. Actually, it follows from (1) that $f(x) = f(\sqrt{x})^2 \geq 0$ for all $x \geq 0$, so that the given equation for $t = x$ and $z = y$ yields $f(x^2+y^2) = (f(x)+f(y))^2 \geq f(x^2)$ for all $x, y \geq 0$.
Using (1) it is easy to get $f(1) = 1$. Now plugging $t = y$ into the given equation, we are led to

$$2[f(x) + f(z)] = f(x - z) + f(x + z) \quad \text{for all } x, z. \tag{2}$$

In particular, $f(z) = f(-z)$. Further, it is easy to get by induction from (2) that $f(nx) = n^2 f(x)$ for all integers n (and consequently for all rational numbers as well). Therefore $f(q) = f(-q) = q^2$ for all $q \in \mathbb{Q}$. But f is increasing for $x > 0$, so we must have $f(x) = x^2$ for all x.
It is easy to verify that $f(x) = 0$, $f(x) = \frac{1}{2}$ and $f(x) = x^2$ are indeed solutions.

19. Write $m = [\sqrt[3]{n}]$. To simplify the calculation, we shall assume that $[b] = 1$. Then $a = \sqrt[3]{n}$, $b = \frac{1}{\sqrt[3]{n}-m} = \frac{1}{n-m^3}\left(m^2 + m\sqrt[3]{n} + \sqrt[3]{n^2}\right)$, $c = \frac{1}{b-1} = u + v\sqrt[3]{n} + w\sqrt[3]{n^2}$ for certain rational numbers u, v, w. Obviously, integers r, s, t with $ra + sb + tc = 0$ exist if (and only if) $u = m^2 w$, i.e., if $(b-1)(m^2 w + v\sqrt[3]{n} + w\sqrt[3]{n^2}) = 1$ for some rational v, w.
When the last equality is expanded and simplified, comparing the coefficients at $1, \sqrt[3]{n}, \sqrt[3]{n^2}$ one obtains

$$\begin{array}{lrr} 1: & v+ & ((m^2+m^3-n)m^2+m)w = n-m^3, \\ \sqrt[3]{n}: & (m^2+m^3-n)v+ & (m^3+n)w = 0, \\ \sqrt[3]{n^2}: & mv+ & (2m^2+m^3-n)w = 0. \end{array} \tag{1}$$

In order for the system (1) to have a solution v, w, we must have $(2m^2 + m^3 - n)(m^2+m^3-n) = m(m^3+n)$. This quadratic equation has solutions $n = m^3$ and $n = m^3 + 3m^2 + m$. The former is not possible, but the latter gives $a - [a] > \frac{1}{2}$, so $[b] = 1$, and the system (1) in v, w is solvable. Hence every number $n = m^3 + 3m^2 + m$, $m \in \mathbb{N}$, satisfies the condition of the problem.

20. Assume to the contrary that $\frac{1}{b_1} + \cdots + \frac{1}{b_n} > 1$. Certainly $n \geq 2$ and A is infinite. Define $f_i : A \to A$ as $f_i(x) = b_i x + c_i$ for each i. By condition (ii), $f_i(x) = f_j(y)$ implies $i = j$ and $x = y$; iterating this argument, we deduce that $f_{i_1}(\dots f_{i_m}(x)\dots) = f_{j_1}(\dots f_{j_m}(x)\dots)$ implies $i_1 = j_1, \dots, i_m = j_m$ and $x = y$.
As an illustration, we shall consider the case $b_1 = b_2 = b_3 = 2$ first. If a is large enough, then for any $i_1, \dots, i_m \in \{1, 2, 3\}$ we have $f_{i_1} \circ \cdots \circ f_{i_m}(a) \leq 2.1^m a$. However, we obtain 3^m values in this way, so they cannot be all distinct if m is sufficiently large, a contradiction.
In the general case, let real numbers $d_i > b_i$, $i = 1, 2 \dots, n$, be chosen such that $\frac{1}{d_1} + \cdots + \frac{1}{d_n} > 1$: for a large enough, $f_i(x) < d_i a$ for each $x \geq a$.

Also, let $k_i > 0$ be arbitrary rational numbers with sum 1; denote by N_0 the least common multiple of their denominators.

Let N be a fixed multiple of N_0, so that each $k_j N$ is an integer. Consider all combinations $f_{i_1} \circ \cdots \circ f_{i_N}$ of N functions, among which each f_i appears exactly $k_i N$ times. There are $F_N = \frac{N!}{(k_1 N)! \cdots (k_n N)!}$ such combinations, so they give F_N distinct values when applied to a. On the other hand, $f_{i_1} \circ \cdots \circ f_{i_N}(a) \le (d_1^{k_1} \cdots d_n^{k_n})^N a$. Therefore

$$(d_1^{k_1} \cdots d_n^{k_n})^N a \ge F_N \qquad \text{for all } N,\ N_0 \mid N. \tag{1}$$

It remains to find a lower estimate for F_N. In fact, it is straightforward to verify that F_{N+N_0}/F_N tends to Q^{N_0}, where $Q = 1/\left(k_1^{k_1} \cdots k_n^{k_n}\right)$. Consequently, for every $q < Q$ there exists $p > 0$ such that $F_N > pq^N$. Then (1) implies that

$$\left(\frac{d_1^{k_1} \cdots d_n^{k_n}}{q}\right)^N > \frac{p}{a} \quad \text{for every multiple } N \text{ of } N_0,$$

and hence $d_1^{k_1} \cdots d_n^{k_n}/q \ge 1$. This must hold for every $q < Q$, and so we have $d_1^{k_1} \cdots d_n^{k_n} \ge Q$, i.e.,

$$(k_1 d_1)^{k_1} \cdots (k_n d_n)^{k_n} \ge 1.$$

However, if we choose $k_1, \dots, k_n$ such that $k_1 d_1 = \cdots = k_n d_n = u$, then we must have $u \ge 1$. Therefore $\frac{1}{d_1} + \cdots + \frac{1}{d_n} \le k_1 + \cdots + k_n = 1$, a contradiction.

21. Let a_i be the number of blue points with x-coordinate i, and b_i the number of blue points with y-coordinate i. Our task is to show that $a_0 a_1 \cdots a_{n-1} = b_0 b_1 \cdots b_{n-1}$. Moreover, we claim that $a_0, \dots, a_{n-1}$ is a permutation of $b_0, \dots, b_{n-1}$, and to show this we use induction on the number of red points.

    The result is trivial if all the points are blue. So, choose a red point (x, y) with $x+y$ maximal: clearly $a_x = b_y = n-x-y-1$. If we change this point to blue, a_x and b_y will decrease by 1. Then by the induction hypothesis, $a_0, \dots, a_{n-1}$ with a_x decreased by 1 is a permutation of $b_0, \dots, b_{n-1}$ with b_y decreased by 1. However, $a_x = b_y$, and the claim follows.

    Remark. One can also use induction on n: it is not more difficult.

22. Write $n = 2k+1$. Consider the black squares at an odd height: there are $(k+1)^2$ of them in total and no two can be covered by one trimino. Thus, we always need at least $(k+1)^2$ triminoes, which cover $3(k+1)^2$ squares in total. However, $3(k+1)^2$ is greater than n^2 for $n = 1, 3, 5$, so we must have $n \ge 7$.

    The case $n = 7$ admits such a covering, as shown in Figure 1. For $n > 7$ this is possible as well: it follows by induction from Figure 2.

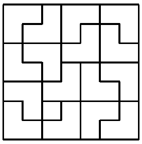


Fig. 1

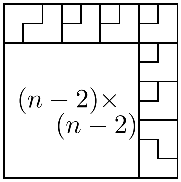


Fig. 2

23. We claim that there are $n!$ full sequences. To show this, we construct a bijection with the set of permutations of $\{1, 2, \dots, n\}$.
Consider a full sequence $(a_1, a_2, \dots, a_n)$, and let m be the greatest of the numbers $a_1, \dots, a_n$. Let S_k, $1 \le k \le m$, be the set of those indices i for which $a_i = k$. Then $S_1, \dots S_m$ are nonempty and form a partition of the set $\{1, 2, \dots, n\}$. Now we write down the elements of S_1 in descending order, then the elements of S_2 in descending order and so on. This maps the full sequence to a permutation of $\{1, 2, \dots, n\}$. Moreover, this map is reversible, since each permutation uniquely breaks apart into decreasing sequences $S_1', S_2', \dots, S_m'$, so that $\max S_i' > \min S_{i-1}'$. Therefore the full sequences are in bijection with the permutations of $\{1, 2, \dots, n\}$.

Second solution. Let there be given a full sequence of length n. Removing from it the first occurrence of the highest number, we obtain a full sequence of length $n-1$. On the other hand, each full sequence of length $n-1$ can be obtained from exactly n full sequences of length n. Therefore, if x_n is the number of full sequences of length n, we deduce $x_n = nx_{n-1}$.

24. Two moves are not sufficient. Indeed, the answer to each move is an even number between 0 and 54, so the answer takes at most 28 distinct values. Consequently, two moves give at most $28^2 = 784$ distinct outcomes, which is less than $10^3 = 1000$.
We now show that three moves are sufficient. With the first move $(0,0,0)$, we get the reply $2(x+y+z)$, so we now know the value of $s = x+y+z$. Now there are several cases:
(i) $s \le 9$. Then we ask $(9,0,0)$ as the second move and get $(9-x-y) + (9-x-z) + (y+z) = 18 - 2x$, so we come to know x. Asking $(0,9,0)$ we obtain y, which is enough, since $z = s - x - y$.
(ii) $10 \le s \le 17$. In this case the second move is $(9, s-9, 0)$. The answer is $z + (9-x) + |x+z-9| = 2k$, where $k = z$ if $x+z \ge 9$, or $k = 9-x$ if $x+z < 9$. In both cases we have $z \le k \le y+z \le s$.
Let $s - k \le 9$. Then in the third move we ask $(s-k, 0, k)$ and obtain $|z-k| + |k-y-z| + y$, which is actually $(k-z) + (y+z-k) + y = 2y$. Thus we also find out $x+z$, and thus deduce whether k is z or $9-x$. Consequently we determine both x and z.
Let $s - k > 9$. In this case, the third move is $(9, s-k-9, k)$. The answer is $|s-k-x-y| + |s-9-y-z| + |k+9-z-x| = (k-z) + (9-x) + (9-x+k-z) = 18 + 2k - 2(x+z)$, from which we find out again whether k is z or $9-x$. Now we are easily done.

(iii) $18 \le s \le 27$. Then as in the first case, asking $(0, 9, 9)$ and $(9, 0, 9)$ we obtain x and y.

25. Assume to the contrary that no set of size less than r meets all sets in $\mathcal{F}$. Consider any set A of size less than r that is contained in infinitely many sets of $\mathcal{F}$. By the assumption, A is disjoint from some set $B \in \mathcal{F}$. Then of the infinitely many sets that contain A, each must meet B, so some element b of B belongs to infinitely many of them. But then the set $A \cup \{b\}$ is contained in infinitely many sets of $\mathcal{F}$ as well.
Such a set A exists: for example, the empty set. Now taking for A the largest such set we come to a contradiction.

26. Write $n = 2m$. We shall define a directed graph G with vertices $1, \dots, m$ and edges labelled $1, 2, \dots, 2m$ in such a way that the edges issuing from i are labelled $2i - 1$ and $2i$, and those entering it are labelled i and $i + m$. What we need is an Euler circuit in G, namely a closed path that passes each edge exactly once. Indeed, if x_i is the ith edge in such a circuit, then x_i enters some vertex j and x_{i+1} leaves it, so $x_i \equiv j \pmod{m}$ and $x_{i+1} = 2j - 1$ or $2j$. Hence $2x_i \equiv 2j$ and $x_{i+1} \equiv 2x_i$ or $2x_i - 1 \pmod{n}$, as required.
The graph G is connected: by induction on k there is a path from 1 to k, since 1 is connected to j with $2j = k$ or $2j - 1 = k$, and there is an edge from j to k. Also, the in-degree and out-degree of each vertex of G are equal (to 2), and thus by a known result, G contains an Euler circuit.

27. For a graph G on 120 vertices (i.e., people at the party), write $q(G)$ for the number of weak quartets in G. Our solution will consist of three parts. First, we prove that some graph G with maximal $q(G)$ breaks up as a disjoint union of complete graphs. This will follow if we show that any two adjacent vertices x, y have the same neighbors (apart from themselves). Let G_x be the graph obtained from G by "copying" x to y (i.e., for each $z \ne x, y$, we add the edge zy if zx is an edge, and delete zy if zx is not an edge). Similarly G_y is the graph obtained from G by copying y to x. We claim that $2q(G) \le q(G_x) + q(G_y)$. Indeed, the number of weak quartets containing neither x nor y is the same in G, G_x, and G_y, while the number of those containing both x and y is not less in G_x and G_y than in G. Also, the number containing exactly one of x and y in G_x is at least twice the number in G containing x but not y, while the number containing exactly one of x and y in G_y is at least twice the number in G containing y but not x. This justifies our claim by adding. It follows that for an extremal graph G we must have $q(G) = q(G_x) = q(G_y)$. Repeating the copying operation pair by pair (y to x, then their common neighbor z to both x, y, etc.) we eventually obtain an extremal graph consisting of disjoint complete graphs.
Second, suppose the complete graphs in G have sizes $a_1, a_2, \dots, a_n$. Then

$$q(G) = \sum_{i=1}^{n} \binom{a_i}{2} \sum_{\substack{j<k \\ j,k \neq i}} a_j a_k.$$

If we fix all the a_i except two, say p, q, then $p + q = s$ is fixed, and for some constants C_i, $q(G) = C_1 + C_2 pq + C_3 \left(\binom{p}{2} + \binom{q}{2}\right) + C_4 \left(q\binom{p}{2} + p\binom{q}{2}\right) = A + Bpq$, where A and B depend only on s. Hence the maximum of $q(G)$ is attained if $|p - q| \leq 1$ or $pq = 0$. Thus if $q(G)$ is maximal, any two nonzero a_i's differ by at most 1.

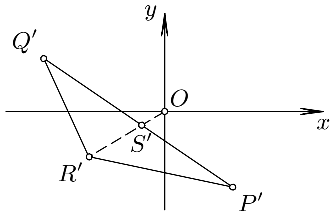
Finally, if G consists of n disjoint complete graphs, then $q(G)$ cannot exceed the value obtained if $a_1 = \cdots = a_n$ (not necessarily integral), which equals

$$Q_n = \frac{120^2}{n} \binom{120/n}{2} \binom{n-1}{2} = 30 \cdot 120^2 \frac{(n-1)(n-2)(120-n)}{n^3}.$$

It is easy to check that Q_n takes its maximum when $n = 5$ and $a_1 = \cdots = a_5 = 24$, and that this maximum equals $15 \cdot 23 \cdot 24^3 = 4769280$.

4.44 Solutions to the Shortlisted Problems of IMO 2003

1. Consider the points $O(0,0,0)$, $P(a_{11},a_{21},a_{31})$, $Q(a_{12},a_{22},a_{32})$, $R(a_{13},a_{23},a_{33})$ in three-dimensional Euclidean space. It is enough to find a point $U(u_1,u_2,u_3)$ in the interior of the triangle PQR whose coordinates are all positive, all negative, or all zero (indeed, then we have $\overrightarrow{OU}=c_1\overrightarrow{OP}+c_2\overrightarrow{OQ}+c_3\overrightarrow{OR}$ for some $c_1,c_2,c_3>0$ with $c_1+c_2+c_3=1$).
   Let $P'(a_{11},a_{21},0)$, $Q'(a_{12},a_{22},0)$, and $R'(a_{13},a_{23},0)$ be the projections of P,Q, and R onto the Oxy plane. We see that P', Q', R' lie in the fourth, second, and third quadrants, respectively. We have the following two cases:
   (i) O is in the exterior of $\triangle P'Q'R'$.
   Set $S'=OR'\cap P'Q'$ and let S be the point of the segment PQ that projects to S'. The point S has its z coordinate negative (because the z coordinates of P and Q are negative). Thus any point of the segment SR sufficiently close to S has all coordinates negative.



   (ii) O is in the interior or on the boundary of $\triangle P'Q'R'$.
   Let T be the point in the plane PQR whose projection is O. If $T=O$, then all coordinates of T are zero, and we are done. Otherwise O is interior to $\triangle P'Q'R'$. Suppose that the z coordinate of T is positive (negative). Since x and y coordinates of T are equal to 0, there is a point U inside PQR close to T with both x and y coordinates positive (respectively negative), and this point U has all coordinates of the same sign.

2. We can rewrite (ii) as $-(f(a)-1)(f(b)-1)=f(-(a-1)(b-1)+1)-1$. So putting $g(x)=f(x+1)-1$, this equation becomes $-g(a-1)g(b-1)=g(-(a-1)(b-1))$ for $a<1<b$. Hence

$$-g(x)g(y)=g(-xy)\text{ for }x<0<y,$$
$$\text{and }g\text{ is nondecreasing with }g(-1)=-1,\ g(0)=0. \tag{1}$$

   Conversely, if g satisfies (1), than f is a solution of our problem.
   Setting $y=1$ in (1) gives $-g(-x)g(1)=g(x)$ for each $x>0$, and therefore (1) reduces to $g(1)g(yz)=g(y)g(z)$ for all $y,z>0$. We have two cases:
   (i) $g(1)=0$. By (1) we have $g(z)=0$ for all $z>0$. Then any nondecreasing function $g:\mathbb{R}\to\mathbb{R}$ with $g(-1)=-1$ and $g(z)=0$ for $z\ge 0$ satisfies (1) and gives a solution: f is nondecreasing, $f(0)=0$ and $f(x)=1$ for every $x\ge 1$
   (ii) $g(1)\neq 0$. Then the function $h(x)=\frac{g(x)}{g(1)}$ is nondecreasing and satisfies $h(0)=0$, $h(1)=1$, and $h(xy)=h(x)h(y)$. Fix $a>0$, and let $h(a)=b=a^k$ for some $k\in\mathbb{R}$. It follows by induction that $h(a^q)=h(a)^q=$

$(a^q)^k$ for every rational number q. But h is nondecreasing, so $k \geq 0$, and since the set $\{a^q \mid q \in \mathbb{Q}\}$ is dense in $\mathbb{R}^+$, we conclude that $h(x) = x^k$ for every $x > 0$. Finally, putting $g(1) = c$, we obtain $g(x) = cx^k$ for all $x > 0$. Then $g(-x) = -x^k$ for all $x > 0$. This g obviously satisfies (1). Hence

$$f(x) = \begin{cases} c(x-1)^k, & \text{if } x > 1; \\ 1, & \text{if } x = 1; \\ 1-(1-x)^k, & \text{if } x < 1, \end{cases} \qquad \text{where } c > 0 \text{ and } k \geq 0.$$

3. (a) Given any sequence c_n (in particular, such that C_n converges), we shall construct a_n and b_n such that A_n and B_n diverge.
First, choose n_1 such that $n_1 c_1 > 1$ and set $a_1 = a_2 = \cdots = a_{n_1} = c_1$: this uniquely determines $b_2 = c_2, \ldots, b_{n_1} = c_{n_1}$. Next, choose n_2 such that $(n_2 - n_1)c_{n_1+1} > 1$ and set $b_{n_1+1} = \cdots = b_{n_2} = c_{n_1+1}$; again $a_{n_1+1}, \ldots, a_{n_2}$ is hereby determined. Then choose n_3 with $(n_3 - n_2)c_{n_2+1} > 1$ and set $a_{n_2+1} = \cdots = a_{n_3} = c_{n_2+1}$, and so on. It is plain that in this way we construct decreasing sequences a_n, b_n such that $\sum a_n$ and $\sum b_n$ diverge, since they contain an infinity of subsums that exceed 1; on the other hand, $c_n = \min(a_n, b_n)$ and C_n is convergent.
(b) The answer changes in this situation. Suppose to the contrary that there is such a pair of sequences (a_n) and (b_n). There are infinitely many indices i such that $c_i = b_i$ (otherwise all but finitely many terms of the sequence (c_n) would be equal to the terms of the sequence (a_n), which has an unbounded sum). Thus for any $n_0 \in \mathbb{N}$ there is $j \geq 2n_0$ such that $c_j = b_j$. Then we have

$$\sum_{k=n_0}^{j} c_k \geq \sum_{k=n_0}^{j} c_j = (j - n_0)\frac{1}{j} \geq \frac{1}{2}.$$

Hence the sequence (C_n) is unbounded, a contradiction.

4. By the Cauchy–Schwarz inequality we have

$$\left(\sum_{i,j=1}^{n} (i-j)^2\right)\left(\sum_{i,j=1}^{n} (x_i - x_j)^2\right) \geq \left(\sum_{i,j=1}^{n} |i-j| \cdot |x_i - x_j|\right)^2. \qquad (1)$$

On the other hand, it is easy to prove (for example by induction) that

$$\sum_{i,j=1}^{n} (i-j)^2 = (2n-2)\cdot 1^2 + (2n-4)\cdot 2^2 + \cdots + 2\cdot(n-1)^2 = \frac{n^2(n^2-1)}{6}$$

and that

$$\sum_{i,j=1}^{n} |i-j| \cdot |x_i - x_j| = \frac{n}{2}\sum_{i,j=1}^{n} |x_i - x_j|.$$

Thus the inequality (1) becomes

$$\frac{n^2(n^2-1)}{6}\left(\sum_{i,j=1}^{n}(x_i-x_j)^2\right) \geq \frac{n^2}{4}\left(\sum_{i,j=1}^{n}|x_i-x_j|\right)^2,$$

which is equivalent to the required one.

5. Placing $x = y = z = 1$ in (i) leads to $4f(1) = f(1)^3$, so by the condition $f(1) > 0$ we get $f(1) = 2$. Also putting $x = ts$, $y = \frac{t}{s}$, $z = \frac{s}{t}$ in (i) gives

$$f(t)f(s) = f(ts) + f(t/s). \tag{1}$$

In particular, for $s = 1$ the last equality yields $f(t) = f(1/t)$; hence $f(t) \geq f(1) = 2$ for each t. It follows that there exists $g(t) \geq 1$ such that $f(t) = g(t) + \frac{1}{g(t)}$. Now it follows by induction from (1) that $g(t^n) = g(t)^n$ for every integer n, and therefore $g(t^q) = g(t)^q$ for every rational q. Consequently, if $t > 1$ is fixed, we have $f(t^q) = a^q + a^{-q}$, where $a = g(t)$. But since the set of a^q ($q \in \mathbb{Q}$) is dense in $\mathbb{R}^+$ and f is monotone on $(0,1]$ and $[1,\infty)$, it follows that $f(t^r) = a^r + a^{-r}$ for every real r. Therefore, if k is such that $t^k = a$, we have

$$f(x) = x^k + x^{-k} \qquad \text{for every } x \in \mathbb{R}.$$

6. Set $X = \max\{x_1, \dots, x_n\}$ and $Y = \max\{y_1, \dots, y_n\}$. By replacing x_i by $x_i' = \frac{x_i}{X}$, y_i by $y_i' = \frac{y_i}{Y}$ and z_i by $z_i' = \frac{z_i}{\sqrt{XY}}$, we may assume that $X = Y = 1$. It is sufficient to prove that

$$M + z_2 + \dots + z_{2n} \geq x_1 + \dots + x_n + y_1 + \dots + y_n, \tag{1}$$

because this implies the result by the A-G mean inequality.
To prove (1) it is enough to prove that for any r, *the number of terms greater than* r *on the left-hand side of* (1) *is at least that number on the right-hand side of* (1).
If $r \geq 1$, then there are no terms on the right-hand side greater than r. Suppose that $r < 1$ and consider the sets $A = \{i \mid 1 \leq i \leq n,\ x_i > r\}$ and $B = \{i \mid 1 \leq i \leq n,\ y_i > r\}$. Set $a = |A|$ and $b = |B|$. If $x_i > r$ and $y_j > r$, then $z_{i+j} \geq \sqrt{x_i y_j} > r$; hence

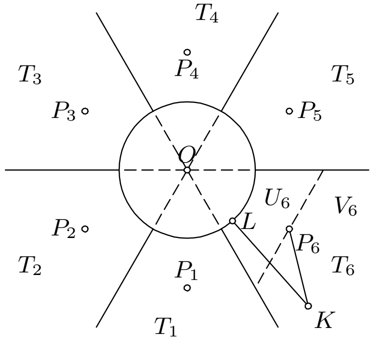
$$C = \{k \mid 2 \leq k \leq 2n,\ z_k > r\} \supseteq A + B = \{\alpha + \beta \mid \alpha \in A, \beta \in B\}.$$

It is easy to verify that $|A+B| \geq |A| + |B| - 1$. It follows that the number of z_k's greater than r is $\geq a + b - 1$. But in that case $M > r$, implying that at least $a + b$ elements of the left-hand side of (1) is greater than r, which completes the proof.

7. Consider the set $D = \{x - y \mid x, y \in A\}$. Obviously, the number of elements of the set D is less than or equal to $101 \cdot 100 + 1$. The sets $A + t_i$ and $A + t_j$

are disjoint if and only if $t_i - t_j \notin D$. Now we shall choose inductively 100 elements $t_1, \dots, t_{100}$.
Let t_1 be any element of the set $S \setminus D$ (such an element exists, since the number of elements of S is greater than the number of elements of D). Suppose now that we have chosen k ($k \le 99$) elements $t_1, \dots, t_k$ from D such that the difference of any two of the chosen elements does not belong to D. We can select t_{k+1} to be an element of S that does not belong to any of the sets $t_1 + D, t_2 + D, \dots, t_k + D$ (this is possible to do, since each of the previous sets has at most $101 \cdot 100 + 1$ elements; hence their union has at most $99(101 \cdot 100 + 1) = 999999 < 1000000$ elements).

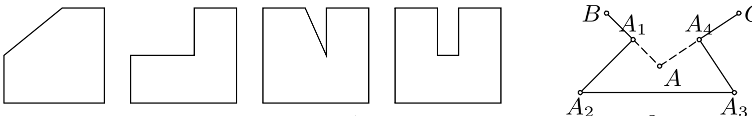
8. Let S be the disk with the smallest radius, say s, and O the center of that disk. Divide the plane into 7 regions: one bounded by disk s and 6 regions $T_1, \dots, T_6$ shown in the figure.
Any of the disks different from S, say D_k, has its center in one of the seven regions. If its center is inside S then D_k contains point O. Hence the number of disks different from S having their centers in S is at most 2002.



Consider a disk D_k that intersects S and whose center is in the region T_i. Let P_i be the point such that OP_i bisects the region T_i and $OP_i = s\sqrt{3}$.
We claim that D_k contains P_i. Divide the region T_i by a line l_i through P_i perpendicular to OP_i into two regions U_i and V_i, where O and U_i are on the same side of l_i. Let K be the center of D_k. Consider two cases:
(i) $K \in U_i$. Since the disk with the center P_i and radius s contains U_i, we see that $KP_i \le s$. Hence D_k contains P_i.
(ii) $K \in V_i$. Denote by L the intersection point of the segment KO with the circle s.
We want to prove that $KL > KP_i$. It is enough to prove that $\angle KP_iL > \angle KLP_i$. However, it is obvious that $\angle LP_iO \le 30^\circ$ and $\angle LOP_i \le 30^\circ$, hence $\angle KLP_i \le 60^\circ$, while $\angle NP_iL = 90^\circ - \angle LP_iO \ge 60^\circ$. This implies that $\angle KP_iL \ge \angle NP_iL \ge 60^\circ \ge \angle KLP_i$ (N is the point on the edge of T_i as shown in the figure). Our claim is thus proved.
Now we see that the number of disks with centers in T_i that intersect S is less than or equal to 2003, and the total number of disks that intersect S is not greater than $2002 + 6 \cdot 2003 = 7 \cdot 2003 - 1$.

9. Suppose that k of the angles of an n-gon are right. Since the other $n-k$ angles are less than 360° and the sum of the angles is $(n-2)180^\circ$, we have

the inequality $k \cdot 90^\circ + (n-k)360^\circ > (n-2)180^\circ$, which is equivalent to $k < \frac{2n+4}{3}$. Since n and k are integers, it follows that $k \le \left[\frac{2n}{3}\right] + 1$.
If $n = 5$, then $\left[\frac{2n}{3}\right] + 1 = 4$, but if a pentagon has four right angles, the other angle is equal to 180°, which is impossible. Hence for $n = 5$, $k \le 3$. It is easy to construct a pentagon with 3 right angles, e.g., as in the picture below.
Now we shall show by induction that for $n \ge 6$ there is an n-gon with $\left[\frac{2n}{3}\right] + 1$ internal right angles. For $n = 6, 7, 8$ examples are presented in the picture. Assume that there is a $(n-3)$gon with $\left[\frac{2(n-3)}{3}\right] + 1 = \left[\frac{2n}{3}\right] - 1$ internal right angles. Then one of the internal angles, say $\angle BAC$, is not convex. Interchange the vertex A with four new vertices A_1, A_2, A_3, A_4 as shown in the picture such that $\angle BA_1A_2 = \angle A_3A_4C = 90^\circ$.



$n = 5$ $n = 6$ $n = 7$ $n = 8$ $n - 3 \to n$

10. Denote by b_{ij} the entries of the matrix B. Suppose the contrary, i.e., that there is a pair (i_0, j_0) such that $a_{i_0,j_0} \ne b_{i_0,j_0}$. We may assume without loss of generality that $a_{i_0,j_0} = 0$ and $b_{i_0,j_0} = 1$.
Since the sums of elements in the i_0th rows of the matrices A and B are equal, there is some j_1 for which $a_{i_0,j_1} = 1$ and $b_{i_0,j_1} = 0$. Similarly, from the fact that the sums in the j_1th columns of the matrices A and B are equal, we conclude that there exists i_1 such that $a_{i_1,j_1} = 0$ and $b_{i_1,j_1} = 1$. Continuing this procedure, we construct two sequences i_k, j_k such that $a_{i_k,j_k} = 0$, $b_{i_k,j_k} = 1$, $a_{i_k,j_{k+1}} = 1$, $b_{i_k,j_{k+1}} = 0$. Since the set of the pairs (i_k, j_k) is finite, there are two different numbers t, s such that $(i_t, j_t) = (i_s, j_s)$. From the given condition we have that $x_{i_k} + y_{i_k} < 0$ and $x_{i_{k+1}} + y_{i_{k+1}} \ge 0$. But $j_t = j_s$, and hence $0 \le \sum_{k=s}^{t-1}(x_{i_k} + y_{j_{k+1}}) = \sum_{k=s}^{t-1}(x_{i_k} + y_{j_k}) < 0$, a contradiction.

11. (a) By the pigeonhole principle there are two different integers x_1, x_2, $x_1 > x_2$, such that $|\{x_1\sqrt{3}\} - \{x_2\sqrt{3}\}| < 0.001$. Set $a = x_1 - x_2$. Consider the equilateral triangle with vertices $(0,0), (2a,0), (a, a\sqrt{3})$. The points $(0,0)$ and $(2a,0)$ are lattice points, and we claim that the point $(a, a\sqrt{3})$ is at distance less than 0.001 from a lattice point. Indeed, since $0.001 > |\{x_1\sqrt{3}\} - \{x_2\sqrt{3}\}| = |a\sqrt{3} - ([x_1\sqrt{3}] - [x_2\sqrt{3}])|$, we see that the distance between the points $(a, a\sqrt{3})$ and $(a, [x_1\sqrt{3}] - [x_2\sqrt{3}])$ is less than 0.001, and the point $(a, [x_1\sqrt{3}] - [x_2\sqrt{3}])$ is with integer coefficients.
(b) Suppose that $P'Q'R'$ is an equilateral triangle with side length $l \le 96$ such that each of its vertices P', Q', R' lies in a disk of radius 0.001 centered at a lattice point. Denote by P, Q, R the centers of these disks. Then we have $l - 0.002 \le PQ, QR, RP \le l + 0.002$. Since PQR is not an equilateral triangle, two of its sides are different, say

$PQ \neq QR$. On the other hand, PQ^2, QR^2 are integers, so we have $1 \leq |PQ^2 - QR^2| = (PQ+QR)|PQ-QR| \leq 0.004(PQ+QR) \leq (2l+0.004)\cdot 0.004 \leq 2\cdot 96.002 \cdot 0.004 < 1$, which is a contradiction.

12. Denote by $\overline{a_{k-1}a_{k-2}\dots a_0}$ the decimal representation of a number whose digits are $a_{k-1},\dots,a_0$. We will use the following well-known fact:

$$\overline{a_{k-1}a_{k-2}\dots a_0} \equiv i \pmod{11} \iff \sum_{l=0}^{k-1}(-1)^l a_l \equiv i \pmod{11}.$$

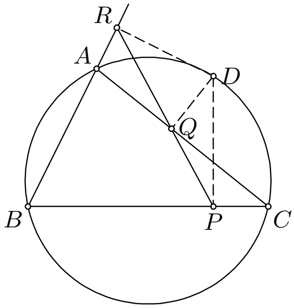
Let m be a positive integer. Define A as the set of integers n $(0 \leq n < 10^{2m})$ whose right $2m-1$ digits can be so permuted to yield an integer divisible by 11, and B as the set of integers n $(0 \leq n < 10^{2m-1})$ whose digits can be permuted resulting in an integer divisible by 11.
Suppose that $a = \overline{a_{2m-1}\dots a_0} \in A$. Then there that satisfies

$$\sum_{l=0}^{2m-1}(-1)^l b_l \equiv 0 \pmod{11}. \tag{1}$$

The $2m$-tuple $(b_{2m-1},\dots,b_0)$ satisfies (1) if and only if the $2m$-tuple $(kb_{2m-1}+l,\dots,kb_0+l)$ satisfies (1), where $k,l \in \mathbb{Z}$, $11 \nmid k$.
Since $a_0+1 \not\equiv 0 \pmod{11}$, we can choose k from the set $\{1,\dots,10\}$ such that $(a_0+1)k \equiv 1 \pmod{11}$. Thus there is a permutation of the $2m$-tuple $((a_{2m-1}+1)k-1,\dots,(a_1+1)k-1,0)$ satisfying (1). Interchanging odd and even positions if necessary, we may assume that this permutation keeps the 0 at the last position. Since $(a_i+1)k$ is not divisible by 11 for any i, there is a unique $b_i \in \{0,1,\dots,9\}$ such that $b_i \equiv (a_i+1)k-1 \pmod{11}$. Hence the number $\overline{b_{2m-1}\dots b_1}$ belongs to B.
Thus for fixed $a_0 \in \{0,1,2,\dots,9\}$, to each $a \in A$ such that the last digit of a is a_0 we associate a unique $b \in B$. Conversely, having $a_0 \in \{0,1,2,\dots,9\}$ fixed, from any number $\overline{b_{2m-1}\dots b_1} \in B$ we can reconstruct $\overline{a_{2m-1}\dots a_1 a_0} \in A$. Hence $|A| = 10|B|$, i.e., $f(2m) = 10f(2m-1)$.

13. Denote by K and L the intersections of the bisectors of $\angle ABC$ and $\angle ADC$ with the line AC, respectively. Since $AB : BC = AK : KC$ and $AD : DC = AL : LC$, we have to prove that

$$PQ = QR \Leftrightarrow \frac{AB}{BC} = \frac{AD}{DC}. \tag{1}$$



Since the quadrilaterals $AQDR$ and $QPCD$ are cyclic, we see that $\angle RDQ = \angle BAC$ and $\angle QDP = \angle ACB$. By the law of sines it follows that $\frac{AB}{BC} = \frac{\sin(\angle ACB)}{\sin(\angle BAC)}$ and that $QR = AD\sin(\angle RDQ)$, $QP = CD\sin(\angle QDP)$. Now we have

$$\frac{AB}{BC} = \frac{\sin(\angle ACB)}{\sin(\angle BAC)} = \frac{\sin(\angle QDP)}{\sin(\angle RDQ)} = \frac{AD \cdot QP}{QR \cdot CD}.$$

The statement (1) follows directly.

14. Denote by R the intersection point of the bisector of $\angle AQC$ and the line AC. From ΔACQ we get

$$\frac{AR}{RC} = \frac{AQ}{QC} = \frac{\sin \angle QCA}{\sin \angle QAC}.$$

By the sine version of Ceva's theorem we have $\frac{\sin \angle APB}{\sin \angle BPC} \cdot \frac{\sin \angle QAC}{\sin \angle PAQ} \cdot \frac{\sin \angle QCP}{\sin \angle QCA} = 1$, which is equivalent to

$$\frac{\sin \angle APB}{\sin \angle BPC} = \left(\frac{\sin \angle QCA}{\sin \angle QAC}\right)^2$$

because $\angle QCA = \angle PAQ$ and $\angle QAC = \angle QCP$. Denote by $S(XYZ)$ the area of a triangle XYZ. Then

$$\frac{\sin \angle APB}{\sin \angle BPC} = \frac{AP \cdot BP \cdot \sin \angle APB}{BP \cdot CP \cdot \sin \angle BPC} = \frac{S(\Delta ABP)}{S(\Delta BCP)} = \frac{AB}{BC},$$

which implies that $\left(\frac{AR}{RC}\right)^2 = \frac{AB}{BC}$. Hence R does not depend on Γ.

15. From the given equality we see that $0 = (BP^2 + PE^2) - (CP^2 + PF^2) = BF^2 - CE^2$, so $BF = CE = x$ for some x. Similarly, there are y and z such that $CD = AF = y$ and $BD = AE = z$. It is easy to verify that D, E, and F must lie on the segments BC, CA, AB.
Denote by a, b, c the length of the segments BC, CA, AB. It follows that $a = z + y$, $b = z + x$, $c = x + y$, so D, E, F are the points where the excircles touch the sides of $\triangle ABC$. Hence P, D, and I_A are collinear and

$$\angle PI_AC = \angle DI_AC = 90^\circ - \frac{180^\circ - \angle ACB}{2} = \frac{\angle ACB}{2}.$$

In the same way we obtain that $\angle PI_BC = \frac{\angle ACB}{2}$ and $PI_B = PI_A$. Analogously, we get $PI_C = PI_B$, which implies that P is the circumcenter of the triangle $I_AI_BI_C$.

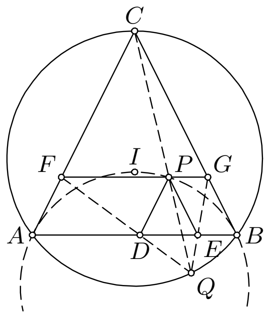
16. Apply an inversion with center at P and radius r; let $\widehat{X}$ denote the image of X. The circles $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are transformed into lines $\widehat{\Gamma_1}, \widehat{\Gamma_2}, \widehat{\Gamma_3}, \widehat{\Gamma_4}$, where $\widehat{\Gamma_1} \parallel \widehat{\Gamma_3}$ and $\widehat{\Gamma_2} \parallel \widehat{\Gamma_4}$, and therefore $\widehat{A}\widehat{B}\widehat{C}\widehat{D}$ is a parallelogram.
Further, we have $AB = \frac{r^2}{P\widehat{A} \cdot P\widehat{B}} \widehat{A}\widehat{B}$, $BC = \frac{r^2}{P\widehat{B} \cdot P\widehat{C}} \widehat{B}\widehat{C}$, $CD = \frac{r^2}{P\widehat{C} \cdot P\widehat{D}} \widehat{C}\widehat{D}$, $DA = \frac{r^2}{P\widehat{D} \cdot P\widehat{A}} \widehat{D}\widehat{A}$ and $PB = \frac{r^2}{P\widehat{B}}$, $PD = \frac{r^2}{P\widehat{D}}$. The equality to be proven becomes

$$\frac{P\widehat{D}^2}{P\widehat{B}^2} \cdot \frac{\widehat{A}\widehat{B} \cdot \widehat{B}\widehat{C}}{\widehat{A}\widehat{D} \cdot \widehat{D}\widehat{C}} = \frac{P\widehat{D}^2}{P\widehat{B}^2},$$

which holds because $\widehat{A}\widehat{B} = \widehat{C}\widehat{D}$ and $\widehat{B}\widehat{C} = \widehat{D}\widehat{A}$.

17. The triangles PDE and CFG are homothetic; hence lines FD, GE, and CP intersect at one point. Let Q be the intersection point of the line CP and the circumcircle of $\triangle ABC$. The required statement will follow if we show that Q lies on the lines GE and FD.
Since $\angle CFG = \angle CBA = \angle CQA$, the quadrilateral $AQPF$ is cyclic. Analogously, $BQPG$ is cyclic. However, the isosceles trapezoid $BDPG$ is also cyclic; it follows that B, Q, D, P, G lie on a circle. Therefore we get

$$\angle PQF = \angle PAC, \ \angle PQD = \angle PBA. \tag{1}$$



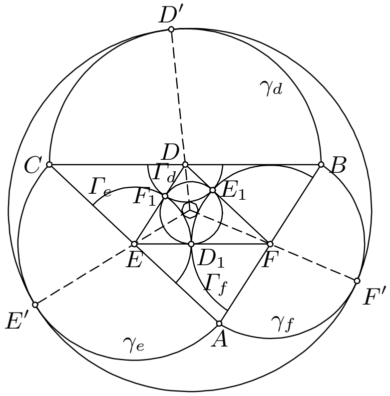
Since I is the incenter of $\triangle ABC$, we have $\angle CAI = \frac{1}{2}\angle CAB = \frac{1}{2}\angle CBA = \angle IBA$; hence CA is the tangent at A to the circumcircle of $\triangle ABI$. This implies that $\angle PAC = \angle PBA$, and it follows from (1) that $\angle PQF = \angle PQD$, i.e., that F, D, Q are also collinear. Similarly, G, E, Q are collinear and the claim is thus proved.

18. Let $ABCDEF$ be the given hexagon. We shall use the following lemma.
Lemma. If $\angle XZY \geq 60^\circ$ and if M is the midpoint of XY, then $MZ \leq \frac{\sqrt{3}}{2}XY$, with equality if and only if $\triangle XYZ$ is equilateral.
Proof. Let Z' be the point such that $\triangle XYZ'$ is equilateral. Then Z is inside the circle circumscribed about $\triangle XYZ'$. Consequently $MZ \leq MZ' = \frac{\sqrt{3}}{2}XY$, with equality if and only if $Z = Z'$.
Set $AD \cap BE = P$, $BE \cap CF = Q$, and $CF \cap AD = R$. Suppose $\angle APB = \angle DPE > 60^\circ$, and let K, L be the midpoints of the segments AB and DE respectively. Then by the lemma,

$$\frac{\sqrt{3}}{2}(AB + DE) = KL \leq PK + PL < \frac{\sqrt{3}}{2}(AB + DE),$$

which is impossible. Therefore $\angle APB \leq 60^\circ$ and similarly $\angle BQC \leq 60^\circ$, $\angle CRD \leq 60^\circ$. But the sum of the angles APB, BQC, CRD is 180°, from which we conclude that these angles are all equal to 60°, and moreover that the triangles APB, BQC, CRD are equilateral. Thus $\angle ABC = \angle ABP + \angle QBC = 120^\circ$, and in the same way all angles of the hexagon are equal to 120°.

19. Let D, E, F be the midpoints of BC, CA, AB, respectively. We construct smaller semicircles $\Gamma_d, \Gamma_e, \Gamma_f$ inside $\triangle ABC$ with centers D, E, F and radii $d = \frac{s-a}{2}$, $e = \frac{s-b}{2}$, $f = \frac{s-c}{2}$ respectively. Since $DE = d + e$, $DF = d + f$, and $EF = e + f$, we deduce that Γ_d, Γ_e, and Γ_f touch each other at the points D_1, E_1, F_1 of tangency of the incircle γ of $\triangle DEF$ with its sides ($D_1 \in EF$, etc.). Consider the circle Γ_g with center O and radius g that lies inside $\triangle DEF$ and tangents $\Gamma_d, \Gamma_e, \Gamma_f$.



Now let OD, OE, OF meet the semicircles $\Gamma_d, \Gamma_e, \Gamma_f$ at D', E', F' respectively. We have $OD' = OD + DD' = g + d + \frac{a}{2} = g + \frac{s}{2}$ and similarly $OE' = OF' = g + \frac{s}{2}$. It follows that the circle with center O and radius $g + \frac{s}{2}$ touches all three semicircles, and consequently $t = g + \frac{s}{2} > \frac{s}{2}$. Now set the coordinate system such that we have the points $D_1(0,0)$, $E(-e,0)$, $F(f,0)$ and such that the y coordinate of D is positive.
Apply the inversion with center D_1 and unit radius. This inversion maps the circles Γ_e and Γ_f to the lines $\widehat{\Gamma_e}\left[x = -\frac{1}{2e}\right]$ and $\widehat{\Gamma_e}\left[x = \frac{1}{2f}\right]$ respectively, and the circle γ goes to the line $\widehat{\gamma}\left[y = \frac{1}{r}\right]$. The images $\widehat{\Gamma_d}$ and $\widehat{\Gamma_g}$ of Γ_d, Γ_g are the circles that touch the lines $\widehat{\Gamma_e}$ and $\widehat{\Gamma_f}$. Since $\widehat{\Gamma_d}$, $\widehat{\Gamma_g}$ are perpendicular to γ, they have radii equal to $R = \frac{1}{4e} + \frac{1}{4f}$ and centers at $\left(-\frac{1}{4e} + \frac{1}{4f}, \frac{1}{r}\right)$ and $\left(-\frac{1}{4e} + \frac{1}{4f}, \frac{1}{r} + 2R\right)$ respectively. Let p and P be the distances from $D_1(0,0)$ to the centers of Γ_g and $\widehat{\Gamma_g}$ respectively. We have that $P^2 = \left(\frac{1}{4e} - \frac{1}{4f}\right)^2 + \left(\frac{1}{r} + 2R\right)^2$, and that the circles Γ_g and $\widehat{\Gamma_g}$ are homothetic with center of homothety D_1; hence $p/P = g/R$. On the other hand, $\widehat{\Gamma_g}$ is the image of Γ_g under inversion; hence the product of the tangents from D_1 to these two circles is equal to 1. In other words, we obtain $\sqrt{p^2 - g^2} \cdot \sqrt{P^2 - R^2} = 1$. Using the relation $p/P = g/R$ we get $g = \frac{R}{P^2 - R^2}$.
The inequality we have to prove is equivalent to $(4 + 2\sqrt{3})g \le r$. This can be proved as follows:

$$\begin{aligned} r - (4+2\sqrt{3})g &= \frac{r(P^2 - R^2 - (4+2\sqrt{3})R/r)}{P^2 - R^2} \\ &= \frac{r\left(\left(\frac{1}{r} + 2R\right)^2 + \left(\frac{1}{4e} - \frac{1}{4f}\right)^2 - R^2 - (4+2\sqrt{3})\frac{R}{r}\right)}{P^2 - R^2} \\ &= \frac{r}{P^2 - R^2}\left(\left(R\sqrt{3} - \frac{1}{r}\right)^2 + \left(\frac{1}{4e} - \frac{1}{4f}\right)^2\right) \ge 0. \end{aligned}$$

Remark. One can obtain a symmetric formula for g:

$$\frac{1}{2g} = \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{2}{r}\,.$$

20. Let r_i be the remainder when x_i is divided by m. Since there are at most m^m types of m-consecutive blocks in the sequence (r_i), some type will

repeat at least twice. Then since the entire sequence is determined by one m-consecutive block, the entire sequence will be periodic.
The formula works both forward and backward; hence using the rule $x_i = x_{i+m} - \sum_{j=1}^{m-1} x_{i+j}$ we can define $x_{-1}, x_{-2}, \dots$. Thus we obtain that

$$(r_{-m}, \dots, r_{-1}) = (0, 0, \dots, 0, 1).$$

Hence there are $m-1$ consecutive terms in the sequence (x_i) that are divisible by m.
If there were m consecutive terms in the sequence (x_i) divisible by m, then by the recurrence relation all the terms of (x_i) would be divisible by m, which is impossible.

21. Let a be a positive integer for which $d(a) = a^2$. Suppose that a has $n+1$ digits, $n \geq 0$. Denote by s the last digit of a and by f the first digit of c. Then $a = \overline{* \dots * s}$, where $*$ stands for a digit that is not important to us at the moment. We have $\overline{* \dots * s}^2 = a^2 = d = \overline{* \dots * f}$ and $b^2 = \overline{s * \dots *}^2 = c = \overline{f * \dots *}$.
We cannot have $s = 0$, since otherwise c would have at most $2n$ digits, while a^2 has either $2n+1$ or $2n+2$ digits. The following table gives all possibilities for s and f:

s	1	2	3	4	5	6	7	8	9
$f =$ last digit of $\overline{* \dots * s}^2$	1	4	9	6	5	6	9	4	1
$f =$ first digit of $\overline{s * \dots *}^2$	$1,2,3$	$4-8$	$9,1$	$1,2$	$2,3$	$3,4$	$4,5,6$	$6,7,8$	$8,9$

We obtain from the table that $s \in \{1,2,3\}$ and $f = s^2$, and consequently $c = b^2$ and d have exactly $2n+1$ digits each. Put $a = 10x + s$, where $x < 10^n$. Then $b = 10^n s + x$, $c = 10^{2n} s^2 + 2 \cdot 10^n sx + x^2$, and $d = 2 \cdot 10^{n+1} sx + 10x^2 + s^2$, so from $d = a^2$ it follows that $x = 2s \cdot \frac{10^n - 1}{9}$. Thus $a = \underbrace{6\dots6}_{n}3$, $a = \underbrace{4\dots4}_{n}2$ or $a = \underbrace{2\dots2}_{n}1$. For $n \geq 1$ we see that a cannot be $a = 6\dots63$ or $a = 4\dots42$ (otherwise a^2 would have $2n+2$ digits). Therefore a equals 1, 2, 3 or $\underbrace{2\dots2}_{n}1$ for $n \geq 0$. It is easy to verify that these numbers have the required property.

22. Let a and b be positive integers for which $\frac{a^2}{2ab^2 - b^3 + 1} = k$ is a positive integer. Since $k > 0$, it follows that $2ab^2 \geq b^3$, so $2a \geq b$. If $2a > b$, then from $2ab^2 - b^3 + 1 > 0$ we see that $a^2 > b^2(2a - b) + 1 > b^2$, i.e. $a > b$. Therefore, if $a \leq b$, then $a = b/2$.
We can rewrite the given equation as a quadratic equation in a, $a^2 - 2kb^2 a + k(b^3 - 1) = 0$, which has two solutions, say a_1 and a_2, one of which is in $\mathbb{N}_0$. From $a_1 + a_2 = 2kb^2$ and $a_1 a_2 = k(b^3 - 1)$ it follows that the other solution is also in $\mathbb{N}_0$. Suppose w.l.o.g. that $a_1 \geq a_2$. Then $a_1 \geq kb^2$ and

$$0 \leq a_2 = \frac{k(b^3 - 1)}{a_1} \leq \frac{k(b^3 - 1)}{kb^2} < b.$$

By the above considerations we have either $a_2 = 0$ or $a_2 = b/2$. If $a_2 = 0$, then $b^3 - 1 = 0$ and hence $a_1 = 2k$, $b = 1$. If $a_2 = b/2$, then $b = 2t$ for some t, and $k = b^2/4$, $a_1 = b^4/2 - b/2$. Therefore the only solutions are

$$(a, b) \in \{(2t, 1),\ (t, 2t),\ (8t^4 - t, 2t) \mid t \in \mathbb{N}\}.$$

It is easy to show that all of these pairs satisfy the given condition.

23. Assume that $b \geq 6$ has the required property. Consider the sequence $y_n = (b-1)x_n$. From the definition of x_n we easily find that $y_n = b^{2n} + b^{n+1} + 3b - 5$. Then $y_n y_{n+1} = (b-1)^2 x_n x_{n+1}$ is a perfect square for all $n > M$. Also, straightforward calculation implies

$$\left(b^{2n+1} + \frac{b^{n+2} + b^{n+1}}{2} - b^3\right)^2 < y_n y_{n+1} < \left(b^{2n+1} + \frac{b^{n+2} + b^{n+1}}{2} + b^3\right)^2.$$

Hence for every $n > M$ there is an integer a_n such that $|a_n| < b^3$ and

$$\begin{aligned} y_n y_{n+1} &= \left(b^{2n} + b^{n+1} + 3b - 5\right)\left(b^{2n+2} + b^{n+2} + 3b - 5\right) \\ &= \left(b^{2n+1} + \frac{b^{n+1}(b+1)}{2} + a_n\right)^2 . \end{aligned} \tag{1}$$

Now considering this equation modulo b^n we obtain $(3b-5)^2 \equiv a_n^2$, so that assuming that $n > 3$ we get $a_n = \pm(3b - 5)$.
If $a_n = 3b - 5$, then substituting in (1) yields $\frac{1}{4}b^{2n}(b^4 - 14b^3 + 45b^2 - 52b + 20) = 0$, with the unique positive integer solution $b = 10$. Also, if $a_n = -3b + 5$, we similarly obtain $\frac{1}{4}b^{2n}(b^4 - 14b^3 - 3b^2 + 28b + 20) - 2b^{n+1}(3b^2 - 2b - 5) = 0$ for each n, which is impossible.
For $b = 10$ it is easy to show that $x_n = \left(\frac{10^n + 5}{3}\right)^2$ for all n. This proves the statement.

Second solution. In problems of this type, computing $z_n = \sqrt{x_n}$ asymptotically usually works.
From $\lim_{n\to\infty} \frac{b^{2n}}{(b-1)x_n} = 1$ we infer that $\lim_{n\to\infty} \frac{b^n}{z_n} = \sqrt{b-1}$. Furthermore, from $(bz_n + z_{n+1})(bz_n - z_{n+1}) = b^2 x_n - x_{n+1} = b^{n+2} + 3b^2 - 2b - 5$ we obtain

$$\lim_{n\to\infty} (bz_n - z_{n+1}) = \frac{b\sqrt{b-1}}{2}.$$

Since the z_n's are integers for all $n \geq M$, we conclude that $bz_n - z_{n+1} = \frac{b\sqrt{b-1}}{2}$ for all n sufficiently large. Hence $b - 1$ is a perfect square, and moreover b divides $2z_{n+1}$ for all large n. It follows that $b \mid 10$; hence the only possibility is $b = 10$.

24. Suppose that $m = u + v + w$ where u, v, w are good integers whose product is a perfect square of an odd integer. Since uvw is an odd perfect square, we have that $uvw \equiv 1 \pmod 4$. Thus either two or none of the numbers

u, v, w are congruent to 3 modulo 4. In both cases $u+v+w \equiv 3 \pmod 4$. Hence $m \equiv 3 \pmod 4$.

Now we shall prove the converse: every $m \equiv 3 \pmod 4$ has infinitely many representations of the desired type. Let $m = 4k+3$. We shall represent m in the form

$$4k+3 = xy + yz + zx, \qquad \text{for } x, y, z \text{ odd.} \tag{1}$$

The product of the summands is an odd square. Set $x = 1 + 2l$ and $y = 1 - 2l$. In order to satisfy (1), z must satisfy $z = 2l^2 + 2k + 1$. The summands xy, yz, zx are distinct except for finitely many l, so it remains only to prove that for infinitely many integers l, $|xy|$, $|yz|$, and $|zx|$ are not perfect squares. First, observe that $|xy| = 4l^2 - 1$ is not a perfect square for any $l \neq 0$.

Let $p, q > m$ be fixed different prime numbers. The system of congruences $1+2l \equiv p \pmod{p^2}$ and $1-2l \equiv q \pmod{q^2}$ has infinitely many solutions l by the Chinese remainder theorem. For any such l, the number $z = 2l^2 + 2k + 1$ is divisible by neither p nor q, and hence $|xz|$ (respectively $|yz|$) is divisible by p, but not by p^2 (respectively by q, but not by q^2). Thus xz and yz are also good numbers.

25. Suppose that for every prime q, there exists an n for which $n^p \equiv p \pmod q$. Assume that $q = kp + 1$. By Fermat's theorem we deduce that $p^k \equiv n^{kp} = n^{q-1} \equiv 1 \pmod q$, so $q \mid p^k - 1$.

It is known that any prime q such that $q \mid \frac{p^p-1}{p-1}$ must satisfy $q \equiv 1 \pmod p$. Indeed, from $q \mid p^{q-1} - 1$ it follows that $q \mid p^{\gcd(p,q-1)} - 1$; but $q \nmid p-1$ because $\frac{p^p-1}{p-1} \equiv 1 \pmod{p-1}$, so $\gcd(p, q-1) \neq 1$. Hence $\gcd(p, q-1) = p$. Now suppose q is any prime divisor of $\frac{p^p-1}{p-1}$. Then $q \mid \gcd(p^k-1, p^p-1) = p^{\gcd(p,k)} - 1$, which implies that $\gcd(p,k) > 1$, so $p \mid k$. Consequently $q \equiv 1 \pmod{p^2}$. However, the number $\frac{p^p-1}{p-1} = p^{p-1} + \cdots + p + 1$ must have at least one prime divisor that is not congruent to 1 modulo p^2. Thus we arrived at a contradiction.

Remark. Taking $q \equiv 1 \pmod p$ is natural, because for every other q, n^p takes all possible residues modulo q (including p too). Indeed, if $p \nmid q-1$, then there is an $r \in \mathbb{N}$ satisfying $pr \equiv 1 \pmod{q-1}$; hence for any a the congruence $n^p \equiv a \pmod q$ has the solution $n \equiv a^r \pmod q$.

The statement of the problem itself is a special case of the Chebotarev's theorem.

26. Define the sequence x_k of positive reals by $a_k = \cosh x_k$ (cosh is the hyperbolic cosine defined by $\cosh t = \frac{e^t + e^{-t}}{2}$). Since $\cosh(2x_k) = 2a_k^2 - 1 = \cosh x_{k+1}$, it follows that $x_{k+1} = 2x_k$ and thus $x_k = \lambda \cdot 2^k$ for some $\lambda > 0$. From the condition $a_0 = 2$ we obtain $\lambda = \log(2 + \sqrt{3})$. Therefore

$$a_n = \frac{(2+\sqrt{3})^{2^n} + (2-\sqrt{3})^{2^n}}{2}.$$

Let p be a prime number such that $p \mid a_n$. We distinguish the following two cases:

(i) There exists an $m \in \mathbb{Z}$ such that $m^2 \equiv 3 \pmod{p}$. Then we have

$$(2+m)^{2^n} + (2-m)^{2^n} \equiv 0 \pmod{p}. \tag{1}$$

Since $(2+m)(2-m) = 4 - m^2 \equiv 1 \pmod{p}$, multiplying both sides of (1) by $(2+m)^{2^n}$ gives $(2+m)^{2^{n+1}} \equiv -1 \pmod{p}$. It follows that the multiplicative order of $(2+m)$ modulo p is 2^{n+2}, or $2^{n+2} \mid p-1$, which implies that $2^{n+3} \mid (p-1)(p+1) = p^2 - 1$.

(ii) $m^2 \equiv 3 \pmod{p}$ has no integer solutions. We will work in the algebraic extension $\mathbb{Z}_p(\sqrt{3})$ of the field $\mathbb{Z}_p$. In this field $\sqrt{3}$ plays the role of m, so as in the previous case we obtain $(2+\sqrt{3})^{2^{n+1}} = -1$; i.e., the order of $2+\sqrt{3}$ in the multiplicative group $\mathbb{Z}_p(\sqrt{3})^*$ is 2^{n+2}. We cannot finish the proof as in the previous case: in fact, we would conclude only that 2^{n+2} divides the order p^2-1 of the group. However, it will be enough to find a $u \in \mathbb{Z}_p(\sqrt{3})$ such that $u^2 = 2+\sqrt{3}$, since then the order of u is equal to 2^{n+3}.
Note that $(1+\sqrt{3})^2 = 2(2+\sqrt{3})$. Thus it is sufficient to prove that $\frac{1}{2}$ is a perfect square in $\mathbb{Z}_p(\sqrt{3})$. But we know that in this field $a_n = 0 = 2a_{n-1}^2 - 1$, and hence $2a_{n-1}^2 = 1$ which implies $\frac{1}{2} = a_{n-1}^2$. This completes the proof.

27. Let $p_1, p_2, \dots, p_r$ be distinct primes, where $r = p-1$. Consider the sets $B_i = \{p_i, p_i^{p+1}, \dots, p_i^{(r-1)p+1}\}$ and $B = \bigcup_{i=1}^r B_i$. Then B has $(p-1)^2$ elements and satisfies (i) and (ii).
Now suppose that $|A| \ge r^2+1$ and that A satisfies (i) and (ii), and let $\{t_1, \dots, t_{r^2+1}\}$ be distinct elements of A, where $t_j = p_1^{\alpha_{j1}} \cdot p_2^{\alpha_{j2}} \cdots p_r^{\alpha_{jr}}$. We shall show that the product of some elements of A is a perfect pth power, i.e., that there exist $\tau_j \in \{0,1\}$ $(1 \le j \le r^2+1)$, not all equal to 0, such that $T = t_1^{\tau_1} \cdot t_2^{\tau_2} \cdots t_{r^2+1}^{\tau_{r^2+1}}$ is a pth power. This is equivalent to the condition that

$$\sum_{j=1}^{r^2+1} \alpha_{ij}\tau_j \equiv 0 \pmod{p}$$

holds for all $i = 1, \dots, r$.
By Fermat's theorem it is sufficient to find integers $x_1, \dots, x_{r^2+1}$, not all zero, such that the relation

$$\sum_{j=1}^{r^2+1} \alpha_{ij} x_j^r \equiv 0 \pmod{p}$$

is satisfied for all $i \in \{1, \dots, r\}$. Set $F_i = \sum_{j=1}^{r^2+1} \alpha_{ij} x_j^r$. We want to find $x_1, \dots, x_r$ such that $F_1 \equiv F_2 \equiv \cdots \equiv F_r \equiv 0 \pmod{p}$, which is by Fermat's theorem equivalent to

$$F(x_1, \ldots, x_r) = F_1^r + F_2^r + \cdots + F_r^r \equiv 0 \pmod{p}. \tag{1}$$

Of course, one solution of (1) is $(0, \ldots, 0)$: we are not satisfied with it because it generates the empty subset of A, but it tells us that (1) has at least one solution.

We shall prove that the number of solutions of (1) is divisible by p, which will imply the existence of a nontrivial solution and thus complete the proof. To do this, consider the sum $\sum F(x_1, \ldots, x_{r^2+1})^r$ taken over all vectors $(x_1, \ldots, x_{r^2+1})$ in the vector space $\mathbb{Z}_p^{r^2+1}$. Our statement is equivalent to

$$\sum F(x_1, \ldots, x_{r^2+1})^r \equiv 0 \pmod{p}. \tag{2}$$

Since the degree of F^r is r^2, in each monomial in F^r at least one of the variables is missing. Consider any of these monomials, say $bx_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$. Then the sum $\sum bx_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$, taken over the set of all vectors $(x_1, \ldots, x_{r^2+1}) \in \mathbb{Z}_p^{r^2+1}$, is equal to

$$p^{r^2+1-u} \cdot \sum_{(x_{i_1}, \ldots, x_{i_k}) \in \mathbb{Z}_p^k} bx_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k},$$

which is divisible by p, so that (2) is proved. Thus the answer is $(p-1)^2$.

4.45 Solutions to the Shortlisted Problems of IMO 2004

1. By symmetry, it is enough to prove that $t_1 + t_2 > t_3$. We have

$$\left(\sum_{i=1}^{n} t_i\right)\left(\sum_{i=1}^{n} \frac{1}{t_i}\right) = n^2 + \sum_{i<j}\left(\frac{t_i}{t_j} + \frac{t_j}{t_i} - 2\right). \tag{1}$$

All the summands on the RHS are positive, and therefore the RHS is not smaller than $n^2 + T$, where $T = (t_1/t_3 + t_3/t_1 - 2) + (t_2/t_3 + t_3/t_2 - 2)$. We note that T is increasing as a function in t_3 for $t_3 \geq \max\{t_1, t_2\}$. If $t_1 + t_2 = t_3$, then $T = (t_1+t_2)(1/t_1+1/t_2) - 1 \geq 3$ by the Cauchy–Schwarz inequality. Hence, if $t_1 + t_2 \leq t_3$, we have $T \geq 1$, and consequently the RHS in (1) is greater than or equal to $n^2 + 1$, a contradiction.

Remark. In can be proved, for example using Lagrange multipliers, that if $n^2 + 1$ in the problem is replaced by $(n + \sqrt{10} - 3)^2$, then the statement remains true. This estimate is the best possible.

2. We claim that the sequence $\{a_n\}$ must be unbounded.
The condition of the sequence is equivalent to $a_n > 0$ and $a_{n+1} = a_n + a_{n-1}$ or $a_n - a_{n-1}$. In particular, if $a_n < a_{n-1}$, then $a_{n+1} > \max\{a_n, a_{n-1}\}$. Let us remove all a_n such that $a_n < a_{n-1}$. The obtained sequence $(b_m)_{m\in\mathbb{N}}$ is strictly increasing. Thus the statement of the problem will follow if we prove that $b_{m+1} - b_m \geq b_m - b_{m-1}$ for all $m \geq 2$.
Let $b_{m+1} = a_{n+2}$ for some n. Then $a_{n+2} > a_{n+1}$. We distinguish two cases:
   (i) If $a_{n+1} > a_n$, we have $b_m = a_{n+1}$ and $b_{m-1} \geq a_{n-1}$ (since b_{m-1} is either a_{n-1} or a_n). Then $b_{m+1} - b_m = a_{n+2} - a_{n+1} = a_n = a_{n+1} - a_{n-1} = b_m - a_{n-1} \geq b_m - b_{m-1}$.
   (ii) If $a_{n+1} < a_n$, we have $b_m = a_n$ and $b_{m-1} \geq a_{n-1}$. Consequently, $b_{m+1} - b_m = a_{n+2} - a_n = a_{n+1} = a_n - a_{n-1} = b_m - a_{n-1} \geq b_m - b_{m-1}$.

3. The answer is *yes*. Every rational number $x > 0$ can be uniquely expressed as a continued fraction of the form $a_0 + 1/(a_1 + 1/(a_2 + 1/(\cdots + 1/a_n)))$ (where $a_0 \in \mathbb{N}_0$, $a_1, \dots, a_n \in \mathbb{N}$). Then we write $x = [a_0; a_1, a_2, \dots, a_n]$. Since n depends only on x, the function $s(x) = (-1)^n$ is well-defined. For $x < 0$ we define $s(x) = -s(-x)$, and set $s(0) = 1$. We claim that this $s(x)$ satisfies the requirements of the problem.
The equality $s(x)s(y) = -1$ trivially holds if $x + y = 0$.
Suppose that $xy = 1$. We may assume w.l.o.g. that $x > y > 0$. Then $x > 1$, so if $x = [a_0; a_1, a_2, \dots, a_n]$, then $a_0 \geq 1$ and $y = 0 + 1/x = [0; a_0, a_1, a_2, \dots, a_n]$. It follows that $s(x) = (-1)^n$, $s(y) = (-1)^{n+1}$, and hence $s(x)s(y) = -1$.
Finally, suppose that $x + y = 1$. We consider two cases:
   (i) Let $x, y > 0$. We may assume w.l.o.g. that $x > 1/2$. Then there exist natural numbers $a_2, \dots, a_n$ such that $x = [0; 1, a_2, \dots, a_n] = 1/(1 + 1/t)$, where $t = [a_2, \dots, a_n]$. Since $y = 1 - x = 1/(1 + t) =$

$[0; 1 + a_2, a_3, \ldots, a_n]$, we have $s(x) = (-1)^n$ and $s(y) = (-1)^{n-1}$, giving us $s(x)s(y) = -1$.

(ii) Let $x > 0 > y$. If $a_0, \ldots, a_n \in \mathbb{N}$ are such that $-y = [a_0; a_1, \ldots, a_n]$, then $x = [1 + a_0; a_1, \ldots, a_n]$. Thus $s(y) = -s(-y) = -(-1)^n$ and $s(x) = (-1)^n$, so again $s(x)s(y) = -1$.

4. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$. For every $x \in \mathbb{R}$ the triple $(a, b, c) = (6x, 3x, -2x)$ satisfies the condition $ab + bc + ca = 0$. Then the condition on P gives us $P(3x) + P(5x) + P(-8x) = 2P(7x)$ for all x, implying that for all $i = 0, 1, 2, \ldots, n$ the following equality holds:

$$\left(3^i + 5^i + (-8)^i - 2 \cdot 7^i\right) a_i = 0.$$

Suppose that $a_i \neq 0$. Then $K(i) = 3^i + 5^i + (-8)^i - 2 \cdot 7^i = 0$. But $K(i)$ is negative for i odd and positive for $i = 0$ or $i \geq 6$ even. Only for $i = 2$ and $i = 4$ do we have $K(i) = 0$. It follows that $P(x) = a_2 x^2 + a_4 x^4$ for some real numbers a_2, a_4.
It is easily verified that all such $P(x)$ satisfy the required condition.

5. By the general mean inequality ($M_1 \leq M_3$), the LHS of the inequality to be proved does not exceed

$$E = \frac{3}{\sqrt[3]{3}} \sqrt[3]{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 6(a + b + c)}.$$

From $ab + bc + ca = 1$ we obtain that $3abc(a + b + c) = 3(ab \cdot ac + ab \cdot bc + ac \cdot bc) \leq (ab + ac + bc)^2 = 1$; hence $6(a + b + c) \leq \frac{2}{abc}$. Since $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc} = \frac{1}{abc}$, it follows that

$$E \leq \frac{3}{\sqrt[3]{3}} \sqrt[3]{\frac{3}{abc}} \leq \frac{1}{abc},$$

where the last inequality follows from the AM–GM inequality $1 = ab + bc + ca \geq 3\sqrt[3]{(abc)^2}$, i.e., $abc \leq 1/(3\sqrt{3})$. The desired inequality now follows. Equality holds if and only if $a = b = c = 1/\sqrt{3}$.

6. Let us make the substitution $z = x + y$, $t = xy$. Given $z, t \in \mathbb{R}$, x, y are real if and only if $4t \leq z^2$. Define $g(x) = 2(f(x) - x)$. Now the given functional equation transforms into

$$f\left(z^2 + g(t)\right) = (f(z))^2 \quad \text{for all} \quad t, z \in \mathbb{R} \quad \text{with} \quad z^2 \geq 4t. \tag{1}$$

Let us set $c = g(0) = 2f(0)$. Substituting $t = 0$ into (1) gives us

$$f(z^2 + c) = (f(z))^2 \quad \text{for all} \quad z \in \mathbb{R}. \tag{2}$$

If $c < 0$, then taking z such that $z^2 + c = 0$, we obtain from (2) that $f(z)^2 = c/2$, which is impossible; hence $c \geq 0$. We also observe that

$$x > c \quad \text{implies} \quad f(x) \geq 0. \tag{3}$$

If g is a constant function, we easily find that $c = 0$ and therefore $f(x) = x$, which is indeed a solution.

Suppose g is nonconstant, and let $a, b \in \mathbb{R}$ be such that $g(a) - g(b) = d > 0$. For some sufficiently large K and each $u, v \geq K$ with $v^2 - u^2 = d$ the equality $u^2 + g(a) = v^2 + g(b)$ by (1) and (3) implies $f(u) = f(v)$. This further leads to $g(u) - g(v) = 2(v - u) = \frac{d}{u + \sqrt{u^2 + d}}$. Therefore every value from some suitably chosen segment $[\delta, 2\delta]$ can be expressed as $g(u) - g(v)$, with u and v bounded from above by some M.

Consider any x, y with $y > x \geq 2\sqrt{M}$ and $\delta < y^2 - x^2 < 2\delta$. By the above considerations, there exist $u, v \leq M$ such that $g(u) - g(v) = y^2 - x^2$, i.e., $x^2 + g(u) = y^2 + g(v)$. Since $x^2 \geq 4u$ and $y^2 \geq 4v$, (1) leads to $f(x)^2 = f(y)^2$. Moreover, if we assume w.l.o.g. that $4M \geq c^2$, we conclude from (3) that $f(x) = f(y)$. Since this holds for any $x, y \geq 2\sqrt{M}$ with $y^2 - x^2 \in [\delta, 2\delta]$, it follows that $f(x)$ is eventually constant, say $f(x) = k$ for $x \geq N = 2\sqrt{M}$. Setting $x > N$ in (2) we obtain $k^2 = k$, so $k = 0$ or $k = 1$.

By (2) we have $f(-z) = \pm f(z)$, and thus $|f(z)| \leq 1$ for all $z \leq -N$. Hence $g(u) = 2f(u) - 2u \geq -2 - 2u$ for $u \leq -N$, which implies that g is unbounded. Hence for each z there exists t such that $z^2 + g(t) > N$, and consequently $f(z)^2 = f(z^2 + g(t)) = k = k^2$. Therefore $f(z) = \pm k$ for each z.

If $k = 0$, then $f(x) \equiv 0$, which is clearly a solution. Assume $k = 1$. Then $c = 2f(0) = 2$ (because $c \geq 0$), which together with (3) implies $f(x) = 1$ for all $x \geq 2$. Suppose that $f(t) = -1$ for some $t < 2$. Then $t - g(t) = 3t + 2 > 4t$. If also $t - g(t) \geq 0$, then for some $z \in \mathbb{R}$ we have $z^2 = t - g(t) > 4t$, which by (1) leads to $f(z)^2 = f(z^2 + g(t)) = f(t) = -1$, which is impossible. Hence $t - g(t) < 0$, giving us $t < -2/3$. On the other hand, if X is any subset of $(-\infty, -2/3)$, the function f defined by $f(x) = -1$ for $x \in X$ and $f(x) = 1$ satisfies the requirements of the problem.

To sum up, the solutions are $f(x) = x$, $f(x) = 0$ and all functions of the form

$$f(x) = \begin{cases} 1, & x \notin X, \\ -1, & x \in X, \end{cases}$$

where $X \subset (-\infty, -2/3)$.

7. Let us set $c_k = A_{k-1}/A_k$ for $k = 1, 2, \ldots, n$, where we define $A_0 = 0$. We observe that $a_k/A_k = (kA_k - (k-1)A_{k-1})/A_k = k - (k-1)c_k$. Now we can write the LHS of the inequality to be proved in terms of c_k, as follows:

$$\sqrt[n]{\frac{G_n}{A_n}} = \sqrt[n^2]{c_2 c_3^2 \cdots c_n^{n-1}} \quad \text{and} \quad \frac{g_n}{G_n} = \sqrt[n]{\prod_{k=1}^{n} (k - (k-1)c_k)}.$$

By the $AM - GM$ inequality we have

$$n \sqrt[n^2]{1^{n(n+1)/2} c_2 c_3^2 \dots c_n^{n-1}} \le \frac{1}{n}\left(\frac{n(n+1)}{2} + \sum_{k=2}^{n}(k-1)c_k\right) \\ = \frac{n+1}{2} + \frac{1}{n}\sum_{k=1}^{n}(k-1)c_k. \tag{1}$$

Also by the AM–GM inequality, we have

$$\sqrt[n]{\prod_{k=1}^{n}(k-(k-1)c_k)} \le \frac{n+1}{2} - \frac{1}{n}\sum_{k=1}^{n}(k-1)c_k. \tag{2}$$

Adding (1) and (2), we obtain the desired inequality. Equality holds if and only if $a_1 = a_2 = \dots = a_n$.

8. Let us write $n = 10001$. Denote by $\mathcal{T}$ the set of ordered triples $(a, C, \mathcal{S})$, where a is a student, C a club, and $\mathcal{S}$ a society such that $a \in C$ and $C \in \mathcal{S}$. We shall count $|\mathcal{T}|$ in two different ways.
Fix a student a and a society $\mathcal{S}$. By (ii), there is a unique club C such that $(a, C, \mathcal{S}) \in \mathcal{T}$. Since the ordered pair $(a, \mathcal{S})$ can be chosen in nk ways, we have that $|\mathcal{T}| = nk$.
Now fix a club C. By (iii), C is in exactly $(|C|-1)/2$ societies, so there are $|C|(|C|-1)/2$ triples from $\mathcal{T}$ with second coordinate C. If $\mathcal{C}$ is the set of all clubs, we obtain $|\mathcal{T}| = \sum_{C\in\mathcal{C}} \frac{|C|(|C|-1)}{2}$. But we also conclude from (i) that
$$\sum_{C\in\mathcal{C}} \frac{|C|(|C|-1)}{2} = \frac{n(n-1)}{2}.$$
Therefore $n(n-1)/2 = nk$, i.e., $k = (n-1)/2 = 5000$.
On the other hand, for $k = (n-1)/2$ there is a desired configuration with only one club C that contains all students and k identical societies with only one element (the club C). It is easy to verify that (i)–(iii) hold.

9. Obviously we must have $2 \le k \le n$. We shall prove that the possible values for k and n are $2 \le k \le n \le 3$ and $3 \le k \le n$. Denote all colors and circles by $1, \dots, n$. Let $F(i, j)$ be the set of colors of the common points of circles i and j.
Suppose that $k = 2 < n$. Consider the ordered pairs (i, j) such that color j appears on the circle i. Since $k = 2$, clearly there are exactly $2n$ such pairs. On the other hand, each of the n colors appears on at least two circles, so there are at least $2n$ pairs (i, j), and equality holds only if each color appears on exactly 2 circles. But then at most two points receive each of the n colors and there are $n(n-1)$ points, implying that $n(n-1) = 2n$, i.e., $n = 3$. It is easy to find examples for $k = 2$ and $n = 2$ or 3.
Next, let $k = 3$. An example for $n = 3$ is given by $F(i, j) = \{i, j\}$ for each $1 \le i < j \le 3$. Assume $n \ge 4$. Then an example is given by $F(1, 2) =$

$\{1,2\}$, $F(i,i+1) = \{i\}$ for $i = 2,\dots,n-2$, $F(n-1,n) = \{n-2,n-1\}$ and $F(i,j) = n$ for all other $i,j > i$.

We now prove by induction on k that a desired coloring exists for each $n \ge k \ge 3$. Let there be given n circles. By the inductive hypothesis, circles $1,2,\dots,n-1$ can be colored in $n-1$ colors, k of which appear on each circle, such that color i appears on circle i. Then we set $F(i,n) = \{i,n\}$ for $i = 1,\dots,k$ and $F(i,n) = \{n\}$ for $i > n$. We thus obtain a coloring of the n circles in n colors, such that $k+1$ colors (including color i) appear on each circle i.

10. The least number of edges of such a graph is n.

    We note that deleting edge AB of a 4-cycle $ABCD$ from a connected and nonbipartite graph G yields a connected and nonbipartite graph, say H. Indeed, the connectedness is obvious; also, if H were bipartite with partition of the set of vertices into P_1 and P_2, then w.l.o.g. $A, C \in P_1$ and $B, D \in P_2$, so $G = H \cup \{AB\}$ would also be bipartite with the same partition, a contradiction.

    Any graph that can be obtained from the complete n-graph in the described way is connected and has at least one cycle (otherwise it would be bipartite); hence it must have at least n edges.

    Now consider a complete graph with vertices $V_1, V_2, \dots, V_n$. Let us remove every edge V_iV_j with $3 \le i < j < n$ from the cycle $V_2V_iV_jV_n$. Then for $i = 3,\dots,n-1$ we remove edges V_2V_i and V_iV_n from the cycles $V_1V_iV_2V_n$ and $V_1V_iV_nV_2$ respectively, thus obtaining a graph with exactly n edges: V_1V_i $(i = 2,\dots,n)$ and V_2V_n.

11. Consider the matrix $A = (a_{ij})_{i,j=1}^n$ such that a_{ij} is equal to 1 if $i, j \le n/2$, -1 if $i, j > n/2$, and 0 otherwise. This matrix satisfies the conditions from the problem and all row sums and column sums are equal to $\pm n/2$. Hence $C \ge n/2$.

    Let us show that $C = n/2$. Assume to the contrary that there is a matrix $B = (b_{ij})_{i,j=1}^n$ all of whose row sums and column sums are either greater than $n/2$ or smaller than $-n/2$. We may assume w.l.o.g. that at least $n/2$ row sums are positive and, permuting rows if necessary, that the first $n/2$ rows have positive sums. The sum of entries in the $n/2 \times n$ submatrix B' consisting of first $n/2$ rows is greater than $n^2/4$, and since each column of B' has sum at most $n/2$, it follows that more than $n/2$ column sums of B', and therefore also of B, are positive. Again, suppose w.l.o.g. that the first $n/2$ column sums are positive. Thus the sums R^+ and C^+ of entries in the first $n/2$ rows and in the first $n/2$ columns respectively are greater than $n^2/4$. Now the sum of all entries of B can be written as

    $$\sum a_{ij} = R^+ + C^+ + \sum_{\substack{i>n/2\\ j>n/2}} a_{ij} - \sum_{\substack{i\le n/2\\ j\le n/2}} a_{ij} > \frac{n^2}{2} - \frac{n^2}{4} - \frac{n^2}{4} = 0,$$

    a contradiction. Hence $C = n/2$, as claimed.

12. We say that a number $n \in \{1, 2, \dots, N\}$ is *winning* if the player who is on turn has a winning strategy, and *losing* otherwise. The game is of type A if and only if 1 is a losing number.

Let us define $n_0 = N$, $n_{i+1} = [n_i/2]$ for $i = 0, 1, \dots$ and let k be such that $n_k = 1$. Consider the sets $A_i = \{n_{i+1} + 1, \dots, n_i\}$. We call a set A_i *all-winning* if all numbers from A_i are winning, *even-winning* if even numbers are winning and odd are losing, and *odd-winning* if odd numbers are winning and even are losing.

(i) Suppose A_i is even-winning and consider A_{i+1}. Multiplying any number from A_{i+1} by 2 yields an even number from A_i, which is a losing number. Thus $x \in A_{i+1}$ is winning if and only if $x + 1$ is losing, i.e., if and only if it is even. Hence A_{i+1} is also even-winning.

(ii) Suppose A_i is odd-winning. Then each $k \in A_{i+1}$ is winning, since $2k$ is losing. Hence A_{i+1} is all-winning.

(iii) Suppose A_i is all-winning. Multiplying $x \in A_{i+1}$ by two is then a losing move, so x is winning if and only if $x+1$ is losing. Since n_{i+1} is losing, A_{i+1} is odd-winning if n_{i+1} is even and even-winning otherwise.

We observe that A_0 is even-winning if N is odd and odd-winning otherwise. Also, if some A_i is even-winning, then all $A_{i+1}, A_{i+2}, \dots$ are even-winning and thus 1 is losing; i.e., the game is of type A. The game is of type B if and only if the sets $A_0, A_1, \dots$ are alternately odd-winning and all-winning with A_0 odd-winning, which is equivalent to $N = n_0, n_2, n_4, \dots$ all being even. Thus N is of type B if and only if all digits at the odd positions in the binary representation of N are zeros.

Since $2004 = \overline{11111010100}$ in the binary system, 2004 is of type A. The least $N > 2004$ that is of type B is $\overline{100000000000} = 2^{11} = 2048$. Thus the answer to part (b) is 2048.

13. Since X_i, Y_i, $i = 1, \dots, 2004$, are 4008 distinct subsets of the set $S_n = \{1, 2, \dots, n\}$, it follows that $2^n \ge 4008$, i.e. $n \ge 12$.

Suppose $n = 12$. Let $\mathcal{X} = \{X_1, \dots, X_{2004}\}$, $\mathcal{Y} = \{Y_1, \dots, Y_{2004}\}$, $\mathcal{A} = \mathcal{X} \cup \mathcal{Y}$. Exactly $2^{12} - 4008 = 88$ subsets of S_n do not occur in $\mathcal{A}$.

Since each row intersects each column, we have $X_i \cap Y_j \neq \emptyset$ for all i, j. Suppose $|X_i|, |Y_j| \le 3$ for some indices i, j. Since then $|X_i \cup Y_j| \le 5$, any of at least $2^7 > 88$ subsets of $S_n \setminus (X_i \cap Y_j)$ can occur in neither $\mathcal{X}$ nor $\mathcal{Y}$, which is impossible. Hence either in $\mathcal{X}$ or in $\mathcal{Y}$ all subsets are of size at least 4. Suppose w.l.o.g. that $k = |X_l| = \min_i |X_i| \ge 4$. There are

$$n_k = \binom{12-k}{0} + \binom{12-k}{1} + \cdots + \binom{12-k}{k-1}$$

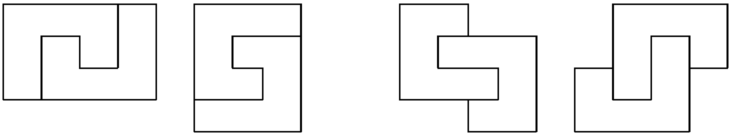
subsets of $S \setminus X_l$ with fewer than k elements, and none of them can be either in $\mathcal{X}$ (because $|X_l|$ is minimal in $\mathcal{X}$) or in $\mathcal{Y}$. Hence we must have $n_k \le 88$. Since $n_4 = 93$ and $n_5 = 99$, it follows that $k \ge 6$. But then none of the $\binom{12}{0} + \cdots + \binom{12}{5} = 1586$ subsets of S_n is in $\mathcal{X}$, hence at least $1586 - 88 = 1498$ of them are in $\mathcal{Y}$. The 1498 complements of these subsets also do not occur in $\mathcal{X}$, which adds to 3084 subsets of S_n not occurring in $\mathcal{X}$. This is clearly a contradiction.

Now we construct a golden matrix for $n = 13$. Let

$$A_1 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A_m = \begin{bmatrix} A_{m-1} & A_{m-1} \\ A_{m-1} & B_{m-1} \end{bmatrix} \text{ for } m = 2, 3, \ldots,$$

where B_{m-1} is the $2^{m-1} \times 2^{m-1}$ matrix with all entries equal to $m+2$. It can be easily proved by induction that each of the matrices A_m is golden. Moreover, every upper-left square submatrix of A_m of size greater than 2^{m-1} is also golden. Since $2^{10} < 2004 < 2^{11}$, we thus obtain a golden matrix of size 2004 with entries in S_{13}.

14. Suppose that an $m \times n$ rectangle can be covered by "hooks". For any hook H there is a unique hook K that covers its " inside" square. Then also H covers the inside square of K, so the set of hooks can be partitioned into pairs of type $\{H, K\}$, each of which forms one of the following two figures consisting of 12 squares:



A_1 $\quad B_1 \quad$ A_2 $\quad B_2$

Thus the $m \times n$ rectangle is covered by these tiles. It immediately follows that $12 \mid mn$.

Suppose one of m, n is divisible by 4. Let w.l.o.g. $4 \mid m$. If $3 \mid n$, one can easily cover the rectangle by 3×4 rectangles and therefore by hooks. Also, if $12 \mid m$ and $n \notin \{1, 2, 5\}$, then there exist $k, l \in \mathbb{N}_0$ such that $n = 3k + 4l$, and thus the rectangle $m \times n$ can be partitioned into 3×12 and 4×12 rectangles all of which can be covered by hooks. If $12 \mid m$ and $n = 1, 2$, or 5, then it is easy to see that covering by hooks is not possible.

Now suppose that $4 \nmid m$ and $4 \nmid n$. Then m, n are even and the number of tiles is odd. Assume that the total number of tiles of types A_1 and B_1 is odd (otherwise the total number of tiles of types A_2 and B_2 is odd, which is analogous). If we color in black all columns whose indices are divisible by 4, we see that each tile of type A_1 or B_1 covers three black squares, which yields an odd number in total. Hence the total number of black squares covered by the tiles of types A_2 and B_2 must be odd. This is impossible, since each such tile covers two or four black squares.

15. Denote by $V_1, \ldots, V_n$ the vertices of a graph G and by E the set of its edges. For each $i = 1, \ldots, n$, let A_i be the set of vertices connected to V_i by an edge, G_i the subgraph of G whose set of vertices is A_i, and E_i the set of edges of G_i. Also, let v_i, e_i, and $t_i = f(G_i)$ be the numbers of vertices, edges, and triangles in G_i respectively.

The numbers of tetrahedra and triangles one of whose vertices is V_i are respectively equal to t_i and e_i. Hence

$$\sum_{i=1}^{n} v_i = 2|E|, \qquad \sum_{i=1}^{n} e_i = 3f(G) \quad \text{and} \quad \sum_{i=1}^{n} t_i = 4g(G).$$

Since $e_i \le v_i(v_i-1)/2 \le v_i^2/2$ and $e_i \le |E|$, we obtain $e_i^2 \le v_i^2|E|/2$, i.e., $e_i \le v_i\sqrt{|E|/2}$. Summing over all i yields $3f(G) \le 2|E|\sqrt{|E|/2}$, or equivalently $f(G)^2 \le 2|E|^3/9$. Since this relation holds for each graph G_i, it follows that

$$t_i = f(G_i) = f(G_i)^{1/3}f(G_i)^{2/3} \le \left(\frac{2}{9}\right)^{1/3} f(G)^{1/3}e_i.$$

Summing the last inequality for $i = 1, \dots, n$ gives us

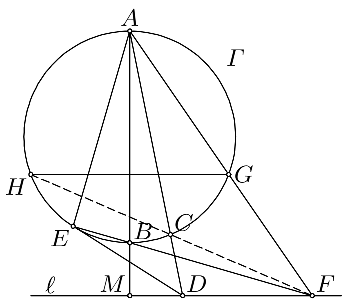
$$4g(G) \le 3\left(\frac{2}{9}\right)^{1/3} f(G)^{1/3}\cdot f(G), \qquad \text{i.e.} \qquad g(G)^3 \le \frac{3}{32}f(G)^4.$$

The constant $c = 3/32$ is the best possible. Indeed, in a complete graph C_n it holds that $g(K_n)^3/f(K_n)^4 = \binom{n}{4}^3\binom{n}{3}^{-4} \to \frac{3}{32}$ as $n \to \infty$.

Remark. Let N_k be the number of complete k-subgraphs in a finite graph G. Continuing inductively, one can prove that $N_{k+1}^k \le \frac{k!}{(k+1)^k}N_k^{k+1}$.

16. Note that $\triangle ANM \sim \triangle ABC$ and consequently $AM \ne AN$. Since $OM = ON$, it follows that OR is a perpendicular bisector of MN. Thus, R is the common point of the median of MN and the bisector of $\angle MAN$. Then it follows from a well-known fact that R lies on the circumcircle of $\triangle AMN$. Let K be the intersection of AR and BC. We then have $\angle MRA = \angle MNA = \angle ABK$ and $\angle NRA = \angle NMA = \angle ACK$, from which we conclude that $RMBK$ and $RNCK$ are cyclic. Thus K is the desired intersection of the circumcircles of $\triangle BMR$ and $\triangle CNR$ and it indeed lies on BC.

17. Let H be the reflection of G about AB ($GH \parallel \ell$). Let M be the intersection of AB and ℓ. Since $\angle FEA = \angle FMA = 90^\circ$, it follows that $AEMF$ is cyclic and hence $\angle DFE = \angle BAE = \angle DEF$. The last equality holds because DE is tangent to Γ. It follows that $DE = DF$ and hence $DF^2 = DE^2 = DC \cdot DA$ (the power of D with respect to Γ). It then follows that $\angle DCF = \angle DFA = \angle HGA = \angle HCA$. Thus it follows that H lies on CF as desired.



18. It is important to note that since $\beta < \gamma$, $\angle ADC = 90^\circ - \gamma + \beta$ is acute. It is elementary that $\angle CAO = 90^\circ - \beta$. Let X and Y respectively be the intersections of FE and GH with AD. We trivially get $X \in EF \perp AD$ and $\triangle AGH \cong \triangle ACB$. Consequently, $\angle GAY = \angle OAB = 90^\circ - \gamma = 90^\circ - \angle AGY$. Hence, $GH \perp AD$ and thus $GH \parallel FE$. That $EFGH$ is a rectangle is now equivalent to $FX = GY$ and $EX = HY$.
We have that $GY = AG \sin\gamma = AC \sin\gamma$ and $FX = AF \sin\gamma$ (since $\angle AFX = \gamma$). Thus,

$$FX = GY \Leftrightarrow CF = AF = AC \Leftrightarrow \angle AFC = 60^\circ \Leftrightarrow \angle ADC = 30^\circ.$$

Since $\angle ADC = 180^\circ - \angle DCA - \angle DAC = 180^\circ - \gamma - (90^\circ - \beta)$, it immediately follows that $FX = GY \Leftrightarrow \gamma - \beta = 60^\circ$. We similarly obtain $EX = HY \Leftrightarrow \gamma - \beta = 60^\circ$, proving the statement of the problem.

19. Assume first that the points A, B, C, D are concyclic. Let the lines BP and DP meet the circumcircle of $ABCD$ again at E and F, respectively. Then it follows from the given conditions that $\widehat{AB} = \widehat{CF}$ and $\widehat{AD} = \widehat{CE}$; hence $BF \parallel AC$ and $DE \parallel AC$. Therefore $BFED$ and $BFAC$ are isosceles trapezoids and thus $P = BE \cap DF$ lies on the common bisector of segments BF, ED, AC. Hence $AP = CP$.
Assume in turn that $AP = CP$. Let P w.l.o.g. lie in the triangles ACD and BCD. Let BP and DP meet AC at K and L, respectively. The points A and C are isogonal conjugates with respect to $\triangle BDP$, which implies that $\angle APK = \angle CPL$. Since $AP = CP$, we infer that K and L are symmetric with respect to the perpendicular bisector p of AC. Let E be the reflection of D in p. Then E lies on the line BP, and the triangles APD and CPE are congruent. Thus $\angle BDC = \angle ADP = \angle BEC$, which means that the points B, C, E, D are concyclic. Moreover, A, C, E, D are also concyclic. Hence, $ABCD$ is a cyclic quadrilateral.

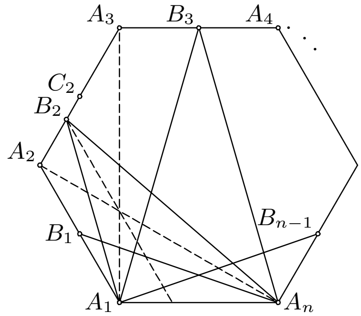
20. We first establish the following lemma.
Lemma. Let $ABCD$ be an isosceles trapezoid with bases AB and CD. The diagonals AC and BD intersect at S. Let M be the midpoint of BC, and let the bisector of the angle BSC intersect BC at N. Then $\angle AMD = \angle AND$.
Proof. It suffices to show that the points A, D, M, N are concyclic. The statement is trivial for $AD \parallel BC$. Let us now assume that AD and BC meet at X, and let $XA = XB = a$, $XC = XD = b$. Since SN is the bisector of $\angle CSB$, we have

$$\frac{a - XN}{XN - b} = \frac{BN}{CN} = \frac{BS}{CS} = \frac{AB}{CD} = \frac{a}{b},$$

and an easy computation yields $XN = \frac{2ab}{a+b}$. We also have $XM = \frac{a+b}{2}$; hence $XM \cdot XN = XA \cdot XD$. Therefore A, D, M, N are concyclic, as needed.

Denote by C_i the midpoint of the side A_iA_{i+1}, $i = 1, \ldots, n-1$. By definition $C_1 = B_1$ and $C_{n-1} = B_{n-1}$. Since $A_1A_iA_{i+1}A_n$ is an isosceles trapezoid with $A_1A_i \parallel A_{i+1}A_n$ for $i = 2, \ldots, n-2$, it follows from the lemma that $\angle A_1B_iA_n = \angle A_1C_iA_n$ for all i.

The sum in consideration thus equals $\angle A_1C_1A_n + \angle A_1C_2A_n + \cdots + \angle A_1C_{n-1}A_n$. Moreover, the triangles $A_1C_iA_n$ and $A_{n+2-i}C_1A_{n+1-i}$ are congruent (a rotation about the center of the n-gon carries the first one to the second), and consequently



$$\angle A_1C_iA_n = \angle A_{n+2-i}C_1A_{n+1-i}$$

for $i = 2, \ldots, n-1$.

Hence $\Sigma = \angle A_1C_1A_n + \angle A_nC_1A_{n-1} + \cdots + \angle A_3C_1A_2 = \angle A_1C_1A_2 = 180°$.

21. Let ABC be the triangle of maximum area S contained in $\mathcal{P}$ (it exists because of compactness of $\mathcal{P}$). Draw parallels to BC, CA, AB through A, B, C, respectively, and denote the triangle thus obtained by $A_1B_1C_1$ ($A \in B_1C_1$, etc.). Since each triangle with vertices in $\mathcal{P}$ has area at most S, the entire polygon $\mathcal{P}$ is contained in $A_1B_1C_1$.

    Next, draw lines of support of $\mathcal{P}$ parallel to BC, CA, AB and not intersecting the triangle ABC. They determine a convex hexagon $U_aV_aU_bV_bU_cV_c$ containing $\mathcal{P}$, with $V_b, U_c \in B_1C_1$, $V_c, U_a \in C_1A_1$, $V_a, U_b \in A_1B_1$. Each of the line segments U_aV_a, U_bV_b, U_cV_c contains points of $\mathcal{P}$. Choose such points A_0, B_0, C_0 on U_aV_a, U_bV_b, U_cV_c, respectively. The convex hexagon $AC_0BA_0CB_0$ is contained in $\mathcal{P}$, because the latter is convex. We prove that $AC_0BA_0CB_0$ has area at least $3/4$ the area of $\mathcal{P}$.

    Let x, y, z denote the areas of triangles U_aBC, U_bCA, and U_cAB. Then $S_1 = S_{AC_0BA_0CB_0} = S + x + y + z$. On the other hand, the triangle $A_1U_aV_a$ is similar to $\triangle A_1BC$ with similitude $\tau = (S-x)/S$, and hence its area is $\tau^2 S = (S-x)^2/S$. Thus the area of quadrilateral U_aV_aCB is $S - (S-x)^2/S = 2z - z^2/S$. Analogous formulas hold for quadrilaterals U_bV_bAC and U_cV_cBA. Therefore

$$\begin{aligned} S_{\mathcal{P}} \le S_{U_aV_aU_bV_bU_cV_c} &= S + S_{U_aV_aCB} + S_{U_bV_bAC} + S_{U_cV_cBA} \\ &= S + 2(x+y+z) - \frac{x^2+y^2+z^2}{S} \\ &\le S + 2(x+y+z) - \frac{(x+y+z)^2}{3S}. \end{aligned}$$

    Now $4S_1 - 3S_{\mathcal{P}} \ge= S - 2(x+y+z) + (x+y+z)^2/S = (S-x-y-z)^2/S \ge 0$; i.e., $S_1 \ge 3S_{\mathcal{P}}/4$, as claimed.

22. The proof uses the following observation:

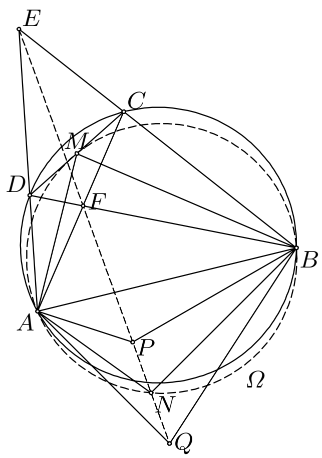
Lemma. In a triangle ABC, let K, L be the midpoints of the sides AC, AB, respectively, and let the incircle of the triangle touch BC, CA at D, E, respectively. Then the lines KL and DE intersect on the bisector of the angle ABC.

Proof. Let the bisector ℓ_b of $\angle ABC$ meet DE at T. One can assume that $AB \neq BC$, or else $T \equiv K \in KL$. Note that the incenter I of $\triangle ABC$ is between B and T, and also $T \neq E$. From the triangles BDT and DEC we obtain $\angle ITD = \alpha/2 = \angle IAE$, which implies that A, I, T, E are concyclic. Then $\angle ATB = \angle AEI = 90^\circ$. Thus L is the circumcenter of $\triangle ATB$ from which $\angle LTB = \angle LBT = \angle TBC \Rightarrow LT \parallel BC \Rightarrow T \in KL$, which is what we were supposed to prove.

Let the incircles of $\triangle ABX$ and $\triangle ACX$ touch BX at D and F, respectively, and let them touch AX at E and G, respectively. Clearly, DE and FG are parallel. If the line PQ intersects BX and AX at M and N, respectively, then $MD^2 = MP \cdot MQ = MF^2$, i.e., $MD = MF$ and analogously $NE = NG$. It follows that PQ is parallel to DE and FG and equidistant from them.

The midpoints of AB, AC, and AX lie on the same line m, parallel to BC. Applying the lemma to $\triangle ABX$, we conclude that DE passes through the common point U of m and the bisector of $\angle ABX$. Analogously, FG passes through the common point V of m and the bisector of $\angle ACX$. Therefore PQ passes through the midpoint W of the line segment UV. Since U, V do not depend on X, neither does W.

23. To start with, note that point N is uniquely determined by the imposed properties. Indeed, $f(X) = AX/BX$ is a monotone function on both arcs AB of the circumcircle of $\triangle ABM$.



Denote by P and Q respectively the second points of intersection of the line EF with the circumcircles of $\triangle ABE$ and $\triangle ABF$. The problem is equivalent to showing that $N \in PQ$. In fact, we shall prove that N coincides with the midpoint $\overline{N}$ of segment PQ.

The cyclic quadrilaterals $APBE$, $AQBF$, and $ABCD$ yield $\angle APQ = 180^\circ - \angle APE = 180^\circ - \angle ABE = \angle ADC$ and $\angle AQP = \angle AQF = \angle ABF = \angle ACD$. It follows that $\triangle APQ \sim \triangle ADC$, and consequently $\triangle A\overline{N}P \sim \triangle AMD$. Analogously $\triangle B\overline{N}P \sim \triangle BMC$. Therefore $A\overline{N}/AM = PQ/DC = B\overline{N}/BM$, i.e., $A\overline{N}/B\overline{N} = AM/BM$. Moreover, $\angle A\overline{N}B = \angle A\overline{N}P + \angle P\overline{N}B = \angle AMD + \angle BMC = 180^\circ - \angle AMB$, which means that point $\overline{N}$ lies on the circumcircle of $\triangle AMB$. By the uniqueness of N, we conclude that $\overline{N} \equiv N$, which completes the solution.

24. Setting $m = an$ we reduce the given equation to $m/\tau(m) = a$.
Let us show that for $a = p^{p-1}$ the above equation has no solutions in $\mathbb{N}$ if $p > 3$ is a prime. Assume to the contrary that $m \in \mathbb{N}$ is such that $m = p^{p-1}\tau(m)$. Then $p^{p-1} \mid m$, so we may set $m = p^\alpha k$, where $\alpha, k \in \mathbb{N}$, $\alpha \geq p-1$, and $p \nmid k$. Let $k = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the decomposition of k into primes. Then $\tau(k) = (\alpha_1 + 1) \cdots (\alpha_r + 1)$ and $\tau(m) = (\alpha + 1)\tau(k)$. Our equation becomes
$$p^{\alpha-p+1}k = (\alpha+1)\tau(k). \tag{1}$$
We observe that $\alpha \neq p-1$: otherwise the RHS would be divisible by p and the LHS would not be so. It follows that $\alpha \geq p$, which also easily implies that $p^{\alpha-p+1} \geq \frac{p}{p+1}(\alpha+1)$.
Furthermore, since $\alpha + 1$ cannot be divisible by $p^{\alpha-p+1}$ for any $\alpha \geq p$, it follows that $p \mid \tau(k)$. Thus if $p \mid \tau(k)$, then at least one $\alpha_i + 1$ is divisible by p and consequently $\alpha_i \geq p-1$ for some i. Hence $k \geq \frac{p_i^{\alpha_i}}{\alpha_i+1}\tau(k) \geq \frac{2^{p-1}}{p}\tau(k)$. But then we have
$$p^{\alpha-p+1}k \geq \frac{p}{p+1}(\alpha+1) \cdot \frac{2^{p-1}}{p}\tau(k) > (\alpha+1)\tau(k),$$
contradicting (1). Therefore (1) has no solutions in $\mathbb{N}$.
Remark. There are many other values of a for which the considered equation has no solutions in $\mathbb{N}$: for example, $a = 6p$ for a prime $p \geq 5$.

25. Let n be a natural number. For each $k = 1, 2, \ldots, n$, the number (k, n) is a divisor of n. Consider any divisor d of n. If $(k, n) = n/d$, then $k = nl/d$ for some $l \in \mathbb{N}$, and $(k, n) = (l, d)n/d$, which implies that l is coprime to d and $l \leq d$. It follows that (k, n) is equal to n/d for exactly $\varphi(d)$ natural numbers $k \leq n$. Therefore
$$\psi(n) = \sum_{k=1}^{n}(k,n) = \sum_{d\mid n}\varphi(d)\frac{n}{d} = n\sum_{d\mid n}\frac{\varphi(d)}{d}. \tag{1}$$
(a) Let n, m be coprime. Then each divisor f of mn can be uniquely expressed as $f = de$, where $d \mid n$ and $e \mid m$. We now have by (1)
$$\begin{aligned}\psi(mn) &= mn\sum_{f\mid mn}\frac{\varphi(f)}{f} = mn\sum_{d\mid n,\ e\mid m}\frac{\varphi(de)}{de} \\ &= mn\sum_{d\mid n,\ e\mid m}\frac{\varphi(d)}{d}\frac{\varphi(e)}{e} = \left(n\sum_{d\mid n}\frac{\varphi(d)}{d}\right)\left(m\sum_{e\mid m}\frac{\varphi(e)}{e}\right) \\ &= \psi(m)\psi(n).\end{aligned}$$

(b) Let $n = p^k$, where p is a prime and k a positive integer. According to (1),

$$\frac{\psi(n)}{n} = \sum_{i=0}^{k} \frac{\varphi(p^i)}{p^i} = 1 + \frac{k(p-1)}{p}.$$

Setting $p = 2$ and $k = 2(a-1)$ we obtain $\psi(n) = an$ for $n = 2^{2(a-1)}$.

(c) We note that $\psi(p^p) = p^{p+1}$ if p is a prime. Hence, if a has an odd prime factor p and $a_1 = a/p$, then $x = p^p 2^{2a_1-2}$ is a solution of $\psi(x) = ax$ different from $x = 2^{2a-2}$.

Now assume that $a = 2^k$ for some $k \in \mathbb{N}$. Suppose $x = 2^\alpha y$ is a positive integer such that $\psi(x) = 2^k x$. Then $2^{\alpha+k} y = \psi(x) = \psi(2^\alpha)\psi(y) = (\alpha+2)2^{\alpha-1}\psi(y)$, i.e., $2^{k+1} y = (\alpha+2)\psi(y)$. We notice that for each odd y, $\psi(y)$ is (by definition) the sum of an odd number of odd summands and therefore odd. It follows that $\psi(y) \mid y$. On the other hand, $\psi(y) > y$ for $y > 1$, so we must have $y = 1$. Consequently $\alpha = 2^{k+1}-2 = 2a-2$, giving us the unique solution $x = 2^{2a-2}$.

Thus $\psi(x) = ax$ has a unique solution if and only if a is a power of 2.

26. For $m = n = 1$ we obtain that $f(1)^2 + f(1)$ divides $(1^2+1)^2 = 4$, from which we find that $f(1) = 1$.

Next, we show that $f(p-1) = p-1$ for each prime p. By the hypothesis for $m = 1$ and $n = p-1$, $f(p-1)+1$ divides p^2, so $f(p-1)$ equals either $p-1$ or p^2-1. If $f(p-1) = p^2-1$, then $f(1)+f(p-1)^2 = p^4-2p^2+2$ divides $(1+(p-1)^2)^2 < p^4-2p^2+2$, giving a contradiction. Hence $f(p-1) = p-1$.

Let us now consider an arbitrary $n \in \mathbb{N}$. By the hypothesis for $m = p-1$, $A = f(n) + (p-1)^2$ divides $(n+(p-1)^2)^2 \equiv (n-f(n))^2 \pmod{A}$, and hence A divides $(n-f(n))^2$ for any prime p. Taking p large enough, we can obtain A to be greater than $(n-f(n))^2$, which implies that $(n-f(n))^2 = 0$, i.e., $f(n) = n$ for every n.

27. Set $a = 1$ and assume that $b \in \mathbb{N}$ is such that $b^2 \equiv b+1 \pmod{m}$. An easy induction gives us $x_n \equiv b^n \pmod{m}$ for all $n \in \mathbb{N}_0$. Moreover, b is obviously coprime to m, and hence each x_n is coprime to m.

It remains to show the existence of b. The congruence $b^2 \equiv b+1 \pmod{m}$ is equivalent to $(2b-1)^2 \equiv 5 \pmod{m}$. Taking $2b-1 \equiv 2k$, i.e., $b \equiv 2k^2+k-2 \pmod{m}$, does the job.

Remark. A desired b exists whenever 5 is a quadratic residue modulo m, in particular, when m is a prime of the form $10k \pm 1$.

28. If n is divisible by 20, then every multiple of n has two last digits even and hence it is not alternate. We shall show that any other n has an alternate multiple.

(i) Let n be coprime to 10. For each k there exists a number $A_k(n) = \overline{10\dots010\dots01\dots0\dots01} = \frac{10^{mk}-1}{10^k-1}$ $(m \in \mathbb{N})$ that is divisible by n (by Euler's theorem, choose $m = \varphi[n(10^k-1)]$). In particular, $A_2(n)$ is alternate.

(ii) Let $n = 2 \cdot 5^r \cdot n_1$, where $r \geq 1$ and $(n_1, 10) = 1$. We shall show by induction that, for each k, there exists an alternative k-digit odd number M_k that is divisible by 5^k. Choosing the number $10A_{2r}(n_1)M_{2r}$ will then solve this case, since it is clearly alternate and divisible by n.

We can trivially choose $M_1 = 5$. Let there be given an alternate r-digit multiple M_r of 5^r, and let $c \in \{0,1,2,3,4\}$ be such that $M_r/5^r \equiv -c \cdot 2^r \pmod 5$. Then the $(r+1)$digit numbers $M_r + c \cdot 10^r$ and $M_r + (5+c) \cdot 10^r$ are respectively equal to $5^r(M_r/5^r + 2^r \cdot c)$ and $5^r(M_r/5^r + 2^r \cdot c + 5 \cdot 2^r)$, and hence they are divisible by 5^{r+1} and exactly one of them is alternate: we set it to be M_{r+1}.

(iii) Let $n = 2^r \cdot n_1$, where $r \geq 1$ and $(n_1, 10) = 1$. We show that there exists an alternate $2r$-digit number N_r that is divisible by 2^{2r+1}. Choosing the number $A_{2r}(n_1)N_r$ will then solve this case.

We choose $N_1 = 16$, and given N_r, we can prove that one of $N_r + m \cdot 10^{2r}$, for $m \in \{10, 12, 14, 16\}$, is divisible by 2^{2r+3} and therefore suitable for N_{r+1}. Indeed, for $N_r = 2^{2r+1}d$ we have $N_r + m \cdot 10^{2r} = 2^{2r+1}(d + 5^r m/2)$ and $d + 5^r m/2 \equiv 0 \pmod 4$ has a solution $m/2 \in \{5,6,7,8\}$ for each d and r.

Remark. The idea is essentially the same as in (SL94-24).

29. Let $S_n = \{x \in \mathbb{N} \mid x \leq n,\ n \mid x^2 - 1\}$. It is easy to check that $P_n \equiv 1 \pmod n$ for $n = 2$ and $P_n \equiv -1 \pmod n$ for $n \in \{3,4\}$, so from now on we assume $n > 4$.

We note that if $x \in S_n$, then also $n - x \in S_n$ and $(x, n) = 1$. Thus S_n splits into pairs $\{x, n-x\}$, where $x \in S_n$ and $x \leq n/2$. In each of these pairs the product of elements gives remainder -1 upon division by n. Therefore $P_n \equiv (-1)^m$, where S_n has $2m$ elements. It remains to find the parity of m.

Suppose first that $n > 4$ is divisible by 4. Whenever $x \in S_n$, the numbers $|n/2 - x|$, $n - x$, $n - |n/2 - x|$ also belong to S_n (indeed, $n \mid (n/2 - x)^2 - 1 = n^2/4 - nx + x^2 - 1$ because $n \mid n^2/4$, etc.). In this way the set S_n splits into four-element subsets $\{x, n/2 - x, n/2 + x, n - x\}$, where $x \in S_n$ and $x < n/4$ (elements of these subsets are different for $x \neq n/4$, and $n/4$ doesn't belong to S_n for $n > 4$). Therefore $m = |S_n|/2$ is even and $P_n \equiv 1 \pmod m$.

Now let n be odd. If $n \mid x^2 - 1 = (x-1)(x+1)$, then there exist natural numbers a, b such that $ab = n$, $a \mid x - 1$, $b \mid x + 1$. Obviously a and b are coprime. Conversely, given any odd $a, b \in \mathbb{N}$ such that $(a, b) = 1$ and $ab = n$, by the Chinese remainder theorem there exists $x \in \{1, 2, \dots, n-1\}$ such that $a \mid x - 1$ and $b \mid x + 1$. This gives a bijection between all ordered pairs (a, b) with $ab = n$ and $(a, b) = 1$ and the elements of S_n. Now if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the decomposition of n into primes, the number of pairs (a, b) is equal to 2^k (since for every i, either $p_i^{\alpha_i} \mid a$ or $p_i^{\alpha_i} \mid b$), and hence

$m = 2^{k-1}$. Thus $P_n \equiv -1 \pmod{n}$ if n is a power of an odd prime, and $P_n \equiv 1$ otherwise.
Finally, let n be even but not divisible by 4. Then $x \in S_n$ if and only if x or $n - x$ belongs to $S_{n/2}$ and x is odd. Since $n/2$ is odd, for each $x \in S_{n/2}$ either x or $x + n/2$ belongs to S_n, and by the case of n odd we have $S_n \equiv \pm 1 \pmod{n/2}$, depending on whether or not $n/2$ is a power of a prime. Since S_n is odd, it follows that $P_n \equiv -1 \pmod{n}$ if $n/2$ is a power of a prime, and $P_n \equiv 1$ otherwise.

Second solution. Obviously S_n is closed under multiplication modulo n. This implies that S_n with multiplication modulo n is a subgroup of $\mathbb{Z}_n$, and therefore there exist elements $a_1 = -1, a_2, \dots, a_k \in S_n$ that generate S_n. In other words, since the a_i are of order two, S_n consists of products $\prod_{i \in A} a_i$, where A runs over all subsets of $\{1, 2, \dots, k\}$. Thus S_n has 2^k elements, and the product of these elements equals $P_n \equiv (a_1 a_2 \cdots a_k)^{2^{k-1}} \pmod{n}$. Since $a_i^2 \equiv 1 \pmod{n}$, it follows that $P_n \equiv 1$ if $k \geq 2$, i.e., if $|S_n| > 2$. Otherwise $P_n \equiv -1 \pmod{n}$.
We note that $|S_n| > 2$ is equivalent to the existence of $a \in S_n$ with $1 < a < n-1$. It is easy to find that such an a exists if and only if neither of n, $n/2$ is a power of an odd prime.

30. We shall denote by k the given circle with diameter p^n.
Let A, B be lattice points (i.e., points with integer coordinates). We shall denote by $\mu(AB)$ the exponent of the highest power of p that divides the integer AB^2. We observe that if S is the area of a triangle ABC where A, B, C are lattice points, then $2S$ is an integer. According to Heron's formula and the formula for the circumradius, a triangle ABC whose circumcenter has diameter p^n satisfies

$$2AB^2BC^2 + 2BC^2CA^2 + 2CA^2AB^2 - AB^4 - BC^4 - CA^4 = 16S^2 \quad (1)$$

and
$$AB^2 \cdot BC^2 \cdot CA^2 = (2S)^2 p^{2n}. \quad (2)$$

Lemma 1. Let A, B, and C be lattice points on k. If none of AB^2, BC^2, CA^2 is divisible by p^{n+1}, then $\mu(AB), \mu(BC), \mu(CA)$ are $0, n, n$ in some order.

Proof. Let $k = \min\{\mu(AB), \mu(BC), \mu(CA)\}$. By (1), $(2S)^2$ is divisible by p^{2k}. Together with (2), this gives us $\mu(AB) + \mu(BC) + \mu(CA) = 2k + 2n$. On the other hand, if none of AB^2, BC^2, CA^2 is divisible by p^{n+1}, then $\mu(AB) + \mu(BC) + \mu(CA) \leq k + 2n$. Therefore $k = 0$ and the remaining two of $\mu(AB), \mu(BC), \mu(CA)$ are equal to n.

Lemma 2. Among every four lattice points on k, there exist two, say M, N, such that $\mu(MN) \geq n + 1$.

Proof. Assume that this doesn't hold for some points A, B, C, D on k. By Lemma 1, μ for some of the segments $AB, AC, \dots, CD$ is 0, say $\mu(AC) = 0$. It easily follows by Lemma 1 that then $\mu(BD) = 0$ and $\mu(AB) = \mu(BC) = \mu(CD) = \mu(DA) = n$. Let $a, b, c, d, e, f \in \mathbb{N}$ be such that $AB^2 = p^n a$, $BC^2 = p^n b$, $CD^2 = p^n c$, $DA^2 = p^n d$, $AC^2 = e$, $BD^2 = f$. By Ptolemy's theorem we have $\sqrt{ef} = p^n\left(\sqrt{ac}+\sqrt{bd}\right)$. Taking squares, we get that $\frac{ef}{p^{2n}} = \left(\sqrt{ac}+\sqrt{bd}\right)^2 = ac+bd+2\sqrt{abcd}$ is rational and hence an integer. It follows that ef is divisible by p^{2n}, a contradiction.

Now we consider eight lattice points $A_1, A_2, \dots, A_8$ on k. We color each segment A_iA_j red if $\mu(A_iA_j) > n$ and black otherwise, and thus obtain a graph G. The degree of a point X will be the number of red segments with an endpoint in X. We distinguish three cases:

(i) There is a point, say A_8, whose degree is at most 1. We may suppose w.l.o.g. that A_8A_7 is red and $A_8A_1, \dots, A_8A_6$ black. By a well-known fact, the segments joining vertices $A_1, A_2, \dots, A_6$ determine either a red triangle, in which case there is nothing to prove, or a black triangle, say $A_1A_2A_3$. But in the latter case the four points A_1, A_2, A_3, A_8 do not determine any red segment, a contradiction to Lemma 2.

(ii) All points have degree 2. Then the set of red segments partitions into cycles. If one of these cycles has length 3, then the proof is complete. If all the cycles have length at least 4, then we have two possibilities: two 4-cycles, say $A_1A_2A_3A_4$ and $A_5A_6A_7A_8$, or one 8-cycle, $A_1A_2\dots A_8$. In both cases, the four points A_1, A_3, A_5, A_7 do not determine any red segment, a contradiction.

(iii) There is a point of degree at least 3, say A_1. Suppose that A_1A_2, A_1A_3, and A_1A_4 are red. We claim that A_2, A_3, A_4 determine at least one red segment, which will complete the solution. If not, by Lemma 1, $\mu(A_2A_3), \mu(A_3A_4), \mu(A_4A_2)$ are $n, n, 0$ in some order. Assuming w.l.o.g. that $\mu(A_2A_3) = 0$, denote by S the area of triangle $A_1A_2A_3$. Now by formula (1), $2S$ is not divisible by p. On the other hand, since $\mu(A_1A_2) \ge n+1$ and $\mu(A_1A_3) \ge n+1$, it follows from (2) that $2S$ is divisible by p, a contradiction.

1

Problems

1.1 The Forty-Sixth IMO
Mérida, Mexico, July 8–19, 2005

1.1.1 Contest Problems

First Day (July 13)

1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC; B_1, B_2 on CA; C_1, C_2 on AB. These points are vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders on division by n. Prove that each integer occurs exactly once in the sequence.

3. Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that
$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

Second Day (July 14)

4. Consider the sequence $a_1, a_2, \ldots$ defined by
$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \ldots).$$
Determine all positive integers that are relatively prime to every term of the sequence.

5. Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF

meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

6. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $2/5$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

1.1.2 Shortlisted Problems

1. **A1 (ROM)** Find all monic polynomials $p(x)$ with integer coefficients of degree two for which there exists a polynomial $q(x)$ with integer coefficients such that $p(x)q(x)$ is a polynomial having all coefficients ± 1.
2. **A2 (BUL)** Let $\mathbb{R}^+$ denote the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f(x)f(y) = 2f(x + yf(x))$$
for all positive real numbers x and y.
3. **A3 (CZE)** Four real numbers p, q, r, s satisfy
$$p + q + r + s = 9 \quad \text{and} \quad p^2 + q^2 + r^2 + s^2 = 21.$$
Prove that $ab - cd \geq 2$ holds for some permutation (a, b, c, d) of (p, q, r, s).
4. **A4 (IND)** Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation
$$f(x + y) + f(x)f(y) = f(xy) + 2xy + 1$$
for all real x and y.
5. **A5 (KOR)**[IMO3] Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that
$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$
6. **C1 (AUS)** A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.
7. **C2 (IRN)** Let k be a fixed positive integer. A company has a special method to sell sombreros. Each customer can convince two persons to buy a sombrero after he/she buys one; convincing someone already convinced does not count. Each

of these new customers can convince two others and so on. If each one of the two customers convinced by someone makes at least k persons buy sombreros (directly or indirectly), then that someone wins a free instructional video. Prove that if n persons bought sombreros, then at most $n/(k+2)$ of them got videos.

8. **C3 (IRN)** In an $m \times n$ rectangular board of mn unit squares, *adjacent* squares are ones with a common edge, and a *path* is a sequence of squares in which any two consecutive squares are adjacent. Each square of the board can be colored black or white. Let N denote the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge, and let M denote the number of colorings in which there exist at least two non-intersecting black paths from the left edge to the right edge. Prove that $N^2 \geq 2^{mn}M$.

9. **C4 (COL)** Let $n \geq 3$ be a given positive integer. We wish to label each side and each diagonal of a regular n-gon $P_1 \dots P_n$ with a positive integer less than or equal to r so that:
   (i) every integer between 1 and r occurs as a label;
   (ii) in each triangle $P_iP_jP_k$ two of the labels are equal and greater than the third.
   Given these conditions:
   (a) Determine the largest positive integer r for which this can be done.
   (b) For that value of r, how many such labellings are there?

10. **C5 (SMN)** There are n markers, each with one side white and the other side black, aligned in a row so that their white sides are up. In each step, if possible, we choose a marker with the white side up (but not one of outermost markers), remove it and reverse the closest marker to the left and the closest marker to the right of it. Prove that one can achieve the state with only two markers remaining if and only if $n-1$ is not divisible by 3.

11. **C6 (ROM)**[IMO6] In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $2/5$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

12. **C7 (USA)** Let $n \geq 1$ be a given integer, and let $a_1, \dots, a_n$ be a sequence of integers such that n divides the sum $a_1 + \cdots + a_n$. Show that there exist permutations σ and τ of $1, 2, \dots, n$ such that $\sigma(i) + \tau(i) \equiv a_i \pmod{n}$ for all $i = 1, \dots, n$.

13. **C8 (BUL)** Let M be a convex n-gon, $n \geq 4$. Some $n-3$ of its diagonals are colored green and some other $n-3$ diagonals are colored red, so that no two diagonals of the same color meet inside M. Find the maximum possible number of intersection points of green and red diagonals inside M.

14. **G1 (GRE)** In a triangle ABC satisfying $AB + BC = 3AC$ the incircle has center I and touches the sides AB and BC at D and E, respectively. Let K and L be the symmetric points of D and E with respect to I. Prove that the quadrilateral $ACKL$ is cyclic.

15. **G2 (ROM)**IMO1 Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC; B_1, B_2 on CA; C_1, C_2 on AB. These points are vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

16. **G3 (UKR)** Let $ABCD$ be a parallelogram. A variable line l passing through the point A intersects the rays BC and DC at points X and Y, respectively. Let K and L be the centers of the excircles of triangles ABX and ADY, touching the sides BX and DY, respectively. Prove that the size of angle KCL does not depend on the choice of the line l.

17. **G4 (POL)**IMO5 Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

18. **G5 (ROM)** Let ABC be an acute-angled triangle with $AB \neq AC$, let H be its orthocenter and M the midpoint of BC. Points D on AB and E on AC are such that $AE = AD$ and D, H, E are collinear. Prove that HM is orthogonal to the common chord of the circumcircles of triangles ABC and ADE.

19. **G6 (RUS)** The median AM of a triangle ABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y. The lines AX and AY intersect BC at P and Q. Prove that $BP = CQ$.

20. **G7 (KOR)** In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that $p(ABC)p(PQR) \geq p(DEF)^2$, where $p(T)$ denotes the perimeter of triangle T.

21. **N1 (POL)**IMO4 Consider the sequence $a_1, a_2, \ldots$ defined by

$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \ldots).$$

Determine all positive integers that are relatively prime to every term of the sequence.

22. **N2 (NET)**IMO2 Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders on division by n. Prove that each integer occurs exactly once in the sequence.

23. **N3 (MON)** Let a, b, c, d, e and f be positive integers. Suppose that the sum $S = a+b+c+d+e+f$ divides both $abc+def$ and $ab+bc+ca-de-ef-fd$. Prove that S is composite.

24. **N4 (COL)** Find all positive integers $n > 1$ for which there exists a unique integer a with $0 < a \leq n!$ such that $a^n + 1$ is divisible by $n!$.

25. **N5 (NET)** Denote by $d(n)$ the number of divisors of the positive integer n. A positive integer n is called *highly divisible* if $d(n) > d(m)$ for all positive integers $m < n$. Two highly divisible integers m and n with $m < n$ are called consecutive if there exists no highly divisible integer s satisfying $m < s < n$.
    (a) Show that there are only finitely many pairs of consecutive highly divisible integers of the form (a,b) with $a|b$.
    (b) Show that for every prime number p there exist infinitely many positive highly divisible integers r such that pr is also highly divisible.
26. **N6 (IRN)** Let a and b be positive integers such that $a^n + n$ divides $b^n + n$ for every positive integer n. Show that $a = b$.
27. **N7 (RUS)** Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_0, \ldots, a_n$ are integers, $a_n > 0$, $n \geq 2$. Prove that there exists a positive integer m such that $P(m!)$ is a composite number.

2

Solutions

2.1 Solutions to the Shortlisted Problems of IMO 2005

1. Clearly, $p(x)$ has to be of the form $p(x) = x^2 + ax \pm 1$ where a is an integer. For $a = \pm 1$ and $a = 0$ polynomial p has the required property: it suffices to take $q = 1$ and $q = x + 1$, respectively.
   Suppose now that $|a| \geq 2$. Then $p(x)$ has two real roots, say x_1, x_2, which are also roots of $p(x)q(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_i = \pm 1$. Thus
   $$1 = \left|\frac{a_{n-1}}{x_i} + \cdots + \frac{a_0}{x_i^n}\right| \leq \frac{1}{|x_i|} + \cdots + \frac{1}{|x_i|^n} < \frac{1}{|x_i| - 1}$$
   which implies $|x_1|, |x_2| < 2$. This immediately rules out the case $|a| \geq 3$ and the polynomials $p(x) = x^2 \pm 2x - 1$. The remaining two polynomials $x^2 \pm 2x + 1$ satisfy the condition for $q(x) = x \mp 1$.
   Summing all, the polynomials $p(x)$ with the desired property are $x^2 \pm x \pm 1$, $x^2 \pm 1$ and $x^2 \pm 2x + 1$.

2. Given $y > 0$, consider the function $\varphi(x) = x + yf(x)$, $x > 0$. This function is injective: indeed, if $\varphi(x_1) = \varphi(x_2)$ then $f(x_1)f(y) = f(\varphi(x_1)) = f(\varphi(x_2)) = f(x_2)f(y)$, so $f(x_1) = f(x_2)$, so $x_1 = x_2$ by the definition of φ. Now if $x_1 > x_2$ and $f(x_1) < f(x_2)$, we have $\varphi(x_1) = \varphi(x_2)$ for $y = \frac{x_1 - x_2}{f(x_2) - f(x_1)} > 0$, which is impossible; hence f is non-decreasing. The functional equation now yields $f(x)f(y) = 2f(x + yf(x)) \geq 2f(x)$ and consequently $f(y) \geq 2$ for $y > 0$. Therefore
   $$f(x + yf(x)) = f(xy) = f(y + xf(y)) \geq f(2x)$$
   holds for arbitrarily small $y > 0$, implying that f is constant on the interval $(x, 2x]$ for each $x > 0$. But then f is constant on the union of all intervals $(x, 2x]$ over all $x > 0$, that is, on all of $\mathbb{R}^+$. Now the functional equation gives us $f(x) = 2$ for all x, which is clearly a solution.

   Second Solution. In the same way as above we prove that f is non-decreasing, hence its discontinuity set is at most countable. We can extend f to $\mathbb{R} \cup \{0\}$ by defining $f(0) = \inf_x f(x) = \lim_{x \to 0} f(x)$ and the new function f is continuous at 0 as well. If x is a point of continuity of f we have $f(x)f(0) = \lim_{y \to 0} f(x)f(y) = \lim_{y \to 0} 2f(x + yf(x)) = 2f(x)$, hence $f(0) = 2$. Now, if f is continuous at $2y$ then $2f(y) = \lim_{x \to 0} f(x)f(y) = \lim_{x \to 0} 2f(x + yf(x)) = 2f(2y)$. Thus $f(y) = f(2y)$, for all but countably many values of y. Being non-decreasing f is a constant, hence $f(x) = 2$.

3. Assume w.l.o.g. that $p \geq q \geq r \geq s$. We have
   $$(pq + rs) + (pr + qs) + (ps + qr) = \frac{(p + q + r + s)^2 - p^2 - q^2 - r^2 - s^2}{2} = 30.$$
   It is easy to see that $pq + rs \geq pr + qs \geq ps + qr$ which gives us $pq + rs \geq 10$. Now setting $p + q = x$ we obtain $x^2 + (9 - x)^2 = (p + q)^2 + (r + s)^2 = 21 + 2(pq + rs) \geq 41$ which is equivalent to $(x - 4)(x - 5) \geq 0$. Since $x = p + q \geq r + s$ we conclude that $x \geq 5$. Thus

$$25 \le p^2+q^2+2pq = 21-(r^2+s^2)+2pq \le 21+2(pq-rs),$$

or $pq-rs \ge 2$, as desired.
Remark. The quadruple $(p,q,r,s) = (3,2,2,2)$ shows that the estimate 2 is the best possible.

4. Setting $y=0$ yields $(f(0)+1)(f(x)-1)=0$, and since $f(x)=1$ for all x is impossible, we get $f(0)=-1$. Now plugging in $x=1$ and $y=-1$ gives us $f(1)=1$ or $f(-1)=0$. In the first case setting $x=1$ in the functional equation yields $f(y+1)=2y+1$, i.e. $f(x)=2x-1$ which is one solution.
Suppose now that $f(1)=a\neq 1$ and $f(-1)=0$. Plugging $(x,y)=(z,1)$ and $(x,y)=(-z,-1)$ in the functional equation yields

$$\begin{aligned} f(z+1) &= (1-a)f(z)+2z+1 \\ f(-z-1) &= f(z)+2z+1. \end{aligned} \qquad (*)$$

It follows that $f(z+1)=(1-a)f(-z-1)+a(2z+1)$, i.e. $f(x)=(1-a)f(-x)+a(2x-1)$. Analogously $f(-x)=(1-a)f(x)+a(-2x-1)$, which together with the previous equation yields

$$(a^2-2a)f(x) = -2a^2x-(a^2-2a).$$

Now $a=2$ is clearly impossible. For $a\notin\{0,2\}$ we get $f(x)=\frac{-2ax}{a-2}-1$. This function satisfies the requirements only for $a=-2$, giving the solution $f(x)=-x-1$. In the remaining case, when $a=0$, we have $f(x)=f(-x)$. Setting $y=z$ and $y=-z$ in the functional equation and subtracting yields $f(2z)=4z^2-1$, so $f(x)=x^2-1$ which satisfies the equation.
Thus the solutions are $f(x)=2x-1$, $f(x)=-x-1$ and $f(x)=x^2-1$.

5. The desired inequality is equivalent to

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2} \le 3. \qquad (*)$$

By the Cauchy inequality we have $(x^5+y^2+z^2)(yz+y^2+z^2) \ge (x^{5/2}(yz)^{1/2}+y^2+z^2)^2 \ge (x^2+y^2+z^2)^2$ and therefore

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2} \le \frac{yz+y^2+z^2}{x^2+y^2+z^2}.$$

We get analogous inequalities for the other two summands in $(*)$. Summing these up yields

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2} \le 2+\frac{xy+yz+zx}{x^2+y^2+z^2},$$

which together with the well-known inequality $x^2+y^2+z^2 \ge xy+yz+zx$ gives us the result.

Second solution. Multiplying the both sides with the common denominator and using the notation as in Chapter 2 (*Muirhead's inequality*) we get

$$T_{5,5,5}+4T_{7,5,0}+T_{5,2,2}+T_{9,0,0}\geq T_{5,5,2}+T_{6,0,0}+2T_{5,4,0}+2T_{4,2,0}+T_{2,2,2}.$$

By Schur's and Muirhead's inequalities we have that $T_{9,0,0}+T_{5,2,2}\geq 2T_{7,2,0}\geq 2T_{7,1,1}$. Since $xyz\geq 1$ we have that $T_{7,1,1}\geq T_{6,0,0}$. Therefore

$$T_{9,0,0}+T_{5,2,2}\geq 2T_{6,0,0}\geq T_{6,0,0}+T_{4,2,0}. \tag{1}$$

Moreover, Muirhead's inequality combined with $xyz\geq 1$ gives us $T_{7,5,0}\geq T_{5,5,2}$, $2T_{7,5,0}\geq 2T_{6,5,1}\geq 2T_{5,4,0}$, $T_{7,5,0}\geq T_{6,4,2}\geq T_{4,2,0}$, and $T_{5,5,5}\geq T_{2,2,2}$. Adding these four inequalities to (1) yields the desired result.

6. A room will be called *economic* if some of its lamps are on and some are off. Two lamps sharing a switch will be called *twins*. The twin of a lamp l will be denoted $\bar{l}$.
   Suppose we have arrived at a state with the minimum possible number of uneconomic rooms, and that this number is strictly positive. Let us choose any uneconomic room, say R_0, and a lamp l_0 in it. Let $\bar{l_0}$ be in a room R_1. Switching l_0 we make R_0 economic; thereby, since the number of uneconomic rooms cannot be decreased, this change must make room R_1 uneconomic. Now choose a lamp l_1 in R_1 having the twin $\bar{l_1}$ in a room R_2. Switching l_1 makes R_1 economic, and thus must make R_2 uneconomic. Continuing in this manner we obtain a sequence $l_0,l_1,\dots$ of lamps with l_i in a room R_i and $\bar{l_i}\neq l_{i+1}$ in R_{i+1} for all i. The lamps $l_0,l_1,\dots$ are switched in this order. This sequence has the property that switching l_i and $\bar{l_i}$ makes room R_i economic and room R_{i+1} uneconomic.
   Let $R_m=R_k$ with $m>k$ be the first repetition in the sequence (R_i). Let us stop switching the lamps at l_{m-1}. The room R_k was uneconomic prior to switching l_k. Thereafter lamps l_k and $\bar{l}_{m-1}$ have been switched in R_k, but since these two lamps are distinct (indeed, their twins $\bar{l_k}$ and l_{m-1} are distinct), the room R_k is now economic as well as all the rooms $R_0,R_1,\dots,R_{m-1}$. This decreases the number of uneconomic rooms, contradicting our assumption.

7. Let v be the number of video winners. One easily finds that for $v=1$ and $v=2$, the number n of customers is at least $2k+3$ and $3k+5$ respectively. We prove by induction on v that if $n\geq k+1$ then $n\geq(k+2)(v+1)-1$.
   We can assume w.l.o.g. that the total number n of customers is minimum possible for given $v>0$. Consider a person P who was convinced by nobody but himself. Then P must have won a video; otherwise P could be removed from the group without decreasing the number of video winners. Let Q and R be the two persons convinced by P. We denote by $\mathscr{C}$ the set of persons made by P through Q to buy a sombrero, including Q, and by $\mathscr{D}$ the set of all other customers excluding P. Let x be the number of video winners in $\mathscr{C}$. Then there are $v-x-1$ video winners in $\mathscr{D}$. We have $|\mathscr{C}|\geq(k+2)(x+1)-1$, by induction hypothesis if $x>0$ and because P is a winner if $x=0$. Similarly, $|\mathscr{D}|\geq(k+2)(v-x)-1$. Thus $n\geq 1+(k+2)(x+1)-1+(k+2)(v-x)-1$, i.e. $n\geq(k+2)(v+1)-1$.

8. Suppose that a two-sided $m \times n$ board T is considered, where exactly k of the squares are transparent. A transparent square is colored only on one side (then it looks the same from the other side), while a non-transparent one needs to be colored on both sides, not necessarily in the same color.
Let $C = C(T)$ be the set of colorings of the board in which there exist two black paths from the left edge to the right edge, one on top and one underneath, not intersecting at any transparent square. If $k = 0$ then $|C| = N^2$. We prove by induction on k that $2^k|C| \le N^2$: this will imply the statement of the problem, as $|C| = M$ for $k = mn$.
Let q be a fixed transparent square. Consider any coloring B in C: If q is converted into a non-transparent square, a new board T' with $k-1$ transparent squares is obtained, so by the induction hypothesis $2^{k-1}|C(T')| \le N^2$. Since B contains two black paths at most one of which passes through q, coloring q in either color on the other side will result in a coloring in C'; hence $|C(T')| \ge 2|C(T)|$, implying $2^k|C(T)| \le N^2$ and finishing the induction.

Second solution. By *path* we shall mean a black path from the left edge to the right edge. Let $\mathscr{A}$ denote the set of pairs of $m \times n$ boards each of which has a path. Let $\mathscr{B}$ denote the set of pairs of boards such that the first board has two non-intersecting paths. Obviously, $|\mathscr{A}| = N^2$ and $|\mathscr{B}| = 2^{mn}M$. To show $|\mathscr{A}| \ge |\mathscr{B}|$ we will construct an injection $f : \mathscr{B} \to \mathscr{A}$.
Among paths on a given board we define path x to be *lower* than y if the set of squares "under" x is a subset of the squares under y. This relation is a relation of incomplete order. However, for each board with at least one path there exists the lowest path (comparing two intersecting paths, we can always take the "lower branch" on each non-intersecting segment). Now, for a given element of $\mathscr{B}$, we "swap" the lowest path and all squares underneath on the first board with the corresponding points on the other board. This swapping operation is the desired injection f. Indeed, since the first board still contains the highest path (which didn't intersect the lowest one), the new configuration belongs to $\mathscr{A}$. On the other hand, this configuration uniquely determines the lowest path on the original element of $\mathscr{B}$; hence no two different elements of $\mathscr{B}$ can go to the same element of $\mathscr{A}$. This completes the proof.

9. Let $[XY]$ denote the label of segment XY, where X and Y are vertices of the polygon. Consider any segment MN with the maximum label $[MN] = r$. By condition (ii), for any $P_i \ne M,N$, exactly one of P_iM and P_iN is labelled by r. Thus the set of all vertices of the n-gon splits into two complementary groups: $\mathscr{A} = \{P_i \mid [P_iM] = r\}$ and $\mathscr{B} = \{P_i \mid [P_iN] = r\}$. We claim that a segment XY is labelled by r if and only if it joins two points from different groups. Assume w.l.o.g. that $X \in \mathscr{A}$. If $Y \in \mathscr{A}$, then $[XM] = [YM] = r$, so $[XY] < r$. If $Y \in \mathscr{B}$, then $[XM] = r$ and $[YM] < r$, so $[XY] = r$ by (ii), as we claimed.
We conclude that a labelling satisfying (ii) is uniquely determined by groups $\mathscr{A}$ and $\mathscr{B}$ and labellings satisfying (ii) within A and B.
   (a) We prove by induction on n that the greatest possible value of r is $n-1$. The degenerate cases $n = 1,2$ are trivial. If $n \ge 3$, the number of different labels

of segments joining vertices in $\mathscr{A}$ (resp. $\mathscr{B}$) does not exceed $|\mathscr{A}|-1$ (resp. $|\mathscr{B}|-1$), while all segments joining a vertex in $\mathscr{A}$ and a vertex in $\mathscr{B}$ are labelled by r. Therefore $r \le (|\mathscr{A}|-1)+(|\mathscr{B}|-1)+1 = n-1$. The equality is achieved if all the mentioned labels are different.

(b) Let a_n be the number of labellings with $r = n-1$. We prove by induction that $a_n = \frac{n!(n-1)!}{2^{n-1}}$. This is trivial for $n=1$, so let $n \ge 2$. If $|\mathscr{A}| = k$ is fixed, the groups $\mathscr{A}$ and $\mathscr{B}$ can be chosen in $\binom{n}{k}$ ways. The set of labels used within $\mathscr{A}$ can be selected among $1,2,\dots,n-2$ in $\binom{n-2}{k-1}$ ways. Now the segments within groups $\mathscr{A}$ and $\mathscr{B}$ can be labelled so as to satisfy (ii) in a_k and a_{n-k} ways, respectively. This way every labelling has been counted twice, since choosing $\mathscr{A}$ is equivalent to choosing $\mathscr{B}$. It follows that

$$\begin{aligned} a_n &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\binom{n-2}{k-1}a_k a_{n-k} \\ &= \frac{n!(n-1)!}{2(n-1)}\sum_{k=1}^{n-1}\frac{a_k}{k!(k-1)!}\cdot\frac{a_{n-k}}{(n-k)!(n-k-1)!} \\ &= \frac{n!(n-1)!}{2(n-1)}\sum_{k=1}^{n-1}\frac{1}{2^{k-1}}\cdot\frac{1}{2^{n-k-1}} = \frac{n!(n-1)!}{2^{n-1}}. \end{aligned}$$

10. Denote by L the leftmost and by R the rightmost marker. To start with, note that the parity of the number of black-side-up markers remains unchanged. Hence, if only two markers remain, these markers must have the same color up.

We 'll show by induction on n that the game can be successfully finished if and only if $n \equiv 0$ or $n \equiv 2 \pmod 3$, and that the upper sides of L and R will be black in the first case and white in the second case.

The statement is clear for $n = 2,3$. Assume that we finished the game for some n, and denote by k the position of the marker X (counting from the left) that was last removed. Having finished the game, we have also finished the subgames with the k markers from L to X and with the $n-k+1$ markers from X to R (inclusive). Thereby, before X was removed, the upper side of L had been black if $k \equiv 0$ and white if $k \equiv 2 \pmod 3$, while the upper side of R had been black if $n-k+1 \equiv 0$ and white if $n-k+1 \equiv 2 \pmod 3$. Markers L and R were reversed upon the removal of X. Therefore, in the final position L and R are white if and only if $k \equiv n-k+1 \equiv 0$, which yields $n \equiv 2 \pmod 3$, and black if and only if $k \equiv n-k+1 \equiv 2$, which yields $n \equiv 0 \pmod 3$.

On the other hand, a game with n markers can be reduced to a game with $n-3$ markers by removing the second, fourth, and third marker in this order. This finishes the induction.

Second solution. An invariant can be defined as follows. To each white marker with k black markers to its left we assign the number $(-1)^k$. Let S be the sum of the assigned numbers. Then it is easy to verify that the remainder of S modulo 3 remains unchanged throughout the game: For example, when a white marker with two white neighbors and k black markers to its left is removed, S decreases by $3(-1)^t$.

Initially, $S = n$. In the final position with two markers remained S equals 0 if the two markers are black and 2 if these are white (note that, as before, the two markers must be of the same color). Thus $n \equiv 0$ or 2 (mod 3).
Conversely, a game with n markers is reduced to $n-3$ markers as in the first solution.

11. Assume there were n contestants, a_i of whom solved exactly i problems, where $a_0 + \cdots + a_5 = n$. Let us count the number N of pairs (C,P), where contestant C solved the pair of problems P. Each of the 15 pairs of problems was solved by at least $\frac{2n+1}{5}$ contestants, implying $N \geq 15 \cdot \frac{2n+1}{5} = 6n+3$. On the other hand, a_i students solved $\frac{i(i-1)}{2}$ pairs; hence

$$6n+3 \leq N \leq a_2 + 3a_3 + 6a_4 + 10a_5 = 6n + 4a_5 - (3a_3 + 5a_2 + 6a_1 + 6a_0).$$

Consequently $a_5 \geq 1$. Assume that $a_5 = 1$. Then we must have $N = 6n+4$, which is only possible if 14 of the pairs of problems were solved by exactly $\frac{2n+1}{5}$ students and the remaining one by $\frac{2n+1}{5} + 1$ students, and all students but the winner solved 4 problems.
The problem t not solved by the winner will be called *tough* and the pair of problems solved by $\frac{2n+1}{5} + 1$ students *special*.
Let us count the number M_p of pairs (C,P) for which P contains a fixed problem p. Let b_p be the number of contestants who solved p. Then $M_t = 3b_t$ (each of the b_t students solved three pairs of problems containing t), and $M_p = 3b_p + 1$ for $p \neq t$ (the winner solved four such pairs). On the other hand, each of the five pairs containing p was solved by $\frac{2n+1}{5}$ or $\frac{2n+1}{5} + 1$ students, so $M_p = 2n+2$ if the special pair contains p, and $M_p = 2n+1$ otherwise.
Now since $M_t = 3b_t = 2n+1$ or $2n+2$, we have $2n+1 \equiv 0$ or 2 (mod 3). But if $p \neq t$ is a problem not contained in the special pair, we have $M_p = 3b_p + 1 = 2n+1$; hence $2n+1 \equiv 1$ (mod 3), which is a contradiction.

12. Suppose that there exist desired permutations σ and τ for some sequence $a_1, \ldots, a_n$. Given a sequence (b_i) with sum divisible by n which differs modulo n from (a_i) only in two positions, say i_1 and i_2, we show how to construct desired permutations σ' and τ' for sequence (b_i). In this way, starting from an arbitrary sequence (a_i) for which σ and τ exist, we can construct desired permutations for any other sequence with sum divisible by n. All congruences below are modulo n.
We know that $\sigma(i) + \tau(i) \equiv b_i$ for all $i \neq i_1, i_2$. We construct the sequence $i_1, i_2, i_3, \ldots$ as follows: for each $k \geq 2$, i_{k+1} is the unique index such that

$$\sigma(i_{k-1}) + \tau(i_{k+1}) \equiv b_{i_k}. \tag{$*$}$$

Let $i_p = i_q$ be the repetition in the sequence with the smallest q. We claim that $p = 1$ or $p = 2$. Assume on the contrary that $p > 2$. Summing up $(*)$ for $k = p, p+1, \ldots, q-1$ and taking the equalities $\sigma(i_k) + \tau(i_k) = b_{i_k}$ for $i_k \neq i_1, i_2$ into account we obtain $\sigma(i_{p-1}) + \sigma(i_p) + \tau(i_{q-1}) + \tau(i_q) \equiv b_p + b_{q-1}$. Since $i_q = i_p$, it follows that $\sigma(i_{p-1})+\tau(i_{q-1})\equiv b_{q-1}$ and therefore $i_{p-1}=i_{q-1}$, a contradiction. Thus $p=1$ or $p=2$ as claimed.

Now we define the following permutations:

$$\sigma'(i_k)=\sigma(i_{k-1}) \text{ for } k=2,3,\dots,q-1 \text{ and } \sigma'(i_1)=\sigma(i_{q-1}),$$
$$\tau'(i_k)=\tau(i_{k+1}) \text{ for } k=2,3,\dots,q-1 \text{ and } \tau'(i_1)=\begin{cases}\tau(i_2) \text{ if } p=1,\\ \tau(i_1) \text{ if } p=2;\end{cases}$$
$$\sigma'(i)=\sigma(i) \text{ and } \tau'(i)=\tau(i) \quad \text{for } i\notin\{i_1,\dots,i_{q-1}\}.$$

Permutations σ' and τ' have the desired property. Indeed, $\sigma'(i)+\tau'(i)=b_i$ obviously holds for all $i\neq i_1$, but then it must also hold for $i=i_1$.

13. For every green diagonal d, let C_d denote the number of green-red intersection points on d. The task is to find the maximum possible value of the sum $\sum_d C_d$ over all green diagonals.

Let d_i and d_j be two green diagonals and let the part of polygon M lying between d_i and d_j have m vertices. There are at most $n-m-1$ red diagonals intersecting both d_i and d_j, while each of the remaining $m-2$ diagonals meets at most one of d_i, d_j. It follows that

$$C_{d_i}+C_{d_j}\leq 2(n-m-1)+(m-2)=2n-m-4. \qquad (*)$$

We now arrange the green diagonals in a sequence $d_1,d_2,\dots,d_{n-3}$ as follows. It is easily seen that there are two green diagonals d_1 and d_2 that divide M into two triangles and an $(n-2)$-gon; then there are two green diagonals d_3 and d_4 that divide the $(n-2)$-gon into two triangles and an $(n-4)$-gon, and so on. We continue this procedure until we end up with a triangle or a quadrilateral. Now the part of M between d_{2k-1} and d_{2k} has at least $n-2k$ vertices for $1\leq k\leq r$, where $n-3=2r+e$, $e\in\{0,1\}$; hence, by $(*)$, $C_{d_{2k-1}}+C_{d_{2k}}\leq n+2k-4$. Moreover, $C_{d_{n-3}}\leq n-3$. Summing up yields

$$\begin{aligned}C_{d_1}+C_{d_2}+\cdots+C_{d_{n-3}} &\leq \sum_{k=1}^{r}(n+2k-4)+e(n-3)\\ &=3r^2+e(3r+1)=\left\lceil\frac{3}{4}(n-3)^2\right\rceil.\end{aligned}$$

This value is attained in the following example. Let $A_1A_2\dots A_n$ be the n-gon M and let $l=\left[\frac{n}{2}\right]+1$. The diagonals A_1A_i, $i=3,\dots,l$ and A_lA_j, $j=l+2,\dots,n$ are colored in green, whereas the diagonals A_2A_i, $i=l+1,\dots,n$, and $A_{l+1}A_j$, $j=3,\dots,l-1$ are colored in red.

Thus the answer is $\lceil\frac{3}{4}(n-3)^2\rceil$.

14. Let F be the point of tangency of the incircle with AC and let M and N be the respective points of tangency of AB and BC with the corresponding excircles. If I is the incenter and I_a and P respectively the center and the tangency point with ray AC of the excircle corresponding to A, we have $\frac{AI}{IL}=\frac{AI}{IF}=\frac{AI_a}{I_aP}=\frac{AI_a}{I_aN}$, which implies that $\triangle AIL\sim\triangle AI_aN$. Thus L lies on AN, and analogously K lies on CM. Denote $x=AF$ and $y=CF$. Since $BD=BE$, $AD=BM=x$, and $CE=BN=y$,

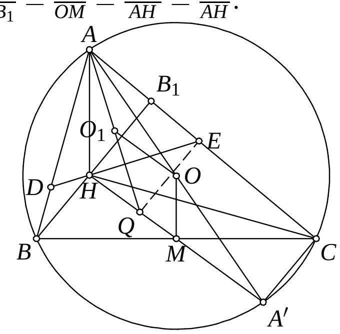
the condition $AB+BC=3AC$ gives us $DM=y$ and $EN=x$. Now the triangles CLN and MKA are congruent since their altitudes KD and LE satisfy $DK=EL$, $DM=CE$, and $AD=EN$. Thus $\angle AKM=\angle CLN$, implying that $ACKL$ is cyclic.

15. Let P be the fourth vertex of the rhombus $C_2A_1A_2P$. Since $\triangle C_2PC_1$ is equilateral, we easily conclude that $B_1B_2C_1P$ is also a rhombus. Thus $\triangle PB_1A_2$ is equilateral and $\angle(C_2A_1,C_1B_2)=\angle A_2PB_1=60^\circ$. It easily follows that $\triangle AC_1B_2\cong\triangle BA_1C_2$ and consequently $AC_1=BA_1$; similarly $BA_1=CB_1$. Therefore triangle $A_1B_1C_1$ is equilateral. Now it follows from $B_1B_2=B_2C_1$ that A_1B_2 bisects $\angle C_1A_1B_1$. Similarly, B_1C_2 and C_1A_2 bisect $\angle A_1B_1C_1$ and $\angle B_1C_1A_1$; hence A_1B_2, B_1C_2, C_1A_2 meet at the incenter of $A_1B_1C_1$, i.e. at the center of ABC.

16. Since $\angle ADL=\angle KBA=180^\circ-\frac{1}{2}\angle BCD$ and $\angle ALD=\frac{1}{2}\angle AYD=\angle KAB$, triangles ABK and LDA are similar. Thus $\frac{BK}{BC}=\frac{BK}{AD}=\frac{AB}{DL}=\frac{DC}{DL}$, which together with $\angle LDC=\angle CBK$ gives us $\triangle LDC\sim\triangle CBK$. Therefore $\angle KCL=360^\circ-\angle BCD-(\angle LCD+\angle KCB)=360^\circ-\angle BCD-(\angle CKB+\angle KCB)=180^\circ-\angle CBK$, which is constant.

17. To start with, we note that points B,E,C are the images of D,F,A respectively under the rotation around point O for the angle $\omega=\angle DOB$, where O is the intersection of the perpendicular bisectors of AC and BD. Then $OE=OF$ and $\angle OFE=\angle OAC=90-\frac{\omega}{2}$; hence the points A,F,R,O are on a circle and $\angle ORP=180^\circ-\angle OFA$. Analogously, the points B,E,Q,O are on a circle and $\angle OQP=180^\circ-\angle OEB=\angle OEC=\angle OFA$. This shows that $\angle ORP=180^\circ-\angle OQP$, i.e. the point O lies on the circumcircle of $\triangle PQR$, thus being the desired point.

18. Let O and O_1 be the circumcenters of triangles ABC and ADE, respectively. It is enough to show that $HM\parallel OO_1$. Let AA' be the diameter of the circumcircle of ABC. We note that if B_1 is the foot of the altitude from B, then HE bisects $\angle CHB_1$. Since the triangles COM and CHB_1 are similar (indeed, $\angle CHB=\angle COM=\angle A$), we have $\frac{CE}{EB_1}=\frac{CH}{HB_1}=\frac{CO}{OM}=\frac{2CO}{AH}=\frac{A'A}{AH}$.



Thus, if Q is the intersection point of the bisector of $\angle A'AH$ with HA', we obtain $\frac{CE}{EB_1}=\frac{A'Q}{QH}$, which together with $A'C\perp AC$ and $HB_1\perp AC$ gives us $QE\perp AC$. Analogously, $QD\perp AB$. Therefore AQ is a diameter of the circumcircle of $\triangle ADE$ and O_1 is the midpoint of AQ. It follows that OO_1 is a middle line in $\triangle A'AQ$ which is parallel to HM.

Second solution. We again prove that $OO_1\parallel HM$. Since $AA'=2AO$, it suffices to prove $AQ=2AO_1$.
Elementary calculations of angles give us $\angle ADE=\angle AED=90^\circ-\frac{\alpha}{2}$. Applying the law of sines to $\triangle DAH$ and $\triangle EAH$ we now have $DE=DH+EH=\frac{AH\cos\beta}{\cos\frac{\alpha}{2}}+$

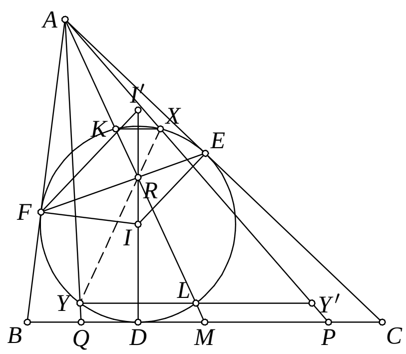
$\frac{AH\cos\gamma}{\cos\frac{\alpha}{2}}$. Since $AH = 2OM = 2R\cos\alpha$, we obtain

$$AO_1 = \frac{DE}{2\sin\alpha} = \frac{AH(\cos\beta + \cos\gamma)}{2\sin\alpha\cos\frac{\alpha}{2}} = \frac{2R\cos\alpha\sin\frac{\alpha}{2}\cos(\frac{\beta-\gamma}{2})}{\sin\alpha\cos\frac{\alpha}{2}}.$$

We now calculate AQ. Let N be the intersection of AQ with the circumcircle. Since $\angle NAO = \frac{\beta-\gamma}{2}$, we have $AN = 2R\cos(\frac{\beta-\gamma}{2})$. Noting that $\triangle QAH \sim \triangle QNM$ (and that $MN = R - OM$), we have

$$AQ = \frac{AN \cdot AH}{MN + AH} = \frac{2R\cos(\frac{\beta-\gamma}{2}) \cdot 2\cos\alpha}{1+\cos\alpha} = \frac{2R\cos(\frac{\beta-\gamma}{2})\cos\alpha}{\cos^2\frac{\alpha}{2}} = 2AO_1.$$

19. We denote by D,E,F the points of tangency of the incircle with BC,CA,AB, respectively, by I the incenter, and by Y' the intersection of AX and LY. Since EF is the polar line to the point A with respect to the incircle, it meets AL at point R such that $A,R;K,L$ are conjugated, i.e. $\frac{KR}{RL} = \frac{KA}{AL}$. Then $\frac{KX}{LY'} = \frac{KA}{AL} = \frac{KR}{RL} = \frac{KX}{L\overline{Y}}$ and therefore $LY = L\overline{Y}$, where $\overline{Y}$ is the intersection of XR and LY. Thus showing that $LY = LY'$ (which is the same as showing that $PM = MQ$, i.e. $CP = QC$) is equivalent to showing that XY contains R. Since $XKYL$ is an inscribed trapezoid, it is enough to show that R lies on its axis of symmetry, that is, DI.



Since AM is the median, the triangles ARB and ARC have equal areas and since $\angle(RF,AB) = \angle(RE,AC)$ we have that $1 = \frac{S_{\triangle ABR}}{S_{\triangle ACR}} = \frac{(AB\cdot FR)}{(AC\cdot ER)}$. Hence $\frac{AB}{AC} = \frac{ER}{FR}$. Let I' be the point of intersction of the line through F parallel to IE with the line IR. Then $\frac{FI'}{EI} = \frac{FR}{RE} = \frac{AC}{AB}$ and $\angle I'FI = \angle BAC$ (angles with orthogonal rays). Thus the triangles ABC and FII' are similar, implying that $\angle FII' = \angle ABC$. Since $\angle FID = 180^\circ - \angle ABC$, it follows that R, I, and D are collinear.

20. We shall show the inequalities $p(ABC) \geq 2p(DEF)$ and $p(PQR) \geq \frac{1}{2}p(DEF)$. The statement of the problem will immediately follow.
Let D_b and D_c be the reflections of D in AB and AC, and let A_1,B_1,C_1 be the midpoints of BC,CA,AB, respectively. It is easy to see that D_b,F,E,D_c are collinear. Hence $p(DEF) = D_bF + FE + ED_c = D_bD_c \leq D_bC_1 + C_1B_1 + B_1D_c = \frac{1}{2}(AB+BC+CA) = \frac{1}{2}p(ABC)$.
To prove the second inequality we observe that P, Q, and R are the points of tangency of the excircles with the sides of $\triangle DEF$. Let $FQ = ER = x$, $DR = FP = y$, and $DQ = EP = z$, and let $\delta,\varepsilon,\varphi$ be the angles of $\triangle DEF$ at D,E,F, respectively. Let Q' and R' be the projections of Q and R onto EF, respectively. Then $QR \geq Q'R' = EF - FQ' - R'E = EF - x(\cos\varphi + \cos\varepsilon)$. Summing this with the analogous inequalities for FD and DE we obtain

$$p(PQR) \geq p(DEF) - x(\cos\varphi + \cos\varepsilon) - y(\cos\delta + \cos\varphi) - z(\cos\delta + \cos\varepsilon).$$

Assuming w.l.o.g. that $x \leq y \leq z$ we also have $DE \leq FD \leq FE$ and consequently $\cos\varphi + \cos\varepsilon \geq \cos\delta + \cos\varphi \geq \cos\delta + \cos\varepsilon$. Now Chebyshev's inequality gives us $p(PQR) \geq p(DEF) - \frac{2}{3}(x+y+z)(\cos\varepsilon + \cos\varphi + \cos\delta) \geq p(DEF) - (x+y+z) = \frac{1}{2}p(DEF)$, where we used $x+y+z = \frac{1}{2}p(DEF)$ and the fact that the sum of the cosines of the angles in a triangle does not exceed $\frac{3}{2}$. This finishes the proof.

21. We will show that 1 is the only such number. It is sufficient to prove that for every prime number p there exists some a_m such that $p \mid a_m$. For $p = 2,3$ we have $p \mid a_2 = 48$. Assume now that $p > 3$. Appyling Fermat's theorem, we have:

$$6a_{p-2} = 3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} - 6 \equiv 3 + 2 + 1 - 6 = 0 \pmod{p}.$$

Hence $p \mid a_{p-2}$, i.e. $\gcd(p, a_{p-2}) = p > 1$. This completes the proof.

22. It immediately follows from the condition of the problem that all the terms of the sequence are distinct. We also note that $|a_i - a_n| \leq n-1$ for all integers i,n where $i < n$, because if $d = |a_i - a_n| \geq n$ then $\{a_1, \dots, a_d\}$ contains two elements congruent to each other modulo d, which is a contradiction. It easily follows by induction that for every $n \in \mathbb{N}$ the set $\{a_1, \dots, a_n\}$ consists of consecutive integers. Thus, if we assumed some integer k did not appear in the sequence $a_1, a_2, \dots$, the same would have to hold for all integers either larger or smaller than k, which contradicts the condition that infinitely many positive and negative integers appear in the sequence. Thus, the sequence contains all integers.

23. Let us consider the polynomial

$$P(x) = (x+a)(x+b)(x+c) - (x-d)(x-e)(x-f) = Sx^2 + Qx + R,$$

where $Q = ab + bc + ca - de - ef - fd$ and $R = abc + def$.
Since $S \mid Q, R$, it follows that $S \mid P(x)$ for every $x \in \mathbb{Z}$. Hence, $S \mid P(d) = (d+a)(d+b)(d+c)$. Since $S > d+a, d+b, d+c$ and thus cannot divide any of them, it follows that S must be composite.

24. We will show that n has the desired property if and only if it is prime.
For $n = 2$ we can take only $a = 1$. For $n > 2$ and even, $4 \mid n!$, but $a^n + 1 \equiv 1,2 \pmod 4$, which is impossible. Now we assume that n is odd. Obviously $(n!-1)^n + 1 \equiv (-1)^n + 1 = 0 \pmod{n!}$. If n is composite and d its prime divisor, then $\left(\frac{n!}{d} - 1\right)^n + 1 = \sum_{k=1}^{n} \binom{n}{k} \frac{n!^k}{d^k}$, where each summand is divisible by $n!$ because $d^2 \mid n!$; therefore $n!$ divides $\left(\frac{n!}{d} - 1\right)^n + 1$. Thus, all composite numbers are ruled out.
It remains to show that if n is an odd prime and $n! \mid a^n + 1$, then $n! \mid a+1$ and therefore $a = n! - 1$ is the only relevant value for which $n! \mid a^n + 1$. Consider any prime number $p \leq n$. If $p \mid \frac{a^n+1}{a+1}$, we have $p \mid (-a)^n - 1$ and by Fermat's theorem $p \mid (-a)^{p-1} - 1$. Therefore $p \mid (-a)^{(n,p-1)} - 1 = -a - 1$, i.e. $a \equiv -1 \pmod{p}$. But then $\frac{a^n+1}{a+1} = a^{n-1} - a^{n-2} + \cdots - a + 1 \equiv n \pmod{p}$, implying that $p = n$. It

follows that $\frac{a^n+1}{a+1}$ is coprime to $(n-1)!$ and consequently $(n-1)!$ divides $a+1$. Moreover, the above consideration shows that n must divide $a+1$. Thus $n! \mid a+1$ as claimed. This finishes our proof.

25. We will use the abbreviation HD to denote a "highly divisible integer". Let $n = 2^{\alpha_2(n)}3^{\alpha_3(n)}\cdots p^{\alpha_p(n)}$ be the factorization of n into primes. We have $d(n) = (\alpha_2(n)+1)\cdots(\alpha_p(n)+1)$. We start with the following two lemmas.

Lemma 1. If n is a HD and p,q primes with $p^k < q^l$ ($k,l \in \mathbb{N}$), then

$$k\alpha_q(n) \le l\alpha_p(n) + (k+1)(l-1).$$

Proof. The inequality is trivial if $\alpha_q(n) < l$. Suppose that $\alpha_q(n) \ge l$. Then np^k/q^l is an integer less than q, and $d(np^k/q^l) < d(n)$, which is equivalent to $(\alpha_q(n)+1)(\alpha_p(n)+1) > (\alpha_q(n)-l+1)(\alpha_p(n)+k+1)$ implying the desired inequality.

Lemma 2. For each p and k there exist only finitely many HD's n such that $\alpha_p(n) \le k$.

Proof. It follows from Lemma 1 that if n is a HD with $\alpha_p(n) \le k$, then $\alpha_q(n)$ is bounded for each prime q and $\alpha_q(n) = 0$ for $q > p^{k+1}$. Therefore there are only finitely many possibilities for n.

We are now ready to prove both parts of the problem.

(a) Suppose that there are infinitely many pairs (a,b) of consecutive HD's with $a \mid b$. Since $d(2a) > d(a)$, we must have $b = 2a$. In particular, $d(s) \le d(a)$ for all $s < 2a$. All but finitely many HD's a are divisible by 2 and by 3^7. Then $d(8a/9) < d(a)$ and $d(3a/2) < d(a)$ yield

$$(\alpha_2(a)+4)(\alpha_3(a)-1) < (\alpha_2(a)+1)(\alpha_3(a)+1) \Rightarrow 3\alpha_3(a) - 5 < 2\alpha_2(a),$$
$$\alpha_2(a)(\alpha_3(a)+2) \le (\alpha_2(a)+1)(\alpha_3(a)+1) \Rightarrow \alpha_2(a) \le \alpha_3(a)+1.$$

We now have $3\alpha_3(a) - 5 < 2\alpha_2(a) \le 2\alpha_3(a) + 2 \Rightarrow \alpha_3(a) < 7$, which is a contradiction.

(b) Assume for a given prime p and positive integer k that n is the smallest HD with $\alpha_p \ge k$. We show that $\frac{n}{p}$ is also a HD. Assume the opposite, i.e. that there exists a HD $m < \frac{n}{p}$ such that $d(m) \ge d(\frac{n}{p})$. By assumption, m must also satisfy $\alpha_p(m) + 1 \le \alpha_p(n)$. Then

$$d(mp) = d(m)\frac{\alpha_p(m)+2}{\alpha_p(m)+1} \ge d(n/p)\frac{\alpha_p(n)+1}{\alpha_p(n)} = d(n),$$

contradicting the initial assumption that n is a HD (since $mp < n$). This proves that $\frac{n}{p}$ is a HD. Since this is true for every positive integer k the proof is complete.

26. Assuming $b \ne a$, it trivially follows that $b > a$. Let $p > b$ be a prime number and let $n = (a+1)(p-1)+1$. We note that $n \equiv 1 \pmod{p-1}$ and $n \equiv -a \pmod{p}$. It follows that $r^n = r \cdot (r^{p-1})^{a+1} \equiv r \pmod{p}$ for every integer r. We now have $a^n + n \equiv a - a = 0 \pmod{p}$. Thus, $a^n + n$ is divisible by p, and hence by the condition

of the problem b^n+n is also divisible by p. However, we also have $b^n+n \equiv b-a \pmod{p}$, i.e. $p \,|\, b-a$, which contradicts $p>b$. Hence, it must follow that $b=a$. We note that $b=a$ trivially fulfills the conditions of the problem for all $a \in \mathbb{N}$.

27. Let p be a prime and $k<p$ an even number. We note that $(p-k)!(k-1)! \equiv (-1)^{k-1}(p-k)!(p-k+1)\dots(p-1) = (-1)^{k-1}(p-1)! \equiv 1 \pmod{p}$ by Wilson's theorem. Therefore

$$\begin{aligned}(k-1)!^n P((p-k)!) &= \textstyle\sum_{i=0}^n a_i[(k-1)!]^{n-i}[(p-k)!(k-1)!]^i \\ &\equiv \textstyle\sum_{i=0}^n a_i[(k-1)!]^{n-i} = S((k-1)!) \pmod{p},\end{aligned}$$

where $S(x)=a_n+a_{n-1}x+\cdots+a_0x^n$. Hence $p \mid P((p-k)!)$ if and only if $p \mid S((k-1)!)$. Note that $S((k-1)!)$ depends only on k. Let $k>2a_n+1$. Then, $s=(k-1)!/a_n$ is an integer which is divisible by all primes smaller than k. Hence $S((k-1)!) = a_nb_k$ for some $b_k \equiv 1 \pmod{s}$. It follows that b_k is divisible only by primes larger than k. For large enough k we have $|b_k|>1$. Thus for every prime divisor p of b_k we have $p \mid P((p-k)!)$.
It remains to select a large enough k for which $|P((p-k)!)|>p$. We take $k=(q-1)!$, where q is a large prime. All the numbers $k+i$ for $i=1,2,\dots,q-1$ are composite (by Wilson's theorem, $q \mid k+1$). Thus $p=k+q+r$, for some $r \geq 0$. We now have $|P((p-k)!)| = |P((q+r)!)| > (q+r)! > (q-1)!+q+r = p$, for large enough q, since $n=\deg P \geq 2$. This completes the proof.
Remark. The above solution actually also works for all linear polynomials P other than $P(x)=x+a_0$. Nevertheless, these particular cases are easily handled. If $|a_0|>1$, then $P(m!)$ is composite for $m>|a_0|$, whereas $P(x)=x+1$ and $P(x)=x-1$ are both composite for, say, $x=5!$. Thus the condition $n \geq 2$ was redundant.

Algebra

A1. A sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined by the formula

$$a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle \qquad \text{for} \quad i \geq 0;$$

here a_0 is an arbitrary real number, $\lfloor a_i \rfloor$ denotes the greatest integer not exceeding a_i, and $\langle a_i \rangle = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for i sufficiently large.

(Estonia)

Solution. First note that if $a_0 \geq 0$, then all $a_i \geq 0$. For $a_i \geq 1$ we have (in view of $\langle a_i \rangle < 1$ and $\lfloor a_i \rfloor > 0$)

$$\lfloor a_{i+1} \rfloor \leq a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle < \lfloor a_i \rfloor;$$

the sequence $\lfloor a_i \rfloor$ is strictly decreasing as long as its terms are in $[1, \infty)$. Eventually there appears a number from the interval $[0, 1)$ and all subsequent terms are 0.

Now pass to the more interesting situation where $a_0 < 0$; then all $a_i \leq 0$. Suppose the sequence never hits 0. Then we have $\lfloor a_i \rfloor \leq -1$ for all i, and so

$$1 + \lfloor a_{i+1} \rfloor > a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle > \lfloor a_i \rfloor;$$

this means that the sequence $\lfloor a_i \rfloor$ is nondecreasing. And since all its terms are integers from $(-\infty, -1]$, this sequence must be constant from some term on:

$$\lfloor a_i \rfloor = c \qquad \text{for} \quad i \geq i_0\,; \qquad c \;\; \text{a negative integer.}$$

The defining formula becomes

$$a_{i+1} = c \cdot \langle a_i \rangle = c(a_i - c) = ca_i - c^2.$$

Consider the sequence

$$b_i = a_i - \frac{c^2}{c-1}\,. \tag{1}$$

It satisfies the recursion rule

$$b_{i+1} = a_{i+1} - \frac{c^2}{c-1} = ca_i - c^2 - \frac{c^2}{c-1} = cb_i,$$

implying

$$b_i = c^{i-i_0} b_{i_0} \qquad \text{for} \quad i \geq i_0. \tag{2}$$

Since all the numbers a_i (for $i \geq i_0$) lie in $[c, c+1)$, the sequence (b_i) is bounded. The equation (2) can be satisfied only if either $b_{i_0} = 0$ or $|c| = 1$, i.e., $c = -1$.

In the first case, $b_i = 0$ for all $i \geq i_0$, so that

$$a_i = \frac{c^2}{c-1} \qquad \text{for} \quad i \geq i_0.$$

In the second case, $c = -1$, equations (1) and (2) say that

$$a_i = -\frac{1}{2} + (-1)^{i-i_0} b_{i_0} = \begin{cases} a_{i_0} & \text{for} \quad i = i_0, i_0+2, i_0+4, \dots, \\ 1 - a_{i_0} & \text{for} \quad i = i_0+1, i_0+3, i_0+5, \dots. \end{cases}$$

Summarising, we see that (from some point on) the sequence (a_i) either is constant or takes alternately two values from the interval $(-1, 0)$. The result follows.

Comment. There is nothing mysterious in introducing the sequence (b_i). The sequence (a_i) arises by iterating the function $x \mapsto cx - c^2$ whose unique fixed point is $c^2/(c-1)$.

A2. The sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined recursively by

$$a_0 = -1, \qquad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \quad \text{for} \quad n \geq 1.$$

Show that $a_n > 0$ for $n \geq 1$.

(Poland)

Solution. The proof goes by induction. For $n = 1$ the formula yields $a_1 = 1/2$. Take $n \geq 1$, assume $a_1, \ldots, a_n > 0$ and write the recurrence formula for n and $n+1$, respectively as

$$\sum_{k=0}^{n} \frac{a_k}{n-k+1} = 0 \qquad \text{and} \qquad \sum_{k=0}^{n+1} \frac{a_k}{n-k+2} = 0.$$

Subtraction yields

$$\begin{aligned} 0 = (n+2) \sum_{k=0}^{n+1} \frac{a_k}{n-k+2} - (n+1) \sum_{k=0}^{n} \frac{a_k}{n-k+1} \\ = (n+2)a_{n+1} + \sum_{k=0}^{n} \left(\frac{n+2}{n-k+2} - \frac{n+1}{n-k+1} \right) a_k. \end{aligned}$$

The coefficient of a_0 vanishes, so

$$a_{n+1} = \frac{1}{n+2} \sum_{k=1}^{n} \left(\frac{n+1}{n-k+1} - \frac{n+2}{n-k+2} \right) a_k = \frac{1}{n+2} \sum_{k=1}^{n} \frac{k}{(n-k+1)(n-k+2)} a_k.$$

The coefficients of $a_1, , \ldots, a_n$ are all positive. Therefore, $a_1, \ldots, a_n > 0$ implies $a_{n+1} > 0$.

Comment. Students familiar with the technique of generating functions will immediately recognise $\sum a_n x^n$ as the power series expansion of $x/\ln(1-x)$ (with value -1 at 0). But this can be a trap; attempts along these lines lead to unpleasant differential equations and integrals hard to handle. Using only tools from real analysis (e.g. computing the coefficients from the derivatives) seems very difficult.

On the other hand, the coefficients can be approached applying complex contour integrals and some other techniques from complex analysis and an attractive formula can be obtained for the coefficients:

$$a_n = \int_1^\infty \frac{\mathrm{d}x}{x^n \big(\pi^2 + \log^2(x-1)\big)} \qquad (n \geq 1)$$

which is evidently positive.

A3. The sequence $c_0, c_1, \ldots, c_n, \ldots$ is defined by $c_0 = 1$, $c_1 = 0$ and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set S of ordered pairs (x, y) for which there is a finite set J of positive integers such that $x = \sum_{j \in J} c_j$, $y = \sum_{j \in J} c_{j-1}$. Prove that there exist real numbers α, β and m, M with the following property: An ordered pair of nonnegative integers (x, y) satisfies the inequality

$$m < \alpha x + \beta y < M$$

if and only if $(x, y) \in S$.

N. B. A sum over the elements of the empty set is assumed to be 0.

(Russia)

Solution. Let $\varphi = (1 + \sqrt{5})/2$ and $\psi = (1 - \sqrt{5})/2$ be the roots of the quadratic equation $t^2 - t - 1 = 0$. So $\varphi\psi = -1$, $\varphi + \psi = 1$ and $1 + \psi = \psi^2$. An easy induction shows that the general term c_n of the given sequence satisfies

$$c_n = \frac{\varphi^{n-1} - \psi^{n-1}}{\varphi - \psi} \qquad \text{for } n \geq 0.$$

Suppose that the numbers α and β have the stated property, for appropriately chosen m and M. Since $(c_n, c_{n-1}) \in S$ for each n, the expression

$$\alpha c_n + \beta c_{n-1} = \frac{\alpha}{\sqrt{5}} \left(\varphi^{n-1} - \psi^{n-1}\right) + \frac{\beta}{\sqrt{5}} \left(\varphi^{n-2} - \psi^{n-2}\right) = \frac{1}{\sqrt{5}} \left[(\alpha\varphi + \beta)\varphi^{n-2} - (\alpha\psi + \beta)\psi^{n-2}\right]$$

is bounded as n grows to infinity. Because $\varphi > 1$ and $-1 < \psi < 0$, this implies $\alpha\varphi + \beta = 0$.

To satisfy $\alpha\varphi + \beta = 0$, one can set for instance $\alpha = \psi$, $\beta = 1$. We now find the required m and M for this choice of α and β.

Note first that the above displayed equation gives $c_n\psi + c_{n-1} = \psi^{n-1}$, $n \geq 1$. In the sequel, we denote the pairs in S by (a_J, b_J), where J is a finite subset of the set $\mathbb{N}$ of positive integers and $a_J = \sum_{j \in J} c_j$, $b_J = \sum_{j \in J} c_{j-1}$. Since $\psi a_J + b_J = \sum_{j \in J}(c_j\psi + c_{j-1})$, we obtain

$$\psi a_J + b_J = \sum_{j \in J} \psi^{j-1} \qquad \text{for each } (a_J, b_J) \in S. \tag{1}$$

On the other hand, in view of $-1 < \psi < 0$,

$$-1 = \frac{\psi}{1 - \psi^2} = \sum_{j=0}^{\infty} \psi^{2j+1} < \sum_{j \in J} \psi^{j-1} < \sum_{j=0}^{\infty} \psi^{2j} = \frac{1}{1 - \psi^2} = 1 - \psi = \varphi.$$

Therefore, according to (1),

$$-1 < \psi a_J + b_J < \varphi \qquad \text{for each } (a_J, b_J) \in S.$$

Thus $m = -1$ and $M = \varphi$ is an appropriate choice.

Conversely, we prove that if an ordered pair of nonnegative integers (x, y) satisfies the inequality $-1 < \psi x + y < \varphi$ then $(x, y) \in S$.

Lemma. Let x, y be nonnegative integers such that $-1 < \psi x + y < \varphi$. Then there exists a subset J of $\mathbb{N}$ such that

$$\psi x + y = \sum_{j \in J} \psi^{j-1} \tag{2}$$

Proof. For $x = y = 0$ it suffices to choose the empty subset of $\mathbb{N}$ as J, so let at least one of x, y be nonzero. There exist representations of $\psi x + y$ of the form

$$\psi x + y = \psi^{i_1} + \cdots + \psi^{i_k}$$

where $i_1 \leq \cdots \leq i_k$ is a sequence of nonnegative integers, not necessarily distinct. For instance, we can take x summands $\psi^1 = \psi$ and y summands $\psi^0 = 1$. Consider all such representations of minimum length k and focus on the ones for which i_1 has the minimum possible value j_1. Among them, consider the representations where i_2 has the minimum possible value j_2. Upon choosing $j_3, \ldots, j_k$ analogously, we obtain a sequence $j_1 \leq \cdots \leq j_k$ which clearly satisfies $\psi x + y = \sum_{r=1}^{k} \psi^{j_r}$. To prove the lemma, it suffices to show that $j_1, \ldots, j_k$ are pairwise distinct.

Suppose on the contrary that $j_r = j_{r+1}$ for some $r = 1, \ldots, k-1$. Let us consider the case $j_r \geq 2$ first. Observing that $2\psi^2 = 1 + \psi^3$, we replace j_r and j_{r+1} by $j_r - 2$ and $j_r + 1$, respectively. Since

$$\psi^{j_r} + \psi^{j_{r+1}} = 2\psi^{j_r} = \psi^{j_r - 2}(1 + \psi^3) = \psi^{j_r - 2} + \psi^{j_r + 1},$$

the new sequence also represents $\psi x + y$ as needed, and the value of i_r in it contradicts the minimum choice of j_r.

Let $j_r = j_{r+1} = 0$. Then the sum $\psi x + y = \sum_{r=1}^{k} \psi^{j_r}$ contains at least two summands equal to $\psi^0 = 1$. On the other hand $j_s \neq 1$ for all s, because the equality $1 + \psi = \psi^2$ implies that a representation of minimum length cannot contain consecutive i_r's. It follows that

$$\psi x + y = \sum_{r=1}^{k} \psi^{j_r} > 2 + \psi^3 + \psi^5 + \psi^7 + \cdots = 2 - \psi^2 = \varphi,$$

contradicting the condition of the lemma.

Let $j_r = j_{r+1} = 1$; then $\sum_{r=1}^{k} \psi^{j_r}$ contains at least two summands equal to $\psi^1 = \psi$. Like in the case $j_r = j_{r+1} = 0$, we also infer that $j_s \neq 0$ and $j_s \neq 2$ for all s. Therefore

$$\psi x + y = \sum_{r=1}^{k} \psi^{j_r} < 2\psi + \psi^4 + \psi^6 + \psi^8 + \cdots = 2\psi - \psi^3 = -1,$$

which is a contradiction again. The conclusion follows. $\square$

Now let the ordered pair (x, y) satisfy $-1 < \psi x + y < \varphi$; hence the lemma applies to (x, y). Let $J \subset \mathbb{N}$ be such that (2) holds. Comparing (1) and (2), we conclude that $\psi x + y = \psi a_J + b_J$. Now, x, y, a_J and b_J are integers, and ψ is irrational. So the last equality implies $x = a_J$ and $y = b_J$. This shows that the numbers $\alpha = \psi$, $\beta = 1$, $m = -1$, $M = \varphi$ meet the requirements.

Comment. We present another way to prove the lemma, constructing the set J inductively. For $x = y = 0$, choose $J = \emptyset$. We induct on $n = 3x + 2y$. Suppose that an appropriate set J exists when $3x + 2y < n$. Now assume $3x + 2y = n > 0$. The current set J should be

$$\text{either} \quad 1 \leq j_1 < j_2 < \cdots < j_k \quad \text{or} \quad j_1 = 0,\ 1 \leq j_2 < \cdots < j_k.$$

These sets fulfil the condition if

$$\frac{\psi x + y}{\psi} = \psi^{i_1 - 1} + \cdots + \psi^{i_k - 1} \quad \text{or} \quad \frac{\psi x + y - 1}{\psi} = \psi^{i_2 - 1} + \cdots + \psi^{i_k - 1},$$

respectively; therefore it suffices to find an appropriate set for $\frac{\psi x+y}{\psi}$ or $\frac{\psi x+y-1}{\psi}$, respectively.

Consider $\frac{\psi x+y}{\psi}$. Knowing that

$$\frac{\psi x+y}{\psi} = x + (\psi-1)y = \psi y + (x-y),$$

let $x' = y$, $y' = x - y$ and test the induction hypothesis on these numbers. We require $\frac{\psi x+y}{\psi} \in (-1, \varphi)$ which is equivalent to

$$\psi x + y \in (\varphi \cdot \psi, (-1) \cdot \psi) = (-1, -\psi). \tag{3}$$

Relation (3) implies $y' = x - y \geq -\psi x - y > \psi > -1$; therefore $x', y' \geq 0$. Moreover, we have $3x' + 2y' = 2x + y \leq \frac{2}{3}n$; therefore, if (3) holds then the induction applies: the numbers x', y' are represented in the form as needed, hence x, y also.

Now consider $\frac{\psi x+y-1}{\psi}$. Since

$$\frac{\psi x+y-1}{\psi} = x + (\psi-1)(y-1) = \psi(y-1) + (x-y+1),$$

we set $x' = y - 1$ and $y' = x - y + 1$. Again we require that $\frac{\psi x+y-1}{\psi} \in (-1, \varphi)$, i.e.

$$\psi x + y \in (\varphi \cdot \psi + 1, (-1) \cdot \psi + 1) = (0, \varphi). \tag{4}$$

If (4) holds then $y - 1 \geq \psi x + y - 1 > -1$ and $x - y + 1 \geq -\psi x - y + 1 > -\varphi + 1 > -1$, therefore $x', y' \geq 0$. Moreover, $3x' + 2y' = 2x + y - 1 < \frac{2}{3}n$ and the induction works.

Finally, $(-1, -\psi) \cup (0, \varphi) = (-1, \varphi)$ so at least one of (3) and (4) holds and the induction step is justified.

A4. Prove the inequality

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \le \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j$$

for positive real numbers $a_1, a_2, \ldots, a_n$.

(Serbia)

Solution 1. Let $S = \sum_i a_i$. Denote by L and R the expressions on the left and right hand side of the proposed inequality. We transform L and R using the identity

$$\sum_{i<j} (a_i + a_j) = (n-1) \sum_i a_i. \tag{1}$$

And thus:

$$L = \sum_{i<j} \frac{a_i a_j}{a_i + a_j} = \sum_{i<j} \frac{1}{4}\Big(a_i + a_j - \frac{(a_i - a_j)^2}{a_i + a_j}\Big) = \frac{n-1}{4} \cdot S - \frac{1}{4} \sum_{i<j} \frac{(a_i - a_j)^2}{a_i + a_j}. \tag{2}$$

To represent R we express the sum $\sum\limits_{i<j} a_i a_j$ in two ways; in the second transformation identity (1) will be applied to the squares of the numbers a_i:

$$\sum_{i<j} a_i a_j = \frac{1}{2}\Big(S^2 - \sum_i a_i^2\Big);$$

$$\sum_{i<j} a_i a_j = \frac{1}{2} \sum_{i<j} \Big(a_i^2 + a_j^2 - (a_i - a_j)^2\Big) = \frac{n-1}{2} \cdot \sum_i a_i^2 - \frac{1}{2} \sum_{i<j} (a_i - a_j)^2.$$

Multiplying the first of these equalities by $n-1$ and adding the second one we obtain

$$n \sum_{i<j} a_i a_j = \frac{n-1}{2} \cdot S^2 - \frac{1}{2} \sum_{i<j} (a_i - a_j)^2.$$

Hence

$$R = \frac{n}{2S} \sum_{i<j} a_i a_j = \frac{n-1}{4} \cdot S - \frac{1}{4} \sum_{i<j} \frac{(a_i - a_j)^2}{S}. \tag{3}$$

Now compare (2) and (3). Since $S \ge a_i + a_j$ for any $i < j$, the claim $L \ge R$ results.

Solution 2. Let $S = a_1 + a_2 + \cdots + a_n$. For any $i \neq j$,

$$4\frac{a_i a_j}{a_i + a_j} = a_i + a_j - \frac{(a_i - a_j)^2}{a_i + a_j} \leq a_i + a_j - \frac{(a_i - a_j)^2}{a_1 + a_2 + \cdots + a_n} = \frac{\sum\limits_{k \neq i} a_i a_k + \sum\limits_{k \neq j} a_j a_k + 2a_i a_j}{S}.$$

The statement is obtained by summing up these inequalities for all pairs i, j:

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{a_i a_j}{a_i + a_j} \leq \frac{1}{8S} \sum_i \sum_{j \neq i} \left(\sum_{k \neq i} a_i a_k + \sum_{k \neq j} a_j a_k + 2a_i a_j \right)$$

$$= \frac{1}{8S} \left(\sum_k \sum_{i \neq k} \sum_{j \neq i} a_i a_k + \sum_k \sum_{j \neq k} \sum_{i \neq j} a_j a_k + \sum_i \sum_{j \neq i} 2a_i a_j \right)$$

$$= \frac{1}{8S} \left(\sum_k \sum_{i \neq k} (n-1) a_i a_k + \sum_k \sum_{j \neq k} (n-1) a_j a_k + \sum_i \sum_{j \neq i} 2a_i a_j \right)$$

$$= \frac{n}{4S} \sum_i \sum_{j \neq i} a_i a_j = \frac{n}{2S} \sum_{i<j} a_i a_j.$$

Comment. Here is an outline of another possible approach. Examine the function $R - L$ subject to constraints $\sum_i a_i = S$, $\sum_{i<j} a_i a_j = U$ for fixed constants $S, U > 0$ (which can jointly occur as values of these symmetric forms). Suppose that among the numbers a_i there are some three, say a_k, a_l, a_m such that $a_k < a_l \leq a_m$. Then it is possible to decrease the value of $R - L$ by perturbing this triple so that in the new triple a'_k, a'_l, a'_m one has $a'_k = a'_l \leq a'_m$, without touching the remaining a_is and without changing the values of S and U; this requires some skill in algebraic manipulations. It follows that the constrained minimum can be only attained for $n - 1$ of the a_is equal and a single one possibly greater. In this case, $R - L \geq 0$ holds almost trivially.

A5. Let a, b, c be the sides of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

(Korea)

Solution 1. Note first that the denominators are all positive, e.g. $\sqrt{a}+\sqrt{b}>\sqrt{a+b}>\sqrt{c}$.

Let $x=\sqrt{b}+\sqrt{c}-\sqrt{a}$, $y=\sqrt{c}+\sqrt{a}-\sqrt{b}$ and $z=\sqrt{a}+\sqrt{b}-\sqrt{c}$. Then

$$b+c-a=\left(\frac{z+x}{2}\right)^2+\left(\frac{x+y}{2}\right)^2-\left(\frac{y+z}{2}\right)^2=\frac{x^2+xy+xz-yz}{2}=x^2-\frac{1}{2}(x-y)(x-z)$$

and

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}=\sqrt{1-\frac{(x-y)(x-z)}{2x^2}} \leq 1-\frac{(x-y)(x-z)}{4x^2},$$

applying $\sqrt{1+2u} \leq 1+u$ in the last step. Similarly we obtain

$$\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \leq 1-\frac{(z-x)(z-y)}{4z^2} \quad \text{and} \quad \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 1-\frac{(y-z)(y-x)}{4y^2}.$$

Substituting these quantities into the statement, it is sufficient to prove that

$$\frac{(x-y)(x-z)}{x^2}+\frac{(y-z)(y-x)}{y^2}+\frac{(z-x)(z-y)}{z^2} \geq 0. \tag{1}$$

By symmetry we can assume $x \leq y \leq z$. Then

$$\frac{(x-y)(x-z)}{x^2}=\frac{(y-x)(z-x)}{x^2} \geq \frac{(y-x)(z-y)}{y^2}=-\frac{(y-z)(y-x)}{y^2},$$

$$\frac{(z-x)(z-y)}{z^2} \geq 0$$

and (1) follows.

Comment 1. Inequality (1) is a special case of the well-known inequality

$$x^t(x-y)(x-z)+y^t(y-z)(y-x)+z^t(z-x)(z-y) \geq 0$$

which holds for all positive numbers x, y, z and real t; in our case $t=-2$. Case $t>0$ is called Schur's inequality. More generally, if $x \leq y \leq z$ are real numbers and p, q, r are nonnegative numbers such that $q \leq p$ or $q \leq r$ then

$$p(x-y)(x-z)+q(y-z)(y-x)+r(z-x)(z-y) \geq 0.$$

Comment 2. One might also start using Cauchy–Schwarz' inequality (or the root mean square vs. arithmetic mean inequality) to the effect that

$$\left(\sum \frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}\right)^2 \le 3 \cdot \sum \frac{b+c-a}{\left(\sqrt{b}+\sqrt{c}-\sqrt{a}\,\right)^2}, \tag{2}$$

in cyclic sum notation. There are several ways to prove that the right-hand side of (2) never exceeds 9 (and this is just what we need). One of them is to introduce new variables x, y, z, as in Solution 1, which upon some manipulation brings the problem again to inequality (1).

Alternatively, the claim that right-hand side of (2) is not greater than 9 can be expressed in terms of the symmetric forms $\sigma_1 = \sum x$, $\sigma_2 = \sum xy$, $\sigma_3 = xyz$ equivalently as

$$4\sigma_1\sigma_2\sigma_3 \le \sigma_2^3 + 9\sigma_3^2, \tag{3}$$

which is a known inequality. A yet different method to deal with the right-hand expression in (2) is to consider $\sqrt{a}, \sqrt{b}, \sqrt{c}$ as sides of a triangle. Through standard trigonometric formulas the problem comes down to showing that

$$p^2 \le 4R^2 + 4Rr + 3r^2, \tag{4}$$

p, R and r standing for the semiperimeter, the circumradius and the inradius of that triangle. Again, (4) is another known inequality. Note that the inequalities (1), (3), (4) are equivalent statements about the same mathematical situation.

Solution 2. Due to the symmetry of variables, it can be assumed that $a \ge b \ge c$. We claim that

$$\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \le 1 \quad \text{and} \quad \frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \le 2.$$

The first inequality follows from

$$\sqrt{a+b-c} - \sqrt{a} = \frac{(a+b-c)-a}{\sqrt{a+b-c}+\sqrt{a}} \le \frac{b-c}{\sqrt{b}+\sqrt{c}} = \sqrt{b} - \sqrt{c}.$$

For proving the second inequality, let $p = \sqrt{a} + \sqrt{b}$ and $q = \sqrt{a} - \sqrt{b}$. Then $a - b = pq$ and the inequality becomes

$$\frac{\sqrt{c-pq}}{\sqrt{c}-q} + \frac{\sqrt{c+pq}}{\sqrt{c}+q} \le 2.$$

From $a \ge b \ge c$ we have $p \ge 2\sqrt{c}$. Applying the Cauchy-Schwarz inequality,

$$\left(\frac{\sqrt{c-pq}}{\sqrt{c}-q} + \frac{\sqrt{c+pq}}{\sqrt{c}+q}\right)^2 \le \left(\frac{c-pq}{\sqrt{c}-q} + \frac{c+pq}{\sqrt{c}+q}\right)\left(\frac{1}{\sqrt{c}-q} + \frac{1}{\sqrt{c}+q}\right)$$

$$= \frac{2(c\sqrt{c}-pq^2)}{c-q^2} \cdot \frac{2\sqrt{c}}{c-q^2} = 4 \cdot \frac{c^2 - \sqrt{c}pq^2}{(c-q^2)^2} \le 4 \cdot \frac{c^2 - 2cq^2}{(c-q^2)^2} \le 4.$$

A6. Determine the smallest number M such that the inequality

$$\left|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)\right| \le M\left(a^2+b^2+c^2\right)^2$$

holds for all real numbers a, b, c.

(Ireland)

Solution. We first consider the cubic polynomial

$$P(t)=tb(t^2-b^2)+bc(b^2-c^2)+ct(c^2-t^2).$$

It is easy to check that $P(b)=P(c)=P(-b-c)=0$, and therefore

$$P(t)=(b-c)(t-b)(t-c)(t+b+c),$$

since the cubic coefficient is $b-c$. The left-hand side of the proposed inequality can therefore be written in the form

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)|=|P(a)|=|(b-c)(a-b)(a-c)(a+b+c)|.$$

The problem comes down to finding the smallest number M that satisfies the inequality

$$|(b-c)(a-b)(a-c)(a+b+c)| \le M\cdot(a^2+b^2+c^2)^2. \tag{1}$$

Note that this expression is symmetric, and we can therefore assume $a\le b\le c$ without loss of generality. With this assumption,

$$|(a-b)(b-c)|=(b-a)(c-b)\le\left(\frac{(b-a)+(c-b)}{2}\right)^2=\frac{(c-a)^2}{4}, \tag{2}$$

with equality if and only if $b-a=c-b$, i.e. $2b=a+c$. Also

$$\left(\frac{(c-b)+(b-a)}{2}\right)^2\le\frac{(c-b)^2+(b-a)^2}{2},$$

or equivalently,

$$3(c-a)^2\le 2\cdot[(b-a)^2+(c-b)^2+(c-a)^2], \tag{3}$$

again with equality only for $2b=a+c$. From (2) and (3) we get

$$\begin{aligned}
&|(b-c)(a-b)(a-c)(a+b+c)|\\
\le\ &\frac{1}{4}\cdot|(c-a)^3(a+b+c)|\\
=\ &\frac{1}{4}\cdot\sqrt{(c-a)^6(a+b+c)^2}\\
\le\ &\frac{1}{4}\cdot\sqrt{\left(\frac{2\cdot[(b-a)^2+(c-b)^2+(c-a)^2]}{3}\right)^3\cdot(a+b+c)^2}\\
=\ &\frac{\sqrt{2}}{2}\cdot\left(\sqrt[4]{\left(\frac{(b-a)^2+(c-b)^2+(c-a)^2}{3}\right)^3\cdot(a+b+c)^2}\right)^2.
\end{aligned}$$

By the weighted AM-GM inequality this estimate continues as follows:

$$\begin{aligned}
& |(b-c)(a-b)(a-c)(a+b+c)| \\
\le\ & \frac{\sqrt{2}}{2}\cdot\left(\frac{(b-a)^2+(c-b)^2+(c-a)^2+(a+b+c)^2}{4}\right)^2 \\
=\ & \frac{9\sqrt{2}}{32}\cdot(a^2+b^2+c^2)^2.
\end{aligned}$$

We see that the inequality (1) is satisfied for $M=\frac{9}{32}\sqrt{2}$, with equality if and only if $2b=a+c$ and

$$\frac{(b-a)^2+(c-b)^2+(c-a)^2}{3}=(a+b+c)^2.$$

Plugging $b=(a+c)/2$ into the last equation, we bring it to the equivalent form

$$2(c-a)^2=9(a+c)^2.$$

The conditions for equality can now be restated as

$$2b=a+c \qquad \text{and} \qquad (c-a)^2=18b^2.$$

Setting $b=1$ yields $a=1-\frac{3}{2}\sqrt{2}$ and $c=1+\frac{3}{2}\sqrt{2}$. We see that $M=\frac{9}{32}\sqrt{2}$ is indeed the smallest constant satisfying the inequality, with equality for any triple (a,b,c) proportional to $\left(1-\frac{3}{2}\sqrt{2},1,1+\frac{3}{2}\sqrt{2}\right)$, up to permutation.

Comment. With the notation $x=b-a$, $y=c-b$, $z=a-c$, $s=a+b+c$ and $r^2=a^2+b^2+c^2$, the inequality (1) becomes just $|sxyz|\le Mr^4$ (with suitable constraints on s and r). The original asymmetric inequality turns into a standard symmetric one; from this point on the solution can be completed in many ways. One can e.g. use the fact that, for fixed values of $\sum x$ and $\sum x^2$, the product xyz is a maximum/minimum only if some of x,y,z are equal, thus reducing one degree of freedom, etc.

As observed by the proposer, a specific attraction of the problem is that the maximum is attained at a point (a,b,c) with all coordinates distinct.

Combinatorics

C1. We have $n \geq 2$ lamps $L_1, \ldots, L_n$ in a row, each of them being either *on* or *off*. Every second we simultaneously modify the state of each lamp as follows:
— if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off;
— otherwise, L_i is switched on.
Initially all the lamps are off except the leftmost one which is on.

(a) Prove that there are infinitely many integers n for which all the lamps will eventually be off.

(b) Prove that there are infinitely many integers n for which the lamps will never be all off.

(France)

Solution. (a) Experiments with small n lead to the guess that every n of the form 2^k should be good. This is indeed the case, and more precisely: let A_k be the $2^k \times 2^k$ matrix whose rows represent the evolution of the system, with entries $0, 1$ (for *off* and *on* respectively). The top row shows the initial state $[1, 0, 0, \ldots, 0]$; the bottom row shows the state after $2^k - 1$ steps. The claim is that:

The bottom row of A_k *is* $[1, 1, 1, \ldots, 1]$.

This will of course suffice because one more move then produces $[0, 0, 0, \ldots, 0]$, as required.

The proof is by induction on k. The base $k = 1$ is obvious. Assume the claim to be true for a $k \geq 1$ and write the matrix A_{k+1} in the block form $\begin{pmatrix} A_k & O_k \\ B_k & C_k \end{pmatrix}$ with four $2^k \times 2^k$ matrices. After m steps, the last 1 in a row is at position $m + 1$. Therefore O_k is the zero matrix. According to the induction hypothesis, the bottom row of $[A_k \ O_k]$ is $[1, \ldots, 1, 0, \ldots, 0]$, with 2^k ones and 2^k zeros. The next row is thus

$$[\underbrace{0, \ldots, 0}_{2^k - 1}, 1, 1, \underbrace{0, \ldots, 0}_{2^k - 1}]$$

It is symmetric about its midpoint, and this symmetry is preserved in all subsequent rows because the procedure described in the problem statement is left/right symmetric. Thus B_k is the mirror image of C_k. In particular, the rightmost column of B_k is identical with the leftmost column of C_k.

Imagine the matrix C_k in isolation from the rest of A_{k+1}. Suppose it is subject to evolution as defined in the problem: the first (leftmost) term in a row depends only on the two first terms in the preceding row, according as they are equal or not. Now embed C_k again in A_k. The 'leftmost' terms in the rows of C_k now have neighbours on their left side—but these neighbours are their exact copies. Consequently the actual evolution within C_k is the same, whether or not C_k is considered as a piece of A_{k+1} or in isolation. And since the top row of C_k is $[1, 0, \ldots, 0]$, it follows that C_k is identical with A_k.

The bottom row of A_k is $[1, 1, \ldots, 1]$; the same is the bottom row of C_k, hence also of B_k, which mirrors C_k. So the bottom row of A_{k+1} consists of ones only and the induction is complete.

(b) There are many ways to produce an infinite sequence of those n for which the state $[0, 0, \ldots, 0]$ will never be achieved. As an example, consider $n = 2^k + 1$ (for $k \geq 1$). The evolution of the system can be represented by a matrix $\mathcal{A}$ of width $2^k + 1$ with infinitely many rows. The top 2^k rows form the matrix A_k discussed above, with one column of zeros attached at its right.

In the next row we then have the vector $[0, 0, \ldots, 0, 1, 1]$. But this is just the second row of $\mathcal{A}$ reversed. Subsequent rows will be mirror copies of the foregoing ones, starting from the second one. So the configuration $[1, 1, 0, \ldots, 0, 0]$, i.e. the second row of $\mathcal{A}$, will reappear. Further rows will periodically repeat this pattern and there will be no row of zeros.

C2. A diagonal of a regular 2006-gon is called *odd* if its endpoints divide the boundary into two parts, each composed of an odd number of sides. Sides are also regarded as odd diagonals.

Suppose the 2006-gon has been dissected into triangles by 2003 nonintersecting diagonals. Find the maximum possible number of isosceles triangles with two odd sides.

(Serbia)

Solution 1. Call an isosceles triangle *odd* if it has two odd sides. Suppose we are given a dissection as in the problem statement. A triangle in the dissection which is odd and isosceles will be called *iso-odd* for brevity.

Lemma. Let AB be one of dissecting diagonals and let $\mathcal{L}$ be the shorter part of the boundary of the 2006-gon with endpoints A, B. Suppose that $\mathcal{L}$ consists of n segments. Then the number of iso-odd triangles with vertices on $\mathcal{L}$ does not exceed $n/2$.

Proof. This is obvious for $n = 2$. Take n with $2 < n \leq 1003$ and assume the claim to be true for every $\mathcal{L}$ of length less than n. Let now $\mathcal{L}$ (endpoints A, B) consist of n segments. Let PQ be the longest diagonal which is a side of an iso-odd triangle PQS with all vertices on $\mathcal{L}$ (if there is no such triangle, there is nothing to prove). Every triangle whose vertices lie on $\mathcal{L}$ is obtuse or right-angled; thus S is the summit of PQS. We may assume that the five points A, P, S, Q, B lie on $\mathcal{L}$ in this order and partition $\mathcal{L}$ into four pieces $\mathcal{L}_{AP}$, $\mathcal{L}_{PS}$, $\mathcal{L}_{SQ}$, $\mathcal{L}_{QB}$ (the outer ones possibly reducing to a point).

By the definition of PQ, an iso-odd triangle cannot have vertices on both $\mathcal{L}_{AP}$ and $\mathcal{L}_{QB}$. Therefore every iso-odd triangle within $\mathcal{L}$ has all its vertices on just one of the four pieces. Applying to each of these pieces the induction hypothesis and adding the four inequalities we get that the number of iso-odd triangles within $\mathcal{L}$ other than PQS does not exceed $n/2$. And since each of $\mathcal{L}_{PS}$, $\mathcal{L}_{SQ}$ consists of an odd number of sides, the inequalities for these two pieces are actually strict, leaving a $1/2 + 1/2$ in excess. Hence the triangle PSQ is also covered by the estimate $n/2$. This concludes the induction step and proves the lemma. $\square$

The remaining part of the solution in fact repeats the argument from the above proof. Consider the longest dissecting diagonal XY. Let $\mathcal{L}_{XY}$ be the shorter of the two parts of the boundary with endpoints X, Y and let XYZ be the triangle in the dissection with vertex Z not on $\mathcal{L}_{XY}$. Notice that XYZ is acute or right-angled, otherwise one of the segments XZ, YZ would be longer than XY. Denoting by $\mathcal{L}_{XZ}$, $\mathcal{L}_{YZ}$ the two pieces defined by Z and applying the lemma to each of $\mathcal{L}_{XY}$, $\mathcal{L}_{XZ}$, $\mathcal{L}_{YZ}$ we infer that there are no more than 2006/2 iso-odd triangles in all, unless XYZ is one of them. But in that case XZ and YZ are odd diagonals and the corresponding inequalities are strict. This shows that also in this case the total number of iso-odd triangles in the dissection, including XYZ, is not greater than 1003.

This bound can be achieved. For this to happen, it just suffices to select a vertex of the 2006-gon and draw a broken line joining every second vertex, starting from the selected one. Since 2006 is even, the line closes. This already gives us the required 1003 iso-odd triangles. Then we can complete the triangulation in an arbitrary fashion.

Solution 2. Let the terms *odd triangle* and *iso-odd triangle* have the same meaning as in the first solution.

Let ABC be an iso-odd triangle, with AB and BC odd sides. This means that there are an odd number of sides of the 2006-gon between A and B and also between B and C. We say that these sides *belong* to the iso-odd triangle ABC.

At least one side in each of these groups does not belong to any other iso-odd triangle. This is so because any odd triangle whose vertices are among the points between A and B has two sides of equal length and therefore has an even number of sides belonging to it in total. Eliminating all sides belonging to any other iso-odd triangle in this area must therefore leave one side that belongs to no other iso-odd triangle. Let us *assign* these two sides (one in each group) to the triangle ABC.

To each iso-odd triangle we have thus assigned a pair of sides, with no two triangles sharing an assigned side. It follows that at most 1003 iso-odd triangles can appear in the dissection.

This value can be attained, as shows the example from the first solution.

C3. Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. Prove that for every real number x

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in S.

NB. A line segment, a point and the empty set are considered as convex polygons of 2, 1 and 0 vertices, respectively.

(Colombia)

Solution 1. For each convex polygon P whose vertices are in S, let $c(P)$ be the number of points of S which are inside P, so that $a(P)+b(P)+c(P)=n$, the total number of points in S. Denoting $1-x$ by y,

$$\sum_P x^{a(P)}y^{b(P)} = \sum_P x^{a(P)}y^{b(P)}(x+y)^{c(P)} = \sum_P \sum_{i=0}^{c(P)} \binom{c(P)}{i} x^{a(P)+i} y^{b(P)+c(P)-i}.$$

View this expression as a homogeneous polynomial of degree n in two independent variables x, y. In the expanded form, it is the sum of terms $x^r y^{n-r}$ $(0 \le r \le n)$ multiplied by some nonnegative integer coefficients.

For a fixed r, the coefficient of $x^r y^{n-r}$ represents the number of ways of choosing a convex polygon P and then choosing some of the points of S inside P so that the number of vertices of P and the number of chosen points inside P jointly add up to r.

This corresponds to just choosing an r-element subset of S. The correspondence is bijective because every set T of points from S splits in exactly one way into the union of two disjoint subsets, of which the first is the set of vertices of a convex polygon — namely, the convex hull of T — and the second consists of some points inside that polygon.

So the coefficient of $x^r y^{n-r}$ equals $\binom{n}{r}$. The desired result follows:

$$\sum_P x^{a(P)}y^{b(P)} = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = (x+y)^n = 1.$$

Solution 2. Apply induction on the number n of points. The case $n=0$ is trivial. Let $n>0$ and assume the statement for less than n points. Take a set S of n points.

Let C be the set of vertices of the convex hull of S, let $m=|C|$.

Let $X\subset C$ be an arbitrary nonempty set. For any convex polygon P with vertices in the set $S\setminus X$, we have $b(P)$ points of S outside P. Excluding the points of X — all outside P — the set $S\setminus X$ contains exactly $b(P)-|X|$ of them. Writing $1-x=y$, by the induction hypothesis

$$\sum_{P\subset S\setminus X} x^{a(P)}y^{b(P)-|X|}=1$$

(where $P\subset S\setminus X$ means that the vertices of P belong to the set $S\setminus X$). Therefore

$$\sum_{P\subset S\setminus X} x^{a(P)}y^{b(P)}=y^{|X|}.$$

All convex polygons appear at least once, except the convex hull C itself. The convex hull adds x^m. We can use the inclusion-exclusion principle to compute the sum of the other terms:

$$\sum_{P\neq C} x^{a(P)}y^{b(P)}=\sum_{k=1}^{m}(-1)^{k-1}\sum_{|X|=k}\sum_{P\subset S\setminus X} x^{a(P)}y^{b(P)}=\sum_{k=1}^{m}(-1)^{k-1}\sum_{|X|=k} y^k$$

$$=\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}y^k=-\big((1-y)^m-1\big)=1-x^m$$

and then

$$\sum_{P} x^{a(P)}y^{b(P)}=\sum_{P=C}+\sum_{P\neq C}=x^m+(1-x^m)=1.$$

C4. A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of *switches*, defined as follows:

A *switch* consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

(Taiwan)

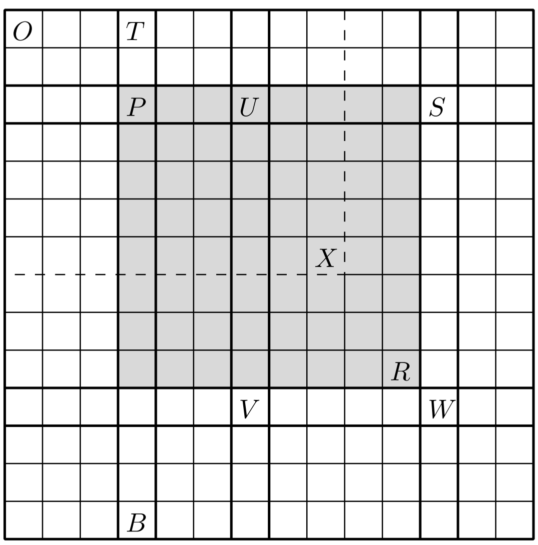
Solution. We use capital letters to denote unit squares; O is the top left corner square. For any two squares X and Y let $[XY]$ be the smallest grid rectangle containing these two squares. Strawberries lie on some squares in arrangement $\mathcal{A}$. Put a plum on each square of the target configuration $\mathcal{B}$. For a square X denote by $a(X)$ and $b(X)$ respectively the number of strawberries and the number of plums in $[OX]$. By hypothesis $a(X) \le b(X)$ for each X, with strict inequality for some X (otherwise the two arrangements coincide and there is nothing to prove).

The idea is to show that by a legitimate switch one can obtain an arrangement $\mathcal{A}'$ such that

$$a(X) \le a'(X) \le b(X) \quad \text{for each } X; \qquad \sum_X a(X) < \sum_X a'(X) \tag{1}$$

(with $a'(X)$ defined analogously to $a(X)$; the sums range over all unit squares X). This will be enough because the same reasoning then applies to $\mathcal{A}'$, giving rise to a new arrangement $\mathcal{A}''$, and so on (induction). Since $\sum a(X) < \sum a'(X) < \sum a''(X) < \ldots$ and all these sums do not exceed $\sum b(X)$, we eventually obtain a sum with all summands equal to the respective $b(X)$s; all strawberries will meet with plums.

Consider the uppermost row in which the plum and the strawberry lie on different squares P and S (respectively); clearly P must be situated left to S. In the column passing through P, let T be the top square and B the bottom square. The strawberry in that column lies below the plum (because there is no plum in that column above P, and the positions of strawberries and plums coincide everywhere above the row of P). Hence there is at least one strawberry in the region $[BS]$ below $[PS]$. Let V be the position of the uppermost strawberry in that region.



Denote by W the square at the intersection of the row through V with the column through S and let R be the square vertex-adjacent to W up-left. We claim that

$$a(X) < b(X) \qquad \text{for all} \quad X \in [PR]. \tag{2}$$

This is so because if $X \in [PR]$ then the portion of $[OX]$ left to column $[TB]$ contains at least as many plums as strawberries (the hypothesis of the problem); in the portion above the row through P and S we have perfect balance; and in the remaining portion, i.e. rectangle $[PX]$ we have a plum on square P and no strawberry at all.

Now we are able to perform the required switch. Let U be the square at the intersection of the row through P with the column through V (some of P, U, R can coincide). We move strawberries from squares S and V to squares U and W. Then

$$a'(X) = a(X) + 1 \quad \text{for} \quad X \in [UR]; \qquad\qquad a'(X) = a(X) \quad \text{for other } X.$$

And since the rectangle $[UR]$ is contained in $[PR]$, we still have $a'(X) \leq b(X)$ for all S, in view of (2); conditions (1) are satisfied and the proof is complete.

C5. An (n,k)-tournament is a contest with n players held in k rounds such that:

(i) Each player plays in each round, and every two players meet at most once.

(ii) If player A meets player B in round i, player C meets player D in round i, and player A meets player C in round j, then player B meets player D in round j.

Determine all pairs (n,k) for which there exists an (n,k)-tournament.

(Argentina)

Solution. For each k, denote by t_k the unique integer such that $2^{t_k-1} < k+1 \le 2^{t_k}$. We show that an (n,k)-tournament exists if and only if 2^{t_k} divides n.

First we prove that if $n=2^t$ for some t then there is an (n,k)-tournament for all $k \le 2^t-1$. Let S be the set of $0-1$ sequences with length t. We label the 2^t players with the elements of S in an arbitrary fashion (which is possible as there are exactly 2^t sequences in S). Players are identified with their labels in the construction below. If $\alpha, \beta \in S$, let $\alpha+\beta \in S$ be the result of the modulo 2 term-by-term addition of α and β (with rules $0+0=0$, $0+1=1+0=1$, $1+1=0$; there is no carryover). For each $i=1,\ldots,2^t-1$ let $\omega(i) \in S$ be the sequence of base 2 digits of i, completed with leading zeros if necessary to achieve length t.

Now define a tournament with $n=2^t$ players in $k \le 2^t-1$ rounds as follows: For all $i=1,\ldots,k$, let player α meet player $\alpha+\omega(i)$ in round i. The tournament is well-defined as $\alpha+\omega(i) \in S$ and $\alpha+\omega(i)=\beta+\omega(i)$ implies $\alpha=\beta$; also $[\alpha+\omega(i)]+\omega(i)=\alpha$ for each $\alpha \in S$ (meaning that player $\alpha+\omega(i)$ meets player α in round i, as needed). Each player plays in each round. Next, every two players meet at most once (exactly once if $k=2^t-1$), since $\omega(i) \neq \omega(j)$ if $i \neq j$. Thus condition (i) holds true, and condition (ii) is also easy to check.

Let player α meet player β in round i, player γ meet player δ in round i, and player α meet player γ in round j. Then $\beta=\alpha+\omega(i)$, $\delta=\gamma+\omega(i)$ and $\gamma=\alpha+\omega(j)$. By definition, β will play in round j with

$$\beta+\omega(j)=[\alpha+\omega(i)]+\omega(j)=[\alpha+\omega(j)]+\omega(i)=\gamma+\omega(i)=\delta,$$

as required by (ii).

So there exists an (n,k)-tournament for pairs (n,k) such that $n=2^t$ and $k \le 2^t-1$. The same conclusion is straightforward for n of the form $n=2^ts$ and $k \le 2^t-1$. Indeed, consider s different $(2^t,k)$-tournaments $T_1,\ldots,T_s$, no two of them having players in common. Their union can be regarded as a $(2^ts,k)$-tournament T where each round is the union of the respective rounds in $T_1,\ldots,T_s$.

In summary, the condition that 2^{t_k} divides n is sufficient for an (n,k)-tournament to exist. We prove that it is also necessary.

Consider an arbitrary (n,k)-tournament. Represent each player by a point and after each round, join by an edge every two players who played in this round. Thus to a round $i=1,\ldots,k$ there corresponds a graph G_i. We say that player Q is an *i-neighbour* of player P if there is a path of edges in G_i from P to Q; in other words, if there are players $P=X_1,X_2,\ldots,X_m=Q$ such that player X_j meets player X_{j+1} in one of the first i rounds, $j=1,2\ldots,m-1$. The set of i-neighbours of a player will be called its *i-component*. Clearly two i-components are either disjoint or coincide.

Hence after each round i the set of players is partitioned into pairwise disjoint i-components. So, to achieve our goal, it suffices to show that all k-components have size divisible by 2^{t_k}.

To this end, let us see how the i-component Γ of a player A changes after round $i+1$. Suppose that A meets player B with i-component Δ in round $i+1$ (components Γ and Δ are not necessarily distinct). We claim that then *in round* $i+1$ *each player from* Γ *meets a player from* Δ*, and vice versa.*

Indeed, let C be any player in Γ, and let C meet D in round $i+1$. Since C is an i-neighbour of A, there is a sequence of players $A = X_1, X_2, \ldots, X_m = C$ such that X_j meets X_{j+1} in one of the first i rounds, $j = 1, 2 \ldots, m-1$. Let X_j meet Y_j in round $i+1$, for $j = 1, 2 \ldots, m$; in particular $Y_1 = B$ and $Y_m = D$. Players Y_j exists in view of condition (i). Suppose that X_j and X_{j+1} met in round r, where $r \leq i$. Then condition (ii) implies that and Y_j and Y_{j+1} met in round r, too. Hence $B = Y_1, Y_2, \ldots, Y_m = D$ is a path in G_i from B to D. This is to say, D is in the i-component Δ of B, as claimed. By symmetry, each player from Δ meets a player from Γ in round $i+1$. It follows in particular that Γ and Δ have the same cardinality.

It is straightforward now that the $(i+1)$-component of A is $\Gamma \cup \Delta$, the union of two sets with the same size. Since Γ and Δ are either disjoint or coincide, we have either $|\Gamma \cup \Delta| = 2|\Gamma|$ or $|\Gamma \cup \Delta| = |\Gamma|$; as usual, $|\cdots|$ denotes the cardinality of a finite set.

Let $\Gamma_1, \ldots, \Gamma_k$ be the consecutive components of a given player A. We obtained that either $|\Gamma_{i+1}| = 2|\Gamma_i|$ or $|\Gamma_{i+1}| = |\Gamma_i|$ for $i = 1, \ldots, k-1$. Because $|\Gamma_1| = 2$, each $|\Gamma_i|$ is a power of 2, $i = 1, \ldots, k-1$. In particular $|\Gamma_k| = 2^u$ for some u.

On the other hand, player A has played with k different opponents by (i). All of them belong to Γ_k, therefore $|\Gamma_k| \geq k+1$.

Thus $2^u \geq k+1$, and since t_k is the least integer satisfying $2^{t_k} \geq k+1$, we conclude that $u \geq t_k$. So the size of each k-component is divisible by 2^{t_k}, which completes the argument.

C6. A *holey triangle* is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A *diamond* is a $60^\circ - 120^\circ$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \leq k \leq n$.

(Colombia)

Solution. Let T be a holey triangle. The unit triangles in it will be called *cells.* We say simply "triangle" instead of "upward equilateral triangle" and "size" instead of "side length."

The necessity will be proven first. Assume that a holey triangle T can be tiled with diamonds and consider such a tiling. Let T' be a triangle of size k in T containing h holes. Focus on the diamonds which cover (one or two) cells in T'. Let them form a figure R. The boundary of T' consists of upward cells, so R is a triangle of size k with h upward holes cut out and possibly some downward cells sticking out. Hence there are exactly $(k^2+k)/2 - h$ upward cells in R, and at least $(k^2-k)/2$ downward cells (not counting those sticking out). On the other hand each diamond covers one upward and one downward cell, which implies $(k^2+k)/2 - h \geq (k^2-k)/2$. It follows that $h \leq k$, as needed.

We pass on to the sufficiency. For brevity, let us say that a set of holes in a given triangle T is *spread out* if every triangle of size k in T contains at most k holes. For any set S of spread out holes, a triangle of size k will be called *full of S* if it contains exactly k holes of S. The proof is based on the following observation.

Lemma. Let S be a set of spread out holes in T. Suppose that two triangles T' and T'' are full of S, and that they touch or intersect. Let $T' + T''$ denote the smallest triangle in T containing them. Then $T' + T''$ is also full of S.

Proof. Let triangles T', T'', $T' \cap T''$ and $T' + T''$ have sizes a, b, c and d, and let them contain a, b, x and y holes of S, respectively. (Note that $T' \cap T''$ could be a point, in which case $c = 0$.) Since S is spread out, we have $x \leq c$ and $y \leq d$. The geometric configuration of triangles clearly satisfies $a + b = c + d$. Furthermore, $a + b \leq x + y$, since $a + b$ counts twice the holes in $T' \cap T''$. These conclusions imply $x = c$ and $y = d$, as we wished to show. $\square$

Now let T_n be a holey triangle of size n, and let the set H of its holes be spread out. We show by induction on n that T_n can be tiled with diamonds. The base $n = 1$ is trivial. Suppose that $n \geq 2$ and that the claim holds for holey triangles of size less than n.

Denote by B the bottom row of T_n and by T' the triangle formed by its top $n-1$ rows. There is at least one hole in B as T' contains at most $n-1$ holes. If this hole is only one, there is a unique way to tile B with diamonds. Also, T' contains exactly $n-1$ holes, making it a holey triangle of size $n-1$, and these holes are spread out. Hence it remains to apply the induction hypothesis.

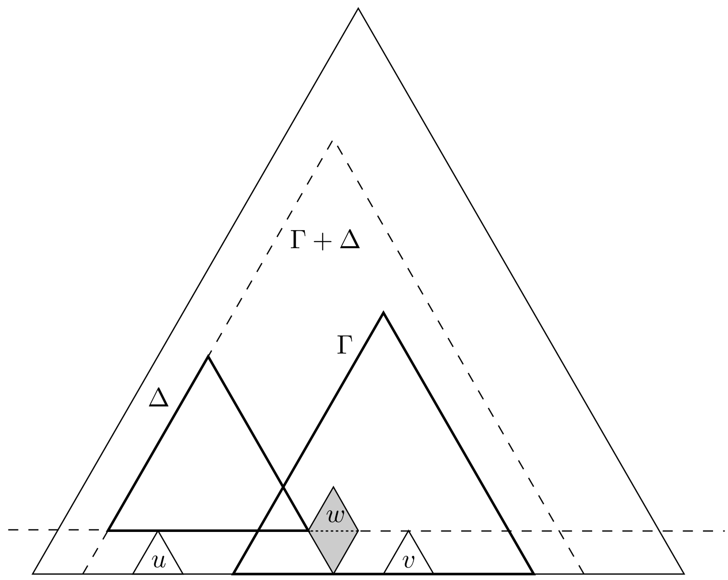
So suppose that there are $m \geq 2$ holes in B and label them $a_1, \ldots, a_m$ from left to right. Let ℓ be the line separating B from T'. For each $i = 1, \ldots, m-1$, pick an upward cell b_i between a_i and a_{i+1}, with base on ℓ. Place a diamond to cover b_i and its lower neighbour, a downward cell in B. The remaining part of B can be tiled uniquely with diamonds. Remove from T_n row B and the cells $b_1, \ldots, b_{m-1}$ to obtain a holey triangle T_{n-1} of size $n-1$. The conclusion will follow by induction if the choice of $b_1, \ldots, b_{m-1}$ guarantees that the following condition is satisfied: If the holes $a_1, \ldots, a_{m-1}$ are replaced by $b_1, \ldots, b_{m-1}$ then the new set of holes is spread out again.

We show that such a choice is possible. The cells $b_1, \ldots, b_{m-1}$ can be defined one at a time in this order, making sure that the above condition holds at each step. Thus it suffices to prove that there is an appropriate choice for b_1, and we set $a_1 = u$, $a_2 = v$ for clarity.

Let Δ be the triangle of maximum size which is full of H, contains the top vertex of the hole u, and has base on line ℓ. Call Δ the *associate of* u. Observe that Δ does not touch v. Indeed, if Δ has size r then it contains r holes of T_n. Extending its slanted sides downwards produces a triangle Δ' of size $r+1$ containing at least one more hole, namely u. Since there are at most $r+1$ holes in Δ', it cannot contain v. Consequently, Δ does not contain the top vertex of v.

Let w be the upward cell with base on ℓ which is to the right of Δ and shares a common vertex with it. The observation above shows that w is to the left of v. Note that w is not a hole, or else Δ could be extended to a larger triangle full of H.

We prove that if the hole u is replaced by w then the new set of holes is spread out again. To verify this, we only need to check that if a triangle Γ in T_n contains w but not u then Γ is not full of H. Suppose to the contrary that Γ is full of H. Consider the minimum triangle $\Gamma+\Delta$ containing Γ and the associate Δ of u. Clearly $\Gamma+\Delta$ is larger than Δ, because Γ contains w but Δ does not. Next, $\Gamma+\Delta$ is full of $H \setminus \{u\}$ by the lemma, since Γ and Δ have a common point and neither of them contains u.



If Γ is above line ℓ then so is $\Gamma+\Delta$, which contradicts the maximum choice of Δ. If Γ contains cells from row B, observe that $\Gamma+\Delta$ contains u. Let s be the size of $\Gamma+\Delta$. Being full of $H \setminus \{u\}$, $\Gamma+\Delta$ contains s holes other than u. But it also contains u, contradicting the assumption that H is spread out.

The claim follows, showing that $b_1 = w$ is an appropriate choice for $a_1 = u$ and $a_2 = v$. As explained above, this is enough to complete the induction.

C7. Consider a convex polyhedron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it.

Call a pair of points of the polyhedron *antipodal* if there exist two parallel planes passing through these points and such that the polyhedron is contained between these planes.

Let A be the number of antipodal pairs of vertices, and let B be the number of antipodal pairs of midpoints of edges. Determine the difference $A-B$ in terms of the numbers of vertices, edges and faces.

(Japan)

Solution 1. Denote the polyhedron by Γ; let its vertices, edges and faces be $V_1, V_2, \ldots, V_n$, $E_1, E_2, \ldots, E_m$ and $F_1, F_2, \ldots, F_\ell$, respectively. Denote by Q_i the midpoint of edge E_i.

Let S be the unit sphere, the set of all unit vectors in three-dimensional space. Map the boundary elements of Γ to some objects on S as follows.

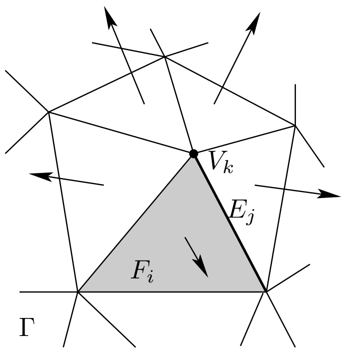
For a face F_i, let $S^+(F_i)$ and $S^-(F_i)$ be the unit normal vectors of face F_i, pointing outwards from Γ and inwards to Γ, respectively. These points are diametrically opposite.

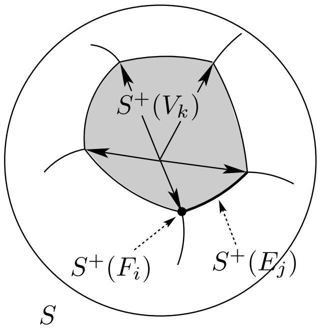
For an edge E_j, with neighbouring faces F_{i_1} and F_{i_2}, take all support planes of Γ (planes which have a common point with Γ but do not intersect it) containing edge E_j, and let $S^+(E_j)$ be the set of their outward normal vectors. The set $S^+(E_j)$ is an arc of a great circle on S. Arc $S^+(E_j)$ is perpendicular to edge E_j and it connects points $S^+(F_{i_1})$ and $S^+(F_{i_2})$.

Define also the set of inward normal vectors $S^-(E_i)$ which is the reflection of $S^+(E_i)$ across the origin.

For a vertex V_k, which is the common endpoint of edges $E_{j_1}, \ldots, E_{j_h}$ and shared by faces $F_{i_1}, \ldots, F_{i_h}$, take all support planes of Γ through point V_k and let $S^+(V_k)$ be the set of their outward normal vectors. This is a region on S, a spherical polygon with vertices $S^+(F_{i_1}), \ldots, S^+(F_{i_h})$ bounded by arcs $S^+(E_{j_1}), \ldots, S^+(E_{j_h})$. Let $S^-(V_k)$ be the reflection of $S^+(V_k)$, the set of inward normal vectors.

Note that region $S^+(V_k)$ is convex in the sense that it is the intersection of several half spheres.





Now translate the conditions on Γ to the language of these objects.

(a) Polyhedron Γ has no parallel edges — the great circles of arcs $S^+(E_i)$ and $S^-(E_j)$ are different for all $i \neq j$.

(b) If an edge E_i does not belong to a face F_j then they are not parallel — the great circle which contains arcs $S^+(E_i)$ and $S^-(E_i)$ does not pass through points $S^+(F_j)$ and $S^-(F_j)$.

(c) Polyhedron Γ has no parallel faces — points $S^+(F_i)$ and $S^-(F_j)$ are pairwise distinct.

The regions $S^+(V_k)$, arcs $S^+(E_j)$ and points $S^+(F_i)$ provide a decomposition of the surface of the sphere. Regions $S^-(V_k)$, arcs $S^-(E_j)$ and points $S^-(F_i)$ provide the reflection of this decomposition. These decompositions are closely related to the problem.

Lemma 1. For any $1 \le i, j \le n$, regions $S^-(V_i)$ and $S^+(V_j)$ overlap if and only if vertices V_i and V_j are antipodal.

Lemma 2. For any $1 \le i, j \le m$, arcs $S^-(E_i)$ and $S^+(E_j)$ intersect if and only if the midpoints Q_i and Q_j of edges E_i and E_j are antipodal.

Proof of lemma 1. First note that by properties (a,b,c) above, the two regions cannot share only a single point or an arc. They are either disjoint or they overlap.

Assume that the two regions have a common interior point u. Let P_1 and P_2 be two parallel support planes of Γ through points V_i and V_j, respectively, with normal vector u. By the definition of regions $S^-(V_i)$ and $S^+(V_j)$, u is the inward normal vector of P_1 and the outward normal vector of P_2. Therefore polyhedron Γ lies between the two planes; vertices V_i and V_j are antipodal.

To prove the opposite direction, assume that V_i and V_j are antipodal. Then there exist two parallel support planes P_1 and P_2 through V_i and V_j, respectively, such that Γ is between them. Let u be the inward normal vector of P_1; then u is the outward normal vector of P_2, therefore $u \in S^-(V_i) \cap S^+(V_j)$. The two regions have a common point, so they overlap. $\Box$

Proof of lemma 2. Again, by properties (a,b) above, the endpoints of arc $S^-(E_i)$ cannot belong to $S^+(E_j)$ and vice versa. The two arcs are either disjoint or intersecting.

Assume that arcs $S^-(E_i)$ and $S^+(E_j)$ intersect at point u. Let P_1 and P_2 be the two support planes through edges E_i and E_j, respectively, with normal vector u. By the definition of arcs $S^-(E_i)$ and $S^+(E_j)$, vector u points inwards from P_1 and outwards from P_2. Therefore Γ is between the planes. Since planes P_1 and P_2 pass through Q_i and Q_j, these points are antipodal.

For the opposite direction, assume that points Q_i and Q_j are antipodal. Let P_1 and P_2 be two support planes through these points, respectively. An edge cannot intersect a support plane, therefore E_i and E_j lie in the planes P_1 and P_2, respectively. Let u be the inward normal vector of P_1, which is also the outward normal vector of P_2. Then $u \in S^-(E_i) \cap S^+(E_j)$. So the two arcs are not disjoint; they therefore intersect. $\Box$

Now create a new decomposition of sphere S. Draw all arcs $S^+(E_i)$ and $S^-(E_j)$ on sphere S and put a knot at each point where two arcs meet. We have ℓ knots at points $S^+(F_i)$ and another ℓ knots at points $S^-(F_i)$, corresponding to the faces of Γ; by property (c) they are different. We also have some pairs $1 \le i, j \le m$ where arcs $S^-(E_i)$ and $S^+(E_j)$ intersect. By Lemma 2, each antipodal pair (Q_i, Q_j) gives rise to two such intersections; hence, the number of all intersections is $2B$ and we have $2\ell + 2B$ knots in all.

Each intersection knot splits two arcs, increasing the number of arcs by 2. Since we started with $2m$ arcs, corresponding the edges of Γ, the number of the resulting curve segments is $2m + 4B$.

The network of these curve segments divides the sphere into some "new" regions. Each new region is the intersection of some overlapping sets $S^-(V_i)$ and $S^+(V_j)$. Due to the convexity, the intersection of two overlapping regions is convex and thus contiguous. By Lemma 1, each pair of overlapping regions corresponds to an antipodal vertex pair and each antipodal vertex pair gives rise to two different overlaps, which are symmetric with respect to the origin. So the number of new regions is $2A$.

The result now follows from Euler's polyhedron theorem. We have $n + l = m + 2$ and

$$(2\ell + 2B) + 2A = (2m + 4B) + 2,$$

therefore

$$A - B = m - \ell + 1 = n - 1.$$

Therefore $A - B$ is by one less than the number of vertices of Γ.

Solution 2. Use the same notations for the polyhedron and its vertices, edges and faces as in Solution 1. We regard points as vectors starting from the origin. Polyhedron Γ is regarded as a closed convex set, including its interior. In some cases the edges and faces of Γ are also regarded as sets of points. The symbol ∂ denotes the boundary of the certain set; e.g. $\partial\Gamma$ is the surface of Γ.

Let $\Delta = \Gamma - \Gamma = \{U - V : U, V \in \Gamma\}$ be the set of vectors between arbitrary points of Γ. Then Δ, being the sum of two bounded convex sets, is also a bounded convex set and, by construction, it is also centrally symmetric with respect to the origin. We will prove that Δ is also a polyhedron and express the numbers of its faces, edges and vertices in terms n, m, ℓ, A and B.

Lemma 1. For points $U, V \in \Gamma$, point $W = U - V$ is a boundary point of Δ if and only if U and V are antipodal. Moreover, for each boundary point $W \in \partial\Delta$ there exists exactly one pair of points $U, V \in \Gamma$ such that $W = U - V$.

Proof. Assume first that U and V are antipodal points of Γ. Let parallel support planes P_1 and P_2 pass through them such that Γ is in between. Consider plane $P = P_1 - U = P_2 - V$. This plane separates the interiors of $\Gamma - U$ and $\Gamma - V$. After reflecting one of the sets, e.g. $\Gamma - V$, the sets $\Gamma - U$ and $-\Gamma + V$ lie in the same half space bounded by P. Then $(\Gamma - U) + (-\Gamma + V) = \Delta - W$ lies in that half space, so $0 \in P$ is a boundary point of the set $\Delta - W$. Translating by W we obtain that W is a boundary point of Δ.

To prove the opposite direction, let $W = U - V$ be a boundary point of Δ, and let $\Psi = (\Gamma - U) \cap (\Gamma - V)$. We claim that $\Psi = \{0\}$. Clearly Ψ is a bounded convex set and $0 \in \Psi$. For any two points $X, Y \in \Psi$, we have $U + X, V + Y \in \Gamma$ and $W + (X - Y) = (U + X) - (V + Y) \in \Delta$. Since W is a boundary point of Δ, the vector $X - Y$ cannot have the same direction as W. This implies that the interior of Ψ is empty. Now suppose that Ψ contains a line segment S. Then $S + U$ and $S + V$ are subsets of some faces or edges of Γ and these faces/edges are parallel to S. In all cases, we find two faces, two edges, or a face and an edge which are parallel, contradicting the conditions of the problem. Therefore, $\Psi = \{0\}$ indeed.

Since $\Psi = (\Gamma - U) \cap (\Gamma - V)$ consists of a single point, the interiors of bodies $\Gamma - U$ and $\Gamma - V$ are disjoint and there exists a plane P which separates them. Let u be the normal vector of P pointing into that half space bounded by P which contains $\Gamma - U$. Consider the planes $P + U$ and $P + V$; they are support planes of Γ, passing through U and V, respectively. From plane $P + U$, the vector u points into that half space which contains Γ. From plane $P + V$, vector u points into the opposite half space containing Γ. Therefore, we found two proper support through points U and V such that Γ is in between.

For the uniqueness part, assume that there exist points $U_1, V_1 \in \Gamma$ such that $U_1 - V_1 = U - V$. The points $U_1 - U$ and $V_1 - V$ lie in the sets $\Gamma - U$ and $\Gamma - V$ separated by P. Since $U_1 - U = V_1 - V$, this can happen only if both are in P; but the only such point is 0. Therefore, $U_1 - V_1 = U - V$ implies $U_1 = U$ and $V_1 = V$. The lemma is complete. □

Lemma 2. Let U and V be two antipodal points and assume that plane P, passing through 0, separates the interiors of $\Gamma - U$ and $\Gamma - V$. Let $\Psi_1 = (\Gamma - U) \cap P$ and $\Psi_2 = (\Gamma - V) \cap P$. Then $\Delta \cap (P + U - V) = \Psi_1 - \Psi_2 + U - V$.

Proof. The sets $\Gamma - U$ and $-\Gamma + V$ lie in the same closed half space bounded by P. Therefore, for any points $X \in (\Gamma - U)$ and $Y \in (-\Gamma + V)$, we have $X + Y \in P$ if and only if $X, Y \in P$. Then

$$(\Delta - (U - V)) \cap P = \big((\Gamma - U) + (-\Gamma + V)\big) \cap P = \big((\Gamma - U) \cap P\big) + \big((-\Gamma + V) \cap P\big) = \Psi_1 - \Psi_2.$$

Now a translation by $(U - V)$ completes the lemma. □

Now classify the boundary points $W = U - V$ of Δ, according to the types of points U and V. In all cases we choose a plane P through 0 which separates the interiors of $\Gamma - U$ and $\Gamma - V$. We will use the notation $\Psi_1 = (\Gamma - U) \cap P$ and $\Psi_2 = (\Gamma - V) \cap P$ as well.

Case 1: Both U and V are vertices of Γ. Bodies $\Gamma - U$ and $\Gamma - V$ have a common vertex which is 0. Choose plane P in such a way that $\Psi_1 = \Psi_2 = \{0\}$. Then Lemma 2 yields $\Delta \cap (P + W) = \{W\}$. Therefore $P + W$ is a support plane of Δ such that they have only one common point so no line segment exists on $\partial\Delta$ which would contain W in its interior.

Since this case occurs for antipodal vertex pairs and each pair is counted twice, the number of such boundary points on Δ is $2A$.

Case 2: Point U is an interior point of an edge E_i and V is a vertex of Γ. Choose plane P such that $\Psi_1 = E_i - U$ and $\Psi_2 = \{0\}$. By Lemma 2, $\Delta \cap (P + W) = E_i - V$. Hence there exists a line segment in $\partial\Delta$ which contains W in its interior, but there is no planar region in $\partial\Delta$ with the same property.

We obtain a similar result if V belongs to an edge of Γ and U is a vertex.

Case 3: Points U and V are interior points of edges E_i and E_j, respectively. Let P be the plane of $E_i - U$ and $E_j - V$. Then $\Psi_1 = E_i - U$, $\Psi_2 = E_j - V$ and $\Delta \cap (P + W) = E_i - E_j$. Therefore point W belongs to a parallelogram face on $\partial\Delta$.

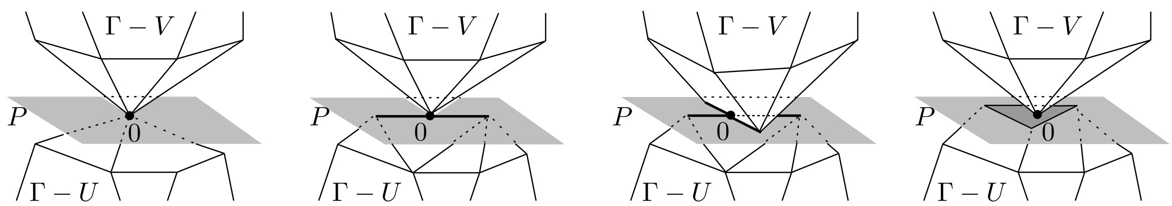
The centre of the parallelogram is $Q_i - Q_j$, the vector between the midpoints. Therefore an edge pair (E_i, E_j) occurs if and only if Q_i and Q_j are antipodal which happens $2B$ times.

Case 4: Point U lies in the interior of a face F_i and V is a vertex of Γ. The only choice for P is the plane of $F_i - U$. Then we have $\Psi_1 = F_i - U$, $\Psi_2 = \{0\}$ and $\Delta \cap (P + W) = F_i - V$. This is a planar face of $\partial\Delta$ which is congruent to F_i.

For each face F_i, the only possible vertex V is the farthest one from the plane of F_i.

If U is a vertex and V belongs to face F_i then we obtain the same way that W belongs to a face $-F_i + U$ which is also congruent to F_i. Therefore, each face of Γ has two copies on $\partial\Delta$, a translated and a reflected copy.

Case 5: Point U belongs to a face F_i of Γ and point V belongs to an edge or a face G. In this case objects F_i and G must be parallel which is not allowed.



Now all points in $\partial\Delta$ belong to some planar polygons (cases 3 and 4), finitely many line segments (case 2) and points (case 1). Therefore Δ is indeed a polyhedron. Now compute the numbers of its vertices, edges and faces.

The vertices are obtained in case 1, their number is $2A$.

Faces are obtained in cases 3 and 4. Case 3 generates $2B$ parallelogram faces. Case 4 generates 2ℓ faces.

We compute the number of edges of Δ from the degrees (number of sides) of faces of Γ. Let d_i be the the degree of face F_i. The sum of degrees is twice as much as the number of edges, so $d_1 + d_2 + \ldots + d_l = 2m$. The sum of degrees of faces of Δ is $2B \cdot 4 + 2(d_1 + d_2 + \cdots + d_l) = 8B + 4m$, so the number of edges on Δ is $4B + 2m$.

Applying Euler's polyhedron theorem on Γ and Δ, we have $n + l = m + 2$ and $2A + (2B + 2\ell) = (4B + 2m) + 2$. Then the conclusion follows:

$$A - B = m - \ell + 1 = n - 1.$$

Geometry

G1. Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies

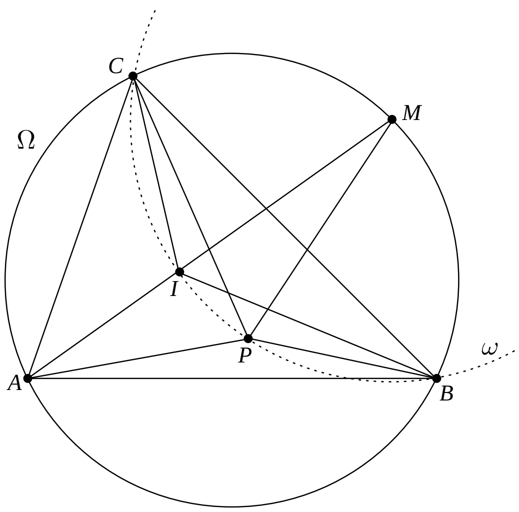
$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$ and that equality holds if and only if P coincides with I.

(Korea)

Solution. Let $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$. Since $\angle PBA + \angle PCA + \angle PBC + \angle PCB = \beta + \gamma$, the condition from the problem statement is equivalent to $\angle PBC + \angle PCB = (\beta + \gamma)/2$, i. e. $\angle BPC = 90^\circ + \alpha/2$.

On the other hand $\angle BIC = 180^\circ - (\beta + \gamma)/2 = 90^\circ + \alpha/2$. Hence $\angle BPC = \angle BIC$, and since P and I are on the same side of BC, the points B, C, I and P are concyclic. In other words, P lies on the circumcircle ω of triangle BCI.



Let Ω be the circumcircle of triangle ABC. It is a well-known fact that the centre of ω is the midpoint M of the arc BC of Ω. This is also the point where the angle bisector AI intersects Ω.

From triangle APM we have

$$AP + PM \geq AM = AI + IM = AI + PM.$$

Therefore $AP \geq AI$. Equality holds if and only if P lies on the line segment AI, which occurs if and only if $P = I$.

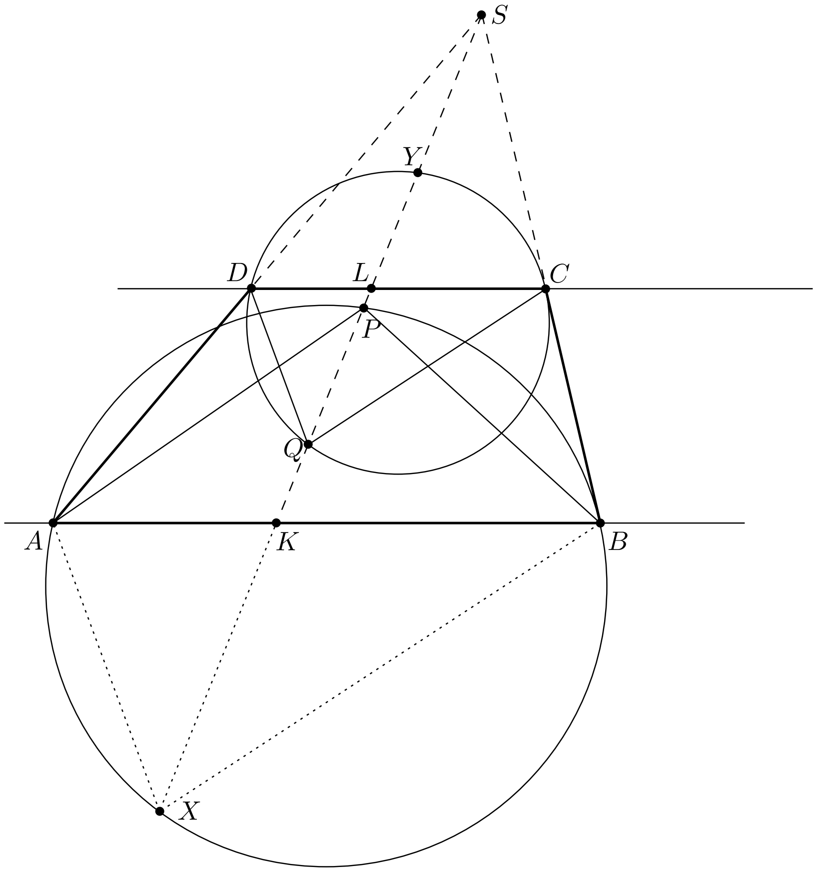
G2. Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points P, Q, B and C are concyclic.

(Ukraine)

Solution 1. Because $AB \parallel CD$, the relation $AK/KB = DL/LC$ readily implies that the lines AD, BC and KL have a common point S.



Consider the second intersection points X and Y of the line SK with the circles (ABP) and (CDQ), respectively. Since $APBX$ is a cyclic quadrilateral and $AB \parallel CD$, one has

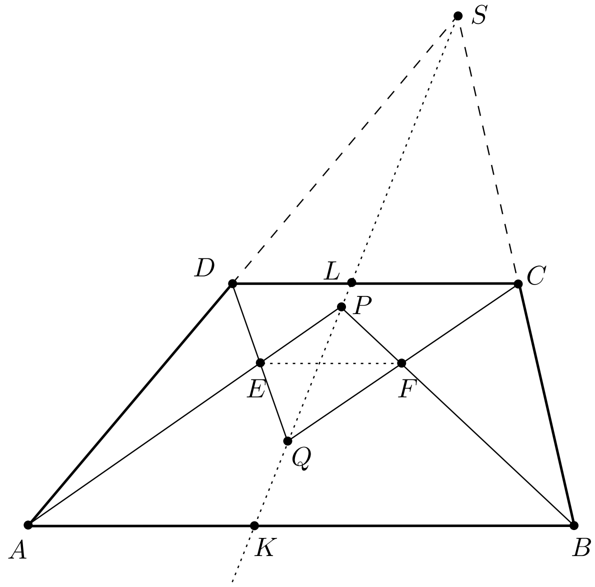
$$\angle AXB = 180^\circ - \angle APB = 180^\circ - \angle BCD = \angle ABC.$$

This shows that BC is tangent to the circle (ABP) at B. Likewise, BC is tangent to the circle (CDQ) at C. Therefore $SP \cdot SX = SB^2$ and $SQ \cdot SY = SC^2$.

Let h be the homothety with centre S and ratio SC/SB. Since $h(B) = C$, the above conclusion about tangency implies that h takes circle (ABP) to circle (CDQ). Also, h takes AB to CD, and it easily follows that $h(P) = Y$, $h(X) = Q$, yielding $SP/SY = SB/SC = SX/SQ$.

Equalities $SP \cdot SX = SB^2$ and $SQ/SX = SC/SB$ imply $SP \cdot SQ = SB \cdot SC$, which is equivalent to P, Q, B and C being concyclic.

Solution 2. The case where $P = Q$ is trivial. Thus assume that P and Q are two distinct points. As in the first solution, notice that the lines AD, BC and KL concur at a point S.



Let the lines AP and DQ meet at E, and let BP and CQ meet at F. Then $\angle EPF = \angle BCD$ and $\angle FQE = \angle ABC$ by the condition of the problem. Since the angles BCD and ABC add up to $180°$, it follows that $PEQF$ is a cyclic quadrilateral.

Applying Menelaus' theorem, first to triangle ASP and line DQ and then to triangle BSP and line CQ, we have

$$\frac{AD}{DS} \cdot \frac{SQ}{QP} \cdot \frac{PE}{EA} = 1 \quad \text{and} \quad \frac{BC}{CS} \cdot \frac{SQ}{QP} \cdot \frac{PF}{FB} = 1.$$

The first factors in these equations are equal, as $AB \| CD$. Thus the last factors are also equal, which implies that EF is parallel to AB and CD. Using this and the cyclicity of $PEQF$, we obtain

$$\angle BCD = \angle BCF + \angle FCD = \angle BCQ + \angle EFQ = \angle BCQ + \angle EPQ.$$

On the other hand,

$$\angle BCD = \angle APB = \angle EPF = \angle EPQ + \angle QPF,$$

and consequently $\angle BCQ = \angle QPF$. The latter angle either coincides with $\angle QPB$ or is supplementary to $\angle QPB$, depending on whether Q lies between K and P or not. In either case it follows that P, Q, B and C are concyclic.

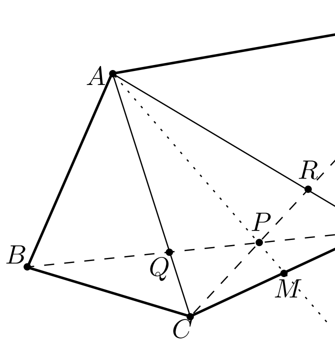
G3. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

(USA)

Solution. Let the diagonals AC and BD meet at Q, the diagonals AD and CE meet at R, and let the ray AP meet the side CD at M. We want to prove that $CM = MD$ holds.



The idea is to show that Q and R divide AC and AD in the same ratio, or more precisely

$$\frac{AQ}{QC} = \frac{AR}{RD} \tag{1}$$

(which is equivalent to $QR \| CD$). The given angle equalities imply that the triangles ABC, ACD and ADE are similar. We therefore have

$$\frac{AB}{AC} = \frac{AC}{AD} = \frac{AD}{AE}.$$

Since $\angle BAD = \angle BAC + \angle CAD = \angle CAD + \angle DAE = \angle CAE$, it follows from $AB/AC = AD/AE$ that the triangles ABD and ACE are also similar. Their angle bisectors in A are AQ and AR, respectively, so that

$$\frac{AB}{AC} = \frac{AQ}{AR}.$$

Because $AB/AC = AC/AD$, we obtain $AQ/AR = AC/AD$, which is equivalent to (1). Now Ceva's theorem for the triangle ACD yields

$$\frac{AQ}{QC} \cdot \frac{CM}{MD} \cdot \frac{DR}{RA} = 1.$$

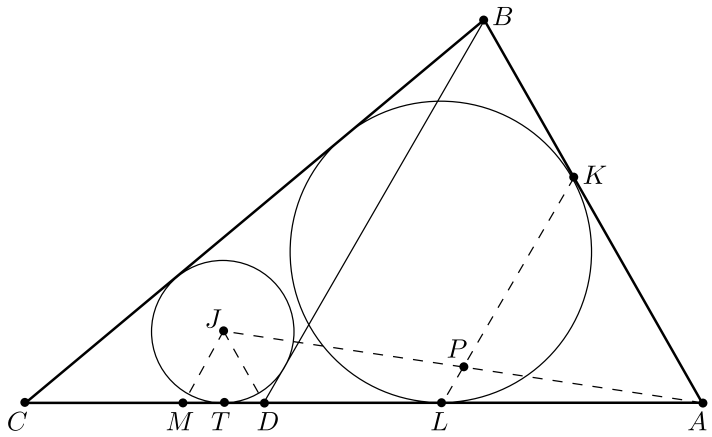
In view of (1), this reduces to $CM = MD$, which completes the proof.

Comment. Relation (1) immediately follows from the fact that quadrilaterals $ABCD$ and $ACDE$ are similar.

G4. A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incentre of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.

(Russia)

Solution. Denote by P be the common point of AJ and KL. Let the parallel to KL through J meet AC at M. Then P is the midpoint of AJ if and only if $AM = 2 \cdot AL$, which we are about to show.



Denoting $\angle BAC = 2\alpha$, the equalities $BA = BD$ and $AK = AL$ imply $\angle ADB = 2\alpha$ and $\angle ALK = 90^\circ - \alpha$. Since DJ bisects $\angle BDC$, we obtain $\angle CDJ = \frac{1}{2} \cdot (180^\circ - \angle ADB) = 90^\circ - \alpha$. Also $\angle DMJ = \angle ALK = 90^\circ - \alpha$ since $JM \| KL$. It follows that $JD = JM$.

Let the incircle of triangle BCD touch its side CD at T. Then $JT \perp CD$, meaning that JT is the altitude to the base DM of the isosceles triangle DMJ. It now follows that $DT = MT$, and we have

$$DM = 2 \cdot DT = BD + CD - BC.$$

Therefore

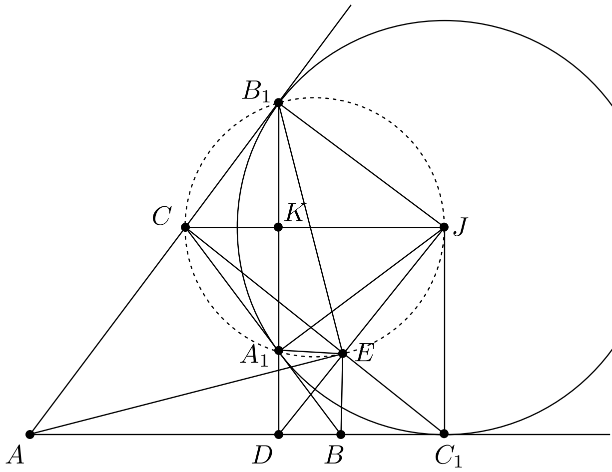
$$\begin{aligned} AM &= AD + (BD + CD - BC) \\ &= AD + AB + DC - BC \\ &= AC + AB - BC \\ &= 2 \cdot AL, \end{aligned}$$

which completes the proof.

G5. In triangle ABC, let J be the centre of the excircle tangent to side BC at A_1 and to the extensions of sides AC and AB at B_1 and C_1, respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

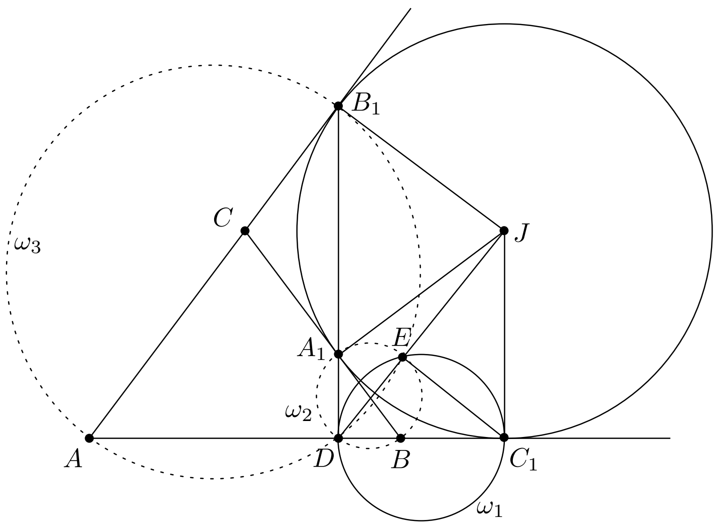
(Greece)

Solution 1. Let K be the intersection point of lines JC and A_1B_1. Obviously $JC \perp A_1B_1$ and since $A_1B_1 \perp AB$, the lines JK and C_1D are parallel and equal. From the right triangle B_1CJ we obtain $JC_1^2 = JB_1^2 = JC \cdot JK = JC \cdot C_1D$ from which we infer that $DC_1/C_1J = C_1J/JC$ and the right triangles DC_1J and C_1JC are similar. Hence $\angle C_1DJ = \angle JC_1C$, which implies that the lines DJ and C_1C are perpendicular, i.e. the points C_1, E, C are collinear.



Since $\angle CA_1J = \angle CB_1J = \angle CEJ = 90°$, points A_1, B_1 and E lie on the circle of diameter CJ. Then $\angle DBA_1 = \angle A_1CJ = \angle DEA_1$, which implies that quadrilateral BEA_1D is cyclic; therefore $\angle A_1EB = 90°$.

Quadrilateral $ADEB_1$ is also cyclic because $\angle EB_1A = \angle EJC = \angle EDC_1$, therefore we obtain $\angle AEB_1 = \angle ADB = 90°$.



Solution 2. Consider the circles ω_1, ω_2 and ω_3 of diameters C_1D, A_1B and AB_1, respectively. Line segments JC_1, JB_1 and JA_1 are tangents to those circles and, due to the right angle at D, ω_2 and ω_3 pass through point D. Since $\angle C_1ED$ is a right angle, point E lies on circle ω_1, therefore

$$JC_1^2 = JD \cdot JE.$$

Since $JA_1 = JB_1 = JC_1$ are all radii of the excircle, we also have

$$JA_1^2 = JD \cdot JE \qquad \text{and} \qquad JB_1^2 = JD \cdot JE.$$

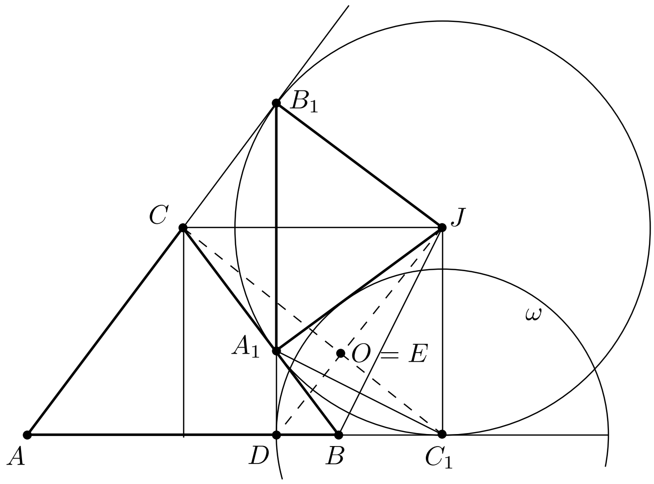
These equalities show that E lies on circles ω_2 and ω_3 as well, so $\angle BEA_1 = \angle AEB_1 = 90°$.

Solution 3. First note that A_1B_1 is perpendicular to the external angle bisector CJ of $\angle BCA$ and parallel to the internal angle bisector of that angle. Therefore, A_1B_1 is perpendicular to AB if and only if triangle ABC is isosceles, $AC = BC$. In that case the external bisector CJ is parallel to AB.

Triangles ABC and B_1A_1J are similar, as their corresponding sides are perpendicular. In particular, we have $\angle DA_1J = \angle C_1BA_1$; moreover, from cyclic deltoid JA_1BC_1,

$$\angle C_1A_1J = \angle C_1BJ = \frac{1}{2}\angle C_1BA_1 = \frac{1}{2}\angle DA_1J.$$

Therefore, A_1C_1 bisects angle $\angle DA_1J$.



In triangle B_1A_1J, line JC_1 is the external bisector at vertex J. The point C_1 is the intersection of two external angle bisectors (at A_1 and J) so C_1 is the centre of the excircle ω, tangent to side A_1J, and to the extension of B_1A_1 at point D.

Now consider the similarity transform φ which moves B_1 to A, A_1 to B and J to C. This similarity can be decomposed into a rotation by 90° around a certain point O and a homothety from the same centre. This similarity moves point C_1 (the centre of excircle ω) to J and moves D (the point of tangency) to C_1.

Since the rotation angle is 90°, we have $\angle XO\varphi(X) = 90°$ for an arbitrary point $X \neq O$. For $X = D$ and $X = C_1$ we obtain $\angle DOC_1 = \angle C_1OJ = 90°$. Therefore O lies on line segment DJ and C_1O is perpendicular to DJ. This means that $O = E$.

For $X = A_1$ and $X = B_1$ we obtain $\angle A_1OB = \angle B_1OA = 90°$, i.e.

$$\angle BEA_1 = \angle AEB_1 = 90°.$$

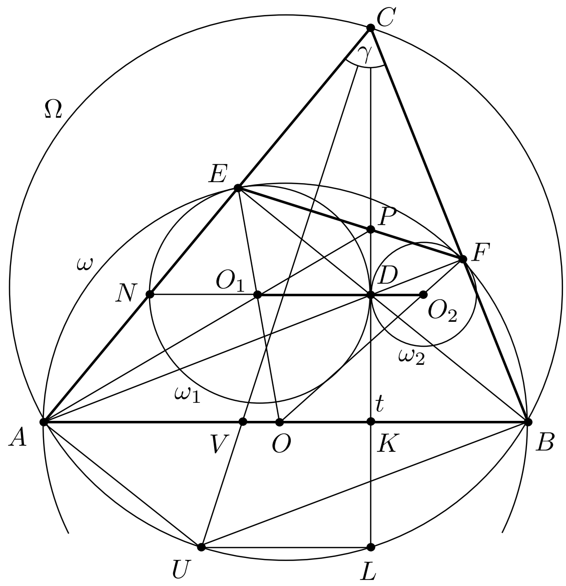
Comment. Choosing $X = J$, it also follows that $\angle JEC = 90°$ which proves that lines DJ and CC_1 intersect at point E. However, this is true more generally, without the assumption that A_1B_1 and AB are perpendicular, because points C and D are conjugates with respect to the excircle. The last observation could replace the first paragraph of Solution 1.

G6. Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F, respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E and O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

(Brasil)

Solution 1. Point E is the centre of a homothety h which takes circle ω_1 to circle ω. The radii O_1D and OB of these circles are parallel as both are perpendicular to line t. Also, O_1D and OB are on the same side of line EO, hence h takes O_1D to OB. Consequently, points E, D and B are collinear. Likewise, points F, D and A are collinear as well.

Let lines AE and BF intersect at C. Since AF and BE are altitudes in triangle ABC, their common point D is the orthocentre of this triangle. So CD is perpendicular to AB, implying that C lies on line t. Note that triangle ABC is acute-angled. We mention the well-known fact that triangles FEC and ABC are similar in ratio $\cos\gamma$, where $\gamma = \angle ACB$. In addition, points C, E, D and F lie on the circle with diameter CD.



Let P be the common point of lines EF and t. We are going to prove that P lies on line AO_1. Denote by N the second common point of circle ω_1 and AC; this is the point of ω_1 diametrically opposite to D. By Menelaus' theorem for triangle DCN, points A, O_1 and P are collinear if and only if

$$\frac{CA}{AN} \cdot \frac{NO_1}{O_1D} \cdot \frac{DP}{PC} = 1.$$

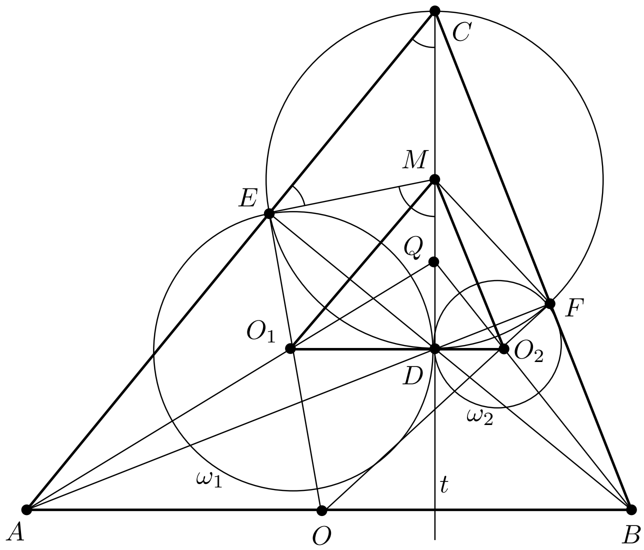
Because $NO_1 = O_1D$, this reduces to $CA/AN = CP/PD$. Let line t meet AB at K. Then $CA/AN = CK/KD$, so it suffices to show that

$$\frac{CP}{PD} = \frac{CK}{KD}. \qquad (1)$$

To verify (1), consider the circumcircle Ω of triangle ABC. Draw its diameter CU through C, and let CU meet AB at V. Extend CK to meet Ω at L. Since AB is parallel to UL, we have $\angle ACU = \angle BCL$. On the other hand $\angle EFC = \angle BAC$, $\angle FEC = \angle ABC$ and $EF/AB = \cos\gamma$, as stated above. So reflection in the bisector of $\angle ACB$ followed by a homothety with centre C and ratio $1/\cos\gamma$ takes triangle FEC to triangle ABC. Consequently, this transformation takes CD to CU, which implies $CP/PD = CV/VU$. Next, we have $KL = KD$, because D is the orthocentre of triangle ABC. Hence $CK/KD = CK/KL$. Finally, $CV/VU = CK/KL$ because AB is parallel to UL. Relation (1) follows, proving that P lies on line AO_1. By symmetry, P also lies on line AO_2 which completes the solution.

Solution 2. We proceed as in the first solution to define a triangle ABC with orthocentre D, in which AF and BE are altitudes.

Denote by M the midpoint of CD. The quadrilateral $CEDF$ is inscribed in a circle with centre M, hence $MC = ME = MD = MF$.



Consider triangles ABC and O_1O_2M. Lines O_1O_2 and AB are parallel, both of them being perpendicular to line t. Next, MO_1 is the line of centres of circles (CEF) and ω_1 whose common chord is DE. Hence MO_1 bisects $\angle DME$ which is the external angle at M in the isosceles triangle CEM. It follows that $\angle DMO_1 = \angle DCA$, so that MO_1 is parallel to AC. Likewise, MO_2 is parallel to BC.

Thus the respective sides of triangles ABC and O_1O_2M are parallel; in addition, these triangles are not congruent. Hence there is a homothety taking ABC to O_1O_2M. The lines AO_1, BO_2 and $CM = t$ are concurrent at the centre Q of this homothety.

Finally, apply Pappus' theorem to the triples of collinear points A, O, B and O_2, D, O_1. The theorem implies that the points $AD \cap OO_2 = F$, $AO_1 \cap BO_2 = Q$ and $OO_1 \cap BD = E$ are collinear. In other words, line EF passes through the common point Q of AO_1, BO_2 and t.

Comment. Relation (1) from Solution 1 expresses the well-known fact that points P and K are harmonic conjugates with respect to points C and D. It is also easy to justify it by direct computation. Denoting $\angle CAB = \alpha$, $\angle ABC = \beta$, it is straightforward to obtain $CP/PD = CK/KD = \tan\alpha \tan\beta$.

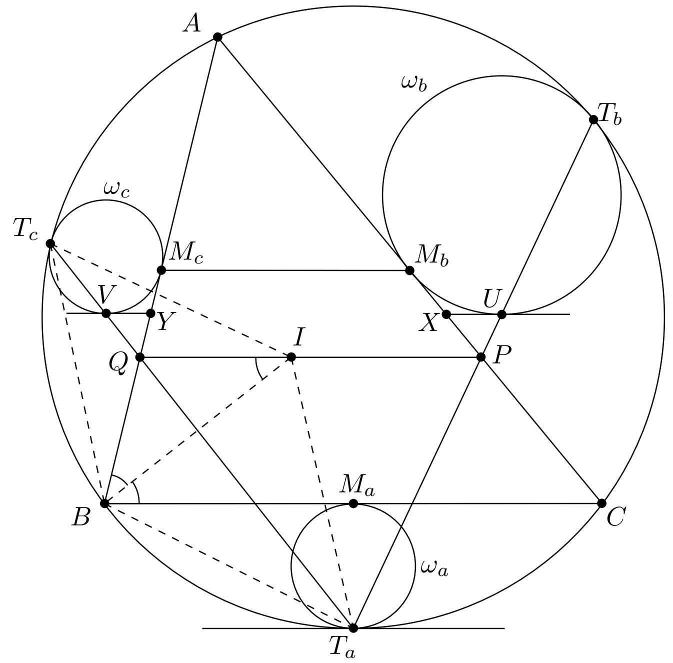
G7. In a triangle ABC, let M_a, M_b, M_c be respectively the midpoints of the sides BC, CA, AB and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the opposite vertices. For $i \in \{a, b, c\}$, let ω_i be the circle with M_iT_i as diameter. Let p_i be the common external tangent to ω_j, ω_k ($\{i, j, k\} = \{a, b, c\}$) such that ω_i lies on the opposite side of p_i than ω_j, ω_k do. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.

(Slovakia)

Solution. Let T_aT_b intersect circle ω_b at T_b and U, and let T_aT_c intersect circle ω_c at T_c and V. Further, let UX be the tangent to ω_b at U, with X on AC, and let VY be the tangent to ω_c at V, with Y on AB. The homothety with centre T_b and ratio T_bT_a/T_bU maps the circle ω_b onto the circumcircle of ABC and the line UX onto the line tangent to the circumcircle at T_a, which is parallel to BC; thus $UX \| BC$. The same is true of VY, so that $UX \| BC \| VY$.

Let T_aT_b cut AC at P and let T_aT_c cut AB at Q. The point X lies on the hypotenuse PM_b of the right triangle PUM_b and is equidistant from U and M_b. So X is the midpoint of M_bP. Similarly Y is the midpoint of M_cQ.

Denote the incentre of triangle ABC as usual by I. It is a known fact that $T_aI = T_aB$ and $T_cI = T_cB$. Therefore the points B and I are symmetric across T_aT_c, and consequently $\angle QIB = \angle QBI = \angle IBC$. This implies that BC is parallel to the line IQ, and likewise, to IP. In other words, PQ is the line parallel to BC passing through I.



Clearly $M_bM_c \| BC$. So PM_bM_cQ is a trapezoid and the segment XY connects the midpoints of its nonparallel sides; hence $XY \| BC$. This combined with the previously established relations $UX \| BC \| VY$ shows that all the four points U, X, Y, V lie on a line which is the common tangent to circles ω_b, ω_c. Since it leaves these two circles on one side and the circle ω_a on the other, this line is just the line p_a from the problem statement.

Line p_a runs midway between I and M_bM_c. Analogous conclusions hold for the lines p_b and p_c. So these three lines form a triangle homothetic from centre I to triangle $M_aM_bM_c$ in ratio $1/2$, hence similar to ABC in ratio $1/4$.

G8. Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that

$$\angle PAB + \angle PDC \leq 90^\circ \qquad \text{and} \qquad \angle PBA + \angle PCD \leq 90^\circ.$$

Prove that $AB + CD \geq BC + AD$.

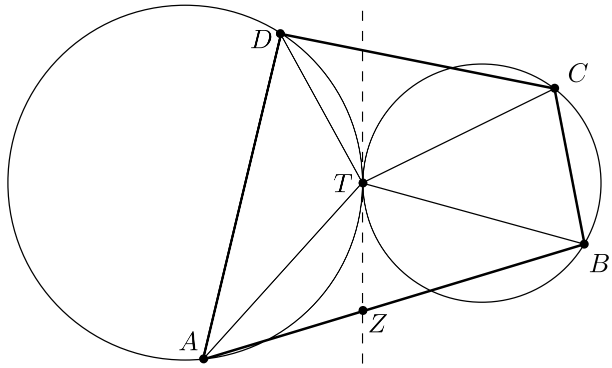
(Poland)

Solution. We start with a preliminary observation. Let T be a point inside the quadrilateral $ABCD$. Then:

$$\begin{array}{l}\textit{Circles } (BCT) \textit{ and } (DAT) \textit{ are tangent at } T \\ \textit{if and only if} \quad \angle ADT + \angle BCT = \angle ATB.\end{array} \tag{1}$$

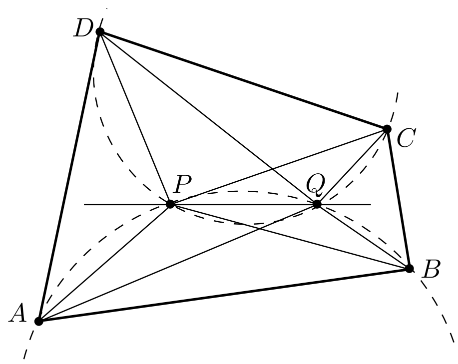
Indeed, if the two circles touch each other then their common tangent at T intersects the segment AB at a point Z, and so $\angle ADT = \angle ATZ$, $\angle BCT = \angle BTZ$, by the tangent-chord theorem. Thus $\angle ADT + \angle BCT = \angle ATZ + \angle BTZ = \angle ATB$.

And conversely, if $\angle ADT + \angle BCT = \angle ATB$ then one can draw from T a ray TZ with Z on AB so that $\angle ADT = \angle ATZ$, $\angle BCT = \angle BTZ$. The first of these equalities implies that TZ is tangent to the circle (DAT); by the second equality, TZ is tangent to the circle (BCT), so the two circles are tangent at T.



So the equivalence (1) is settled. It will be used later on. Now pass to the actual solution. Its key idea is to introduce the circumcircles of triangles ABP and CDP and to consider their second intersection Q (assume for the moment that they indeed meet at two distinct points P and Q).

Since the point A lies outside the circle (BCP), we have $\angle BCP + \angle BAP < 180^\circ$. Therefore the point C lies outside the circle (ABP). Analogously, D also lies outside that circle. It follows that P and Q lie on the same arc CD of the circle (BCP).



By symmetry, P and Q lie on the same arc AB of the circle (ABP). Thus the point Q lies either inside the angle BPC or inside the angle APD. Without loss of generality assume that Q lies inside the angle BPC. Then

$$\angle AQD = \angle PQA + \angle PQD = \angle PBA + \angle PCD \leq 90^\circ, \tag{2}$$

by the condition of the problem.

In the cyclic quadrilaterals $APQB$ and $DPQC$, the angles at vertices A and D are acute. So their angles at Q are obtuse. This implies that Q lies not only inside the angle BPC but in fact inside the triangle BPC, hence also inside the quadrilateral $ABCD$.

Now an argument similar to that used in deriving (2) shows that

$$\angle BQC = \angle PAB + \angle PDC \leq 90^\circ. \tag{3}$$

Moreover, since $\angle PCQ = \angle PDQ$, we get

$$\angle ADQ + \angle BCQ = \angle ADP + \angle PDQ + \angle BCP - \angle PCQ = \angle ADP + \angle BCP.$$

The last sum is equal to $\angle APB$, according to the observation (1) applied to $T = P$. And because $\angle APB = \angle AQB$, we obtain

$$\angle ADQ + \angle BCQ = \angle AQB.$$

Applying now (1) to $T = Q$ we conclude that the circles (BCQ) and (DAQ) are externally tangent at Q. (We have assumed $P \neq Q$; but if $P = Q$ then the last conclusion holds trivially.)

Finally consider the halfdiscs with diameters BC and DA constructed inwardly to the quadrilateral $ABCD$. They have centres at M and N, the midpoints of BC and DA respectively. In view of (2) and (3), these two halfdiscs lie entirely inside the circles (BQC) and (AQD); and since these circles are tangent, the two halfdiscs cannot overlap. Hence $MN \geq \frac{1}{2}BC + \frac{1}{2}DA$.

On the other hand, since $\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{CD})$, we have $MN \leq \frac{1}{2}(AB + CD)$. Thus indeed $AB + CD \geq BC + DA$, as claimed.

G9. Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2, respectively ($A_2 \neq A$, $B_2 \neq B$, $C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

(Russia)

Solution. We will work with oriented angles between lines. For two straight lines ℓ, m in the plane, $\angle(\ell, m)$ denotes the angle of counterclockwise rotation which transforms line ℓ into a line parallel to m (the choice of the rotation centre is irrelevant). This is a signed quantity; values differing by a multiple of π are identified, so that

$$\angle(\ell, m) = -\angle(m, \ell), \qquad \angle(\ell, m) + \angle(m, n) = \angle(\ell, n).$$

If ℓ is the line through points K, L and m is the line through M, N, one writes $\angle(KL, MN)$ for $\angle(\ell, m)$; the characters K, L are freely interchangeable; and so are M, N.

The counterpart of the classical theorem about cyclic quadrilaterals is the following: *If K, L, M, N are four noncollinear points in the plane then*

$$K, L, M, N \text{ are concyclic if and only if } \angle(KM, LM) = \angle(KN, LN). \tag{1}$$

Passing to the solution proper, we first show that the three circles (AB_1C_1), (BC_1A_1), (CA_1B_1) have a common point. So, let (AB_1C_1) and (BC_1A_1) intersect at the points C_1 and P. Then by (1)

$$\angle(PA_1, CA_1) = \angle(PA_1, BA_1) = \angle(PC_1, BC_1)$$

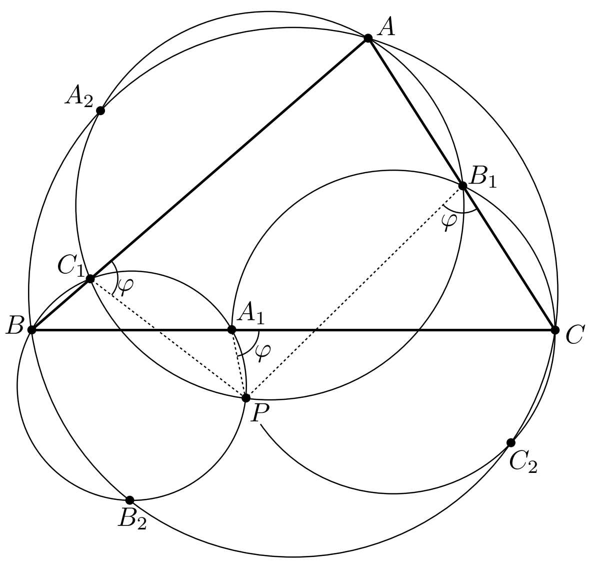
$$= \angle(PC_1, AC_1) = \angle(PB_1, AB_1) = \angle(PB_1, CB_1).$$

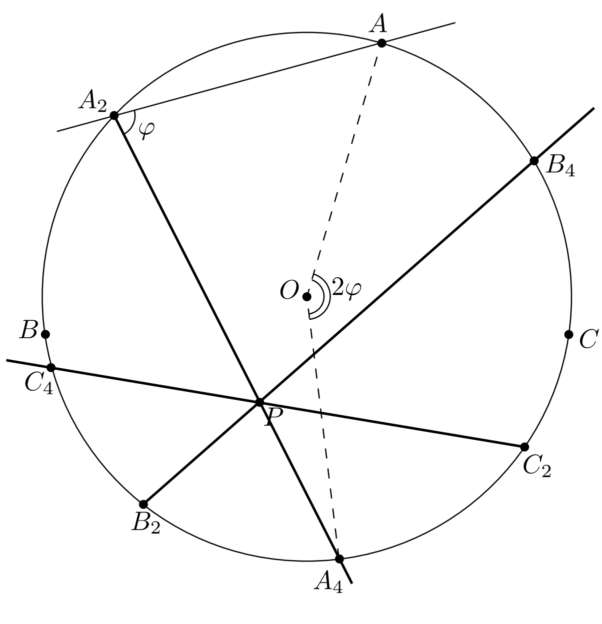
Denote this angle by φ.

The equality between the outer terms shows, again by (1), that the points A_1, B_1, P, C are concyclic. Thus P is the common point of the three mentioned circles.

From now on the basic property (1) will be used without explicit reference. We have

$$\varphi = \angle(PA_1, BC) = \angle(PB_1, CA) = \angle(PC_1, AB). \tag{2}$$





Let lines A_2P, B_2P, C_2P meet the circle (ABC) again at A_4, B_4, C_4, respectively. As

$$\angle(A_4A_2, AA_2) = \angle(PA_2, AA_2) = \angle(PC_1, AC_1) = \angle(PC_1, AB) = \varphi,$$

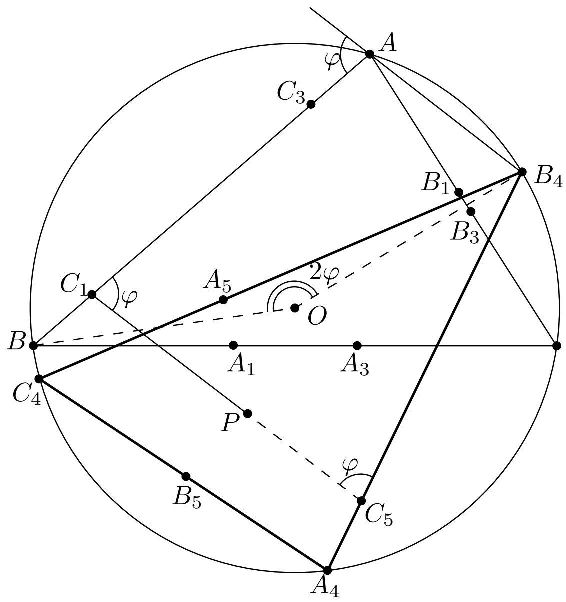
we see that line A_2A is the image of line A_2A_4 under rotation about A_2 by the angle φ. Hence the point A is the image of A_4 under rotation by 2φ about O, the centre of (ABC). The same rotation sends B_4 to B and C_4 to C. Triangle ABC is the image of $A_4B_4C_4$ in this map. Thus

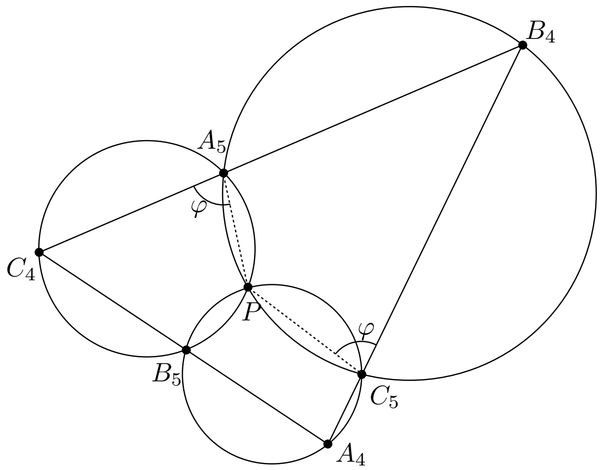
$$\angle(A_4B_4, AB) = \angle(B_4C_4, BC) = \angle(C_4A_4, CA) = 2\varphi. \tag{3}$$

Since the rotation by 2φ about O takes B_4 to B, we have $\angle(AB_4, AB) = \varphi$. Hence by (2)

$$\angle(AB_4, PC_1) = \angle(AB_4, AB) + \angle(AB, PC_1) = \varphi + (-\varphi) = 0,$$

which means that $AB_4 \| PC_1$.





Let C_5 be the intersection of lines PC_1 and A_4B_4; define A_5, B_5 analogously. So $AB_4 \| C_1C_5$ and, by (3) and (2),

$$\angle(A_4B_4, PC_1) = \angle(A_4B_4, AB) + \angle(AB, PC_1) = 2\varphi + (-\varphi) = \varphi; \tag{4}$$

i.e., $\angle(B_4C_5, C_5C_1) = \varphi$. This combined with $\angle(C_5C_1, C_1A) = \angle(PC_1, AB) = \varphi$ (see (2)) proves that the quadrilateral $AB_4C_5C_1$ is an isosceles trapezoid with $AC_1 = B_4C_5$.

Interchanging the roles of A and B we infer that also $BC_1 = A_4C_5$. And since $AC_1 + BC_1 = AB = A_4B_4$, it follows that the point C_5 lies on the line segment A_4B_4 and partitions it into segments A_4C_5, B_4C_5 of lengths BC_1 $(= AC_3)$ and AC_1 $(= BC_3)$. In other words, the rotation which maps triangle $A_4B_4C_4$ onto ABC carries C_5 onto C_3. Likewise, it sends A_5 to A_3 and B_5 to B_3. So the triangles $A_3B_3C_3$ and $A_5B_5C_5$ are congruent. It now suffices to show that the latter is similar to $A_2B_2C_2$.

Lines B_4C_5 and PC_5 coincide respectively with A_4B_4 and PC_1. Thus by (4)

$$\angle(B_4C_5, PC_5) = \varphi.$$

Analogously (by cyclic shift) $\varphi = \angle(C_4A_5, PA_5)$, which rewrites as

$$\varphi = \angle(B_4A_5, PA_5).$$

These relations imply that the points P, B_4, C_5, A_5 are concyclic. Analogously, P, C_4, A_5, B_5 and P, A_4, B_5, C_5 are concyclic quadruples. Therefore

$$\angle(A_5B_5, C_5B_5) = \angle(A_5B_5, PB_5) + \angle(PB_5, C_5B_5) = \angle(A_5C_4, PC_4) + \angle(PA_4, C_5A_4). \tag{5}$$

On the other hand, since the points $A_2, B_2, C_2, A_4, B_4, C_4$ all lie on the circle (ABC), we have

$$\angle(A_2B_2, C_2B_2) = \angle(A_2B_2, B_4B_2) + \angle(B_4B_2, C_2B_2) = \angle(A_2A_4, B_4A_4) + \angle(B_4C_4, C_2C_4). \tag{6}$$

But the lines A_2A_4, B_4A_4, B_4C_4, C_2C_4 coincide respectively with PA_4, C_5A_4, A_5C_4, PC_4. So the sums on the right-hand sides of (5) and (6) are equal, leading to equality between their left-hand sides: $\angle(A_5B_5, C_5B_5) = \angle(A_2B_2, C_2B_2)$. Hence (by cyclic shift, once more) also $\angle(B_5C_5, A_5C_5) = \angle(B_2C_2, A_2C_2)$ and $\angle(C_5A_5, B_5A_5) = \angle(C_2A_2, B_2A_2)$. This means that the triangles $A_5B_5C_5$ and $A_2B_2C_2$ have their corresponding angles equal, and consequently they are similar.

Comment 1. This is the way in which the proof has been presented by the proposer. Trying to work it out in the language of classical geometry, so as to avoid oriented angles, one is led to difficulties due to the fact that the reasoning becomes heavily case-dependent. Disposition of relevant points can vary in many respects. Angles which are equal in one case become supplementary in another. Although it seems not hard to translate all formulas from the shapes they have in one situation to the one they have in another, the real trouble is to identify all cases possible and rigorously verify that the key conclusions retain validity in each case.

The use of oriented angles is a very efficient method to omit this trouble. It seems to be the most appropriate environment in which the solution can be elaborated.

Comment 2. Actually, the fact that the circles (AB_1C_1), (BC_1A_1) and (CA_1B_1) have a common point does not require a proof; it is known as Miquel's theorem.

G10. To each side a of a convex polygon we assign the maximum area of a triangle contained in the polygon and having a as one of its sides. Show that the sum of the areas assigned to all sides of the polygon is not less than twice the area of the polygon.

(Serbia)

Solution 1.

Lemma. Every convex $(2n)$-gon, of area S, has a side and a vertex that jointly span a triangle of area not less than S/n.

Proof. By *main diagonals* of the $(2n)$-gon we shall mean those which partition the $(2n)$-gon into two polygons with equally many sides. For any side b of the $(2n)$-gon denote by Δ_b the triangle ABP where A, B are the endpoints of b and P is the intersection point of the main diagonals AA', BB'. We claim that the union of triangles Δ_b, taken over all sides, covers the whole polygon.

To show this, choose any side AB and consider the main diagonal AA' as a directed segment. Let X be any point in the polygon, not on any main diagonal. For definiteness, let X lie on the left side of the ray AA'. Consider the sequence of main diagonals AA', BB', CC', $\ldots$, where $A, B, C, \ldots$ are consecutive vertices, situated right to AA'.

The n-th item in this sequence is the diagonal $A'A$ (i.e. AA' reversed), having X on its right side. So there are two successive vertices K, L in the sequence $A, B, C, \ldots$ before A' such that X still lies to the left of KK' but to the right of LL'. And this means that X is in the triangle $\Delta_{\ell'}$, $\ell' = K'L'$. Analogous reasoning applies to points X on the right of AA' (points lying on main diagonals can be safely ignored). Thus indeed the triangles Δ_b jointly cover the whole polygon.

The sum of their areas is no less than S. So we can find two opposite sides, say $b = AB$ and $b' = A'B'$ (with AA', BB' main diagonals) such that $[\Delta_b] + [\Delta_{b'}] \geq S/n$, where $[\cdots]$ stands for the area of a region. Let AA', BB' intersect at P; assume without loss of generality that $PB \geq PB'$. Then

$$[ABA'] = [ABP] + [PBA'] \geq [ABP] + [PA'B'] = [\Delta_b] + [\Delta_{b'}] \geq S/n,$$

proving the lemma. $\square$

Now, let $\mathcal{P}$ be any convex polygon, of area S, with m sides $a_1, \ldots, a_m$. Let S_i be the area of the greatest triangle in $\mathcal{P}$ with side a_i. Suppose, contrary to the assertion, that

$$\sum_{i=1}^{m} \frac{S_i}{S} < 2.$$

Then there exist rational numbers $q_1, \ldots, q_m$ such that $\sum q_i = 2$ and $q_i > S_i/S$ for each i.

Let n be a common denominator of the m fractions $q_1, \ldots, q_m$. Write $q_i = k_i/n$; so $\sum k_i = 2n$. Partition each side a_i of $\mathcal{P}$ into k_i equal segments, creating a convex $(2n)$-gon of area S (with some angles of size $180°$), to which we apply the lemma. Accordingly, this refined polygon has a side b and a vertex H spanning a triangle T of area $[T] \geq S/n$. If b is a piece of a side a_i of $\mathcal{P}$, then the triangle W with base a_i and summit H has area

$$[W] = k_i \cdot [T] \geq k_i \cdot S/n = q_i \cdot S > S_i,$$

in contradiction with the definition of S_i. This ends the proof.

Solution 2. As in the first solution, we allow again angles of size 180° at some vertices of the convex polygons considered.

To each convex n-gon $\mathcal{P} = A_1A_2\dots A_n$ we assign a centrally symmetric convex $(2n)$-gon $\mathcal{Q}$ with side vectors $\pm\overrightarrow{A_iA_{i+1}}$, $1 \le i \le n$. The construction is as follows. Attach the $2n$ vectors $\pm\overrightarrow{A_iA_{i+1}}$ at a common origin and label them $\overrightarrow{\mathbf{b}_1}, \overrightarrow{\mathbf{b}_2}, \dots, \overrightarrow{\mathbf{b}_{2n}}$ in counterclockwise direction; the choice of the first vector $\overrightarrow{\mathbf{b}_1}$ is irrelevant. The order of labelling is well-defined if $\mathcal{P}$ has neither parallel sides nor angles equal to 180°. Otherwise several collinear vectors with the same direction are labelled consecutively $\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}, \dots, \overrightarrow{\mathbf{b}_{j+r}}$. One can assume that in such cases the respective opposite vectors occur in the order $-\overrightarrow{\mathbf{b}_j}, -\overrightarrow{\mathbf{b}_{j+1}}, \dots, -\overrightarrow{\mathbf{b}_{j+r}}$, ensuring that $\overrightarrow{\mathbf{b}_{j+n}} = -\overrightarrow{\mathbf{b}_j}$ for $j = 1, \dots, 2n$. Indices are taken cyclically here and in similar situations below.

Choose points $B_1, B_2, \dots, B_{2n}$ satisfying $\overrightarrow{B_jB_{j+1}} = \overrightarrow{\mathbf{b}_j}$ for $j = 1, \dots, 2n$. The polygonal line $\mathcal{Q} = B_1B_2\dots B_{2n}$ is closed, since $\sum_{j=1}^{2n}\overrightarrow{\mathbf{b}_j} = \overrightarrow{0}$. Moreover, $\mathcal{Q}$ is a convex $(2n)$-gon due to the arrangement of the vectors $\overrightarrow{\mathbf{b}_j}$, possibly with 180°-angles. The side vectors of $\mathcal{Q}$ are $\pm\overrightarrow{A_iA_{i+1}}$, $1 \le i \le n$. So in particular $\mathcal{Q}$ is centrally symmetric, because it contains as side vectors $\overrightarrow{A_iA_{i+1}}$ and $-\overrightarrow{A_iA_{i+1}}$ for each $i = 1, \dots, n$. Note that B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ are opposite sides of $\mathcal{Q}$, $1 \le j \le n$. We call $\mathcal{Q}$ the *associate* of $\mathcal{P}$.

Let S_i be the maximum area of a triangle with side A_iA_{i+1} in $\mathcal{P}$, $1 \le i \le n$. We prove that

$$[B_1B_2\dots B_{2n}] = 2\sum_{i=1}^{n} S_i \tag{1}$$

and

$$[B_1B_2\dots B_{2n}] \ge 4\,[A_1A_2\dots A_n]\,. \tag{2}$$

It is clear that (1) and (2) imply the conclusion of the original problem.

Lemma. For a side A_iA_{i+1} of $\mathcal{P}$, let h_i be the maximum distance from a point of $\mathcal{P}$ to line A_iA_{i+1}, $i = 1, \dots, n$. Denote by B_jB_{j+1} the side of $\mathcal{Q}$ such that $\overrightarrow{A_iA_{i+1}} = \overrightarrow{B_jB_{j+1}}$. Then the distance between B_jB_{j+1} and its opposite side in $\mathcal{Q}$ is equal to $2h_i$.

Proof. Choose a vertex A_k of $\mathcal{P}$ at distance h_i from line A_iA_{i+1}. Let $\mathbf{u}$ be the unit vector perpendicular to A_iA_{i+1} and pointing inside $\mathcal{P}$. Denoting by $\mathbf{x}\cdot\mathbf{y}$ the dot product of vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$h = \mathbf{u}\cdot\overrightarrow{A_iA_k} = \mathbf{u}\cdot(\overrightarrow{A_iA_{i+1}} + \dots + \overrightarrow{A_{k-1}A_k}) = \mathbf{u}\cdot(\overrightarrow{A_iA_{i-1}} + \dots + \overrightarrow{A_{k+1}A_k}).$$

In $\mathcal{Q}$, the distance H_i between the opposite sides B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ is given by

$$H_i = \mathbf{u}\cdot(\overrightarrow{B_jB_{j+1}} + \dots + \overrightarrow{B_{j+n-1}B_{j+n}}) = \mathbf{u}\cdot(\overrightarrow{\mathbf{b}_j} + \overrightarrow{\mathbf{b}_{j+1}} + \dots + \overrightarrow{\mathbf{b}_{j+n-1}}).$$

The choice of vertex A_k implies that the n consecutive vectors $\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}, \dots, \overrightarrow{\mathbf{b}_{j+n-1}}$ are precisely $\overrightarrow{A_iA_{i+1}}, \dots, \overrightarrow{A_{k-1}A_k}$ and $\overrightarrow{A_iA_{i-1}}, \dots, \overrightarrow{A_{k+1}A_k}$, taken in some order. This implies $H_i = 2h_i$. $\square$

For a proof of (1), apply the lemma to each side of $\mathcal{P}$. If O the centre of $\mathcal{Q}$ then, using the notation of the lemma,

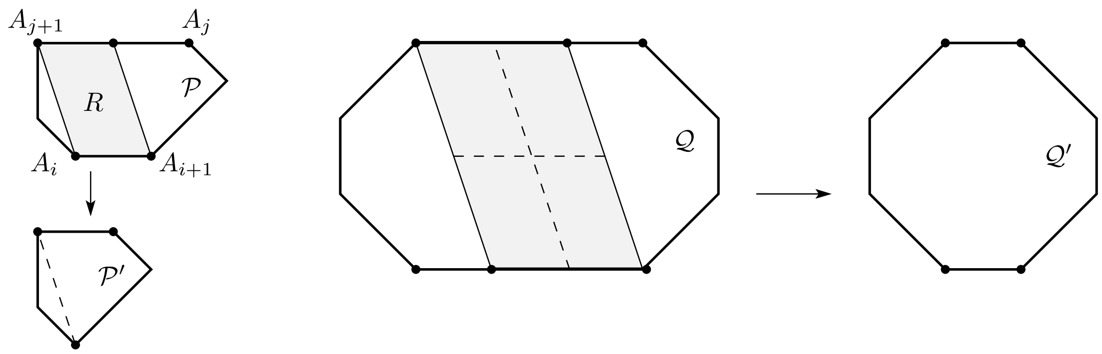
$$[B_jB_{j+1}O] = [B_{j+n}B_{j+n+1}O] = [A_iA_{i+1}A_k] = S_i\,.$$

Summation over all sides of $\mathcal{P}$ yields (1).

Set $d(\mathcal{P}) = [\mathcal{Q}] - 4[\mathcal{P}]$ for a convex polygon $\mathcal{P}$ with associate $\mathcal{Q}$. Inequality (2) means that $d(\mathcal{P}) \ge 0$ for each convex polygon $\mathcal{P}$. The last inequality will be proved by induction on the number ℓ of side directions of $\mathcal{P}$, i. e. the number of pairwise nonparallel lines each containing a side of $\mathcal{P}$.

We choose to start the induction with $\ell = 1$ as a base case, meaning that certain degenerate polygons are allowed. More exactly, we regard as *degenerate* convex polygons all closed polygonal lines of the form $X_1X_2 \dots X_kY_1Y_2 \dots Y_mX_1$, where $X_1, X_2, \dots, X_k$ are points in this order on a line segment X_1Y_1, and so are $Y_m, Y_{m-1}, \dots, Y_1$. The initial construction applies to degenerate polygons; their associates are also degenerate, and the value of d is zero. For the inductive step, consider a convex polygon $\mathcal{P}$ which determines ℓ side directions, assuming that $d(\mathcal{P}) \geq 0$ for polygons with smaller values of ℓ.

Suppose first that $\mathcal{P}$ *has a pair of parallel sides*, i. e. sides on distinct parallel lines. Let A_iA_{i+1} and A_jA_{j+1} be such a pair, and let $A_iA_{i+1} \leq A_jA_{j+1}$. Remove from $\mathcal{P}$ the parallelogram R determined by vectors $\overrightarrow{A_iA_{i+1}}$ and $\overrightarrow{A_iA_{j+1}}$. Two polygons are obtained in this way. Translating one of them by vector $\overrightarrow{A_iA_{i+1}}$ yields a new convex polygon $\mathcal{P}'$, of area $[\mathcal{P}] - [R]$ and with value of ℓ not exceeding the one of $\mathcal{P}$. The construction just described will be called operation **A**.



The associate of $\mathcal{P}'$ is obtained from $\mathcal{Q}$ upon decreasing the lengths of two opposite sides by an amount of $2A_iA_{i+1}$. By the lemma, the distance between these opposite sides is twice the distance between A_iA_{i+1} and A_jA_{j+1}. Thus operation **A** decreases $[\mathcal{Q}]$ by the area of a parallelogram with base and respective altitude twice the ones of R, i. e. by $4[R]$. Hence **A** leaves the difference $d(\mathcal{P}) = [\mathcal{Q}] - 4[\mathcal{P}]$ unchanged.

Now, if $\mathcal{P}'$ also has a pair of parallel sides, apply operation **A** to it. Keep doing so with the subsequent polygons obtained for as long as possible. Now, **A** decreases the number p of pairs of parallel sides in $\mathcal{P}$. Hence its repeated applications gradually reduce p to 0, and further applications of **A** will be impossible after several steps. For clarity, let us denote by $\mathcal{P}$ again the polygon obtained at that stage.

The inductive step is complete if $\mathcal{P}$ is degenerate. Otherwise $\ell > 1$ and $p = 0$, i. e. there are no parallel sides in $\mathcal{P}$. Observe that then $\ell \geq 3$. Indeed, $\ell = 2$ means that the vertices of $\mathcal{P}$ all lie on the boundary of a parallelogram, implying $p > 0$.

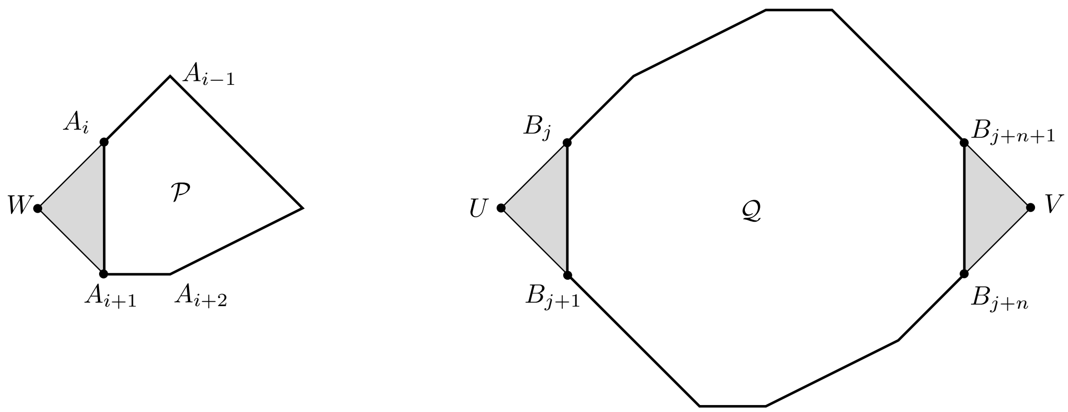
Furthermore, since $\mathcal{P}$ has no parallel sides, consecutive collinear vectors in the sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$ (if any) correspond to consecutive $180°$-angles in $\mathcal{P}$. Removing the vertices of such angles, we obtain a convex polygon with the same value of $d(\mathcal{P})$.

In summary, if operation **A** is impossible for a nondegenerate polygon $\mathcal{P}$, then $\ell \geq 3$. In addition, one may assume that $\mathcal{P}$ has no angles of size $180°$.

The last two conditions then also hold for the associate $\mathcal{Q}$ of $\mathcal{P}$, and we perform the following construction. Since $\ell \geq 3$, there is a side B_jB_{j+1} of $\mathcal{Q}$ such that the sum of the angles at B_j and B_{j+1} is greater than $180°$. (Such a side exists in each convex k-gon for $k > 4$.) Naturally, $B_{j+n}B_{j+n+1}$ is a side with the same property. Extend the pairs of sides $B_{j-1}B_j, B_{j+1}B_{j+2}$

and $B_{j+n-1}B_{j+n}, B_{j+n+1}B_{j+n+2}$ to meet at U and V, respectively. Let $\mathcal{Q}'$ be the centrally symmetric convex $2(n+1)$-gon obtained from $\mathcal{Q}$ by inserting U and V into the sequence $B_1, \ldots, B_{2n}$ as new vertices between B_j, B_{j+1} and B_{j+n}, B_{j+n+1}, respectively. Informally, we adjoin to $\mathcal{Q}$ the congruent triangles $B_jB_{j+1}U$ and $B_{j+n}B_{j+n+1}V$. Note that B_j, B_{j+1}, B_{j+n} and B_{j+n+1} are kept as vertices of $\mathcal{Q}'$, although B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ are no longer its sides.

Let A_iA_{i+1} be the side of $\mathcal{P}$ such that $\overrightarrow{A_iA_{i+1}} = \overrightarrow{B_jB_{j+1}} = \overrightarrow{\mathbf{b}_j}$. Consider the point W such that triangle $A_iA_{i+1}W$ is congruent to triangle $B_jB_{j+1}U$ and exterior to $\mathcal{P}$. Insert W into the sequence $A_1, A_2, \ldots, A_n$ as a new vertex between A_i and A_{i+1} to obtain an $(n+1)$-gon $\mathcal{P}'$. We claim that $\mathcal{P}'$ is convex and its associate is $\mathcal{Q}'$.



Vectors $\overrightarrow{A_iW}$ and $\overrightarrow{\mathbf{b}_{j-1}}$ are collinear and have the same direction, as well as vectors $\overrightarrow{WA_{i+1}}$ and $\overrightarrow{\mathbf{b}_{j+1}}$. Since $\overrightarrow{\mathbf{b}_{j-1}}, \overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}$ are consecutive terms in the sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$, the angle inequalities $\angle(\overrightarrow{\mathbf{b}_{j-1}}, \overrightarrow{\mathbf{b}_j}) \le \angle(\overrightarrow{A_{i-1}A_i}, \overrightarrow{\mathbf{b}_j})$ and $\angle(\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}) \le \angle(\overrightarrow{\mathbf{b}_j}, \overrightarrow{A_{i+1}A_{i+2}})$ hold true. They show that $\mathcal{P}'$ is a convex polygon. To construct its associate, vectors $\pm\overrightarrow{A_iA_{i+1}} = \pm\overrightarrow{\mathbf{b}_j}$ must be deleted from the defining sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$ of $\mathcal{Q}$, and the vectors $\pm\overrightarrow{A_iW}$, $\pm\overrightarrow{WA_{i+1}}$ must be inserted appropriately into it. The latter can be done as follows:

$$\ldots,\ \overrightarrow{\mathbf{b}_{j-1}},\ \overrightarrow{A_iW},\ \overrightarrow{WA_{i+1}},\ \overrightarrow{\mathbf{b}_{j+1}},\ \ldots,\ -\overrightarrow{\mathbf{b}_{j-1}},\ -\overrightarrow{A_iW},\ -\overrightarrow{WA_{i+1}},\ -\overrightarrow{\mathbf{b}_{j+1}},\ \ldots.$$

This updated sequence produces $\mathcal{Q}'$ as the associate of $\mathcal{P}'$.

It follows from the construction that $[\mathcal{P}'] = [\mathcal{P}] + [A_iA_{i+1}W]$ and $[\mathcal{Q}'] = [\mathcal{Q}] + 2[A_iA_{i+1}W]$. Therefore $d(\mathcal{P}') = d(\mathcal{P}) - 2[A_iA_{i+1}W] < d(\mathcal{P})$.

To finish the induction, it remains to notice that the value of ℓ for $\mathcal{P}'$ is less than the one for $\mathcal{P}$. This is because side A_iA_{i+1} was removed. The newly added sides A_iW and WA_{i+1} do not introduce new side directions. Each one of them is either parallel to a side of $\mathcal{P}$ or lies on the line determined by such a side. The proof is complete.

Number Theory

N1. Determine all pairs (x, y) of integers satisfying the equation

$$1 + 2^x + 2^{2x+1} = y^2.$$

(USA)

Solution. If (x, y) is a solution then obviously $x \geq 0$ and $(x, -y)$ is a solution too. For $x = 0$ we get the two solutions $(0, 2)$ and $(0, -2)$.

Now let (x, y) be a solution with $x > 0$; without loss of generality confine attention to $y > 0$. The equation rewritten as

$$2^x(1 + 2^{x+1}) = (y - 1)(y + 1)$$

shows that the factors $y - 1$ and $y + 1$ are even, exactly one of them divisible by 4. Hence $x \geq 3$ and one of these factors is divisible by 2^{x-1} but not by 2^x. So

$$y = 2^{x-1}m + \epsilon, \qquad m \text{ odd}, \qquad \epsilon = \pm 1. \tag{1}$$

Plugging this into the original equation we obtain

$$2^x \left(1 + 2^{x+1}\right) = \left(2^{x-1}m + \epsilon\right)^2 - 1 = 2^{2x-2}m^2 + 2^x m\epsilon,$$

or, equivalently

$$1 + 2^{x+1} = 2^{x-2}m^2 + m\epsilon.$$

Therefore

$$1 - \epsilon m = 2^{x-2}(m^2 - 8). \tag{2}$$

For $\epsilon = 1$ this yields $m^2 - 8 \leq 0$, i.e., $m = 1$, which fails to satisfy (2).

For $\epsilon = -1$ equation (2) gives us

$$1 + m = 2^{x-2}(m^2 - 8) \geq 2(m^2 - 8),$$

implying $2m^2 - m - 17 \leq 0$. Hence $m \leq 3$; on the other hand m cannot be 1 by (2). Because m is odd, we obtain $m = 3$, leading to $x = 4$. From (1) we get $y = 23$. These values indeed satisfy the given equation. Recall that then $y = -23$ is also good. Thus we have the complete list of solutions (x, y): $(0, 2)$, $(0, -2)$, $(4, 23)$, $(4, -23)$.

N2. For $x \in (0,1)$ let $y \in (0,1)$ be the number whose nth digit after the decimal point is the (2^n)th digit after the decimal point of x. Show that if x is rational then so is y.

(Canada)

Solution. Since x is rational, its digits repeat periodically starting at some point. We wish to show that this is also true for the digits of y, implying that y is rational.

Let d be the length of the period of x and let $d = 2^u \cdot v$, where v is odd. There is a positive integer w such that

$$2^w \equiv 1 \pmod{v}.$$

(For instance, one can choose w to be $\varphi(v)$, the value of Euler's function at v.) Therefore

$$2^{n+w} = 2^n \cdot 2^w \equiv 2^n \pmod{v}$$

for each n. Also, for $n \geq u$ we have

$$2^{n+w} \equiv 2^n \equiv 0 \pmod{2^u}.$$

It follows that, for all $n \geq u$, the relation

$$2^{n+w} \equiv 2^n \pmod{d}$$

holds. Thus, for n sufficiently large, the 2^{n+w}th digit of x is in the same spot in the cycle of x as its 2^nth digit, and so these digits are equal. Hence the $(n+w)$th digit of y is equal to its nth digit. This means that the digits of y repeat periodically with period w from some point on, as required.

N3. The sequence $f(1), f(2), f(3), \ldots$ is defined by

$$f(n) = \frac{1}{n}\left(\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n}\right\rfloor\right),$$

where $\lfloor x \rfloor$ denotes the integer part of x.

(a) Prove that $f(n+1) > f(n)$ infinitely often.

(b) Prove that $f(n+1) < f(n)$ infinitely often.

(South Africa)

Solution. Let $g(n) = nf(n)$ for $n \geq 1$ and $g(0) = 0$. We note that, for $k = 1, \ldots, n$,

$$\left\lfloor\frac{n}{k}\right\rfloor - \left\lfloor\frac{n-1}{k}\right\rfloor = 0$$

if k is not a divisor of n and

$$\left\lfloor\frac{n}{k}\right\rfloor - \left\lfloor\frac{n-1}{k}\right\rfloor = 1$$

if k divides n. It therefore follows that if $d(n)$ is the number of positive divisors of $n \geq 1$ then

$$\begin{aligned} g(n) &= \left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n-1}\right\rfloor + \left\lfloor\frac{n}{n}\right\rfloor \\ &= \left\lfloor\frac{n-1}{1}\right\rfloor + \left\lfloor\frac{n-1}{2}\right\rfloor + \cdots + \left\lfloor\frac{n-1}{n-1}\right\rfloor + \left\lfloor\frac{n-1}{n}\right\rfloor + d(n) \\ &= g(n-1) + d(n). \end{aligned}$$

Hence

$$g(n) = g(n-1) + d(n) = g(n-2) + d(n-1) + d(n) = \cdots = d(1) + d(2) + \cdots + d(n),$$

meaning that

$$f(n) = \frac{d(1) + d(2) + \cdots + d(n)}{n}.$$

In other words, $f(n)$ is equal to the arithmetic mean of $d(1), d(2), \ldots, d(n)$. In order to prove the claims, it is therefore sufficient to show that $d(n+1) > f(n)$ and $d(n+1) < f(n)$ both hold infinitely often.

We note that $d(1) = 1$. For $n > 1$, $d(n) \geq 2$ holds, with equality if and only if n is prime. Since $f(6) = 7/3 > 2$, it follows that $f(n) > 2$ holds for all $n \geq 6$.

Since there are infinitely many primes, $d(n+1) = 2$ holds for infinitely many values of n, and for each such $n \geq 6$ we have $d(n+1) = 2 < f(n)$. This proves claim (b).

To prove (a), notice that the sequence $d(1), d(2), d(3), \ldots$ is unbounded (e. g. $d(2^k) = k+1$ for all k). Hence $d(n+1) > \max\{d(1), d(2), \ldots, d(n)\}$ for infinitely many n. For all such n, we have $d(n+1) > f(n)$. This completes the solution.

N4. Let P be a polynomial of degree $n>1$ with integer coefficients and let k be any positive integer. Consider the polynomial $Q(x)=P(P(\ldots P(P(x))\ldots))$, with k pairs of parentheses. Prove that Q has no more than n integer fixed points, i.e. integers satisfying the equation $Q(x)=x$.

(Romania)

Solution. The claim is obvious if every integer fixed point of Q is a fixed point of P itself. For the sequel assume that this is not the case. Take any integer x_0 such that $Q(x_0)=x_0$, $P(x_0)\neq x_0$ and define inductively $x_{i+1}=P(x_i)$ for $i=0,1,2,\ldots$; then $x_k=x_0$.

It is evident that

$$P(u)-P(v) \text{ is divisible by } u-v \text{ for distinct integers } u,v. \tag{1}$$

(Indeed, if $P(x)=\sum a_i x^i$ then each $a_i(u^i-v^i)$ is divisible by $u-v$.) Therefore each term in the chain of (nonzero) differences

$$x_0-x_1,\quad x_1-x_2,\quad \ldots,\quad x_{k-1}-x_k,\quad x_k-x_{k+1} \tag{2}$$

is a divisor of the next one; and since $x_k-x_{k+1}=x_0-x_1$, all these differences have equal absolute values. For $x_m=\min(x_1,\ldots,x_k)$ this means that $x_{m-1}-x_m=-(x_m-x_{m+1})$. Thus $x_{m-1}=x_{m+1}(\neq x_m)$. It follows that consecutive differences in the sequence (2) have opposite signs. Consequently, $x_0,x_1,x_2,\ldots$ is an alternating sequence of two distinct values. In other words, every integer fixed point of Q is a fixed point of the polynomial $P(P(x))$. Our task is to prove that there are at most n such points.

Let a be one of them so that $b=P(a)\neq a$ (we have assumed that such an a exists); then $a=P(b)$. Take any other integer fixed point α of $P(P(x))$ and let $P(\alpha)=\beta$, so that $P(\beta)=\alpha$; the numbers α and β need not be distinct (α can be a fixed point of P), but each of α,β is different from each of a,b. Applying property (1) to the four pairs of integers (α,a), (β,b), (α,b), (β,a) we get that the numbers $\alpha-a$ and $\beta-b$ divide each other, and also $\alpha-b$ and $\beta-a$ divide each other. Consequently

$$\alpha-b=\pm(\beta-a),\qquad \alpha-a=\pm(\beta-b). \tag{3}$$

Suppose we have a plus in both instances: $\alpha-b=\beta-a$ and $\alpha-a=\beta-b$. Subtraction yields $a-b=b-a$, a contradiction, as $a\neq b$. Therefore at least one equality in (3) holds with a minus sign. For each of them this means that $\alpha+\beta=a+b$; equivalently $a+b-\alpha-P(\alpha)=0$.

Denote $a+b$ by C. We have shown that every integer fixed point of Q other that a and b is a root of the polynomial $F(x)=C-x-P(x)$. This is of course true for a and b as well. And since P has degree $n>1$, the polynomial F has the same degree, so it cannot have more than n roots. Hence the result.

Comment. The first part of the solution, showing that integer fixed points of any iterate of P are in fact fixed points of the second iterate $P\circ P$ is standard; moreover, this fact has already appeared in contests. We however do not consider this as a major drawback to the problem because the only tricky moment comes up only at the next stage of the reasoning—to apply the divisibility property (1) to points from distinct 2-orbits of P. Yet maybe it would be more appropriate to state the problem in a version involving $k=2$ only.

N5. Find all integer solutions of the equation

$$\frac{x^7 - 1}{x - 1} = y^5 - 1.$$

(Russia)

Solution. The equation has no integer solutions. To show this, we first prove a lemma.

Lemma. If x is an integer and p is a prime divisor of $\frac{x^7 - 1}{x - 1}$ then either $p \equiv 1 \pmod 7$ or $p = 7$.

Proof. Both $x^7 - 1$ and $x^{p-1} - 1$ are divisible by p, by hypothesis and by Fermat's little theorem, respectively. Suppose that 7 does not divide $p - 1$. Then $\gcd(p-1, 7) = 1$, so there exist integers k and m such that $7k + (p-1)m = 1$. We therefore have

$$x \equiv x^{7k+(p-1)m} \equiv (x^7)^k \cdot (x^{p-1})^m \equiv 1 \pmod p,$$

and so

$$\frac{x^7 - 1}{x - 1} = 1 + x + \cdots + x^6 \equiv 7 \pmod p.$$

It follows that p divides 7, hence $p = 7$ must hold if $p \equiv 1 \pmod 7$ does not, as stated. $\square$

The lemma shows that each positive divisor d of $\frac{x^7 - 1}{x - 1}$ satisfies either $d \equiv 0 \pmod 7$ or $d \equiv 1 \pmod 7$.

Now assume that (x, y) is an integer solution of the original equation. Notice that $y - 1 > 0$, because $\frac{x^7 - 1}{x - 1} > 0$ for all $x \neq 1$. Since $y - 1$ divides $\frac{x^7 - 1}{x - 1} = y^5 - 1$, we have $y \equiv 1 \pmod 7$ or $y \equiv 2 \pmod 7$ by the previous paragraph. In the first case, $1 + y + y^2 + y^3 + y^4 \equiv 5 \pmod 7$, and in the second $1 + y + y^2 + y^3 + y^4 \equiv 3 \pmod 7$. Both possibilities contradict the fact that the positive divisor $1 + y + y^2 + y^3 + y^4$ of $\frac{x^7 - 1}{x - 1}$ is congruent to 0 or 1 modulo 7. So the given equation has no integer solutions.

N6. Let $a > b > 1$ be relatively prime positive integers. Define the *weight* of an integer c, denoted by $w(c)$, to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers x and y such that

$$ax + by = c.$$

An integer c is called a *local champion* if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$.

Find all local champions and determine their number.

(USA)

Solution. Call the pair of integers (x, y) a *representation* of c if $ax + by = c$ and $|x| + |y|$ has the smallest possible value, i.e. $|x| + |y| = w(c)$.

We characterise the local champions by the following three observations.

Lemma 1. If (x, y) a representation of a local champion c then $xy < 0$.

Proof. Suppose indirectly that $x \geq 0$ and $y \geq 0$ and consider the values $w(c)$ and $w(c+a)$. All representations of the numbers c and $c + a$ in the form $au + bv$ can be written as

$$c = a(x - kb) + b(y + ka), \qquad c + a = a(x + 1 - kb) + b(y + ka)$$

where k is an arbitrary integer.

Since $|x| + |y|$ is minimal, we have

$$x + y = |x| + |y| \leq |x - kb| + |y + ka|$$

for all k. On the other hand, $w(c + a) \leq w(c)$, so there exists a k for which

$$|x + 1 - kb| + |y + ka| \leq |x| + |y| = x + y.$$

Then

$$(x + 1 - kb) + (y + ka) \leq |x + 1 - kb| + |y + ka| \leq x + y \leq |x - kb| + |y + ka|.$$

Comparing the first and the third expressions, we find $k(a - b) + 1 \leq 0$ implying $k < 0$. Comparing the second and fourth expressions, we get $|x + 1 - kb| \leq |x - kb|$, therefore $kb > x$; this is a contradiction.

If $x, y \leq 0$ then we can switch to $-c$, $-x$ and $-y$. $\square$

From this point, write $c = ax - by$ instead of $c = ax + by$ and consider only those cases where x and y are nonzero and have the same sign. By Lemma 1, there is no loss of generality in doing so.

Lemma 2. Let $c = ax - by$ where $|x| + |y|$ is minimal and x, y have the same sign. The number c is a local champion if and only if $|x| < b$ and $|x| + |y| = \left\lfloor \frac{a+b}{2} \right\rfloor$.

Proof. Without loss of generality we may assume $x, y > 0$.

The numbers $c - a$ and $c + b$ can be written as

$$c - a = a(x - 1) - by \quad \text{and} \quad c + b = ax - b(y - 1)$$

and trivially $w(c - a) \leq (x - 1) + y < w(c)$ and $w(c + b) \leq x + (y - 1) < w(c)$ in all cases.

Now assume that c is a local champion and consider $w(c + a)$. Since $w(c + a) \leq w(c)$, there exists an integer k such that

$$c + a = a(x + 1 - kb) - b(y - ka) \quad \text{and} \quad |x + 1 - kb| + |y - ka| \leq x + y.$$

This inequality cannot hold if $k \le 0$, therefore $k > 0$. We prove that we can choose $k = 1$.

Consider the function $f(t) = |x+1-bt| + |y-at| - (x+y)$. This is a convex function and we have $f(0) = 1$ and $f(k) \le 0$. By Jensen's inequality, $f(1) \le \left(1 - \frac{1}{k}\right) f(0) + \frac{1}{k} f(k) < 1$. But $f(1)$ is an integer. Therefore $f(1) \le 0$ and

$$|x+1-b| + |y-a| \le x+y.$$

Knowing $c = a(x-b) - b(y-a)$, we also have

$$x+y \le |x-b| + |y-a|.$$

Combining the two inequalities yields $|x+1-b| \le |x-b|$ which is equivalent to $x < b$.

Considering $w(c-b)$, we obtain similarly that $y < a$.

Now $|x-b| = b-x$, $|x+1-b| = b-x-1$ and $|y-a| = a-y$, therefore we have

$$(b-x-1) + (a-y) \le x+y \le (b-x) + (a-y),$$

$$\frac{a+b-1}{2} \le x+y \le \frac{a+b}{2}.$$

Hence $x+y = \left\lfloor \frac{a+b}{2} \right\rfloor$.

To prove the opposite direction, assume $0 < x < b$ and $x+y = \left\lfloor \frac{a+b}{2} \right\rfloor$. Since $a > b$, we also have $0 < y < a$. Then

$$w(c+a) \le |x+1-b| + |y-a| = a+b-1-(x+y) \le x+y = w(c)$$

and

$$w(c-b) \le |x-b| + |y+1-a| = a+b-1-(x+y) \le x+y = w(c)$$

therefore c is a local champion indeed. $\square$

Lemma 3. Let $c = ax - by$ and assume that x and y have the same sign, $|x| < b$, $|y| < a$ and $|x| + |y| = \left\lfloor \frac{a+b}{2} \right\rfloor$. Then $w(c) = x+y$.

Proof. By definition $w(c) = \min\{|x-kb| + |y-ka| : \ k \in \mathbb{Z}\}$. If $k \le 0$ then obviously $|x-kb| + |y-ka| \ge x+y$. If $k \ge 1$ then

$$|x-kb| + |y-ka| = (kb-x) + (ka-y) = k(a+b) - (x+y) \ge (2k-1)(x+y) \ge x+y.$$

Therefore $w(c) = x+y$ indeed. $\square$

Lemmas 1, 2 and 3 together yield that the set of local champions is

$$C = \left\{ \pm (ax - by) : \ 0 < x < b, \ x+y = \left\lfloor \frac{a+b}{2} \right\rfloor \right\}.$$

Denote by C^+ and C^- the two sets generated by the expressions $+(ax-by)$ and $-(ax-by)$, respectively. It is easy to see that both sets are arithmetic progressions of length $b-1$, with difference $a+b$.

If a and b are odd, then $C^+ = C^-$, because $a(-x) - b(-y) = a(b-x) - b(a-y)$ and $x+y = \frac{a+b}{2}$ is equivalent to $(b-x) + (a-y) = \frac{a+b}{2}$. In this case there exist $b-1$ local champions.

If a and b have opposite parities then the answer is different. For any $c_1 \in C^+$ and $c_2 \in C^-$,

$$2c_1 \equiv -2c_2 \equiv 2\left(a\frac{a+b-1}{2} - b \cdot 0\right) \equiv -a \pmod{a+b}$$

and

$$2c_1 - 2c_2 \equiv -2a \pmod{a+b}.$$

The number $a+b$ is odd and relatively prime to a, therefore the elements of C^+ and C^- belong to two different residue classes modulo $a+b$. Hence, the set C is the union of two disjoint arithmetic progressions and the number of all local champions is $2(b-1)$.

So the number of local champions is $b-1$ if both a and b are odd and $2(b-1)$ otherwise.

Comment. The original question, as stated by the proposer, was:

(a) Show that there exists only finitely many local champions;

(b) Show that there exists at least one local champion.

N7. Prove that, for every positive integer n, there exists an integer m such that $2^m + m$ is divisible by n.

(Estonia)

Solution. We will prove by induction on d that, for every positive integer N, there exist positive integers $b_0, b_1, \ldots, b_{d-1}$ such that, for each $i = 0, 1, 2, \ldots, d-1$, we have $b_i > N$ and

$$2^{b_i} + b_i \equiv i \pmod{d}.$$

This yields the claim for $m = b_0$.

The base case $d = 1$ is trivial. Take an $a > 1$ and assume that the statement holds for all $d < a$. Note that the remainders of 2^i modulo a repeat periodically starting with some exponent M. Let k be the length of the period; this means that $2^{M+k'} \equiv 2^M \pmod{a}$ holds only for those k' which are multiples of k. Note further that the period cannot contain all the a remainders, since 0 either is missing or is the only number in the period. Thus $k < a$.

Let $d = \gcd(a, k)$ and let $a' = a/d$, $k' = k/d$. Since $0 < k < a$, we also have $0 < d < a$. By the induction hypothesis, there exist positive integers $b_0, b_1, \ldots, b_{d-1}$ such that $b_i > \max(2^M, N)$ and

$$2^{b_i} + b_i \equiv i \pmod{d} \quad \text{for} \quad i = 0, 1, 2, \ldots, d-1. \tag{1}$$

For each $i = 0, 1, \ldots, d-1$ consider the sequence

$$2^{b_i} + b_i, \;\; 2^{b_i+k} + (b_i + k), \;\; \ldots, \;\; 2^{b_i+(a'-1)k} + (b_i + (a'-1)k). \tag{2}$$

Modulo a, these numbers are congruent to

$$2^{b_i} + b_i, \;\; 2^{b_i} + (b_i + k), \;\; \ldots, \;\; 2^{b_i} + (b_i + (a'-1)k),$$

respectively. The d sequences contain $a'd = a$ numbers altogether. We shall now prove that no two of these numbers are congruent modulo a.

Suppose that

$$2^{b_i} + (b_i + mk) \equiv 2^{b_j} + (b_j + nk) \pmod{a} \tag{3}$$

for some values of $i, j \in \{0, 1, \ldots, d-1\}$ and $m, n \in \{0, 1, \ldots, a'-1\}$. Since d is a divisor of a, we also have

$$2^{b_i} + (b_i + mk) \equiv 2^{b_j} + (b_j + nk) \pmod{d}.$$

Because d is a divisor of k and in view of (1), we obtain $i \equiv j \pmod{d}$. As $i, j \in \{0, 1, \ldots, d-1\}$, this just means that $i = j$. Substituting this into (3) yields $mk \equiv nk \pmod{a}$. Therefore $mk' \equiv nk' \pmod{a'}$; and since a' and k' are coprime, we get $m \equiv n \pmod{a'}$. Hence also $m = n$.

It follows that the a numbers that make up the d sequences (2) satisfy all the requirements; they are certainly all greater than N because we chose each $b_i > \max(2^M, N)$. So the statement holds for a, completing the induction.

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Shortlisted Problems *with Solutions*

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Shortlisted Problems with Solutions

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Problem Selection Committee

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Algebra

A1. Given a sequence $a_1, a_2, \ldots, a_n$ of real numbers. For each i $(1 \le i \le n)$ define

$$d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : i \le j \le n\}$$

and let

$$d = \max\{d_i : 1 \le i \le n\}.$$

(a) Prove that for arbitrary real numbers $x_1 \le x_2 \le \ldots \le x_n$,

$$\max\bigl\{|x_i - a_i| : 1 \le i \le n\bigr\} \ge \frac{d}{2}. \tag{1}$$

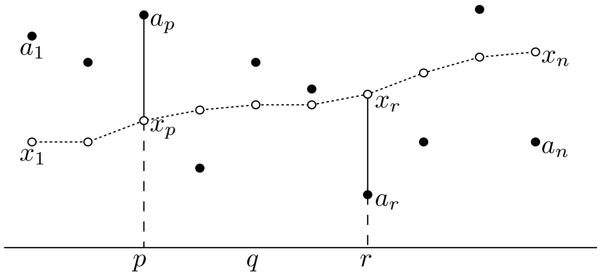
(b) Show that there exists a sequence $x_1 \le x_2 \le \ldots \le x_n$ of real numbers such that we have equality in (1).

(New Zealand)

Solution 1. (a) Let $1 \le p \le q \le r \le n$ be indices for which

$$d = d_q, \qquad a_p = \max\{a_j : 1 \le j \le q\}, \quad a_r = \min\{a_j : q \le j \le n\}$$

and thus $d = a_p - a_r$. (These indices are not necessarily unique.)



For arbitrary real numbers $x_1 \le x_2 \le \ldots \le x_n$, consider just the two quantities $|x_p - a_p|$ and $|x_r - a_r|$. Since

$$(a_p - x_p) + (x_r - a_r) = (a_p - a_r) + (x_r - x_p) \ge a_p - a_r = d,$$

we have either $a_p - x_p \ge \dfrac{d}{2}$ or $x_r - a_r \ge \dfrac{d}{2}$. Hence,

$$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \max\bigl\{|x_p - a_p|, |x_r - a_r|\bigr\} \ge \max\{a_p - x_p, x_r - a_r\} \ge \frac{d}{2}.$$

(b) Define the sequence (x_k) as

$$x_1 = a_1 - \frac{d}{2}, \qquad x_k = \max\left\{x_{k-1},\ a_k - \frac{d}{2}\right\} \quad \text{for } 2 \le k \le n.$$

We show that we have equality in (1) for this sequence.

By the definition, sequence (x_k) is non-decreasing and $x_k - a_k \ge -\dfrac{d}{2}$ for all $1 \le k \le n$. Next we prove that

$$x_k - a_k \le \frac{d}{2} \quad \text{for all } 1 \le k \le n. \tag{2}$$

Consider an arbitrary index $1 \le k \le n$. Let $\ell \le k$ be the smallest index such that $x_k = x_\ell$. We have either $\ell = 1$, or $\ell \ge 2$ and $x_\ell > x_{\ell-1}$. In both cases,

$$x_k = x_\ell = a_\ell - \frac{d}{2}. \tag{3}$$

Since

$$a_\ell - a_k \le \max\{a_j : 1 \le j \le k\} - \min\{a_j :\ k \le j \le n\} = d_k \le d,$$

equality (3) implies

$$x_k - a_k = a_\ell - a_k - \frac{d}{2} \le d - \frac{d}{2} = \frac{d}{2}.$$

We obtained that $-\dfrac{d}{2} \le x_k - a_k \le \dfrac{d}{2}$ for all $1 \le k \le n$, so

$$\max\Big\{|x_i - a_i| :\ 1 \le i \le n\Big\} \le \frac{d}{2}.$$

We have equality because $|x_1 - a_1| = \dfrac{d}{2}$.

Solution 2. We present another construction of a sequence (x_i) for part (b).

For each $1 \le i \le n$, let

$$M_i = \max\{a_j : 1 \le j \le i\} \quad \text{and} \quad m_i = \min\{a_j :\ i \le j \le n\}.$$

For all $1 \le i < n$, we have

$$M_i = \max\{a_1, \dots, a_i\} \le \max\{a_1, \dots, a_i, a_{i+1}\} = M_{i+1}$$

and

$$m_i = \min\{a_i, a_{i+1}, \dots, a_n\} \le \min\{a_{i+1}, \dots, a_n\} = m_{i+1}.$$

Therefore sequences (M_i) and (m_i) are non-decreasing. Moreover, since a_i is listed in both definitions,

$$m_i \le a_i \le M_i.$$

To achieve equality in (1), set

$$x_i = \frac{M_i + m_i}{2}.$$

Since sequences (M_i) and (m_i) are non-decreasing, this sequence is non-decreasing as well.

From $d_i = M_i - m_i$ we obtain that

$$-\frac{d_i}{2} = \frac{m_i - M_i}{2} = x_i - M_i \leq x_i - a_i \leq x_i - m_i = \frac{M_i - m_i}{2} = \frac{d_i}{2}.$$

Therefore

$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \leq \max\left\{\frac{d_i}{2} : 1 \leq i \leq n\right\} = \frac{d}{2}.$$

Since the opposite inequality has been proved in part (a), we must have equality.

A2. Consider those functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f\big(f(n)\big) - 1 \tag{1}$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

($\mathbb{N}$ denotes the set of all positive integers.)

(Bulgaria)

Answer. 1, 2, …, 2008.

Solution. Suppose that a function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies (1). For arbitrary positive integers $m > n$, by (1) we have

$$f(m) = f\big(n + (m-n)\big) \geq f(n) + f\big(f(m-n)\big) - 1 \geq f(n),$$

so f is nondecreasing.

Function $f \equiv 1$ is an obvious solution. To find other solutions, assume that $f \not\equiv 1$ and take the smallest $a \in \mathbb{N}$ such that $f(a) > 1$. Then $f(b) \geq f(a) > 1$ for all integer $b \geq a$.

Suppose that $f(n) > n$ for some $n \in \mathbb{N}$. Then we have

$$f\big(f(n)\big) = f\Big(\big(f(n)-n\big) + n\Big) \geq f\big(f(n)-n\big) + f\big(f(n)\big) - 1,$$

so $f\big(f(n)-n\big) \leq 1$ and hence $f(n)-n < a$. Then there exists a maximal value of the expression $f(n)-n$; denote this value by c, and let $f(k)-k = c \geq 1$. Applying the monotonicity together with (1), we get

$$\begin{aligned} 2k + c \geq f(2k) = f(k+k) &\geq f(k) + f\big(f(k)\big) - 1 \\ &\geq f(k) + f(k) - 1 = 2(k+c) - 1 = 2k + (2c-1), \end{aligned}$$

hence $c \leq 1$ and $f(n) \leq n+1$ for all $n \in \mathbb{N}$. In particular, $f(2007) \leq 2008$.

Now we present a family of examples showing that all values from 1 to 2008 can be realized. Let

$$f_j(n) = \max\{1, n+j-2007\} \quad \text{for } j = 1, 2, \ldots, 2007; \qquad f_{2008}(n) = \begin{cases} n, & 2007 \nmid n, \\ n+1, & 2007 \mid n. \end{cases}$$

We show that these functions satisfy the condition (1) and clearly $f_j(2007) = j$.

To check the condition (1) for the function f_j ($j \leq 2007$), note first that f_j is nondecreasing and $f_j(n) \leq n$, hence $f_j\big(f_j(n)\big) \leq f_j(n) \leq n$ for all $n \in \mathbb{N}$. Now, if $f_j(m) = 1$, then the inequality (1) is clear since $f_j(m+n) \geq f_j(n) \geq f_j\big(f_j(n)\big) = f_j(m) + f_j\big(f_j(n)\big) - 1$. Otherwise,

$$f_j(m) + f_j\big(f_j(n)\big) - 1 \leq (m+j-2007) + n = (m+n) + j - 2007 = f_j(m+n).$$

In the case $j = 2008$, clearly $n+1 \geq f_{2008}(n) \geq n$ for all $n \in \mathbb{N}$; moreover, $n+1 \geq f_{2008}\big(f_{2008}(n)\big)$ as well. Actually, the latter is trivial if $f_{2008}(n) = n$; otherwise, $f_{2008}(n) = n+1$, which implies $2007 \nmid n+1$ and hence $n+1 = f_{2008}(n+1) = f_{2008}\big(f_{2008}(n)\big)$.

So, if $2007 \mid m+n$, then

$$f_{2008}(m+n) = m+n+1 = (m+1) + (n+1) - 1 \geq f_{2008}(m) + f_{2008}\big(f_{2008}(n)\big) - 1.$$

Otherwise, $2007 \nmid m+n$, hence $2007 \nmid m$ or $2007 \nmid n$. In the former case we have $f_{2008}(m) = m$, while in the latter one $f_{2008}\big(f_{2008}(n)\big) = f_{2008}(n) = n$, providing

$$f_{2008}(m) + f_{2008}\big(f_{2008}(n)\big) - 1 \leq (m+n+1) - 1 = f_{2008}(m+n).$$

Comment. The examples above are not unique. The values $1, 2, \dots, 2008$ can be realized in several ways. Here we present other two constructions for $j \le 2007$, without proof:

$$g_j(n) = \begin{cases} 1, & n < 2007, \\ j, & n = 2007, \\ n, & n > 2007; \end{cases} \qquad h_j(n) = \max\left\{1, \left\lfloor \frac{jn}{2007} \right\rfloor \right\}.$$

Also the example for $j = 2008$ can be generalized. In particular, choosing a divisor $d > 1$ of 2007, one can set

$$f_{2008,d}(n) = \begin{cases} n, & d \nmid n, \\ n+1, & d \mid n. \end{cases}$$

A3. Let n be a positive integer, and let x and y be positive real numbers such that $x^n+y^n=1$. Prove that

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{1}{(1-x)(1-y)}.$$

(Estonia)

Solution 1. For each real $t\in(0,1)$,

$$\frac{1+t^2}{1+t^4}=\frac{1}{t}-\frac{(1-t)(1-t^3)}{t(1+t^4)}<\frac{1}{t}.$$

Substituting $t=x^k$ and $t=y^k$,

$$0<\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}<\sum_{k=1}^{n}\frac{1}{x^k}=\frac{1-x^n}{x^n(1-x)}\quad\text{and}\quad 0<\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}<\sum_{k=1}^{n}\frac{1}{y^k}=\frac{1-y^n}{y^n(1-y)}.$$

Since $1-y^n=x^n$ and $1-x^n=y^n$,

$$\frac{1-x^n}{x^n(1-x)}=\frac{y^n}{x^n(1-x)},\qquad \frac{1-y^n}{y^n(1-y)}=\frac{x^n}{y^n(1-y)}$$

and therefore

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{y^n}{x^n(1-x)}\cdot\frac{x^n}{y^n(1-y)}=\frac{1}{(1-x)(1-y)}.$$

Solution 2. We prove

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{\left(\frac{1+\sqrt{2}}{2}\ln 2\right)^2}{(1-x)(1-y)}<\frac{0.7001}{(1-x)(1-y)}. \tag{1}$$

The idea is to estimate each term on the left-hand side with the same constant. To find the upper bound for the expression $\dfrac{1+x^{2k}}{1+x^{4k}}$, consider the function $f(t)=\dfrac{1+t}{1+t^2}$ in interval $(0,1)$. Since

$$f'(t)=\frac{1-2t-t^2}{(1+t^2)^2}=\frac{(\sqrt{2}+1+t)(\sqrt{2}-1-t)}{(1+t^2)^2},$$

the function increases in interval $(0,\sqrt{2}-1]$ and decreases in $[\sqrt{2}-1,1)$. Therefore the maximum is at point $t_0=\sqrt{2}-1$ and

$$f(t)=\frac{1+t}{1+t^2}\le f(t_0)=\frac{1+\sqrt{2}}{2}=\alpha.$$

Applying this to each term on the left-hand side of (1), we obtain

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)\le n\alpha\cdot n\alpha=(n\alpha)^2. \tag{2}$$

To estimate $(1-x)(1-y)$ on the right-hand side, consider the function

$$g(t)=\ln(1-t^{1/n})+\ln\bigl(1-(1-t)^{1/n}\bigr).$$

Substituting s for $1-t$, we have

$$-ng'(t) = \frac{t^{1/n-1}}{1-t^{1/n}} - \frac{s^{1/n-1}}{1-s^{1/n}} = \frac{1}{st}\left(\frac{(1-t)t^{1/n}}{1-t^{1/n}} - \frac{(1-s)s^{1/n}}{1-s^{1/n}}\right) = \frac{h(t)-h(s)}{st}.$$

The function

$$h(t) = t^{1/n}\frac{1-t}{1-t^{1/n}} = \sum_{i=1}^{n} t^{i/n}$$

is obviously increasing for $t \in (0,1)$, hence for these values of t we have

$$g'(t) > 0 \iff h(t) < h(s) \iff t < s = 1-t \iff t < \frac{1}{2}.$$

Then, the maximum of $g(t)$ in $(0,1)$ is attained at point $t_1 = 1/2$ and therefore

$$g(t) \le g\left(\frac{1}{2}\right) = 2\ln(1-2^{-1/n}), \qquad t \in (0,1).$$

Substituting $t = x^n$, we have $1-t = y^n$, $(1-x)(1-y) = \exp g(t)$ and hence

$$(1-x)(1-y) = \exp g(t) \le (1-2^{-1/n})^2. \tag{3}$$

Combining (2) and (3), we get

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right) \le (\alpha n)^2 \cdot 1 \le (\alpha n)^2\frac{(1-2^{-1/n})^2}{(1-x)(1-y)} = \frac{\left(\alpha n(1-2^{-1/n})\right)^2}{(1-x)(1-y)}.$$

Applying the inequality $1-\exp(-t) < t$ for $t = \dfrac{\ln 2}{n}$, we obtain

$$\alpha n(1-2^{-1/n}) = \alpha n\left(1-\exp\left(-\frac{\ln 2}{n}\right)\right) < \alpha n \cdot \frac{\ln 2}{n} = \alpha \ln 2 = \frac{1+\sqrt{2}}{2}\ln 2.$$

Hence,

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{\left(\frac{1+\sqrt{2}}{2}\ln 2\right)^2}{(1-x)(1-y)}.$$

Comment. It is a natural idea to compare the sum $S_n(x) = \displaystyle\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}$ with the integral $I_n(x) = \displaystyle\int_0^n \frac{1+x^{2t}}{1+x^{4t}}\mathrm{d}t$. Though computing the integral is quite standard, many difficulties arise. First, the integrand $\dfrac{1+x^{2k}}{1+x^{4k}}$ has an increasing segment and, depending on x, it can have a decreasing segment as well. So comparing $S_n(x)$ and $I_n(x)$ is not completely obvious. We can add a term to fix the estimate, e.g. $S_n \le I_n + (\alpha - 1)$, but then the final result will be weak for the small values of n. Second, we have to minimize $(1-x)(1-y)I_n(x)I_n(y)$ which leads to very unpleasant computations.

However, by computer search we found that the maximum of $I_n(x)I_n(y)$ is at $x = y = 2^{-1/n}$, as well as the maximum of $S_n(x)S_n(y)$, and the latter is less. Hence, one can conjecture that the exact constant which can be put into the numerator on the right-hand side of (1) is

$$\left(\ln 2 \cdot \int_0^1 \frac{1+4^{-t}}{1+16^{-t}}\mathrm{d}t\right)^2 = \frac{1}{4}\left(\frac{1}{2}\ln\frac{17}{2} + \arctan 4 - \frac{\pi}{4}\right)^2 \approx 0.6484.$$

A4. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\big(x+f(y)\big) = f(x+y) + f(y) \tag{1}$$

for all $x, y \in \mathbb{R}^+$. (Symbol $\mathbb{R}^+$ denotes the set of all positive real numbers.)

(Thaliand)

Answer. $f(x) = 2x$.

Solution 1. First we show that $f(y) > y$ for all $y \in \mathbb{R}^+$. Functional equation (1) yields $f\big(x+f(y)\big) > f(x+y)$ and hence $f(y) \neq y$ immediately. If $f(y) < y$ for some y, then setting $x = y - f(y)$ we get

$$f(y) = f\Big(\big(y-f(y)\big)+f(y)\Big) = f\Big(\big(y-f(y)\big)+y\Big) + f(y) > f(y),$$

contradiction. Therefore $f(y) > y$ for all $y \in \mathbb{R}^+$.

For $x \in \mathbb{R}^+$ define $g(x) = f(x) - x$; then $f(x) = g(x) + x$ and, as we have seen, $g(x) > 0$. Transforming (1) for function $g(x)$ and setting $t = x + y$,

$$f\big(t+g(y)\big) = f(t) + f(y),$$

$$g\big(t+g(y)\big) + t + g(y) = \big(g(t)+t\big) + \big(g(y)+y\big)$$

and therefore

$$g\big(t+g(y)\big) = g(t) + y \qquad \text{for all } t > y > 0. \tag{2}$$

Next we prove that function $g(x)$ is injective. Suppose that $g(y_1) = g(y_2)$ for some numbers $y_1, y_2 \in \mathbb{R}^+$. Then by (2),

$$g(t) + y_1 = g\big(t+g(y_1)\big) = g\big(t+g(y_2)\big) = g(t) + y_2$$

for all $t > \max\{y_1, y_2\}$. Hence, $g(y_1) = g(y_2)$ is possible only if $y_1 = y_2$.

Now let u, v be arbitrary positive numbers and $t > u + v$. Applying (2) three times,

$$g\big(t+g(u)+g(v)\big) = g\big(t+g(u)\big) + v = g(t) + u + v = g\big(t+g(u+v)\big).$$

By the injective property we conclude that $t + g(u) + g(v) = t + g(u+v)$, hence

$$g(u) + g(v) = g(u+v). \tag{3}$$

Since function $g(v)$ is positive, equation (3) also shows that g is an increasing function.

Finally we prove that $g(x) = x$. Combining (2) and (3), we obtain

$$g(t) + y = g\big(t+g(y)\big) = g(t) + g\big(g(y)\big)$$

and hence

$$g\big(g(y)\big) = y.$$

Suppose that there exists an $x \in \mathbb{R}^+$ such that $g(x) \neq x$. By the monotonicity of g, if $x > g(x)$ then $g(x) > g\big(g(x)\big) = x$. Similarly, if $x < g(x)$ then $g(x) < g\big(g(x)\big) = x$. Both cases lead to contradiction, so there exists no such x.

We have proved that $g(x) = x$ and therefore $f(x) = g(x) + x = 2x$ for all $x \in \mathbb{R}^+$. This function indeed satisfies the functional equation (1).

Comment. It is well-known that the additive property (3) together with $g(x) \geq 0$ (for $x > 0$) imply $g(x) = cx$. So, after proving (3), it is sufficient to test functions $f(x) = (c+1)x$.

Solution 2. We prove that $f(y) > y$ and introduce function $g(x) = f(x) - x > 0$ in the same way as in Solution 1.

For arbitrary $t > y > 0$, substitute $x = t - y$ into (1) to obtain

$$f\big(t + g(y)\big) = f(t) + f(y)$$

which, by induction, implies

$$f\big(t + ng(y)\big) = f(t) + nf(y) \qquad \text{for all } t > y > 0,\ n \in \mathbb{N}. \tag{4}$$

Take two arbitrary positive reals y and z and a third fixed number $t > \max\{y, z\}$. For each positive integer k, let $\ell_k = \left\lfloor k\frac{g(y)}{g(z)} \right\rfloor$. Then $t + kg(y) - \ell_k g(z) \geq t > z$ and, applying (4) twice,

$$f\big(t + kg(y) - \ell_k g(z)\big) + \ell_k f(z) = f\big(t + kg(y)\big) = f(t) + kf(y),$$
$$0 < \frac{1}{k} f\big(t + kg(y) - \ell_k g(z)\big) = \frac{f(t)}{k} + f(y) - \frac{\ell_k}{k} f(z).$$

As $k \to \infty$ we get

$$0 \leq \lim_{k\to\infty} \left(\frac{f(t)}{k} + f(y) - \frac{\ell_k}{k} f(z) \right) = f(y) - \frac{g(y)}{g(z)} f(z) = f(y) - \frac{f(y) - y}{f(z) - z} f(z)$$

and therefore

$$\frac{f(y)}{y} \leq \frac{f(z)}{z}.$$

Exchanging variables y and z, we obtain the reverse inequality. Hence, $\frac{f(y)}{y} = \frac{f(z)}{z}$ for arbitrary y and z; so function $\frac{f(x)}{x}$ is constant, $f(x) = cx$.

Substituting back into (1), we find that $f(x) = cx$ is a solution if and only if $c = 2$. So the only solution for the problem is $f(x) = 2x$.

A5. Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \le 2a(m) + 2a(n) \quad \text{for all } m, n \ge 1, \tag{1}$$

and

$$a(2^k) \le \frac{1}{(k+1)^c} \quad \text{for all } k \ge 0. \tag{2}$$

Prove that the sequence $a(n)$ is bounded.

(Croatia)

Solution 1. For convenience, define $a(0) = 0$; then condition (1) persists for all pairs of nonnegative indices.

Lemma 1. For arbitrary nonnegative indices $n_1, \ldots, n_k$, we have

$$a\left(\sum_{i=1}^{k} n_i\right) \le \sum_{i=1}^{k} 2^i a(n_i) \tag{3}$$

and

$$a\left(\sum_{i=1}^{k} n_i\right) \le 2k \sum_{i=1}^{k} a(n_i). \tag{4}$$

Proof. Inequality (3) is proved by induction on k. The base case $k = 1$ is trivial, while the induction step is provided by

$$a\left(\sum_{i=1}^{k+1} n_i\right) = a\left(n_1 + \sum_{i=2}^{k+1} n_i\right) \le 2a(n_1) + 2a\left(\sum_{i=1}^{k} n_{i+1}\right) \le 2a(n_1) + 2\sum_{i=1}^{k} 2^i a(n_{i+1}) = \sum_{i=1}^{k+1} 2^i a(n_i).$$

To establish (4), first the inequality

$$a\left(\sum_{i=1}^{2^d} n_i\right) \le 2^d \sum_{i=1}^{2^d} a(n_i)$$

can be proved by an obvious induction on d. Then, turning to (4), we find an integer d such that $2^{d-1} < k \le 2^d$ to obtain

$$a\left(\sum_{i=1}^{k} n_i\right) = a\left(\sum_{i=1}^{k} n_i + \sum_{i=k+1}^{2^d} 0\right) \le 2^d \left(\sum_{i=1}^{k} a(n_i) + \sum_{i=k+1}^{2^d} a(0)\right) = 2^d \sum_{i=1}^{k} a(n_i) \le 2k \sum_{i=1}^{k} a(n_i).$$

□

Fix an increasing unbounded sequence $0 = M_0 < M_1 < M_2 < \ldots$ of real numbers; the exact values will be defined later. Let n be an arbitrary positive integer and write

$$n = \sum_{i=0}^{d} \varepsilon_i \cdot 2^i, \qquad \text{where } \varepsilon_i \in \{0, 1\}.$$

Set $\varepsilon_i = 0$ for $i > d$, and take some positive integer f such that $M_f > d$. Applying (3), we get

$$a(n) = a\left(\sum_{k=1}^{f} \sum_{M_{k-1} \le i < M_k} \varepsilon_i \cdot 2^i\right) \le \sum_{k=1}^{f} 2^k a\left(\sum_{M_{k-1} \le i < M_k} \varepsilon_i \cdot 2^i\right).$$

Note that there are less than $M_k - M_{k-1} + 1$ integers in interval $[M_{k-1}, M_k)$; hence, using (4) we have

$$\begin{aligned} a(n) &\le \sum_{k=1}^{f} 2^k \cdot 2(M_k - M_{k-1} + 1) \sum_{M_{k-1} \le i < M_k} \varepsilon_i \cdot a(2^i) \\ &\le \sum_{k=1}^{f} 2^k \cdot 2(M_k - M_{k-1} + 1)^2 \max_{M_{k-1} \le i < M_k} a(2^i) \\ &\le \sum_{k=1}^{f} 2^{k+1}(M_k + 1)^2 \cdot \frac{1}{(M_{k-1}+1)^c} = \sum_{k=1}^{f} \left(\frac{M_k + 1}{M_{k-1}+1} \right)^2 \frac{2^{k+1}}{(M_{k-1}+1)^{c-2}}. \end{aligned}$$

Setting $M_k = 4^{k/(c-2)} - 1$, we obtain

$$a(n) \le \sum_{k=1}^{f} 4^{2/(c-2)} \frac{2^{k+1}}{(4^{(k-1)/(c-2)})^{c-2}} = 8 \cdot 4^{2/(c-2)} \sum_{k=1}^{f} \left(\frac{1}{2}\right)^k < 8 \cdot 4^{2/(c-2)},$$

and the sequence $a(n)$ is bounded.

Solution 2.

Lemma 2. Suppose that $s_1, \ldots, s_k$ are positive integers such that

$$\sum_{i=1}^{k} 2^{-s_i} \le 1.$$

Then for arbitrary positive integers $n_1, \ldots, n_k$ we have

$$a\left(\sum_{i=1}^{k} n_i\right) \le \sum_{i=1}^{k} 2^{s_i} a(n_i).$$

Proof. Apply an induction on k. The base cases are $k = 1$ (trivial) and $k = 2$ (follows from the condition (1)). Suppose that $k > 2$. We can assume that $s_1 \le s_2 \le \cdots \le s_k$. Note that

$$\sum_{i=1}^{k-1} 2^{-s_i} \le 1 - 2^{-s_{k-1}},$$

since the left-hand side is a fraction with the denominator $2^{s_{k-1}}$, and this fraction is less than 1. Define $s'_{k-1} = s_{k-1} - 1$ and $n'_{k-1} = n_{k-1} + n_k$; then we have

$$\sum_{i=1}^{k-2} 2^{-s_i} + 2^{-s'_{k-1}} \le (1 - 2 \cdot 2^{-s_{k-1}}) + 2^{1-s_{k-1}} = 1.$$

Now, the induction hypothesis can be applied to achieve

$$\begin{aligned} a\left(\sum_{i=1}^{k} n_i\right) = a\left(\sum_{i=1}^{k-2} n_i + n'_{k-1}\right) &\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s'_{k-1}} a(n'_{k-1}) \\ &\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s_{k-1}-1} \cdot 2\big(a(n_{k-1}) + a(n_k)\big) \\ &\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s_{k-1}} a(n_{k-1}) + 2^{s_k} a(n_k). \end{aligned}$$

□

Let $q = c/2 > 1$. Take an arbitrary positive integer n and write

$$n = \sum_{i=1}^{k} 2^{u_i}, \qquad 0 \le u_1 < u_2 < \cdots < u_k.$$

Choose $s_i = \lfloor \log_2 (u_i + 1)^q \rfloor + d$ $(i = 1, \ldots, k)$ for some integer d. We have

$$\sum_{i=1}^{k} 2^{-s_i} = 2^{-d} \sum_{i=1}^{k} 2^{-\lfloor \log_2 (u_i+1)^q \rfloor},$$

and we choose d in such a way that

$$\frac{1}{2} < \sum_{i=1}^{k} 2^{-s_i} \le 1.$$

In particular, this implies

$$2^d < 2 \sum_{i=1}^{k} 2^{-\lfloor \log_2 (u_i+1)^q \rfloor} < 4 \sum_{i=1}^{k} \frac{1}{(u_i+1)^q}.$$

Now, by Lemma 2 we obtain

$$\begin{aligned} a(n) = a\left(\sum_{i=1}^{k} 2^{u_i}\right) &\le \sum_{i=1}^{k} 2^{s_i} a(2^{u_i}) \le \sum_{i=1}^{k} 2^d (u_i+1)^q \cdot \frac{1}{(u_i+1)^{2q}} \\ &= 2^d \sum_{i=1}^{k} \frac{1}{(u_i+1)^q} < 4 \left(\sum_{i=1}^{k} \frac{1}{(u_i+1)^q}\right)^2, \end{aligned}$$

which is bounded since $q > 1$.

Comment 1. In fact, Lemma 2 (applied to the case $n_i = 2^{u_i}$ only) provides a sharp bound for any $a(n)$. Actually, let $b(k) = \dfrac{1}{(k+1)^c}$ and consider the sequence

$$a(n) = \min\left\{ \sum_{i=1}^{k} 2^{s_i} b(u_i) \;\middle|\; k \in \mathbb{N}, \quad \sum_{i=1}^{k} 2^{-s_i} \le 1, \quad \sum_{i=1}^{k} 2^{u_i} = n \right\}. \tag{5}$$

We show that this sequence satisfies the conditions of the problem. Take two arbitrary indices m and n. Let

$$\begin{aligned} a(m) &= \sum_{i=1}^{k} 2^{s_i} b(u_i), \qquad & \sum_{i=1}^{k} 2^{-s_i} \le 1, \quad & \sum_{i=1}^{k} 2^{u_i} = m; \\ a(n) &= \sum_{i=1}^{l} 2^{r_i} b(w_i), \qquad & \sum_{i=1}^{l} 2^{-r_i} \le 1, \quad & \sum_{i=1}^{l} 2^{w_i} = n. \end{aligned}$$

Then we have

$$\sum_{i=1}^{k} 2^{-1-s_i} + \sum_{i=1}^{l} 2^{-1-r_i} \le \frac{1}{2} + \frac{1}{2} = 1, \qquad \sum_{i=1}^{k} 2^{u_i} + \sum_{i=1}^{l} 2^{w_i} = m + n,$$

so by (5) we obtain

$$a(n+m) \le \sum_{i=1}^{k} 2^{1+s_i} b(u_i) + \sum_{i=1}^{l} 2^{1+r_i} b(w_i) = 2a(m) + 2a(n).$$

Comment 2. The condition $c > 2$ is sharp; we show that the sequence (5) is not bounded if $c \leq 2$.

First, we prove that for an arbitrary n the minimum in (5) is attained with a sequence (u_i) consisting of distinct numbers. To the contrary, assume that $u_{k-1} = u_k$. Replace u_{k-1} and u_k by a single number $u'_{k-1} = u_k + 1$, and s_{k-1} and s_k by $s'_{k-1} = \min\{s_{k-1}, s_k\}$. The modified sequences provide a better bound since

$$2^{s'_{k-1}} b(u'_{k-1}) = 2^{s'_{k-1}} b(u_k + 1) < 2^{s_{k-1}} b(u_{k-1}) + 2^{s_k} b(u_k)$$

(we used the fact that $b(k)$ is decreasing). This is impossible.

Hence, the claim is proved, and we can assume that the minimum is attained with $u_1 < \cdots < u_k$; then

$$n = \sum_{i=1}^{k} 2^{u_i}$$

is simply the binary representation of n. (In particular, it follows that $a(2^n) = b(n)$ for each n.)

Now we show that the sequence $\big(a(2^k - 1)\big)$ is not bounded. For some $s_1, \ldots, s_k$ we have

$$a(2^k - 1) = a\left(\sum_{i=1}^{k} 2^{i-1}\right) = \sum_{i=1}^{k} 2^{s_i} b(i-1) = \sum_{i=1}^{k} \frac{2^{s_i}}{i^c}.$$

By the Cauchy–Schwarz inequality we get

$$a(2^k - 1) = a(2^k - 1) \cdot 1 \geq \left(\sum_{i=1}^{k} \frac{2^{s_i}}{i^c}\right)\left(\sum_{i=1}^{k} \frac{1}{2^{s_i}}\right) \geq \left(\sum_{i=1}^{k} \frac{1}{i^{c/2}}\right)^2,$$

which is unbounded.

For $c \leq 2$, it is also possible to show a concrete counterexample. Actually, one can prove that the sequence

$$a\left(\sum_{i=1}^{k} 2^{u_i}\right) = \sum_{i=1}^{k} \frac{i}{(u_i + 1)^2} \qquad (0 \leq u_1 < \ldots < u_k)$$

satisfies (1) and (2) but is not bounded.

A6. Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that $a_1^2+a_2^2+\ldots+a_{100}^2=1$. Prove that

$$a_1^2a_2+a_2^2a_3+\ldots+a_{100}^2a_1<\frac{12}{25}.$$

(Poland)

Solution. Let $S=\sum\limits_{k=1}^{100}a_k^2a_{k+1}$. (As usual, we consider the indices modulo 100, e.g. we set $a_{101}=a_1$ and $a_{102}=a_2$.)

Applying the Cauchy-Schwarz inequality to sequences (a_{k+1}) and $(a_k^2+2a_{k+1}a_{k+2})$, and then the AM-GM inequality to numbers a_{k+1}^2 and a_{k+2}^2,

$$\begin{aligned}(3S)^2&=\left(\sum_{k=1}^{100}a_{k+1}(a_k^2+2a_{k+1}a_{k+2})\right)^2\le\left(\sum_{k=1}^{100}a_{k+1}^2\right)\left(\sum_{k=1}^{100}(a_k^2+2a_{k+1}a_{k+2})^2\right)\qquad(1)\\&=1\cdot\sum_{k=1}^{100}(a_k^2+2a_{k+1}a_{k+2})^2=\sum_{k=1}^{100}(a_k^4+4a_k^2a_{k+1}a_{k+2}+4a_{k+1}^2a_{k+2}^2)\\&\le\sum_{k=1}^{100}\bigl(a_k^4+2a_k^2(a_{k+1}^2+a_{k+2}^2)+4a_{k+1}^2a_{k+2}^2\bigr)=\sum_{k=1}^{100}\bigl(a_k^4+6a_k^2a_{k+1}^2+2a_k^2a_{k+2}^2\bigr).\end{aligned}$$

Applying the trivial estimates

$$\sum_{k=1}^{100}(a_k^4+2a_k^2a_{k+1}^2+2a_k^2a_{k+2}^2)\le\left(\sum_{k=1}^{100}a_k^2\right)^2\quad\text{and}\quad\sum_{k=1}^{100}a_k^2a_{k+1}^2\le\left(\sum_{i=1}^{50}a_{2i-1}^2\right)\left(\sum_{j=1}^{50}a_{2j}^2\right),$$

we obtain that

$$(3S)^2\le\left(\sum_{k=1}^{100}a_k^2\right)^2+4\left(\sum_{i=1}^{50}a_{2i-1}^2\right)\left(\sum_{j=1}^{50}a_{2j}^2\right)\le1+\left(\sum_{i=1}^{50}a_{2i-1}^2+\sum_{j=1}^{50}a_{2j}^2\right)^2=2,$$

hence

$$S\le\frac{\sqrt2}{3}\approx0.4714<\frac{12}{25}=0.48.$$

Comment 1. By applying the Lagrange multiplier method, one can see that the maximum is attained at values of a_i satisfying

$$a_{k-1}^2+2a_ka_{k+1}=2\lambda a_k\qquad(2)$$

for all $k=1,2,\ldots,100$. Though this system of equations seems hard to solve, it can help to find the estimate above; it may suggest to have a closer look at the expression $a_{k-1}^2a_k+2a_k^2a_{k+1}$.

Moreover, if the numbers $a_1,\ldots,a_{100}$ satisfy (2), we have equality in (1). (See also Comment 3.)

Comment 2. It is natural to ask what is the best constant c_n in the inequality

$$a_1^2a_2+a_2^2a_3+\ldots+a_n^2a_1\le c_n\bigl(a_1^2+a_2^2+\ldots+a_n^2\bigr)^{3/2}.\qquad(3)$$

For $1\le n\le4$ one may prove $c_n=1/\sqrt{n}$ which is achieved when $a_1=a_2=\ldots=a_n$. However, the situation changes completely if $n\ge5$. In this case we do not know the exact value of c_n. By computer search it can be found that $c_n\approx0.4514$ and it is realized for example if

$$a_1\approx0.5873,\quad a_2\approx0.6771,\quad a_3\approx0.4224,\quad a_4\approx0.1344,\quad a_5\approx0.0133$$

and $a_k\approx0$ for $k\ge6$. This example also proves that $c_n>0.4513$.

Comment 3. The solution can be improved in several ways to give somewhat better bounds for c_n. Here we show a variant which proves $c_n < 0.4589$ for $n \geq 5$.

The value of c_n does not change if negative values are also allowed in (3). So the problem is equivalent to maximizing

$$f(a_1, a_2, \ldots, a_n) = a_1^2 a_2 + a_2^2 a_3 + \ldots + a_n^2 a_1$$

on the unit sphere $a_1^2 + a_2^2 + \ldots + a_n^2 = 1$ in $\mathbb{R}^n$. Since the unit sphere is compact, the function has a maximum and we can apply the Lagrange multiplier method; for each maximum point there exists a real number λ such that

$$a_{k-1}^2 + 2a_k a_{k+1} = \lambda \cdot 2a_k \qquad \text{for all } k = 1, 2, \ldots, n.$$

Then

$$3S = \sum_{k=1}^{n} \left(a_{k-1}^2 a_k + 2a_k^2 a_{k+1}\right) = \sum_{k=1}^{n} 2\lambda a_k^2 = 2\lambda$$

and therefore

$$a_{k-1}^2 + 2a_k a_{k+1} = 3S a_k \qquad \text{for all } k = 1, 2, \ldots, n. \tag{4}$$

From (4) we can derive

$$9S^2 = \sum_{k=1}^{n} (3S a_k)^2 = \sum_{k=1}^{n} \left(a_{k-1}^2 + 2a_k a_{k+1}\right)^2 = \sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + 4\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2} \tag{5}$$

and

$$3S^2 = \sum_{k=1}^{n} 3S a_{k-1}^2 a_k = \sum_{k=1}^{n} a_{k-1}^2 \left(a_{k-1}^2 + 2a_k a_{k+1}\right) = \sum_{k=1}^{n} a_k^4 + 2\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2}. \tag{6}$$

Let p be a positive number. Combining (5) and (6) and applying the AM-GM inequality,

$$\begin{aligned}
(9+3p)S^2 &= (1+p)\sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + (4+2p)\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2} \\
&\leq (1+p)\sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + \sum_{k=1}^{n}\left(2(1+p)a_k^2 a_{k+2}^2 + \frac{(2+p)^2}{2(1+p)} a_k^2 a_{k+1}^2\right) \\
&= (1+p)\sum_{k=1}^{n}(a_k^4 + 2a_k^2 a_{k+1}^2 + 2a_k^2 a_{k+2}^2) + \left(4 + \frac{(2+p)^2}{2(1+p)} - 2(1+p)\right)\sum_{k=1}^{n} a_k^2 a_{k+1}^2 \\
&\leq (1+p)\left(\sum_{k=1}^{n} a_k^2\right)^2 + \frac{8+4p-3p^2}{2(1+p)}\sum_{k=1}^{n} a_k^2 a_{k+1}^2 \\
&= (1+p) + \frac{8+4p-3p^2}{2(1+p)}\sum_{k=1}^{n} a_k^2 a_{k+1}^2.
\end{aligned}$$

Setting $p = \dfrac{2+2\sqrt{7}}{3}$ which is the positive root of $8 + 4p - 3p^2 = 0$, we obtain

$$S \leq \sqrt{\frac{1+p}{9+3p}} = \sqrt{\frac{5+2\sqrt{7}}{33+6\sqrt{7}}} \approx 0.458879.$$

A7. Let $n>1$ be an integer. In the space, consider the set

$$S=\{(x,y,z)\mid x,y,z\in\{0,1,\ldots,n\},\ x+y+z>0\}.$$

Find the smallest number of planes that jointly contain all $(n+1)^3-1$ points of S but none of them passes through the origin.

(Netherlands)

Answer. $3n$ planes.

Solution. It is easy to find $3n$ such planes. For example, planes $x=i$, $y=i$ or $z=i$ $(i=1,2,\ldots,n)$ cover the set S but none of them contains the origin. Another such collection consists of all planes $x+y+z=k$ for $k=1,2,\ldots,3n$.

We show that $3n$ is the smallest possible number.

Lemma 1. Consider a nonzero polynomial $P(x_1,\ldots,x_k)$ in k variables. Suppose that P vanishes at all points $(x_1,\ldots,x_k)$ such that $x_1,\ldots,x_k\in\{0,1,\ldots,n\}$ and $x_1+\cdots+x_k>0$, while $P(0,0,\ldots,0)\neq 0$. Then $\deg P\geq kn$.

Proof. We use induction on k. The base case $k=0$ is clear since $P\neq 0$. Denote for clarity $y=x_k$.

Let $R(x_1,\ldots,x_{k-1},y)$ be the residue of P modulo $Q(y)=y(y-1)\ldots(y-n)$. Polynomial $Q(y)$ vanishes at each $y=0,1,\ldots,n$, hence $P(x_1,\ldots,x_{k-1},y)=R(x_1,\ldots,x_{k-1},y)$ for all $x_1,\ldots,x_{k-1},y\in\{0,1,\ldots,n\}$. Therefore, R also satisfies the condition of the Lemma; moreover, $\deg_y R\leq n$. Clearly, $\deg R\leq\deg P$, so it suffices to prove that $\deg R\geq nk$.

Now, expand polynomial R in the powers of y:

$$R(x_1,\ldots,x_{k-1},y)=R_n(x_1,\ldots,x_{k-1})y^n+R_{n-1}(x_1,\ldots,x_{k-1})y^{n-1}+\cdots+R_0(x_1,\ldots,x_{k-1}).$$

We show that polynomial $R_n(x_1,\ldots,x_{k-1})$ satisfies the condition of the induction hypothesis.

Consider the polynomial $T(y)=R(0,\ldots,0,y)$ of degree $\leq n$. This polynomial has n roots $y=1,\ldots,n$; on the other hand, $T(y)\not\equiv 0$ since $T(0)\neq 0$. Hence $\deg T=n$, and its leading coefficient is $R_n(0,0,\ldots,0)\neq 0$. In particular, in the case $k=1$ we obtain that coefficient R_n is nonzero.

Similarly, take any numbers $a_1,\ldots,a_{k-1}\in\{0,1,\ldots,n\}$ with $a_1+\cdots+a_{k-1}>0$. Substituting $x_i=a_i$ into $R(x_1,\ldots,x_{k-1},y)$, we get a polynomial in y which vanishes at all points $y=0,\ldots,n$ and has degree $\leq n$. Therefore, this polynomial is null, hence $R_i(a_1,\ldots,a_{k-1})=0$ for all $i=0,1,\ldots,n$. In particular, $R_n(a_1,\ldots,a_{k-1})=0$.

Thus, the polynomial $R_n(x_1,\ldots,x_{k-1})$ satisfies the condition of the induction hypothesis. So, we have $\deg R_n\geq(k-1)n$ and $\deg P\geq\deg R\geq\deg R_n+n\geq kn$. $\square$

Now we can finish the solution. Suppose that there are N planes covering all the points of S but not containing the origin. Let their equations be $a_ix+b_iy+c_iz+d_i=0$. Consider the polynomial

$$P(x,y,z)=\prod_{i=1}^{N}(a_ix+b_iy+c_iz+d_i).$$

It has total degree N. This polynomial has the property that $P(x_0,y_0,z_0)=0$ for any $(x_0,y_0,z_0)\in S$, while $P(0,0,0)\neq 0$. Hence by Lemma 1 we get $N=\deg P\geq 3n$, as desired.

Comment 1. There are many other collections of $3n$ planes covering the set S but not covering the origin.

Solution 2. We present a different proof of the main Lemma 1. Here we confine ourselves to the case $k=3$, which is applied in the solution, and denote the variables by x, y and z. (The same proof works for the general statement as well.)

The following fact is known with various proofs; we provide one possible proof for the completeness.

Lemma 2. For arbitrary integers $0 \le m < n$ and for an arbitrary polynomial $P(x)$ of degree m,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(k)=0. \tag{1}$$

Proof. We use an induction on n. If $n=1$, then $P(x)$ is a constant polynomial, hence $P(1)-P(0)=0$, and the base is proved.

For the induction step, define $P_1(x)=P(x+1)-P(x)$. Then clearly $\deg P_1=\deg P-1=m-1<n-1$, hence by the induction hypothesis we get

$$\begin{aligned}
0 &= -\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}P_1(k)=\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\big(P(k)-P(k+1)\big)\\
&=\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}P(k)-\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}P(k+1)\\
&=\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}P(k)+\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}P(k)\\
&=P(0)+\sum_{k=1}^{n-1}(-1)^k\left(\binom{n-1}{k-1}+\binom{n-1}{k}\right)P(k)+(-1)^nP(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(k). \quad \square
\end{aligned}$$

Now return to the proof of Lemma 1. Suppose, to the contrary, that $\deg P=N<3n$. Consider the sum

$$\Sigma=\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}(-1)^{i+j+k}\binom{n}{i}\binom{n}{j}\binom{n}{k}P(i,j,k).$$

The only nonzero term in this sum is $P(0,0,0)$ and its coefficient is $\binom{n}{0}^3=1$; therefore $\Sigma=P(0,0,0)\neq 0$.

On the other hand, if $P(x,y,z)=\sum\limits_{\alpha+\beta+\gamma\le N}p_{\alpha,\beta,\gamma}x^\alpha y^\beta z^\gamma$, then

$$\begin{aligned}
\Sigma &= \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}(-1)^{i+j+k}\binom{n}{i}\binom{n}{j}\binom{n}{k}\sum_{\alpha+\beta+\gamma\le N}p_{\alpha,\beta,\gamma}i^\alpha j^\beta k^\gamma\\
&=\sum_{\alpha+\beta+\gamma\le N}p_{\alpha,\beta,\gamma}\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^\alpha\right)\left(\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^\beta\right)\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^\gamma\right).
\end{aligned}$$

Consider an arbitrary term in this sum. We claim that it is zero. Since $N<3n$, one of three inequalities $\alpha<n$, $\beta<n$ or $\gamma<n$ is valid. For the convenience, suppose that $\alpha<n$. Applying Lemma 2 to polynomial x^α, we get $\sum\limits_{i=0}^{n}(-1)^i\binom{n}{i}i^\alpha=0$, hence the term is zero as required.

This yields $\Sigma=0$ which is a contradiction. Therefore, $\deg P\ge 3n$.

Comment 2. The proof does not depend on the concrete coefficients in Lemma 2. Instead of this Lemma, one can simply use the fact that there exist numbers $\alpha_0, \alpha_1, \dots, \alpha_n$ ($\alpha_0 \neq 0$) such that

$$\sum_{k=0}^{n} \alpha_k k^m = 0 \qquad \text{for every } 0 \leq m < n.$$

This is a system of homogeneous linear equations in variables α_i. Since the number of equations is less than the number of variables, the only nontrivial thing is that there exists a solution with $\alpha_0 \neq 0$. It can be shown in various ways.

Combinatorics

C1. Let $n>1$ be an integer. Find all sequences $a_1, a_2, \ldots, a_{n^2+n}$ satisfying the following conditions:

(a) $a_i \in \{0,1\}$ for all $1 \leq i \leq n^2+n$;

(b) $a_{i+1}+a_{i+2}+\ldots+a_{i+n} < a_{i+n+1}+a_{i+n+2}+\ldots+a_{i+2n}$ for all $0 \leq i \leq n^2-n$.

(Serbia)

Answer. Such a sequence is unique. It can be defined as follows:

$$a_{u+vn} = \begin{cases} 0, & u+v \leq n, \\ 1, & u+v \geq n+1 \end{cases} \qquad \text{for all } 1 \leq u \leq n \text{ and } 0 \leq v \leq n. \tag{1}$$

The terms can be arranged into blocks of length n as

$$(\underbrace{0\ \ldots\ 0}_{n})\ (\underbrace{0\ \ldots\ 0}_{n-1}\ 1)\ (\underbrace{0\ \ldots\ 0}_{n-2}\ 1\ 1) \ldots (\underbrace{0\ \ldots\ 0}_{n-v}\ \underbrace{1\ \ldots\ 1}_{v}) \ldots (0\ \underbrace{1\ \ldots\ 1}_{n-1})\ (\underbrace{1\ \ldots\ 1}_{n}).$$

Solution 1. Consider a sequence (a_i) satisfying the conditions. For arbitrary integers $0 \leq k \leq l \leq n^2+n$ denote $S(k,l] = a_{k+1}+\cdots+a_l$. (If $k=l$ then $S(k,l]=0$.) Then condition (b) can be rewritten as $S(i,i+n] < S(i+n,i+2n]$ for all $0 \leq i \leq n^2-n$. Notice that for $0 \leq k \leq l \leq m \leq n^2+n$ we have $S(k,m] = S(k,l]+S(l,m]$.

By condition (b),

$$0 \leq S(0,n] < S(n,2n] < \cdots < S(n^2, n^2+n] \leq n.$$

We have only $n+1$ distinct integers in the interval $[0,n]$; hence,

$$S\big(vn, (v+1)n\big] = v \qquad \text{for all } 0 \leq v \leq n. \tag{2}$$

In particular, $S(0,n]=0$ and $S(n^2,n^2+n]=n$, therefore

$$a_1 = a_2 = \ldots = a_n = 0, \tag{3}$$

$$a_{n^2+1} = a_{n^2+2} = \ldots = a_{n^2+n} = 1. \tag{4}$$

Subdivide sequence (a_i) into $n+1$ blocks, each consisting of n consecutive terms, and number them from 0 to n. We show by induction on v that the vth blocks has the form

$$(\underbrace{0\ \ldots\ 0}_{n-v}\ \underbrace{1\ \ldots\ 1}_{v}).$$

The base case $v=0$ is provided by (3).

Consider the vth block for $v > 0$. By (2), it contains some "ones". Let the first "one" in this block be at the uth position (that is, $a_{u+vn} = 1$). By the induction hypothesis, the $(v-1)$th and vth blocks of (a_i) have the form

$$(\overbrace{\underbrace{0\ \dots\ 0\ \dots\ 0}_{n-v+1}\ \underbrace{1\ \dots\ 1}_{v-1})\ (\underbrace{0\ \dots\ 0}_{u-1}\ 1}^{\sum = v} * \ \dots\ *),$$

where each star can appear to be any binary digit. Observe that $u \le n - v + 1$, since the sum in this block is v. Then, the fragment of length n bracketed above has exactly $(v-1)+1$ ones, i. e. $S\big(u + (v-1)n, u + vn\big] = v$. Hence,

$$v = S\big(u + (v-1)n, u + vn\big] < S\big(u + vn, u + (v+1)n\big] < \dots < S\big(u + (n-1)n, u + n^2\big] \le n;$$

we have $n - v + 1$ distinct integers in the interval $[v, n]$, therefore $S(u + (t-1)n, u + tn] = t$ for each $t = v, \dots, n$.

Thus, the end of sequence (a_i) looks as following:

$$(\underbrace{\overbrace{0\ \dots\ 0}^{u\text{ zeroes}}\ \overbrace{0\ \dots\ 0\ 1\ \dots\ 1}^{\sum=v}}_{\sum=v-1})\ (\underbrace{\overbrace{0\ \dots\ 0\ 1 * \dots *}^{\sum=v+1}}_{\sum=v})\ (\underbrace{\overbrace{* \dots *}^{\cdots} * \dots *}_{\sum=v+1}) \dots (\underbrace{\overbrace{1\ \dots\ 1}^{\sum=n}\ \overbrace{1\ \dots\ 1}^{n-u\text{ ones}}}_{\sum=n})$$

(each bracketed fragment contains n terms). Computing in two ways the sum of all digits above, we obtain $n - u = v - 1$ and $u = n - v + 1$. Then, the first $n - v$ terms in the vth block are zeroes, and the next v terms are ones, due to the sum of all terms in this block. The statement is proved.

We are left to check that the sequence obtained satisfies the condition. Notice that $a_i \le a_{i+n}$ for all $1 \le i \le n^2$. Moreover, if $1 \le u \le n$ and $0 \le v \le n-1$, then $a_{u+vn} < a_{u+vn+n}$ exactly when $u + v = n$. In this case we have $u + vn = n + v(n-1)$.

Consider now an arbitrary index $0 \le i \le n^2 - n$. Clearly, there exists an integer v such that $n + v(n-1) \in [i+1, i+n]$. Then, applying the above inequalities we obtain that condition (b) is valid.

Solution 2. Similarly to Solution 1, we introduce the notation $S(k, l]$ and obtain (2), (3), and (4) in the same way. The sum of all elements of the sequence can be computed as

$$S(0, n^2 + n] = S(0, n] + S(n, 2n] + \ldots + S(n^2, n^2 + n] = 0 + 1 + \ldots + n.$$

For an arbitrary integer $0 \le u \le n$, consider the numbers

$$S(u, u+n] < S(u+n, u+2n] < \ldots < S\big(u + (n-1)n, u + n^2\big]. \tag{5}$$

They are n distinct integers from the $n+1$ possible values $0, 1, 2, \dots, n$. Denote by m the "missing" value which is not listed. We determine m from $S(0, n^2 + n]$. Write this sum as

$$S(0, n^2{+}n] = S(0, u]{+}S(u, u{+}n]{+}S(u{+}n, u{+}2n]{+}\ldots{+}S(u{+}(n{-}1)n, u{+}n^2]{+}S(u{+}n^2, n^2{+}n].$$

Since $a_1 = a_2 = \ldots = a_u = 0$ and $a_{u+n^2+1} = \ldots = a_{n^2+n} = 1$, we have $S(0, u] = 0$ and $S(u + n^2, n + n^2] = n - u$. Then

$$0 + 1 + \ldots + n = S(0, n^2 + n] = 0 + \big((0 + 1 + \ldots + n) - m\big) + (n - u),$$

so $m = n - u$.

Hence, the numbers listed in (5) are $0, 1, \ldots, n-u-1$ and $n-u+1, \ldots, n$, respectively, therefore

$$S\big(u+vn, u+(v+1)n\big] = \begin{cases} v, & v \le n-u-1, \\ v+1, & v \ge n-u \end{cases} \quad \text{for all } 0 \le u \le n,\ 0 \le v \le n-1. \quad (6)$$

Conditions (6), together with (3), provide a system of linear equations in variables a_i. Now we solve this system and show that the solution is unique and satisfies conditions (a) and (b).

First, observe that any solution of the system (3), (6) satisfies the condition (b). By the construction, equations (6) immediately imply (5). On the other hand, all inequalities mentioned in condition (b) are included into the chain (5) for some value of u.

Next, note that the system (3), (6) is redundant. The numbers $S\big(kn, (k+1)n\big]$, where $1 \le k \le n-1$, appear twice in (6). For $u = 0$ and $v = k$ we have $v \le n-u-1$, and (6) gives $S\big(kn, (k+1)n\big] = v = k$. For $u = n$ and $v = k-1$ we have $v \ge n-u$ and we obtain the same value, $S\big(kn, (k+1)n\big] = v+1 = k$. Therefore, deleting one equation from each redundant pair, we can make every sum $S(k, k+n]$ appear exactly once on the left-hand side in (6).

Now, from (3), (6), the sequence (a_i) can be reconstructed inductively by

$$a_1 = a_2 = \ldots = a_{n-1} = 0, \qquad a_{k+n} = S(k, k+n] - (a_{k+1} + a_{k+2} + \ldots + a_{k+n-1}) \quad (0 \le k \le n^2),$$

taking the values of $S(k, k+n]$ from (6). This means first that there exists at most one solution of our system. Conversely, the constructed sequence obviously satisfies all equations (3), (6) (the only missing equation is $a_n = 0$, which follows from $S(0, n] = 0$). Hence it satisfies condition (b), and we are left to check condition (a) only.

For arbitrary integers $1 \le u, t \le n$ we get

$$\begin{aligned} a_{u+tn} - a_{u+(t-1)n} &= S\big(u+(t-1)n, u+tn\big] - S\big((u-1)+(t-1)n, (u-1)+tn\big] \\ &= \begin{cases} (t-1)-(t-1) = 0, & t \le n-u, \\ t-(t-1) = 1, & t = n-u+1, \\ t-t = 0, & t \ge n-u+2. \end{cases} \end{aligned}$$

Since $a_u = 0$, we have

$$a_{u+vn} = a_{u+vn} - a_u = \sum_{t=1}^{v} (a_{u+tn} - a_{u+(t-1)n})$$

for all $1 \le u, v \le n$. If $v < n-u+1$ then all terms are 0 on the right-hand side. If $v \ge n-u+1$, then variable t attains the value $n-u+1$ once. Hence,

$$a_{u+vn} = \begin{cases} 0, & u+v \le n, \\ 1, & u+v \ge n+1, \end{cases}$$

according with (1). Note that the formula is valid for $v = 0$ as well.

Finally, we presented the direct formula for (a_i), and we have proved that it satisfies condition (a). So, the solution is complete.

C2. A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that in this dissection, there exists a rectangle having no point on the boundary of the square.

(Japan)

Solution 1. Call the directions of the sides of the square horizontal and vertical. A horizontal or vertical line, which intersects the interior of the square but does not intersect the interior of any rectangle, will be called a *splitting line*. A rectangle having no point on the boundary of the square will be called an *interior rectangle*.

Suppose, to the contrary, that there exists a dissection of the square into more than one rectangle, such that no interior rectangle and no splitting line appear. Consider such a dissection with the least possible number of rectangles. Notice that this number of rectangles is greater than 2, otherwise their common side provides a splitting line.

If there exist two rectangles having a common side, then we can replace them by their union (see Figure 1). The number of rectangles was greater than 2, so in a new dissection it is greater than 1. Clearly, in the new dissection, there is also no splitting line as well as no interior rectangle. This contradicts the choice of the original dissection.

Denote the initial square by $ABCD$, with A and B being respectively the lower left and lower right vertices. Consider those two rectangles a and b containing vertices A and B, respectively. (Note that $a \neq b$, otherwise its top side provides a splitting line.) We can assume that the height of a is not greater than that of b. Then consider the rectangle c neighboring to the lower right corner of a (it may happen that $c = b$). By aforementioned, the heights of a and c are distinct. Then two cases are possible.

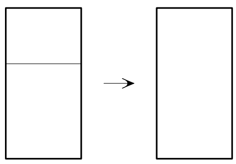


Figure 1

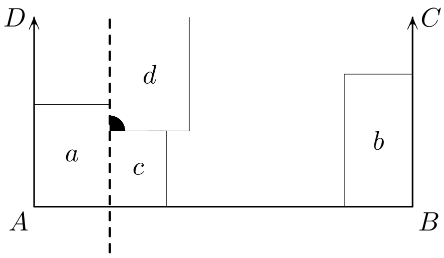


Figure 2

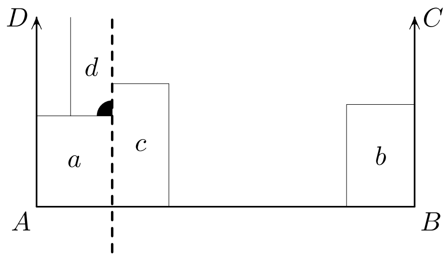


Figure 3

Case 1. The height of c is less than that of a. Consider the rectangle d which is adjacent to both a and c, i. e. the one containing the angle marked in Figure 2. This rectangle has no common point with BC (since a is not higher than b), as well as no common point with AB or with AD (obviously). Then d has a common point with CD, and its left side provides a splitting line. Contradiction.

Case 2. The height of c is greater than that of a. Analogously, consider the rectangle d containing the angle marked on Figure 3. It has no common point with AD (otherwise it has a common side with a), as well as no common point with AB or with BC (obviously). Then d has a common point with CD. Hence its right side provides a splitting line, and we get the contradiction again.

Solution 2. Again, we suppose the contrary. Consider an arbitrary counterexample. Then we know that each rectangle is attached to at least one side of the square. Observe that a rectangle cannot be attached to two opposite sides, otherwise one of its sides lies on a splitting line.

We say that two rectangles are *opposite* if they are attached to opposite sides of $ABCD$. We claim that there exist two opposite rectangles having a common point.

Consider the union L of all rectangles attached to the left. Assume, to the contrary, that L has no common point with the rectangles attached to the right. Take a polygonal line p connecting the top and the bottom sides of the square and passing close from the right to the boundary of L (see Figure 4). Then all its points belong to the rectangles attached either to the top or to the bottom. Moreover, the upper end-point of p belongs to a rectangle attached to the top, and the lower one belongs to an other rectangle attached to the bottom. Hence, there is a point on p where some rectangles attached to the top and to the bottom meet each other. So, there always exists a pair of neighboring opposite rectangles.

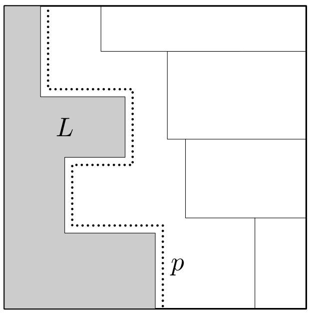


Figure 4

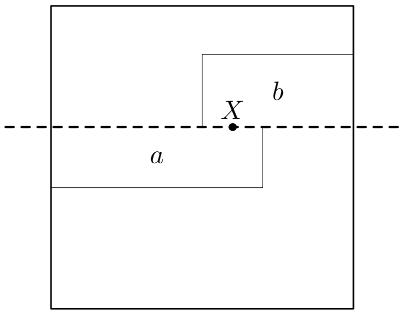


Figure 5

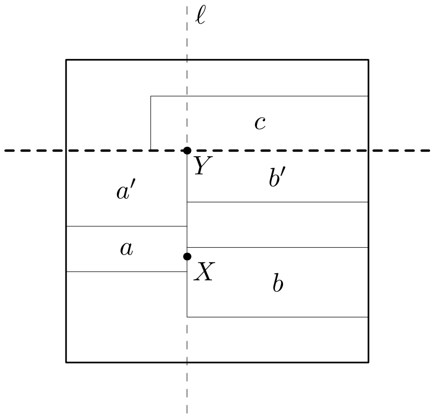


Figure 6

Now, take two opposite neighboring rectangles a and b. We can assume that a is attached to the left and b is attached to the right. Let X be their common point. If X belongs to their horizontal sides (in particular, X may appear to be a common vertex of a and b), then these sides provide a splitting line (see Figure 5). Otherwise, X lies on the vertical sides. Let ℓ be the line containing these sides.

Since ℓ is not a splitting line, it intersects the interior of some rectangle. Let c be such a rectangle, closest to X; we can assume that c lies above X. Let Y be the common point of ℓ and the bottom side of c (see Figure 6). Then Y is also a vertex of two rectangles lying below c.

So, let Y be the upper-right and upper-left corners of the rectangles a' and b', respectively. Then a' and b' are situated not lower than a and b, respectively (it may happen that $a = a'$ or $b = b'$). We claim that a' is attached to the left. If $a = a'$ then of course it is. If $a \neq a'$ then a' is above a, below c and to the left from b'. Hence, it can be attached to the left only.

Analogously, b' is attached to the right. Now, the top sides of these two rectangles pass through Y, hence they provide a splitting line again. This last contradiction completes the proof.

C3. Find all positive integers n, for which the numbers in the set $S = \{1, 2, \ldots, n\}$ can be colored red and blue, with the following condition being satisfied: the set $S \times S \times S$ contains exactly 2007 ordered triples (x, y, z) such that (i) x, y, z are of the same color and (ii) $x + y + z$ is divisible by n.

(Netherlands)

Answer. $n = 69$ and $n = 84$.

Solution. Suppose that the numbers $1, 2, \ldots, n$ are colored red and blue. Denote by R and B the sets of red and blue numbers, respectively; let $|R| = r$ and $|B| = b = n - r$. Call a triple $(x, y, z) \in S \times S \times S$ *monochromatic* if x, y, z have the same color, and *bichromatic* otherwise. Call a triple (x, y, z) *divisible* if $x + y + z$ is divisible by n. We claim that there are exactly $r^2 - rb + b^2$ divisible monochromatic triples.

For any pair $(x, y) \in S \times S$ there exists a unique $z_{x,y} \in S$ such that the triple $(x, y, z_{x,y})$ is divisible; so there are exactly n^2 divisible triples. Furthermore, if a divisible triple (x, y, z) is bichromatic, then among x, y, z there are either one blue and two red numbers, or vice versa. In both cases, exactly one of the pairs (x, y), (y, z) and (z, x) belongs to the set $R \times B$. Assign such pair to the triple (x, y, z).

Conversely, consider any pair $(x, y) \in R \times B$, and denote $z = z_{x,y}$. Since $x \neq y$, the triples (x, y, z), (y, z, x) and (z, x, y) are distinct, and (x, y) is assigned to each of them. On the other hand, if (x, y) is assigned to some triple, then this triple is clearly one of those mentioned above. So each pair in $R \times B$ is assigned exactly three times.

Thus, the number of bichromatic divisible triples is three times the number of elements in $R \times B$, and the number of monochromatic ones is $n^2 - 3rb = (r + b)^2 - 3rb = r^2 - rb + b^2$, as claimed.

So, to find all values of n for which the desired coloring is possible, we have to find all n, for which there exists a decomposition $n = r + b$ with $r^2 - rb + b^2 = 2007$. Therefore, $9 \mid r^2 - rb + b^2 = (r + b)^2 - 3rb$. From this it consequently follows that $3 \mid r + b$, $3 \mid rb$, and then $3 \mid r$, $3 \mid b$. Set $r = 3s$, $b = 3c$. We can assume that $s \geq c$. We have $s^2 - sc + c^2 = 223$.

Furthermore,

$$892 = 4(s^2 - sc + c^2) = (2c - s)^2 + 3s^2 \geq 3s^2 \geq 3s^2 - 3c(s - c) = 3(s^2 - sc + c^2) = 669,$$

so $297 \geq s^2 \geq 223$ and $17 \geq s \geq 15$. If $s = 15$ then

$$c(15 - c) = c(s - c) = s^2 - (s^2 - sc + c^2) = 15^2 - 223 = 2$$

which is impossible for an integer c. In a similar way, if $s = 16$ then $c(16 - c) = 33$, which is also impossible. Finally, if $s = 17$ then $c(17 - c) = 66$, and the solutions are $c = 6$ and $c = 11$. Hence, $(r, b) = (51, 18)$ or $(r, b) = (51, 33)$, and the possible values of n are $n = 51 + 18 = 69$ and $n = 51 + 33 = 84$.

Comment. After the formula for the number of monochromatic divisible triples is found, the solution can be finished in various ways. The one presented is aimed to decrease the number of considered cases.

C4. Let $A_0 = (a_1, \dots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \dots, x_n)$ we construct a new sequence A_{k+1} in the following way.

1. We choose a partition $\{1, \dots, n\} = I \cup J$, where I and J are two disjoint sets, such that the expression

$$\left|\sum_{i \in I} x_i - \sum_{j \in J} x_j\right|$$

attains the smallest possible value. (We allow the sets I or J to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.

2. We set $A_{k+1} = (y_1, \dots, y_n)$, where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

Prove that for some k, the sequence A_k contains an element x such that $|x| \geq n/2$.

(Iran)

Solution.

Lemma. Suppose that all terms of the sequence $(x_1, \dots, x_n)$ satisfy the inequality $|x_i| < a$. Then there exists a partition $\{1, 2, \dots, n\} = I \cup J$ into two disjoint sets such that

$$\left|\sum_{i \in I} x_i - \sum_{j \in J} x_j\right| < a. \tag{1}$$

Proof. Apply an induction on n. The base case $n = 1$ is trivial. For the induction step, consider a sequence $(x_1, \dots, x_n)$ $(n > 1)$. By the induction hypothesis there exists a splitting $\{1, \dots, n-1\} = I' \cup J'$ such that

$$\left|\sum_{i \in I'} x_i - \sum_{j \in J'} x_j\right| < a.$$

For convenience, suppose that $\sum\limits_{i \in I'} x_i \geq \sum\limits_{j \in J'} x_j$. If $x_n \geq 0$ then choose $I = I'$, $J = J \cup \{n\}$; otherwise choose $I = I' \cup \{n\}$, $J = J'$. In both cases, we have $\sum\limits_{i \in I'} x_i - \sum\limits_{j \in J'} x_j \in [0, a)$ and $|x_n| \in [0, a)$; hence

$$\sum_{i \in I} x_i - \sum_{j \in J} x_j = \sum_{i \in I'} x_i - \sum_{j \in J'} x_j - |x_n| \in (-a, a),$$

as desired. $\square$

Let us turn now to the problem. To the contrary, assume that for all k, all the numbers in A_k lie in interval $(-n/2, n/2)$. Consider an arbitrary sequence $A_k = (b_1, \dots, b_n)$. To obtain the term b_i, we increased and decreased number a_i by one several times. Therefore $b_i - a_i$ is always an integer, and there are not more than n possible values for b_i. So, there are not more than n^n distinct possible sequences A_k, and hence two of the sequences $A_1, A_2, \dots, A_{n^n+1}$ should be identical, say $A_p = A_q$ for some $p < q$.

For any positive integer k, let S_k be the sum of squares of elements in A_k. Consider two consecutive sequences $A_k = (x_1, \dots, x_n)$ and $A_{k+1} = (y_1, \dots, y_n)$. Let $\{1, 2, \dots, n\} = I \cup J$ be the partition used in this step — that is, $y_i = x_i + 1$ for all $i \in I$ and $y_j = x_j - 1$ for all $j \in J$. Since the value of $\left|\sum\limits_{i \in I} x_i - \sum\limits_{j \in J} x_j\right|$ is the smallest possible, the Lemma implies that it is less than $n/2$. Then we have

$$S_{k+1} - S_k = \sum_{i \in I}\left((x_i+1)^2 - x_i^2\right) + \sum_{j \in J}\left((x_j-1)^2 - x_j^2\right) = n + 2\left(\sum_{i \in I} x_i - \sum_{j \in J} x_j\right) > n - 2 \cdot \frac{n}{2} = 0.$$

Thus we obtain $S_q > S_{q-1} > \dots > S_p$. This is impossible since $A_p = A_q$ and hence $S_p = S_q$.

C5. In the Cartesian coordinate plane define the strip $S_n = \{(x, y) \mid n \leq x < n+1\}$ for every integer n. Assume that each strip S_n is colored either red or blue, and let a and b be two distinct positive integers. Prove that there exists a rectangle with side lengths a and b such that its vertices have the same color.

(Romania)

Solution. If S_n and S_{n+a} have the same color for some integer n, then we can choose the rectangle with vertices $(n, 0) \in S_n$, $(n, b) \in S_n$, $(n+a, 0) \in S_{n+a}$, and $(n+a, b) \in S_{n+a}$, and we are done. So it can be assumed that S_n and S_{n+a} have opposite colors for each n.

Similarly, it also can be assumed that S_n and S_{n+b} have opposite colors. Then, by induction on $|p|+|q|$, we obtain that for arbitrary integers p and q, strips S_n and $S_{n+pa+qb}$ have the same color if $p+q$ is even, and these two strips have opposite colors if $p+q$ is odd.

Let $d = \gcd(a, b)$, $a_1 = a/d$ and $b_1 = b/d$. Apply the result above for $p = b_1$ and $q = -a_1$. The strips S_0 and $S_{0+b_1 a - a_1 b}$ are identical and therefore they have the same color. Hence, $a_1 + b_1$ is even. By the construction, a_1 and b_1 are coprime, so this is possible only if both are odd.

Without loss of generality, we can assume $a > b$. Then $a_1 > b_1 \geq 1$, so $a_1 \geq 3$.

Choose integers k and ℓ such that $ka_1 - \ell b_1 = 1$ and therefore $ka - \ell b = d$. Since a_1 and b_1 are odd, $k + \ell$ is odd as well. Hence, for every integer n, strips S_n and $S_{n+ka-\ell b} = S_{n+d}$ have opposite colors. This also implies that the coloring is periodic with period $2d$, i.e. strips S_n and S_{n+2d} have the same color for every n.

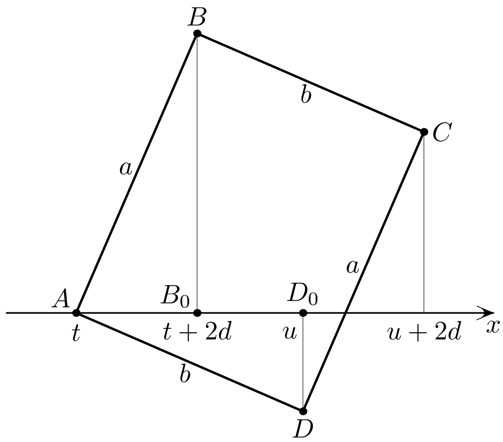


Figure 1

We will construct the desired rectangle $ABCD$ with $AB = CD = a$ and $BC = AD = b$ in a position such that vertex A lies on the x-axis, and the projection of side AB onto the x-axis is of length $2d$ (see Figure 1). This is possible since $a = a_1 d > 2d$. The coordinates of the vertices will have the forms

$$A = (t, 0), \quad B = (t+2d, y_1), \quad C = (u+2d, y_2), \quad D = (u, y_3).$$

Let $\varphi = \sqrt{a_1^2 - 4}$. By Pythagoras' theorem,

$$y_1 = BB_0 = \sqrt{a^2 - 4d^2} = d\sqrt{a_1^2 - 4} = d\varphi.$$

So, by the similar triangles ADD_0 and BAB_0, we have the constraint

$$u - t = AD_0 = \frac{AD}{AB} \cdot BB_0 = \frac{bd}{a}\varphi \tag{1}$$

for numbers t and u. Computing the numbers y_2 and y_3 is not required since they have no effect to the colors.

Observe that the number φ is irrational, because φ^2 is an integer, but φ is not: $a_1 > \varphi \geq \sqrt{a_1^2 - 2a_1 + 2} > a_1 - 1$.

By the periodicity, points A and B have the same color; similarly, points C and D have the same color. Furthermore, these colors depend only on the values of t and u. So it is sufficient to choose numbers t and u such that vertices A and D have the same color.

Let w be the largest positive integer such that there exist w consecutive strips $S_{n_0}, S_{n_0+1}, \ldots, S_{n_0+w-1}$ with the same color, say red. (Since S_{n_0+d} must be blue, we have $w \leq d$.) We will choose t from the interval $(n_0, n_0 + w)$.

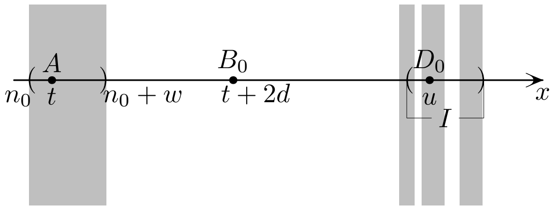


Figure 2

Consider the interval $I = \left(n_0 + \dfrac{bd}{a}\varphi, n_0 + \dfrac{bd}{a}\varphi + w\right)$ on the x-axis (see Figure 2). Its length is w, and the end-points are irrational. Therefore, this interval intersects $w + 1$ consecutive strips. Since at most w consecutive strips may have the same color, interval I must contain both red and blue points. Choose $u \in I$ such that the line $x = u$ is red and set $t = u - \dfrac{bd}{a}\varphi$, according to the constraint (1). Then $t \in (n_0, n_0 + w)$ and $A = (t, 0)$ is red as well as $D = (u, y_3)$.

Hence, variables u and t can be set such that they provide a rectangle with four red vertices.

Comment. The statement is false for squares, i.e. in the case $a = b$. If strips S_{2ka}, S_{2ka+1}, ..., $S_{(2k+1)a-1}$ are red, and strips $S_{(2k+1)a}$, $S_{(2k+1)a+1}$, ..., $S_{(2k+2)a-1}$ are blue for every integer k, then each square of size $a \times a$ has at least one red and at least one blue vertex as well.

C6. In a mathematical competition some competitors are friends; friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. The number of members in a clique is called its *size*.

It is known that the largest size of cliques is even. Prove that the competitors can be arranged in two rooms such that the largest size of cliques in one room is the same as the largest size of cliques in the other room.

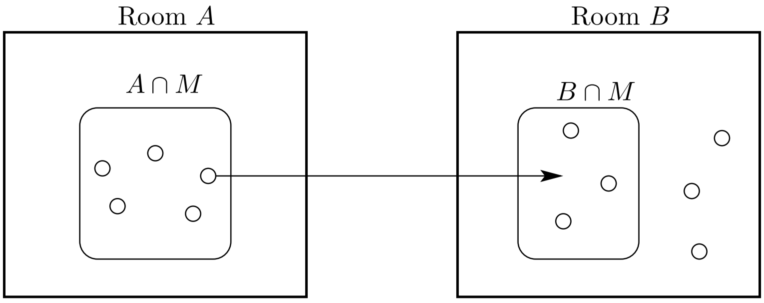
(Russia)

Solution. We present an algorithm to arrange the competitors. Let the two rooms be *Room A* and *Room B*. We start with an initial arrangement, and then we modify it several times by sending one person to the other room. At any state of the algorithm, A and B denote the sets of the competitors in the rooms, and $c(A)$ and $c(B)$ denote the largest sizes of cliques in the rooms, respectively.

Step 1. *Let M be one of the cliques of largest size, $|M| = 2m$. Send all members of M to Room A and all other competitors to Room B.*

Since M is a clique of the largest size, we have $c(A) = |M| \geq c(B)$.

Step 2. *While $c(A) > c(B)$, send one person from Room A to Room B.*



Note that $c(A) > c(B)$ implies that Room A is not empty.

In each step, $c(A)$ decreases by one and $c(B)$ increases by at most one. So at the end we have $c(A) \leq c(B) \leq c(A) + 1$.

We also have $c(A) = |A| \geq m$ at the end. Otherwise we would have at least $m+1$ members of M in Room B and at most $m-1$ in Room A, implying $c(B) - c(A) \geq (m+1) - (m-1) = 2$.

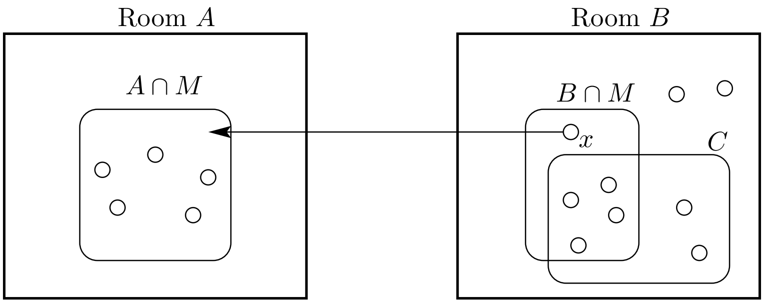
Step 3. *Let $k = c(A)$. If $c(B) = k$ then STOP.*

If we reached $c(A) = c(B) = k$ then we have found the desired arrangement.

In all other cases we have $c(B) = k + 1$.

From the estimate above we also know that $k = |A| = |A \cap M| \geq m$ and $|B \cap M| \leq m$.

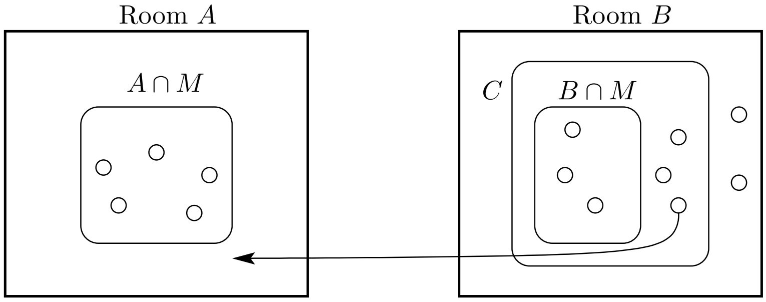
Step 4. *If there exists a competitor $x \in B \cap M$ and a clique $C \subset B$ such that $|C| = k + 1$ and $x \notin C$, then move x to Room A and STOP.*



After moving x back to Room A, we will have $k + 1$ members of M in Room A, thus $c(A) = k + 1$. Due to $x \notin C$, $c(B) = |C|$ is not decreased, and after this step we have $c(A) = c(B) = k + 1$.

If there is no such competitor x, then in Room B, all cliques of size $k+1$ contain $B \cap M$ as a subset.

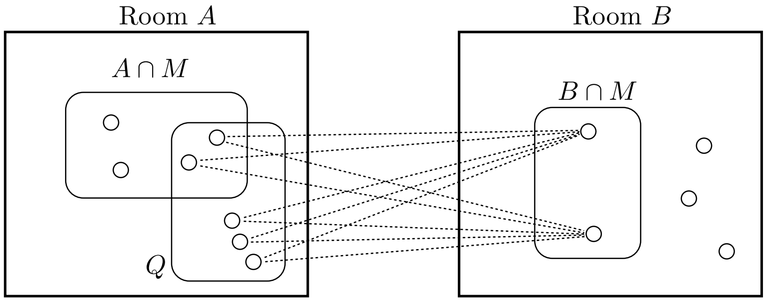
Step 5. *While $c(B) = k+1$, choose a clique $C \subset B$ such that $|C| = k+1$ and move one member of $C \setminus M$ to Room A.*



Note that $|C| = k+1 > m \geq |B \cap M|$, so $C \setminus M$ cannot be empty.

Every time we move a single person from Room B to Room A, so $c(B)$ decreases by at most 1. Hence, at the end of this loop we have $c(B) = k$.

In Room A we have the clique $A \cap M$ with size $|A \cap M| = k$ thus $c(A) \geq k$. We prove that there is no clique of larger size there. Let $Q \subset A$ be an arbitrary clique. We show that $|Q| \leq k$.



In Room A, and specially in set Q, there can be two types of competitors:

– Some members of M. Since M is a clique, they are friends with all members of $B \cap M$.

– Competitors which were moved to Room A in Step 5. Each of them has been in a clique with $B \cap M$ so they are also friends with all members of $B \cap M$.

Hence, all members of Q are friends with all members of $B \cap M$. Sets Q and $B \cap M$ are cliques themselves, so $Q \cup (B \cap M)$ is also a clique. Since M is a clique of the largest size,

$$|M| \geq |Q \cup (B \cap M)| = |Q| + |B \cap M| = |Q| + |M| - |A \cap M|,$$

therefore

$$|Q| \leq |A \cap M| = k.$$

Finally, after Step 5 we have $c(A) = c(B) = k$.

Comment. Obviously, the statement is false without the assumption that the largest clique size is even.

C7. Let $\alpha < \frac{3-\sqrt{5}}{2}$ be a positive real number. Prove that there exist positive integers n and $p > \alpha \cdot 2^n$ for which one can select $2p$ pairwise distinct subsets $S_1, \ldots, S_p, T_1, \ldots, T_p$ of the set $\{1, 2, \ldots, n\}$ such that $S_i \cap T_j \neq \varnothing$ for all $1 \leq i, j \leq p$.

(Austria)

Solution. Let k and m be positive integers (to be determined later) and set $n = km$. Decompose the set $\{1, 2, \ldots, n\}$ into k disjoint subsets, each of size m; denote these subsets by $A_1, \ldots, A_k$. Define the following families of sets:

$$\mathcal{S} = \big\{S \subset \{1, 2, \ldots, n\} : \ \forall i \ S \cap A_i \neq \varnothing\big\},$$
$$\mathcal{T}_1 = \big\{T \subset \{1, 2, \ldots, n\} : \ \exists i \ A_i \subset T\big\}, \qquad \mathcal{T} = \mathcal{T}_1 \setminus \mathcal{S}.$$

For each set $T \in \mathcal{T} \subset \mathcal{T}_1$, there exists an index $1 \leq i \leq k$ such that $A_i \subset T$. Then for all $S \in \mathcal{S}$, $S \cap T \supset S \cap A_i \neq \varnothing$. Hence, each $S \in \mathcal{S}$ and each $T \in \mathcal{T}$ have at least one common element.

Below we show that the numbers m and k can be chosen such that $|\mathcal{S}|, |\mathcal{T}| > \alpha \cdot 2^n$. Then, choosing $p = \min\big\{|\mathcal{S}|, |\mathcal{T}|\big\}$, one can select the desired $2p$ sets $S_1, \ldots, S_p$ and $T_1, \ldots, T_p$ from families $\mathcal{S}$ and $\mathcal{T}$, respectively. Since families $\mathcal{S}$ and $\mathcal{T}$ are disjoint, sets S_i and T_j will be pairwise distinct.

To count the sets $S \in \mathcal{S}$, observe that each A_i has $2^m - 1$ nonempty subsets so we have $2^m - 1$ choices for $S \cap A_i$. These intersections uniquely determine set S, so

$$|\mathcal{S}| = (2^m - 1)^k. \tag{1}$$

Similarly, if a set $H \subset \{1, 2, \ldots, n\}$ does not contain a certain set A_i then we have $2^m - 1$ choices for $H \cap A_i$: all subsets of A_i, except A_i itself. Therefore, the complement of $\mathcal{T}_1$ contains $(2^m - 1)^k$ sets and

$$|\mathcal{T}_1| = 2^{km} - (2^m - 1)^k. \tag{2}$$

Next consider the family $\mathcal{S} \setminus \mathcal{T}_1$. If a set S intersects all A_i but does not contain any of them, then there exists $2^m - 2$ possible values for each $S \cap A_i$: all subsets of A_i except $\varnothing$ and A_i. Therefore the number of such sets S is $(2^m - 2)^k$, so

$$|\mathcal{S} \setminus \mathcal{T}_1| = (2^m - 2)^k. \tag{3}$$

From (1), (2), and (3) we obtain

$$|\mathcal{T}| = |\mathcal{T}_1| - |\mathcal{S} \cap \mathcal{T}_1| = |\mathcal{T}_1| - \big(|\mathcal{S}| - |\mathcal{S} \setminus \mathcal{T}_1|\big) = 2^{km} - 2(2^m - 1)^k + (2^m - 2)^k.$$

Let $\delta = \frac{3-\sqrt{5}}{2}$ and $k = k(m) = \big[2^m \log \frac{1}{\delta}\big]$. Then

$$\lim_{m\to\infty} \frac{|\mathcal{S}|}{2^{km}} = \lim_{m\to\infty} \left(1 - \frac{1}{2^m}\right)^k = \exp\left(-\lim_{m\to\infty} \frac{k}{2^m}\right) = \delta$$

and similarly

$$\lim_{m\to\infty} \frac{|\mathcal{T}|}{2^{km}} = 1 - 2\lim_{m\to\infty} \left(1 - \frac{1}{2^m}\right)^k + \lim_{m\to\infty} \left(1 - \frac{2}{2^m}\right)^k = 1 - 2\delta + \delta^2 = \delta.$$

Hence, if m is sufficiently large then $\frac{|\mathcal{S}|}{2^{mk}}$ and $\frac{|\mathcal{T}|}{2^{mk}}$ are greater than α (since $\alpha < \delta$). So $|\mathcal{S}|, |\mathcal{T}| > \alpha \cdot 2^{mk} = \alpha \cdot 2^n$.

Comment. It can be proved that the constant $\frac{3-\sqrt{5}}{2}$ is sharp. Actually, if $S_1, \ldots, S_p, T_1, \ldots, T_p$ are distinct subsets of $\{1, 2, \ldots, n\}$ such that each S_i intersects each T_j, then $p < \frac{3-\sqrt{5}}{2} \cdot 2^n$.

C8. Given a convex n-gon P in the plane. For every three vertices of P, consider the triangle determined by them. Call such a triangle *good* if all its sides are of unit length.

Prove that there are not more than $\frac{2}{3}n$ good triangles.

(Ukraine)

Solution. Consider all good triangles containing a certain vertex A. The other two vertices of any such triangle lie on the circle ω_A with unit radius and center A. Since P is convex, all these vertices lie on an arc of angle less than 180°. Let L_AR_A be the shortest such arc, oriented clockwise (see Figure 1). Each of segments AL_A and AR_A belongs to a unique good triangle. We say that the good triangle with side AL_A is *assigned counterclockwise* to A, and the second one, with side AR_A, is *assigned clockwise* to A. In those cases when there is a single good triangle containing vertex A, this triangle is assigned to A twice.

There are at most two assignments to each vertex of the polygon. (Vertices which do not belong to any good triangle have no assignment.) So the number of assignments is at most $2n$.

Consider an arbitrary good triangle ABC, with vertices arranged clockwise. We prove that ABC is assigned to its vertices at least three times. Then, denoting the number of good triangles by t, we obtain that the number K of all assignments is at most $2n$, while it is not less than $3t$. Then $3t \le K \le 2n$, as required.

Actually, we prove that triangle ABC is assigned either counterclockwise to C or clockwise to B. Then, by the cyclic symmetry of the vertices, we obtain that triangle ABC is assigned either counterclockwise to A or clockwise to C, and either counterclockwise to B or clockwise to A, providing the claim.

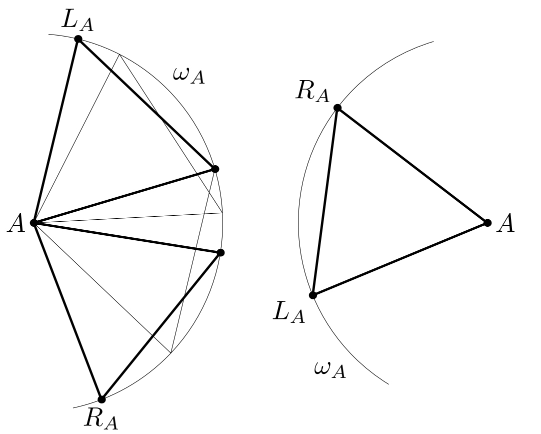


Figure 1

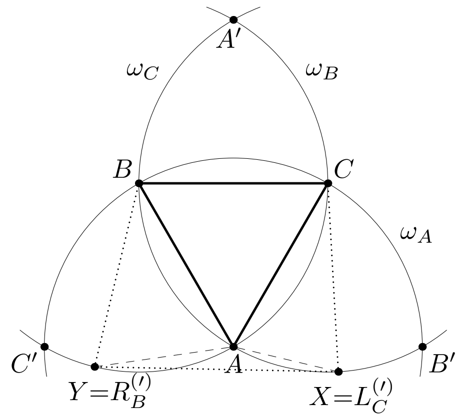


Figure 2

Assume, to the contrary, that $L_C \neq A$ and $R_B \neq A$. Denote by A', B', C' the intersection points of circles ω_A, ω_B and ω_C, distinct from A, B, C (see Figure 2). Let $CL_CL'_C$ be the good triangle containing CL_C. Observe that the angle of arc L_CA is less than 120°. Then one of the points L_C and L'_C belongs to arc $B'A$ of ω_C; let this point be X. In the case when $L_C = B'$ and $L'_C = A$, choose $X = B'$.

Analogously, considering the good triangle BR'_BR_B which contains BR_B as an edge, we see that one of the points R_B and R'_B lies on arc AC' of ω_B. Denote this point by Y, $Y \neq A$. Then angles XAY, YAB, BAC and CAX (oriented clockwise) are not greater than 180°. Hence, point A lies in quadrilateral $XYBC$ (either in its interior or on segment XY). This is impossible, since all these five points are vertices of P.

Hence, each good triangle has at least three assignments, and the statement is proved.

Comment 1. Considering a diameter AB of the polygon, one can prove that every good triangle containing either A or B has at least four assignments. This observation leads to $t \le \left\lfloor \frac{2}{3}(n-1) \right\rfloor$.

Comment 2. The result $t \leq \left\lfloor \frac{2}{3}(n-1) \right\rfloor$ is sharp. To construct a polygon with $n = 3k+1$ vertices and $t = 2k$ triangles, take a rhombus $AB_1C_1D_1$ with unit side length and $\angle B_1 = 60^\circ$. Then rotate it around A by small angles obtaining rhombi $AB_2C_2D_2, \ldots, AB_kC_kD_k$ (see Figure 3). The polygon $AB_1 \ldots B_kC_1 \ldots C_kD_1 \ldots D_k$ has $3k+1$ vertices and contains $2k$ good triangles.

The construction for $n = 3k$ and $n = 3k-1$ can be obtained by deleting vertices D_n and D_{n-1}.

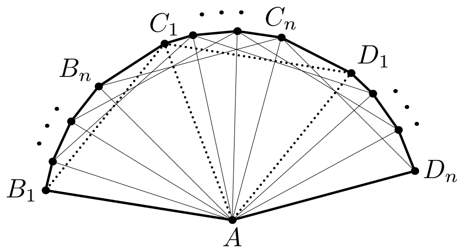


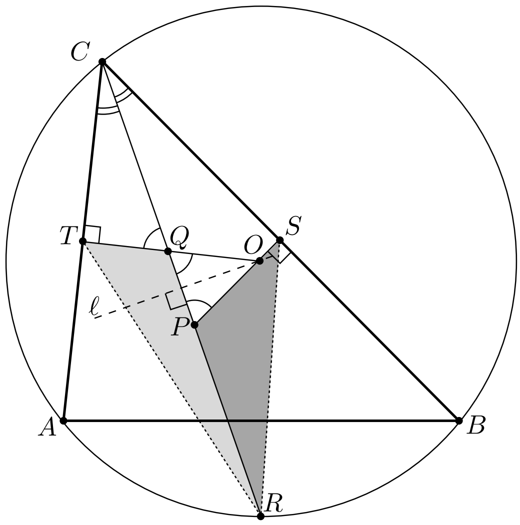
Figure 3

Geometry

G1. In triangle ABC, the angle bisector at vertex C intersects the circumcircle and the perpendicular bisectors of sides BC and CA at points R, P, and Q, respectively. The midpoints of BC and CA are S and T, respectively. Prove that triangles RQT and RPS have the same area.

(Czech Republic)

Solution 1. If $AC = BC$ then triangle ABC is isosceles, triangles RQT and RPS are symmetric about the bisector CR and the statement is trivial. If $AC \neq BC$ then it can be assumed without loss of generality that $AC < BC$.



Denote the circumcenter by O. The right triangles CTQ and CSP have equal angles at vertex C, so they are similar, $\angle CPS = \angle CQT = \angle OQP$ and

$$\frac{QT}{PS} = \frac{CQ}{CP}. \tag{1}$$

Let ℓ be the perpendicular bisector of chord CR; of course, ℓ passes through the circumcenter O. Due to the equal angles at P and Q, triangle OPQ is isosceles with $OP = OQ$. Then line ℓ is the axis of symmetry in this triangle as well. Therefore, points P and Q lie symmetrically on line segment CR,

$$RP = CQ \quad \text{and} \quad RQ = CP. \tag{2}$$

Triangles RQT and RPS have equal angles at vertices Q and P, respectively. Then

$$\frac{\operatorname{area}(RQT)}{\operatorname{area}(RPS)} = \frac{\frac{1}{2} \cdot RQ \cdot QT \cdot \sin \angle RQT}{\frac{1}{2} \cdot RP \cdot PS \cdot \sin \angle RPS} = \frac{RQ}{RP} \cdot \frac{QT}{PS}.$$

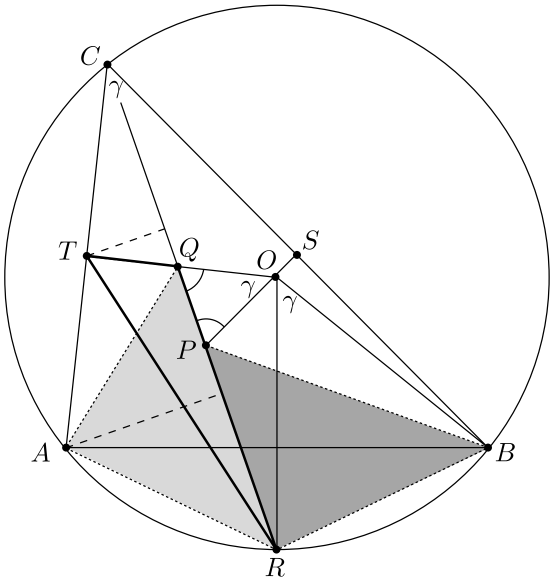
Substituting (1) and (2),

$$\frac{\operatorname{area}(RQT)}{\operatorname{area}(RPS)} = \frac{RQ}{RP} \cdot \frac{QT}{PS} = \frac{CP}{CQ} \cdot \frac{CQ}{CP} = 1.$$

Hence, $\operatorname{area}(RQT) = \operatorname{area}(RSP)$.

Solution 2. Assume again $AC < BC$. Denote the circumcenter by O, and let γ be the angle at C. Similarly to the first solution, from right triangles CTQ and CSP we obtain that $\angle OPQ = \angle OQP = 90^\circ - \frac{\gamma}{2}$. Then triangle OPQ is isosceles, $OP = OQ$ and moreover $\angle POQ = \gamma$.

As is well-known, point R is the midpoint of arc AB and $\angle ROA = \angle BOR = \gamma$.



Consider the rotation around point O by angle γ. This transform moves A to R, R to B and Q to P; hence triangles RQA and BPR are congruent and they have the same area.

Triangles RQT and RQA have RQ as a common side, so the ratio between their areas is

$$\frac{\text{area}(RQT)}{\text{area}(RQA)} = \frac{d(T, CR)}{d(A, CR)} = \frac{CT}{CA} = \frac{1}{2}.$$

($d(X, YZ)$ denotes the distance between point X and line YZ).

It can be obtained similarly that

$$\frac{\text{area}(RPS)}{\text{area}(BPR)} = \frac{CS}{CB} = \frac{1}{2}.$$

Now the proof can be completed as

$$\text{area}(RQT) = \frac{1}{2}\,\text{area}(RQA) = \frac{1}{2}\,\text{area}(BPR) = \text{area}(RPS).$$

G2. Given an isosceles triangle ABC with $AB = AC$. The midpoint of side BC is denoted by M. Let X be a variable point on the shorter arc MA of the circumcircle of triangle ABM. Let T be the point in the angle domain BMA, for which $\angle TMX = 90^\circ$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on X.

(Canada)

Solution 1. Let N be the midpoint of segment BT (see Figure 1). Line XN is the axis of symmetry in the isosceles triangle BXT, thus $\angle TNX = 90^\circ$ and $\angle BXN = \angle NXT$. Moreover, in triangle BCT, line MN is the midline parallel to CT; hence $\angle CTM = \angle NMT$.

Due to the right angles at points M and N, these points lie on the circle with diameter XT. Therefore,

$$\angle MTB = \angle MTN = \angle MXN \qquad \text{and} \qquad \angle CTM = \angle NMT = \angle NXT = \angle BXN.$$

Hence

$$\angle MTB - \angle CTM = \angle MXN - \angle BXN = \angle MXB = \angle MAB$$

which does not depend on X.

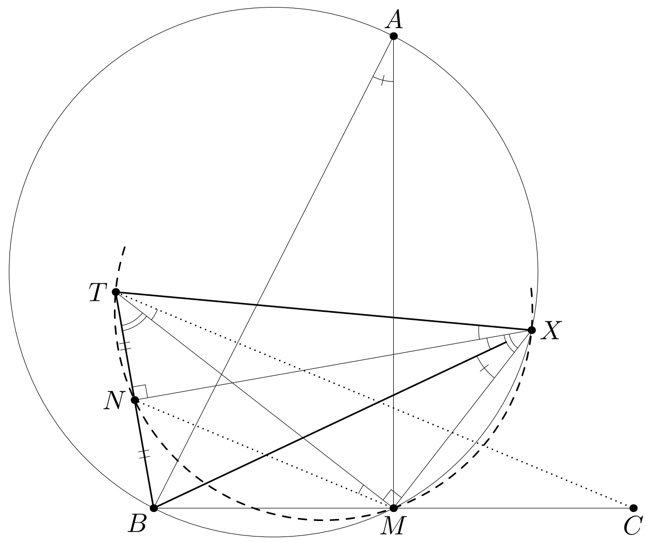


Figure 1

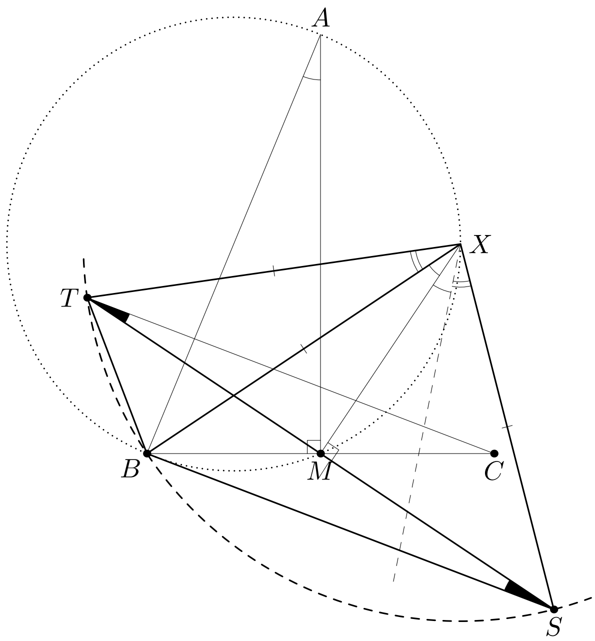


Figure 2

Solution 2. Let S be the reflection of point T over M (see Figure 2). Then XM is the perpendicular bisector of TS, hence $XB = XT = XS$, and X is the circumcenter of triangle BST. Moreover, $\angle BSM = \angle CTM$ since they are symmetrical about M. Then

$$\angle MTB - \angle CTM = \angle STB - \angle BST = \frac{\angle SXB - \angle BXT}{2}.$$

Observe that $\angle SXB = \angle SXT - \angle BXT = 2\angle MXT - \angle BXT$, so

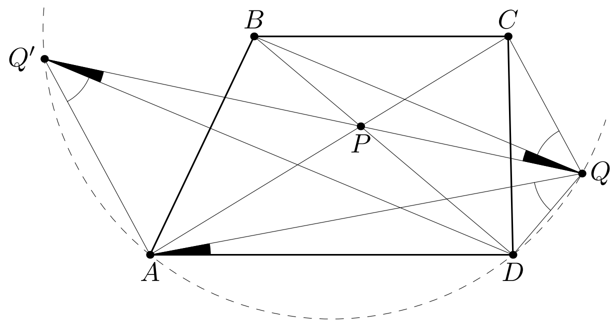
$$\angle MTB - \angle CTM = \frac{2\angle MXT - 2\angle BXT}{2} = \angle MXB = \angle MAB,$$

which is constant.

G3. The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

(Ukraine)

Solution. Let $t = \dfrac{AD}{BC}$. Consider the homothety h with center P and scale $-t$. Triangles PDA and PBC are similar with ratio t, hence $h(B) = D$ and $h(C) = A$.



Let $Q' = h(Q)$ (see Figure 1). Then points Q, P and Q' are obviously collinear. Points Q and P lie on the same side of AD, as well as on the same side of BC; hence Q' and P are also on the same side of $h(BC) = AD$, and therefore Q and Q' are on the same side of AD. Moreover, points Q and C are on the same side of BD, while Q' and A are on the opposite side (see Figure above).

By the homothety, $\angle AQ'D = \angle CQB = \angle AQD$, hence quadrilateral $AQ'QD$ is cyclic. Then

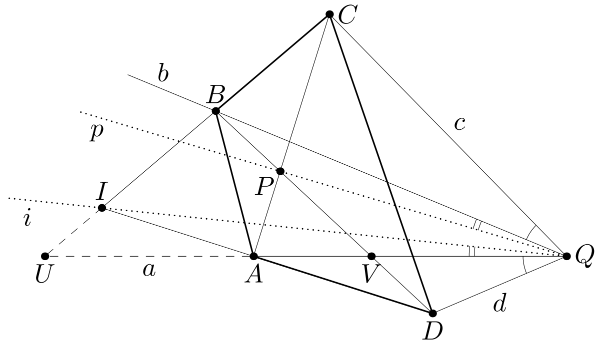
$$\angle DAQ = \angle DQ'Q = \angle DQ'P = \angle BQP$$

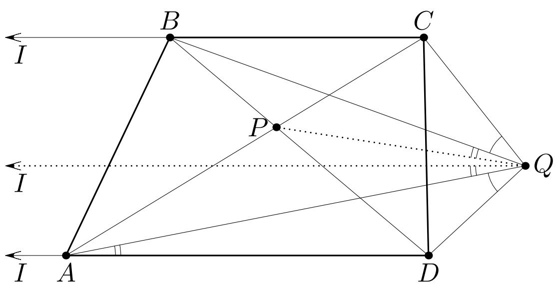
(the latter equality is valid by the homothety again).

Comment. The statement of the problem is a limit case of the following result.

In an arbitrary quadrilateral $ABCD$, let $P = AC \cap BD$, $I = AD \cap BC$, and let Q be an arbitrary point which is not collinear with any two of points A, B, C, D. Then $\angle AQD = \angle CQB$ if and only if $\angle BQP = \angle IQA$ (angles are oriented; see Figure below to the left).

In the special case of the trapezoid, I is an ideal point and $\angle DAQ = \angle IQA = \angle BQP$.





Let $a = QA$, $b = QB$, $c = QC$, $d = QD$, $i = QI$ and $p = QP$. Let line QA intersect lines BC and BD at points U and V, respectively. On lines BC and BD we have

$$(abci) = (UBCI) \qquad \text{and} \qquad (badp) = (abpd) = (VBPD).$$

Projecting from A, we get

$$(abci) = (UBCI) = (VBPD) = (badp).$$

Suppose that $\angle AQD = \angle CQB$. Let line p' be the reflection of line i about the bisector of angle AQB. Then by symmetry we have $(badp') = (abci) = (badp)$. Hence $p = p'$, as desired.

The converse statement can be proved analogously.

G4. Consider five points A, B, C, D, E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A, and let ℓ intersect segment DC and line BC at points F and G, respectively. Suppose that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

(Luxembourg)

Solution. If $CF = CG$, then $\angle FGC = \angle GFC$, hence $\angle GAB = \angle GFC = \angle FGC = \angle FAD$, and ℓ is a bisector.

Assume that $CF < GC$. Let EK and EL be the altitudes in the isosceles triangles ECF and EGC, respectively. Then in the right triangles EKF and ELC we have $EF = EC$ and

$$KF = \frac{CF}{2} < \frac{GC}{2} = LC,$$

so

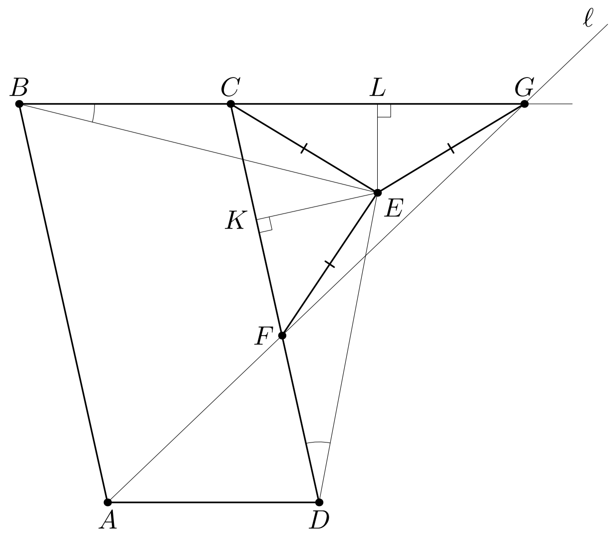
$$KE = \sqrt{EF^2 - KF^2} > \sqrt{EC^2 - LC^2} = LE.$$

Since quadrilateral $BCED$ is cyclic, we have $\angle EDC = \angle EBC$, so the right triangles BEL and DEK are similar. Then $KE > LE$ implies $DK > BL$, and hence

$$DF = DK - KF > BL - LC = BC = AD.$$

But triangles ADF and GCF are similar, so we have $1 > \dfrac{AD}{DF} = \dfrac{GC}{CF}$; this contradicts our assumption.

The case $CF > GC$ is completely similar. We consequently obtain the converse inequalities $KF > LC$, $KE < LE$, $DK < BL$, $DF < AD$, hence $1 < \dfrac{AD}{DF} = \dfrac{GC}{CF}$; a contradiction.



G5. Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C' respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

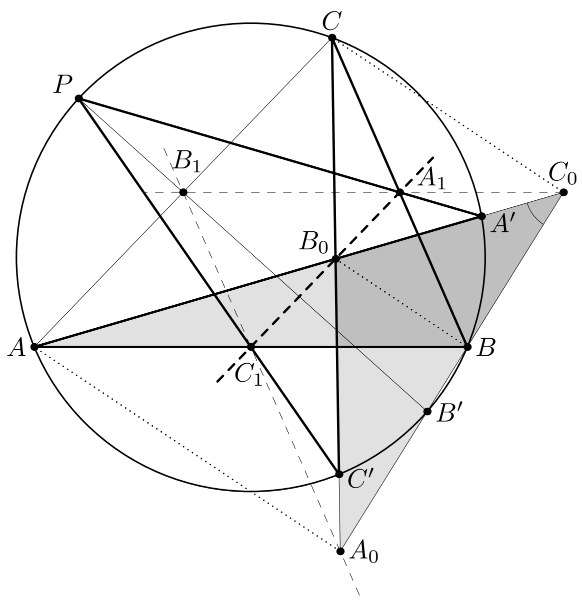
(United Kingdom)

Solution 1. Let A_0, B_0, C_0 be the points of intersection of the lines AA', BB' and CC' (see Figure). We claim that $\operatorname{area}(A_0B_0C_0) = \frac{1}{2}\operatorname{area}(ABC)$, hence it is constant.

Consider the inscribed hexagon $ABCC'PA'$. By Pascal's theorem, the points of intersection of its opposite sides (or of their extensions) are collinear. These points are $AB \cap C'P = C_1$, $BC \cap PA' = A_1$, $CC' \cap A'A = B_0$. So point B_0 lies on the midline A_1C_1 of triangle ABC. Analogously, points A_0 and C_0 lie on lines B_1C_1 and A_1B_1, respectively.

Lines AC and A_1C_1 are parallel, so triangles $B_0C_0A_1$ and AC_0B_1 are similar; hence we have

$$\frac{B_0C_0}{AC_0} = \frac{A_1C_0}{B_1C_0}.$$



Analogously, from $BC \parallel B_1C_1$ we obtain

$$\frac{A_1C_0}{B_1C_0} = \frac{BC_0}{A_0C_0}.$$

Combining these equalities, we get

$$\frac{B_0C_0}{AC_0} = \frac{BC_0}{A_0C_0},$$

or

$$A_0C_0 \cdot B_0C_0 = AC_0 \cdot BC_0.$$

Hence we have

$$\operatorname{area}(A_0B_0C_0) = \frac{1}{2}A_0C_0 \cdot B_0C_0 \sin\angle A_0C_0B_0 = \frac{1}{2}AC_0 \cdot BC_0 \sin\angle AC_0B = \operatorname{area}(ABC_0).$$

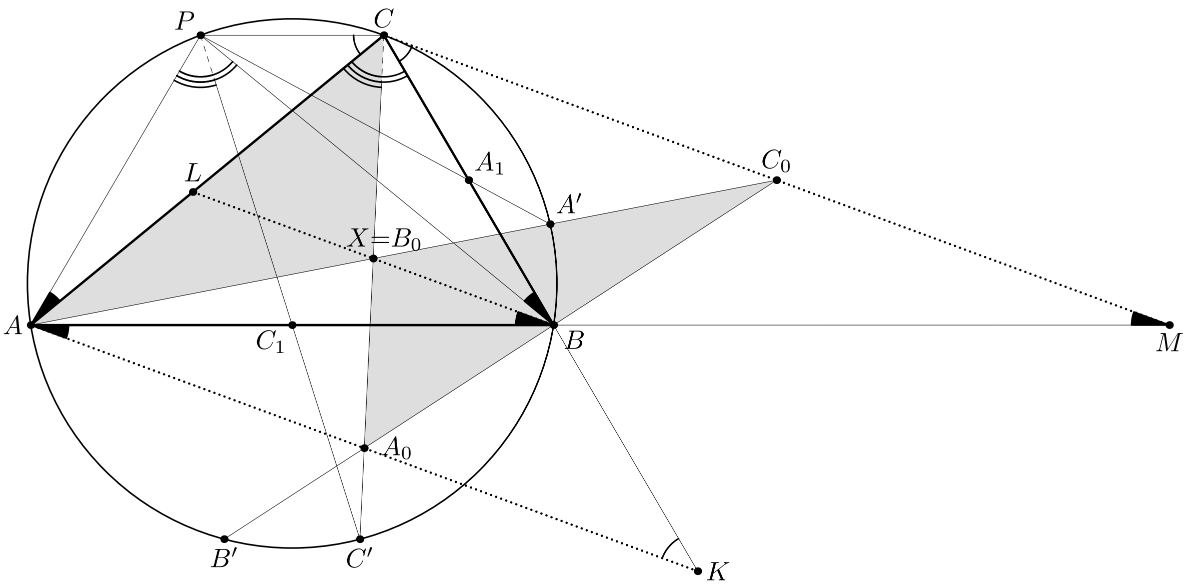
Since C_0 lies on the midline, we have $d(C_0, AB) = \frac{1}{2}d(C, AB)$ (we denote by $d(X, YZ)$ the distance between point X and line YZ). Then we obtain

$$\operatorname{area}(A_0B_0C_0) = \operatorname{area}(ABC_0) = \frac{1}{2}AB \cdot d(C_0, AB) = \frac{1}{4}AB \cdot d(C, AB) = \frac{1}{2}\operatorname{area}(ABC).$$

Solution 2. Again, we prove that $\operatorname{area}(A_0B_0C_0) = \frac{1}{2}\operatorname{area}(ABC)$.

We can assume that P lies on arc AC. Mark a point L on side AC such that $\angle CBL = \angle PBA$; then $\angle LBA = \angle CBA - \angle CBL = \angle CBA - \angle PBA = \angle CBP$. Note also that $\angle BAL = \angle BAC = \angle BPC$ and $\angle LCB = \angle APB$. Hence, triangles BAL and BPC are similar, and so are triangles LCB and APB.

Analogously, mark points K and M respectively on the extensions of sides CB and AB beyond point B, such that $\angle KAB = \angle CAP$ and $\angle BCM = \angle PCA$. For analogous reasons, $\angle KAC = \angle BAP$ and $\angle ACM = \angle PCB$. Hence $\triangle ABK \sim \triangle APC \sim \triangle MBC$, $\triangle ACK \sim \triangle APB$, and $\triangle MAC \sim \triangle BPC$. From these similarities, we have $\angle CMB = \angle KAB = \angle CAP$, while we have seen that $\angle CAP = \angle CBP = \angle LBA$. Hence, $AK \parallel BL \parallel CM$.



Let line CC' intersect BL at point X. Note that $\angle LCX = \angle ACC' = \angle APC' = \angle APC_1$, and PC_1 is a median in triangle APB. Since triangles APB and LCB are similar, CX is a median in triangle LCB, and X is a midpoint of BL. For the same reason, AA' passes through this midpoint, so $X = B_0$. Analogously, A_0 and C_0 are the midpoints of AK and CM.

Now, from $AA_0 \parallel CC_0$, we have

$$\operatorname{area}(A_0B_0C_0) = \operatorname{area}(AC_0A_0) - \operatorname{area}(AB_0A_0) = \operatorname{area}(ACA_0) - \operatorname{area}(AB_0A_0) = \operatorname{area}(ACB_0).$$

Finally,

$$\operatorname{area}(A_0B_0C_0) = \operatorname{area}(ACB_0) = \frac{1}{2}B_0L \cdot AC \sin ALB_0 = \frac{1}{4}BL \cdot AC \sin ALB = \frac{1}{2}\operatorname{area}(ABC).$$

Comment 1. The equality $\operatorname{area}(A_0B_0C_0) = \operatorname{area}(ACB_0)$ in Solution 2 does not need to be proved since the following fact is frequently known.

Suppose that the lines KL and MN are parallel, while the lines KM and LN intersect in a point E. Then $\operatorname{area}(KEN) = \operatorname{area}(MEL)$.

Comment 2. It follows immediately from both solutions that $AA_0 \parallel BB_0 \parallel CC_0$. These lines pass through an ideal point which is isogonally conjugate to P. It is known that they are parallel to the Simson line of point Q which is opposite to P on the circumcircle.

Comment 3. If $A = A'$, then one can define the line AA' to be the tangent to the circumcircle at point A. Then the statement of the problem is also valid in this case.

G6. Determine the smallest positive real number k with the following property.

Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1 and D_1 lie on sides AB, BC, CD and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \geq S$.

(U.S.A.)

Answer. $k = 1$.

Solution. Throughout the solution, triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1 will be referred to as *border triangles*. We will denote by $[\mathcal{R}]$ the area of a region $\mathcal{R}$.

First, we show that $k \geq 1$. Consider a triangle ABC with unit area; let A_1, B_1, K be the midpoints of its sides AB, BC, AC, respectively. Choose a point D on the extension of BK, close to K. Take points C_1 and D_1 on sides CD and DA close to D (see Figure 1). We have $[BB_1A_1] = \frac{1}{4}$. Moreover, as $C_1, D_1, D \to K$, we get $[A_1B_1C_1D_1] \to [A_1B_1K] = \frac{1}{4}$, $[AA_1D_1] \to [AA_1K] = \frac{1}{4}$, $[CC_1B_1] \to [CKB_1] = \frac{1}{4}$ and $[DD_1C_1] \to 0$. Hence, the sum of the two smallest areas of border triangles tends to $\frac{1}{4}$, as well as $[A_1B_1C_1D_1]$; therefore, their ratio tends to 1, and $k \geq 1$.

We are left to prove that $k = 1$ satisfies the desired property.

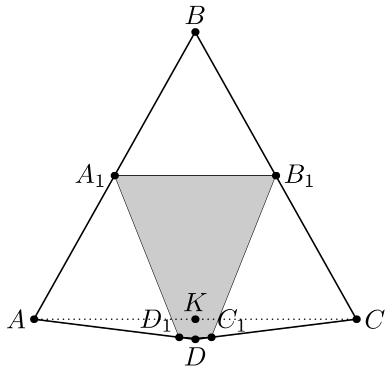


Figure 1

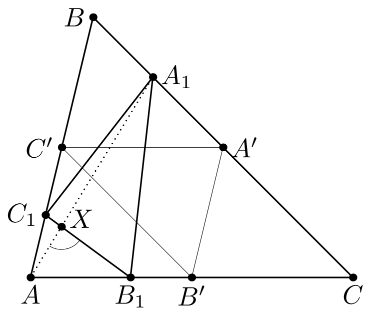


Figure 2

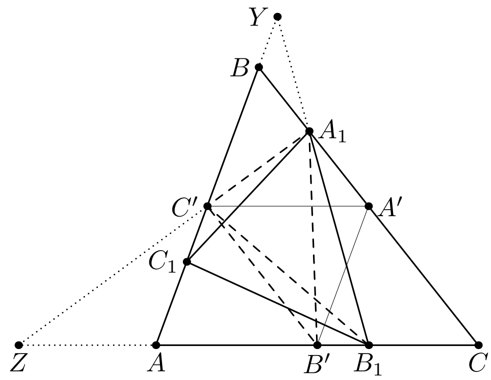


Figure 3

Lemma. Let points A_1, B_1, C_1 lie respectively on sides BC, CA, AB of a triangle ABC. Then $[A_1B_1C_1] \geq \min\{[AC_1B_1], [BA_1C_1], [CB_1A_1]\}$.

Proof. Let A', B', C' be the midpoints of sides BC, CA and AB, respectively.

Suppose that two of points A_1, B_1, C_1 lie in one of triangles $AC'B'$, $BA'C'$ and $CB'A'$ (for convenience, let points B_1 and C_1 lie in triangle $AC'B'$; see Figure 2). Let segments B_1C_1 and AA_1 intersect at point X. Then X also lies in triangle $AC'B'$. Hence $A_1X \geq AX$, and we have

$$\frac{[A_1B_1C_1]}{[AC_1B_1]} = \frac{\frac{1}{2}A_1X \cdot B_1C_1 \cdot \sin\angle A_1XC_1}{\frac{1}{2}AX \cdot B_1C_1 \cdot \sin\angle AXB_1} = \frac{A_1X}{AX} \geq 1,$$

as required.

Otherwise, each one of triangles $AC'B'$, $BA'C'$, $CB'A'$ contains exactly one of points A_1, B_1, C_1, and we can assume that $BA_1 < BA'$, $CB_1 < CB'$, $AC_1 < AC'$ (see Figure 3). Then lines B_1A_1 and AB intersect at a point Y on the extension of AB beyond point B, hence $\frac{[A_1B_1C_1]}{[A_1B_1C']} = \frac{C_1Y}{C'Y} > 1$; also, lines A_1C' and CA intersect at a point Z on the extension of CA beyond point A, hence $\frac{[A_1B_1C']}{[A_1B'C']} = \frac{B_1Z}{B'Z} > 1$. Finally, since $A_1A' \parallel B'C'$, we have $[A_1B_1C_1] > [A_1B_1C'] > [A_1B'C'] = [A'B'C'] = \frac{1}{4}[ABC]$.

Now, from $[A_1B_1C_1] + [AC_1B_1] + [BA_1C_1] + [CB_1A_1] = [ABC]$ we obtain that one of the remaining triangles AC_1B_1, BA_1C_1, CB_1A_1 has an area less than $\frac{1}{4}[ABC]$, so it is less than $[A_1B_1C_1]$. $\square$

Now we return to the problem. We say that triangle $A_1B_1C_1$ is *small* if $[A_1B_1C_1]$ is less than each of $[BB_1A_1]$ and $[CC_1B_1]$; otherwise this triangle is *big* (the similar notion is introduced for triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$). If both triangles $A_1B_1C_1$ and $C_1D_1A_1$ are big, then $[A_1B_1C_1]$ is not less than the area of some border triangle, and $[C_1D_1A_1]$ is not less than the area of another one; hence, $S_1 = [A_1B_1C_1] + [C_1D_1A_1] \geq S$. The same is valid for the pair of $B_1C_1D_1$ and $D_1A_1B_1$. So it is sufficient to prove that in one of these pairs both triangles are big.

Suppose the contrary. Then there is a small triangle in each pair. Without loss of generality, assume that triangles $A_1B_1C_1$ and $D_1A_1B_1$ are small. We can assume also that $[A_1B_1C_1] \leq [D_1A_1B_1]$. Note that in this case ray D_1C_1 intersects line BC.

Consider two cases.

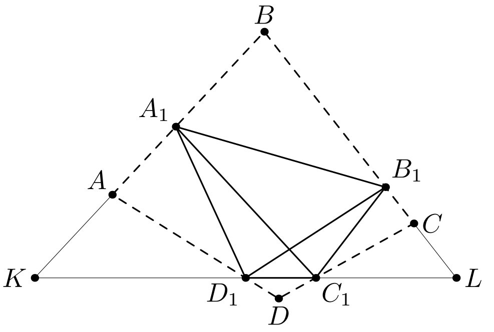


Figure 4

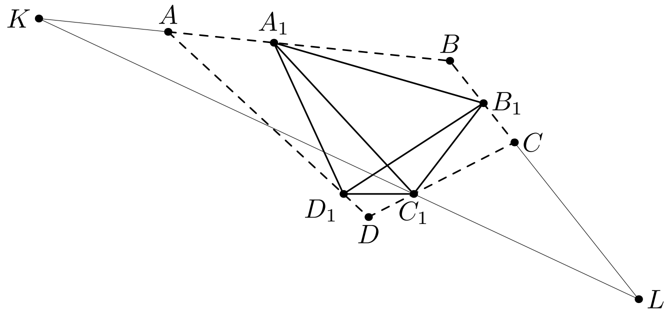


Figure 5

Case 1. Ray C_1D_1 intersects line AB at some point K. Let ray D_1C_1 intersect line BC at point L (see Figure 4). Then we have $[A_1B_1C_1] < [CC_1B_1] < [LC_1B_1]$, $[A_1B_1C_1] < [BB_1A_1]$ (both — since $[A_1B_1C_1]$ is small), and $[A_1B_1C_1] \leq [D_1A_1B_1] < [AA_1D_1] < [KA_1D_1] < [KA_1C_1]$ (since triangle $D_1A_1B_1$ is small). This contradicts the Lemma, applied for triangle $A_1B_1C_1$ inside LKB.

Case 2. Ray C_1D_1 does not intersect AB. Then choose a "sufficiently far" point K on ray BA such that $[KA_1C_1] > [A_1B_1C_1]$, and that ray KC_1 intersects line BC at some point L (see Figure 5). Since ray C_1D_1 does not intersect line AB, the points A and D_1 are on different sides of KL; then A and D are also on different sides, and C is on the same side as A and B. Then analogously we have $[A_1B_1C_1] < [CC_1B_1] < [LC_1B_1]$ and $[A_1B_1C_1] < [BB_1A_1]$ since triangle $A_1B_1C_1$ is small. This (together with $[A_1B_1C_1] < [KA_1C_1]$) contradicts the Lemma again.

G7. Given an acute triangle ABC with angles α, β and γ at vertices A, B and C, respectively, such that $\beta > \gamma$. Point I is the incenter, and R is the circumradius. Point D is the foot of the altitude from vertex A. Point K lies on line AD such that $AK = 2R$, and D separates A and K. Finally, lines DI and KI meet sides AC and BC at E and F, respectively.

Prove that if $IE = IF$ then $\beta \leq 3\gamma$.

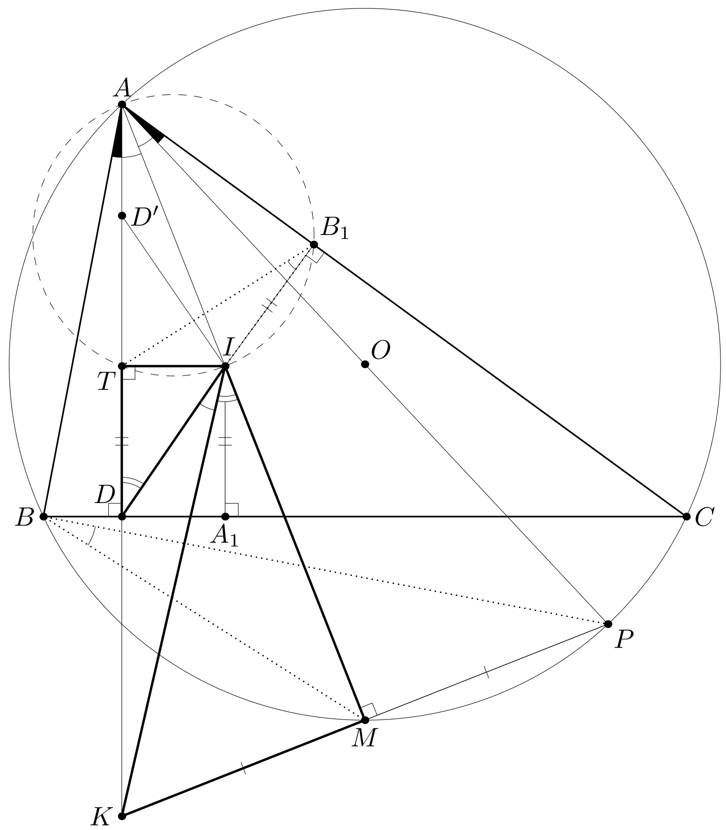
(Iran)

Solution 1. We first prove that

$$\angle KID = \frac{\beta - \gamma}{2} \tag{1}$$

even without the assumption that $IE = IF$. Then we will show that the statement of the problem is a consequence of this fact.

Denote the circumcenter by O. On the circumcircle, let P be the point opposite to A, and let the angle bisector AI intersect the circle again at M. Since $AK = AP = 2R$, triangle AKP is isosceles. It is known that $\angle BAD = \angle CAO$, hence $\angle DAI = \angle BAI - \angle BAD = \angle CAI - \angle CAO = \angle OAI$, and AM is the bisector line in triangle AKP. Therefore, points K and P are symmetrical about AM, and $\angle AMK = \angle AMP = 90°$. Thus, M is the midpoint of KP, and AM is the perpendicular bisector of KP.



Denote the perpendicular feet of incenter I on lines BC, AC, and AD by A_1, B_1, and T, respectively. Quadrilateral DA_1IT is a rectangle, hence $TD = IA_1 = IB_1$.

Due to the right angles at T and B_1, quadrilateral AB_1IT is cyclic. Hence $\angle B_1TI = \angle B_1AI = \angle CAM = \angle BAM = \angle BPM$ and $\angle IB_1T = \angle IAT = \angle MAK = \angle MAP = \angle MBP$. Therefore, triangles B_1TI and BPM are similar and $\dfrac{IT}{IB_1} = \dfrac{MP}{MB}$.

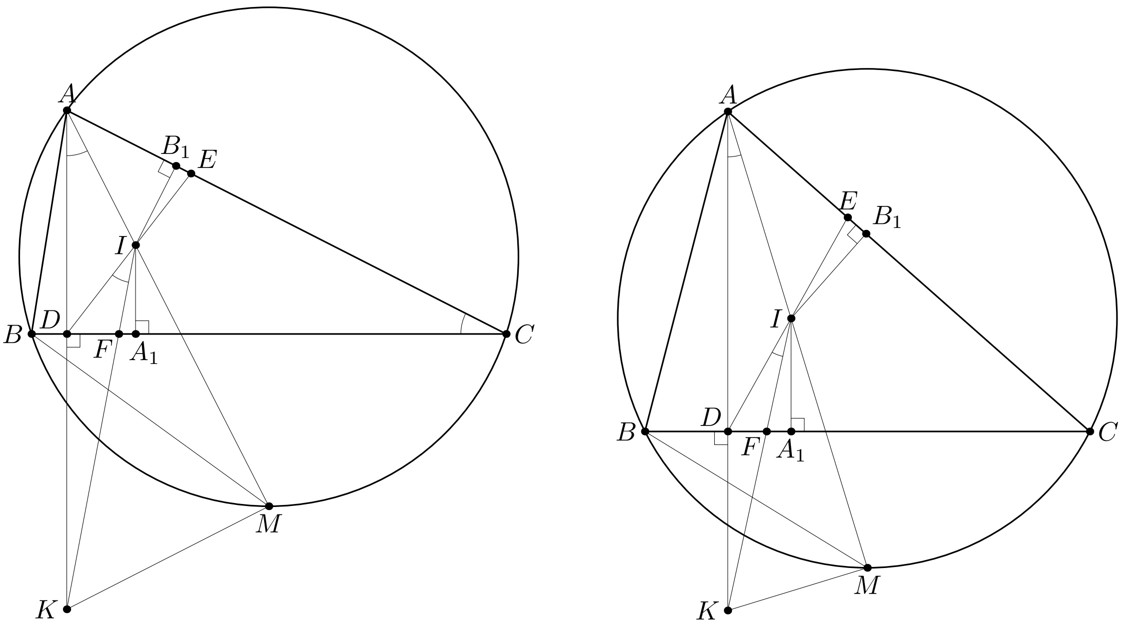
It is well-known that $MB = MC = MI$. Then right triangles ITD and KMI are also similar, because $\dfrac{IT}{TD} = \dfrac{IT}{IB_1} = \dfrac{MP}{MB} = \dfrac{KM}{MI}$. Hence, $\angle KIM = \angle IDT = \angle IDA$, and

$$\angle KID = \angle MID - \angle KIM = (\angle IAD + \angle IDA) - \angle IDA = \angle IAD.$$

Finally, from the right triangle ADB we can compute

$$\angle KID = \angle IAD = \angle IAB - \angle DAB = \frac{\alpha}{2} - (90^\circ - \beta) = \frac{\alpha}{2} - \frac{\alpha+\beta+\gamma}{2} + \beta = \frac{\beta-\gamma}{2}.$$

Now let us turn to the statement and suppose that $IE = IF$. Since $IA_1 = IB_1$, the right triangles IEB_1 and IFA_1 are congruent and $\angle IEB_1 = \angle IFA_1$. Since $\beta > \gamma$, A_1 lies in the interior of segment CD and F lies in the interior of A_1D. Hence, $\angle IFC$ is acute. Then two cases are possible depending on the order of points A, C, B_1 and E.



If point E lies between C and B_1 then $\angle IFC = \angle IEA$, hence quadrilateral $CEIF$ is cyclic and $\angle FCE = 180^\circ - \angle EIF = \angle KID$. By (1), in this case we obtain $\angle FCE = \gamma = \angle KID = \dfrac{\beta-\gamma}{2}$ and $\beta = 3\gamma$.

Otherwise, if point E lies between A and B_1, quadrilateral $CEIF$ is a deltoid such that $\angle IEC = \angle IFC < 90^\circ$. Then we have $\angle FCE > 180^\circ - \angle EIF = \angle KID$. Therefore, $\angle FCE = \gamma > \angle KID = \dfrac{\beta-\gamma}{2}$ and $\beta < 3\gamma$.

Comment 1. In the case when quadrilateral $CEIF$ is a deltoid, one can prove the desired inequality without using (1). Actually, from $\angle IEC = \angle IFC < 90^\circ$ it follows that $\angle ADI = 90^\circ - \angle EDC < \angle AED - \angle EDC = \gamma$. Since the incircle lies inside triangle ABC, we have $AD > 2r$ (here r is the inradius), which implies $DT < TA$ and $DI < AI$; hence $\dfrac{\beta-\gamma}{2} = \angle IAD < \angle ADI < \gamma$.

Solution 2. We give a different proof for (1). Then the solution can be finished in the same way as above.

Define points M and P again; it can be proved in the same way that AM is the perpendicular bisector of KP. Let J be the center of the excircle touching side BC. It is well-known that points B, C, I, J lie on a circle with center M; denote this circle by ω_1.

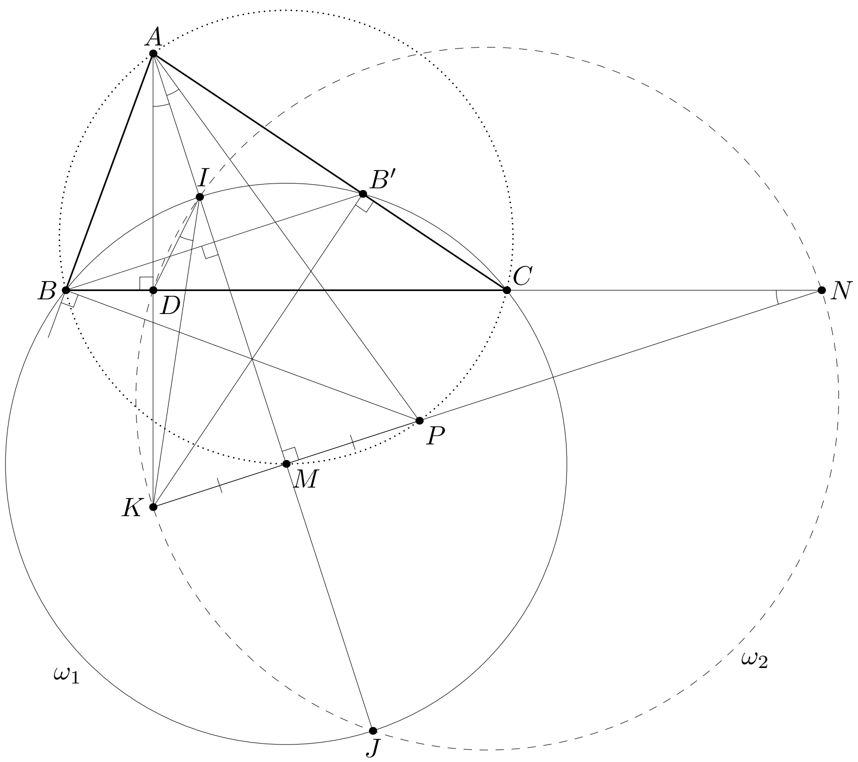
Let B' be the reflection of point B about the angle bisector AM. By the symmetry, B' is the second intersection point of circle ω_1 and line AC. Triangles PBA and $KB'A$ are symmetrical with respect to line AM, therefore $\angle KB'A = \angle PBA = 90°$. By the right angles at D and B', points K, D, B', C are concyclic and

$$AD \cdot AK = AB' \cdot AC.$$

From the cyclic quadrilateral $IJCB'$ we obtain $AB' \cdot AC = AI \cdot AJ$ as well, therefore

$$AD \cdot AK = AB' \cdot AC = AI \cdot AJ$$

and points I, J, K, D are also concyclic. Denote circle $IDKJ$ by ω_2.



Let N be the point on circle ω_2 which is opposite to K. Since $\angle NDK = 90° = \angle CDK$, point N lies on line BC. Point M, being the center of circle ω_1, is the midpoint of segment IJ, and KM is perpendicular to IJ. Therefore, line KM is the perpendicular bisector of IJ and hence it passes through N.

From the cyclic quadrilateral $IDKN$ we obtain

$$\angle KID = \angle KND = 90° - \angle DKN = 90° - \angle AKM = \angle MAK = \frac{\beta - \gamma}{2}.$$

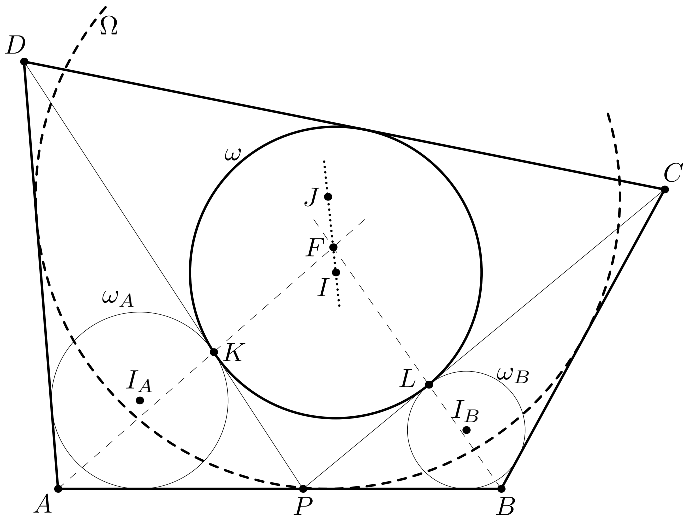
Comment 2. The main difficulty in the solution is finding (1). If someone can guess this fact, he or she can compute it in a relatively short way.

One possible way is finding and applying the relation $AI^2 = 2R(h_a - 2r)$, where $h_a = AD$ is the length of the altitude. Using this fact, one can see that triangles AKI and AID' are similar (here D' is the point symmetrical to D about T). Hence, $\angle MIK = \angle DD'I = \angle IDD'$. The proof can be finished as in Solution 1.

G8. Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

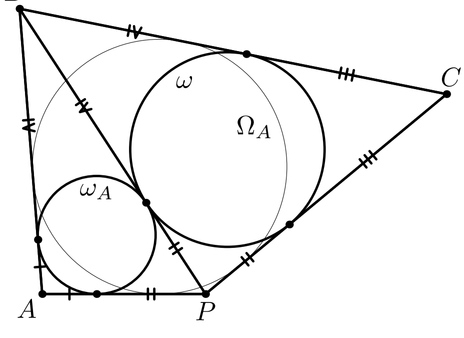
(Poland)

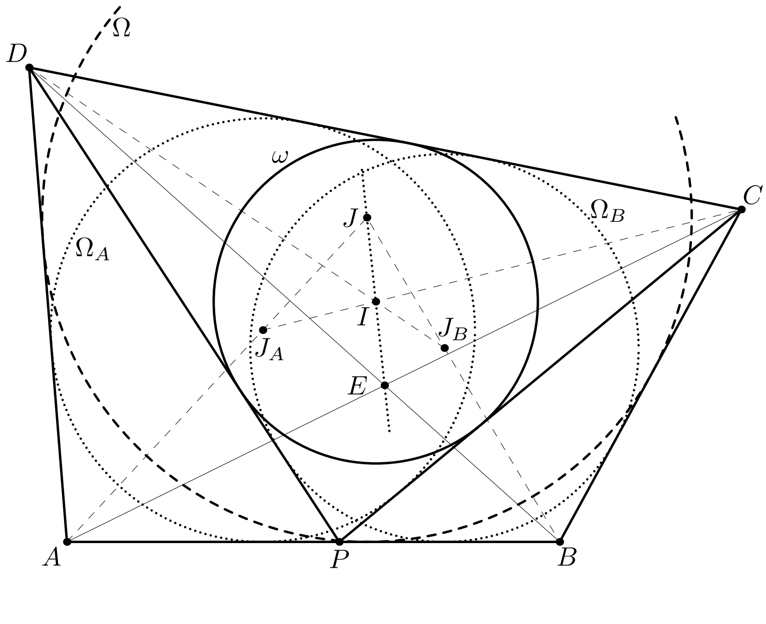
Solution. Let Ω be the circle tangent to segment AB and to rays AD and BC; let J be its center. We prove that points E and F lie on line IJ.



Denote the incircles of triangles ADP and BCP by ω_A and ω_B. Let h_1 be the homothety with a negative scale taking ω to Ω. Consider this homothety as the composition of two homotheties: one taking ω to ω_A (with a negative scale and center K), and another one taking ω_A to Ω (with a positive scale and center A). It is known that in such a case the three centers of homothety are collinear (this theorem is also referred to as the theorem on the three similitude centers). Hence, the center of h_1 lies on line AK. Analogously, it also lies on BL, so this center is F. Hence, F lies on the line of centers of ω and Ω, i.e. on IJ (if $I = J$, then $F = I$ as well, and the claim is obvious).

Consider quadrilateral $APCD$ and mark the equal segments of tangents to ω and ω_A (see the figure below to the left). Since circles ω and ω_A have a common point of tangency with PD, one can easily see that $AD + PC = AP + CD$. So, quadrilateral $APCD$ is circumscribed; analogously, circumscribed is also quadrilateral $BCDP$. Let Ω_A and Ω_B respectively be their incircles.





Consider the homothety h_2 with a positive scale taking ω to Ω. Consider h_2 as the composition of two homotheties: taking ω to Ω_A (with a positive scale and center C), and taking Ω_A to Ω (with a positive scale and center A), respectively. So the center of h_2 lies on line AC. By analogous reasons, it lies also on BD, hence this center is E. Thus, E also lies on the line of centers IJ, and the claim is proved.

Comment. In both main steps of the solution, there can be several different reasonings for the same claims. For instance, one can mostly use Desargues' theorem instead of the three homotheties theorem. Namely, if I_A and I_B are the centers of ω_A and ω_B, then lines I_AI_B, KL and AB are concurrent (by the theorem on three similitude centers applied to ω, ω_A and ω_B). Then Desargues' theorem, applied to triangles AI_AK and BI_BL, yields that the points $J = AI_A \cap BI_B$, $I = I_AK \cap I_BL$ and $F = AK \cap BL$ are collinear.

For the second step, let J_A and J_B be the centers of Ω_A and Ω_B. Then lines J_AJ_B, AB and CD are concurrent, since they appear to be the two common tangents and the line of centers of Ω_A and Ω_B. Applying Desargues' theorem to triangles AJ_AC and BJ_BD, we obtain that the points $J = AJ_A \cap BJ_B$, $I = CJ_A \cap DJ_B$ and $E = AC \cap BD$ are collinear.

Number Theory

N1. Find all pairs (k, n) of positive integers for which $7^k - 3^n$ divides $k^4 + n^2$.

(Austria)

Answer. $(2, 4)$.

Solution. Suppose that a pair (k, n) satisfies the condition of the problem. Since $7^k - 3^n$ is even, $k^4 + n^2$ is also even, hence k and n have the same parity. If k and n are odd, then $k^4 + n^2 \equiv 1 + 1 = 2 \pmod 4$, while $7^k - 3^n \equiv 7 - 3 \equiv 0 \pmod 4$, so $k^4 + n^2$ cannot be divisible by $7^k - 3^n$. Hence, both k and n must be even.

Write $k = 2a$, $n = 2b$. Then $7^k - 3^n = 7^{2a} - 3^{2b} = \dfrac{7^a - 3^b}{2} \cdot 2(7^a + 3^b)$, and both factors are integers. So $2(7^a + 3^b) \mid 7^k - 3^n$ and $7^k - 3^n \mid k^4 + n^2 = 2(8a^4 + 2b^2)$, hence

$$7^a + 3^b \leq 8a^4 + 2b^2. \tag{1}$$

We prove by induction that $8a^4 < 7^a$ for $a \geq 4$, $2b^2 < 3^b$ for $b \geq 1$ and $2b^2 + 9 \leq 3^b$ for $b \geq 3$. In the initial cases $a = 4$, $b = 1$, $b = 2$ and $b = 3$ we have $8 \cdot 4^4 = 2048 < 7^4 = 2401$, $2 < 3$, $2 \cdot 2^2 = 8 < 3^2 = 9$ and $2 \cdot 3^2 + 9 = 3^3 = 27$, respectively.

If $8a^4 < 7^a$ $(a \geq 4)$ and $2b^2 + 9 \leq 3^b$ $(b \geq 3)$, then

$$8(a+1)^4 = 8a^4 \left(\frac{a+1}{a}\right)^4 < 7^a \left(\frac{5}{4}\right)^4 = 7^a \frac{625}{256} < 7^{a+1} \qquad \text{and}$$
$$2(b+1)^2 + 9 < (2b^2 + 9)\left(\frac{b+1}{b}\right)^2 \leq 3^b \left(\frac{4}{3}\right)^2 = 3^b \frac{16}{9} < 3^{b+1},$$

as desired.

For $a \geq 4$ we obtain $7^a + 3^b > 8a^4 + 2b^2$ and inequality (1) cannot hold. Hence $a \leq 3$, and three cases are possible.

Case 1: $a = 1$. Then $k = 2$ and $8 + 2b^2 \geq 7 + 3^b$, thus $2b^2 + 1 \geq 3^b$. This is possible only if $b \leq 2$. If $b = 1$ then $n = 2$ and $\dfrac{k^4 + n^2}{7^k - 3^n} = \dfrac{2^4 + 2^2}{7^2 - 3^2} = \dfrac{1}{2}$, which is not an integer. If $b = 2$ then $n = 4$ and $\dfrac{k^4 + n^2}{7^k - 3^n} = \dfrac{2^4 + 4^2}{7^2 - 3^4} = -1$, so $(k, n) = (2, 4)$ is a solution.

Case 2: $a = 2$. Then $k = 4$ and $k^4 + n^2 = 256 + 4b^2 \geq |7^4 - 3^n| = |49 - 3^b| \cdot (49 + 3^b)$. The smallest value of the first factor is 22, attained at $b = 3$, so $128 + 2b^2 \geq 11(49 + 3^b)$, which is impossible since $3^b > 2b^2$.

Case 3: $a = 3$. Then $k = 6$ and $k^4 + n^2 = 1296 + 4b^2 \geq |7^6 - 3^n| = |343 - 3^b| \cdot (343 + 3^b)$. Analogously, $|343 - 3^b| \geq 100$ and we have $324 + b^2 \geq 25(343 + 3^b)$, which is impossible again.

We find that there exists a unique solution $(k, n) = (2, 4)$.

N2. Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

(Canada)

Solution. Let the prime factorization of b be $b = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, where $p_1, \dots, p_s$ are distinct primes. Our goal is to show that all exponents α_i are divisible by n, then we can set $A = p_1^{\alpha_1/n} \dots p_s^{\alpha_s/n}$.

Apply the condition for $k = b^2$. The number $b - a_k^n$ is divisible by b^2 and hence, for each $1 \leq i \leq s$, it is divisible by $p_i^{2\alpha_i} > p_i^{\alpha_i}$ as well. Therefore

$$a_k^n \equiv b \equiv 0 \pmod{p_i^{\alpha_i}}$$

and

$$a_k^n \equiv b \not\equiv 0 \pmod{p_i^{\alpha_i+1}},$$

which implies that the largest power of p_i dividing a_k^n is $p_i^{\alpha_i}$. Since a_k^n is a complete nth power, this implies that α_i is divisible by n.

Comment. If $n = 8$ and $b = 16$, then for each *prime* p there exists an integer a_p such that $b - a_p^n$ is divisible by p. Actually, the congruency $x^8 - 16 \equiv 0 \pmod{p}$ expands as

$$(x^2 - 2)(x^2 + 2)(x^2 - 2x + 2)(x^2 + 2x + 2) \equiv 0 \pmod{p}.$$

Hence, if -1 is a quadratic residue modulo p, then congruency $x^2 + 2x + 2 = (x + 1)^2 + 1 \equiv 0$ has a solution. Otherwise, one of congruencies $x^2 \equiv 2$ and $x^2 \equiv -2$ has a solution.

Thus, the solution cannot work using only prime values of k.

N3. Let X be a set of 10 000 integers, none of them is divisible by 47. Prove that there exists a 2007-element subset Y of X such that $a-b+c-d+e$ is not divisible by 47 for any $a,b,c,d,e\in Y$.

(Netherlands)

Solution. Call a set M of integers *good* if $47 \nmid a-b+c-d+e$ for any $a,b,c,d,e\in M$.

Consider the set $J=\{-9,-7,-5,-3,-1,1,3,5,7,9\}$. We claim that J is good. Actually, for any $a,b,c,d,e\in J$ the number $a-b+c-d+e$ is odd and

$$-45=(-9)-9+(-9)-9+(-9)\leq a-b+c-d+e\leq 9-(-9)+9-(-9)+9=45.$$

But there is no odd number divisible by 47 between -45 and 45.

For any $k=1,\dots,46$ consider the set

$$A_k=\{x\in X \mid \exists j\in J:\quad kx\equiv j \pmod{47}\}.$$

If A_k is not good, then $47 \mid a-b+c-d+e$ for some $a,b,c,d,e\in A_k$, hence $47 \mid ka-kb+kc-kd+ke$. But set J contains numbers with the same residues modulo 47, so J also is not good. This is a contradiction; therefore each A_k is a good subset of X.

Then it suffices to prove that there exists a number k such that $|A_k|\geq 2007$. Note that each $x\in X$ is contained in exactly 10 sets A_k. Then

$$\sum_{k=1}^{46}|A_k|=10|X|=100\,000,$$

hence for some value of k we have

$$|A_k|\geq\frac{100\,000}{46}>2173>2007.$$

This completes the proof.

Comment. For the solution, it is essential to find a good set consisting of 10 different residues. Actually, consider a set X containing almost uniform distribution of the nonzero residues (i. e. each residue occurs 217 or 218 times). Let $Y\subset X$ be a good subset containing 2007 elements. Then the set K of all residues appearing in Y contains not less than 10 residues, and obviously this set is good.

On the other hand, there is no good set K consisting of 11 different residues. The Cauchy–Davenport theorem claims that for any sets A, B of residues modulo a prime p,

$$|A+B|\geq\min\{p,|A|+|B|-1\}.$$

Hence, if $|K|\geq 11$, then $|K+K|\geq 21$, $|K+K+K|\geq 31>47-|K+K|$, hence $|K+K+K+(-K)+(-K)|=47$, and $0\equiv a+c+e-b-d \pmod{47}$ for some $a,b,c,d,e\in K$.

From the same reasoning, one can see that a good set K containing 10 residues should satisfy equalities $|K+K|=19=2|K|-1$ and $|K+K+K|=28=|K+K|+|K|-1$. It can be proved that in this case set K consists of 10 residues forming an arithmetic progression. As an easy consequence, one obtains that set K has the form aJ for some nonzero residue a.

N4. For every integer $k \geq 2$, prove that 2^{3k} divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}} \tag{1}$$

but 2^{3k+1} does not.

(Poland)

Solution. We use the notation $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$ and $(2n)!! = 2 \cdot 4 \cdots (2n) = 2^n n!$ for any positive integer n. Observe that $(2n)! = (2n)!!\,(2n-1)!! = 2^n n!\,(2n-1)!!$.

For any positive integer n we have

$$\binom{4n}{2n} = \frac{(4n)!}{(2n)!^2} = \frac{2^{2n}(2n)!\,(4n-1)!!}{(2n)!^2} = \frac{2^{2n}}{(2n)!}(4n-1)!!,$$

$$\binom{2n}{n} = \frac{1}{(2n)!}\left(\frac{(2n)!}{n!}\right)^2 = \frac{1}{(2n)!}\left(2^n(2n-1)!!\right)^2 = \frac{2^{2n}}{(2n)!}(2n-1)!!^2.$$

Then expression (1) can be rewritten as follows:

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}} = \frac{2^{2^k}}{(2^k)!}(2^{k+1}-1)!! - \frac{2^{2^k}}{(2^k)!}(2^k-1)!!^2$$

$$= \frac{2^{2^k}(2^k-1)!!}{(2^k)!} \cdot \Big((2^k+1)(2^k+3)\ldots(2^k+2^k-1) - (2^k-1)(2^k-3)\ldots(2^k-2^k+1)\Big). \tag{2}$$

We compute the exponent of 2 in the prime decomposition of each factor (the first one is a rational number but not necessarily an integer; it is not important).

First, we show by induction on n that the exponent of 2 in $(2^n)!$ is $2^n - 1$. The base case $n = 1$ is trivial. Suppose that $(2^n)! = 2^{2^n-1}(2d+1)$ for some integer d. Then we have

$$(2^{n+1})! = 2^{2^n}(2^n)!\,(2^{n+1}-1)!! = 2^{2^n}2^{2^n-1} \cdot (2d+1)(2^{n+1}-1)!! = 2^{2^{n+1}-1} \cdot (2q+1)$$

for some integer q. This finishes the induction step.

Hence, the exponent of 2 in the first factor in (2) is $2^k - (2^k - 1) = 1$.

The second factor in (2) can be considered as the value of the polynomial

$$P(x) = (x+1)(x+3)\ldots(x+2^k-1) - (x-1)(x-3)\ldots(x-2^k+1). \tag{3}$$

at $x = 2^k$. Now we collect some information about $P(x)$.

Observe that $P(-x) = -P(x)$, since $k \geq 2$. So $P(x)$ is an odd function, and it has nonzero coefficients only at odd powers of x. Hence $P(x) = x^3 Q(x) + cx$, where $Q(x)$ is a polynomial with integer coefficients.

Compute the exponent of 2 in c. We have

$$c = 2(2^k-1)!!\sum_{i=1}^{2^{k-1}} \frac{1}{2i-1} = (2^k-1)!!\sum_{i=1}^{2^{k-1}}\left(\frac{1}{2i-1} + \frac{1}{2^k-2i+1}\right)$$

$$= (2^k-1)!!\sum_{i=1}^{2^{k-1}} \frac{2^k}{(2i-1)(2^k-2i+1)} = 2^k \sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)(2^k-2i+1)} = 2^k S.$$

For any integer $i = 1, \dots, 2^{k-1}$, denote by a_{2i-1} the residue inverse to $2i-1$ modulo 2^k. Clearly, when $2i-1$ runs through all odd residues, so does a_{2i-1}, hence

$$\begin{aligned} S = \sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)(2^k-2i+1)} &\equiv -\sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)^2} \equiv -\sum_{i=1}^{2^{k-1}} (2^k-1)!!\, a_{2i-1}^2 \\ &= -(2^k-1)!! \sum_{i=1}^{2^{k-1}} (2i-1)^2 = -(2^k-1)!! \frac{2^{k-1}(2^{2k}-1)}{3} \pmod{2^k}. \end{aligned}$$

Therefore, the exponent of 2 in S is $k-1$, so $c = 2^k S = 2^{2k-1}(2t+1)$ for some integer t.

Finally we obtain that

$$P(2^k) = 2^{3k} Q(2^k) + 2^k c = 2^{3k} Q(2^k) + 2^{3k-1}(2t+1),$$

which is divisible exactly by 2^{3k-1}. Thus, the exponent of 2 in (2) is $1 + (3k-1) = 3k$.

Comment. The fact that (1) is divisible by 2^{2k} is known; but it does not help in solving this problem.

N5. Find all surjective functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime p, the number $f(m+n)$ is divisible by p if and only if $f(m)+f(n)$ is divisible by p.

($\mathbb{N}$ is the set of all positive integers.)

(Iran)

Answer. $f(n)=n$.

Solution. Suppose that function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies the problem conditions.

Lemma. For any prime p and any $x, y \in \mathbb{N}$, we have $x \equiv y \pmod{p}$ if and only if $f(x) \equiv f(y) \pmod{p}$. Moreover, $p \mid f(x)$ if and only if $p \mid x$.

Proof. Consider an arbitrary prime p. Since f is surjective, there exists some $x \in \mathbb{N}$ such that $p \mid f(x)$. Let

$$d = \min\bigl\{x \in \mathbb{N} : p \mid f(x)\bigr\}.$$

By induction on k, we obtain that $p \mid f(kd)$ for all $k \in \mathbb{N}$. The base is true since $p \mid f(d)$. Moreover, if $p \mid f(kd)$ and $p \mid f(d)$ then, by the problem condition, $p \mid f(kd+d) = f\bigl((k+1)d\bigr)$ as required.

Suppose that there exists an $x \in \mathbb{N}$ such that $d \nmid x$ but $p \mid f(x)$. Let

$$y = \min\bigl\{x \in \mathbb{N} : d \nmid x,\ p \mid f(x)\bigr\}.$$

By the choice of d, we have $y > d$, and $y-d$ is a positive integer not divisible by d. Then $p \nmid f(y-d)$, while $p \mid f(d)$ and $p \mid f\bigl(d+(y-d)\bigr) = f(y)$. This contradicts the problem condition. Hence, there is no such x, and

$$p \mid f(x) \iff d \mid x. \tag{1}$$

Take arbitrary $x, y \in \mathbb{N}$ such that $x \equiv y \pmod{d}$. We have $p \mid f\bigl(x+(2xd-x)\bigr) = f(2xd)$; moreover, since $d \mid 2xd+(y-x) = y+(2xd-x)$, we get $p \mid f\bigl(y+(2xd-x)\bigr)$. Then by the problem condition $p \mid f(x)+f(2xd-x)$, $p \mid f(y)+f(2xd-x)$, and hence $f(x) \equiv -f(2xd-x) \equiv f(y) \pmod{p}$.

On the other hand, assume that $f(x) \equiv f(y) \pmod{p}$. Again we have $p \mid f(x)+f(2xd-x)$ which by our assumption implies that $p \mid f(x)+f(2xd-x)+\bigl(f(y)-f(x)\bigr) = f(y)+f(2xd-x)$. Hence by the problem condition $p \mid f\bigl(y+(2xd-x)\bigr)$. Using (1) we get $0 \equiv y+(2xd-x) \equiv y-x \pmod{d}$.

Thus, we have proved that

$$x \equiv y \pmod{d} \iff f(x) \equiv f(y) \pmod{p}. \tag{2}$$

We are left to show that $p=d$: in this case (1) and (2) provide the desired statements.

The numbers $1, 2, \ldots, d$ have distinct residues modulo d. By (2), numbers $f(1), f(2), \ldots, f(d)$ have distinct residues modulo p; hence there are at least d distinct residues, and $p \geq d$. On the other hand, by the surjectivity of f, there exist $x_1, \ldots, x_p \in \mathbb{N}$ such that $f(x_i)=i$ for any $i = 1, 2, \ldots, p$. By (2), all these x_i's have distinct residues modulo d. For the same reasons, $d \geq p$. Hence, $d=p$. □

Now we prove that $f(n)=n$ by induction on n. If $n=1$ then, by the Lemma, $p \nmid f(1)$ for any prime p, so $f(1)=1$, and the base is established. Suppose that $n>1$ and denote $k=f(n)$. Note that there exists a prime $q \mid n$, so by the Lemma $q \mid k$ and $k>1$.

If $k>n$ then $k-n+1>1$, and there exists a prime $p \mid k-n+1$; we have $k \equiv n-1 \pmod{p}$. By the induction hypothesis we have $f(n-1) = n-1 \equiv k = f(n) \pmod{p}$. Now, by the Lemma we obtain $n-1 \equiv n \pmod{p}$ which cannot be true.

Analogously, if $k < n$, then $f(k-1) = k-1$ by induction hypothesis. Moreover, $n-k+1 > 1$, so there exists a prime $p \mid n-k+1$ and $n \equiv k-1 \pmod{p}$. By the Lemma again, $k = f(n) \equiv f(k-1) = k-1 \pmod{p}$, which is also false. The only remaining case is $k = n$, so $f(n) = n$.

Finally, the function $f(n) = n$ obviously satisfies the condition.

N6. Let k be a positive integer. Prove that the number $(4k^2-1)^2$ has a positive divisor of the form $8kn-1$ if and only if k is even.

(United Kingdom)

Solution. The statement follows from the following fact.

Lemma. For arbitrary positive integers x and y, the number $4xy-1$ divides $(4x^2-1)^2$ if and only if $x=y$.

Proof. If $x=y$ then $4xy-1=4x^2-1$ obviously divides $(4x^2-1)^2$ so it is sufficient to consider the opposite direction.

Call a pair (x,y) of positive integers *bad* if $4xy-1$ divides $(4x^2-1)^2$ but $x\neq y$. In order to prove that bad pairs do not exist, we present two properties of them which provide an infinite descent.

Property (i). *If (x,y) is a bad pair and $x<y$ then there exists a positive integer $z<x$ such that (x,z) is also bad.*

Let $r=\dfrac{(4x^2-1)^2}{4xy-1}$. Then

$$r=-r\cdot(-1)\equiv -r(4xy-1)=-(4x^2-1)^2\equiv -1 \pmod{4x}$$

and $r=4xz-1$ with some positive integer z. From $x<y$ we obtain that

$$4xz-1=\frac{(4x^2-1)^2}{4xy-1}<4x^2-1$$

and therefore $z<x$. By the construction, the number $4xz-1$ is a divisor of $(4x^2-1)^2$ so (x,z) is a bad pair.

Property (ii). *If (x,y) is a bad pair then (y,x) is also bad.*

Since $1=1^2\equiv(4xy)^2 \pmod{4xy-1}$, we have

$$(4y^2-1)^2\equiv\left(4y^2-(4xy)^2\right)^2=16y^4(4x^2-1)^2\equiv 0 \pmod{4xy-1}.$$

Hence, the number $4xy-1$ divides $(4y^2-1)^2$ as well.

Now suppose that there exists at least one bad pair. Take a bad pair (x,y) such that $2x+y$ attains its smallest possible value. If $x<y$ then property (i) provides a bad pair (x,z) with $z<y$ and thus $2x+z<2x+y$. Otherwise, if $y<x$, property (ii) yields that pair (y,x) is also bad while $2y+x<2x+y$. Both cases contradict the assumption that $2x+y$ is minimal; the Lemma is proved. □

To prove the problem statement, apply the Lemma for $x=k$ and $y=2n$; the number $8kn-1$ divides $(4k^2-1)^2$ if and only if $k=2n$. Hence, there is no such n if k is odd and $n=k/2$ is the only solution if k is even.

Comment. The constant 4 in the Lemma can be replaced with an arbitrary integer greater than 1: if $a>1$ and $axy-1$ divides $(ax^2-1)^2$ then $x=y$.

N7. For a prime p and a positive integer n, denote by $\nu_p(n)$ the exponent of p in the prime factorization of $n!$. Given a positive integer d and a finite set $\{p_1, \dots, p_k\}$ of primes. Show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n)$ for all $1 \le i \le k$.

(India)

Solution 1. For arbitrary prime p and positive integer n, denote by $\operatorname{ord}_p(n)$ the exponent of p in n. Thus,

$$\nu_p(n) = \operatorname{ord}_p(n!) = \sum_{i=1}^{n} \operatorname{ord}_p(i).$$

Lemma. Let p be a prime number, q be a positive integer, k and r be positive integers such that $p^k > r$. Then $\nu_p(qp^k + r) = \nu_p(qp^k) + \nu_p(r)$.

Proof. We claim that $\operatorname{ord}_p(qp^k + i) = \operatorname{ord}_p(i)$ for all $0 < i < p^k$. Actually, if $d = \operatorname{ord}_p(i)$ then $d < k$, so $qp^k + i$ is divisible by p^d, but only the first term is divisible by p^{d+1}; hence the sum is not.

Using this claim, we obtain

$$\nu_p(qp^k + r) = \sum_{i=1}^{qp^k} \operatorname{ord}_p(i) + \sum_{i=qp^k+1}^{qp^k+r} \operatorname{ord}_p(i) = \sum_{i=1}^{qp^k} \operatorname{ord}_p(i) + \sum_{i=1}^{r} \operatorname{ord}_p(i) = \nu_p(qp^k) + \nu_p(r). \quad \square$$

For any integer a, denote by $\overline{a}$ its residue modulo d. The addition of residues will also be performed modulo d, i. e. $\overline{a} + \overline{b} = \overline{a+b}$. For any positive integer n, let $f(n) = \big(f_1(n), \dots, f_k(n)\big)$, where $f_i(n) = \overline{\nu_{p_i}(n)}$.

Define the sequence $n_1 = 1$, $n_{\ell+1} = (p_1 p_2 \dots p_k)^{n_\ell}$. We claim that

$$f(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2}) + \ldots + f(n_{\ell_m})$$

for any $\ell_1 < \ell_2 < \ldots < \ell_m$. (The addition of k-tuples is componentwise.) The base case $m = 1$ is trivial.

Suppose that $m > 1$. By the construction of the sequence, $p_i^{n_{\ell_1}}$ divides $n_{\ell_2} + \ldots + n_{\ell_m}$; clearly, $p_i^{n_{\ell_1}} > n_{\ell_1}$ for all $1 \le i \le k$. Therefore the Lemma can be applied for $p = p_i$, $k = r = n_{\ell_1}$ and $qp^k = n_{\ell_2} + \ldots + n_{\ell_m}$ to obtain

$$f_i(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f_i(n_{\ell_1}) + f_i(n_{\ell_2} + \ldots + n_{\ell_m}) \qquad \text{for all } 1 \le i \le k,$$

and hence

$$f(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2}) + \ldots + f(n_{\ell_m})$$

by the induction hypothesis.

Now consider the values $f(n_1)$, $f(n_2)$, ... There exist finitely many possible values of f. Hence, there exists an infinite sequence of indices $\ell_1 < \ell_2 < \ldots$ such that $f(n_{\ell_1}) = f(n_{\ell_2}) = \ldots$ and thus

$$f(n_{\ell_{m+1}} + n_{\ell_{m+2}} + \ldots + n_{\ell_{m+d}}) = f(n_{\ell_{m+1}}) + \ldots + f(n_{\ell_{m+d}}) = d \cdot f(n_{\ell_1}) = (\overline{0}, \ldots, \overline{0})$$

for all m. We have found infinitely many suitable numbers.

Solution 2. We use the same Lemma and definition of the function f.

Let $S = \{f(n) : n \in \mathbb{N}\}$. Obviously, set S is finite. For every $s \in S$ choose the minimal n_s such that $f(n_s) = s$. Denote $N = \max_{s \in S} n_s$. Moreover, let g be an integer such that $p_i^g > N$ for each $i = 1, 2, \ldots, k$. Let $P = (p_1 p_2 \ldots p_k)^g$.

We claim that

$$\bigl\{f(n) \mid n \in [mP, mP + N]\bigr\} = S \tag{1}$$

for every positive integer m. In particular, since $(\overline{0}, \ldots, \overline{0}) = f(1) \in S$, it follows that for an arbitrary m there exists $n \in [mP, mP + N]$ such that $f(n) = (\overline{0}, \ldots, \overline{0})$. So there are infinitely many suitable numbers.

To prove (1), let $a_i = f_i(mP)$. Consider all numbers of the form $n_{m,s} = mP + n_s$ with $s = (s_1, \ldots, s_k) \in S$ (clearly, all $n_{m,s}$ belong to $[mP, mP+N]$). Since $n_s \leq N < p_i^g$ and $p_i^g \bigm| mP$, we can apply the Lemma for the values $p = p_i$, $r = n_s$, $k = g$, $qp^k = mP$ to obtain

$$f_i(n_{m,s}) = f_i(mP) + f_i(n_s) = a_i + s_i;$$

hence for distinct $s, t \in S$ we have $f(n_{m,s}) \neq f(n_{m,t})$.

Thus, the function f attains at least $|S|$ distinct values in $[mP, mP + N]$. Since all these values belong to S, f should attain all possible values in $[mP, mP + N]$.

Comment. Both solutions can be extended to prove the following statements.

Claim 1. For any K there exist infinitely many n divisible by K, such that $d \bigm| \nu_{p_i}(n)$ for each i.

Claim 2. For any $s \in S$, there exist infinitely many $n \in \mathbb{N}$ such that $f(n) = s$.

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SHORT LIST

WITH SOLUTIONS

49^{th} International Mathematical Olympiad

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Shortlisted Problems with Solutions

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Algebra

A1. Find all functions $f\colon (0,\infty) \to (0,\infty)$ such that

$$\frac{f(p)^2 + f(q)^2}{f(r^2) + f(s^2)} = \frac{p^2 + q^2}{r^2 + s^2}$$

for all $p, q, r, s > 0$ with $pq = rs$.

Solution. Let f satisfy the given condition. Setting $p = q = r = s = 1$ yields $f(1)^2 = f(1)$ and hence $f(1) = 1$. Now take any $x > 0$ and set $p = x$, $q = 1$, $r = s = \sqrt{x}$ to obtain

$$\frac{f(x)^2 + 1}{2f(x)} = \frac{x^2 + 1}{2x}\,.$$

This recasts into

$$\begin{gathered} xf(x)^2 + x = x^2 f(x) + f(x), \\ \bigl(xf(x) - 1\bigr)\bigl(f(x) - x\bigr) = 0. \end{gathered}$$

And thus,

$$\text{for every } \ x > 0, \quad \text{either } \ f(x) = x \ \text{ or } \ f(x) = \frac{1}{x}\,. \tag{1}$$

Obviously, if

$$f(x) = x \quad \text{for all } x > 0 \qquad \text{or} \qquad f(x) = \frac{1}{x} \quad \text{for all } x > 0 \tag{2}$$

then the condition of the problem is satisfied. We show that actually these two functions are the only solutions.

So let us assume that there exists a function f satisfying the requirement, other than those in (2). Then $f(a) \neq a$ and $f(b) \neq 1/b$ for some $a, b > 0$. By (1), these values must be $f(a) = 1/a$, $f(b) = b$. Applying now the equation with $p = a$, $q = b$, $r = s = \sqrt{ab}$ we obtain $(a^{-2} + b^2)/2f(ab) = (a^2 + b^2)/2ab$; equivalently,

$$f(ab) = \frac{ab(a^{-2} + b^2)}{a^2 + b^2}\,. \tag{3}$$

We know however (see (1)) that $f(ab)$ must be either ab or $1/ab$. If $f(ab) = ab$ then by (3) $a^{-2} + b^2 = a^2 + b^2$, so that $a = 1$. But, as $f(1) = 1$, this contradicts the relation $f(a) \neq a$. Likewise, if $f(ab) = 1/ab$ then (3) gives $a^2b^2(a^{-2} + b^2) = a^2 + b^2$, whence $b = 1$, in contradiction to $f(b) \neq 1/b$. Thus indeed the functions listed in (2) are the only two solutions.

Comment. The equation has as many as four variables with only one constraint $pq = rs$, leaving three degrees of freedom and providing a lot of information. Various substitutions force various useful properties of the function searched. We sketch one more method to reach conclusion (1); certainly there are many others.

Noticing that $f(1) = 1$ and setting, first, $p = q = 1$, $r = \sqrt{x}$, $s = 1/\sqrt{x}$, and then $p = x$, $q = 1/x$, $r = s = 1$, we obtain two relations, holding for every $x > 0$,

$$f(x) + f\left(\frac{1}{x}\right) = x + \frac{1}{x} \qquad \text{and} \qquad f(x)^2 + f\left(\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2}\,. \tag{4}$$

Squaring the first and subtracting the second gives $2f(x)f(1/x) = 2$. Subtracting this from the second relation of (4) leads to

$$\left(f(x) - f\left(\frac{1}{x}\right)\right)^2 = \left(x - \frac{1}{x}\right)^2 \qquad \text{or} \qquad f(x) - f\left(\frac{1}{x}\right) = \pm\left(x - \frac{1}{x}\right).$$

The last two alternatives combined with the first equation of (4) imply the two alternatives of (1).

A2. (a) Prove the inequality

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1$$

for real numbers $x, y, z \neq 1$ satisfying the condition $xyz = 1$.

(b) Show that there are infinitely many triples of rational numbers x, y, z for which this inequality turns into equality.

Solution 1. (a) We start with the substitution

$$\frac{x}{x-1}=a, \qquad \frac{y}{y-1}=b, \qquad \frac{z}{z-1}=c, \qquad \text{i.e.,} \qquad x=\frac{a}{a-1}, \qquad y=\frac{b}{b-1}, \qquad z=\frac{c}{c-1}\,.$$

The inequality to be proved reads $a^2+b^2+c^2 \geq 1$. The new variables are subject to the constraints $a, b, c \neq 1$ and the following one coming from the condition $xyz = 1$,

$$(a-1)(b-1)(c-1)=abc.$$

This is successively equivalent to

$$\begin{aligned}
a+b+c-1 &= ab+bc+ca,\\
2(a+b+c-1) &= (a+b+c)^2-(a^2+b^2+c^2),\\
a^2+b^2+c^2-2 &= (a+b+c)^2-2(a+b+c),\\
a^2+b^2+c^2-1 &= (a+b+c-1)^2.
\end{aligned}$$

Thus indeed $a^2+b^2+c^2 \geq 1$, as desired.

(b) From the equation $a^2+b^2+c^2-1=(a+b+c-1)^2$ we see that the proposed inequality becomes an equality if and only if both sums $a^2+b^2+c^2$ and $a+b+c$ have value 1. The first of them is equal to $(a+b+c)^2-2(ab+bc+ca)$. So the instances of equality are described by the system of two equations

$$a+b+c=1, \qquad\qquad ab+bc+ca=0$$

plus the constraint $a, b, c \neq 1$. Elimination of c leads to $a^2+ab+b^2=a+b$, which we regard as a quadratic equation in b,

$$b^2+(a-1)b+a(a-1)=0,$$

with discriminant

$$\Delta=(a-1)^2-4a(a-1)=(1-a)(1+3a).$$

We are looking for rational triples (a, b, c); it will suffice to have a rational such that $1-a$ and $1+3a$ are both squares of rational numbers (then Δ will be so too). Set $a = k/m$. We want $m-k$ and $m+3k$ to be squares of integers. This is achieved for instance by taking $m=k^2-k+1$ (clearly nonzero); then $m-k=(k-1)^2$, $m+3k=(k+1)^2$. Note that distinct integers k yield distinct values of $a = k/m$.

And thus, if k is any integer and $m=k^2-k+1$, $a=k/m$ then $\Delta=(k^2-1)^2/m^2$ and the quadratic equation has rational roots $b=(m-k\pm k^2\mp 1)/(2m)$. Choose e.g. the larger root,

$$b=\frac{m-k+k^2-1}{2m}=\frac{m+(m-2)}{2m}=\frac{m-1}{m}.$$

Computing c from $a+b+c=1$ then gives $c=(1-k)/m$. The condition $a,b,c\neq 1$ eliminates only $k=0$ and $k=1$. Thus, as k varies over integers greater than 1, we obtain an infinite family of rational triples (a,b,c)—and coming back to the original variables ($x=a/(a-1)$ etc.)—an infinite family of rational triples (x,y,z) with the needed property. (A short calculation shows that the resulting triples are $x=-k/(k-1)^2$, $y=k-k^2$, $z=(k-1)/k^2$; but the proof was complete without listing them.)

Comment 1. There are many possible variations in handling the equation system $a^2+b^2+c^2=1$, $a+b+c=1$ $(a,b,c\neq 1)$ which of course describes a circle in the (a,b,c)-space (with three points excluded), and finding infinitely many rational points on it.

Also the initial substitution $x=a/(a-1)$ (etc.) can be successfully replaced by other similar substitutions, e.g. $x=1-1/\alpha$ (etc.); or $x=x'-1$ (etc.); or $1-yz=u$ (etc.)—eventually reducing the inequality to $(\cdots)^2\geq 0$, the expression in the parentheses depending on the actual substitution.

Depending on the method chosen, one arrives at various sequences of rational triples (x,y,z) as needed; let us produce just one more such example: $x=(2r-2)/(r+1)^2$, $y=(2r+2)/(r-1)^2$, $z=(r^2-1)/4$ where r can be any rational number different from 1 or -1.

Solution 2 (an outline)**.** (a) Without changing variables, just setting $z=1/xy$ and clearing fractions, the proposed inequality takes the form

$$(xy-1)^2\big(x^2(y-1)^2+y^2(x-1)^2\big)+(x-1)^2(y-1)^2\geq (x-1)^2(y-1)^2(xy-1)^2.$$

With the notation $p=x+y$, $q=xy$ this becomes, after lengthy routine manipulation and a lot of cancellation

$$q^4-6q^3+2pq^2+9q^2-6pq+p^2\geq 0.$$

It is not hard to notice that the expression on the left is just $(q^2-3q+p)^2$, hence nonnegative.

(Without introducing p and q, one is of course led with some more work to the same expression, just written in terms of x and y; but then it is not that easy to see that it is a square.)

(b) To have equality, one needs $q^2-3q+p=0$. Note that x and y are the roots of the quadratic trinomial (in a formal variable t): t^2-pt+q. When $q^2-3q+p=0$, the discriminant equals

$$\delta=p^2-4q=(3q-q^2)^2-4q=q(q-1)^2(q-4).$$

Now it suffices to have both q and $q-4$ squares of rational numbers (then $p=3q-q^2$ and $\sqrt{\delta}$ are also rational, and so are the roots of the trinomial). On setting $q=(n/m)^2=4+(l/m)^2$ the requirement becomes $4m^2+l^2=n^2$ (with l,m,n being integers). This is just the Pythagorean equation, known to have infinitely many integer solutions.

Comment 2. Part (a) alone might also be considered as a possible contest problem (in the category of easy problems).

A3. Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair (f, g) of functions from S into S is a *Spanish Couple* on S, if they satisfy the following conditions:

(i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all $x, y \in S$ with $x < y$;

(ii) The inequality $f(g(g(x))) < g(f(x))$ holds for all $x \in S$.

Decide whether there exists a Spanish Couple

(a) on the set $S = \mathbb{N}$ of positive integers;

(b) on the set $S = \{a - 1/b : \ a, b \in \mathbb{N}\}$.

Solution. We show that the answer is NO for part (a), and YES for part (b).

(a) Throughout the solution, we will use the notation $g_k(x) = \overbrace{g(g(\ldots g}^{k}(x)\ldots))$, including $g_0(x) = x$ as well.

Suppose that there exists a Spanish Couple (f, g) on the set $\mathbb{N}$. From property (i) we have $f(x) \geq x$ and $g(x) \geq x$ for all $x \in \mathbb{N}$.

We claim that $g_k(x) \leq f(x)$ for all $k \geq 0$ and all positive integers x. The proof is done by induction on k. We already have the base case $k = 0$ since $x \leq f(x)$. For the induction step from k to $k+1$, apply the induction hypothesis on $g_2(x)$ instead of x, then apply (ii):

$$g(g_{k+1}(x)) = g_k\big(g_2(x)\big) \leq f\big(g_2(x)\big) < g(f(x)).$$

Since g is increasing, it follows that $g_{k+1}(x) < f(x)$. The claim is proven.

If $g(x) = x$ for all $x \in \mathbb{N}$ then $f(g(g(x))) = f(x) = g(f(x))$, and we have a contradiction with (ii). Therefore one can choose an $x_0 \in S$ for which $x_0 < g(x_0)$. Now consider the sequence $x_0, x_1, \ldots$ where $x_k = g_k(x_0)$. The sequence is increasing. Indeed, we have $x_0 < g(x_0) = x_1$, and $x_k < x_{k+1}$ implies $x_{k+1} = g(x_k) < g(x_{k+1}) = x_{k+2}$.

Hence, we obtain a strictly increasing sequence $x_0 < x_1 < \ldots$ of positive integers which on the other hand has an upper bound, namely $f(x_0)$. This cannot happen in the set $\mathbb{N}$ of positive integers, thus no Spanish Couple exists on $\mathbb{N}$.

(b) We present a Spanish Couple on the set $S = \{a - 1/b : \ a, b \in \mathbb{N}\}$.

Let

$$\begin{aligned} f(a - 1/b) &= a + 1 - 1/b, \\ g(a - 1/b) &= a - 1/(b + 3^a). \end{aligned}$$

These functions are clearly increasing. Condition (ii) holds, since

$$f(g(g(a - 1/b))) = (a + 1) - 1/(b + 2 \cdot 3^a) < (a + 1) - 1/(b + 3^{a+1}) = g(f(a - 1/b)).$$

Comment. Another example of a Spanish couple is $f(a - 1/b) = 3a - 1/b$, $g(a - 1/b) = a - 1/(a+b)$. More generally, postulating $f(a - 1/b) = h(a) - 1/b$, $g(a - 1/b) = a - 1/G(a, b)$ with h increasing and G increasing in both variables, we get that $f \circ g \circ g < g \circ f$ holds if $G\big(a, G(a, b)\big) < G\big(h(a), b\big)$. A search just among linear functions $h(a) = Ca$, $G(a, b) = Aa + Bb$ results in finding that any integers $A > 0$, $C > 2$ and $B = 1$ produce a Spanish couple (in the example above, $A = 1$, $C = 3$). The proposer's example results from taking $h(a) = a + 1$, $G(a, b) = 3^a + b$.

A4. For an integer m, denote by $t(m)$ the unique number in $\{1,2,3\}$ such that $m+t(m)$ is a multiple of 3. A function $f\colon \mathbb{Z}\to\mathbb{Z}$ satisfies $f(-1)=0$, $f(0)=1$, $f(1)=-1$ and

$$f(2^n+m)=f(2^n-t(m))-f(m) \quad \text{for all integers } m,n\geq 0 \text{ with } 2^n>m.$$

Prove that $f(3p)\geq 0$ holds for all integers $p\geq 0$.

Solution. The given conditions determine f uniquely on the positive integers. The signs of $f(1), f(2), \ldots$ seem to change quite erratically. However values of the form $f(2^n-t(m))$ are sufficient to compute directly any functional value. Indeed, let $n>0$ have base 2 representation $n=2^{a_0}+2^{a_1}+\cdots+2^{a_k}$, $a_0>a_1>\cdots>a_k\geq 0$, and let $n_j=2^{a_j}+2^{a_{j-1}}+\cdots+2^{a_k}$, $j=0,\ldots,k$. Repeated applications of the recurrence show that $f(n)$ is an alternating sum of the quantities $f(2^{a_j}-t(n_{j+1}))$ plus $(-1)^{k+1}$. (The exact formula is not needed for our proof.)

So we focus attention on the values $f(2^n-1)$, $f(2^n-2)$ and $f(2^n-3)$. Six cases arise; more specifically,

$$t(2^{2k}-3)=2,\ t(2^{2k}-2)=1,\ t(2^{2k}-1)=3,\ t(2^{2k+1}-3)=1,\ t(2^{2k+1}-2)=3,\ t(2^{2k+1}-1)=2.$$

Claim. For all integers $k\geq 0$ the following equalities hold:

$$\begin{array}{lll} f(2^{2k+1}-3)=0, & f(2^{2k+1}-2)=3^k, & f(2^{2k+1}-1)=-3^k, \\ f(2^{2k+2}-3)=-3^k, & f(2^{2k+2}-2)=-3^k, & f(2^{2k+2}-1)=2\cdot 3^k. \end{array}$$

Proof. By induction on k. The base $k=0$ comes down to checking that $f(2)=-1$ and $f(3)=2$; the given values $f(-1)=0$, $f(0)=1$, $f(1)=-1$ are also needed. Suppose the claim holds for $k-1$. For $f(2^{2k+1}-t(m))$, the recurrence formula and the induction hypothesis yield

$$\begin{aligned} f(2^{2k+1}-3)&=f(2^{2k}+(2^{2k}-3))=f(2^{2k}-2)-f(2^{2k}-3)=-3^{k-1}+3^{k-1}=0, \\ f(2^{2k+1}-2)&=f(2^{2k}+(2^{2k}-2))=f(2^{2k}-1)-f(2^{2k}-2)=2\cdot 3^{k-1}+3^{k-1}=3^k, \\ f(2^{2k+1}-1)&=f(2^{2k}+(2^{2k}-1))=f(2^{2k}-3)-f(2^{2k}-1)=-3^{k-1}-2\cdot 3^{k-1}=-3^k. \end{aligned}$$

For $f(2^{2k+2}-t(m))$ we use the three equalities just established:

$$\begin{aligned} f(2^{2k+2}-3)&=f(2^{2k+1}+(2^{2k+1}-3))=f(2^{2k+1}-1)-f(2^{2k+1}-3)=-3^k-0=-3^k, \\ f(2^{2k+2}-2)&=f(2^{2k+1}+(2^{2k+1}-2))=f(2^{2k+1}-3)-f(2^{2k}-2)=0-3^k=-3^k, \\ f(2^{2k+2}-1)&=f(2^{2k+1}+(2^{2k+1}-1))=f(2^{2k+1}-2)-f(2^{2k+1}-1)=3^k+3^k=2\cdot 3^k. \end{aligned}$$

The claim follows.

A closer look at the six cases shows that $f(2^n-t(m))\geq 3^{(n-1)/2}$ if $2^n-t(m)$ is divisible by 3, and $f(2^n-t(m))\leq 0$ otherwise. On the other hand, note that $2^n-t(m)$ is divisible by 3 if and only if 2^n+m is. Therefore, for all nonnegative integers m and n,

(i) $f(2^n-t(m))\geq 3^{(n-1)/2}$ if 2^n+m is divisible by 3;

(ii) $f(2^n-t(m))\leq 0$ if 2^n+m is not divisible by 3.

One more (direct) consequence of the claim is that $|f(2^n-t(m))|\leq \frac{2}{3}\cdot 3^{n/2}$ for all $m,n\geq 0$.

The last inequality enables us to find an upper bound for $|f(m)|$ for m less than a given power of 2. We prove by induction on n that $|f(m)|\leq 3^{n/2}$ holds true for all integers $m,n\geq 0$ with $2^n>m$.

The base $n = 0$ is clear as $f(0) = 1$. For the inductive step from n to $n+1$, let m and n satisfy $2^{n+1} > m$. If $m < 2^n$, we are done by the inductive hypothesis. If $m \geq 2^n$ then $m = 2^n + k$ where $2^n > k \geq 0$. Now, by $|f(2^n - t(k))| \leq \frac{2}{3} \cdot 3^{n/2}$ and the inductive assumption,

$$|f(m)| = |f(2^n - t(k)) - f(k)| \leq |f(2^n - t(k))| + |f(k)| \leq \frac{2}{3} \cdot 3^{n/2} + 3^{n/2} < 3^{(n+1)/2}.$$

The induction is complete.

We proceed to prove that $f(3p) \geq 0$ for all integers $p \geq 0$. Since $3p$ is not a power of 2, its binary expansion contains at least two summands. Hence one can write $3p = 2^a + 2^b + c$ where $a > b$ and $2^b > c \geq 0$. Applying the recurrence formula twice yields

$$f(3p) = f(2^a + 2^b + c) = f(2^a - t(2^b + c)) - f(2^b - t(c)) + f(c).$$

Since $2^a + 2^b + c$ is divisible by 3, we have $f(2^a - t(2^b + c)) \geq 3^{(a-1)/2}$ by (i). Since $2^b + c$ is not divisible by 3, we have $f(2^b - t(c)) \leq 0$ by (ii). Finally $|f(c)| \leq 3^{b/2}$ as $2^b > c \geq 0$, so that $f(c) \geq -3^{b/2}$. Therefore $f(3p) \geq 3^{(a-1)/2} - 3^{b/2}$ which is nonnegative because $a > b$.

A5. Let a, b, c, d be positive real numbers such that

$$abcd = 1 \quad \text{and} \quad a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}.$$

Prove that

$$a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}.$$

Solution. We show that if $abcd = 1$, the sum $a+b+c+d$ cannot exceed a certain weighted mean of the expressions $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$ and $\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}$.

By applying the AM-GM inequality to the numbers $\frac{a}{b}$, $\frac{a}{b}$, $\frac{b}{c}$ and $\frac{a}{d}$, we obtain

$$a = \sqrt[4]{\frac{a^4}{abcd}} = \sqrt[4]{\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{a}{d}} \le \frac{1}{4}\left(\frac{a}{b}+\frac{a}{b}+\frac{b}{c}+\frac{a}{d}\right).$$

Analogously,

$$b \le \frac{1}{4}\left(\frac{b}{c}+\frac{b}{c}+\frac{c}{d}+\frac{b}{a}\right), \quad c \le \frac{1}{4}\left(\frac{c}{d}+\frac{c}{d}+\frac{d}{a}+\frac{c}{b}\right) \quad \text{and} \quad d \le \frac{1}{4}\left(\frac{d}{a}+\frac{d}{a}+\frac{a}{b}+\frac{d}{c}\right).$$

Summing up these estimates yields

$$a+b+c+d \le \frac{3}{4}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right) + \frac{1}{4}\left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right).$$

In particular, if $a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$ then $a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}$.

Comment. The estimate in the above solution was obtained by applying the AM-GM inequality to each column of the 4×4 array

$$\begin{array}{cccc} a/b & b/c & c/d & d/a \\ a/b & b/c & c/d & d/a \\ b/c & c/d & d/a & a/b \\ a/d & b/a & c/b & d/c \end{array}$$

and adding up the resulting inequalities. The same table yields a stronger bound: If $a, b, c, d > 0$ and $abcd = 1$ then

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)^3\left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right) \ge (a+b+c+d)^4.$$

It suffices to apply Hölder's inequality to the sequences in the four rows, with weights $1/4$:

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)^{1/4}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)^{1/4}\left(\frac{b}{c}+\frac{c}{d}+\frac{d}{a}+\frac{a}{b}\right)^{1/4}\left(\frac{a}{d}+\frac{b}{a}+\frac{c}{b}+\frac{d}{c}\right)^{1/4}$$
$$\ge \left(\frac{aaba}{bbcd}\right)^{1/4} + \left(\frac{bbcb}{ccda}\right)^{1/4} + \left(\frac{ccdc}{ddab}\right)^{1/4} + \left(\frac{ddad}{aabc}\right)^{1/4} = a+b+c+d.$$

A6. Let $f\colon \mathbb{R}\to\mathbb{N}$ be a function which satisfies

$$f\left(x+\frac{1}{f(y)}\right)=f\left(y+\frac{1}{f(x)}\right)\qquad \text{for all } x,y\in\mathbb{R}. \tag{1}$$

Prove that there is a positive integer which is not a value of f.

Solution. Suppose that the statement is false and $f(\mathbb{R})=\mathbb{N}$. We prove several properties of the function f in order to reach a contradiction.

To start with, observe that one can assume $f(0)=1$. Indeed, let $a\in\mathbb{R}$ be such that $f(a)=1$, and consider the function $g(x)=f(x+a)$. By substituting $x+a$ and $y+a$ for x and y in (1), we have

$$g\left(x+\frac{1}{g(y)}\right)=f\left(x+a+\frac{1}{f(y+a)}\right)=f\left(y+a+\frac{1}{f(x+a)}\right)=g\left(y+\frac{1}{g(x)}\right).$$

So g satisfies the functional equation (1), with the additional property $g(0)=1$. Also, g and f have the same set of values: $g(\mathbb{R})=f(\mathbb{R})=\mathbb{N}$. Henceforth we assume $f(0)=1$.

Claim 1. For an arbitrary fixed $c\in\mathbb{R}$ we have $\left\{f\left(c+\frac{1}{n}\right): n\in\mathbb{N}\right\}=\mathbb{N}$.

Proof. Equation (1) and $f(\mathbb{R})=\mathbb{N}$ imply

$$f(\mathbb{R})=\left\{f\left(x+\frac{1}{f(c)}\right): x\in\mathbb{R}\right\}=\left\{f\left(c+\frac{1}{f(x)}\right): x\in\mathbb{R}\right\}\subset\left\{f\left(c+\frac{1}{n}\right): n\in\mathbb{N}\right\}\subset f(\mathbb{R}).$$

The claim follows.

We will use Claim 1 in the special cases $c=0$ and $c=1/3$:

$$\left\{f\left(\frac{1}{n}\right): n\in\mathbb{N}\right\}=\left\{f\left(\frac{1}{3}+\frac{1}{n}\right): n\in\mathbb{N}\right\}=\mathbb{N}. \tag{2}$$

Claim 2. If $f(u)=f(v)$ for some $u,v\in\mathbb{R}$ then $f(u+q)=f(v+q)$ for all nonnegative rational q. Furthermore, if $f(q)=1$ for some nonnegative rational q then $f(kq)=1$ for all $k\in\mathbb{N}$.

Proof. For all $x\in\mathbb{R}$ we have by (1)

$$f\left(u+\frac{1}{f(x)}\right)=f\left(x+\frac{1}{f(u)}\right)=f\left(x+\frac{1}{f(v)}\right)=f\left(v+\frac{1}{f(x)}\right).$$

Since $f(x)$ attains all positive integer values, this yields $f(u+1/n)=f(v+1/n)$ for all $n\in\mathbb{N}$. Let $q=k/n$ be a positive rational number. Then k repetitions of the last step yield

$$f(u+q)=f\left(u+\frac{k}{n}\right)=f\left(v+\frac{k}{n}\right)=f(v+q).$$

Now let $f(q)=1$ for some nonnegative rational q, and let $k\in\mathbb{N}$. As $f(0)=1$, the previous conclusion yields successively $f(q)=f(2q), f(2q)=f(3q),\ldots,f((k-1)q)=f(kq)$, as needed.

Claim 3. The equality $f(q)=f(q+1)$ holds for all nonnegative rational q.

Proof. Let m be a positive integer such that $f(1/m)=1$. Such an m exists by (2). Applying the second statement of Claim 2 with $q=1/m$ and $k=m$ yields $f(1)=1$.

Given that $f(0)=f(1)=1$, the first statement of Claim 2 implies $f(q)=f(q+1)$ for all nonnegative rational q.

Claim 4. The equality $f\left(\dfrac{1}{n}\right) = n$ holds for every $n \in \mathbb{N}$.

Proof. For a nonnegative rational q we set $x = q$, $y = 0$ in (1) and use Claim 3 to obtain

$$f\left(\frac{1}{f(q)}\right) = f\left(q + \frac{1}{f(0)}\right) = f(q+1) = f(q).$$

By (2), for each $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f(1/k) = n$. Applying the last equation with $q = 1/k$, we have

$$n = f\left(\frac{1}{k}\right) = f\left(\frac{1}{f(1/k)}\right) = f\left(\frac{1}{n}\right).$$

Now we are ready to obtain a contradiction. Let $n \in \mathbb{N}$ be such that $f(1/3 + 1/n) = 1$. Such an n exists by (2). Let $1/3 + 1/n = s/t$, where $s, t \in \mathbb{N}$ are coprime. Observe that $t > 1$ as $1/3 + 1/n$ is not an integer. Choose $k, l \in \mathbb{N}$ so that that $ks - lt = 1$.

Because $f(0) = f(s/t) = 1$, Claim 2 implies $f(ks/t) = 1$. Now $f(ks/t) = f(1/t + l)$; on the other hand $f(1/t + l) = f(1/t)$ by l successive applications of Claim 3. Finally, $f(1/t) = t$ by Claim 4, leading to the impossible $t = 1$. The solution is complete.

A7. Prove that for any four positive real numbers a, b, c, d the inequality

$$\frac{(a-b)(a-c)}{a+b+c}+\frac{(b-c)(b-d)}{b+c+d}+\frac{(c-d)(c-a)}{c+d+a}+\frac{(d-a)(d-b)}{d+a+b}\geq 0$$

holds. Determine all cases of equality.

Solution 1. Denote the four terms by

$$A=\frac{(a-b)(a-c)}{a+b+c}, \quad B=\frac{(b-c)(b-d)}{b+c+d}, \quad C=\frac{(c-d)(c-a)}{c+d+a}, \quad D=\frac{(d-a)(d-b)}{d+a+b}.$$

The expression $2A$ splits into two summands as follows,

$$2A=A'+A'' \qquad \text{where} \qquad A'=\frac{(a-c)^2}{a+b+c}, \qquad A''=\frac{(a-c)(a-2b+c)}{a+b+c};$$

this is easily verified. We analogously represent $2B=B'+B''$, $2C=C'+C''$, $2B=D'+D''$ and examine each of the sums $A'+B'+C'+D'$ and $A''+B''+C''+D''$ separately.

Write $s=a+b+c+d$; the denominators become $s-d$, $s-a$, $s-b$, $s-c$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\left(\frac{|a-c|}{\sqrt{s-d}}\cdot\sqrt{s-d}+\frac{|b-d|}{\sqrt{s-a}}\cdot\sqrt{s-a}+\frac{|c-a|}{\sqrt{s-b}}\cdot\sqrt{s-b}+\frac{|d-b|}{\sqrt{s-c}}\cdot\sqrt{s-c}\right)^2\\ &\qquad\leq\left(\frac{(a-c)^2}{s-d}+\frac{(b-d)^2}{s-a}+\frac{(c-a)^2}{s-b}+\frac{(d-b)^2}{s-c}\right)\big(4s-s\big)=3s\big(A'+B'+C'+D'\big).\end{aligned}$$

Hence

$$A'+B'+C'+D'\geq\frac{\big(2|a-c|+2|b-d|\big)^2}{3s}\geq\frac{16\cdot|a-c|\cdot|b-d|}{3s}. \tag{1}$$

Next we estimate the absolute value of the other sum. We couple A'' with C'' to obtain

$$\begin{aligned}A''+C'' &= \frac{(a-c)(a+c-2b)}{s-d}+\frac{(c-a)(c+a-2d)}{s-b}\\ &= \frac{(a-c)(a+c-2b)(s-b)+(c-a)(c+a-2d)(s-d)}{(s-d)(s-b)}\\ &= \frac{(a-c)\big(-2b(s-b)-b(a+c)+2d(s-d)+d(a+c)\big)}{s(a+c)+bd}\\ &= \frac{3(a-c)(d-b)(a+c)}{M}, \qquad \text{with} \qquad M=s(a+c)+bd.\end{aligned}$$

Hence by cyclic shift

$$B''+D''=\frac{3(b-d)(a-c)(b+d)}{N}, \qquad \text{with} \qquad N=s(b+d)+ca.$$

Thus

$$A''+B''+C''+D''=3(a-c)(b-d)\left(\frac{b+d}{N}-\frac{a+c}{M}\right)=\frac{3(a-c)(b-d)W}{MN} \tag{2}$$

where

$$W=(b+d)M-(a+c)N=bd(b+d)-ac(a+c). \tag{3}$$

Note that

$$MN > \big(ac(a+c) + bd(b+d)\big)s \geq |W| \cdot s. \tag{4}$$

Now (2) and (4) yield

$$|A'' + B'' + C'' + D''| \leq \frac{3 \cdot |a-c| \cdot |b-d|}{s}. \tag{5}$$

Combined with (1) this results in

$$\begin{aligned} 2(A+B+C+D) &= (A'+B'+C'+D') + (A''+B''+C''+D'') \\ &\geq \frac{16 \cdot |a-c| \cdot |b-d|}{3s} - \frac{3 \cdot |a-c| \cdot |b-d|}{s} = \frac{7 \cdot |a-c| \cdot |b-d|}{3(a+b+c+d)} \geq 0. \end{aligned}$$

This is the required inequality. From the last line we see that equality can be achieved only if either $a = c$ or $b = d$. Since we also need equality in (1), this implies that actually $a = c$ and $b = d$ must hold simultaneously, which is obviously also a sufficient condition.

Solution 2. We keep the notations A, B, C, D, s, and also M, N, W from the preceding solution; the definitions of M, N, W and relations (3), (4) in that solution did not depend on the foregoing considerations. Starting from

$$2A = \frac{(a-c)^2 + 3(a+c)(a-c)}{a+b+c} - 2a + 2c,$$

we get

$$\begin{aligned} 2(A+C) &= (a-c)^2\left(\frac{1}{s-d} + \frac{1}{s-b}\right) + 3(a+c)(a-c)\left(\frac{1}{s-d} - \frac{1}{s-b}\right) \\ &= (a-c)^2\,\frac{2s-b-d}{M} + 3(a+c)(a-c) \cdot \frac{d-b}{M} = \frac{p(a-c)^2 - 3(a+c)(a-c)(b-d)}{M} \end{aligned}$$

where $p = 2s - b - d = s + a + c$. Similarly, writing $q = s + b + d$ we have

$$2(B+D) = \frac{q(b-d)^2 - 3(b+d)(b-d)(c-a)}{N};$$

specific grouping of terms in the numerators has its aim. Note that $pq > 2s^2$. By adding the fractions expressing $2(A+C)$ and $2(B+D)$,

$$2(A+B+C+D) = \frac{p(a-c)^2}{M} + \frac{3(a-c)(b-d)W}{MN} + \frac{q(b-d)^2}{N}$$

with W defined by (3).

Substitution $x = (a-c)/M$, $y = (b-d)/N$ brings the required inequality to the form

$$2(A+B+C+D) = Mpx^2 + 3Wxy + Nqy^2 \geq 0. \tag{6}$$

It will be enough to verify that the discriminant $\Delta = 9W^2 - 4MNpq$ of the quadratic trinomial $Mpt^2 + 3Wt + Nq$ is negative; on setting $t = x/y$ one then gets (6). The first inequality in (4) together with $pq > 2s^2$ imply $4MNpq > 8s^3\big(ac(a+c) + bd(b+d)\big)$. Since

$$(a+c)s^3 > (a+c)^4 \geq 4ac(a+c)^2 \quad \text{and likewise} \quad (b+d)s^3 > 4bd(b+d)^2,$$

the estimate continues as follows,

$$4MNpq > 8\big(4(ac)^2(a+c)^2 + 4(bd)^2(b+d)^2\big) > 32\big(bd(b+d) - ac(a+c)\big)^2 = 32W^2 \geq 9W^2.$$

Thus indeed $\Delta < 0$. The desired inequality (6) hence results. It becomes an equality if and only if $x = y = 0$; equivalently, if and only if $a = c$ and simultaneously $b = d$.

Comment. The two solutions presented above do not differ significantly; large portions overlap. The properties of the number W turn out to be crucial in both approaches. The Cauchy-Schwarz inequality, applied in the first solution, is avoided in the second, which requires no knowledge beyond quadratic trinomials.

The estimates in the proof of $\Delta < 0$ in the second solution seem to be very wasteful. However, they come close to sharp when the terms in one of the pairs (a, c), (b, d) are equal and much bigger than those in the other pair.

In attempts to prove the inequality by just considering the six cases of arrangement of the numbers a, b, c, d on the real line, one soon discovers that the cases which create real trouble are precisely those in which a and c are both greater or both smaller than b and d.

Solution 3.

$$(a-b)(a-c)(a+b+d)(a+c+d)(b+c+d) =$$
$$= \Big((a-b)(a+b+d)\Big)\Big((a-c)(a+c+d)\Big)(b+c+d) =$$
$$= (a^2+ad-b^2-bd)(a^2+ad-c^2-cd)(b+c+d) =$$
$$= \left(a^4+2a^3d-a^2b^2-a^2bd-a^2c^2-a^2cd+a^2d^2-ab^2d-abd^2-ac^2d-acd^2+b^2c^2+b^2cd+bc^2d+bcd^2\right)(b+c+d) =$$
$$= a^4b+a^4c+a^4d+(b^3c^2+a^2d^3)-a^2c^3+(2a^3d^2-b^3a^2+c^3b^2)+$$
$$+(b^3cd-c^3da-d^3ab)+(2a^3bd+c^3db-d^3ac)+(2a^3cd-b^3da+d^3bc)$$
$$+(-a^2b^2c+3b^2c^2d-2ac^2d^2)+(-2a^2b^2d+2bc^2d^2)+(-a^2bc^2-2a^2c^2d-2ab^2d^2+2b^2cd^2)+$$
$$+(-2a^2bcd-ab^2cd-abc^2d-2abcd^2)$$

Introducing the notation $S_{xyzw} = \sum\limits_{cyc} a^x b^y c^z d^w$, one can write

$$\sum_{cyc}(a-b)(a-c)(a+b+d)(a+c+d)(b+c+d) =$$
$$= S_{4100}+S_{4010}+S_{4001}+2S_{3200}-S_{3020}+2S_{3002}-S_{3110}+2S_{3101}+2S_{3011}-3S_{2120}-6S_{2111} =$$
$$+\left(S_{4100}+S_{4001}+\frac{1}{2}S_{3110}+\frac{1}{2}S_{3011}-3S_{2120}\right)+$$
$$+\left(S_{4010}-S_{3020}-\frac{3}{2}S_{3110}+\frac{3}{2}S_{3011}+\frac{9}{16}S_{2210}+\frac{9}{16}S_{2201}-\frac{9}{8}S_{2111}\right)+$$
$$+\frac{9}{16}\Big(S_{3200}-S_{2210}-S_{2201}+S_{3002}\Big)+\frac{23}{16}\Big(S_{3200}-2S_{3101}+S_{3002}\Big)+\frac{39}{8}\Big(S_{3101}-S_{2111}\Big),$$

where the expressions

$$S_{4100}+S_{4001}+\frac{1}{2}S_{3110}+\frac{1}{2}S_{3011}-3S_{2120} = \sum_{cyc}\left(a^4b+bc^4+\frac{1}{2}a^3bc+\frac{1}{2}abc^3-3a^2bc^2\right),$$

$$S_{4010}-S_{3020}-\frac{3}{2}S_{3110}+\frac{3}{2}S_{3011}+\frac{9}{16}S_{2210}+\frac{9}{16}S_{2201}-\frac{9}{8}S_{2111} = \sum_{cyc}a^2c\left(a-c-\frac{3}{4}b+\frac{3}{4}d\right)^2,$$

$$S_{3200}-S_{2210}-S_{2201}+S_{3002} = \sum_{cyc}b^2(a^3-a^2c-ac^2+c^3) = \sum_{cyc}b^2(a+c)(a-c)^2,$$

$$S_{3200}-2S_{3101}+S_{3002} = \sum_{cyc}a^3(b-d)^2 \qquad \text{and} \qquad S_{3101}-S_{2111} = \frac{1}{3}\sum_{cyc}bd(2a^3+c^3-3a^2c)$$

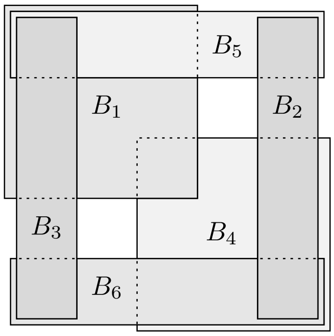
are all nonnegative.

Combinatorics

C1. In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a *box*. Two boxes *intersect* if they have a common point in their interior or on their boundary.

Find the largest n for which there exist n boxes $B_1, \ldots, B_n$ such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

Solution. The maximum number of such boxes is 6. One example is shown in the figure.



Now we show that 6 is the maximum. Suppose that boxes $B_1, \ldots, B_n$ satisfy the condition. Let the closed intervals I_k and J_k be the projections of B_k onto the x- and y-axis, for $1 \leq k \leq n$.

If B_i and B_j intersect, with a common point (x, y), then $x \in I_i \cap I_j$ and $y \in J_i \cap J_j$. So the intersections $I_i \cap I_j$ and $J_i \cap J_j$ are nonempty. Conversely, if $x \in I_i \cap I_j$ and $y \in J_i \cap J_j$ for some real numbers x, y, then (x, y) is a common point of B_i and B_j. Putting it around, B_i and B_j are disjoint if and only if their projections on at least one coordinate axis are disjoint.

For brevity we call two boxes or intervals adjacent if their indices differ by 1 modulo n, and nonadjacent otherwise.

The adjacent boxes B_k and B_{k+1} do not intersect for each $k = 1, \ldots, n$. Hence (I_k, I_{k+1}) or (J_k, J_{k+1}) is a pair of disjoint intervals, $1 \leq k \leq n$. So there are at least n pairs of disjoint intervals among $(I_1, I_2), \ldots, (I_{n-1}, I_n), (I_n, I_1); (J_1, J_2), \ldots, (J_{n-1}, J_n), (J_n, J_1)$.

Next, every two nonadjacent boxes intersect, hence their projections on both axes intersect, too. Then the claim below shows that at most 3 pairs among $(I_1, I_2), \ldots, (I_{n-1}, I_n), (I_n, I_1)$ are disjoint, and the same holds for $(J_1, J_2), \ldots, (J_{n-1}, J_n), (J_n, J_1)$. Consequently $n \leq 3 + 3 = 6$, as stated. Thus we are left with the claim and its justification.

Claim. Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be intervals on a straight line such that every two nonadjacent intervals intersect. Then Δ_k and Δ_{k+1} are disjoint for at most three values of $k = 1, \ldots, n$.

Proof. Denote $\Delta_k = [a_k, b_k]$, $1 \leq k \leq n$. Let $\alpha = \max(a_1, \ldots, a_n)$ be the rightmost among the left endpoints of $\Delta_1, \ldots, \Delta_n$, and let $\beta = \min(b_1, \ldots, b_n)$ be the leftmost among their right endpoints. Assume that $\alpha = a_2$ without loss of generality.

If $\alpha \leq \beta$ then $a_i \leq \alpha \leq \beta \leq b_i$ for all i. Every Δ_i contains α, and thus no disjoint pair (Δ_i, Δ_{i+1}) exists.

If $\beta < \alpha$ then $\beta = b_i$ for some i such that $a_i < b_i = \beta < \alpha = a_2 < b_2$, hence Δ_2 and Δ_i are disjoint. Now Δ_2 intersects all remaining intervals except possibly Δ_1 and Δ_3, so Δ_2 and Δ_i can be disjoint only if $i = 1$ or $i = 3$. Suppose by symmetry that $i = 3$; then $\beta = b_3$. Since each of the intervals $\Delta_4, \ldots, \Delta_n$ intersects Δ_2, we have $a_i \le \alpha \le b_i$ for $i = 4, \ldots, n$. Therefore $\alpha \in \Delta_4 \cap \ldots \cap \Delta_n$, in particular $\Delta_4 \cap \ldots \cap \Delta_n \ne \emptyset$. Similarly, $\Delta_5, \ldots, \Delta_n, \Delta_1$ all intersect Δ_3, so that $\Delta_5 \cap \ldots \cap \Delta_n \cap \Delta_1 \ne \emptyset$ as $\beta \in \Delta_5 \cap \ldots \cap \Delta_n \cap \Delta_1$. This leaves (Δ_1, Δ_2), (Δ_2, Δ_3) and (Δ_3, Δ_4) as the only candidates for disjoint interval pairs, as desired.

Comment. The problem is a two-dimensional version of the original proposal which is included below. The extreme shortage of easy and appropriate submissions forced the Problem Selection Committee to shortlist a simplified variant. The same one-dimensional Claim is used in both versions.

Original proposal. We consider parallelepipeds in three-dimensional space, with edges parallel to the coordinate axes and of positive length. Such a parallelepiped will be called a *box*. Two boxes *intersect* if they have a common point in their interior or on their boundary.

Find the largest n for which there exist n boxes $B_1, \ldots, B_n$ such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

The maximum number of such boxes is 9. Suppose that boxes $B_1, \ldots, B_n$ satisfy the condition. Let the closed intervals I_k, J_k and K_k be the projections of box B_k onto the x-, y- and z-axis, respectively, for $1 \le k \le n$. As before, B_i and B_j are disjoint if and only if their projections on at least one coordinate axis are disjoint.

We call again two boxes or intervals adjacent if their indices differ by 1 modulo n, and nonadjacent otherwise.

The adjacent boxes B_i and B_{i+1} do not intersect for each $i = 1, \ldots, n$. Hence at least one of the pairs (I_i, I_{i+1}), (J_i, J_{i+1}) and (K_i, K_{i+1}) is a pair of disjoint intervals. So there are at least n pairs of disjoint intervals among $(I_i, I_{i+1}), (J_i, J_{i+1}), (K_i, K_{i+1})$, $1 \le i \le n$.

Next, every two nonadjacent boxes intersect, hence their projections on the three axes intersect, too. Referring to the Claim in the solution of the two-dimensional version, we cocnclude that at most 3 pairs among $(I_1, I_2), \ldots, (I_{n-1}, I_n), (I_n, I_1)$ are disjoint; the same holds for $(J_1, J_2), \ldots, (J_{n-1}, J_n), (J_n, J_1)$ and $(K_1, K_2), \ldots, (K_{n-1}, K_n), (K_n, K_1)$. Consequently $n \le 3 + 3 + 3 = 9$, as stated.

For $n = 9$, the desired system of boxes exists. Consider the intervals in the following table:

i	I_i	J_i	K_i
1	$[1,4]$	$[1,6]$	$[3,6]$
2	$[5,6]$	$[1,6]$	$[1,6]$
3	$[1,2]$	$[1,6]$	$[1,6]$
4	$[3,6]$	$[1,4]$	$[1,6]$
5	$[1,6]$	$[5,6]$	$[1,6]$
6	$[1,6]$	$[1,2]$	$[1,6]$
7	$[1,6]$	$[3,6]$	$[1,4]$
8	$[1,6]$	$[1,6]$	$[5,6]$
9	$[1,6]$	$[1,6]$	$[1,2]$

We have $I_1 \cap I_2 = I_2 \cap I_3 = I_3 \cap I_4 = \emptyset$, $J_4 \cap J_5 = J_5 \cap J_6 = J_6 \cap J_7 = \emptyset$, and finally $K_7 \cap K_8 = K_8 \cap K_9 = K_9 \cap K_1 = \emptyset$. The intervals in each column intersect in all other cases. It follows that the boxes $B_i = I_i \times J_i \times K_i$, $i = 1, \ldots, 9$, have the stated property.

C2. For every positive integer n determine the number of permutations $(a_1, a_2, \dots, a_n)$ of the set $\{1, 2, \dots, n\}$ with the following property:

$$2(a_1 + a_2 + \cdots + a_k) \quad \text{is divisible by } k \quad \text{for } k = 1, 2, \dots, n.$$

Solution. For each n let F_n be the number of permutations of $\{1, 2, \dots, n\}$ with the required property; call them *nice*. For $n = 1, 2, 3$ every permutation is nice, so $F_1 = 1$, $F_2 = 2$, $F_3 = 6$.

Take an $n > 3$ and consider any nice permutation $(a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$. Then $n - 1$ must be a divisor of the number

$$\begin{aligned} 2(a_1 + a_2 + \cdots + a_{n-1}) &= 2\big((1 + 2 + \cdots + n) - a_n\big) \\ &= n(n+1) - 2a_n = (n+2)(n-1) + (2 - 2a_n). \end{aligned}$$

So $2a_n - 2$ must be divisible by $n - 1$, hence equal to 0 or $n - 1$ or $2n - 2$. This means that

$$a_n = 1 \qquad \text{or} \qquad a_n = \frac{n+1}{2} \qquad \text{or} \qquad a_n = n.$$

Suppose that $a_n = (n+1)/2$. Since the permutation is nice, taking $k = n - 2$ we get that $n - 2$ has to be a divisor of

$$\begin{aligned} 2(a_1 + a_2 + \cdots + a_{n-2}) &= 2\big((1 + 2 + \cdots + n) - a_n - a_{n-1}\big) \\ &= n(n+1) - (n+1) - 2a_{n-1} = (n+2)(n-2) + (3 - 2a_{n-1}). \end{aligned}$$

So $2a_{n-1} - 3$ should be divisible by $n - 2$, hence equal to 0 or $n - 2$ or $2n - 4$. Obviously 0 and $2n - 4$ are excluded because $2a_{n-1} - 3$ is odd. The remaining possibility ($2a_{n-1} - 3 = n - 2$) leads to $a_{n-1} = (n+1)/2 = a_n$, which also cannot hold. This eliminates $(n+1)/2$ as a possible value of a_n. Consequently $a_n = 1$ or $a_n = n$.

If $a_n = n$ then $(a_1, a_2, \dots, a_{n-1})$ is a nice permutation of $\{1, 2, \dots, n-1\}$. There are F_{n-1} such permutations. Attaching n to any one of them at the end creates a nice permutation of $\{1, 2, \dots, n\}$.

If $a_n = 1$ then $(a_1-1, a_2-1, \dots, a_{n-1}-1)$ is a permutation of $\{1, 2, \dots, n-1\}$. It is also nice because the number

$$2\big((a_1-1) + \cdots + (a_k-1)\big) = 2(a_1 + \cdots + a_k) - 2k$$

is divisible by k, for any $k \le n - 1$. And again, any one of the F_{n-1} nice permutations $(b_1, b_2, \dots, b_{n-1})$ of $\{1, 2, \dots, n-1\}$ gives rise to a nice permutation of $\{1, 2, \dots, n\}$ whose last term is 1, namely $(b_1+1, b_2+1, \dots, b_{n-1}+1, 1)$.

The bijective correspondences established in both cases show that there are F_{n-1} nice permutations of $\{1, 2, \dots, n\}$ with the last term 1 and also F_{n-1} nice permutations of $\{1, 2, \dots, n\}$ with the last term n. Hence follows the recurrence $F_n = 2F_{n-1}$. With the base value $F_3 = 6$ this gives the outcome formula $F_n = 3 \cdot 2^{n-2}$ for $n \ge 3$.

C3. In the coordinate plane consider the set S of all points with integer coordinates. For a positive integer k, two distinct points $A, B \in S$ will be called *k-friends* if there is a point $C \in S$ such that the area of the triangle ABC is equal to k. A set $T \subset S$ will be called a *k-clique* if every two points in T are k-friends. Find the least positive integer k for which there exists a k-clique with more than 200 elements.

Solution. To begin, let us describe those points $B \in S$ which are k-friends of the point $(0,0)$. By definition, $B = (u, v)$ satisfies this condition if and only if there is a point $C = (x, y) \in S$ such that $\frac{1}{2}|uy - vx| = k$. (This is a well-known formula expressing the area of triangle ABC when A is the origin.)

To say that there exist integers x, y for which $|uy - vx| = 2k$, is equivalent to saying that the greatest common divisor of u and v is also a divisor of $2k$. Summing up, a point $B = (u, v) \in S$ is a k-friend of $(0,0)$ if and only if $\gcd(u, v)$ divides $2k$.

Translation by a vector with integer coordinates does not affect k-friendship; if two points are k-friends, so are their translates. It follows that two points $A, B \in S$, $A = (s,t)$, $B = (u, v)$, are k-friends if and only if the point $(u - s, v - t)$ is a k-friend of $(0,0)$; i.e., if $\gcd(u - s, v - t)|2k$.

Let n be a positive integer which does not divide $2k$. We claim that a k-clique cannot have more than n^2 elements.

Indeed, all points $(x, y) \in S$ can be divided into n^2 classes determined by the remainders that x and y leave in division by n. If a set T has more than n^2 elements, some two points $A, B \in T$, $A = (s,t)$, $B = (u, v)$, necessarily fall into the same class. This means that $n|u - s$ and $n|v - t$. Hence $n|d$ where $d = \gcd(u - s, v - t)$. And since n does not divide $2k$, also d does not divide $2k$. Thus A and B are not k-friends and the set T is not a k-clique.

Now let $M(k)$ be the *least* positive integer which does not divide $2k$. Write $M(k) = m$ for the moment and consider the set T of all points (x, y) with $0 \leq x, y < m$. There are m^2 of them. If $A = (s,t)$, $B = (u, v)$ are two distinct points in T then both differences $|u - s|$, $|v - t|$ are integers less than m and at least one of them is positive. By the definition of m, every positive integer less than m divides $2k$. Therefore $u - s$ (if nonzero) divides $2k$, and the same is true of $v - t$. So $2k$ is divisible by $\gcd(u - s, v - t)$, meaning that A, B are k-friends. Thus T is a k-clique.

It follows that the maximum size of a k-clique is $M(k)^2$, with $M(k)$ defined as above. We are looking for the minimum k such that $M(k)^2 > 200$.

By the definition of $M(k)$, $2k$ is divisible by the numbers $1, 2, \ldots, M(k)-1$, but not by $M(k)$ itself. If $M(k)^2 > 200$ then $M(k) \geq 15$. Trying to hit $M(k) = 15$ we get a contradiction immediately ($2k$ would have to be divisible by 3 and 5, but not by 15).

So let us try $M(k) = 16$. Then $2k$ is divisible by the numbers $1, 2, \ldots, 15$, hence also by their least common multiple L, but not by 16. And since L is not a multiple of 16, we infer that $k = L/2$ is the least k with $M(k) = 16$.

Finally, observe that if $M(k) \geq 17$ then $2k$ must be divisible by the least common multiple of $1, 2, \ldots, 16$, which is equal to $2L$. Then $2k \geq 2L$, yielding $k > L/2$.

In conclusion, the least k with the required property is equal to $L/2 = 180180$.

C4. Let n and k be fixed positive integers of the same parity, $k \geq n$. We are given $2n$ lamps numbered 1 through $2n$; each of them can be *on* or *off*. At the beginning all lamps are *off*. We consider sequences of k steps. At each step one of the lamps is switched (from *off* to *on* or from *on* to *off*).

Let N be the number of k-step sequences ending in the state: lamps $1, \ldots, n$ *on*, lamps $n+1, \ldots, 2n$ *off*.

Let M be the number of k-step sequences leading to the same state and not touching lamps $n+1, \ldots, 2n$ at all.

Find the ratio N/M.

Solution. A sequence of k switches ending in the state as described in the problem statement (lamps $1, \ldots, n$ *on*, lamps $n+1, \ldots, 2n$ *off*) will be called an *admissible process*. If, moreover, the process does not touch the lamps $n+1, \ldots, 2n$, it will be called *restricted*. So there are N admissible processes, among which M are restricted.

In every admissible process, restricted or not, each one of the lamps $1, \ldots, n$ goes from *off* to *on*, so it is switched an odd number of times; and each one of the lamps $n+1, \ldots, 2n$ goes from *off* to *off*, so it is switched an even number of times.

Notice that $M > 0$; i.e., restricted admissible processes do exist (it suffices to switch each one of the lamps $1, \ldots, n$ just once and then choose one of them and switch it $k - n$ times, which by hypothesis is an even number).

Consider any restricted admissible process $\mathbf{p}$. Take any lamp ℓ, $1 \leq \ell \leq n$, and suppose that it was switched k_ℓ times. As noticed, k_ℓ must be odd. Select arbitrarily an even number of these k_ℓ switches and replace each of them by the switch of lamp $n+\ell$. This can be done in $2^{k_\ell - 1}$ ways (because a k_ℓ-element set has $2^{k_\ell - 1}$ subsets of even cardinality). Notice that $k_1 + \cdots + k_n = k$.

These actions are independent, in the sense that the action involving lamp ℓ does not affect the action involving any other lamp. So there are $2^{k_1 - 1} \cdot 2^{k_2 - 1} \cdots 2^{k_n - 1} = 2^{k-n}$ ways of combining these actions. In any of these combinations, each one of the lamps $n+1, \ldots, 2n$ gets switched an even number of times and each one of the lamps $1, \ldots, n$ remains switched an odd number of times, so the final state is the same as that resulting from the original process $\mathbf{p}$.

This shows that every restricted admissible process $\mathbf{p}$ can be modified in 2^{k-n} ways, giving rise to 2^{k-n} distinct admissible processes (with all lamps allowed).

Now we show that every admissible process $\mathbf{q}$ can be achieved in that way. Indeed, it is enough to replace every switch of a lamp with a label $\ell > n$ that occurs in $\mathbf{q}$ by the switch of the corresponding lamp $\ell - n$; in the resulting process $\mathbf{p}$ the lamps $n+1, \ldots, 2n$ are not involved.

Switches of each lamp with a label $\ell > n$ had occurred in $\mathbf{q}$ an even number of times. So the performed replacements have affected each lamp with a label $\ell \leq n$ also an even number of times; hence in the overall effect the final state of each lamp has remained the same. This means that the resulting process $\mathbf{p}$ is admissible—and clearly restricted, as the lamps $n+1, \ldots, 2n$ are not involved in it any more.

If we now take process $\mathbf{p}$ and reverse all these replacements, then we obtain process $\mathbf{q}$. These reversed replacements are nothing else than the modifications described in the foregoing paragraphs.

Thus there is a one–to–(2^{k-n}) correspondence between the M restricted admissible processes and the total of N admissible processes. Therefore $N/M = 2^{k-n}$.

C5. Let $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$ be a $(k+\ell)$-element set of real numbers contained in the interval $[0,1]$; k and ℓ are positive integers. A k-element subset $A \subset S$ is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{\ell} \sum_{x_j \in S \setminus A} x_j \right| \le \frac{k+\ell}{2k\ell}.$$

Prove that the number of nice subsets is at least $\dfrac{2}{k+\ell}\dbinom{k+\ell}{k}$.

Solution. For a k-element subset $A \subset S$, let $f(A) = \dfrac{1}{k} \displaystyle\sum_{x_i \in A} x_i - \dfrac{1}{\ell} \sum_{x_j \in S \setminus A} x_j$. Denote $\dfrac{k+\ell}{2k\ell} = d$. By definition a subset A is nice if $|f(A)| \le d$.

To each permutation $(y_1, y_2, \ldots, y_{k+\ell})$ of the set $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$ we assign $k+\ell$ subsets of S with k elements each, namely $A_i = \{y_i, y_{i+1}, \ldots, y_{i+k-1}\}$, $i = 1, 2, \ldots, k+\ell$. Indices are taken modulo $k+\ell$ here and henceforth. In other words, if $y_1, y_2, \ldots, y_{k+\ell}$ are arranged around a circle in this order, the sets in question are all possible blocks of k consecutive elements.

Claim. At least two nice sets are assigned to every permutation of S.

Proof. Adjacent sets A_i and A_{i+1} differ only by the elements y_i and y_{i+k}, $i = 1, \ldots, k+\ell$. By the definition of f, and because $y_i, y_{i+k} \in [0,1]$,

$$|f(A_{i+1}) - f(A_i)| = \left| \left(\frac{1}{k} + \frac{1}{\ell} \right) (y_{i+k} - y_i) \right| \le \frac{1}{k} + \frac{1}{\ell} = 2d.$$

Each element $y_i \in S$ belongs to exactly k of the sets $A_1, \ldots, A_{k+\ell}$. Hence in k of the expressions $f(A_1), \ldots, f(A_{k+\ell})$ the coefficient of y_i is $1/k$; in the remaining ℓ expressions, its coefficient is $-1/\ell$. So the contribution of y_i to the sum of all $f(A_i)$ equals $k \cdot 1/k - \ell \cdot 1/\ell = 0$. Since this holds for all i, it follows that $f(A_1) + \cdots + f(A_{k+\ell}) = 0$.

If $f(A_p) = \min f(A_i)$, $f(A_q) = \max f(A_i)$, we obtain in particular $f(A_p) \le 0$, $f(A_q) \ge 0$. Let $p < q$ (the case $p > q$ is analogous; and the claim is true for $p = q$ as $f(A_i) = 0$ for all i).

We are ready to prove that at least two of the sets $A_1, \ldots, A_{k+\ell}$ are nice. The interval $[-d, d]$ has length $2d$, and we saw that adjacent numbers in the circular arrangement $f(A_1), \ldots, f(A_{k+\ell})$ differ by at most $2d$. Suppose that $f(A_p) < -d$ and $f(A_q) > d$. Then one of the numbers $f(A_{p+1}), \ldots, f(A_{q-1})$ lies in $[-d, d]$, and also one of the numbers $f(A_{q+1}), \ldots, f(A_{p-1})$ lies there. Consequently, one of the sets $A_{p+1}, \ldots, A_{q-1}$ is nice, as well as one of the sets $A_{q+1}, \ldots, A_{p-1}$. If $-d \le f(A_p)$ and $f(A_q) \le d$ then A_p and A_q are nice.

Let now $f(A_p) < -d$ and $f(A_q) \le d$. Then $f(A_p) + f(A_q) < 0$, and since $\sum f(A_i) = 0$, there is an $r \ne q$ such that $f(A_r) > 0$. We have $0 < f(A_r) \le f(A_q) \le d$, so the sets $f(A_r)$ and $f(A_q)$ are nice. The only case remaining, $-d \le f(A_p)$ and $d < f(A_q)$, is analogous.

Apply the claim to each of the $(k+\ell)!$ permutations of $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$. This gives at least $2(k+\ell)!$ nice sets, counted with repetitions: each nice set is counted as many times as there are permutations to which it is assigned.

On the other hand, each k-element set $A \subset S$ is assigned to exactly $(k+\ell)\, k!\, \ell!$ permutations. Indeed, such a permutation $(y_1, y_2, \ldots, y_{k+\ell})$ is determined by three independent choices: an index $i \in \{1, 2, \ldots, k+\ell\}$ such that $A = \{y_i, y_{i+1}, \ldots, y_{i+k-1}\}$, a permutation $(y_i, y_{i+1}, \ldots, y_{i+k-1})$ of the set A, and a permutation $(y_{i+k}, y_{i+k+1}, \ldots, y_{i-1})$ of the set $S \setminus A$.

In summary, there are at least $\dfrac{2(k+\ell)!}{(k+\ell)\, k!\, \ell!} = \dfrac{2}{k+\ell}\dbinom{k+\ell}{k}$ nice sets.

C6. For $n \geq 2$, let $S_1, S_2, \ldots, S_{2^n}$ be 2^n subsets of $A = \{1, 2, 3, \ldots, 2^{n+1}\}$ that satisfy the following property: There do not exist indices a and b with $a < b$ and elements $x, y, z \in A$ with $x < y < z$ such that $y, z \in S_a$ and $x, z \in S_b$. Prove that at least one of the sets $S_1, S_2, \ldots, S_{2^n}$ contains no more than $4n$ elements.

Solution 1. We prove that there exists a set S_a with at most $3n + 1$ elements.

Given a $k \in \{1, \ldots, n\}$, we say that an element $z \in A$ is k-*good* to a set S_a if $z \in S_a$ and S_a contains two other elements x and y with $x < y < z$ such that $z - y < 2^k$ and $z - x \geq 2^k$. Also, $z \in A$ will be called *good* to S_a if z is k-good to S_a for some $k = 1, \ldots, n$.

We claim that each $z \in A$ can be k-good to at most one set S_a. Indeed, suppose on the contrary that z is k-good simultaneously to S_a and S_b, with $a < b$. Then there exist $y_a \in S_a$, $y_a < z$, and $x_b \in S_b$, $x_b < z$, such that $z - y_a < 2^k$ and $z - x_b \geq 2^k$. On the other hand, since $z \in S_a \cap S_b$, by the condition of the problem there is no element of S_a strictly between x_b and z. Hence $y_a \leq x_b$, implying $z - y_a \geq z - x_b$. However this contradicts $z - y_a < 2^k$ and $z - x_b \geq 2^k$. The claim follows.

As a consequence, a fixed $z \in A$ can be good to at most n of the given sets (no more than one of them for each $k = 1, \ldots, n$).

Furthermore, let $u_1 < u_2 < \cdots < u_m < \cdots < u_p$ be all elements of a fixed set S_a that are not good to S_a. We prove that $u_m - u_1 > 2(u_{m-1} - u_1)$ for all $m \geq 3$.

Indeed, assume that $u_m - u_1 \leq 2(u_{m-1} - u_1)$ holds for some $m \geq 3$. This inequality can be written as $2(u_m - u_{m-1}) \leq u_m - u_1$. Take the unique k such that $2^k \leq u_m - u_1 < 2^{k+1}$. Then $2(u_m - u_{m-1}) \leq u_m - u_1 < 2^{k+1}$ yields $u_m - u_{m-1} < 2^k$. However the elements $z = u_m$, $x = u_1$, $y = u_{m-1}$ of S_a then satisfy $z - y < 2^k$ and $z - x \geq 2^k$, so that $z = u_m$ is k-good to S_a.

Thus each term of the sequence $u_2 - u_1, u_3 - u_1, \ldots, u_p - u_1$ is more than twice the previous one. Hence $u_p - u_1 > 2^{p-1}(u_2 - u_1) \geq 2^{p-1}$. But $u_p \in \{1, 2, 3, \ldots, 2^{n+1}\}$, so that $u_p \leq 2^{n+1}$. This yields $p - 1 \leq n$, i. e. $p \leq n + 1$.

In other words, each set S_a contains at most $n + 1$ elements that are not good to it.

To summarize the conclusions, mark with red all elements in the sets S_a that are good to the respective set, and with blue the ones that are not good. Then the total number of red elements, counting multiplicities, is at most $n \cdot 2^{n+1}$ (each $z \in A$ can be marked red in at most n sets). The total number of blue elements is at most $(n + 1)2^n$ (each set S_a contains at most $n + 1$ blue elements). Therefore the sum of cardinalities of $S_1, S_2, \ldots, S_{2^n}$ does not exceed $(3n + 1)2^n$. By averaging, the smallest set has at most $3n + 1$ elements.

Solution 2. We show that one of the sets S_a has at most $2n + 1$ elements. In the sequel $|\cdot|$ denotes the cardinality of a (finite) set.

Claim. For $n \geq 2$, suppose that k subsets $S_1, \ldots, S_k$ of $\{1, 2, \ldots, 2^n\}$ (not necessarily different) satisfy the condition of the problem. Then

$$\sum_{i=1}^{k} (|S_i| - n) \leq (2n - 1)2^{n-2}.$$

Proof. Observe that if the sets S_i $(1 \leq i \leq k)$ satisfy the condition then so do their arbitrary subsets T_i $(1 \leq i \leq k)$. The condition also holds for the sets $t + S_i = \{t + x \mid x \in S_i\}$ where t is arbitrary.

Note also that a set may occur more than once among $S_1, \ldots, S_k$ only if its cardinality is less than 3, in which case its contribution to the sum $\sum_{i=1}^{k}(|S_i| - n)$ is nonpositive (as $n \geq 2$).

The proof is by induction on n. In the base case $n = 2$ we have subsets S_i of $\{1, 2, 3, 4\}$. Only the ones of cardinality 3 and 4 need to be considered by the remark above; each one of them occurs at most once among $S_1, \ldots, S_k$. If $S_i = \{1,2,3,4\}$ for some i then no S_j is a 3-element subset in view of the condition, hence $\sum_{i=1}^{k}(|S_i|-2) \le 2$. By the condition again, it is impossible that $S_i = \{1,3,4\}$ and $S_j = \{2,3,4\}$ for some i, j. So if $|S_i| \le 3$ for all i then at most 3 summands $|S_i| - 2$ are positive, corresponding to 3-element subsets. This implies $\sum_{i=1}^{k}(|S_i|-2) \le 3$, therefore the conclusion is true for $n = 2$.

Suppose that the claim holds for some $n \ge 2$, and let the sets $S_1, \ldots, S_k \subseteq \{1, 2, \ldots, 2^{n+1}\}$ satisfy the given property. Denote $U_i = S_i \cap \{1, 2, \ldots, 2^n\}$, $V_i = S_i \cap \{2^n + 1, \ldots, 2^{n+1}\}$. Let

$$I = \{i \mid 1 \le i \le k,\ |U_i| \ne 0\}, \qquad J = \{1, \ldots, k\} \setminus I.$$

The sets S_j with $j \in J$ are all contained in $\{2^n + 1, \ldots, 2^{n+1}\}$, so the induction hypothesis applies to their translates $-2^n + S_j$ which have the same cardinalities. Consequently, this gives $\sum_{j \in J}(|S_j| - n) \le (2n-1)2^{n-2}$, so that

$$\sum_{j\in J}(|S_j| - (n+1)) \le \sum_{j \in J}(|S_j| - n) \le (2n-1)2^{n-2}. \tag{1}$$

For $i \in I$, denote by v_i the least element of V_i. Observe that if V_a and V_b intersect, with $a < b$, $a, b \in I$, then v_a is their unique common element. Indeed, let $z \in V_a \cap V_b \subseteq S_a \cap S_b$ and let m be the least element of S_b. Since $b \in I$, we have $m \le 2^n$. By the condition, there is no element of S_a strictly between $m \le 2^n$ and $z > 2^n$, which implies $z = v_a$.

It follows that if the element v_i is removed from each V_i, a family of pairwise disjoint sets $W_i = V_i \setminus \{v_i\}$ is obtained, $i \in I$ (we assume $W_i = \emptyset$ if $V_i = \emptyset$). As $W_i \subseteq \{2^n + 1, \ldots, 2^{n+1}\}$ for all i, we infer that $\sum_{i \in I}|W_i| \le 2^n$. Therefore $\sum_{i\in I}(|V_i| - 1) \le \sum_{i \in I}|W_i| \le 2^n$.

On the other hand, the induction hypothesis applies directly to the sets U_i, $i \in I$, so that $\sum_{i \in \mathcal{I}}(|U_i| - n) \le (2n-1)2^{n-2}$. In summary,

$$\sum_{i \in I}(|S_i| - (n+1)) = \sum_{i\in I}(|U_i| - n) + \sum_{i \in I}(|V_i| - 1) \le (2n-1)2^{n-2} + 2^n. \tag{2}$$

The estimates (1) and (2) are sufficient to complete the inductive step:

$$\begin{aligned}\sum_{i=1}^{k}(|S_i| - (n+1)) &= \sum_{i \in I}(|S_i| - (n+1)) + \sum_{j \in J}(|S_j| - (n+1)) \\ &\le (2n-1)2^{n-2} + 2^n + (2n-1)2^{n-2} = (2n+1)2^{n-1}.\end{aligned}$$

Returning to the problem, consider $k = 2^n$ subsets $S_1, S_2, \ldots, S_{2^n}$ of $\{1, 2, 3, \ldots, 2^{n+1}\}$. If they satisfy the given condition, the claim implies $\sum_{i=1}^{2^n}(|S_i| - (n+1)) \le (2n+1)2^{n-1}$. By averaging again, we see that the smallest set has at most $2n + 1$ elements.

Comment. It can happen that each set S_i has cardinality at least $n + 1$. Here is an example by the proposer.

For $i = 1, \ldots, 2^n$, let $S_i = \{i + 2^k \mid 0 \le k \le n\}$. Then $|S_i| = n + 1$ for all i. Suppose that there exist $a < b$ and $x < y < z$ such that $y, z \in S_a$ and $x, z \in S_b$. Hence $z = a + 2^k = b + 2^l$ for some $k > l$. Since $y \in S_a$ and $y < z$, we have $y \le a + 2^{k-1}$. So the element $x \in S_b$ satisfies

$$x < y \le a + 2^{k-1} = z - 2^{k-1} \le z - 2^l = b.$$

However the least element of S_b is $b + 1$, a contradiction.

Geometry

G1. In an acute-angled triangle ABC, point H is the orthocentre and A_0, B_0, C_0 are the midpoints of the sides BC, CA, AB, respectively. Consider three circles passing through H: ω_a around A_0, ω_b around B_0 and ω_c around C_0. The circle ω_a intersects the line BC at A_1 and A_2; ω_b intersects CA at B_1 and B_2; ω_c intersects AB at C_1 and C_2. Show that the points A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

Solution 1. The perpendicular bisectors of the segments A_1A_2, B_1B_2, C_1C_2 are also the perpendicular bisectors of BC, CA, AB. So they meet at O, the circumcentre of ABC. Thus O is the only point that can possibly be the centre of the desired circle.

From the right triangle OA_0A_1 we get

$$OA_1^2 = OA_0^2 + A_0A_1^2 = OA_0^2 + A_0H^2. \tag{1}$$

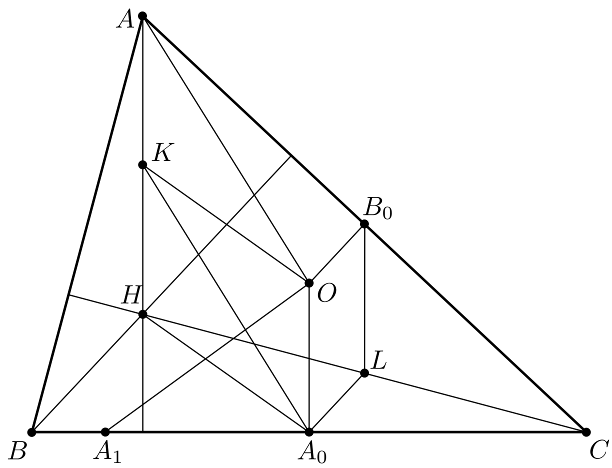
Let K be the midpoint of AH and let L be the midpoint of CH. Since A_0 and B_0 are the midpoints of BC and CA, we see that $A_0L \| BH$ and $B_0L \| AH$. Thus the segments A_0L and B_0L are perpendicular to AC and BC, hence parallel to OB_0 and OA_0, respectively. Consequently OA_0LB_0 is a parallelogram, so that OA_0 and B_0L are equal and parallel. Also, the midline B_0L of triangle AHC is equal and parallel to AK and KH.

It follows that AKA_0O and HA_0OK are parallelograms. The first one gives $A_0K = OA = R$, where R is the circumradius of ABC. From the second one we obtain

$$2(OA_0^2 + A_0H^2) = OH^2 + A_0K^2 = OH^2 + R^2. \tag{2}$$

(In a parallelogram, the sum of squares of the diagonals equals the sum of squares of the sides).

From (1) and (2) we get $OA_1^2 = (OH^2 + R^2)/2$. By symmetry, the same holds for the distances OA_2, OB_1, OB_2, OC_1 and OC_2. Thus A_1, A_2, B_1, B_2, C_1, C_2 all lie on a circle with centre at O and radius $(OH^2 + R^2)/2$.

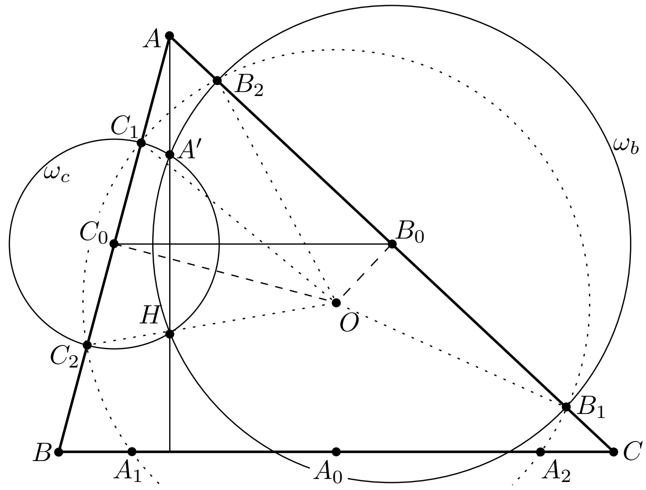


Solution 2. We are going to show again that the circumcentre O is equidistant from the six points in question.

Let A' be the second intersection point of ω_b and ω_c. The line B_0C_0, which is the line of centers of circles ω_b and ω_c, is a midline in triangle ABC, parallel to BC and perpendicular to the altitude AH. The points A' and H are symmetric with respect to the line of centers. Therefore A' lies on the line AH.

From the two circles ω_b and ω_c we obtain $AC_1 \cdot AC_2 = AA' \cdot AH = AB_1 \cdot AB_2$. So the quadrilateral $B_1B_2C_1C_2$ is cyclic. The perpendicular bisectors of the sides B_1B_2 and C_1C_2 meet at O. Hence O is the circumcentre of $B_1B_2C_1C_2$ and so $OB_1 = OB_2 = OC_1 = OC_2$.

Analogous arguments yield $OA_1 = OA_2 = OB_1 = OB_2$ and $OA_1 = OA_2 = OC_1 = OC_2$. Thus A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle centred at O.



Comment. The problem can be solved without much difficulty in many ways by calculation, using trigonometry, coordinate geometry or complex numbers. As an example we present a short proof using vectors.

Solution 3. Let again O and R be the circumcentre and circumradius. Consider the vectors

$$\overrightarrow{OA} = \mathbf{a}, \quad \overrightarrow{OB} = \mathbf{b}, \quad \overrightarrow{OC} = \mathbf{c}, \quad \text{where} \quad \mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = R^2.$$

It is well known that $\overrightarrow{OH} = \mathbf{a} + \mathbf{b} + \mathbf{c}$. Accordingly,

$$\overrightarrow{A_0H} = \overrightarrow{OH} - \overrightarrow{OA_0} = (\mathbf{a} + \mathbf{b} + \mathbf{c}) - \frac{\mathbf{b} + \mathbf{c}}{2} = \frac{2\mathbf{a} + \mathbf{b} + \mathbf{c}}{2},$$

and

$$OA_1^2 = OA_0^2 + A_0A_1^2 = OA_0^2 + A_0H^2 = \left(\frac{\mathbf{b} + \mathbf{c}}{2}\right)^2 + \left(\frac{2\mathbf{a} + \mathbf{b} + \mathbf{c}}{2}\right)^2$$

$$= \frac{1}{4}(\mathbf{b}^2 + 2\mathbf{bc} + \mathbf{c}^2) + \frac{1}{4}(4\mathbf{a}^2 + 4\mathbf{ab} + 4\mathbf{ac} + \mathbf{b}^2 + 2\mathbf{bc} + \mathbf{c}^2) = 2R^2 + (\mathbf{ab} + \mathbf{ac} + \mathbf{bc});$$

here $\mathbf{ab}$, $\mathbf{bc}$, etc. denote dot products of vectors. We get the same for the distances OA_2, OB_1, OB_2, OC_1 and OC_2.

G2. Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD, such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF; assume it does not lie on line AB.

Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.

Solution. Assume that the disposition of points is as in the diagram.

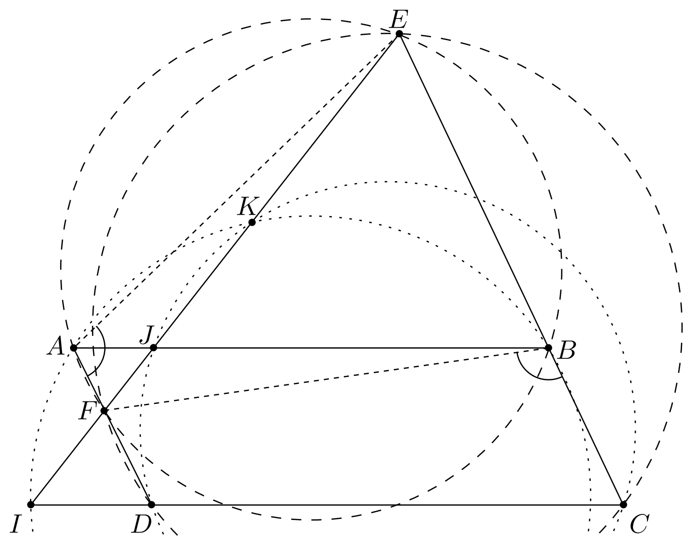
Since $\angle EBF = 180° - \angle CBF = 180° - \angle EAF$ by hypothesis, the quadrilateral $AEBF$ is cyclic. Hence $AJ \cdot JB = FJ \cdot JE$. In view of this equality, I belongs to the circumcircle of ABK if and only if $IJ \cdot JK = FJ \cdot JE$. Expressing $IJ = IF + FJ$, $JE = FE - FJ$, and $JK = \frac{1}{2}FE - FJ$, we find that I belongs to the circumcircle of ABK if and only if

$$FJ = \frac{IF \cdot FE}{2IF + FE}.$$

Since $AEBF$ is cyclic and AB, CD are parallel, $\angle FEC = \angle FAB = 180° - \angle CDF$. Then $CDFE$ is also cyclic, yielding $ID \cdot IC = IF \cdot IE$. It follows that K belongs to the circumcircle of CDJ if and only if $IJ \cdot IK = IF \cdot IE$. Expressing $IJ = IF + FJ$, $IK = IF + \frac{1}{2}FE$, and $IE = IF + FE$, we find that K is on the circumcircle of CDJ if and only if

$$FJ = \frac{IF \cdot FE}{2IF + FE}.$$

The conclusion follows.



Comment. While the figure shows B inside segment CE, it is possible that C is inside segment BE. Consequently, I would be inside segment EF and J outside segment EF. The position of point K on line EF with respect to points I, J may also vary.

Some case may require that an angle φ be replaced by $180° - \varphi$, and in computing distances, a sum may need to become a difference. All these cases can be covered by the proposed solution if it is clearly stated that *signed* distances and angles are used.

G3. Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

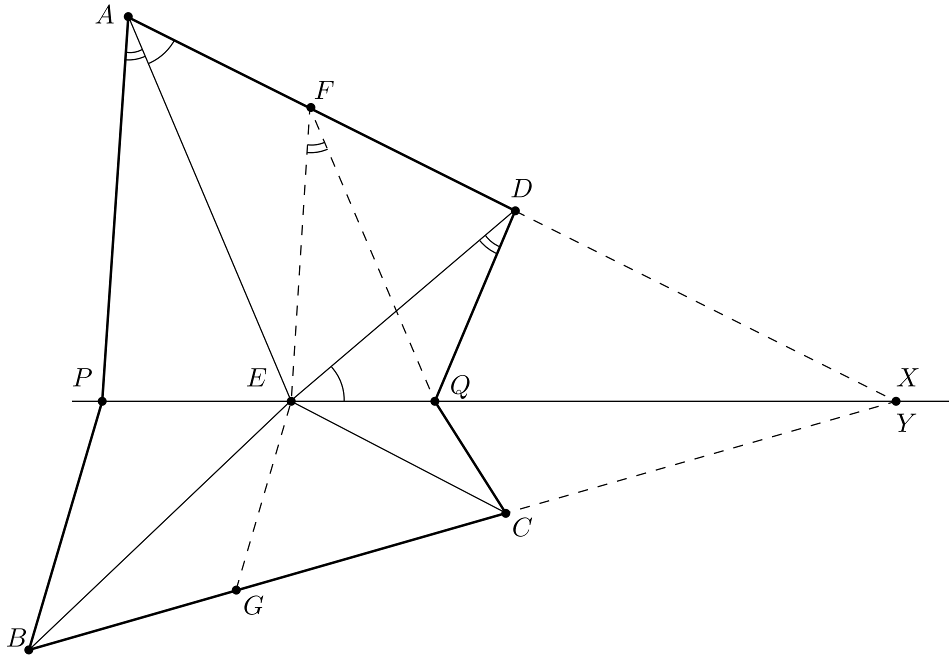
Solution 1. Let F be the point on the line AD such that $EF \| PA$. By hypothesis, the quadrilateral $PQDA$ is cyclic. So if F lies between A and D then $\angle EFD = \angle PAD = 180^\circ - \angle EQD$; the points F and Q are on distinct sides of the line DE and we infer that $EFDQ$ is a cyclic quadrilateral. And if D lies between A and F then a similar argument shows that $\angle EFD = \angle EQD$; but now the points F and Q lie on the same side of DE, so that $EDFQ$ is a cyclic quadrilateral.

In either case we obtain the equality $\angle EFQ = \angle EDQ = \angle PAE$ which implies that $FQ \| AE$. So the triangles EFQ and PAE are either homothetic or parallel-congruent. More specifically, triangle EFQ is the image of PAE under the mapping f which carries the points P, E respectively to E, Q and is either a homothety or translation by a vector. Note that f is uniquely determined by these conditions and the position of the points P, E, Q alone.

Let now G be the point on the line BC such that $EG \| PB$. The same reasoning as above applies to points B, C in place of A, D, implying that the triangle EGQ is the image of PBE under the same mapping f. So f sends the four points A, P, B, E respectively to F, E, G, Q.

If $PE \neq QE$, so that f is a homothety with a centre X, then the lines AF, PE, BG—i.e. the lines AD, PQ, BC—are concurrent at X. And since $PQDA$ and $QPBC$ are cyclic quadrilaterals, the equalities $XA \cdot XD = XP \cdot XQ = XB \cdot XC$ hold, showing that the quadrilateral $ABCD$ is cyclic.

Finally, if $PE = QE$, so that f is a translation, then $AD \| PQ \| BC$. Thus $PQDA$ and $QPBC$ are isosceles trapezoids. Then also $ABCD$ is an isosceles trapezoid, hence a cyclic quadrilateral.



Solution 2. Here is another way to reach the conclusion that the lines AD, BC and PQ are either concurrent or parallel. From the cyclic quadrilateral $PQDA$ we get

$$\angle PAD = 180^\circ - \angle PQD = \angle QDE + \angle QED = \angle PAE + \angle QED.$$

Hence $\angle QED = \angle PAD - \angle PAE = \angle EAD$. This in view of the tangent-chord theorem means that the circumcircle of triangle EAD is tangent to the line PQ at E. Analogously, the circumcircle of triangle EBC is tangent to PQ at E.

Suppose that the line AD intersects PQ at X. Since XE is tangent to the circle (EAD), $XE^2 = XA \cdot XD$. Also, $XA \cdot XD = XP \cdot XQ$ because P, Q, D, A lie on a circle. Therefore $XE^2 = XP \cdot XQ$.

It is not hard to see that this equation determines the position of the point X on the line PQ uniquely. Thus, if BC also cuts PQ, say at Y, then the analogous equation for Y yields $X = Y$, meaning that the three lines indeed concur. In this case, as well as in the case where $AD \| PQ \| BC$, the concluding argument is the same as in the first solution.

It remains to eliminate the possibility that e.g. AD meets PQ at X while $BC \| PQ$. Indeed, $QPBC$ would then be an isosceles trapezoid and the angle equality $\angle PBE = \angle QCE$ would force that E is the midpoint of PQ. So the length of XE, which is the geometric mean of the lengths of XP and XQ, should also be their arithmetic mean—impossible, as $XP \neq XQ$. The proof is now complete.

Comment. After reaching the conclusion that the circles (EDA) and (EBC) are tangent to PQ one may continue as follows. Denote the circles $(PQDA)$, (EDA), (EBC), $(QPBC)$ by ω_1, ω_2, ω_3, ω_4 respectively. Let ℓ_{ij} be the radical axis of the pair (ω_i, ω_j) for $i < j$. As is well-known, the lines ℓ_{12}, ℓ_{13}, ℓ_{23} concur, possibly at infinity (let this be the meaning of the word *concur* in this comment). So do the lines ℓ_{12}, ℓ_{14}, ℓ_{24}. Note however that ℓ_{23} and ℓ_{14} both coincide with the line PQ. Hence the pair ℓ_{12}, PQ is in both triples; thus the four lines ℓ_{12}, ℓ_{13}, ℓ_{24} and PQ are concurrent.

Similarly, ℓ_{13}, ℓ_{14}, ℓ_{34} concur, ℓ_{23}, ℓ_{24}, ℓ_{34} concur, and since $\ell_{14} = \ell_{23} = PQ$, the four lines ℓ_{13}, ℓ_{24}, ℓ_{34} and PQ are concurrent. The lines ℓ_{13} and ℓ_{24} are present in both quadruples, therefore all the lines ℓ_{ij} are concurrent. Hence the result.

G4. In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the points A and F are tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that the lines PE and QF intersect on the circumcircle of triangle AEF.

Solution 1. To approach the desired result we need some information about the slopes of the lines PE and QF; this information is provided by formulas (1) and (2) which we derive below.

The tangents BP and BQ to the two circles passing through A and F are equal, as $BP^2 = BA \cdot BF = BQ^2$. Consider the altitude AD of triangle ABC and its orthocentre H. From the cyclic quadrilaterals $CDFA$ and $CDHE$ we get $BA \cdot BF = BC \cdot BD = BE \cdot BH$. Thus $BP^2 = BE \cdot BH$, or $BP/BH = BE/BP$, implying that the triangles BPH and BEP are similar. Hence

$$\angle BPE = \angle BHP. \tag{1}$$

The point P lies between D and C; this follows from the equality $BP^2 = BC \cdot BD$. In view of this equality, and because $BP = BQ$,

$$DP \cdot DQ = (BP - BD) \cdot (BP + BD) = BP^2 - BD^2 = BD \cdot (BC - BD) = BD \cdot DC.$$

Also $AD \cdot DH = BD \cdot DC$, as is seen from the similar triangles BDH and ADC. Combining these equalities we obtain $AD \cdot DH = DP \cdot DQ$. Therefore $DH/DP = DQ/DA$, showing that the triangles HDP and QDA are similar. Hence $\angle HPD = \angle QAD$, which can be rewritten as $\angle BPH = \angle BAD + \angle BAQ$. And since BQ is tangent to the circumcircle of triangle FAQ,

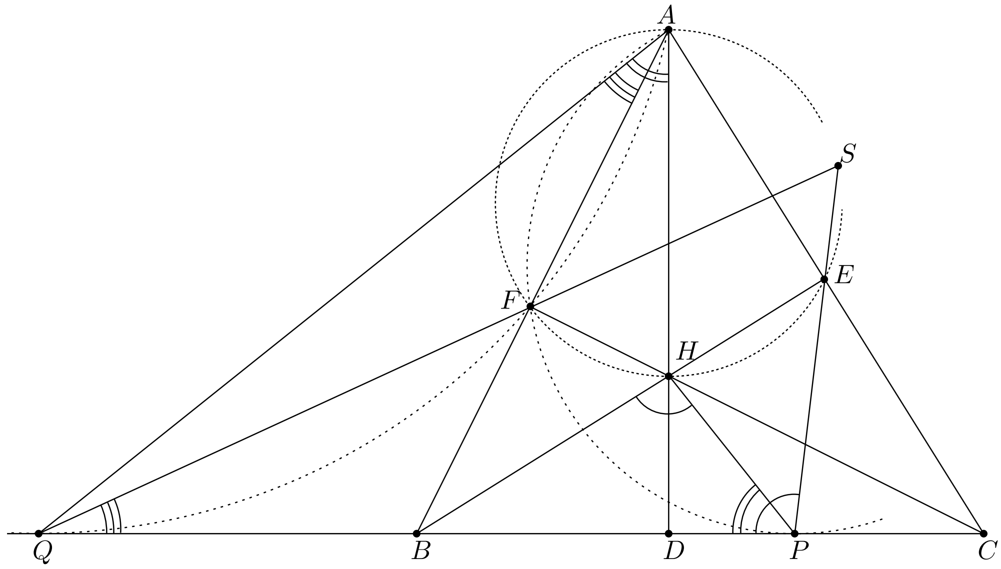
$$\angle BQF = \angle BAQ = \angle BPH - \angle BAD. \tag{2}$$

From (1) and (2) we deduce

$$\begin{aligned}\angle BPE + \angle BQF &= (\angle BHP + \angle BPH) - \angle BAD = (180^\circ - \angle PBH) - \angle BAD \\ &= (90^\circ + \angle BCA) - (90^\circ - \angle ABC) = \angle BCA + \angle ABC = 180^\circ - \angle CAB.\end{aligned}$$

Thus $\angle BPE + \angle BQF < 180^\circ$, which means that the rays PE and QF meet. Let S be the point of intersection. Then $\angle PSQ = 180^\circ - (\angle BPE + \angle BQF) = \angle CAB = \angle EAF$.

If S lies between P and E then $\angle PSQ = 180^\circ - \angle ESF$; and if E lies between P and S then $\angle PSQ = \angle ESF$. In either case the equality $\angle PSQ = \angle EAF$ which we have obtained means that S lies on the circumcircle of triangle AEF.



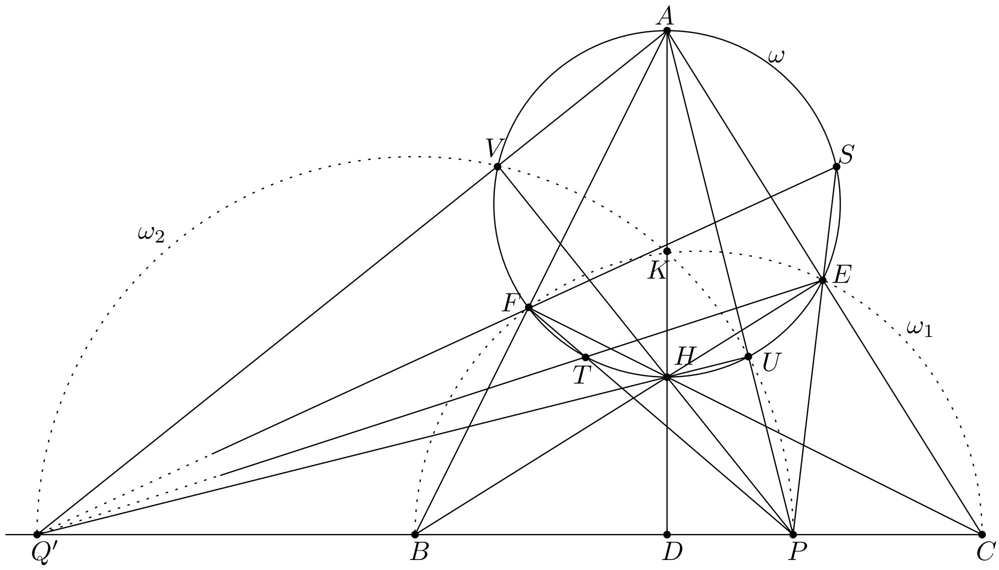
Solution 2. Let H be the orthocentre of triangle ABC and let ω be the circle with diameter AH, passing through E and F. Introduce the points of intersection of ω with the following lines emanating from P: $PA \cap \omega = \{A, U\}$, $PH \cap \omega = \{H, V\}$, $PE \cap \omega = \{E, S\}$. The altitudes of triangle AHP are contained in the lines AV, HU, BC, meeting at its orthocentre Q'.

By Pascal's theorem applied to the (tied) hexagon $AESFHV$, the points $AE \cap FH = C$, $ES \cap HV = P$ and $SF \cap VA$ are collinear, so FS passes through Q'.

Denote by ω_1 and ω_2 the circles with diameters BC and PQ', respectively. Let D be the foot of the altitude from A in triangle ABC. Suppose that AD meets the circles ω_1 and ω_2 at the respective points K and L.

Since H is the orthocentre of ABC, the triangles BDH and ADC are similar, and so $DA \cdot DH = DB \cdot DC = DK^2$; the last equality holds because BKC is a right triangle. Since H is the orthocentre also in triangle $AQ'P$, we analogously have $DL^2 = DA \cdot DH$. Therefore $DK = DL$ and $K = L$.

Also, $BD \cdot BC = BA \cdot BF$, from the similar triangles ABD, CBF. In the right triangle BKC we have $BK^2 = BD \cdot BC$. Hence, and because $BA \cdot BF = BP^2 = BQ^2$ (by the definition of P and Q in the problem statement), we obtain $BK = BP = BQ$. It follows that B is the centre of ω_2 and hence $Q' = Q$. So the lines PE and QF meet at the point S lying on the circumcircle of triangle AEF.



Comment 1. If T is the point defined by $PF \cap \omega = \{F, T\}$, Pascal's theorem for the hexagon $AFTEHV$ will analogously lead to the conclusion that the line ET goes through Q'. In other words, the lines PF and QE also concur on ω.

Comment 2. As is known from algebraic geometry, the points of the circle ω form a commutative groups with the operation defined as follows. Choose any point $0 \in \omega$ (to be the neutral element of the group) and a line ℓ exterior to the circle. For $X, Y \in \omega$, draw the line from the point $XY \cap \ell$ through 0 to its second intersection with ω and define this point to be $X + Y$.

In our solution we have chosen H to be the neutral element in this group and line BC to be ℓ. The fact that the lines AV, HU, ET, FS are concurrent can be deduced from the identities $A + A = 0$, $F = E + A$, $V = U + A = S + E = T + F$.

Comment 3. The problem was submitted in the following equivalent formulation:

Let BE and CF be altitudes of an acute triangle ABC. We choose P on the side BC and Q on the extension of CB beyond B such that $BQ^2 = BP^2 = BF \cdot AB$. If QF and PE intersect at S, prove that $ESAF$ is cyclic.

G5. Let k and n be integers with $0 \le k \le n-2$. Consider a set L of n lines in the plane such that no two of them are parallel and no three have a common point. Denote by I the set of intersection points of lines in L. Let O be a point in the plane not lying on any line of L.

A point $X \in I$ is colored red if the open line segment OX intersects at most k lines in L. Prove that I contains at least $\frac{1}{2}(k+1)(k+2)$ red points.

Solution. There are at least $\frac{1}{2}(k+1)(k+2)$ points in the intersection set I in view of the condition $n \ge k+2$.

For each point $P \in I$, define its *order* as the number of lines that intersect the open line segment OP. By definition, P is red if its order is at most k. Note that there is always at least one point $X \in I$ of order 0. Indeed, the lines in L divide the plane into regions, bounded or not, and O belongs to one of them. Clearly any corner of this region is a point of I with order 0.

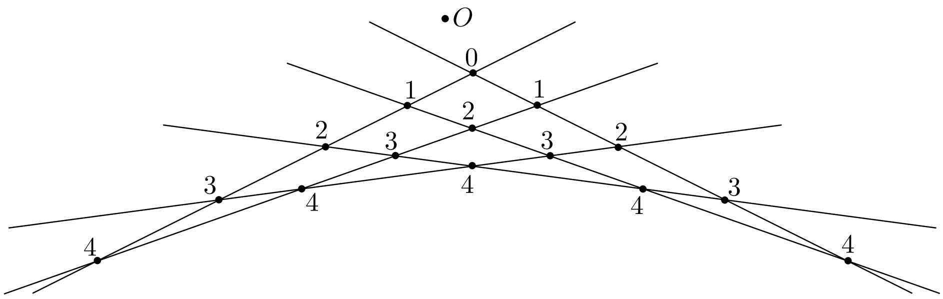
Claim. Suppose that two points $P, Q \in I$ lie on the same line of L, and no other line of L intersects the open line segment PQ. Then the orders of P and Q differ by at most 1.

Proof. Let P and Q have orders p and q, respectively, with $p \ge q$. Consider triangle OPQ. Now p equals the number of lines in L that intersect the interior of side OP. None of these lines intersects the interior of side PQ, and at most one can pass through Q. All remaining lines must intersect the interior of side OQ, implying that $q \ge p-1$. The conclusion follows.

We prove the main result by induction on k. The base $k=0$ is clear since there is a point of order 0 which is red. Assuming the statement true for $k-1$, we pass on to the inductive step. Select a point $P \in I$ of order 0, and consider one of the lines $\ell \in L$ that pass through P. There are $n-1$ intersection points on ℓ, one of which is P. Out of the remaining $n-2$ points, the k closest to P have orders not exceeding k by the Claim. It follows that there are at least $k+1$ red points on ℓ.

Let us now consider the situation with ℓ removed (together with all intersection points it contains). By hypothesis of induction, there are at least $\frac{1}{2}k(k+1)$ points of order not exceeding $k-1$ in the resulting configuration. Restoring ℓ back produces at most one new intersection point on each line segment joining any of these points to O, so their order is at most k in the original configuration. The total number of points with order not exceeding k is therefore at least $(k+1) + \frac{1}{2}k(k+1) = \frac{1}{2}(k+1)(k+2)$. This completes the proof.

Comment. The steps of the proof can be performed in reverse order to obtain a configuration of n lines such that equality holds simultaneously for all $0 \le k \le n-2$. Such a set of lines is illustrated in the Figure.



G6. There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ \qquad (1)$$

if and only if the diagonals AC and BD are perpendicular.

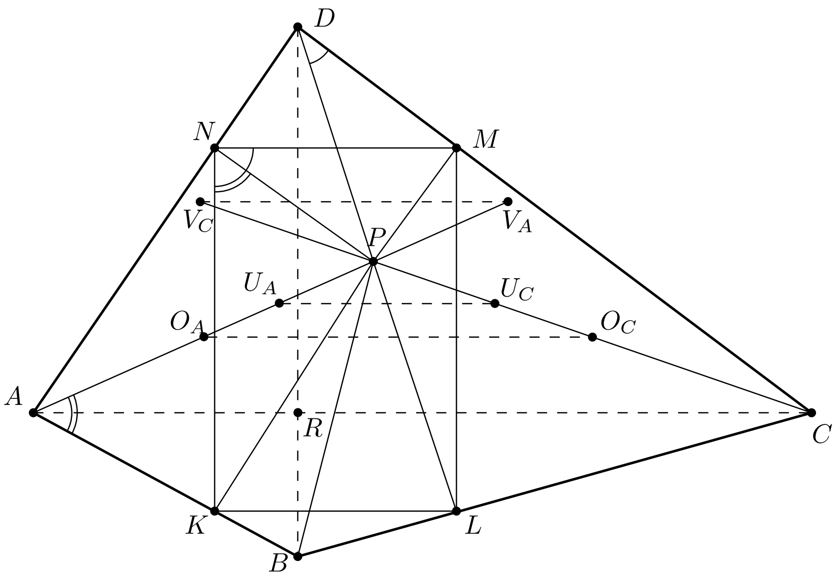
Solution 1. For a point P in $ABCD$ which satisfies (1), let K, L, M, N be the feet of perpendiculars from P to lines AB, BC, CD, DA, respectively. Note that K, L, M, N are interior to the sides as all angles in (1) are acute. The cyclic quadrilaterals $AKPN$ and $DNPM$ give

$$\angle PAB + \angle PDC = \angle PNK + \angle PNM = \angle KNM.$$

Analogously, $\angle PBC + \angle PAD = \angle LKN$ and $\angle PCD + \angle PBA = \angle MLK$. Hence the equalities (1) imply $\angle KNM = \angle LKN = \angle MLK = 90^\circ$, so that $KLMN$ is a rectangle. The converse also holds true, provided that K, L, M, N are interior to sides AB, BC, CD, DA.

(i) Suppose that there exists a point P in $ABCD$ such that $KLMN$ is a rectangle. We show that AC and BD are parallel to the respective sides of $KLMN$.

Let O_A and O_C be the circumcentres of the cyclic quadrilaterals $AKPN$ and $CMPL$. Line O_AO_C is the common perpendicular bisector of LM and KN, therefore O_AO_C is parallel to KL and MN. On the other hand, O_AO_C is the midline in the triangle ACP that is parallel to AC. Therefore the diagonal AC is parallel to the sides KL and MN of the rectangle. Likewise, BD is parallel to KN and LM. Hence AC and BD are perpendicular.



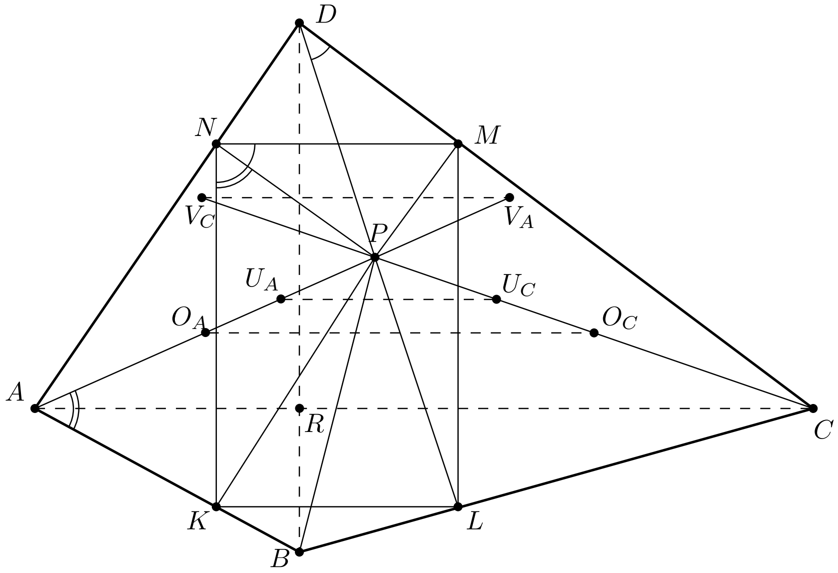
(ii) Suppose that AC and BD are perpendicular and meet at R. If $ABCD$ is a rhombus, P can be chosen to be its centre. So assume that $ABCD$ is not a rhombus, and let $BR < DR$ without loss of generality.

Denote by U_A and U_C the circumcentres of the triangles ABD and CDB, respectively. Let AV_A and CV_C be the diameters through A and C of the two circumcircles. Since AR is an altitude in triangle ADB, lines AC and AV_A are isogonal conjugates, i. e. $\angle DAV_A = \angle BAC$. Now $BR < DR$ implies that ray AU_A lies in $\angle DAC$. Similarly, ray CU_C lies in $\angle DCA$. Both diameters AV_A and CV_C intersect BD as the angles at B and D of both triangles are acute. Also U_AU_C is parallel to AC as it is the perpendicular bisector of BD. Hence V_AV_C is parallel to AC, too. We infer that AV_A and CV_C intersect at a point P inside triangle ACD, hence inside $ABCD$.

Construct points K, L, M, N, O_A and O_C in the same way as in the introduction. It follows from the previous paragraph that K, L, M, N are interior to the respective sides. Now O_AO_C is a midline in triangle ACP again. Therefore lines AC, O_AO_C and U_AU_C are parallel.

The cyclic quadrilateral $AKPN$ yields $\angle NKP = \angle NAP$. Since $\angle NAP = \angle DAU_A = \angle BAC$, as specified above, we obtain $\angle NKP = \angle BAC$. Because PK is perpendicular to AB, it follows that NK is perpendicular to AC, hence parallel to BD. Likewise, LM is parallel to BD.

Consider the two homotheties with centres A and C which transform triangles ABD and CDB into triangles AKN and CML, respectively. The images of points U_A and U_C are O_A and O_C, respectively. Since U_AU_C and O_AO_C are parallel to AC, the two ratios of homothety are the same, equal to $\lambda = AN/AD = AK/AB = AO_A/AU_A = CO_C/CU_C = CM/CD = CL/CB$. It is now straightforward that $DN/DA = DM/DC = BK/BA = BL/BC = 1-\lambda$. Hence KL and MN are parallel to AC, implying that $KLMN$ is a rectangle and completing the proof.



Solution 2. For a point P distinct from A, B, C, D, let circles (APD) and (BPC) intersect again at Q ($Q = P$ if the circles are tangent). Next, let circles (AQB) and (CQD) intersect again at R. We show that if P lies in $ABCD$ and satisfies (1) then AC and BD intersect at R and are perpendicular; the converse is also true. It is convenient to use directed angles. Let $\measuredangle(UV, XY)$ denote the angle of counterclockwise rotation that makes line UV parallel to line XY. Recall that four noncollinear points U, V, X, Y are concyclic if and only if $\measuredangle(UX, VX) = \measuredangle(UY, VY)$.

The definitions of points P, Q and R imply

$$\begin{aligned}
\measuredangle(AR, BR) &= \measuredangle(AQ, BQ) = \measuredangle(AQ, PQ) + \measuredangle(PQ, BQ) = \measuredangle(AD, PD) + \measuredangle(PC, BC),\\
\measuredangle(CR, DR) &= \measuredangle(CQ, DQ) = \measuredangle(CQ, PQ) + \measuredangle(PQ, DQ) = \measuredangle(CB, PB) + \measuredangle(PA, DA),\\
\measuredangle(BR, CR) &= \measuredangle(BR, RQ) + \measuredangle(RQ, CR) = \measuredangle(BA, AQ) + \measuredangle(DQ, CD)\\
&= \measuredangle(BA, AP) + \measuredangle(AP, AQ) + \measuredangle(DQ, DP) + \measuredangle(DP, CD)\\
&= \measuredangle(BA, AP) + \measuredangle(DP, CD).
\end{aligned}$$

Observe that the whole construction is reversible. One may start with point R, define Q as the second intersection of circles (ARB) and (CRD), and then define P as the second intersection of circles (AQD) and (BQC). The equalities above will still hold true.

Assume in addition that P is interior to $ABCD$. Then

$$\measuredangle(AD,PD)=\angle PDA,\ \measuredangle(PC,BC)=\angle PCB,\ \measuredangle(CB,PB)=\angle PBC,\ \measuredangle(PA,DA)=\angle PAD,$$
$$\measuredangle(BA,AP)=\angle PAB,\ \measuredangle(DP,CD)=\angle PDC.$$

(i) Suppose that P lies in $ABCD$ and satisfies (1). Then $\measuredangle(AR,BR)=\angle PDA+\angle PCB=90^\circ$ and similarly $\measuredangle(BR,CR)=\measuredangle(CR,DR)=90^\circ$. It follows that R is the common point of lines AC and BD, and that these lines are perpendicular.
(ii) Suppose that AC and BD are perpendicular and intersect at R. We show that the point P defined by the reverse construction (starting with R and ending with P) lies in $ABCD$. This is enough to finish the solution, because then the angle equalities above will imply (1).

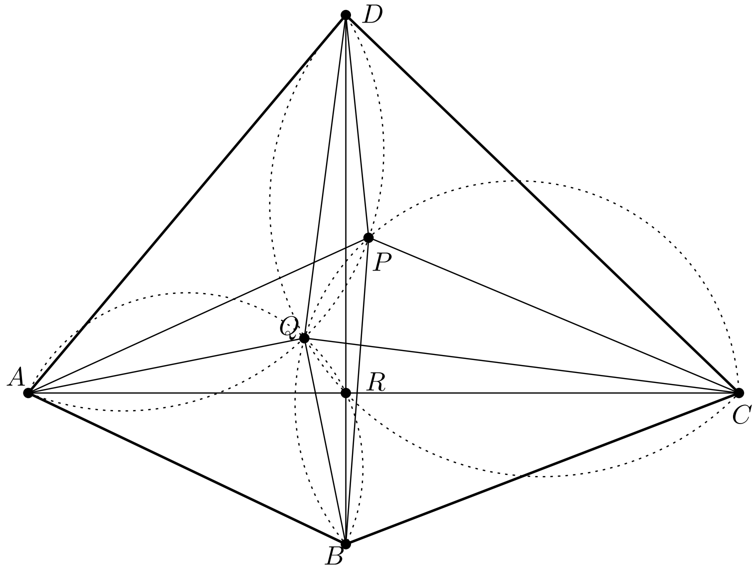
One can assume that Q, the second common point of circles (ABR) and (CDR), lies in $\angle ARD$. Then in fact Q lies in triangle ADR as angles AQR and DQR are obtuse. Hence $\angle AQD$ is obtuse, too, so that B and C are outside circle (ADQ) ($\angle ABD$ and $\angle ACD$ are acute).

Now $\angle CAB+\angle CDB=\angle BQR+\angle CQR=\angle CQB$ implies $\angle CAB<\angle CQB$ and $\angle CDB<\angle CQB$. Hence A and D are outside circle (BCQ). In conclusion, the second common point P of circles (ADQ) and (BCQ) lies on their arcs ADQ and BCQ.

We can assume that P lies in $\angle CQD$. Since

$$\angle QPC+\angle QPD=(180^\circ-\angle QBC)+(180^\circ-\angle QAD)=$$
$$=360^\circ-(\angle RBC+\angle QBR)-(\angle RAD-\angle QAR)=360^\circ-\angle RBC-\angle RAD>180^\circ,$$

point P lies in triangle CDQ, and hence in $ABCD$. The proof is complete.



G7. Let $ABCD$ be a convex quadrilateral with $AB \neq BC$. Denote by ω_1 and ω_2 the incircles of triangles ABC and ADC. Suppose that there exists a circle ω inscribed in angle ABC, tangent to the extensions of line segments AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

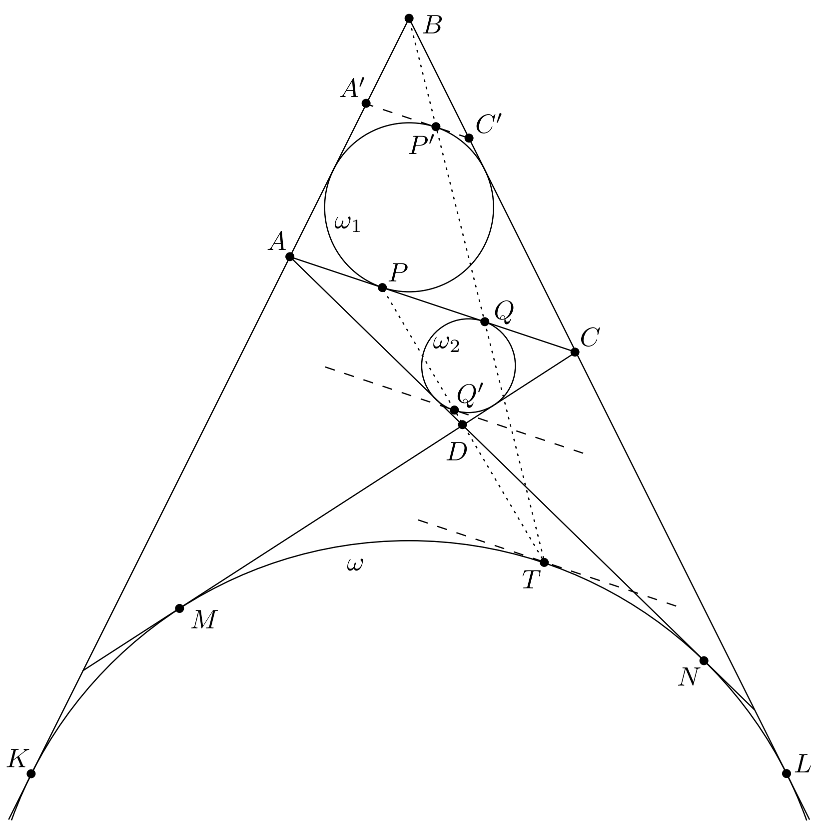
Solution. The proof below is based on two known facts.

Lemma 1. Given a convex quadrilateral $ABCD$, suppose that there exists a circle which is inscribed in angle ABC and tangent to the extensions of line segments AD and CD. Then $AB + AD = CB + CD$.

Proof. The circle in question is tangent to each of the lines AB, BC, CD, DA, and the respective points of tangency K, L, M, N are located as with circle ω in the figure. Then

$$AB + AD = (BK - AK) + (AN - DN), \quad CB + CD = (BL - CL) + (CM - DM).$$

Also $BK = BL$, $DN = DM$, $AK = AN$, $CL = CM$ by equalities of tangents. It follows that $AB + AD = CB + CD$.



For brevity, in the sequel we write "excircle AC" for the excircle of a triangle with side AC which is tangent to line segment AC and the extensions of the other two sides.

Lemma 2. The incircle of triangle ABC is tangent to its side AC at P. Let PP' be the diameter of the incircle through P, and let line BP' intersect AC at Q. Then Q is the point of tangency of side AC and excircle AC.

Proof. Let the tangent at P' to the incircle ω_1 meet BA and BC at A' and C'. Now ω_1 is the excircle $A'C'$ of triangle $A'BC'$, and it touches side $A'C'$ at P'. Since $A'C' \parallel AC$, the homothety with centre B and ratio BQ/BP' takes ω_1 to the excircle AC of triangle ABC. Because this homothety takes P' to Q, the lemma follows.

Recall also that if the incircle of a triangle touches its side AC at P, then the tangency point Q of the same side and excircle AC is the unique point on line segment AC such that $AP = CQ$.

We pass on to the main proof. Let ω_1 and ω_2 touch AC at P and Q, respectively; then $AP = (AC + AB - BC)/2$, $CQ = (CA + CD - AD)/2$. Since $AB - BC = CD - AD$ by Lemma 1, we obtain $AP = CQ$. It follows that in triangle ABC side AC and excircle AC are tangent at Q. Likewise, in triangle ADC side AC and excircle AC are tangent at P. Note that $P \neq Q$ as $AB \neq BC$.

Let PP' and QQ' be the diameters perpendicular to AC of ω_1 and ω_2, respectively. Then Lemma 2 shows that points B, P' and Q are collinear, and so are points D, Q' and P.

Consider the diameter of ω perpendicular to AC and denote by T its endpoint that is closer to AC. The homothety with centre B and ratio BT/BP' takes ω_1 to ω. Hence B, P' and T are collinear. Similarly, D, Q' and T are collinear since the homothety with centre D and ratio $-DT/DQ'$ takes ω_2 to ω.

We infer that points T, P' and Q are collinear, as well as T, Q' and P. Since $PP' \parallel QQ'$, line segments PP' and QQ' are then homothetic with centre T. The same holds true for circles ω_1 and ω_2 because they have PP' and QQ' as diameters. Moreover, it is immediate that T lies on the same side of line PP' as Q and Q', hence the ratio of homothety is positive. In particular ω_1 and ω_2 are not congruent.

In summary, T is the centre of a homothety with positive ratio that takes circle ω_1 to circle ω_2. This completes the solution, since the only point with the mentioned property is the intersection of the the common external tangents of ω_1 and ω_2.

Number Theory

N1. Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa,$$

then $a = b = c$.

Solution 1. If two of a, b, c are equal, it is immediate that all the three are equal. So we may assume that $a \neq b \neq c \neq a$. Subtracting the equations we get $a^n - b^n = -p(b-c)$ and two cyclic copies of this equation, which upon multiplication yield

$$\frac{a^n - b^n}{a-b} \cdot \frac{b^n - c^n}{b-c} \cdot \frac{c^n - a^n}{c-a} = -p^3. \tag{1}$$

If n is odd then the differences $a^n - b^n$ and $a - b$ have the same sign and the product on the left is positive, while $-p^3$ is negative. So n must be even.

Let d be the greatest common divisor of the three differences $a-b$, $b-c$, $c-a$, so that $a - b = du,\ \ b - c = dv,\ \ c - a = dw;\ \ \gcd(u, v, w) = 1,\ \ u + v + w = 0$.

From $a^n - b^n = -p(b-c)$ we see that $(a-b)|p(b-c)$, i.e., $u|pv$; and cyclically $v|pw$, $w|pu$. As $\gcd(u, v, w) = 1$ and $u + v + w = 0$, at most one of u, v, w can be divisible by p. Supposing that the prime p does not divide any one of them, we get $u|v$, $v|w$, $w|u$, whence $|u| = |v| = |w| = 1$; but this quarrels with $u + v + w = 0$.

Thus p must divide exactly one of these numbers. Let e.g. $p|u$ and write $u = pu_1$. Now we obtain, similarly as before, $u_1|v$, $v|w$, $w|u_1$ so that $|u_1| = |v| = |w| = 1$. The equation $pu_1 + v + w = 0$ forces that the prime p must be even; i.e. $p = 2$. Hence $v + w = -2u_1 = \pm 2$, implying $v = w\ (= \pm 1)$ and $u = -2v$. Consequently $a - b = -2(b - c)$.

Knowing that n is even, say $n = 2k$, we rewrite the equation $a^n - b^n = -p(b - c)$ with $p = 2$ in the form

$$(a^k + b^k)(a^k - b^k) = -2(b - c) = a - b.$$

The second factor on the left is divisible by $a - b$, so the first factor $(a^k + b^k)$ must be ± 1. Then exactly one of a and b must be odd; yet $a - b = -2(b - c)$ is even. Contradiction ends the proof.

Solution 2. The beginning is as in the first solution. Assuming that a, b, c are not all equal, hence are all distinct, we derive equation (1) with the conclusion that n is even. Write $n = 2k$.

Suppose that p is odd. Then the integer

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + b^{n-1},$$

which is a factor in (1), must be odd as well. This sum of $n = 2k$ summands is odd only if a and b have different parities. The same conclusion holding for b, c and for c, a, we get that a, b, c, a alternate in their parities, which is clearly impossible.

Thus $p = 2$. The original system shows that a, b, c must be of the same parity. So we may divide (1) by p^3, i.e. 2^3, to obtain the following product of six integer factors:

$$\frac{a^k + b^k}{2} \cdot \frac{a^k - b^k}{a - b} \cdot \frac{b^k + c^k}{2} \cdot \frac{b^k - c^k}{b - c} \cdot \frac{c^k + a^k}{2} \cdot \frac{c^k - a^k}{c - a} = -1. \tag{2}$$

Each one of the factors must be equal to ± 1. In particular, $a^k + b^k = \pm 2$. If k is even, this becomes $a^k + b^k = 2$ and yields $|a| = |b| = 1$, whence $a^k - b^k = 0$, contradicting (2).

Let now k be odd. Then the sum $a^k + b^k$, with value ± 2, has $a + b$ as a factor. Since a and b are of the same parity, this means that $a + b = \pm 2$; and cyclically, $b + c = \pm 2$, $c + a = \pm 2$. In some two of these equations the signs must coincide, hence some two of a, b, c are equal. This is the desired contradiction.

Comment. Having arrived at the equation (1) one is tempted to write down all possible decompositions of $-p^3$ (cube of a prime) into a product of three integers. This leads to cumbersome examination of many cases, some of which are unpleasant to handle. One may do that just for $p = 2$, having earlier in some way eliminated odd primes from consideration.

However, the second solution shows that the condition of p being a prime is far too strong. What is actually being used in that solution, is that p is either a positive odd integer or $p = 2$.

N2. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1, 3a_2, \ldots, 3a_n$.

Solution. Without loss of generality, let $0 < a_1 < a_2 < \cdots < a_n$. One can also assume that $a_1, a_2, \ldots, a_n$ are coprime. Otherwise division by their greatest common divisor reduces the question to the new sequence whose terms are coprime integers.

Suppose that the claim is false. Then for each $i < n$ there exists a j such that $a_n + a_i$ divides $3a_j$. If $a_n + a_i$ is not divisible by 3 then $a_n + a_i$ divides a_j which is impossible as $0 < a_j \leq a_n < a_n + a_i$. Thus $a_n + a_i$ is a multiple of 3 for $i = 1, \ldots, n-1$, so that $a_1, a_2, \ldots, a_{n-1}$ are all congruent (to $-a_n$) modulo 3.

Now a_n is not divisible by 3 or else so would be all remaining a_i's, meaning that $a_1, a_2, \ldots, a_n$ are not coprime. Hence $a_n \equiv r \pmod 3$ where $r \in \{1, 2\}$, and $a_i \equiv 3 - r \pmod 3$ for all $i = 1, \ldots, n-1$.

Consider a sum $a_{n-1} + a_i$ where $1 \leq i \leq n-2$. There is at least one such sum as $n \geq 3$. Let j be an index such that $a_{n-1} + a_i$ divides $3a_j$. Observe that $a_{n-1} + a_i$ is not divisible by 3 since $a_{n-1} + a_i \equiv 2a_i \not\equiv 0 \pmod 3$. It follows that $a_{n-1} + a_i$ divides a_j, in particular $a_{n-1} + a_i \leq a_j$. Hence $a_{n-1} < a_j \leq a_n$, implying $j = n$. So a_n is divisible by all sums $a_{n-1} + a_i$, $1 \leq i \leq n-2$. In particular $a_{n-1} + a_i \leq a_n$ for $i = 1, \ldots, n-2$.

Let j be such that $a_n + a_{n-1}$ divides $3a_j$. If $j \leq n-2$ then $a_n + a_{n-1} \leq 3a_j < a_j + 2a_{n-1}$. This yields $a_n < a_{n-1} + a_j$; however $a_{n-1} + a_j \leq a_n$ for $j \leq n-2$. Therefore $j = n-1$ or $j = n$.

For $j = n-1$ we obtain $3a_{n-1} = k(a_n + a_{n-1})$ with k an integer, and it is straightforward that $k = 1$ ($k \leq 0$ and $k \geq 3$ contradict $0 < a_{n-1} < a_n$; $k = 2$ leads to $a_{n-1} = 2a_n > a_{n-1}$). Thus $3a_{n-1} = a_n + a_{n-1}$, i. e. $a_n = 2a_{n-1}$.

Similarly, if $j = n$ then $3a_n = k(a_n + a_{n-1})$ for some integer k, and only $k = 2$ is possible. Hence $a_n = 2a_{n-1}$ holds true in both cases remaining, $j = n-1$ and $j = n$.

Now $a_n = 2a_{n-1}$ implies that the sum $a_{n-1} + a_1$ is strictly between $a_n/2$ and a_n. But a_{n-1} and a_1 are distinct as $n \geq 3$, so it follows from the above that $a_{n-1} + a_1$ divides a_n. This provides the desired contradiction.

N3. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \ge 2^n$ for all $n \ge 0$.

Solution. Since $a_i \ge \gcd(a_i, a_{i+1}) > a_{i-1}$, the sequence is strictly increasing. In particular $a_0 \ge 1$, $a_1 \ge 2$. For each $i \ge 1$ we also have $a_{i+1} - a_i \ge \gcd(a_i, a_{i+1}) > a_{i-1}$, and consequently $a_{i+1} \ge a_i + a_{i-1} + 1$. Hence $a_2 \ge 4$ and $a_3 \ge 7$. The equality $a_3 = 7$ would force equalities in the previous estimates, leading to $\gcd(a_2, a_3) = \gcd(4, 7) > a_1 = 2$, which is false. Thus $a_3 \ge 8$; the result is valid for $n = 0, 1, 2, 3$. These are the base cases for a proof by induction.

Take an $n \ge 3$ and assume that $a_i \ge 2^i$ for $i = 0, 1, \ldots, n$. We must show that $a_{n+1} \ge 2^{n+1}$. Let $\gcd(a_n, a_{n+1}) = d$. We know that $d > a_{n-1}$. The induction claim is reached immediately in the following cases:

$$\begin{array}{lll} \text{if } a_{n+1} \ge 4d & \text{then } a_{n+1} > 4a_{n-1} \ge 4 \cdot 2^{n-1} = 2^{n+1}\,; \\ \text{if } a_n \ge 3d & \text{then } a_{n+1} \ge a_n + d \ge 4d > 4a_{n-1} \ge 4 \cdot 2^{n-1} = 2^{n+1}\,; \\ \text{if } a_n = d & \text{then } a_{n+1} \ge a_n + d = 2a_n \ge 2 \cdot 2^n = 2^{n+1}. \end{array}$$

The only remaining possibility is that $a_n = 2d$ and $a_{n+1} = 3d$, which we assume for the sequel. So $a_{n+1} = \frac{3}{2}a_n$.

Let now $\gcd(a_{n-1}, a_n) = d'$; then $d' > a_{n-2}$. Write $a_n = md'$ (m an integer). Keeping in mind that $d' \le a_{n-1} < d$ and $a_n = 2d$, we get that $m \ge 3$. Also $a_{n-1} < d = \frac{1}{2}md'$, $a_{n+1} = \frac{3}{2}md'$. Again we single out the cases which imply the induction claim immediately:

$$\begin{array}{ll} \text{if } m \ge 6 & \text{then } a_{n+1} = \frac{3}{2}md' \ge 9d' > 9a_{n-2} \ge 9 \cdot 2^{n-2} > 2^{n+1}\,; \\ \text{if } 3 \le m \le 4 & \text{then } a_{n-1} < \frac{1}{2} \cdot 4d', \text{ and hence } a_{n-1} = d', \\ & \qquad a_{n+1} = \frac{3}{2}ma_{n-1} \ge \frac{3}{2} \cdot 3a_{n-1} \ge \frac{9}{2} \cdot 2^{n-1} > 2^{n+1}. \end{array}$$

So we are left with the case $m = 5$, which means that $a_n = 5d'$, $a_{n+1} = \frac{15}{2}d'$, $a_{n-1} < d = \frac{5}{2}d'$. The last relation implies that a_{n-1} is either d' or $2d'$. Anyway, $a_{n-1} | 2d'$.

The same pattern repeats once more. We denote $\gcd(a_{n-2}, a_{n-1}) = d''$; then $d'' > a_{n-3}$. Because d'' is a divisor of a_{n-1}, hence also of $2d'$, we may write $2d' = m'd''$ (m' an integer). Since $d'' \le a_{n-2} < d'$, we get $m' \ge 3$. Also, $a_{n-2} < d' = \frac{1}{2}m'd''$, $a_{n+1} = \frac{15}{2}d' = \frac{15}{4}m'd''$. As before, we consider the cases:

$$\begin{array}{ll} \text{if } m' \ge 5 & \text{then } a_{n+1} = \frac{15}{4}m'd'' \ge \frac{75}{4}d'' > \frac{75}{4}a_{n-3} \ge \frac{75}{4} \cdot 2^{n-3} > 2^{n+1}\,; \\ \text{if } 3 \le m' \le 4 & \text{then } a_{n-2} < \frac{1}{2} \cdot 4d'', \text{ and hence } a_{n-2} = d'', \\ & \qquad a_{n+1} = \frac{15}{4}m'a_{n-2} \ge \frac{15}{4} \cdot 3a_{n-2} \ge \frac{45}{4} \cdot 2^{n-2} > 2^{n+1}. \end{array}$$

Both of them have produced the induction claim. But now there are no cases left. Induction is complete; the inequality $a_n \ge 2^n$ holds for all n.

N4. Let n be a positive integer. Show that the numbers

$$\binom{2^n-1}{0}, \quad \binom{2^n-1}{1}, \quad \binom{2^n-1}{2}, \quad \ldots, \quad \binom{2^n-1}{2^{n-1}-1}$$

are congruent modulo 2^n to $1, 3, 5, \ldots, 2^n-1$ in some order.

Solution 1. It is well-known that all these numbers are odd. So the assertion that their remainders $(\bmod\ 2^n)$ make up a permutation of $\{1, 3, \ldots, 2^n-1\}$ is equivalent just to saying that these remainders are all distinct. We begin by showing that

$$\binom{2^n-1}{2k} + \binom{2^n-1}{2k+1} \equiv 0 \pmod{2^n} \quad \text{and} \quad \binom{2^n-1}{2k} \equiv (-1)^k \binom{2^{n-1}-1}{k} \pmod{2^n}. \quad (1)$$

The first relation is immediate, as the sum on the left is equal to $\binom{2^n}{2k+1} = \frac{2^n}{2k+1}\binom{2^n-1}{2k}$, hence is divisible by 2^n. The second relation:

$$\binom{2^n-1}{2k} = \prod_{j=1}^{2k} \frac{2^n-j}{j} = \prod_{i=1}^{k} \frac{2^n-(2i-1)}{2i-1} \cdot \prod_{i=1}^{k} \frac{2^{n-1}-i}{i} \equiv (-1)^k \binom{2^{n-1}-1}{k} \pmod{2^n}.$$

This prepares ground for a proof of the required result by induction on n. The base case $n=1$ is obvious. Assume the assertion is true for $n-1$ and pass to n, denoting $a_k = \binom{2^{n-1}-1}{k}$, $b_m = \binom{2^n-1}{m}$. The induction hypothesis is that all the numbers a_k $(0 \le k < 2^{n-2})$ are distinct $(\bmod\ 2^{n-1})$; the claim is that all the numbers b_m $(0 \le m < 2^{n-1})$ are distinct $(\bmod\ 2^n)$.

The congruence relations (1) are restated as

$$b_{2k} \equiv (-1)^k a_k \equiv -b_{2k+1} \pmod{2^n}. \quad (2)$$

Shifting the exponent in the first relation of (1) from n to $n-1$ we also have the congruence $a_{2i+1} \equiv -a_{2i} \pmod{2^{n-1}}$. We hence conclude:

If, for some $j, k < 2^{n-2}$, $a_k \equiv -a_j \pmod{2^{n-1}}$, then $\{j, k\} = \{2i, 2i+1\}$ for some i. (3)

This is so because in the sequence $(a_k\colon k < 2^{n-2})$ each term a_j is complemented to $0 \pmod{2^{n-1}}$ by only one other term a_k, according to the induction hypothesis.

From (2) we see that $b_{4i} \equiv a_{2i}$ and $b_{4i+3} \equiv a_{2i+1} \pmod{2^n}$. Let

$$M = \{m\colon 0 \le m < 2^{n-1},\ m \equiv 0 \text{ or } 3 \pmod 4\}, \quad L = \{l\colon 0 \le l < 2^{n-1},\ l \equiv 1 \text{ or } 2 \pmod 4\}.$$

The last two congruences take on the unified form

$$b_m \equiv a_{\lfloor m/2 \rfloor} \pmod{2^n} \quad \text{for all} \quad m \in M. \quad (4)$$

Thus all the numbers b_m for $m \in M$ are distinct $(\bmod\ 2^n)$ because so are the numbers a_k (they are distinct $(\bmod\ 2^{n-1})$, hence also $(\bmod\ 2^n)$).

Every $l \in L$ is paired with a unique $m \in M$ into a pair of the form $\{2k, 2k+1\}$. So (2) implies that also all the b_l for $l \in L$ are distinct $(\bmod\ 2^n)$. It remains to eliminate the possibility that $b_m \equiv b_l \pmod{2^n}$ for some $m \in M$, $l \in L$.

Suppose that such a situation occurs. Let $m' \in M$ be such that $\{m', l\}$ is a pair of the form $\{2k, 2k+1\}$, so that (see (2)) $b_{m'} \equiv -b_l \pmod{2^n}$. Hence $b_{m'} \equiv -b_m \pmod{2^n}$. Since both m' and m are in M, we have by (4) $b_{m'} \equiv a_j$, $b_m \equiv a_k \pmod{2^n}$ for $j = \lfloor m'/2 \rfloor$, $k = \lfloor m/2 \rfloor$.

Then $a_j \equiv -a_k \pmod{2^n}$. Thus, according to (3), $j = 2i$, $k = 2i+1$ for some i (or vice versa). The equality $a_{2i+1} \equiv -a_{2i} \pmod{2^n}$ now means that $\binom{2^{n-1}-1}{2i} + \binom{2^{n-1}-1}{2i+1} \equiv 0 \pmod{2^n}$. However, the sum on the left is equal to $\binom{2^{n-1}}{2i+1}$. A number of this form cannot be divisible by 2^n. This is a contradiction which concludes the induction step and proves the result.

Solution 2. We again proceed by induction, writing for brevity $N = 2^{n-1}$ and keeping notation $a_k = \binom{N-1}{k}$, $b_m = \binom{2N-1}{m}$. Assume that the result holds for the sequence $(a_0, a_1, a_2, \ldots, a_{N/2-1})$. In view of the symmetry $a_{N-1-k} = a_k$ this sequence is a permutation of $(a_0, a_2, a_4, \ldots, a_{N-2})$. So the induction hypothesis says that this latter sequence, taken (mod N), is a permutation of $(1, 3, 5, \ldots, N-1)$. Similarly, the induction claim is that $(b_0, b_2, b_4, \ldots, b_{2N-2})$, taken (mod $2N$), is a permutation of $(1, 3, 5, \ldots, 2N-1)$.

In place of the congruence relations (2) we now use the following ones,

$$b_{4i} \equiv a_{2i} \pmod N \qquad \text{and} \qquad b_{4i+2} \equiv b_{4i} + N \pmod{2N}. \tag{5}$$

Given this, the conclusion is immediate: the first formula of (5) together with the induction hypothesis tells us that $(b_0, b_4, b_8, \ldots, b_{2N-4})$ (mod N) is a permutation of $(1, 3, 5, \ldots, N-1)$. Then the second formula of (5) shows that $(b_2, b_6, b_{10}, \ldots, b_{2N-2})$ (mod N) is exactly the same permutation; moreover, this formula distinguishes (mod $2N$) each b_{4i} from b_{4i+2}.

Consequently, these two sequences combined represent (mod $2N$) a permutation of the sequence $(1, 3, 5, \ldots, N-1,\ N+1, N+3, N+5, \ldots, N+N-1)$, and this is precisely the induction claim.

Now we prove formulas (5); we begin with the second one. Since $b_{m+1} = b_m \cdot \frac{2N-m-1}{m+1}$,

$$b_{4i+2} = b_{4i} \cdot \frac{2N-4i-1}{4i+1} \cdot \frac{2N-4i-2}{4i+2} = b_{4i} \cdot \frac{2N-4i-1}{4i+1} \cdot \frac{N-2i-1}{2i+1}.$$

The desired congruence $b_{4i+2} \equiv b_{4i} + N$ may be multiplied by the odd number $(4i+1)(2i+1)$, giving rise to a chain of successively equivalent congruences:

$$\begin{aligned}
b_{4i}(2N-4i-1)(N-2i-1) &\equiv (b_{4i}+N)(4i+1)(2i+1) &&\pmod{2N},\\
b_{4i}(2i+1-N) &\equiv (b_{4i}+N)(2i+1) &&\pmod{2N},\\
(b_{4i}+2i+1)N &\equiv 0 &&\pmod{2N};
\end{aligned}$$

and the last one is satisfied, as b_{4i} is odd. This settles the second relation in (5).

The first one is proved by induction on i. It holds for $i = 0$. Assume $b_{4i} \equiv a_{2i}$ (mod $2N$) and consider $i+1$:

$$b_{4i+4} = b_{4i+2} \cdot \frac{2N-4i-3}{4i+3} \cdot \frac{2N-4i-4}{4i+4}; \qquad a_{2i+2} = a_{2i} \cdot \frac{N-2i-1}{2i+1} \cdot \frac{N-2i-2}{2i+2}.$$

Both expressions have the fraction $\frac{N-2i-2}{2i+2}$ as the last factor. Since $2i+2 < N = 2^{n-1}$, this fraction reduces to ℓ/m with ℓ and m odd. In showing that $b_{4i+4} \equiv a_{2i+2}$ (mod $2N$), we may ignore this common factor ℓ/m. Clearing other odd denominators reduces the claim to

$$b_{4i+2}(2N-4i-3)(2i+1) \equiv a_{2i}(N-2i-1)(4i+3) \pmod{2N}.$$

By the inductive assumption (saying that $b_{4i} \equiv a_{2i}$ (mod $2N$)) and by the second relation of (5), this is equivalent to

$$(b_{4i}+N)(2i+1) \equiv b_{4i}(2i+1-N) \pmod{2N},$$

a congruence which we have already met in the preceding proof a few lines above. This completes induction (on i) and the proof of (5), hence also the whole solution.

Comment. One can avoid the words *congruent modulo* in the problem statement by rephrasing the assertion into: *Show that these numbers leave distinct remainders in division by* 2^n.

N5. For every $n \in \mathbb{N}$ let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ with the following properties:

(i) $d(f(x)) = x$ for all $x \in \mathbb{N}$;

(ii) $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all $x, y \in \mathbb{N}$.

Solution. There is a unique solution: the function $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(1) = 1$ and

$$f(n) = p_1^{p_1^{a_1}-1} p_2^{p_2^{a_2}-1} \cdots p_k^{p_k^{a_k}-1} \text{ where } n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \text{ is the prime factorization of } n > 1. \quad (1)$$

Direct verification shows that this function meets the requirements.

Conversely, let $f\colon \mathbb{N} \to \mathbb{N}$ satisfy (i) and (ii). Applying (i) for $x = 1$ gives $d(f(1)) = 1$, so $f(1) = 1$. In the sequel we prove that (1) holds for all $n > 1$. Notice that $f(m) = f(n)$ implies $m = n$ in view of (i). The formula $d\left(p_1^{b_1} \cdots p_k^{b_k}\right) = (b_1 + 1) \cdots (b_k + 1)$ will be used throughout.

Let p be a prime. Since $d(f(p)) = p$, the formula just mentioned yields $f(p) = q^{p-1}$ for some prime q; in particular $f(2) = q^{2-1} = q$ is a prime. We prove that $f(p) = p^{p-1}$ for all primes p.

Suppose that p is odd and $f(p) = q^{p-1}$ for a prime q. Applying (ii) first with $x = 2$, $y = p$ and then with $x = p$, $y = 2$ shows that $f(2p)$ divides both $(2-1)p^{2p-1}f(2) = p^{2p-1}f(2)$ and $(p-1)2^{2p-1}f(p) = (p-1)2^{2p-1}q^{p-1}$. If $q \neq p$ then the odd prime p does not divide $(p-1)2^{2p-1}q^{p-1}$, hence the greatest common divisor of $p^{2p-1}f(2)$ and $(p-1)2^{2p-1}q^{p-1}$ is a divisor of $f(2)$. Thus $f(2p)$ divides $f(2)$ which is a prime. As $f(2p) > 1$, we obtain $f(2p) = f(2)$ which is impossible. So $q = p$, i. e. $f(p) = p^{p-1}$.

For $p = 2$ the same argument with $x = 2$, $y = 3$ and $x = 3$, $y = 2$ shows that $f(6)$ divides both $3^5 f(2)$ and $2^6 f(3) = 2^6 3^2$. If the prime $f(2)$ is odd then $f(6)$ divides $3^2 = 9$, so $f(6) \in \{1, 3, 9\}$. However then $6 = d(f(6)) \in \{d(1), d(3), d(9)\} = \{1, 2, 3\}$ which is false. In conclusion $f(2) = 2$.

Next, for each $n > 1$ the prime divisors of $f(n)$ are among the ones of n. Indeed, let p be the least prime divisor of n. Apply (ii) with $x = p$ and $y = n/p$ to obtain that $f(n)$ divides $(p-1)y^{n-1}f(p) = (p-1)y^{n-1}p^{p-1}$. Write $f(n) = \ell P$ where ℓ is coprime to n and P is a product of primes dividing n. Since ℓ divides $(p-1)y^{n-1}p^{p-1}$ and is coprime to $y^{n-1}p^{p-1}$, it divides $p-1$; hence $d(\ell) \leq \ell < p$. But (i) gives $n = d(f(n)) = d(\ell P)$, and $d(\ell P) = d(\ell)d(P)$ as ℓ and P are coprime. Therefore $d(\ell)$ is a divisor of n less than p, meaning that $\ell = 1$ and proving the claim.

Now (1) is immediate for prime powers. If p is a prime and $a \geq 1$, by the above the only prime factor of $f(p^a)$ is p (a prime factor does exist as $f(p^a) > 1$). So $f(p^a) = p^b$ for some $b \geq 1$, and (i) yields $p^a = d(f(p^a)) = d\left(p^b\right) = b + 1$. Hence $f(p^a) = p^{p^a - 1}$, as needed.

Let us finally show that (1) is true for a general $n > 1$ with prime factorization $n = p_1^{a_1} \cdots p_k^{a_k}$. We saw that the prime factorization of $f(n)$ has the form $f(n) = p_1^{b_1} \cdots p_k^{b_k}$. For $i = 1, \ldots, k$, set $x = p_i^{a_i}$ and $y = n/x$ in (ii) to infer that $f(n)$ divides $(p_i^{a_i} - 1)\, y^{n-1} f(p_i^{a_i})$. Hence $p_i^{b_i}$ divides $(p_i^{a_i} - 1)\, y^{n-1} f(p_i^{a_i})$, and because $p_i^{b_i}$ is coprime to $(p_i^{a_i} - 1)\, y^{n-1}$, it follows that $p_i^{b_i}$ divides $f(p_i^{a_i}) = p_i^{p_i^{a_i}-1}$. So $b_i \leq p_i^{a_i} - 1$ for all $i = 1, \ldots, k$. Combined with (i), these conclusions imply

$$p_1^{a_1} \cdots p_k^{a_k} = n = d(f(n)) = d\left(p_1^{b_1} \cdots p_k^{b_k}\right) = (b_1 + 1) \cdots (b_k + 1) \leq p_1^{a_1} \cdots p_k^{a_k}.$$

Hence all inequalities $b_i \leq p_i^{a_i} - 1$ must be equalities, $i = 1, \ldots, k$, implying that (1) holds true. The proof is complete.

N6. Prove that there exist infinitely many positive integers n such that n^2+1 has a prime divisor greater than $2n+\sqrt{2n}$.

Solution. Let $p \equiv 1 \pmod 8$ be a prime. The congruence $x^2 \equiv -1 \pmod p$ has two solutions in $[1, p-1]$ whose sum is p. If n is the smaller one of them then p divides n^2+1 and $n \leq (p-1)/2$. We show that $p > 2n+\sqrt{10n}$.

Let $n = (p-1)/2 - \ell$ where $\ell \geq 0$. Then $n^2 \equiv -1 \pmod p$ gives

$$\left(\frac{p-1}{2}-\ell\right)^2 \equiv -1 \pmod p \qquad \text{or} \qquad (2\ell+1)^2+4 \equiv 0 \pmod p.$$

Thus $(2\ell+1)^2+4 = rp$ for some $r \geq 0$. As $(2\ell+1)^2 \equiv 1 \equiv p \pmod 8$, we have $r \equiv 5 \pmod 8$, so that $r \geq 5$. Hence $(2\ell+1)^2+4 \geq 5p$, implying $\ell \geq \left(\sqrt{5p-4}-1\right)/2$. Set $\sqrt{5p-4} = u$ for clarity; then $\ell \geq (u-1)/2$. Therefore

$$n = \frac{p-1}{2} - \ell \leq \frac{1}{2}(p-u).$$

Combined with $p = (u^2+4)/5$, this leads to $u^2 - 5u - 10n + 4 \geq 0$. Solving this quadratic inequality with respect to $u \geq 0$ gives $u \geq \left(5+\sqrt{40n+9}\right)/2$. So the estimate $n \leq \left(p-u\right)/2$ leads to

$$p \geq 2n + u \geq 2n + \frac{1}{2}\left(5+\sqrt{40n+9}\right) > 2n + \sqrt{10n}.$$

Since there are infinitely many primes of the form $8k+1$, it follows easily that there are also infinitely many n with the stated property.

Comment. By considering the prime factorization of the product $\prod_{n=1}^{N}(n^2+1)$, it can be obtained that its greatest prime divisor is at least $cN\log N$. This could improve the statement as $p > n\log n$.

However, the proof applies some advanced information about the distribution of the primes of the form $4k+1$, which is inappropriate for high schools contests.

50th International Mathematical Olympiad

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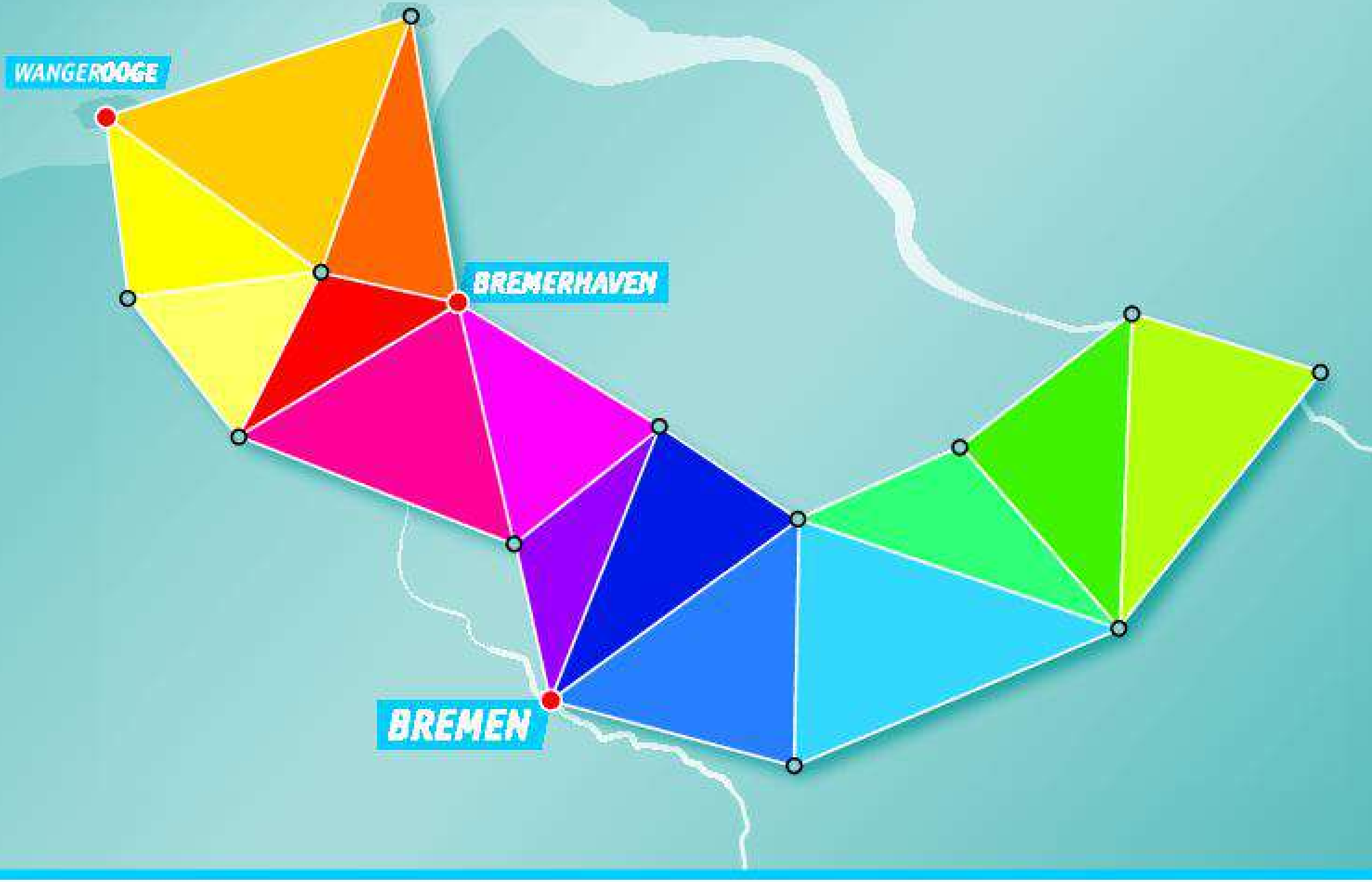
Problem Shortlist with solutions



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Problem Shortlist with Solutions

The Problem Selection Committee

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We insistently ask everybody to consider the following IMO Regulations rule:

These Shortlist Problems have to be kept strictly confidential until IMO 2010.

The Problem Selection Committee

Konrad Engel, Karl Fegert, Andreas Felgenhauer, Hans-Dietrich Gronau, Roger Labahn, Bernd Mulansky, Jürgen Prestin, Christian Reiher, Peter Scholze, Eckard Specht, Robert Strich, Martin Welk

gratefully received

132 problem proposals submitted by 39 countries:

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Layout: Roger Labahn with LaTeX & TeX
Drawings: Eckard Specht with `nicefig 2.0`

: The Problem Selection Committee

Contents

Algebra

A1 CZE (Czech Republic)

Find the largest possible integer k, such that the following statement is true:

Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are colored, such that one is blue, one is red and one is white. Now, for every color separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{rll} & b_1 \leq b_2 \leq \ldots \leq b_{2009} & \text{the lengths of the blue sides,} \\ & r_1 \leq r_2 \leq \ldots \leq r_{2009} & \text{the lengths of the red sides,} \\ \text{and} & w_1 \leq w_2 \leq \ldots \leq w_{2009} & \text{the lengths of the white sides.} \end{array}$$

Then there exist k indices j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j.

A2 EST (Estonia)

Let a, b, c be positive real numbers such that $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=a+b+c$. Prove that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2} \leq \frac{3}{16}.$$

A3 FRA (France)

Determine all functions f from the set of positive integers into the set of positive integers such that for all x and y there exists a non degenerated triangle with sides of lengths

$$x, \quad f(y) \quad \text{and} \quad f(y+f(x)-1).$$

A4 BLR (Belarus)

Let a, b, c be positive real numbers such that $ab+bc+ca \leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3 \leq \sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right).$$

A5 BLR (Belarus)

Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f\left(x-f(y)\right) > yf(x)+x.$$

A6 USA (United States of America)

Suppose that $s_1, s_2, s_3, \dots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \dots \qquad \text{and} \qquad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \dots$$

are both arithmetic progressions. Prove that $s_1, s_2, s_3, \dots$ is itself an arithmetic progression.

A7 JPN (Japan)

Find all functions f from the set of real numbers into the set of real numbers which satisfy for all real x, y the identity

$$f(xf(x+y)) = f(yf(x)) + x^2.$$

Combinatorics

C1 NZL (New Zealand)

Consider 2009 cards, each having one gold side and one black side, lying in parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

C2 ROU (Romania)

For any integer $n \geq 2$, let $N(n)$ be the maximal number of triples (a_i, b_i, c_i), $i = 1, \ldots, N(n)$, consisting of nonnegative integers a_i, b_i and c_i such that the following two conditions are satisfied:

(1) $a_i + b_i + c_i = n$ for all $i = 1, \ldots, N(n)$,

(2) If $i \neq j$, then $a_i \neq a_j, b_i \neq b_j$ and $c_i \neq c_j$.

Determine $N(n)$ for all $n \geq 2$.

Comment. The original problem was formulated for m-tuples instead for triples. The numbers $N(m, n)$ are then defined similarly to $N(n)$ in the case $m = 3$. The numbers $N(3, n)$ and $N(n, n)$ should be determined. The case $m = 3$ is the same as in the present problem. The upper bound for $N(n, n)$ can be proved by a simple generalization. The construction of a set of triples attaining the bound can be easily done by induction from n to $n + 2$.

C3 RUS (Russian Federation)

Let n be a positive integer. Given a sequence $\varepsilon_1, \ldots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \ldots, n-1$, the sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \qquad \text{for each } i = 1, \ldots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \qquad \text{for each } i = 1, \ldots, n-1.$$

Prove that $a_n = b_n$.

C4 NLD (Netherlands)

For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of the chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

C5 NLD (Netherlands)

Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

C6 BGR (Bulgaria)

On a 999×999 board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.

How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

C7 RUS (Russian Federation)

Variant 1. A grasshopper jumps along the real axis. He starts at point 0 and makes 2009 jumps to the right with lengths $1, 2, \ldots, 2009$ in an arbitrary order. Let M be a set of 2008 positive integers less than $1005 \cdot 2009$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

Variant 2. Let n be a nonnegative integer. A grasshopper jumps along the real axis. He starts at point 0 and makes $n+1$ jumps to the right with pairwise different positive integral lengths $a_1, a_2, \ldots, a_{n+1}$ in an arbitrary order. Let M be a set of n positive integers in the interval $(0, s)$, where $s = a_1 + a_2 + \cdots + a_{n+1}$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

C8 AUT (Austria)

For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let r be the rightmost digit of n.

(1) If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of n by removing this rightmost digit 0.

(2) If $1 \leq r \leq 9$ we split the decimal representation of n into a maximal right part R that solely consists of digits not less than r and into a left part L that either is empty or ends with a digit strictly smaller than r. Then the decimal representation of $h(n)$ consists of the decimal representation of L, followed by two copies of the decimal representation of $R-1$. For instance, for the number $n = 17{,}151{,}345{,}543$, we will have $L = 17{,}151$, $R = 345{,}543$ and $h(n) = 17{,}151{,}345{,}542{,}345{,}542$.

Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of h produces the integer 1 after finitely many steps.

Geometry

G1 BEL (Belgium)

Let ABC be a triangle with $AB = AC$. The angle bisectors of A and B meet the sides BC and AC in D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle BAC$.

G2 RUS (Russian Federation)

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. The circle k passes through the midpoints of the segments BP, CQ, and PQ. Prove that if the line PQ is tangent to circle k then $OP = OQ$.

G3 IRN (Islamic Republic of Iran)

Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms.

Prove that $GR = GS$.

G4 UNK (United Kingdom)

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G, and H.

G5 POL (Poland)

Let P be a polygon that is convex and symmetric to some point O. Prove that for some parallelogram R satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets R and P, respectively.

G6 UKR (Ukraine)

Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are the circumcenters and points H_1 and H_2 are the orthocenters of triangles ABP and DCP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the line H_1H_2 are concurrent.

G7 IRN (Islamic Republic of Iran)

Let ABC be a triangle with incenter I and let X, Y and Z be the incenters of the triangles BIC, CIA and AIB, respectively. Let the triangle XYZ be equilateral. Prove that ABC is equilateral too.

G8 BGR (Bulgaria)

Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2, and I_3 the incenters of $\triangle ABM$, $\triangle MNC$, and $\triangle NDA$, respectively. Show that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

Number Theory

N1 AUS (Australia)

A social club has n members. They have the membership numbers $1, 2, \dots, n$, respectively. From time to time members send presents to other members, including items they have already received as presents from other members. In order to avoid the embarrassing situation that a member might receive a present that he or she has sent to other members, the club adds the following rule to its statutes at one of its annual general meetings:

"A member with membership number a is permitted to send a present to a member with membership number b if and only if $a(b-1)$ is a multiple of n."

Prove that, if each member follows this rule, none will receive a present from another member that he or she has already sent to other members.

Alternative formulation: Let G be a directed graph with n vertices $v_1, v_2, \dots, v_n$, such that there is an edge going from v_a to v_b if and only if a and b are distinct and $a(b-1)$ is a multiple of n. Prove that this graph does not contain a directed cycle.

N2 PER (Peru)

A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x + a)(x + b)$.

(a) Prove that there exist distinct positive integers a and b such that all the numbers $P(1), P(2), \dots, P(50)$ are balanced.

(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

N3 EST (Estonia)

Let f be a non-constant function from the set of positive integers into the set of positive integers, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

N4 PRK (Democratic People's Republic of Korea)

Find all positive integers n such that there exists a sequence of positive integers $a_1, a_2, \dots, a_n$ satisfying

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \le k \le n - 1$.

N5 HUN (Hungary)

Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \ge 1$, where T^n denotes the n-fold application of T.

N6 TUR (Turkey)

Let k be a positive integer. Show that if there exists a sequence $a_0, a_1, \ldots$ of integers satisfying the condition

$$a_n = \frac{a_{n-1} + n^k}{n} \quad \text{for all } n \geq 1,$$

then $k - 2$ is divisible by 3.

N7 MNG (Mongolia)

Let a and b be distinct integers greater than 1. Prove that there exists a positive integer n such that $(a^n - 1)(b^n - 1)$ is not a perfect square.

Algebra

A1 CZE (Czech Republic)

Find the largest possible integer k, such that the following statement is true:

Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are colored, such that one is blue, one is red and one is white. Now, for every color separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{lll} & b_1 \leq b_2 \leq \ldots \leq b_{2009} & \text{the lengths of the blue sides,} \\ & r_1 \leq r_2 \leq \ldots \leq r_{2009} & \text{the lengths of the red sides,} \\ \text{and} & w_1 \leq w_2 \leq \ldots \leq w_{2009} & \text{the lengths of the white sides.} \end{array}$$

Then there exist k indices j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j.

Solution. We will prove that the largest possible number k of indices satisfying the given condition is one.

Firstly we prove that b_{2009}, r_{2009}, w_{2009} are always lengths of the sides of a triangle. Without loss of generality we may assume that $w_{2009} \geq r_{2009} \geq b_{2009}$. We show that the inequality $b_{2009} + r_{2009} > w_{2009}$ holds. Evidently, there exists a triangle with side lengths w, b, r for the white, blue and red side, respectively, such that $w_{2009} = w$. By the conditions of the problem we have $b + r > w$, $b_{2009} \geq b$ and $r_{2009} \geq r$. From these inequalities it follows

$$b_{2009} + r_{2009} \geq b + r > w = w_{2009}.$$

Secondly we will describe a sequence of triangles for which w_j, b_j, r_j with $j < 2009$ are not the lengths of the sides of a triangle. Let us define the sequence Δ_j, $j = 1, 2, \ldots, 2009$, of triangles, where Δ_j has

a blue side of length $2j$,

a red side of length j for all $j \leq 2008$ and 4018 for $j = 2009$,

and a white side of length $j + 1$ for all $j \leq 2007$, 4018 for $j = 2008$ and 1 for $j = 2009$.

Since

$$\begin{array}{rll} (j+1) + j > 2j \quad \geq j + 1 > j, & \text{if} & j \leq 2007, \\ 2j + j > 4018 > 2j \quad > j, & \text{if} & j = 2008, \\ 4018 + 1 > 2j \quad = 4018 > 1, & \text{if} & j = 2009, \end{array}$$

such a sequence of triangles exists. Moreover, $w_j = j$, $r_j = j$ and $b_j = 2j$ for $1 \leq j \leq 2008$. Then

$$w_j + r_j = j + j = 2j = b_j,$$

i.e., b_j, r_j and w_j are not the lengths of the sides of a triangle for $1 \leq j \leq 2008$.

A2 EST (Estonia)

Let a, b, c be positive real numbers such that $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=a+b+c$. Prove that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2}\leq\frac{3}{16}.$$

Solution 1. For positive real numbers x, y, z, from the arithmetic-geometric-mean inequality,

$$2x+y+z=(x+y)+(x+z)\geq 2\sqrt{(x+y)(x+z)},$$

we obtain

$$\frac{1}{(2x+y+z)^2}\leq\frac{1}{4(x+y)(x+z)}.$$

Applying this to the left-hand side terms of the inequality to prove, we get

$$\begin{aligned}&\frac{1}{(2a+b+c)^2}+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2}\\ &\qquad\leq\frac{1}{4(a+b)(a+c)}+\frac{1}{4(b+c)(b+a)}+\frac{1}{4(c+a)(c+b)}\\ &\qquad=\frac{(b+c)+(c+a)+(a+b)}{4(a+b)(b+c)(c+a)}=\frac{a+b+c}{2(a+b)(b+c)(c+a)}.\end{aligned}\tag{1}$$

A second application of the inequality of the arithmetic-geometric mean yields

$$a^2b+a^2c+b^2a+b^2c+c^2a+c^2b\geq 6abc,$$

or, equivalently,

$$9(a+b)(b+c)(c+a)\geq 8(a+b+c)(ab+bc+ca).\tag{2}$$

The supposition $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$ can be written as

$$ab+bc+ca=abc(a+b+c).\tag{3}$$

Applying the arithmetic-geometric-mean inequality $x^2y^2+x^2z^2\geq 2x^2yz$ thrice, we get

$$a^2b^2+b^2c^2+c^2a^2\geq a^2bc+ab^2c+abc^2,$$

which is equivalent to

$$(ab+bc+ca)^2\geq 3abc(a+b+c).\tag{4}$$

Combining (1), (2), (3), and (4), we will finish the proof:

$$\begin{aligned}\frac{a+b+c}{2(a+b)(b+c)(c+a)} &= \frac{(a+b+c)(ab+bc+ca)}{2(a+b)(b+c)(c+a)} \cdot \frac{ab+bc+ca}{abc(a+b+c)} \cdot \frac{abc(a+b+c)}{(ab+bc+ca)^2} \\ &\le \frac{9}{2\cdot 8}\cdot 1 \cdot \frac{1}{3} = \frac{3}{16}.\end{aligned}$$

Solution 2. Equivalently, we prove the homogenized inequality

$$\frac{(a+b+c)^2}{(2a+b+c)^2} + \frac{(a+b+c)^2}{(a+2b+c)^2} + \frac{(a+b+c)^2}{(a+b+2c)^2} \le \frac{3}{16}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

for all positive real numbers a, b, c. Without loss of generality we choose $a+b+c=1$. Thus, the problem is equivalent to prove for all $a,b,c>0$, fulfilling this condition, the inequality

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} \le \frac{3}{16}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right). \tag{5}$$

Applying JENSEN's inequality to the function $f(x) = \dfrac{x}{(1+x)^2}$, which is concave for $0 \le x \le 2$ and increasing for $0 \le x \le 1$, we obtain

$$\alpha\frac{a}{(1+a)^2} + \beta\frac{b}{(1+b)^2} + \gamma\frac{c}{(1+c)^2} \le (\alpha+\beta+\gamma)\frac{A}{(1+A)^2}, \quad \text{where} \quad A = \frac{\alpha a + \beta b + \gamma c}{\alpha+\beta+\gamma}.$$

Choosing $\alpha = \dfrac{1}{a}$, $\beta = \dfrac{1}{b}$, and $\gamma = \dfrac{1}{c}$, we can apply the harmonic-arithmetic-mean inequality

$$A = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \le \frac{a+b+c}{3} = \frac{1}{3} < 1.$$

Finally we prove (5):

$$\begin{aligned}\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} &\le \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\frac{A}{(1+A)^2} \\ &\le \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\frac{\frac{1}{3}}{\left(1+\frac{1}{3}\right)^2} = \frac{3}{16}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).\end{aligned}$$

A3 FRA (France)

Determine all functions f from the set of positive integers into the set of positive integers such that for all x and y there exists a non degenerated triangle with sides of lengths

$$x, \quad f(y) \quad \text{and} \quad f(y+f(x)-1).$$

Solution. The identity function $f(x)=x$ is the only solution of the problem.

If $f(x)=x$ for all positive integers x, the given three lengths are x, $y=f(y)$ and $z=f(y+f(x)-1)=x+y-1$. Because of $x \geq 1$, $y \geq 1$ we have $z \geq \max\{x,y\} > |x-y|$ and $z < x+y$. From this it follows that a triangle with these side lengths exists and does not degenerate. We prove in several steps that there is no other solution.

Step 1. We show $f(1)=1$.
If we had $f(1)=1+m>1$ we would conclude $f(y)=f(y+m)$ for all y considering the triangle with the side lengths 1, $f(y)$ and $f(y+m)$. Thus, f would be m-periodic and, consequently, bounded. Let B be a bound, $f(x) \leq B$. If we choose $x>2B$ we obtain the contradiction $x > 2B \geq f(y)+f(y+f(x)-1)$.

Step 2. For all positive integers z, we have $f(f(z))=z$.
Setting $x=z$ and $y=1$ this follows immediately from Step 1.

Step 3. For all integers $z \geq 1$, we have $f(z) \leq z$.
Let us show, that the contrary leads to a contradiction. Assume $w+1=f(z)>z$ for some z. From Step 1 we know that $w \geq z \geq 2$. Let $M=\max\{f(1),f(2),\ldots,f(w)\}$ be the largest value of f for the first w integers. First we show, that no positive integer t exists with

$$f(t) > \frac{z-1}{w} \cdot t + M, \tag{1}$$

otherwise we decompose the smallest value t as $t=wr+s$ where r is an integer and $1 \leq s \leq w$. Because of the definition of M, we have $t>w$. Setting $x=z$ and $y=t-w$ we get from the triangle inequality

$$z+f(t-w) > f((t-w)+f(z)-1) = f(t-w+w) = f(t).$$

Hence,

$$f(t-w) \geq f(t)-(z-1) > \frac{z-1}{w}(t-w)+M,$$

a contradiction to the minimality of t.

Therefore the inequality (1) fails for all $t \geq 1$, we have proven

$$f(t) \leq \frac{z-1}{w} \cdot t + M, \tag{2}$$

instead.

Now, using (2), we finish the proof of Step 3. Because of $z \leq w$ we have $\frac{z-1}{w} < 1$ and we can choose an integer t sufficiently large to fulfill the condition

$$\left(\frac{z-1}{w}\right)^2 t + \left(\frac{z-1}{w} + 1\right) M < t.$$

Applying (2) twice we get

$$f\left(f(t)\right) \leq \frac{z-1}{w} f(t) + M \leq \frac{z-1}{w}\left(\frac{z-1}{w} t + M\right) + M < t$$

in contradiction to Step 2, which proves Step 3.

Final step. Thus, following Step 2 and Step 3, we obtain

$$z = f(f(z)) \leq f(z) \leq z$$

and $f(z) = z$ for all positive integers z is proven.

A4 BLR (Belarus)

Let a, b, c be positive real numbers such that $ab+bc+ca \leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3 \leq \sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right).$$

Solution. Starting with the terms of the right-hand side, the quadratic-arithmetic-mean inequality yields

$$\begin{aligned}\sqrt{2}\sqrt{a+b} &= 2\sqrt{\frac{ab}{a+b}}\sqrt{\frac{1}{2}\left(2+\frac{a^2+b^2}{ab}\right)} \\ &\geq 2\sqrt{\frac{ab}{a+b}}\cdot\frac{1}{2}\left(\sqrt{2}+\sqrt{\frac{a^2+b^2}{ab}}\right)=\sqrt{\frac{2ab}{a+b}}+\sqrt{\frac{a^2+b^2}{a+b}}\end{aligned}$$

and, analogously,

$$\sqrt{2}\sqrt{b+c} \geq \sqrt{\frac{2bc}{b+c}}+\sqrt{\frac{b^2+c^2}{b+c}}, \qquad \sqrt{2}\sqrt{c+a} \geq \sqrt{\frac{2ca}{c+a}}+\sqrt{\frac{c^2+a^2}{c+a}}.$$

Applying the inequality between the arithmetic mean and the squared harmonic mean will finish the proof:

$$\sqrt{\frac{2ab}{a+b}}+\sqrt{\frac{2bc}{b+c}}+\sqrt{\frac{2ca}{c+a}} \geq 3\cdot\sqrt{\frac{3}{\sqrt{\frac{a+b}{2ab}}^2+\sqrt{\frac{b+c}{2bc}}^2+\sqrt{\frac{c+a}{2ca}}^2}}=3\cdot\sqrt{\frac{3abc}{ab+bc+ca}} \geq 3.$$

A5 BLR (Belarus)

Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f\left(x - f(y)\right) > yf(x) + x.$$

Solution 1. Assume that

$$f(x - f(y)) \le yf(x) + x \quad \text{for all real } x, y. \tag{1}$$

Let $a = f(0)$. Setting $y = 0$ in (1) gives $f(x - a) \le x$ for all real x and, equivalently,

$$f(y) \le y + a \quad \text{for all real } y. \tag{2}$$

Setting $x = f(y)$ in (1) yields in view of (2)

$$a = f(0) \le yf(f(y)) + f(y) \le yf(f(y)) + y + a.$$

This implies $0 \le y(f(f(y)) + 1)$ and thus

$$f(f(y)) \ge -1 \quad \text{for all } y > 0. \tag{3}$$

From (2) and (3) we obtain $-1 \le f(f(y)) \le f(y) + a$ for all $y > 0$, so

$$f(y) \ge -a - 1 \quad \text{for all } y > 0. \tag{4}$$

Now we show that

$$f(x) \le 0 \quad \text{for all real } x. \tag{5}$$

Assume the contrary, i.e. there is some x such that $f(x) > 0$. Take any y such that

$$y < x - a \quad \text{and} \quad y < \frac{-a - x - 1}{f(x)}\,.$$

Then in view of (2)

$$x - f(y) \ge x - (y + a) > 0$$

and with (1) and (4) we obtain

$$yf(x) + x \ge f(x - f(y)) \ge -a - 1,$$

whence

$$y \ge \frac{-a - x - 1}{f(x)}$$

contrary to our choice of y. Thereby, we have established (5).

Setting $x = 0$ in (5) leads to $a = f(0) \le 0$ and (2) then yields

$$f(x) \le x \quad \text{for all real } x. \tag{6}$$

Now choose y such that $y > 0$ and $y > -f(-1) - 1$ and set $x = f(y) - 1$. From (1), (5) and

(6) we obtain

$$f(-1)=f(x-f(y))\le yf(x)+x=yf(f(y)-1)+f(y)-1\le y(f(y)-1)-1\le -y-1,$$

i.e. $y\le -f(-1)-1$, a contradiction to the choice of y.

Solution 2. Assume that

$$f(x-f(y))\le yf(x)+x \quad \text{for all real } x,y. \tag{7}$$

Let $a=f(0)$. Setting $y=0$ in (7) gives $f(x-a)\le x$ for all real x and, equivalently,

$$f(y)\le y+a \quad \text{for all real } y. \tag{8}$$

Now we show that

$$f(z)\ge 0 \quad \text{for all } z\ge 1. \tag{9}$$

Let $z\ge 1$ be fixed, set $b=f(z)$ and assume that $b<0$. Setting $x=w+b$ and $y=z$ in (7) gives

$$f(w)-zf(w+b)\le w+b \quad \text{for all real } w. \tag{10}$$

Applying (10) to $w,w+b,\dots,w+(n-1)b$, where $n=1,2,\dots$, leads to

$$\begin{aligned} f(w)-z^nf(w+nb)=&\,(f(w)-zf(w+b))+z\,(f(w+b)-zf(w+2b))\\ &+\ \cdots\ +z^{n-1}\,(f(w+(n-1)b)-zf(w+nb))\\ \le&\,(w+b)+z(w+2b)+\ \cdots\ +z^{n-1}(w+nb). \end{aligned}$$

From (8) we obtain

$$f(w+nb)\le w+nb+a$$

and, thus, we have for all positive integers n

$$f(w)\le (1+z+\cdots+z^{n-1}+z^n)w+(1+2z+\cdots+nz^{n-1}+nz^n)b+z^na. \tag{11}$$

With $w=0$ we get

$$a\le (1+2z+\cdots+nz^{n-1}+nz^n)b+az^n. \tag{12}$$

In view of the assumption $b<0$ we find some n such that

$$a>(nb+a)z^n \tag{13}$$

because the right hand side tends to $-\infty$ as $n\to\infty$. Now (12) and (13) give the desired contradiction and (9) is established. In addition, we have for $z=1$ the strict inequality

$$f(1)>0. \tag{14}$$

Indeed, assume that $f(1)=0$. Then setting $w=-1$ and $z=1$ in (11) leads to

$$f(-1)\le -(n+1)+a$$

which is false if n is sufficiently large.

To complete the proof we set $t=\min\{-a,-2/f(1)\}$. Setting $x=1$ and $y=t$ in (7) gives

$$f(1-f(t))\le tf(1)+1\le -2+1=-1. \tag{15}$$

On the other hand, by (8) and the choice of t we have $f(t) \le t + a \le 0$ and hence $1 - f(t) \ge 1$. The inequality (9) yields

$$f(1 - f(t)) \ge 0,$$

which contradicts (15).

A6 USA (United States of America)

Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \ldots \qquad \text{and} \qquad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \ldots$$

are both arithmetic progressions. Prove that $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Solution 1. Let D be the common difference of the progression $s_{s_1}, s_{s_2}, \ldots$. Let for $n = 1, 2, \ldots$

$$d_n = s_{n+1} - s_n.$$

We have to prove that d_n is constant. First we show that the numbers d_n are bounded. Indeed, by supposition $d_n \geq 1$ for all n. Thus, we have for all n

$$d_n = s_{n+1} - s_n \leq d_{s_n} + d_{s_n+1} + \cdots + d_{s_{n+1}-1} = s_{s_{n+1}} - s_{s_n} = D.$$

The boundedness implies that there exist

$$m = \min\{d_n : n = 1, 2, \ldots\} \quad \text{and} \quad M = \max\{d_n : n = 1, 2, \ldots\}.$$

It suffices to show that $m = M$. Assume that $m < M$. Choose n such that $d_n = m$. Considering a telescoping sum of $m = d_n = s_{n+1} - s_n$ items not greater than M leads to

$$D = s_{s_{n+1}} - s_{s_n} = s_{s_n+m} - s_{s_n} = d_{s_n} + d_{s_n+1} + \cdots + d_{s_n+m-1} \leq mM \tag{1}$$

and equality holds if and only if all items of the sum are equal to M. Now choose n such that $d_n = M$. In the same way, considering a telescoping sum of M items not less than m we obtain

$$D = s_{s_{n+1}} - s_{s_n} = s_{s_n+M} - s_{s_n} = d_{s_n} + d_{s_n+1} + \cdots + d_{s_n+M-1} \geq Mm \tag{2}$$

and equality holds if and only if all items of the sum are equal to m. The inequalities (1) and (2) imply that $D = Mm$ and that

$$\begin{aligned} d_{s_n} = d_{s_n+1} = \cdots = d_{s_{n+1}-1} = M \quad &\text{if } d_n = m, \\ d_{s_n} = d_{s_n+1} = \cdots = d_{s_{n+1}-1} = m \quad &\text{if } d_n = M. \end{aligned}$$

Hence, $d_n = m$ implies $d_{s_n} = M$. Note that $s_n \geq s_1 + (n-1) \geq n$ for all n and moreover $s_n > n$ if $d_n = n$, because in the case $s_n = n$ we would have $m = d_n = d_{s_n} = M$ in contradiction to the assumption $m < M$. In the same way $d_n = M$ implies $d_{s_n} = m$ and $s_n > n$. Consequently, there is a strictly increasing sequence $n_1, n_2, \ldots$ such that

$$d_{s_{n_1}} = M, \quad d_{s_{n_2}} = m, \quad d_{s_{n_3}} = M, \quad d_{s_{n_4}} = m, \quad \ldots .$$

The sequence $d_{s_1}, d_{s_2}, \ldots$ is the sequence of pairwise differences of $s_{s_1+1}, s_{s_2+1}, \ldots$ and $s_{s_1}, s_{s_2}, \ldots$, hence also an arithmetic progression. Thus $m = M$.

Solution 2. Let the integers D and E be the common differences of the progressions $s_{s_1}, s_{s_2}, \ldots$ and $s_{s_1+1}, s_{s_2+1}, \ldots$, respectively. Let briefly $A = s_{s_1} - D$ and $B = s_{s_1+1} - E$. Then, for all positive integers n,

$$s_{s_n} = A + nD, \qquad s_{s_n+1} = B + nE.$$

Since the sequence $s_1, s_2, \ldots$ is strictly increasing, we have for all positive integers n

$$s_{s_n} < s_{s_n+1} \leq s_{s_{n+1}},$$

which implies

$$A + nD < B + nE \leq A + (n+1)D,$$

and thereby

$$0 < B - A + n(E - D) \leq D,$$

which implies $D - E = 0$ and thus

$$0 \leq B - A \leq D. \tag{3}$$

Let $m = \min\{s_{n+1} - s_n : n = 1, 2, \ldots\}$. Then

$$B - A = (s_{s_1+1} - E) - (s_{s_1} - D) = s_{s_1+1} - s_{s_1} \geq m \tag{4}$$

and

$$D = A + (s_1 + 1)D - (A + s_1 D) = s_{s_{s_1+1}} - s_{s_{s_1}} = s_{B+D} - s_{A+D} \geq m(B - A). \tag{5}$$

From (3) we consider two cases.

Case 1. $B - A = D$.
Then, for each positive integer n, $s_{s_n+1} = B + nD = A + (n+1)D = s_{s_{n+1}}$, hence $s_{n+1} = s_n + 1$ and $s_1, s_2, \ldots$ is an arithmetic progression with common difference 1.

Case 2. $B - A < D$. Choose some positive integer N such that $s_{N+1} - s_N = m$. Then

$$\begin{aligned}
m(A - B + D - 1) &= m((A + (N+1)D) - (B + ND + 1)) \\
&\leq s_{A+(N+1)D} - s_{B+ND+1} = s_{s_{s_{N+1}}} - s_{s_{s_N}+1} \\
&= (A + s_{N+1}D) - (B + (s_N + 1)D) = (s_{N+1} - s_N)D + A - B - D \\
&= mD + A - B - D,
\end{aligned}$$

i.e.,

$$(B - A - m) + (D - m(B - A)) \leq 0. \tag{6}$$

The inequalities (4)-(6) imply that

$$B - A = m \quad \text{and} \quad D = m(B - A).$$

Assume that there is some positive integer n such that $s_{n+1} > s_n + m$. Then

$$m(m+1) \leq m(s_{n+1} - s_n) \leq s_{s_{n+1}} - s_{s_n} = (A + (n+1)D) - (A + nD)) = D = m(B - A) = m^2,$$

a contradiction. Hence $s_1, s_2, \ldots$ is an arithmetic progression with common difference m.

A7 JPN (Japan)

Find all functions f from the set of real numbers into the set of real numbers which satisfy for all real x, y the identity

$$f(xf(x+y)) = f(yf(x)) + x^2.$$

Solution 1. It is no hard to see that the two functions given by $f(x) = x$ and $f(x) = -x$ for all real x respectively solve the functional equation. In the sequel, we prove that there are no further solutions.
Let f be a function satisfying the given equation. It is clear that f cannot be a constant. Let us first show that $f(0) = 0$. Suppose that $f(0) \neq 0$. For any real t, substituting $(x, y) = (0, \frac{t}{f(0)})$ into the given functional equation, we obtain

$$f(0) = f(t), \tag{1}$$

contradicting the fact that f is not a constant function. Therefore, $f(0) = 0$. Next for any t, substituting $(x, y) = (t, 0)$ and $(x, y) = (t, -t)$ into the given equation, we get

$$f(tf(t)) = f(0) + t^2 = t^2,$$

and

$$f(tf(0)) = f(-tf(t)) + t^2,$$

respectively. Therefore, we conclude that

$$f(tf(t)) = t^2, \quad f(-tf(t)) = -t^2, \qquad \text{for every real } t. \tag{2}$$

Consequently, for every real v, there exists a real u, such that $f(u) = v$. We also see that if $f(t) = 0$, then $0 = f(tf(t)) = t^2$ so that $t = 0$, and thus 0 is the only real number satisfying $f(t) = 0$.
We next show that for any real number s,

$$f(-s) = -f(s). \tag{3}$$

This is clear if $f(s) = 0$. Suppose now $f(s) < 0$, then we can find a number t for which $f(s) = -t^2$. As $t \neq 0$ implies $f(t) \neq 0$, we can also find number a such that $af(t) = s$. Substituting $(x, y) = (t, a)$ into the given equation, we get

$$f(tf(t+a)) = f(af(t)) + t^2 = f(s) + t^2 = 0,$$

and therefore, $tf(t+a) = 0$, which implies $t + a = 0$, and hence $s = -tf(t)$. Consequently, $f(-s) = f(tf(t)) = t^2 = -(-t^2) = -f(s)$ holds in this case.
Finally, suppose $f(s) > 0$ holds. Then there exists a real number $t \neq 0$ for which $f(s) = t^2$. Choose a number a such that $tf(a) = s$. Substituting $(x, y) = (t, a-t)$ into the given equation, we get $f(s) = f(tf(a)) = f((a-t)f(t)) + t^2 = f((a-t)f(t)) + f(s)$. So we have $f((a-t)f(t)) = 0$, from which we conclude that $(a-t)f(t) = 0$. Since $f(t) \neq 0$, we get $a = t$ so that $s = tf(t)$ and thus we see $f(-s) = f(-tf(t)) = -t^2 = -f(s)$ holds in this case also. This observation finishes the proof of (3).
By substituting $(x, y) = (s, t)$, $(x, y) = (t, -s-t)$ and $(x, y) = (-s-t, s)$ into the given equation,

we obtain

$$f(sf(s+t)) = f(tf(s)) + s^2,$$
$$f(tf(-s)) = f((-s-t)f(t)) + t^2,$$

and

$$f((-s-t)f(-t)) = f(sf(-s-t)) + (s+t)^2,$$

respectively. Using the fact that $f(-x) = -f(x)$ holds for all x to rewrite the second and the third equation, and rearranging the terms, we obtain

$$\begin{aligned} f(tf(s)) - f(sf(s+t)) &= -s^2, \\ f(tf(s)) - f((s+t)f(t)) &= -t^2, \\ f((s+t)f(t)) + f(sf(s+t)) &= (s+t)^2. \end{aligned}$$

Adding up these three equations now yields $2f(tf(s)) = 2ts$, and therefore, we conclude that $f(tf(s)) = ts$ holds for every pair of real numbers s, t. By fixing s so that $f(s) = 1$, we obtain $f(x) = sx$. In view of the given equation, we see that $s = \pm 1$. It is easy to check that both functions $f(x) = x$ and $f(x) = -x$ satisfy the given functional equation, so these are the desired solutions.

Solution 2. As in Solution 1 we obtain (1), (2) and (3).

Now we prove that f is injective. For this purpose, let us assume that $f(r) = f(s)$ for some $r \neq s$. Then, by (2)

$$r^2 = f(rf(r)) = f(rf(s)) = f((s-r)f(r)) + r^2,$$

where the last statement follows from the given functional equation with $x = r$ and $y = s - r$. Hence, $h = (s-r)f(r)$ satisfies $f(h) = 0$ which implies $h^2 = f(hf(h)) = f(0) = 0$, i.e., $h = 0$. Then, by $s \neq r$ we have $f(r) = 0$ which implies $r = 0$, and finally $f(s) = f(r) = f(0) = 0$. Analogously, it follows that $s = 0$ which gives the contradiction $r = s$.

To prove $|f(1)| = 1$ we apply (2) with $t = 1$ and also with $t = f(1)$ and obtain $f(f(1)) = 1$ and $(f(1))^2 = f(f(1) \cdot f(f(1))) = f(f(1)) = 1$.

Now we choose $\eta \in \{-1, 1\}$ with $f(1) = \eta$. Using that f is odd and the given equation with $x = 1, y = z$ (second equality) and with $x = -1, y = z + 2$ (fourth equality) we obtain

$$\begin{aligned} f(z) + 2\eta &= \eta(f(z\eta) + 2) = \eta(f(f(z+1)) + 1) = \eta(-f(-f(z+1)) + 1) \\ &= -\eta f((z+2)f(-1)) = -\eta f((z+2)(-\eta)) = \eta f((z+2)\eta) = f(z+2). \end{aligned} \tag{4}$$

Hence,

$$f(z + 2\eta) = \eta f(\eta z + 2) = \eta(f(\eta z) + 2\eta) = f(z) + 2.$$

Using this argument twice we obtain

$$f(z + 4\eta) = f(z + 2\eta) + 2 = f(z) + 4.$$

Substituting $z = 2f(x)$ we have

$$f(2f(x)) + 4 = f(2f(x) + 4\eta) = f(2f(x+2)),$$

where the last equality follows from (4). Applying the given functional equation we proceed to

$$f(2f(x+2)) = f(xf(2)) + 4 = f(2\eta x) + 4$$

where the last equality follows again from (4) with $z = 0$, i.e., $f(2) = 2\eta$. Finally, $f(2f(x)) = f(2\eta x)$ and by injectivity of f we get $2f(x) = 2\eta x$ and hence the two solutions.

Combinatorics

C1 NZL (New Zealand)

Consider 2009 cards, each having one gold side and one black side, lying in parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

Solution. (a) We interpret a card showing black as the digit 0 and a card showing gold as the digit 1. Thus each position of the 2009 cards, read from left to right, corresponds bijectively to a nonnegative integer written in binary notation of 2009 digits, where leading zeros are allowed. Each move decreases this integer, so the game must end.

(b) We show that there is no winning strategy for the starting player. We label the cards from right to left by $1, \ldots, 2009$ and consider the set S of cards with labels $50i$, $i = 1, 2, \ldots, 40$. Let g_n be the number of cards from S showing gold after n moves. Obviously, $g_0 = 40$. Moreover, $|g_n - g_{n+1}| = 1$ as long as the play goes on. Thus, after an odd number of moves, the non-starting player finds a card from S showing gold and hence can make a move. Consequently, this player always wins.

C2 ROU (Romania)

For any integer $n \geq 2$, let $N(n)$ be the maximal number of triples (a_i, b_i, c_i), $i = 1, \dots, N(n)$, consisting of nonnegative integers a_i, b_i and c_i such that the following two conditions are satisfied:

(1) $a_i + b_i + c_i = n$ for all $i = 1, \dots, N(n)$,

(2) If $i \neq j$, then $a_i \neq a_j, b_i \neq b_j$ and $c_i \neq c_j$.

Determine $N(n)$ for all $n \geq 2$.

Comment. The original problem was formulated for m-tuples instead for triples. The numbers $N(m, n)$ are then defined similarly to $N(n)$ in the case $m = 3$. The numbers $N(3, n)$ and $N(n, n)$ should be determined. The case $m = 3$ is the same as in the present problem. The upper bound for $N(n, n)$ can be proved by a simple generalization. The construction of a set of triples attaining the bound can be easily done by induction from n to $n + 2$.

Solution. Let $n \geq 2$ be an integer and let $\{T_1, \dots, T_N\}$ be any set of triples of nonnegative integers satisfying the conditions (1) and (2). Since the a-coordinates are pairwise distinct we have

$$\sum_{i=1}^{N} a_i \geq \sum_{i=1}^{N} (i-1) = \frac{N(N-1)}{2}.$$

Analogously,

$$\sum_{i=1}^{N} b_i \geq \frac{N(N-1)}{2} \quad \text{and} \quad \sum_{i=1}^{N} c_i \geq \frac{N(N-1)}{2}.$$

Summing these three inequalities and applying (1) yields

$$3\frac{N(N-1)}{2} \leq \sum_{i=1}^{N} a_i + \sum_{i=1}^{N} b_i + \sum_{i=1}^{N} c_i = \sum_{i=1}^{N} (a_i + b_i + c_i) = nN,$$

hence $3\frac{N-1}{2} \leq n$ and, consequently,

$$N \leq \left\lfloor \frac{2n}{3} \right\rfloor + 1.$$

By constructing examples, we show that this upper bound can be attained, so $N(n) = \lfloor \frac{2n}{3} \rfloor + 1$.

We distinguish the cases $n = 3k-1$, $n = 3k$ and $n = 3k+1$ for $k \geq 1$ and present the extremal examples in form of a table.

$n = 3k-1$, $\lfloor \frac{2n}{3} \rfloor + 1 = 2k$		
a_i	b_i	c_i
0	$k+1$	$2k-2$
1	$k+2$	$2k-4$
$\vdots$	$\vdots$	$\vdots$
$k-1$	$2k$	0
k	0	$2k-1$
$k+1$	1	$2k-3$
$\vdots$	$\vdots$	$\vdots$
$2k-1$	$k-1$	1

$n = 3k$, $\lfloor \frac{2n}{3} \rfloor + 1 = 2k+1$		
a_i	b_i	c_i
0	k	$2k$
1	$k+1$	$2k-2$
$\vdots$	$\vdots$	$\vdots$
k	$2k$	0
$k+1$	0	$2k-1$
$k+2$	1	$2k-3$
$\vdots$	$\vdots$	$\vdots$
$2k$	$k-1$	1

$n = 3k+1$, $\lfloor \frac{2n}{3} \rfloor + 1 = 2k+1$		
a_i	b_i	c_i
0	k	$2k+1$
1	$k+1$	$2k-1$
$\vdots$	$\vdots$	$\vdots$
k	$2k$	1
$k+1$	0	$2k$
$k+2$	1	$2k-2$
$\vdots$	$\vdots$	$\vdots$
$2k$	$k-1$	2

It can be easily seen that the conditions (1) and (2) are satisfied and that we indeed have $\lfloor \frac{2n}{3} \rfloor + 1$ triples in each case.

Comment. A cute combinatorial model is given by an equilateral triangle, partitioned into n^2 congruent equilateral triangles by $n-1$ equidistant parallels to each of its three sides. Two chess-like bishops placed at any two vertices of the small triangles are said to *menace* one another if they lie on a same parallel. The problem is to determine the largest number of bishops that can be placed so that none menaces another. A bishop may be assigned three coordinates a, b, c, namely the numbers of sides of small triangles they are off each of the sides of the big triangle. It is readily seen that the sum of these coordinates is always n, therefore fulfilling the requirements.

C3 RUS (Russian Federation)

Let n be a positive integer. Given a sequence $\varepsilon_1, \dots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \dots, n-1$, the sequences $a_0, \dots, a_n$ and $b_0, \dots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \quad \text{for each } i = 1, \dots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \quad \text{for each } i = 1, \dots, n-1.$$

Prove that $a_n = b_n$.

Solution. For a binary word $w = \sigma_1 \dots \sigma_n$ of length n and a letter $\sigma \in \{0,1\}$ let $w\sigma = \sigma_1 \dots \sigma_n \sigma$ and $\sigma w = \sigma \sigma_1 \dots \sigma_n$. Moreover let $\overline{w} = \sigma_n \dots \sigma_1$ and let $\emptyset$ be the empty word (of length 0 and with $\overline{\emptyset} = \emptyset$). Let (u,v) be a pair of two real numbers. For binary words w we define recursively the numbers $(u,v)^w$ as follows:

$$(u,v)^{\emptyset} = v, \quad (u,v)^0 = 2u + 3v, \quad (u,v)^1 = 3u + v,$$

$$(u,v)^{w\sigma\varepsilon} = \begin{cases} 2(u,v)^w + 3(u,v)^{w\sigma}, & \text{if } \varepsilon = 0, \\ 3(u,v)^w + (u,v)^{w\sigma}, & \text{if } \varepsilon = 1. \end{cases}$$

It easily follows by induction on the length of w that for all real numbers $u_1, v_1, u_2, v_2, \lambda_1$ and λ_2

$$(\lambda_1 u_1 + \lambda_2 u_2, \lambda_1 v_1 + \lambda_2 v_2)^w = \lambda_1 (u_1, v_1)^w + \lambda_2 (u_2, v_2)^w \tag{1}$$

and that for $\varepsilon \in \{0,1\}$

$$(u,v)^{\varepsilon w} = (v, (u,v)^{\varepsilon})^w. \tag{2}$$

Obviously, for $n \geq 1$ and $w = \varepsilon_1 \dots \varepsilon_{n-1}$, we have $a_n = (1,7)^w$ and $b_n = (1,7)^{\overline{w}}$. Thus it is sufficient to prove that

$$(1,7)^w = (1,7)^{\overline{w}} \tag{3}$$

for each binary word w. We proceed by induction on the length of w. The assertion is obvious if w has length 0 or 1. Now let $w\sigma\varepsilon$ be a binary word of length $n \geq 2$ and suppose that the assertion is true for all binary words of length at most $n-1$.

Note that $(2,1)^{\sigma} = 7 = (1,7)^{\emptyset}$ for $\sigma \in \{0,1\}$, $(1,7)^0 = 23$, and $(1,7)^1 = 10$.

First let $\varepsilon = 0$. Then in view of the induction hypothesis and the equalities (1) and (2), we obtain

$$(1,7)^{w\sigma 0} = 2(1,7)^w + 3(1,7)^{w\sigma} = 2(1,7)^{\overline{w}} + 3(1,7)^{\sigma\overline{w}} = 2(2,1)^{\sigma\overline{w}} + 3(1,7)^{\sigma\overline{w}}$$
$$= (7,23)^{\sigma\overline{w}} = (1,7)^{0\sigma\overline{w}}.$$

Now let $\varepsilon = 1$. Analogously, we obtain

$$(1,7)^{w\sigma 1} = 3(1,7)^{w} + (1,7)^{w\sigma} = 3(1,7)^{\overline{w}} + (1,7)^{\sigma\overline{w}} = 3(2,1)^{\sigma\overline{w}} + (1,7)^{\sigma\overline{w}}$$
$$= (7,10)^{\sigma\overline{w}} = (1,7)^{1\sigma\overline{w}}.$$

Thus the induction step is complete, (3) and hence also $a_n = b_n$ are proved.

Comment. The original solution uses the relation

$$(1,7)^{\alpha\beta w} = ((1,7)^{w}, (1,7)^{\beta w})^{\alpha}, \quad \alpha, \beta \in \{0,1\},$$

which can be proved by induction on the length of w. Then (3) also follows by induction on the length of w:

$$(1,7)^{\alpha\beta w} = ((1,7)^{w}, (1,7)^{\beta w})^{\alpha} = ((1,7)^{\overline{w}}, (1,7)^{\overline{w}\beta})^{\alpha} = (1,7)^{\overline{w}\beta\alpha}.$$

Here w may be the empty word.

C4 NLD (Netherlands)

For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of the chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

Solution 1. For a $k \times k$ chessboard, we introduce in a standard way coordinates of the vertices of the cells and assume that the cell C_{ij} in row i and column j has vertices $(i-1, j-1), (i-1, j), (i, j-1), (i, j)$, where $i, j \in \{1, \ldots, k\}$. Without loss of generality assume that the cells C_{ii}, $i = 1, \ldots, k$, form a separate rectangle. Then we may consider the boards $B_k = \bigcup_{1 \leq i < j \leq k} C_{ij}$ below that diagonal and the congruent board $B'_k = \bigcup_{1 \leq j < i \leq k} C_{ij}$ above that diagonal separately because no rectangle can simultaneously cover cells from B_k and B'_k. We will show that for $k = 2^m$ the smallest total perimeter of a rectangular partition of B_k is $m2^{m+1}$. Then the overall answer to the problem is $2 \cdot m2^{m+1} + 4 \cdot 2^m = (m+1)2^{m+2}$.

First we inductively construct for $m \geq 1$ a partition of B_{2^m} with total perimeter $m2^{m+1}$. If $m = 0$, the board B_{2^m} is empty and the total perimeter is 0. For $m \geq 0$, the board $B_{2^{m+1}}$ consists of a $2^m \times 2^m$ square in the lower right corner with vertices $(2^m, 2^m), (2^m, 2^{m+1}), (2^{m+1}, 2^m)$, $(2^{m+1}, 2^{m+1})$ to which two boards congruent to B_{2^m} are glued along the left and the upper margin. The square together with the inductive partitions of these two boards yield a partition with total perimeter $4 \cdot 2^m + 2 \cdot m2^{m+1} = (m+1)2^{m+2}$ and the induction step is complete.

Let

$$D_k = 2k \log_2 k.$$

Note that $D_k = m2^{m+1}$ if $k = 2^m$. Now we show by induction on k that the total perimeter of a rectangular partition of B_k is at least D_k. The case $k = 1$ is trivial (see $m = 0$ from above). Let the assertion be true for all positive integers less than k. We investigate a fixed rectangular partition of B_k that attains the minimal total perimeter. Let R be the rectangle that covers the cell C_{1k} in the lower right corner. Let (i, j) be the upper left corner of R. First we show that $i = j$. Assume that $i < j$. Then the line from (i, j) to $(i+1, j)$ or from (i, j) to $(i, j-1)$ must belong to the boundary of some rectangle in the partition. Without loss of generality assume that this is the case for the line from (i, j) to $(i+1, j)$.

Case 1. No line from (i, l) to $(i+1, l)$ where $j < l < k$ belongs to the boundary of some rectangle of the partition.
Then there is some rectangle R' of the partition that has with R the common side from (i, j) to (i, k). If we join these two rectangles to one rectangle we get a partition with smaller total perimeter, a contradiction.

Case 2. There is some l such that $j < l < k$ and the line from (i, l) to $(i+1, l)$ belongs to the boundary of some rectangle of the partition.
Then we replace the upper side of R by the line $(i+1, j)$ to $(i+1, k)$ and for the rectangles whose lower side belongs to the line from (i, j) to (i, k) we shift the lower side upwards so that the new lower side belongs to the line from $(i+1, j)$ to $(i+1, k)$. In such a way we obtain a rectangular partition of B_k with smaller total perimeter, a contradiction.

Now the fact that the upper left corner of R has the coordinates (i, i) is established. Consequently, the partition consists of R, of rectangles of a partition of a board congruent to B_i and of rectangles of a partition of a board congruent to B_{k-i}. By the induction hypothesis, its total perimeter is at least

$$2(k-i)+2i+D_i+D_{k-i} \geq 2k+2i\log_2 i+2(k-i)\log_2(k-i). \tag{1}$$

Since the function $f(x)=2x\log_2 x$ is convex for $x>0$, JENSEN's inequality immediately shows that the minimum of the right hand sight of (1) is attained for $i=k/2$. Hence the total perimeter of the optimal partition of B_k is at least $2k+2k/2\log_2 k/2+2(k/2)\log_2(k/2)=D_k$.

Solution 2. We start as in Solution 1 and present another proof that $m2^{m+1}$ is a lower bound for the total perimeter of a partition of B_{2^m} into n rectangles. Let briefly $M=2^m$. For $1\leq i\leq M$, let r_i denote the number of rectangles in the partition that cover some cell from row i and let c_j be the number of rectangles that cover some cell from column j. Note that the total perimeter p of all rectangles in the partition is

$$p=2\left(\sum_{i=1}^{M} r_i+\sum_{i=1}^{M} c_i\right).$$

No rectangle can simultaneously cover cells from row i and from column i since otherwise it would also cover the cell C_{ii}. We classify subsets S of rectangles of the partition as follows. We say that S is of type i, $1\leq i\leq M$, if S contains all r_i rectangles that cover some cell from row i, but none of the c_i rectangles that cover some cell from column i. Altogether there are $2^{n-r_i-c_i}$ subsets of type i. Now we show that no subset S can be simultaneously of type i and of type j if $i\neq j$. Assume the contrary and let without loss of generality $i<j$. The cell C_{ij} must be covered by some rectangle R. The subset S is of type i, hence R is contained in S. S is of type j, thus R does not belong to S, a contradiction. Since there are 2^n subsets of rectangles of the partition, we infer

$$2^n \geq \sum_{i=1}^{M} 2^{n-r_i-c_i}=2^n\sum_{i=1}^{M} 2^{-(r_i+c_i)}. \tag{2}$$

By applying JENSEN'S inequality to the convex function $f(x)=2^{-x}$ we derive

$$\frac{1}{M}\sum_{i=1}^{M} 2^{-(r_i+c_i)} \geq 2^{-\frac{1}{M}\sum_{i=1}^{M}(r_i+c_i)}=2^{-\frac{p}{2M}}. \tag{3}$$

From (2) and (3) we obtain

$$1\geq M2^{-\frac{p}{2M}}$$

and equivalently

$$p\geq m2^{m+1}.$$

C5 NLD (Netherlands)

Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

Solution 1. No, the Stepmother cannot enforce a bucket overflow and Cinderella can keep playing forever. Throughout we denote the five buckets by B_0, B_1, B_2, B_3, and B_4, where B_k is adjacent to bucket B_{k-1} and B_{k+1} ($k = 0, 1, 2, 3, 4$) and all indices are taken modulo 5. Cinderella enforces that the following three conditions are satisfied at the beginning of every round:

(1) Two adjacent buckets (say B_1 and B_2) are empty.

(2) The two buckets standing next to these adjacent buckets (here B_0 and B_3) have total contents at most 1.

(3) The remaining bucket (here B_4) has contents at most 1.

These conditions clearly hold at the beginning of the first round, when all buckets are empty.

Assume that Cinderella manages to maintain them until the beginning of the r-th round ($r \geq 1$). Denote by x_k ($k = 0, 1, 2, 3, 4$) the contents of bucket B_k at the beginning of this round and by y_k the corresponding contents after the Stepmother has distributed her liter of water in this round.

By the conditions, we can assume $x_1 = x_2 = 0$, $x_0 + x_3 \leq 1$ and $x_4 \leq 1$. Then, since the Stepmother adds one liter, we conclude $y_0 + y_1 + y_2 + y_3 \leq 2$. This inequality implies $y_0 + y_2 \leq 1$ or $y_1 + y_3 \leq 1$. For reasons of symmetry, we only consider the second case.

Then Cinderella empties buckets B_0 and B_4.

At the beginning of the next round B_0 and B_4 are empty (condition (1) is fulfilled), due to $y_1 + y_3 \leq 1$ condition (2) is fulfilled and finally since $x_2 = 0$ we also must have $y_2 \leq 1$ (condition (3) is fulfilled).

Therefore, Cinderella can indeed manage to maintain the three conditions (1)–(3) also at the beginning of the $(r+1)$-th round. By induction, she thus manages to maintain them at the beginning of every round. In particular she manages to keep the contents of every single bucket at most 1 liter. Therefore, the buckets of 2-liter capacity will never overflow.

Solution 2. We prove that Cinderella can maintain the following two conditions and hence she can prevent the buckets from overflow:

(1′) Every two non-adjacent buckets contain a total of at most 1.

(2′) The total contents of all five buckets is at most $\frac{3}{2}$.

We use the same notations as in the first solution. The two conditions again clearly hold at the beginning. Assume that Cinderella maintained these two conditions until the beginning of the r-th round. A pair of non-neighboring buckets (B_i, B_{i+2}), $i = 0, 1, 2, 3, 4$ is called *critical* if $y_i + y_{i+2} > 1$. By condition (2′), after the Stepmother has distributed her water we have $y_0 + y_1 + y_2 + y_3 + y_4 \le \frac{5}{2}$. Therefore,

$$(y_0 + y_2) + (y_1 + y_3) + (y_2 + y_4) + (y_3 + y_0) + (y_4 + y_1) = 2(y_0 + y_1 + y_2 + y_3 + y_4) \le 5,$$

and hence there is a pair of non-neighboring buckets which is not critical, say (B_0, B_2). Now, if both of the pairs (B_3, B_0) and (B_2, B_4) are critical, we must have $y_1 < \frac{1}{2}$ and Cinderella can empty the buckets B_3 and B_4. This clearly leaves no critical pair of buckets and the total contents of all the buckets is then $y_1 + (y_0 + y_2) \le \frac{3}{2}$. Therefore, conditions (1′) and (2′) are fulfilled.

Now suppose that without loss of generality the pair (B_3, B_0) is not critical. If in this case $y_0 \le \frac{1}{2}$, then one of the inequalities $y_0 + y_1 + y_2 \le \frac{3}{2}$ and $y_0 + y_3 + y_4 \le \frac{3}{2}$ must hold. But then Cinderella can empty B_3 and B_4 or B_1 and B_2, respectively and clearly fulfill the conditions.

Finally consider the case $y_0 > \frac{1}{2}$. By $y_0 + y_1 + y_2 + y_3 + y_4 \le \frac{5}{2}$, at least one of the pairs (B_1, B_3) and (B_2, B_4) is not critical. Without loss of generality let this be the pair (B_1, B_3). Since the pair (B_3, B_0) is not critical and $y_0 > \frac{1}{2}$, we must have $y_3 \le \frac{1}{2}$. But then, as before, Cinderella can maintain the two conditions at the beginning of the next round by either emptying B_1 and B_2 or B_4 and B_0.

Comments on GREEDY approaches. A natural approach for Cinderella would be a GREEDY strategy as for example: *Always remove as much water as possible from the system.* It is straightforward to prove that GREEDY can avoid buckets of capacity $\frac{5}{2}$ from overflowing: If before the Stepmothers move one has $x_0 + x_1 + x_2 + x_3 + x_4 \le \frac{3}{2}$ then after her move the inequality $Y = y_0 + y_1 + y_2 + y_3 + y_4 \le \frac{5}{2}$ holds. If now Cinderella removes the two adjacent buckets with maximum total contents she removes at least $\frac{2Y}{5}$ and thus the remaining buckets contain at most $\frac{3}{5} \cdot Y \le \frac{3}{2}$.

But GREEDY is in general not strong enough to settle this problem as can be seen in the following example:

- In an initial phase, the Stepmother brings all the buckets (after her move) to contents of at least $\frac{1}{2} - 2\epsilon$, where ϵ is an arbitrary small positive number. This can be done by always splitting the 1 liter she has to distribute so that all buckets have the same contents. After her r-th move the total contents of each of the buckets is then c_r with $c_1 = 1$ and $c_{r+1} = 1 + \frac{3}{5} \cdot c_r$ and hence $c_r = \frac{5}{2} - \frac{3}{2} \cdot \left(\frac{3}{5}\right)^{r-1}$. So the contents of each single bucket indeed approaches $\frac{1}{2}$ (from below). In particular, any two adjacent buckets have total contents strictly less than 1 which enables the Stepmother to always refill the buckets that Cinderella just emptied and then distribute the remaining water evenly over all buckets.
- After that phase GREEDY faces a situation like this $(\frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon)$ and leaves a situation of the form $(x_0, x_1, x_2, x_3, x_4) = (\frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, 0, 0)$.
- Then the Stepmother can add the amounts $(0, \frac{1}{4} + \epsilon, \epsilon, \frac{3}{4} - 2\epsilon, 0)$ to achieve a situation like this: $(y_0, y_1, y_2, y_3, y_4) = (\frac{1}{2} - 2\epsilon, \frac{3}{4} - \epsilon, \frac{1}{2} - \epsilon, \frac{3}{4} - 2\epsilon, 0)$.
- Now B_1 and B_2 are the adjacent buckets with the maximum total contents and thus GREEDY empties them to yield $(x_0, x_1, x_2, x_3, x_4) = (\frac{1}{2} - 2\epsilon, 0, 0, \frac{3}{4} - 2\epsilon, 0)$.
- Then the Stepmother adds $(\frac{5}{8}, 0, 0, \frac{3}{8}, 0)$, which yields $(\frac{9}{8} - 2\epsilon, 0, 0, \frac{9}{8} - 2\epsilon, 0)$.
- Now GREEDY can only empty one of the two nonempty buckets and in the next step the Stepmother adds her liter to the other bucket and brings it to $\frac{17}{8} - 2\epsilon$, i.e. an overflow.

A harder variant. Five identical empty buckets of capacity b stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Determine all bucket capacities b for which the Stepmother can enforce a bucket to overflow.

Solution to the harder variant. The answer is $b < 2$.

The previous proof shows that for all $b \geq 2$ the Stepmother cannot enforce overflowing. Now if $b < 2$, let R be a positive integer such that $b < 2 - 2^{1-R}$. In the first R rounds the Stepmother now ensures that at least one of the (nonadjacent) buckets B_1 and B_3 have contents of at least $1 - 2^{1-r}$ at the beginning of round r ($r = 1, 2, \ldots, R$). This is trivial for $r = 1$ and if it holds at the beginning of round r, she can fill the bucket which contains at least $1 - 2^{1-r}$ liters with another 2^{-r} liters and put the rest of her water – $1 - 2^{-r}$ liters – in the other bucket. As Cinderella now can remove the water of at most one of the two buckets, the other bucket carries its contents into the next round.

At the beginning of the R-th round there are $1 - 2^{1-R}$ liters in B_1 or B_3. The Stepmother puts the entire liter into that bucket and produces an overflow since $b < 2 - 2^{1-R}$.

C6 BGR (Bulgaria)

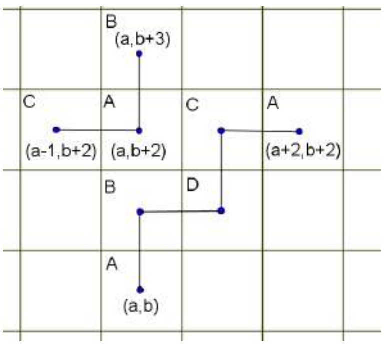
On a 999×999 board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.

How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

Solution. The answer is $998^2 - 4 = 4 \cdot (499^2 - 1)$ squares.

First we show that this number is an upper bound for the number of cells a limp rook can visit. To do this we color the cells with four colors A, B, C and D in the following way: for $(i,j) \equiv (0,0) \mod 2$ use A, for $(i,j) \equiv (0,1) \mod 2$ use B, for $(i,j) \equiv (1,0) \mod 2$ use C and for $(i,j) \equiv (1,1) \mod 2$ use D. From an A-cell the rook has to move to a B-cell or a C-cell. In the first case, the order of the colors of the cells visited is given by $A, B, D, C, A, B, D, C, A, \ldots$, in the second case it is $A, C, D, B, A, C, D, B, A, \ldots$. Since the route is closed it must contain the same number of cells of each color. There are only 499^2 A-cells. In the following we will show that the rook cannot visit all the A-cells on its route and hence the maximum possible number of cells in a route is $4 \cdot (499^2 - 1)$.

Assume that the route passes through every single A-cell. Color the A-cells in black and white in a chessboard manner, i.e. color any two A-cells at distance 2 in different color. Since the number of A-cells is odd the rook cannot always alternate between visiting black and white A-cells along its route. Hence there are two A-cells of the same color which are four rook-steps apart that are visited directly one after the other. Let these two A-cells have row and column numbers (a,b) and $(a+2, b+2)$ respectively.



There is up to reflection only one way the rook can take from (a,b) to $(a+2,b+2)$. Let this way be $(a,b) \to (a,b+1) \to (a+1,b+1) \to (a+1,b+2) \to (a+2,b+2)$. Also let without loss of generality the color of the cell $(a,b+1)$ be B (otherwise change the roles of columns and rows).

Now consider the A-cell $(a,b+2)$. The only way the rook can pass through it is via $(a-1,b+2) \to (a,b+2) \to (a,b+3)$ in this order, since according to our assumption after every A-cell the rook passes through a B-cell. Hence, to connect these two parts of the path, there must be a path connecting the cell $(a, b+3)$ and (a, b) and also a path connecting $(a+2, b+2)$ and $(a-1, b+2)$.

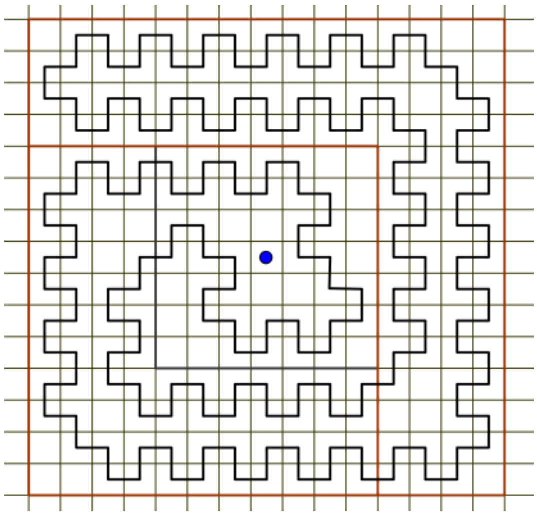
But these four cells are opposite vertices of a convex quadrilateral and the paths are outside of that quadrilateral and hence they must intersect. This is due to the following fact:

The path from (a, b) to $(a, b+3)$ together with the line segment joining these two cells form a closed loop that has one of the cells $(a-1, b+2)$ and $(a+2, b+2)$ in its inside and the other one on the outside. Thus the path between these two points must cross the previous path.

But an intersection is only possible if a cell is visited twice. This is a contradiction.

Hence the number of cells visited is at most $4 \cdot (499^2 - 1)$.

The following picture indicates a recursive construction for all $n \times n$-chessboards with $n \equiv 3 \mod 4$ which clearly yields a path that misses exactly one A-cell (marked with a dot, the center cell of the 15×15-chessboard) and hence, in the case of $n = 999$ crosses exactly $4 \cdot (499^2 - 1)$ cells.



C7 RUS (Russian Federation)

Variant 1. A grasshopper jumps along the real axis. He starts at point 0 and makes 2009 jumps to the right with lengths $1, 2, \ldots, 2009$ in an arbitrary order. Let M be a set of 2008 positive integers less than $1005 \cdot 2009$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

Variant 2. Let n be a nonnegative integer. A grasshopper jumps along the real axis. He starts at point 0 and makes $n+1$ jumps to the right with pairwise different positive integral lengths $a_1, a_2, \ldots, a_{n+1}$ in an arbitrary order. Let M be a set of n positive integers in the interval $(0, s)$, where $s = a_1 + a_2 + \cdots + a_{n+1}$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

Solution of Variant 1. We construct the set of landing points of the grasshopper.

Case 1. M does not contain numbers divisible by 2009.
We fix the numbers $2009k$ as landing points, $k = 1, 2, \ldots, 1005$. Consider the open intervals $I_k = (2009(k-1), 2009k)$, $k = 1, 2, \ldots, 1005$. We show that we can choose exactly one point outside of M as a landing point in 1004 of these intervals such that all lengths from 1 to 2009 are realized. Since there remains one interval without a chosen point, the length 2009 indeed will appear. Each interval has length 2009, hence a new landing point in an interval yields with a length d also the length $2009 - d$. Thus it is enough to implement only the lengths from $D = \{1, 2, \ldots, 1004\}$. We will do this in a greedy way. Let n_k, $k = 1, 2, \ldots, 1005$, be the number of elements of M that belong to the interval I_k. We order these numbers in a decreasing way, so let $p_1, p_2, \ldots, p_{1005}$ be a permutation of $\{1, 2, \ldots, 1005\}$ such that $n_{p_1} \geq n_{p_2} \geq \cdots \geq n_{p_{1005}}$. In I_{p_1} we do not choose a landing point. Assume that landing points have already been chosen in the intervals $I_{p_2}, \ldots, I_{p_m}$ and the lengths $d_2, \ldots, d_m$ from D are realized, $m = 1, \ldots, 1004$. We show that there is some $d \in D \setminus \{d_2, \ldots, d_m\}$ that can be implemented with a new landing point in $I_{p_{m+1}}$. Assume the contrary. Then the $1004 - (m-1)$ other lengths are obstructed by the $n_{p_{m+1}}$ points of M in $I_{p_{m+1}}$. Each length d can be realized by two landing points, namely $2009(p_{m+1} - 1) + d$ and $2009p_{m+1} - d$, hence

$$n_{p_{m+1}} \geq 2(1005 - m). \tag{1}$$

Moreover, since $|M| = 2008 = n_1 + \cdots + n_{1005}$,

$$2008 \geq n_{p_1} + n_{p_2} + \cdots + n_{p_{m+1}} \geq (m+1)n_{p_{m+1}}. \tag{2}$$

Consequently, by (1) and (2),

$$2008 \geq 2(m+1)(1005 - m).$$

The right hand side of the last inequality obviously attains its minimum for $m = 1004$ and this minimum value is greater than 2008, a contradiction.

Case 2. M does contain a number μ divisible by 2009.
By the pigeonhole principle there exists some $r \in \{1, \ldots, 2008\}$ such that M does not contain numbers with remainder r modulo 2009. We fix the numbers $2009(k-1) + r$ as landing points, $k = 1, 2, \ldots, 1005$. Moreover, $1005 \cdot 2009$ is a landing point. Consider the open intervals

$I_k = (2009(k-1)+r, 2009k+r)$, $k = 1, 2, \ldots, 1004$. Analogously to Case 1, it is enough to show that we can choose in 1003 of these intervals exactly one landing point outside of $M \setminus \{\mu\}$ such that each of the lengths of $D = \{1, 2, \ldots, 1004\} \setminus \{r\}$ are implemented. Note that r and $2009 - r$ are realized by the first and last jump and that choosing μ would realize these two differences again. Let n_k, $k = 1, 2, \ldots, 1004$, be the number of elements of $M \setminus \{\mu\}$ that belong to the interval I_k and $p_1, p_2, \ldots, p_{1004}$ be a permutation of $\{1, 2, \ldots, 1004\}$ such that $n_{p_1} \geq n_{p_2} \geq \cdots \geq n_{p_{1004}}$. With the same reasoning as in Case 1 we can verify that a greedy choice of the landing points in $I_{p_2}, I_{p_3}, \ldots, I_{p_{1004}}$ is possible. We only have to replace (1) by

$$n_{p_{m+1}} \geq 2(1004 - m)$$

(D has one element less) and (2) by

$$2007 \geq n_{p_1} + n_{p_2} + \cdots + n_{p_{m+1}} \geq (m+1)n_{p_{m+1}}.$$

Comment. The cardinality 2008 of M in the problem is the maximum possible value. For $M = \{1, 2, \ldots, 2009\}$, the grasshopper necessarily lands on a point from M.

Solution of Variant 2. First of all we remark that the statement in the problem implies a strengthening of itself: Instead of $|M| = n$ it is sufficient to suppose that $|M \cap (0, s - \overline{a}]| \leq n$, where $\overline{a} = \min\{a_1, a_2, \ldots, a_{n+1}\}$. This fact will be used in the proof.

We prove the statement by induction on n. The case $n = 0$ is obvious. Let $n > 0$ and let the assertion be true for all nonnegative integers less than n. Moreover let $a_1, a_2, \ldots, a_{n+1}, s$ and M be given as in the problem. Without loss of generality we may assume that $a_{n+1} < a_n < \cdots < a_2 < a_1$. Set

$$T_k = \sum_{i=1}^{k} a_i \quad \text{for } k = 0, 1, \ldots, n+1.$$

Note that $0 = T_0 < T_1 < \cdots < T_{n+1} = s$. We will make use of the induction hypothesis as follows:

Claim 1. It suffices to show that for some $m \in \{1, 2, \ldots, n+1\}$ the grasshopper is able to do at least m jumps without landing on a point of M and, in addition, after these m jumps he has jumped over at least m points of M.

Proof. Note that $m = n+1$ is impossible by $|M| = n$. Now set $n' = n - m$. Then $0 \leq n' < n$. The remaining $n' + 1$ jumps can be carried out without landing on one of the remaining at most n' forbidden points by the induction hypothesis together with a shift of the origin. This proves the claim.

An integer $k \in \{1, 2, \ldots, n+1\}$ is called *smooth*, if the grasshopper is able to do k jumps with the lengths $a_1, a_2, \ldots, a_k$ in such a way that he never lands on a point of M except for the very last jump, when he may land on a point of M.

Obviously, 1 is smooth. Thus there is a largest number k^*, such that all the numbers $1, 2, \ldots, k^*$ are smooth. If $k^* = n+1$, the proof is complete. In the following let $k^* \leq n$.

Claim 2. We have

$$T_{k^*} \in M \quad \text{and} \quad |M \cap (0, T_{k^*})| \geq k^*. \tag{3}$$

Proof. In the case $T_{k^*} \notin M$ any sequence of jumps that verifies the smoothness of k^* can be extended by appending a_{k^*+1}, which is a contradiction to the maximality of k^*. Therefore we have $T_{k^*} \in M$. If $|M \cap (0, T_{k^*})| < k^*$, there exists an $l \in \{1, 2, \ldots, k^*\}$ with $T_{k^*+1} - a_l \notin M$. By the induction hypothesis with $k^* - 1$ instead of n, the grasshopper is able to reach $T_{k^*+1} - a_l$

with k^* jumps of lengths from $\{a_1, a_2, \ldots, a_{k^*+1}\} \setminus \{a_l\}$ without landing on any point of M. Therefore k^*+1 is also smooth, which is a contradiction to the maximality of k^*. Thus Claim 2 is proved.

Now, by Claim 2, there exists a smallest integer $\overline{k} \in \{1, 2, \ldots, k^*\}$ with

$$T_{\overline{k}} \in M \quad \text{and} \quad |M \cap (0, T_{\overline{k}})| \geq \overline{k}.$$

Claim 3. It is sufficient to consider the case

$$|M \cap (0, T_{\overline{k}-1}]| \leq \overline{k} - 1. \tag{4}$$

Proof. If $\overline{k} = 1$, then (4) is clearly satisfied. In the following let $\overline{k} > 1$. If $T_{\overline{k}-1} \in M$, then (4) follows immediately by the minimality of $\overline{k}$. If $T_{\overline{k}-1} \notin M$, by the smoothness of $\overline{k} - 1$, we obtain a situation as in Claim 1 with $m = \overline{k} - 1$ provided that $|M \cap (0, T_{\overline{k}-1}]| \geq \overline{k} - 1$. Hence, we may even restrict ourselves to $|M \cap (0, T_{\overline{k}-1}]| \leq \overline{k} - 2$ in this case and Claim 3 is proved.

Choose an integer $v \geq 0$ with $|M \cap (0, T_{\overline{k}})| = \overline{k} + v$. Let $r_1 > r_2 > \cdots > r_l$ be exactly those indices r from $\{\overline{k}+1, \overline{k}+2, \ldots, n+1\}$ for which $T_{\overline{k}} + a_r \notin M$. Then

$$n = |M| = |M \cap (0, T_{\overline{k}})| + 1 + |M \cap (T_{\overline{k}}, s)| \geq \overline{k} + v + 1 + (n + 1 - \overline{k} - l)$$

and consequently $l \geq v + 2$. Note that

$$T_{\overline{k}} + a_{r_1} - a_1 < T_{\overline{k}} + a_{r_1} - a_2 < \cdots < T_{\overline{k}} + a_{r_1} - a_{\overline{k}} < T_{\overline{k}} + a_{r_2} - a_{\overline{k}} < \cdots < T_{\overline{k}} + a_{r_{v+2}} - a_{\overline{k}} < T_{\overline{k}}$$

and that this are $\overline{k} + v + 1$ numbers from $(0, T_{\overline{k}})$. Therefore we find some $r \in \{\overline{k}+1, \overline{k}+2, \ldots, n+1\}$ and some $s \in \{1, 2, \ldots, \overline{k}\}$ with $T_{\overline{k}} + a_r \notin M$ and $T_{\overline{k}} + a_r - a_s \notin M$. Consider the set of jump lengths $B = \{a_1, a_2, \ldots, a_{\overline{k}}, a_r\} \setminus \{a_s\}$. We have

$$\sum_{x \in B} x = T_{\overline{k}} + a_r - a_s$$

and

$$T_{\overline{k}} + a_r - a_s - \min(B) = T_{\overline{k}} - a_s \leq T_{\overline{k}-1}.$$

By (4) and the strengthening, mentioned at the very beginning with $\overline{k} - 1$ instead of n, the grasshopper is able to reach $T_{\overline{k}} + a_r - a_s$ by $\overline{k}$ jumps with lengths from B without landing on any point of M. From there he is able to jump to $T_{\overline{k}} + a_r$ and therefore we reach a situation as in Claim 1 with $m = \overline{k} + 1$, which completes the proof.

C8 AUT (Austria)

For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let r be the rightmost digit of n.

(1) If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of n by removing this rightmost digit 0.

(2) If $1 \leq r \leq 9$ we split the decimal representation of n into a maximal right part R that solely consists of digits not less than r and into a left part L that either is empty or ends with a digit strictly smaller than r. Then the decimal representation of $h(n)$ consists of the decimal representation of L, followed by two copies of the decimal representation of $R-1$. For instance, for the number $n = 17{,}151{,}345{,}543$, we will have $L = 17{,}151$, $R = 345{,}543$ and $h(n) = 17{,}151{,}345{,}542{,}345{,}542$.

Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of h produces the integer 1 after finitely many steps.

Solution 1. We identify integers $n \geq 2$ with the digit-strings, briefly strings, of their decimal representation and extend the definition of h to all non-empty strings with digits from 0 to 9. We recursively define ten functions $f_0, \ldots, f_9$ that map some strings into integers for $k = 9, 8, \ldots, 1, 0$. The function f_9 is only defined on strings x (including the empty string ε) that entirely consist of nines. If x consists of m nines, then $f_9(x) = m+1$, $m = 0, 1, \ldots$. For $k \leq 8$, the domain of $f_k(x)$ is the set of all strings consisting only of digits that are $\geq k$. We write x in the form $x_0 k x_1 k x_2 k \ldots x_{m-1} k x_m$ where the strings x_s only consist of digits $\geq k+1$. Note that some of these strings might equal the empty string ε and that $m = 0$ is possible, i.e. the digit k does not appear in x. Then we define

$$f_k(x) = \sum_{s=0}^{m} 4^{f_{k+1}(x_s)}.$$

We will use the following obvious fact:

Fact 1. If x does not contain digits smaller than k, then $f_i(x) = 4^{f_{i+1}(x)}$ for all $i = 0, \ldots, k-1$. In particular, $f_i(\varepsilon) = 4^{9-i}$ for all $i = 0, 1, \ldots, 9$.

Moreover, by induction on $k = 9, 8, \ldots, 0$ it follows easily:

Fact 2. If the nonempty string x does not contain digits smaller than k, then $f_i(x) > f_i(\varepsilon)$ for all $i = 0, \ldots, k$.

We will show the essential fact:

Fact 3. $f_0(n) > f_0(h(n))$.

Then the empty string will necessarily be reached after a finite number of applications of h. But starting from a string without leading zeros, ε can only be reached via the strings $1 \to 00 \to 0 \to \varepsilon$. Hence also the number 1 will appear after a finite number of applications of h.

Proof of Fact 3. If the last digit r of n is 0, then we write $n = x_0 0 \ldots 0 x_{m-1} 0 \varepsilon$ where the x_i do not contain the digit 0. Then $h(n) = x_0 0 \ldots 0 x_{m-1}$ and $f_0(n) - f_0(h(n)) = f_0(\varepsilon) > 0$.

So let the last digit r of n be at least 1. Let $L = yk$ and $R = zr$ be the corresponding left and right parts where y is some string, $k \leq r-1$ and the string z consists only of digits not less than r. Then $n = ykzr$ and $h(n) = ykz(r-1)z(r-1)$. Let $d(y)$ be the smallest digit of y. We consider two cases which do not exclude each other.

Case 1. $d(y) \geq k$.
Then

$$f_k(n) - f_k(h(n)) = f_k(zr) - f_k(z(r-1)z(r-1)).$$

In view of Fact 1 this difference is positive if and only if

$$f_{r-1}(zr) - f_{r-1}(z(r-1)z(r-1)) > 0.$$

We have, using Fact 2,

$$f_{r-1}(zr) = 4^{f_r(zr)} = 4^{f_r(z)+4^{f_{r+1}(\varepsilon)}} \geq 4 \cdot 4^{f_r(z)} > 4^{f_r(z)} + 4^{f_r(z)} + 4^{f_r(\varepsilon)} = f_{r-1}(z(r-1)z(r-1)).$$

Here we use the additional definition $f_{10}(\varepsilon) = 0$ if $r = 9$. Consequently, $f_k(n) - f_k(h(n)) > 0$ and according to Fact 1, $f_0(n) - f_0(h(n)) > 0$.

Case 2. $d(y) \leq k$.
We prove by induction on $d(y) = k, k-1, \ldots, 0$ that $f_i(n) - f_i(h(n)) > 0$ for all $i = 0, \ldots, d(y)$. By Fact 1, it suffices to do so for $i = d(y)$. The initialization $d(y) = k$ was already treated in Case 1. Let $t = d(y) < k$. Write y in the form utv where v does not contain digits $\leq t$. Then, in view of the induction hypothesis,

$$f_t(n) - f_t(h(n)) = f_t(vkzr) - f_t(vkz(r-1)z(r-1)) = 4^{f_{t+1}(vkzr)} - 4^{f_{t+1}(vkz(r-1)z(r-1))} > 0.$$

Thus the inequality $f_{d(y)}(n) - f_{d(y)}(h(n)) > 0$ is established and from Fact 1 it follows that $f_0(n) - f_0(h(n)) > 0$.

Solution 2. We identify integers $n \geq 2$ with the digit-strings, briefly strings, of their decimal representation and extend the definition of h to all non-empty strings with digits from 0 to 9. Moreover, let us define that the empty string, ε, is being mapped to the empty string. In the following all functions map the set of strings into the set of strings. For two functions f and g let $g \circ f$ be defined by $(g \circ f)(x) = g(f(x))$ for all strings x and let, for non-negative integers n, f^n denote the n-fold application of f. For any string x let $s(x)$ be the smallest digit of x, and for the empty string let $s(\varepsilon) = \infty$. We define nine functions $g_1, \ldots, g_9$ as follows: Let $k \in \{1, \ldots, 9\}$ and let x be a string. If $x = \varepsilon$ then $g_k(x) = \varepsilon$. Otherwise, write x in the form $x = yzr$ where y is either the empty string or ends with a digit smaller than k, $s(z) \geq k$ and r is the rightmost digit of x. Then $g_k(x) = zr$.

Lemma 1. We have $g_k \circ h = g_k \circ h \circ g_k$ for all $k = 1, \ldots, 9$.

Proof of Lemma 1. Let $x = yzr$ be as in the definition of g_k. If $y = \varepsilon$, then $g_k(x) = x$, whence

$$g_k(h(x)) = g_k(h(g_k(x))). \tag{1}$$

So let $y \neq \varepsilon$.

Case 1. z contains a digit smaller than r.
Let $z = uav$ where $a < r$ and $s(v) \geq r$. Then

$$h(x) = \begin{cases} yuav & \text{if } r = 0, \\ yuav(r-1)v(r-1) & \text{if } r > 0 \end{cases}$$

and

$$h(g_k(x)) = h(zr) = h(uavr) = \begin{cases} uav & \text{if } r = 0, \\ uav(r-1)v(r-1) & \text{if } r > 0. \end{cases}$$

Since y ends with a digit smaller than k, (1) is obviously true.

Case 2. z does not contain a digit smaller than r.
Let $y = uv$ where u is either the empty string or ends with a digit smaller than r and $s(v) \geq r$. We have

$$h(x) = \begin{cases} uvz & \text{if } r = 0, \\ uvz(r-1)vz(r-1) & \text{if } r > 0 \end{cases}$$

and

$$h(g_k(x)) = h(zr) = \begin{cases} z & \text{if } r = 0, \\ z(r-1)z(r-1) & \text{if } r > 0. \end{cases}$$

Recall that y and hence v ends with a digit smaller than k, but all digits of v are at least r. Now if $r > k$, then $v = \varepsilon$, whence the terminal digit of u is smaller than k, which entails

$$g_k(h(x)) = z(r-1)z(r-1) = g_k(h(g_k(x))).$$

If $r \leq k$, then

$$g_k(h(x)) = z(r-1) = g_k(h(g_k(x))) \ ,$$

so that in both cases (1) is true. Thus Lemma 1 is proved.

Lemma 2. Let $k \in \{1, \ldots, 9\}$, let x be a non-empty string and let n be a positive integer. If $h^n(x) = \varepsilon$ then $(g_k \circ h)^n(x) = \varepsilon$.

Proof of Lemma 2. We proceed by induction on n. If $n = 1$ we have

$$\varepsilon = h(x) = g_k(h(x)) = (g_k \circ h)(x).$$

Now consider the step from $n-1$ to n where $n \geq 2$. Let $h^n(x) = \varepsilon$ and let $y = h(x)$. Then $h^{n-1}(y) = \varepsilon$ and by the induction hypothesis $(g_k \circ h)^{n-1}(y) = \varepsilon$. In view of Lemma 1,

$$\begin{aligned} \varepsilon = (g_k \circ h)^{n-2}((g_k \circ h)(y)) &= (g_k \circ h)^{n-2}(g_k(h(y)) \\ &= (g_k \circ h)^{n-2}(g_k(h(g_k(y))) = (g_k \circ h)^{n-2}(g_k(h(g_k(h(x)))) = (g_k \circ h)^n(x). \end{aligned}$$

Thus the induction step is complete and Lemma 2 is proved.

We say that the non-empty string x *terminates* if $h^n(x) = \varepsilon$ for some non-negative integer n.

Lemma 3. Let $x = yzr$ where $s(y) \geq k$, $s(z) \geq k$, y ends with the digit k and z is possibly empty. If y and zr terminate then also x terminates.

Proof of Lemma 3. Suppose that y and zr terminate. We proceed by induction on k. Let $k = 0$. Obviously, $h(yw) = yh(w)$ for any non-empty string w. Let $h^n(zr) = \epsilon$. It follows easily by induction on m that $h^m(yzr) = yh^m(zr)$ for $m = 1, \ldots, n$. Consequently, $h^n(yzr) = y$. Since y terminates, also $x = yzr$ terminates.

Now let the assertion be true for all nonnegative integers less than k and let us prove it for k where $k \geq 1$. It turns out that it is sufficient to prove that $yg_k(h(zr))$ terminates. Indeed:

Case 1. $r = 0$.
Then $h(yzr) = yz = yg_k(h(zr))$.

Case 2. $0 < r \leq k$.
We have $h(zr) = z(r-1)z(r-1)$ and $g_k(h(zr)) = z(r-1)$. Then $h(yzr) = yz(r-1)yz(r-$

$1) = yg_k(h(zr))yg_k(h(zr))$ and we may apply the induction hypothesis to see that if $yg_kh(zr))$ terminates, then $h(yzr)$ terminates.

Case 3. $r > k$.
Then $h(yzr) = yh(zr) = yg_k(h(zr))$.

Note that $yg_k(h(zr))$ has the form $yz'r'$ where $s(z') \geq k$. By the same arguments it is sufficient to prove that $yg_k(h(z'r')) = y(g_k \circ h)^2(zr)$ terminates and, by induction, that $y(g_k \circ h)^m(zr)$ terminates for some positive integer m. In view of Lemma 2 there is some m such that $(g_k \circ h)^m(zr) = \epsilon$, so $x = yzr$ terminates if y terminates. Thus Lemma 3 is proved.

Now assume that there is some string x that does not terminate. We choose x minimal. If $x \geq 10$, we can write x in the form $x = yzr$ of Lemma 3 and by this lemma x terminates since y and zr are smaller than x. If $x \leq 9$, then $h(x) = (x-1)(x-1)$ and $h(x)$ terminates again by Lemma 3 and the minimal choice of x.

Solution 3. We commence by introducing some terminology. Instead of integers, we will consider the set S of all strings consisting of the digits $0, 1, \ldots, 9$, including the empty string ϵ. If $(a_1, a_2, \ldots, a_n)$ is a nonempty string, we let $\rho(a) = a_n$ denote the *terminal digit of* a and $\lambda(a)$ be the string with the last digit removed. We also define $\lambda(\epsilon) = \epsilon$ and denote the set of non-negative integers by $\mathbb{N}_0$.

Now let $k \in \{0, 1, 2, \ldots, 9\}$ denote any digit. We define a function $f_k : S \longrightarrow S$ on the set of strings: First, if the terminal digit of n belongs to $\{0, 1, \ldots, k\}$, then $f_k(n)$ is obtained from n by deleting this terminal digit, i.e $f_k(n) = \lambda(n)$. Secondly, if the terminal digit of n belongs to $\{k+1, \ldots, 9\}$, then $f_k(n)$ is obtained from n by the process described in the problem. We also define $f_k(\epsilon) = \epsilon$. Note that up to the definition for integers $n \leq 1$, the function f_0 coincides with the function h in the problem, through interpreting integers as digit strings. The argument will be roughly as follows. We begin by introducing a straightforward generalization of our claim about f_0. Then it will be easy to see that f_9 has all these stronger properties, which means that is suffices to show for $k \in \{0, 1, \ldots, 8\}$ that f_k possesses these properties provided that f_{k+1} does.

We continue to use k to denote any digit. The operation f_k is said to be *separating*, if the followings holds: Whenever a is an initial segment of b, there is some $N \in \mathbb{N}_0$ such that $f_k^N(b) = a$. The following two notions only apply to the case where f_k is indeed separating, otherwise they remain undefined. For every $a \in S$ we denote the least $N \in \mathbb{N}_0$ for which $f_k^N(a) = \epsilon$ occurs by $g_k(a)$ (because ϵ is an initial segment of a, such an N exists if f_k is separating). If for every two strings a and b such that a is a terminal segment of b one has $g_k(a) \leq g_k(b)$, we say that f_k is *coherent*. In case that f_k is separating and coherent we call the digit k *seductive*.

As $f_9(a) = \lambda(a)$ for all a, it is obvious that 9 is seductive. Hence in order to show that 0 is seductive, which clearly implies the statement of the problem, it suffices to take any $k \in \{0, 1, \ldots, 8\}$ such that $k+1$ is seductive and to prove that k has to be seductive as well. Note that in doing so, we have the function g_{k+1} at our disposal. We have to establish two things and we begin with

Step 1. f_k is separating.

Before embarking on the proof of this, we record a useful observation which is easily proved by induction on M.

Claim 1. For any strings A, B and any positive integer M such that $f_k^{M-1}(B) \neq \epsilon$, we have

$$f_k^M(AkB) = Akf_k^M(B).$$

Now we call a pair (a, b) of strings *wicked* provided that a is an initial segment of b, but there is no $N \in \mathbb{N}_0$ such that $f_k^N(b) = a$. We need to show that there are none, so assume that there were such pairs. Choose a wicked pair (a, b) for which $g_{k+1}(b)$ attains its minimal possible value. Obviously $b \neq \epsilon$ for any wicked pair (a, b). Let z denote the terminal digit of b. Observe that $a \neq b$, which means that a is also an initial segment of $\lambda(b)$. To facilitate the construction of the eventual contradiction, we prove

Claim 2. There cannot be an $N \in \mathbb{N}_0$ such that

$$f_k^N(b) = \lambda(b).$$

Proof of Claim 2. For suppose that such an N existed. Because $g_{k+1}(\lambda(b)) < g_{k+1}(b)$ in view of the coherency of f_{k+1}, the pair $(a, \lambda(b))$ is not wicked. But then there is some N' for which $f_k^{N'}(\lambda(b)) = a$ which entails $f_k^{N+N'}(b) = a$, contradiction. Hence Claim 2 is proved.

It follows that $z \leq k$ is impossible, for otherwise $N = 1$ violated Claim 2.

Also $z > k+1$ is impossible: Set $B = f_k(b)$. Then also $f_{k+1}(b) = B$, but $g_{k+1}(B) < g_{k+1}(b)$ and a is an initial segment of B. Thus the pair (a, B) is not wicked. Hence there is some $N \in \mathbb{N}_0$ with $a = f_k^N(B)$, which, however, entails $a = f_k^{N+1}(b)$.

We are left with the case $z = k+1$. Let L denote the left part and $R = R^*(k+1)$ the right part of b. Then we have symbolically

$$f_k(b) = LR^*kR^*k\ ,\ f_k^2(b) = LR^*kR^* \quad \text{and} \quad f_{k+1}(b) = LR^*.$$

Using that R^* is a terminal segment of LR^* and the coherency of f_{k+1}, we infer

$$g_{k+1}(R^*) \leq g_{k+1}(LR^*) < g_{k+1}(b).$$

Hence the pair (ϵ, R^*) is not wicked, so there is some minimal $M \in \mathbb{N}_0$ with $f_k^M(R^*) = \epsilon$ and by Claim 1 it follows that $f_k^{2+M}(b) = LR^*k$. Finally, we infer that $\lambda(b) = LR^* = f_k(LR^*k) = f_k^{3+M}(b)$, which yields a contradiction to Claim 2.

This final contradiction establishes that f_k is indeed separating.

Step 2. f_k is coherent.

To prepare the proof of this, we introduce some further pieces of terminology. A nonempty string $(a_1, a_2, \ldots, a_n)$ is called a *hypostasis*, if $a_n < a_i$ for all $i = 1, \ldots, n-1$. Reading an arbitrary string a backwards, we easily find a, possibly empty, sequence $(A_1, A_2, \ldots, A_m)$ of hypostases such that $\rho(A_1) \leq \rho(A_2) \leq \cdots \leq \rho(A_m)$ and, symbolically, $a = A_1A_2 \ldots A_m$. The latter sequence is referred to as the *decomposition* of a. So, for instance, $(20, 0, 9)$ is the decomposition of 2009 and the string 50 is a hypostasis. Next we explain when we say about two strings a and b that a is *injectible* into b. The definition is by induction on the length of b. Let $(B_1, B_2, \ldots, B_n)$ be the decomposition of b into hypostases. Then a is injectible into b if for the decomposition $(A_1, A_2, \ldots, A_m)$ of a there is a strictly increasing function $H : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$ satisfying

$\rho(A_i) = \rho(B_{H(i)})$ for all $i = 1, \ldots, m$;

$\lambda(A_i)$ is injectible into $\lambda(B_{H(i)})$ for all $i = 1, \ldots, m$.

If one can choose H with $H(m) = n$, then we say that a is *strongly injectible* into b. Obviously, if a is a terminal segment of b, then a is strongly injectible into b.

Claim 3. If a and b are two nonempty strings such that a is strongly injectible into b, then $\lambda(a)$ is injectible into $\lambda(b)$.

Proof of Claim 3. Let $(B_1, B_2, \ldots, B_n)$ be the decomposition of b and let $(A_1, A_2, \ldots, A_m)$ be the decomposition of a. Take a function H exemplifying that a is strongly injectible into b. Let $(C_1, C_2, \ldots, C_r)$ be the decomposition of $\lambda(A_m)$ and let $(D_1, D_2, \ldots, D_s)$ be the decomposition of $\lambda(B_n)$. Choose a strictly increasing $H' : \{1, 2, \ldots, r\} \longrightarrow \{1, 2, \ldots s\}$ witnessing that $\lambda(A_m)$ is injectible into $\lambda(B_n)$. Clearly, $(A_1, A_2, \ldots, A_{m-1}, C_1, C_2, \ldots, C_r)$ is the decomposition of $\lambda(a)$ and $(B_1, B_2, \ldots, B_{n-1}, D_1, D_2, \ldots, D_s)$ is the decomposition of $\lambda(b)$. Then the function $H'' : \{1, 2, \ldots, m+r-1\} \longrightarrow \{1, 2, \ldots, n+s-1\}$ given by $H''(i) = H(i)$ for $i = 1, 2, \ldots, m-1$ and $H''(m-1+i) = n-1+H'(i)$ for $i = 1, 2, \ldots, r$ exemplifies that $\lambda(a)$ is injectible into $\lambda(b)$, which finishes the proof of the claim.

A pair (a, b) of strings is called *aggressive* if a is injectible into b and nevertheless $g_k(a) > g_k(b)$. Observe that if f_k was incoherent, which we shall assume from now on, then such pairs existed. Now among all aggressive pairs we choose one, say (a, b), for which $g_k(b)$ attains its least possible value. Obviously $f_k(a)$ cannot be injectible into $f_k(b)$, for otherwise the pair $(f_k(a), f_k(b))$ was aggressive and contradicted our choice of (a, b). Let $(A_1, A_2, \ldots, A_m)$ and $(B_1, B_2, \ldots, B_n)$ be the decompositions of a and b and take a function $H : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$ exemplifying that a is indeed injectible into b. If we had $H(m) < n$, then a was also injectible into the number b' whose decomposition is $(B_1, B_2, \ldots, B_{n-1})$ and by separativity of f_k we obtained $g_k(b') < g_k(b)$, whence the pair (a, b') was also aggressive, contrary to the minimality condition imposed on b. Therefore a is strongly injectible into b. In particular, a and b have a common terminal digit, say z. If we had $z \leq k$, then $f_k(a) = \lambda(a)$ and $f_k(b) = \lambda(b)$, so that by Claim 3, $f_k(a)$ was injectible into $f_k(b)$, which is a contradiction. Hence, $z \geq k+1$.

Now let r be the minimal element of $\{1, 2, \ldots, m\}$ for which $\rho(A_r) = z$. Then the maximal right part of a consisting of digits $\geq z$ is equal to R_a, the string whose decomposition is $(A_r, A_{r+1}, \ldots, A_m)$. Then $R_a - 1$ is a hypostasis and $(A_1, \ldots, A_{r-1}, R_a - 1, R_a - 1)$ is the decomposition of $f_k(a)$. Defining s and R_b in a similar fashion with respect to b, we see that $(B_1, \ldots, B_{s-1}, R_b - 1, R_b - 1)$ is the decomposition of $f_k(b)$. The definition of injectibility then easily entails that R_a is strongly injectible into R_b. It follows from Claim 3 that $\lambda(R_a) = \lambda(R_a - 1)$ is injectible into $\lambda(R_b) = \lambda(R_b - 1)$, whence the function $H' : \{1, 2, \ldots, r+1\} \longrightarrow \{1, 2, \ldots, s+1\}$, given by $H'(i) = H(i)$ for $i = 1, 2, \ldots, r-1$, $H'(r) = s$ and $H'(r+1) = s+1$ exemplifies that $f_k(a)$ is injectible into $f_k(b)$, which yields a contradiction as before.

This shows that aggressive pairs cannot exist, whence f_k is indeed coherent, which finishes the proof of the seductivity of k, whereby the problem is finally solved.

Geometry

G1 BEL (Belgium)

Let ABC be a triangle with $AB = AC$. The angle bisectors of A and B meet the sides BC and AC in D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle BAC$.

Solution 1. Answer: $\angle BAC = 60°$ or $\angle BAC = 90°$ are possible values and the only possible values.

Let I be the incenter of triangle ABC, then K lies on the line CI. Let F be the point, where the incircle of triangle ABC touches the side AC; then the segments IF and ID have the same length and are perpendicular to AC and BC, respectively.

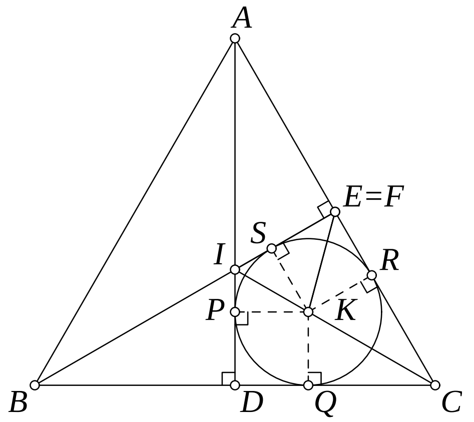


Figure 1

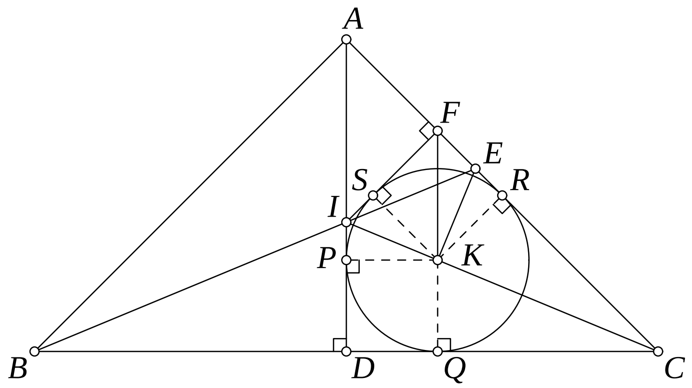


Figure 2

Let P, Q and R be the points where the incircle of triangle ADC touches the sides AD, DC and CA, respectively. Since K and I lie on the angle bisector of $\angle ACD$, the segments ID and IF are symmetric with respect to the line IC. Hence there is a point S on IF where the incircle of triangle ADC touches the segment IF. Then segments KP, KQ, KR and KS all have the same length and are perpendicular to AD, DC, CA and IF, respectively. So – regardless of the value of $\angle BEK$ – the quadrilateral $KRFS$ is a square and $\angle SFK = \angle KFC = 45°$.

Consider the case $\angle BAC = 60°$ (see Figure 1). Then triangle ABC is equilateral. Furthermore we have $F = E$, hence $\angle BEK = \angle IFK = \angle SEK = 45°$. So $60°$ is a possible value for $\angle BAC$.

Now consider the case $\angle BAC = 90°$ (see Figure 2). Then $\angle CBA = \angle ACB = 45°$. Furthermore, $\angle KIE = \frac{1}{2}\angle CBA + \frac{1}{2}\angle ACB = 45°$, $\angle AEB = 180° - 90° - 22.5° = 67.5°$ and $\angle EIA = \angle BID = 180° - 90° - 22.5° = 67.5°$. Hence triangle IEA is isosceles and a reflection of the bisector of $\angle IAE$ takes I to E and K to itself. So triangle IKE is symmetric with respect to this axis, i.e. $\angle KIE = \angle IEK = \angle BEK = 45°$. So $90°$ is a possible value for $\angle BAC$, too.

If, on the other hand, $\angle BEK = 45°$ then $\angle BEK = \angle IEK = \angle IFK = 45°$. Then

- either $F = E$, which makes the angle bisector BI be an altitude, i.e., which makes triangle ABC isosceles with base AC and hence equilateral and so $\angle BAC = 60°$,
- or E lies between F and C, which makes the points K, E, F and I concyclic, so $45° = \angle KFC = \angle KFE = \angle KIE = \angle CBI + \angle ICB = 2 \cdot \angle ICB = 90° - \frac{1}{2}\angle BAC$, and so $\angle BAC = 90°$,

- or F lies between E and C, then again, K, E, F and I are concyclic, so $45^\circ = \angle KFC = 180^\circ - \angle KFE = \angle KIE$, which yields the same result $\angle BAC = 90^\circ$. (However, for $\angle BAC = 90^\circ$ E lies, in fact, between F and C, see Figure 2. So this case does not occur.)

This proves 90° and 60° to be the only possible values for $\angle BAC$.

Solution 2. Denote angles at A, B and C as usual by α, β and γ. Since triangle ABC is isosceles, we have $\beta = \gamma = 90^\circ - \frac{\alpha}{2} < 90^\circ$, so $\angle ECK = 45^\circ - \frac{\alpha}{4} = \angle KCD$. Since K is the incenter of triangle ADC, we have $\angle CDK = \angle KDA = 45^\circ$; furthermore $\angle DIC = 45^\circ + \frac{\alpha}{4}$. Now, if $\angle BEK = 45^\circ$, easy calculations within triangles BCE and KCE yield

$$\begin{aligned}\angle KEC &= 180^\circ - \tfrac{\beta}{2} - 45^\circ - \beta = 135^\circ - \tfrac{3}{2}\beta = \tfrac{3}{2}(90^\circ - \beta) = \tfrac{3}{4}\alpha,\\ \angle IKE &= \tfrac{3}{4}\alpha + 45^\circ - \tfrac{\alpha}{4} = 45^\circ + \tfrac{\alpha}{2}.\end{aligned}$$

So in triangles ICE, IKE, IDK and IDC we have (see Figure 3)

$$\frac{IC}{IE} = \frac{\sin\angle IEC}{\sin\angle ECI} = \frac{\sin(45^\circ + \frac{3}{4}\alpha)}{\sin(45^\circ - \frac{\alpha}{4})}, \qquad \frac{IE}{IK} = \frac{\sin\angle EKI}{\sin\angle IEK} = \frac{\sin(45^\circ + \frac{\alpha}{2})}{\sin 45^\circ},$$

$$\frac{IK}{ID} = \frac{\sin\angle KDI}{\sin\angle IKD} = \frac{\sin 45^\circ}{\sin(90^\circ - \frac{\alpha}{4})}, \qquad \frac{ID}{IC} = \frac{\sin\angle ICD}{\sin\angle CDI} = \frac{\sin(45^\circ - \frac{\alpha}{4})}{\sin 90^\circ}.$$

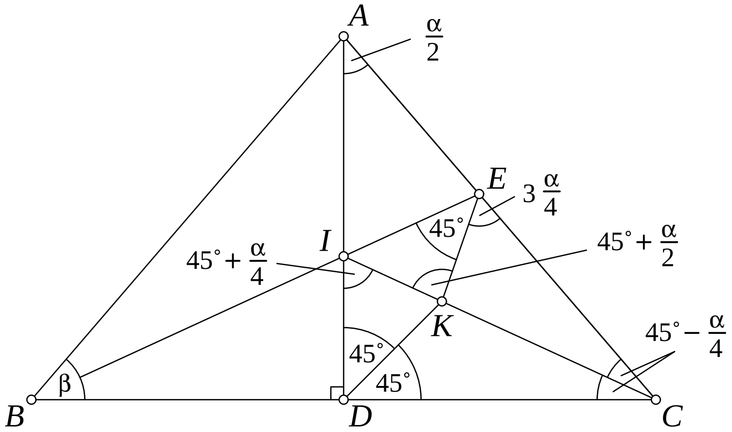


Figure 3

Multiplication of these four equations yields

$$1 = \frac{\sin(45^\circ + \frac{3}{4}\alpha)\sin(45^\circ + \frac{\alpha}{2})}{\sin(90^\circ - \frac{\alpha}{4})}.$$

But, since

$$\begin{aligned}\sin(90^\circ - \tfrac{\alpha}{4}) &= \cos\tfrac{\alpha}{4} = \cos\big((45^\circ + \tfrac{3}{4}\alpha) - (45^\circ + \tfrac{\alpha}{2})\big)\\ &= \cos\big(45^\circ + \tfrac{3}{4}\alpha\big)\cos(45^\circ + \tfrac{\alpha}{2}) + \sin(45^\circ + \tfrac{3}{4}\alpha)\sin(45^\circ + \tfrac{\alpha}{2}),\end{aligned}$$

this is equivalent to

$$\sin(45^\circ + \tfrac{3}{4}\alpha)\sin(45^\circ + \tfrac{\alpha}{2}) = \cos(45^\circ + \tfrac{3}{4}\alpha)\cos(45^\circ + \tfrac{\alpha}{2}) + \sin(45^\circ + \tfrac{3}{4}\alpha)\sin(45^\circ + \tfrac{\alpha}{2})$$

and finally

$$\cos(45^\circ + \tfrac{3}{4}\alpha)\cos(45^\circ + \tfrac{\alpha}{2}) = 0.$$

But this means $\cos\left(45^\circ + \frac{3}{4}\alpha\right) = 0$, hence $45^\circ + \frac{3}{4}\alpha = 90^\circ$, i.e. $\alpha = 60^\circ$ or $\cos\left(45^\circ + \frac{\alpha}{2}\right) = 0$, hence $45^\circ + \frac{\alpha}{2} = 90^\circ$, i.e. $\alpha = 90^\circ$. So these values are the only two possible values for α.

On the other hand, both $\alpha = 90^\circ$ and $\alpha = 60^\circ$ yield $\angle BEK = 45^\circ$, this was shown in Solution 1.

G2 RUS (Russian Federation)

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. The circle k passes through the midpoints of the segments BP, CQ, and PQ. Prove that if the line PQ is tangent to circle k then $OP = OQ$.

Solution 1. Let K, L, M, B', C' be the midpoints of BP, CQ, PQ, CA, and AB, respectively (see Figure 1). Since $CA \parallel LM$, we have $\angle LMP = \angle QPA$. Since k touches the segment PQ at M, we find $\angle LMP = \angle LKM$. Thus $\angle QPA = \angle LKM$. Similarly it follows from $AB \parallel MK$ that $\angle PQA = \angle KLM$. Therefore, triangles APQ and MKL are similar, hence

$$\frac{AP}{AQ} = \frac{MK}{ML} = \frac{\frac{QB}{2}}{\frac{PC}{2}} = \frac{QB}{PC}. \tag{1}$$

Now (1) is equivalent to $AP \cdot PC = AQ \cdot QB$ which means that the power of points P and Q with respect to the circumcircle of $\triangle ABC$ are equal, hence $OP = OQ$.

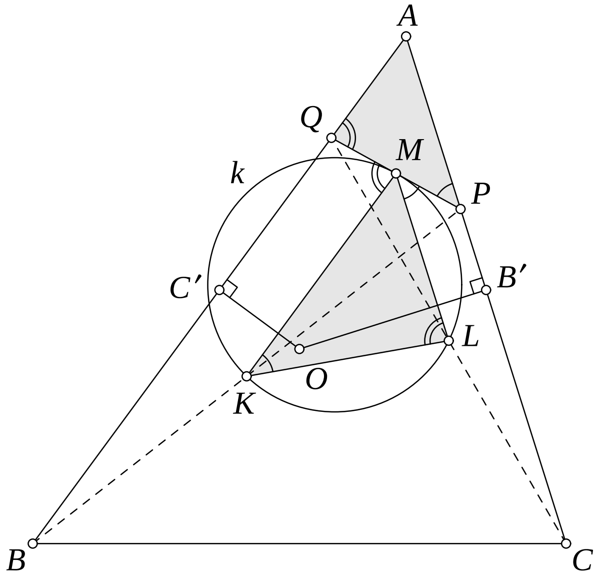


Figure 1

Comment. The last argument can also be established by the following calculation:

$$\begin{aligned} OP^2 - OQ^2 &= OB'^2 + B'P^2 - OC'^2 - C'Q^2 \\ &= (OA^2 - AB'^2) + B'P^2 - (OA^2 - AC'^2) - C'Q^2 \\ &= (AC'^2 - C'Q^2) - (AB'^2 - B'P^2) \\ &= (AC' - C'Q)(AC' + C'Q) - (AB' - B'P)(AB' + B'P) \\ &= AQ \cdot QB - AP \cdot PC. \end{aligned}$$

With (1), we conclude $OP^2 - OQ^2 = 0$, as desired.

Solution 2. Again, denote by K, L, M the midpoints of segments BP, CQ, and PQ, respectively. Let O, S, T be the circumcenters of triangles ABC, KLM, and APQ, respectively (see Figure 2). Note that MK and LM are the midlines in triangles BPQ and CPQ, respectively, so $\overrightarrow{MK} = \frac{1}{2}\overrightarrow{QB}$ and $\overrightarrow{ML} = \frac{1}{2}\overrightarrow{PC}$. Denote by $\mathrm{pr}_l(\overrightarrow{v})$ the projection of vector $\overrightarrow{v}$ onto line l. Then $\mathrm{pr}_{AB}(\overrightarrow{OT}) = \mathrm{pr}_{AB}(\overrightarrow{OA} - \overrightarrow{TA}) = \frac{1}{2}\overrightarrow{BA} - \frac{1}{2}\overrightarrow{QA} = \frac{1}{2}\overrightarrow{BQ} = \overrightarrow{KM}$ and $\mathrm{pr}_{AB}(\overrightarrow{SM}) = \mathrm{pr}_{MK}(\overrightarrow{SM}) = \frac{1}{2}\overrightarrow{KM} = \frac{1}{2}\mathrm{pr}_{AB}(\overrightarrow{OT})$. Analogously we get $\mathrm{pr}_{CA}(\overrightarrow{SM}) = \frac{1}{2}\mathrm{pr}_{CA}(\overrightarrow{OT})$. Since AB and CA are not parallel, this implies that $\overrightarrow{SM} = \frac{1}{2}\overrightarrow{OT}$.

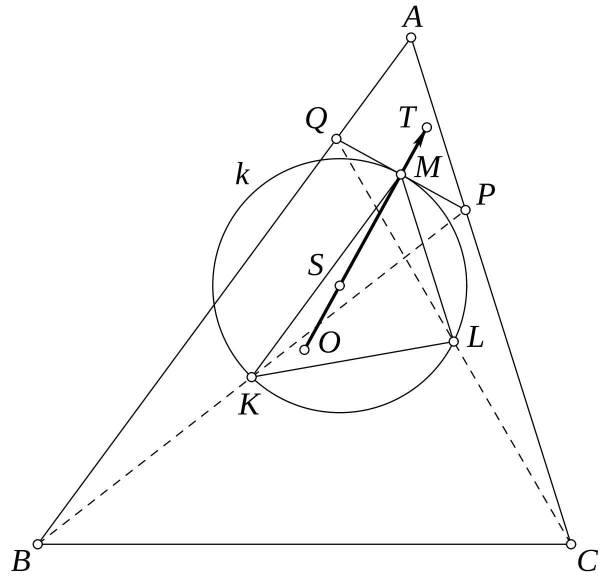


Figure 2

Now, since the circle k touches PQ at M, we get $SM \perp PQ$, hence $OT \perp PQ$. Since T is equidistant from P and Q, the line OT is a perpendicular bisector of segment PQ, and hence O is equidistant from P and Q which finishes the proof.

G3 IRN (Islamic Republic of Iran)

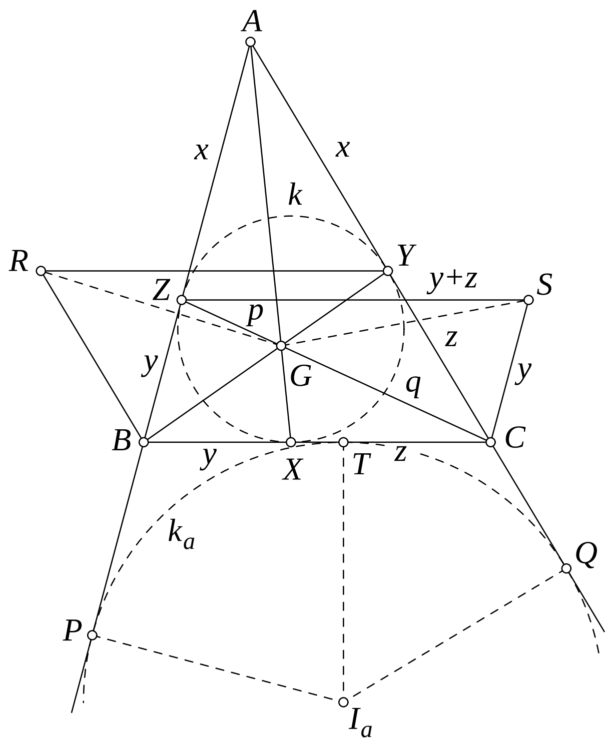
Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms.

Prove that $GR = GS$.

Solution 1. Denote by k the incircle and by k_a the excircle opposite to A of triangle ABC. Let k and k_a touch the side BC at the points X and T, respectively, let k_a touch the lines AB and AC at the points P and Q, respectively. We use several times the fact that opposing sides of a parallelogram are of equal length, that points of contact of the excircle and incircle to a side of a triangle lie symmetric with respect to the midpoint of this side and that segments on two tangents to a circle defined by the points of contact and their point of intersection have the same length. So we conclude

$$ZP = ZB + BP = XB + BT = BX + CX = ZS \quad \text{and}$$

$$CQ = CT = BX = BZ = CS.$$



So for each of the points Z, C, their distances to S equal the length of a tangent segment from this point to k_a. It is well-known, that all points with this property lie on the line ZC, which is the radical axis of S and k_a. Similar arguments yield that BY is the radical axis of R and k_a. So the point of intersection of ZC and BY, which is G by definition, is the radical center of R, S and k_a, from which the claim $GR = GS$ follows immediately.

Solution 2. Denote $x = AZ = AY$, $y = BZ = BX$, $z = CX = CY$, $p = ZG$, $q = GC$. Several lengthy calculations (Menelaos' theorem in triangle AZC, law of Cosines in triangles ABC and AZC and Stewart's theorem in triangle ZCS) give four equations for p, q, $\cos\alpha$

and GS in terms of x, y, and z that can be resolved for GS. The result is symmetric in y and z, so $GR = GS$. More in detail this means:

The line BY intersects the sides of triangle AZC, so MENELAOS' theorem yields $\frac{p}{q} \cdot \frac{z}{x} \cdot \frac{x+y}{y} = 1$, hence

$$\frac{p}{q} = \frac{xy}{yz + zx}. \tag{1}$$

Since we only want to show that the term for GS is symmetric in y and z, we abbreviate terms that are symmetric in y and z by capital letters, starting with $N = xy + yz + zx$. So (1) implies

$$\frac{p}{p+q} = \frac{xy}{xy+yz+zx} = \frac{xy}{N} \qquad \text{and} \qquad \frac{q}{p+q} = \frac{yz+zx}{xy+yz+zx} = \frac{yz+zx}{N}. \tag{2}$$

Now the law of Cosines in triangle ABC yields

$$\cos\alpha = \frac{(x+y)^2 + (x+z)^2 - (y+z)^2}{2(x+y)(x+z)} = \frac{2x^2 + 2xy + 2xz - 2yz}{2(x+y)(x+z)} = 1 - \frac{2yz}{(x+y)(x+z)}.$$

We use this result to apply the law of Cosines in triangle AZC:

$$\begin{aligned}(p+q)^2 &= x^2 + (x+z)^2 - 2x(x+z)\cos\alpha \\ &= x^2 + (x+z)^2 - 2x(x+z) \cdot \left(1 - \frac{2yz}{(x+y)(x+z)}\right) \\ &= z^2 + \frac{4xyz}{x+y}. \end{aligned} \tag{3}$$

Now in triangle ZCS the segment GS is a cevian, so with STEWART's theorem we have $py^2 + q(y+z)^2 = (p+q)(GS^2 + pq)$, hence

$$GS^2 = \frac{p}{p+q} \cdot y^2 + \frac{q}{p+q} \cdot (y+z)^2 - \frac{p}{p+q} \cdot \frac{q}{p+q} \cdot (p+q)^2.$$

Replacing the p's and q's herein by (2) and (3) yields

$$\begin{aligned} GS^2 &= \frac{xy}{N}y^2 + \frac{yz+zx}{N}(y+z)^2 - \frac{xy}{N} \cdot \frac{yz+zx}{N} \cdot \left(z^2 + \frac{4xyz}{x+y}\right) \\ &= \frac{xy^3}{N} + \underbrace{\frac{yz(y+z)^2}{N}}_{M_1} + \frac{zx(y+z)^2}{N} - \frac{xyz^3(x+y)}{N^2} - \underbrace{\frac{4x^2y^2z^2}{N^2}}_{M_2} \\ &= \frac{xy^3 + zx(y+z)^2}{N} - \frac{xyz^3(x+y)}{N^2} + M_1 - M_2 \\ &= \underbrace{\frac{x(y^3 + y^2z + yz^2 + z^3)}{N}}_{M_3} + \frac{xyz^2N}{N^2} - \frac{xyz^3(x+y)}{N^2} + M_1 - M_2 \\ &= \frac{x^2y^2z^2 + xy^2z^3 + x^2yz^3 - x^2yz^3 - xy^2z^3}{N^2} + M_1 - M_2 + M_3 \\ &= \frac{x^2y^2z^2}{N^2} + M_1 - M_2 + M_3, \end{aligned}$$

a term that is symmetric in y and z, indeed.

Comment. G is known as GERGONNE's point of $\triangle ABC$.

G4 UNK (United Kingdom)

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G, and H.

Solution 1. It suffices to show that $\angle HEF = \angle HGE$ (see Figure 1), since in circle EGH the angle over the chord EH at G equals the angle between the tangent at E and EH.

First, $\angle BAD = 180° - \angle DCB = \angle FCD$. Since triangles FAB and FCD have also a common interior angle at F, they are similar.

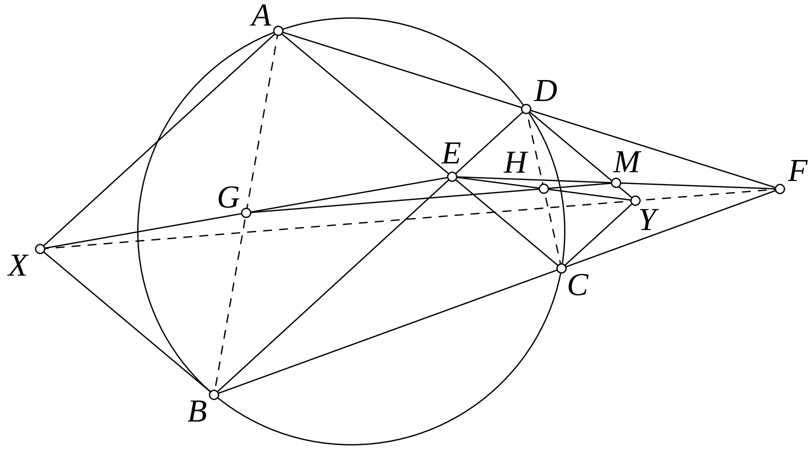


Figure 1

Denote by $\mathcal{T}$ the transformation consisting of a reflection at the bisector of $\angle DFC$ followed by a dilation with center F and factor of $\frac{FA}{FC}$. Then $\mathcal{T}$ maps F to F, C to A, D to B, and H to G. To see this, note that $\triangle FCA \sim \triangle FDB$, so $\frac{FA}{FC} = \frac{FB}{FD}$. Moreover, as $\angle ADB = \angle ACB$, the image of the line DE under $\mathcal{T}$ is parallel to AC (and passes through B) and similarly the image of CE is parallel to DB and passes through A. Hence E is mapped to the point X which is the fourth vertex of the parallelogram $BEAX$. Thus, in particular $\angle HEF = \angle FXG$.

As G is the midpoint of the diagonal AB of the parallelogram $BEAX$, it is also the midpoint of EX. In particular, E, G, X are collinear, and $EX = 2 \cdot EG$.

Denote by Y the fourth vertex of the parallelogram $DECY$. By an analogous reasoning as before, it follows that $\mathcal{T}$ maps Y to E, thus E, H, Y are collinear with $EY = 2 \cdot EH$. Therefore, by the intercept theorem, $HG \parallel XY$.

From the construction of $\mathcal{T}$ it is clear that the lines FX and FE are symmetric with respect to the bisector of $\angle DFC$, as are FY and FE. Thus, F, X, Y are collinear, which together with $HG \parallel XY$ implies $\angle FXE = \angle HGE$. This completes the proof.

Solution 2. We use the following

Lemma (Gauß). Let $ABCD$ be a quadrilateral. Let AB and CD intersect at P, and BC and DA intersect at Q. Then the midpoints K, L, M of AC, BD, and PQ, respectively, are collinear.

Proof: Let us consider the points Z that fulfill the equation

$$(ABZ) + (CDZ) = (BCZ) + (DAZ), \tag{1}$$

where (RST) denotes the oriented area of the triangle RST (see Figure 2).

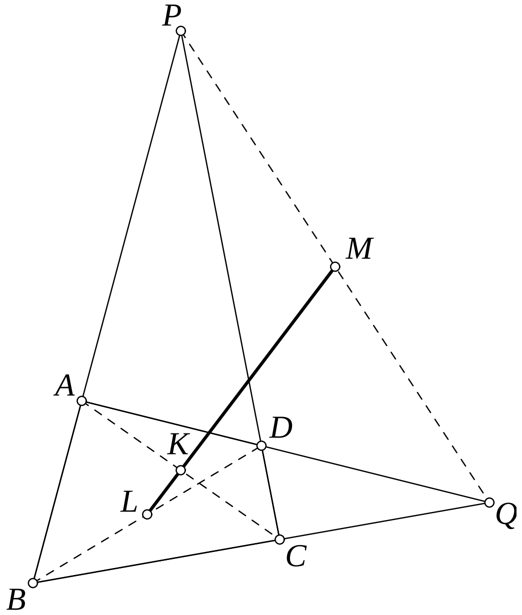


Figure 2

As (1) is linear in Z, it can either characterize a line, or be contradictory, or be trivially fulfilled for all Z in the plane. If (1) was fulfilled for all Z, then it would hold for $Z = A$, $Z = B$, which gives $(CDA) = (BCA)$, $(CDB) = (DAB)$, respectively, i.e. the diagonals of $ABCD$ would bisect each other, thus $ABCD$ would be a parallelogram. This contradicts the hypothesis that AD and BC intersect. Since E, F, G fulfill (1), it is the equation of a line which completes the proof of the lemma.

Now consider the parallelograms $EAXB$ and $ECYD$ (see Figure 1). Then G, H are the midpoints of EX, EY, respectively. Let M be the midpoint of EF. By applying the Lemma to the (re-entrant) quadrilateral $ADBC$, it is evident that G, H, and M are collinear. A dilation by a factor of 2 with center E shows that X, Y, F are collinear. Since $AX \parallel DE$ and $BX \parallel CE$, we have pairwise equal interior angles in the quadrilaterals $FDEC$ and $FBXA$. Since we have also $\angle EBA = \angle DCA = \angle CDY$, the quadrilaterals are similar. Thus, $\angle FXA = \angle CEF$.

Clearly the parallelograms $ECYD$ and $EBXA$ are similar, too, thus $\angle EXA = \angle CEY$. Consequently, $\angle FXE = \angle FXA - \angle EXA = \angle CEF - \angle CEY = \angle YEF$. By the converse of the tangent-chord angle theorem EF is tangent to the circle XEY. A dilation by a factor of $\frac{1}{2}$ completes the proof.

Solution 3. As in Solution 2, G, H, M are proven to be collinear. It suffices to show that $ME^2 = MG \cdot MH$. If $\boldsymbol{p} = \overrightarrow{OP}$ denotes the vector from circumcenter O to point P, the claim becomes

$$\left(\frac{\boldsymbol{e}-\boldsymbol{f}}{2}\right)^2 = \left(\frac{\boldsymbol{e}+\boldsymbol{f}}{2} - \frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right)\left(\frac{\boldsymbol{e}+\boldsymbol{f}}{2} - \frac{\boldsymbol{c}+\boldsymbol{d}}{2}\right),$$

or equivalently

$$4\,\boldsymbol{ef} - (\boldsymbol{e}+\boldsymbol{f})(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}) + (\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{c}+\boldsymbol{d}) = 0. \tag{2}$$

With R as the circumradius of $ABCD$, we obtain for the powers $\mathcal{P}(E)$ and $\mathcal{P}(F)$ of E and F, respectively, with respect to the circumcircle

$$\begin{aligned}\mathcal{P}(E) &= (\boldsymbol{e}-\boldsymbol{a})(\boldsymbol{e}-\boldsymbol{c}) = (\boldsymbol{e}-\boldsymbol{b})(\boldsymbol{e}-\boldsymbol{d}) = \boldsymbol{e}^2 - R^2,\\ \mathcal{P}(F) &= (\boldsymbol{f}-\boldsymbol{a})(\boldsymbol{f}-\boldsymbol{d}) = (\boldsymbol{f}-\boldsymbol{b})(\boldsymbol{f}-\boldsymbol{c}) = \boldsymbol{f}^2 - R^2,\end{aligned}$$

hence

$$(\boldsymbol{e}-\boldsymbol{a})(\boldsymbol{e}-\boldsymbol{c}) = \boldsymbol{e}^2 - R^2, \tag{3}$$
$$(\boldsymbol{e}-\boldsymbol{b})(\boldsymbol{e}-\boldsymbol{d}) = \boldsymbol{e}^2 - R^2, \tag{4}$$
$$(\boldsymbol{f}-\boldsymbol{a})(\boldsymbol{f}-\boldsymbol{d}) = \boldsymbol{f}^2 - R^2, \tag{5}$$
$$(\boldsymbol{f}-\boldsymbol{b})(\boldsymbol{f}-\boldsymbol{c}) = \boldsymbol{f}^2 - R^2. \tag{6}$$

Since F lies on the polar to E with respect to the circumcircle, we have

$$4\,\boldsymbol{ef} = 4R^2. \tag{7}$$

Adding up (3) to (7) yields (2), as desired.

G5 POL (Poland)

Let P be a polygon that is convex and symmetric to some point O. Prove that for some parallelogram R satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets R and P, respectively.

Solution 1. We will construct two parallelograms R_1 and R_3, each of them containing P, and prove that at least one of the inequalities $|R_1| \leq \sqrt{2}\,|P|$ and $|R_3| \leq \sqrt{2}\,|P|$ holds (see Figure 1).

First we will construct a parallelogram $R_1 \supseteq P$ with the property that the midpoints of the sides of R_1 are points of the boundary of P.

Choose two points A and B of P such that the triangle OAB has maximal area. Let a be the line through A parallel to OB and b the line through B parallel to OA. Let A', B', a' and b' be the points or lines, that are symmetric to A, B, a and b, respectively, with respect to O. Now let R_1 be the parallelogram defined by a, b, a' and b'.

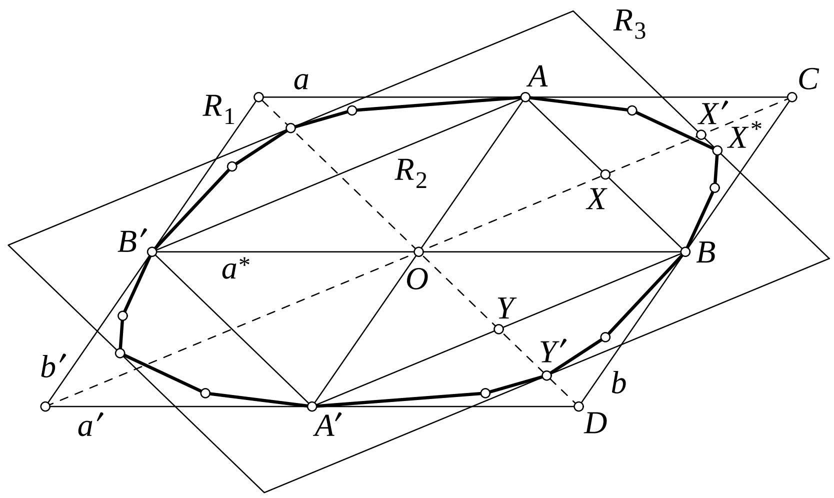


Figure 1

Obviously, A and B are located on the boundary of the polygon P, and A, B, A' and B' are midpoints of the sides of R_1. We note that $P \subseteq R_1$. Otherwise, there would be a point $Z \in P$ but $Z \notin R_1$, i.e., one of the lines a, b, a' or b' were between O and Z. If it is a, we have $|OZB| > |OAB|$, which is contradictory to the choice of A and B. If it is one of the lines b, a' or b' almost identical arguments lead to a similar contradiction.

Let R_2 be the parallelogram $ABA'B'$. Since A and B are points of P, segment $AB \subset P$ and so $R_2 \subset R_1$. Since A, B, A' and B' are midpoints of the sides of R_1, an easy argument yields

$$|R_1| = 2 \cdot |R_2|. \tag{1}$$

Let R_3 be the smallest parallelogram enclosing P defined by lines parallel to AB and BA'. Obviously $R_2 \subset R_3$ and every side of R_3 contains at least one point of the boundary of P. Denote by C the intersection point of a and b, by X the intersection point of AB and OC, and by X' the intersection point of XC and the boundary of R_3. In a similar way denote by D

the intersection point of b and a', by Y the intersection point of $A'B$ and OD, and by Y' the intersection point of YD and the boundary of R_3.

Note that $OC = 2 \cdot OX$ and $OD = 2 \cdot OY$, so there exist real numbers x and y with $1 \le x, y \le 2$ and $OX' = x \cdot OX$ and $OY' = y \cdot OY$. Corresponding sides of R_3 and R_2 are parallel which yields

$$|R_3| = xy \cdot |R_2|. \tag{2}$$

The side of R_3 containing X' contains at least one point X^* of P; due to the convexity of P we have $AX^*B \subset P$. Since this side of the parallelogram R_3 is parallel to AB we have $|AX^*B| = |AX'B|$, so $|OAX'B|$ does not exceed the area of P confined to the sector defined by the rays OB and OA. In a similar way we conclude that $|OB'Y'A'|$ does not exceed the area of P confined to the sector defined by the rays OB and OA'. Putting things together we have $|OAX'B| = x \cdot |OAB|$, $|OBDA'| = y \cdot |OBA'|$. Since $|OAB| = |OBA'|$, we conclude that $|P| \ge 2 \cdot |AX'BY'A'| = 2 \cdot (x \cdot |OAB| + y \cdot |OBA'|) = 4 \cdot \frac{x+y}{2} \cdot |OAB| = \frac{x+y}{2} \cdot R_2$; this is in short

$$\frac{x+y}{2} \cdot |R_2| \le |P|. \tag{3}$$

Since all numbers concerned are positive, we can combine (1)–(3). Using the arithmetic-geometric-mean inequality we obtain

$$|R_1| \cdot |R_3| = 2 \cdot |R_2| \cdot xy \cdot |R_2| \le 2 \cdot |R_2|^2 \left(\frac{x+y}{2}\right)^2 \le 2 \cdot |P|^2.$$

This implies immediately the desired result $|R_1| \le \sqrt{2} \cdot |P|$ or $|R_3| \le \sqrt{2} \cdot |P|$.

Solution 2. We construct the parallelograms R_1, R_2 and R_3 in the same way as in Solution 1 and will show that $\frac{|R_1|}{|P|} \le \sqrt{2}$ or $\frac{|R_3|}{|P|} \le \sqrt{2}$.

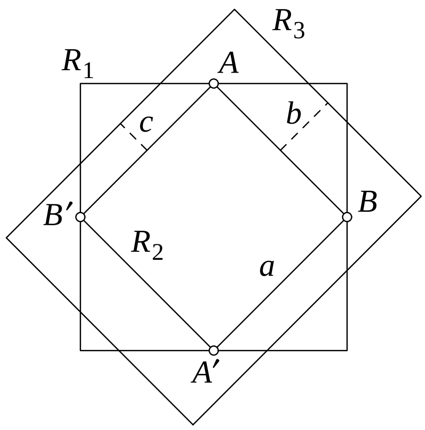


Figure 2

Recall that affine one-to-one maps of the plane preserve the ratio of areas of subsets of the plane. On the other hand, every parallelogram can be transformed with an affine map onto a square. It follows that without loss of generality we may assume that R_1 is a square (see Figure 2).

Then R_2, whose vertices are the midpoints of the sides of R_1, is a square too, and R_3, whose sides are parallel to the diagonals of R_1, is a rectangle.

Let $a > 0$, $b \ge 0$ and $c \ge 0$ be the distances introduced in Figure 2. Then $|R_1| = 2a^2$ and

$|R_3| = (a+2b)(a+2c)$.

Points A, A', B and B' are in the convex polygon P. Hence the square $ABA'B'$ is a subset of P. Moreover, each of the sides of the rectangle R_3 contains a point of P, otherwise R_3 would not be minimal. It follows that

$$|P| \geq a^2 + 2 \cdot \frac{ab}{2} + 2 \cdot \frac{ac}{2} = a(a+b+c).$$

Now assume that both $\frac{|R_1|}{|P|} > \sqrt{2}$ and $\frac{|R_3|}{|P|} > \sqrt{2}$, then

$$2a^2 = |R_1| > \sqrt{2} \cdot |P| \geq \sqrt{2} \cdot a(a+b+c)$$

and

$$(a+2b)(a+2c) = |R_3| > \sqrt{2} \cdot |P| \geq \sqrt{2} \cdot a(a+b+c).$$

All numbers concerned are positive, so after multiplying these inequalities we get

$$2a^2(a+2b)(a+2c) > 2a^2(a+b+c)^2.$$

But the arithmetic-geometric-mean inequality implies the contradictory result

$$2a^2(a+2b)(a+2c) \leq 2a^2\left(\frac{(a+2b)+(a+2c)}{2}\right)^2 = 2a^2(a+b+c)^2.$$

Hence $\frac{|R_1|}{|P|} \leq \sqrt{2}$ or $\frac{|R_3|}{|P|} \leq \sqrt{2}$, as desired.

G6 UKR (Ukraine)

Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are the circumcenters and points H_1 and H_2 are the orthocenters of triangles ABP and DCP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the line H_1H_2 are concurrent.

Solution 1. We keep triangle ABP fixed and move the line CD parallel to itself uniformly, i.e. linearly dependent on a single parameter λ (see Figure 1). Then the points C and D also move uniformly. Hence, the points O_2, H_2 and E_2 move uniformly, too. Therefore also the perpendicular from E_2 on AB moves uniformly. Obviously, the points O_1, H_1, E_1 and the perpendicular from E_1 on CD do not move at all. Hence, the intersection point S of these two perpendiculars moves uniformly. Since H_1 does not move, while H_2 and S move uniformly along parallel lines (both are perpendicular to CD), it is sufficient to prove their collinearity for two different positions of CD.

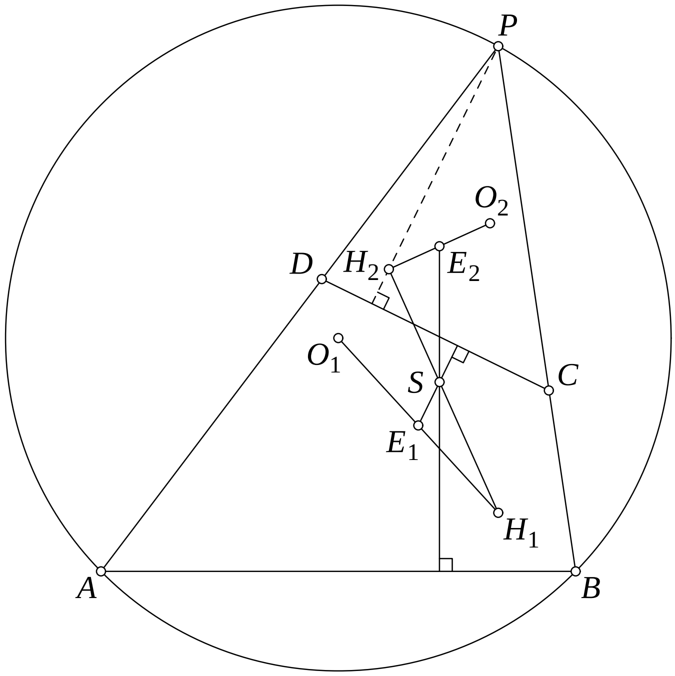


Figure 1

Let CD pass through either point A or point B. Note that by hypothesis these two cases are different. We will consider the case $A \in CD$, i.e. $A = D$. So we have to show that the perpendiculars from E_1 on AC and from E_2 on AB intersect on the altitude AH of triangle ABC (see Figure 2).

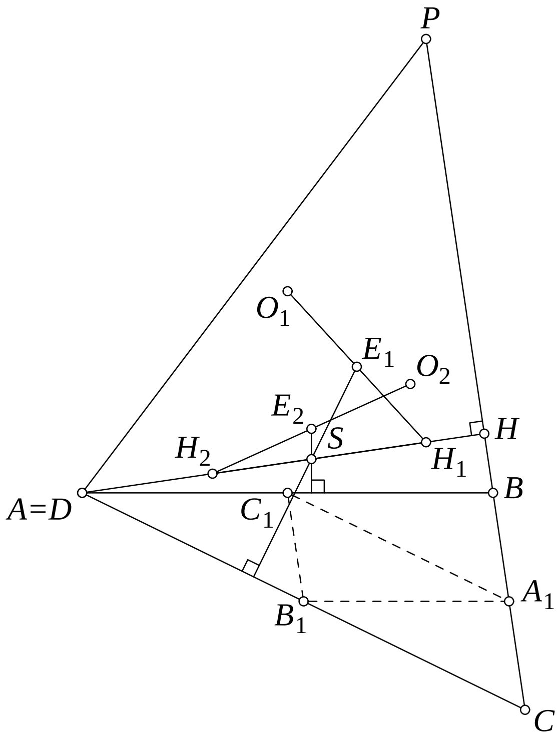


Figure 2

To this end, we consider the midpoints A_1, B_1, C_1 of BC, CA, AB, respectively. As E_1 is the center of FEUERBACH's circle (nine-point circle) of $\triangle ABP$, we have $E_1C_1 = E_1H$. Similarly, $E_2B_1 = E_2H$. Note further that a point X lies on the perpendicular from E_1 on A_1C_1 if and only if

$$XC_1^2 - XA_1^2 = E_1C_1^2 - E_1A_1^2.$$

Similarly, the perpendicular from E_2 on A_1B_1 is characterized by

$$XA_1^2 - XB_1^2 = E_2A_1^2 - E_2B_1^2.$$

The line H_1H_2, which is perpendicular to B_1C_1 and contains A, is given by

$$XB_1^2 - XC_1^2 = AB_1^2 - AC_1^2.$$

The three lines are concurrent if and only if

$$\begin{aligned} 0 &= XC_1^2 - XA_1^2 + XA_1^2 - XB_1^2 + XB_1^2 - XC_1^2 \\ &= E_1C_1^2 - E_1A_1^2 + E_2A_1^2 - E_2B_1^2 + AB_1^2 - AC_1^2 \\ &= -E_1A_1^2 + E_2A_1^2 + E_1H^2 - E_2H^2 + AB_1^2 - AC_1^2, \end{aligned}$$

i.e. it suffices to show that

$$E_1A_1^2 - E_2A_1^2 - E_1H^2 + E_2H^2 = \frac{AC^2 - AB^2}{4}.$$

We have

$$\frac{AC^2 - AB^2}{4} = \frac{HC^2 - HB^2}{4} = \frac{(HC + HB)(HC - HB)}{4} = \frac{HA_1 \cdot BC}{2}.$$

Let F_1, F_2 be the projections of E_1, E_2 on BC. Obviously, these are the midpoints of HP_1,

HP_2, where P_1, P_2 are the midpoints of PB and PC respectively. Then

$$\begin{aligned}
&E_1A_1^2 - E_2A_1^2 - E_1H^2 + E_2H^2 \\
&= F_1A_1^2 - F_1H^2 - F_2A_1^2 + F_2H^2 \\
&= (F_1A_1 - F_1H)(F_1A_1 + F_1H) - (F_2A_1 - F_2H)(F_2A_1 + F_2H) \\
&= A_1H \cdot (A_1P_1 - A_1P_2) \\
&= \frac{A_1H \cdot BC}{2} \\
&= \frac{AC^2 - AB^2}{4},
\end{aligned}$$

which proves the claim.

Solution 2. Let the perpendicular from E_1 on CD meet PH_1 at X, and the perpendicular from E_2 on AB meet PH_2 at Y (see Figure 3). Let φ be the intersection angle of AB and CD. Denote by M, N the midpoints of PH_1, PH_2 respectively.

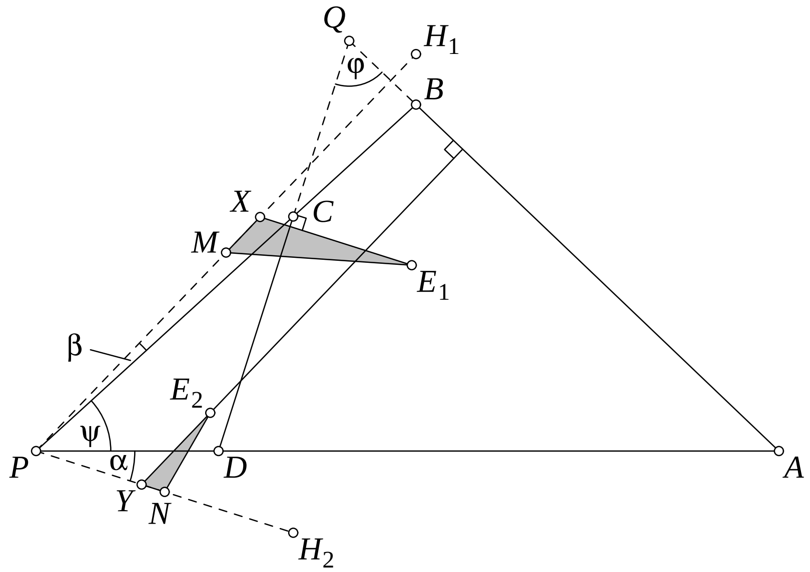


Figure 3

We will prove now that triangles E_1XM and E_2YN have equal angles at E_1, E_2, and supplementary angles at X, Y.

In the following, angles are understood as oriented, and equalities of angles modulo 180°.

Let $\alpha = \angle H_2PD$, $\psi = \angle DPC$, $\beta = \angle CPH_1$. Then $\alpha + \psi + \beta = \varphi$, $\angle E_1XH_1 = \angle H_2YE_2 = \varphi$, thus $\angle MXE_1 + \angle NYE_2 = 180°$.

By considering the FEUERBACH circle of $\triangle ABP$ whose center is E_1 and which goes through M, we have $\angle E_1MH_1 = \psi + 2\beta$. Analogous considerations with the FEUERBACH circle of $\triangle DCP$ yield $\angle H_2NE_2 = \psi + 2\alpha$. Hence indeed $\angle XE_1M = \varphi - (\psi + 2\beta) = (\psi + 2\alpha) - \varphi = \angle YE_2N$.

It follows now that

$$\frac{XM}{ME_1} = \frac{YN}{NE_2}\,.$$

Furthermore, ME_1 is half the circumradius of $\triangle ABP$, while PH_1 is the distance of P to the orthocenter of that triangle, which is twice the circumradius times the cosine of ψ. Together with analogous reasoning for $\triangle DCP$ we have

$$\frac{ME_1}{PH_1} = \frac{1}{4\cos\psi} = \frac{NE_2}{PH_2}\ .$$

By multiplication,

$$\frac{XM}{PH_1} = \frac{YN}{PH_2},$$

and therefore

$$\frac{PX}{XH_1} = \frac{H_2Y}{YP}.$$

Let E_1X, E_2Y meet H_1H_2 in R, S respectively.

Applying the intercept theorem to the parallels E_1X, PH_2 and center H_1 gives

$$\frac{H_2R}{RH_1} = \frac{PX}{XH_1},$$

while with parallels E_2Y, PH_1 and center H_2 we obtain

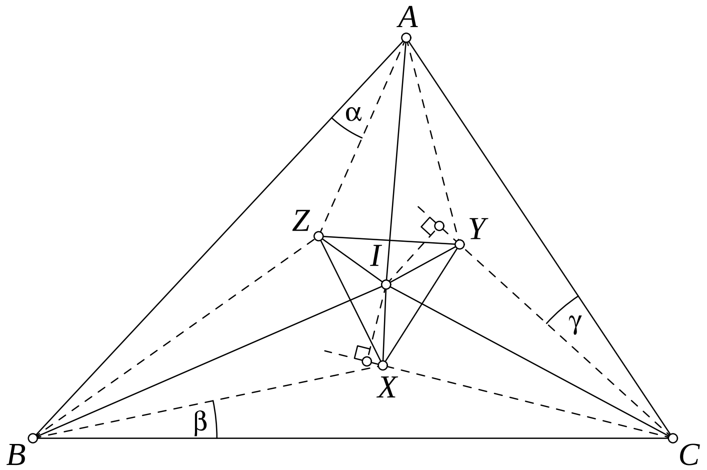
$$\frac{H_2S}{SH_1} = \frac{H_2Y}{YP}.$$

Combination of the last three equalities yields that R and S coincide.

G7 IRN (Islamic Republic of Iran)

Let ABC be a triangle with incenter I and let X, Y and Z be the incenters of the triangles BIC, CIA and AIB, respectively. Let the triangle XYZ be equilateral. Prove that ABC is equilateral too.

Solution. AZ, AI and AY divide $\angle BAC$ into four equal angles; denote them by α. In the same way we have four equal angles β at B and four equal angles γ at C. Obviously $\alpha+\beta+\gamma=\frac{180^\circ}{4}=45^\circ$; and $0^\circ<\alpha,\beta,\gamma<45^\circ$.



Easy calculations in various triangles yield $\angle BIC=180^\circ-2\beta-2\gamma=180^\circ-(90^\circ-2\alpha)=90^\circ+2\alpha$, hence (for X is the incenter of triangle BCI, so IX bisects $\angle BIC$) we have $\angle XIC=\angle BIX=\frac{1}{2}\angle BIC=45^\circ+\alpha$ and with similar aguments $\angle CIY=\angle YIA=45^\circ+\beta$ and $\angle AIZ=\angle ZIB=45^\circ+\gamma$. Furthermore, we have $\angle XIY=\angle XIC+\angle CIY=(45^\circ+\alpha)+(45^\circ+\beta)=135^\circ-\gamma$, $\angle YIZ=135^\circ-\alpha$, and $\angle ZIX=135^\circ-\beta$.

Now we calculate the lengths of IX, IY and IZ in terms of α, β and γ. The perpendicular from I on CX has length $IX\cdot\sin\angle CXI=IX\cdot\sin(90^\circ+\beta)=IX\cdot\cos\beta$. But CI bisects $\angle YCX$, so the perpendicular from I on CY has the same length, and we conclude

$$IX\cdot\cos\beta=IY\cdot\cos\alpha.$$

To make calculations easier we choose a length unit that makes $IX=\cos\alpha$. Then $IY=\cos\beta$ and with similar arguments $IZ=\cos\gamma$.

Since XYZ is equilateral we have $ZX=ZY$. The law of Cosines in triangles XYI, YZI yields

$$\begin{aligned}
&ZX^2=ZY^2\\
\Longrightarrow\quad &IZ^2+IX^2-2\cdot IZ\cdot IX\cdot\cos\angle ZIX=IZ^2+IY^2-2\cdot IZ\cdot IY\cdot\cos\angle YIZ\\
\Longrightarrow\quad &IX^2-IY^2=2\cdot IZ\cdot(IX\cdot\cos\angle ZIX-IY\cdot\cos\angle YIZ)\\
\Longrightarrow\quad &\underbrace{\cos^2\alpha-\cos^2\beta}_{\text{L.H.S.}}=\underbrace{2\cdot\cos\gamma\cdot(\cos\alpha\cdot\cos(135^\circ-\beta)-\cos\beta\cdot\cos(135^\circ-\alpha))}_{\text{R.H.S.}}.
\end{aligned}$$

A transformation of the left-hand side (L.H.S.) yields

$$\begin{aligned}
\text{L.H.S.}&=\cos^2\alpha\cdot\left(\sin^2\beta+\cos^2\beta\right)-\cos^2\beta\cdot\left(\sin^2\alpha+\cos^2\alpha\right)\\
&=\cos^2\alpha\cdot\sin^2\beta-\cos^2\beta\cdot\sin^2\alpha
\end{aligned}$$

$$
\begin{aligned}
&= (\cos\alpha \cdot \sin\beta + \cos\beta \cdot \sin\alpha) \cdot (\cos\alpha \cdot \sin\beta - \cos\beta \cdot \sin\alpha) \\
&= \sin(\beta+\alpha) \cdot \sin(\beta-\alpha) = \sin(45^\circ - \gamma) \cdot \sin(\beta - \alpha)
\end{aligned}
$$

whereas a transformation of the right-hand side (R.H.S.) leads to

$$
\begin{aligned}
\text{R.H.S.} &= 2 \cdot \cos\gamma \cdot (\cos\alpha \cdot (-\cos(45^\circ+\beta)) - \cos\beta \cdot (-\cos(45^\circ+\alpha))) \\
&= 2 \cdot \frac{\sqrt{2}}{2} \cdot \cos\gamma \cdot (\cos\alpha \cdot (\sin\beta - \cos\beta) + \cos\beta \cdot (\cos\alpha - \sin\alpha)) \\
&= \sqrt{2} \cdot \cos\gamma \cdot (\cos\alpha \cdot \sin\beta - \cos\beta \cdot \sin\alpha) \\
&= \sqrt{2} \cdot \cos\gamma \cdot \sin(\beta-\alpha).
\end{aligned}
$$

Equating L.H.S. and R.H.S. we obtain

$$
\begin{aligned}
& \sin(45^\circ-\gamma) \cdot \sin(\beta-\alpha) = \sqrt{2} \cdot \cos\gamma \cdot \sin(\beta-\alpha) \\
\Longrightarrow\quad & \sin(\beta-\alpha) \cdot \left(\sqrt{2} \cdot \cos\gamma - \sin(45^\circ-\gamma)\right) = 0 \\
\Longrightarrow\quad & \alpha = \beta \quad \text{or} \quad \sqrt{2} \cdot \cos\gamma = \sin(45^\circ - \gamma).
\end{aligned}
$$

But $\gamma < 45^\circ$; so $\sqrt{2} \cdot \cos\gamma > \cos\gamma > \cos 45^\circ = \sin 45^\circ > \sin(45^\circ - \gamma)$. This leaves $\alpha = \beta$. With similar reasoning we have $\alpha = \gamma$, which means triangle ABC must be equilateral.

G8 BGR (Bulgaria)

Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2, and I_3 the incenters of $\triangle ABM$, $\triangle MNC$, and $\triangle NDA$, respectively. Show that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

Solution 1. Let k_1, k_2 and k_3 be the incircles of triangles ABM, MNC, and NDA, respectively (see Figure 1). We shall show that the tangent h from C to k_1 which is different from CB is also tangent to k_3.

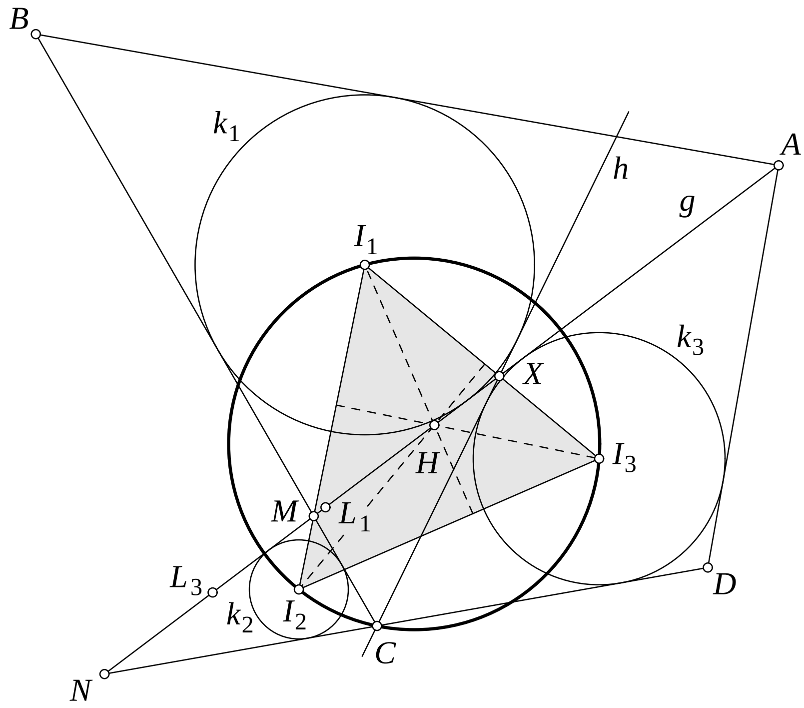


Figure 1

To this end, let X denote the point of intersection of g and h. Then $ABCX$ and $ABCD$ are circumscribed quadrilaterals, whence

$$CD - CX = (AB + CD) - (AB + CX) = (BC + AD) - (BC + AX) = AD - AX,$$

i.e.

$$AX + CD = CX + AD$$

which in turn reveals that the quadrilateral $AXCD$ is also circumscribed. Thus h touches indeed the circle k_3.

Moreover, we find that $\angle I_3CI_1 = \angle I_3CX + \angle XCI_1 = \frac{1}{2}(\angle DCX + \angle XCB) = \frac{1}{2}\angle DCB = \frac{1}{2}(180^\circ - \angle MCN) = 180^\circ - \angle MI_2N = \angle I_3I_2I_1$, from which we conclude that C, I_1, I_2, I_3 are concyclic.

Let now L_1 and L_3 be the reflection points of C with respect to the lines I_2I_3 and I_1I_2 respectively. Since I_1I_2 is the angle bisector of $\angle NMC$, it follows that L_3 lies on g. By analogous reasoning, L_1 lies on g.

Let H be the orthocenter of $\triangle I_1I_2I_3$. We have $\angle I_2L_3I_1 = \angle I_1CI_2 = \angle I_1I_3I_2 = 180^\circ - \angle I_1HI_2$, which entails that the quadrilateral $I_2HI_1L_3$ is cyclic. Analogously, $I_3HL_1I_2$ is cyclic.

Then, working with oriented angles modulo 180°, we have

$$\angle L_3HI_2 = \angle L_3I_1I_2 = \angle I_2I_1C = \angle I_2I_3C = \angle L_1I_3I_2 = \angle L_1HI_2,$$

whence L_1, L_3, and H are collinear. By $L_1 \neq L_3$, the claim follows.

Comment. The last part of the argument essentially reproves the following fact: The Simson line of a point P lying on the circumcircle of a triangle ABC with respect to that triangle bisects the line segment connecting P with the orthocenter of ABC.

Solution 2. We start by proving that C, I_1, I_2, and I_3 are concyclic.

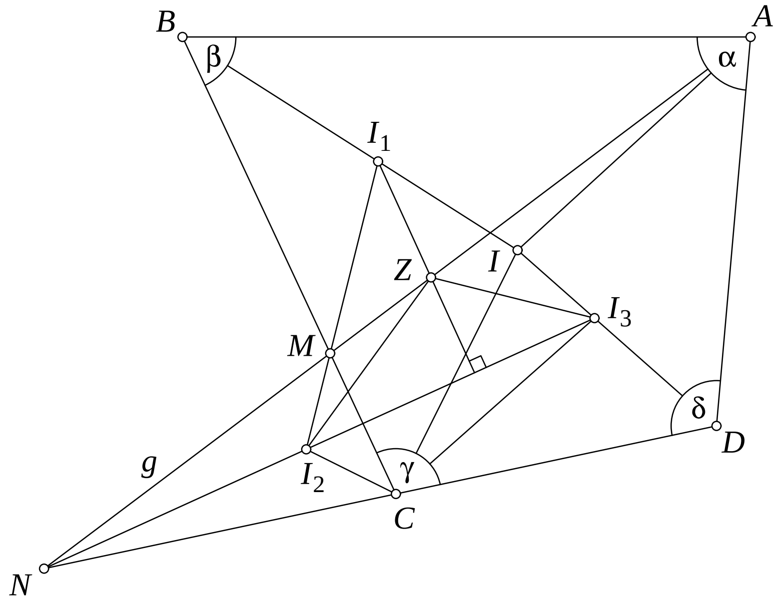


Figure 2

To this end, notice first that I_2, M, I_1 are collinear, as are N, I_2, I_3 (see Figure 2). Denote by α, β, γ, δ the internal angles of $ABCD$. By considerations in triangle CMN, it follows that $\angle I_3I_2I_1 = \frac{\gamma}{2}$. We will show that $\angle I_3CI_1 = \frac{\gamma}{2}$, too. Denote by I the incenter of $ABCD$. Clearly, $I_1 \in BI$, $I_3 \in DI$, $\angle I_1AI_3 = \frac{\alpha}{2}$.

Using the abbreviation $[X, YZ]$ for the distance from point X to the line YZ, we have because of $\angle BAI_1 = \angle IAI_3$ and $\angle I_1AI = \angle I_3AD$ that

$$\frac{[I_1, AB]}{[I_1, AI]} = \frac{[I_3, AI]}{[I_3, AD]} .$$

Furthermore, consideration of the angle sums in AIB, BIC, CID and DIA implies $\angle AIB + \angle CID = \angle BIC + \angle DIA = 180°$, from which we see

$$\frac{[I_1, AI]}{[I_3, CI]} = \frac{I_1I}{I_3I} = \frac{[I_1, CI]}{[I_3, AI]} .$$

Because of $[I_1, AB] = [I_1, BC]$, $[I_3, AD] = [I_3, CD]$, multiplication yields

$$\frac{[I_1, BC]}{[I_3, CI]} = \frac{[I_1, CI]}{[I_3, CD]} .$$

By $\angle DCI = \angle ICB = \gamma/2$ it follows that $\angle I_1CB = \angle I_3CI$ which concludes the proof of the above statement.

Let the perpendicular from I_1 on I_2I_3 intersect g at Z. Then $\angle MI_1Z = 90^\circ - \angle I_3I_2I_1 = 90^\circ - \gamma/2 = \angle MCI_2$. Since we have also $\angle ZMI_1 = \angle I_2MC$, triangles MZI_1 and MI_2C are similar. From this one easily proves that also MI_2Z and MCI_1 are similar. Because C, I_1, I_2, and I_3 are concyclic, $\angle MZI_2 = \angle MI_1C = \angle NI_3C$, thus NI_2Z and NCI_3 are similar, hence NCI_2 and NI_3Z are similar. We conclude $\angle ZI_3I_2 = \angle I_2CN = 90^\circ - \gamma/2$, hence $I_1I_2 \perp ZI_3$. This completes the proof.

Number Theory

N1 AUS (Australia)

A social club has n members. They have the membership numbers $1, 2, \dots, n$, respectively. From time to time members send presents to other members, including items they have already received as presents from other members. In order to avoid the embarrassing situation that a member might receive a present that he or she has sent to other members, the club adds the following rule to its statutes at one of its annual general meetings:

"A member with membership number a is permitted to send a present to a member with membership number b if and only if $a(b-1)$ is a multiple of n."

Prove that, if each member follows this rule, none will receive a present from another member that he or she has already sent to other members.

Alternative formulation: Let G be a directed graph with n vertices $v_1, v_2, \dots, v_n$, such that there is an edge going from v_a to v_b if and only if a and b are distinct and $a(b-1)$ is a multiple of n. Prove that this graph does not contain a directed cycle.

Solution 1. Suppose there is an edge from v_i to v_j. Then $i(j-1) = ij - i = kn$ for some integer k, which implies $i = ij - kn$. If $\gcd(i, n) = d$ and $\gcd(j, n) = e$, then e divides $ij - kn = i$ and thus e also divides d. Hence, if there is an edge from v_i to v_j, then $\gcd(j, n) \mid \gcd(i, n)$.

If there is a cycle in G, say $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_r} \to v_{i_1}$, then we have

$$\gcd(i_1, n) \mid \gcd(i_r, n) \mid \gcd(i_{r-1}, n) \mid \dots \mid \gcd(i_2, n) \mid \gcd(i_1, n),$$

which implies that all these greatest common divisors must be equal, say be equal to t.

Now we pick any of the i_k, without loss of generality let it be i_1. Then $i_r(i_1 - 1)$ is a multiple of n and hence also (by dividing by t), $i_1 - 1$ is a multiple of $\frac{n}{t}$. Since i_1 and $i_1 - 1$ are relatively prime, also t and $\frac{n}{t}$ are relatively prime. So, by the Chinese remainder theorem, the value of i_1 is uniquely determined modulo $n = t \cdot \frac{n}{t}$ by the value of t. But, as i_1 was chosen arbitrarily among the i_k, this implies that all the i_k have to be equal, a contradiction.

Solution 2. If a, b, c are integers such that $ab - a$ and $bc - b$ are multiples of n, then also $ac - a = a(bc - b) + (ab - a) - (ab - a)c$ is a multiple of n. This implies that if there is an edge from v_a to v_b and an edge from v_b to v_c, then there also must be an edge from v_a to v_c. Therefore, if there are any cycles at all, the smallest cycle must have length 2. But suppose the vertices v_a and v_b form such a cycle, i. e., $ab - a$ and $ab - b$ are both multiples of n. Then $a - b$ is also a multiple of n, which can only happen if $a = b$, which is impossible.

Solution 3. Suppose there was a cycle $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_r} \to v_{i_1}$. Then $i_1(i_2 - 1)$ is a multiple of n, i. e., $i_1 \equiv i_1 i_2 \bmod n$. Continuing in this manner, we get $i_1 \equiv i_1 i_2 \equiv i_1 i_2 i_3 \equiv i_1 i_2 i_3 \dots i_r \bmod n$. But the same holds for all i_k, i. e., $i_k \equiv i_1 i_2 i_3 \dots i_r \bmod n$. Hence $i_1 \equiv i_2 \equiv \cdots \equiv i_r \bmod n$, which means $i_1 = i_2 = \cdots = i_r$, a contradiction.

Solution 4. Let $n = k$ be the smallest value of n for which the corresponding graph has a cycle. We show that k is a prime power.
If k is not a prime power, it can be written as a product $k = de$ of relatively prime integers greater than 1. Reducing all the numbers modulo d yields a single vertex or a cycle in the corresponding graph on d vertices, because if $a(b-1) \equiv 0 \bmod k$ then this equation also holds modulo d. But since the graph on d vertices has no cycles, by the minimality of k, we must have that all the indices of the cycle are congruent modulo d. The same holds modulo e and hence also modulo $k = de$. But then all the indices are equal, which is a contradiction.
Thus k must be a prime power $k = p^m$. There are no edges ending at v_k, so v_k is not contained in any cycle. All edges not starting at v_k end at a vertex belonging to a non-multiple of p, and all edges starting at a non-multiple of p must end at v_1. But there is no edge starting at v_1. Hence there is no cycle.

Solution 5. Suppose there was a cycle $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_r} \to v_{i_1}$. Let $q = p^m$ be a prime power dividing n. We claim that either $i_1 \equiv i_2 \equiv \cdots \equiv i_r \equiv 0 \bmod q$ or $i_1 \equiv i_2 \equiv \cdots \equiv i_r \equiv 1 \bmod q$.

Suppose that there is an i_s not divisible by q. Then, as $i_s(i_{s+1} - 1)$ is a multiple of q, $i_{s+1} \equiv 1 \bmod p$. Similarly, we conclude $i_{s+2} \equiv 1 \bmod p$ and so on. So none of the labels is divisible by p, but since $i_s(i_{s+1} - 1)$ is a multiple of $q = p^m$ for all s, all i_{s+1} are congruent to 1 modulo q. This proves the claim.

Now, as all the labels are congruent modulo all the prime powers dividing n, they must all be equal by the Chinese remainder theorem. This is a contradiction.

N2 PER (Peru)

A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x+a)(x+b)$.

(a) Prove that there exist distinct positive integers a and b such that all the numbers $P(1), P(2), \dots, P(50)$ are balanced.

(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

Solution. Define a function f on the set of positive integers by $f(n) = 0$ if n is balanced and $f(n) = 1$ otherwise. Clearly, $f(nm) \equiv f(n) + f(m) \mod 2$ for all positive integers n, m.

(a) Now for each positive integer n consider the binary sequence $(f(n+1), f(n+2), \dots, f(n+50))$. As there are only 2^{50} different such sequences there are two different positive integers a and b such that

$$(f(a+1), f(a+2), \dots, f(a+50)) = (f(b+1), f(b+2), \dots, f(b+50)).$$

But this implies that for the polynomial $P(x) = (x+a)(x+b)$ all the numbers $P(1)$, $P(2)$, $\dots$, $P(50)$ are balanced, since for all $1 \le k \le 50$ we have $f(P(k)) \equiv f(a+k) + f(b+k) \equiv 2f(a+k) \equiv 0 \mod 2$.

(b) Now suppose $P(n)$ is balanced for all positive integers n and $a < b$. Set $n = k(b-a) - a$ for sufficiently large k, such that n is positive. Then $P(n) = k(k+1)(b-a)^2$, and this number can only be balanced, if $f(k) = f(k+1)$ holds. Thus, the sequence $f(k)$ must become constant for sufficiently large k. But this is not possible, as for every prime p we have $f(p) = 1$ and for every square t^2 we have $f(t^2) = 0$.

Hence $a = b$.

Comment. Given a positive integer k, a computer search for the pairs of positive integers (a, b), for which $P(1), P(2), \dots, P(k)$ are all balanced yields the following results with minimal sum $a + b$ and $a < b$:

k	3	4	5	10	20
(a,b)	$(2,4)$	$(6,11)$	$(8,14)$	$(20,34)$	$(1751,3121)$

Therefore, trying to find a and b in part (a) of the problem cannot be done by elementary calculations.

N3 EST (Estonia)

Let f be a non-constant function from the set of positive integers into the set of positive integers, such that $a-b$ divides $f(a)-f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

Solution 1. Denote by $v_p(a)$ the exponent of the prime p in the prime decomposition of a.

Assume that there are only finitely many primes $p_1, p_2, \ldots, p_m$ that divide some function value produced of f.

There are infinitely many positive integers a such that $v_{p_i}(a) > v_{p_i}(f(1))$ for all $i = 1, 2, \ldots, m$, e.g. $a = (p_1 p_2 \ldots p_m)^\alpha$ with α sufficiently large. Pick any such a. The condition of the problem then yields $a \mid (f(a+1) - f(1))$. Assume $f(a+1) \neq f(1)$. Then we must have $v_{p_i}(f(a+1)) \neq v_{p_i}(f(1))$ for at least one i. This yields $v_{p_i}(f(a+1) - f(1)) = \min\{v_{p_i}(f(a+1)), v_{p_i}(f(1))\} \leq v_{p_1}(f(1)) < v_{p_i}(a)$. But this contradicts the fact that $a \mid (f(a+1) - f(1))$.

Hence we must have $f(a+1) = f(1)$ for all such a.

Now, for any positive integer b and all such a, we have $(a+1-b) \mid (f(a+1) - f(b))$, i.e., $(a+1-b) \mid (f(1) - f(b))$. Since this is true for infinitely many positive integers a we must have $f(b) = f(1)$. Hence f is a constant function, a contradiction. Therefore, our initial assumption was false and there are indeed infinitely many primes p dividing $f(c)$ for some positive integer c.

Solution 2. Assume that there are only finitely many primes $p_1, p_2, \ldots, p_m$ that divide some function value of f. Since f is not identically 1, we must have $m \geq 1$.

Then there exist non-negative integers $\alpha_1, \ldots, \alpha_m$ such that

$$f(1) = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}.$$

We can pick a positive integer r such that $f(r) \neq f(1)$. Let

$$M = 1 + p_1^{\alpha_1+1} p_2^{\alpha_2+1} \ldots p_m^{\alpha_m+1} \cdot (f(r) + r).$$

Then for all $i \in \{1, \ldots, m\}$ we have that $p_i^{\alpha_i+1}$ divides $M-1$ and hence by the condition of the problem also $f(M) - f(1)$. This implies that $f(M)$ is divisible by $p_i^{\alpha_i}$ but not by $p_i^{\alpha_i+1}$ for all i and therefore $f(M) = f(1)$.

Hence

$$\begin{aligned} M - r &> p_1^{\alpha_1+1} p_2^{\alpha_2+1} \ldots p_m^{\alpha_m+1} \cdot (f(r) + r) - r \\ &\geq p_1^{\alpha_1+1} p_2^{\alpha_2+1} \ldots p_m^{\alpha_m+1} + (f(r) + r) - r \\ &> p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m} + f(r) \\ &\geq |f(M) - f(r)|. \end{aligned}$$

But since $M - r$ divides $f(M) - f(r)$ this can only be true if $f(r) = f(M) = f(1)$, which contradicts the choice of r.

Comment. In the case that f is a polynomial with integer coefficients the result is well-known, see e.g. W. Schwarz, Einführung in die Methoden der Primzahltheorie, 1969.

N4 PRK (Democratic People's Republic of Korea)

Find all positive integers n such that there exists a sequence of positive integers $a_1, a_2, \ldots, a_n$ satisfying

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \leq k \leq n - 1$.

Solution 1. Such a sequence exists for $n = 1, 2, 3, 4$ and no other n. Since the existence of such a sequence for some n implies the existence of such a sequence for all smaller n, it suffices to prove that $n = 5$ is not possible and $n = 4$ is possible.

Assume first that for $n = 5$ there exists a sequence of positive integers $a_1, a_2, \ldots, a_5$ satisfying the conditions

$$\begin{aligned} a_2^2 + 1 &= (a_1 + 1)(a_3 + 1), \\ a_3^2 + 1 &= (a_2 + 1)(a_4 + 1), \\ a_4^2 + 1 &= (a_3 + 1)(a_5 + 1). \end{aligned}$$

Assume a_1 is odd, then a_2 has to be odd as well and as then $a_2^2 + 1 \equiv 2 \mod 4$, a_3 has to be even. But this is a contradiction, since then the even number $a_2 + 1$ cannot divide the odd number $a_3^2 + 1$.

Hence a_1 is even.

If a_2 is odd, $a_3^2 + 1$ is even (as a multiple of $a_2 + 1$) and hence a_3 is odd, too. Similarly we must have a_4 odd as well. But then $a_3^2 + 1$ is a product of two even numbers $(a_2 + 1)(a_4 + 1)$ and thus is divisible by 4, which is a contradiction as for odd a_3 we have $a_3^2 + 1 \equiv 2 \mod 4$.

Hence a_2 is even. Furthermore $a_3 + 1$ divides the odd number $a_2^2 + 1$ and so a_3 is even. Similarly, a_4 and a_5 are even as well.

Now set $x = a_2$ and $y = a_3$. From the given condition we get $(x+1) | (y^2+1)$ and $(y+1) | (x^2+1)$. We will prove that there is no pair of positive even numbers (x, y) satisfying these two conditions, thus yielding a contradiction to the assumption.

Assume there exists a pair (x_0, y_0) of positive even numbers satisfying the two conditions $(x_0 + 1) | (y_0^2 + 1)$ and $(y_0 + 1) | (x_0^2 + 1)$.

Then one has $(x_0 + 1) | (y_0^2 + 1 + x_0^2 - 1)$, i.e., $(x_0 + 1) | (x_0^2 + y_0^2)$, and similarly $(y_0 + 1) | (x_0^2 + y_0^2)$. Any common divisor d of $x_0 + 1$ and $y_0 + 1$ must hence also divide the number $(x_0^2 + 1) + (y_0^2 + 1) - (x_0^2 + y_0^2) = 2$. But as $x_0 + 1$ and $y_0 + 1$ are both odd, we must have $d = 1$. Thus $x_0 + 1$ and $y_0 + 1$ are relatively prime and therefore there exists a positive integer k such that

$$k(x + 1)(y + 1) = x^2 + y^2$$

has the solution (x_0, y_0). We will show that the latter equation has no solution (x, y) in positive even numbers.

Assume there is a solution. Pick the solution (x_1, y_1) with the smallest sum $x_1 + y_1$ and assume $x_1 \geq y_1$. Then x_1 is a solution to the quadratic equation

$$x^2 - k(y_1 + 1)x + y_1^2 - k(y_1 + 1) = 0.$$

Let x_2 be the second solution, which by VIETA's theorem fulfills $x_1 + x_2 = k(y_1 + 1)$ and $x_1x_2 = y_1^2 - k(y_1 + 1)$. If $x_2 = 0$, the second equation implies $y_1^2 = k(y_1 + 1)$, which is impossible, as $y_1 + 1 > 1$ cannot divide the relatively prime number y_1^2. Therefore $x_2 \neq 0$.

Also we get $(x_1 + 1)(x_2 + 1) = x_1x_2 + x_1 + x_2 + 1 = y_1^2 + 1$ which is odd, and hence x_2 must be even and positive. Also we have $x_2 + 1 = \frac{y_1^2+1}{x_1+1} \leq \frac{y_1^2+1}{y_1+1} \leq y_1 \leq x_1$. But this means that the pair (x', y') with $x' = y_1$ and $y' = x_2$ is another solution of $k(x + 1)(y + 1) = x^2 + y^2$ in even positive numbers with $x' + y' < x_1 + y_1$, a contradiction.

Therefore we must have $n \leq 4$.

When $n = 4$, a possible example of a sequence is $a_1 = 4$, $a_2 = 33$, $a_3 = 217$ and $a_4 = 1384$.

Solution 2. It is easy to check that for $n = 4$ the sequence $a_1 = 4$, $a_2 = 33$, $a_3 = 217$ and $a_4 = 1384$ is possible.

Now assume there is a sequence with $n \geq 5$. Then we have in particular

$$\begin{aligned} a_2^2 + 1 &= (a_1 + 1)(a_3 + 1), \\ a_3^2 + 1 &= (a_2 + 1)(a_4 + 1), \\ a_4^2 + 1 &= (a_3 + 1)(a_5 + 1). \end{aligned}$$

Also assume without loss of generality that among all such quintuples $(a_1, a_2, a_3, a_4, a_5)$ we have chosen one with minimal a_1.

One shows quickly the following fact:

> If three positive integers x, y, z fulfill $y^2 + 1 = (x + 1)(z + 1)$ and if y is even, then x and z are even as well and either $x < y < z$ or $z < y < x$ holds. (1)

Indeed, the first part is obvious and from $x < y$ we conclude

$$z + 1 = \frac{y^2 + 1}{x + 1} \geq \frac{y^2 + 1}{y} > y,$$

and similarly in the other case.

Now, if a_3 was odd, then $(a_2 + 1)(a_4 + 1) = a_3^2 + 1 \equiv 2 \mod 4$ would imply that one of a_2 or a_4 is even, this contradicts (1). Thus a_3 and hence also a_1, a_2, a_4 and a_5 are even. According to (1), one has $a_1 < a_2 < a_3 < a_4 < a_5$ or $a_1 > a_2 > a_3 > a_4 > a_5$ but due to the minimality of a_1 the first series of inequalities must hold.

Consider the identity

$$(a_3+1)(a_1+a_3) = a_3^2-1+(a_1+1)(a_3+1) = a_2^2+a_3^2 = a_2^2-1+(a_2+1)(a_4+1) = (a_2+1)(a_2+a_4).$$

Any common divisor of the two odd numbers $a_2 + 1$ and $a_3 + 1$ must also divide $(a_2 + 1)(a_4 + 1) - (a_3 + 1)(a_3 - 1) = 2$, so these numbers are relatively prime. Hence the last identity shows that $a_1 + a_3$ must be a multiple of $a_2 + 1$, i.e. there is an integer k such that

$$a_1 + a_3 = k(a_2 + 1). \tag{2}$$

Now set $a_0 = k(a_1 + 1) - a_2$. This is an integer and we have

$$\begin{aligned} (a_0 + 1)(a_2 + 1) &= k(a_1 + 1)(a_2 + 1) - (a_2 - 1)(a_2 + 1) \\ &= (a_1 + 1)(a_1 + a_3) - (a_1 + 1)(a_3 + 1) + 2 \\ &= (a_1 + 1)(a_1 - 1) + 2 = a_1^2 + 1. \end{aligned}$$

Thus $a_0 \geq 0$. If $a_0 > 0$, then by (1) we would have $a_0 < a_1 < a_2$ and then the quintuple $(a_0, a_1, a_2, a_3, a_4)$ would contradict the minimality of a_1.

Hence $a_0 = 0$, implying $a_2 = a_1^2$. But also $a_2 = k(a_1 + 1)$, which finally contradicts the fact that $a_1 + 1 > 1$ is relatively prime to a_1^2 and thus cannot be a divisor of this number.

Hence $n \geq 5$ is not possible.

Comment 1. Finding the example for $n = 4$ is not trivial and requires a tedious calculation, but it can be reduced to checking a few cases. The equations $(a_1 + 1)(a_3 + 1) = a_2^2 + 1$ and $(a_2+1)(a_4+1) = a_3^2+1$ imply, as seen in the proof, that a_1 is even and a_2, a_3, a_4 are odd. The case $a_1 = 2$ yields $a_2^2 \equiv -1 \mod 3$ which is impossible. Hence $a_1 = 4$ is the smallest possibility. In this case $a_2^2 \equiv -1 \mod 5$ and a_2 is odd, which implies $a_2 \equiv 3$ or $a_2 \equiv 7 \mod 10$. Hence we have to start checking $a_2 = 7, 13, 17, 23, 27, 33$ and in the last case we succeed.

Comment 2. The choice of $a_0 = k(a_1+1) - a_2$ in the second solution appears more natural if one considers that by the previous calculations one has $a_1 = k(a_2+1) - a_3$ and $a_2 = k(a_3+1) - a_4$.

Alternatively, one can solve the equation (2) for a_3 and use $a_2^2 + 1 = (a_1 + 1)(a_3 + 1)$ to get $a_2^2 - k(a_1+1)a_2 + a_1^2 - k(a_1+1) = 0$. Now a_0 is the second solution to this quadratic equation in a_2 (Vieta jumping).

N5 HUN (Hungary)

Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.

Solution 1. Assume there is a polynomial P of degree at least 1 with the desired property for a given function T. Let $A(n)$ denote the set of all $x \in \mathbb{Z}$ such that $T^n(x) = x$ and let $B(n)$ denote the set of all $x \in \mathbb{Z}$ for which $T^n(x) = x$ and $T^k(x) \neq x$ for all $1 \leq k < n$. Both sets are finite under the assumption made. For each $x \in A(n)$ there is a smallest $k \geq 1$ such that $T^k(x) = x$, i.e., $x \in B(k)$. Let $d = \gcd(k, n)$. There are positive integers r, s such that $rk - sn = d$ and hence $x = T^{rk}(x) = T^{sn+d}(x) = T^d(T^{sn}(x)) = T^d(x)$. The minimality of k implies $d = k$, i.e., $k|n$. On the other hand one clearly has $B(k) \subset A(n)$ if $k|n$ and thus we have $A(n) = \bigcup_{d|n} B(d)$ as a disjoint union and hence

$$|A(n)| = \sum_{d|n} |B(d)|.$$

Furthermore, for every $x \in B(n)$ the elements $x, T^1(x), T^2(x), \ldots, T^{n-1}(x)$ are n distinct elements of $B(n)$. The fact that they are in $A(n)$ is obvious. If for some $k < n$ and some $0 \leq i < n$ we had $T^k(T^i(x)) = T^i(x)$, i.e. $T^{k+i}(x) = T^i(x)$, that would imply $x = T^n(x) = T^{n-i}(T^i(x)) = T^{n-i}(T^{k+i}(x)) = T^k(T^n(x)) = T^k(x)$ contradicting the minimality of n. Thus $T^i(x) \in B(n)$ and $T^i(x) \neq T^j(x)$ for $0 \leq i < j \leq n-1$.

So indeed, T permutes the elements of $B(n)$ in (disjoint) cycles of length n and in particular one has $n \big| |B(n)|$.

Now let $P(x) = \sum_{i=0}^{k} a_i x^i$, $a_i \in \mathbb{Z}$, $k \geq 1$, $a_k \neq 0$ and suppose that $|A(n)| = P(n)$ for all $n \geq 1$. Let p be any prime. Then

$$p^2 \big| |B(p^2)| = |A(p^2)| - |A(p)| = a_1(p^2 - p) + a_2(p^4 - p^2) + \ldots$$

Hence $p|a_1$ and since this is true for all primes we must have $a_1 = 0$.

Now consider any two different primes p and q. Since $a_1 = 0$ we have that

$$|A(p^2q)| - |A(pq)| = a_2(p^4q^2 - p^2q^2) + a_3(p^6q^3 - p^3q^3) + \ldots$$

is a multiple of p^2q. But we also have

$$p^2q \big| |B(p^2q)| = |A(p^2q)| - |A(pq)| - |B(p^2)|.$$

This implies

$$p^2q \big| |B(p^2)| = |A(p^2)| - |A(p)| = a_2(p^4 - p^2) + a_3(p^6 - p^3) + \cdots + a_k(p^{2k} - p^k).$$

Since this is true for every prime q we must have $a_2(p^4 - p^2) + a_3(p^6 - p^3) + \cdots + a_k(p^{2k} - p^k) = 0$ for every prime p. Since this expression is a polynomial in p of degree $2k$ (because $a_k \neq 0$) this is a contradiction, as such a polynomial can have at most $2k$ zeros.

Comment. The last contradiction can also be reached via

$$a_k = \lim_{p\to\infty} \frac{1}{p^{2k}} \left(a_2(p^4 - p^2) + a_3(p^6 - p^3) + \cdots + a_k(p^{2k} - p^k)\right) = 0.$$

Solution 2. As in the first solution define $A(n)$ and $B(n)$ and assume that a polynomial P with the required property exists. This again implies that $|A(n)|$ and $|B(n)|$ is finite for all positive integers n and that

$$P(n) = |A(n)| = \sum_{d|n} |B(d)| \qquad \text{and} \qquad n \big| |B(n)|.$$

Now, for any two distinct primes p and q, we have

$$P(0) \equiv P(pq) \equiv |B(1)| + |B(p)| + |B(q)| + |B(pq)| \equiv |B(1)| + |B(p)| \mod q.$$

Thus, for any fixed p, the expression $P(0) - |B(1)| - |B(p)|$ is divisible by arbitrarily large primes q which means that $P(0) = |B(1)| + |B(p)| = P(p)$ for any prime p. This implies that the polynomial P is constant, a contradiction.

N6 TUR (Turkey)

Let k be a positive integer. Show that if there exists a sequence $a_0, a_1, \ldots$ of integers satisfying the condition

$$a_n = \frac{a_{n-1} + n^k}{n} \quad \text{for all } n \geq 1,$$

then $k - 2$ is divisible by 3.

Solution 1. *Part A.* For each positive integer k, there exists a polynomial P_k of degree $k-1$ with integer coefficients, i. e., $P_k \in \mathbb{Z}[x]$, and an integer q_k such that the polynomial identity

$$xP_k(x) = x^k + P_k(x-1) + q_k \tag{I_k}$$

is satisfied. To prove this, for fixed k we write

$$P_k(x) = b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$$

and determine the coefficients $b_{k-1}, b_{k-2}, \ldots, b_0$ and the number q_k successively. Obviously, we have $b_{k-1} = 1$. For $m = k-1, k-2, \ldots, 1$, comparing the coefficients of x^m in the identity (I_k) results in an expression of b_{m-1} as an integer linear combination of $b_{k-1}, \ldots, b_m$, and finally $q_k = -P_k(-1)$.

Part B. Let k be a positive integer, and let $a_0, a_1, \ldots$ be a sequence of real numbers satisfying the recursion given in the problem. This recursion can be written as

$$a_n - P_k(n) = \frac{a_{n-1} - P_k(n-1)}{n} - \frac{q_k}{n} \quad \text{for all } n \geq 1,$$

which by induction gives

$$a_n - P_k(n) = \frac{a_0 - P_k(0)}{n!} - q_k \sum_{i=0}^{n-1} \frac{i!}{n!} \quad \text{for all } n \geq 1.$$

Therefore, the numbers a_n are integers for all $n \geq 1$ only if

$$a_0 = P_k(0) \quad \text{and} \quad q_k = 0.$$

Part C. Multiplying the identity (I_k) by $x^2 + x$ and subtracting the identities (I_{k+1}), (I_{k+2}) and $q_k x^2 = q_k x^2$ therefrom, we obtain

$$xT_k(x) = T_k(x-1) + 2x\big(P_k(x-1) + q_k\big) - (q_{k+2} + q_{k+1} + q_k),$$

where the polynomials $T_k \in \mathbb{Z}[x]$ are defined by $T_k(x) = (x^2+x)P_k(x) - P_{k+1}(x) - P_{k+2}(x) - q_k x$. Thus

$$xT_k(x) \equiv T_k(x-1) + q_{k+2} + q_{k+1} + q_k \bmod 2, \quad k = 1, 2, \ldots$$

Comparing the degrees, we easily see that this is only possible if T_k is the zero polynomial modulo 2, and

$$q_{k+2} \equiv q_{k+1} + q_k \bmod 2 \quad \text{for } k = 1, 2, \ldots$$

Since $q_1 = -1$ and $q_2 = 0$, these congruences finish the proof.

Solution 2. *Part A and B.* Let k be a positive integer, and suppose there is a sequence $a_0, a_1, \ldots$ as required. We prove: There exists a polynomial $P \in \mathbb{Z}[x]$, i. e., with integer coefficients, such that $a_n = P(n)$, $n = 0, 1, \ldots$, and $xP(x) = x^k + P(x-1)$.
To prove this, we write $P(x) = b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$ and determine the coefficients $b_{k-1}, b_{k-2}, \ldots, b_0$ successively such that

$$xP(x) - x^k - P(x-1) = q,$$

where $q = q_k$ is an integer. Comparing the coefficients of x^m results in an expression of b_{m-1} as an integer linear combination of $b_{k-1}, \ldots, b_m$.
Defining $c_n = a_n - P(n)$, we get

$$\begin{aligned} P(n) + c_n &= \frac{P(n-1) + c_{n-1} + n^k}{n}, \quad \text{i. e.,} \\ q + nc_n &= c_{n-1}, \end{aligned}$$

hence

$$c_n = \frac{c_0}{n!} - q \cdot \frac{0! + 1! + \cdots + (n-1)!}{n!}.$$

We conclude $\lim_{n\to\infty} c_n = 0$, which, using $c_n \in \mathbb{Z}$, implies $c_n = 0$ for sufficiently large n. Therefore, we get $q = 0$ and $c_n = 0$, $n = 0, 1, \ldots$.

Part C. Suppose that $q = q_k = 0$, i. e. $xP(x) = x^k + P(x-1)$. To consider this identity for arguments $x \in \mathbb{F}_4$, we write $\mathbb{F}_4 = \{0, 1, \alpha, \alpha+1\}$. Then we get

$$\begin{aligned} \alpha P_k(\alpha) &= \alpha^k + P_k(\alpha+1) \quad \text{and} \\ (\alpha+1)P_k(\alpha+1) &= (\alpha+1)^k + P_k(\alpha), \end{aligned}$$

hence

$$\begin{aligned} P_k(\alpha) &= 1 \cdot P_k(\alpha) = (\alpha+1)\alpha P_k(\alpha) \\ &= (\alpha+1)P_k(\alpha+1) + (\alpha+1)\alpha^k \\ &= P_k(\alpha) + (\alpha+1)^k + (\alpha+1)\alpha^k. \end{aligned}$$

Now, $(\alpha+1)^{k-1} = \alpha^k$ implies $k \equiv 2 \bmod 3$.

Comment 1. For $k = 2$, the sequence given by $a_n = n+1$, $n = 0, 1, \ldots$, satisfies the conditions of the problem.

Comment 2. The first few polynomials P_k and integers q_k are

$$\begin{aligned}
&P_1(x) = 1, \quad q_1 = -1,\\
&P_2(x) = x+1, \quad q_2 = 0,\\
&P_3(x) = x^2+x-1, \quad q_3 = 1,\\
&P_4(x) = x^3+x^2-2x-1, \quad q_4 = -1,\\
&P_5(x) = x^4+x^3-3x^2+5, \quad q_5 = -2,\\
&P_6(x) = x^5+x^4-4x^3+2x^2+10x-5, \quad q_6 = 9,\\
&q_7 = -9, \quad q_8 = -50, \quad q_9 = 267, \quad q_{10} = -413, \quad q_{11} = -2180.
\end{aligned}$$

A lookup in the On-Line Encyclopedia of Integer Sequences (A000587) reveals that the sequence $q_1, -q_2, q_3, -q_4, q_5, \ldots$ is known as UPPULURI-CARPENTER numbers. The result that $q_k = 0$ implies $k \equiv 2 \bmod 3$ is contained in
MURTY, SUMMER: On the p-adic series $\sum_{n=0}^{\infty} n^k \cdot n!$. CRM Proc. and Lecture Notes 36, 2004.
As shown by ALEXANDER (Non-Vanishing of Uppuluri-Carpenter Numbers, Preprint 2006), UPPULURI-CARPENTER numbers are zero at most twice.

Comment 3. The numbers q_k can be written in terms of the STIRLING numbers of the second kind. To show this, we fix the notation such that

$$\begin{aligned}
x^k =& S_{k-1,k-1}x(x-1)\cdots(x-k+1)\\
&+ S_{k-1,k-2}x(x-1)\cdots(x-k+2)\\
&+\cdots+S_{k-1,0}x,
\end{aligned} \tag{$*$}$$

e. g., $S_{2,2} = 1$, $S_{2,1} = 3$, $S_{2,0} = 1$, and we define

$$\Omega_k = S_{k-1,k-1} - S_{k-1,k-2} + - \cdots .$$

Replacing x by $-x$ in $(*)$ results in

$$\begin{aligned}
x^k =& S_{k-1,k-1}x(x+1)\cdots(x+k-1)\\
&- S_{k-1,k-2}x(x+1)\cdots(x+k-2)\\
&+-\cdots \pm S_{k-1,0}x.
\end{aligned}$$

Defining

$$\begin{aligned}
P(x) =& S_{k-1,k-1}(x+1)\cdots(x+k-1)\\
&+ (S_{k-1,k-1} - S_{k-1,k-2})(x+1)\cdots(x+k-2)\\
&+ (S_{k-1,k-1} - S_{k-1,k-2} + S_{k-1,k-3})(x+1)\cdots(x+k-3)\\
&+\cdots+\Omega_k,
\end{aligned}$$

we obtain

$$\begin{aligned}
xP(x) - P(x-1) &= S_{k-1,k-1}x(x+1)\cdots(x+k-1)\\
&\quad - S_{k-1,k-2}x(x+1)\cdots(x+k-2)\\
&\quad +-\cdots\pm S_{k-1,0}x - \Omega_k\\
&= x^k - \Omega_k,
\end{aligned}$$

hence $q_k = -\Omega_k$.

N7 MNG (Mongolia)

Let a and b be distinct integers greater than 1. Prove that there exists a positive integer n such that $(a^n-1)(b^n-1)$ is not a perfect square.

Solution 1. At first we notice that

$$(1-\alpha)^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}} = \left(1-\tfrac{1}{2}\cdot\alpha-\tfrac{1}{8}\cdot\alpha^2-\cdots\right)\left(1-\tfrac{1}{2}\cdot\beta-\tfrac{1}{8}\cdot\beta^2-\cdots\right)$$
$$= \sum_{k,\ell\ge 0} c_{k,\ell}\cdot\alpha^k\beta^\ell \quad \text{for all } \alpha,\beta\in(0,1), \tag{1}$$

where $c_{0,0}=1$ and $c_{k,\ell}$ are certain coefficients.

For an indirect proof, we suppose that $x_n=\sqrt{(a^n-1)(b^n-1)}\in\mathbb{Z}$ for all positive integers n. Replacing a by a^2 and b by b^2 if necessary, we may assume that a and b are perfect squares, hence $\sqrt{ab}$ is an integer.

At first we shall assume that $a^\mu\ne b^\nu$ for all positive integers μ,ν. We have

$$x_n=(\sqrt{ab})^n\left(1-\frac{1}{a^n}\right)^{\frac{1}{2}}\left(1-\frac{1}{b^n}\right)^{\frac{1}{2}} = \sum_{k,\ell\ge 0} c_{k,\ell}\left(\frac{\sqrt{ab}}{a^k b^\ell}\right)^n. \tag{2}$$

Choosing k_0 and ℓ_0 such that $a^{k_0}>\sqrt{ab}$, $b^{\ell_0}>\sqrt{ab}$, we define the polynomial

$$P(x)=\prod_{k=0,\ell=0}^{k_0-1,\ell_0-1}\left(a^kb^\ell x-\sqrt{ab}\right) =: \sum_{i=0}^{k_0\cdot\ell_0} d_i x^i$$

with integer coefficients d_i. By our assumption, the zeros

$$\frac{\sqrt{ab}}{a^kb^\ell},\quad k=0,\ldots,k_0-1,\quad \ell=0,\ldots,\ell_0-1,$$

of P are pairwise distinct.

Furthermore, we consider the integer sequence

$$y_n=\sum_{i=0}^{k_0\cdot\ell_0} d_i x_{n+i},\quad n=1,2,\ldots \tag{3}$$

By the theory of linear recursions, we obtain

$$y_n=\sum_{\substack{k,\ell\ge 0\\ k\ge k_0 \text{ or } \ell\ge\ell_0}} e_{k,\ell}\left(\frac{\sqrt{ab}}{a^kb^\ell}\right)^n,\quad n=1,2,\ldots, \tag{4}$$

with real numbers $e_{k,\ell}$. We have

$$|y_n|\le \sum_{\substack{k,\ell\ge 0\\ k\ge k_0 \text{ or } \ell\ge\ell_0}} |e_{k,\ell}|\left(\frac{\sqrt{ab}}{a^kb^\ell}\right)^n =: M_n.$$

Because the series in (4) is obtained by a finite linear combination of the absolutely convergent series (1), we conclude that in particular $M_1 < \infty$. Since

$$\frac{\sqrt{ab}}{a^k b^\ell} \leq \lambda := \max\left\{\frac{\sqrt{ab}}{a^{k_0}}, \frac{\sqrt{ab}}{b^{\ell_0}}\right\} \quad \text{for all } k, \ell \geq 0 \text{ such that } k \geq k_0 \text{ or } \ell \geq \ell_0,$$

we get the estimates $M_{n+1} \leq \lambda M_n$, $n = 1, 2, \ldots$ Our choice of k_0 and ℓ_0 ensures $\lambda < 1$, which implies $M_n \to 0$ and consequently $y_n \to 0$ as $n \to \infty$. It follows that $y_n = 0$ for all sufficiently large n.

So, equation (3) reduces to $\sum_{i=0}^{k_0 \cdot \ell_0} d_i x_{n+i} = 0$.

Using the theory of linear recursions again, for sufficiently large n we have

$$x_n = \sum_{k=0,\ell=0}^{k_0-1,\ell_0-1} f_{k,\ell}\left(\frac{\sqrt{ab}}{a^k b^\ell}\right)^n$$

for certain real numbers $f_{k,\ell}$.

Comparing with (2), we see that $f_{k,\ell} = c_{k,\ell}$ for all $k, \ell \geq 0$ with $k < k_0$, $\ell < \ell_0$, and $c_{k,\ell} = 0$ if $k \geq k_0$ or $\ell \geq \ell_0$, since we assumed that $a^\mu \neq b^\nu$ for all positive integers μ, ν.

In view of (1), this means

$$(1-\alpha)^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}} = \sum_{k=0,\ell=0}^{k_0-1,\ell_0-1} c_{k,\ell} \cdot \alpha^k \beta^\ell \tag{5}$$

for all real numbers $\alpha, \beta \in (0,1)$. We choose $k^* < k_0$ maximal such that there is some i with $c_{k^*,i} \neq 0$. Squaring (5) and comparing coefficients of $\alpha^{2k^*}\beta^{2i^*}$, where i^* is maximal with $c_{k^*,i^*} \neq 0$, we see that $k^* = 0$. This means that the right hand side of (5) is independent of α, which is clearly impossible.

We are left with the case that $a^\mu = b^\nu$ for some positive integers μ and ν. We may assume that μ and ν are relatively prime. Then there is some positive integer c such that $a = c^\nu$ and $b = c^\mu$. Now starting with the expansion (2), i. e.,

$$x_n = \sum_{j \geq 0} g_j \left(\frac{\sqrt{c^{\mu+\nu}}}{c^j}\right)^n$$

for certain coefficients g_j, and repeating the arguments above, we see that $g_j = 0$ for sufficiently large j, say $j > j_0$. But this means that

$$(1-x^\mu)^{\frac{1}{2}}(1-x^\nu)^{\frac{1}{2}} = \sum_{j=0}^{j_0} g_j x^j$$

for all real numbers $x \in (0,1)$. Squaring, we see that

$$(1-x^\mu)(1-x^\nu)$$

is the square of a polynomial in x. In particular, all its zeros are of order at least 2, which implies $\mu = \nu$ by looking at roots of unity. So we obtain $\mu = \nu = 1$, i. e., $a = b$, a contradiction.

Solution 2. We set $a^2 = A$, $b^2 = B$, and $z_n = \sqrt{(A^n - 1)(B^n - 1)}$. Let us assume that z_n is an integer for $n = 1, 2, \ldots$ Without loss of generality, we may suppose that $b < a$. We determine an integer $k \geq 2$ such that $b^{k-1} \leq a < b^k$, and define a sequence $\gamma_1, \gamma_2, \ldots$ of rational numbers such that

$$2\gamma_1 = 1 \qquad \text{and} \qquad 2\gamma_{n+1} = \sum_{i=1}^{n} \gamma_i \gamma_{n-i} \quad \text{for} \quad n = 1, 2, \ldots$$

It could easily be shown that $\gamma_n = \frac{1\cdot 1\cdot 3 \ldots (2n-3)}{2\cdot 4\cdot 6\ldots 2n}$, for instance by reading Vandermondes convolution as an equation between polynomials, but we shall have no use for this fact.

Using Landaus O–Notation in the usual way, we have

$$\begin{aligned}
&\left\{(ab)^n - \gamma_1\left(\frac{a}{b}\right)^n - \gamma_2\left(\frac{a}{b^3}\right)^n - \cdots - \gamma_k\left(\frac{a}{b^{2k-1}}\right)^n + O\left(\frac{b}{a}\right)^n\right\}^2 \\
&= A^nB^n - 2\gamma_1 A^n - \sum_{i=2}^{k}\left(2\gamma_i - \sum_{j=1}^{i-1}\gamma_j\gamma_{i-j}\right)\left(\frac{A}{B^{i-1}}\right)^n + O\left(\frac{A}{B^k}\right)^n + O\left(B^n\right) \\
&= A^nB^n - A^n + O\left(B^n\right),
\end{aligned}$$

whence

$$z_n = (ab)^n - \gamma_1\left(\frac{a}{b}\right)^n - \gamma_2\left(\frac{a}{b^3}\right)^n - \cdots - \gamma_k\left(\frac{a}{b^{2k-1}}\right)^n + O\left(\frac{b}{a}\right)^n.$$

Now choose rational numbers $r_1, r_2, \ldots, r_{k+1}$ such that

$$(x - ab)\cdot(x - \tfrac{a}{b})\ldots(x - \tfrac{a}{b^{2k-1}}) = x^{k+1} - r_1x^k + - \cdots \pm r_{k+1},$$

and then a natural number M for which $Mr_1, Mr_2, \ldots Mr_{k+1}$ are integers. For known reasons,

$$M(z_{n+k+1} - r_1z_{n+k} + - \cdots \pm r_{k+1}z_n) = O\left(\frac{b}{a}\right)^n$$

for all $n \in \mathbb{N}$ and thus there is a natural number N which is so large, that

$$z_{n+k+1} = r_1z_{n+k} - r_2z_{n+k-1} + - \cdots \mp r_{k+1}z_n$$

holds for all $n \geqslant N$. Now the theory of linear recursions reveals that there are some rational numbers $\delta_0, \delta_1, \delta_2, \ldots, \delta_k$ such that

$$z_n = \delta_0(ab)^n - \delta_1\left(\frac{a}{b}\right)^n - \delta_2\left(\frac{a}{b^3}\right)^n - \cdots - \delta_k\left(\frac{a}{b^{2k-1}}\right)^n$$

for sufficiently large n, where $\delta_0 > 0$ as $z_n > 0$. As before, one obtains

$$\begin{aligned}
&A^nB^n - A^n - B^n + 1 = z_n^2 \\
&= \left\{\delta_0(ab)^n - \delta_1\left(\frac{a}{b}\right)^n - \delta_2\left(\frac{a}{b^3}\right)^n - \cdots - \delta_k\left(\frac{a}{b^{2k-1}}\right)^n\right\}^2 \\
&= \delta_0^2A^nB^n - 2\delta_0\delta_1A^n - \sum_{i=2}^{i=k}\left(2\delta_0\delta_i - \sum_{j=1}^{j=i-1}\delta_j\delta_{i-j}\right)\left(\frac{A}{B^{i-1}}\right)^n + O\left(\frac{A}{B^k}\right)^n.
\end{aligned}$$

Easy asymptotic calculations yield $\delta_0 = 1$, $\delta_1 = \frac{1}{2}$, $\delta_i = \frac{1}{2}\sum_{j=1}^{j=i-1}\delta_j\delta_{i-j}$ for $i = 2, 3, \ldots, k-2$, and then $a = b^{k-1}$. It follows that $k > 2$ and there is some $P \in \mathbb{Q}[X]$ for which $(X-1)(X^{k-1}-1) = P(X)^2$. But this cannot occur, for instance as $X^{k-1} - 1$ has no double zeros. Thus our assumption that z_n was an integer for $n = 1, 2, \dots$ turned out to be wrong, which solves the problem.

Original formulation of the problem. a, b are positive integers such that $a \cdot b$ is not a square of an integer. Prove that there exists a (infinitely many) positive integer n such that $(a^n - 1)(b^n - 1)$ is not a square of an integer.

Solution. *Lemma.* Let c be a positive integer, which is not a perfect square. Then there exists an odd prime p such that c is not a quadratic residue modulo p.
Proof. Denoting the square-free part of c by c', we have the equality $\left(\frac{c'}{p}\right) = \left(\frac{c}{p}\right)$ of the corresponding Legendre symbols. Suppose that $c' = q_1 \cdots q_m$, where $q_1 < \cdots < q_m$ are primes. Then we have

$$\left(\frac{c'}{p}\right) = \left(\frac{q_1}{p}\right) \cdots \left(\frac{q_m}{p}\right).$$

Case 1. Let q_1 be odd. We choose a quadratic nonresidue r_1 modulo q_1 and quadratic residues r_i modulo q_i for $i = 2, \dots, m$. By the Chinese remainder theorem and the Dirichlet theorem, there exists a (infinitely many) prime p such that

$$\begin{aligned} p &\equiv r_1 \bmod q_1, \\ p &\equiv r_2 \bmod q_2, \\ &\vdots \quad \vdots \\ p &\equiv r_m \bmod q_m, \\ p &\equiv 1 \bmod 4. \end{aligned}$$

By our choice of the residues, we obtain

$$\left(\frac{p}{q_i}\right) = \left(\frac{r_i}{q_i}\right) = \begin{cases} -1, & i = 1, \\ 1, & i = 2, \dots, m. \end{cases}$$

The congruence $p \equiv 1 \bmod 4$ implies that $\left(\frac{q_i}{p}\right) = \left(\frac{p}{q_i}\right)$, $i = 1, \dots, m$, by the law of quadratic reciprocity. Thus

$$\left(\frac{c'}{p}\right) = \left(\frac{q_1}{p}\right) \cdots \left(\frac{q_m}{p}\right) = -1.$$

Case 2. Suppose $q_1 = 2$. We choose quadratic residues r_i modulo q_i for $i = 2, \dots, m$. Again, by the Chinese remainder theorem and the Dirichlet theorem, there exists a prime p such that

$$\begin{aligned} p &\equiv r_2 \bmod q_2, \\ &\vdots \quad \vdots \\ p &\equiv r_m \bmod q_m, \\ p &\equiv 5 \bmod 8. \end{aligned}$$

By the choice of the residues, we obtain $\left(\frac{p}{q_i}\right) = \left(\frac{r_i}{q_i}\right) = 1$ for $i = 2, \dots, m$. Since $p \equiv 1 \bmod 4$ we have $\left(\frac{q_i}{p}\right) = \left(\frac{p}{q_i}\right)$, $i = 2, \dots, m$, by the law of quadratic reciprocity. The congruence $p \equiv 5 \bmod 8$

implies that $\left(\frac{2}{p}\right)=-1$. Thus

$$\left(\frac{c'}{p}\right)=\left(\frac{2}{p}\right)\left(\frac{q_2}{p}\right)\cdots\left(\frac{q_m}{p}\right)=-1,$$

and the lemma is proved.

Applying the lemma for $c=a\cdot b$, we find an odd prime p such that

$$\left(\frac{ab}{p}\right)=\left(\frac{a}{p}\right)\cdot\left(\frac{b}{p}\right)=-1.$$

This implies either

$$a^{\frac{p-1}{2}}\equiv 1 \bmod p,\quad b^{\frac{p-1}{2}}\equiv -1 \bmod p,\qquad \text{or}\qquad a^{\frac{p-1}{2}}\equiv -1 \bmod p,\quad b^{\frac{p-1}{2}}\equiv 1 \bmod p.$$

Without loss of generality, suppose that $a^{\frac{p-1}{2}}\equiv 1 \bmod p$ and $b^{\frac{p-1}{2}}\equiv -1 \bmod p$. The second congruence implies that $b^{\frac{p-1}{2}}-1$ is not divisible by p. Hence, if the exponent $\nu_p(a^{\frac{p-1}{2}}-1)$ of p in the prime decomposition of $(a^{\frac{p-1}{2}}-1)$ is odd, then $(a^{\frac{p-1}{2}}-1)(b^{\frac{p-1}{2}}-1)$ is not a perfect square. If $\nu_p(a^{\frac{p-1}{2}}-1)$ is even, then $\nu_p(a^{\frac{p-1}{2}p}-1)$ is odd by the well-known power lifting property

$$\nu_p\left(a^{\frac{p-1}{2}p}-1\right)=\nu_p\left(a^{\frac{p-1}{2}}-1\right)+1.$$

In this case, $(a^{\frac{p-1}{2}p}-1)(b^{\frac{p-1}{2}p}-1)$ is not a perfect square.

Comment 1. In 1998, the following problem appeared in Crux Mathematicorum:
Problem 2344. Find all positive integers N that are quadratic residues modulo all primes greater than N.
The published solution (Crux Mathematicorum, 25(1999)4) is the same as the proof of the lemma given above, see also http://www.mathlinks.ro/viewtopic.php?t=150495.

Comment 2. There is also an elementary proof of the lemma. We cite Theorem 3 of Chapter 5 and its proof from the book
Ireland, Rosen: A Classical Introduction to Modern Number Theory, Springer 1982.

Theorem. Let a be a nonsquare integer. Then there are infinitely many primes p for which a is a quadratic nonresidue.

Proof. It is easily seen that we may assume that a is square-free. Let $a=2^e q_1 q_2\cdots q_n$, where q_i are distinct odd primes and $e=0$ or 1. The case $a=2$ has to be dealt with separately. We shall assume to begin with that $n\geq 1$, i.e., that a is divisible by an odd prime.

Let $\ell_1,\ell_2,\dots,\ell_k$ be a finite set of odd primes not including any q_i. Let s be any quadratic nonresidue modq_n, and find a simultaneous solution to the congruences

$$\begin{aligned} x&\equiv 1 \bmod \ell_i,\quad i=1,\dots,k,\\ x&\equiv 1 \bmod 8,\\ x&\equiv 1 \bmod q_i,\quad i=1,\dots,n-1,\\ x&\equiv s \bmod q_n.\end{aligned}$$

Call the solution b. b is odd. Suppose that $b=p_1p_2\cdots p_m$ is its prime decomposition. Since

$b \equiv 1 \bmod 8$ we have $\left(\frac{2}{b}\right) = 1$ and $\left(\frac{q_i}{b}\right) = \left(\frac{b}{q_i}\right)$ by a result on JACOBI symbols. Thus

$$\left(\frac{a}{b}\right) = \left(\frac{2}{b}\right)^e \left(\frac{q_1}{b}\right) \cdots \left(\frac{q_{n-1}}{b}\right)\left(\frac{q_n}{b}\right) = \left(\frac{b}{q_1}\right) \cdots \left(\frac{b}{q_{n-1}}\right)\left(\frac{b}{q_n}\right) = \left(\frac{1}{q_1}\right) \cdots \left(\frac{1}{q_{n-1}}\right)\left(\frac{s}{q_n}\right) = -1.$$

On the other hand, by the definition of $\left(\frac{a}{b}\right)$, we have $\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right) \cdots \left(\frac{a}{p_m}\right)$. It follows that $\left(\frac{a}{p_i}\right) = -1$ for some i.

Notice that ℓ_j does not divide b. Thus $p_i \notin \{\ell_1, \ell_2, \ldots, \ell_k\}$.

To summarize, if a is a nonsquare, divisible by an odd prime, we have found a prime p, outside of a given finite set of primes $\{2, \ell_1, \ell_2, \ldots, \ell_k\}$, such that $\left(\frac{a}{p}\right) = -1$. This proves the theorem in this case.

It remains to consider the case $a = 2$. Let $\ell_1, \ell_2, \ldots, \ell_k$ be a finite set of primes, excluding 3, for which $\left(\frac{2}{\ell_i}\right) = -1$. Let $b = 8\ell_1\ell_2 \cdots \ell_k + 3$. b is not divisible by 3 or any ℓ_i. Since $b \equiv 3 \bmod 8$ we have $\left(\frac{2}{b}\right) = (-1)^{\frac{b^2-1}{8}} = -1$. Suppose that $b = p_1 p_2 \cdots p_m$ is the prime decomposition of b. Then, as before, we see that $\left(\frac{2}{p_i}\right) = -1$ for some i. $p_i \notin \{3, \ell_1, \ell_2, \ldots, \ell_k\}$. This proves the theorem for $a = 2$.

This proof has also been posted to `mathlinks`, see `http://www.mathlinks.ro/viewtopic.php?t=150495` again.

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Shortlisted Problems with Solutions

Contents

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Algebra

A1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality

$$f([x]y) = f(x)[f(y)]. \tag{1}$$

holds for all $x, y \in \mathbb{R}$. Here, by $[x]$ we denote the greatest integer not exceeding x.

(France)

Answer. $f(x) = \text{const} = C$, where $C = 0$ or $1 \le C < 2$.

Solution 1. First, setting $x = 0$ in (1) we get

$$f(0) = f(0)[f(y)] \tag{2}$$

for all $y \in \mathbb{R}$. Now, two cases are possible.

Case 1. Assume that $f(0) \neq 0$. Then from (2) we conclude that $[f(y)] = 1$ for all $y \in \mathbb{R}$. Therefore, equation (1) becomes $f([x]y) = f(x)$, and substituting $y = 0$ we have $f(x) = f(0) = C \neq 0$. Finally, from $[f(y)] = 1 = [C]$ we obtain that $1 \le C < 2$.

Case 2. Now we have $f(0) = 0$. Here we consider two subcases.

Subcase 2a. Suppose that there exists $0 < \alpha < 1$ such that $f(\alpha) \neq 0$. Then setting $x = \alpha$ in (1) we obtain $0 = f(0) = f(\alpha)[f(y)]$ for all $y \in \mathbb{R}$. Hence, $[f(y)] = 0$ for all $y \in \mathbb{R}$. Finally, substituting $x = 1$ in (1) provides $f(y) = 0$ for all $y \in \mathbb{R}$, thus contradicting the condition $f(\alpha) \neq 0$.

Subcase 2b. Conversely, we have $f(\alpha) = 0$ for all $0 \le \alpha < 1$. Consider any real z; there exists an integer N such that $\alpha = \dfrac{z}{N} \in [0, 1)$ (one may set $N = [z] + 1$ if $z \ge 0$ and $N = [z] - 1$ otherwise). Now, from (1) we get $f(z) = f([N]\alpha) = f(N)[f(\alpha)] = 0$ for all $z \in \mathbb{R}$.

Finally, a straightforward check shows that all the obtained functions satisfy (1).

Solution 2. Assume that $[f(y)] = 0$ for some y; then the substitution $x = 1$ provides $f(y) = f(1)[f(y)] = 0$. Hence, if $[f(y)] = 0$ for all y, then $f(y) = 0$ for all y. This function obviously satisfies the problem conditions.

So we are left to consider the case when $[f(a)] \neq 0$ for some a. Then we have

$$f([x]a) = f(x)[f(a)], \qquad \text{or} \quad f(x) = \frac{f([x]a)}{[f(a)]}. \tag{3}$$

This means that $f(x_1) = f(x_2)$ whenever $[x_1] = [x_2]$, hence $f(x) = f([x])$, and we may assume that a is an integer.

Now we have

$$f(a) = f\left(2a \cdot \tfrac{1}{2}\right) = f(2a)\left[f\left(\tfrac{1}{2}\right)\right] = f(2a)[f(0)];$$

this implies $[f(0)] \neq 0$, so we may even assume that $a = 0$. Therefore equation (3) provides

$$f(x) = \frac{f(0)}{[f(0)]} = C \neq 0$$

for each x. Now, condition (1) becomes equivalent to the equation $C = C[C]$ which holds exactly when $[C] = 1$.

A2. Let the real numbers a, b, c, d satisfy the relations $a+b+c+d=6$ and $a^2+b^2+c^2+d^2=12$. Prove that

$$36 \leq 4(a^3+b^3+c^3+d^3)-(a^4+b^4+c^4+d^4) \leq 48.$$

(Ukraine)

Solution 1. Observe that

$$\begin{aligned}4(a^3+b^3+c^3+d^3)-(a^4+b^4+c^4+d^4) &= -\big((a-1)^4+(b-1)^4+(c-1)^4+(d-1)^4\big)\\ &\quad+6(a^2+b^2+c^2+d^2)-4(a+b+c+d)+4\\ &= -\big((a-1)^4+(b-1)^4+(c-1)^4+(d-1)^4\big)+52.\end{aligned}$$

Now, introducing $x=a-1$, $y=b-1$, $z=c-1$, $t=d-1$, we need to prove the inequalities

$$16 \geq x^4+y^4+z^4+t^4 \geq 4,$$

under the constraint

$$x^2+y^2+z^2+t^2=(a^2+b^2+c^2+d^2)-2(a+b+c+d)+4=4 \tag{1}$$

(we will not use the value of $x+y+z+t$ though it can be found).

Now the rightmost inequality in (1) follows from the power mean inequality:

$$x^4+y^4+z^4+t^4 \geq \frac{(x^2+y^2+z^2+t^2)^2}{4}=4.$$

For the other one, expanding the brackets we note that

$$(x^2+y^2+z^2+t^2)^2=(x^4+y^4+z^4+t^4)+q,$$

where q is a nonnegative number, so

$$x^4+y^4+z^4+t^4 \leq (x^2+y^2+z^2+t^2)^2=16,$$

and we are done.

Comment 1. The estimates are sharp; the lower and upper bounds are attained at $(3,1,1,1)$ and $(0,2,2,2)$, respectively.

Comment 2. After the change of variables, one can finish the solution in several different ways. The latter estimate, for instance, it can be performed by moving the variables – since we need only the second of the two shifted conditions.

Solution 2. First, we claim that $0 \leq a,b,c,d \leq 3$. Actually, we have

$$a+b+c=6-d, \quad a^2+b^2+c^2=12-d^2,$$

hence the power mean inequality

$$a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3}$$

rewrites as

$$12-d^2 \geq \frac{(6-d)^2}{3} \quad\iff\quad 2d(d-3) \leq 0,$$

which implies the desired inequalities for d; since the conditions are symmetric, we also have the same estimate for the other variables.

Now, to prove the rightmost inequality, we use the obvious inequality $x^2(x-2)^2 \geq 0$ for each real x; this inequality rewrites as $4x^3 - x^4 \leq 4x^2$. It follows that

$$(4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) \leq 4(a^2 + b^2 + c^2 + d^2) = 48,$$

as desired.

Now we prove the leftmost inequality in an analogous way. For each $x \in [0,3]$, we have $(x+1)(x-1)^2(x-3) \leq 0$ which is equivalent to $4x^3 - x^4 \geq 2x^2 + 4x - 3$. This implies that

$$(4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) \geq 2(a^2 + b^2 + c^2 + d^2) + 4(a+b+c+d) - 12 = 36,$$

as desired.

Comment. It is easy to guess the extremal points $(0,2,2,2)$ and $(3,1,1,1)$ for this inequality. This provides a method of finding the polynomials used in Solution 2. Namely, these polynomials should have the form $x^4 - 4x^3 + ax^2 + bx + c$; moreover, the former polynomial should have roots at 2 (with an even multiplicity) and 0, while the latter should have roots at 1 (with an even multiplicity) and 3. These conditions determine the polynomials uniquely.

Solution 3. First, expanding $48 = 4(a^2 + b^2 + c^2 + d^2)$ and applying the AM–GM inequality, we have

$$\begin{aligned} a^4 + b^4 + c^4 + d^4 + 48 &= (a^4 + 4a^2) + (b^4 + 4b^2) + (c^4 + 4c^2) + (d^4 + 4d^2) \\ &\geq 2\left(\sqrt{a^4 \cdot 4a^2} + \sqrt{b^4 \cdot 4b^2} + \sqrt{c^4 \cdot 4c^2} + \sqrt{d^4 \cdot 4d^2}\right) \\ &= 4(|a^3| + |b^3| + |c^3| + |d^3|) \geq 4(a^3 + b^3 + c^3 + d^3), \end{aligned}$$

which establishes the rightmost inequality.

To prove the leftmost inequality, we first show that $a, b, c, d \in [0,3]$ as in the previous solution. Moreover, we can assume that $0 \leq a \leq b \leq c \leq d$. Then we have $a + b \leq b + c \leq \frac{2}{3}(b+c+d) \leq \frac{2}{3} \cdot 6 = 4$.

Next, we show that $4b - b^2 \leq 4c - c^2$. Actually, this inequality rewrites as $(c-b)(b+c-4) \leq 0$, which follows from the previous estimate. The inequality $4a - a^2 \leq 4b - b^2$ can be proved analogously.

Further, the inequalities $a \leq b \leq c$ together with $4a - a^2 \leq 4b - b^2 \leq 4c - c^2$ allow us to apply the Chebyshev inequality obtaining

$$\begin{aligned} a^2(4a - a^2) + b^2(4b - b^2) + c^2(4c - c^2) &\geq \frac{1}{3}(a^2 + b^2 + c^2)\left(4(a+b+c) - (a^2 + b^2 + c^2)\right) \\ &= \frac{(12 - d^2)(4(6-d) - (12 - d^2))}{3}. \end{aligned}$$

This implies that

$$\begin{aligned} (4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) &\geq \frac{(12 - d^2)(d^2 - 4d + 12)}{3} + 4d^3 - d^4 \\ = \frac{144 - 48d + 16d^3 - 4d^4}{3} &= 36 + \frac{4}{3}(3-d)(d-1)(d^2 - 3). \end{aligned} \tag{2}$$

Finally, we have $d^2 \geq \frac{1}{4}(a^2 + b^2 + c^2 + d^2) = 3$ (which implies $d > 1$); so, the expression $\frac{4}{3}(3-d)(d-1)(d^2-3)$ in the right-hand part of (2) is nonnegative, and the desired inequality is proved.

Comment. The rightmost inequality is easier than the leftmost one. In particular, Solutions 2 and 3 show that only the condition $a^2 + b^2 + c^2 + d^2 = 12$ is needed for the former one.

A3. Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \le 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1$, $x_{102} = x_2$). Find the maximal possible value of the sum

$$S = \sum_{i=1}^{100} x_i x_{i+2}.$$

(Russia)

Answer. $\dfrac{25}{2}$.

Solution 1. Let $x_{2i} = 0$, $x_{2i-1} = \frac{1}{2}$ for all $i = 1, \ldots, 50$. Then we have $S = 50 \cdot \left(\frac{1}{2}\right)^2 = \frac{25}{2}$. So, we are left to show that $S \le \frac{25}{2}$ for all values of x_i's satisfying the problem conditions.

Consider any $1 \le i \le 50$. By the problem condition, we get $x_{2i-1} \le 1 - x_{2i} - x_{2i+1}$ and $x_{2i+2} \le 1 - x_{2i} - x_{2i+1}$. Hence by the AM–GM inequality we get

$$\begin{aligned} x_{2i-1}x_{2i+1} + x_{2i}x_{2i+2} &\le (1 - x_{2i} - x_{2i+1})x_{2i+1} + x_{2i}(1 - x_{2i} - x_{2i+1}) \\ &= (x_{2i} + x_{2i+1})(1 - x_{2i} - x_{2i+1}) \le \left(\frac{(x_{2i} + x_{2i+1}) + (1 - x_{2i} - x_{2i+1})}{2}\right)^2 = \frac{1}{4}. \end{aligned}$$

Summing up these inequalities for $i = 1, 2, \ldots, 50$, we get the desired inequality

$$\sum_{i=1}^{50} (x_{2i-1}x_{2i+1} + x_{2i}x_{2i+2}) \le 50 \cdot \frac{1}{4} = \frac{25}{2}.$$

Comment. This solution shows that a bit more general fact holds. Namely, consider $2n$ nonnegative numbers $x_1, \ldots, x_{2n}$ in a row (with no cyclic notation) and suppose that $x_i + x_{i+1} + x_{i+2} \le 1$ for all $i = 1, 2, \ldots, 2n-2$. Then $\sum\limits_{i=1}^{2n-2} x_i x_{i+2} \le \dfrac{n-1}{4}$.

The proof is the same as above, though if might be easier to find it (for instance, applying induction). The original estimate can be obtained from this version by considering the sequence $x_1, x_2, \ldots, x_{100}, x_1, x_2$.

Solution 2. We present another proof of the estimate. From the problem condition, we get

$$\begin{aligned} S = \sum_{i=1}^{100} x_i x_{i+2} \le \sum_{i=1}^{100} x_i(1 - x_i - x_{i+1}) &= \sum_{i=1}^{100} x_i - \sum_{i=1}^{100} x_i^2 - \sum_{i=1}^{100} x_i x_{i+1} \\ &= \sum_{i=1}^{100} x_i - \frac{1}{2}\sum_{i=1}^{100} (x_i + x_{i+1})^2. \end{aligned}$$

By the AM–QM inequality, we have $\sum (x_i + x_{i+1})^2 \ge \frac{1}{100}\left(\sum (x_i + x_{i+1})\right)^2$, so

$$\begin{aligned} S \le \sum_{i=1}^{100} x_i - \frac{1}{200}\left(\sum_{i=1}^{100} (x_i + x_{i+1})\right)^2 &= \sum_{i=1}^{100} x_i - \frac{2}{100}\left(\sum_{i=1}^{100} x_i\right)^2 \\ &= \frac{2}{100}\left(\sum_{i=1}^{100} x_i\right)\left(\frac{100}{2} - \sum_{i=1}^{100} x_i\right). \end{aligned}$$

And finally, by the AM–GM inequality

$$S \le \frac{2}{100} \cdot \left(\frac{1}{2}\left(\sum_{i=1}^{100} x_i + \frac{100}{2} - \sum_{i=1}^{100} x_i\right)\right)^2 = \frac{2}{100} \cdot \left(\frac{100}{4}\right)^2 = \frac{25}{2}.$$

Comment. These solutions are not as easy as they may seem at the first sight. There are two different optimal configurations in which the variables have different values, and not all of sums of three consecutive numbers equal 1. Although it is easy to find the value $\frac{25}{2}$, the estimates must be done with care to preserve equality in the optimal configurations.

A4. A sequence $x_1, x_2, \dots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k$, $x_{2k-1} = (-1)^{k+1}x_k$ for all $k \geq 1$. Prove that $x_1 + x_2 + \dots + x_n \geq 0$ for all $n \geq 1$.

(Austria)

Solution 1. We start with some observations. First, from the definition of x_i it follows that for each positive integer k we have

$$x_{4k-3} = x_{2k-1} = -x_{4k-2} \quad \text{and} \quad x_{4k-1} = x_{4k} = -x_{2k} = x_k. \tag{1}$$

Hence, denoting $S_n = \sum_{i=1}^n x_i$, we have

$$S_{4k} = \sum_{i=1}^{k}\big((x_{4k-3} + x_{4k-2}) + (x_{4k-1} + x_{4k})\big) = \sum_{i=1}^{k}(0 + 2x_k) = 2S_k, \tag{2}$$

$$S_{4k+2} = S_{4k} + (x_{4k+1} + x_{4k+2}) = S_{4k}. \tag{3}$$

Observe also that $S_n = \sum_{i=1}^n x_i \equiv \sum_{i=1}^n 1 = n \pmod 2$.

Now we prove by induction on k that $S_i \geq 0$ for all $i \leq 4k$. The base case is valid since $x_1 = x_3 = x_4 = 1$, $x_2 = -1$. For the induction step, assume that $S_i \geq 0$ for all $i \leq 4k$. Using the relations (1)–(3), we obtain

$$S_{4k+4} = 2S_{k+1} \geq 0, \quad S_{4k+2} = S_{4k} \geq 0, \quad S_{4k+3} = S_{4k+2} + x_{4k+3} = \frac{S_{4k+2} + S_{4k+4}}{2} \geq 0.$$

So, we are left to prove that $S_{4k+1} \geq 0$. If k is odd, then $S_{4k} = 2S_k \geq 0$; since k is odd, S_k is odd as well, so we have $S_{4k} \geq 2$ and hence $S_{4k+1} = S_{4k} + x_{4k+1} \geq 1$.

Conversely, if k is even, then we have $x_{4k+1} = x_{2k+1} = x_{k+1}$, hence $S_{4k+1} = S_{4k} + x_{4k+1} = 2S_k + x_{k+1} = S_k + S_{k+1} \geq 0$. The step is proved.

Solution 2. We will use the notation of S_n and the relations (1)–(3) from the previous solution.

Assume the contrary and consider the minimal n such that $S_{n+1} < 0$; surely $n \geq 1$, and from $S_n \geq 0$ we get $S_n = 0$, $x_{n+1} = -1$. Hence, we are especially interested in the set $M = \{n : S_n = 0\}$; our aim is to prove that $x_{n+1} = 1$ whenever $n \in M$ thus coming to a contradiction.

For this purpose, we first describe the set M inductively. We claim that (i) M consists only of even numbers, (ii) $2 \in M$, and (iii) for every even $n \geq 4$ we have $n \in M \iff [n/4] \in M$. Actually, (i) holds since $S_n \equiv n \pmod 2$, (ii) is straightforward, while (iii) follows from the relations $S_{4k+2} = S_{4k} = 2S_k$.

Now, we are left to prove that $x_{n+1} = 1$ if $n \in M$. We use the induction on n. The base case is $n = 2$, that is, the minimal element of M; here we have $x_3 = 1$, as desired.

For the induction step, consider some $4 \leq n \in M$ and let $m = [n/4] \in M$; then m is even, and $x_{m+1} = 1$ by the induction hypothesis. We prove that $x_{n+1} = x_{m+1} = 1$. If $n = 4m$ then we have $x_{n+1} = x_{2m+1} = x_{m+1}$ since m is even; otherwise, $n = 4m+2$, and $x_{n+1} = -x_{2m+2} = x_{m+1}$, as desired. The proof is complete.

Comment. Using the inductive definition of set M, one can describe it explicitly. Namely, M consists exactly of all positive integers not containing digits 1 and 3 in their 4-base representation.

A5. Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\big(f(x)^2 y\big) = x^3 f(xy). \tag{1}$$

(Switzerland)

Answer. The only such function is $f(x) = \dfrac{1}{x}$.

Solution. By substituting $y = 1$, we get

$$f\big(f(x)^2\big) = x^3 f(x). \tag{2}$$

Then, whenever $f(x) = f(y)$, we have

$$x^3 = \frac{f\big(f(x)^2\big)}{f(x)} = \frac{f\big(f(y)^2\big)}{f(y)} = y^3$$

which implies $x = y$, so the function f is injective.

Now replace x by xy in (2), and apply (1) twice, second time to $\big(y, f(x)^2\big)$ instead of (x, y):

$$f\big(f(xy)^2\big) = (xy)^3 f(xy) = y^3 f\big(f(x)^2 y\big) = f\big(f(x)^2 f(y)^2\big).$$

Since f is injective, we get

$$f(xy)^2 = f(x)^2 f(y)^2,$$
$$f(xy) = f(x) f(y).$$

Therefore, f is multiplicative. This also implies $f(1) = 1$ and $f(x^n) = f(x)^n$ for all integers n.

Then the function equation (1) can be re-written as

$$f\big(f(x)\big)^2 f(y) = x^3 f(x) f(y),$$
$$f\big(f(x)\big) = \sqrt{x^3 f(x)}. \tag{3}$$

Let $g(x) = x f(x)$. Then, by (3), we have

$$g\big(g(x)\big) = g\big(x f(x)\big) = x f(x) \cdot f\big(x f(x)\big) = x f(x)^2 f\big(f(x)\big) =$$
$$= x f(x)^2 \sqrt{x^3 f(x)} = \big(x f(x)\big)^{5/2} = \big(g(x)\big)^{5/2},$$

and, by induction,

$$\underbrace{g\Big(g\big(\dots g}_{n+1}(x)\dots\big)\Big) = \big(g(x)\big)^{(5/2)^n} \tag{4}$$

for every positive integer n.

Consider (4) for a fixed x. The left-hand side is always rational, so $\big(g(x)\big)^{(5/2)^n}$ must be rational for every n. We show that this is possible only if $g(x) = 1$. Suppose that $g(x) \neq 1$, and let the prime factorization of $g(x)$ be $g(x) = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where $p_1, \dots, p_k$ are distinct primes and $\alpha_1, \dots, \alpha_k$ are nonzero integers. Then the unique prime factorization of (4) is

$$\underbrace{g\Big(g\big(\dots g}_{n+1}(x)\dots\big)\Big) = \big(g(x)\big)^{(5/2)^n} = p_1^{(5/2)^n \alpha_1} \cdots p_k^{(5/2)^n \alpha_k}$$

where the exponents should be integers. But this is not true for large values of n, for example $(\frac{5}{2})^n\alpha_1$ cannot be a integer number when $2^n \not| \alpha_1$. Therefore, $g(x) \neq 1$ is impossible.

Hence, $g(x) = 1$ and thus $f(x) = \dfrac{1}{x}$ for all x.

The function $f(x) = \dfrac{1}{x}$ satisfies the equation (1):

$$f(f(x)^2y) = \frac{1}{f(x)^2y} = \frac{1}{\left(\frac{1}{x}\right)^2y} = \frac{x^3}{xy} = x^3f(xy).$$

Comment. Among $\mathbb{R}^+ \to \mathbb{R}^+$ functions, $f(x) = \dfrac{1}{x}$ is not the only solution. Another solution is $f_1(x) = x^{3/2}$. Using transfinite tools, infinitely many other solutions can be constructed.

A6. Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

(Germany)

Solution 1. Throughout the solution, by $\mathbb{N}$ we denote the set of all positive integers. For any function $h\colon \mathbb{N} \to \mathbb{N}$ and for any positive integer k, define $h^k(x) = \underbrace{h\big(h\big(\dots h}_{k}(x)\dots\big)\big)$ (in particular, $h^0(x) = x$).

Observe that $f\big(g^k(x)\big) = f\big(g^{k-1}(x)\big) + 1 = \dots = f(x) + k$ for any positive integer k, and similarly $g\big(f^k(x)\big) = g(x) + k$. Now let a and b are the minimal values attained by f and g, respectively; say $f(n_f) = a$, $g(n_g) = b$. Then we have $f\big(g^k(n_f)\big) = a + k$, $g\big(f^k(n_g)\big) = b + k$, so the function f attains all values from the set $N_f = \{a, a+1, \dots\}$, while g attains all the values from the set $N_g = \{b, b+1, \dots\}$.

Next, note that $f(x) = f(y)$ implies $g(x) = g\big(f(x)\big) - 1 = g\big(f(y)\big) - 1 = g(y)$; surely, the converse implication also holds. Now, we say that x and y are *similar* (and write $x \sim y$) if $f(x) = f(y)$ (equivalently, $g(x) = g(y)$). For every $x \in \mathbb{N}$, we define $[x] = \{y \in \mathbb{N} : x \sim y\}$; surely, $y_1 \sim y_2$ for all $y_1, y_2 \in [x]$, so $[x] = [y]$ whenever $y \in [x]$.

Now we investigate the structure of the sets $[x]$.

Claim 1. Suppose that $f(x) \sim f(y)$; then $x \sim y$, that is, $f(x) = f(y)$. Consequently, each class $[x]$ contains at most one element from N_f, as well as at most one element from N_g.

Proof. If $f(x) \sim f(y)$, then we have $g(x) = g\big(f(x)\big) - 1 = g\big(f(y)\big) - 1 = g(y)$, so $x \sim y$. The second statement follows now from the sets of values of f and g. $\square$

Next, we clarify which classes do not contain large elements.

Claim 2. For any $x \in \mathbb{N}$, we have $[x] \subseteq \{1, 2, \dots, b-1\}$ if and only if $f(x) = a$. Analogously, $[x] \subseteq \{1, 2, \dots, a-1\}$ if and only if $g(x) = b$.

Proof. We will prove that $[x] \nsubseteq \{1, 2, \dots, b-1\} \iff f(x) > a$; the proof of the second statement is similar.

Note that $f(x) > a$ implies that there exists some y satisfying $f(y) = f(x) - 1$, so $f\big(g(y)\big) = f(y) + 1 = f(x)$, and hence $x \sim g(y) \geq b$. Conversely, if $b \leq c \sim x$ then $c = g(y)$ for some $y \in \mathbb{N}$, which in turn follows $f(x) = f\big(g(y)\big) = f(y) + 1 \geq a + 1$, and hence $f(x) > a$. $\square$

Claim 2 implies that there exists exactly one class contained in $\{1, \dots, a-1\}$ (that is, the class $[n_g]$), as well as exactly one class contained in $\{1, \dots, b-1\}$ (the class $[n_f]$). Assume for a moment that $a \leq b$; then $[n_g]$ is contained in $\{1, \dots, b-1\}$ as well, hence it coincides with $[n_g]$. So, we get that

$$f(x) = a \iff g(x) = b \iff x \sim n_f \sim n_g. \tag{1}$$

Claim 3. $a = b$.

Proof. By Claim 2, we have $[a] \neq [n_f]$, so $[a]$ should contain some element $a' \geq b$ by Claim 2 again. If $a \neq a'$, then $[a]$ contains two elements $\geq a$ which is impossible by Claim 1. Therefore, $a = a' \geq b$. Similarly, $b \geq a$. $\square$

Now we are ready to prove the problem statement. First, we establish the following

Claim 4. For every integer $d \geq 0$, $f^{d+1}(n_f) = g^{d+1}(n_f) = a + d$.

Proof. Induction on d. For $d = 0$, the statement follows from (1) and Claim 3. Next, for $d > 1$ from the induction hypothesis we have $f^{d+1}(n_f) = f\big(f^d(n_f)\big) = f\big(g^d(n_f)\big) = f(n_f) + d = a + d$. The equality $g^{d+1}(n_f) = a + d$ is analogous. $\square$

Finally, for each $x \in \mathbb{N}$, we have $f(x) = a + d$ for some $d \geq 0$, so $f(x) = f\big(g^d(n_f)\big)$ and hence $x \sim g^d(n_f)$. It follows that $g(x) = g\big(g^d(n_f)\big) = g^{d+1}(n_f) = a + d = f(x)$ by Claim 4.

Solution 2. We start with the same observations, introducing the relation $\sim$ and proving Claim 1 from the previous solution.

Note that $f(a) > a$ since otherwise we have $f(a) = a$ and hence $g(a) = g\big(f(a)\big) = g(a) + 1$, which is false.

Claim 2′. $a = b$.

Proof. We can assume that $a \leq b$. Since $f(a) \geq a + 1$, there exists some $x \in \mathbb{N}$ such that $f(a) = f(x) + 1$, which is equivalent to $f(a) = f\big(g(x)\big)$ and $a \sim g(x)$. Since $g(x) \geq b \geq a$, by Claim 1 we have $a = g(x) \geq b$, which together with $a \leq b$ proves the Claim. $\square$

Now, almost the same method allows to find the values $f(a)$ and $g(a)$.

Claim 3′. $f(a) = g(a) = a + 1$.

Proof. Assume the contrary; then $f(a) \geq a + 2$, hence there exist some $x, y \in \mathbb{N}$ such that $f(x) = f(a) - 2$ and $f(y) = g(x)$ (as $g(x) \geq a = b$). Now we get $f(a) = f(x) + 2 = f\big(g^2(x)\big)$, so $a \sim g^2(x) \geq a$, and by Claim 1 we get $a = g^2(x) = g\big(f(y)\big) = 1 + g(y) \geq 1 + a$; this is impossible. The equality $g(a) = a + 1$ is similar.

Now, we are prepared for the proof of the problem statement. First, we prove it for $n \geq a$.

Claim 4′. For each integer $x \geq a$, we have $f(x) = g(x) = x + 1$.

Proof. Induction on x. The base case $x = a$ is provided by Claim 3′, while the induction step follows from $f(x+1) = f\big(g(x)\big) = f(x) + 1 = (x+1) + 1$ and the similar computation for $g(x+1)$.

Finally, for an arbitrary $n \in \mathbb{N}$ we have $g(n) \geq a$, so by Claim 4′ we have $f(n) + 1 = f\big(g(n)\big) = g(n) + 1$, hence $f(n) = g(n)$.

Comment. It is not hard now to describe all the functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the property $f(f(n)) = f(n) + 1$. For each such function, there exists $n_0 \in \mathbb{N}$ such that $f(n) = n + 1$ for all $n \geq n_0$, while for each $n < n_0$, $f(n)$ is an arbitrary number greater than of equal to n_0 (these numbers may be different for different $n < n_0$).

A7. Let $a_1, \ldots, a_r$ be positive real numbers. For $n > r$, we inductively define

$$a_n = \max_{1 \le k \le n-1} (a_k + a_{n-k}). \tag{1}$$

Prove that there exist positive integers $\ell \le r$ and N such that $a_n = a_{n-\ell} + a_\ell$ for all $n \ge N$.

(Iran)

Solution 1. First, from the problem conditions we have that each a_n $(n > r)$ can be expressed as $a_n = a_{j_1} + a_{j_2}$ with $j_1, j_2 < n$, $j_1 + j_2 = n$. If, say, $j_1 > r$ then we can proceed in the same way with a_{j_1}, and so on. Finally, we represent a_n in a form

$$a_n = a_{i_1} + \cdots + a_{i_k}, \tag{2}$$

$$1 \le i_j \le r, \quad i_1 + \cdots + i_k = n. \tag{3}$$

Moreover, if a_{i_1} and a_{i_2} are the numbers in (2) obtained on the last step, then $i_1 + i_2 > r$. Hence we can adjust (3) as

$$1 \le i_j \le r, \quad i_1 + \cdots + i_k = n, \quad i_1 + i_2 > r. \tag{4}$$

On the other hand, suppose that the indices $i_1, \ldots, i_k$ satisfy the conditions (4). Then, denoting $s_j = i_1 + \cdots + i_j$, from (1) we have

$$a_n = a_{s_k} \ge a_{s_{k-1}} + a_{i_k} \ge a_{s_{k-2}} + a_{i_{k-1}} + a_{i_k} \ge \cdots \ge a_{i_1} + \cdots + a_{i_k}.$$

Summarizing these observations we get the following

Claim. For every $n > r$, we have

$$a_n = \max\{a_{i_1} + \cdots + a_{i_k} : \text{the collection } (i_1, \ldots, i_k) \text{ satisfies } (4)\}. \qquad \square$$

Now we denote

$$s = \max_{1 \le i \le r} \frac{a_i}{i}$$

and fix some index $\ell \le r$ such that $s = \dfrac{a_\ell}{\ell}$.

Consider some $n \ge r^2\ell + 2r$ and choose an expansion of a_n in the form (2), (4). Then we have $n = i_1 + \cdots + i_k \le rk$, so $k \ge n/r \ge r\ell + 2$. Suppose that none of the numbers $i_3, \ldots, i_k$ equals ℓ. Then by the pigeonhole principle there is an index $1 \le j \le r$ which appears among $i_3, \ldots, i_k$ at least ℓ times, and surely $j \ne \ell$. Let us delete these ℓ occurrences of j from $(i_1, \ldots, i_k)$, and add j occurrences of ℓ instead, obtaining a sequence $(i_1, i_2, i'_3, \ldots, i'_{k'})$ also satisfying (4). By Claim, we have

$$a_{i_1} + \cdots + a_{i_k} = a_n \ge a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}},$$

or, after removing the coinciding terms, $\ell a_j \ge j a_\ell$, so $\dfrac{a_\ell}{\ell} \le \dfrac{a_j}{j}$. By the definition of ℓ, this means that $\ell a_j = j a_\ell$, hence

$$a_n = a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}}.$$

Thus, for every $n \ge r^2\ell + 2r$ we have found a representation of the form (2), (4) with $i_j = \ell$ for some $j \ge 3$. Rearranging the indices we may assume that $i_k = \ell$.

Finally, observe that in this representation, the indices $(i_1, \ldots, i_{k-1})$ satisfy the conditions (4) with n replaced by $n - \ell$. Thus, from the Claim we get

$$a_{n-\ell} + a_\ell \ge (a_{i_1} + \cdots + a_{i_{k-1}}) + a_\ell = a_n,$$

which by (1) implies

$$a_n = a_{n-\ell} + a_\ell \qquad \text{for each } n \ge r^2\ell + 2r,$$

as desired.

Solution 2. As in the previous solution, we involve the expansion (2), (3), and we fix some index $1 \le \ell \le r$ such that

$$\frac{a_\ell}{\ell} = s = \max_{1\le i\le r} \frac{a_i}{i}.$$

Now, we introduce the sequence (b_n) as $b_n = a_n - sn$; then $b_\ell = 0$.

We prove by induction on n that $b_n \le 0$, and (b_n) satisfies the same recurrence relation as (a_n). The base cases $n \le r$ follow from the definition of s. Now, for $n > r$ from the induction hypothesis we have

$$b_n = \max_{1\le k\le n-1}(a_k + a_{n-k}) - ns = \max_{1\le k\le n-1}(b_k + b_{n-k} + ns) - ns = \max_{1\le k\le n-1}(b_k + b_{n-k}) \le 0,$$

as required.

Now, if $b_k = 0$ for all $1 \le k \le r$, then $b_n = 0$ for all n, hence $a_n = sn$, and the statement is trivial. Otherwise, define

$$M = \max_{1\le i\le r} |b_i|, \quad \varepsilon = \min\{|b_i| : 1 \le i \le r,\ b_i < 0\}.$$

Then for $n > r$ we obtain

$$b_n = \max_{1\le k\le n-1}(b_k + b_{n-k}) \ge b_\ell + b_{n-\ell} = b_{n-\ell},$$

so

$$0 \ge b_n \ge b_{n-\ell} \ge b_{n-2\ell} \ge \cdots \ge -M.$$

Thus, in view of the expansion (2), (3) applied to the sequence (b_n), we get that each b_n is contained in a set

$$T = \{b_{i_1} + b_{i_2} + \cdots + b_{i_k} :\ i_1, \ldots, i_k \le r\} \cap [-M, 0]$$

We claim that this set is finite. Actually, for any $x \in T$, let $x = b_{i_1} + \cdots + b_{i_k}$ $(i_1, \ldots, i_k \le r)$. Then among b_{i_j}'s there are at most $\frac{M}{\varepsilon}$ nonzero terms (otherwise $x < \frac{M}{\varepsilon} \cdot (-\varepsilon) < -M$). Thus x can be expressed in the same way with $k \le \frac{M}{\varepsilon}$, and there is only a finite number of such sums.

Finally, for every $t = 1, 2, \ldots, \ell$ we get that the sequence

$$b_{r+t}, b_{r+t+\ell}, b_{r+t+2\ell}, \ldots$$

is non-decreasing and attains the finite number of values; therefore it is constant from some index. Thus, the sequence (b_n) is periodic with period ℓ from some index N, which means that

$$b_n = b_{n-\ell} = b_{n-\ell} + b_\ell \qquad \text{for all } n > N + \ell,$$

and hence

$$a_n = b_n + ns = (b_{n-\ell} + (n-\ell)s) + (b_\ell + \ell s) = a_{n-\ell} + a_\ell \qquad \text{for all } n > N + \ell,$$

as desired.

A8. Given six positive numbers a, b, c, d, e, f such that $a < b < c < d < e < f$. Let $a+c+e = S$ and $b + d + f = T$. Prove that

$$2ST > \sqrt{3(S+T)\big(S(bd+bf+df) + T(ac+ae+ce)\big)}. \tag{1}$$

(South Korea)

Solution 1. We define also $\sigma = ac + ce + ae$, $\tau = bd + bf + df$. The idea of the solution is to interpret (1) as a natural inequality on the roots of an appropriate polynomial.

Actually, consider the polynomial

$$\begin{aligned} P(x) &= (b+d+f)(x-a)(x-c)(x-e) + (a+c+e)(x-b)(x-d)(x-f) \\ &= T(x^3 - Sx^2 + \sigma x - ace) + S(x^3 - Tx^2 + \tau x - bdf). \end{aligned} \tag{2}$$

Surely, P is cubic with leading coefficient $S + T > 0$. Moreover, we have

$$\begin{aligned} P(a) &= S(a-b)(a-d)(a-f) < 0, & \qquad P(c) &= S(c-b)(c-d)(c-f) > 0, \\ P(e) &= S(e-b)(e-d)(e-f) < 0, & \qquad P(f) &= T(f-a)(f-c)(f-e) > 0. \end{aligned}$$

Hence, each of the intervals (a, c), (c, e), (e, f) contains at least one root of $P(x)$. Since there are at most three roots at all, we obtain that there is exactly one root in each interval (denote them by $\alpha \in (a, c)$, $\beta \in (c, e)$, $\gamma \in (e, f)$). Moreover, the polynomial P can be factorized as

$$P(x) = (T+S)(x-\alpha)(x-\beta)(x-\gamma). \tag{3}$$

Equating the coefficients in the two representations (2) and (3) of $P(x)$ provides

$$\alpha + \beta + \gamma = \frac{2TS}{T+S}, \qquad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{S\tau + T\sigma}{T+S}.$$

Now, since the numbers α, β, γ are distinct, we have

$$0 < (\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\beta-\gamma)^2 = 2(\alpha+\beta+\gamma)^2 - 6(\alpha\beta + \alpha\gamma + \beta\gamma),$$

which implies

$$\frac{4S^2T^2}{(T+S)^2} = (\alpha+\beta+\gamma)^2 > 3(\alpha\beta + \alpha\gamma + \beta\gamma) = \frac{3(S\tau + T\sigma)}{T+S},$$

or

$$4S^2T^2 > 3(T+S)(T\sigma + S\tau),$$

which is exactly what we need.

Comment 1. In fact, one can locate the roots of $P(x)$ more narrowly: they should lie in the intervals (a, b), (c, d), (e, f).

Surely, if we change all inequality signs in the problem statement to non-strict ones, the (non-strict) inequality will also hold by continuity. One can also find when the equality is achieved. This happens in that case when $P(x)$ is a perfect cube, which immediately implies that $b = c = d = e (= \alpha = \beta = \gamma)$, together with the additional condition that $P''(b) = 0$. Algebraically,

$$\begin{aligned} 6(T+S)b - 4TS = 0 \qquad &\iff \qquad 3b(a+4b+f) = 2(a+2b)(2b+f) \\ &\iff \qquad f = \frac{b(4b-a)}{2a+b} = b\left(1 + \frac{3(b-a)}{2a+b}\right) > b. \end{aligned}$$

This means that for every pair of numbers a, b such that $0 < a < b$, there exists $f > b$ such that the point (a, b, b, b, b, f) is a point of equality.

Solution 2. Let
$$U=\frac{1}{2}\big((e-a)^2+(c-a)^2+(e-c)^2\big)=S^2-3(ac+ae+ce)$$
and
$$V=\frac{1}{2}\big((f-b)^2+(f-d)^2+(d-b)^2\big)=T^2-3(bd+bf+df).$$
Then
$$\begin{aligned}(\text{L.H.S.})^2-(\text{R.H.S.})^2&=(2ST)^2-(S+T)\big(S\cdot 3(bd+bf+df)+T\cdot 3(ac+ae+ce)\big)=\\ &=4S^2T^2-(S+T)\big(S(T^2-V)+T(S^2-U)\big)=(S+T)(SV+TU)-ST(T-S)^2,\end{aligned}$$
and the statement is equivalent with
$$(S+T)(SV+TU)>ST(T-S)^2. \tag{4}$$

By the Cauchy-Schwarz inequality,
$$(S+T)(TU+SV)\ge\big(\sqrt{S\cdot TU}+\sqrt{T\cdot SV}\big)^2=ST\big(\sqrt{U}+\sqrt{V}\big)^2. \tag{5}$$
Estimate the quantities $\sqrt{U}$ and $\sqrt{V}$ by the QM–AM inequality with the positive terms $(e-c)^2$ and $(d-b)^2$ being omitted:
$$\begin{aligned}\sqrt{U}+\sqrt{V}&>\sqrt{\frac{(e-a)^2+(c-a)^2}{2}}+\sqrt{\frac{(f-b)^2+(f-d)^2}{2}}\\ &>\frac{(e-a)+(c-a)}{2}+\frac{(f-b)+(f-d)}{2}=\left(f-\frac{d}{2}-\frac{b}{2}\right)+\left(\frac{e}{2}+\frac{c}{2}-a\right)\\ &=(T-S)+\frac{3}{2}(e-d)+\frac{3}{2}(c-b)>T-S.\end{aligned} \tag{6}$$

The estimates (5) and (6) prove (4) and hence the statement.

Solution 3. We keep using the notations σ and τ from Solution 1. Moreover, let $s=c+e$. Note that
$$(c-b)(c-d)+(e-f)(e-d)+(e-f)(c-b)<0,$$
since each summand is negative. This rewrites as
$$\begin{gathered}(bd+bf+df)-(ac+ce+ae)<(c+e)(b+d+f-a-c-e),\ \text{ or}\\ \tau-\sigma<s(T-S).\end{gathered} \tag{7}$$
Then we have
$$\begin{aligned}S\tau+T\sigma&=S(\tau-\sigma)+(S+T)\sigma<Ss(T-S)+(S+T)(ce+as)\\ &\le Ss(T-S)+(S+T)\left(\frac{s^2}{4}+(S-s)s\right)=s\left(2ST-\frac{3}{4}(S+T)s\right).\end{aligned}$$
Using this inequality together with the AM–GM inequality we get
$$\begin{aligned}\sqrt{\frac{3}{4}(S+T)(S\tau+T\sigma)}&<\sqrt{\frac{3}{4}(S+T)s\left(2ST-\frac{3}{4}(S+T)s\right)}\\ &\le\frac{\frac{3}{4}(S+T)s+2ST-\frac{3}{4}(S+T)s}{2}=ST.\end{aligned}$$
Hence,
$$2ST>\sqrt{3(S+T)\big(S(bd+bf+df)+T(ac+ae+ce)\big)}.$$

Comment 2. The expression (7) can be found by considering the sum of the roots of the quadratic polynomial $q(x) = (x-b)(x-d)(x-f) - (x-a)(x-c)(x-e)$.

Solution 4. We introduce the expressions σ and τ as in the previous solutions. The idea of the solution is to change the values of variables $a, \ldots, f$ keeping the left-hand side unchanged and increasing the right-hand side; it will lead to a simpler inequality which can be proved in a direct way.

Namely, we change the variables (i) keeping the (non-strict) inequalities $a \le b \le c \le d \le e \le f$; (ii) keeping the values of sums S and T unchanged; and finally (iii) increasing the values of σ and τ. Then the left-hand side of (1) remains unchanged, while the right-hand side increases. Hence, the inequality (1) (and even a non-strict version of (1)) for the changed values would imply the same (strict) inequality for the original values.

First, we find the sufficient conditions for (ii) and (iii) to be satisfied.

Lemma. Let $x, y, z > 0$; denote $U(x,y,z) = x+y+z$, $\upsilon(x,y,z) = xy+xz+yz$. Suppose that $x'+y' = x+y$ but $|x-y| \ge |x'-y'|$; then we have $U(x',y',z) = U(x,y,z)$ and $\upsilon(x',y',z) \ge \upsilon(x,y,z)$ with equality achieved only when $|x-y| = |x'-y'|$.

Proof. The first equality is obvious. For the second, we have

$$\begin{aligned}\upsilon(x',y',z) = z(x'+y') + x'y' &= z(x'+y') + \frac{(x'+y')^2 - (x'-y')^2}{4}\\ &\ge z(x+y) + \frac{(x+y)^2 - (x-y)^2}{4} = \upsilon(x,y,z),\end{aligned}$$

with the equality achieved only for $(x'-y')^2 = (x-y)^2 \iff |x'-y'| = |x-y|$, as desired. $\square$

Now, we apply Lemma several times making the following changes. For each change, we denote the new values by the same letters to avoid cumbersome notations.

1. Let $k = \dfrac{d-c}{2}$. Replace (b,c,d,e) by $(b+k, c+k, d-k, e-k)$. After the change we have $a < b < c = d < e < f$, the values of S, T remain unchanged, but σ, τ strictly increase by Lemma.

2. Let $\ell = \dfrac{e-d}{2}$. Replace (c,d,e,f) by $(c+\ell, d+\ell, e-\ell, f-\ell)$. After the change we have $a < b < c = d = e < f$, the values of S, T remain unchanged, but σ, τ strictly increase by the Lemma.

3. Finally, let $m = \dfrac{c-b}{3}$. Replace (a,b,c,d,e,f) by $(a+2m, b+2m, c-m, d-m, e-m, f-m)$. After the change, we have $a < b = c = d = e < f$ and S, T are unchanged. To check (iii), we observe that our change can be considered as a composition of two changes: $(a,b,c,d) \to (a+m, b+m, c-m, d-m)$ and $(a,b,e,f) \to (a+m, b+m, e-m, f-m)$. It is easy to see that each of these two consecutive changes satisfy the conditions of the Lemma, hence the values of σ and τ increase.

Finally, we come to the situation when $a < b = c = d = e < f$, and we need to prove the inequality

$$\begin{aligned}2(a+2b)(2b+f) &\ge \sqrt{3(a+4b+f)\big((a+2b)(b^2+2bf) + (2b+f)(2ab+b^2)\big)}\\ &= \sqrt{3b(a+4b+f)\cdot\big((a+2b)(b+2f) + (2b+f)(2a+b)\big)}. \qquad (8)\end{aligned}$$

Now, observe that

$$2\cdot 2(a+2b)(2b+f) = 3b(a+4b+f) + \big((a+2b)(b+2f) + (2a+b)(2b+f)\big).$$

Hence (4) rewrites as

$$\begin{aligned}3b(a+4b+f) &+ \big((a+2b)(b+2f)+(2a+b)(2b+f)\big) \\ &\geq 2\sqrt{3b(a+4b+f)\cdot\big((a+2b)(b+2f)+(2b+f)(2a+b)\big)},\end{aligned}$$

which is simply the AM–GM inequality.

Comment 3. Here, we also can find all the cases of equality. Actually, it is easy to see that if some two numbers among b, c, d, e are distinct then one can use Lemma to increase the right-hand side of (1). Further, if $b = c = d = e$, then we need equality in (4); this means that we apply AM–GM to equal numbers, that is,

$$3b(a+4b+f) = (a+2b)(b+2f)+(2a+b)(2b+f),$$

which leads to the same equality as in Comment 1.

Combinatorics

C1. In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

(Austria)

Answer. Yes, such an example exists.

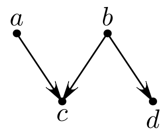
Solution. We say that an order of singers is *good* if it satisfied all their wishes. Next, we say that a number N is *realizable* by k singers (or k-*realizable*) if for some set of wishes of these singers there are exactly N good orders. Thus, we have to prove that a number 2010 is 20-realizable.

We start with the following simple

Lemma. Suppose that numbers n_1, n_2 are realizable by respectively k_1 and k_2 singers. Then the number $n_1 n_2$ is $(k_1 + k_2)$-realizable.

Proof. Let the singers $A_1, \dots, A_{k_1}$ (with some wishes among them) realize n_1, and the singers B_1, $\dots$, B_{k_2} (with some wishes among them) realize n_2. Add to each singer B_i the wish to perform later than all the singers A_j. Then, each good order of the obtained set of singers has the form $(A_{i_1}, \dots, A_{i_{k_1}}, B_{j_1}, \dots, B_{j_{k_2}})$, where $(A_{i_1}, \dots, A_{i_{k_1}})$ is a good order for A_i's and $(B_{j_1}, \dots, B_{j_{k_2}})$ is a good order for B_j's. Conversely, each order of this form is obviously good. Hence, the number of good orders is $n_1 n_2$. $\square$

In view of Lemma, we show how to construct sets of singers containing 4, 3 and 13 singers and realizing the numbers 5, 6 and 67, respectively. Thus the number $2010 = 6 \cdot 5 \cdot 67$ will be realizable by $4 + 3 + 13 = 20$ singers. These companies of singers are shown in Figs. 1–3; the wishes are denoted by arrows, and the number of good orders for each Figure stands below in the brackets.



(5)

Fig. 1

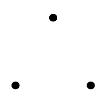

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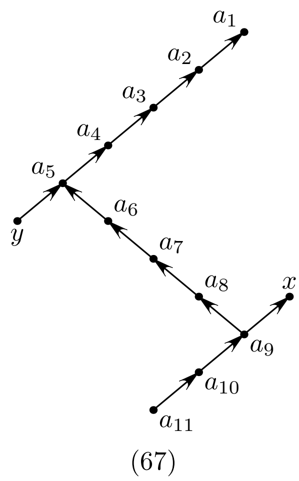
Fig. 2



(67)

Fig. 3

For Fig. 1, there are exactly 5 good orders (a, b, c, d), (a, b, d, c), (b, a, c, d), (b, a, d, c), (b, d, a, c). For Fig. 2, each of 6 orders is good since there are no wishes.

Finally, for Fig. 3, the order of $a_1, \dots, a_{11}$ is fixed; in this line, singer x can stand before each of a_i ($i \le 9$), and singer y can stand after each of a_j ($j \ge 5$), thus resulting in $9 \cdot 7 = 63$ cases. Further, the positions of x and y in this line determine the whole order uniquely unless both of them come between the same pair (a_i, a_{i+1}) (thus $5 \le i \le 8$); in the latter cases, there are two orders instead of 1 due to the order of x and y. Hence, the total number of good orders is $63 + 4 = 67$, as desired.

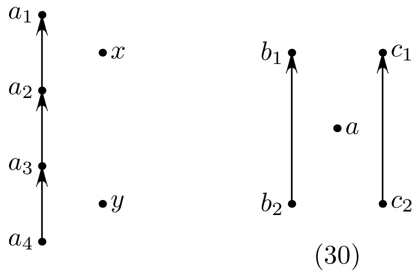
Comment. The number 20 in the problem statement is not sharp and is put there to respect the original formulation. So, if necessary, the difficulty level of this problem may be adjusted by replacing 20 by a smaller number. Here we present some improvements of the example leading to a smaller number of singers.

Surely, each example with < 20 singers can be filled with some "super-stars" who should perform at the very end in a fixed order. Hence each of these improvements provides a different solution for the problem. Moreover, the large variety of ideas standing behind these examples allows to suggest that there are many other examples.

1. Instead of building the examples realizing 5 and 6, it is more economic to make an example realizing 30; it may seem even simpler. Two possible examples consisting of 5 and 6 singers are shown in Fig. 4; hence the number 20 can be decreased to 19 or 18.

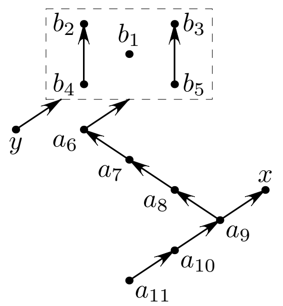
For Fig. 4a, the order of $a_1, \dots, a_4$ is fixed, there are 5 ways to add x into this order, and there are 6 ways to add y into the resulting order of $a_1, \dots, a_4, x$. Hence there are $5 \cdot 6 = 30$ good orders.

On Fig. 4b, for 5 singers a, b_1, b_2, c_1, c_2 there are $5! = 120$ orders at all. Obviously, exactly one half of them satisfies the wish $b_1 \leftarrow b_2$, and exactly one half of these orders satisfies another wish $c_1 \leftarrow c_2$; hence, there are exactly $5!/4 = 30$ good orders.



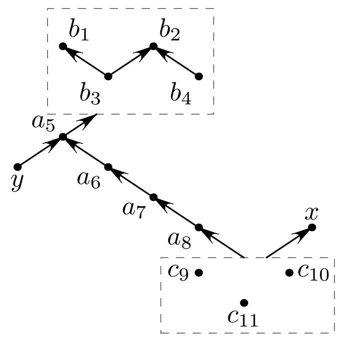
(30) *a)* (30) *b)*

Fig. 4



(2010)

Fig. 5



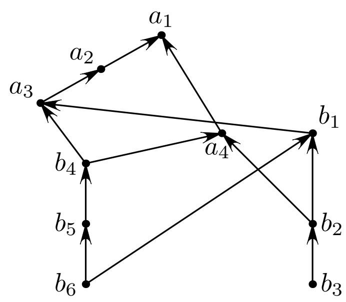
(2010)

Fig. 6

2. One can merge the examples for 30 and 67 shown in Figs. 4b and 3 in a smarter way, obtaining a set of 13 singers representing 2010. This example is shown in Fig. 5; an arrow from/to group $\{b_1, \dots, b_5\}$ means that there exists such arrow from each member of this group.

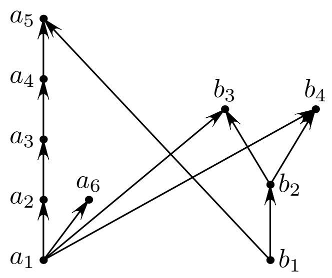
Here, as in Fig. 4b, one can see that there are exactly 30 orders of $b_1, \dots, b_5, a_6, \dots, a_{11}$ satisfying all their wishes among themselves. Moreover, one can prove in the same way as for Fig. 3 that each of these orders can be complemented by x and y in exactly 67 ways, hence obtaining $30 \cdot 67 = 2010$ good orders at all.

Analogously, one can merge the examples in Figs. 1–3 to represent 2010 by 13 singers as is shown in Fig. 6.



(67)

Fig. 7



(2010)

Fig. 8

3. Finally, we will present two other improvements; the proofs are left to the reader. The graph in Fig. 7 shows how 10 singers can represent 67. Moreover, even a company of 10 singers representing 2010 can be found; this company is shown in Fig. 8.

C2. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag.

We say that a set of N flags is *diverse* if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

(Croatia)

Answer. $M = 2^{N-2} + 1$.

Solution. When speaking about the diagonal of a square, we will always mean the main diagonal.

Let M_N be the smallest positive integer satisfying the problem condition. First, we show that $M_N > 2^{N-2}$. Consider the collection of all 2^{N-2} flags having yellow first squares and blue second ones. Obviously, both colors appear on the diagonal of each $N \times N$ square formed by these flags.

We are left to show that $M_N \leq 2^{N-2} + 1$, thus obtaining the desired answer. We start with establishing this statement for $N = 4$.

Suppose that we have 5 flags of length 4. We decompose each flag into two parts of 2 squares each; thereby, we denote each flag as LR, where the 2×1 flags $L, R \in \mathcal{S} = \{\mathrm{BB}, \mathrm{BY}, \mathrm{YB}, \mathrm{YY}\}$ are its left and right parts, respectively. First, we make two easy observations on the flags 2×1 which can be checked manually.

(i) For each $A \in \mathcal{S}$, there exists only one 2×1 flag $C \in \mathcal{S}$ (possibly $C = A$) such that A and C cannot form a 2×2 square with monochrome diagonal (for part BB, that is YY, and for BY that is YB).

(ii) Let $A_1, A_2, A_3 \in \mathcal{S}$ be three distinct elements; then two of them can form a 2×2 square with yellow diagonal, and two of them can form a 2×2 square with blue diagonal (for all parts but BB, a pair (BY, YB) fits for both statements, while for all parts but BY, these pairs are (YB, YY) and (BB, YB)).

Now, let ℓ and r be the numbers of distinct left and right parts of our 5 flags, respectively. The total number of flags is $5 \leq r\ell$, hence one of the factors (say, r) should be at least 3. On the other hand, $\ell, r \leq 4$, so there are two flags with coinciding right part; let them be L_1R_1 and L_2R_1 ($L_1 \neq L_2$).

Next, since $r \geq 3$, there exist some flags L_3R_3 and L_4R_4 such that R_1, R_3, R_4 are distinct. Let $L'R'$ be the remaining flag. By (i), one of the pairs (L', L_1) and (L', L_2) can form a 2×2 square with monochrome diagonal; we can assume that L', L_2 form a square with a blue diagonal. Finally, the right parts of two of the flags L_1R_1, L_3R_3, L_4R_4 can also form a 2×2 square with a blue diagonal by (ii). Putting these 2×2 squares on the diagonal of a 4×4 square, we find a desired arrangement of four flags.

We are ready to prove the problem statement by induction on N; actually, above we have proved the base case $N = 4$. For the induction step, assume that $N > 4$, consider any $2^{N-2} + 1$ flags of length N, and arrange them into a large flag of size $(2^{N-2} + 1) \times N$. This flag contains a non-monochrome column since the flags are distinct; we may assume that this column is the first one. By the pigeonhole principle, this column contains at least $\left\lceil \dfrac{2^{N-2} + 1}{2} \right\rceil = 2^{N-3} + 1$ squares of one color (say, blue). We call the flags with a blue first square *good*.

Consider all the good flags and remove the first square from each of them. We obtain at least $2^{N-3} + 1 \geq M_{N-1}$ flags of length $N - 1$; by the induction hypothesis, $N - 1$ of them can form a square Q with the monochrome diagonal. Now, returning the removed squares, we obtain a rectangle $(N-1)\times N$, and our aim is to supplement it on the top by one more flag.

If Q has a yellow diagonal, then we can take each flag with a yellow first square (it exists by a choice of the first column; moreover, it is not used in Q). Conversely, if the diagonal of Q is blue then we can take any of the $\geq 2^{N-3}+1-(N-1)>0$ remaining good flags. So, in both cases we get a desired $N\times N$ square.

Solution 2. We present a different proof of the estimate $M_N \leq 2^{N-2}+1$. We do not use the induction, involving Hall's lemma on matchings instead.

Consider arbitrary $2^{N-2}+1$ distinct flags and arrange them into a large $(2^{N-2}+1)\times N$ flag. Construct two bipartite graphs $G_{\mathrm{y}}=(V\cup V', E_{\mathrm{y}})$ and $G_{\mathrm{b}}=(V\cup V', E_{\mathrm{b}})$ with the common set of vertices as follows. Let V and V' be the set of columns and the set of flags under consideration, respectively. Next, let the edge (c,f) appear in E_{y} if the intersection of column c and flag f is yellow, and $(c,f)\in E_{\mathrm{b}}$ otherwise. Then we have to prove exactly that one of the graphs G_{y} and G_{b} contains a matching with all the vertices of V involved.

Assume that these matchings do not exist. By Hall's lemma, it means that there exist two sets of columns $S_{\mathrm{y}}, S_{\mathrm{b}}\subset V$ such that $|E_{\mathrm{y}}(S_{\mathrm{y}})|\leq |S_{\mathrm{y}}|-1$ and $|E_{\mathrm{b}}(S_{\mathrm{b}})|\leq |S_{\mathrm{b}}|-1$ (in the left-hand sides, $E_{\mathrm{y}}(S_{\mathrm{y}})$ and $E_{\mathrm{b}}(S_{\mathrm{b}})$ denote respectively the sets of all vertices connected to S_{y} and S_{b} in the corresponding graphs). Our aim is to prove that this is impossible. Note that $S_{\mathrm{y}}, S_{\mathrm{b}}\neq V$ since $N\leq 2^{N-2}+1$.

First, suppose that $S_{\mathrm{y}}\cap S_{\mathrm{b}}\neq\varnothing$, so there exists some $c\in S_{\mathrm{y}}\cap S_{\mathrm{b}}$. Note that each flag is connected to c either in G_{y} or in G_{b}, hence $E_{\mathrm{y}}(S_{\mathrm{y}})\cup E_{\mathrm{b}}(S_{\mathrm{b}})=V'$. Hence we have $2^{N-2}+1=|V'|\leq |E_{\mathrm{y}}(S_{\mathrm{y}})|+|E_{\mathrm{b}}(S_{\mathrm{b}})|\leq |S_{\mathrm{y}}|+|S_{\mathrm{b}}|-2\leq 2N-4$; this is impossible for $N\geq 4$.

So, we have $S_{\mathrm{y}}\cap S_{\mathrm{b}}=\varnothing$. Let $y=|S_{\mathrm{y}}|$, $b=|S_{\mathrm{b}}|$. From the construction of our graph, we have that all the flags in the set $V''=V'\backslash\big(E_{\mathrm{y}}(S_{\mathrm{y}})\cup E_{\mathrm{b}}(S_{\mathrm{b}})\big)$ have blue squares in the columns of S_{y} and yellow squares in the columns of S_{b}. Hence the only undetermined positions in these flags are the remaining $N-y-b$ ones, so $2^{N-y-b}\geq |V''|\geq |V'|-|E_{\mathrm{y}}(S_{\mathrm{y}})|-|E_{\mathrm{b}}(S_{\mathrm{b}})|\geq 2^{N-2}+1-(y-1)-(b-1)$, or, denoting $c=y+b$, $2^{N-c}+c>2^{N-2}+2$. This is impossible since $N\geq c\geq 2$.

C3. 2500 chess kings have to be placed on a 100×100 chessboard so that

(i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);

(ii) each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

(Russia)

Answer. There are two such arrangements.

Solution. Suppose that we have an arrangement satisfying the problem conditions. Divide the board into 2×2 pieces; we call these pieces *blocks*. Each block can contain not more than one king (otherwise these two kings would attack each other); hence, by the pigeonhole principle each block must contain exactly one king.

Now assign to each block a letter T or B if a king is placed in its top or bottom half, respectively. Similarly, assign to each block a letter L or R if a king stands in its left or right half. So we define *T-blocks*, *B-blocks*, *L-blocks*, and *R-blocks*. We also combine the letters; we call a block *a TL-block* if it is simultaneously T-block and L-block. Similarly we define *TR-blocks*, *BL-blocks*, and *BR-blocks*. The arrangement of blocks determines uniquely the arrangement of kings; so in the rest of the solution we consider the 50×50 system of blocks (see Fig. 1). We identify the blocks by their coordinate pairs; the pair (i, j), where $1 \leq i, j \leq 50$, refers to the jth block in the ith row (or the ith block in the jth column). The upper-left block is $(1, 1)$.

The system of blocks has the following properties..

(i$'$) If (i, j) is a B-block then $(i+1, j)$ is a B-block: otherwise the kings in these two blocks can take each other. Similarly: if (i, j) is a T-block then $(i-1, j)$ is a T-block; if (i, j) is an L-block then $(i, j-1)$ is an L-block; if (i, j) is an R-block then $(i, j+1)$ is an R-block.

(ii$'$) Each column contains exactly 25 L-blocks and 25 R-blocks, and each row contains exactly 25 T-blocks and 25 B-blocks. In particular, the total number of L-blocks (or R-blocks, or T-blocks, or B-blocks) is equal to $25 \cdot 50 = 1250$.

Consider any B-block of the form $(1, j)$. By (i$'$), all blocks in the jth column are B-blocks; so we call such a column *B-column*. By (ii$'$), we have 25 B-blocks in the first row, so we obtain 25 B-columns. These 25 B-columns contain 1250 B-blocks, hence all blocks in the remaining columns are T-blocks, and we obtain 25 *T-columns*. Similarly, there are exactly 25 *L-rows* and exactly 25 *R-rows*.

Now consider an arbitrary pair of a T-column and a neighboring B-column (columns with numbers j and $j+1$).

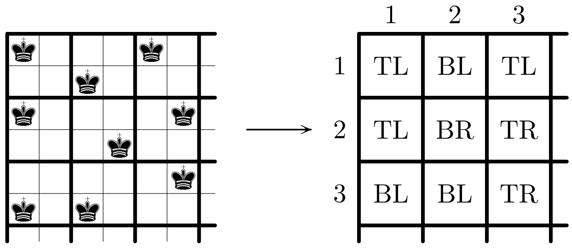


Fig. 1

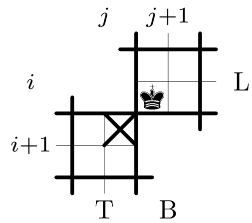


Fig. 2

Case 1. Suppose that the jth column is a T-column, and the $(j+1)$th column is a B-column. Consider some index i such that the ith row is an L-row; then $(i, j+1)$ is a BL-block. Therefore, $(i+1, j)$ cannot be a TR-block (see Fig. 2), hence $(i+1, j)$ is a TL-block, thus the

$(i+1)$th row is an L-row. Now, choosing the ith row to be the topmost L-row, we successively obtain that all rows from the ith to the 50th are L-rows. Since we have exactly 25 L-rows, it follows that the rows from the 1st to the 25th are R-rows, and the rows from the 26th to the 50th are L-rows.

Now consider the neighboring R-row and L-row (that are the rows with numbers 25 and 26). Replacing in the previous reasoning rows by columns and vice versa, the columns from the 1st to the 25th are T-columns, and the columns from the 26th to the 50th are B-columns. So we have a unique arrangement of blocks that leads to the arrangement of kings satisfying the condition of the problem (see Fig. 3).

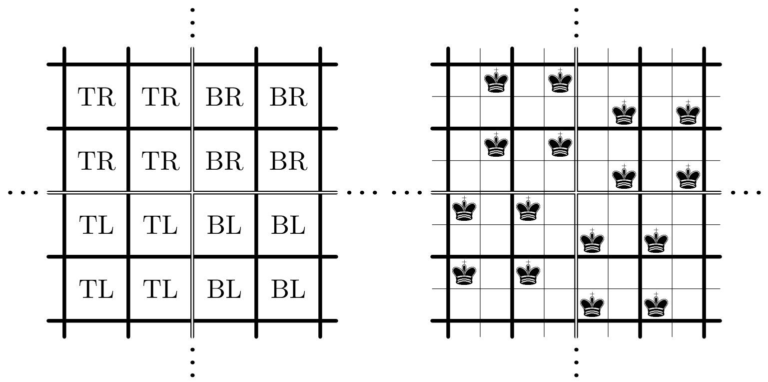


Fig. 3

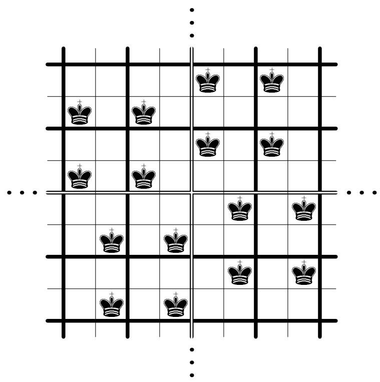


Fig. 4

Case 2. Suppose that the jth column is a B-column, and the $(j+1)$th column is a T-column. Repeating the arguments from Case 1, we obtain that the rows from the 1st to the 25th are L-rows (and all other rows are R-rows), the columns from the 1st to the 25th are B-columns (and all other columns are T-columns), so we find exactly one more arrangement of kings (see Fig. 4).

C4. Six stacks $S_1, \ldots, S_6$ of coins are standing in a row. In the beginning every stack contains a single coin. There are two types of allowed moves:

Move 1: If stack S_k with $1 \le k \le 5$ contains at least one coin, you may remove one coin from S_k and add two coins to S_{k+1}.

Move 2: If stack S_k with $1 \le k \le 4$ contains at least one coin, then you may remove one coin from S_k and exchange stacks S_{k+1} and S_{k+2}.

Decide whether it is possible to achieve by a sequence of such moves that the first five stacks are empty, whereas the sixth stack S_6 contains exactly $2010^{2010^{2010}}$ coins.

C4′. Same as Problem C4, but the constant $2010^{2010^{2010}}$ is replaced by 2010^{2010}.

(Netherlands)

Answer. Yes (in both variants of the problem). There exists such a sequence of moves.

Solution. Denote by $(a_1, a_2, \ldots, a_n) \to (a'_1, a'_2, \ldots, a'_n)$ the following: if some consecutive stacks contain $a_1, \ldots, a_n$ coins, then it is possible to perform several allowed moves such that the stacks contain $a'_1, \ldots, a'_n$ coins respectively, whereas the contents of the other stacks remain unchanged.

Let $A = 2010^{2010}$ or $A = 2010^{2010^{2010}}$, respectively. Our goal is to show that

$$(1,1,1,1,1,1) \to (0,0,0,0,0,A).$$

First we prove two auxiliary observations.

Lemma 1. $(a,0,0) \to (0,2^a,0)$ for every $a \ge 1$.

Proof. We prove by induction that $(a,0,0) \to (a-k, 2^k, 0)$ for every $1 \le k \le a$. For $k=1$, apply Move 1 to the first stack:

$$(a,0,0) \to (a-1,2,0) = (a-1,2^1,0).$$

Now assume that $k<a$ and the statement holds for some $k<a$. Starting from $(a-k,2^k,0)$, apply Move 1 to the middle stack 2^k times, until it becomes empty. Then apply Move 2 to the first stack:

$$(a-k,2^k,0) \to (a-k,2^k-1,2) \to \cdots \to (a-k,0,2^{k+1}) \to (a-k-1,2^{k+1},0).$$

Hence,

$$(a,0,0) \to (a-k,2^k,0) \to (a-k-1,2^{k+1},0). \qquad \square$$

Lemma 2. For every positive integer n, let $P_n = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n}$ (e.g. $P_3 = 2^{2^2} = 16$). Then $(a,0,0,0) \to (0,P_a,0,0)$ for every $a \ge 1$.

Proof. Similarly to Lemma 1, we prove that $(a,0,0,0) \to (a-k,P_k,0,0)$ for every $1 \le k \le a$.

For $k=1$, apply Move 1 to the first stack:

$$(a,0,0,0) \to (a-1,2,0,0) = (a-1,P_1,0,0).$$

Now assume that the lemma holds for some $k<a$. Starting from $(a-k,P_k,0,0)$, apply Lemma 1, then apply Move 1 to the first stack:

$$(a-k,P_k,0,0) \to (a-k,0,2^{P_k},0) = (a-k,0,P_{k+1},0) \to (a-k-1,P_{k+1},0,0).$$

Therefore,

$$(a,0,0,0) \to (a-k,P_k,0,0) \to (a-k-1,P_{k+1},0,0). \qquad \square$$

Now we prove the statement of the problem.

First apply Move 1 to stack 5, then apply Move 2 to stacks S_4, S_3, S_2 and S_1 in this order. Then apply Lemma 2 twice:

$$\begin{aligned}(1,1,1,1,1,1) &\to (1,1,1,1,0,3) \to (1,1,1,0,3,0) \to (1,1,0,3,0,0) \to (1,0,3,0,0,0) \to \\ &\to (0,3,0,0,0,0) \to (0,0,P_3,0,0,0) = (0,0,16,0,0,0) \to (0,0,0,P_{16},0,0).\end{aligned}$$

We already have more than A coins in stack S_4, since

$$A \le 2010^{2010^{2010}} < (2^{11})^{2010^{2010}} = 2^{11\cdot 2010^{2010}} < 2^{2010^{2011}} < 2^{(2^{11})^{2011}} = 2^{2^{11\cdot 2011}} < 2^{2^{2^{15}}} < P_{16}.$$

To decrease the number of coins in stack S_4, apply Move 2 to this stack repeatedly until its size decreases to $A/4$. (In every step, we remove a coin from S_4 and exchange the empty stacks S_5 and S_6.)

$$\begin{aligned}(0,0,0,P_{16},0,0) &\to (0,0,0,P_{16}-1,0,0) \to (0,0,0,P_{16}-2,0,0) \to \\ &\to \cdots \to (0,0,0,A/4,0,0).\end{aligned}$$

Finally, apply Move 1 repeatedly to empty stacks S_4 and S_5:

$$(0,0,0,A/4,0,0) \to \cdots \to (0,0,0,0,A/2,0) \to \cdots \to (0,0,0,0,0,A).$$

Comment 1. Starting with only 4 stack, it is not hard to check manually that we can achieve at most 28 coins in the last position. However, around 5 and 6 stacks the maximal number of coins explodes. With 5 stacks it is possible to achieve more than $2^{2^{14}}$ coins. With 6 stacks the maximum is greater than $P_{P_{2^{14}}}$.

It is not hard to show that the numbers 2010^{2010} and $2010^{2010^{2010}}$ in the problem can be replaced by any nonnegative integer up to $P_{P_{2^{14}}}$.

Comment 2. The simpler variant C4$'$ of the problem can be solved without Lemma 2:

$$\begin{aligned}(1,1,1,1,1,1) &\to (0,3,1,1,1,1) \to (0,1,5,1,1,1) \to (0,1,1,9,1,1) \to \\ &\to (0,1,1,1,17,1) \to (0,1,1,1,0,35) \to (0,1,1,0,35,0) \to (0,1,0,35,0,0) \to \\ &\to (0,0,35,0,0,0) \to (0,0,1,2^{34},0,0) \to (0,0,1,0,2^{2^{34}},0) \to (0,0,0,2^{2^{34}},0,0) \\ &\to (0,0,0,2^{2^{34}}-1,0,0) \to \ldots \to (0,0,0,A/4,0,0) \to (0,0,0,0,A/2,0) \to (0,0,0,0,0,A).\end{aligned}$$

For this reason, the PSC suggests to consider the problem C4 as well. Problem C4 requires more invention and technical care. On the other hand, the problem statement in C4$'$ hides the fact that the resulting amount of coins can be such incredibly huge and leaves this discovery to the students.

C5. $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and ℓ_i be respectively the number of wins and losses of the ith player. Prove that

$$\sum_{i=1}^{n}(w_i - \ell_i)^3 \geq 0. \tag{1}$$

(South Korea)

Solution. For any tournament T satisfying the problem condition, denote by $S(T)$ sum under consideration, namely

$$S(T) = \sum_{i=1}^{n}(w_i - \ell_i)^3.$$

First, we show that the statement holds if a tournament T has only 4 players. Actually, let $A = (a_1, a_2, a_3, a_4)$ be the number of wins of the players; we may assume that $a_1 \geq a_2 \geq a_3 \geq a_4$. We have $a_1 + a_2 + a_3 + a_4 = \binom{4}{2} = 6$, hence $a_4 \leq 1$. If $a_4 = 0$, then we cannot have $a_1 = a_2 = a_3 = 2$, otherwise the company of all players is bad. Hence we should have $A = (3, 2, 1, 0)$, and $S(T) = 3^3 + 1^3 + (-1)^3 + (-3)^3 = 0$. On the other hand, if $a_4 = 1$, then only two possibilities, $A = (3, 1, 1, 1)$ and $A = (2, 2, 1, 1)$ can take place. In the former case we have $S(T) = 3^3 + 3 \cdot (-2)^3 > 0$, while in the latter one $S(T) = 1^3 + 1^3 + (-1)^3 + (-1)^3 = 0$, as desired.

Now we turn to the general problem. Consider a tournament T with no bad companies and enumerate the players by the numbers from 1 to n. For every 4 players i_1, i_2, i_3, i_4 consider a "sub-tournament" $T_{i_1 i_2 i_3 i_4}$ consisting of only these players and the games which they performed with each other. By the abovementioned, we have $S(T_{i_1 i_2 i_3 i_4}) \geq 0$. Our aim is to prove that

$$S(T) = \sum_{i_1, i_2, i_3, i_4} S(T_{i_1 i_2 i_3 i_4}), \tag{2}$$

where the sum is taken over all 4-tuples of distinct numbers from the set $\{1, \dots, n\}$. This way the problem statement will be established.

We interpret the number $(w_i - \ell_i)^3$ as following. For $i \neq j$, let $\varepsilon_{ij} = 1$ if the ith player wins against the jth one, and $\varepsilon_{ij} = -1$ otherwise. Then

$$(w_i - \ell_i)^3 = \left(\sum_{j \neq i} \varepsilon_{ij}\right)^3 = \sum_{j_1, j_2, j_3 \neq i} \varepsilon_{ij_1}\varepsilon_{ij_2}\varepsilon_{ij_3}.$$

Hence,

$$S(T) = \sum_{i \notin \{j_1, j_2, j_3\}} \varepsilon_{ij_1}\varepsilon_{ij_2}\varepsilon_{ij_3}.$$

To simplify this expression, consider all the terms in this sum where two indices are equal. If, for instance, $j_1 = j_2$, then the term contains $\varepsilon_{ij_1}^2 = 1$, so we can replace this term by ε_{ij_3}. Make such replacements for each such term; obviously, after this change each term of the form ε_{ij_3} will appear $P(T)$ times, hence

$$S(T) = \sum_{|\{i, j_1, j_2, j_3\}| = 4} \varepsilon_{ij_1}\varepsilon_{ij_2}\varepsilon_{ij_3} + P(T)\sum_{i \neq j} \varepsilon_{ij} = S_1(T) + P(T)S_2(T).$$

We show that $S_2(T)=0$ and hence $S(T)=S_1(T)$ for each tournament. Actually, note that $\varepsilon_{ij}=-\varepsilon_{ji}$, and the whole sum can be split into such pairs. Since the sum in each pair is 0, so is $S_2(T)$.

Thus the desired equality (2) rewrites as

$$S_1(T)=\sum_{i_1,i_2,i_3,i_4} S_1(T_{i_1i_2i_3i_4}). \tag{3}$$

Now, if all the numbers j_1,j_2,j_3 are distinct, then the set $\{i,j_1,j_2,j_3\}$ is contained in exactly one 4-tuple, hence the term $\varepsilon_{ij_1}\varepsilon_{ij_2}\varepsilon_{ij_3}$ appears in the right-hand part of (3) exactly once, as well as in the left-hand part. Clearly, there are no other terms in both parts, so the equality is established.

Solution 2. Similarly to the first solution, we call the subsets of players as *companies*, and the k-element subsets will be called as k*-companies*.

In any company of the players, call a player *the local champion of the company* if he defeated all other members of the company. Similarly, if a player lost all his games against the others in the company then call him *the local loser of the company*. Obviously every company has at most one local champion and at most one local loser. By the condition of the problem, whenever a 4-company has a local loser, then this company has a local champion as well.

Suppose that k is some positive integer, and let us count all cases when a player is the local champion of some k-company. The ith player won against w_i other player. To be the local champion of a k-company, he must be a member of the company, and the other $k-1$ members must be chosen from those whom he defeated. Therefore, the ith player is the local champion of $\binom{w_i}{k-1}$ k-companies. Hence, the total number of local champions of all k-companies is $\sum_{i=1}^{n}\binom{w_i}{k-1}$.

Similarly, the total number of local losers of the k-companies is $\sum_{i=1}^{n}\binom{\ell_i}{k-1}$.

Now apply this for $k=2,3$ and 4.

Since every game has a winner and a loser, we have $\sum_{i=1}^{n} w_i=\sum_{i=1}^{n}\ell_i=\binom{n}{2}$, and hence

$$\sum_{i=1}^{n}\big(w_i-\ell_i\big)=0. \tag{4}$$

In every 3-company, either the players defeated one another in a cycle or the company has both a local champion and a local loser. Therefore, the total number of local champions and local losers in the 3-companies is the same, $\sum_{i=1}^{n}\binom{w_i}{2}=\sum_{i=1}^{n}\binom{\ell_i}{2}$. So we have

$$\sum_{i=1}^{n}\left(\binom{w_i}{2}-\binom{\ell_i}{2}\right)=0. \tag{5}$$

In every 4-company, by the problem's condition, the number of local losers is less than or equal to the number of local champions. Then the same holds for the total numbers of local champions and local losers in all 4-companies, so $\sum_{i=1}^{n}\binom{w_i}{3} \geq \sum_{i=1}^{n}\binom{\ell_i}{3}$. Hence,

$$\sum_{i=1}^{n}\left(\binom{w_i}{3}-\binom{\ell_i}{3}\right) \geq 0. \tag{6}$$

Now we establish the problem statement (1) as a linear combination of (4), (5) and (6). It is easy check that

$$(x-y)^3 = 24\left(\binom{x}{3}-\binom{y}{3}\right) + 24\left(\binom{x}{2}-\binom{y}{2}\right) - \big(3(x+y)^2-4\big)(x-y).$$

Apply this identity to $x = w_1$ and $y = \ell_i$. Since every player played $n-1$ games, we have $w_i + \ell_i = n-1$, and thus

$$(w_i-\ell_i)^3 = 24\left(\binom{w_i}{3}-\binom{\ell_i}{3}\right) + 24\left(\binom{w_i}{2}-\binom{\ell_i}{2}\right) - \big(3(n-1)^2-4\big)\Big(w_i-\ell_i\Big).$$

Then

$$\sum_{i=1}^{n}(w_i-\ell_i)^3 = 24\underbrace{\sum_{i=1}^{n}\left(\binom{w_i}{3}-\binom{\ell_i}{3}\right)}_{\geq 0} + 24\underbrace{\sum_{i=1}^{n}\left(\binom{w_i}{2}-\binom{\ell_i}{2}\right)}_{0} - \big(3(n-1)^2-4\big)\underbrace{\sum_{i=1}^{n}\Big(w_i-\ell_i\Big)}_{0} \geq 0.$$

C6. Given a positive integer k and other two integers $b > w > 1$. There are two strings of pearls, a string of b black pearls and a string of w white pearls. The *length* of a string is the number of pearls on it.

One cuts these strings in some steps by the following rules. In each step:

(i) The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then k first ones (if they consist of more than one pearl) are chosen; if there are less than k strings longer than 1, then one chooses all of them.

(ii) Next, one cuts each chosen string into two parts differing in length by at most one.

(For instance, if there are strings of 5, 4, 4, 2 black pearls, strings of 8, 4, 3 white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4)$, $(3, 2)$, $(2, 2)$ and $(2, 2)$, respectively.)

The process stops immediately after the step when a first isolated white pearl appears. Prove that at this stage, there will still exist a string of at least two black pearls.

(Canada)

Solution 1. Denote the situation after the ith step by A_i; hence A_0 is the initial situation, and $A_{i-1} \to A_i$ is the ith step. We call a string containing m pearls an m-string; it is an *m-w-string* or a *m-b-string* if it is white or black, respectively.

We continue the process until every string consists of a single pearl. We will focus on three moments of the process: (a) the first stage A_s when the first 1-string (no matter black or white) appears; (b) the first stage A_t where the total number of strings is greater than k (if such moment does not appear then we put $t = \infty$); and (c) the first stage A_f when all black pearls are isolated. It is sufficient to prove that in A_{f-1} (or earlier), a 1-w-string appears.

We start with some easy properties of the situations under consideration. Obviously, we have $s \leq f$. Moreover, all b-strings from A_{f-1} become single pearls in the fth step, hence all of them are 1- or 2-b-strings.

Next, observe that in each step $A_i \to A_{i+1}$ with $i \leq t-1$, all (>1)-strings were cut since there are not more than k strings at all; if, in addition, $i < s$, then there were no 1-string, so all the strings were cut in this step.

Now, let B_i and b_i be the lengths of the longest and the shortest b-strings in A_i, and let W_i and w_i be the same for w-strings. We show by induction on $i \leq \min\{s, t\}$ that (i) the situation A_i contains exactly 2^i black and 2^i white strings, (ii) $B_i \geq W_i$, and (iii) $b_i \geq w_i$. The base case $i = 0$ is obvious. For the induction step, if $i \leq \min\{s, t\}$ then in the ith step, each string is cut, thus the claim (i) follows from the induction hypothesis; next, we have $B_i = \lceil B_{i-1}/2 \rceil \geq \lceil W_{i-1}/2 \rceil = W_i$ and $b_i = \lfloor b_{i-1}/2 \rfloor \geq \lfloor w_{i-1}/2 \rfloor = w_i$, thus establishing (ii) and (iii).

For the numbers s, t, f, two cases are possible.

Case 1. Suppose that $s \leq t$ or $f \leq t+1$ (and hence $s \leq t+1$); in particular, this is true when $t = \infty$. Then in A_{s-1} we have $B_{s-1} \geq W_{s-1}$, $b_{s-1} \geq w_{s-1} > 1$ as $s - 1 \leq \min\{s, t\}$. Now, if $s = f$, then in A_{s-1}, there is no 1-w-string as well as no (>2)-b-string. That is, $2 = B_{s-1} \geq W_{s-1} \geq b_{s-1} \geq w_{s-1} > 1$, hence all these numbers equal 2. This means that in A_{s-1}, all strings contain 2 pearls, and there are 2^{s-1} black and 2^{s-1} white strings, which means $b = 2 \cdot 2^{s-1} = w$. This contradicts the problem conditions.

Hence we have $s \leq f - 1$ and thus $s \leq t$. Therefore, in the sth step each string is cut into two parts. Now, if a 1-b-string appears in this step, then from $w_{s-1} \leq b_{s-1}$ we see that a

1-w-string appears as well; so, in each case in the sth step a 1-w-string appears, while not all black pearls become single, as desired.

Case 2. Now assume that $t+1 \leq s$ and $t+2 \leq f$. Then in A_t we have exactly 2^t white and 2^t black strings, all being larger than 1, and $2^{t+1} > k \geq 2^t$ (the latter holds since 2^t is the total number of strings in A_{t-1}). Now, in the $(t+1)$st step, exactly k strings are cut, not more than 2^t of them being black; so the number of w-strings in A_{t+1} is at least $2^t + (k - 2^t) = k$. Since the number of w-strings does not decrease in our process, in A_{f-1} we have at least k white strings as well.

Finally, in A_{f-1}, all b-strings are not larger than 2, and at least one 2-b-string is cut in the fth step. Therefore, at most $k-1$ white strings are cut in this step, hence there exists a w-string $\mathcal{W}$ which is not cut in the fth step. On the other hand, since a 2-b-string is cut, all (≥ 2)-w-strings should also be cut in the fth step; hence $\mathcal{W}$ should be a single pearl. This is exactly what we needed.

Comment. In this solution, we used the condition $b \neq w$ only to avoid the case $b = w = 2^t$. Hence, if a number $b = w$ is not a power of 2, then the problem statement is also valid.

Solution 2. We use the same notations as introduced in the first paragraph of the previous solution. We claim that at every stage, there exist a u-b-string and a v-w-string such that either

(i) $u > v \geq 1$, or

(ii) $2 \leq u \leq v < 2u$, and there also exist $k-1$ of $(>v/2)$-strings other than considered above.

First, we notice that this statement implies the problem statement. Actually, in both cases (i) and (ii) we have $u > 1$, so at each stage there exists a (≥ 2)-b-string, and for the last stage it is exactly what we need.

Now, we prove the claim by induction on the number of the stage. Obviously, for A_0 the condition (i) holds since $b > w$. Further, we suppose that the statement holds for A_i, and prove it for A_{i+1}. Two cases are possible.

Case 1. Assume that in A_i, there are a u-b-string and a v-w-string with $u > v$. We can assume that v is the length of the shortest w-string in A_i; since we are not at the final stage, we have $v \geq 2$. Now, in the $(i+1)$st step, two subcases may occur.

Subcase 1a. Suppose that either no u-b-string is cut, or both some u-b-string and some v-w-string are cut. Then in A_{i+1}, we have either a u-b-string and a $(\leq v)$-w-string (and (i) is valid), or we have a $\lceil u/2 \rceil$-b-string and a $\lfloor v/2 \rfloor$-w-string. In the latter case, from $u > v$ we get $\lceil u/2 \rceil > \lfloor v/2 \rfloor$, and (i) is valid again.

Subcase 1b. Now, some u-b-string is cut, and no v-w-string is cut (and hence all the strings which are cut are longer than v). If $u' = \lceil u/2 \rceil > v$, then the condition (i) is satisfied since we have a u'-b-string and a v-w-string in A_{i+1}. Otherwise, notice that the inequality $u > v \geq 2$ implies $u' \geq 2$. Furthermore, besides a fixed u-b-string, other $k-1$ of $(\geq v+1)$-strings should be cut in the $(i+1)$st step, hence providing at least $k-1$ of $(\geq \lceil (v+1)/2 \rceil)$-strings, and $\lceil (v+1)/2 \rceil > v/2$. So, we can put $v' = v$, and we have $u' \leq v < u \leq 2u'$, so the condition (ii) holds for A_{i+1}.

Case 2. Conversely, assume that in A_i there exist a u-b-string, a v-w-string $(2 \leq u \leq v < 2u)$ and a set S of $k-1$ other strings larger than $v/2$ (and hence larger than 1). In the $(i+1)$st step, three subcases may occur.

Subcase 2a. Suppose that some u-b-string is not cut, and some v-w-string is cut. The latter one results in a $\lfloor v/2 \rfloor$-w-string, we have $v' = \lfloor v/2 \rfloor < u$, and the condition (i) is valid.

Subcase 2b. Next, suppose that no v-w-string is cut (and therefore no u-b-string is cut as $u \leq v$). Then all k strings which are cut have the length $> v$, so each one results in a $(>v/2)$-string. Hence in A_{i+1}, there exist $k \geq k-1$ of $(>v/2)$-strings other than the considered u- and v-strings, and the condition (ii) is satisfied.

Subcase 2c. In the remaining case, all u-b-strings are cut. This means that all $(\geq u)$-strings are cut as well, hence our v-w-string is cut. Therefore in A_{i+1} there exists a $\lceil u/2 \rceil$-b-string together with a $\lfloor v/2 \rfloor$-w-string. Now, if $u' = \lceil u/2 \rceil > \lfloor v/2 \rfloor = v'$ then the condition (i) is fulfilled. Otherwise, we have $u' \leq v' < u \leq 2u'$. In this case, we show that $u' \geq 2$. If, to the contrary, $u' = 1$ (and hence $u = 2$), then all black and white (≥ 2)-strings should be cut in the $(i+1)$st step, and among these strings there are at least a u-b-string, a v-w-string, and $k-1$ strings in S ($k+1$ strings altogether). This is impossible.

Hence, we get $2 \leq u' \leq v' < 2u'$. To reach (ii), it remains to check that in A_{i+1}, there exists a set S' of $k-1$ other strings larger than $v'/2$. These will be exactly the strings obtained from the elements of S. Namely, each $s \in S$ was either cut in the $(i+1)$st step, or not. In the former case, let us include into S' the largest of the strings obtained from s; otherwise we include s itself into S'. All $k-1$ strings in S' are greater than $v/2 \geq v'$, as desired.

C7. Let $P_1, \dots, P_s$ be arithmetic progressions of integers, the following conditions being satisfied:

(i) each integer belongs to at least one of them;

(ii) each progression contains a number which does not belong to other progressions.

Denote by n the least common multiple of steps of these progressions; let $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be its prime factorization. Prove that

$$s \geq 1 + \sum_{i=1}^{k} \alpha_i (p_i - 1).$$

(Germany)

Solution 1. First, we prove the key lemma, and then we show how to apply it to finish the solution.

Let $n_1, \dots, n_k$ be positive integers. By an $n_1 \times n_2 \times \dots \times n_k$ *grid* we mean the set $N = \{(a_1, \dots, a_k) : a_i \in \mathbb{Z},\ 0 \leq a_i \leq n_i - 1\}$; the elements of N will be referred to as *points*. In this grid, we define *a subgrid* as a subset of the form

$$L = \{(b_1, \dots, b_k) \in N : b_{i_1} = x_{i_1}, \dots, b_{i_t} = x_{i_t}\}, \tag{1}$$

where $I = \{i_1, \dots, i_t\}$ is an arbitrary nonempty set of indices, and $x_{i_j} \in [0, n_{i_j} - 1]$ $(1 \leq j \leq t)$ are fixed integer numbers. Further, we say that a subgrid (1) is *orthogonal* to the ith coordinate axis if $i \in I$, and that it is *parallel* to the ith coordinate axis otherwise.

Lemma. Assume that the grid N is covered by subgrids $L_1, L_2, \dots, L_s$ (this means $N = \bigcup_{i=1}^{s} L_i$) so that

(ii′) each subgrid contains a point which is not covered by other subgrids;

(iii) for each coordinate axis, there exists a subgrid L_i orthogonal to this axis.

Then

$$s \geq 1 + \sum_{i=1}^{k} (n_i - 1).$$

Proof. Assume to the contrary that $s \leq \sum_i (n_i - 1) = s'$. Our aim is to find a point that is not covered by $L_1, \dots, L_s$.

The idea of the proof is the following. Imagine that we expand each subgrid to some maximal subgrid so that for the ith axis, there will be at most $n_i - 1$ maximal subgrids orthogonal to this axis. Then the desired point can be found easily: its ith coordinate should be that not covered by the maximal subgrids orthogonal to the ith axis. Surely, the conditions for existence of such expansion are provided by Hall's lemma on matchings. So, we will follow this direction, although we will apply Hall's lemma to some subgraph instead of the whole graph.

Construct a bipartite graph $G = (V \cup V', E)$ as follows. Let $V = \{L_1, \dots, L_s\}$, and let $V' = \{v_{ij} : 1 \leq i \leq s,\ 1 \leq j \leq n_i - 1\}$ be some set of s' elements. Further, let the edge (L_m, v_{ij}) appear iff L_m is orthogonal to the ith axis.

For each subset $W \subset V$, denote

$$f(W) = \{v \in V' : (L, v) \in E \text{ for some } L \in W\}.$$

Notice that $f(V) = V'$ by (iii).

Now, consider the set $W \subset V$ containing the maximal number of elements such that $|W| > |f(W)|$; if there is no such set then we set $W = \varnothing$. Denote $W' = f(W)$, $U = V \backslash W$, $U' = V' \backslash W'$.

By our assumption and the Lemma condition, $|f(V)| = |V'| \geq |V|$, hence $W \neq V$ and $U \neq \varnothing$. Permuting the coordinates, we can assume that $U' = \{v_{ij} : 1 \leq i \leq \ell\}$, $W' = \{v_{ij} : \ell+1 \leq i \leq k\}$.

Consider the induced subgraph G' of G on the vertices $U \cup U'$. We claim that for every $X \subset U$, we get $|f(X) \cap U'| \geq |X|$ (so G' satisfies the conditions of Hall's lemma). Actually, we have $|W| \geq |f(W)|$, so if $|X| > |f(X) \cap U'|$ for some $X \subset U$, then we have

$$|W \cup X| = |W| + |X| > |f(W)| + |f(X) \cap U'| = |f(W) \cup (f(X) \cap U')| = |f(W \cup X)|.$$

This contradicts the maximality of $|W|$.

Thus, applying Hall's lemma, we can assign to each $L \in U$ some vertex $v_{ij} \in U'$ so that to distinct elements of U, distinct vertices of U' are assigned. In this situation, we say that $L \in U$ corresponds to the ith axis, and write $g(L) = i$. Since there are $n_i - 1$ vertices of the form v_{ij}, we get that for each $1 \leq i \leq \ell$, not more than $n_i - 1$ subgrids correspond to the ith axis.

Finally, we are ready to present the desired point. Since $W \neq V$, there exists a point $b = (b_1, b_2, \ldots, b_k) \in N \backslash (\cup_{L \in W} L)$. On the other hand, for every $1 \leq i \leq \ell$, consider any subgrid $L \in U$ with $g(L) = i$. This means exactly that L is orthogonal to the ith axis, and hence all its elements have the same ith coordinate c_L. Since there are at most $n_i - 1$ such subgrids, there exists a number $0 \leq a_i \leq n_i - 1$ which is not contained in a set $\{c_L : g(L) = i\}$. Choose such number for every $1 \leq i \leq \ell$. Now we claim that point $a = (a_1, \ldots, a_\ell, b_{\ell+1}, \ldots, b_k)$ is not covered, hence contradicting the Lemma condition.

Surely, point a cannot lie in some $L \in U$, since all the points in L have $g(L)$th coordinate $c_L \neq a_{g(L)}$. On the other hand, suppose that $a \in L$ for some $L \in W$; recall that $b \notin L$. But the points a and b differ only at first ℓ coordinates, so L should be orthogonal to at least one of the first ℓ axes, and hence our graph contains some edge (L, v_{ij}) for $i \leq \ell$. It contradicts the definition of W'. The Lemma is proved. □

Now we turn to the problem. Let d_j be the step of the progression P_j. Note that since $n = \text{l.c.m.}(d_1, \ldots, d_s)$, for each $1 \leq i \leq k$ there exists an index $j(i)$ such that $p_i^{\alpha_i} \mid d_{j(i)}$. We assume that $n > 1$; otherwise the problem statement is trivial.

For each $0 \leq m \leq n-1$ and $1 \leq i \leq k$, let m_i be the residue of m modulo $p_i^{\alpha_i}$, and let $m_i = \overline{r_{i\alpha_i} \ldots r_{i1}}$ be the base p_i representation of m_i (possibly, with some leading zeroes). Now, we put into correspondence to m the sequence $r(m) = (r_{11}, \ldots, r_{1\alpha_1}, r_{21}, \ldots, r_{k\alpha_k})$. Hence $r(m)$ lies in a $\underbrace{p_1 \times \cdots \times p_1}_{\alpha_1 \text{ times}} \times \cdots \times \underbrace{p_k \times \cdots \times p_k}_{\alpha_k \text{ times}}$ grid N.

Surely, if $r(m) = r(m')$ then $p_i^{\alpha_i} \mid m_i - m_i'$, which follows $p_i^{\alpha_i} \mid m - m'$ for all $1 \leq i \leq k$; consequently, $n \mid m - m'$. So, when m runs over the set $\{0, \ldots, n-1\}$, the sequences $r(m)$ do not repeat; since $|N| = n$, this means that r is a bijection between $\{0, \ldots, n-1\}$ and N. Now we will show that for each $1 \leq i \leq s$, the set $L_i = \{r(m) : m \in P_i\}$ is a subgrid, and that for each axis there exists a subgrid orthogonal to this axis. Obviously, these subgrids cover N, and the condition (ii′) follows directly from (ii). Hence the Lemma provides exactly the estimate we need.

Consider some $1 \leq j \leq s$ and let $d_j = p_1^{\gamma_1} \ldots p_k^{\gamma_k}$. Consider some $q \in P_j$ and let $r(q) = (r_{11}, \ldots, r_{k\alpha_k})$. Then for an arbitrary q', setting $r(q') = (r'_{11}, \ldots, r'_{k\alpha_k})$ we have

$$q' \in P_j \quad \iff \quad p_i^{\gamma_i} \mid q - q' \text{ for each } 1 \leq i \leq k \quad \iff \quad r_{i,t} = r'_{i,t} \text{ for all } t \leq \gamma_i.$$

Hence $L_j = \{(r'_{11}, \ldots, r'_{k\alpha_k}) \in N : r_{i,t} = r'_{i,t} \text{ for all } t \leq \gamma_i\}$ which means that L_j is a subgrid containing $r(q)$. Moreover, in $L_{j(i)}$, all the coordinates corresponding to p_i are fixed, so it is orthogonal to all of their axes, as desired.

Comment 1. The estimate in the problem is sharp for every n. One of the possible examples is the following one. For each $1 \le i \le k$, $0 \le j \le \alpha_i - 1$, $1 \le k \le p-1$, let

$$P_{i,j,k} = kp_i^j + p_i^{j+1}\mathbb{Z},$$

and add the progression $P_0 = n\mathbb{Z}$. One can easily check that this set satisfies all the problem conditions. There also exist other examples.

On the other hand, the estimate can be adjusted in the following sense. For every $1 \le i \le k$, let $0 = \alpha_{i0}, \alpha_{i1}, \dots, \alpha_{ih_i}$ be all the numbers of the form $\operatorname{ord}_{p_i}(d_j)$ in an increasing order (we delete the repeating occurences of a number, and add a number $0 = \alpha_{i0}$ if it does not occur). Then, repeating the arguments from the solution one can obtain that

$$s \ge 1 + \sum_{i=1}^{k}\sum_{j=1}^{h_i}(p^{\alpha_j - \alpha_{j-1}} - 1).$$

Note that $p^\alpha - 1 \ge \alpha(p-1)$, and the equality is achieved only for $\alpha = 1$. Hence, for reaching the minimal number of the progressions, one should have $\alpha_{i,j} = j$ for all i, j. In other words, for each $1 \le j \le \alpha_i$, there should be an index t such that $\operatorname{ord}_{p_i}(d_t) = j$.

Solution 2. We start with introducing some notation. For positive integer r, we denote $[r] = \{1, 2, \dots, r\}$. Next, we say that a set of progressions $\mathcal{P} = \{P_1, \dots, P_s\}$ *cover* $\mathbb{Z}$ if each integer belongs to some of them; we say that this covering is *minimal* if no proper subset of $\mathcal{P}$ covers $\mathbb{Z}$. Obviously, each covering contains a minimal subcovering.

Next, for a minimal covering $\{P_1, \dots, P_s\}$ and for every $1 \le i \le s$, let d_i be the step of progression P_i, and h_i be some number which is contained in P_i but in none of the other progressions. We assume that $n > 1$, otherwise the problem is trivial. This implies $d_i > 1$, otherwise the progression P_i covers all the numbers, and $n = 1$.

We will prove a more general statement, namely the following

Claim. Assume that the progressions $P_1, \dots, P_s$ and number $n = p_1^{\alpha_1} \dots p_k^{\alpha_k} > 1$ are chosen as in the problem statement. Moreover, choose some nonempty set of indices $I = \{i_1, \dots, i_t\} \subseteq [k]$ and some positive integer $\beta_i \le \alpha_i$ for every $i \in I$. Consider the set of indices

$$T = \left\{ j : 1 \le j \le s, \text{and } p_i^{\alpha_i - \beta_i + 1} \mid d_j \text{ for some } i \in I \right\}.$$

Then

$$|T| \ge 1 + \sum_{i \in I} \beta_i (p_i - 1). \tag{2}$$

Observe that the Claim for $I = [k]$ and $\beta_i = \alpha_i$ implies the problem statement, since the left-hand side in (2) is not greater than s. Hence, it suffices to prove the Claim.

1. First, we prove the Claim assuming that all d_j's are prime numbers. If for some $1 \le i \le k$ we have at least p_i progressions with the step p_i, then they do not intersect and hence cover all the integers; it means that there are no other progressions, and $n = p_i$; the Claim is trivial in this case.

Now assume that for every $1 \le i \le k$, there are not more than $p_i - 1$ progressions with step p_i; each such progression covers the numbers with a fixed residue modulo p_i, therefore there exists a residue $q_i \mod p_i$ which is not touched by these progressions. By the Chinese Remainder Theorem, there exists a number q such that $q \equiv q_i \pmod{p_i}$ for all $1 \le i \le k$; this number cannot be covered by any progression with step p_i, hence it is not covered at all. A contradiction.

2. Now, we assume that the general Claim is not valid, and hence we consider a counterexample $\{P_1,\dots,P_s\}$ for the Claim; we can choose it to be minimal in the following sense:

• the number n is minimal possible among all the counterexamples;

• the sum $\sum_i d_i$ is minimal possible among all the counterexamples having the chosen value of n.

As was mentioned above, not all numbers d_i are primes; hence we can assume that d_1 is composite, say $p_1 \bigm| d_1$ and $d_1' = \frac{d_1}{p_1} > 1$. Consider a progression P_1' having the step d_1', and containing P_1. We will focus on two coverings constructed as follows.

(i) Surely, the progressions $P_1', P_2, \dots, P_s$ cover $\mathbb{Z}$, though this covering in not necessarily minimal. So, choose some minimal subcovering $\mathcal{P}'$ in it; surely $P_1' \in \mathcal{P}'$ since h_1 is not covered by $P_2,\dots,P_s$, so we may assume that $\mathcal{P}' = \{P_1', P_2, \dots, P_{s'}\}$ for some $s' \le s$. Furthermore, the period of the covering $\mathcal{P}'$ can appear to be less than n; so we denote this period by

$$n' = p_1^{\alpha_1-\sigma_1}\dots p_k^{\alpha_k-\sigma_k} = \text{l.c.m.}\big(d_1', d_2, \dots, d_{s'}\big).$$

Observe that for each $P_j \notin \mathcal{P}'$, we have $h_j \in P_1'$, otherwise h_j would not be covered by $\mathcal{P}$.

(ii) On the other hand, each nonempty set of the form $R_i = P_i \cap P_1'$ $(1 \le i \le s)$ is also a progression with a step $r_i = \text{l.c.m.}(d_i, d_1')$, and such sets cover P_1'. Scaling these progressions with the ratio $1/d_1'$, we obtain the progressions Q_i with steps $q_i = r_i/d_1'$ which cover $\mathbb{Z}$. Now we choose a minimal subcovering $\mathcal{Q}$ of this covering; again we should have $Q_1 \in \mathcal{Q}$ by the reasons of h_1. Now, denote the period of $\mathcal{Q}$ by

$$n'' = \text{l.c.m.}\{q_i : Q_i \in \mathcal{Q}\} = \frac{\text{l.c.m.}\{r_i : Q_i \in \mathcal{Q}\}}{d_1'} = \frac{p_1^{\gamma_1}\dots p_k^{\gamma_k}}{d_1'}.$$

Note that if $h_j \in P_1'$, then the image of h_j under the scaling can be covered by Q_j only; so, in this case we have $Q_j \in \mathcal{Q}$.

Our aim is to find the desired number of progressions in coverings $\mathcal{P}$ and $\mathcal{Q}$. First, we have $n \ge n'$, and the sum of the steps in $\mathcal{P}'$ is less than that in $\mathcal{P}$; hence the Claim is valid for $\mathcal{P}'$. We apply it to the set of indices $I' = \{i \in I : \beta_i > \sigma_i\}$ and the exponents $\beta_i' = \beta_i - \sigma_i$; hence the set under consideration is

$$T' = \Big\{j : 1 \le j \le s', \text{and } p_i^{(\alpha_i-\sigma_i)-\beta_i'+1} = p_i^{\alpha_i-\beta_i+1} \bigm| d_j \text{ for some } i \in I'\Big\} \subseteq T \cap [s'],$$

and we obtain that

$$|T \cap [s']| \ge |T'| \ge 1 + \sum_{i\in I'}(\beta_i - \sigma_i)(p_i - 1) = 1 + \sum_{i\in I}(\beta_i - \sigma_i)_+(p_i - 1),$$

where $(x)_+ = \max\{x, 0\}$; the latter equality holds as for $i \notin I'$ we have $\beta_i \le \sigma_i$.

Observe that $x = (x-y)_+ + \min\{x, y\}$ for all x, y. So, if we find at least

$$G = \sum_{i\in I}\min\{\beta_i,\sigma_i\}(p_i - 1)$$

indices in $T \cap \{s'+1,\dots,s\}$, then we would have

$$|T| = |T\cap[s']| + |T\cap\{s'+1,\dots,s\}| \ge 1+\sum_{i\in I}\big((\beta_i-\sigma_i)_+ + \min\{\beta_i,\sigma_i\}\big)(p_i-1) = 1+\sum_{i\in I}\beta_i(p_i-1),$$

thus leading to a contradiction with the choice of $\mathcal{P}$. We will find those indices among the indices of progressions in $\mathcal{Q}$.

3. Now denote $I'' = \{i \in I : \sigma_i > 0\}$ and consider some $i \in I''$; then $p_i^{\alpha_i} \nmid n'$. On the other hand, there exists an index $j(i)$ such that $p_i^{\alpha_i} \mid d_{j(i)}$; this means that $d_{j(i)} \nmid n'$ and hence $P_{j(i)}$ cannot appear in $\mathcal{P}'$, so $j(i) > s'$. Moreover, we have observed before that in this case $h_{j(i)} \in P'_1$, hence $Q_{j(i)} \in \mathcal{Q}$. This means that $q_{j(i)} \mid n''$, therefore $\gamma_i = \alpha_i$ for each $i \in I''$ (recall here that $q_i = r_i/d'_1$ and hence $d_{j(i)} \mid r_{j(i)} \mid d'_1 n''$).

Let $d'_1 = p_1^{\tau_1} \dots p_k^{\tau_k}$. Then $n'' = p_1^{\gamma_1-\tau_1} \dots p_k^{\gamma_i-\tau_i}$. Now, if $i \in I''$, then for every β the condition $p_i^{(\gamma_i-\tau_i)-\beta+1} \mid q_j$ is equivalent to $p_i^{\alpha_i-\beta+1} \mid r_j$.

Note that $n'' \le n/d'_1 < n$, hence we can apply the Claim to the covering $\mathcal{Q}$. We perform this with the set of indices I'' and the exponents $\beta''_i = \min\{\beta_i, \sigma_i\} > 0$. So, the set under consideration is

$$T'' = \left\{ j : Q_j \in \mathcal{Q}, \text{and } p_i^{(\gamma_i-\tau_i)-\min\{\beta_i,\sigma_i\}+1} \mid q_j \text{ for some } i \in I'' \right\}$$
$$= \left\{ j : Q_j \in \mathcal{Q}, \text{and } p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid r_j \text{ for some } i \in I'' \right\},$$

and we obtain $|T''| \ge 1 + G$. Finally, we claim that $T'' \subseteq T \cap \big(\{1\} \cup \{s'+1, \dots, s\}\big)$; then we will obtain $|T \cap \{s'+1, \dots, s\}| \ge G$, which is exactly what we need.

To prove this, consider any $j \in T''$. Observe first that $\alpha_i - \min\{\beta_i, \sigma_i\} + 1 > \alpha_i - \sigma_i \ge \tau_i$, hence from $p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid r_j = \text{l.c.m.}(d'_1, d_j)$ we have $p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid d_j$, which means that $j \in T$. Next, the exponent of p_i in d_j is greater than that in n', which means that $P_j \notin \mathcal{P}'$. This may appear only if $j = 1$ or $j > s'$, as desired. This completes the proof.

Comment 2. A grid analogue of the Claim is also valid. It reads as following.

Claim. Assume that the grid N is covered by subgrids $L_1, L_2, \dots, L_s$ so that

(ii′) each subgrid contains a point which is not covered by other subgrids;

(iii) for each coordinate axis, there exists a subgrid L_i orthogonal to this axis.

Choose some set of indices $I = \{i_1, \dots, i_t\} \subset [k]$, and consider the set of indices

$$T = \{j : 1 \le j \le s, \text{and } L_j \text{ is orthogonal to the } i\text{th axis for some } i \in I\}.$$

Then
$$|T| \ge 1 + \sum_{i \in I}(n_i - 1).$$

This Claim may be proved almost in the same way as in Solution 1.

Geometry

G1. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

(United Kingdom)

Solution 1. The line EF intersects the circumcircle at two points. Depending on the choice of P, there are two different cases to consider.

Case 1: The point P lies on the ray EF (see Fig. 1).

Let $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. The quadrilaterals $BCEF$ and $CAFD$ are cyclic due to the right angles at D, E and F. So,

$$\begin{aligned}\angle BDF &= 180^\circ - \angle FDC = \angle CAF = \alpha,\\ \angle AFE &= 180^\circ - \angle EFB = \angle BCE = \gamma,\\ \angle DFB &= 180^\circ - \angle AFD = \angle DCA = \gamma.\end{aligned}$$

Since P lies on the arc AB of the circumcircle, $\angle PBA < \angle BCA = \gamma$. Hence, we have

$$\angle PBD + \angle BDF = \angle PBA + \angle ABD + \angle BDF < \gamma + \beta + \alpha = 180^\circ,$$

and the point Q must lie on the extensions of BP and DF beyond the points P and F, respectively.

From the cyclic quadrilateral $APBC$ we get

$$\angle QPA = 180^\circ - \angle APB = \angle BCA = \gamma = \angle DFB = \angle QFA.$$

Hence, the quadrilateral $AQPF$ is cyclic. Then $\angle AQP = 180^\circ - \angle PFA = \angle AFE = \gamma$.

We obtained that $\angle AQP = \angle QPA = \gamma$, so the triangle AQP is isosceles, $AP = AQ$.

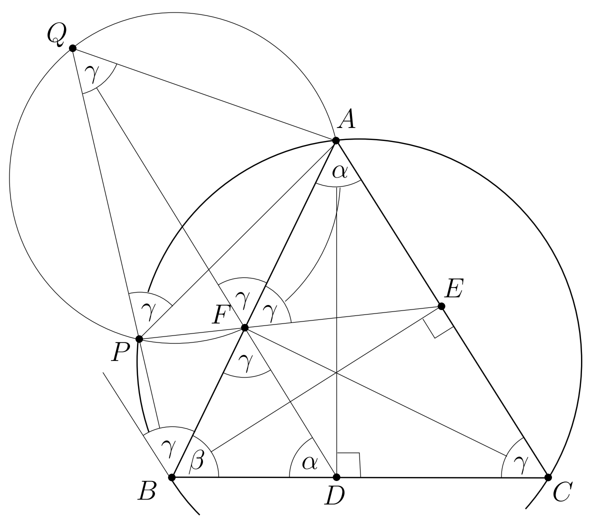


Fig. 1

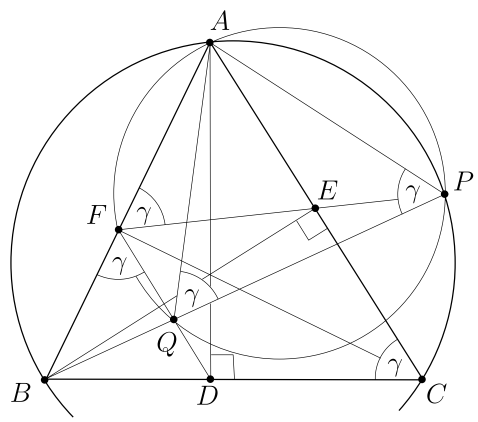


Fig. 2

Case 2: The point P lies on the ray FE (see Fig. 2). In this case the point Q lies inside the segment FD.

Similarly to the first case, we have

$$\angle QPA = \angle BCA = \gamma = \angle DFB = 180^\circ - \angle AFQ.$$

Hence, the quadrilateral $AFQP$ is cyclic.

Then $\angle AQP = \angle AFP = \angle AFE = \gamma = \angle QPA$. The triangle AQP is isosceles again, $\angle AQP = \angle QPA$ and thus $AP = AQ$.

Comment. Using signed angles, the two possible configurations can be handled simultaneously, without investigating the possible locations of P and Q.

Solution 2. For arbitrary points X, Y on the circumcircle, denote by $\widehat{XY}$ the central angle of the arc XY.

Let P and P' be the two points where the line EF meets the circumcircle; let P lie on the arc AB and let P' lie on the arc CA. Let BP and BP' meet the line DF and Q and Q', respectively (see Fig. 3). We will prove that $AP = AP' = AQ = AQ'$.

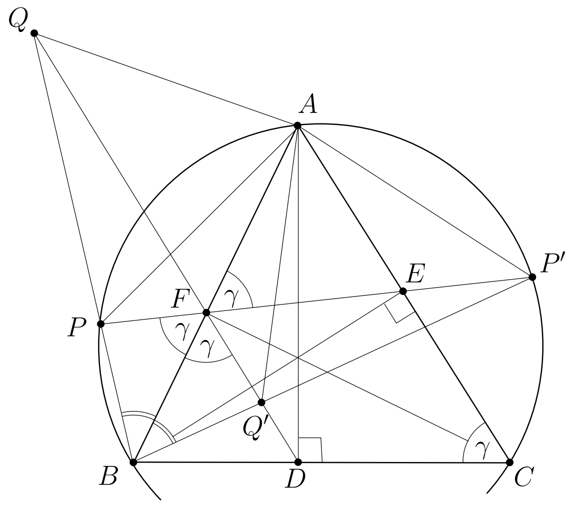


Fig. 3

Like in the first solution, we have $\angle AFE = \angle BFP = \angle DFB = \angle BCA = \gamma$ from the cyclic quadrilaterals $BCEF$ and $CAFD$.

By $\widehat{PB} + \widehat{P'A} = 2\angle AFP' = 2\gamma = 2\angle BCA = \widehat{AP} + \widehat{PB}$, we have

$$\widehat{AP} = \widehat{P'A}, \quad \angle PBA = \angle ABP' \qquad \text{and} \quad AP = AP'. \tag{1}$$

Due to $\widehat{AP} = \widehat{P'A}$, the lines BP and BQ' are symmetrical about line AB.

Similarly, by $\angle BFP = \angle Q'FB$, the lines FP and FQ' are symmetrical about AB. It follows that also the points P and P' are symmetrical to Q' and Q, respectively. Therefore,

$$AP = AQ' \quad \text{and} \quad AP' = AQ. \tag{2}$$

The relations (1) and (2) together prove $AP = AP' = AQ = AQ'$.

G2. Point P lies inside triangle ABC. Lines AP, BP, CP meet the circumcircle of ABC again at points K, L, M, respectively. The tangent to the circumcircle at C meets line AB at S. Prove that $SC = SP$ if and only if $MK = ML$.

(Poland)

Solution 1. We assume that $CA > CB$, so point S lies on the ray AB.

From the similar triangles $\triangle PKM \sim \triangle PCA$ and $\triangle PLM \sim \triangle PCB$ we get $\dfrac{PM}{KM} = \dfrac{PA}{CA}$ and $\dfrac{LM}{PM} = \dfrac{CB}{PB}$. Multiplying these two equalities, we get

$$\frac{LM}{KM} = \frac{CB}{CA} \cdot \frac{PA}{PB}.$$

Hence, the relation $MK = ML$ is equivalent to $\dfrac{CB}{CA} = \dfrac{PB}{PA}$.

Denote by E the foot of the bisector of angle B in triangle ABC. Recall that the locus of points X for which $\dfrac{XA}{XB} = \dfrac{CA}{CB}$ is the Apollonius circle Ω with the center Q on the line AB, and this circle passes through C and E. Hence, we have $MK = ML$ if and only if P lies on Ω, that is $QP = QC$.

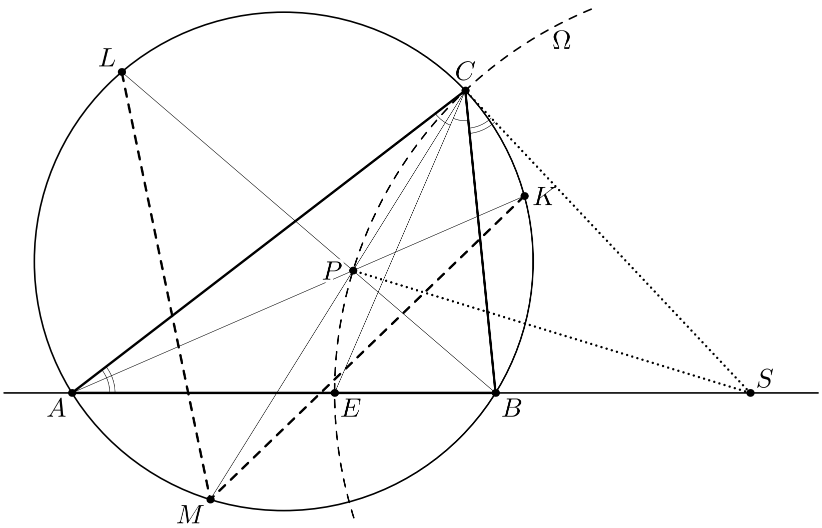


Fig. 1

Now we prove that $S = Q$, thus establishing the problem statement. We have $\angle CES = \angle CAE + \angle ACE = \angle BCS + \angle ECB = \angle ECS$, so $SC = SE$. Hence, the point S lies on AB as well as on the perpendicular bisector of CE and therefore coincides with Q.

Solution 2. As in the previous solution, we assume that S lies on the ray AB.

1. Let P be an arbitrary point inside both the circumcircle ω of the triangle ABC and the angle ASC, the points K, L, M defined as in the problem. We claim that $SP = SC$ implies $MK = ML$.

Let E and F be the points of intersection of the line SP with ω, point E lying on the segment SP (see Fig. 2).

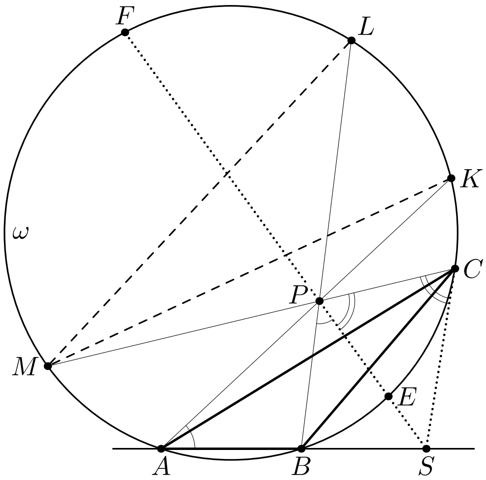


Fig. 2

We have $SP^2 = SC^2 = SA \cdot SB$, so $\dfrac{SP}{SB} = \dfrac{SA}{SP}$, and hence $\triangle PSA \sim \triangle BSP$. Then $\angle BPS = \angle SAP$. Since $2\angle BPS = \overset{\frown}{BE} + \overset{\frown}{LF}$ and $2\angle SAP = \overset{\frown}{BE} + \overset{\frown}{EK}$ we have

$$\overset{\frown}{LF} = \overset{\frown}{EK}. \tag{1}$$

On the other hand, from $\angle SPC = \angle SCP$ we have $\overset{\frown}{EC} + \overset{\frown}{MF} = \overset{\frown}{EC} + \overset{\frown}{EM}$, or

$$\overset{\frown}{MF} = \overset{\frown}{EM}. \tag{2}$$

From (1) and (2) we get $\overset{\frown}{MFL} = \overset{\frown}{MF} + \overset{\frown}{FL} = \overset{\frown}{ME} + \overset{\frown}{EK} = \overset{\frown}{MEK}$ and hence $MK = ML$. The claim is proved.

2. We are left to prove the converse. So, assume that $MK = ML$, and introduce the points E and F as above. We have $SC^2 = SE \cdot SF$; hence, there exists a point P' lying on the segment EF such that $SP' = SC$ (see Fig. 3).

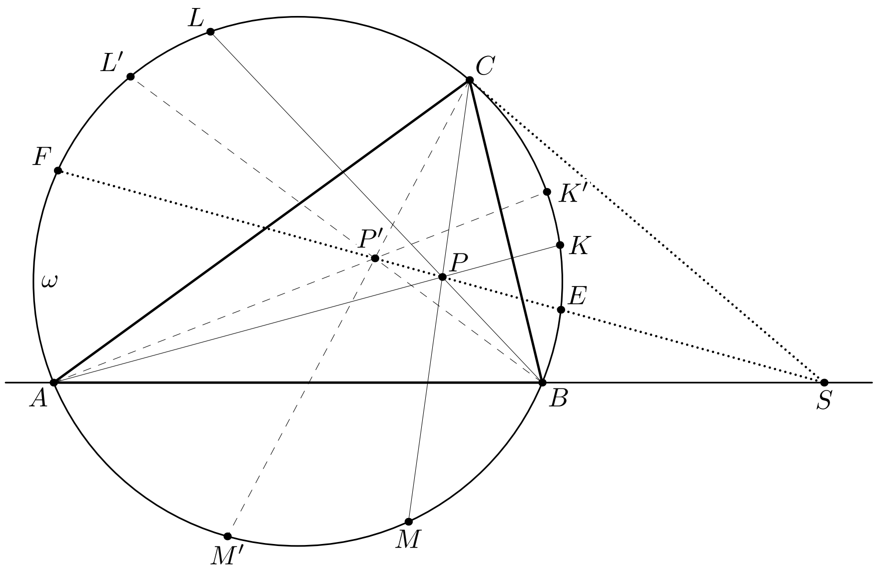


Fig. 3

Assume that $P \neq P'$. Let the lines AP', BP', CP' meet ω again at points K', L', M' respectively. Now, if P' lies on the segment PF then by the first part of the solution we have $\widehat{M'FL'} = \widehat{M'EK'}$. On the other hand, we have $\widehat{MFL} > \widehat{M'FL'} = \widehat{M'EK'} > \widehat{MEK}$, therefore $\widehat{MFL} > \widehat{MEK}$ which contradicts $MK = ML$.

Similarly, if point P' lies on the segment EP then we get $\widehat{MFL} < \widehat{MEK}$ which is impossible. Therefore, the points P and P' coincide and hence $SP = SP' = SC$.

Solution 3. We present a different proof of the converse direction, that is, $MK = ML \Rightarrow SP = SC$. As in the previous solutions we assume that $CA > CB$, and the line SP meets ω at E and F.

From $ML = MK$ we get $\widehat{MEK} = \widehat{MFL}$. Now we claim that $\widehat{ME} = \widehat{MF}$ and $\widehat{EK} = \widehat{FL}$.

To the contrary, suppose first that $\widehat{ME} > \widehat{MF}$; then $\widehat{EK} = \widehat{MEK} - \widehat{ME} < \widehat{MFL} - \widehat{MF} = \widehat{FL}$. Now, the inequality $\widehat{ME} > \widehat{MF}$ implies $2\angle SCM = \widehat{EC} + \widehat{ME} > \widehat{EC} + \widehat{MF} = 2\angle SPC$ and hence $SP > SC$. On the other hand, the inequality $\widehat{EK} < \widehat{FL}$ implies $2\angle SPK = \widehat{EK} + \widehat{AF} < \widehat{FL} + \widehat{AF} = 2\angle ABL$, hence

$$\angle SPA = 180^\circ - \angle SPK > 180^\circ - \angle ABL = \angle SBP.$$

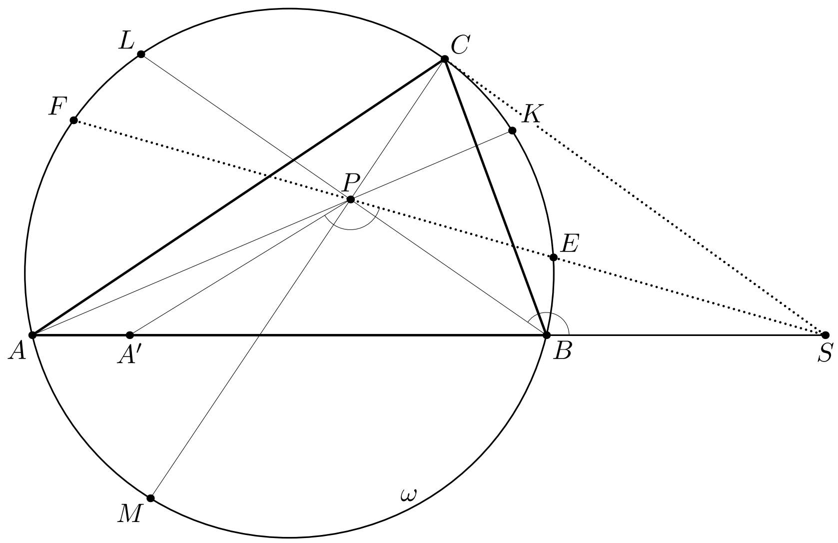


Fig. 4

Consider the point A' on the ray SA for which $\angle SPA' = \angle SBP$; in our case, this point lies on the segment SA (see Fig. 4). Then $\triangle SBP \sim \triangle SPA'$ and $SP^2 = SB \cdot SA' < SB \cdot SA = SC^2$. Therefore, $SP < SC$ which contradicts $SP > SC$.

Similarly, one can prove that the inequality $\widehat{ME} < \widehat{MF}$ is also impossible. So, we get $\widehat{ME} = \widehat{MF}$ and therefore $2\angle SCM = \widehat{EC} + \widehat{ME} = \widehat{EC} + \widehat{MF} = 2\angle SPC$, which implies $SC = SP$.

G3. Let $A_1A_2\dots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \dots, P_n$ onto lines $A_1A_2, \dots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \dots, X_n$ on sides $A_1A_2, \dots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \dots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

(Armenia)

Solution 1. Denote $P_{n+1} = P_1$, $X_{n+1} = X_1$, $A_{n+1} = A_1$.

Lemma. Let point Q lies inside $A_1A_2\dots A_n$. Then it is contained in at least one of the circumcircles of triangles $X_1A_2X_2, \dots, X_nA_1X_1$.

Proof. If Q lies in one of the triangles $X_1A_2X_2, \dots, X_nA_1X_1$, the claim is obvious. Otherwise Q lies inside the polygon $X_1X_2\dots X_n$ (see Fig. 1). Then we have

$$\begin{aligned}(\angle X_1A_2X_2 + \angle X_1QX_2) + \dots + (\angle X_nA_1X_1 + \angle X_nQX_1)\\ = (\angle X_1A_1X_2 + \dots + \angle X_nA_1X_1) + (\angle X_1QX_2 + \dots + \angle X_nQX_1) = (n-2)\pi + 2\pi = n\pi,\end{aligned}$$

hence there exists an index i such that $\angle X_iA_{i+1}X_{i+1} + \angle X_iQX_{i+1} \geq \frac{\pi n}{n} = \pi$. Since the quadrilateral $QX_iA_{i+1}X_{i+1}$ is convex, this means exactly that Q is contained the circumcircle of $\triangle X_iA_{i+1}X_{i+1}$, as desired. $\square$

Now we turn to the solution. Applying lemma, we get that P lies inside the circumcircle of triangle $X_iA_{i+1}X_{i+1}$ for some i. Consider the circumcircles ω and Ω of triangles $P_iA_{i+1}P_{i+1}$ and $X_iA_{i+1}X_{i+1}$ respectively (see Fig. 2); let r and R be their radii. Then we get $2r = A_{i+1}P \leq 2R$ (since P lies inside Ω), hence

$$P_iP_{i+1} = 2r\sin\angle P_iA_{i+1}P_{i+1} \leq 2R\sin\angle X_iA_{i+1}X_{i+1} = X_iX_{i+1},$$

QED.

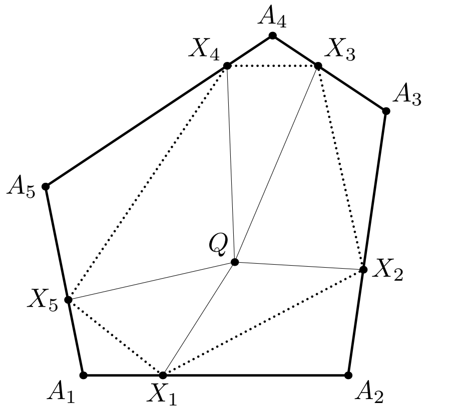


Fig. 1

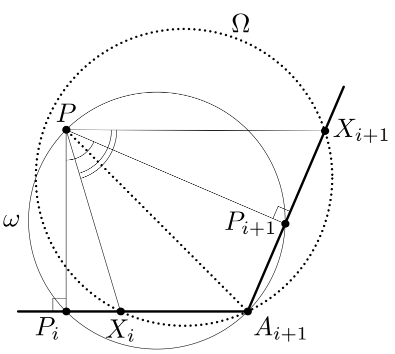


Fig. 2

Solution 2. As in Solution 1, we assume that all indices of points are considered modulo n.

We will prove a bit stronger inequality, namely

$$\max\left\{\frac{X_1X_2}{P_1P_2}\cos\alpha_1, \ldots, \frac{X_nX_1}{P_nP_1}\cos\alpha_n\right\} \geq 1,$$

where α_i $(1 \leq i \leq n)$ is the angle between lines X_iX_{i+1} and P_iP_{i+1}. We denote $\beta_i = \angle A_iP_iP_{i-1}$ and $\gamma_i = \angle A_{i+1}P_iP_{i+1}$ for all $1 \leq i \leq n$.

Suppose that for some $1 \leq i \leq n$, point X_i lies on the segment A_iP_i, while point X_{i+1} lies on the segment $P_{i+1}A_{i+2}$. Then the projection of the segment X_iX_{i+1} onto the line P_iP_{i+1} contains segment P_iP_{i+1}, since γ_i and β_{i+1} are acute angles (see Fig. 3). Therefore, $X_iX_{i+1}\cos\alpha_i \geq P_iP_{i+1}$, and in this case the statement is proved.

So, the only case left is when point X_i lies on segment P_iA_{i+1} for all $1 \leq i \leq n$ (the case when each X_i lies on segment A_iP_i is completely analogous).

Now, assume to the contrary that the inequality

$$X_iX_{i+1}\cos\alpha_i < P_iP_{i+1} \tag{1}$$

holds for every $1 \leq i \leq n$. Let Y_i and Y'_{i+1} be the projections of X_i and X_{i+1} onto P_iP_{i+1}. Then inequality (1) means exactly that $Y_iY'_{i+1} < P_iP_{i+1}$, or $P_iY_i > P_{i+1}Y'_{i+1}$ (again since γ_i and β_{i+1} are acute; see Fig. 4). Hence, we have

$$X_iP_i\cos\gamma_i > X_{i+1}P_{i+1}\cos\beta_{i+1}, \qquad 1 \leq i \leq n.$$

Multiplying these inequalities, we get

$$\cos\gamma_1\cos\gamma_2\cdots\cos\gamma_n > \cos\beta_1\cos\beta_2\cdots\cos\beta_n. \tag{2}$$

On the other hand, the sines theorem applied to triangle PP_iP_{i+1} provides

$$\frac{PP_i}{PP_{i+1}} = \frac{\sin\left(\frac{\pi}{2}-\beta_{i+1}\right)}{\sin\left(\frac{\pi}{2}-\gamma_i\right)} = \frac{\cos\beta_{i+1}}{\cos\gamma_i}.$$

Multiplying these equalities we get

$$1 = \frac{\cos\beta_2}{\cos\gamma_1}\cdot\frac{\cos\beta_3}{\cos\gamma_2}\cdots\frac{\cos\beta_1}{\cos\gamma_n}$$

which contradicts (2).

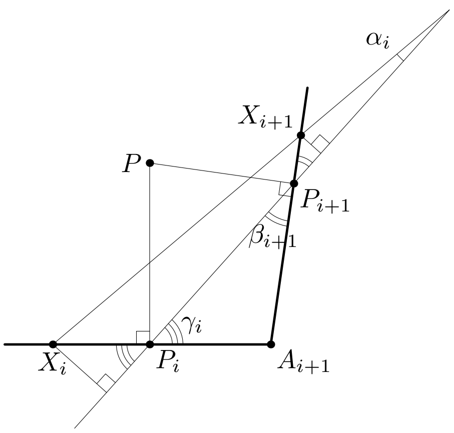


Fig. 3

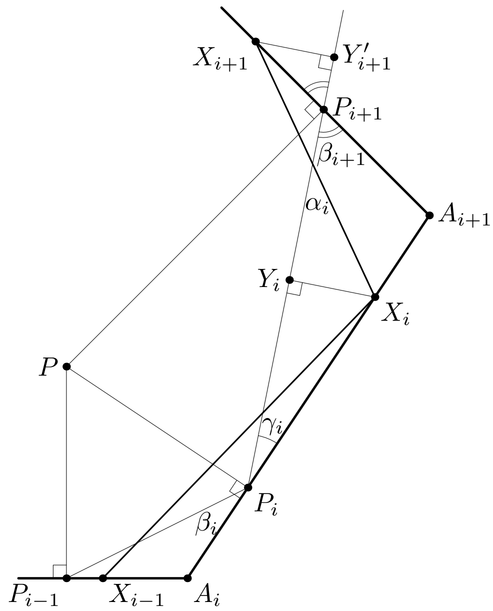


Fig. 4

G4. Let I be the incenter of a triangle ABC and Γ be its circumcircle. Let the line AI intersect Γ at a point $D \neq A$. Let F and E be points on side BC and arc BDC respectively such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.

(Hong Kong)

Solution 1. Let X be the second point of intersection of line EI with Γ, and L be the foot of the bisector of angle BAC. Let G' and T be the points of intersection of segment DX with lines IF and AF, respectively. We are to prove that $G = G'$, or $IG' = G'F$. By the Menelaus theorem applied to triangle AIF and line DX, it means that we need the relation

$$1 = \frac{G'F}{IG'} = \frac{TF}{AT} \cdot \frac{AD}{ID}, \qquad \text{or} \quad \frac{TF}{AT} = \frac{ID}{AD}.$$

Let the line AF intersect Γ at point $K \neq A$ (see Fig. 1); since $\angle BAK = \angle CAE$ we have $\widehat{BK} = \widehat{CE}$, hence $KE \parallel BC$. Notice that $\angle IAT = \angle DAK = \angle EAD = \angle EXD = \angle IXT$, so the points I, A, X, T are concyclic. Hence we have $\angle ITA = \angle IXA = \angle EXA = \angle EKA$, so $IT \parallel KE \parallel BC$. Therefore we obtain $\dfrac{TF}{AT} = \dfrac{IL}{AI}$.

Since CI is the bisector of $\angle ACL$, we get $\dfrac{IL}{AI} = \dfrac{CL}{AC}$. Furthermore, $\angle DCL = \angle DCB = \angle DAB = \angle CAD = \frac{1}{2}\angle BAC$, hence the triangles DCL and DAC are similar; therefore we get $\dfrac{CL}{AC} = \dfrac{DC}{AD}$. Finally, it is known that the midpoint D of arc BC is equidistant from points I, B, C, hence $\dfrac{DC}{AD} = \dfrac{ID}{AD}$.

Summarizing all these equalities, we get

$$\frac{TF}{AT} = \frac{IL}{AI} = \frac{CL}{AC} = \frac{DC}{AD} = \frac{ID}{AD},$$

as desired.

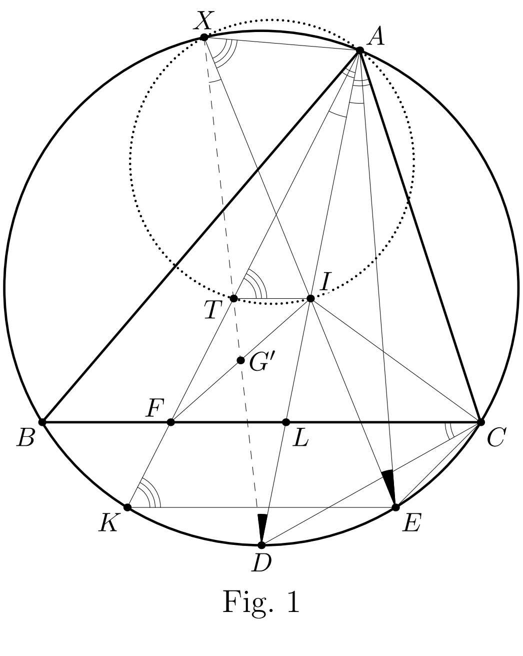


Fig. 1

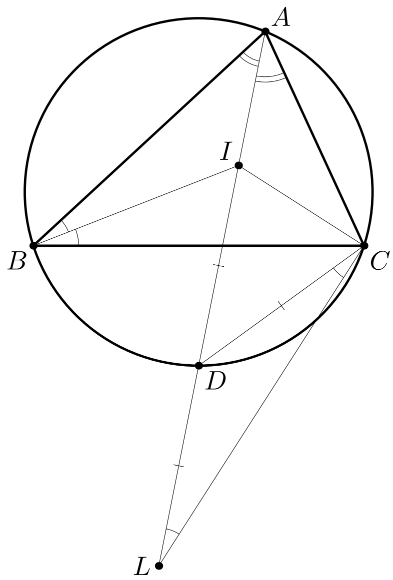


Fig. 2

Comment. The equality $\dfrac{AI}{IL} = \dfrac{AD}{DI}$ is known and can be obtained in many different ways. For instance, one can consider the inversion with center D and radius $DC = DI$. This inversion takes $\widehat{BAC}$ to the segment BC, so point A goes to L. Hence $\dfrac{IL}{DI} = \dfrac{AI}{AD}$, which is the desired equality.

Solution 2. As in the previous solution, we introduce the points X, T and K and note that it suffice to prove the equality

$$\frac{TF}{AT} = \frac{DI}{AD} \quad \Longleftrightarrow \quad \frac{TF + AT}{AT} = \frac{DI + AD}{AD} \quad \Longleftrightarrow \quad \frac{AT}{AD} = \frac{AF}{DI + AD}.$$

Since $\angle FAD = \angle EAI$ and $\angle TDA = \angle XDA = \angle XEA = \angle IEA$, we get that the triangles ATD and AIE are similar, therefore $\dfrac{AT}{AD} = \dfrac{AI}{AE}$.

Next, we also use the relation $DB = DC = DI$. Let J be the point on the extension of segment AD over point D such that $DJ = DI = DC$ (see Fig. 2). Then $\angle DJC = \angle JCD = \frac{1}{2}(\pi - \angle JDC) = \frac{1}{2}\angle ADC = \frac{1}{2}\angle ABC = \angle ABI$. Moreover, $\angle BAI = \angle JAC$, hence triangles ABI and AJC are similar, so $\dfrac{AB}{AJ} = \dfrac{AI}{AC}$, or $AB \cdot AC = AJ \cdot AI = (DI + AD) \cdot AI$.

On the other hand, we get $\angle ABF = \angle ABC = \angle AEC$ and $\angle BAF = \angle CAE$, so triangles ABF and AEC are also similar, which implies $\dfrac{AF}{AC} = \dfrac{AB}{AE}$, or $AB \cdot AC = AF \cdot AE$.

Summarizing we get

$$(DI + AD) \cdot AI = AB \cdot AC = AF \cdot AE \quad \Rightarrow \quad \frac{AI}{AE} = \frac{AF}{AD + DI} \quad \Rightarrow \quad \frac{AT}{AD} = \frac{AF}{AD + DI},$$

as desired.

Comment. In fact, point J is an excenter of triangle ABC.

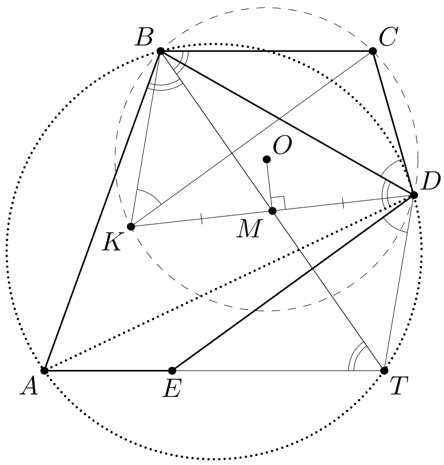
G5. Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

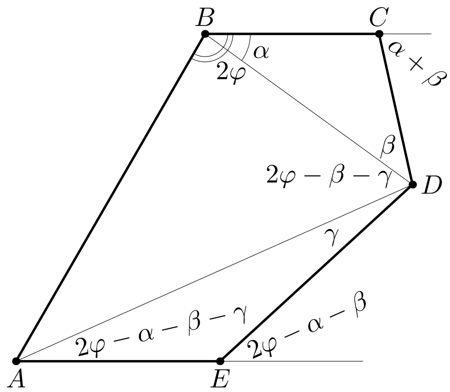
(Ukraine)

Solution 1. Choose point T on ray AE such that $AT = AB$; then from $AE \parallel BC$ we have $\angle CBT = \angle ATB = \angle ABT$, so BT is the bisector of $\angle ABC$. On the other hand, we have $ET = AT - AE = AB - AE = BC$, hence quadrilateral $BCTE$ is a parallelogram, and the midpoint M of its diagonal CE is also the midpoint of the other diagonal BT.

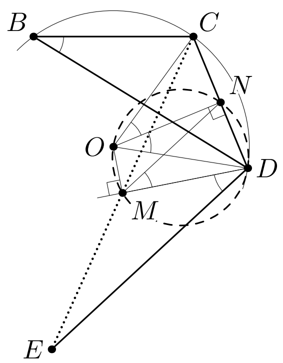
Next, let point K be symmetrical to D with respect to M. Then OM is the perpendicular bisector of segment DK, and hence $OD = OK$, which means that point K lies on the circumcircle of triangle BCD. Hence we have $\angle BDC = \angle BKC$. On the other hand, the angles BKC and TDE are symmetrical with respect to M, so $\angle TDE = \angle BKC = \angle BDC$.

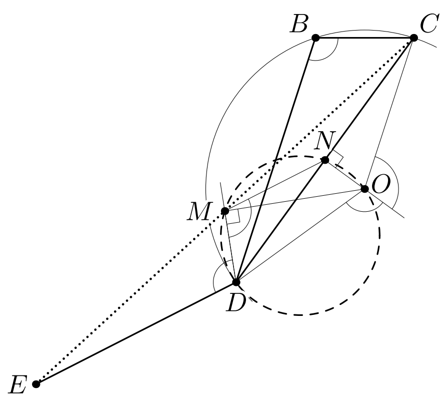
Therefore, $\angle BDT = \angle BDE + \angle EDT = \angle BDE + \angle BDC = \angle CDE = \angle ABC = 180^\circ - \angle BAT$. This means that the points A, B, D, T are concyclic, and hence $\angle ADB = \angle ATB = \frac{1}{2}\angle ABC = \frac{1}{2}\angle CDE$, as desired.





Solution 2. Let$\angle CBD = \alpha$, $\angle BDC = \beta$, $\angle ADE = \gamma$, and $\angle ABC = \angle CDE = 2\varphi$. Then we have $\angle ADB = 2\varphi - \beta - \gamma$, $\angle BCD = 180^\circ - \alpha - \beta$, $\angle AED = 360^\circ - \angle BCD - \angle CDE = 180^\circ - 2\varphi + \alpha + \beta$, and finally $\angle DAE = 180^\circ - \angle ADE - \angle AED = 2\varphi - \alpha - \beta - \gamma$.





Let N be the midpoint of CD; then $\angle DNO = 90^\circ = \angle DMO$, hence points M, N lie on the circle with diameter OD. Now, if points O and M lie on the same side of CD, we have $\angle DMN = \angle DON = \frac{1}{2}\angle DOC = \alpha$; in the other case, we have $\angle DMN = 180^\circ - \angle DON = \alpha$;

so, in both cases $\angle DMN = \alpha$ (see Figures). Next, since MN is a midline in triangle CDE, we have $\angle MDE = \angle DMN = \alpha$ and $\angle NDM = 2\varphi - \alpha$.

Now we apply the sine rule to the triangles ABD, ADE (twice), BCD and MND obtaining

$$\frac{AB}{AD} = \frac{\sin(2\varphi - \beta - \gamma)}{\sin(2\varphi - \alpha)}, \quad \frac{AE}{AD} = \frac{\sin\gamma}{\sin(2\varphi - \alpha - \beta)}, \quad \frac{DE}{AD} = \frac{\sin(2\varphi - \alpha - \beta - \gamma)}{\sin(2\varphi - \alpha - \beta)},$$
$$\frac{BC}{CD} = \frac{\sin\beta}{\sin\alpha}, \quad \frac{CD}{DE} = \frac{CD/2}{DE/2} = \frac{ND}{NM} = \frac{\sin\alpha}{\sin(2\varphi - \alpha)},$$

which implies

$$\frac{BC}{AD} = \frac{BC}{CD} \cdot \frac{CD}{DE} \cdot \frac{DE}{AD} = \frac{\sin\beta \cdot \sin(2\varphi - \alpha - \beta - \gamma)}{\sin(2\varphi - \alpha) \cdot \sin(2\varphi - \alpha - \beta)}.$$

Hence, the condition $AB = AE + BC$, or equivalently $\dfrac{AB}{AD} = \dfrac{AE + BC}{AD}$, after multiplying by the common denominator rewrites as

$$\begin{aligned}
&\sin(2\varphi - \alpha - \beta) \cdot \sin(2\varphi - \beta - \gamma) = \sin\gamma \cdot \sin(2\varphi - \alpha) + \sin\beta \cdot \sin(2\varphi - \alpha - \beta - \gamma) \\
\iff\ &\cos(\gamma - \alpha) - \cos(4\varphi - 2\beta - \alpha - \gamma) = \cos(2\varphi - \alpha - 2\beta - \gamma) - \cos(2\varphi + \gamma - \alpha) \\
\iff\ &\cos(\gamma - \alpha) + \cos(2\varphi + \gamma - \alpha) = \cos(2\varphi - \alpha - 2\beta - \gamma) + \cos(4\varphi - 2\beta - \alpha - \gamma) \\
\iff\ &\cos\varphi \cdot \cos(\varphi + \gamma - \alpha) = \cos\varphi \cdot \cos(3\varphi - 2\beta - \alpha - \gamma) \\
\iff\ &\cos\varphi \cdot \big(\cos(\varphi + \gamma - \alpha) - \cos(3\varphi - 2\beta - \alpha - \gamma)\big) = 0 \\
\iff\ &\cos\varphi \cdot \sin(2\varphi - \beta - \alpha) \cdot \sin(\varphi - \beta - \gamma) = 0.
\end{aligned}$$

Since $2\varphi - \beta - \alpha = 180^\circ - \angle AED < 180^\circ$ and $\varphi = \frac{1}{2}\angle ABC < 90^\circ$, it follows that $\varphi = \beta + \gamma$, hence $\angle BDA = 2\varphi - \beta - \gamma = \varphi = \frac{1}{2}\angle CDE$, as desired.

G6. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC. Prove that the incenter of triangle ABC lies inside triangle XYZ.

G6′. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Prove that if the incenter of triangle ABC lies outside triangle XYZ, then one of the angles of triangle ABC is greater than $120°$.

(Bulgaria)

Solution 1 for G6. We will prove a stronger fact; namely, we will show that the incenter I of triangle ABC lies inside the incircle of triangle XYZ (and hence surely inside triangle XYZ itself). We denote by $d(U, VW)$ the distance between point U and line VW.

Denote by O the incenter of $\triangle XYZ$ and by r, r' and R' the inradii of triangles ABC, XYZ and the circumradius of XYZ, respectively. Then we have $R' = 2r'$, and the desired inequality is $OI \le r'$. We assume that $O \ne I$; otherwise the claim is trivial.

Let the incircle of $\triangle ABC$ touch its sides BC, AC, AB at points A_1, B_1, C_1 respectively. The lines IA_1, IB_1, IC_1 cut the plane into 6 acute angles, each one containing one of the points A_1, B_1, C_1 on its border. We may assume that O lies in an angle defined by lines IA_1, IC_1 and containing point C_1 (see Fig. 1). Let A' and C' be the projections of O onto lines IA_1 and IC_1, respectively.

Since $OX = R'$, we have $d(O, BC) \le R'$. Since $OA' \parallel BC$, it follows that $d(A', BC) = A'I + r \le R'$, or $A'I \le R' - r$. On the other hand, the incircle of $\triangle XYZ$ lies inside $\triangle ABC$, hence $d(O, AB) \ge r'$, and analogously we get $d(O, AB) = C'C_1 = r - IC' \ge r'$, or $IC' \le r - r'$.

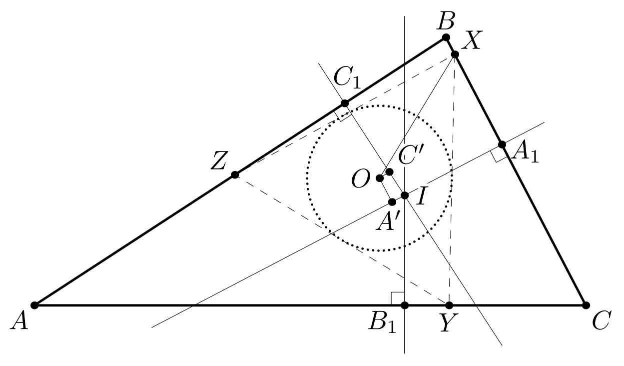


Fig. 1

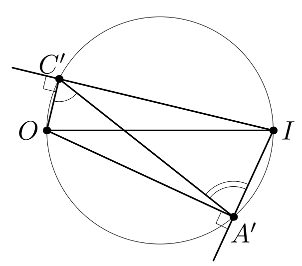


Fig. 2

Finally, the quadrilateral $IA'OC'$ is circumscribed due to the right angles at A' and C' (see Fig. 2). On its circumcircle, we have $\widehat{A'OC'} = 2\angle A'IC' < 180° = \widehat{OC'I}$, hence $180° \ge \widehat{IC'} > \widehat{A'O}$. This means that $IC' > A'O$. Finally, we have $OI \le IA' + A'O < IA' + IC' \le (R' - r) + (r - r') = R' - r' = r'$, as desired.

Solution 2 for G6. Assume the contrary. Then the incenter I should lie in one of triangles AYZ, BXZ, CXY — assume that it lies in $\triangle AYZ$. Let the incircle ω of $\triangle ABC$ touch sides BC, AC at point A_1, B_1 respectively. Without loss of generality, assume that point A_1 lies on segment CX. In this case we will show that $\angle C > 90°$ thus leading to a contradiction.

Note that ω intersects each of the segments XY and YZ at two points; let U, U' and V, V' be the points of intersection of ω with XY and YZ, respectively ($UY > U'Y$, $VY > V'Y$; see Figs. 3 and 4). Note that $60° = \angle XYZ = \frac{1}{2}(\widehat{UV} - \widehat{U'V'}) \le \frac{1}{2}\widehat{UV}$, hence $\widehat{UV} \ge 120°$.

On the other hand, since I lies in $\triangle AYZ$, we get $\wideparen{VUV'} < 180^\circ$, hence $\wideparen{UA_1U'} \leq \wideparen{UA_1V'} < 180^\circ - \wideparen{UV} \leq 60^\circ$.

Now, two cases are possible due to the order of points Y, B_1 on segment AC.

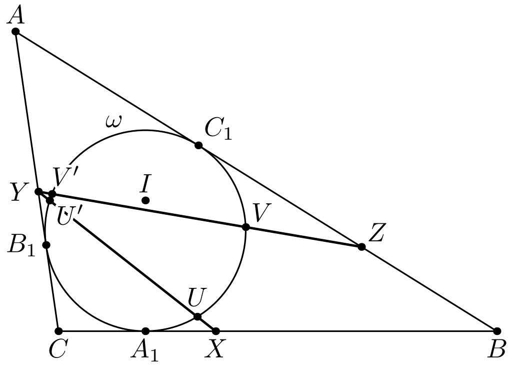


Fig. 3

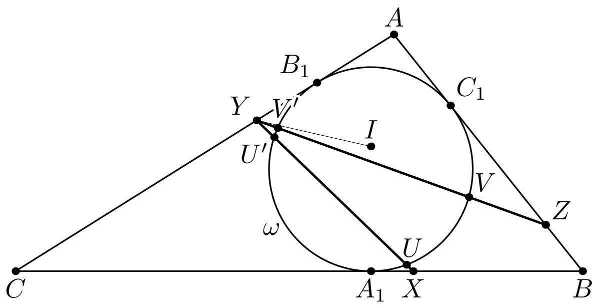


Fig. 4

Case 1. Let point Y lie on the segment AB_1 (see Fig. 3). Then we have $\angle YXC = \frac{1}{2}\left(\wideparen{A_1U'} - \wideparen{A_1U}\right) \leq \frac{1}{2}\wideparen{UA_1U'} < 30^\circ$; analogously, we get $\angle XYC \leq \frac{1}{2}\wideparen{UA_1U'} < 30^\circ$. Therefore, $\angle YCX = 180^\circ - \angle YXC - \angle XYC > 120^\circ$, as desired.

Case 2. Now let point Y lie on the segment CB_1 (see Fig. 4). Analogously, we obtain $\angle YXC < 30^\circ$. Next, $\angle IYX > \angle ZYX = 60^\circ$, but $\angle IYX < \angle IYB_1$, since YB_1 is a tangent and YX is a secant line to circle ω from point Y. Hence, we get $120^\circ < \angle IYB_1 + \angle IYX = \angle B_1YX = \angle YXC + \angle YCX < 30^\circ + \angle YCX$, hence $\angle YCX > 120^\circ - 30^\circ = 90^\circ$, as desired.

Comment. In the same way, one can prove a more general

Claim. Let the vertices X, Y, Z of a triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Suppose that the incenter of triangle ABC lies outside triangle XYZ, and α is the least angle of $\triangle XYZ$. Then one of the angles of triangle ABC is greater than $3\alpha - 90^\circ$.

Solution for G6′. Assume the contrary. As in Solution 2, we assume that the incenter I of $\triangle ABC$ lies in $\triangle AYZ$, and the tangency point A_1 of ω and BC lies on segment CX. Surely, $\angle YZA \leq 180^\circ - \angle YZX = 120^\circ$, hence points I and Y lie on one side of the perpendicular bisector to XY; therefore $IX > IY$. Moreover, ω intersects segment XY at two points, and therefore the projection M of I onto XY lies on the segment XY. In this case, we will prove that $\angle C > 120^\circ$.

Let YK, YL be two tangents from point Y to ω (points K and A_1 lie on one side of XY; if Y lies on ω, we say $K = L = Y$); one of the points K and L is in fact a tangency point B_1 of ω and AC. From symmetry, we have $\angle YIK = \angle YIL$. On the other hand, since $IX > IY$, we get $XM < XY$ which implies $\angle A_1XY < \angle KYX$.

Next, we have $\angle MIY = 90^\circ - \angle IYX < 90^\circ - \angle ZYX = 30^\circ$. Since $IA_1 \perp A_1X$, $IM \perp XY$, $IK \perp YK$ we get $\angle MIA_1 = \angle A_1XY < \angle KYX = \angle MIK$. Finally, we get

$$\begin{aligned}\angle A_1IK < \angle A_1IL &= (\angle A_1IM + \angle MIK) + (\angle KIY + \angle YIL)\\ &< 2\angle MIK + 2\angle KIY = 2\angle MIY < 60^\circ.\end{aligned}$$

Hence, $\angle A_1IB_1 < 60^\circ$, and therefore $\angle ACB = 180^\circ - \angle A_1IB_1 > 120^\circ$, as desired.

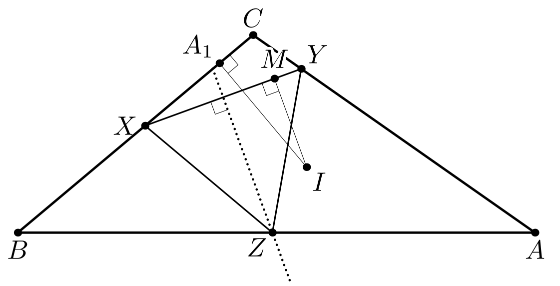


Fig. 5

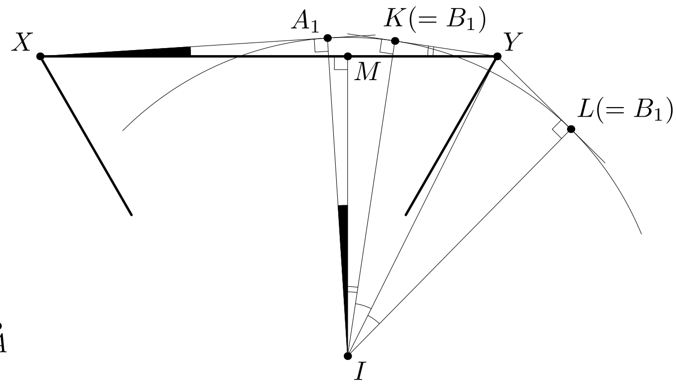


Fig. 6

Comment 1. The estimate claimed in G6′ is sharp. Actually, if $\angle BAC > 120°$, one can consider an equilateral triangle XYZ with $Z = A$, $Y \in AC$, $X \in BC$ (such triangle exists since $\angle ACB < 60°$). It intersects with the angle bisector of $\angle BAC$ only at point A, hence it does not contain I.

Comment 2. As in the previous solution, there is a generalization for an arbitrary triangle XYZ, but here we need some additional condition. The statement reads as follows.

Claim. Let the vertices X, Y, Z of a triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Suppose that the incenter of triangle ABC lies outside triangle XYZ, α is the least angle of $\triangle XYZ$, and all sides of triangle XYZ are greater than $2r\cot\alpha$, where r is the inradius of $\triangle ABC$. Then one of the angles of triangle ABC is greater than 2α.

The additional condition is needed to verify that $XM > YM$ since it cannot be shown in the original way. Actually, we have $\angle MYI > \alpha$, $IM < r$, hence $YM < r\cot\alpha$. Now, if we have $XY = XM + YM > 2r\cot\alpha$, then surely $XM > YM$.

On the other hand, this additional condition follows easily from the conditions of the original problem. Actually, if $I \in \triangle AYZ$, then the diameter of ω parallel to YZ is contained in $\triangle AYZ$ and is thus shorter than YZ. Hence $YZ > 2r > 2r\cot 60°$.

G7. Three circular arcs γ_1, γ_2, and γ_3 connect the points A and C. These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3. Point B lies on the segment AC. Let h_1, h_2, and h_3 be three rays starting at B, lying in the same half-plane, h_2 being between h_1 and h_3. For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j (see the Figure below).

Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{k\ell}V_{i\ell}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{i\ell}$, $V_{kj}V_{k\ell}$ and arcs $V_{ij}V_{kj}$ and $V_{i\ell}V_{k\ell}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs.

Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}$, $\widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}$, $\widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.

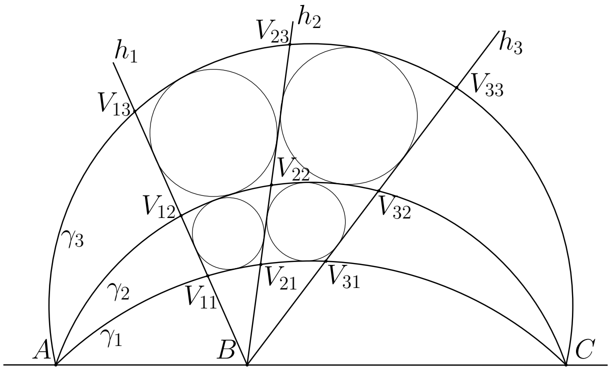


Fig. 1

(Hungary)

Solution. Denote by O_i and R_i the center and the radius of γ_i, respectively. Denote also by H the half-plane defined by AC which contains the whole configuration. For every point P in the half-plane H, denote by $d(P)$ the distance between P and line AC. Furthermore, for any $r > 0$, denote by $\Omega(P, r)$ the circle with center P and radius r.

Lemma 1. For every $1 \le i < j \le 3$, consider those circles $\Omega(P, r)$ in the half-plane H which are tangent to h_i and h_j.

(a) The locus of the centers of these circles is the angle bisector β_{ij} between h_i and h_j.

(b) There is a constant u_{ij} such that $r = u_{ij} \cdot d(P)$ for all such circles.

Proof. Part (a) is obvious. To prove part (b), notice that the circles which are tangent to h_i and h_j are homothetic with the common homothety center B (see Fig. 2). Then part (b) also becomes trivial. □

Lemma 2. For every $1 \le i < j \le 3$, consider those circles $\Omega(P, r)$ in the half-plane H which are externally tangent to γ_i and internally tangent to γ_j.

(a) The locus of the centers of these circles is an ellipse arc ε_{ij} with end-points A and C.

(b) There is a constant v_{ij} such that $r = v_{ij} \cdot d(P)$ for all such circles.

Proof. (a) Notice that the circle $\Omega(P, r)$ is externally tangent to γ_i and internally tangent to γ_j if and only if $O_iP = R_i + r$ and $O_j = R_j - r$. Therefore, for each such circle we have

$$O_iP + O_jP = O_iA + O_jA = O_iC + O_jC = R_i + R_j.$$

Such points lie on an ellipse with foci O_i and O_j; the diameter of this ellipse is $R_i + R_j$, and it passes through the points A and C. Let ε_{ij} be that arc AC of the ellipse which runs inside the half plane H (see Fig. 3.)

This ellipse arc lies between the arcs γ_i and γ_j. Therefore, if some point P lies on ε_{ij}, then $O_iP > R_i$ and $O_jP < R_j$. Now, we choose $r = O_iP - R_i = R_j - O_jP > 0$; then the

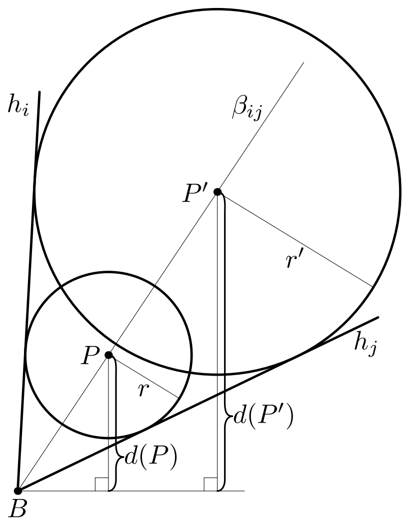


Fig. 2

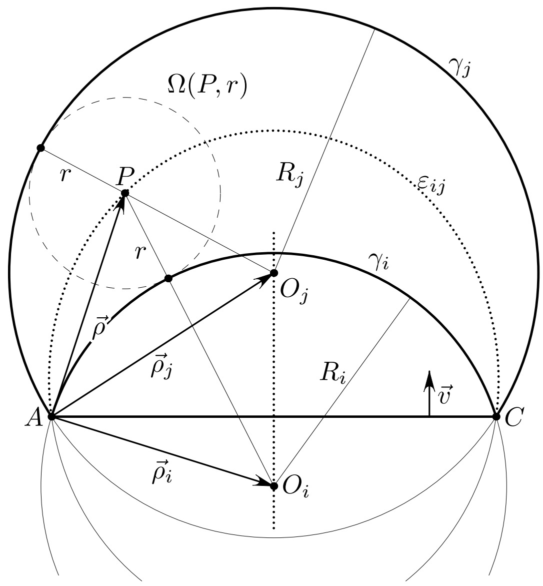


Fig. 3

circle $\Omega(P,r)$ touches γ_i externally and touches γ_j internally, so P belongs to the locus under investigation.

(b) Let $\vec{\rho} = \overrightarrow{AP}$, $\vec{\rho}_i = \overrightarrow{AO_i}$, and $\vec{\rho}_j = \overrightarrow{AO_j}$; let $d_{ij} = O_iO_j$, and let $\vec{v}$ be a unit vector orthogonal to AC and directed toward H. Then we have $|\vec{\rho}_i| = R_i$, $|\vec{\rho}_j| = R_j$, $|\overrightarrow{O_iP}| = |\vec{\rho} - \vec{\rho}_i| = R_i + r$, $|\overrightarrow{O_jP}| = |\vec{\rho} - \vec{\rho}_j| = R_j - r$, hence

$$(\vec{\rho} - \vec{\rho}_i)^2 - (\vec{\rho} - \vec{\rho}_j)^2 = (R_i + r)^2 - (R_j - r)^2,$$
$$(\vec{\rho}_i^{\,2} - \vec{\rho}_j^{\,2}) + 2\vec{\rho}\cdot(\vec{\rho}_j - \vec{\rho}_i) = (R_i^2 - R_j^2) + 2r(R_i + R_j),$$
$$d_{ij} \cdot d(P) = d_{ij}\vec{v}\cdot\vec{\rho} = (\vec{\rho}_j - \vec{\rho}_i)\cdot\vec{\rho} = r(R_i + R_j).$$

Therefore,

$$r = \frac{d_{ij}}{R_i + R_j} \cdot d(P),$$

and the value $v_{ij} = \dfrac{d_{ij}}{R_i + R_j}$ does not depend on P. □

Lemma 3. The curved quadrilateral $\mathcal{Q}_{ij} = V_{i,j}\overset{\frown}{V_{i+1,j}}V_{i+1,j+1}\overset{\frown}{V_{i,j+1}}$ is circumscribed if and only if $u_{i,i+1} = v_{j,j+1}$.

Proof. First suppose that the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed and $\Omega(P,r)$ is its inscribed circle. By Lemma 1 and Lemma 2 we have $r = u_{i,i+1} \cdot d(P)$ and $r = v_{j,j+1} \cdot d(P)$ as well. Hence, $u_{i,i+1} = v_{j,j+1}$.

To prove the opposite direction, suppose $u_{i,i+1} = v_{j,j+1}$. Let P be the intersection of the angle bisector $\beta_{i,i+1}$ and the ellipse arc $\varepsilon_{j,j+1}$. Choose $r = u_{i,i+1} \cdot d(P) = v_{j,j+1} \cdot d(P)$. Then the circle $\Omega(P,r)$ is tangent to the half lines h_i and h_{i+1} by Lemma 1, and it is tangent to the arcs γ_j and γ_{j+1} by Lemma 2. Hence, the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed. □

By Lemma 3, the statement of the problem can be reformulated to an obvious fact: If the equalities $u_{12} = v_{12}$, $u_{12} = v_{23}$, and $u_{23} = v_{12}$ hold, then $u_{23} = v_{23}$ holds as well.

Comment 1. Lemma 2(b) (together with the easy Lemma 1(b)) is the key tool in this solution. If one finds this fact, then the solution can be finished in many ways. That is, one can find a circle touching three of h_2, h_3, γ_2, and γ_3, and then prove that it is tangent to the fourth one in either synthetic or analytical way. Both approaches can be successful.

Here we present some discussion about this key Lemma.

1. In the solution above we chose an analytic proof for Lemma 2(b) because we expect that most students will use coordinates or vectors to examine the locus of the centers, and these approaches are less case-sensitive.

Here we outline a synthetic proof. We consider only the case when P does not lie in the line O_iO_j. The other case can be obtained as a limit case, or computed in a direct way.

Let S be the internal homothety center between the circles of γ_i and γ_j, lying on O_iO_j; this point does not depend on P. Let U and V be the points of tangency of circle $\sigma = \Omega(P, r)$ with γ_i and γ_j, respectively (then $r = PU = PV$); in other words, points U and V are the intersection points of rays O_iP, O_jP with arcs γ_i, γ_j respectively (see Fig. 4).

Due to the theorem on three homothety centers (or just to the Menelaus theorem applied to triangle O_iO_jP), the points U, V and S are collinear. Let T be the intersection point of line AC and the common tangent to σ and γ_i at U; then T is the radical center of σ, γ_i and γ_j, hence TV is the common tangent to σ and γ_j.

Let Q be the projection of P onto the line AC. By the right angles, the points U, V and Q lie on the circle with diameter PT. From this fact and the equality $PU = PV$ we get $\angle UQP = \angle UVP = \angle VUP = \angle SUO_i$. Since $O_iS \parallel PQ$, we have $\angle SO_iU = \angle QPU$. Hence, the triangles SO_iU and UPQ are similar and thus $\dfrac{r}{d(P)} = \dfrac{PU}{PQ} = \dfrac{O_iS}{O_iU} = \dfrac{O_iS}{R_i}$; the last expression is constant since S is a constant point. □

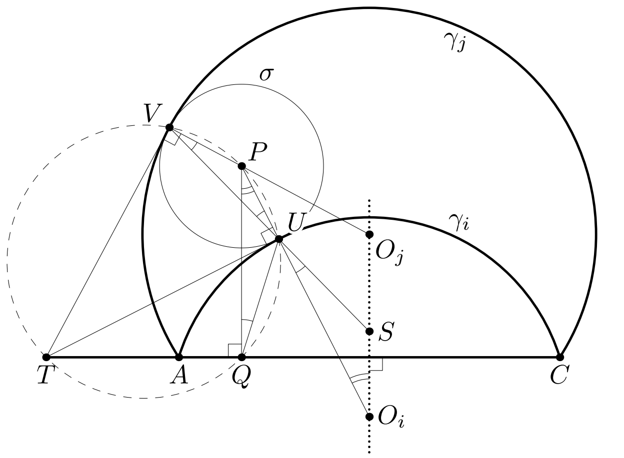


Fig. 4

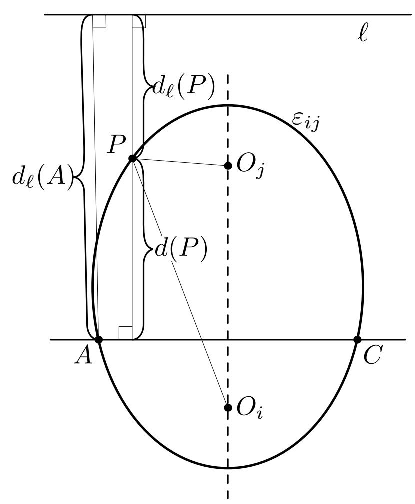


Fig. 5

2. Using some known facts about conics, the same statement can be proved in a very short way. Denote by ℓ the directrix of ellipse of ε_{ij} related to the focus O_j; since ε_{ij} is symmetrical about O_iO_j, we have $\ell \parallel AC$. Recall that for each point $P \in \varepsilon_{ij}$, we have $PO_j = \epsilon \cdot d_\ell(P)$, where $d_\ell(P)$ is the distance from P to ℓ, and ϵ is the eccentricity of ε_{ij} (see Fig. 5).

Now we have

$$r = R_j - (R_j - r) = AO_j - PO_j = \epsilon\big(d_\ell(A) - d_\ell(P)\big) = \epsilon\big(d(P) - d(A)\big) = \epsilon \cdot d(P),$$

and ϵ does not depend on P. □

Comment 2. One can find a spatial interpretations of the problem and the solution.

For every point (x, y) and radius $r > 0$, represent the circle $\Omega\big((x, y), r\big)$ by the point (x, y, r) in space. This point is the apex of the cone with base circle $\Omega\big((x, y), r\big)$ and height r. According to Lemma 1, the circles which are tangent to h_i and h_j correspond to the points of a half line β'_{ij}, starting at B.

Now we translate Lemma 2. Take some $1 \le i < j \le 3$, and consider those circles which are internally tangent to γ_j. It is easy to see that the locus of the points which represent these circles is a subset of a cone, containing γ_j. Similarly, the circles which are externally tangent to γ_i correspond to the points on the extension of another cone, which has its apex on the opposite side of the base plane Π. (See Fig. 6; for this illustration, the z-coordinates were multiplied by 2.)

The two cones are symmetric to each other (they have the same aperture, and their axes are parallel). As is well-known, it follows that the common points of the two cones are co-planar. So the intersection of the two cones is a a conic section — which is an ellipse, according to Lemma 2(a). The points which represent the circles touching γ_i and γ_j is an ellipse arc ε'_{ij} with end-points A and C.

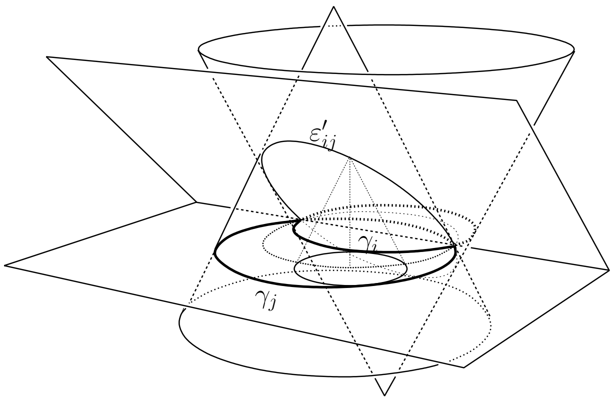


Fig. 6

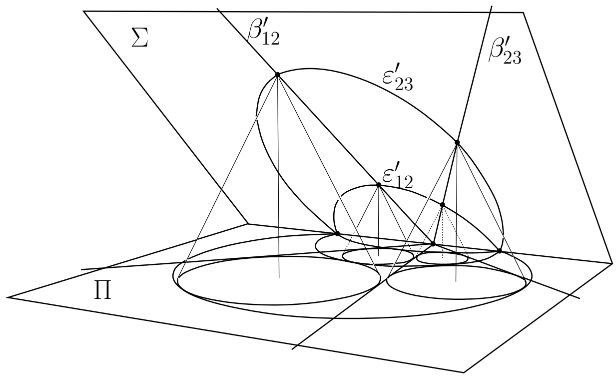


Fig. 7

Thus, the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed if and only if $\beta'_{i,i+1}$ and $\varepsilon'_{j,j+1}$ intersect, i.e. if they are coplanar. If three of the four curved quadrilaterals are circumscribed, it means that ε'_{12}, ε'_{23}, β'_{12} and β'_{23} lie in the same plane Σ, and the fourth intersection comes to existence, too (see Fig. 7).

A connection between mathematics and real life:
the Palace of Creativity "Shabyt" ("Inspiration") in Astana

Number Theory

N1. Find the least positive integer n for which there exists a set $\{s_1, s_2, \dots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\dots\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$

N1′. Same as Problem N1, but the constant $\dfrac{51}{2010}$ is replaced by $\dfrac{42}{2010}$.

(Canada)

Answer for Problem N1. $n = 39$.

Solution for Problem N1. Suppose that for some n there exist the desired numbers; we may assume that $s_1 < s_2 < \dots < s_n$. Surely $s_1 > 1$ since otherwise $1 - \dfrac{1}{s_1} = 0$. So we have $2 \le s_1 \le s_2 - 1 \le \dots \le s_n - (n-1)$, hence $s_i \ge i+1$ for each $i = 1, \dots, n$. Therefore

$$\begin{aligned}\frac{51}{2010} &= \left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\dots\left(1-\frac{1}{s_n}\right)\\ &\ge \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{n+1}\right)=\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n}{n+1}=\frac{1}{n+1},\end{aligned}$$

which implies

$$n+1 \ge \frac{2010}{51}=\frac{670}{17}>39,$$

so $n \ge 39$.

Now we are left to show that $n = 39$ fits. Consider the set $\{2, 3, \dots, 33, 35, 36, \dots, 40, 67\}$ which contains exactly 39 numbers. We have

$$\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{32}{33}\cdot\frac{34}{35}\cdots\frac{39}{40}\cdot\frac{66}{67}=\frac{1}{33}\cdot\frac{34}{40}\cdot\frac{66}{67}=\frac{17}{670}=\frac{51}{2010}, \tag{1}$$

hence for $n = 39$ there exists a desired example.

Comment. One can show that the example (1) is unique.

Answer for Problem N1′. $n = 48$.

Solution for Problem N1′. Suppose that for some n there exist the desired numbers. In the same way we obtain that $s_i \ge i+1$. Moreover, since the denominator of the fraction $\dfrac{42}{2010} = \dfrac{7}{335}$ is divisible by 67, some of s_i's should be divisible by 67, so $s_n \ge s_i \ge 67$. This means that

$$\frac{42}{2010} \ge \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n-1}{n}\cdot\left(1-\frac{1}{67}\right)=\frac{66}{67n},$$

which implies
$$n \geq \frac{2010 \cdot 66}{42 \cdot 67} = \frac{330}{7} > 47,$$
so $n \geq 48$.

Now we are left to show that $n = 48$ fits. Consider the set $\{2, 3, \ldots, 33, 36, 37, \ldots, 50, 67\}$ which contains exactly 48 numbers. We have
$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{32}{33} \cdot \frac{35}{36} \cdots \frac{49}{50} \cdot \frac{66}{67} = \frac{1}{33} \cdot \frac{35}{50} \cdot \frac{66}{67} = \frac{7}{335} = \frac{42}{2010},$$
hence for $n = 48$ there exists a desired example.

Comment 1. In this version of the problem, the estimate needs one more step, hence it is a bit harder. On the other hand, the example in this version is not unique. Another example is
$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{46}{47} \cdot \frac{66}{67} \cdot \frac{329}{330} = \frac{1}{67} \cdot \frac{66}{330} \cdot \frac{329}{47} = \frac{7}{67 \cdot 5} = \frac{42}{2010}.$$

Comment 2. N1′ was the Proposer's formulation of the problem. We propose N1 according to the number of current IMO.

N2. Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m\bigl(2^{n+1} - 1\bigr)\,. \tag{1}$$

(Australia)

Answer. $(6,3)$, $(9,3)$, $(9,5)$, $(54,5)$.

Solution. For fixed values of n, the equation (1) is a simple quadratic equation in m. For $n \leq 5$ the solutions are listed in the following table.

case	equation	discriminant	integer roots
$n = 0$	$m^2 - m + 2 = 0$	-7	none
$n = 1$	$m^2 - 3m + 6 = 0$	-15	none
$n = 2$	$m^2 - 7m + 18 = 0$	-23	none
$n = 3$	$m^2 - 15m + 54 = 0$	9	$m = 6$ and $m = 9$
$n = 4$	$m^2 - 31m + 162 = 0$	313	none
$n = 5$	$m^2 - 63m + 486 = 0$	$2025 = 45^2$	$m = 9$ and $m = 54$

We prove that there is no solution for $n \geq 6$.

Suppose that (m, n) satisfies (1) and $n \geq 6$. Since $m \mid 2 \cdot 3^n = m\bigl(2^{n+1} - 1\bigr) - m^2$, we have $m = 3^p$ with some $0 \leq p \leq n$ or $m = 2 \cdot 3^q$ with some $0 \leq q \leq n$.

In the first case, let $q = n - p$; then

$$2^{n+1} - 1 = m + \frac{2 \cdot 3^n}{m} = 3^p + 2 \cdot 3^q.$$

In the second case let $p = n - q$. Then

$$2^{n+1} - 1 = m + \frac{2 \cdot 3^n}{m} = 2 \cdot 3^q + 3^p.$$

Hence, in both cases we need to find the nonnegative integer solutions of

$$3^p + 2 \cdot 3^q = 2^{n+1} - 1, \quad p + q = n. \tag{2}$$

Next, we prove bounds for p, q. From (2) we get

$$3^p < 2^{n+1} = 8^{\frac{n+1}{3}} < 9^{\frac{n+1}{3}} = 3^{\frac{2(n+1)}{3}}$$

and

$$2 \cdot 3^q < 2^{n+1} = 2 \cdot 8^{\frac{n}{3}} < 2 \cdot 9^{\frac{n}{3}} = 2 \cdot 3^{\frac{2n}{3}} < 2 \cdot 3^{\frac{2(n+1)}{3}},$$

so $p, q < \frac{2(n+1)}{3}$. Combining these inequalities with $p + q = n$, we obtain

$$\frac{n-2}{3} < p, q < \frac{2(n+1)}{3}. \tag{3}$$

Now let $h = \min(p, q)$. By (3) we have $h > \frac{n-2}{3}$; in particular, we have $h > 1$. On the left-hand side of (2), both terms are divisible by 3^h, therefore $9 \mid 3^h \mid 2^{n+1} - 1$. It is easy check that $\operatorname{ord}_9(2) = 6$, so $9 \mid 2^{n+1} - 1$ if and only if $6 \mid n + 1$. Therefore, $n + 1 = 6r$ for some positive integer r, and we can write

$$2^{n+1} - 1 = 4^{3r} - 1 = (4^{2r} + 4^r + 1)(2^r - 1)(2^r + 1). \tag{4}$$

Notice that the factor $4^{2r} + 4^r + 1 = (4^r - 1)^2 + 3 \cdot 4^r$ is divisible by 3, but it is never divisible by 9. The other two factors in (4), $2^r - 1$ and $2^r + 1$ are coprime: both are odd and their difference is 2. Since the whole product is divisible by 3^h, we have either $3^{h-1} \mid 2^r - 1$ or $3^{h-1} \mid 2^r + 1$. In any case, we have $3^{h-1} \leq 2^r + 1$. Then

$$3^{h-1} \leq 2^r + 1 \leq 3^r = 3^{\frac{n+1}{6}},$$
$$\frac{n-2}{3} - 1 < h - 1 \leq \frac{n+1}{6},$$
$$n < 11.$$

But this is impossible since we assumed $n \geq 6$, and we proved $6 \mid n + 1$.

N3. Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \cdots + f_n(x)^2.$$

(Poland)

Answer. The smallest n is 5.

Solution 1. The equality $x^2 + 7 = x^2 + 2^2 + 1^2 + 1^2 + 1^2$ shows that $n \leq 5$. It remains to show that $x^2 + 7$ is not a sum of four (or less) squares of polynomials with rational coefficients.

Suppose by way of contradiction that $x^2 + 7 = f_1(x)^2 + f_2(x)^2 + f_3(x)^2 + f_4(x)^2$, where the coefficients of polynomials f_1, f_2, f_3 and f_4 are rational (some of these polynomials may be zero).

Clearly, the degrees of f_1, f_2, f_3 and f_4 are at most 1. Thus $f_i(x) = a_i x + b_i$ for $i = 1, 2, 3, 4$ and some rationals a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4. It follows that $x^2 + 7 = \sum_{i=1}^{4}(a_i x + b_i)^2$ and hence

$$\sum_{i=1}^{4} a_i^2 = 1, \quad \sum_{i=1}^{4} a_i b_i = 0, \quad \sum_{i=1}^{4} b_i^2 = 7. \tag{1}$$

Let $p_i = a_i + b_i$ and $q_i = a_i - b_i$ for $i = 1, 2, 3, 4$. Then

$$\sum_{i=1}^{4} p_i^2 = \sum_{i=1}^{4} a_i^2 + 2\sum_{i=1}^{4} a_i b_i + \sum_{i=1}^{4} b_i^2 = 8,$$
$$\sum_{i=1}^{4} q_i^2 = \sum_{i=1}^{4} a_i^2 - 2\sum_{i=1}^{4} a_i b_i + \sum_{i=1}^{4} b_i^2 = 8$$
$$\text{and} \quad \sum_{i=1}^{4} p_i q_i = \sum_{i=1}^{4} a_i^2 - \sum_{i=1}^{4} b_i^2 = -6,$$

which means that there exist a solution in integers x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4 and $m > 0$ of the system of equations

$$\text{(i)} \sum_{i=1}^{4} x_i^2 = 8m^2, \qquad \text{(ii)} \sum_{i=1}^{4} y_i^2 = 8m^2, \qquad \text{(iii)} \sum_{i=1}^{4} x_i y_i = -6m^2.$$

We will show that such a solution does not exist.

Assume the contrary and consider a solution with minimal m. Note that if an integer x is odd then $x^2 \equiv 1 \pmod 8$. Otherwise (i.e., if x is even) we have $x^2 \equiv 0 \pmod 8$ or $x^2 \equiv 4 \pmod 8$. Hence, by (i), we get that x_1, x_2, x_3 and x_4 are even. Similarly, by (ii), we get that y_1, y_2, y_3 and y_4 are even. Thus the LHS of (iii) is divisible by 4 and m is also even. It follows that $(\frac{x_1}{2}, \frac{y_1}{2}, \frac{x_2}{2}, \frac{y_2}{2}, \frac{x_3}{2}, \frac{y_3}{2}, \frac{x_4}{2}, \frac{y_4}{2}, \frac{m}{2})$ is a solution of the system of equations (i), (ii) and (iii), which contradicts the minimality of m.

Solution 2. We prove that $n \leq 4$ is impossible. Define the numbers a_i, b_i for $i = 1, 2, 3, 4$ as in the previous solution.

By Euler's identity we have

$$\begin{aligned}(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) =& (a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4)^2 + (a_1 b_2 - a_2 b_1 + a_3 b_4 - a_4 b_3)^2 \\ &+ (a_1 b_3 - a_3 b_1 + a_4 b_2 - a_2 b_4)^2 + (a_1 b_4 - a_4 b_1 + a_2 b_3 - a_3 b_2)^2.\end{aligned}$$

So, using the relations (1) from the Solution 1 we get that

$$7 = \left(\frac{m_1}{m}\right)^2 + \left(\frac{m_2}{m}\right)^2 + \left(\frac{m_3}{m}\right)^2, \tag{2}$$

where

$$\begin{aligned} \frac{m_1}{m} &= a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3, \\ \frac{m_2}{m} &= a_1b_3 - a_3b_1 + a_4b_2 - a_2b_4, \\ \frac{m_3}{m} &= a_1b_4 - a_4b_1 + a_2b_3 - a_3b_2 \end{aligned}$$

and $m_1, m_2, m_3 \in \mathbb{Z}, m \in \mathbb{N}$.

Let m be a minimum positive integer number for which (2) holds. Then

$$8m^2 = m_1^2 + m_2^2 + m_3^2 + m^2.$$

As in the previous solution, we get that m_1, m_2, m_3, m are all even numbers. Then $\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{2}, \frac{m}{2}\right)$ is also a solution of (2) which contradicts the minimality of m. So, we have $n \geq 5$. The example with $n = 5$ is already shown in Solution 1.

N4. Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is *n-good* if $n \mid P(m) - P(k)$ implies $n \mid m - k$ for all integers m, k. We say that (a, b) is *very good* if (a, b) is n-good for infinitely many positive integers n.

(a) Find a pair (a, b) which is 51-good, but not very good.

(b) Show that all 2010-good pairs are very good.

(Turkey)

Solution. (a) We show that the pair $(1, -51^2)$ is good but not very good. Let $P(x) = x^3 - 51^2 x$. Since $P(51) = P(0)$, the pair $(1, -51^2)$ is not n-good for any positive integer that does not divide 51. Therefore, $(1, -51^2)$ is not very good.

On the other hand, if $P(m) \equiv P(k) \pmod{51}$, then $m^3 \equiv k^3 \pmod{51}$. By Fermat's theorem, from this we obtain

$$m \equiv m^3 \equiv k^3 \equiv k \pmod{3} \quad \text{and} \quad m \equiv m^{33} \equiv k^{33} \equiv k \pmod{17}.$$

Hence we have $m \equiv k \pmod{51}$. Therefore $(1, -51^2)$ is 51-good.

(b) We will show that if a pair (a, b) is 2010-good then (a, b) is 67^i-good for all positive integer i.

Claim 1. If (a, b) is 2010-good then (a, b) is 67-good.

Proof. Assume that $P(m) = P(k) \pmod{67}$. Since 67 and 30 are coprime, there exist integers m' and k' such that $k' \equiv k \pmod{67}$, $k' \equiv 0 \pmod{30}$, and $m' \equiv m \pmod{67}$, $m' \equiv 0 \pmod{30}$. Then we have $P(m') \equiv P(0) \equiv P(k') \pmod{30}$ and $P(m') \equiv P(m) \equiv P(k) \equiv P(k') \pmod{67}$, hence $P(m') \equiv P(k') \pmod{2010}$. This implies $m' \equiv k' \pmod{2010}$ as (a, b) is 2010-good. It follows that $m \equiv m' \equiv k' \equiv k \pmod{67}$. Therefore, (a, b) is 67-good. $\square$

Claim 2. If (a, b) is 67-good then $67 \mid a$.

Proof. Suppose that $67 \nmid a$. Consider the sets $\{at^2 \pmod{67} : 0 \le t \le 33\}$ and $\{-3as^2 - b \mod 67 : 0 \le s \le 33\}$. Since $a \not\equiv 0 \pmod{67}$, each of these sets has 34 elements. Hence they have at least one element in common. If $at^2 \equiv -3as^2 - b \pmod{67}$ then for $m = t \pm s$, $k = \mp 2s$ we have

$$\begin{aligned} P(m) - P(k) = a(m^3 - k^3) + b(m - k) &= (m - k)\big(a(m^2 + mk + k^2) + b\big) \\ &= (t \pm 3s)(at^2 + 3as^2 + b) \equiv 0 \pmod{67}. \end{aligned}$$

Since (a, b) is 67-good, we must have $m \equiv k \pmod{67}$ in both cases, that is, $t \equiv 3s \pmod{67}$ and $t \equiv -3s \pmod{67}$. This means $t \equiv s \equiv 0 \pmod{67}$ and $b \equiv -3as^2 - at^2 \equiv 0 \pmod{67}$. But then $67 \mid P(7) - P(2) = 67 \cdot 5a + 5b$ and $67 \nmid 7 - 2$, contradicting that (a, b) is 67-good. $\square$

Claim 3. If (a, b) is 2010-good then (a, b) is 67^i-good all $i \ge 1$.

Proof. By Claim 2, we have $67 \mid a$. If $67 \mid b$, then $P(x) \equiv P(0) \pmod{67}$ for all x, contradicting that (a, b) is 67-good. Hence, $67 \nmid b$.

Suppose that $67^i \mid P(m) - P(k) = (m - k)\big(a(m^2 + mk + k^2) + b\big)$. Since $67 \mid a$ and $67 \nmid b$, the second factor $a(m^2 + mk + k^2) + b$ is coprime to 67 and hence $67^i \mid m - k$. Therefore, (a, b) is 67^i-good. $\square$

Comment 1. In the proof of Claim 2, the following reasoning can also be used. Since 3 is not a quadratic residue modulo 67, either $au^2 \equiv -b \pmod{67}$ or $3av^2 \equiv -b \pmod{67}$ has a solution. The settings $(m, k) = (u, 0)$ in the first case and $(m, k) = (v, -2v)$ in the second case lead to $b \equiv 0 \pmod{67}$.

Comment 2. The pair $(67, 30)$ is n-good if and only if $n = d \cdot 67^i$, where $d \mid 30$ and $i \ge 0$. It shows that in part (b), one should deal with the large powers of 67 to reach the solution. The key property of number 67 is that it has the form $3k + 1$, so there exists a nontrivial cubic root of unity modulo 67.

N5. Let $\mathbb{N}$ be the set of all positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that the number $\big(f(m)+n\big)\big(m+f(n)\big)$ is a square for all $m, n \in \mathbb{N}$.

(U.S.A.)

Answer. All functions of the form $f(n) = n + c$, where $c \in \mathbb{N} \cup \{0\}$.

Solution. First, it is clear that all functions of the form $f(n) = n + c$ with a constant nonnegative integer c satisfy the problem conditions since $\big(f(m)+n\big)\big(f(n)+m\big) = (n+m+c)^2$ is a square.

We are left to prove that there are no other functions. We start with the following

Lemma. Suppose that $p \mid f(k) - f(\ell)$ for some prime p and positive integers k, ℓ. Then $p \mid k - \ell$.

Proof. Suppose first that $p^2 \mid f(k) - f(\ell)$, so $f(\ell) = f(k) + p^2 a$ for some integer a. Take some positive integer $D > \max\{f(k), f(\ell)\}$ which is not divisible by p and set $n = pD - f(k)$. Then the positive numbers $n + f(k) = pD$ and $n + f(\ell) = pD + \big(f(\ell) - f(k)\big) = p(D + pa)$ are both divisible by p but not by p^2. Now, applying the problem conditions, we get that both the numbers $\big(f(k)+n\big)\big(f(n)+k\big)$ and $\big(f(\ell)+n\big)\big(f(n)+\ell\big)$ are squares divisible by p (and thus by p^2); this means that the multipliers $f(n)+k$ and $f(n)+\ell$ are also divisible by p, therefore $p \mid \big(f(n)+k\big) - \big(f(n)+\ell\big) = k - \ell$ as well.

On the other hand, if $f(k) - f(\ell)$ is divisible by p but not by p^2, then choose the same number D and set $n = p^3 D - f(k)$. Then the positive numbers $f(k) + n = p^3 D$ and $f(\ell) + n = p^3 D + \big(f(\ell) - f(k)\big)$ are respectively divisible by p^3 (but not by p^4) and by p (but not by p^2). Hence in analogous way we obtain that the numbers $f(n)+k$ and $f(n)+\ell$ are divisible by p, therefore $p \mid \big(f(n)+k\big) - \big(f(n)+\ell\big) = k - \ell$. $\square$

We turn to the problem. First, suppose that $f(k) = f(\ell)$ for some $k, \ell \in \mathbb{N}$. Then by Lemma we have that $k - \ell$ is divisible by every prime number, so $k - \ell = 0$, or $k = \ell$. Therefore, the function f is injective.

Next, consider the numbers $f(k)$ and $f(k+1)$. Since the number $(k+1) - k = 1$ has no prime divisors, by Lemma the same holds for $f(k+1) - f(k)$; thus $|f(k+1) - f(k)| = 1$.

Now, let $f(2) - f(1) = q$, $|q| = 1$. Then we prove by induction that $f(n) = f(1) + q(n-1)$. The base for $n = 1, 2$ holds by the definition of q. For the step, if $n > 1$ we have $f(n+1) = f(n) \pm q = f(1) + q(n-1) \pm q$. Since $f(n) \neq f(n-2) = f(1) + q(n-2)$, we get $f(n) = f(1) + qn$, as desired.

Finally, we have $f(n) = f(1) + q(n-1)$. Then q cannot be -1 since otherwise for $n \geq f(1)+1$ we have $f(n) \leq 0$ which is impossible. Hence $q = 1$ and $f(n) = (f(1) - 1) + n$ for each $n \in \mathbb{N}$, and $f(1) - 1 \geq 0$, as desired.

N6. The rows and columns of a $2^n \times 2^n$ table are numbered from 0 to $2^n - 1$. The cells of the table have been colored with the following property being satisfied: for each $0 \le i, j \le 2^n - 1$, the jth cell in the ith row and the $(i+j)$th cell in the jth row have the same color. (The indices of the cells in a row are considered modulo 2^n.)

Prove that the maximal possible number of colors is 2^n.

(Iran)

Solution. Throughout the solution we denote the cells of the table by coordinate pairs; (i, j) refers to the jth cell in the ith row.

Consider the directed graph, whose vertices are the cells of the board, and the edges are the arrows $(i, j) \to (j, i + j)$ for all $0 \le i, j \le 2^n - 1$. From each vertex (i, j), exactly one edge passes (to $(j, i + j \mod 2^n)$); conversely, to each cell (j, k) exactly one edge is directed (from the cell $(k - j \mod 2^n, j)$). Hence, the graph splits into cycles.

Now, in any coloring considered, the vertices of each cycle should have the same color by the problem condition. On the other hand, if each cycle has its own color, the obtained coloring obviously satisfies the problem conditions. Thus, the maximal possible number of colors is the same as the number of cycles, and we have to prove that this number is 2^n.

Next, consider any cycle $(i_1, j_1), (i_2, j_2), \dots$; we will describe it in other terms. Define a sequence $(a_0, a_1, \dots)$ by the relations $a_0 = i_1$, $a_1 = j_1$, $a_{n+1} = a_n + a_{n-1}$ for all $n \ge 1$ (we say that such a sequence is *a Fibonacci-type sequence*). Then an obvious induction shows that $i_k \equiv a_{k-1} \pmod{2^n}$, $j_k \equiv a_k \pmod{2^n}$. Hence we need to investigate the behavior of Fibonacci-type sequences modulo 2^n.

Denote by $F_0, F_1, \dots$ the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \ge 0$. We also set $F_{-1} = 1$ according to the recurrence relation.

For every positive integer m, denote by $\nu(m)$ the exponent of 2 in the prime factorization of m, i.e. for which $2^{\nu(m)} \mid m$ but $2^{\nu(m)+1} \nmid m$.

Lemma 1. For every Fibonacci-type sequence $a_0, a_1, a_2, \dots$, and every $k \ge 0$, we have $a_k = F_{k-1}a_0 + F_k a_1$.

Proof. Apply induction on k. The base cases $k = 0, 1$ are trivial. For the step, from the induction hypothesis we get

$$a_{k+1} = a_k + a_{k-1} = (F_{k-1}a_0 + F_k a_1) + (F_{k-2}a_0 + F_{k-1}a_1) = F_k a_0 + F_{k+1}a_1. \qquad \square$$

Lemma 2. For every $m \ge 3$,

(a) we have $\nu(F_{3 \cdot 2^{m-2}}) = m$;

(b) $d = 3 \cdot 2^{m-2}$ is the least positive index for which $2^m \mid F_d$;

(c) $F_{3 \cdot 2^{m-2}+1} \equiv 1 + 2^{m-1} \pmod{2^m}$.

Proof. Apply induction on m. In the base case $m = 3$ we have $\nu(F_{3 \cdot 2^{m-2}}) = F_6 = 8$, so $\nu(F_{3 \cdot 2^{m-2}}) = \nu(8) = 3$, the preceding Fibonacci-numbers are not divisible by 8, and indeed $F_{3 \cdot 2^{m-2}+1} = F_7 = 13 \equiv 1 + 4 \pmod 8$.

Now suppose that $m > 3$ and let $k = 3 \cdot 2^{m-3}$. By applying Lemma 1 to the Fibonacci-type sequence $F_k, F_{k+1}, \dots$ we get

$$F_{2k} = F_{k-1}F_k + F_k F_{k+1} = (F_{k+1} - F_k)F_k + F_{k+1}F_k = 2F_{k+1}F_k - F_k^2,$$
$$F_{2k+1} = F_k \cdot F_k + F_{k+1} \cdot F_{k+1} = F_k^2 + F_{k+1}^2.$$

By the induction hypothesis, $\nu(F_k) = m - 1$, and F_{k+1} is odd. Therefore we get $\nu(F_k^2) = 2(m-1) > (m-1) + 1 = \nu(2F_k F_{k+1})$, which implies $\nu(F_{2k}) = m$, establishing statement (a).

Moreover, since $F_{k+1} = 1 + 2^{m-2} + a2^{m-1}$ for some integer a, we get

$$F_{2k+1} = F_k^2 + F_{k+1}^2 \equiv 0 + (1 + 2^{m-2} + a2^{m-1})^2 \equiv 1 + 2^{m-1} \pmod{2^m},$$

as desired in statement (c).

We are left to prove that $2^m \nmid F_\ell$ for $\ell < 2k$. Assume the contrary. Since $2^{m-1} \mid F_\ell$, from the induction hypothesis it follows that $\ell > k$. But then we have $F_\ell = F_{k-1}F_{\ell-k} + F_kF_{\ell-k+1}$, where the second summand is divisible by 2^{m-1} but the first one is not (since F_{k-1} is odd and $\ell - k < k$). Hence the sum is not divisible even by 2^{m-1}. A contradiction. □

Now, for every pair of integers $(a, b) \neq (0, 0)$, let $\mu(a, b) = \min\{\nu(a), \nu(b)\}$. By an obvious induction, for every Fibonacci-type sequence $A = (a_0, a_1, \ldots)$ we have $\mu(a_0, a_1) = \mu(a_1, a_2) = \ldots$; denote this common value by $\mu(A)$. Also denote by $p_n(A)$ the period of this sequence modulo 2^n, that is, the least $p > 0$ such that $a_{k+p} \equiv a_k \pmod{2^n}$ for all $k \geq 0$.

Lemma 3. Let $A = (a_0, a_1, \ldots)$ be a Fibonacci-type sequence such that $\mu(A) = k < n$. Then $p_n(A) = 3 \cdot 2^{n-1-k}$.

Proof. First, we note that the sequence $(a_0, a_1, \ldots)$ has period p modulo 2^n if and only if the sequence $(a_0/2^k, a_1/2^k, \ldots)$ has period p modulo 2^{n-k}. Hence, passing to this sequence we can assume that $k = 0$.

We prove the statement by induction on n. It is easy to see that for $n = 1, 2$ the claim is true; actually, each Fibonacci-type sequence A with $\mu(A) = 0$ behaves as $0, 1, 1, 0, 1, 1, \ldots$ modulo 2, and as $0, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, \ldots$ modulo 4 (all pairs of residues from which at least one is odd appear as a pair of consecutive terms in this sequence).

Now suppose that $n \geq 3$ and consider an arbitrary Fibonacci-type sequence $A = (a_0, a_1, \ldots)$ with $\mu(A) = 0$. Obviously we should have $p_{n-1}(A) \mid p_n(A)$, or, using the induction hypothesis, $s = 3 \cdot 2^{n-2} \mid p_n(A)$. Next, we may suppose that a_0 is even; hence a_1 is odd, and $a_0 = 2b_0$, $a_1 = 2b_1 + 1$ for some integers b_0, b_1.

Consider the Fibonacci-type sequence $B = (b_0, b_1, \ldots)$ starting with (b_0, b_1). Since $a_0 = 2b_0 + F_0$, $a_1 = 2b_1 + F_1$, by an easy induction we get $a_k = 2b_k + F_k$ for all $k \geq 0$. By the induction hypothesis, we have $p_{n-1}(B) \mid s$, hence the sequence $(2b_0, 2b_1, \ldots)$ is s-periodic modulo 2^n. On the other hand, by Lemma 2 we have $F_{s+1} \equiv 1 + 2^{n-1} \pmod{2^n}$, $F_{2s} \equiv 0 \pmod{2^n}$, $F_{2s+1} \equiv 1 \pmod{2^n}$, hence

$$\begin{aligned} a_{s+1} = 2b_{s+1} + F_{s+1} &\equiv 2b_1 + 1 + 2^{n-1} \not\equiv 2b_1 + 1 = a_1 \pmod{2^n}, \\ a_{2s} = 2b_{2s} + F_{2s} &\equiv 2b_0 + 0 = a_0 \pmod{2^n}, \\ a_{2s+1} = 2b_{2s+1} + F_{2s+1} &\equiv 2b_1 + 1 = a_1 \pmod{2^n}. \end{aligned}$$

The first line means that A is not s-periodic, while the other two provide that $a_{2s} \equiv a_0$, $a_{2s+1} \equiv a_1$ and hence $a_{2s+t} \equiv a_t$ for all $t \geq 0$. Hence $s \mid p_n(A) \mid 2s$ and $p_n(A) \neq s$, which means that $p_n(A) = 2s$, as desired. □

Finally, Lemma 3 provides a straightforward method of counting the number of cycles. Actually, take any number $0 \leq k \leq n-1$ and consider all the cells (i, j) with $\mu(i, j) = k$. The total number of such cells is $2^{2(n-k)} - 2^{2(n-k-1)} = 3 \cdot 2^{2n-2k-2}$. On the other hand, they are split into cycles, and by Lemma 3 the length of each cycle is $3 \cdot 2^{n-1-k}$. Hence the number of cycles consisting of these cells is exactly $\dfrac{3 \cdot 2^{2n-2k-2}}{3 \cdot 2^{n-1-k}} = 2^{n-k-1}$. Finally, there is only one cell $(0, 0)$ which is not mentioned in the previous computation, and it forms a separate cycle. So the total number of cycles is

$$1 + \sum_{k=0}^{n-1} 2^{n-1-k} = 1 + (1 + 2 + 4 + \cdots + 2^{n-1}) = 2^n.$$

Comment. We outline a different proof for the essential part of Lemma 3. That is, we assume that $k = 0$ and show that in this case the period of (a_i) modulo 2^n coincides with the period of the Fibonacci numbers modulo 2^n; then the proof can be finished by the arguments from Lemma 2..

Note that p is a (not necessarily minimal) period of the sequence (a_i) modulo 2^n if and only if we have $a_0 \equiv a_p \pmod{2^n}$, $a_1 \equiv a_{p+1} \pmod{2^n}$, that is,

$$\begin{aligned} a_0 &\equiv a_p \equiv F_{p-1}a_0 + F_p a_1 = F_p(a_1 - a_0) + F_{p+1}a_0 \pmod{2^n}, \\ a_1 &\equiv a_{p+1} = F_p a_0 + F_{p+1}a_1 \pmod{2^n}. \end{aligned} \tag{1}$$

Now, If p is a period of (F_i) then we have $F_p \equiv F_0 = 0 \pmod{2^n}$ and $F_{p+1} \equiv F_1 = 1 \pmod{2^n}$, which by (1) implies that p is a period of (a_i) as well.

Conversely, suppose that p is a period of (a_i). Combining the relations of (1) we get

$$\begin{aligned} 0 = a_1 \cdot a_0 - a_0 \cdot a_1 &\equiv a_1\big(F_p(a_1 - a_0) + F_{p+1}a_0\big) - a_0(F_p a_0 + F_{p+1}a_1) \\ &= F_p(a_1^2 - a_1 a_0 - a_0^2) \pmod{2^n}, \\ a_1^2 - a_1 a_0 - a_0^2 = (a_1 - a_0)a_1 - a_0 \cdot a_0 &\equiv (a_1 - a_0)(F_p a_0 + F_{p+1}a_1) - a_0\big(F_p(a_1 - a_0) + F_{p+1}a_0\big) \\ &= F_{p+1}(a_1^2 - a_1 a_0 - a_0^2) \pmod{2^n}. \end{aligned}$$

Since at least one of the numbers a_0, a_1 is odd, the number $a_1^2 - a_1 a_0 - a_0^2$ is odd as well. Therefore the previous relations are equivalent with $F_p \equiv 0 \pmod{2^n}$ and $F_{p+1} \equiv 1 \pmod{2^n}$, which means exactly that p is a period of $(F_0, F_1, \dots)$ modulo 2^n.

So, the sets of periods of (a_i) and (F_i) coincide, and hence the minimal periods coincide as well.

52nd International Mathematical Olympiad

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Problem Shortlist with Solutions

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Problem shortlist with solutions

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Algebra

A1

A1

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1 + a_2 + a_3 + a_4$, let p_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

A2

A2

Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$

A3

A3

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x + y)) = f(x) + (2x + y)g(y)$$

for all real numbers x and y.

A4

A4

Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n + 1) - g(n + 1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

A5

A5

Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n + 1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

A6

A6

Let f be a function from the set of real numbers to itself that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

A7

A7

Let a, b, and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}.$$

Combinatorics

C1

C1

Let $n > 0$ be an integer. We are given a balance and n weights of weight 2^0, $2^1, \dots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

C2

C2

Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

C3

C3

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. By a *windmill* we mean a process as follows. Start with a line ℓ going through a point $P \in \mathcal{S}$. Rotate ℓ clockwise around the *pivot* P until the line contains another point Q of $\mathcal{S}$. The point Q now takes over as the new pivot. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

Show that for a suitable $P \in \mathcal{S}$ and a suitable starting line ℓ containing P, the resulting windmill will visit each point of $\mathcal{S}$ as a pivot infinitely often.

C4

C4

Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \dots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \dots, k\}$ there exist two distinct elements of A_i whose sum is n.

C5

C5

Let m be a positive integer and consider a checkerboard consisting of m by m unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in opposite directions meet, they both turn $90°$ clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard or prove that such a moment does not necessarily exist.

C6

C6

Let n be a positive integer and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word consisting of the letters a and b. Suppose that the minimal period N of W is greater than 2^n.

A finite nonempty word U is said to *appear* in W if there exist indices $k \leq \ell$ such that $U = x_k x_{k+1} \ldots x_\ell$. A finite word U is called *ubiquitous* if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

C7

C7

On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?

Geometry

G1

G1

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to AB at B' and to AC at C'. Suppose also that the circumcenter O of the triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

G2

G2

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

G3

G3

Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC, and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA, and AB. Prove that the midpoint of the segment EF lies on the line through the two intersection points of ω_E and ω_F.

G4

G4

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G, and X are collinear.

G5

G5

Let ABC be a triangle with incenter I and circumcircle ω. Let D and E be the second intersection points of ω with the lines AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and at B meet at a point K. Prove that the three lines AE, BD, and KP are either parallel or concurrent.

G6

G6

Let ABC be a triangle with $AB = AC$, and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C in a point E inside the triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

G7

G7

Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

G8

G8

Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to ω. Let t_a, t_b, and t_c be the lines obtained by reflecting t in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

Number Theory

N1

N1

For any integer $d > 0$, let $f(d)$ be the smallest positive integer that has exactly d positive divisors (so for example we have $f(1) = 1$, $f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f(2^k)$ divides $f(2^{k+1})$.

N2

N2

Consider a polynomial $P(x) = (x + d_1)(x + d_2) \cdot \ldots \cdot (x + d_9)$, where $d_1, d_2, \ldots, d_9$ are nine distinct integers. Prove that there exists an integer N such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

N3

N3

Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

N4

N4

For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n such that all the differences

$$t(n+a) - t(n), \quad t(n+a+1) - t(n+1), \quad \ldots, \quad t(n+2a-1) - t(n+a-1)$$

are divisible by 4.

N5

N5

Let f be a function from the set of integers to the set of positive integers. Suppose that for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that for all integers m, n with $f(m) \leq f(n)$ the number $f(n)$ is divisible by $f(m)$.

N6

N6

Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)} - 1$ divides $3^{P(n)} - 1$. Prove that $Q(x)$ is a constant polynomial.

N7

N7

Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Prove that p divides m.

N8

N8

Let k be a positive integer and set $n = 2^k + 1$. Prove that n is a prime number if and only if the following holds: there is a permutation $a_1, \ldots, a_{n-1}$ of the numbers $1, 2, \ldots, n-1$ and a sequence of integers $g_1, g_2, \ldots, g_{n-1}$ such that n divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$, where we set $a_n = a_1$.

A1

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1+a_2+a_3+a_4$, let p_A denote the number of pairs (i, j) with $1 \le i < j \le 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

Answer. The sets A for which p_A is maximal are the sets the form $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer. For all these sets p_A is 4.

Solution. Firstly, we will prove that the maximum value of p_A is at most 4. Without loss of generality, we may assume that $a_1 < a_2 < a_3 < a_4$. We observe that for each pair of indices (i, j) with $1 \le i < j \le 4$, the sum $a_i + a_j$ divides s_A if and only if $a_i + a_j$ divides $s_A - (a_i + a_j) = a_k + a_l$, where k and l are the other two indices. Since there are 6 distinct pairs, we have to prove that at least two of them do not satisfy the previous condition. We claim that two such pairs are (a_2, a_4) and (a_3, a_4). Indeed, note that $a_2 + a_4 > a_1 + a_3$ and $a_3 + a_4 > a_1 + a_2$. Hence $a_2 + a_4$ and $a_3 + a_4$ do not divide s_A. This proves $p_A \le 4$.

Now suppose $p_A = 4$. By the previous argument we have

$$\begin{array}{lcl} a_1 + a_4 \mid a_2 + a_3 & \text{and} & a_2 + a_3 \mid a_1 + a_4, \\ a_1 + a_2 \mid a_3 + a_4 & \text{and} & a_3 + a_4 \nmid a_1 + a_2, \\ a_1 + a_3 \mid a_2 + a_4 & \text{and} & a_2 + a_4 \nmid a_1 + a_3. \end{array}$$

Hence, there exist positive integers m and n with $m > n \ge 2$ such that

$$\begin{cases} a_1 + a_4 = a_2 + a_3 \\ m(a_1 + a_2) = a_3 + a_4 \\ n(a_1 + a_3) = a_2 + a_4. \end{cases}$$

Adding up the first equation and the third one, we get $n(a_1 + a_3) = 2a_2 + a_3 - a_1$. If $n \ge 3$, then $n(a_1 + a_3) > 3a_3 > 2a_2 + a_3 > 2a_2 + a_3 - a_1$. This is a contradiction. Therefore $n = 2$. If we multiply by 2 the sum of the first equation and the third one, we obtain

$$6a_1 + 2a_3 = 4a_2,$$

while the sum of the first one and the second one is

$$(m + 1)a_1 + (m - 1)a_2 = 2a_3.$$

Adding up the last two equations we get

$$(m + 7)a_1 = (5 - m)a_2.$$

It follows that $5 - m \geq 1$, because the left-hand side of the last equation and a_2 are positive. Since we have $m > n = 2$, the integer m can be equal only to either 3 or 4. Substituting $(3, 2)$ and $(4, 2)$ for (m, n) and solving the previous system of equations, we find the families of solutions $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer.

A2

Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$

Answer. The only sequence that satisfies the condition is

$$(x_1, \ldots, x_{2011}) = (1, k, \ldots, k) \quad \text{with } k = 2 + 3 + \cdots + 2011 = 2023065.$$

Solution. Throughout this solution, the set of positive integers will be denoted by $\mathbb{Z}_+$.

Put $k = 2 + 3 + \cdots + 2011 = 2023065$. We have

$$1^n + 2k^n + \cdots 2011k^n = 1 + k \cdot k^n = k^{n+1} + 1$$

for all n, so $(1, k, \ldots, k)$ is a valid sequence. We shall prove that it is the only one.

Let a valid sequence $(x_1, \ldots, x_{2011})$ be given. For each $n \in \mathbb{Z}_+$ we have some $y_n \in \mathbb{Z}_+$ with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = y_n^{n+1} + 1.$$

Note that $x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n < (x_1 + 2x_2 + \cdots + 2011x_{2011})^{n+1}$, which implies that the sequence (y_n) is bounded. In particular, there is some $y \in \mathbb{Z}_+$ with $y_n = y$ for infinitely many n.

Let m be the maximum of all the x_i. Grouping terms with equal x_i together, the sum $x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n$ can be written as

$$x_1^n + 2x_2^n + \cdots + x_{2011}^n = a_m m^n + a_{m-1}(m-1)^n + \cdots + a_1$$

with $a_i \geq 0$ for all i and $a_1 + \cdots + a_m = 1 + 2 + \cdots + 2011$. So there exist arbitrarily large values of n, for which

$$a_m m^n + \cdots + a_1 - 1 - y \cdot y^n = 0. \tag{1}$$

The following lemma will help us to determine the a_i and y:

Lemma. Let integers $b_1, \ldots, b_N$ be given and assume that there are arbitrarily large positive integers n with $b_1 + b_2 2^n + \cdots + b_N N^n = 0$. Then $b_i = 0$ for all i.

Proof. Suppose that not all b_i are zero. We may assume without loss of generality that $b_N \neq 0$.

Dividing through by N^n gives

$$|b_N| = \left| b_{N-1}\left(\frac{N-1}{N}\right)^n + \cdots + b_1\left(\frac{1}{N}\right)^n \right| \leq \left(|b_{N-1}| + \cdots + |b_1|\right)\left(\frac{N-1}{N}\right)^n .$$

The expression $\left(\frac{N-1}{N}\right)^n$ can be made arbitrarily small for n large enough, contradicting the assumption that b_N be non-zero. $\square$

We obviously have $y > 1$. Applying the lemma to (1) we see that $a_m = y = m$, $a_1 = 1$, and all the other a_i are zero. This implies $(x_1, \ldots, x_{2011}) = (1, m, \ldots, m)$. But we also have $1 + m = a_1 + \cdots + a_m = 1 + \cdots + 2011 = 1 + k$ so $m = k$, which is what we wanted to show.

A3

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

Answer. Either both f and g vanish identically, or there exists a real number C such that $f(x) = x^2 + C$ and $g(x) = x$ for all real numbers x.

Solution. Clearly all these pairs of functions satisfy the functional equation in question, so it suffices to verify that there cannot be any further ones. Substituting $-2x$ for y in the given functional equation we obtain

$$g(f(-x)) = f(x). \tag{1}$$

Using this equation for $-x-y$ in place of x we obtain

$$f(-x-y) = g(f(x+y)) = f(x) + (2x+y)g(y). \tag{2}$$

Now for any two real numbers a and b, setting $x = -b$ and $y = a+b$ we get

$$f(-a) = f(-b) + (a-b)g(a+b).$$

If c denotes another arbitrary real number we have similarly

$$f(-b) = f(-c) + (b-c)g(b+c)$$

as well as

$$f(-c) = f(-a) + (c-a)g(c+a).$$

Adding all these equations up, we obtain

$$\big((a+c)-(b+c)\big)g(a+b) + \big((a+b)-(a+c)\big)g(b+c) + \big((b+c)-(a+b)\big)g(a+c) = 0.$$

Now given any three real numbers x, y, and z one may determine three reals a, b, and c such that $x = b+c$, $y = c+a$, and $z = a+b$, so that we get

$$(y-x)g(z) + (z-y)g(x) + (x-z)g(y) = 0.$$

This implies that the three points $(x, g(x))$, $(y, g(y))$, and $(z, g(z))$ from the graph of g are collinear. Hence that graph is a line, i.e., g is either a constant or a linear function.

Let us write $g(x) = Ax + B$, where A and B are two real numbers. Substituting $(0, -y)$ for (x, y) in (2) and denoting $C = f(0)$, we have $f(y) = Ay^2 - By + C$. Now, comparing the coefficients of x^2 in (1) we see that $A^2 = A$, so $A = 0$ or $A = 1$.

If $A = 0$, then (1) becomes $B = -Bx + C$ and thus $B = C = 0$, which provides the first of the two solutions mentioned above.

Now suppose $A = 1$. Then (1) becomes $x^2 - Bx + C + B = x^2 - Bx + C$, so $B = 0$. Thus, $g(x) = x$ and $f(x) = x^2 + C$, which is the second solution from above.

Comment. Another way to show that $g(x)$ is either a constant or a linear function is the following. If we interchange x and y in the given functional equation and subtract this new equation from the given one, we obtain

$$f(x) - f(y) = (2y + x)g(x) - (2x + y)g(y).$$

Substituting $(x, 0)$, $(1, x)$, and $(0, 1)$ for (x, y), we get

$$\begin{aligned}
f(x) - f(0) &= xg(x) - 2xg(0), \\
f(1) - f(x) &= (2x + 1)g(1) - (x + 2)g(x), \\
f(0) - f(1) &= 2g(0) - g(1).
\end{aligned}$$

Taking the sum of these three equations and dividing by 2, we obtain

$$g(x) = x\big(g(1) - g(0)\big) + g(0).$$

This proves that $g(x)$ is either a constant of a linear function.

A4

Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

Answer. The only pair (f, g) of functions that satisfies the equation is given by $f(n) = n$ and $g(n) = 1$ for all n.

Solution. The given relation implies

$$f\left(f^{g(n)}(n)\right) < f(n+1) \quad \text{for all } n, \tag{1}$$

which will turn out to be sufficient to determine f.

Let $y_1 < y_2 < \ldots$ be all the values attained by f (this sequence might be either finite or infinite). We will prove that for every positive n the function f attains at least n values, and we have $(\mathrm{i})_n$: $f(x) = y_n$ if and only if $x = n$, and $(\mathrm{ii})_n$: $y_n = n$. The proof will follow the scheme

$$(\mathrm{i})_1,\ (\mathrm{ii})_1,\ (\mathrm{i})_2,\ (\mathrm{ii})_2,\ \ldots,\ (\mathrm{i})_n,\ (\mathrm{ii})_n,\ \ldots \tag{2}$$

To start, consider any x such that $f(x) = y_1$. If $x > 1$, then (1) reads $f\big(f^{g(x-1)}(x-1)\big) < y_1$, contradicting the minimality of y_1. So we have that $f(x) = y_1$ is equivalent to $x = 1$, establishing $(\mathrm{i})_1$.

Next, assume that for some n statement $(\mathrm{i})_n$ is established, as well as all the previous statements in (2). Note that these statements imply that for all $k \geq 1$ and $a < n$ we have $f^k(x) = a$ if and only if $x = a$.

Now, each value y_i with $1 \leq i \leq n$ is attained at the unique integer i, so y_{n+1} exists. Choose an arbitrary x such that $f(x) = y_{n+1}$; we necessarily have $x > n$. Substituting $x - 1$ into (1) we have $f\big(f^{g(x-1)}(x-1)\big) < y_{n+1}$, which implies

$$f^{g(x-1)}(x-1) \in \{1, \ldots, n\} \tag{3}$$

Set $b = f^{g(x-1)}(x-1)$. If $b < n$ then we would have $x - 1 = b$ which contradicts $x > n$. So $b = n$, and hence $y_n = n$, which proves $(\mathrm{ii})_n$. Next, from $(\mathrm{i})_n$ we now get $f(k) = n \iff k = n$, so removing all the iterations of f in (3) we obtain $x - 1 = b = n$, which proves $(\mathrm{i})_{n+1}$.

So, all the statements in (2) are valid and hence $f(n) = n$ for all n. The given relation between f and g now reads $n + g^n(n) = n + 1 - g(n+1) + 1$ or $g^n(n) + g(n+1) = 2$, from which it immediately follows that we have $g(n) = 1$ for all n.

Comment. Several variations of the above solution are possible. For instance, one may first prove by induction that the smallest n values of f are exactly $f(1) < \cdots < f(n)$ and proceed as follows. We certainly have $f(n) \geq n$ for all n. If there is an n with $f(n) > n$, then $f(x) > x$ for all $x \geq n$. From this we conclude $f^{g(n)+1}(n) > f^{g(n)}(n) > \cdots > f(n)$. But we also have $f^{g(n)+1} < f(n+1)$. Having squeezed in a function value between $f(n)$ and $f(n+1)$, we arrive at a contradiction.

In any case, the inequality (1) plays an essential rôle.

A5

Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n + 1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

Solution. Throughout the solution, we denote by $[a, b]$ the set $\{a, a + 1, \ldots, b\}$. We say that $\{a, b, c\}$ is an *obtuse triple* if a, b, c are the sides of some obtuse triangle.

We prove by induction on n that there exists a partition of $[2, 3n + 1]$ into n obtuse triples A_i $(2 \le i \le n + 1)$ having the form $A_i = \{i, a_i, b_i\}$. For the base case $n = 1$, one can simply set $A_2 = \{2, 3, 4\}$. For the induction step, we need the following simple lemma.

Lemma. Suppose that the numbers $a < b < c$ form an obtuse triple, and let x be any positive number. Then the triple $\{a, b + x, c + x\}$ is also obtuse.

Proof. The numbers $a < b + x < c + x$ are the sides of a triangle because $(c + x) - (b + x) = c - b < a$. This triangle is obtuse since $(c+x)^2 - (b+x)^2 = (c-b)(c+b+2x) > (c-b)(c+b) > a^2$. $\square$

Now we turn to the induction step. Let $n > 1$ and put $t = \lfloor n/2 \rfloor < n$. By the induction hypothesis, there exists a partition of the set $[2, 3t + 1]$ into t obtuse triples $A'_i = \{i, a'_i, b'_i\}$ $(i \in [2, t + 1])$. For the same values of i, define $A_i = \{i, a'_i + (n - t), b'_i + (n - t)\}$. The constructed triples are obviously disjoint, and they are obtuse by the lemma. Moreover, we have

$$\bigcup_{i=2}^{t+1} A_i = [2, t + 1] \cup [n + 2, n + 2t + 1].$$

Next, for each $i \in [t + 2, n + 1]$, define $A_i = \{i, n + t + i, 2n + i\}$. All these sets are disjoint, and

$$\bigcup_{i=t+2}^{n+1} A_i = [t + 2, n + 1] \cup [n + 2t + 2, 2n + t + 1] \cup [2n + t + 2, 3n + 1],$$

so

$$\bigcup_{i=2}^{n+1} A_i = [2, 3n + 1].$$

Thus, we are left to prove that the triple A_i is obtuse for each $i \in [t + 2, n + 1]$.

Since $(2n + i) - (n + t + i) = n - t < t + 2 \le i$, the elements of A_i are the sides of a triangle. Next, we have

$$(2n + i)^2 - (n + t + i)^2 = (n - t)(3n + t + 2i) \ge \frac{n}{2} \cdot (3n + 3(t + 1) + 1) > \frac{n}{2} \cdot \frac{9n}{2} \ge (n + 1)^2 \ge i^2,$$

so this triangle is obtuse. The proof is completed.

A6

Let f be a function from the set of real numbers to itself that satisfies

$$f(x+y) \leq yf(x) + f(f(x)) \tag{1}$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Solution 1. Substituting $y = t - x$, we rewrite (1) as

$$f(t) \leq tf(x) - xf(x) + f(f(x)). \tag{2}$$

Consider now some real numbers a, b and use (2) with $t = f(a)$, $x = b$ as well as with $t = f(b)$, $x = a$. We get

$$f(f(a)) - f(f(b)) \leq f(a)f(b) - bf(b),$$
$$f(f(b)) - f(f(a)) \leq f(a)f(b) - af(a).$$

Adding these two inequalities yields

$$2f(a)f(b) \geq af(a) + bf(b).$$

Now, substitute $b = 2f(a)$ to obtain $2f(a)f(b) \geq af(a) + 2f(a)f(b)$, or $af(a) \leq 0$. So, we get

$$f(a) \geq 0 \quad \text{for all } a < 0. \tag{3}$$

Now suppose $f(x) > 0$ for some real number x. From (2) we immediately get that for every $t < \dfrac{xf(x) - f(f(x))}{f(x)}$ we have $f(t) < 0$. This contradicts (3); therefore

$$f(x) \leq 0 \quad \text{for all real } x, \tag{4}$$

and by (3) again we get $f(x) = 0$ for all $x < 0$.

We are left to find $f(0)$. Setting $t = x < 0$ in (2) we get

$$0 \leq 0 - 0 + f(0),$$

so $f(0) \geq 0$. Combining this with (4) we obtain $f(0) = 0$.

Solution 2. We will also use the condition of the problem in form (2). For clarity we divide the argument into four steps.

Step 1. We begin by proving that f attains nonpositive values only. Assume that there exist some real number z with $f(z) > 0$. Substituting $x = z$ into (2) and setting $A = f(z)$, $B = -zf(z) - f(f(z))$ we get $f(t) \le At + B$ for all real t. Hence, if for any positive real number t we substitute $x = -t$, $y = t$ into (1), we get

$$\begin{aligned} f(0) &\le tf(-t) + f(f(-t)) \le t(-At + B) + Af(-t) + B \\ &\le -t(At - B) + A(-At + B) + B = -At^2 - (A^2 - B)t + (A+1)B. \end{aligned}$$

But surely this cannot be true if we take t to be large enough. This contradiction proves that we have indeed $f(x) \le 0$ for all real numbers x. Note that for this reason (1) entails

$$f(x+y) \le yf(x) \tag{5}$$

for all real numbers x and y.

Step 2. We proceed by proving that f has at least one zero. If $f(0) = 0$, we are done. Otherwise, in view of Step 1 we get $f(0) < 0$. Observe that (5) tells us now $f(y) \le yf(0)$ for all real numbers y. Thus we can specify a positive real number a that is so large that $f(a)^2 > -f(0)$. Put $b = f(a)$ and substitute $x = b$ and $y = -b$ into (5); we learn $-b^2 < f(0) \le -bf(b)$, i.e. $b < f(b)$. Now we apply (2) to $x = b$ and $t = f(b)$, which yields

$$f(f(b)) \le \big(f(b) - b\big)f(b) + f(f(b)),$$

i.e. $f(b) \ge 0$. So in view of Step 1, b is a zero of f.

Step 3. Next we show that if $f(a) = 0$ and $b < a$, then $f(b) = 0$ as well. To see this, we just substitute $x = b$ and $y = a - b$ into (5), thus getting $f(b) \ge 0$, which suffices by Step 1.

Step 4. By Step 3, the solution of the problem is reduced to showing $f(0) = 0$. Pick any zero r of f and substitute $x = r$ and $y = -1$ into (1). Because of $f(r) = f(r-1) = 0$ this gives $f(0) \ge 0$ and hence $f(0) = 0$ by Step 1 again.

Comment 1. Both of these solutions also show $f(x) \le 0$ for all real numbers x. As one can see from Solution 1, this task gets much easier if one already knows that f takes nonnegative values for sufficiently small arguments. Another way of arriving at this statement, suggested by the proposer, is as follows:

Put $a = f(0)$ and substitute $x = 0$ into (1). This gives $f(y) \le ay + f(a)$ for all real numbers y. Thus if for any real number x we plug $y = a - x$ into (1), we obtain

$$f(a) \le (a-x)f(x) + f(f(x)) \le (a-x)f(x) + af(x) + f(a)$$

and hence $0 \le (2a - x)f(x)$. In particular, if $x < 2a$, then $f(x) \ge 0$.

Having reached this point, one may proceed almost exactly as in the first solution to deduce $f(x) \le 0$ for all x. Afterwards the problem can be solved in a few lines as shown in steps 3 and 4 of the second solution.

Comment 2. The original problem also contained the question whether a nonzero function satisfying the problem condition exists. Here we present a family of such functions.

Notice first that if $g : (0, \infty) \longrightarrow [0, \infty)$ denotes any function such that

$$g(x + y) \geq yg(x) \tag{6}$$

for all positive real numbers x and y, then the function f given by

$$f(x) = \begin{cases} -g(x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{7}$$

automatically satisfies (1). Indeed, we have $f(x) \leq 0$ and hence also $f(f(x)) = 0$ for all real numbers x. So (1) reduces to (5); moreover, this inequality is nontrivial only if x and y are positive. In this last case it is provided by (6).

Now it is not hard to come up with a nonzero function g obeying (6). E.g. $g(z) = Ce^z$ (where C is a positive constant) fits since the inequality $e^y > y$ holds for all (positive) real numbers y. One may also consider the function $g(z) = e^z - 1$; in this case, we even have that f is continuous.

A7

Let a, b, and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}. \tag{1}$$

Throughout both solutions, we denote the sums of the form $f(a,b,c) + f(b,c,a) + f(c,a,b)$ by $\sum f(a,b,c)$.

Solution 1. The condition $b+c > \sqrt{2}$ implies $b^2+c^2 > 1$, so $a^2 = 3 - (b^2+c^2) < 2$, i.e. $a < \sqrt{2} < b+c$. Hence we have $b+c-a > 0$, and also $c+a-b > 0$ and $a+b-c > 0$ for similar reasons.

We will use the variant of Hölder's inequality

$$\frac{x_1^{p+1}}{y_1^p} + \frac{x_1^{p+1}}{y_1^p} + \ldots + \frac{x_n^{p+1}}{y_n^p} \geq \frac{(x_1+x_2+\ldots+x_n)^{p+1}}{(y_1+y_2+\ldots+y_n)^p},$$

which holds for all positive real numbers $p, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$. Applying it to the left-hand side of (1) with $p=2$ and $n=3$, we get

$$\sum \frac{a}{(b+c-a)^2} = \sum \frac{(a^2)^3}{a^5(b+c-a)^2} \geq \frac{(a^2+b^2+c^2)^3}{\left(\sum a^{5/2}(b+c-a)\right)^2} = \frac{27}{\left(\sum a^{5/2}(b+c-a)\right)^2}. \tag{2}$$

To estimate the denominator of the right-hand part, we use an instance of Schur's inequality, namely

$$\sum a^{3/2}(a-b)(a-c) \geq 0,$$

which can be rewritten as

$$\sum a^{5/2}(b+c-a) \leq abc(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

Moreover, by the inequality between the arithmetic mean and the fourth power mean we also have

$$\left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3}\right)^4 \leq \frac{a^2+b^2+c^2}{3} = 1,$$

i.e., $\sqrt{a}+\sqrt{b}+\sqrt{c} \leq 3$. Hence, (2) yields

$$\sum \frac{a}{(b+c-a)^2} \geq \frac{27}{\left(abc(\sqrt{a}+\sqrt{b}+\sqrt{c})\right)^2} \geq \frac{3}{a^2b^2c^2},$$

thus solving the problem.

Comment. In this solution, one may also start from the following version of HÖLDER's inequality

$$\left(\sum_{i=1}^{n} a_i^3\right)\left(\sum_{i=1}^{n} b_i^3\right)\left(\sum_{i=1}^{n} c_i^3\right) \geq \left(\sum_{i=1}^{n} a_i b_i c_i\right)^3$$

applied as

$$\sum \frac{a}{(b+c-a)^2} \cdot \sum a^3(b+c-a) \cdot \sum a^2(b+c-a) \geq 27.$$

After doing that, one only needs the slightly better known instances

$$\sum a^3(b+c-a) \leq (a+b+c)abc \qquad \text{and} \qquad \sum a^2(b+c-a) \leq 3abc$$

of SCHUR's Inequality.

Solution 2. As in Solution 1, we mention that all the numbers $b+c-a$, $a+c-b$, $a+b-c$ are positive. We will use only this restriction and the condition

$$a^5 + b^5 + c^5 \geq 3, \tag{3}$$

which is weaker than the given one. Due to the symmetry we may assume that $a \geq b \geq c$.

In view of (3), it suffices to prove the inequality

$$\sum \frac{a^3 b^2 c^2}{(b+c-a)^2} \geq \sum a^5,$$

or, moving all the terms into the left-hand part,

$$\sum \frac{a^3}{(b+c-a)^2}\left((bc)^2 - (a(b+c-a))^2\right) \geq 0. \tag{4}$$

Note that the signs of the expressions $(yz)^2 - (x(y+z-x))^2$ and $yz - x(y+z-x) = (x-y)(x-z)$ are the same for every positive x, y, z satisfying the triangle inequality. So the terms in (4) corresponding to a and c are nonnegative, and hence it is sufficient to prove that the sum of the terms corresponding to a and b is nonnegative. Equivalently, we need the relation

$$\frac{a^3}{(b+c-a)^2}(a-b)(a-c)(bc+a(b+c-a)) \geq \frac{b^3}{(a+c-b)^2}(a-b)(b-c)(ac+b(a+c-b)).$$

Obviously, we have

$$a^3 \geq b^3 \geq 0, \quad 0 < b+c-a \leq a+c-b, \quad \text{and} \quad a-c \geq b-c \geq 0,$$

hence it suffices to prove that

$$\frac{ab+ac+bc-a^2}{b+c-a} \geq \frac{ab+ac+bc-b^2}{c+a-b}.$$

Since all the denominators are positive, it is equivalent to

$$(c+a-b)(ab+ac+bc-a^2)-(ab+ac+bc-b^2)(b+c-a)\geq 0,$$

or

$$(a-b)(2ab-a^2-b^2+ac+bc)\geq 0.$$

Since $a\geq b$, the last inequality follows from

$$c(a+b)>(a-b)^2$$

which holds since $c>a-b\geq 0$ and $a+b>a-b\geq 0$.

C1

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

Answer. The number $f(n)$ of ways of placing the n weights is equal to the product of all odd positive integers less than or equal to $2n-1$, i.e. $f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$.

Solution 1. Assume $n \geq 2$. We claim

$$f(n) = (2n-1)f(n-1). \tag{1}$$

Firstly, note that after the first move the left pan is always at least 1 heavier than the right one. Hence, any valid way of placing the n weights on the scale gives rise, by not considering weight 1, to a valid way of placing the weights $2, 2^2, \ldots, 2^{n-1}$.

If we divide the weight of each weight by 2, the answer does not change. So these $n-1$ weights can be placed on the scale in $f(n-1)$ valid ways. Now we look at weight 1. If it is put on the scale in the first move, then it has to be placed on the left side, otherwise it can be placed either on the left or on the right side, because after the first move the difference between the weights on the left pan and the weights on the right pan is at least 2. Hence, there are exactly $2n-1$ different ways of inserting weight 1 in each of the $f(n-1)$ valid sequences for the $n-1$ weights in order to get a valid sequence for the n weights. This proves the claim.

Since $f(1) = 1$, by induction we obtain for all positive integers n

$$f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1).$$

Comment 1. The word "compute" in the statement of the problem is probably too vague. An alternative but more artificial question might ask for the smallest n for which the number of valid ways is divisible by 2011. In this case the answer would be 1006.

Comment 2. It is useful to remark that the answer is the same for any set of weights where each weight is heavier than the sum of the lighter ones. Indeed, in such cases the given condition is equivalent to asking that during the process the heaviest weight on the balance is always on the left pan.

Comment 3. Instead of considering the lightest weight, one may also consider the last weight put on the balance. If this weight is 2^{n-1} then it should be put on the left pan. Otherwise it may be put on any pan; the inequality would not be violated since at this moment the heaviest weight is already put onto the left pan. In view of the previous comment, in each of these $2n-1$ cases the number of ways to place the previous weights is exactly $f(n-1)$, which yields (1).

Solution 2. We present a different way of obtaining (1). Set $f(0)=1$. Firstly, we find a recurrent formula for $f(n)$.

Assume $n \geq 1$. Suppose that weight 2^{n-1} is placed on the balance in the i-th move with $1 \leq i \leq n$. This weight has to be put on the left pan. For the previous moves we have $\binom{n-1}{i-1}$ choices of the weights and from Comment 2 there are $f(i-1)$ valid ways of placing them on the balance. For later moves there is no restriction on the way in which the weights are to be put on the pans. Therefore, all $(n-i)!2^{n-i}$ ways are possible. This gives

$$f(n)=\sum_{i=1}^{n}\binom{n-1}{i-1}f(i-1)(n-i)!2^{n-i}=\sum_{i=1}^{n}\frac{(n-1)!f(i-1)2^{n-i}}{(i-1)!}. \tag{2}$$

Now we are ready to prove (1). Using $n-1$ instead of n in (2) we get

$$f(n-1)=\sum_{i=1}^{n-1}\frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!}.$$

Hence, again from (2) we get

$$\begin{aligned}f(n)&=2(n-1)\sum_{i=1}^{n-1}\frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!}+f(n-1)\\&=(2n-2)f(n-1)+f(n-1)=(2n-1)f(n-1),\end{aligned}$$

QED.

Comment. There exist different ways of obtaining the formula (2). Here we show one of them.

Suppose that in the first move we use weight 2^{n-i+1}. Then the lighter $n-i$ weights may be put on the balance at any moment and on either pan. This gives $2^{n-i}\cdot(n-1)!/(i-1)!$ choices for the moves (moments and choices of pan) with the lighter weights. The remaining $i-1$ moves give a valid sequence for the $i-1$ heavier weights and this is the only requirement for these moves, so there are $f(i-1)$ such sequences. Summing over all $i=1,2,\ldots,n$ we again come to (2).

C2

Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \le k \le 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

Solution. Number the students consecutively from 1 to 1000. Let $a_i = 1$ if the ith student is a girl, and $a_i = 0$ otherwise. We expand this notion for all integers i by setting $a_{i+1000} = a_{i-1000} = a_i$. Next, let

$$S_k(i) = a_i + a_{i+1} + \cdots + a_{i+k-1}.$$

Now the statement of the problem can be reformulated as follows:

There exist an integer k with $100 \le k \le 300$ and an index i such that $S_k(i) = S_k(i+k)$.

Assume now that this statement is false. Choose an index i such that $S_{100}(i)$ attains the maximal possible value. In particular, we have $S_{100}(i-100) - S_{100}(i) < 0$ and $S_{100}(i) - S_{100}(i+100) > 0$, for if we had an equality, then the statement would hold. This means that the function $S(j) - S(j+100)$ changes sign somewhere on the segment $[i-100, i]$, so there exists some index $j \in [i-100, i-1]$ such that

$$S_{100}(j) \le S_{100}(j+100) - 1, \quad \text{but} \quad S_{100}(j+1) \ge S_{100}(j+101) + 1. \tag{1}$$

Subtracting the first inequality from the second one, we get $a_{j+100} - a_j \ge a_{j+200} - a_{j+100} + 2$, so

$$a_j = 0, \quad a_{j+100} = 1, \quad a_{j+200} = 0.$$

Substituting this into the inequalities of (1), we also obtain $S_{99}(j+1) \le S_{99}(j+101) \le S_{99}(j+1)$, which implies

$$S_{99}(j+1) = S_{99}(j+101). \tag{2}$$

Now let k and ℓ be the least positive integers such that $a_{j-k} = 1$ and $a_{j+200+\ell} = 1$. By symmetry, we may assume that $k \ge \ell$. If $k \ge 200$ then we have $a_j = a_{j-1} = \cdots = a_{j-199} = 0$, so $S_{100}(j-199) = S_{100}(j-99) = 0$, which contradicts the initial assumption. Hence $\ell \le k \le 199$.

Finally, we have

$$S_{100+\ell}(j-\ell+1) = (a_{j-\ell+1} + \cdots + a_j) + S_{99}(j+1) + a_{j+100} = S_{99}(j+1) + 1,$$
$$S_{100+\ell}(j+101) = S_{99}(j+101) + (a_{j+200} + \cdots + a_{j+200+\ell-1}) + a_{j+200+\ell} = S_{99}(j+101) + 1.$$

Comparing with (2) we get $S_{100+\ell}(j-\ell+1) = S_{100+\ell}(j+101)$ and $100 + \ell \le 299$, which again contradicts our assumption.

Comment. It may be seen from the solution that the number 300 from the problem statement can be replaced by 299. Here we consider some improvements of this result. Namely, we investigate which interval can be put instead of $[100, 300]$ in order to keep the problem statement valid.

First of all, the two examples

$$\underbrace{1,1,\ldots,1}_{167},\underbrace{0,0,\ldots,0}_{167},\underbrace{1,1,\ldots,1}_{167},\underbrace{0,0,\ldots,0}_{167},\underbrace{1,1,\ldots,1}_{167},\underbrace{0,0,\ldots,0}_{165}$$

and

$$\underbrace{1,1,\ldots,1}_{249},\underbrace{0,0,\ldots,0}_{251},\underbrace{1,1,\ldots,1}_{249},\underbrace{0,0,\ldots,0}_{251}$$

show that the interval can be changed neither to $[84, 248]$ nor to $[126, 374]$.

On the other hand, we claim that this interval can be changed to $[125, 250]$. Note that this statement is invariant under replacing all 1's by 0's and vice versa. Assume, to the contrary, that there is no admissible $k \in [125, 250]$. The arguments from the solution easily yield the following lemma.

Lemma. Under our assumption, suppose that for some indices $i < j$ we have $S_{125}(i) \leq S_{125}(i+125)$ but $S_{125}(j) \geq S_{125}(j+125)$. Then there exists some $t \in [i, j-1]$ such that $a_t = a_{t-1} = \cdots = a_{t-125} = 0$ and $a_{t+250} = a_{t+251} = \cdots = a_{t+375} = 0$. □

Let us call a segment $[i, j]$ of indices a *crowd*, if (a) $a_i = a_{i+1} = \cdots = a_j$, but $a_{i-1} \neq a_i \neq a_{j+1}$, and (b) $j - i \geq 125$. Now, using the lemma, one can get in the same way as in the solution that there exists some crowd. Take all the crowds in the circle, and enumerate them in cyclic order as $A_1, \ldots, A_d$. We also assume always that $A_{s+d} = A_{s-d} = A_s$.

Consider one of the crowds, say A_1. We have $A_1 = [i, i+t]$ with $125 \leq t \leq 248$ (if $t \geq 249$, then $a_i = a_{i+1} = \cdots = a_{i+249}$ and therefore $S_{125}(i) = S_{125}(i+125)$, which contradicts our assumption). We may assume that $a_i = 1$. Then we have $S_{125}(i+t-249) \leq 125 = S_{125}(i+t-124)$ and $S_{125}(i) = 125 \geq S_{125}(i+125)$, so by the lemma there exists some index $j \in [i+t-249, i-1]$ such that the segments $[j-125, j]$ and $[j+250, j+375]$ are contained in some crowds.

Let us fix such j and denote the segment $[j+1, j+249]$ by B_1. Clearly, $A_1 \subseteq B_1$. Moreover, B_1 cannot contain any crowd other than A_1 since $|B_1| = 249 < 2 \cdot 126$. Hence it is clear that $j \in A_d$ and $j + 250 \in A_2$. In particular, this means that the genders of A_d and A_2 are different from that of A_1.

Performing this procedure for every crowd A_s, we find segments $B_s = [j_s + 1, j_s + 249]$ such that $|B_s| = 249$, $A_s \subseteq B_s$, and $j_s \in A_{s-1}$, $j_s + 250 \in A_{s+1}$. So, B_s covers the whole segment between A_{s-1} and A_{s+1}, hence the sets $B_1, \ldots, B_d$ cover some 1000 consecutive indices. This implies $249d \geq 1000$, and $d \geq 5$. Moreover, the gender of A_i is alternating, so d is even; therefore $d \geq 6$.

Consider now three segments $A_1 = [i_1, i'_1]$, $B_2 = [j_2 + 1, j_2 + 249]$, $A_3 = [i_3, i'_3]$. By construction, we have $[j_2 - 125, j_2] \subseteq A_1$ and $[j_2 + 250, j_2 + 375] \subseteq A_3$, whence $i_1 \leq j_2 - 125$, $i'_3 \geq j_2 + 375$. Therefore $i'_3 - i_1 \geq 500$. Analogously, if $A_4 = [i_4, i'_4]$, $A_6 = [i_6, i'_6]$ then $i'_6 - i_4 \geq 500$. But from $d \geq 6$ we get $i_1 < i'_3 < i_4 < i'_6 < i_1 + 1000$, so $1000 > (i'_3 - i_1) + (i'_6 - i_4) \geq 500 + 500$. This final contradiction shows that our claim holds.

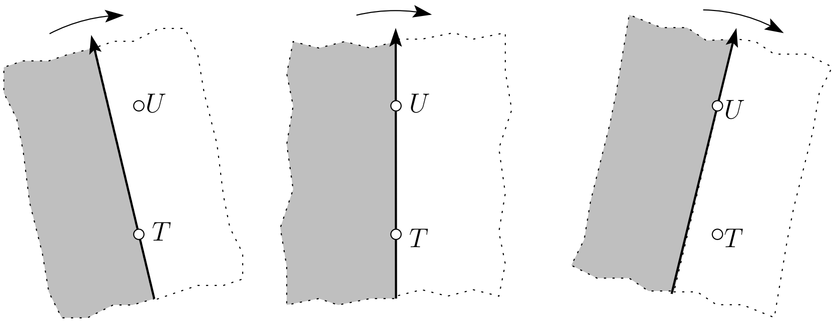
One may even show that the interval in the statement of the problem may be replaced by $[125, 249]$ (both these numbers cannot be improved due to the examples above). But a proof of this fact is a bit messy, and we do not present it here.

C3

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. By a *windmill* we mean a process as follows. Start with a line ℓ going through a point $P \in \mathcal{S}$. Rotate ℓ clockwise around the *pivot* P until the line contains another point Q of $\mathcal{S}$. The point Q now takes over as the new pivot. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

Show that for a suitable $P \in \mathcal{S}$ and a suitable starting line ℓ containing P, the resulting windmill will visit each point of $\mathcal{S}$ as a pivot infinitely often.

Solution. Give the rotating line an orientation and distinguish its sides as the *oranje side* and the *blue side*. Notice that whenever the pivot changes from some point T to another point U, after the change, T is on the same side as U was before. Therefore, the number of elements of $\mathcal{S}$ on the oranje side and the number of those on the blue side remain the same throughout the whole process (except for those moments when the line contains two points).



First consider the case that $|\mathcal{S}| = 2n+1$ is odd. We claim that through any point $T \in \mathcal{S}$, there is a line that has n points on each side. To see this, choose an oriented line through T containing no other point of $\mathcal{S}$ and suppose that it has $n+r$ points on its oranje side. If $r=0$ then we have established the claim, so we may assume that $r \neq 0$. As the line rotates through $180°$ around T, the number of points of $\mathcal{S}$ on its oranje side changes by 1 whenever the line passes through a point; after $180°$, the number of points on the oranje side is $n-r$. Therefore there is an intermediate stage at which the oranje side, and thus also the blue side, contains n points.

Now select the point P arbitrarily, and choose a line through P that has n points of $\mathcal{S}$ on each side to be the initial state of the windmill. We will show that during a rotation over $180°$, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and select a line ℓ through T that separates $\mathcal{S}$ into equal halves. The point T is the unique point of $\mathcal{S}$ through which a line in this direction can separate the points of $\mathcal{S}$ into equal halves (parallel translation would disturb the balance). Therefore, when the windmill line is parallel to ℓ, it must be ℓ itself, and so pass through T.

Next suppose that $|\mathcal{S}| = 2n$. Similarly to the odd case, for every $T \in \mathcal{S}$ there is an oriented line through T with $n-1$ points on its oranje side and n points on its blue side. Select such an oriented line through an arbitrary P to be the initial state of the windmill.

We will now show that during a rotation over $360°$, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and an oriented line ℓ through T that separates $\mathcal{S}$ into two subsets with $n-1$ points on its oranje and n points on its blue side. Again, parallel translation would change the numbers of points on the two sides, so when the windmill line is parallel to ℓ with the same orientation, the windmill line must pass through T.

Comment. One may shorten this solution in the following way.

Suppose that $|\mathcal{S}| = 2n+1$. Consider any line ℓ that separates $\mathcal{S}$ into equal halves; this line is unique given its direction and contains some point $T \in \mathcal{S}$. Consider the windmill starting from this line. When the line has made a rotation of $180°$, it returns to the same location but the oranje side becomes blue and vice versa. So, for each point there should have been a moment when it appeared as pivot, as this is the only way for a point to pass from on side to the other.

Now suppose that $|\mathcal{S}| = 2n$. Consider a line having $n-1$ and n points on the two sides; it contains some point T. Consider the windmill starting from this line. After having made a rotation of $180°$, the windmill line contains some different point R, and each point different from T and R has changed the color of its side. So, the windmill should have passed through all the points.

C4

Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

Answer. The greatest such number k is 3.

Solution 1. There are various examples showing that $k = 3$ does indeed have the property under consideration. E.g. one can take

$$\begin{aligned} A_1 &= \{1, 2, 3\} \cup \{3m \mid m \geq 4\}, \\ A_2 &= \{4, 5, 6\} \cup \{3m - 1 \mid m \geq 4\}, \\ A_3 &= \{7, 8, 9\} \cup \{3m - 2 \mid m \geq 4\}. \end{aligned}$$

To check that this partition fits, we notice first that the sums of two distinct elements of A_i obviously represent all numbers $n \geq 1 + 12 = 13$ for $i = 1$, all numbers $n \geq 4 + 11 = 15$ for $i = 2$, and all numbers $n \geq 7 + 10 = 17$ for $i = 3$. So, we are left to find representations of the numbers 15 and 16 as sums of two distinct elements of A_3. These are $15 = 7+8$ and $16 = 7+9$.

Let us now suppose that for some $k \geq 4$ there exist sets $A_1, A_2, \ldots, A_k$ satisfying the given property. Obviously, the sets $A_1, A_2, A_3, A_4 \cup \cdots \cup A_k$ also satisfy the same property, so one may assume $k = 4$.

Put $B_i = A_i \cap \{1, 2, \ldots, 23\}$ for $i = 1, 2, 3, 4$. Now for any index i each of the ten numbers $15, 16, \ldots, 24$ can be written as sum of two distinct elements of B_i. Therefore this set needs to contain at least five elements. As we also have $|B_1| + |B_2| + |B_3| + |B_4| = 23$, there has to be some index j for which $|B_j| = 5$. Let $B_j = \{x_1, x_2, x_3, x_4, x_5\}$. Finally, now the sums of two distinct elements of A_j representing the numbers $15, 16, \ldots, 24$ should be exactly all the pairwise sums of the elements of B_j. Calculating the sum of these numbers in two different ways, we reach

$$4(x_1 + x_2 + x_3 + x_4 + x_5) = 15 + 16 + \ldots + 24 = 195.$$

Thus the number 195 should be divisible by 4, which is false. This contradiction completes our solution.

Comment. There are several variation of the proof that k should not exceed 3. E.g., one may consider the sets $C_i = A_i \cap \{1, 2, \ldots, 19\}$ for $i = 1, 2, 3, 4$. As in the previous solution one can show that for some index j one has $|C_j| = 4$, and the six pairwise sums of the elements of C_j should represent all numbers $15, 16, \ldots, 20$. Let $C_j = \{y_1, y_2, y_3, y_4\}$ with $y_1 < y_2 < y_3 < y_4$. It is not hard to deduce

$C_j = \{7, 8, 9, 11\}$, so in particular we have $1 \notin C_j$. Hence it is impossible to represent 21 as sum of two distinct elements of A_j, which completes our argument.

Solution 2. Again we only prove that $k \leq 3$. Assume that $A_1, A_2, \ldots, A_k$ is a partition satisfying the given property. We construct a graph $\mathcal{G}$ on the set $V = \{1, 2, \ldots, 18\}$ of *vertices* as follows. For each $i \in \{1, 2, \ldots, k\}$ and each $d \in \{15, 16, 17, 19\}$ we choose one pair of distinct elements $a, b \in A_i$ with $a + b = d$, and we draw an *edge* in the i^{th} color connecting a with b. By hypothesis, $\mathcal{G}$ has exactly 4 edges of each color.

Claim. The graph $\mathcal{G}$ contains at most one circuit.

Proof. Note that all the connected components of $\mathcal{G}$ are monochromatic and hence contain at most four edges. Thus also all circuits of $\mathcal{G}$ are monochromatic and have length at most four. Moreover, each component contains at most one circuit since otherwise it should contain at least five edges.

Suppose that there is a 4-cycle in $\mathcal{G}$, say with vertices a, b, c, and d in order. Then $\{a + b, b + c, c + d, d + a\} = \{15, 16, 17, 19\}$. Taking sums we get $2(a + b + c + d) = 15 + 16 + 17 + 19$ which is impossible for parity reasons. Thus all circuits of $\mathcal{G}$ are triangles.

Now if the vertices a, b, and c form such a triangle, then by a similar reasoning the set $\{a+b, b+c, c+a\}$ coincides with either $\{15, 16, 17\}$, or $\{15, 16, 19\}$, or $\{16, 17, 19\}$, or $\{15, 17, 19\}$. The last of these alternatives can be excluded for parity reasons again, whilst in the first three cases the set $\{a, b, c\}$ appears to be either $\{7, 8, 9\}$, or $\{6, 9, 10\}$, or $\{7, 9, 10\}$, respectively. Thus, a component containing a circuit should contain 9 as a vertex. Therefore there is at most one such component and hence at most one circuit. □

By now we know that $\mathcal{G}$ is a graph with $4k$ edges, at least k components and at most one circuit. Consequently, $\mathcal{G}$ must have at least $4k + k - 1$ vertices. Thus $5k - 1 \leq 18$, and $k \leq 3$.

C5

Let m be a positive integer and consider a checkerboard consisting of m by m unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in opposite directions meet, they both turn 90° clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard or prove that such a moment does not necessarily exist.

Antswer. The latest possible moment for the last ant to fall off is $\frac{3m}{2} - 1$.

Solution. For $m = 1$ the answer is clearly correct, so assume $m > 1$. In the sequel, the word *collision* will be used to denote meeting of exactly two ants, moving in opposite directions.

If at the beginning we place an ant on the southwest corner square facing east and an ant on the southeast corner square facing west, then they will meet in the middle of the bottom row at time $\frac{m-1}{2}$. After the collision, the ant that moves to the north will stay on the board for another $m - \frac{1}{2}$ time units and thus we have established an example in which the last ant falls off at time $\frac{m-1}{2} + m - \frac{1}{2} = \frac{3m}{2} - 1$. So, we are left to prove that this is the latest possible moment.

Consider any collision of two ants a and a'. Let us change the rule for this collision, and enforce these two ants to turn anticlockwise. Then the succeeding behavior of all the ants does not change; the only difference is that a and a' swap their positions. These arguments may be applied to any collision separately, so we may assume that at any collision, either both ants rotate clockwise or both of them rotate anticlockwise by our own choice.

For instance, we may assume that there are only two types of ants, depending on their initial direction: *NE-ants*, which move only north or east, and *SW-ants*, moving only south and west. Then we immediately obtain that all ants will have fallen off the board after $2m - 1$ time units. However, we can get a better bound by considering the last moment at which a given ant collides with another ant.

Choose a coordinate system such that the corners of the checkerboard are $(0, 0)$, $(m, 0)$, (m, m) and $(0, m)$. At time t, there will be no NE-ants in the region $\{(x, y) : x + y < t + 1\}$ and no SW-ants in the region $\{(x, y) : x + y > 2m - t - 1\}$. So if two ants collide at (x, y) at time t, we have

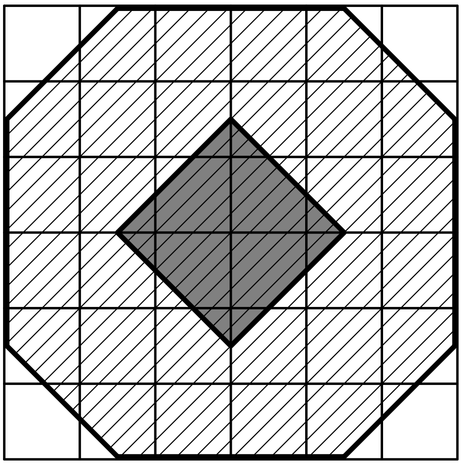
$$t + 1 \le x + y \le 2m - t - 1. \tag{1}$$

Analogously, we may change the rules so that each ant would move either alternatingly north and west, or alternatingly south and east. By doing so, we find that apart from (1) we also have $|x-y| \le m-t-1$ for each collision at point (x, y) and time t.

To visualize this, put

$$B(t) = \left\{(x,y) \in [0,m]^2 \colon t+1 \le x+y \le 2m-t-1 \text{ and } |x-y| \le m-t-1\right\}.$$

An ant can thus only collide with another ant at time t if it happens to be in the region $B(t)$. The following figure displays $B(t)$ for $t = \frac{1}{2}$ and $t = \frac{7}{2}$ in the case $m = 6$:



Now suppose that an NE-ant has its last collision at time t and that it does so at the point (x, y) (if the ant does not collide at all, it will fall off the board within $m-\frac{1}{2} < \frac{3m}{2}-1$ time units, so this case can be ignored). Then we have $(x,y) \in B(t)$ and thus $x+y \ge t+1$ and $x-y \ge -(m-t-1)$. So we get

$$x \ge \frac{(t+1)-(m-t-1)}{2} = t+1-\frac{m}{2}.$$

By symmetry we also have $y \ge t+1-\frac{m}{2}$, and hence $\min\{x,y\} \ge t+1-\frac{m}{2}$. After this collision, the ant will move directly to an edge, which will take at most $m - \min\{x,y\}$ units of time. In sum, the total amount of time the ant stays on the board is at most

$$t + (m - \min\{x,y\}) \le t + m - \left(t+1-\frac{m}{2}\right) = \frac{3m}{2} - 1.$$

By symmetry, the same bound holds for SW-ants as well.

C6

Let n be a positive integer and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word consisting of the letters a and b. Suppose that the minimal period N of W is greater than 2^n.

A finite nonempty word U is said to *appear* in W if there exist indices $k \le \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A finite word U is called *ubiquitous* if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

Solution. Throughout the solution, all the words are nonempty. For any word R of length m, we call the number of indices $i \in \{1, 2, \ldots, N\}$ for which R coincides with the subword $x_{i+1}x_{i+2}\ldots x_{i+m}$ of W the *multiplicity* of R and denote it by $\mu(R)$. Thus a word R appears in W if and only if $\mu(R) > 0$. Since each occurrence of a word in W is both succeeded by either the letter a or the letter b and similarly preceded by one of those two letters, we have

$$\mu(R) = \mu(Ra) + \mu(Rb) = \mu(aR) + \mu(bR) \tag{1}$$

for all words R.

We claim that the condition that N is in fact the minimal period of W guarantees that each word of length N has multiplicity 1 or 0 depending on whether it appears or not. Indeed, if the words $x_{i+1}x_{i+2}\ldots x_{i+N}$ and $x_{j+1}\ldots x_{j+N}$ are equal for some $1 \le i < j \le N$, then we have $x_{i+a} = x_{j+a}$ for every integer a, and hence $j - i$ is also a period.

Moreover, since $N > 2^n$, at least one of the two words a and b has a multiplicity that is strictly larger than 2^{n-1}.

For each $k = 0, 1, \ldots, n-1$, let U_k be a subword of W whose multiplicity is strictly larger than 2^k and whose length is maximal subject to this property. Note that such a word exists in view of the two observations made in the two previous paragraphs.

Fix some index $k \in \{0, 1, \ldots, n-1\}$. Since the word U_kb is longer than U_k, its multiplicity can be at most 2^k, so in particular $\mu(U_kb) < \mu(U_k)$. Therefore, the word U_ka has to appear by (1). For a similar reason, the words U_kb, aU_k, and bU_k have to appear as well. Hence, the word U_k is ubiquitous. Moreover, if the multiplicity of U_k were strictly greater than 2^{k+1}, then by (1) at least one of the two words U_ka and U_kb would have multiplicity greater than 2^k and would thus violate the maximality condition imposed on U_k.

So we have $\mu(U_0) \le 2 < \mu(U_1) \le 4 < \ldots \le 2^{n-1} < \mu(U_{n-1})$, which implies in particular that the words $U_0, U_1, \ldots, U_{n-1}$ have to be distinct. As they have been proved to be ubiquitous as well, the problem is solved.

Comment 1. There is an easy construction for obtaining ubiquitous words from appearing words whose multiplicity is at least two. Starting with any such word U we may simply extend one of its occurrences in W forwards and backwards as long as its multiplicity remains fixed, thus arriving at a word that one might call the *ubiquitous prolongation* $p(U)$ of U.

There are several variants of the argument in the second half of the solution using the concept of prolongation. For instance, one may just take all ubiquitous words $U_1, U_2, \ldots, U_\ell$ ordered by increasing multiplicity and then prove for $i \in \{1, 2, \ldots, \ell\}$ that $\mu(U_i) \leq 2^i$. Indeed, assume that i is a minimal counterexample to this statement; then by the arguments similar to those presented above, the ubiquitous prolongation of one of the words $U_i a$, $U_i b$, aU_i or bU_i violates the definition of U_i.

Now the multiplicity of one of the two letters a and b is strictly greater than 2^{n-1}, so passing to ubiquitous prolongations once more we obtain $2^{n-1} < \mu(U_\ell) \leq 2^\ell$, which entails $\ell \geq n$, as needed.

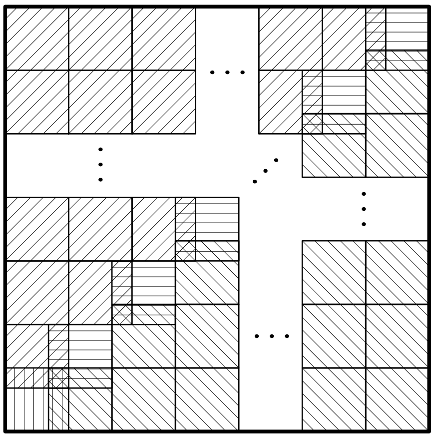
Comment 2. The bound n for the number of ubiquitous subwords in the problem statement is not optimal, but it is close to an optimal one in the following sense. There is a universal constant $C > 0$ such that for each positive integer n there exists an infinite periodic word W whose minimal period is greater than 2^n but for which there exist fewer than Cn ubiquitous words.

C7

On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?

Answer. $2011^2 - \big((52^2 - 35^2) \cdot 39 - 17^2\big) = 4044121 - 57392 = 3986729.$

Solution 1. Let $m = 39$, then $2011 = 52m - 17$. We begin with an example showing that there can exist 3986729 cells carrying the same positive number.



To describe it, we number the columns from the left to the right and the rows from the bottom to the top by $1, 2, \dots, 2011$. We will denote each napkin by the coordinates of its lower-left cell. There are four kinds of napkins: first, we take all napkins $(52i + 36, 52j + 1)$ with $0 \le j \le i \le m - 2$; second, we use all napkins $(52i + 1, 52j + 36)$ with $0 \le i \le j \le m - 2$; third, we use all napkins $(52i + 36, 52i + 36)$ with $0 \le i \le m - 2$; and finally the napkin $(1, 1)$. Different groups of napkins are shown by different types of hatchings in the picture.

Now except for those squares that carry two or more different hatchings, all squares have the number 1 written into them. The number of these exceptional cells is easily computed to be $(52^2 - 35^2)m - 17^2 = 57392$.

We are left to prove that 3986729 is an upper bound for the number of cells containing the same number. Consider any configuration of napkins and any positive integer M. Suppose there are g cells with a number different from M. Then it suffices to show $g \ge 57392$. Throughout the solution, a *line* will mean either a row or a column.

Consider any line ℓ. Let $a_1, \dots, a_{52m-17}$ be the numbers written into its consecutive cells. For $i = 1, 2, \dots, 52$, let $s_i = \sum_{t \equiv i \pmod{52}} a_t$. Note that $s_1, \dots, s_{35}$ have m terms each, while $s_{36}, \dots, s_{52}$ have $m - 1$ terms each. Every napkin intersecting ℓ contributes exactly 1 to each s_i;

hence the number s of all those napkins satisfies $s_1 = \cdots = s_{52} = s$. Call the line ℓ *rich* if $s > (m-1)M$ and *poor* otherwise.

Suppose now that ℓ is rich. Then in each of the sums $s_{36}, \ldots, s_{52}$ there exists a term greater than M; consider all these terms and call the corresponding cells the *rich bad cells* for this line. So, each rich line contains at least 17 cells that are bad for this line.

If, on the other hand, ℓ is poor, then certainly $s < mM$ so in each of the sums $s_1, \ldots, s_{35}$ there exists a term less than M; consider all these terms and call the corresponding cells the *poor bad cells* for this line. So, each poor line contains at least 35 cells that are bad for this line.

Let us call all indices congruent to $1, 2, \ldots,$ or 35 modulo 52 *small*, and all other indices, i.e. those congruent to $36, 37, \ldots,$ or 52 modulo 52, *big*. Recall that we have numbered the columns from the left to the right and the rows from the bottom to the top using the numbers $1, 2, \ldots, 52m-17$; we say that a line is *big* or *small* depending on whether its index is big or small. By definition, all rich bad cells for the rows belong to the big columns, while the poor ones belong to the small columns, and vice versa.

In each line, we put a *strawberry* on each cell that is bad for this line. In addition, for each **small rich** line we put an *extra* strawberry on each of its (rich) bad cells. A cell gets the strawberries from its row and its column independently.

Notice now that a cell with a strawberry on it contains a number different from M. If this cell gets a strawberry by the extra rule, then it contains a number greater than M. Moreover, it is either in a small row and in a big column, or vice versa. Suppose that it is in a small row, then it is not bad for its column. So it has not more than two strawberries in this case. On the other hand, if the extra rule is not applied to some cell, then it also has not more than two strawberries. So, the total number N of strawberries is at most $2g$.

We shall now estimate N in a different way. For each of the $2 \cdot 35m$ small lines, we have introduced at least 34 strawberries if it is rich and at least 35 strawberries if it is poor, so at least 34 strawberries in any case. Similarly, for each of the $2 \cdot 17(m-1)$ big lines, we put at least $\min(17, 35) = 17$ strawberries. Summing over all lines we obtain

$$2g \geq N \geq 2(35m \cdot 34 + 17(m-1) \cdot 17) = 2(1479m - 289) = 2 \cdot 57392,$$

as desired.

Comment. The same reasoning applies also if we replace 52 by R and 2011 by $Rm - H$, where m, R, and H are integers with $m, R \geq 1$ and $0 \leq H \leq \frac{1}{3}R$. More detailed information is provided after the next solution.

Solution 2. We present a different proof of the estimate which is the hard part of the problem. Let $S = 35$, $H = 17$, $m = 39$; so the table size is $2011 = Sm + H(m-1)$, and the napkin size is $52 = S + H$. Fix any positive integer M and call a cell *vicious* if it contains a number distinct from M. We will prove that there are at least $H^2(m-1)+2SHm$ vicious cells.

Firstly, we introduce some terminology. As in the previous solution, we number rows and columns and we use the same notions of *small* and *big* indices and lines; so, an index is small if it is congruent to one of the numbers $1, 2, \ldots, S$ modulo $(S+H)$. The numbers $1, 2, \ldots, S+H$ will be known as *residues*. For two residues i and j, we say that a cell is of *type* (i, j) if the index of its row is congruent to i and the index of its column to j modulo $(S+H)$. The number of vicious cells of this type is denoted by v_{ij}.

Let s, s' be two variables ranging over small residues and let h, h' be two variables ranging over big residues. A cell is said to be of *class* A, B, C, or D if its type is of shape (s, s'), (s, h), (h, s), or (h, h'), respectively. The numbers of vicious cells belonging to these classes are denoted in this order by a, b, c, and d. Observe that each cell belongs to exactly one class.

Claim 1. We have

$$m \leq \frac{a}{S^2} + \frac{b+c}{2SH}. \tag{1}$$

Proof. Consider an arbitrary small row r. Denote the numbers of vicious cells on r belonging to the classes A and B by α and β, respectively. As in the previous solution, we obtain that $\alpha \geq S$ or $\beta \geq H$. So in each case we have $\frac{\alpha}{S} + \frac{\beta}{H} \geq 1$.

Performing this argument separately for each small row and adding up all the obtained inequalities, we get $\frac{a}{S} + \frac{b}{H} \geq mS$. Interchanging rows and columns we similarly get $\frac{a}{S} + \frac{c}{H} \geq mS$. Summing these inequalities and dividing by $2S$ we get what we have claimed. $\square$

Claim 2. Fix two small residue s, s' and two big residues h, h'. Then $2m-1 \leq v_{ss'} + v_{sh'} + v_{hh'}$.

Proof. Each napkin covers exactly one cell of type (s, s'). Removing all napkins covering a vicious cell of this type, we get another collection of napkins, which covers each cell of type (s, s') either 0 or M times depending on whether the cell is vicious or not. Hence $(m^2 - v_{ss'})M$ napkins are left and throughout the proof of Claim 2 we will consider only these remaining napkins. Now, using a red pen, write in each cell the number of napkins covering it. Notice that a cell containing a red number greater than M is surely vicious.

We call two cells *neighbors* if they can be simultaneously covered by some napkin. So, each cell of type (h, h') has not more than four neighbors of type (s, s'), while each cell of type (s, h') has not more than two neighbors of each of the types (s, s') and (h, h'). Therefore, each red number at a cell of type (h, h') does not exceed $4M$, while each red number at a cell of type (s, h') does not exceed $2M$.

Let x, y, and z be the numbers of cells of type (h, h') whose red number belongs to $(M, 2M]$, $(2M, 3M]$, and $(3M, 4M]$, respectively. All these cells are vicious, hence $x + y + z \leq v_{hh'}$. The red numbers appearing in cells of type (h, h') clearly sum up to $(m^2 - v_{ss'})M$. Bounding each of these numbers by a multiple of M we get

$$(m^2 - v_{ss'})M \leq \big((m-1)^2 - (x+y+z)\big)\, M + 2xM + 3yM + 4zM,$$

i.e.

$$2m-1 \le v_{ss'} + x + 2y + 3z \le v_{ss'} + v_{hh'} + y + 2z.$$

So, to prove the claim it suffices to prove that $y + 2z \le v_{sh'}$.

For a cell δ of type (h, h') and a cell β of type (s, h') we say that δ *forces* β if there are more than M napkins covering both of them. Since each red number in a cell of type (s, h') does not exceed $2M$, it cannot be forced by more than one cell.

On the other hand, if a red number in a (h, h')-cell belongs to $(2M, 3M]$, then it forces at least one of its neighbors of type (s, h') (since the sum of red numbers in their cells is greater than $2M$). Analogously, an (h, h')-cell with the red number in $(3M, 4M]$ forces both its neighbors of type (s, h'), since their red numbers do not exceed $2M$. Therefore there are at least $y + 2z$ forced cells and clearly all of them are vicious, as desired. □

Claim 3. We have

$$2m-1 \le \frac{a}{S^2} + \frac{b+c}{2SH} + \frac{d}{H^2}. \tag{2}$$

Proof. Averaging the previous result over all S^2H^2 possibilities for the quadruple (s, s', h, h'), we get $2m - 1 \le \frac{a}{S^2} + \frac{b}{SH} + \frac{d}{H^2}$. Due to the symmetry between rows and columns, the same estimate holds with b replaced by c. Averaging these two inequalities we arrive at our claim. □

Now let us multiply (2) by H^2, multiply (1) by $(2SH - H^2)$ and add them; we get

$$H^2(2m-1)+(2SH-H^2)m \le a\cdot\frac{H^2+2SH-H^2}{S^2}+(b+c)\frac{H^2+2SH-H^2}{2SH}+d = a\cdot\frac{2H}{S}+b+c+d.$$

The left-hand side is exactly $H^2(m-1) + 2SHm$, while the right-hand side does not exceed $a + b + c + d$ since $2H \le S$. Hence we come to the desired inequality.

Comment 1. Claim 2 is the key difference between the two solutions, because it allows to get rid of the notions of rich and poor cells. However, one may prove it by the "strawberry method" as well. It suffices to put a strawberry on each cell which is bad for an s-row, and a strawberry on each cell which is bad for an h'-column. Then each cell would contain not more than one strawberry.

Comment 2. Both solutions above work if the residue of the table size T modulo the napkin size R is at least $\frac{2}{3}R$, or equivalently if $T = Sm + H(m-1)$ and $R = S + H$ for some positive integers S, H, m such that $S \ge 2H$. Here we discuss all other possible combinations.

Case 1. If $2H \ge S \ge H/2$, then the sharp bound for the number of vicious cells is $mS^2 + (m-1)H^2$; it can be obtained by the same methods as in any of the solutions. To obtain an example showing that the bound is sharp, one may simply remove the napkins of the third kind from the example in Solution 1 (with an obvious change in the numbers).

Case 2. If $2S \le H$, the situation is more difficult. If $(S + H)^2 > 2H^2$, then the answer and the example are the same as in the previous case; otherwise the answer is $(2m-1)S^2 + 2SH(m-1)$, and the example is provided simply by $(m-1)^2$ nonintersecting napkins.

Now we sketch the proof of both estimates for Case 2. We introduce a more appropriate notation based on that from Solution 2. Denote by a_- and a_+ the number of cells of class A that contain the number which is strictly less than M and strictly greater than M, respectively. The numbers $b_\pm$, $c_\pm$, and $d_\pm$ are defined in a similar way. One may notice that the proofs of Claim 1 and Claims 2, 3 lead in fact to the inequalities

$$m-1 \le \frac{b_- + c_-}{2SH} + \frac{d_+}{H^2} \quad \text{and} \quad 2m-1 \le \frac{a}{S^2} + \frac{b_+ + c_+}{2SH} + \frac{d_+}{H^2}$$

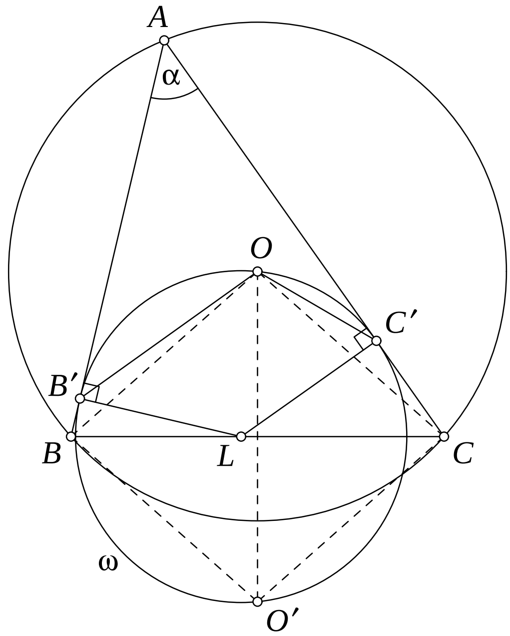
(to obtain the first one, one needs to look at the big lines instead of the small ones). Combining these inequalities, one may obtain the desired estimates.

These estimates can also be proved in some different ways, e.g. without distinguishing rich and poor cells.

G1

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to AB at B' and to AC at C'. Suppose also that the circumcenter O of the triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

Solution. The point B', being the perpendicular foot of L, is an interior point of side AB. Analogously, C' lies in the interior of AC. The point O is located inside the triangle $AB'C'$, hence $\angle COB < \angle C'OB'$.



Let $\alpha = \angle CAB$. The angles $\angle CAB$ and $\angle C'OB'$ are inscribed into the two circles with centers O and L, respectively, so $\angle COB = 2\angle CAB = 2\alpha$ and $2\angle C'OB' = 360^\circ - \angle C'LB'$. From the kite $AB'LC'$ we have $\angle C'LB' = 180^\circ - \angle C'AB' = 180^\circ - \alpha$. Combining these, we get

$$2\alpha = \angle COB < \angle C'OB' = \frac{360^\circ - \angle C'LB'}{2} = \frac{360^\circ - (180^\circ - \alpha)}{2} = 90^\circ + \frac{\alpha}{2},$$

so

$$\alpha < 60^\circ.$$

Let O' be the reflection of O in the line BC. In the quadrilateral $ABO'C$ we have

$$\angle CO'B + \angle CAB = \angle COB + \angle CAB = 2\alpha + \alpha < 180^\circ,$$

so the point O' is outside the circle ABC. Hence, O and O' are two points of ω such that one of them lies inside the circumcircle, while the other one is located outside. Therefore, the two circles intersect.

Comment. There are different ways of reducing the statement of the problem to the case $\alpha < 60^\circ$. E.g., since the point O lies in the interior of the isosceles triangle $AB'C'$, we have $OA < AB'$. So, if $AB' \leq 2LB'$ then $OA < 2LO$, which means that ω intersects the circumcircle of ABC. Hence the only interesting case is $AB' > 2LB'$, and this condition implies $\angle CAB = 2\angle B'AL < 2 \cdot 30^\circ = 60^\circ$.

G2

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2-r_1^2}+\frac{1}{O_2A_2^2-r_2^2}+\frac{1}{O_3A_3^2-r_3^2}+\frac{1}{O_4A_4^2-r_4^2}=0.$$

Solution 1. Let M be the point of intersection of the diagonals A_1A_3 and A_2A_4. On each diagonal choose a direction and let x, y, z, and w be the signed distances from M to the points A_1, A_2, A_3, and A_4, respectively.

Let ω_1 be the circumcircle of the triangle $A_2A_3A_4$ and let B_1 be the second intersection point of ω_1 and A_1A_3 (thus, $B_1=A_3$ if and only if A_1A_3 is tangent to ω_1). Since the expression $O_1A_1^2-r_1^2$ is the power of the point A_1 with respect to ω_1, we get

$$O_1A_1^2-r_1^2=A_1B_1\cdot A_1A_3.$$

On the other hand, from the equality $MB_1\cdot MA_3=MA_2\cdot MA_4$ we obtain $MB_1=yw/z$. Hence, we have

$$O_1A_1^2-r_1^2=\left(\frac{yw}{z}-x\right)(z-x)=\frac{z-x}{z}(yw-xz).$$

Substituting the analogous expressions into the sought sum we get

$$\sum_{i=1}^{4}\frac{1}{O_iA_i^2-r_i^2}=\frac{1}{yw-xz}\left(\frac{z}{z-x}-\frac{w}{w-y}+\frac{x}{x-z}-\frac{y}{y-w}\right)=0,$$

as desired.

Comment. One might reformulate the problem by assuming that the quadrilateral $A_1A_2A_3A_4$ is convex. This should not really change the difficulty, but proofs that distinguish several cases may become shorter.

Solution 2. Introduce a Cartesian coordinate system in the plane. Every circle has an equation of the form $p(x,y)=x^2+y^2+l(x,y)=0$, where $l(x,y)$ is a polynomial of degree at most 1. For any point $A=(x_A,y_A)$ we have $p(x_A,y_A)=d^2-r^2$, where d is the distance from A to the center of the circle and r is the radius of the circle.

For each i in $\{1,2,3,4\}$ let $p_i(x,y)=x^2+y^2+l_i(x,y)=0$ be the equation of the circle with center O_i and radius r_i and let d_i be the distance from A_i to O_i. Consider the equation

$$\sum_{i=1}^{4}\frac{p_i(x,y)}{d_i^2-r_i^2}=1. \tag{1}$$

Since the coordinates of the points A_1, A_2, A_3, and A_4 satisfy (1) but these four points do not lie on a circle or on an line, equation (1) defines neither a circle, nor a line. Hence, the equation is an identity and the coefficient of the quadratic term x^2+y^2 also has to be zero, i.e.

$$\sum_{i=1}^{4}\frac{1}{d_i^2-r_i^2}=0.$$

Comment. Using the determinant form of the equation of the circle through three given points, the same solution can be formulated as follows.

For $i=1,2,3,4$ let (u_i,v_i) be the coordinates of A_i and define

$$\Delta=\begin{vmatrix} u_1^2+v_1^2 & u_1 & v_1 & 1\\ u_2^2+v_2^2 & u_2 & v_2 & 1\\ u_3^2+v_3^2 & u_3 & v_3 & 1\\ u_4^2+v_4^2 & u_4 & v_4 & 1\end{vmatrix} \qquad \text{and} \qquad \Delta_i=\begin{vmatrix} u_{i+1} & v_{i+1} & 1\\ u_{i+2} & v_{i+2} & 1\\ u_{i+3} & v_{i+3} & 1\end{vmatrix},$$

where $i+1$, $i+2$, and $i+3$ have to be read modulo 4 as integers in the set $\{1,2,3,4\}$.

Expanding $\begin{vmatrix} u_1 & v_1 & 1 & 1\\ u_2 & v_2 & 1 & 1\\ u_3 & v_3 & 1 & 1\\ u_4 & v_4 & 1 & 1\end{vmatrix}=0$ along the third column, we get $\Delta_1-\Delta_2+\Delta_3-\Delta_4=0$.

The circle through A_{i+1}, A_{i+2}, and A_{i+3} is given by the equation

$$\frac{1}{\Delta_i}\begin{vmatrix} x^2+y^2 & x & y & 1\\ u_{i+1}^2+v_{i+1}^2 & u_{i+1} & v_{i+1} & 1\\ u_{i+2}^2+v_{i+2}^2 & u_{i+2} & v_{i+2} & 1\\ u_{i+3}^2+v_{i+3}^2 & u_{i+3} & v_{i+3} & 1\end{vmatrix}=0. \tag{2}$$

On the left-hand side, the coefficient of x^2+y^2 is equal to 1. Substituting (u_i,v_i) for (x,y) in (2) we obtain the power of point A_i with respect to the circle through A_{i+1}, A_{i+2}, and A_{i+3}:

$$d_i^2-r_i^2=\frac{1}{\Delta_i}\begin{vmatrix} u_i^2+v_i^2 & u_i & v_i & 1\\ u_{i+1}^2+v_{i+1}^2 & u_{i+1} & v_{i+1} & 1\\ u_{i+2}^2+v_{i+2}^2 & u_{i+2} & v_{i+2} & 1\\ u_{i+3}^2+v_{i+3}^2 & u_{i+3} & v_{i+3} & 1\end{vmatrix}=(-1)^{i+1}\frac{\Delta}{\Delta_i}.$$

Thus, we have

$$\sum_{i=1}^{4}\frac{1}{d_i^2-r_i^2}=\frac{\Delta_1-\Delta_2+\Delta_3-\Delta_4}{\Delta}=0.$$

G3

Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC, and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA, and AB. Prove that the midpoint of the segment EF lies on the line through the two intersection points of ω_E and ω_F.

Solution. Denote by P, Q, R, and S the projections of E on the lines DA, AB, BC, and CD respectively. The points P and Q lie on the circle with diameter AE, so $\angle QPE = \angle QAE$; analogously, $\angle QRE = \angle QBE$. So $\angle QPE + \angle QRE = \angle QAE + \angle QBE = 90^\circ$. By similar reasons, we have $\angle SPE + \angle SRE = 90^\circ$, hence we get $\angle QPS + \angle QRS = 90^\circ + 90^\circ = 180^\circ$, and the quadrilateral $PQRS$ is inscribed in ω_E. Analogously, all four projections of F onto the sides of $ABCD$ lie on ω_F.

Denote by K the meeting point of the lines AD and BC. Due to the arguments above, there is no loss of generality in assuming that A lies on segment DK. Suppose that $\angle CKD \geq 90^\circ$; then the circle with diameter CD covers the whole quadrilateral $ABCD$, so the points E, F cannot lie inside this quadrilateral. Hence our assumption is wrong. Therefore, the lines EP and BC intersect at some point P', while the lines ER and AD intersect at some point R'.

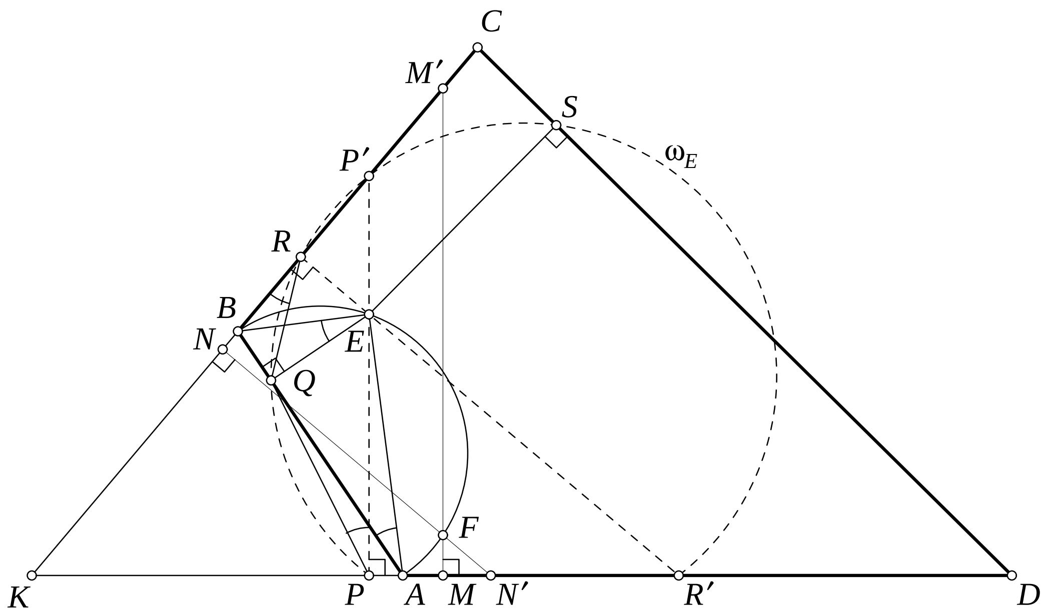


Figure 1

We claim that the points P' and R' also belong to ω_E. Since the points R, E, Q, B are concyclic, $\angle QRK = \angle QEB = 90^\circ - \angle QBE = \angle QAE = \angle QPE$. So $\angle QRK = \angle QPP'$, which means that the point P' lies on ω_E. Analogously, R' also lies on ω_E.

In the same manner, denote by M and N the projections of F on the lines AD and BC

respectively, and let $M' = FM \cap BC$, $N' = FN \cap AD$. By the same arguments, we obtain that the points M' and N' belong to ω_F.

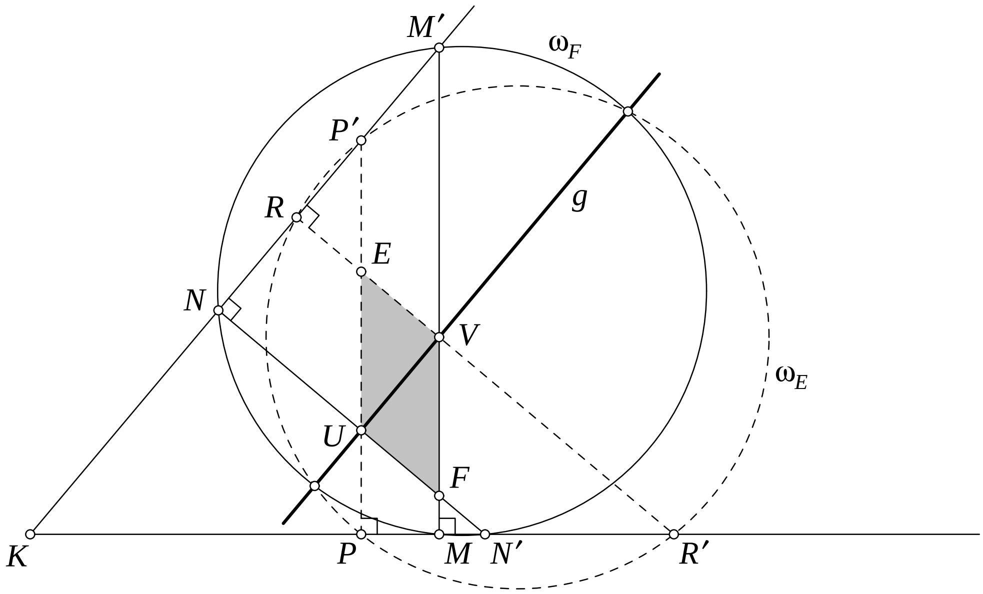


Figure 2

Now we concentrate on Figure 2, where all unnecessary details are removed. Let $U = NN' \cap PP'$, $V = MM' \cap RR'$. Due to the right angles at N and P, the points N, N', P, P' are concyclic, so $UN \cdot UN' = UP \cdot UP'$ which means that U belongs to the radical axis g of the circles ω_E and ω_F. Analogously, V also belongs to g.

Finally, since $EUFV$ is a parallelogram, the radical axis UV of ω_E and ω_F bisects EF.

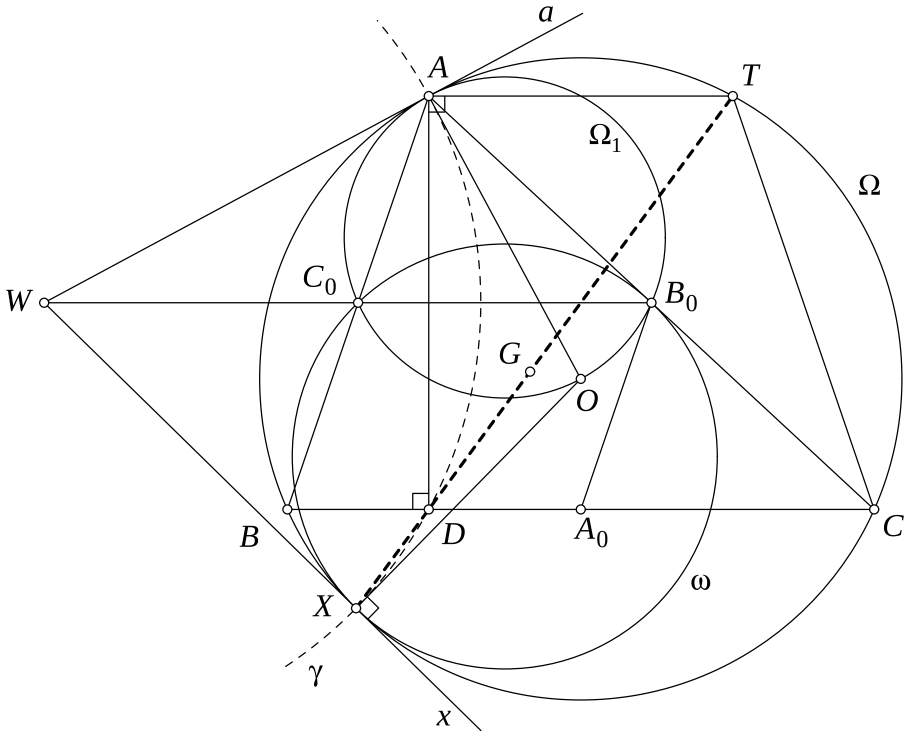
G4

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G, and X are collinear.

Solution 1. If $AB = AC$, then the statement is trivial. So without loss of generality we may assume $AB < AC$. Denote the tangents to Ω at points A and X by a and x, respectively.

Let Ω_1 be the circumcircle of triangle AB_0C_0. The circles Ω and Ω_1 are homothetic with center A, so they are tangent at A, and a is their radical axis. Now, the lines a, x, and B_0C_0 are the three radical axes of the circles Ω, Ω_1, and ω. Since $a \not\parallel B_0C_0$, these three lines are concurrent at some point W.

The points A and D are symmetric with respect to the line B_0C_0; hence $WX = WA = WD$. This means that W is the center of the circumcircle γ of triangle ADX. Moreover, we have $\angle WAO = \angle WXO = 90°$, where O denotes the center of Ω. Hence $\angle AWX + \angle AOX = 180°$.



Denote by T the second intersection point of Ω and the line DX. Note that O belongs to Ω_1. Using the circles γ and Ω, we find $\angle DAT = \angle ADX - \angle ATD = \frac{1}{2}(360° - \angle AWX) - \frac{1}{2}\angle AOX = 180° - \frac{1}{2}(\angle AWX + \angle AOX) = 90°$. So, $AD \perp AT$, and hence $AT \parallel BC$. Thus, $ATCB$ is an isosceles trapezoid inscribed in Ω.

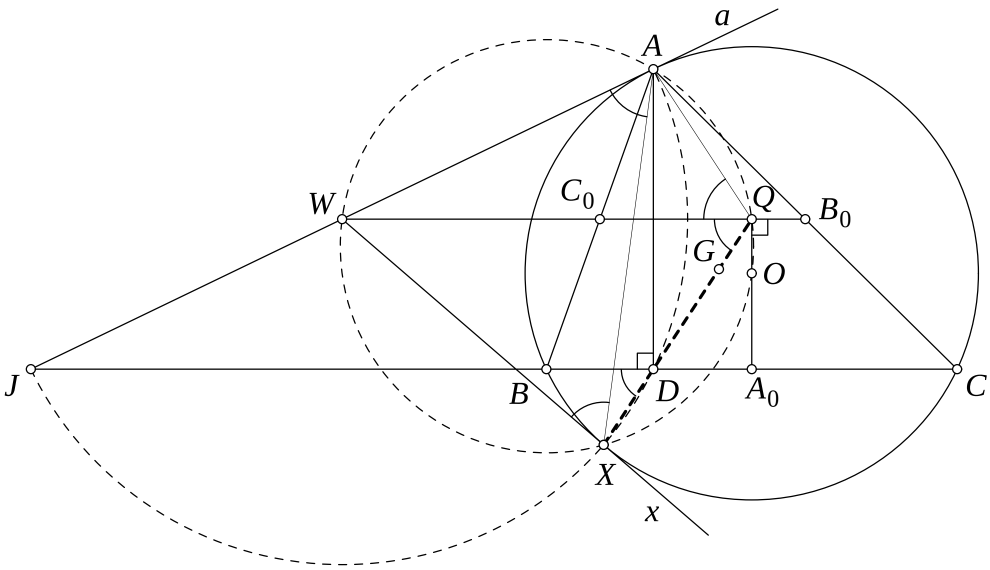
Denote by A_0 the midpoint of BC, and consider the image of $ATCB$ under the homothety h with center G and factor $-\frac{1}{2}$. We have $h(A) = A_0$, $h(B) = B_0$, and $h(C) = C_0$. From the symmetry about B_0C_0, we have $\angle TCB = \angle CBA = \angle B_0C_0A = \angle DC_0B_0$. Using $AT \parallel DA_0$, we conclude $h(T) = D$. Hence the points D, G, and T are collinear, and X lies on the same line.

Solution 2. We define the points A_0, O, and W as in the previous solution and we concentrate on the case $AB < AC$. Let Q be the perpendicular projection of A_0 on B_0C_0.

Since $\angle WAO = \angle WQO = \angle OXW = 90°$, the five points A, W, X, O, and Q lie on a common circle. Furthermore, the reflections with respect to B_0C_0 and OW map A to D and X, respectively. For these reasons, we have

$$\angle WQD = \angle AQW = \angle AXW = \angle WAX = \angle WQX.$$

Thus the three points Q, D, and X lie on a common line, say ℓ.



To complete the argument, we note that the homothety centered at G sending the triangle ABC to the triangle $A_0B_0C_0$ maps the altitude AD to the altitude A_0Q. Therefore it maps D to Q, so the points D, G, and Q are collinear. Hence G lies on ℓ as well.

Comment. There are various other ways to prove the collinearity of Q, D, and X obtained in the middle part of Solution 2. Introduce for instance the point J where the lines AW and BC intersect. Then the four points A, D, X, and J lie at the same distance from W, so the quadrilateral $ADXJ$ is cyclic. In combination with the fact that $AWXQ$ is cyclic as well, this implies

$$\angle JDX = \angle JAX = \angle WAX = \angle WQX.$$

Since $BC \parallel WQ$, it follows that Q, D, and X are indeed collinear.

G5

Let ABC be a triangle with incenter I and circumcircle ω. Let D and E be the second intersection points of ω with the lines AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and at B meet at a point K. Prove that the three lines AE, BD, and KP are either parallel or concurrent.

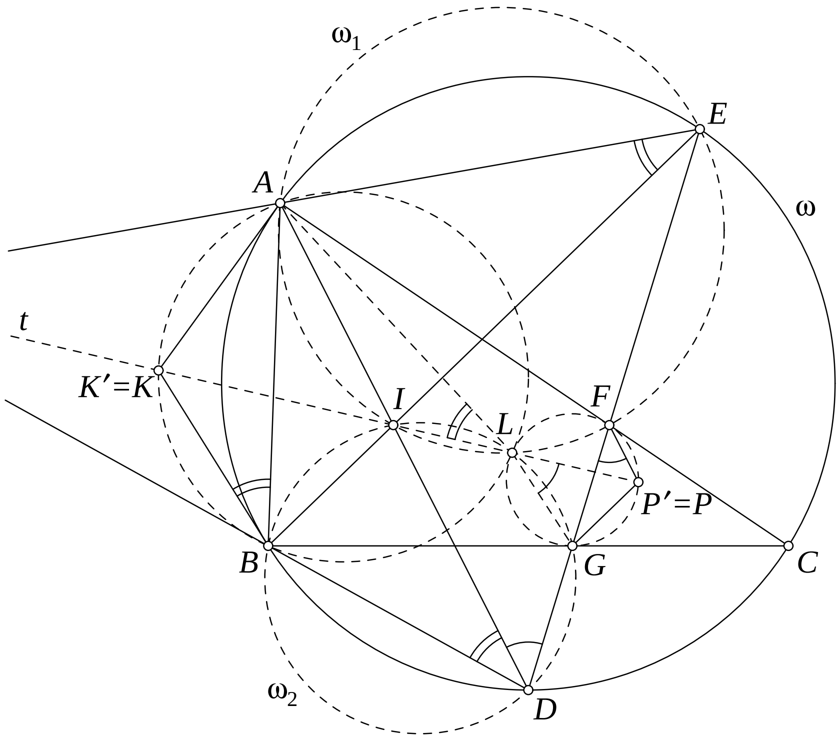
Solution 1. Since

$$\angle IAF = \angle DAC = \angle BAD = \angle BED = \angle IEF$$

the quadrilateral $AIFE$ is cyclic. Denote its circumcircle by ω_1. Similarly, the quadrilateral $BDGI$ is cyclic; denote its circumcircle by ω_2.

The line AE is the radical axis of ω and ω_1, and the line BD is the radical axis of ω and ω_2. Let t be the radical axis of ω_1 and ω_2. These three lines meet at the radical center of the three circles, or they are parallel to each other. We will show that t is in fact the line PK.

Let L be the second intersection point of ω_1 and ω_2, so $t = IL$. (If the two circles are tangent to each other then $L = I$ and t is the common tangent.)



Let the line t meet the circumcircles of the triangles ABL and FGL at $K' \neq L$ and $P' \neq L$, respectively. Using oriented angles we have

$$\angle(AB, BK') = \angle(AL, LK') = \angle(AL, LI) = \angle(AE, EI) = \angle(AE, EB) = \angle(AB, BK),$$

so $BK' \parallel BK$. Similarly we have $AK' \parallel AK$, and therefore $K' = K$. Next, we have

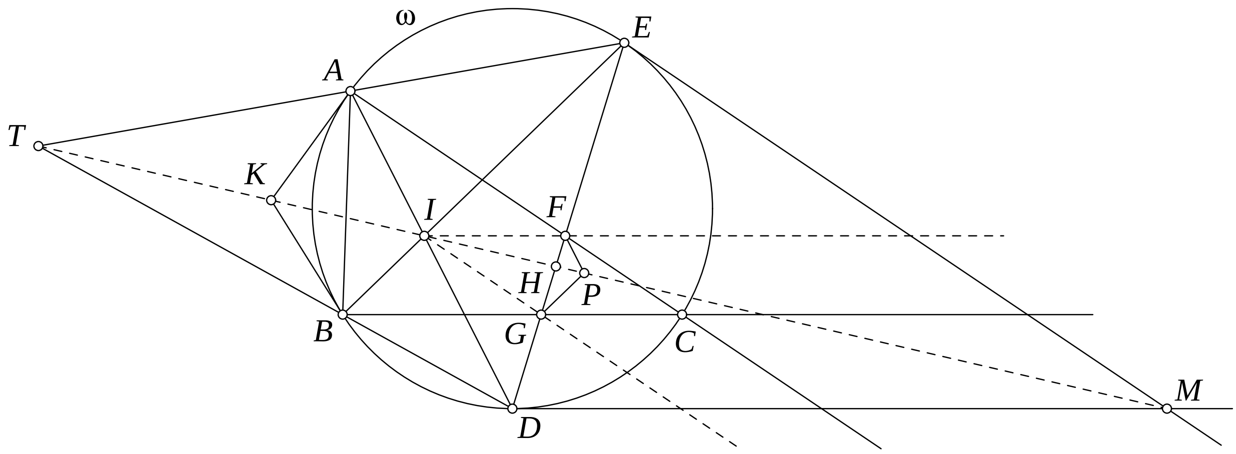
$$\angle(P'F, FG) = \angle(P'L, LG) = \angle(IL, LG) = \angle(ID, DG) = \angle(AD, DE) = \angle(PF, FG),$$

hence $P'F \parallel PF$ and similarly $P'G \parallel PG$. Therefore $P' = P$. This means that t passes through K and P, which finishes the proof.

Solution 2. Let M be the intersection point of the tangents to ω at D and E, and let the lines AE and BD meet at T; if AE and BD are parallel, then let T be their common ideal point. It is well-known that the points K and M lie on the line TI (as a consequence of Pascal's theorem, applied to the inscribed degenerate hexagons $AADBBE$ and $ADDBEE$).

The lines AD and BE are the angle bisectors of the angles $\angle CAB$ and $\angle ABC$, respectively, so D and E are the midpoints of the arcs BC and CA of the circle ω, respectively. Hence, DM is parallel to BC and EM is parallel to AC.

Apply Pascal's theorem to the degenerate hexagon $CADDEB$. By the theorem, the points $CA \cap DE = F$, $AD \cap EB = I$ and the common ideal point of lines DM and BC are collinear, therefore FI is parallel to BC and DM. Analogously, the line GI is parallel to AC and EM.



Now consider the homothety with scale factor $-\frac{FG}{ED}$ which takes E to G and D to F. Since the triangles EDM and GFI have parallel sides, the homothety takes M to I. Similarly, since the triangles DEI and FGP have parallel sides, the homothety takes I to P. Hence, the points M, I, P and the homothety center H must lie on the same line. Therefore, the point P also lies on the line $TKIM$.

Comment. One may prove that $IF \parallel BC$ and $IG \parallel AC$ in a more elementary way. Since $\angle ADE = \angle EDC$ and $\angle DEB = \angle CED$, the points I and C are symmetric about DE. Moreover, since the arcs AE and EC are equal and the arcs CD and DB are equal, we have $\angle CFG = \angle FGC$, so the triangle CFG is isosceles. Hence, the quadrilateral $IFCG$ is a rhombus.

G6

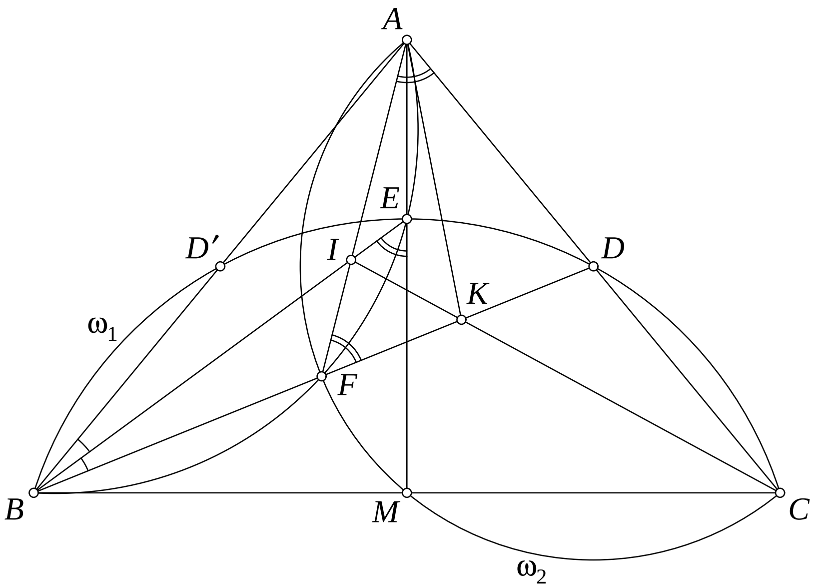
Let ABC be a triangle with $AB = AC$, and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C in a point E inside the triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

Solution 1. Let D' be the midpoint of the segment AB, and let M be the midpoint of BC. By symmetry at line AM, the point D' has to lie on the circle BCD. Since the arcs $D'E$ and ED of that circle are equal, we have $\angle ABI = \angle D'BE = \angle EBD = IBK$, so I lies on the angle bisector of $\angle ABK$. For this reason it suffices to prove in the sequel that the ray AI bisects the angle $\angle BAK$.

From

$$\angle DFA = 180^\circ - \angle BFA = 180^\circ - \angle BEA = \angle MEB = \frac{1}{2}\angle CEB = \frac{1}{2}\angle CDB$$

we derive $\angle DFA = \angle DAF$ so the triangle AFD is isosceles with $AD = DF$.



Applying Menelaus's theorem to the triangle ADF with respect to the line CIK, and applying the angle bisector theorem to the triangle ABF, we infer

$$1 = \frac{AC}{CD}\cdot\frac{DK}{KF}\cdot\frac{FI}{IA} = 2\cdot\frac{DK}{KF}\cdot\frac{BF}{AB} = 2\cdot\frac{DK}{KF}\cdot\frac{BF}{2\cdot AD} = \frac{DK}{KF}\cdot\frac{BF}{AD}$$

and therefore

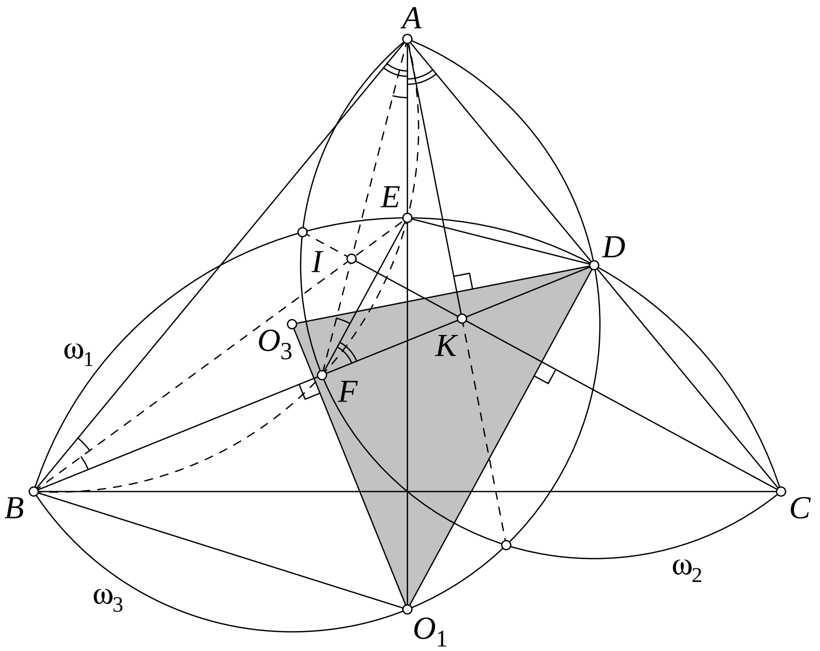
$$\frac{BD}{AD} = \frac{BF+FD}{AD} = \frac{BF}{AD} + 1 = \frac{KF}{DK} + 1 = \frac{DF}{DK} = \frac{AD}{DK}.$$

It follows that the triangles ADK and BDA are similar, hence $\angle DAK = \angle ABD$. Then

$$\angle IAB = \angle AFD - \angle ABD = \angle DAF - \angle DAK = \angle KAI$$

shows that the point K is indeed lying on the angle bisector of $\angle BAK$.

Solution 2. It can be shown in the same way as in the first solution that I lies on the angle bisector of $\angle ABK$. Here we restrict ourselves to proving that KI bisects $\angle AKB$.



Denote the circumcircle of triangle BCD and its center by ω_1 and by O_1, respectively. Since the quadrilateral $ABFE$ is cyclic, we have $\angle DFE = \angle BAE = \angle DAE$. By the same reason, we have $\angle EAF = \angle EBF = \angle ABE = \angle AFE$. Therefore $\angle DAF = \angle DFA$, and hence $DF = DA = DC$. So triangle AFC is inscribed in a circle ω_2 with center D.

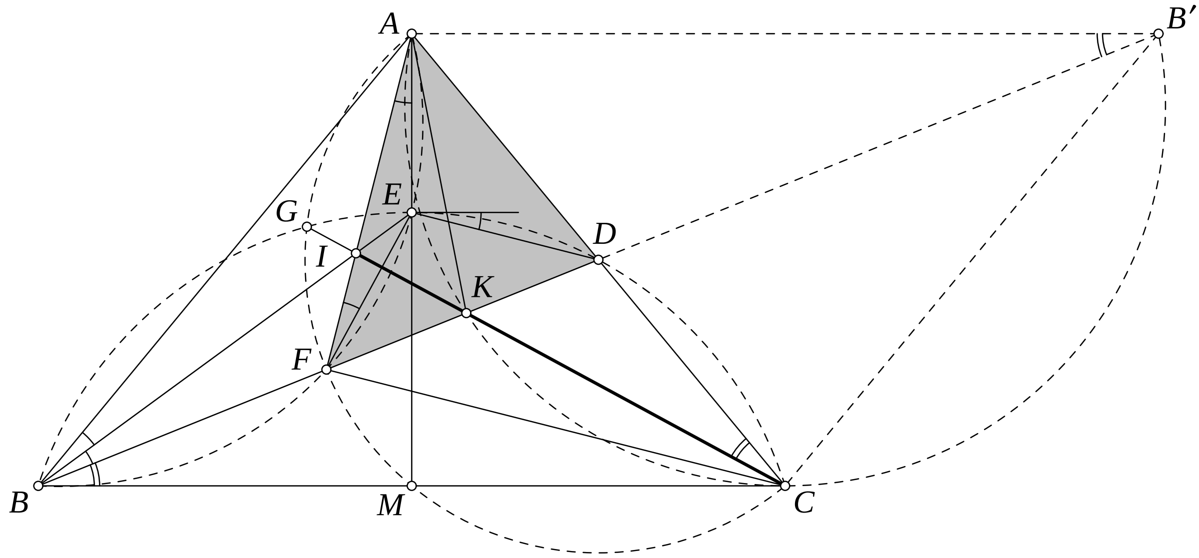
Denote the circumcircle of triangle ABD by ω_3, and let its center be O_3. Since the arcs BE and EC of circle ω_1 are equal, and the triangles ADE and FDE are congruent, we have $\angle AO_1B = 2\angle BDE = \angle BDA$, so O_1 lies on ω_3. Hence $\angle O_3O_1D = \angle O_3DO_1$.

The line BD is the radical axis of ω_1 and ω_3. Point C belongs to the radical axis of ω_1 and ω_2, and I also belongs to it since $AI \cdot IF = BI \cdot IE$. Hence $K = BD \cap CI$ is the radical center of ω_1, ω_2, and ω_3, and AK is the radical axis of ω_2 and ω_3. Now, the radical axes AK, BK and IK are perpendicular to the central lines O_3D, O_3O_1 and O_1D, respectively. By $\angle O_3O_1D = \angle O_3DO_1$, we get that KI is the angle bisector of $\angle AKB$.

Solution 3. Again, let M be the midpoint of BC. As in the previous solutions, we can deduce $\angle ABI = \angle IBK$. We show that the point I lies on the angle bisector of $\angle KAB$.

Let G be the intersection point of the circles AFC and BCD, different from C. The lines

CG, AF, and BE are the radical axes of the three circles $AGFC$, CDB, and $ABFE$, so $I = AF \cap BE$ is the radical center of the three circles and CG also passes through I.



The angle between line DE and the tangent to the circle BCD at E is equal to $\angle EBD = \angle EAF = \angle ABE = \angle AFE$. As the tangent at E is perpendicular to AM, the line DE is perpendicular to AF. The triangle AFE is isosceles, so DE is the perpendicular bisector of AF and thus $AD = DF$. Hence, the point D is the center of the circle AFC, and this circle passes through M as well since $\angle AMC = 90°$.

Let B' be the reflection of B in the point D, so $ABCB'$ is a parallelogram. Since $DC = DG$ we have $\angle GCD = \angle DBC = \angle KB'A$. Hence, the quadrilateral $AKCB'$ is cyclic and thus $\angle CAK = \angle CB'K = \angle ABD = 2\angle MAI$. Then

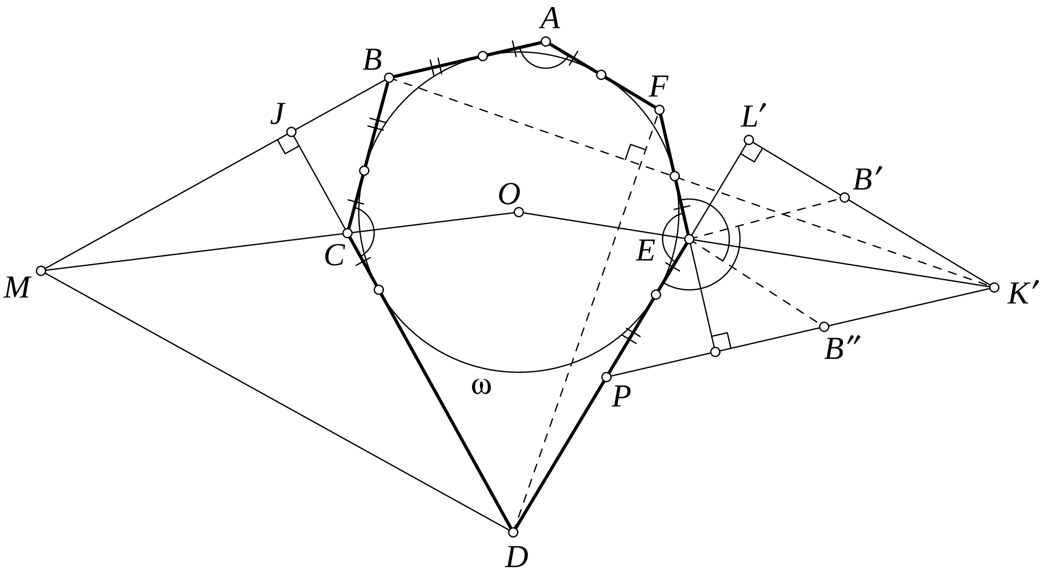
$$\angle IAB = \angle MAB - \angle MAI = \frac{1}{2}\angle CAB - \frac{1}{2}\angle CAK = \frac{1}{2}\angle KAB$$

and therefore AI is the angle bisector of $\angle KAB$.

G7

Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

Solution 1. Since ω and the circumcircle of triangle ACE are concentric, the tangents from A, C, and E to ω have equal lengths; that means that $AB = BC$, $CD = DE$, and $EF = FA$. Moreover, we have $\angle BCD = \angle DEF = \angle FAB$.



Consider the rotation around point D mapping C to E; let B' and L' be the images of the points B and J, respectively, under this rotation. Then one has $DJ = DL'$ and $B'L' \perp DE$; moreover, the triangles $B'ED$ and BCD are congruent. Since $\angle DEO < 90°$, the lines EO and $B'L'$ intersect at some point K'. We intend to prove that $K'B \perp DF$; this would imply $K = K'$, therefore $L = L'$, which proves the problem statement.

Analogously, consider the rotation around F mapping A to E; let B'' be the image of B under this rotation. Then the triangles FAB and FEB'' are congruent. We have $EB'' = AB = BC = EB'$ and $\angle FEB'' = \angle FAB = \angle BCD = \angle DEB'$, so the points B' and B'' are symmetrical with respect to the angle bisector EO of $\angle DEF$. So, from $K'B' \perp DE$ we get $K'B'' \perp EF$.

From these two relations we obtain

$$K'D^2 - K'E^2 = B'D^2 - B'E^2 \quad \text{and} \quad K'E^2 - K'F^2 = B''E^2 - B''F^2.$$

Adding these equalities and taking into account that $B'E = B''E$ we obtain

$$K'D^2 - K'F^2 = B'D^2 - B''F^2 = BD^2 - BF^2,$$

which means exactly that $K'B \perp DF$.

Comment. There are several variations of this solution. For instance, let us consider the intersection point M of the lines BJ and OC. Define the point K' as the reflection of M in the line DO. Then one has

$$DK'^2 - DB^2 = DM^2 - DB^2 = CM^2 - CB^2.$$

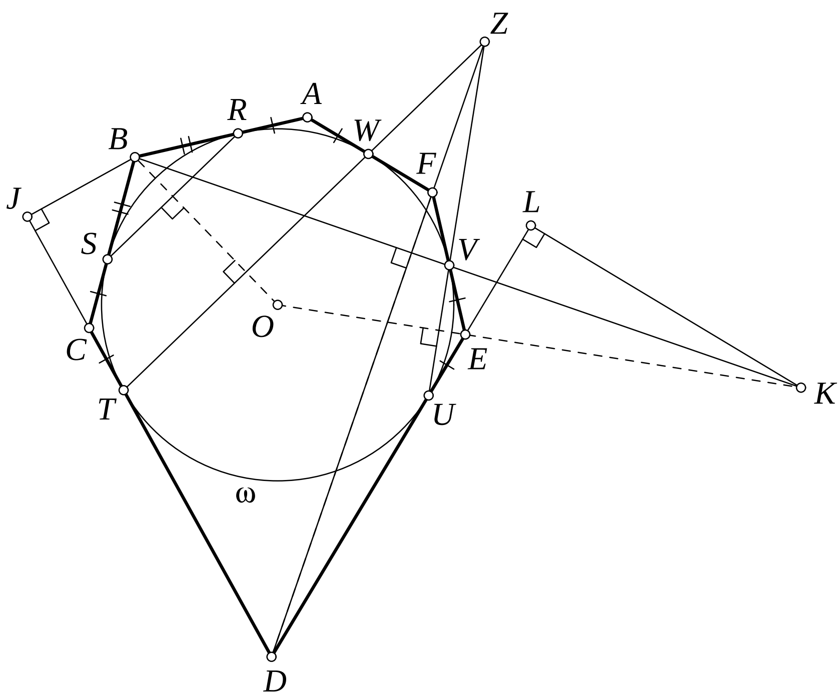
Next, consider the rotation around O which maps CM to EK'. Let P be the image of B under this rotation; so P lies on ED. Then $EF \perp K'P$, so

$$CM^2 - CB^2 = EK'^2 - EP^2 = FK'^2 - FP^2 = FK'^2 - FB^2,$$

since the triangles FEP and FAB are congruent.

Solution 2. Let us denote the points of tangency of AB, BC, CD, DE, EF, and FA to ω by R, S, T, U, V, and W, respectively. As in the previous solution, we mention that $AR = AW = CS = CT = EU = EV$.

The reflection in the line BO maps R to S, therefore A to C and thus also W to T. Hence, both lines RS and WT are perpendicular to OB, therefore they are parallel. On the other hand, the lines UV and WT are not parallel, since otherwise the hexagon $ABCDEF$ is symmetric with respect to the line BO and the lines defining the point K coincide, which contradicts the conditions of the problem. Therefore we can consider the intersection point Z of UV and WT.



Next, we recall a well-known fact that the points D, F, Z are collinear. Actually, D is the pole of the line UT, F is the pole of VW, and $Z = TW \cap UV$; so all these points belong to the polar line of $TU \cap VW$.

Now, we put O into the origin, and identify each point (say X) with the vector $\overrightarrow{OX}$. So, from now on all the products of points refer to the scalar products of the corresponding vectors.

Since $OK \perp UZ$ and $OB \perp TZ$, we have $K \cdot (Z-U) = 0 = B \cdot (Z-T)$. Next, the condition $BK \perp DZ$ can be written as $K \cdot (D-Z) = B \cdot (D-Z)$. Adding these two equalities we get

$$K \cdot (D-U) = B \cdot (D-T).$$

By symmetry, we have $D \cdot (D-U) = D \cdot (D-T)$. Subtracting this from the previous equation, we obtain $(K-D) \cdot (D-U) = (B-D) \cdot (D-T)$ and rewrite it in vector form as

$$\overrightarrow{DK} \cdot \overrightarrow{UD} = \overrightarrow{DB} \cdot \overrightarrow{TD}.$$

Finally, projecting the vectors $\overrightarrow{DK}$ and $\overrightarrow{DB}$ onto the lines UD and TD respectively, we can rewrite this equality in terms of segment lengths as $DL \cdot UD = DJ \cdot TD$, thus $DL = DJ$.

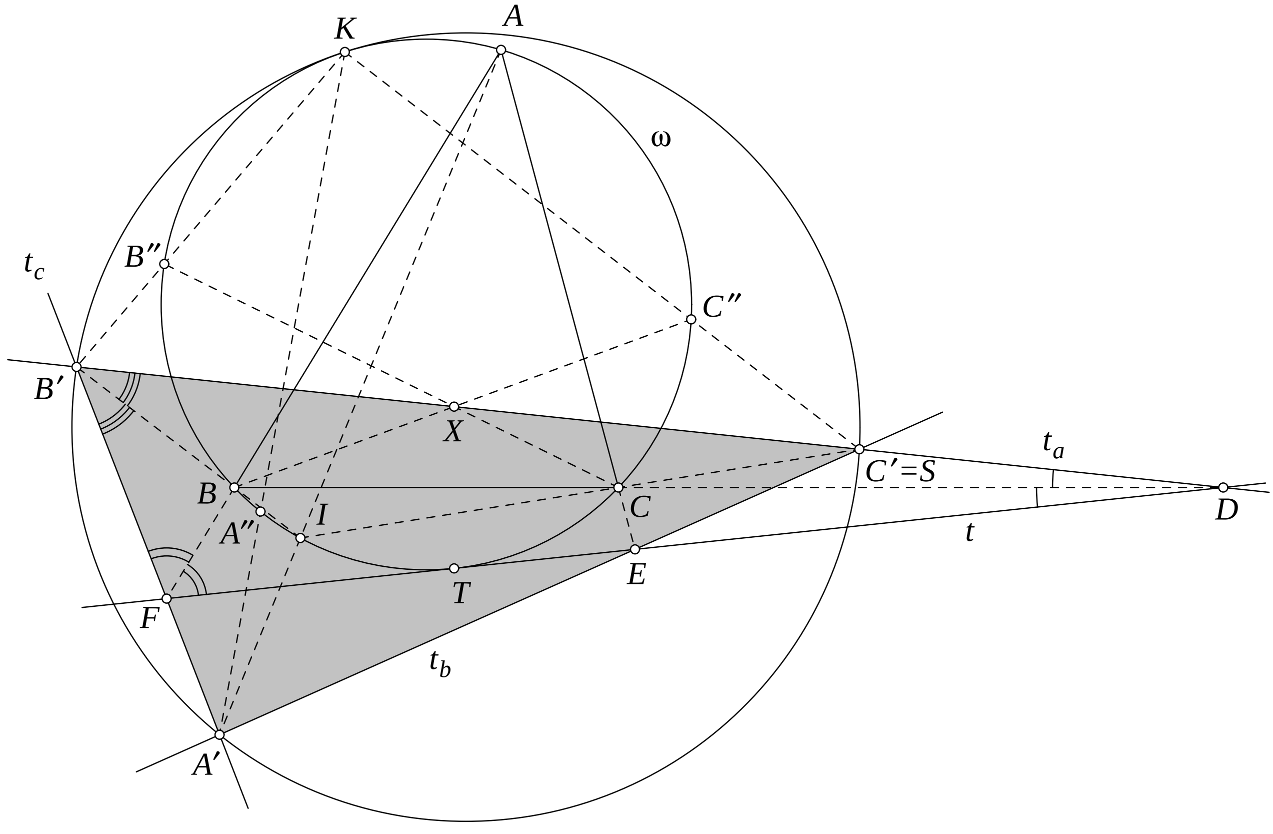
Comment. The collinearity of Z, F, and D may be shown in various more elementary ways. For instance, applying the sine theorem to the triangles DTZ and DUZ, one gets $\dfrac{\sin\angle DZT}{\sin\angle DZU} = \dfrac{\sin\angle DTZ}{\sin\angle DUZ}$; analogously, $\dfrac{\sin\angle FZW}{\sin\angle FZV} = \dfrac{\sin\angle FWZ}{\sin\angle FVZ}$. The right-hand sides are equal, hence so are the left-hand sides, which implies the collinearity of the points D, F, and Z.

There also exist purely synthetic proofs of this fact. E.g., let Q be the point of intersection of the circumcircles of the triangles ZTV and ZWU different from Z. Then QZ is the bisector of $\angle VQW$ since $\angle VQZ = \angle VTZ = \angle VUW = \angle ZQW$. Moreover, all these angles are equal to $\frac{1}{2}\angle VOW$, so $\angle VQW = \angle VOW$, hence the quadrilateral $VWOQ$ is cyclic. On the other hand, the points O, V, W lie on the circle with diameter OF due to the right angles; so Q also belongs to this circle. Since $FV = FW$, QF is also the bisector of $\angle VQW$, so F lies on QZ. Analogously, D lies on the same line.

G8

Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to ω. Let t_a, t_b, and t_c be the lines obtained by reflecting t in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

To avoid a large case distinction, we will use the notion of *oriented angles*. Namely, for two lines ℓ and m, we denote by $\angle(\ell, m)$ the angle by which one may rotate ℓ anticlockwise to obtain a line parallel to m. Thus, all oriented angles are considered modulo 180°.



Solution 1. Denote by T the point of tangency of t and ω. Let $A' = t_b \cap t_c$, $B' = t_a \cap t_c$, $C' = t_a \cap t_b$. Introduce the point A'' on ω such that $TA = AA''$ ($A'' \neq T$ unless TA is a diameter). Define the points B'' and C'' in a similar way.

Since the points C and B are the midpoints of arcs TC'' and TB'', respectively, we have

$$\begin{aligned}\angle(t, B''C'') &= \angle(t, TC'') + \angle(TC'', B''C'') = 2\angle(t, TC) + 2\angle(TC'', BC'') \\ &= 2\big(\angle(t, TC) + \angle(TC, BC)\big) = 2\angle(t, BC) = \angle(t, t_a).\end{aligned}$$

It follows that t_a and $B''C''$ are parallel. Similarly, $t_b \parallel A''C''$ and $t_c \parallel A''B''$. Thus, either the triangles $A'B'C'$ and $A''B''C''$ are homothetic, or they are translates of each other. Now we will prove that they are in fact homothetic, and that the center K of the homothety belongs to ω. It would then follow that their circumcircles are also homothetic with respect to K and are therefore tangent at this point, as desired.

We need the two following claims.

Claim 1. The point of intersection X of the lines $B''C$ and BC'' lies on t_a.

Proof. Actually, the points X and T are symmetric about the line BC, since the lines CT and CB'' are symmetric about this line, as are the lines BT and BC''. $\square$

Claim 2. The point of intersection I of the lines BB' and CC' lies on the circle ω.

Proof. We consider the case that t is not parallel to the sides of ABC; the other cases may be regarded as limit cases. Let $D = t \cap BC$, $E = t \cap AC$, and $F = t \cap AB$.

Due to symmetry, the line DB is one of the angle bisectors of the lines $B'D$ and FD; analogously, the line FB is one of the angle bisectors of the lines $B'F$ and DF. So B is either the incenter or one of the excenters of the triangle $B'DF$. In any case we have $\angle(BD, DF) + \angle(DF, FB) + \angle(B'B, B'D) = 90^\circ$, so

$$\angle(B'B, B'C') = \angle(B'B, B'D) = 90^\circ - \angle(BC, DF) - \angle(DF, BA) = 90^\circ - \angle(BC, AB).$$

Analogously, we get $\angle(C'C, B'C') = 90^\circ - \angle(BC, AC)$. Hence,

$$\angle(BI, CI) = \angle(B'B, B'C') + \angle(B'C', C'C) = \angle(BC, AC) - \angle(BC, AB) = \angle(AB, AC),$$

which means exactly that the points A, B, I, C are concyclic. $\square$

Now we can complete the proof. Let K be the second intersection point of $B'B''$ and ω. Applying Pascal's theorem to hexagon $KB''CIBC''$ we get that the points $B' = KB'' \cap IB$ and $X = B''C \cap BC''$ are collinear with the intersection point S of CI and $C''K$. So $S = CI \cap B'X = C'$, and the points C', C'', K are collinear. Thus K is the intersection point of $B'B''$ and $C'C''$ which implies that K is the center of the homothety mapping $A'B'C'$ to $A''B''C''$, and it belongs to ω.

Solution 2. Define the points T, A', B', and C' in the same way as in the previous solution. Let X, Y, and Z be the symmetric images of T about the lines BC, CA, and AB, respectively. Note that the projections of T on these lines form a Simson line of T with respect to ABC, therefore the points X, Y, Z are also collinear. Moreover, we have $X \in B'C'$, $Y \in C'A'$, $Z \in A'B'$.

Denote $\alpha = \angle(t, TC) = \angle(BT, BC)$. Using the symmetry in the lines AC and BC, we get

$$\angle(BC, BX) = \angle(BT, BC) = \alpha \quad \text{and} \quad \angle(XC, XC') = \angle(t, TC) = \angle(YC, YC') = \alpha.$$

Since $\angle(XC, XC') = \angle(YC, YC')$, the points X, Y, C, C' lie on some circle ω_c. Define the circles ω_a and ω_b analogously. Let ω' be the circumcircle of triangle $A'B'C'$.

Now, applying MIQUEL's theorem to the four lines $A'B'$, $A'C'$, $B'C'$, and XY, we obtain that the circles ω', ω_a, ω_b, ω_c intersect at some point K. We will show that K lies on ω, and that the tangent lines to ω and ω' at this point coincide; this implies the problem statement.

Due to symmetry, we have $XB = TB = ZB$, so the point B is the midpoint of one of the arcs XZ of circle ω_b. Therefore $\angle(KB, KX) = \angle(XZ, XB)$. Analogously, $\angle(KX, KC) = \angle(XC, XY)$. Adding these equalities and using the symmetry in the line BC we get

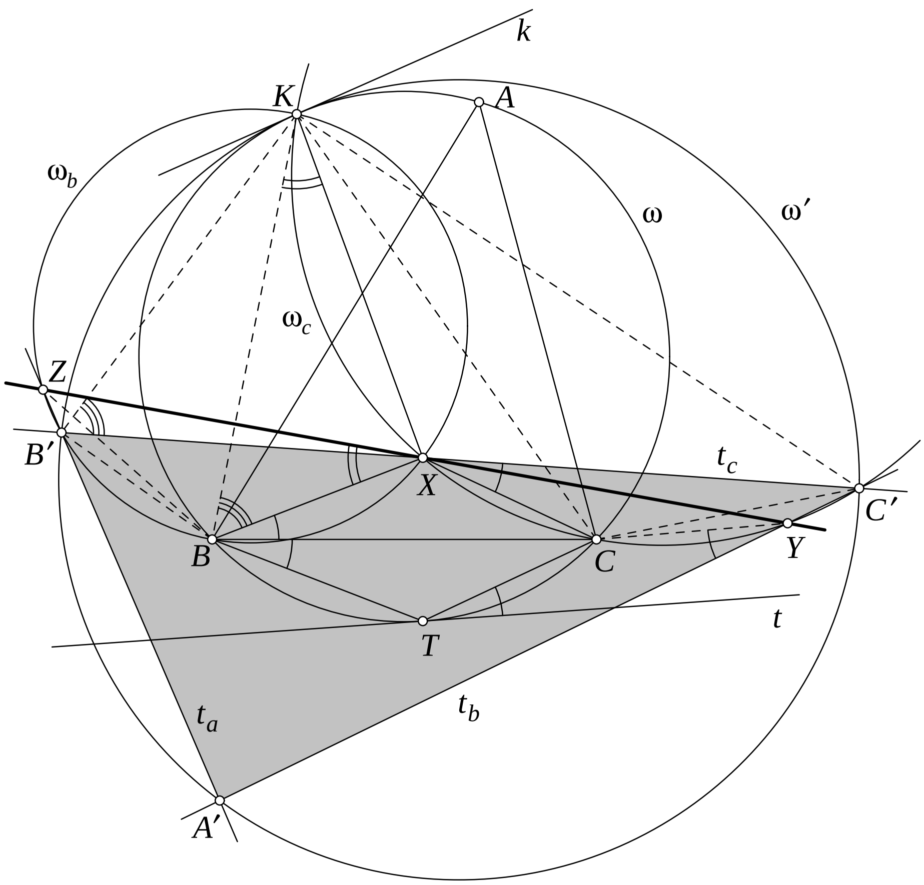
$$\angle(KB, KC) = \angle(XZ, XB) + \angle(XC, XZ) = \angle(XC, XB) = \angle(TB, TC).$$

Therefore, K lies on ω.

Next, let k be the tangent line to ω at K. We have

$$\begin{aligned}\angle(k, KC') &= \angle(k, KC) + \angle(KC, KC') = \angle(KB, BC) + \angle(XC, XC') \\ &= \big(\angle(KB, BX) - \angle(BC, BX)\big) + \alpha = \angle(KB', B'X) - \alpha + \alpha = \angle(KB', B'C'),\end{aligned}$$

which means exactly that k is tangent to ω'.



Comment. There exist various solutions combining the ideas from the two solutions presented above. For instance, one may define the point X as the reflection of T with respect to the line BC, and then introduce the point K as the second intersection point of the circumcircles of $BB'X$ and $CC'X$. Using the fact that BB' and CC' are the bisectors of $\angle(A'B', B'C')$ and $\angle(A'C', B'C')$ one can show successively that $K \in \omega$, $K \in \omega'$, and that the tangents to ω and ω' at K coincide.

N1

For any integer $d > 0$, let $f(d)$ be the smallest positive integer that has exactly d positive divisors (so for example we have $f(1) = 1$, $f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f(2^k)$ divides $f(2^{k+1})$.

Solution 1. For any positive integer n, let $d(n)$ be the number of positive divisors of n. Let $n = \prod_p p^{a(p)}$ be the prime factorization of n where p ranges over the prime numbers, the integers $a(p)$ are nonnegative and all but finitely many $a(p)$ are zero. Then we have $d(n) = \prod_p (a(p)+1)$. Thus, $d(n)$ is a power of 2 if and only if for every prime p there is a nonnegative integer $b(p)$ with $a(p) = 2^{b(p)} - 1 = 1 + 2 + 2^2 + \cdots + 2^{b(p)-1}$. We then have

$$n = \prod_p \prod_{i=0}^{b(p)-1} p^{2^i}, \qquad \text{and} \quad d(n) = 2^k \quad \text{with} \quad k = \sum_p b(p).$$

Let $\mathcal{S}$ be the set of all numbers of the form p^{2^r} with p prime and r a nonnegative integer. Then we deduce that $d(n)$ is a power of 2 if and only if n is the product of the elements of some finite subset $\mathcal{T}$ of $\mathcal{S}$ that satisfies the following condition: for all $t \in \mathcal{T}$ and $s \in \mathcal{S}$ with $s \mid t$ we have $s \in \mathcal{T}$. Moreover, if $d(n) = 2^k$ then the corresponding set $\mathcal{T}$ has k elements.

Note that the set $\mathcal{T}_k$ consisting of the smallest k elements from $\mathcal{S}$ obviously satisfies the condition above. Thus, given k, the smallest n with $d(n) = 2^k$ is the product of the elements of $\mathcal{T}_k$. This n is $f(2^k)$. Since obviously $\mathcal{T}_k \subset \mathcal{T}_{k+1}$, it follows that $f(2^k) \mid f(2^{k+1})$.

Solution 2. This is an alternative to the second part of the Solution 1. Suppose k is a nonnegative integer. From the first part of Solution 1 we see that $f(2^k) = \prod_p p^{a(p)}$ with $a(p) = 2^{b(p)} - 1$ and $\sum_p b(p) = k$. We now claim that for any two distinct primes p, q with $b(q) > 0$ we have

$$m = p^{2^{b(p)}} > q^{2^{b(q)-1}} = \ell. \tag{1}$$

To see this, note first that ℓ divides $f(2^k)$. With the first part of Solution 1 one can see that the integer $n = f(2^k)m/\ell$ also satisfies $d(n) = 2^k$. By the definition of $f(2^k)$ this implies that $n \geq f(2^k)$ so $m \geq \ell$. Since $p \neq q$ the inequality (1) follows.

Let the prime factorization of $f(2^{k+1})$ be given by $f(2^{k+1}) = \prod_p p^{r(p)}$ with $r(p) = 2^{s(p)} - 1$. Since we have $\sum_p s(p) = k + 1 > k = \sum_p b(p)$ there is a prime p with $s(p) > b(p)$. For any prime $q \neq p$ with $b(q) > 0$ we apply inequality (1) twice and get

$$q^{2^{s(q)}} > p^{2^{s(p)-1}} \geq p^{2^{b(p)}} > q^{2^{b(q)-1}},$$

which implies $s(q) \geq b(q)$. It follows that $s(q) \geq b(q)$ for all primes q, so $f(2^k) \mid f(2^{k+1})$.

N2

Consider a polynomial $P(x) = (x + d_1)(x + d_2) \cdot \ldots \cdot (x + d_9)$, where $d_1, d_2, \ldots, d_9$ are nine distinct integers. Prove that there exists an integer N such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

Solution 1. Note that the statement of the problem is invariant under translations of x; hence without loss of generality we may suppose that the numbers $d_1, d_2, \ldots, d_9$ are positive.

The key observation is that there are only eight primes below 20, while $P(x)$ involves more than eight factors.

We shall prove that $N = d^8$ satisfies the desired property, where $d = \max\{d_1, d_2, \ldots, d_9\}$. Suppose for the sake of contradiction that there is some integer $x \geq N$ such that $P(x)$ is composed of primes below 20 only. Then for every index $i \in \{1, 2, \ldots, 9\}$ the number $x + d_i$ can be expressed as product of powers of the first 8 primes.

Since $x + d_i > x \geq d^8$ there is some prime power $f_i > d$ that divides $x + d_i$. Invoking the pigeonhole principle we see that there are two distinct indices i and j such that f_i and f_j are powers of the same prime number. For reasons of symmetry, we may suppose that $f_i \leq f_j$. Now both of the numbers $x + d_i$ and $x + d_j$ are divisible by f_i and hence so is their difference $d_i - d_j$. But as

$$0 < |d_i - d_j| \leq \max(d_i, d_j) \leq d < f_i,$$

this is impossible. Thereby the problem is solved.

Solution 2. Observe that for each index $i \in \{1, 2, \ldots, 9\}$ the product

$$D_i = \prod_{1 \leq j \leq 9, j \neq i} |d_i - d_j|$$

is positive. We claim that $N = \max\{D_1 - d_1, D_2 - d_2, \ldots, D_9 - d_9\} + 1$ satisfies the statement of the problem. Suppose there exists an integer $x \geq N$ such that all primes dividing $P(x)$ are smaller than 20. For each index i we reduce the fraction $(x + d_i)/D_i$ to lowest terms. Since $x + d_i > D_i$ the numerator of the fraction we thereby get cannot be 1, and hence it has to be divisible by some prime number $p_i < 20$.

By the pigeonhole principle, there are a prime number p and two distinct indices i and j such that $p_i = p_j = p$. Let p^{α_i} and p^{α_j} be the greatest powers of p dividing $x + d_i$ and $x + d_j$, respectively. Due to symmetry we may suppose $\alpha_i \leq \alpha_j$. But now p^{α_i} divides $d_i - d_j$ and hence also D_i, which means that all occurrences of p in the numerator of the fraction $(x + d_i)/D_i$ cancel out, contrary to the choice of $p = p_i$. This contradiction proves our claim.

Solution 3. Given a nonzero integer N as well as a prime number p we write $v_p(N)$ for the exponent with which p occurs in the prime factorization of $|N|$.

Evidently, if the statement of the problem were not true, then there would exist an infinite sequence (x_n) of positive integers tending to infinity such that for each $n \in \mathbb{Z}_+$ the integer $P(x_n)$ is not divisible by any prime number > 20. Observe that the numbers $-d_1, -d_2, \ldots, -d_9$ do not appear in this sequence.

Now clearly there exists a prime $p_1 < 20$ for which the sequence $v_{p_1}(x_n + d_1)$ is not bounded; thinning out the sequence (x_n) if necessary we may even suppose that

$$v_{p_1}(x_n + d_1) \longrightarrow \infty.$$

Repeating this argument eight more times we may similarly choose primes $p_2, \ldots, p_9 < 20$ and suppose that our sequence (x_n) has been thinned out to such an extent that $v_{p_i}(x_n + d_i) \longrightarrow \infty$ holds for $i = 2, \ldots, 9$ as well. In view of the pigeonhole principle, there are distinct indices i and j as well as a prime $p < 20$ such that $p_i = p_j = p$. Setting $k = v_p(d_i - d_j)$ there now has to be some n for which both $v_p(x_n + d_i)$ and $v_p(x_n + d_j)$ are greater than k. But now the numbers $x_n + d_i$ and $x_n + d_j$ are divisible by p^{k+1} whilst their difference $d_i - d_j$ is not – a contradiction.

Comment. This problem is supposed to be a relatively easy one, so one might consider adding the hypothesis that the numbers $d_1, d_2, \ldots, d_9$ be positive. Then certain merely technical issues are not going to arise while the main ideas required to solve the problems remain the same.

N3

Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

Answer. All functions f of the form $f(x) = \varepsilon x^d + c$, where ε is in $\{1, -1\}$, the integer d is a positive divisor of n, and c is an integer.

Solution. Obviously, all functions in the answer satisfy the condition of the problem. We will show that there are no other functions satisfying that condition.

Let f be a function satisfying the given condition. For each integer n, the function g defined by $g(x) = f(x) + n$ also satisfies the same condition. Therefore, by subtracting $f(0)$ from $f(x)$ we may assume that $f(0) = 0$.

For any prime p, the condition on f with $(x, y) = (p, 0)$ states that $f(p)$ divides p^n. Since the set of primes is infinite, there exist integers d and ε with $0 \leq d \leq n$ and $\varepsilon \in \{1, -1\}$ such that for infinitely many primes p we have $f(p) = \varepsilon p^d$. Denote the set of these primes by P. Since a function g satisfies the given condition if and only if $-g$ satisfies the same condition, we may suppose $\varepsilon = 1$.

The case $d = 0$ is easily ruled out, because 0 does not divide any nonzero integer. Suppose $d \geq 1$ and write n as $md + r$, where m and r are integers such that $m \geq 1$ and $0 \leq r \leq d - 1$. Let x be an arbitrary integer. For each prime p in P, the difference $f(p) - f(x)$ divides $p^n - x^n$. Using the equality $f(p) = p^d$, we get

$$p^n - x^n = p^r (p^d)^m - x^n \equiv p^r f(x)^m - x^n \equiv 0 \pmod{p^d - f(x)}$$

Since we have $r < d$, for large enough primes $p \in P$ we obtain

$$|p^r f(x)^m - x^n| < p^d - f(x).$$

Hence $p^r f(x)^m - x^n$ has to be zero. This implies $r = 0$ and $x^n = (x^d)^m = f(x)^m$. Since m is odd, we obtain $f(x) = x^d$.

Comment. If n is an even positive integer, then the functions f of the form

$$f(x) = \begin{cases} x^d + c \text{ for some integers,} \\ -x^d + c \text{ for the rest of integers,} \end{cases}$$

where d is a positive divisor of $n/2$ and c is an integer, also satisfy the condition of the problem. Together with the functions in the answer, they are all functions that satisfy the condition when n is even.

N4

For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n such that all the differences

$$t(n+a)-t(n), \quad t(n+a+1)-t(n+1), \quad \ldots, \quad t(n+2a-1)-t(n+a-1)$$

are divisible by 4.

Answer. $a=1$, 3, or 5.

Solution. A pair (a, n) satisfying the condition of the problem will be called a *winning pair*. It is straightforward to check that the pairs $(1,1)$, $(3,1)$, and $(5,4)$ are winning pairs.

Now suppose that a is a positive integer not equal to 1, 3, and 5. We will show that there are no winning pairs (a, n) by distinguishing three cases.

Case 1: a is even. In this case we have $a=2^\alpha d$ for some positive integer α and some odd d. Since $a \geq 2^\alpha$, for each positive integer n there exists an $i \in \{0,1,\ldots,a-1\}$ such that $n+i=2^{\alpha-1}e$, where e is some odd integer. Then we have $t(n+i)=t(2^{\alpha-1}e)=e$ and

$$t(n+a+i)=t(2^\alpha d+2^{\alpha-1}e)=2d+e\equiv e+2 \pmod 4.$$

So we get $t(n+i)-t(n+a+i)\equiv 2 \pmod 4$, and (a,n) is not a winning pair.

Case 2: a is odd and $a>8$. For each positive integer n, there exists an $i\in\{0,1,\ldots,a-5\}$ such that $n+i=2d$ for some odd d. We get

$$t(n+i)=d\not\equiv d+2=t(n+i+4) \pmod 4$$

and

$$t(n+a+i)=n+a+i\equiv n+a+i+4=t(n+a+i+4) \pmod 4.$$

Therefore, the integers $t(n+a+i)-t(n+i)$ and $t(n+a+i+4)-t(n+i+4)$ cannot be both divisible by 4, and therefore there are no winning pairs in this case.

Case 3: $a=7$. For each positive integer n, there exists an $i\in\{0,1,\ldots,6\}$ such that $n+i$ is either of the form $8k+3$ or of the form $8k+6$, where k is a nonnegative integer. But we have

$$t(8k+3)\equiv 3\not\equiv 1\equiv 4k+5=t(8k+3+7) \pmod 4$$

and

$$t(8k+6)=4k+3\equiv 3\not\equiv 1\equiv t(8k+6+7) \pmod 4.$$

Hence, there are no winning pairs of the form $(7,n)$.

N5

Let f be a function from the set of integers to the set of positive integers. Suppose that for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that for all integers m, n with $f(m) \leq f(n)$ the number $f(n)$ is divisible by $f(m)$.

Solution 1. Suppose that x and y are two integers with $f(x) < f(y)$. We will show that $f(x) \mid f(y)$. By taking $m = x$ and $n = y$ we see that

$$f(x-y) \mid |f(x) - f(y)| = f(y) - f(x) > 0,$$

so $f(x-y) \leq f(y) - f(x) < f(y)$. Hence the number $d = f(x) - f(x-y)$ satisfies

$$-f(y) < -f(x-y) < d < f(x) < f(y).$$

Taking $m = x$ and $n = x - y$ we see that $f(y) \mid d$, so we deduce $d = 0$, or in other words $f(x) = f(x-y)$. Taking $m = x$ and $n = y$ we see that $f(x) = f(x-y) \mid f(x) - f(y)$, which implies $f(x) \mid f(y)$.

Solution 2. We split the solution into a sequence of claims; in each claim, the letters m and n denote arbitrary integers.

Claim 1. $f(n) \mid f(mn)$.

Proof. Since trivially $f(n) \mid f(1 \cdot n)$ and $f(n) \mid f((k+1)n) - f(kn)$ for all integers k, this is easily seen by using induction on m in both directions. □

Claim 2. $f(n) \mid f(0)$ and $f(n) = f(-n)$.

Proof. The first part follows by plugging $m = 0$ into Claim 1. Using Claim 1 twice with $m = -1$, we get $f(n) \mid f(-n) \mid f(n)$, from which the second part follows. □

From Claim 1, we get $f(1) \mid f(n)$ for all integers n, so $f(1)$ is the minimal value attained by f. Next, from Claim 2, the function f can attain only a finite number of values since all these values divide $f(0)$.

Now we prove the statement of the problem by induction on the number N_f of values attained by f. In the base case $N_f \leq 2$, we either have $f(0) \neq f(1)$, in which case these two numbers are the only values attained by f and the statement is clear, or we have $f(0) = f(1)$, in which case we have $f(1) \mid f(n) \mid f(0)$ for all integers n, so f is constant and the statement is obvious again.

For the induction step, assume that $N_f \geq 3$, and let a be the least positive integer with $f(a) > f(1)$. Note that such a number exists due to the symmetry of f obtained in Claim 2.

Claim 3. $f(n) \neq f(1)$ if and only if $a \mid n$.

Proof. Since $f(1) = \cdots = f(a-1) < f(a)$, the claim follows from the fact that

$$f(n) = f(1) \iff f(n+a) = f(1).$$

So it suffices to prove this fact.

Assume that $f(n) = f(1)$. Then $f(n+a) \mid f(a) - f(-n) = f(a) - f(n) > 0$, so $f(n+a) \leq f(a) - f(n) < f(a)$; in particular the difference $f(n+a) - f(n)$ is stricly smaller than $f(a)$. Furthermore, this difference is divisible by $f(a)$ and nonnegative since $f(n) = f(1)$ is the least value attained by f. So we have $f(n+a) - f(n) = 0$, as desired. For the converse direction we only need to remark that $f(n+a) = f(1)$ entails $f(-n-a) = f(1)$, and hence $f(n) = f(-n) = f(1)$ by the forward implication. $\square$

We return to the induction step. So let us take two arbitrary integers m and n with $f(m) \leq f(n)$. If $a \nmid m$, then we have $f(m) = f(1) \mid f(n)$. On the other hand, suppose that $a \mid m$; then by Claim 3 $a \mid n$ as well. Now define the function $g(x) = f(ax)$. Clearly, g satisfies the conditions of the problem, but $N_g < N_f - 1$, since g does not attain $f(1)$. Hence, by the induction hypothesis, $f(m) = g(m/a) \mid g(n/a) = f(n)$, as desired.

Comment. After the fact that f attains a finite number of values has been established, there are several ways of finishing the solution. For instance, let $f(0) = b_1 > b_2 > \cdots > b_k$ be all these values. One may show (essentially in the same way as in Claim 3) that the set $S_i = \{n : f(n) \geq b_i\}$ consists exactly of all numbers divisible by some integer $a_i \geq 0$. One obviously has $a_i \mid a_{i-1}$, which implies $f(a_i) \mid f(a_{i-1})$ by Claim 1. So, $b_k \mid b_{k-1} \mid \cdots \mid b_1$, thus proving the problem statement.

Moreover, now it is easy to describe all functions satisfying the conditions of the problem. Namely, all these functions can be constructed as follows. Consider a sequence of nonnegative integers $a_1, a_2, \ldots, a_k$ and another sequence of positive integers $b_1, b_2, \ldots, b_k$ such that $|a_k| = 1$, $a_i \neq a_j$ and $b_i \neq b_j$ for all $1 \leq i < j \leq k$, and $a_i \mid a_{i-1}$ and $b_i \mid b_{i-1}$ for all $i = 2, \ldots, k$. Then one may introduce the function

$$f(n) = b_{i(n)}, \qquad \text{where} \quad i(n) = \min\{i : a_i \mid n\}.$$

These are all the functions which satisfy the conditions of the problem.

N6

Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)}-1$ divides $3^{P(n)}-1$. Prove that $Q(x)$ is a constant polynomial.

Solution. First we show that there exists an integer d such that for all positive integers n we have $\gcd\bigl(P(n), Q(n)\bigr) \le d$.

Since $P(x)$ and $Q(x)$ are coprime (over the polynomials with rational coefficients), EUCLID's algorithm provides some polynomials $R_0(x), S_0(x)$ with rational coefficients such that $P(x)R_0(x)-Q(x)S_0(x)=1$. Multiplying by a suitable positive integer d, we obtain polynomials $R(x)=d\cdot R_0(x)$ and $S(x)=d\cdot S_0(x)$ with integer coefficients for which $P(x)R(x)-Q(x)S(x)=d$. Then we have $\gcd\bigl(P(n), Q(n)\bigr) \le d$ for any integer n.

To prove the problem statement, suppose that $Q(x)$ is not constant. Then the sequence $Q(n)$ is not bounded and we can choose a positive integer m for which

$$M = 2^{Q(m)} - 1 \ge 3^{\max\{P(1),P(2),\ldots,P(d)\}}. \tag{1}$$

Since $M = 2^{Q(n)}-1 \mid 3^{P(n)}-1$, we have $2, 3 \nmid M$. Let a and b be the multiplicative orders of 2 and 3 modulo M, respectively. Obviously, $a = Q(m)$ since the lower powers of 2 do not reach M. Since M divides $3^{P(m)}-1$, we have $b \mid P(m)$. Then $\gcd(a,b) \le \gcd\bigl(P(m), Q(m)\bigr) \le d$. Since the expression $ax - by$ attains all integer values divisible by $\gcd(a,b)$ when x and y run over all nonnegative integer values, there exist some nonnegative integers x, y such that $1 \le m + ax - by \le d$.

By $Q(m+ax) \equiv Q(m) \pmod{a}$ we have

$$2^{Q(m+ax)} \equiv 2^{Q(m)} \equiv 1 \pmod{M}$$

and therefore

$$M \bigm| 2^{Q(m+ax)} - 1 \bigm| 3^{P(m+ax)} - 1.$$

Then, by $P(m+ax-by) \equiv P(m+ax) \pmod{b}$ we have

$$3^{P(m+ax-by)} \equiv 3^{P(m+ax)} \equiv 1 \pmod{M}.$$

Since $P(m+ax-by) > 0$ this implies $M \le 3^{P(m+ax-by)} - 1$. But $P(m+ax-by)$ is listed among $P(1), P(2), \ldots, P(d)$, so

$$M < 3^{P(m+ax-by)} \le 3^{\max\{P(1),P(2),\ldots,P(d)\}}$$

which contradicts (1).

Comment. We present another variant of the solution above.

Denote the degree of P by k and its leading coefficient by p. Consider any positive integer n and let $a = Q(n)$. Again, denote by b the multiplicative order of 3 modulo $2^a - 1$. Since $2^a - 1 \mid 3^{P(n)} - 1$, we have $b \mid P(n)$. Moreover, since $2^{Q(n+at)} - 1 \mid 3^{P(n+at)} - 1$ and $a = Q(n) \mid Q(n+at)$ for each positive integer t, we have $2^a - 1 \mid 3^{P(n+at)} - 1$, hence $b \mid P(n+at)$ as well.

Therefore, b divides $\gcd\{P(n+at) : t \geq 0\}$; hence it also divides the number

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} P(n+ai) = p \cdot k! \cdot a^k.$$

Finally, we get $b \,\big|\, \gcd\big(P(n), k! \cdot p \cdot Q(n)^k\big)$, which is bounded by the same arguments as in the beginning of the solution. So $3^b - 1$ is bounded, and hence $2^{Q(n)} - 1$ is bounded as well.

N7

Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Prove that p divides m.

Solution 1. For rational numbers p_1/q_1 and p_2/q_2 with the denominators q_1, q_2 not divisible by p, we write $p_1/q_1 \equiv p_2/q_2 \pmod p$ if the numerator $p_1q_2 - p_2q_1$ of their difference is divisible by p.

We start with finding an explicit formula for the residue of S_a modulo p. Note first that for every $k = 1, \ldots, p-1$ the number $\binom{p}{k}$ is divisible by p, and

$$\frac{1}{p}\binom{p}{k} = \frac{(p-1)(p-2)\cdots(p-k+1)}{k!} \equiv \frac{(-1)\cdot(-2)\cdots(-k+1)}{k!} = \frac{(-1)^{k-1}}{k} \pmod p$$

Therefore, we have

$$S_a = -\sum_{k=1}^{p-1} \frac{(-a)^k(-1)^{k-1}}{k} \equiv -\sum_{k=1}^{p-1}(-a)^k \cdot \frac{1}{p}\binom{p}{k} \pmod p.$$

The number on the right-hand side is integer. Using the binomial formula we express it as

$$-\sum_{k=1}^{p-1}(-a)^k \cdot \frac{1}{p}\binom{p}{k} = -\frac{1}{p}\left(-1-(-a)^p + \sum_{k=0}^{p}(-a)^k\binom{p}{k}\right) = \frac{(a-1)^p - a^p + 1}{p}$$

since p is odd. So, we have

$$S_a \equiv \frac{(a-1)^p - a^p + 1}{p} \pmod p.$$

Finally, using the obtained formula we get

$$\begin{aligned} S_3 + S_4 - 3S_2 &\equiv \frac{(2^p - 3^p + 1) + (3^p - 4^p + 1) - 3(1^p - 2^p + 1)}{p} \\ &= \frac{4 \cdot 2^p - 4^p - 4}{p} = -\frac{(2^p-2)^2}{p} \pmod p. \end{aligned}$$

By Fermat's theorem, $p \mid 2^p - 2$, so $p^2 \mid (2^p-2)^2$ and hence $S_3 + S_4 - 3S_2 \equiv 0 \pmod p$.

Solution 2. One may solve the problem without finding an explicit formula for S_a. It is enough to find the following property.

Lemma. For every integer a, we have $S_{a+1} \equiv S_{-a} \pmod p$.

Proof. We expand S_{a+1} using the binomial formula as

$$S_{a+1} = \sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} a^j = \sum_{k=1}^{p-1} \left(\frac{1}{k} + \sum_{j=1}^{k} a^j \cdot \frac{1}{k}\binom{k}{j} \right) = \sum_{k=1}^{p-1} \frac{1}{k} + \sum_{j=1}^{p-1} a^j \sum_{k=j}^{p-1} \frac{1}{k}\binom{k}{j} a^k.$$

Note that $\frac{1}{k} + \frac{1}{p-k} = \frac{p}{k(p-k)} \equiv 0 \pmod p$ for all $1 \le k \le p-1$; hence the first sum vanishes modulo p. For the second sum, we use the relation $\frac{1}{k}\binom{k}{j} = \frac{1}{j}\binom{k-1}{j-1}$ to obtain

$$S_{a+1} \equiv \sum_{j=1}^{p-1} \frac{a^j}{j} \sum_{k=1}^{p-1} \binom{k-1}{j-1} \pmod p.$$

Finally, from the relation

$$\sum_{k=1}^{p-1} \binom{k-1}{j-1} = \binom{p-1}{j} = \frac{(p-1)(p-2)\dots(p-j)}{j!} \equiv (-1)^j \pmod p$$

we obtain

$$S_{a+1} \equiv \sum_{j=1}^{p-1} \frac{a^j(-1)^j}{j!} = S_{-a}. \qquad \square$$

Now we turn to the problem. Using the lemma we get

$$S_3 - 3S_2 \equiv S_{-2} - 3S_2 = \sum_{\substack{1 \le k \le p-1 \\ k \text{ is even}}} \frac{-2 \cdot 2^k}{k} + \sum_{\substack{1 \le k \le p-1 \\ k \text{ is odd}}} \frac{-4 \cdot 2^k}{k} \pmod p. \tag{1}$$

The first sum in (1) expands as

$$\sum_{\ell=1}^{(p-1)/2} \frac{-2 \cdot 2^{2\ell}}{2\ell} = -\sum_{\ell=1}^{(p-1)/2} \frac{4^\ell}{\ell}.$$

Next, using FERMAT's theorem, we expand the second sum in (1) as

$$-\sum_{\ell=1}^{(p-1)/2} \frac{2^{2\ell+1}}{2\ell-1} \equiv -\sum_{\ell=1}^{(p-1)/2} \frac{2^{p+2\ell}}{p+2\ell-1} = -\sum_{m=(p+1)/2}^{p-1} \frac{2 \cdot 4^m}{2m} = -\sum_{m=(p+1)/2}^{p-1} \frac{4^m}{m} \pmod p$$

(here we set $m = \ell + \frac{p-1}{2}$). Hence,

$$S_3 - 3S_2 \equiv -\sum_{\ell=1}^{(p-1)/2} \frac{4^\ell}{\ell} - \sum_{m=(p+1)/2}^{p-1} \frac{4^m}{m} = -S_4 \pmod p.$$

N8

Let k be a positive integer and set $n = 2^k + 1$. Prove that n is a prime number if and only if the following holds: there is a permutation $a_1, \dots, a_{n-1}$ of the numbers $1, 2, \dots, n-1$ and a sequence of integers $g_1, g_2, \dots, g_{n-1}$ such that n divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \dots, n-1\}$, where we set $a_n = a_1$.

Solution. Let $N = \{1, 2, \dots, n-1\}$. For $a, b \in N$, we say that b *follows* a if there exists an integer g such that $b \equiv g^a \pmod{n}$ and denote this property as $a \to b$. This way we have a directed graph with N as set of vertices. If $a_1, \dots, a_{n-1}$ is a permutation of $1, 2, \dots, n-1$ such that $a_1 \to a_2 \to \dots \to a_{n-1} \to a_1$ then this is a *Hamiltonian cycle* in the graph.

Step I. First consider the case when n is composite. Let $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$ be its prime factorization. All primes p_i are odd.

Suppose that $\alpha_i > 1$ for some i. For all integers a, g with $a \geq 2$, we have $g^a \not\equiv p_i \pmod{p_i^2}$, because g^a is either divisible by p_i^2 or it is not divisible by p_i. It follows that in any Hamiltonian cycle p_i comes immediately after 1. The same argument shows that $2p_i$ also should come immediately after 1, which is impossible. Hence, there is no Hamiltonian cycle in the graph.

Now suppose that n is square-free. We have $n = p_1 p_2 \dots p_s > 9$ and $s \geq 2$. Assume that there exists a Hamiltonian cycle. There are $\frac{n-1}{2}$ even numbers in this cycle, and each number which follows one of them should be a quadratic residue modulo n. So, there should be at least $\frac{n-1}{2}$ nonzero quadratic residues modulo n. On the other hand, for each p_i there exist exactly $\frac{p_i+1}{2}$ quadratic residues modulo p_i; by the Chinese Remainder Theorem, the number of quadratic residues modulo n is exactly $\frac{p_1+1}{2} \cdot \frac{p_2+1}{2} \cdot \ldots \cdot \frac{p_s+1}{2}$, including 0. Then we have a contradiction by

$$\frac{p_1+1}{2} \cdot \frac{p_2+1}{2} \cdot \ldots \cdot \frac{p_s+1}{2} \leq \frac{2p_1}{3} \cdot \frac{2p_2}{3} \cdot \ldots \cdot \frac{2p_s}{3} = \left(\frac{2}{3}\right)^s n \leq \frac{4n}{9} < \frac{n-1}{2}.$$

This proves the "if"-part of the problem.

Step II. Now suppose that n is prime. For any $a \in N$, denote by $\nu_2(a)$ the exponent of 2 in the prime factorization of a, and let $\mu(a) = \max\{t \in [0, k] \mid 2^t \to a\}$.

Lemma. For any $a, b \in N$, we have $a \to b$ if and only if $\nu_2(a) \leq \mu(b)$.

Proof. Let $\ell = \nu_2(a)$ and $m = \mu(b)$.

Suppose $\ell \leq m$. Since b follows 2^m, there exists some g_0 such that $b \equiv g_0^{2^m} \pmod{n}$. By $\gcd(a, n-1) = 2^\ell$ there exist some integers p and q such that $pa - q(n-1) = 2^\ell$. Choosing $g = g_0^{2^{m-\ell}p}$ we have $g^a = g_0^{2^{m-\ell}pa} = g_0^{2^m + 2^{m-\ell}q(n-1)} \equiv g_0^{2^m} \equiv b \pmod{n}$ by Fermat's theorem. Hence, $a \to b$.

To prove the reverse statement, suppose that $a \to b$, so $b \equiv g^a \pmod{n}$ with some g. Then $b \equiv (g^{a/2^\ell})^{2^\ell}$, and therefore $2^\ell \to b$. By the definition of $\mu(b)$, we have $\mu(b) \geq \ell$. The lemma is proved. $\square$

Now for every i with $0 \le i \le k$, let

$$\begin{aligned} A_i &= \{a \in N \mid \nu_2(a) = i\}, \\ B_i &= \{a \in N \mid \mu(a) = i\}, \\ \text{and } C_i &= \{a \in N \mid \mu(a) \ge i\} = B_i \cup B_{i+1} \cup \ldots \cup B_k. \end{aligned}$$

We claim that $|A_i| = |B_i|$ for all $0 \le i \le k$. Obviously we have $|A_i| = 2^{k-i-1}$ for all $i = 0, \ldots, k-1$, and $|A_k| = 1$. Now we determine $|C_i|$. We have $|C_0| = n - 1$ and by FERMAT's theorem we also have $C_k = \{1\}$, so $|C_k| = 1$. Next, notice that $C_{i+1} = \{x^2 \mod n \mid x \in C_i\}$. For every $a \in N$, the relation $x^2 \equiv a \pmod n$ has at most two solutions in N. Therefore we have $2|C_{i+1}| \le |C_i|$, with the equality achieved only if for every $y \in C_{i+1}$, there exist distinct elements $x, x' \in C_i$ such that $x^2 \equiv x'^2 \equiv y \pmod n$ (this implies $x + x' = n$). Now, since $2^k|C_k| = |C_0|$, we obtain that this equality should be achieved in each step. Hence $|C_i| = 2^{k-i}$ for $0 \le i \le k$, and therefore $|B_i| = 2^{k-i-1}$ for $0 \le i \le k-1$ and $|B_k| = 1$.

From the previous arguments we can see that for each $z \in C_i$ $(0 \le i < k)$ the equation $x^2 \equiv z^2 \pmod n$ has two solutions in C_i, so we have $n - z \in C_i$. Hence, for each $i = 0, 1, \ldots, k-1$, exactly half of the elements of C_i are odd. The same statement is valid for $B_i = C_i \setminus C_{i+1}$ for $0 \le i \le k-2$. In particular, each such B_i contains an odd number. Note that $B_k = \{1\}$ also contains an odd number, and $B_{k-1} = \{2^k\}$ since C_{k-1} consists of the two square roots of 1 modulo n.

Step III. Now we construct a Hamiltonian cycle in the graph. First, for each i with $0 \le i \le k$, connect the elements of A_i to the elements of B_i by means of an arbitrary bijection. After performing this for every i, we obtain a subgraph with all vertices having in-degree 1 and out-degree 1, so the subgraph is a disjoint union of cycles. If there is a unique cycle, we are done. Otherwise, we modify the subgraph in such a way that the previous property is preserved and the number of cycles decreases; after a finite number of steps we arrive at a single cycle.

For every cycle C, let $\lambda(C) = \min_{c \in C} \nu_2(c)$. Consider a cycle C for which $\lambda(C)$ is maximal. If $\lambda(C) = 0$, then for any other cycle C' we have $\lambda(C') = 0$. Take two arbitrary vertices $a \in C$ and $a' \in C'$ such that $\nu_2(a) = \nu_2(a') = 0$; let their direct successors be b and b', respectively. Then we can unify C and C' to a single cycle by replacing the edges $a \to b$ and $a' \to b'$ by $a \to b'$ and $a' \to b$.

Now suppose that $\lambda = \lambda(C) \ge 1$; let $a \in C \cap A_\lambda$. If there exists some $a' \in A_\lambda \setminus C$, then a' lies in another cycle C' and we can merge the two cycles in exactly the same way as above. So, the only remaining case is $A_\lambda \subset C$. Since the edges from A_λ lead to B_λ, we get also $B_\lambda \subset C$. If $\lambda \ne k-1$ then B_λ contains an odd number; this contradicts the assumption $\lambda(C) > 0$. Finally, if $\lambda = k-1$, then C contains 2^{k-1} which is the only element of A_{k-1}. Since $B_{k-1} = \{2^k\} = A_k$ and $B_k = \{1\}$, the cycle C contains the path $2^{k-1} \to 2^k \to 1$ and it contains an odd number again. This completes the proof of the "only if"-part of the problem.

Comment 1. The lemma and the fact $|A_i| = |B_i|$ together show that for every edge $a \to b$ of the Hamiltonian cycle, $\nu_2(a) = \mu(b)$ must hold. After this observation, the Hamiltonian cycle can be built in many ways. For instance, it is possible to select edges from A_i to B_i for $i = k, k-1, \dots, 1$ in such a way that they form disjoint paths; at the end all these paths will have odd endpoints. In the final step, the paths can be closed to form a unique cycle.

Comment 2. Step II is an easy consequence of some basic facts about the multiplicative group modulo the prime $n = 2^k + 1$. The Lemma follows by noting that this group has order 2^k, so the a-th powers are exactly the $2^{\nu_2(a)}$-th powers. Using the existence of a primitive root g modulo n one sees that the map from $\{1, 2, \dots, n-1\}$ to itself that sends a to $g^a \bmod n$ is a bijection that sends A_i to B_i for each $i \in \{0, \dots, k\}$.

Shortlisted Problems with Solutions

53$^{\text{rd}}$ International Mathematical Olympiad

Mar del Plata, Argentina 2012

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2013

Contributing Countries

The Organizing Committee and the Problem Selection Committee of IMO 2012 thank the following 40 countries for contributing 136 problem proposals:

Australia, Austria, Belarus, Belgium, Bulgaria, Canada, Cyprus, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hong Kong, India, Iran, Ireland, Israel, Japan, Kazakhstan, Luxembourg, Malaysia, Montenegro, Netherlands, Norway, Pakistan, Romania, Russia, Serbia, Slovakia, Slovenia, South Africa, South Korea, Sweden, Thailand, Ukraine, United Kingdom, United States of America, Uzbekistan

Problem Selection Committee

Martín Avendaño
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Géza Kós
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Algebra

A1. Find all the functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a)$$

for all integers a, b, c satisfying $a + b + c = 0$.

A2. Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rationals respectively.

a) Does there exist a partition of $\mathbb{Z}$ into three non-empty subsets A, B, C such that the sets $A + B$, $B + C$, $C + A$ are disjoint?

b) Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B, C such that the sets $A + B$, $B + C$, $C + A$ are disjoint?

Here $X + Y$ denotes the set $\{x + y \mid x \in X, y \in Y\}$, for $X, Y \subseteq \mathbb{Z}$ and $X, Y \subseteq \mathbb{Q}$.

A3. Let $a_2, \ldots, a_n$ be $n - 1$ positive real numbers, where $n \geq 3$, such that $a_2a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2(1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

A4. Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

A5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1 + xy) - f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}$$

and $f(-1) \neq 0$.

A6. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

A7. We say that a function $f : \mathbb{R}^k \to \mathbb{R}$ is a metapolynomial if, for some positive integers m and n, it can be represented in the form

$$f(x_1, \ldots, x_k) = \max_{i=1,\ldots,m} \min_{j=1,\ldots,n} P_{i,j}(x_1, \ldots, x_k)$$

where $P_{i,j}$ are multivariate polynomials. Prove that the product of two metapolynomials is also a metapolynomial.

Combinatorics

C1. Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

C2. Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

C3. In a 999×999 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1 and C_3 white and C_2 red. Find the maximum value T can attain.

C4. Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules:

- On every move of his B passes 1 coin from every box to an adjacent box.
- On every move of hers A chooses several coins that were *not* involved in B's previous move and are in different boxes. She passes every chosen coin to an adjacent box.

Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

C5. The columns and the rows of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is 0, 1 or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.

Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

C6. Let k and n be fixed positive integers. In the liar's guessing game, Amy chooses integers x and N with $1 \leq x \leq N$. She tells Ben what N is, but not what x is. Ben may then repeatedly ask Amy whether $x \in S$ for arbitrary sets S of integers. Amy will always answer with *yes* or *no*, but she might lie. The only restriction is that she can lie at most k times in a row. After he has asked as many questions as he wants, Ben must specify a set of at most n positive integers. If x is in this set he wins; otherwise, he loses. Prove that:

a) If $n \geq 2^k$ then Ben can always win.

b) For sufficiently large k there exist $n \geq 1.99^k$ such that Ben cannot guarantee a win.

C7. There are given 2^{500} points on a circle labeled $1, 2, \ldots, 2^{500}$ in some order. Prove that one can choose 100 pairwise disjoint chords joining some of these points so that the 100 sums of the pairs of numbers at the endpoints of the chosen chords are equal.

Geometry

G1. In the triangle ABC the point J is the center of the excircle opposite to A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

G2. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

G3. In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

G4. Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

G5. Let ABC be a triangle with $\angle BCA = 90°$, and let C_0 be the foot of the altitude from C. Choose a point X in the interior of the segment CC_0, and let K, L be the points on the segments AX, BX for which $BK = BC$ and $AL = AC$ respectively. Denote by M the intersection of AL and BK. Show that $MK = ML$.

G6. Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

G7. Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

G8. Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Number Theory

N1. Call admissible a set A of integers that has the following property:

If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k.

Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

N2. Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

N3. Determine all integers $m \geq 2$ such that every n with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

N4. An integer a is called friendly if the equation $(m^2 + n)(n^2 + m) = a(m - n)^3$ has a solution over the positive integers.

a) Prove that there are at least 500 friendly integers in the set $\{1, 2, \ldots, 2012\}$.

b) Decide whether $a = 2$ is friendly.

N5. For a nonnegative integer n define $rad(n) = 1$ if $n = 0$ or $n = 1$, and $rad(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of n. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $rad(f(n))$ divides $rad(f(n^{rad(n)}))$ for every nonnegative integer n.

N6. Let x and y be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer n, prove that $x = 1$.

N7. Find all $n \in \mathbb{N}$ for which there exist nonnegative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

N8. Prove that for every prime $p > 100$ and every integer r there exist two integers a and b such that p divides $a^2 + b^5 - r$.

Algebra

A1. Find all the functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a)$$

for all integers a, b, c satisfying $a + b + c = 0$.

Solution. The substitution $a = b = c = 0$ gives $3f(0)^2 = 6f(0)^2$, hence

$$f(0) = 0. \tag{1}$$

The substitution $b = -a$ and $c = 0$ gives $((f(a) - f(-a))^2 = 0$. Hence f is an even function:

$$f(a) = f(-a) \quad \text{for all } a \in \mathbb{Z}. \tag{2}$$

Now set $b = a$ and $c = -2a$ to obtain $2f(a)^2 + f(2a)^2 = 2f(a)^2 + 4f(a)f(2a)$. Hence

$$f(2a) = 0 \text{ or } f(2a) = 4f(a) \quad \text{for all } a \in \mathbb{Z}. \tag{3}$$

If $f(r) = 0$ for some $r \geq 1$ then the substitution $b = r$ and $c = -a-r$ gives $(f(a+r)-f(a))^2 = 0$. So f is periodic with period r, i. e.

$$f(a + r) = f(a) \quad \text{for all } a \in \mathbb{Z}.$$

In particular, if $f(1) = 0$ then f is constant, thus $f(a) = 0$ for all $a \in \mathbb{Z}$. This function clearly satisfies the functional equation. For the rest of the analysis, we assume $f(1) = k \neq 0$.

By (3) we have $f(2) = 0$ or $f(2) = 4k$. If $f(2) = 0$ then f is periodic of period 2, thus $f(\text{even}) = 0$ and $f(\text{odd}) = k$. This function is a solution for every k. We postpone the verification; for the sequel assume $f(2) = 4k \neq 0$.

By (3) again, we have $f(4) = 0$ or $f(4) = 16k$. In the first case f is periodic of period 4, and $f(3) = f(-1) = f(1) = k$, so we have $f(4n) = 0$, $f(4n+1) = f(4n+3) = k$, and $f(4n+2) = 4k$ for all $n \in \mathbb{Z}$. This function is a solution too, which we justify later. For the rest of the analysis, we assume $f(4) = 16k \neq 0$.

We show now that $f(3) = 9k$. In order to do so, we need two substitutions:

$$a = 1, b = 2, c = -3 \Longrightarrow f(3)^2 - 10kf(3) + 9k^2 = 0 \Longrightarrow f(3) \in \{k, 9k\},$$
$$a = 1, b = 3, c = -4 \Longrightarrow f(3)^2 - 34kf(3) + 225k^2 = 0 \Longrightarrow f(3) \in \{9k, 25k\}.$$

Therefore $f(3) = 9k$, as claimed. Now we prove inductively that the only remaining function is $f(x) = kx^2$, $x \in \mathbb{Z}$. We proved this for $x = 0, 1, 2, 3, 4$. Assume that $n \geq 4$ and that $f(x) = kx^2$ holds for all integers $x \in [0, n]$. Then the substitutions $a = n$, $b = 1$, $c = -n-1$ and $a = n-1$, $b = 2$, $c = -n-1$ lead respectively to

$$f(n+1) \in \{k(n+1)^2, k(n-1)^2\} \qquad \text{and} \qquad f(n+1) \in \{k(n+1)^2, k(n-3)^2\}.$$

Since $k(n-1)^2 \neq k(n-3)^2$ for $n \neq 2$, the only possibility is $f(n+1) = k(n+1)^2$. This completes the induction, so $f(x) = kx^2$ for all $x \geq 0$. The same expression is valid for negative values of x since f is even. To verify that $f(x) = kx^2$ is actually a solution, we need to check the identity $a^4 + b^4 + (a+b)^4 = 2a^2b^2 + 2a^2(a+b)^2 + 2b^2(a+b)^2$, which follows directly by expanding both sides.

Therefore the only possible solutions of the functional equation are the constant function $f_1(x) = 0$ and the following functions:

$$f_2(x) = kx^2 \qquad f_3(x) = \begin{cases} 0 & x \text{ even} \\ k & x \text{ odd} \end{cases} \qquad f_4(x) = \begin{cases} 0 & x \equiv 0 \pmod 4 \\ k & x \equiv 1 \pmod 2 \\ 4k & x \equiv 2 \pmod 4 \end{cases}$$

for any non-zero integer k. The verification that they are indeed solutions was done for the first two. For f_3 note that if $a + b + c = 0$ then either a, b, c are all even, in which case $f(a) = f(b) = f(c) = 0$, or one of them is even and the other two are odd, so both sides of the equation equal $2k^2$. For f_4 we use similar parity considerations and the symmetry of the equation, which reduces the verification to the triples $(0, k, k)$, $(4k, k, k)$, $(0, 0, 0)$, $(0, 4k, 4k)$. They all satisfy the equation.

Comment. We used several times the same fact: For any $a, b \in \mathbb{Z}$ the functional equation is a quadratic equation in $f(a + b)$ whose coefficients depend on $f(a)$ and $f(b)$:

$$f(a+b)^2 - 2(f(a) + f(b))f(a+b) + (f(a) - f(b))^2 = 0.$$

Its discriminant is $16f(a)f(b)$. Since this value has to be non-negative for any $a, b \in \mathbb{Z}$, we conclude that either f or $-f$ is always non-negative. Also, if f is a solution of the functional equation, then $-f$ is also a solution. Therefore we can assume $f(x) \geq 0$ for all $x \in \mathbb{Z}$. Now, the two solutions of the quadratic equation are

$$f(a+b) \in \left\{ \left(\sqrt{f(a)} + \sqrt{f(b)}\right)^2, \left(\sqrt{f(a)} - \sqrt{f(b)}\right)^2 \right\} \quad \text{for all } a, b \in \mathbb{Z}.$$

The computation of $f(3)$ from $f(1)$, $f(2)$ and $f(4)$ that we did above follows immediately by setting $(a, b) = (1, 2)$ and $(a, b) = (1, -4)$. The inductive step, where $f(n + 1)$ is derived from $f(n)$, $f(n - 1)$, $f(2)$ and $f(1)$, follows immediately using $(a, b) = (n, 1)$ and $(a, b) = (n - 1, 2)$.

A2. Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rationals respectively.

a) Does there exist a partition of $\mathbb{Z}$ into three non-empty subsets A, B, C such that the sets $A+B$, $B+C$, $C+A$ are disjoint?

b) Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B, C such that the sets $A+B$, $B+C$, $C+A$ are disjoint?

Here $X+Y$ denotes the set $\{x+y \mid x \in X, y \in Y\}$, for $X, Y \subseteq \mathbb{Z}$ and $X, Y \subseteq \mathbb{Q}$.

Solution 1. a) The residue classes modulo 3 yield such a partition:

$$A = \{3k \mid k \in \mathbb{Z}\}, \quad B = \{3k+1 \mid k \in \mathbb{Z}\}, \quad C = \{3k+2 \mid k \in \mathbb{Z}\}.$$

b) The answer is *no*. Suppose that $\mathbb{Q}$ can be partitioned into non-empty subsets A, B, C as stated. Note that for all $a \in A$, $b \in B$, $c \in C$ one has

$$a+b-c \in C, \quad b+c-a \in A, \quad c+a-b \in B. \tag{1}$$

Indeed $a+b-c \notin A$ as $(A+B) \cap (A+C) = \emptyset$, and similarly $a+b-c \notin B$, hence $a+b-c \in C$. The other two relations follow by symmetry. Hence $A+B \subset C+C$, $B+C \subset A+A$, $C+A \subset B+B$.

The opposite inclusions also hold. Let $a, a' \in A$ and $b \in B$, $c \in C$ be arbitrary. By (1) $a'+c-b \in B$, and since $a \in A$, $c \in C$, we use (1) again to obtain

$$a+a'-b = a+(a'+c-b)-c \in C.$$

So $A+A \subset B+C$ and likewise $B+B \subset C+A$, $C+C \subset A+B$. In summary

$$B+C = A+A, \qquad C+A = B+B, \qquad A+B = C+C.$$

Furthermore suppose that $0 \in A$ without loss of generality. Then $B = \{0\} + B \subset A+B$ and $C = \{0\} + C \subset A+C$. So, since $B+C$ is disjoint with $A+B$ and $A+C$, it is also disjoint with B and C. Hence $B+C$ is contained in $\mathbb{Z} \setminus (B \cup C) = A$. Because $B+C = A+A$, we obtain $A+A \subset A$. On the other hand $A = \{0\} + A \subset A+A$, implying $A = A+A = B+C$.

Therefore $A+B+C = A+A+A = A$, and now $B+B = C+A$ and $C+C = A+B$ yield $B+B+B = A+B+C = A$, $C+C+C = A+B+C = A$. In particular if $r \in \mathbb{Q} = A \cup B \cup C$ is arbitrary then $3r \in A$.

However such a conclusion is impossible. Take any $b \in B$ ($B \neq \emptyset$) and let $r = b/3 \in \mathbb{Q}$. Then $b = 3r \in A$ which is a contradiction.

Solution 2. We prove that the example for $\mathbb{Z}$ from the first solution is unique, and then use this fact to solve part b).

Let $\mathbb{Z} = A \cup B \cup C$ be a partition of $\mathbb{Z}$ with $A, B, C \neq \emptyset$ and $A+B$, $B+C$, $C+A$ disjoint. We need the relations (1) which clearly hold for $\mathbb{Z}$. Fix two consecutive integers from different sets, say $b \in B$ and $c = b+1 \in C$. For every $a \in A$ we have, in view of (1), $a-1 = a+b-c \in C$ and $a+1 = a+c-b \in B$. So every $a \in A$ is preceded by a number from C and followed by a number from B.

In particular there are pairs of the form $c, c+1$ with $c \in C$, $c+1 \in A$. For such a pair and any $b \in B$ analogous reasoning shows that each $b \in B$ is preceded by a number from A and followed by a number from C. There are also pairs $b, b-1$ with $b \in B$, $b-1 \in A$. We use them in a similar way to prove that each $c \in C$ is preceded by a number from B and followed by a number from A.

By putting the observations together we infer that A, B, C are the three congruence classes modulo 3. Observe that all multiples of 3 are in the set of the partition that contains 0.

Now we turn to part b). Suppose that there is a partition of $\mathbb{Q}$ with the given properties. Choose three rationals $r_i = p_i/q_i$ from the three sets A, B, C, $i = 1, 2, 3$, and set $N = 3q_1q_2q_3$.

Let $S \subset \mathbb{Q}$ be the set of fractions with denominators N (irreducible or not). It is obtained through multiplication of every integer by the constant $1/N$, hence closed under sums and differences. Moreover, if we identify each $k \in \mathbb{Z}$ with $k/N \in S$ then S is essentially the set $\mathbb{Z}$ with respect to addition. The numbers r_i belong to S because

$$r_1 = \frac{3p_1q_2q_3}{N}, \qquad r_2 = \frac{3p_2q_3q_1}{N}, \qquad r_3 = \frac{3p_3q_1q_2}{N}.$$

The partition $\mathbb{Q} = A \cup B \cup C$ of $\mathbb{Q}$ induces a partition $S = A' \cup B' \cup C'$ of S, with $A' = A \cap S$, $B' = B \cap S$, $C' = C \cap S$. Clearly $A' + B'$, $B' + C'$, $C' + A'$ are disjoint, so this partition has the properties we consider.

By the uniqueness of the example for $\mathbb{Z}$ the sets A', B', C' are the congruence classes modulo 3, multiplied by $1/N$. Also all multiples of $3/N$ are in the same set, A', B' or C'. This holds for r_1, r_2, r_3 in particular as they are all multiples of $3/N$. However r_1, r_2, r_3 are in different sets A', B', C' since they were chosen from different sets A, B, C. The contradiction ends the proof.

Comment. The uniqueness of the example for $\mathbb{Z}$ can also be deduced from the argument in the first solution.

A3. Let $a_2, \ldots, a_n$ be $n-1$ positive real numbers, where $n \geq 3$, such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

Solution. The substitution $a_2 = \dfrac{x_2}{x_1}$, $a_3 = \dfrac{x_3}{x_2}$, $\ldots$, $a_n = \dfrac{x_1}{x_{n-1}}$ transforms the original problem into the inequality

$$(x_1+x_2)^2(x_2+x_3)^3 \cdots (x_{n-1}+x_1)^n > n^n x_1^2 x_2^3 \cdots x_{n-1}^n \tag{$*$}$$

for all $x_1, \ldots, x_{n-1} > 0$. To prove this, we use the AM-GM inequality for each factor of the left-hand side as follows:

$$\begin{array}{rcccl}
(x_1+x_2)^2 & & & & \geq 2^2 x_1 x_2 \\
(x_2+x_3)^3 & = & \left(2\left(\frac{x_2}{2}\right)+x_3\right)^3 & & \geq 3^3\left(\frac{x_2}{2}\right)^2 x_3 \\
(x_3+x_4)^4 & = & \left(3\left(\frac{x_3}{3}\right)+x_4\right)^4 & & \geq 4^4\left(\frac{x_3}{3}\right)^3 x_4 \\
\vdots & & \vdots & & \vdots \\
(x_{n-1}+x_1)^n & = & \left((n-1)\left(\frac{x_{n-1}}{n-1}\right)+x_1\right)^n & & \geq n^n\left(\frac{x_{n-1}}{n-1}\right)^{n-1} x_1.
\end{array}$$

Multiplying these inequalities together gives (*), with inequality sign $\geq$ instead of $>$. However for the equality to occur it is necessary that $x_1 = x_2$, $x_2 = 2x_3$, $\ldots$, $x_{n-1} = (n-1)x_1$, implying $x_1 = (n-1)!x_1$. This is impossible since $x_1 > 0$ and $n \geq 3$. Therefore the inequality is strict.

Comment. One can avoid the substitution $a_i = x_i/x_{i-1}$. Apply the weighted AM-GM inequality to each factor $(1+a_k)^k$, with the same weights like above, to obtain

$$(1+a_k)^k = \left((k-1)\frac{1}{k-1}+a_k\right)^k \geq \frac{k^k}{(k-1)^{k-1}} a_k.$$

Multiplying all these inequalities together gives

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n \geq n^n a_2 a_3 \cdots a_n = n^n.$$

The same argument as in the proof above shows that the equality cannot be attained.

A4. Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

Solution 1. Since $\deg f > \deg g$, we have $|g(x)/f(x)| < 1$ for sufficiently large x; more precisely, there is a real number R such that $|g(x)/f(x)| < 1$ for all x with $|x| > R$. Then for all such x and all primes p we have

$$\left|pf(x) + g(x)\right| \geq \left|f(x)\right| \left(p - \frac{|g(x)|}{|f(x)|}\right) > 0.$$

Hence all real roots of the polynomials $pf + g$ lie in the interval $[-R, R]$.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$ where $n > m$, $a_n \neq 0$ and $b_m \neq 0$. Upon replacing $f(x)$ and $g(x)$ by $a_n^{n-1} f(x/a_n)$ and $a_n^{n-1} g(x/a_n)$ respectively, we reduce the problem to the case $a_n = 1$. In other words one can assume that f is monic. Then the leading coefficient of $pf + g$ is p, and if $r = u/v$ is a rational root of $pf + g$ with $(u, v) = 1$ and $v > 0$, then either $v = 1$ or $v = p$.

First consider the case when $v = 1$ infinitely many times. If $v = 1$ then $|u| \leq R$, so there are only finitely many possibilities for the integer u. Therefore there exist distinct primes p and q for which we have the same value of u. Then the polynomials $pf + g$ and $qf + g$ share this root, implying $f(u) = g(u) = 0$. So in this case f and g have an integer root in common.

Now suppose that $v = p$ infinitely many times. By comparing the exponent of p in the denominators of $pf(u/p)$ and $g(u/p)$ we get $m = n - 1$ and $pf(u/p) + g(u/p) = 0$ reduces to an equation of the form

$$\left(u^n + a_{n-1}pu^{n-1} + \ldots + a_0 p^n\right) + \left(b_{n-1}u^{n-1} + b_{n-2}pu^{n-2} + \ldots + b_0 p^{n-1}\right) = 0.$$

The equation above implies that $u^n + b_{n-1}u^{n-1}$ is divisible by p and hence, since $(u, p) = 1$, we have $u + b_{n-1} = pk$ with some integer k. On the other hand all roots of $pf + g$ lie in the interval $[-R, R]$, so that

$$\frac{|pk - b_{n-1}|}{p} = \frac{|u|}{p} < R,$$
$$|k| < R + \frac{|b_{n-1}|}{p} < R + |b_{n-1}|.$$

Therefore the integer k can attain only finitely many values. Hence there exists an integer k such that the number $\frac{pk - b_{n-1}}{p} = k - \frac{b_{n-1}}{p}$ is a root of $pf + g$ for infinitely many primes p. For these primes we have

$$f\left(k - b_{n-1}\frac{1}{p}\right) + \frac{1}{p} g\left(k - b_{n-1}\frac{1}{p}\right) = 0.$$

So the equation

$$f\left(k - b_{n-1}x\right) + xg\left(k - b_{n-1}x\right) = 0 \tag{1}$$

has infinitely many solutions of the form $x = 1/p$. Since the left-hand side is a polynomial, this implies that (1) is a polynomial identity, so it holds for all real x. In particular, by substituting $x = 0$ in (1) we get $f(k) = 0$. Thus the integer k is a root of f.

In summary the monic polynomial f obtained after the initial reduction always has an integer root. Therefore the original polynomial f has a rational root.

Solution 2. Analogously to the first solution, there exists a real number R such that the complex roots of all polynomials of the form $pf+g$ lie in the disk $|z| \le R$.

For each prime p such that $pf+g$ has a rational root, by GAUSS' lemma $pf+g$ is the product of two integer polynomials, one with degree 1 and the other with degree $\deg f - 1$. Since p is a prime, the leading coefficient of one of these factors divides the leading coefficient of f. Denote that factor by h_p.

By narrowing the set of the primes used we can assume that all polynomials h_p have the same degree and the same leading coefficient. Their complex roots lie in the disk $|z| \le R$, hence VIETA's formulae imply that all coefficients of all polynomials h_p form a bounded set. Since these coefficients are integers, there are only finitely many possible polynomials h_p. Hence there is a polynomial h such that $h_p = h$ for infinitely many primes p.

Finally, if p and q are distinct primes with $h_p = h_q = h$ then h divides $(p-q)f$. Since $\deg h = 1$ or $\deg h = \deg f - 1$, in both cases f has a rational root.

Comment. Clearly the polynomial h is a common factor of f and g. If $\deg h = 1$ then f and g share a rational root. Otherwise $\deg h = \deg f - 1$ forces $\deg g = \deg f - 1$ and g divides f over the rationals.

Solution 3. Like in the first solution, there is a real number R such that the real roots of all polynomials of the form $pf+g$ lie in the interval $[-R, R]$.

Let $p_1 < p_2 < \cdots$ be an infinite sequence of primes so that for every index k the polynomial $p_k f + g$ has a rational root r_k. The sequence $r_1, r_2, \ldots$ is bounded, so it has a convergent subsequence $r_{k_1}, r_{k_2}, \ldots$. Now replace the sequences $(p_1, p_2, \ldots)$ and $(r_1, r_2, \ldots)$ by $(p_{k_1}, p_{k_2}, \ldots)$ and $(r_{k_1}, r_{k_2}, \ldots)$; after this we can assume that the sequence $r_1, r_2, \ldots$ is convergent. Let $\alpha = \lim\limits_{k\to\infty} r_k$. We show that α is a rational root of f.

Over the interval $[-R, R]$, the polynomial g is bounded, $|g(x)| \le M$ with some fixed M. Therefore

$$|f(r_k)| = \left| f(r_k) - \frac{p_k f(r_k) + g(r_k)}{p_k} \right| = \frac{|g(r_k)|}{p_k} \le \frac{M}{p_k} \to 0,$$

and

$$f(\alpha) = f\Big(\lim_{k\to\infty} r_k \Big) = \lim_{k\to\infty} f(r_k) = 0.$$

So α is a root of f indeed.

Now let u_k, v_k be relative prime integers for which $r_k = \frac{u_k}{v_k}$. Let a be the leading coefficient of f, let $b = f(0)$ and $c = g(0)$ be the constant terms of f and g, respectively. The leading coefficient of the polynomial $p_k f + g$ is $p_k a$, its constant term is $p_k b + c$. So v_k divides $p_k a$ and u_k divides $p_k b + c$. Let $p_k b + c = u_k e_k$ (if $p_k b + c = u_k = 0$ then let $e_k = 1$).

We prove that α is rational by using the following fact. *Let (p_n) and (q_n) be sequences of integers such that the sequence (p_n/q_n) converges. If (p_n) or (q_n) is bounded then* $\lim(p_n/q_n)$ *is rational.*

Case 1: There is an infinite subsequence (k_n) of indices such that v_{k_n} divides a. Then (v_{k_n}) is bounded, so $\alpha = \lim_{n\to\infty}(u_{k_n}/v_{k_n})$ is rational.

Case 2: There is an infinite subsequence (k_n) of indices such that v_{k_n} does not divide a. For such indices we have $v_{k_n} = p_{k_n} d_{k_n}$ where d_{k_n} is a divisor of a. Then

$$\alpha = \lim_{n\to\infty} \frac{u_{k_n}}{v_{k_n}} = \lim_{n\to\infty} \frac{p_{k_n} b + c}{p_{k_n} d_{k_n} e_{k_n}} = \lim_{n\to\infty} \frac{b}{d_{k_n} e_{k_n}} + \lim_{n\to\infty} \frac{c}{p_{k_n} d_{k_n} e_{k_n}} = \lim_{n\to\infty} \frac{b}{d_{k_n} e_{k_n}}.$$

Because the numerator b in the last limit is bounded, α is rational.

A5. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1+xy) - f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}$$

and $f(-1) \neq 0$.

Solution. The only solution is the function $f(x) = x - 1$, $x \in \mathbb{R}$.

We set $g(x) = f(x) + 1$ and show that $g(x) = x$ for all real x. The conditions take the form

$$g(1+xy) - g(x+y) = \big(g(x) - 1\big)\big(g(y) - 1\big) \quad \text{for all } x, y \in \mathbb{R} \text{ and } g(-1) \neq 1. \tag{1}$$

Denote $C = g(-1) - 1 \neq 0$. Setting $y = -1$ in (1) gives

$$g(1-x) - g(x-1) = C(g(x) - 1). \tag{2}$$

Set $x = 1$ in (2) to obtain $C(g(1) - 1) = 0$. Hence $g(1) = 1$ as $C \neq 0$. Now plugging in $x = 0$ and $x = 2$ yields $g(0) = 0$ and $g(2) = 2$ respectively.

We pass on to the key observations

$$g(x) + g(2-x) = 2 \quad \text{for all } x \in \mathbb{R}, \tag{3}$$
$$g(x+2) - g(x) = 2 \quad \text{for all } x \in \mathbb{R}. \tag{4}$$

Replace x by $1 - x$ in (2), then change x to $-x$ in the resulting equation. We obtain the relations $g(x) - g(-x) = C(g(1-x) - 1)$, $g(-x) - g(x) = C(g(1+x) - 1)$. Then adding them up leads to $C(g(1-x) + g(1+x) - 2) = 0$. Thus $C \neq 0$ implies (3).

Let u, v be such that $u + v = 1$. Apply (1) to the pairs (u, v) and $(2-u, 2-v)$:

$$g(1+uv) - g(1) = \big(g(u) - 1\big)\big(g(v) - 1\big), \quad g(3+uv) - g(3) = \big(g(2-u) - 1\big)\big(g(2-v) - 1\big).$$

Observe that the last two equations have equal right-hand sides by (3). Hence $u+v = 1$ implies

$$g(uv+3) - g(uv+1) = g(3) - g(1).$$

Each $x \leq 5/4$ is expressible in the form $x = uv + 1$ with $u + v = 1$ (the quadratic function $t^2 - t + (x-1)$ has real roots for $x \leq 5/4$). Hence $g(x+2) - g(x) = g(3) - g(1)$ whenever $x \leq 5/4$. Because $g(x) = x$ holds for $x = 0, 1, 2$, setting $x = 0$ yields $g(3) = 3$. This proves (4) for $x \leq 5/4$. If $x > 5/4$ then $-x < 5/4$ and so $g(2-x) - g(-x) = 2$ by the above. On the other hand (3) gives $g(x) = 2 - g(2-x)$, $g(x+2) = 2 - g(-x)$, so that $g(x+2) - g(x) = g(2-x) - g(-x) = 2$. Thus (4) is true for all $x \in \mathbb{R}$.

Now replace x by $-x$ in (3) to obtain $g(-x) + g(2+x) = 2$. In view of (4) this leads to $g(x) + g(-x) = 0$, i. e. $g(-x) = -g(x)$ for all x. Taking this into account, we apply (1) to the pairs $(-x, y)$ and $(x, -y)$:

$$g(1-xy) - g(-x+y) = \big(g(x) + 1\big)\big(1 - g(y)\big), \quad g(1-xy) - g(x-y) = \big(1 - g(x)\big)\big(g(y) + 1\big).$$

Adding up yields $g(1-xy) = 1 - g(x)g(y)$. Then $g(1+xy) = 1 + g(x)g(y)$ by (3). Now the original equation (1) takes the form $g(x+y) = g(x) + g(y)$. Hence g is additive.

By additvity $g(1+xy) = g(1) + g(xy) = 1 + g(xy)$; since $g(1+xy) = 1 + g(x)g(y)$ was shown above, we also have $g(xy) = g(x)g(y)$ (g is multiplicative). In particular $y = x$ gives $g(x^2) = g(x)^2 \geq 0$ for all x, meaning that $g(x) \geq 0$ for $x \geq 0$. Since g is additive and bounded from below on $[0, +\infty)$, it is linear; more exactly $g(x) = g(1)x = x$ for all $x \in \mathbb{R}$.

In summary $f(x) = x - 1$, $x \in \mathbb{R}$. It is straightforward that this function satisfies the requirements.

Comment. There are functions that satisfy the given equation but vanish at -1, for instance the constant function 0 and $f(x) = x^2 - 1$, $x \in \mathbb{R}$.

A6. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

Solution. We restrict attention to the set

$$S = \{1, f(1), f^2(1), \ldots\}.$$

Observe that S is unbounded because for every number n in S there exists a $k > 0$ such that $f^{2k}(n) = n + k$ is in S. Clearly f maps S into itself; moreover f is injective on S. Indeed if $f^i(1) = f^j(1)$ with $i \neq j$ then the values $f^m(1)$ start repeating periodically from some point on, and S would be finite.

Define $g : S \to S$ by $g(n) = f^{2k_n}(n) = n + k_n$. We prove that g is injective too. Suppose that $g(a) = g(b)$ with $a < b$. Then $a + k_a = f^{2k_a}(a) = f^{2k_b}(b) = b + k_b$ implies $k_a > k_b$. So, since f is injective on S, we obtain

$$f^{2(k_a - k_b)}(a) = b = a + (k_a - k_b).$$

However this contradicts the minimality of k_a as $0 < k_a - k_b < k_a$.

Let T be the set of elements of S that are not of the form $g(n)$ with $n \in S$. Note that $1 \in T$ by $g(n) > n$ for $n \in S$, so T is non-empty. For each $t \in T$ denote $C_t = \{t, g(t), g^2(t), \ldots\}$; call C_t the chain starting at t. Observe that distinct chains are disjoint because g is injective. Each $n \in S \backslash T$ has the form $n = g(n')$ with $n' < n$, $n' \in S$. Repeated applications of the same observation show that $n \in C_t$ for some $t \in T$, i. e. S is the disjoint union of the chains C_t.

If $f^n(1)$ is in the chain C_t starting at $t = f^{n_t}(1)$ then $n = n_t + 2a_1 + \cdots + 2a_j$ with

$$f^n(1) = g^j(f^{n_t}(1)) = f^{2a_j}(f^{2a_{j-1}}(\cdots f^{2a_1}(f^{n_t}(1)))) = f^{n_t}(1) + a_1 + \cdots + a_j.$$

Hence

$$f^n(1) = f^{n_t}(1) + \frac{n - n_t}{2} = t + \frac{n - n_t}{2}. \tag{1}$$

Now we show that T is infinite. We argue by contradiction. Suppose that there are only finitely many chains $C_{t_1}, \ldots, C_{t_r}$, starting at $t_1 < \cdots < t_r$. Fix N. If $f^n(1)$ with $1 \leq n \leq N$ is in C_t then $f^n(1) = t + \frac{n - n_t}{2} \leq t_r + \frac{N}{2}$ by (1). But then the $N + 1$ distinct natural numbers $1, f(1), \ldots, f^N(1)$ are all less than $t_r + \frac{N}{2}$ and hence $N + 1 \leq t_r + \frac{N}{2}$. This is a contradiction if N is sufficiently large, and hence T is infinite.

To complete the argument, choose any k in $\mathbb{N}$ and consider the $k + 1$ chains starting at the first $k + 1$ numbers in T. Let t be the greatest one among these numbers. Then each of the chains in question contains a number not exceeding t, and at least one of them does not contain any number among $t + 1, \ldots, t + k$. So there is a number n in this chain such that $g(n) - n > k$, i. e. $k_n > k$. In conclusion $k_1, k_2, \ldots$ is unbounded.

A7. We say that a function $f : \mathbb{R}^k \to \mathbb{R}$ is a metapolynomial if, for some positive integers m and n, it can be represented in the form

$$f(x_1, \ldots, x_k) = \max_{i=1,\ldots,m} \min_{j=1,\ldots,n} P_{i,j}(x_1, \ldots, x_k)$$

where $P_{i,j}$ are multivariate polynomials. Prove that the product of two metapolynomials is also a metapolynomial.

Solution. We use the notation $f(x) = f(x_1, \ldots, x_k)$ for $x = (x_1, \ldots, x_k)$ and $[m] = \{1, 2, \ldots, m\}$. Observe that if a metapolynomial $f(x)$ admits a representation like the one in the statement for certain positive integers m and n, then they can be replaced by any $m' \geq m$ and $n' \geq n$. For instance, if we want to replace m by $m+1$ then it is enough to define $P_{m+1,j}(x) = P_{m,j}(x)$ and note that repeating elements of a set do not change its maximum nor its minimum. So one can assume that any two metapolynomials are defined with the same m and n. We reserve letters P and Q for polynomials, so every function called $P, P_{i,j}, Q, Q_{i,j}, \ldots$ is a polynomial function.

We start with a lemma that is useful to change expressions of the form $\min\max f_{i,j}$ to ones of the form $\max\min g_{i,j}$.

Lemma. Let $\{a_{i,j}\}$ be real numbers, for all $i \in [m]$ and $j \in [n]$. Then

$$\min_{i\in[m]} \max_{j\in[n]} a_{i,j} = \max_{j_1,\ldots,j_m\in[n]} \min_{i\in[m]} a_{i,j_i},$$

where the max in the right-hand side is over all vectors $(j_1, \ldots, j_m)$ with $j_1, \ldots, j_m \in [n]$.

Proof. We can assume for all i that $a_{i,n} = \max\{a_{i,1}, \ldots, a_{i,n}\}$ and $a_{m,n} = \min\{a_{1,n}, \ldots, a_{m,n}\}$. The left-hand side is $= a_{m,n}$ and hence we need to prove the same for the right-hand side. If $(j_1, j_2, \ldots, j_m) = (n, n, \ldots, n)$ then $\min\{a_{1,j_1}, \ldots, a_{m,j_m}\} = \min\{a_{1,n}, \ldots, a_{m,n}\} = a_{m,n}$ which implies that the right-hand side is $\geq a_{m,n}$. It remains to prove the opposite inequality and this is equivalent to $\min\{a_{1,j_1}, \ldots, a_{m,j_m}\} \leq a_{m,n}$ for all possible $(j_1, j_2, \ldots, j_m)$. This is true because $\min\{a_{1,j_1}, \ldots, a_{m,j_m}\} \leq a_{m,j_m} \leq a_{m,n}$. □

We need to show that the family $\mathcal{M}$ of metapolynomials is closed under multiplication, but it turns out easier to prove more: that it is also closed under addition, maxima and minima.

First we prove the assertions about the maxima and the minima. If $f_1, \ldots, f_r$ are metapolynomials, assume them defined with the same m and n. Then

$$f = \max\{f_1, \ldots, f_r\} = \max\{\max_{i\in[m]} \min_{j\in[n]} P^1_{i,j}, \ldots, \max_{i\in[m]} \min_{j\in[n]} P^r_{i,j}\} = \max_{s\in[r], i\in[m]} \min_{j\in[n]} P^s_{i,j}.$$

It follows that $f = \max\{f_1, \ldots, f_r\}$ is a metapolynomial. The same argument works for the minima, but first we have to replace *min max* by *max min*, and this is done via the lemma.

Another property we need is that if $f = \max\min P_{i,j}$ is a metapolynomial then so is $-f$. Indeed, $-f = \min(-\min P_{i,j}) = \min\max P_{i,j}$.

To prove $\mathcal{M}$ is closed under addition let $f = \max\min P_{i,j}$ and $g = \max\min Q_{i,j}$. Then

$$f(x) + g(x) = \max_{i\in[m]} \min_{j\in[n]} P_{i,j}(x) + \max_{i\in[m]} \min_{j\in[n]} Q_{i,j}(x)$$

$$= \max_{i_1,i_2\in[m]} (\min_{j\in[n]} P_{i_1,j}(x) + \min_{j\in[n]} Q_{i_2,j}(x)) = \max_{i_1,i_2\in[m]} \min_{j_1,j_2\in[n]} \big(P_{i_1,j_1}(x) + Q_{i_2,j_2}(x)\big),$$

and hence $f(x) + g(x)$ is a metapolynomial.

We proved that $\mathcal{M}$ is closed under sums, maxima and minima, in particular any function that can be expressed by sums, max, min, polynomials or even metapolynomials is in $\mathcal{M}$.

We would like to proceed with multiplication along the same lines like with addition, but there is an essential difference. In general the product of the maxima of two sets is not equal to the maximum of the product of the sets. We need to deal with the fact that $a < b$ and $c < d$ do not imply $ac < bd$. However this is true for $a, b, c, d \geq 0$.

In view of this we decompose each function $f(x)$ into its positive part $f^+(x) = \max\{f(x), 0\}$ and its negative part $f^-(x) = \max\{0, -f(x)\}$. Note that $f = f^+ - f^-$ and $f^+, f^- \in \mathcal{M}$ if $f \in \mathcal{M}$. The whole problem reduces to the claim that if f and g are metapolynomials with $f, g \geq 0$ then fg it is also a metapolynomial.

Assuming this claim, consider arbitrary $f, g \in \mathcal{M}$. We have

$$fg = (f^+ - f^-)(g^+ - g^-) = f^+g^+ - f^+g^- - f^-g^+ + f^-g^-,$$

and hence $fg \in \mathcal{M}$. Indeed, $\mathcal{M}$ is closed under addition, also $f^+g^+, f^+g^-, f^-g^+, f^-g^- \in \mathcal{M}$ because $f^+, f^-, g^+, g^- \geq 0$.

It remains to prove the claim. In this case $f, g \geq 0$, and one can try to repeat the argument for the sum. More precisely, let $f = \max\min P_{ij} \geq 0$ and $g = \max\min Q_{ij} \geq 0$. Then

$$fg = \max\min P_{i,j} \cdot \max\min Q_{i,j} = \max\min P^+_{i,j} \cdot \max\min Q^+_{i,j} = \max\min P^+_{i_1,j_1} \cdot Q^+_{i_2,j_2}.$$

Hence it suffices to check that $P^+Q^+ \in \mathcal{M}$ for any pair of polynomials P and Q. This reduces to the identity

$$u^+v^+ = \max\{0, \min\{uv, u, v\}, \min\{uv, uv^2, u^2v\}, \min\{uv, u, u^2v\}, \min\{uv, uv^2, v\}\},$$

with u replaced by $P(x)$ and v replaced by $Q(x)$. The formula is proved by a case-by-case analysis. If $u \leq 0$ or $v \leq 0$ then both sides equal 0. In case $u, v \geq 0$, the right-hand side is clearly $\leq uv$. To prove the opposite inequality we use that uv equals

$$\begin{array}{ll} \min\{uv, u, v\} & \text{if } 0 \leq u, v \leq 1, \\ \min\{uv, uv^2, u^2v\} & \text{if } 1 \leq u, v, \\ \min\{uv, u, u^2v\} & \text{if } 0 \leq v \leq 1 \leq u, \\ \min\{uv, uv^2, v\} & \text{if } 0 \leq u \leq 1 \leq v. \end{array}$$

Comment. The case $k = 1$ is simpler and can be solved by proving that a function $f : \mathbb{R} \to \mathbb{R}$ is a metapolynomial if and only if it is a piecewise polinomial (and continuos) function.

It is enough to prove that all such functions are metapolynomials, and this easily reduces to the following case. Given a polynomial $P(x)$ with $P(0) = 0$, the function f defined by $f(x) = P(x)$ for $x \geq 0$ and 0 otherwise is a metapolynomial. For this last claim, it suffices to prove that $(x^+)^n$ is a metapolynomial, and this follows from the formula $(x^+)^n = \max\{0, \min\{x^{n-1}, x^n\}, \min\{x^n, x^{n+1}\}\}$.

Combinatorics

C1. Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

Solution 1. Note first that the allowed operation does not change the maximum M of the initial sequence. Let $a_1, a_2, \ldots, a_n$ be the numbers obtained at some point of the process. Consider the sum

$$S = a_1 + 2a_2 + \cdots + na_n.$$

We claim that S increases by a positive integer amount with every operation. Let the operation replace the pair (a_i, a_{i+1}) by a pair (c, a_i), where $a_i > a_{i+1}$ and $c = a_{i+1}+1$ or $c = a_i-1$. Then the new and the old value of S differ by $d = (ic+(i+1)a_i)-(ia_i+(i+1)a_{i+1}) = a_i-a_{i+1}+i(c-a_{i+1})$. The integer d is positive since $a_i - a_{i+1} \geq 1$ and $c - a_{i+1} \geq 0$.

On the other hand $S \leq (1+2+\cdots+n)M$ as $a_i \leq M$ for all $i = 1, \ldots, n$. Since S increases by at least 1 at each step and never exceeds the constant $(1+2+\cdots+n)M$, the process stops after a finite number of iterations.

Solution 2. Like in the first solution note that the operations do not change the maximum M of the initial sequence. Now consider the reverse lexicographical order for n-tuples of integers. We say that $(x_1, \ldots, x_n) < (y_1, \ldots, y_n)$ if $x_n < y_n$, or if $x_n = y_n$ and $x_{n-1} < y_{n-1}$, or if $x_n = y_n$, $x_{n-1} = y_{n-1}$ and $x_{n-2} < y_{n-2}$, etc. Each iteration creates a sequence that is greater than the previous one with respect to this order, and no sequence occurs twice during the process. On the other hand there are finitely many possible sequences because their terms are always positive integers not exceeding M. Hence the process cannot continue forever.

Solution 3. Let the current numbers be $a_1, a_2, \ldots, a_n$. Define the *score* s_i of a_i as the number of a_j's that are less than a_i. Call the sequence $s_1, s_2, \ldots, s_n$ the score sequence of $a_1, a_2, \ldots, a_n$.

Let us say that a sequence $x_1, \ldots, x_n$ dominates a sequence $y_1, \ldots, y_n$ if the first index i with $x_i \neq y_i$ is such that $x_i < y_i$. We show that after each operation the new score sequence dominates the old one. Score sequences do not repeat, and there are finitely many possibilities for them, no more than $(n-1)^n$. Hence the process will terminate.

Consider an operation that replaces (x, y) by (a, x), with $a = y + 1$ or $a = x - 1$. Suppose that x was originally at position i. For each $j < i$ the score s_j does not increase with the change because $y \leq a$ and $x \leq x$. If s_j decreases for some $j < i$ then the new score sequence dominates the old one. Assume that s_j stays the same for all $j < i$ and consider s_i. Since $x > y$ and $y \leq a \leq x$, we see that s_i decreases by at least 1. This concludes the proof.

Comment. All three proofs work if x and y are not necessarily adjacent, and if the pair (x, y) is replaced by any pair (a, x), with a an integer satisfying $y \leq a \leq x$. There is nothing special about the "weights" $1, 2, \ldots, n$ in the definition of $S = \sum_{i=1}^{n} ia_i$ from the first solution. For any sequence $w_1 < w_2 < \cdots < w_n$ of positive integers, the sum $\sum_{i=1}^{n} w_i a_i$ increases by at least 1 with each operation.

Consider the same problem, but letting Alice replace the pair (x, y) by (a, x), where a is any positive integer less than x. The same conclusion holds in this version, i. e. the process stops eventually. The solution using the reverse lexicographical order works without any change. The first solution would require a special set of weights like $w_i = M^i$ for $i = 1, \ldots, n$.

Comment. The first and the second solutions provide upper bounds for the number of possible operations, respectively of order Mn^2 and M^n where M is the maximum of the original sequence. The upper bound $(n-1)^n$ in the third solution does not depend on M.

C2. Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

Solution. Consider x such pairs in $\{1, 2, \ldots, n\}$. The sum S of the $2x$ numbers in them is at least $1+2+\cdots+2x$ since the pairs are disjoint. On the other hand $S \leq n+(n-1)+\cdots+(n-x+1)$ because the sums of the pairs are different and do not exceed n. This gives the inequality

$$\frac{2x(2x+1)}{2} \leq nx - \frac{x(x-1)}{2},$$

which leads to $x \leq \frac{2n-1}{5}$. Hence there are at most $\left\lfloor \frac{2n-1}{5} \right\rfloor$ pairs with the given properties.

We show a construction with exactly $\left\lfloor \frac{2n-1}{5} \right\rfloor$ pairs. First consider the case $n = 5k+3$ with $k \geq 0$, where $\left\lfloor \frac{2n-1}{5} \right\rfloor = 2k+1$. The pairs are displayed in the following table.

Pairs	$3k+1$ 2	$3k$ 4	$\cdots$ $\cdots$	$2k+2$ $2k$	$4k+2$ 1	$4k+1$ 3	$\cdots$ $\cdots$	$3k+3$ $2k-1$	$3k+2$ $2k+1$
Sums	$3k+3$	$3k+4$	$\cdots$	$4k+2$	$4k+3$	$4k+4$	$\cdots$	$5k+2$	$5k+3$

The $2k+1$ pairs involve all numbers from 1 to $4k+2$; their sums are all numbers from $3k+3$ to $5k+3$. The same construction works for $n = 5k+4$ and $n = 5k+5$ with $k \geq 0$. In these cases the required number $\left\lfloor \frac{2n-1}{5} \right\rfloor$ of pairs equals $2k+1$ again, and the numbers in the table do not exceed $5k+3$. In the case $n = 5k+2$ with $k \geq 0$ one needs only $2k$ pairs. They can be obtained by ignoring the last column of the table (thus removing $5k+3$). Finally, $2k$ pairs are also needed for the case $n = 5k+1$ with $k \geq 0$. Now it suffices to ignore the last column of the table and then subtract 1 from each number in the first row.

Comment. The construction above is not unique. For instance, the following table shows another set of $2k+1$ pairs for the cases $n = 5k+3$, $n = 5k+4$, and $n = 5k+5$.

Pairs	1 $4k+1$	2 $4k-1$	$\cdots$ $\cdots$	k $2k+3$	$k+1$ $4k+2$	$k+2$ $4k$	$\cdots$ $\cdots$	$2k+1$ $2k+2$
Sums	$4k+2$	$4k+1$	$\cdots$	$3k+3$	$5k+3$	$5k+2$	$\cdots$	$4k+3$

The table for the case $n = 5k+2$ would be the same, with the pair $(k+1, 4k+2)$ removed. For the case $n = 5k+1$ remove the last column and subtract 2 from each number in the second row.

C3. In a 999×999 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1 and C_3 white and C_2 red. Find the maximum value T can attain.

Solution. We prove that in an $n \times n$ square table there are at most $\frac{4n^4}{27}$ such triples.

Let row i and column j contain a_i and b_j white cells respectively, and let R be the set of red cells. For every red cell (i, j) there are $a_i b_j$ admissible triples (C_1, C_2, C_3) with $C_2 = (i, j)$, therefore

$$T = \sum_{(i,j)\in R} a_i b_j.$$

We use the inequality $2ab \le a^2 + b^2$ to obtain

$$T \le \frac{1}{2} \sum_{(i,j)\in R} (a_i^2 + b_j^2) = \frac{1}{2} \sum_{i=1}^{n} (n - a_i) a_i^2 + \frac{1}{2} \sum_{j=1}^{n} (n - b_j) b_j^2.$$

This is because there are $n - a_i$ red cells in row i and $n - b_j$ red cells in column j. Now we maximize the right-hand side.

By the AM-GM inequality we have

$$(n - x)x^2 = \frac{1}{2}(2n - 2x) \cdot x \cdot x \le \frac{1}{2} \left(\frac{2n}{3} \right)^3 = \frac{4n^3}{27},$$

with equality if and only if $x = \frac{2n}{3}$. By putting everything together, we get

$$T \le \frac{n}{2} \frac{4n^3}{27} + \frac{n}{2} \frac{4n^3}{27} = \frac{4n^4}{27}.$$

If $n = 999$ then any coloring of the square table with $x = \frac{2n}{3} = 666$ white cells in each row and column attains the maximum as all inequalities in the previous argument become equalities. For example color a cell (i, j) white if $i - j \equiv 1, 2, \ldots, 666 \pmod{999}$, and red otherwise.

Therefore the maximum value T can attain is $T = \frac{4 \cdot 999^4}{27}$.

Comment. One can obtain a better preliminary estimate with the Cauchy-Schwarz inequality:

$$T = \sum_{(i,j)\in R} a_i b_j \le \left(\sum_{(i,j)\in R} a_i^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{(i,j)\in R} b_j^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} (n - a_i) a_i^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{n} (n - b_j) b_j^2 \right)^{\frac{1}{2}}.$$

It can be used to reach the same conclusion.

C4. Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules:

- On every move of his B passes 1 coin from every box to an adjacent box.
- On every move of hers A chooses several coins that were *not* involved in B's previous move and are in different boxes. She passes every chosen coin to an adjacent box.

Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

Solution. We argue for a general $n \geq 7$ instead of 2012 and prove that the required minimum N is $2n-2$. For $n = 2012$ this gives $N_{\min} = 4022$.

a) If $N = 2n-2$ player A can achieve her goal. Let her start the game with a *regular* distribution: $n-2$ boxes with 2 coins and 2 boxes with 1 coin. Call the boxes of the two kinds *red* and *white* respectively. We claim that on her first move A can achieve a regular distribution again, regardless of B's first move M. She acts according as the following situation S occurs after M or not: *The initial distribution contains a red box R with 2 white neighbors, and R receives no coins from them on move M.*

Suppose that S does not occur. Exactly one of the coins c_1 and c_2 in a given red box X is involved in M, say c_1. If M passes c_1 to the right neighbor of X, let A pass c_2 to its left neighbor, and vice versa. By doing so with all red boxes A performs a legal move M'. Thus M and M' combined move the 2 coins of every red box in opposite directions. Hence after M and M' are complete each neighbor of a red box X contains exactly 1 coin that was initially in X. So each box with a red neighbor is non-empty after M'. If initially there is a box X with 2 white neighbors (X is red and unique) then X receives a coin from at least one of them on move M since S does not occur. Such a coin is not involved in M', so X is also non-empty after M'. Furthermore each box Y has given away its initial content after M and M'. A red neighbor of Y adds 1 coin to it; a white neighbor adds at most 1 coin because it is not involved in M'. Hence each box contains 1 or 2 coins after M'. Because $N = 2n-2$, such a distribution is regular.

Now let S occur after move M. Then A leaves untouched the exceptional red box R. With all remaining red boxes she proceeds like in the previous case, thus making a legal move M''. Box R receives no coins from its neighbors on either move, so there is 1 coin in it after M''. Like above M and M'' combined pass exactly 1 coin from every red box different from R to each of its neighbors. Every box except R has a red neighbor different from R, hence all boxes are non-empty after M''. Next, each box Y except R loses its initial content after M and M''. A red neighbor of Y adds at most 1 coin to it; a white neighbor also adds at most 1 coin as it does not participate in M''. Thus each box has 1 or 2 coins after M'', and the obtained distribution is regular.

Player A can apply the described strategy indefinitely, so $N = 2n-2$ enables her to succeed.

b) For $N \leq 2n-3$ player B can achieve an empty box after some move of A. Let α be a set of ℓ consecutive boxes containing a total of $N(\alpha)$ coins. We call α an *arc* if $\ell \leq n-2$ and $N(\alpha) \leq 2\ell - 3$. Note that $\ell \geq 2$ by the last condition. Moreover if both extremes of α are non-empty boxes then $N(\alpha) \geq 2$, so that $N(\alpha) \leq 2\ell - 3$ implies $\ell \geq 3$. Observe also that if an extreme X of α has more than 1 coin then ignoring X yields a shorter arc. It follows that every arc contains an arc whose extremes have at most 1 coin each.

Given a clockwise labeling $1, 2, \ldots, n$ of the boxes, suppose that boxes $1, 2, \ldots, \ell$ form an arc α, with $\ell \leq n-2$ and $N(\alpha) \leq 2\ell - 3$. Suppose also that all $n \geq 7$ boxes are non-empty. Then B can move so that an arc α' with $N(\alpha') < N(\alpha)$ will appear after any response of A.

One may assume exactly 1 coin in boxes 1 and ℓ by a previous remark. Let B pass 1 coin in counterclockwise direction from box 1 and box n, and in clockwise direction from each remaining box. This leaves $N(\alpha)-2$ coins in the boxes of α. In addition, due to $3 \le \ell \le n-2$, box ℓ has exactly 1 coin c, the one received from box $\ell-1$.

Let player A's next move M pass $k \le 2$ coins to boxes $1, 2, \ldots, \ell$ from the remaining ones. Only boxes 1 and ℓ can receive such coins, at most 1 each. If $k < 2$ then after move M boxes $1, 2, \ldots, \ell$ form an arc α' with $N(\alpha') < N(\alpha)$. If $k = 2$ then M adds a coin to box ℓ. Also M does not move coin c from ℓ because c is involved in the previous move of B. In summary boxes $1, 2, \ldots, \ell$ contain $N(\alpha)$ coins like before, so they form an arc. However there are 2 coins now in the extreme ℓ of the arc. Ignore ℓ to obtain a shorter arc α' with $N(\alpha') < N(\alpha)$.

Consider any initial distribution without empty boxes. Since $N \le 2n-3$, there are at least 3 boxes in it with exactly 1 coin. It follows from $n \ge 7$ that some 2 of them are the extremes of an arc α. Hence B can make the move described above, which leads to an arc α' with $N(\alpha') < N(\alpha)$ after A's response. If all boxes in the new distribution are non-empty he can repeat the same, and so on. Because $N(\alpha)$ cannot decrease indefinitely, an empty box will occur after some move of A.

C5. The columns and the rows of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is 0, 1 or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.

Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

Solution. Without loss of generality it suffices to prove that the A-tokens can be moved to distinct A-squares in such a way that each A-token is moved to a distance at most $d + 2$ from its original place. This means we need a perfect matching between the $3n^2$ A-squares and the $3n^2$ A-tokens such that the distance in each pair of the matching is at most $d + 2$.

To find the matching, we construct a bipartite graph. The A-squares will be the vertices in one class of the graph; the vertices in the other class will be the A-tokens.

Split the board into 3×1 horizontal triminos; then each trimino contains exactly one A-square. Take a permutation π of the tokens which moves A-tokens to B-tokens, B-tokens to C-tokens, and C-tokens to A-tokens, in each case to a distance at most d. For each A-square S, and for each A-token T, connect S and T by an edge if T, $\pi(T)$ or $\pi^{-1}(T)$ is on the trimino containing S. We allow multiple edges; it is even possible that the same square and the same token are connected with three edges. Obviously the lengths of the edges in the graph do not exceed $d + 2$. By length of an edge we mean the distance between the A-square and the A-token it connects.

Each A-token T is connected with the three A-squares whose triminos contain T, $\pi(T)$ and $\pi^{-1}(T)$. Therefore in the graph all tokens are of degree 3. We show that the same is true for the A-squares. Let S be an arbitrary A-square, and let T_1, T_2, T_3 be the three tokens on the trimino containing S. For $i = 1, 2, 3$, if T_i is an A-token, then S is connected with T_i; if T_i is a B-token then S is connected with $\pi^{-1}(T_i)$; finally, if T_i is a C-token then S is connected with $\pi(T_i)$. Hence in the graph the A-squares also are of degree 3.

Since the A-squares are of degree 3, from every set $\mathcal{S}$ of A-squares exactly $3|\mathcal{S}|$ edges start. These edges end in at least $|\mathcal{S}|$ tokens because the A-tokens also are of degree 3. Hence every set $\mathcal{S}$ of A-squares has at least $|\mathcal{S}|$ neighbors among the A-tokens.

Therefore, by Hall's marriage theorem, the graph contains a perfect matching between the two vertex classes. So there is a perfect matching between the A-squares and A-tokens with edges no longer than $d + 2$. It follows that the tokens can be permuted as specified in the problem statement.

Comment 1. In the original problem proposal the board was infinite and there were only two colors. Having n colors for some positive integer n was an option; we chose $n = 3$. Moreover, we changed the board to a finite one to avoid dealing with infinite graphs (although Hall's theorem works in the infinite case as well).

With only two colors Hall's theorem is not needed. In this case we split the board into 2×1 dominos, and in the resulting graph all vertices are of degree 2. The graph consists of disjoint cycles with even length and infinite paths, so the existence of the matching is trivial.

Having more than three colors would make the problem statement more complicated, because we need a matching between every two color classes of tokens. However, this would not mean a significant increase in difficulty.

Comment 2. According to Wikipedia, the color *asparagus* (hexadecimal code `#87A96B`) is a tone of green that is named after the vegetable. Crayola created this color in 1993 as one of the 16 to be named in the Name The Color Contest. *Byzantium* (`#702963`) is a dark tone of purple. Its first recorded use as a color name in English was in 1926. *Citrine* (`#E4D00A`) is variously described as yellow, greenish-yellow, brownish-yellow or orange. The first known use of citrine as a color name in English was in the 14th century.

C6. Let k and n be fixed positive integers. In the liar's guessing game, Amy chooses integers x and N with $1 \le x \le N$. She tells Ben what N is, but not what x is. Ben may then repeatedly ask Amy whether $x \in S$ for arbitrary sets S of integers. Amy will always answer with *yes* or *no*, but she might lie. The only restriction is that she can lie at most k times in a row. After he has asked as many questions as he wants, Ben must specify a set of at most n positive integers. If x is in this set he wins; otherwise, he loses. Prove that:

a) If $n \ge 2^k$ then Ben can always win.

b) For sufficiently large k there exist $n \ge 1.99^k$ such that Ben cannot guarantee a win.

Solution. Consider an answer $A \in \{yes, no\}$ to a question of the kind "Is x in the set S?" We say that A is inconsistent with a number i if $A = yes$ and $i \notin S$, or if $A = no$ and $i \in S$. Observe that an answer inconsistent with the target number x is a lie.

a) Suppose that Ben has determined a set T of size m that contains x. This is true initially with $m = N$ and $T = \{1, 2, \ldots, N\}$. For $m > 2^k$ we show how Ben can find a number $y \in T$ that is different from x. By performing this step repeatedly he can reduce T to be of size $2^k \le n$ and thus win.

Since only the size $m > 2^k$ of T is relevant, assume that $T = \{0, 1, \ldots, 2^k, \ldots, m-1\}$. Ben begins by asking repeatedly whether x is 2^k. If Amy answers *no* $k+1$ times in a row, one of these answers is truthful, and so $x \ne 2^k$. Otherwise Ben stops asking about 2^k at the first answer *yes*. He then asks, for each $i = 1, \ldots, k$, if the binary representation of x has a 0 in the ith digit. Regardless of what the k answers are, they are all inconsistent with a certain number $y \in \{0, 1, \ldots, 2^k - 1\}$. The preceding answer *yes* about 2^k is also inconsistent with y. Hence $y \ne x$. Otherwise the last $k+1$ answers are not truthful, which is impossible.

Either way, Ben finds a number in T that is different from x, and the claim is proven.

b) We prove that if $1 < \lambda < 2$ and $n = \lfloor (2-\lambda)\lambda^{k+1} \rfloor - 1$ then Ben cannot guarantee a win. To complete the proof, then it suffices to take λ such that $1.99 < \lambda < 2$ and k large enough so that

$$n = \lfloor (2-\lambda)\lambda^{k+1} \rfloor - 1 \ge 1.99^k.$$

Consider the following strategy for Amy. First she chooses $N = n+1$ and $x \in \{1, 2, \ldots, n+1\}$ arbitrarily. After every answer of hers Amy determines, for each $i = 1, 2, \ldots, n+1$, the number m_i of consecutive answers she has given by that point that are inconsistent with i. To decide on her next answer, she then uses the quantity

$$\phi = \sum_{i=1}^{n+1} \lambda^{m_i}.$$

No matter what Ben's next question is, Amy chooses the answer which minimizes ϕ.

We claim that with this strategy ϕ will always stay less than λ^{k+1}. Consequently no exponent m_i in ϕ will ever exceed k, hence Amy will never give more than k consecutive answers inconsistent with some i. In particular this applies to the target number x, so she will never lie more than k times in a row. Thus, given the claim, Amy's strategy is legal. Since the strategy does not depend on x in any way, Ben can make no deductions about x, and therefore he cannot guarantee a win.

It remains to show that $\phi < \lambda^{k+1}$ at all times. Initially each m_i is 0, so this condition holds in the beginning due to $1 < \lambda < 2$ and $n = \lfloor (2-\lambda)\lambda^{k+1} \rfloor - 1$. Suppose that $\phi < \lambda^{k+1}$ at some point, and Ben has just asked if $x \in S$ for some set S. According as Amy answers *yes* or *no*, the new value of ϕ becomes

$$\phi_1 = \sum_{i \in S} 1 + \sum_{i \notin S} \lambda^{m_i+1} \qquad \text{or} \qquad \phi_2 = \sum_{i \in S} \lambda^{m_i+1} + \sum_{i \notin S} 1.$$

Since Amy chooses the option minimizing ϕ, the new ϕ will equal $\min(\phi_1, \phi_2)$. Now we have

$$\min(\phi_1, \phi_2) \leq \frac{1}{2}(\phi_1 + \phi_2) = \frac{1}{2}\left(\sum_{i \in S}\left(1 + \lambda^{m_i+1}\right) + \sum_{i \notin S}\left(\lambda^{m_i+1} + 1\right)\right) = \frac{1}{2}(\lambda\phi + n + 1).$$

Because $\phi < \lambda^{k+1}$, the assumptions $\lambda < 2$ and $n = \left\lfloor (2-\lambda)\lambda^{k+1} \right\rfloor - 1$ lead to

$$\min(\phi_1, \phi_2) < \frac{1}{2}(\lambda^{k+2} + (2-\lambda)\lambda^{k+1}) = \lambda^{k+1}.$$

The claim follows, which completes the solution.

Comment. Given a fixed k, let $f(k)$ denote the minimum value of n for which Ben can guarantee a victory. The problem asks for a proof that for large k

$$1.99^k \leq f(k) \leq 2^k.$$

A computer search shows that $f(k) = 2, 3, 4, 7, 11, 17$ for $k = 1, 2, 3, 4, 5, 6$.

C7. There are given 2^{500} points on a circle labeled $1, 2, \ldots, 2^{500}$ in some order. Prove that one can choose 100 pairwise disjoint chords joining some of these points so that the 100 sums of the pairs of numbers at the endpoints of the chosen chords are equal.

Solution. The proof is based on the following general fact.

Lemma. In a graph G each vertex v has degree d_v. Then G contains an independent set S of vertices such that $|S| \geq f(G)$ where

$$f(G) = \sum_{v \in G} \frac{1}{d_v + 1}.$$

Proof. Induction on $n = |G|$. The base $n = 1$ is clear. For the inductive step choose a vertex v_0 in G of minimum degree d. Delete v_0 and all of its neighbors $v_1, \ldots, v_d$ and also all edges with endpoints $v_0, v_1, \ldots, v_d$. This gives a new graph G'. By the inductive assumption G' contains an independent set S' of vertices such that $|S'| \geq f(G')$. Since no vertex in S' is a neighbor of v_0 in G, the set $S = S' \cup \{v_0\}$ is independent in G.

Let d'_v be the degree of a vertex v in G'. Clearly $d'_v \leq d_v$ for every such vertex v, and also $d_{v_i} \geq d$ for all $i = 0, 1, \ldots, d$ by the minimal choice of v_0. Therefore

$$f(G') = \sum_{v \in G'} \frac{1}{d'_v + 1} \geq \sum_{v \in G'} \frac{1}{d_v + 1} = f(G) - \sum_{i=0}^{d} \frac{1}{d_{v_i} + 1} \geq f(G) - \frac{d+1}{d+1} = f(G) - 1.$$

Hence $|S| = |S'| + 1 \geq f(G') + 1 \geq f(G)$, and the induction is complete. $\square$

We pass on to our problem. For clarity denote $n = 2^{499}$ and draw all chords determined by the given $2n$ points. Color each chord with one of the colors $3, 4, \ldots, 4n - 1$ according to the sum of the numbers at its endpoints. Chords with a common endpoint have different colors. For each color c consider the following graph G_c. Its vertices are the chords of color c, and two chords are neighbors in G_c if they intersect. Let $f(G_c)$ have the same meaning as in the lemma for all graphs G_c.

Every chord ℓ divides the circle into two arcs, and one of them contains $m(\ell) \leq n - 1$ given points. (In particular $m(\ell) = 0$ if ℓ joins two consecutive points.) For each $i = 0, 1, \ldots, n - 2$ there are $2n$ chords ℓ with $m(\ell) = i$. Such a chord has degree at most i in the respective graph. Indeed let $A_1, \ldots, A_i$ be all points on either arc determined by a chord ℓ with $m(\ell) = i$ and color c. Every A_j is an endpoint of at most 1 chord colored c, $j = 1, \ldots, i$. Hence at most i chords of color c intersect ℓ.

It follows that for each $i = 0, 1, \ldots, n - 2$ the $2n$ chords ℓ with $m(\ell) = i$ contribute at least $\frac{2n}{i+1}$ to the sum $\sum_c f(G_c)$. Summation over $i = 0, 1, \ldots, n - 2$ gives

$$\sum_c f(G_c) \geq 2n \sum_{i=1}^{n-1} \frac{1}{i}.$$

Because there are $4n - 3$ colors in all, averaging yields a color c such that

$$f(G_c) \geq \frac{2n}{4n-3} \sum_{i=1}^{n-1} \frac{1}{i} > \frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i}.$$

By the lemma there are at least $\frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i}$ pairwise disjoint chords of color c, i. e. with the same sum c of the pairs of numbers at their endpoints. It remains to show that $\frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i} \geq 100$ for $n = 2^{499}$. Indeed we have

$$\sum_{i=1}^{n-1} \frac{1}{i} > \sum_{i=1}^{2^{400}} \frac{1}{i} = 1 + \sum_{k=1}^{400} \sum_{i=2^{k-1}+1}^{2^k} \frac{1}{i} > 1 + \sum_{k=1}^{400} \frac{2^{k-1}}{2^k} = 201 > 200.$$

This completes the solution.

Geometry

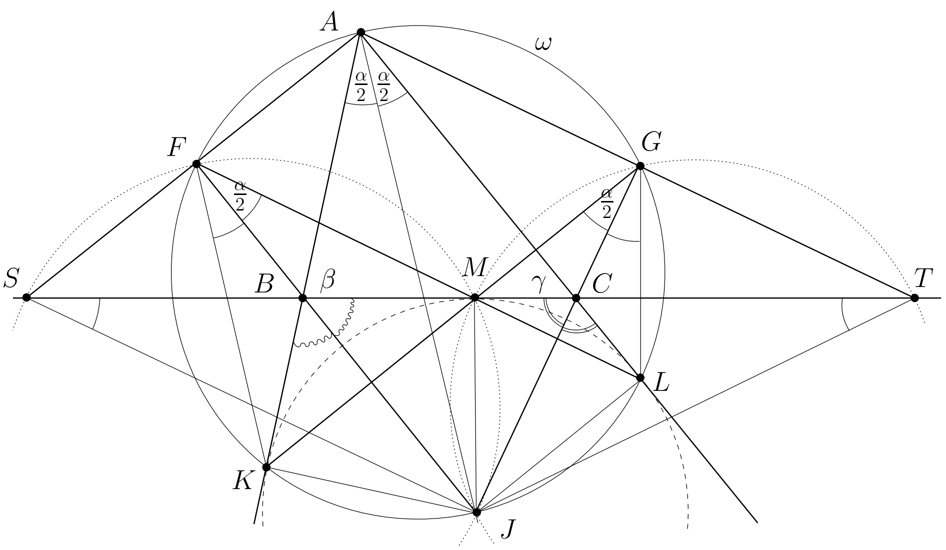
G1. In the triangle ABC the point J is the center of the excircle opposite to A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

Solution. Let $\alpha = \angle CAB$, $\beta = \angle ABC$ and $\gamma = \angle BCA$. The line AJ is the bisector of $\angle CAB$, so $\angle JAK = \angle JAL = \frac{\alpha}{2}$. By $\angle AKJ = \angle ALJ = 90^\circ$ the points K and L lie on the circle ω with diameter AJ.

The triangle KBM is isosceles as BK and BM are tangents to the excircle. Since BJ is the bisector of $\angle KBM$, we have $\angle MBJ = 90^\circ - \frac{\beta}{2}$ and $\angle BMK = \frac{\beta}{2}$. Likewise $\angle MCJ = 90^\circ - \frac{\gamma}{2}$ and $\angle CML = \frac{\gamma}{2}$. Also $\angle BMF = \angle CML$, therefore

$$\angle LFJ = \angle MBJ - \angle BMF = \left(90^\circ - \frac{\beta}{2}\right) - \frac{\gamma}{2} = \frac{\alpha}{2} = \angle LAJ.$$

Hence F lies on the circle ω. (By the angle computation, F and A are on the same side of BC.) Analogously, G also lies on ω. Since AJ is a diameter of ω, we obtain $\angle AFJ = \angle AGJ = 90^\circ$.



The lines AB and BC are symmetric with respect to the external bisector BF. Because $AF \perp BF$ and $KM \perp BF$, the segments SM and AK are symmetric with respect to BF, hence $SM = AK$. By symmetry $TM = AL$. Since AK and AL are equal as tangents to the excircle, it follows that $SM = TM$, and the proof is complete.

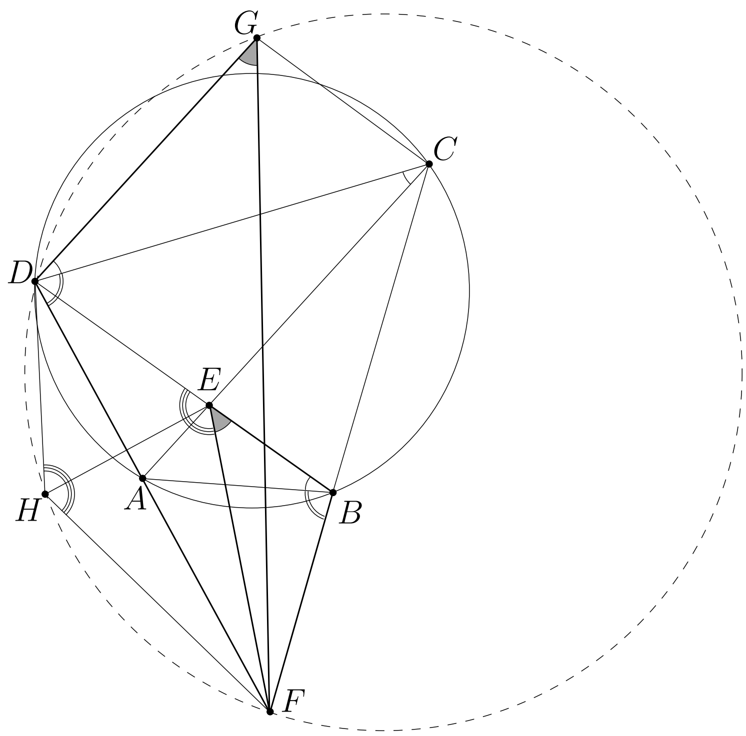
Comment. After discovering the circle $AFKJLG$, there are many other ways to complete the solution. For instance, from the cyclic quadrilaterals $JMFS$ and $JMGT$ one can find $\angle TSJ = \angle STJ = \frac{\alpha}{2}$. Another possibility is to use the fact that the lines AS and GM are parallel (both are perpendicular to the external angle bisector BJ), so $\frac{MS}{MT} = \frac{AG}{GT} = 1$.

G2. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

Solution. We show first that the triangles FDG and FBE are similar. Since $ABCD$ is cyclic, the triangles EAB and EDC are similar, as well as FAB and FCD. The parallelogram $ECGD$ yields $GD = EC$ and $\angle CDG = \angle DCE$; also $\angle DCE = \angle DCA = \angle DBA$ by inscribed angles. Therefore

$$\angle FDG = \angle FDC + \angle CDG = \angle FBA + \angle ABD = \angle FBE,$$
$$\frac{GD}{EB} = \frac{CE}{EB} = \frac{CD}{AB} = \frac{FD}{FB}.$$

It follows that FDG and FBE are similar, and so $\angle FGD = \angle FEB$.



Since H is the reflection of E with respect to FD, we conclude that

$$\angle FHD = \angle FED = 180^\circ - \angle FEB = 180^\circ - \angle FGD.$$

This proves that D, H, F, G are concyclic.

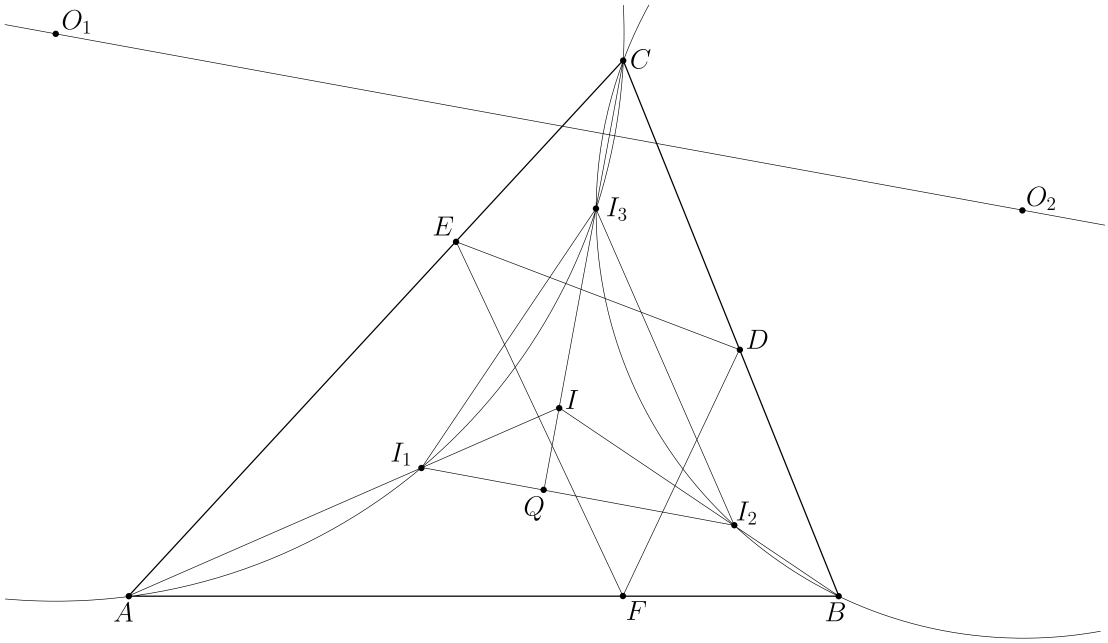
Comment. Points E and G are always in the half-plane determined by the line FD that contains B and C, but H is always in the other half-plane. In particular, $DHFG$ is cyclic if and only if $\angle FHD + \angle FGD = 180^\circ$.

G3. In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Solution. Let $\angle CAB = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$. We start by showing that A, B, I_1 and I_2 are concyclic. Since AI_1 and BI_2 bisect $\angle CAB$ and $\angle ABC$, their extensions beyond I_1 and I_2 meet at the incenter I of the triangle. The points E and F are on the circle with diameter BC, so $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Hence the triangles AEF and ABC are similar with ratio of similitude $\frac{AE}{AB} = \cos\alpha$. Because I_1 and I are their incenters, we obtain $I_1A = IA\cos\alpha$ and $II_1 = IA - I_1A = 2IA\sin^2\frac{\alpha}{2}$. By symmetry $II_2 = 2IB\sin^2\frac{\beta}{2}$. The law of sines in the triangle ABI gives $IA\sin\frac{\alpha}{2} = IB\sin\frac{\beta}{2}$. Hence

$$II_1 \cdot IA = 2\left(IA\sin\tfrac{\alpha}{2}\right)^2 = 2\left(IB\sin\tfrac{\beta}{2}\right)^2 = II_2 \cdot IB.$$

Therefore A, B, I_1 and I_2 are concyclic, as claimed.



In addition $II_1 \cdot IA = II_2 \cdot IB$ implies that I has the same power with respect to the circles (ACI_1), (BCI_2) and (ABI_1I_2). Then CI is the radical axis of (ACI_1) and (BCI_2); in particular CI is perpendicular to the line of centers O_1O_2.

Now it suffices to prove that $CI \perp I_1I_2$. Let CI meet I_1I_2 at Q, then it is enough to check that $\angle II_1Q + \angle I_1IQ = 90^\circ$. Since $\angle I_1IQ$ is external for the triangle ACI, we have

$$\angle II_1Q + \angle I_1IQ = \angle II_1Q + (\angle ACI + \angle CAI) = \angle II_1I_2 + \angle ACI + \angle CAI.$$

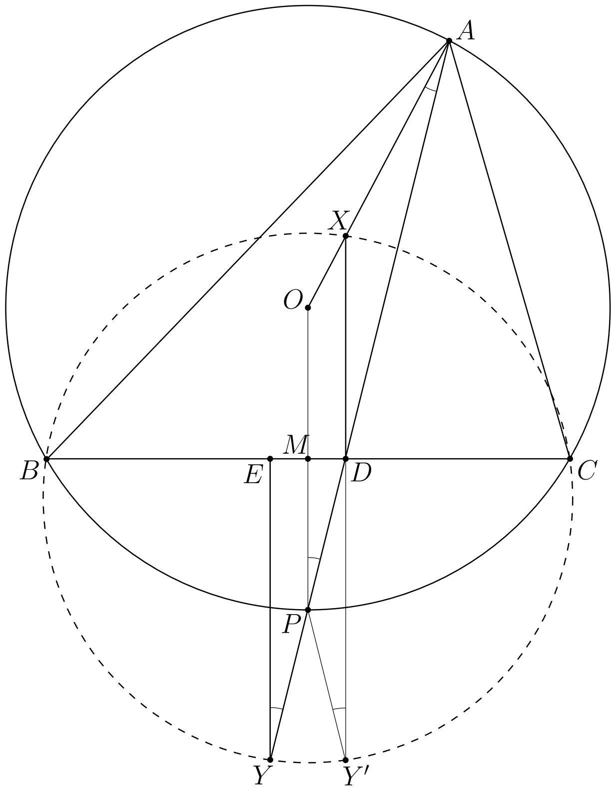
It remains to note that $\angle II_1I_2 = \frac{\beta}{2}$ from the cyclic quadrilateral ABI_1I_2, and $\angle ACI = \frac{\gamma}{2}$, $\angle CAI = \frac{\alpha}{2}$. Therefore $\angle II_1Q + \angle I_1IQ = \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90^\circ$, completing the proof.

Comment. It follows from the first part of the solution that the common point $I_3 \neq C$ of the circles (ACI_1) and (BCI_2) is the incenter of the triangle CDE.

G4. Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

Solution. The bisector of $\angle BAC$ and the perpendicular bisector of BC meet at P, the midpoint of the minor arc $\widehat{BC}$ (they are different lines as $AB \neq AC$). In particular OP is perpendicular to BC and intersects it at M, the midpoint of BC.

Denote by Y' the reflexion of Y with respect to OP. Since $\angle BYC = \angle BY'C$, it suffices to prove that $BXCY'$ is cyclic.



We have

$$\angle XAP = \angle OPA = \angle EYP.$$

The first equality holds because $OA = OP$, and the second one because EY and OP are both perpendicular to BC and hence parallel. But $\{Y, Y'\}$ and $\{E, D\}$ are pairs of symmetric points with respect to OP, it follows that $\angle EYP = \angle DY'P$ and hence

$$\angle XAP = \angle DY'P = \angle XY'P.$$

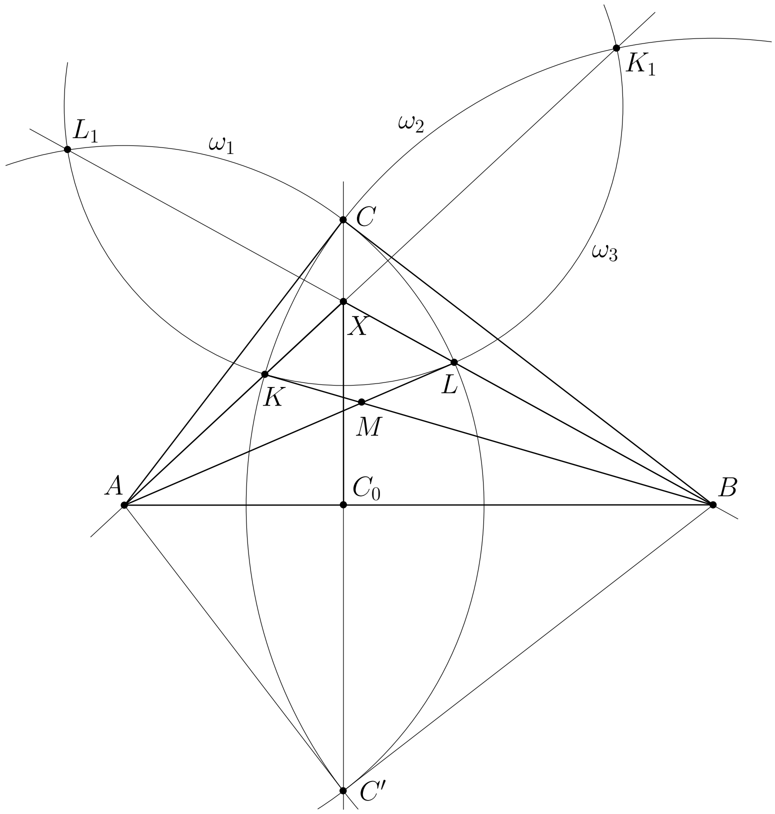
The last equation implies that $XAY'P$ is cyclic. By the powers of D with respect to the circles $(XAY'P)$ and $(ABPC)$ we obtain

$$XD \cdot DY' = AD \cdot DP = BD \cdot DC.$$

It follows that $BXCY'$ is cyclic, as desired.

G5. Let ABC be a triangle with $\angle BCA = 90°$, and let C_0 be the foot of the altitude from C. Choose a point X in the interior of the segment CC_0, and let K, L be the points on the segments AX, BX for which $BK = BC$ and $AL = AC$ respectively. Denote by M the intersection of AL and BK. Show that $MK = ML$.

Solution. Let C' be the reflection of C in the line AB, and let ω_1 and ω_2 be the circles with centers A and B, passing through L and K respectively. Since $AC' = AC = AL$ and $BC' = BC = BK$, both ω_1 and ω_2 pass through C and C'. By $\angle BCA = 90°$, AC is tangent to ω_2 at C, and BC is tangent to ω_1 at C. Let $K_1 \neq K$ be the second intersection of AX and ω_2, and let $L_1 \neq L$ be the second intersection of BX and ω_1.



By the powers of X with respect to ω_2 and ω_1,

$$XK \cdot XK_1 = XC \cdot XC' = XL \cdot XL_1,$$

so the points K_1, L, K, L_1 lie on a circle ω_3.

The power of A with respect to ω_2 gives

$$AL^2 = AC^2 = AK \cdot AK_1,$$

indicating that AL is tangent to ω_3 at L. Analogously, BK is tangent to ω_3 at K. Hence MK and ML are the two tangents from M to ω_3 and therefore $MK = ML$.

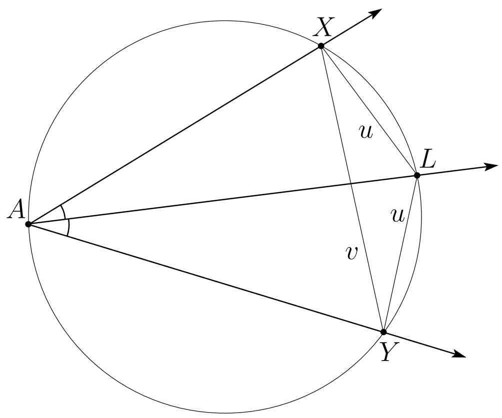
G6. Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

Solution. By MIQUEL's theorem the circles $(AEF) = \omega_A$, $(BFD) = \omega_B$ and $(CDE) = \omega_C$ have a common point, for arbitrary points D, E and F on BC, CA and AB. So ω_A passes through the common point $P \neq D$ of ω_B and ω_C.

Let ω_A, ω_B and ω_C meet the bisectors AI, BI and CI at $A \neq A'$, $B \neq B'$ and $C \neq C'$ respectively. The key observation is that A', B' and C' do not depend on the particular choice of D, E and F, provided that $BD + BF = CA$, $CD + CE = AB$ and $AE + AF = BC$ hold true (the last equality follows from the other two). For a proof we need the following fact.

Lemma. Given is an angle with vertex A and measure α. A circle ω through A intersects the angle bisector at L and sides of the angle at X and Y. Then $AX + AY = 2AL \cos \frac{\alpha}{2}$.

Proof. Note that L is the midpoint of arc $\widehat{XLY}$ in ω and set $XL = YL = u$, $XY = v$. By PTOLEMY's theorem $AX \cdot YL + AY \cdot XL = AL \cdot XY$, which rewrites as $(AX + AY)u = AL \cdot v$. Since $\angle LXY = \frac{\alpha}{2}$ and $\angle XLY = 180° - \alpha$, we have $v = 2 \cos \frac{\alpha}{2} u$ by the law of sines, and the claim follows. $\square$



Apply the lemma to $\angle BAC = \alpha$ and the circle $\omega = \omega_A$, which intersects AI at A'. This gives $2AA' \cos \frac{\alpha}{2} = AE + AF = BC$; by symmetry analogous relations hold for BB' and CC'. It follows that A', B' and C' are independent of the choice of D, E and F, as stated.

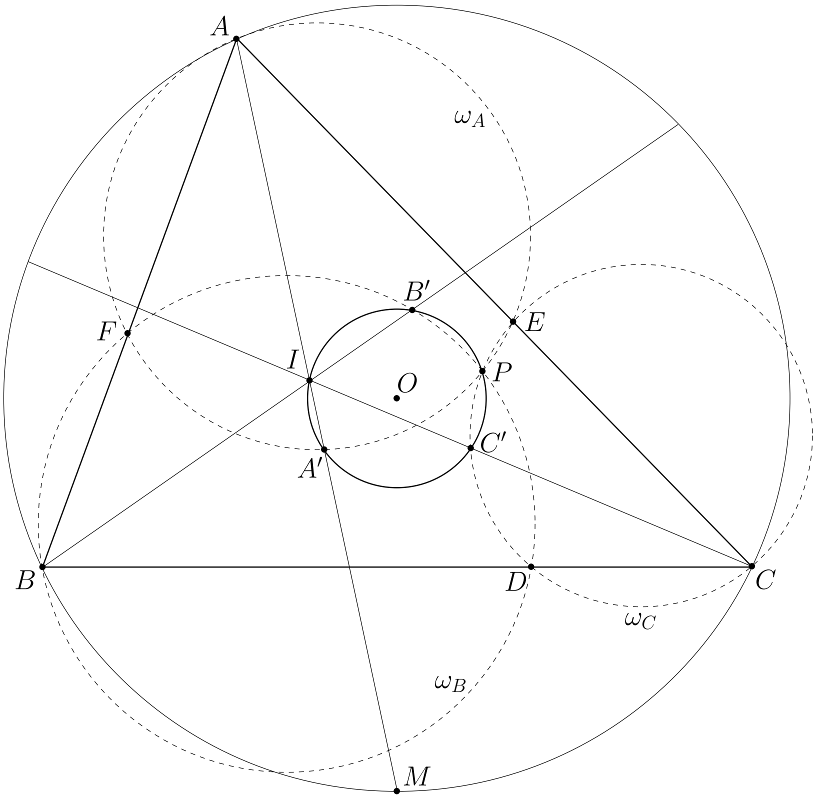
We use the lemma two more times with $\angle BAC = \alpha$. Let ω be the circle with diameter AI. Then X and Y are the tangency points of the incircle of ABC with AB and AC, and hence $AX = AY = \frac{1}{2}(AB + AC - BC)$. So the lemma yields $2AI \cos \frac{\alpha}{2} = AB + AC - BC$. Next, if ω is the circumcircle of ABC and AI intersects ω at $M \neq A$ then $\{X, Y\} = \{B, C\}$, and so $2AM \cos \frac{\alpha}{2} = AB + AC$ by the lemma. To summarize,

$$2AA' \cos \tfrac{\alpha}{2} = BC, \qquad 2AI \cos \tfrac{\alpha}{2} = AB + AC - BC, \qquad 2AM \cos \tfrac{\alpha}{2} = AB + AC. \qquad (*)$$

These equalities imply $AA' + AI = AM$, hence the segments AM and IA' have a common midpoint. It follows that I and A' are equidistant from the circumcenter O. By symmetry $OI = OA' = OB' = OC'$, so I, A', B', C' are on a circle centered at O.

To prove $OP = OI$, now it suffices to show that I, A', B', C' and P are concyclic. Clearly one can assume $P \neq I, A', B', C'$.

We use oriented angles to avoid heavy case distinction. The oriented angle between the lines l and m is denoted by $\angle(l, m)$. We have $\angle(l, m) = -\angle(m, l)$ and $\angle(l, m) + \angle(m, n) = \angle(l, n)$ for arbitrary lines l, m and n. Four distinct non-collinear points U, V, X, Y are concyclic if and only if $\angle(UX, VX) = \angle(UY, VY)$.



Suppose for the moment that A', B', P, I are distinct and noncollinear; then it is enough to check the equality $\angle(A'P, B'P) = \angle(A'I, B'I)$. Because A, F, P, A' are on the circle ω_A, we have $\angle(A'P, FP) = \angle(A'A, FA) = \angle(A'I, AB)$. Likewise $\angle(B'P, FP) = \angle(B'I, AB)$. Therefore

$$\angle(A'P, B'P) = \angle(A'P, FP) + \angle(FP, B'P) = \angle(A'I, AB) - \angle(B'I, AB) = \angle(A'I, B'I).$$

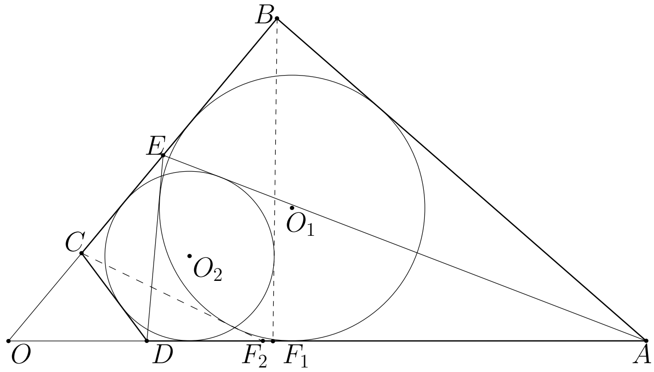
Here we assumed that $P \neq F$. If $P = F$ then $P \neq D, E$ and the conclusion follows similarly (use $\angle(A'F, B'F) = \angle(A'F, EF) + \angle(EF, DF) + \angle(DF, B'F)$ and inscribed angles in $\omega_A, \omega_B, \omega_C$).

There is no loss of generality in assuming A', B', P, I distinct and noncollinear. If ABC is an equilateral triangle then the equalities (*) imply that A', B', C', I, O and P coincide, so $OP = OI$. Otherwise at most one of A', B', C' coincides with I. If say $C' = I$ then $OI \perp CI$ by the previous reasoning. It follows that $A', B' \neq I$ and hence $A' \neq B'$. Finally A', B' and I are noncollinear because I, A', B', C' are concyclic.

Comment. The proposer remarks that the locus γ of the points P is an arc of the circle $(A'B'C'I)$. The reflection I' of I in O belongs to γ; it is obtained by choosing D, E and F to be the tangency points of the three excircles with their respective sides. The rest of the circle $(A'B'C'I)$, except I, can be included in γ by letting D, E and F vary on the extensions of the sides and assuming signed lengths. For instance if B is between C and D then the length BD must be taken with a negative sign. The incenter I corresponds to the limit case where D tends to infinity.

G7. Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

Solution. Let ω_1 and ω_2 be the incircles and O_1 and O_2 the incenters of the quadrilaterals $ABED$ and $AECD$ respectively. A point F with the stated property exists only if ω_1 and ω_2 are also the incircles of the quadrilaterals $ABCF$ and $BCDF$.



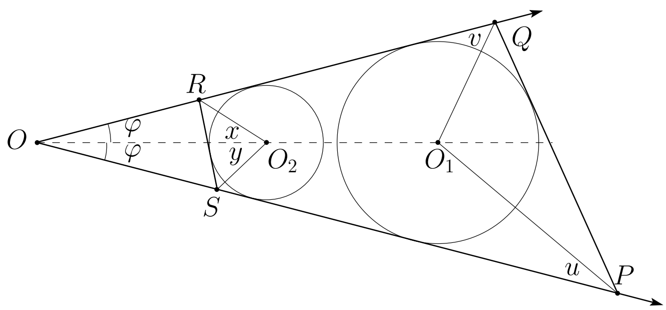
Let the tangents from B to ω_2 and from C to ω_1 (other than BC) meet AD at F_1 and F_2 respectively. We need to prove that $F_1 = F_2$ if and only if $AB \parallel CD$.

Lemma. The circles ω_1 and ω_2 with centers O_1 and O_2 are inscribed in an angle with vertex O. The points P, S on one side of the angle and Q, R on the other side are such that ω_1 is the incircle of the triangle PQO, and ω_2 is the excircle of the triangle RSO opposite to O. Denote $p = OO_1 \cdot OO_2$. Then exactly one of the following relations holds:

$$OP \cdot OR < p < OQ \cdot OS, \qquad OP \cdot OR > p > OQ \cdot OS, \qquad OP \cdot OR = p = OQ \cdot OS.$$

Proof. Denote $\angle OPO_1 = u$, $\angle OQO_1 = v$, $\angle OO_2R = x$, $\angle OO_2S = y$, $\angle POQ = 2\varphi$. Because PO_1, QO_1, RO_2, SO_2 are internal or external bisectors in the triangles PQO and RSO, we have

$$u + v = x + y \ (= 90^\circ - \varphi). \tag{1}$$



By the law of sines

$$\frac{OP}{OO_1} = \frac{\sin(u+\varphi)}{\sin u} \quad \text{and} \quad \frac{OO_2}{OR} = \frac{\sin(x+\varphi)}{\sin x}.$$

Therefore, since x, u and φ are acute,

$$OP \cdot OR \geq p \Leftrightarrow \frac{OP}{OO_1} \geq \frac{OO_2}{OR} \Leftrightarrow \sin x \sin(u+\varphi) \geq \sin u \sin(x+\varphi) \Leftrightarrow \sin(x-u) \geq 0 \Leftrightarrow x \geq u.$$

Thus $OP \cdot OR \geq p$ is equivalent to $x \geq u$, with $OP \cdot OR = p$ if and only if $x = u$.

Analogously, $p \geq OQ \cdot OS$ is equivalent to $v \geq y$, with $p = OQ \cdot OS$ if and only if $v = y$. On the other hand $x \geq u$ and $v \geq y$ are equivalent by (1), with $x = u$ if and only if $v = y$. The conclusion of the lemma follows from here. $\square$

Going back to the problem, apply the lemma to the quadruples $\{B, E, D, F_1\}$, $\{A, B, C, D\}$ and $\{A, E, C, F_2\}$. Assuming $OE \cdot OF_1 > p$, we obtain

$$OE \cdot OF_1 > p \;\Rightarrow\; OB \cdot OD < p \;\Rightarrow\; OA \cdot OC > p \;\Rightarrow\; OE \cdot OF_2 < p.$$

In other words, $OE \cdot OF_1 > p$ implies

$$OB \cdot OD < p < OA \cdot OC \quad \text{and} \quad OE \cdot OF_1 > p > OE \cdot OF_2.$$

Similarly, $OE \cdot OF_1 < p$ implies

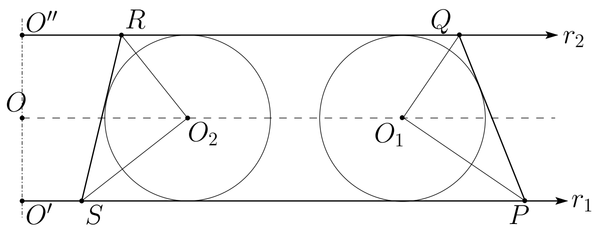
$$OB \cdot OD > p > OA \cdot OC \quad \text{and} \quad OE \cdot OF_1 < p < OE \cdot OF_2.$$

In these cases $F_1 \neq F_2$ and $OB \cdot OD \neq OA \cdot OC$, so the lines AB and CD are not parallel.

There remains the case $OE \cdot OF_1 = p$. Here the lemma leads to $OB \cdot OD = p = OA \cdot OC$ and $OE \cdot OF_1 = p = OE \cdot OF_2$. Therefore $F_1 = F_2$ and $AB \parallel CD$.

Comment. The conclusion is also true if BC and AD are parallel. One can prove a limit case of the lemma for the configuration shown in the figure below, where r_1 and r_2 are parallel rays starting at O' and O'', with $O'O'' \perp r_1, r_2$ and O the midpoint of $O'O''$. Two circles with centers O_1 and O_2 are inscribed in the strip between r_1 and r_2. The lines PQ and RS are tangent to the circles, with P, S on r_1, and Q, R on r_2, so that O, O_1 are on the same side of PQ and O, O_2 are on different sides of RS. Denote $s = OO_1 + OO_2$. Then exactly one of the following relations holds:

$$O'P + O''R < s < O''Q + O'S, \quad O'P + O''R > s > O''Q + O'S, \quad O'P + O''R = s = O''Q + O'S.$$

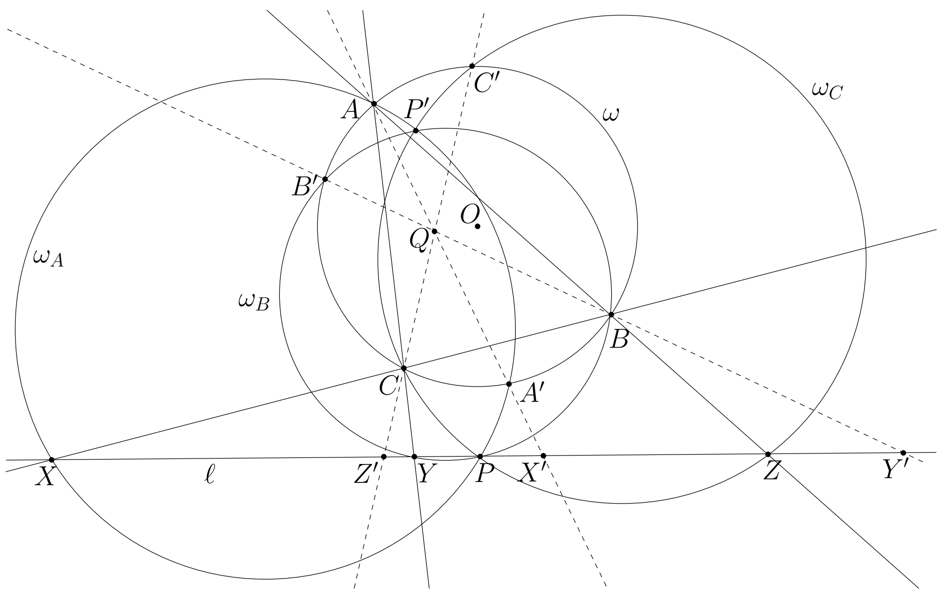


Once this is established, the proof of the original statement for $BC \parallel AD$ is analogous to the one in the intersecting case. One replaces products by sums of relevant segments.

G8. Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Solution 1. Let $\omega_A, \omega_B, \omega_C$ and ω be the circumcircles of triangles AXP, BYP, CZP and ABC respectively. The strategy of the proof is to construct a point Q with the same power with respect to the four circles. Then each of P and Q has the same power with respect to $\omega_A, \omega_B, \omega_C$ and hence the three circles are coaxial. In other words they have another common point P' or the three of them are tangent at P.

We first give a description of the point Q. Let $A' \neq A$ be the second intersection of ω and ω_A; define B' and C' analogously. We claim that AA', BB' and CC' have a common point. Once this claim is established, the point just constructed will be on the radical axes of the three pairs of circles $\{\omega, \omega_A\}, \{\omega, \omega_B\}, \{\omega, \omega_C\}$. Hence it will have the same power with respect to $\omega, \omega_A, \omega_B, \omega_C$.



We proceed to prove that AA', BB' and CC' intersect at one point. Let r be the circumradius of triangle ABC. Define the points X', Y', Z' as the intersections of AA', BB', CC' with ℓ. Observe that X', Y', Z' do exist. If AA' is parallel to ℓ then ω_A is tangent to ℓ; hence $X = P$ which is a contradiction. Similarly, BB' and CC' are not parallel to ℓ.

From the powers of the point X' with respect to the circles ω_A and ω we get

$$X'P \cdot (X'P + PX) = X'P \cdot X'X = X'A' \cdot X'A = X'O^2 - r^2,$$

hence

$$X'P \cdot PX = X'O^2 - r^2 - X'P^2 = OP^2 - r^2.$$

We argue analogously for the points Y' and Z', obtaining

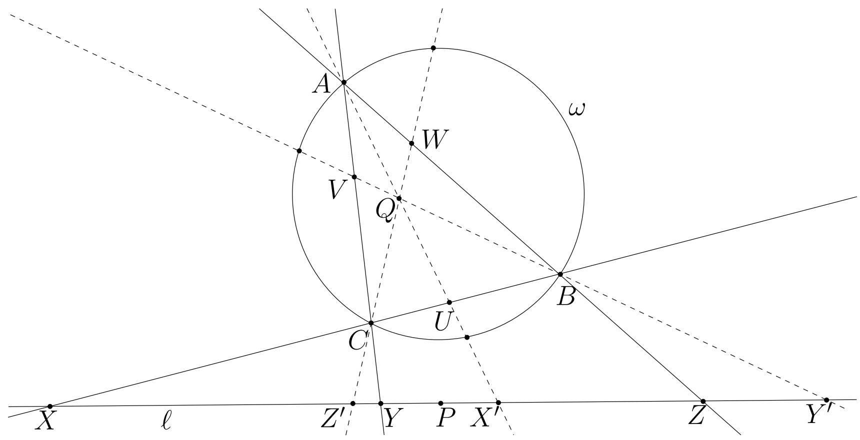
$$X'P \cdot PX = Y'P \cdot PY = Z'P \cdot PZ = OP^2 - r^2 = k^2. \tag{1}$$

In these computations all segments are regarded as directed segments. We keep the same convention for the sequel.

We prove that the lines AA', BB', CC' intersect at one point by CEVA's theorem. To avoid distracting remarks we interpret everything projectively, i. e. whenever two lines are parallel they meet at a point on the line at infinity.

Let U, V, W be the intersections of AA', BB', CC' with BC, CA, AB respectively. The idea is that although it is difficult to calculate the ratio $\frac{BU}{CU}$, it is easier to deal with the cross-ratio $\frac{BU}{CU}/\frac{BX}{CX}$ because we can send it to the line ℓ. With this in mind we apply MENELAUS' theorem to the triangle ABC and obtain $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1$. Hence Ceva's ratio can be expressed as

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = \frac{BU}{CU}\Big/\frac{BX}{CX} \cdot \frac{CV}{AV}\Big/\frac{CY}{AY} \cdot \frac{AW}{BW}\Big/\frac{AZ}{BZ}.$$



Project the line BC to ℓ from A. The cross-ratio between BC and UX equals the cross-ratio between ZY and $X'X$. Repeating the same argument with the lines CA and AB gives

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = \frac{ZX'}{YX'}\Big/\frac{ZX}{YX} \cdot \frac{XY'}{ZY'}\Big/\frac{XY}{ZY} \cdot \frac{YZ'}{XZ'}\Big/\frac{YZ}{XZ}$$

and hence

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = (-1) \cdot \frac{ZX'}{YX'} \cdot \frac{XY'}{ZY'} \cdot \frac{YZ'}{XZ'}.$$

The equations (1) reduce the problem to a straightforward computation on the line ℓ. For instance, the transformation $t \mapsto -k^2/t$ preserves cross-ratio and interchanges the points X, Y, Z with the points X', Y', Z'. Then

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = (-1) \cdot \frac{ZX'}{YX'}\Big/\frac{ZZ'}{YZ'} \cdot \frac{XY'}{ZY'}\Big/\frac{XZ'}{ZZ'} = -1.$$

We proved that CEVA's ratio equals -1, so AA', BB', CC' intersect at one point Q.

Comment 1. There is a nice projective argument to prove that AX', BY', CZ' intersect at one point. Suppose that ℓ and ω intersect at a pair of complex conjugate points D and E. Consider a projective transformation that takes D and E to $[i; 1, 0]$ and $[-i, 1, 0]$. Then ℓ is the line at infinity, and ω is a conic through the special points $[i; 1, 0]$ and $[-i, 1, 0]$, hence it is a circle. So one can assume that AX, BY, CZ are parallel to BC, CA, AB. The involution on ℓ taking X, Y, Z to X', Y', Z' and leaving D, E fixed is the involution changing each direction to its perpendicular one. Hence AX, BY, CZ are also perpendicular to AX', BY', CZ'.

It follows from the above that AX', BY', CZ' intersect at the orthocenter of triangle ABC.

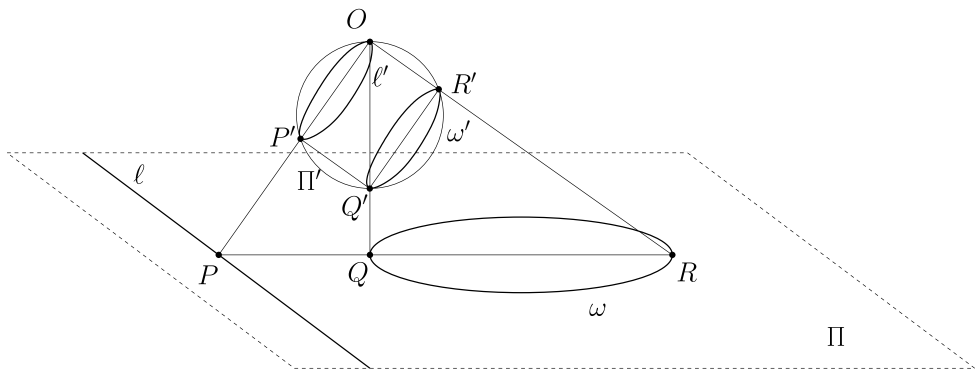
Comment 2. The restriction that the line ℓ does not intersect the circumcricle ω is unnecessary. The proof above works in general. In case ℓ intersects ω at D and E point P is the midpoint of DE, and some equations can be interpreted differently. For instance

$$X'P \cdot X'X = X'A' \cdot X'A = X'D \cdot X'E,$$

and hence the pairs $X'X$ and DE are harmonic conjugates. This means that X', Y', Z' are the harmonic conjugates of X, Y, Z with respect to the segment DE.

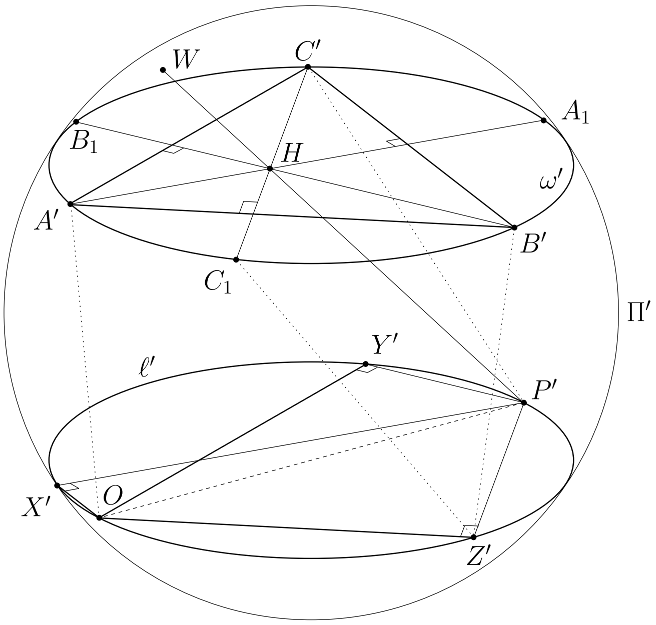
Solution 2. First we prove that there is an inversion in space that takes ℓ and ω to parallel circles on a sphere. Let QR be the diameter of ω whose extension beyond Q passes through P. Let Π be the plane carrying our objects. In space, choose a point O such that the line QO is perpendicular to Π and $\angle POR = 90°$, and apply an inversion with pole O (the radius of the inversion does not matter). For any object $\mathcal{T}$ denote by $\mathcal{T}'$ the image of $\mathcal{T}$ under this inversion.

The inversion takes the plane Π to a sphere Π'. The lines in Π are taken to circles through O, and the circles in Π also are taken to circles on Π'.



Since the line ℓ and the circle ω are perpendicular to the plane OPQ, the circles ℓ' and ω' also are perpendicular to this plane. Hence, the planes of the circles ℓ' and ω' are parallel.

Now consider the circles $A'X'P'$, $B'Y'P'$ and $C'Z'P'$. We want to prove that either they have a common point (on Π'), different from P', or they are tangent to each other.



The point X' is the second intersection of the circles $B'C'O$ and ℓ', other than O. Hence, the lines OX' and $B'C'$ are coplanar. Moreover, they lie in the parallel planes of ℓ' and ω'. Therefore, OX' and $B'C'$ are parallel. Analogously, OY' and OZ' are parallel to $A'C'$ and $A'B'$.

Let A_1 be the second intersection of the circles $A'X'P'$ and ω', other than A'. The segments $A'A_1$ and $P'X'$ are coplanar, and therefore parallel. Now we know that $B'C'$ and $A'A_1$ are parallel to OX' and $X'P'$ respectively, but these two segments are perpendicular because OP' is a diameter in ℓ'. We found that $A'A_1$ and $B'C'$ are perpendicular, hence $A'A_1$ is the altitude in the triangle $A'B'C'$, starting from A.

Analogously, let B_1 and C_1 be the second intersections of ω' with the circles $B'P'Y'$ and $C'P'Z'$, other than B' and C' respectively. Then $B'B_1$ and $C'C_1$ are the other two altitudes in the triangle $A'B'C'$.

Let H be the orthocenter of the triangle $A'B'C'$. Let W be the second intersection of the line $P'H$ with the sphere Π', other than P'. The point W lies on the sphere Π', in the plane of the circle $A'P'X'$, so W lies on the circle $A'P'X'$. Similarly, W lies on the circles $B'P'Y'$ and $C'P'Z'$ as well; indeed W is the second common point of the three circles.

If the line $P'H$ is tangent to the sphere then W coincides with P', and $P'H$ is the common tangent of the three circles.

Number Theory

N1. Call admissible a set A of integers that has the following property:

If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k.

Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

Solution. A pair of integers m, n fulfills the condition if and only if $gcd(m, n) = 1$. Suppose that $gcd(m, n) = d > 1$. The set

$$A = \{\ldots, -2d, -d, 0, d, 2d, \ldots\}$$

is admissible, because if d divides x and y then it divides $x^2 + kxy + y^2$ for every integer k. Also $m, n \in A$ and $A \neq \mathbb{Z}$.

Now let $gcd(m, n) = 1$, and let A be an admissible set containing m and n. We use the following observations to prove that $A = \mathbb{Z}$:

(i) $kx^2 \in A$ for every $x \in A$ and every integer k.

(ii) $(x + y)^2 \in A$ for all $x, y \in A$.

To justify (i) let $y = x$ in the definition of an admissible set; to justify (ii) let $k = 2$.

Since $gcd(m, n) = 1$, we also have $gcd(m^2, n^2) = 1$. Hence one can find integers a, b such that $am^2 + bn^2 = 1$. It follows from (i) that $am^2 \in A$ and $bn^2 \in A$. Now we deduce from (ii) that $1 = (am^2 + bn^2)^2 \in A$. But if $1 \in A$ then (i) implies $k \in A$ for every integer k.

N2. Find all triples (x, y, z) of positive integers such that $x \le y \le z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

Solution. First note that x divides $2012 \cdot 2 = 2^3 \cdot 503$. If $503 \mid x$ then the right-hand side of the equation is divisible by 503^3, and it follows that $503^2 \mid xyz + 2$. This is false as $503 \mid x$. Hence $x = 2^m$ with $m \in \{0, 1, 2, 3\}$. If $m \ge 2$ then $2^6 \mid 2012(xyz + 2)$. However the highest powers of 2 dividing 2012 and $xyz + 2 = 2^m yz + 2$ are 2^2 and 2^1 respectively. So $x = 1$ or $x = 2$, yielding the two equations

$$y^3 + z^3 = 2012(yz + 2), \qquad \text{and} \qquad y^3 + z^3 = 503(yz + 1).$$

In both cases the prime $503 = 3 \cdot 167 + 2$ divides $y^3 + z^3$. We claim that $503 \mid y + z$. This is clear if $503 \mid y$, so let $503 \nmid y$ and $503 \nmid z$. Then $y^{502} \equiv z^{502} \pmod{503}$ by Fermat's little theorem. On the other hand $y^3 \equiv -z^3 \pmod{503}$ implies $y^{3 \cdot 167} \equiv -z^{3 \cdot 167} \pmod{503}$, i. e. $y^{501} \equiv -z^{501} \pmod{503}$. It follows that $y \equiv -z \pmod{503}$ as claimed.

Therefore $y + z = 503k$ with $k \ge 1$. In view of $y^3 + z^3 = (y + z)\big((y - z)^2 + yz\big)$ the two equations take the form

$$k(y - z)^2 + (k - 4)yz = 8, \tag{1}$$
$$k(y - z)^2 + (k - 1)yz = 1. \tag{2}$$

In (1) we have $(k - 4)yz \le 8$, which implies $k \le 4$. Indeed if $k > 4$ then $1 \le (k - 4)yz \le 8$, so that $y \le 8$ and $z \le 8$. This is impossible as $y + z = 503k \ge 503$. Note next that $y^3 + z^3$ is even in the first equation. Hence $y + z = 503k$ is even too, meaning that k is even. Thus $k = 2$ or $k = 4$. Clearly (1) has no integer solutions for $k = 4$. If $k = 2$ then (1) takes the form $(y + z)^2 - 5yz = 4$. Since $y + z = 503k = 503 \cdot 2$, this leads to $5yz = 503^2 \cdot 2^2 - 4$. However $503^2 \cdot 2^2 - 4$ is not a multiple of 5. Therefore (1) has no integer solutions.

Equation (2) implies $0 \le (k - 1)yz \le 1$, so that $k = 1$ or $k = 2$. Also $0 \le k(y - z)^2 \le 1$, hence $k = 2$ only if $y = z$. However then $y = z = 1$, which is false in view of $y + z \ge 503$. Therefore $k = 1$ and (2) takes the form $(y - z)^2 = 1$, yielding $z - y = |y - z| = 1$. Combined with $k = 1$ and $y + z = 503k$, this leads to $y = 251$, $z = 252$.

In summary the triple $(2, 251, 252)$ is the only solution.

N3. Determine all integers $m \geq 2$ such that every n with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

Solution. The integers in question are all prime numbers.

First we check that all primes satisfy the condition, and even a stronger one. Namely, if p is a prime then every n with $1 \leq n \leq \frac{p}{2}$ divides $\binom{n}{p-2n}$. This is true for $p = 2$ where $n = 1$ is the only possibility. For an odd prime p take $n \in [1, \frac{p}{2}]$ and consider the following identity of binomial coefficients:

$$(p-2n) \cdot \binom{n}{p-2n} = n \cdot \binom{n-1}{p-2n-1}.$$

Since $p \geq 2n$ and p is odd, all factors are non-zero. If $d = gcd(p-2n, n)$ then d divides p, but $d \leq n < p$ and hence $d = 1$. It follows that $p - 2n$ and n are relatively prime, and so the factor n in the right-hand side divides the binomial coefficient $\binom{n}{p-2n}$.

Next we show that no composite number m has the stated property. Consider two cases.

- If $m = 2k$ with $k > 1$, pick $n = k$. Then $\frac{m}{3} \leq n \leq \frac{m}{2}$ but $\binom{n}{m-2n} = \binom{k}{0} = 1$ is not divisible by $k > 1$.

- If m is odd then there exist an odd prime p and an integer $k \geq 1$ with $m = p(2k+1)$. Pick $n = pk$, then $\frac{m}{3} \leq n \leq \frac{m}{2}$ by $k \geq 1$. However

$$\frac{1}{n}\binom{n}{m-2n} = \frac{1}{pk}\binom{pk}{p} = \frac{(pk-1)(pk-2)\cdots(pk-(p-1))}{p!}$$

is not an integer, because p divides the denominator but not the numerator.

N4. An integer a is called friendly if the equation $(m^2+n)(n^2+m)=a(m-n)^3$ has a solution over the positive integers.

a) Prove that there are at least 500 friendly integers in the set $\{1,2,\ldots,2012\}$.

b) Decide whether $a=2$ is friendly.

Solution. a) Every a of the form $a=4k-3$ with $k\geq 2$ is friendly. Indeed the numbers $m=2k-1>0$ and $n=k-1>0$ satisfy the given equation with $a=4k-3$:

$$(m^2+n)(n^2+m)=\big((2k-1)^2+(k-1)\big)\big((k-1)^2+(2k-1)\big)=(4k-3)k^3=a(m-n)^3.$$

Hence $5,9,\ldots,2009$ are friendly and so $\{1,2,\ldots,2012\}$ contains at least 502 friendly numbers.

b) We show that $a=2$ is not friendly. Consider the equation with $a=2$ and rewrite its left-hand side as a difference of squares:

$$\frac{1}{4}\left((m^2+n+n^2+m)^2-(m^2+n-n^2-m)^2\right)=2(m-n)^3.$$

Since $m^2+n-n^2-m=(m-n)(m+n-1)$, we can further reformulate the equation as

$$(m^2+n+n^2+m)^2=(m-n)^2\left(8(m-n)+(m+n-1)^2\right).$$

It follows that $8(m-n)+(m+n-1)^2$ is a perfect square. Clearly $m>n$, hence there is an integer $s\geq 1$ such that

$$(m+n-1+2s)^2=8(m-n)+(m+n-1)^2.$$

Subtracting the squares gives $s(m+n-1+s)=2(m-n)$. Since $m+n-1+s>m-n$, we conclude that $s<2$. Therefore the only possibility is $s=1$ and $m=3n$. However then the left-hand side of the given equation (with $a=2$) is greater than $m^3=27n^3$, whereas its right-hand side equals $16n^3$. The contradiction proves that $a=2$ is not friendly.

Comment. A computer search shows that there are 561 friendly numbers in $\{1,2,\ldots,2012\}$.

N5. For a nonnegative integer n define $rad(n) = 1$ if $n = 0$ or $n = 1$, and $rad(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of n. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $rad(f(n))$ divides $rad(f(n^{rad(n)}))$ for every nonnegative integer n.

Solution 1. We are going to prove that $f(x) = ax^m$ for some nonnegative integers a and m. If $f(x)$ is the zero polynomial we are done, so assume that $f(x)$ has at least one positive coefficient. In particular $f(1) > 0$.

Let p be a prime number. The condition is that $f(n) \equiv 0 \pmod{p}$ implies

$$f(n^{rad(n)}) \equiv 0 \pmod{p}. \tag{1}$$

Since $rad(n^{rad(n)^k}) = rad(n)$ for all k, repeated applications of the preceding implication show that if p divides $f(n)$ then

$$f(n^{rad(n)^k}) \equiv 0 \pmod{p} \quad \text{for all } k.$$

The idea is to construct a prime p and a positive integer n such that $p-1$ divides n and p divides $f(n)$. In this case, for k large enough $p-1$ divides $rad(n)^k$. Hence if $(p,n) = 1$ then $n^{rad(n)^k} \equiv 1 \pmod{p}$ by FERMAT's little theorem, so that

$$f(1) \equiv f(n^{rad(n)^k}) \equiv 0 \pmod{p}. \tag{2}$$

Suppose that $f(x) = g(x)x^m$ with $g(0) \neq 0$. Let t be a positive integer, p any prime factor of $g(-t)$ and $n = (p-1)t$. So $p-1$ divides n and $f(n) = f((p-1)t) \equiv f(-t) \equiv 0 \pmod{p}$, hence either $(p,n) > 1$ or (2) holds. If $(p,(p-1)t) > 1$ then p divides t and $g(0) \equiv g(-t) \equiv 0 \pmod{p}$, meaning that p divides $g(0)$.

In conclusion we proved that each prime factor of $g(-t)$ divides $g(0)f(1) \neq 0$, and thus the set of prime factors of $g(-t)$ when t ranges through the positive integers is finite. This is known to imply that $g(x)$ is a constant polynomial, and so $f(x) = ax^m$.

Solution 2. Let $f(x)$ be a polynomial with integer coefficients (not necessarily nonnegative) such that $rad(f(n))$ divides $rad(f(n^{rad(n)}))$ for any nonnegative integer n. We give a complete description of all polynomials with this property. More precisely, we claim that if $f(x)$ is such a polynomial and ξ is a root of $f(x)$ then so is ξ^d for every positive integer d.

Therefore each root of $f(x)$ is zero or a root of unity. In particular, if a root of unity ξ is a root of $f(x)$ then $1 = \xi^d$ is a root too (for some positive integer d). In the original problem $f(x)$ has nonnegative coefficients. Then either $f(x)$ is the zero polynomial or $f(1) > 0$ and $\xi = 0$ is the only possible root. In either case $f(x) = ax^m$ with a and m nonnegative integers.

To prove the claim let ξ be a root of $f(x)$, and let $g(x)$ be an irreducible factor of $f(x)$ such that $g(\xi) = 0$. If 0 or 1 are roots of $g(x)$ then either $\xi = 0$ or $\xi = 1$ (because $g(x)$ is irreducible) and we are done. So assume that $g(0), g(1) \neq 0$. By decomposing d as a product of prime numbers, it is enough to consider the case $d = p$ prime. We argue for $p = 2$. Since $rad(2^k) = 2$ for every k, we have

$$rad(f(2^k)) \mid rad(f(2^{2k})).$$

Now we prove that $g(x)$ divides $f(x^2)$. Suppose that this is not the case. Then, since $g(x)$ is irreducible, there are integer-coefficient polynomials $a(x)$, $b(x)$ and an integer N such that

$$a(x)g(x) + b(x)f(x^2) = N. \tag{3}$$

Each prime factor p of $g(2^k)$ divides $f(2^k)$, so by $rad(f(2^k)) | rad(f(2^{2k}))$ it also divides $f(2^{2k})$. From the equation above with $x = 2^k$ it follows that p divides N.

In summary, each prime divisor of $g(2^k)$ divides N, for all $k \geq 0$. Let $p_1, \dots, p_n$ be the odd primes dividing N, and suppose that

$$g(1) = 2^{\alpha} p_1^{\alpha_1} \cdots p_n^{\alpha_n}.$$

If k is divisible by $\varphi(p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1})$ then

$$2^k \equiv 1 \pmod{p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1}},$$

yielding

$$g(2^k) \equiv g(1) \pmod{p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1}}.$$

It follows that for each i the maximal power of p_i dividing $g(2^k)$ and $g(1)$ is the same, namely $p_i^{\alpha_i}$. On the other hand, for large enough k, the maximal power of 2 dividing $g(2^k)$ and $g(0) \neq 0$ is the same. From the above, for k divisible by $\varphi(p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1})$ and large enough, we obtain that $g(2^k)$ divides $g(0) \cdot g(1)$. This is impossible because $g(0), g(1) \neq 0$ are fixed and $g(2^k)$ is arbitrarily large.

In conclusion, $g(x)$ divides $f(x^2)$. Recall that ξ is a root of $f(x)$ such that $g(\xi) = 0$; then $f(\xi^2) = 0$, i. e. ξ^2 is a root of $f(x)$.

Likewise if ξ is a root of $f(x)$ and p an arbitrary prime then ξ^p is a root too. The argument is completely analogous, in the proof above just replace 2 by p and "odd prime" by "prime different from p."

Comment. The claim in the second solution can be proved by varying $n \pmod{p}$ in (1). For instance, we obtain

$$f(n^{rad(n+pk)}) \equiv 0 \pmod{p}$$

for every positive integer k. One can prove that if $(n,p) = 1$ then $rad(n+pk)$ runs through all residue classes $r \pmod{p-1}$ with $(r, p-1)$ squarefree. Hence if $f(n) \equiv 0 \pmod{p}$ then $f(n^r) \equiv 0 \pmod{p}$ for all integers r. This implies the claim by an argument leading to the identity (3).

N6. Let x and y be positive integers. If $x^{2^n}-1$ is divisible by 2^ny+1 for every positive integer n, prove that $x=1$.

Solution. First we prove the following fact: For every positive integer y there exist infinitely many primes $p \equiv 3 \pmod 4$ such that p divides some number of the form 2^ny+1.

Clearly it is enough to consider the case y odd. Let

$$2y+1 = p_1^{e_1}\cdots p_r^{e_r}$$

be the prime factorization of $2y+1$. Suppose on the contrary that there are finitely many primes $p_{r+1},\dots,p_{r+s} \equiv 3 \pmod 4$ that divide some number of the form 2^ny+1 but do not divide $2y+1$.

We want to find an n such that $p_i^{e_i} || 2^ny+1$ for $1 \le i \le r$ and $p_i \nmid 2^ny+1$ for $r+1 \le i \le r+s$. For this it suffices to take

$$n = 1+\varphi(p_1^{e_1+1}\cdots p_r^{e_r+1}p_{r+1}^1\cdots p_{r+s}^1),$$

because then

$$2^ny+1 \equiv 2y+1 \pmod{p_1^{e_1+1}\cdots p_r^{e_r+1}p_{r+1}^1\cdots p_{r+s}^1}.$$

The last congruence means that $p_1^{e_1},\dots,p_r^{e_r}$ divide exactly 2^ny+1 and no prime $p_{r+1},\dots,p_{r+s}$ divides 2^ny+1. It follows that the prime factorization of 2^ny+1 consists of the prime powers $p_1^{e_1},\dots,p_r^{e_r}$ and powers of primes $\equiv 1 \pmod 4$. Because y is odd, we obtain

$$2^ny+1 \equiv p_1^{e_1}\cdots p_r^{e_r} \equiv 2y+1 \equiv 3 \pmod 4.$$

This is a contradiction since $n>1$, and so $2^ny+1 \equiv 1 \pmod 4$.

Now we proceed to the problem. If p is a prime divisor of 2^ny+1 the problem statement implies that $x^d \equiv 1 \pmod p$ for $d=2^n$. By FERMAT's little theorem the same congruence holds for $d=p-1$, so it must also hold for $d=(2^n,p-1)$. For $p \equiv 3 \pmod 4$ we have $(2^n,p-1)=2$, therefore in this case $x^2 \equiv 1 \pmod p$.

In summary, we proved that every prime $p \equiv 3 \pmod 4$ that divides some number of the form 2^ny+1 also divides x^2-1. This is possible only if $x=1$, otherwise by the above x^2-1 would be a positive integer with infinitely many prime factors.

Comment. For each x and each odd prime p the maximal power of p dividing $x^{2^n}-1$ for some n is bounded and hence the same must be true for the numbers 2^ny+1. We infer that p^2 divides $2^{p-1}-1$ for each prime divisor p of 2^ny+1. However trying to reach a contradiction with this conclusion alone seems hopeless, since it is not even known if there are infinitely many primes p *without* this property.

N7. Find all $n \in \mathbb{N}$ for which there exist nonnegative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

Solution. Such numbers $a_1, a_2, \ldots, a_n$ exist if and only if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$.

Let $\sum_{k=1}^{n} \frac{k}{3^{a_k}} = 1$ with $a_1, a_2, \ldots, a_n$ nonnegative integers. Then $1 \cdot x_1 + 2 \cdot x_2 + \cdots + n \cdot x_n = 3^a$ with $x_1, \ldots, x_n$ powers of 3 and $a \geq 0$. The right-hand side is odd, and the left-hand side has the same parity as $1 + 2 + \cdots + n$. Hence the latter sum is odd, which implies $n \equiv 1, 2 \pmod 4$. Now we prove the converse.

Call *feasible* a sequence $b_1, b_2, \ldots, b_n$ if there are nonnegative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{b_1}{3^{a_1}} + \frac{b_2}{3^{a_2}} + \cdots + \frac{b_n}{3^{a_n}} = 1.$$

Let b_k be a term of a feasible sequence $b_1, b_2, \ldots, b_n$ with exponents $a_1, a_2, \ldots, a_n$ like above, and let u, v be nonnegative integers with sum $3b_k$. Observe that

$$\frac{1}{2^{a_k+1}} + \frac{1}{2^{a_k+1}} = \frac{1}{2^{a_k}} \qquad \text{and} \qquad \frac{u}{3^{a_k+1}} + \frac{v}{3^{a_k+1}} = \frac{b_k}{3^{a_k}}.$$

It follows that the sequence $b_1, \ldots, b_{k-1}, u, v, b_{k+1}, \ldots, b_n$ is feasible. The exponents a_i are the same for the unchanged terms b_i, $i \neq k$; the new terms u, v have exponents $a_k + 1$.

We state the conclusion in reverse. If two terms u, v of a sequence are replaced by one term $\frac{u+v}{3}$ and the obtained sequence is feasible, then the original sequence is feasible too. Denote by α_n the sequence $1, 2, \ldots, n$. To show that α_n is feasible for $n \equiv 1, 2 \pmod 4$, we transform it by $n-1$ replacements $\{u, v\} \mapsto \frac{u+v}{3}$ to the one-term sequence α_1. The latter is feasible, with $a_1 = 0$. Note that if m and $2m$ are terms of a sequence then $\{m, 2m\} \mapsto m$, so $2m$ can be ignored if necessary.

Let $n \geq 16$. We prove that α_n can be reduced to α_{n-12} by 12 operations. Write $n = 12k + r$ where $k \geq 1$ and $0 \leq r \leq 11$. If $0 \leq r \leq 5$ then the last 12 terms of α_n can be partitioned into 2 singletons $\{12k - 6\}$, $\{12k\}$ and the following 5 pairs:

$$\{12k - 6 - i, 12k - 6 + i\}, i = 1, \ldots, 5 - r; \qquad \{12k - j, 12k + j\}, j = 1, \ldots, r.$$

(There is only one kind of pairs if $r \in \{0, 5\}$.) One can ignore $12k - 6$ and $12k$ since α_n contains $6k - 3$ and $6k$. Furthermore the 5 operations $\{12k - 6 - i, 12k - 6 + i\} \mapsto 8k - 4$ and $\{12k - j, 12k + j\} \mapsto 8k$ remove the 10 terms in the pairs and bring in 5 new terms equal to $8k - 4$ or $8k$. All of these can be ignored too as $4k - 2$ and $4k$ are still present in the sequence. Indeed $4k \leq n - 12$ is equivalent to $8k \geq 12 - r$, which is true for $r \in \{4, 5\}$. And if $r \in \{0, 1, 2, 3\}$ then $n \geq 16$ implies $k \geq 2$, so $8k \geq 12 - r$ also holds. Thus α_n reduces to α_{n-12}.

The case $6 \leq r \leq 11$ is analogous. Consider the singletons $\{12k\}$, $\{12k + 6\}$ and the 5 pairs

$$\{12k - i, 12k + i\}, i = 1, \ldots, 11 - r; \qquad \{12k + 6 - j, 12k + 6 + j\}, j = 1, \ldots, r - 6.$$

Ignore the singletons like before, then remove the pairs via operations $\{12k - i, 12k + i\} \mapsto 8k$ and $\{12k + 6 - j, 12k + 6 + j\} \mapsto 8k + 4$. The 5 newly-appeared terms $8k$ and $8k + 4$ can be ignored too since $4k + 2 \leq n - 12$ (this follows from $k \geq 1$ and $r \geq 6$). We obtain α_{n-12} again.

The problem reduces to $2 \leq n \leq 15$. In fact $n \in \{2, 5, 6, 9, 10, 13, 14\}$ by $n \equiv 1, 2 \pmod 4$. The cases $n = 2, 6, 10, 14$ reduce to $n = 1, 5, 9, 13$ respectively because the last even term of α_n can be ignored. For $n = 5$ apply $\{4, 5\} \mapsto 3$, then $\{3, 3\} \mapsto 2$, then ignore the 2 occurrences of 2. For $n = 9$ ignore 6 first, then apply $\{5, 7\} \mapsto 4$, $\{4, 8\} \mapsto 4$, $\{3, 9\} \mapsto 4$. Now ignore the 3 occurrences of 4, then ignore 2. Finally $n = 13$ reduces to $n = 10$ by $\{11, 13\} \mapsto 8$ and ignoring 8 and 12. The proof is complete.

N8. Prove that for every prime $p > 100$ and every integer r there exist two integers a and b such that p divides $a^2 + b^5 - r$.

Solution 1. Throughout the solution, all congruence relations are meant modulo p.

Fix p, and let $\mathcal{P} = \{0, 1, \ldots, p-1\}$ be the set of residue classes modulo p. For every $r \in \mathcal{P}$, let $S_r = \{(a, b) \in \mathcal{P} \times \mathcal{P} \colon a^2 + b^5 \equiv r\}$, and let $s_r = |S_r|$. Our aim is to prove $s_r > 0$ for all $r \in \mathcal{P}$.

We will use the well-known fact that for every residue class $r \in \mathcal{P}$ and every positive integer k, there are at most k values $x \in \mathcal{P}$ such that $x^k \equiv r$.

Lemma. Let N be the number of quadruples $(a, b, c, d) \in \mathcal{P}^4$ for which $a^2 + b^5 \equiv c^2 + d^5$. Then

$$N = \sum_{r \in \mathcal{P}} s_r^2 \qquad (a)$$

and

$$N \leq p(p^2 + 4p - 4). \qquad (b)$$

Proof. (a) For each residue class r there exist exactly s_r pairs (a, b) with $a^2 + b^5 \equiv r$ and s_r pairs (c, d) with $c^2 + d^5 \equiv r$. So there are s_r^2 quadruples with $a^2 + b^5 \equiv c^2 + d^5 \equiv r$. Taking the sum over all $r \in \mathcal{P}$, the statement follows.

(b) Choose an arbitrary pair $(b, d) \in \mathcal{P}$ and look for the possible values of a, c.

1. Suppose that $b^5 \equiv d^5$, and let k be the number of such pairs (b, d). The value b can be chosen in p different ways. For $b \equiv 0$ only $d = 0$ has this property; for the nonzero values of b there are at most 5 possible values for d. So we have $k \leq 1 + 5(p-1) = 5p - 4$.

The values a and c must satisfy $a^2 \equiv c^2$, so $a \equiv \pm c$, and there are exactly $2p - 1$ such pairs (a, c).

2. Now suppose $b^5 \not\equiv d^5$. In this case a and c must be distinct. By $(a - c)(a + c) = d^5 - b^5$, the value of $a - c$ uniquely determines $a + c$ and thus a and c as well. Hence, there are $p - 1$ suitable pairs (a, c).

Thus, for each of the k pairs (b, d) with $b^5 \equiv d^5$ there are $2p - 1$ pairs (a, c), and for each of the other $p^2 - k$ pairs (b, d) there are $p - 1$ pairs (a, c). Hence,

$$N = k(2p-1) + (p^2 - k)(p-1) = p^2(p-1) + kp \leq p^2(p-1) + (5p-4)p = p(p^2 + 4p - 4). \quad \square$$

To prove the statement of the problem, suppose that $S_r = \emptyset$ for some $r \in \mathcal{P}$; obviously $r \not\equiv 0$. Let $T = \{x^{10} \colon x \in \mathcal{P} \setminus \{0\}\}$ be the set of nonzero 10th powers modulo p. Since each residue class is the 10th power of at most 10 elements in $\mathcal{P}$, we have $|T| \geq \frac{p-1}{10} \geq 4$ by $p > 100$.

For every $t \in T$, we have $S_{tr} = \emptyset$. Indeed, if $(x, y) \in S_{tr}$ and $t \equiv z^{10}$ then

$$(z^{-5}x)^2 + (z^{-2}y)^5 \equiv t^{-1}(x^2 + y^5) \equiv r,$$

so $(z^{-5}x, z^{-2}y) \in S_r$. So, there are at least $\frac{p-1}{10} \geq 4$ empty sets among $S_1, \ldots, S_{p-1}$, and there are at most $p - 4$ nonzero values among $s_0, s_2, \ldots, s_{p-1}$. Then by the AM-QM inequality we obtain

$$N = \sum_{r \in \mathcal{P} \setminus rT} s_r^2 \geq \frac{1}{p-4}\left(\sum_{r \in \mathcal{P} \setminus rT} s_r\right)^2 = \frac{|\mathcal{P} \times \mathcal{P}|^2}{p-4} = \frac{p^4}{p-4} > p(p^2 + 4p - 4),$$

which is impossible by the lemma.

Solution 2. If $5 \nmid p-1$, then all modulo p residue classes are complete fifth powers and the statement is trivial. So assume that $p = 10k+1$ where $k \geq 10$. Let g be a primitive root modulo p.

We will use the following facts:

(F1) If some residue class x is not quadratic then $x^{(p-1)/2} \equiv -1 \pmod{p}$.

(F2) For every integer d, as a simple corollary of the summation formula for geometric progressions,

$$\sum_{i=0}^{2k-1} g^{5di} \equiv \begin{cases} 2k & \text{if } 2k \mid d \\ 0 & \text{if } 2k \nmid d \end{cases} \pmod{p}.$$

Suppose that, contrary to the statement, some modulo p residue class r cannot be expressed as a^2+b^5. Of course $r \not\equiv 0 \pmod{p}$. By (F1) we have $(r-b^5)^{(p-1)/2} = (r-b^5)^{5k} \equiv -1 \pmod{p}$ for all residue classes b.

For $t = 1, 2 \ldots, k-1$ consider the sums

$$S(t) = \sum_{i=0}^{2k-1} \left(r - g^{5i}\right)^{5k} g^{5ti}.$$

By the indirect assumption and (F2),

$$S(t) = \sum_{i=0}^{2k-1} \left(r - (g^i)^5\right)^{5k} g^{5ti} \equiv \sum_{i=0}^{2k-1} (-1) g^{5ti} \equiv -\sum_{i=0}^{2k-1} g^{5ti} \equiv 0 \pmod{p}$$

because $2k$ cannot divide t.

On the other hand, by the binomial theorem,

$$S(t) = \sum_{i=0}^{2k-1} \left(\sum_{j=0}^{5k} \binom{5k}{j} r^{5k-j} \left(-g^{5i}\right)^j \right) g^{5ti} = \sum_{j=0}^{5k} (-1)^j \binom{5k}{j} r^{5k-j} \left(\sum_{i=0}^{2k-1} g^{5(j+t)i} \right) \equiv$$

$$\equiv \sum_{j=0}^{5k} (-1)^j \binom{5k}{j} r^{5k-j} \begin{cases} 2k & \text{if } 2k \mid j+t \\ 0 & \text{if } 2k \nmid j+t \end{cases} \pmod{p}.$$

Since $1 \leq j+t < 6k$, the number $2k$ divides $j+t$ only for $j = 2k-t$ and $j = 4k-t$. Hence,

$$0 \equiv S(t) \equiv (-1)^t \left(\binom{5k}{2k-t} r^{3k+t} + \binom{5k}{4k-t} r^{k+t} \right) \cdot 2k \pmod{p},$$

$$\binom{5k}{2k-t} r^{2k} + \binom{5k}{4k-t} \equiv 0 \pmod{p}.$$

Taking this for $t = 1, 2$ and eliminating r, we get

$$\begin{aligned} 0 &\equiv \binom{5k}{2k-2} \left(\binom{5k}{2k-1} r^{2k} + \binom{5k}{4k-1} \right) - \binom{5k}{2k-1} \left(\binom{5k}{2k-2} r^{2k} + \binom{5k}{4k-2} \right) \\ &= \binom{5k}{2k-2}\binom{5k}{4k-1} - \binom{5k}{2k-1}\binom{5k}{4k-2} \\ &= \frac{(5k)!^2}{(2k-1)!(3k+2)!(4k-1)!(k+2)!} \Big((2k-1)(k+2) - (3k+2)(4k-1) \Big) \\ &= \frac{-(5k)!^2 \cdot 2k(5k+1)}{(2k-1)!(3k+2)!(4k-1)!(k+2)!} \pmod{p}. \end{aligned}$$

But in the last expression none of the numbers is divisible by $p = 10k+1$, a contradiction.

Comment 1. The argument in the second solution is valid whenever $k \geq 3$, that is for all primes $p = 10k + 1$ except $p = 11$. This is an exceptional case when the statement is not true; $r = 7$ cannot be expressed as desired.

Comment 2. The statement is true in a more general setting: for every positive integer n, for all sufficiently large p, each residue class modulo p can be expressed as $a^2 + b^n$. Choosing $t = 3$ would allow using the Cauchy-Davenport theorem (together with some analysis on the case of equality).

In the literature more general results are known. For instance, the statement easily follows from the *Hasse-Weil bound*.

Shortlisted Problems with Solutions

54th International Mathematical Olympiad

Santa Marta, Colombia 2013

Note of Confidentiality

The Shortlisted Problems should be kept strictly confidential until IMO 2014.

Contributing Countries

The Organizing Committee and the Problem Selection Committee of IMO 2013 thank the following 50 countries for contributing 149 problem proposals.

Argentina, Armenia, Australia, Austria, Belgium, Belarus, Brazil, Bulgaria, Croatia, Cyprus, Czech Republic, Denmark, El Salvador, Estonia, Finland, France, Georgia, Germany, Greece, Hungary, India, Indonesia, Iran, Ireland, Israel, Italy, Japan, Latvia, Lithuania, Luxembourg, Malaysia, Mexico, Netherlands, Nicaragua, Pakistan, Panama, Poland, Romania, Russia, Saudi Arabia, Serbia, Slovenia, Sweden, Switzerland, Tajikistan, Thailand, Turkey, U.S.A., Ukraine, United Kingdom

Problem Selection Committee

Federico Ardila (chairman)
Ilya I. Bogdanov
Géza Kós
Carlos Gustavo Tamm de Araújo Moreira (Gugu)
Christian Reiher

Problems

Algebra

A1. Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and

$$u_{k+1} = u_k + a_k u_{k-1}, \qquad v_{k+1} = v_k + a_{n-k} v_{k-1} \qquad \text{for } k = 1, \ldots, n-1.$$

Prove that $u_n = v_n$.

(France)

A2. Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that

$$\left| \frac{a-b}{c-d} - 1 \right| < \frac{1}{100000}.$$

(Lithuania)

A3. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f \colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the conditions

$$f(x)f(y) \geqslant f(xy) \quad \text{and} \quad f(x+y) \geqslant f(x) + f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$. Given that $f(a) = a$ for some rational $a > 1$, prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

(Bulgaria)

A4. Let n be a positive integer, and consider a sequence $a_1, a_2, \ldots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \ldots$ by defining $a_{n+i} = a_i$ for all $i \geqslant 1$. If

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant a_1 + n$$

and

$$a_{a_i} \leqslant n + i - 1 \qquad \text{for } i = 1, 2, \ldots, n,$$

prove that

$$a_1 + \cdots + a_n \leqslant n^2.$$

(Germany)

A5. Let $\mathbb{Z}_{\geqslant 0}$ be the set of all nonnegative integers. Find all the functions $f \colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{Z}_{\geqslant 0}$.

(Serbia)

A6. Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3 - mx^2 + 1)P(x+1) + (x^3 + mx^2 + 1)P(x-1) = 2(x^3 - mx + 1)P(x)$$

for all real numbers x.

(Serbia)

Combinatorics

C1. Let n be a positive integer. Find the smallest integer k with the following property: Given any real numbers $a_1, \ldots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leqslant a_i \leqslant 1$ for $i = 1, 2, \ldots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

(Poland)

C2. In the plane, 2013 red points and 2014 blue points are marked so that no three of the marked points are collinear. One needs to draw k lines not passing through the marked points and dividing the plane into several regions. The goal is to do it in such a way that no region contains points of both colors.

Find the minimal value of k such that the goal is attainable for every possible configuration of 4027 points.

(Australia)

C3. A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be *entangled*, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

(Japan)

C4. Let n be a positive integer, and let A be a subset of $\{1, \ldots, n\}$. An *A-partition* of n *into k parts* is a representation of n as a sum $n = a_1 + \cdots + a_k$, where the parts $a_1, \ldots, a_k$ belong to A and are not necessarily distinct. The *number of different parts* in such a partition is the number of (distinct) elements in the set $\{a_1, a_2, \ldots, a_k\}$.

We say that an A-partition of n into k parts is *optimal* if there is no A-partition of n into r parts with $r < k$. Prove that any optimal A-partition of n contains at most $\sqrt[3]{6n}$ different parts.

(Germany)

C5. Let r be a positive integer, and let $a_0, a_1, \ldots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers m and s there exists a positive integer $n \in [m+1, m+r]$ such that

$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}.$$

Prove that the sequence is periodic, i. e. there exists some $p \geqslant 1$ such that $a_{n+p} = a_n$ for all $n \geqslant 0$.

(India)

C6. In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The *distance* between two cities is defined to be the least possible number of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

(Russia)

C7. Let $n \geqslant 2$ be an integer. Consider all circular arrangements of the numbers $0, 1, \ldots, n$; the $n+1$ rotations of an arrangement are considered to be equal. A circular arrangement is called *beautiful* if, for any four distinct numbers $0 \leqslant a, b, c, d \leqslant n$ with $a+c=b+d$, the chord joining numbers a and c does not intersect the chord joining numbers b and d.

Let M be the number of beautiful arrangements of $0, 1, \ldots, n$. Let N be the number of pairs (x, y) of positive integers such that $x+y \leqslant n$ and $\gcd(x, y)=1$. Prove that

$$M = N + 1.$$

(Russia)

C8. Players A and B play a paintful game on the real line. Player A has a pot of paint with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In every round, player A picks some positive integer m and provides $1/2^m$ units of ink from the pot. Player B then picks an integer k and blackens the interval from $k/2^m$ to $(k+1)/2^m$ (some parts of this interval may have been blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0,1]$ is not completely blackened.

Decide whether there exists a strategy for player A to win in a finite number of moves.

(Austria)

Geometry

G1. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on side BC. Denote by M and N the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 which is diametrically opposite to W. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 which is diametrically opposite to W. Prove that X, Y and H are collinear.

(Thaliand)

G2. Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

(Iran)

G3. In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \leqslant \max\{\angle BAC, \angle ABC\}$.

(Serbia)

G4. Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

(Georgia)

G5. Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

(Ukraine)

G6. Let the excircle of the triangle ABC lying opposite to A touch its side BC at the point A_1. Define the points B_1 and C_1 analogously. Suppose that the circumcentre of the triangle $A_1B_1C_1$ lies on the circumcircle of the triangle ABC. Prove that the triangle ABC is right-angled.

(Russia)

Number Theory

N1. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

(Malaysia)

N2. Prove that for any pair of positive integers k and n there exist k positive integers $m_1, m_2, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

(Japan)

N3. Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

(Belgium)

N4. Determine whether there exists an infinite sequence of nonzero digits $a_1, a_2, a_3, \ldots$ and a positive integer N such that for every integer $k > N$, the number $\overline{a_k a_{k-1} \ldots a_1}$ is a perfect square.

(Iran)

N5. Fix an integer $k \geqslant 2$. Two players, called Ana and Banana, play the following *game of numbers*: Initially, some integer $n \geqslant k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leqslant m' < m$ that is coprime to m. The first player who cannot move anymore loses.

An integer $n \geqslant k$ is called *good* if Banana has a winning strategy when the initial number is n, and *bad* otherwise.

Consider two integers $n, n' \geqslant k$ with the property that each prime number $p \leqslant k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

(Italy)

N6. Determine all functions $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfying

$$f\left(\frac{f(x) + a}{b}\right) = f\left(\frac{x + a}{b}\right)$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$. (Here, $\mathbb{Z}_{>0}$ denotes the set of positive integers.)

(Israel)

N7. Let ν be an irrational positive number, and let m be a positive integer. A pair (a, b) of positive integers is called *good* if

$$a\lceil b\nu\rceil - b\lfloor a\nu\rfloor = m.$$

A good pair (a, b) is called *excellent* if neither of the pairs $(a-b, b)$ and $(a, b-a)$ is good. (As usual, by $\lfloor x\rfloor$ and $\lceil x\rceil$ we denote the integer numbers such that $x - 1 < \lfloor x\rfloor \leqslant x$ and $x \leqslant \lceil x\rceil < x + 1$.)

Prove that the number of excellent pairs is equal to the sum of the positive divisors of m.

(U.S.A.)

Solutions

Algebra

A1. Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and

$$u_{k+1} = u_k + a_k u_{k-1}, \qquad v_{k+1} = v_k + a_{n-k} v_{k-1} \qquad \text{for } k = 1, \ldots, n-1.$$

Prove that $u_n = v_n$.

(France)

Solution 1. We prove by induction on k that

$$u_k = \sum_{\substack{0<i_1<\ldots<i_t<k, \\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t}. \tag{1}$$

Note that we have one trivial summand equal to 1 (which corresponds to $t = 0$ and the empty sequence, whose product is 1).

For $k = 0, 1$ the sum on the right-hand side only contains the empty product, so (1) holds due to $u_0 = u_1 = 1$. For $k \geqslant 1$, assuming the result is true for $0, 1, \ldots, k$, we have

$$\begin{aligned} u_{k+1} &= \sum_{\substack{0<i_1<\ldots<i_t<k, \\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t} + \sum_{\substack{0<i_1<\ldots<i_t<k-1, \\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t} \cdot a_k \\ &= \sum_{\substack{0<i_1<\ldots<i_t<k+1, \\ i_{j+1}-i_j \geqslant 2, \\ k \notin \{i_1, \ldots, i_t\}}} a_{i_1} \ldots a_{i_t} + \sum_{\substack{0<i_1<\ldots<i_t<k+1, \\ i_{j+1}-i_j \geqslant 2, \\ k \in \{i_1, \ldots, i_t\}}} a_{i_1} \ldots a_{i_t} \\ &= \sum_{\substack{0<i_1<\ldots<i_t<k+1, \\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t}, \end{aligned}$$

as required.

Applying (1) to the sequence $b_1, \ldots, b_n$ given by $b_k = a_{n-k}$ for $1 \leqslant k \leqslant n$, we get

$$v_k = \sum_{\substack{0<i_1<\ldots<i_t<k, \\ i_{j+1}-i_j \geqslant 2}} b_{i_1} \ldots b_{i_t} = \sum_{\substack{n>i_1>\ldots>i_t>n-k, \\ i_j-i_{j+1} \geqslant 2}} a_{i_1} \ldots a_{i_t}. \tag{2}$$

For $k = n$ the expressions (1) and (2) coincide, so indeed $u_n = v_n$.

Solution 2. Define recursively a sequence of multivariate polynomials by

$$P_0 = P_1 = 1, \qquad P_{k+1}(x_1, \ldots, x_k) = P_k(x_1, \ldots, x_{k-1}) + x_k P_{k-1}(x_1, \ldots, x_{k-2}),$$

so P_n is a polynomial in $n-1$ variables for each $n \geqslant 1$. Two easy inductive arguments show that

$$u_n = P_n(a_1, \ldots, a_{n-1}), \qquad v_n = P_n(a_{n-1}, \ldots, a_1),$$

so we need to prove $P_n(x_1,\dots,x_{n-1}) = P_n(x_{n-1},\dots,x_1)$ for every positive integer n. The cases $n=1,2$ are trivial, and the cases $n=3,4$ follow from $P_3(x,y)=1+x+y$ and $P_4(x,y,z)=1+x+y+z+xz$.

Now we proceed by induction, assuming that $n \geqslant 5$ and the claim hold for all smaller cases. Using $F(a,b)$ as an abbreviation for $P_{|a-b|+1}(x_a,\dots,x_b)$ (where the indices $a,\dots,b$ can be either in increasing or decreasing order),

$$\begin{aligned}
F(n,1) &= F(n,2)+x_1F(n,3) = F(2,n)+x_1F(3,n)\\
&= \big(F(2,n-1)+x_nF(2,n-2)\big)+x_1\big(F(3,n-1)+x_nF(3,n-2)\big)\\
&= \big(F(n-1,2)+x_1F(n-1,3)\big)+x_n\big(F(n-2,2)+x_1F(n-2,3)\big)\\
&= F(n-1,1)+x_nF(n-2,1) = F(1,n-1)+x_nF(1,n-2)\\
&= F(1,n),
\end{aligned}$$

as we wished to show.

Solution 3. Using matrix notation, we can rewrite the recurrence relation as

$$\begin{pmatrix} u_{k+1} \\ u_{k+1}-u_k \end{pmatrix} = \begin{pmatrix} u_k+a_ku_{k-1} \\ a_ku_{k-1} \end{pmatrix} = \begin{pmatrix} 1+a_k & -a_k \\ a_k & -a_k \end{pmatrix} \begin{pmatrix} u_k \\ u_k-u_{k-1} \end{pmatrix}$$

for $1 \leqslant k \leqslant n-1$, and similarly

$$(v_{k+1}; v_k-v_{k+1}) = \big(v_k+a_{n-k}v_{k-1}; -a_{n-k}v_{k-1}\big) = (v_k; v_{k-1}-v_k)\begin{pmatrix} 1+a_{n-k} & -a_{n-k} \\ a_{n-k} & -a_{n-k} \end{pmatrix}$$

for $1 \leqslant k \leqslant n-1$. Hence, introducing the 2×2 matrices $A_k = \begin{pmatrix} 1+a_k & -a_k \\ a_k & -a_k \end{pmatrix}$ we have

$$\begin{pmatrix} u_{k+1} \\ u_{k+1}-u_k \end{pmatrix} = A_k \begin{pmatrix} u_k \\ u_k-u_{k-1} \end{pmatrix} \quad \text{and} \quad (v_{k+1}; v_k-v_{k+1}) = (v_k; v_{k-1}-v_k)A_{n-k}.$$

for $1 \leqslant k \leqslant n-1$. Since $\binom{u_1}{u_1-u_0} = \binom{1}{0}$ and $(v_1; v_0-v_1) = (1;0)$, we get

$$\begin{pmatrix} u_n \\ u_n-u_{n-1} \end{pmatrix} = A_{n-1}A_{n-2}\cdots A_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad (v_n; v_{n-1}-v_n) = (1;0)\cdot A_{n-1}A_{n-2}\cdots A_1.$$

It follows that

$$(u_n) = (1;0)\begin{pmatrix} u_n \\ u_n-u_{n-1} \end{pmatrix} = (1;0)\cdot A_{n-1}A_{n-2}\cdots A_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (v_n; v_{n-1}-v_n)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (v_n).$$

Comment 1. These sequences are related to the Fibonacci sequence; when $a_1=\dots=a_{n-1}=1$, we have $u_k=v_k=F_{k+1}$, the $(k+1)$st Fibonacci number. Also, for every positive integer k, the polynomial $P_k(x_1,\dots,x_{k-1})$ from Solution 2 is the sum of F_{k+1} monomials.

Comment 2. One may notice that the condition is equivalent to

$$\frac{u_{k+1}}{u_k} = 1+\cfrac{a_k}{1+\cfrac{a_{k-1}}{1+\dots+\cfrac{a_2}{1+a_1}}} \qquad \text{and} \qquad \frac{v_{k+1}}{v_k} = 1+\cfrac{a_{n-k}}{1+\cfrac{a_{n-k+1}}{1+\dots+\cfrac{a_{n-2}}{1+a_{n-1}}}}$$

so the problem claims that the corresponding continued fractions for u_n/u_{n-1} and v_n/v_{n-1} have the same numerator.

Comment 3. An alternative variant of the problem is the following.

> *Let n be a positive integer and let $a_1, \dots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \dots, u_n$ and $v_0, \dots, v_n$ inductively by $u_0 = v_0 = 0$, $u_1 = v_1 = 1$, and*
>
> $$u_{k+1} = a_k u_k + u_{k-1}, \qquad v_{k+1} = a_{n-k} v_k + v_{k-1} \qquad \text{for } k = 1, \dots, n-1.$$
>
> *Prove that $u_n = v_n$.*

All three solutions above can be reformulated to prove this statement; one may prove

$$u_n = v_n = \sum_{\substack{0=i_0<i_1<\ldots<i_t=n, \\ i_{j+1}-i_j \text{ is odd}}} a_{i_1} \dots a_{i_{t-1}} \qquad \text{for } n > 0$$

or observe that

$$\begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} = \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_k \\ u_{k-1} \end{pmatrix} \quad \text{and} \quad (v_{k+1}; v_k) = (v_k; v_{k-1}) \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}.$$

Here we have

$$\frac{u_{k+1}}{u_k} = a_k + \cfrac{1}{a_{k-1} + \cfrac{1}{a_{k-2} + \ldots + \cfrac{1}{a_1}}} = [a_k; a_{k-1}, \dots, a_1]$$

and

$$\frac{v_{k+1}}{v_k} = a_{n-k} + \cfrac{1}{a_{n-k+1} + \cfrac{1}{a_{n-k+2} + \ldots + \cfrac{1}{a_{n-1}}}} = [a_{n-k}; a_{n-k+1}, \dots, a_{n-1}],$$

so this alternative statement is equivalent to the known fact that the continued fractions $[a_{n-1}; a_{n-2}, \dots, a_1]$ and $[a_1; a_2, \dots, a_{n-1}]$ have the same numerator.

A2. Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that

$$\left|\frac{a-b}{c-d} - 1\right| < \frac{1}{100000}.$$

(Lithuania)

Solution. For any set S of $n = 2000$ distinct real numbers, let $D_1 \leqslant D_2 \leqslant \cdots \leqslant D_m$ be the distances between them, displayed with their multiplicities. Here $m = n(n-1)/2$. By rescaling the numbers, we may assume that the smallest distance D_1 between two elements of S is $D_1 = 1$. Let $D_1 = 1 = y - x$ for $x, y \in S$. Evidently $D_m = v - u$ is the difference between the largest element v and the smallest element u of S.

If $D_{i+1}/D_i < 1 + 10^{-5}$ for some $i = 1, 2, \ldots, m-1$ then the required inequality holds, because $0 \leqslant D_{i+1}/D_i - 1 < 10^{-5}$. Otherwise, the reverse inequality

$$\frac{D_{i+1}}{D_i} \geqslant 1 + \frac{1}{10^5}$$

holds for each $i = 1, 2, \ldots, m-1$, and therefore

$$v - u = D_m = \frac{D_m}{D_1} = \frac{D_m}{D_{m-1}} \cdots \frac{D_3}{D_2} \cdot \frac{D_2}{D_1} \geqslant \left(1 + \frac{1}{10^5}\right)^{m-1}.$$

From $m - 1 = n(n-1)/2 - 1 = 1000 \cdot 1999 - 1 > 19 \cdot 10^5$, together with the fact that for all $n \geqslant 1$, $\left(1 + \frac{1}{n}\right)^n \geqslant 1 + \binom{n}{1} \cdot \frac{1}{n} = 2$, we get

$$\left(1 + \frac{1}{10^5}\right)^{19 \cdot 10^5} = \left(\left(1 + \frac{1}{10^5}\right)^{10^5}\right)^{19} \geqslant 2^{19} = 2^9 \cdot 2^{10} > 500 \cdot 1000 > 2 \cdot 10^5,$$

and so $v - u = D_m > 2 \cdot 10^5$.

Since the distance of x to at least one of the numbers u, v is at least $(u-v)/2 > 10^5$, we have

$$|x - z| > 10^5.$$

for some $z \in \{u, v\}$. Since $y - x = 1$, we have either $z > y > x$ (if $z = v$) or $y > x > z$ (if $z = u$). If $z > y > x$, selecting $a = z, b = y, c = z$ and $d = x$ (so that $b \neq d$), we obtain

$$\left|\frac{a-b}{c-d} - 1\right| = \left|\frac{z-y}{z-x} - 1\right| = \left|\frac{x-y}{z-x}\right| = \frac{1}{z-x} < 10^{-5}.$$

Otherwise, if $y > x > z$, we may choose $a = y, b = z, c = x$ and $d = z$ (so that $a \neq c$), and obtain

$$\left|\frac{a-b}{c-d} - 1\right| = \left|\frac{y-z}{x-z} - 1\right| = \left|\frac{y-x}{x-z}\right| = \frac{1}{x-z} < 10^{-5}.$$

The desired result follows.

Comment. As the solution shows, the numbers 2000 and $\frac{1}{100000}$ appearing in the statement of the problem may be replaced by any $n \in \mathbb{Z}_{>0}$ and $\delta > 0$ satisfying

$$\delta(1 + \delta)^{n(n-1)/2 - 1} > 2.$$

A3. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f\colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the conditions

$$f(x)f(y) \geqslant f(xy), \tag{1}$$
$$f(x+y) \geqslant f(x) + f(y) \tag{2}$$

for all $x, y \in \mathbb{Q}_{>0}$. Given that $f(a) = a$ for some rational $a > 1$, prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

(Bulgaria)

Solution. Denote by $\mathbb{Z}_{>0}$ the set of positive integers.

Plugging $x = 1$, $y = a$ into (1) we get $f(1) \geqslant 1$. Next, by an easy induction on n we get from (2) that

$$f(nx) \geqslant nf(x) \quad \text{for all } n \in \mathbb{Z}_{>0} \text{ and } x \in \mathbb{Q}_{>0}. \tag{3}$$

In particular, we have

$$f(n) \geqslant nf(1) \geqslant n \quad \text{for all } n \in \mathbb{Z}_{>0}. \tag{4}$$

From (1) again we have $f(m/n)f(n) \geqslant f(m)$, so $f(q) > 0$ for all $q \in \mathbb{Q}_{>0}$.

Now, (2) implies that f is strictly increasing; this fact together with (4) yields

$$f(x) \geqslant f(\lfloor x \rfloor) \geqslant \lfloor x \rfloor > x - 1 \quad \text{for all } x \geqslant 1.$$

By an easy induction we get from (1) that $f(x)^n \geqslant f(x^n)$, so

$$f(x)^n \geqslant f(x^n) > x^n - 1 \quad \Longrightarrow \quad f(x) \geqslant \sqrt[n]{x^n - 1} \quad \text{for all } x > 1 \text{ and } n \in \mathbb{Z}_{>0}.$$

This yields

$$f(x) \geqslant x \quad \text{for every } x > 1. \tag{5}$$

(Indeed, if $x > y > 1$ then $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + y^n) > n(x - y)$, so for a large n we have $x^n - 1 > y^n$ and thus $f(x) > y$.)

Now, (1) and (5) give $a^n = f(a)^n \geqslant f(a^n) \geqslant a^n$, so $f(a^n) = a^n$. Now, for $x > 1$ let us choose $n \in \mathbb{Z}_{>0}$ such that $a^n - x > 1$. Then by (2) and (5) we get

$$a^n = f(a^n) \geqslant f(x) + f(a^n - x) \geqslant x + (a^n - x) = a^n$$

and therefore $f(x) = x$ for $x > 1$. Finally, for every $x \in \mathbb{Q}_{>0}$ and every $n \in \mathbb{Z}_{>0}$, from (1) and (3) we get

$$nf(x) = f(n)f(x) \geqslant f(nx) \geqslant nf(x),$$

which gives $f(nx) = nf(x)$. Therefore $f(m/n) = f(m)/n = m/n$ for all $m, n \in \mathbb{Z}_{>0}$.

Comment. The condition $f(a) = a > 1$ is essential. Indeed, for $b \geqslant 1$ the function $f(x) = bx^2$ satisfies (1) and (2) for all $x, y \in \mathbb{Q}_{>0}$, and it has a unique fixed point $1/b \leqslant 1$.

A4. Let n be a positive integer, and consider a sequence $a_1, a_2, \ldots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \ldots$ by defining $a_{n+i} = a_i$ for all $i \geqslant 1$. If

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant a_1 + n \tag{1}$$

and

$$a_{a_i} \leqslant n + i - 1 \qquad \text{for } i = 1, 2, \ldots, n, \tag{2}$$

prove that

$$a_1 + \cdots + a_n \leqslant n^2.$$

(Germany)

Solution 1. First, we claim that

$$a_i \leqslant n + i - 1 \qquad \text{for } i = 1, 2, \ldots, n. \tag{3}$$

Assume contrariwise that i is the smallest counterexample. From $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_i \geqslant n + i$ and $a_{a_i} \leqslant n + i - 1$, taking into account the periodicity of our sequence, it follows that

$$a_i \text{ cannot be congruent to } i,\ i+1, \ldots, n-1, \text{or } n \pmod{n}. \tag{4}$$

Thus our assumption that $a_i \geqslant n + i$ implies the stronger statement that $a_i \geqslant 2n + 1$, which by $a_1 + n \geqslant a_n \geqslant a_i$ gives $a_1 \geqslant n + 1$. The minimality of i then yields $i = 1$, and (4) becomes contradictory. This establishes our first claim.

In particular we now know that $a_1 \leqslant n$. If $a_n \leqslant n$, then $a_1 \leqslant \cdots \leqslant \cdots a_n \leqslant n$ and the desired inequality holds trivially. Otherwise, consider the number t with $1 \leqslant t \leqslant n-1$ such that

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_t \leqslant n < a_{t+1} \leqslant \ldots \leqslant a_n. \tag{5}$$

Since $1 \leqslant a_1 \leqslant n$ and $a_{a_1} \leqslant n$ by (2), we have $a_1 \leqslant t$ and hence $a_n \leqslant n + t$. Therefore if for each positive integer i we let b_i be the number of indices $j \in \{t+1, \ldots, n\}$ satisfying $a_j \geqslant n + i$, we have

$$b_1 \geqslant b_2 \geqslant \ldots \geqslant b_t \geqslant b_{t+1} = 0.$$

Next we claim that $a_i + b_i \leqslant n$ for $1 \leqslant i \leqslant t$. Indeed, by $n + i - 1 \geqslant a_{a_i}$ and $a_i \leqslant n$, each j with $a_j \geqslant n + i$ (thus $a_j > a_{a_i}$) belongs to $\{a_i + 1, \ldots, n\}$, and for this reason $b_i \leqslant n - a_i$.

It follows from the definition of the b_is and (5) that

$$a_{t+1} + \ldots + a_n \leqslant n(n-t) + b_1 + \ldots + b_t.$$

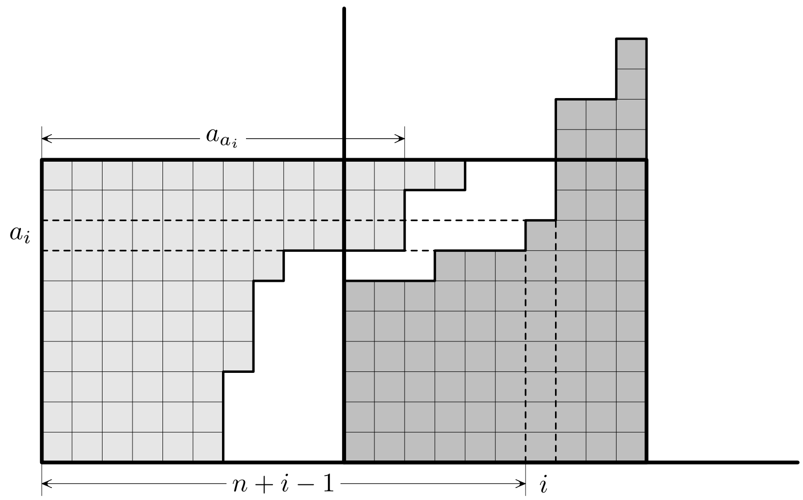
Adding $a_1 + \ldots + a_t$ to both sides and using that $a_i + b_i \leqslant n$ for $1 \leqslant i \leqslant t$, we get

$$a_1 + a_2 + \cdots + a_n \leqslant n(n-t) + nt = n^2$$

as we wished to prove.

Solution 2. In the first quadrant of an infinite grid, consider the increasing "staircase" obtained by shading in dark the bottom a_i cells of the ith column for $1 \leqslant i \leqslant n$. We will prove that there are at most n^2 dark cells.

To do it, consider the $n \times n$ square S in the first quadrant with a vertex at the origin. Also consider the $n \times n$ square directly to the left of S. Starting from its lower left corner, shade in light the leftmost a_j cells of the jth row for $1 \leqslant j \leqslant n$. Equivalently, the light shading is obtained by reflecting the dark shading across the line $x = y$ and translating it n units to the left. The figure below illustrates this construction for the sequence $6, 6, 6, 7, 7, 7, 8, 12, 12, 14$.



We claim that there is no cell in S which is both dark and light. Assume, contrariwise, that there is such a cell in column i. Consider the highest dark cell in column i which is inside S. Since it is above a light cell and inside S, it must be light as well. There are two cases:

Case 1. $a_i \leqslant n$

If $a_i \leqslant n$ then this dark and light cell is (i, a_i), as highlighted in the figure. However, this is the $(n+i)$-th cell in row a_i, and we only shaded $a_{a_i} < n + i$ light cells in that row, a contradiction.

Case 2. $a_i \geqslant n + 1$

If $a_i \geqslant n + 1$, this dark and light cell is (i, n). This is the $(n + i)$-th cell in row n and we shaded $a_n \leqslant a_1 + n$ light cells in this row, so we must have $i \leqslant a_1$. But $a_1 \leqslant a_{a_1} \leqslant n$ by (1) and (2), so $i \leqslant a_1$ implies $a_i \leqslant a_{a_1} \leqslant n$, contradicting our assumption.

We conclude that there are no cells in S which are both dark and light. It follows that the number of shaded cells in S is at most n^2.

Finally, observe that if we had a light cell to the right of S, then by symmetry we would have a dark cell above S, and then the cell (n, n) would be dark and light. It follows that the number of light cells in S equals the number of dark cells outside of S, and therefore the number of shaded cells in S equals $a_1 + \cdots + a_n$. The desired result follows.

Solution 3. As in Solution 1, we first establish that $a_i \leqslant n + i - 1$ for $1 \leqslant i \leqslant n$. Now define $c_i = \max(a_i, i)$ for $1 \leqslant i \leqslant n$ and extend the sequence $c_1, c_2, \ldots$ periodically modulo n. We claim that this sequence also satisfies the conditions of the problem.

For $1 \leqslant i < j \leqslant n$ we have $a_i \leqslant a_j$ and $i < j$, so $c_i \leqslant c_j$. Also $a_n \leqslant a_1 + n$ and $n < 1 + n$ imply $c_n \leqslant c_1 + n$. Finally, the definitions imply that $c_{c_i} \in \{a_{a_i}, a_i, a_i - n, i\}$ so $c_{c_i} \leqslant n + i - 1$ by (2) and (3). This establishes (1) and (2) for $c_1, c_2, \ldots$.

Our new sequence has the additional property that

$$c_i \geqslant i \qquad \text{for } i = 1, 2, \ldots, n, \tag{6}$$

which allows us to construct the following visualization: Consider n equally spaced points on a circle, sequentially labelled $1, 2, \ldots, n \pmod n$, so point k is also labelled $n+k$. We draw arrows from vertex i to vertices $i+1, \ldots, c_i$ for $1 \leqslant i \leqslant n$, keeping in mind that $c_i \geqslant i$ by (6). Since $c_i \leqslant n+i-1$ by (3), no arrow will be drawn twice, and there is no arrow from a vertex to itself. The total number of arrows is

$$\text{number of arrows } = \sum_{i=1}^{n}(c_i - i) = \sum_{i=1}^{n} c_i - \binom{n+1}{2}$$

Now we show that we never draw both arrows $i \to j$ and $j \to i$ for $1 \leqslant i < j \leqslant n$. Assume contrariwise. This means, respectively, that

$$i < j \leqslant c_i \qquad \text{and} \qquad j < n+i \leqslant c_j.$$

We have $n+i \leqslant c_j \leqslant c_1 + n$ by (1), so $i \leqslant c_1$. Since $c_1 \leqslant n$ by (3), this implies that $c_i \leqslant c_{c_1} \leqslant n$ using (1) and (3). But then, using (1) again, $j \leqslant c_i \leqslant n$ implies $c_j \leqslant c_{c_i}$, which combined with $n+i \leqslant c_j$ gives us that $n+i \leqslant c_{c_i}$. This contradicts (2).

This means that the number of arrows is at most $\binom{n}{2}$, which implies that

$$\sum_{i=1}^{n} c_i \leqslant \binom{n}{2} + \binom{n+1}{2} = n^2.$$

Recalling that $a_i \leqslant c_i$ for $1 \leqslant i \leqslant n$, the desired inequality follows.

Comment 1. We sketch an alternative proof by induction. Begin by verifying the initial case $n = 1$ and the simple cases when $a_1 = 1$, $a_1 = n$, or $a_n \leqslant n$. Then, as in Solution 1, consider the index t such that $a_1 \leqslant \cdots \leqslant a_t \leqslant n < a_{t+1} \leqslant \cdots \leqslant a_n$. Observe again that $a_1 \leqslant t$. Define the sequence $d_1, \ldots, d_{n-1}$ by

$$d_i = \begin{cases} a_{i+1} - 1 & \text{if } i \leqslant t-1 \\ a_{i+1} - 2 & \text{if } i \geqslant t \end{cases}$$

and extend it periodically modulo $n-1$. One may verify that this sequence also satisfies the hypotheses of the problem. The induction hypothesis then gives $d_1 + \cdots + d_{n-1} \leqslant (n-1)^2$, which implies that

$$\sum_{i=1}^{n} a_i = a_1 + \sum_{i=2}^{t}(d_{i-1}+1) + \sum_{i=t+1}^{n}(d_{i-1}+2) \leqslant t + (t-1) + 2(n-t) + (n-1)^2 = n^2.$$

Comment 2. One unusual feature of this problem is that there are many different sequences for which equality holds. The discovery of such *optimal sequences* is not difficult, and it is useful in guiding the steps of a proof.

In fact, Solution 2 gives a complete description of the optimal sequences. Start with any lattice path P from the lower left to the upper right corner of the $n \times n$ square S using only steps up and right, such that the total number of steps along the left and top edges of S is at least n. Shade the cells of S below P dark, and the cells of S above P light. Now reflect the light shape across the line $x = y$ and shift it up n units, and shade it dark. As Solution 2 shows, the dark region will then correspond to an optimal sequence, and every optimal sequence arises in this way.

A5. Let $\mathbb{Z}_{\geqslant 0}$ be the set of all nonnegative integers. Find all the functions $f\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ satisfying the relation
$$f(f(f(n))) = f(n+1) + 1 \tag{$*$}$$
for all $n \in \mathbb{Z}_{\geqslant 0}$.

(Serbia)

Answer. There are two such functions: $f(n) = n+1$ for all $n \in \mathbb{Z}_{\geqslant 0}$, and
$$f(n) = \begin{cases} n+1, & n \equiv 0 \pmod 4 \text{ or } n \equiv 2 \pmod 4, \\ n+5, & n \equiv 1 \pmod 4, \\ n-3, & n \equiv 3 \pmod 4 \end{cases} \qquad \text{for all } n \in \mathbb{Z}_{\geqslant 0}. \tag{1}$$

Throughout all the solutions, we write $h^k(x)$ to abbreviate the kth iteration of function h, so h^0 is the identity function, and $h^k(x) = \underbrace{h(\dots h}_{k \text{ times}}(x)\dots))$ for $k \geqslant 1$.

Solution 1. To start, we get from $(*)$ that
$$f^4(n) = f(f^3(n)) = f\big(f(n+1)+1\big) \quad \text{and} \quad f^4(n+1) = f^3(f(n+1)) = f\big(f(n+1)+1\big)+1,$$
thus
$$f^4(n) + 1 = f^4(n+1). \tag{2}$$

I. Let us denote by R_i the range of f^i; note that $R_0 = \mathbb{Z}_{\geqslant 0}$ since f^0 is the identity function. Obviously, $R_0 \supseteq R_1 \supseteq \dots$. Next, from (2) we get that if $a \in R_4$ then also $a+1 \in R_4$. This implies that $\mathbb{Z}_{\geqslant 0} \backslash R_4$ — and hence $\mathbb{Z}_{\geqslant 0} \backslash R_1$ — is finite. In particular, R_1 is unbounded.

Assume that $f(m) = f(n)$ for some distinct m and n. Then from $(*)$ we obtain $f(m+1) = f(n+1)$; by an easy induction we then get that $f(m+c) = f(n+c)$ for every $c \geqslant 0$. So the function $f(k)$ is periodic with period $|m-n|$ for $k \geqslant m$, and thus R_1 should be bounded, which is false. So, f is injective.

II. Denote now $S_i = R_{i-1} \backslash R_i$; all these sets are finite for $i \leqslant 4$. On the other hand, by the injectivity we have $n \in S_i \iff f(n) \in S_{i+1}$. By the injectivity again, f implements a bijection between S_i and S_{i+1}, thus $|S_1| = |S_2| = \dots$; denote this common cardinality by k. If $0 \in R_3$ then $0 = f(f(f(n)))$ for some n, thus from $(*)$ we get $f(n+1) = -1$ which is impossible. Therefore $0 \in R_0 \backslash R_3 = S_1 \cup S_2 \cup S_3$, thus $k \geqslant 1$.

Next, let us describe the elements b of $R_0 \backslash R_3 = S_1 \cup S_2 \cup S_3$. We claim that each such element satisfies at least one of three conditions (i) $b = 0$, (ii) $b = f(0)+1$, and (iii) $b-1 \in S_1$. Otherwise $b-1 \in \mathbb{Z}_{\geqslant 0}$, and there exists some $n > 0$ such that $f(n) = b-1$; but then $f^3(n-1) = f(n)+1 = b$, so $b \in R_3$.

This yields
$$3k = |S_1 \cup S_2 \cup S_3| \leqslant 1 + 1 + |S_1| = k+2,$$
or $k \leqslant 1$. Therefore $k = 1$, and the inequality above comes to equality. So we have $S_1 = \{a\}$, $S_2 = \{f(a)\}$, and $S_3 = \{f^2(a)\}$ for some $a \in \mathbb{Z}_{\geqslant 0}$, and each one of the three options (i), (ii), and (iii) should be realized exactly once, which means that
$$\{a, f(a), f^2(a)\} = \{0, a+1, f(0)+1\}. \tag{3}$$

III. From (3), we get $a+1 \in \{f(a), f^2(a)\}$ (the case $a+1=a$ is impossible). If $a+1=f^2(a)$ then we have $f(a+1)=f^3(a)=f(a+1)+1$ which is absurd. Therefore

$$f(a)=a+1. \tag{4}$$

Next, again from (3) we have $0 \in \{a, f^2(a)\}$. Let us consider these two cases separately.

Case 1. Assume that $a=0$, then $f(0)=f(a)=a+1=1$. Also from (3) we get $f(1)=f^2(a)=f(0)+1=2$. Now, let us show that $f(n)=n+1$ by induction on n; the base cases $n \leqslant 1$ are established. Next, if $n \geqslant 2$ then the induction hypothesis implies

$$n+1=f(n-1)+1=f^3(n-2)=f^2(n-1)=f(n),$$

establishing the step. In this case we have obtained the first of two answers; checking that is satisfies $(*)$ is straightforward.

Case 2. Assume now that $f^2(a)=0$; then by (3) we get $a=f(0)+1$. By (4) we get $f(a+1)=f^2(a)=0$, then $f(0)=f^3(a)=f(a+1)+1=1$, hence $a=f(0)+1=2$ and $f(2)=3$ by (4). To summarize,

$$f(0)=1, \quad f(2)=3, \quad f(3)=0.$$

Now let us prove by induction on m that (1) holds for all $n=4k, 4k+2, 4k+3$ with $k \leqslant m$ and for all $n=4k+1$ with $k<m$. The base case $m=0$ is established above. For the step, assume that $m \geqslant 1$. From $(*)$ we get $f^3(4m-3)=f(4m-2)+1=4m$. Next, by (2) we have

$$f(4m)=f^4(4m-3)=f^4(4m-4)+1=f^3(4m-3)+1=4m+1.$$

Then by the induction hypothesis together with $(*)$ we successively obtain

$$\begin{aligned} f(4m-3)&=f^3(4m-1)=f(4m)+1=4m+2,\\ f(4m+2)&=f^3(4m-4)=f(4m-3)+1=4m+3,\\ f(4m+3)&=f^3(4m-3)=f(4m-2)+1=4m, \end{aligned}$$

thus finishing the induction step.

Finally, it is straightforward to check that the constructed function works:

$$\begin{aligned} f^3(4k)&=4k+7=f(4k+1)+1, &\qquad f^3(4k+1)&=4k+4=f(4k+2)+1,\\ f^3(4k+2)&=4k+1=f(4k+3)+1, &\qquad f^3(4k+3)&=4k+6=f(4k+4)+1. \end{aligned}$$

Solution 2. I. For convenience, let us introduce the function $g(n)=f(n)+1$. Substituting $f(n)$ instead of n into $(*)$ we obtain

$$f^4(n)=f\big(f(n)+1\big)+1, \quad \text{or} \quad f^4(n)=g^2(n). \tag{5}$$

Applying f to both parts of $(*)$ and using (5) we get

$$f^4(n)+1=f\big(f(n+1)+1\big)+1=f^4(n+1). \tag{6}$$

Thus, if $g^2(0)=f^4(0)=c$ then an easy induction on n shows that

$$g^2(n)=f^4(n)=n+c, \qquad n \in \mathbb{Z}_{\geqslant 0}. \tag{7}$$

This relation implies that both f and g are injective: if, say, $f(m) = f(n)$ then $m + c = f^4(m) = f^4(n) = n + c$. Next, since $g(n) \geqslant 1$ for every n, we have $c = g^2(0) \geqslant 1$. Thus from (7) again we obtain $f(n) \neq n$ and $g(n) \neq n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

II. Next, application of f and g to (7) yields

$$f(n+c) = f^5(n) = f^4(f(n)) = f(n) + c \quad \text{and} \quad g(n+c) = g^3(n) = g(n) + c. \tag{8}$$

In particular, this means that if $m \equiv n \pmod c$ then $f(m) \equiv f(n) \pmod c$. Conversely, if $f(m) \equiv f(n) \pmod c$ then we get $m + c = f^4(m) \equiv f^4(n) = n + c \pmod c$. Thus,

$$m \equiv n \pmod c \iff f(m) \equiv f(n) \pmod c \iff g(m) \equiv g(n) \pmod c. \tag{9}$$

Now, let us introduce the function $\delta(n) = f(n) - n = g(n) - n - 1$. Set

$$S = \sum_{n=0}^{c-1} \delta(n).$$

Using (8), we get that for every complete residue system $n_1, \dots, n_c$ modulo c we also have

$$S = \sum_{i=1}^{c} \delta(n_i).$$

By (9), we get that $\{f^k(n) \colon n = 0, \dots, c-1\}$ and $\{g^k(n) \colon n = 0, \dots, c-1\}$ are complete residue systems modulo c for all k. Thus we have

$$c^2 = \sum_{n=0}^{c-1} \big(f^4(n) - n\big) = \sum_{k=0}^{3} \sum_{n=0}^{c-1} \big(f^{k+1}(n) - f^k(n)\big) = \sum_{k=0}^{3} \sum_{n=0}^{c-1} \delta(f^k(n)) = 4S$$

and similarly

$$c^2 = \sum_{n=0}^{c-1} \big(g^2(n) - n\big) = \sum_{k=0}^{1} \sum_{n=0}^{c-1} \big(g^{k+1}(n) - g^k(n)\big) = \sum_{k=0}^{1} \sum_{n=0}^{c-1} \big(\delta(g^k(n)) + 1\big) = 2S + 2c.$$

Therefore $c^2 = 4S = 2 \cdot 2S = 2(c^2 - 2c)$, or $c^2 = 4c$. Since $c \neq 0$, we get $c = 4$. Thus, in view of (8) it is sufficient to determine the values of f on the numbers $0, 1, 2, 3$.

III. Let $d = g(0) \geqslant 1$. Then $g(d) = g^2(0) = 0 + c = 4$. Now, if $d \geqslant 4$, then we would have $g(d-4) = g(d) - 4 = 0$ which is impossible. Thus $d \in \{1, 2, 3\}$. If $d = 1$ then we have $f(0) = g(0) - 1 = 0$ which is impossible since $f(n) \neq n$ for all n. If $d = 3$ then $g(3) = g^2(0) = 4$ and hence $f(3) = 3$ which is also impossible. Thus $g(0) = 2$ and hence $g(2) = g^2(0) = 4$.

Next, if $g(1) = 1 + 4k$ for some integer k, then $5 = g^2(1) = g(1 + 4k) = g(1) + 4k = 1 + 8k$ which is impossible. Thus, since $\{g(n) \colon n = 0, 1, 2, 3\}$ is a complete residue system modulo 4, we get $g(1) = 3 + 4k$ and hence $g(3) = g^2(1) - 4k = 5 - 4k$, leading to $k = 0$ or $k = 1$. So, we obtain iether

$$f(0) = 1,\ f(1) = 2,\ f(2) = 3,\ f(3) = 4, \quad \text{or} \quad f(0) = 1,\ f(1) = 6,\ f(2) = 3,\ f(3) = 0,$$

thus arriving to the two functions listed in the answer.

Finally, one can check that these two function work as in Solution 1. One may simplify the checking by noticing that (8) allows us to reduce it to $n = 0, 1, 2, 3$.

A6. Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3 - mx^2 + 1)P(x+1) + (x^3 + mx^2 + 1)P(x-1) = 2(x^3 - mx + 1)P(x) \tag{1}$$

for all real numbers x.

(Serbia)

Answer. $P(x) = tx$ for any real number t.

Solution. Let $P(x) = a_n x^n + \cdots + a_0 x^0$ with $a_n \neq 0$. Comparing the coefficients of x^{n+1} on both sides gives $a_n(n - 2m)(n-1) = 0$, so $n = 1$ or $n = 2m$.

If $n = 1$, one easily verifies that $P(x) = x$ is a solution, while $P(x) = 1$ is not. Since the given condition is linear in P, this means that the linear solutions are precisely $P(x) = tx$ for $t \in \mathbb{R}$.

Now assume that $n = 2m$. The polynomial $xP(x+1) - (x+1)P(x) = (n-1)a_n x^n + \cdots$ has degree n, and therefore it has at least one (possibly complex) root r. If $r \notin \{0, -1\}$, define $k = P(r)/r = P(r+1)/(r+1)$. If $r = 0$, let $k = P(1)$. If $r = -1$, let $k = -P(-1)$. We now consider the polynomial $S(x) = P(x) - kx$. It also satisfies (1) because $P(x)$ and kx satisfy it. Additionally, it has the useful property that r and $r+1$ are roots.

Let $A(x) = x^3 - mx^2 + 1$ and $B(x) = x^3 + mx^2 + 1$. Plugging in $x = s$ into (1) implies that:

> If $s-1$ and s are roots of S and s is not a root of A, then $s+1$ is a root of S.
>
> If s and $s+1$ are roots of S and s is not a root of B, then $s-1$ is a root of S.

Let $a \geqslant 0$ and $b \geqslant 1$ be such that $r-a, r-a+1, \ldots, r, r+1, \ldots, r+b-1, r+b$ are roots of S, while $r-a-1$ and $r+b+1$ are not. The two statements above imply that $r-a$ is a root of B and $r+b$ is a root of A.

Since $r-a$ is a root of $B(x)$ and of $A(x+a+b)$, it is also a root of their greatest common divisor $C(x)$ as integer polynomials. If $C(x)$ was a non-trivial divisor of $B(x)$, then B would have a rational root α. Since the first and last coefficients of B are 1, α can only be 1 or -1; but $B(-1) = m > 0$ and $B(1) = m + 2 > 0$ since $n = 2m$.

Therefore $B(x) = A(x+a+b)$. Writing $c = a + b \geqslant 1$ we compute

$$0 = A(x+c) - B(x) = (3c - 2m)x^2 + c(3c-2m)x + c^2(c-m).$$

Then we must have $3c - 2m = c - m = 0$, which gives $m = 0$, a contradiction. We conclude that $f(x) = tx$ is the only solution.

Solution 2. Multiplying (1) by x, we rewrite it as

$$x(x^3 - mx^2 + 1)P(x+1) + x(x^3 + mx^2 + 1)P(x-1) = [(x+1) + (x-1)](x^3 - mx + 1)P(x).$$

After regrouping, it becomes

$$(x^3 - mx^2 + 1)Q(x) = (x^3 + mx^2 + 1)Q(x-1), \tag{2}$$

where $Q(x) = xP(x+1) - (x+1)P(x)$. If $\deg P \geqslant 2$ then $\deg Q = \deg P$, so $Q(x)$ has a finite multiset of complex roots, which we denote R_Q. Each root is taken with its multiplicity. Then the multiset of complex roots of $Q(x-1)$ is $R_Q + 1 = \{z + 1 : z \in R_Q\}$.

Let $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ be the multisets of roots of the polynomials $A(x) = x^3 - mx^2 + 1$ and $B(x) = x^3 + mx^2 + 1$, respectively. From (2) we get the equality of multisets

$$\{x_1, x_2, x_3\} \cup R_Q = \{y_1, y_2, y_3\} \cup (R_Q + 1).$$

For every $r \in R_Q$, since $r + 1$ is in the set of the right hand side, we must have $r + 1 \in R_Q$ or $r + 1 = x_i$ for some i. Similarly, since r is in the set of the left hand side, either $r - 1 \in R_Q$ or $r = y_i$ for some i. This implies that, possibly after relabelling y_1, y_2, y_3, all the roots of (2) may be partitioned into three chains of the form $\{y_i,\, y_i + 1, \ldots, y_i + k_i = x_i\}$ for $i = 1, 2, 3$ and some integers $k_1, k_2, k_3 \geqslant 0$.

Now we analyze the roots of the polynomial $A_a(x) = x^3 + ax^2 + 1$. Using calculus or elementary methods, we find that the local extrema of $A_a(x)$ occur at $x = 0$ and $x = -2a/3$; their values are $A_a(0) = 1 > 0$ and $A_a(-2a/3) = 1 + 4a^3/27$, which is positive for integers $a \geqslant -1$ and negative for integers $a \leqslant -2$. So when $a \in \mathbb{Z}$, A_a has three real roots if $a \leqslant -2$ and one if $a \geqslant -1$.

Now, since $y_i - x_i \in \mathbb{Z}$ for $i = 1, 2, 3$, the cubics A_m and A_{-m} must have the same number of real roots. The previous analysis then implies that $m = 1$ or $m = -1$. Therefore the real root α of $A_1(x) = x^3 + x^2 + 1$ and the real root β of $A_{-1}(x) = x^3 - x^2 + 1$ must differ by an integer. But this is impossible, because $A_1\left(-\frac{3}{2}\right) = -\frac{1}{8}$ and $A_1(-1) = 1$ so $-1.5 < \alpha < -1$, while $A_{-1}(-1) = -1$ and $A_{-1}\left(-\frac{1}{2}\right) = \frac{5}{8}$, so $-1 < \beta < -0.5$.

It follows that $\deg P \leqslant 1$. Then, as shown in Solution 1, we conclude that the solutions are $P(x) = tx$ for all real numbers t.

Combinatorics

C1. Let n be a positive integer. Find the smallest integer k with the following property: Given any real numbers $a_1, \ldots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leqslant a_i \leqslant 1$ for $i = 1, 2, \ldots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

(Poland)

Answer. $k = 2n - 1$.

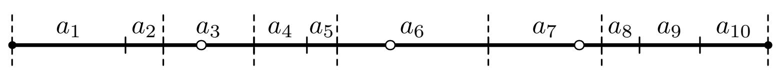
Solution 1. If $d = 2n-1$ and $a_1 = \cdots = a_{2n-1} = n/(2n-1)$, then each group in such a partition can contain at most one number, since $2n/(2n-1) > 1$. Therefore $k \geqslant 2n-1$. It remains to show that a suitable partition into $2n-1$ groups always exists.

We proceed by induction on d. For $d \leqslant 2n-1$ the result is trivial. If $d \geqslant 2n$, then since

$$(a_1 + a_2) + \ldots + (a_{2n-1} + a_{2n}) \leqslant n$$

we may find two numbers a_i, a_{i+1} such that $a_i + a_{i+1} \leqslant 1$. We "merge" these two numbers into one new number $a_i + a_{i+1}$. By the induction hypothesis, a suitable partition exists for the $d-1$ numbers $a_1, \ldots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \ldots, a_d$. This induces a suitable partition for $a_1, \ldots, a_d$.

Solution 2. We will show that it is even possible to split the sequence $a_1, \ldots, a_d$ into $2n-1$ *contiguous* groups so that the sum of the numbers in each groups does not exceed 1. Consider a segment S of length n, and partition it into segments $S_1, \ldots, S_d$ of lengths $a_1, \ldots, a_d$, respectively, as shown below. Consider a second partition of S into n equal parts by $n-1$ "empty dots".



Assume that the $n-1$ empty dots are in segments $S_{i_1}, \ldots, S_{i_{n-1}}$. (If a dot is on the boundary of two segments, we choose the right segment). These $n-1$ segments are distinct because they have length at most 1. Consider the partition:

$$\{a_1, \ldots, a_{i_1-1}\}, \{a_{i_1}\}, \{a_{i_1+1}, \ldots, a_{i_2-1}\}, \{a_{i_2}\}, \ldots \{a_{i_{n-1}}\}, \{a_{i_{n-1}+1}, \ldots, a_d\}.$$

In the example above, this partition is $\{a_1, a_2\}, \{a_3\}, \{a_4, a_5\}, \{a_6\}, \varnothing, \{a_7\}, \{a_8, a_9, a_{10}\}$. We claim that in this partition, the sum of the numbers in this group is at most 1.

For the sets $\{a_{i_t}\}$ this is obvious since $a_{i_t} \leqslant 1$. For the sets $\{a_{i_t} + 1, \ldots, a_{i_{t+1}-1}\}$ this follows from the fact that the corresponding segments lie between two neighboring empty dots, or between an endpoint of S and its nearest empty dot. Therefore the sum of their lengths cannot exceed 1.

Solution 3. First put all numbers greater than $\frac{1}{2}$ in their own groups. Then, form the remaining groups as follows: For each group, add new a_is one at a time until their sum exceeds $\frac{1}{2}$. Since the last summand is at most $\frac{1}{2}$, this group has sum at most 1. Continue this procedure until we have used all the a_is. Notice that the last group may have sum less than $\frac{1}{2}$. If the sum of the numbers in the last two groups is less than or equal to 1, we merge them into one group. In the end we are left with m groups. If $m = 1$ we are done. Otherwise the first $m-2$ have sums greater than $\frac{1}{2}$ and the last two have total sum greater than 1. Therefore $n > (m-2)/2 + 1$ so $m \leqslant 2n-1$ as desired.

Comment 1. The original proposal asked for the minimal value of k when $n = 2$.

Comment 2. More generally, one may ask the same question for real numbers between 0 and 1 whose sum is a real number r. In this case the smallest value of k is $k = \lceil 2r \rceil - 1$, as Solution 3 shows.

Solutions 1 and 2 lead to the slightly weaker bound $k \leqslant 2\lceil r \rceil - 1$. This is actually the optimal bound for partitions into *consecutive* groups, which are the ones contemplated in these two solutions. To see this, assume that r is not an integer and let $c = (r + 1 - \lceil r \rceil)/(1 + \lceil r \rceil)$. One easily checks that $0 < c < \frac{1}{2}$ and $\lceil r \rceil(2c) + (\lceil r \rceil - 1)(1 - c) = r$, so the sequence

$$2c,\ 1-c,\ 2c,\ 1-c,\ \ldots,\ 1-c,\ 2c$$

of $2\lceil r \rceil - 1$ numbers satisfies the given conditions. For this sequence, the only suitable partition into consecutive groups is the trivial partition, which requires $2\lceil r \rceil - 1$ groups.

C2. In the plane, 2013 red points and 2014 blue points are marked so that no three of the marked points are collinear. One needs to draw k lines not passing through the marked points and dividing the plane into several regions. The goal is to do it in such a way that no region contains points of both colors.

Find the minimal value of k such that the goal is attainable for every possible configuration of 4027 points.

(Australia)

Answer. $k = 2013$.

Solution 1. Firstly, let us present an example showing that $k \geqslant 2013$. Mark 2013 red and 2013 blue points on some circle alternately, and mark one more blue point somewhere in the plane. The circle is thus split into 4026 arcs, each arc having endpoints of different colors. Thus, if the goal is reached, then each arc should intersect some of the drawn lines. Since any line contains at most two points of the circle, one needs at least $4026/2 = 2013$ lines.

It remains to prove that one can reach the goal using 2013 lines. First of all, let us mention that for every two points A and B having the same color, one can draw two lines separating these points from all other ones. Namely, it suffices to take two lines parallel to AB and lying on different sides of AB sufficiently close to it: the only two points between these lines will be A and B.

Now, let P be the convex hull of all marked points. Two cases are possible.

Case 1. Assume that P has a red vertex A. Then one may draw a line separating A from all the other points, pair up the other 2012 red points into 1006 pairs, and separate each pair from the other points by two lines. Thus, 2013 lines will be used.

Case 2. Assume now that all the vertices of P are blue. Consider any two consecutive vertices of P, say A and B. One may separate these two points from the others by a line parallel to AB. Then, as in the previous case, one pairs up all the other 2012 blue points into 1006 pairs, and separates each pair from the other points by two lines. Again, 2013 lines will be used.

Comment 1. Instead of considering the convex hull, one may simply take a line containing two marked points A and B such that all the other marked points are on one side of this line. If one of A and B is red, then one may act as in Case 1; otherwise both are blue, and one may act as in Case 2.

Solution 2. Let us present a different proof of the fact that $k = 2013$ suffices. In fact, we will prove a more general statement:

If n points in the plane, no three of which are collinear, are colored in red and blue arbitrarily, then it suffices to draw $\lfloor n/2 \rfloor$ lines to reach the goal.

We proceed by induction on n. If $n \leqslant 2$ then the statement is obvious. Now assume that $n \geqslant 3$, and consider a line ℓ containing two marked points A and B such that all the other marked points are on one side of ℓ; for instance, any line containing a side of the convex hull works.

Remove for a moment the points A and B. By the induction hypothesis, for the remaining configuration it suffices to draw $\lfloor n/2 \rfloor - 1$ lines to reach the goal. Now return the points A and B back. Three cases are possible.

Case 1. If A and B have the same color, then one may draw a line parallel to ℓ and separating A and B from the other points. Obviously, the obtained configuration of $\lfloor n/2 \rfloor$ lines works.

Case 2. If A and B have different colors, but they are separated by some drawn line, then again the same line parallel to ℓ works.

Case 3. Finally, assume that A and B have different colors and lie in one of the regions defined by the drawn lines. By the induction assumption, this region contains no other points of one of the colors — without loss of generality, the only blue point it contains is A. Then it suffices to draw a line separating A from all other points.

Thus the step of the induction is proved.

Comment 2. One may ask a more general question, replacing the numbers 2013 and 2014 by any positive integers m and n, say with $m \leqslant n$. Denote the answer for this problem by $f(m, n)$.

One may show along the lines of Solution 1 that $m \leqslant f(m, n) \leqslant m + 1$; moreover, if m is even then $f(m, n) = m$. On the other hand, for every odd m there exists an N such that $f(m, n) = m$ for all $m \leqslant n \leqslant N$, and $f(m, n) = m + 1$ for all $n > N$.

C3. A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be *entangled*, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

(Japan)

Solution 1. Let us consider a graph with the imons as vertices, and two imons being connected if and only if they are entangled. Recall that a *proper coloring* of a graph G is a coloring of its vertices in several colors so that every two connected vertices have different colors.

Lemma. Assume that a graph G admits a proper coloring in n colors ($n > 1$). Then one may perform a sequence of operations resulting in a graph which admits a proper coloring in $n-1$ colors.

Proof. Let us apply repeatedly operation (i) to any appropriate vertices while it is possible. Since the number of vertices decreases, this process finally results in a graph where all the degrees are even. Surely this graph also admits a proper coloring in n colors $1, \ldots, n$; let us fix this coloring.

Now apply the operation (ii) to this graph. A proper coloring of the resulting graph in n colors still exists: one may preserve the colors of the original vertices and color the vertex I' in a color $k+1 \pmod{n}$ if the vertex I has color k. Then two connected original vertices still have different colors, and so do their two connected copies. On the other hand, the vertices I and I' have different colors since $n > 1$.

All the degrees of the vertices in the resulting graph are odd, so one may apply operation (i) to delete consecutively all the vertices of color n one by one; no two of them are connected by an edge, so their degrees do not change during the process. Thus, we obtain a graph admitting a proper coloring in $n-1$ colors, as required. The lemma is proved. $\square$

Now, assume that a graph G has n vertices; then it admits a proper coloring in n colors. Applying repeatedly the lemma we finally obtain a graph admitting a proper coloring in one color, that is — a graph with no edges, as required.

Solution 2. Again, we will use the graph language.

I. We start with the following observation.

Lemma. Assume that a graph G contains an isolated vertex A, and a graph G° is obtained from G by deleting this vertex. Then, if one can apply a sequence of operations which makes a graph with no edges from G°, then such a sequence also exists for G.

Proof. Consider any operation applicable to G° resulting in a graph G_1°; then there exists a sequence of operations applicable to G and resulting in a graph G_1 differing from G_1° by an addition of an isolated vertex A. Indeed, if this operation is of type (i), then one may simply repeat it in G.

Otherwise, the operation is of type (ii), and one may apply it to G and then delete the vertex A' (it will have degree 1).

Thus one may change the process for G° into a corresponding process for G step by step. $\square$

In view of this lemma, if at some moment a graph contains some isolated vertex, then we may simply delete it; let us call this operation (iii).

II. Let $V = \{A_1^0, \ldots, A_n^0\}$ be the vertices of the initial graph. Let us describe which graphs can appear during our operations. Assume that operation (ii) was applied m times. If these were the only operations applied, then the resulting graph G_n^m has the set of vertices which can be enumerated as

$$V_n^m = \{A_i^j : 1 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant 2^m - 1\},$$

where A_i^0 is the common "ancestor" of all the vertices A_i^j, and the binary expansion of j (adjoined with some zeroes at the left to have m digits) "keeps the history" of this vertex: the dth digit from the right is 0 if at the dth doubling the ancestor of A_i^j was in the original part, and this digit is 1 if it was in the copy.

Next, the two vertices A_i^j and A_k^ℓ in G_n^m are connected with an edge exactly if either (1) $j = \ell$ and there was an edge between A_i^0 and A_k^0 (so these vertices appeared at the same application of operation (ii)); or (2) $i = k$ and the binary expansions of j and ℓ differ in exactly one digit (so their ancestors became connected as a copy and the original vertex at some application of (ii)).

Now, if some operations (i) were applied during the process, then simply some vertices in G_n^m disappeared. So, in any case the resulting graph is some induced subgraph of G_n^m.

III. Finally, we will show that from each (not necessarily induced) subgraph of G_n^m one can obtain a graph with no vertices by applying operations (i), (ii) and (iii). We proceed by induction on n; the base case $n = 0$ is trivial.

For the induction step, let us show how to apply several operations so as to obtain a graph containing no vertices of the form A_n^j for $j \in \mathbb{Z}$. We will do this in three steps.

Step 1. We apply repeatedly operation (i) to any appropriate vertices while it is possible. In the resulting graph, all vertices have even degrees.

Step 2. Apply operation (ii) obtaining a subgraph of G_n^{m+1} with all degrees being odd. In this graph, we delete one by one all the vertices A_n^j where the sum of the binary digits of j is even; it is possible since there are no edges between such vertices, so all their degrees remain odd. After that, we delete all isolated vertices.

Step 3. Finally, consider any remaining vertex A_n^j (then the sum of digits of j is odd). If its degree is odd, then we simply delete it. Otherwise, since A_n^j is not isolated, we consider any vertex adjacent to it. It has the form A_k^j for some $k < n$ (otherwise it would have the form A_n^ℓ, where ℓ has an even digit sum; but any such vertex has already been deleted at Step 2). No neighbor of A_k^j was deleted at Steps 2 and 3, so it has an odd degree. Then we successively delete A_k^j and A_n^j.

Notice that this deletion does not affect the applicability of this step to other vertices, since no two vertices A_i^j and A_k^ℓ for different j, ℓ with odd digit sum are connected with an edge. Thus we will delete all the remaining vertices of the form A_n^j, obtaining a subgraph of G_{n-1}^{m+1}. The application of the induction hypothesis finishes the proof.

Comment. In fact, the graph G_n^m is a Cartesian product of G and the graph of an m-dimensional hypercube.

C4. Let n be a positive integer, and let A be a subset of $\{1, \ldots, n\}$. An *A-partition* of n *into k parts* is a representation of n as a sum $n = a_1 + \cdots + a_k$, where the parts $a_1, \ldots, a_k$ belong to A and are not necessarily distinct. The *number of different parts* in such a partition is the number of (distinct) elements in the set $\{a_1, a_2, \ldots, a_k\}$.

We say that an A-partition of n into k parts is *optimal* if there is no A-partition of n into r parts with $r < k$. Prove that any optimal A-partition of n contains at most $\sqrt[3]{6n}$ different parts.

(Germany)

Solution 1. If there are no A-partitions of n, the result is vacuously true. Otherwise, let $k_{\min}$ be the minimum number of parts in an A-partition of n, and let $n = a_1 + \cdots + a_{k_{\min}}$ be an optimal partition. Denote by s the number of different parts in this partition, so we can write $S = \{a_1, \ldots, a_{k_{\min}}\} = \{b_1, \ldots, b_s\}$ for some pairwise different numbers $b_1 < \cdots < b_s$ in A.

If $s > \sqrt[3]{6n}$, we will prove that there exist subsets X and Y of S such that $|X| < |Y|$ and $\sum_{x \in X} x = \sum_{y \in Y} y$. Then, deleting the elements of Y from our partition and adding the elements of X to it, we obtain an A-partition of n into less than $k_{\min}$ parts, which is the desired contradiction.

For each positive integer $k \leqslant s$, we consider the k-element subset

$$S^k_{1,0} := \{b_1, \ldots, b_k\}$$

as well as the following k-element subsets $S^k_{i,j}$ of S:

$$S^k_{i,j} := \big\{b_1, \ldots, b_{k-i}, b_{k-i+j+1}, b_{s-i+2}, \ldots, b_s\big\}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, s-k.$$

Pictorially, if we represent the elements of S by a sequence of dots in increasing order, and represent a subset of S by shading in the appropriate dots, we have:

$$S^k_{i,j} = \underbrace{\bullet\bullet\bullet\bullet\bullet\bullet\bullet}_{k-i}\underbrace{\circ\circ\circ\circ\circ}_{j}\bullet\underbrace{\circ\circ\circ\circ\circ\circ\circ}_{s-k-j}\underbrace{\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet}_{i-1}$$

Denote by $\Sigma^k_{i,j}$ the sum of elements in $S^k_{i,j}$. Clearly, $\Sigma^k_{1,0}$ is the minimum sum of a k-element subset of S. Next, for all appropriate indices i and j we have

$$\Sigma^k_{i,j} = \Sigma^k_{i,j+1} + b_{k-i+j+1} - b_{k-i+j+2} < \Sigma^k_{i,j+1} \quad \text{and} \quad \Sigma^k_{i,s-k} = \Sigma^k_{i+1,1} + b_{k-i} - b_{k-i+1} < \Sigma^k_{i+1,1}.$$

Therefore

$$1 \leqslant \Sigma^k_{1,0} < \Sigma^k_{1,1} < \Sigma^k_{1,2} < \cdots < \Sigma^k_{1,s-k} < \Sigma^k_{2,1} < \cdots < \Sigma^k_{2,s-k} < \Sigma^k_{3,1} < \cdots < \Sigma^k_{k,s-k} \leqslant n.$$

To see this in the picture, we start with the k leftmost points marked. At each step, we look for the rightmost point which can move to the right, and move it one unit to the right. We continue until the k rightmost points are marked. As we do this, the corresponding sums clearly increase.

For each k we have found $k(s-k)+1$ different integers of the form $\Sigma^k_{i,j}$ between 1 and n. As we vary k, the total number of integers we are considering is

$$\sum_{k=1}^{s} \big(k(s-k)+1\big) = s \cdot \frac{s(s+1)}{2} - \frac{s(s+1)(2s+1)}{6} + s = \frac{s(s^2+5)}{6} > \frac{s^3}{6} > n.$$

Since they are between 1 and n, at least two of these integers are equal. Consequently, there exist $1 \leqslant k < k' \leqslant s$ and $X = S^k_{i,j}$ as well as $Y = S^{k'}_{i',j'}$ such that

$$\sum_{x \in X} x = \sum_{y \in Y} y, \quad \text{but} \quad |X| = k < k' = |Y|,$$

as required. The result follows.

Solution 2. Assume, to the contrary, that the statement is false, and choose the minimum number n for which it fails. So there exists a set $A \subseteq \{1, \ldots, n\}$ together with an optimal A-partition $n = a_1 + \cdots + a_{k_{\min}}$ of n refuting our statement, where, of course, $k_{\min}$ is the minimum number of parts in an A-partition of n. Again, we define $S = \{a_1, \ldots, a_{k_{\min}}\} = \{b_1, \ldots, b_s\}$ with $b_1 < \cdots < b_s$; by our assumption we have $s > \sqrt[3]{6n} > 1$. Without loss of generality we assume that $a_{k_{\min}} = b_s$. Let us distinguish two cases.

Case 1. $b_s \geqslant \frac{s(s-1)}{2} + 1$.

Consider the partition $n - b_s = a_1 + \cdots + a_{k_{\min}-1}$, which is clearly a minimum A-partition of $n - b_s$ with at least $s - 1 \geqslant 1$ different parts. Now, from $n < \frac{s^3}{6}$ we obtain

$$n - b_s \leqslant n - \frac{s(s-1)}{2} - 1 < \frac{s^3}{6} - \frac{s(s-1)}{2} - 1 < \frac{(s-1)^3}{6},$$

so $s - 1 > \sqrt[3]{6(n - b_s)}$, which contradicts the choice of n.

Case 2. $b_s \leqslant \frac{s(s-1)}{2}$.

Set $b_0 = 0$, $\Sigma_{0,0} = 0$, and $\Sigma_{i,j} = b_1 + \cdots + b_{i-1} + b_j$ for $1 \leqslant i \leqslant j < s$. There are $\frac{s(s-1)}{2} + 1 > b_s$ such sums; so at least two of them, say $\Sigma_{i,j}$ and $\Sigma_{i',j'}$, are congruent modulo b_s (where $(i,j) \neq (i',j')$). This means that $\Sigma_{i,j} - \Sigma_{i',j'} = rb_s$ for some integer r. Notice that for $i \leqslant j < k < s$ we have

$$0 < \Sigma_{i,k} - \Sigma_{i,j} = b_k - b_j < b_s,$$

so the indices i and i' are distinct, and we may assume that $i > i'$. Next, we observe that $\Sigma_{i,j} - \Sigma_{i',j'} = (b_{i'} - b_{j'}) + b_j + b_{i'+1} + \cdots + b_{i-1}$ and $b_{i'} \leqslant b_{j'}$ imply

$$-b_s < -b_{j'} < \Sigma_{i,j} - \Sigma_{i',j'} < (i - i')b_s,$$

so $0 \leqslant r \leqslant i - i' - 1$.

Thus, we may remove the i terms of $\Sigma_{i,j}$ in our A-partition, and replace them by the i' terms of $\Sigma_{i',j'}$ and r terms equal to b_s, for a total of $r + i' < i$ terms. The result is an A-partition of n into a smaller number of parts, a contradiction.

Comment. The original proposal also contained a second part, showing that the estimate appearing in the problem has the correct order of magnitude:

For every positive integer n, there exist a set A and an optimal A-partition of n that contains $\lfloor \sqrt[3]{2n} \rfloor$ different parts.

The Problem Selection Committee removed this statement from the problem, since it seems to be less suitable for the competiton; but for completeness we provide an outline of its proof here.

Let $k = \lfloor \sqrt[3]{2n} \rfloor - 1$. The statement is trivial for $n < 4$, so we assume $n \geqslant 4$ and hence $k \geqslant 1$. Let $h = \lfloor \frac{n-1}{k} \rfloor$. Notice that $h \geqslant \frac{n}{k} - 1$.

Now let $A = \{1, \ldots, h\}$, and set $a_1 = h$, $a_2 = h - 1$, ..., $a_k = h - k + 1$, and $a_{k+1} = n - (a_1 + \cdots + a_k)$. It is not difficult to prove that $a_k > a_{k+1} \geqslant 1$, which shows that

$$n = a_1 + \ldots + a_{k+1}$$

is an A-partition of n into $k+1$ different parts. Since $kh < n$, any A-partition of n has at least $k+1$ parts. Therefore our A-partition is optimal, and it has $\lfloor \sqrt[3]{2n} \rfloor$ distinct parts, as desired.

C5. Let r be a positive integer, and let $a_0, a_1, \dots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers m and s there exists a positive integer $n \in [m+1, m+r]$ such that

$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}.$$

Prove that the sequence is periodic, i. e. there exists some $p \geqslant 1$ such that $a_{n+p} = a_n$ for all $n \geqslant 0$.

(India)

Solution. For every indices $m \leqslant n$ we will denote $S(m,n) = a_m + a_{m+1} + \cdots + a_{n-1}$; thus $S(n,n) = 0$. Let us start with the following lemma.

Lemma. Let $b_0, b_1, \dots$ be an infinite sequence. Assume that for every nonnegative integer m there exists a nonnegative integer $n \in [m+1, m+r]$ such that $b_m = b_n$. Then for every indices $k \leqslant \ell$ there exists an index $t \in [\ell, \ell+r-1]$ such that $b_t = b_k$. Moreover, there are at most r distinct numbers among the terms of (b_i).

Proof. To prove the first claim, let us notice that there exists an infinite sequence of indices $k_1 = k, k_2, k_3, \dots$ such that $b_{k_1} = b_{k_2} = \cdots = b_k$ and $k_i < k_{i+1} \leqslant k_i + r$ for all $i \geqslant 1$. This sequence is unbounded from above, thus it hits each segment of the form $[\ell, \ell+r-1]$ with $\ell \geqslant k$, as required.

To prove the second claim, assume, to the contrary, that there exist $r+1$ distinct numbers $b_{i_1}, \dots, b_{i_{r+1}}$. Let us apply the first claim to $k = i_1, \dots, i_{r+1}$ and $\ell = \max\{i_1, \dots, i_{r+1}\}$; we obtain that for every $j \in \{1, \dots, r+1\}$ there exists $t_j \in [s, s+r-1]$ such that $b_{t_j} = b_{i_j}$. Thus the segment $[s, s+r-1]$ should contain $r+1$ distinct integers, which is absurd. $\square$

Setting $s = 0$ in the problem condition, we see that the sequence (a_i) satisfies the condition of the lemma, thus it attains at most r distinct values. Denote by A_i the ordered r-tuple $(a_i, \dots, a_{i+r-1})$; then among A_i's there are at most r^r distinct tuples, so for every $k \geqslant 0$ two of the tuples $A_k, A_{k+1}, \dots, A_{k+r^r}$ are identical. This means that there exists a positive integer $p \leqslant r^r$ such that the equality $A_d = A_{d+p}$ holds infinitely many times. Let D be the set of indices d satisfying this relation.

Now we claim that D coincides with the set of all nonnegative integers. Since D is unbounded, it suffices to show that $d \in D$ whenever $d+1 \in D$. For that, denote $b_k = S(k, p+k)$. The sequence $b_0, b_1, \dots$ satisfies the lemma conditions, so there exists an index $t \in [d+1, d+r]$ such that $S(t, t+p) = S(d, d+p)$. This last relation rewrites as $S(d,t) = S(d+p, t+p)$. Since $A_{d+1} = A_{d+p+1}$, we have $S(d+1, t) = S(d+p+1, t+p)$, therefore we obtain

$$a_d = S(d,t) - S(d+1,t) = S(d+p, t+p) - S(d+p+1, t+p) = a_{d+p}$$

and thus $A_d = A_{d+p}$, as required.

Finally, we get $A_d = A_{d+p}$ for all d, so in particular $a_d = a_{d+p}$ for all d, QED.

Comment 1. In the present proof, the upper bound for the minimal period length is r^r. This bound is not sharp; for instance, one may improve it to $(r-1)^r$ for $r \geqslant 3$..

On the other hand, this minimal length may happen to be greater than r. For instance, it is easy to check that the sequence with period $(3, -3, 3, -3, 3, -1, -1, -1)$ satisfies the problem condition for $r = 7$.

Comment 2. The conclusion remains true even if the problem condition only holds for every $s \geqslant N$ for some positive integer N. To show that, one can act as follows. Firstly, the sums of the form $S(i, i+N)$ attain at most r values, as well as the sums of the form $S(i, i+N+1)$. Thus the terms $a_i = S(i, i+N+1) - S(i+1, i+N+1)$ attain at most r^2 distinct values. Then, among the tuples $A_k, A_{k+N}, \dots, A_{k+r^{2r}N}$ two are identical, so for some $p \leqslant r^{2r}$ the set $D = \{d\colon A_d = A_{d+Np}\}$ is infinite. The further arguments apply almost literally, with p being replaced by Np.

After having proved that such a sequence is also necessarily periodic, one may reduce the bound for the minimal period length to r^r — essentially by verifying that the sequence satisfies the original version of the condition.

C6. In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The *distance* between two cities is defined to be the least possible number of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

(Russia)

Solution. Let us denote by $d(a, b)$ the distance between the cities a and b, and by

$$S_i(a) = \{c\colon d(a, c) = i\}$$

the set of cities at distance exactly i from city a.

Assume that for some city x the set $D = S_4(x)$ has size at least 2551. Let $A = S_1(x)$. A subset A' of A is said to be *substantial*, if every city in D can be reached from x with four flights while passing through some member of A'; in other terms, every city in D has distance 3 from some member of A', or $D \subseteq \bigcup_{a \in A'} S_3(a)$. For instance, A itself is substantial. Now let us fix some substantial subset A^* of A having the minimal cardinality $m = |A^*|$.

Since

$$m(101 - m) \leqslant 50 \cdot 51 = 2550,$$

there has to be a city $a \in A^*$ such that $|S_3(a) \cap D| \geqslant 102 - m$. As $|S_3(a)| \leqslant 100$, we obtain that $S_3(a)$ may contain at most $100 - (102 - m) = m - 2$ cities c with $d(c, x) \leqslant 3$. Let us denote by $T = \{c \in S_3(a)\colon d(x, c) \leqslant 3\}$ the set of all such cities, so $|T| \leqslant m - 2$. Now, to get a contradiction, we will construct $m - 1$ distinct elements in T, corresponding to $m - 1$ elements of the set $A_a = A^* \backslash \{a\}$.

Firstly, due to the minimality of A^*, for each $y \in A_a$ there exists some city $d_y \in D$ which can only be reached with four flights from x by passing through y. So, there is a way to get from x to d_y along x–y–b_y–c_y–d_y for some cities b_y and c_y; notice that $d(x, b_y) = 2$ and $d(x, c_y) = 3$ since this path has the minimal possible length.

Now we claim that all $2(m - 1)$ cities of the form b_y, c_y with $y \in A_a$ are distinct. Indeed, no b_y may coincide with any c_z since their distances from x are different. On the other hand, if one had $b_y = b_z$ for $y \neq z$, then there would exist a path of length 4 from x to d_z via y, namely x–y–b_z–c_z–d_z; this is impossible by the choice of d_z. Similarly, $c_y \neq c_z$ for $y \neq z$.

So, it suffices to prove that for every $y \in A_a$, one of the cities b_y and c_y has distance 3 from a (and thus belongs to T). For that, notice that $d(a, y) \leqslant 2$ due to the path a–x–y, while $d(a, d_y) \geqslant d(x, d_y) - d(x, a) = 3$. Moreover, $d(a, d_y) \neq 3$ by the choice of d_y; thus $d(a, d_y) > 3$. Finally, in the sequence $d(a, y)$, $d(a, b_y)$, $d(a, c_y)$, $d(a, d_y)$ the neighboring terms differ by at most 1, the first term is less than 3, and the last one is greater than 3; thus there exists one which is equal to 3, as required.

Comment 1. The upper bound 2550 is sharp. This can be seen by means of various examples; one of them is the "Roman Empire": it has one capital, called "Rome", that is connected to 51 semicapitals by internally disjoint paths of length 3. Moreover, each of these semicapitals is connected to 50 rural cities by direct flights.

Comment 2. Observe that, under the conditions of the problem, there exists no bound for the size of $S_1(x)$ or $S_2(x)$.

Comment 3. The numbers 100 and 2550 appearing in the statement of the problem may be replaced by n and $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$ for any positive integer n. Still more generally, one can also replace the pair $(3, 4)$ of distances under consideration by any pair (r, s) of positive integers satisfying $r < s \leqslant \frac{3}{2}r$.

To adapt the above proof to this situation, one takes $A = S_{s-r}(x)$ and defines the concept of substantiality as before. Then one takes A^* to be a minimal substantial subset of A, and for each $y \in A^*$ one fixes an element $d_y \in S_s(x)$ which is only reachable from x by a path of length s by passing through y. As before, it suffices to show that for distinct $a, y \in A^*$ and a path $y = y_0 - y_1 - \ldots - y_r = d_y$, at least one of the cities $y_0, \ldots, y_{r-1}$ has distance r from a. This can be done as above; the relation $s \leqslant \frac{3}{2}r$ is used here to show that $d(a, y_0) \leqslant r$.

Moreover, the estimate $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$ is also sharp for every positive integer n and every positive integers r, s with $r < s \leqslant \frac{3}{2}r$. This may be shown by an example similar to that in the previous comment.

C7. Let $n \geqslant 2$ be an integer. Consider all circular arrangements of the numbers $0, 1, \ldots, n$; the $n+1$ rotations of an arrangement are considered to be equal. A circular arrangement is called *beautiful* if, for any four distinct numbers $0 \leqslant a, b, c, d \leqslant n$ with $a+c=b+d$, the chord joining numbers a and c does not intersect the chord joining numbers b and d.

Let M be the number of beautiful arrangements of $0, 1, \ldots, n$. Let N be the number of pairs (x, y) of positive integers such that $x+y \leqslant n$ and $\gcd(x, y)=1$. Prove that

$$M = N + 1.$$

(Russia)

Solution 1. Given a circular arrangement of $[0, n] = \{0, 1, \ldots, n\}$, we define a *$k$-chord* to be a (possibly degenerate) chord whose (possibly equal) endpoints add up to k. We say that three chords of a circle are *aligned* if one of them separates the other two. Say that $m \geqslant 3$ chords are aligned if any three of them are aligned. For instance, in Figure 1, A, B, and C are aligned, while B, C, and D are not.

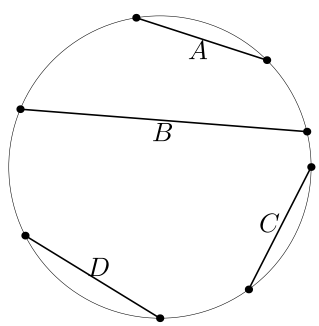


Figure 1

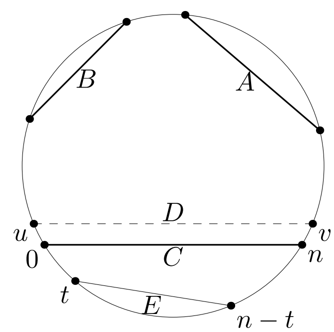


Figure 2

Claim. In a beautiful arrangement, the k–chords are aligned for any integer k.

Proof. We proceed by induction. For $n \leqslant 3$ the statement is trivial. Now let $n \geqslant 4$, and proceed by contradiction. Consider a beautiful arrangement S where the three k–chords A, B, C are not aligned. If n is not among the endpoints of A, B, and C, then by deleting n from S we obtain a beautiful arrangement $S \backslash \{n\}$ of $[0, n-1]$, where A, B, and C are aligned by the induction hypothesis. Similarly, if 0 is not among these endpoints, then deleting 0 and decreasing all the numbers by 1 gives a beautiful arrangement $S \backslash \{0\}$ where A, B, and C are aligned. Therefore both 0 and n are among the endpoints of these segments. If x and y are their respective partners, we have $n \geqslant 0 + x = k = n + y \geqslant n$. Thus 0 and n are the endpoints of one of the chords; say it is C.

Let D be the chord formed by the numbers u and v which are adjacent to 0 and n and on the same side of C as A and B, as shown in Figure 2. Set $t = u + v$. If we had $t = n$, the n–chords A, B, and D would not be aligned in the beautiful arrangement $S \backslash \{0, n\}$, contradicting the induction hypothesis. If $t < n$, then the t-chord from 0 to t cannot intersect D, so the chord C separates t and D. The chord E from t to $n-t$ does not intersect C, so t and $n-t$ are on the same side of C. But then the chords A, B, and E are not aligned in $S \backslash \{0, n\}$, a contradiction. Finally, the case $t > n$ is equivalent to the case $t < n$ via the beauty-preserving relabelling $x \mapsto n - x$ for $0 \leqslant x \leqslant n$, which sends t-chords to $(2n-t)$–chords. This proves the Claim.

Having established the Claim, we prove the desired result by induction. The case $n = 2$ is trivial. Now assume that $n \geqslant 3$. Let S be a beautiful arrangement of $[0, n]$ and delete n to obtain the beautiful arrangement T of $[0, n-1]$. The n–chords of T are aligned, and they contain every point except 0. Say T is of *Type 1* if 0 lies between two of these n–chords, and it is of *Type 2* otherwise; *i.e.*, if 0 is aligned with these n–chords. We will show that each Type 1 arrangement of $[0, n-1]$ arises from a unique arrangement of $[0, n]$, and each Type 2 arrangement of $[0, n-1]$ arises from exactly two beautiful arrangements of $[0, n]$.

If T is of Type 1, let 0 lie between chords A and B. Since the chord from 0 to n must be aligned with A and B in S, n must be on the other arc between A and B. Therefore S can be recovered uniquely from T. In the other direction, if T is of Type 1 and we insert n as above, then we claim the resulting arrangement S is beautiful. For $0 < k < n$, the k–chords of S are also k–chords of T, so they are aligned. Finally, for $n < k < 2n$, notice that the n–chords of S are parallel by construction, so there is an antisymmetry axis ℓ such that x is symmetric to $n - x$ with respect to ℓ for all x. If we had two k–chords which intersect, then their reflections across ℓ would be two $(2n-k)$-chords which intersect, where $0 < 2n - k < n$, a contradiction.

If T is of Type 2, there are two possible positions for n in S, on either side of 0. As above, we check that both positions lead to beautiful arrangements of $[0, n]$.

Hence if we let M_n be the number of beautiful arrangements of $[0, n]$, and let L_n be the number of beautiful arrangements of $[0, n-1]$ of Type 2, we have

$$M_n = (M_{n-1} - L_{n-1}) + 2L_{n-1} = M_{n-1} + L_{n-1}.$$

It then remains to show that L_{n-1} is the number of pairs (x, y) of positive integers with $x + y = n$ and $\gcd(x, y) = 1$. Since $n \geqslant 3$, this number equals $\varphi(n) = \#\{x : 1 \leqslant x \leqslant n, \gcd(x, n) = 1\}$.

To prove this, consider a Type 2 beautiful arrangement of $[0, n-1]$. Label the positions $0, \ldots, n-1 \pmod{n}$ clockwise around the circle, so that number 0 is in position 0. Let $f(i)$ be the number in position i; note that f is a permutation of $[0, n-1]$. Let a be the position such that $f(a) = n - 1$.

Since the n–chords are aligned with 0, and every point is in an n–chord, these chords are all parallel and

$$f(i) + f(-i) = n \qquad \text{for all } i.$$

Similarly, since the $(n-1)$–chords are aligned and every point is in an $(n-1)$–chord, these chords are also parallel and

$$f(i) + f(a-i) = n - 1 \qquad \text{for all } i.$$

Therefore $f(a-i) = f(-i) - 1$ for all i; and since $f(0) = 0$, we get

$$f(-ak) = k \qquad \text{for all } k. \tag{1}$$

Recall that this is an equality modulo n. Since f is a permutation, we must have $(a, n) = 1$. Hence $L_{n-1} \leqslant \varphi(n)$.

To prove equality, it remains to observe that the labeling (1) is beautiful. To see this, consider four numbers w, x, y, z on the circle with $w + y = x + z$. Their positions around the circle satisfy $(-aw) + (-ay) = (-ax) + (-az)$, which means that the chord from w to y and the chord from x to z are parallel. Thus (1) is beautiful, and by construction it has Type 2. The desired result follows.

Solution 2. Notice that there are exactly N irreducible fractions $f_1 < \cdots < f_N$ in $(0,1)$ whose denominator is at most n, since the pair (x,y) with $x+y \leqslant n$ and $(x,y)=1$ corresponds to the fraction $x/(x+y)$. Write $f_i = \frac{a_i}{b_i}$ for $1 \leqslant i \leqslant N$.

We begin by constructing $N+1$ beautiful arrangements. Take any $\alpha \in (0,1)$ which is not one of the above N fractions. Consider a circle of perimeter 1. Successively mark points $0,1,2,\ldots,n$ where 0 is arbitrary, and the clockwise distance from i to $i+1$ is α. The point k will be at clockwise distance $\{k\alpha\}$ from 0, where $\{r\}$ denotes the fractional part of r. Call such a circular arrangement *cyclic* and denote it by $A(\alpha)$. If the clockwise order of the points is the same in $A(\alpha_1)$ and $A(\alpha_2)$, we regard them as the same circular arrangement. Figure 3 shows the cyclic arrangement $A(3/5+\epsilon)$ of $[0,13]$ where $\epsilon > 0$ is very small.

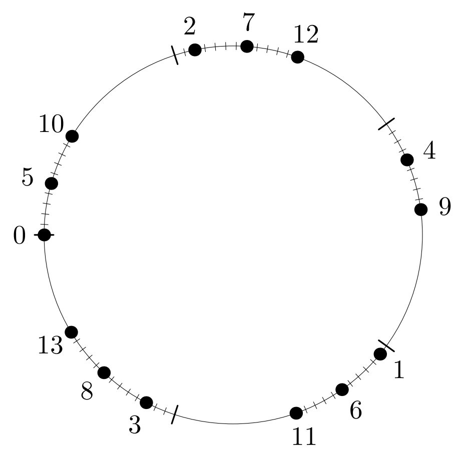


Figure 3

If $0 \leqslant a, b, c, d \leqslant n$ satisfy $a+c=b+d$, then $a\alpha + c\alpha = b\alpha + d\alpha$, so the chord from a to c is parallel to the chord from b to d in $A(\alpha)$. Hence in a cyclic arrangement all k—chords are parallel. In particular every cyclic arrangement is beautiful.

Next we show that there are exactly $N+1$ distinct cyclic arrangements. To see this, let us see how $A(\alpha)$ changes as we increase α from 0 to 1. The order of points p and q changes precisely when we cross a value $\alpha = f$ such that $\{pf\} = \{qf\}$; this can only happen if f is one of the N fractions $f_1,\ldots,f_N$. Therefore there are at most $N+1$ different cyclic arrangements.

To show they are all distinct, recall that $f_i = a_i/b_i$ and let $\epsilon > 0$ be a very small number. In the arrangement $A(f_i+\epsilon)$, point k lands at $\frac{ka_i \pmod{b_i}}{b_i} + k\epsilon$. Therefore the points are grouped into b_i clusters next to the points $0, \frac{1}{b_i}, \ldots, \frac{b_i-1}{b_i}$ of the circle. The cluster following $\frac{k}{b_i}$ contains the numbers congruent to ka_i^{-1} modulo b_i, listed clockwise in increasing order. It follows that the first number after 0 in $A(f_i+\epsilon)$ is b_i, and the first number after 0 which is less than b_i is $a_i^{-1} \pmod{b_i}$, which uniquely determines a_i. In this way we can recover f_i from the cyclic arrangement. Note also that $A(f_i+\epsilon)$ is not the trivial arrangement where we list $0,1,\ldots,n$ in order clockwise. It follows that the $N+1$ cyclic arrangements $A(\epsilon), A(f_1+\epsilon), \ldots, A(f_N+\epsilon)$ are distinct.

Let us record an observation which will be useful later:

$$\text{if } f_i < \alpha < f_{i+1} \text{ then 0 is immediately after } b_{i+1} \text{ and before } b_i \text{ in } A(\alpha). \tag{2}$$

Indeed, we already observed that b_i is the first number after 0 in $A(f_i+\epsilon) = A(\alpha)$. Similarly we see that b_{i+1} is the last number before 0 in $A(f_{i+1}-\epsilon) = A(\alpha)$.

Finally, we show that any beautiful arrangement of $[0, n]$ is cyclic by induction on n. For $n \leqslant 2$ the result is clear. Now assume that all beautiful arrangements of $[0, n-1]$ are cyclic, and consider a beautiful arrangement A of $[0, n]$. The subarrangement $A_{n-1} = A \backslash \{n\}$ of $[0, n-1]$ obtained by deleting n is cyclic; say $A_{n-1} = A_{n-1}(\alpha)$.

Let α be between the consecutive fractions $\frac{p_1}{q_1} < \frac{p_2}{q_2}$ among the irreducible fractions of denominator at most $n-1$. There is at most one fraction $\frac{i}{n}$ in $(\frac{p_1}{q_1}, \frac{p_2}{q_2})$, since $\frac{i}{n} < \frac{i}{n-1} \leqslant \frac{i+1}{n}$ for $0 < i \leqslant n-1$.

Case 1. There is no fraction with denominator n between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$.

In this case the only cyclic arrangement extending $A_{n-1}(\alpha)$ is $A_n(\alpha)$. We know that A and $A_n(\alpha)$ can only differ in the position of n. Assume n is immediately after x and before y in $A_n(\alpha)$. Since the neighbors of 0 are q_1 and q_2 by (2), we have $x, y \geqslant 1$.

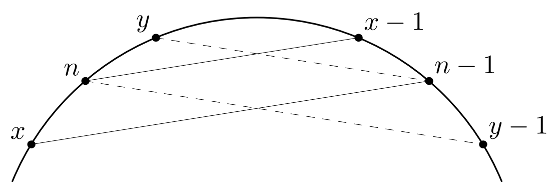


Figure 4

In $A_n(\alpha)$ the chord from $n-1$ to x is parallel and adjacent to the chord from n to $x-1$, so $n-1$ is between $x-1$ and x in clockwise order, as shown in Figure 4. Similarly, $n-1$ is between y and $y-1$. Therefore x, y, $x-1$, $n-1$, and $y-1$ occur in this order in $A_n(\alpha)$ and hence in A (possibly with $y = x-1$ or $x = y-1$).

Now, A may only differ from $A_n(\alpha)$ in the location of n. In A, since the chord from $n-1$ to x and the chord from n to $x-1$ do not intersect, n is between x and $n-1$. Similarly, n is between $n-1$ and y. Then n must be between x and y and $A = A_n(\alpha)$. Therefore A is cyclic as desired.

Case 2. There is exactly one i with $\frac{p_1}{q_1} < \frac{i}{n} < \frac{p_2}{q_2}$.

In this case there exist two cyclic arrangements $A_n(\alpha_1)$ and $A_n(\alpha_2)$ of the numbers $0, \ldots, n$ extending $A_{n-1}(\alpha)$, where $\frac{p_1}{q_1} < \alpha_1 < \frac{i}{n}$ and $\frac{i}{n} < \alpha_2 < \frac{p_2}{q_2}$. In $A_{n-1}(\alpha)$, 0 is the only number between q_2 and q_1 by (2). For the same reason, n is between q_2 and 0 in $A_n(\alpha_1)$, and between 0 and q_1 in $A_n(\alpha_2)$.

Letting $x = q_2$ and $y = q_1$, the argument of Case 1 tells us that n must be between x and y in A. Therefore A must equal $A_n(\alpha_1)$ or $A_n(\alpha_2)$, and therefore it is cyclic.

This concludes the proof that every beautiful arrangement is cyclic. It follows that there are exactly $N + 1$ beautiful arrangements of $[0, n]$ as we wished to show.

C8. Players A and B play a paintful game on the real line. Player A has a pot of paint with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In every round, player A picks some positive integer m and provides $1/2^m$ units of ink from the pot. Player B then picks an integer k and blackens the interval from $k/2^m$ to $(k+1)/2^m$ (some parts of this interval may have been blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0,1]$ is not completely blackened.

Decide whether there exists a strategy for player A to win in a finite number of moves.

(Austria)

Answer. No. Such a strategy for player A does not exist.

Solution. We will present a strategy for player B that guarantees that the interval $[0,1]$ is completely blackened, once the paint pot has become empty.

At the beginning of round r, let x_r denote the largest real number for which the interval between 0 and x_r has already been blackened; for completeness we define $x_1 = 0$. Let m be the integer picked by player A in this round; we define an integer y_r by

$$\frac{y_r}{2^m} \leqslant x_r < \frac{y_r + 1}{2^m}.$$

Note that $I_0^r = [y_r/2^m, (y_r+1)/2^m]$ is the leftmost interval that may be painted in round r and that still contains some uncolored point.

Player B now looks at the *next* interval $I_1^r = [(y_r+1)/2^m, (y_r+2)/2^m]$. If I_1^r still contains an uncolored point, then player B blackens the interval I_1^r; otherwise he blackens the interval I_0^r. We make the convention that, at the beginning of the game, the interval $[1,2]$ is already blackened; thus, if $y_r + 1 = 2^m$, then B blackens I_0^r.

Our aim is to estimate the amount of ink used after each round. Firstly, we will prove by induction that, if before rth round the segment $[0,1]$ is not completely colored, then, before this move,

(i) the amount of ink used for the segment $[0, x_r]$ is at most $3x_r$; and

(ii) for every m, B has blackened at most one interval of length $1/2^m$ to the right of x_r.

Obviously, these conditions are satisfied for $r = 0$. Now assume that they were satisfied before the rth move, and consider the situation after this move; let m be the number A has picked at this move.

If B has blackened the interval I_1^r at this move, then $x_{r+1} = x_r$, and (i) holds by the induction hypothesis. Next, had B blackened before the rth move any interval of length $1/2^m$ to the right of x_r, this interval would necessarily coincide with I_1^r. By our strategy, this cannot happen. So, condition (ii) also remains valid.

Assume now that B has blackened the interval I_0^r at the rth move, but the interval $[0,1]$ still contains uncolored parts (which means that I_1^r is contained in $[0,1]$). Then condition (ii) clearly remains true, and we need to check (i) only. In our case, the intervals I_0^r and I_1^r are completely colored after the rth move, so x_{r+1} either reaches the right endpoint of I_1 or moves even further to the right. So, $x_{r+1} = x_r + \alpha$ for some $\alpha > 1/2^m$.

Next, any interval blackened by B before the rth move which intersects (x_r, x_{r+1}) should be contained in $[x_r, x_{r+1}]$; by (ii), all such intervals have different lengths not exceeding $1/2^m$, so the total amount of ink used for them is less than $2/2^m$. Thus, the amount of ink used for the segment $[0, x_{r+1}]$ does not exceed the sum of $2/2^m$, $3x_r$ (used for $[0, x_r]$), and $1/2^m$ used for the segment I_0^r. In total it gives at most $3(x_r + 1/2^m) < 3(x_r + \alpha) = 3x_{r+1}$. Thus condition (i) is also verified in this case. The claim is proved.

Finally, we can perform the desired estimation. Consider any situation in the game, say after the $(r-1)$st move; assume that the segment $[0, 1]$ is not completely black. By (ii), in the segment $[x_r, 1]$ player B has colored several segments of different lengths; all these lengths are negative powers of 2 not exceeding $1 - x_r$; thus the total amount of ink used for this interval is at most $2(1 - x_r)$. Using (i), we obtain that the total amount of ink used is at most $3x_r + 2(1 - x_r) < 3$. Thus the pot is not empty, and therefore A never wins.

Comment 1. Notice that this strategy works even if the pot contains initially only 3 units of ink.

Comment 2. There exist other strategies for B allowing him to prevent emptying the pot before the whole interval is colored. On the other hand, let us mention some idea which *does not* work.

Player B could try a strategy in which the set of blackened points in each round is an interval of the type $[0, x]$. Such a strategy cannot work (even if there is more ink available). Indeed, under the assumption that B uses such a strategy, let us prove by induction on s the following statement:

For any positive integer s, player A has a strategy picking only positive integers $m \leqslant s$ in which, if player B ever paints a point $x \geqslant 1 - 1/2^s$ then after some move, exactly the interval $[0, 1 - 1/2^s]$ is blackened, and the amount of ink used up to this moment is at least $s/2$.

For the base case $s = 1$, player A just picks $m = 1$ in the first round. If for some positive integer k player A has such a strategy, for $s + 1$ he can first rescale his strategy to the interval $[0, 1/2]$ (sending in each round half of the amount of ink he would give by the original strategy). Thus, after some round, the interval $[0, 1/2 - 1/2^{s+1}]$ becomes blackened, and the amount of ink used is at least $s/4$. Now player A picks $m = 1/2$, and player B spends $1/2$ unit of ink to blacken the interval $[0, 1/2]$. After that, player A again rescales his strategy to the interval $[1/2, 1]$, and player B spends at least $s/4$ units of ink to blacken the interval $[1/2, 1 - 1/2^{s+1}]$, so he spends in total at least $s/4 + 1/2 + s/4 = (s + 1)/2$ units of ink.

Comment 3. In order to avoid finiteness issues, the statement could be replaced by the following one:

> *Players A and B play a paintful game on the real numbers. Player A has a paint pot with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In the beginning of the game, player A chooses (and announces) a positive integer N. In every round, player A picks some positive integer $m \leqslant N$ and provides $1/2^m$ units of ink from the pot. The player B picks an integer k and blackens the interval from $k/2^m$ to $(k + 1)/2^m$ (some parts of this interval may happen to be blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0, 1]$ is not completely blackened.*
>
> *Decide whether there exists a strategy for player A to win.*

However, the Problem Selection Committee believes that this version may turn out to be harder than the original one.

Geometry

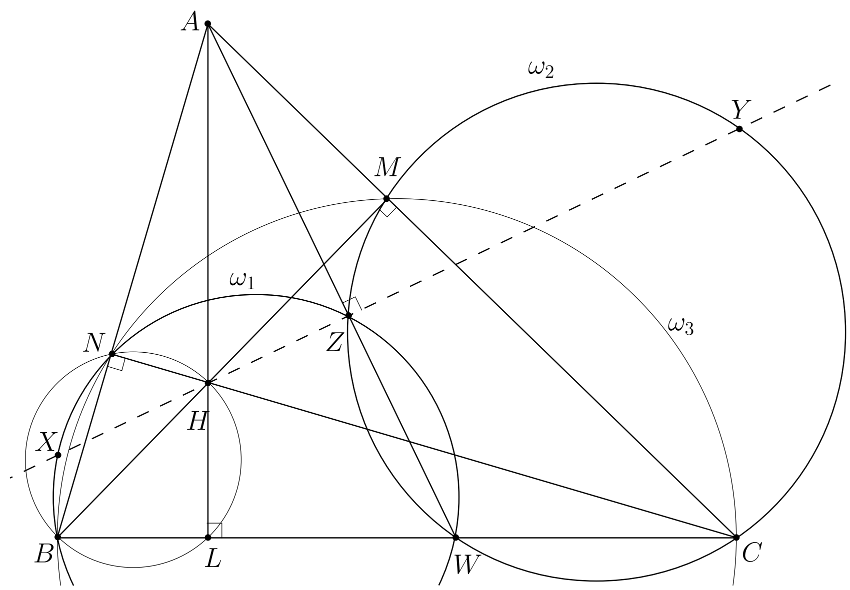
G1. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on side BC. Denote by M and N the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 which is diametrically opposite to W. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 which is diametrically opposite to W. Prove that X, Y and H are collinear.

(Thaliand)

Solution. Let L be the foot of the altitude from A, and let Z be the second intersection point of circles ω_1 and ω_2, other than W. We show that X, Y, Z and H lie on the same line.

Due to $\angle BNC = \angle BMC = 90^\circ$, the points B, C, N and M are concyclic; denote their circle by ω_3. Observe that the line WZ is the radical axis of ω_1 and ω_2; similarly, BN is the radical axis of ω_1 and ω_3, and CM is the radical axis of ω_2 and ω_3. Hence $A = BN \cap CM$ is the radical center of the three circles, and therefore WZ passes through A.

Since WX and WY are diameters in ω_1 and ω_2, respectively, we have $\angle WZX = \angle WZY = 90^\circ$, so the points X and Y lie on the line through Z, perpendicular to WZ.



The quadrilateral $BLHN$ is cyclic, because it has two opposite right angles. From the power of A with respect to the circles ω_1 and $BLHN$ we find $AL \cdot AH = AB \cdot AN = AW \cdot AZ$. If H lies on the line AW then this implies $H = Z$ immediately. Otherwise, by $\frac{AZ}{AH} = \frac{AL}{AW}$ the triangles AHZ and AWL are similar. Then $\angle HZA = \angle WLA = 90^\circ$, so the point H also lies on the line XYZ.

Comment. The original proposal also included a second statement:

> *Let P be the point on ω_1 such that WP is parallel to CN, and let Q be the point on ω_2 such that WQ is parallel to BM. Prove that P, Q and H are collinear if and only if $BW = CW$ or $AW \perp BC$.*

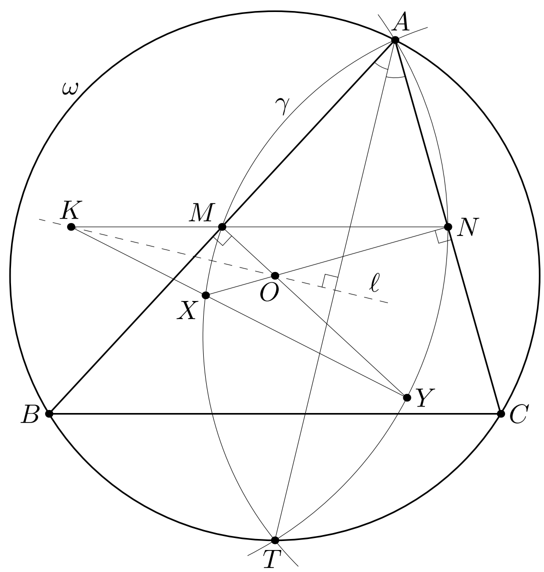
The Problem Selection Committee considered the first part more suitable for the competition.

G2. Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

(Iran)

Solution 1. Let O be the center of ω, thus $O = MY \cap NX$. Let ℓ be the perpendicular bisector of AT (it also passes through O). Denote by r the operation of reflection about ℓ. Since AT is the angle bisector of $\angle BAC$, the line $r(AB)$ is parallel to AC. Since $OM \perp AB$ and $ON \perp AC$, this means that the line $r(OM)$ is parallel to the line ON and passes through O, so $r(OM) = ON$. Finally, the circumcircle γ of the triangle AMT is symmetric about ℓ, so $r(\gamma) = \gamma$. Thus the point M maps to the common point of ON with the arc AMT of γ — that is, $r(M) = X$.

Similarly, $r(N) = Y$. Thus, we get $r(MN) = XY$, and the common point K of MN nd XY lies on ℓ. This means exactly that $KA = KT$.



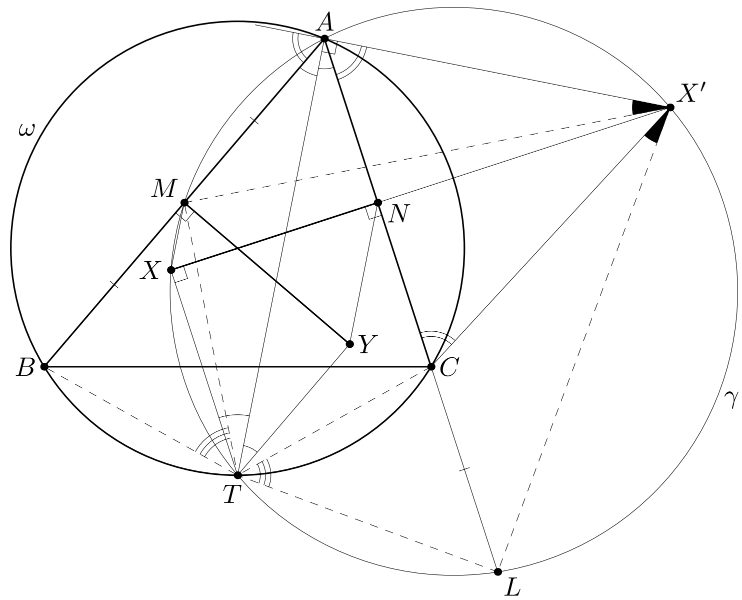
Solution 2. Let L be the second common point of the line AC with the circumcircle γ of the triangle AMT. From the cyclic quadrilaterals $ABTC$ and $AMTL$ we get $\angle BTC = 180^\circ - \angle BAC = \angle MTL$, which implies $\angle BTM = \angle CTL$. Since AT is an angle bisector in these quadrilaterals, we have $BT = TC$ and $MT = TL$. Thus the triangles BTM and CTL are congruent, so $CL = BM = AM$.

Let X' be the common point of the line NX with the external bisector of $\angle BAC$; notice that it lies outside the triangle ABC. Then we have $\angle TAX' = 90^\circ$ and $X'A = X'C$, so we get $\angle X'AM = 90^\circ + \angle BAC/2 = 180^\circ - \angle X'AC = 180^\circ - \angle X'CA = \angle X'CL$. Thus the triangles $X'AM$ and $X'CL$ are congruent, and therefore

$$\angle MX'L = \angle AX'C + (\angle CX'L - \angle AX'M) = \angle AX'C = 180^\circ - 2\angle X'AC = \angle BAC = \angle MAL.$$

This means that X' lies on γ.

Thus we have $\angle TXN = \angle TXX' = \angle TAX' = 90^\circ$, so $TX \parallel AC$. Then $\angle XTA = \angle TAC = \angle TAM$, so the cyclic quadrilateral $MATX$ is an isosceles trapezoid. Similarly, $NATY$ is an isosceles trapezoid, so again the lines MN and XY are the reflections of each other about the perpendicular bisector of AT. Thus K belongs to this perpendicular bisector.



Comment. There are several different ways of showing that the points X and M are symmetrical with respect to ℓ. For instance, one can show that the quadrilaterals $AMON$ and $TXOY$ are congruent. We chose Solution 1 as a simple way of doing it. On the other hand, Solution 2 shows some other interesting properties of the configuration.

Let us define Y', analogously to X', as the common point of MY and the external bisector of $\angle BAC$. One may easily see that in general the lines MN and $X'Y'$ (which is the external bisector of $\angle BAC$) do not intersect on the perpendicular bisector of AT. Thus, any solution should involve some argument using the choice of the intersection points X and Y.

G3. In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \leqslant \max\{\angle BAC, \angle ABC\}$.

(Serbia)

Solution 1. Let K, L, M, and N be the vertices of the rhombus lying on the sides AE, ED, DB, and BA, respectively. Denote by $d(X, YZ)$ the distance from a point X to a line YZ. Since D and E are the feet of the bisectors, we have $d(D, AB) = d(D, AC)$, $d(E, AB) = d(E, BC)$, and $d(D, BC) = d(E, AC) = 0$, which implies

$$d(D, AC) + d(D, BC) = d(D, AB) \quad \text{and} \quad d(E, AC) + d(E, BC) = d(E, AB).$$

Since L lies on the segment DE and the relation $d(X, AC) + d(X, BC) = d(X, AB)$ is linear in X inside the triangle, these two relations imply

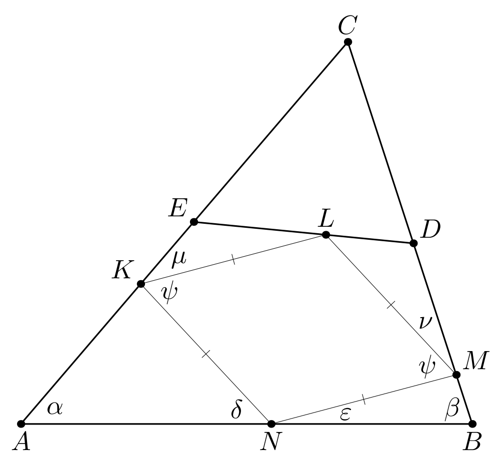
$$d(L, AC) + d(L, BC) = d(L, AB). \tag{1}$$

Denote the angles as in the figure below, and denote $a = KL$. Then we have $d(L, AC) = a \sin \mu$ and $d(L, BC) = a \sin \nu$. Since $KLMN$ is a parallelogram lying on one side of AB, we get

$$d(L, AB) = d(L, AB) + d(N, AB) = d(K, AB) + d(M, AB) = a(\sin \delta + \sin \varepsilon).$$

Thus the condition (1) reads

$$\sin \mu + \sin \nu = \sin \delta + \sin \varepsilon. \tag{2}$$



If one of the angles α and β is non-acute, then the desired inequality is trivial. So we assume that $\alpha, \beta < \pi/2$. It suffices to show then that $\psi = \angle NKL \leqslant \max\{\alpha, \beta\}$.

Assume, to the contrary, that $\psi > \max\{\alpha, \beta\}$. Since $\mu + \psi = \angle CKN = \alpha + \delta$, by our assumption we obtain $\mu = (\alpha - \psi) + \delta < \delta$. Similarly, $\nu < \varepsilon$. Next, since $KN \parallel ML$, we have $\beta = \delta + \nu$, so $\delta < \beta < \pi/2$. Similarly, $\varepsilon < \pi/2$. Finally, by $\mu < \delta < \pi/2$ and $\nu < \varepsilon < \pi/2$, we obtain

$$\sin \mu < \sin \delta \quad \text{and} \quad \sin \nu < \sin \varepsilon.$$

This contradicts (2).

Comment. One can see that the equality is achieved if $\alpha = \beta$ for every rhombus inscribed into the quadrilateral $AEDB$.

G4. Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

(Georgia)

Solution 1. Denote by ω the circumcircle of the triangle ABC, and let $\angle ACB = \gamma$. Note that the condition $\gamma < \angle CBA$ implies $\gamma < 90^\circ$. Since $\angle PBA = \gamma$, the line PB is tangent to ω, so $PA \cdot PC = PB^2 = PD^2$. By $\frac{PA}{PD} = \frac{PD}{PC}$ the triangles PAD and PDC are similar, and $\angle ADP = \angle DCP$.

Next, since $\angle ABQ = \angle ACB$, the triangles ABC and AQB are also similar. Then $\angle AQB = \angle ABC = \angle ARC$, which means that the points D, R, C, and Q are concyclic. Therefore $\angle DRQ = \angle DCQ = \angle ADP$.

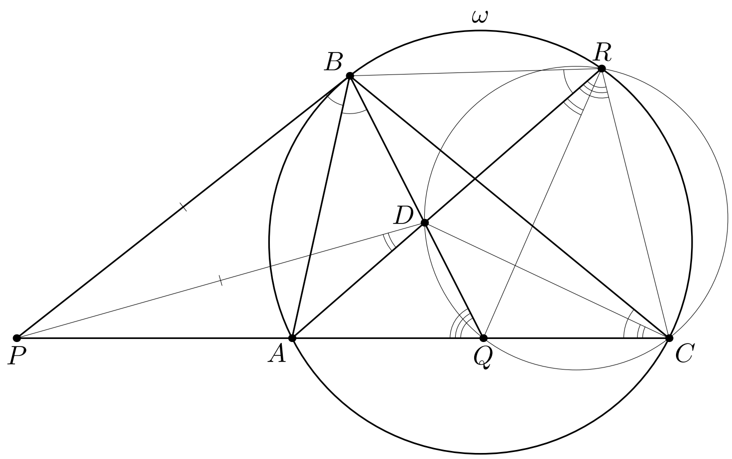


Figure 1

Now from $\angle ARB = \angle ACB = \gamma$ and $\angle PDB = \angle PBD = 2\gamma$ we get

$$\angle QBR = \angle ADB - \angle ARB = \angle ADP + \angle PDB - \angle ARB = \angle DRQ + \gamma = \angle QRB,$$

so the triangle QRB is isosceles, which yields $QB = QR$.

Solution 2. Again, denote by ω the circumcircle of the triangle ABC. Denote $\angle ACB = \gamma$. Since $\angle PBA = \gamma$, the line PB is tangent to ω.

Let E be the second intersection point of BQ with ω. If V' is any point on the ray CE beyond E, then $\angle BEV' = 180^\circ - \angle BEC = 180^\circ - \angle BAC = \angle PAB$; together with $\angle ABQ = \angle PBA$ this shows firstly, that the rays BA and CE intersect at some point V, and secondly that the triangle VEB is similar to the triangle PAB. Thus we have $\angle BVE = \angle BPA$. Next, $\angle AEV = \angle BEV - \gamma = \angle PAB - \angle ABQ = \angle AQB$; so the triangles PBQ and VAE are also similar.

Let PH be an altitude in the isosceles triangle PBD; then $BH = HD$. Let G be the intersection point of PH and AB. By the symmetry with respect to PH, we have $\angle BDG = \angle DBG = \gamma = \angle BEA$; thus $DG \parallel AE$ and hence $\frac{BG}{GA} = \frac{BD}{DE}$. Thus the points G and D correspond to each other in the similar triangles PAB and VEB, so $\angle DVB = \angle GPB = 90^\circ - \angle PBQ = 90^\circ - \angle VAE$. Thus $VD \perp AE$.

Let T be the common point of VD and AE, and let DS be an altitude in the triangle BDR. The points S and T are the feet of corresponding altitudes in the similar triangles ADE and BDR, so $\frac{BS}{SR} = \frac{AT}{TE}$. On the other hand, the points T and H are feet of corresponding altitudes in the similar triangles VAE and PBQ, so $\frac{AT}{TE} = \frac{BH}{HQ}$. Thus $\frac{BS}{SR} = \frac{AT}{TE} = \frac{BH}{HQ}$, and the triangles BHS and BQR are similar.

Finally, SH is a median in the right-angled triangle SBD; so $BH = HS$, and hence $BQ = QR$.

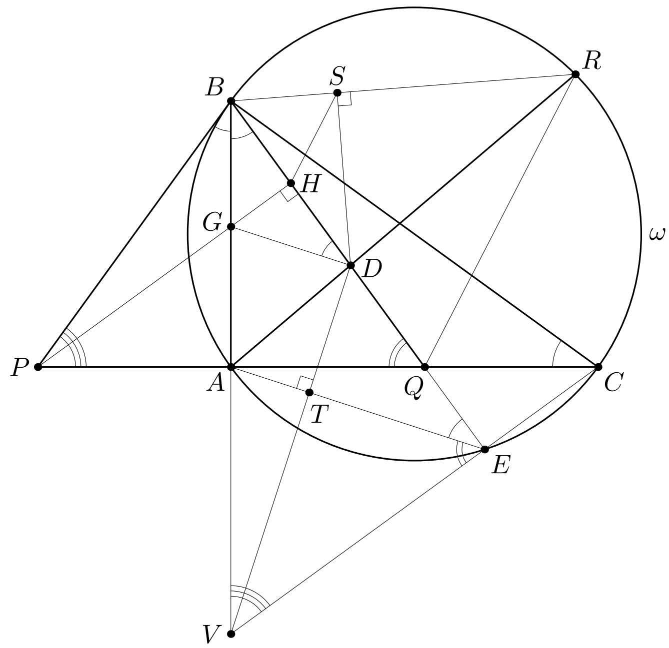


Figure 2

Solution 3. Denote by ω and O the circumcircle of the triangle ABC and its center, respectively. From the condition $\angle PBA = \angle BCA$ we know that BP is tangent to ω.

Let E be the second point of intersection of ω and BD. Due to the isosceles triangle BDP, the tangent of ω at E is parallel to DP and consequently it intersects BP at some point L. Of course, $PD \parallel LE$. Let M be the midpoint of BE, and let H be the midpoint of BR. Notice that $\angle AEB = \angle ACB = \angle ABQ = \angle ABE$, so A lies on the perpendicular bisector of BE; thus the points L, A, M, and O are collinear. Let ω_1 be the circle with diameter BO. Let $Q' = HO \cap BE$; since HO is the perpendicular bisector of BR, the statement of the problem is equivalent to $Q' = Q$.

Consider the following sequence of projections (see Fig. 3).

1. Project the line BE to the line LB through the center A. (This maps Q to P.)
2. Project the line LB to BE in parallel direction with LE. ($P \mapsto D$.)
3. Project the line BE to the circle ω through its point A. ($D \mapsto R$.)
4. Scale ω by the ratio $\frac{1}{2}$ from the point B to the circle ω_1. ($R \mapsto H$.)
5. Project ω_1 to the line BE through its point O. ($H \mapsto Q'$.)

We prove that the composition of these transforms, which maps the line BE to itself, is the identity. To achieve this, it suffices to show three fixed points. An obvious fixed point is B which is fixed by all the transformations above. Another fixed point is M, its path being $M \mapsto L \mapsto E \mapsto E \mapsto M \mapsto M$.

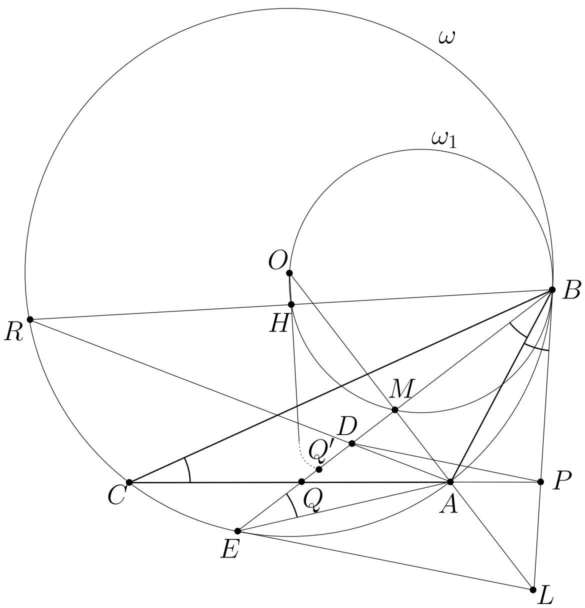


Figure 3

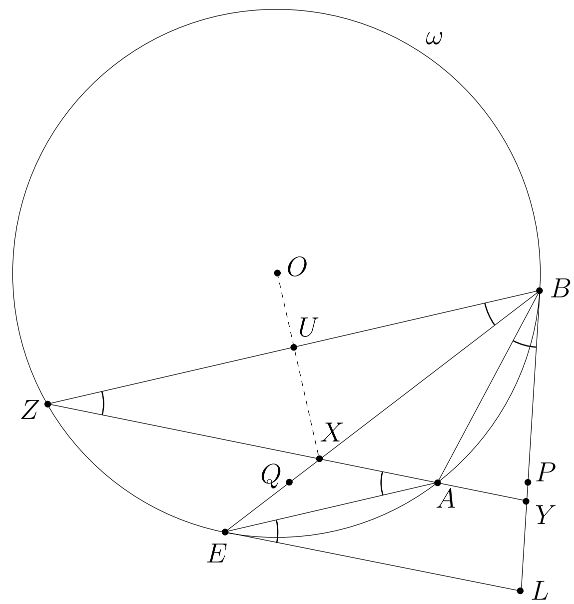


Figure 4

In order to show a third fixed point, draw a line parallel with LE through A; let that line intersect BE, LB and ω at X, Y and $Z \neq A$, respectively (see Fig. 4). We show that X is a fixed point. The images of X at the first three transformations are $X \mapsto Y \mapsto X \mapsto Z$. From $\angle XBZ = \angle EAZ = \angle AEL = \angle LBA = \angle BZX$ we can see that the triangle XBZ is isosceles. Let U be the midpoint of BZ; then the last two transformations do $Z \mapsto U \mapsto X$, and the point X is fixed.

Comment. Verifying that the point E is fixed seems more natural at first, but it appears to be less straightforward. Here we outline a possible proof.

Let the images of E at the first three transforms above be F, G and I. After comparing the angles depicted in Fig. 5 (noticing that the quadrilateral $AFBG$ is cyclic) we can observe that the tangent LE of ω is parallel to BI. Then, similarly to the above reasons, the point E is also fixed.

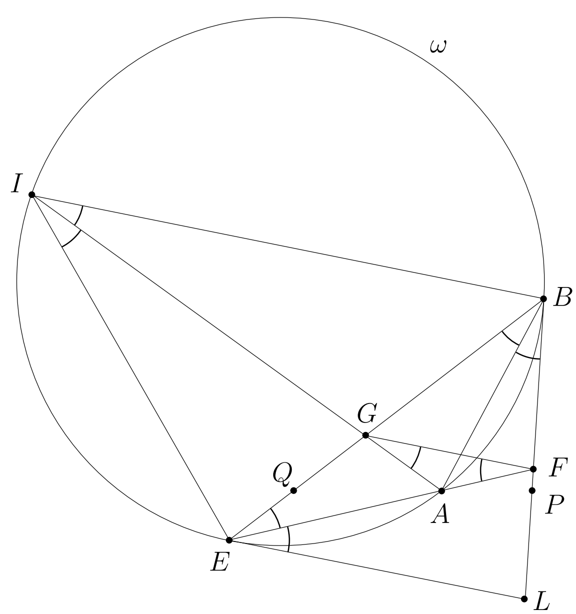


Figure 5

G5. Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

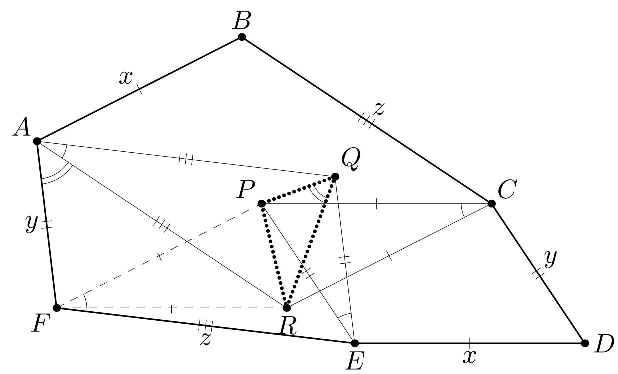
(Ukraine)

In all three solutions, we denote $\theta = \angle A - \angle D = \angle C - \angle F = \angle E - \angle B$ and assume without loss of generality that $\theta \geqslant 0$.

Solution 1. Let $x = AB = DE$, $y = CD = FA$, $z = EF = BC$. Consider the points P, Q, and R such that the quadrilaterals $CDEP$, $EFAQ$, and $ABCR$ are parallelograms. We compute

$$\begin{aligned}\angle PEQ &= \angle FEQ + \angle DEP - \angle E = (180^\circ - \angle F) + (180^\circ - \angle D) - \angle E \\ &= 360^\circ - \angle D - \angle E - \angle F = \tfrac{1}{2}\big(\angle A + \angle B + \angle C - \angle D - \angle E - \angle F\big) = \theta/2.\end{aligned}$$

Similarly, $\angle QAR = \angle RCP = \theta/2$.



If $\theta = 0$, since $\triangle RCP$ is isosceles, $R = P$. Therefore $AB \parallel RC = PC \parallel ED$, so $ABDE$ is a parallelogram. Similarly, $BCEF$ and $CDFA$ are parallelograms. It follows that AD, BE and CF meet at their common midpoint.

Now assume $\theta > 0$. Since $\triangle PEQ$, $\triangle QAR$, and $\triangle RCP$ are isosceles and have the same angle at the apex, we have $\triangle PEQ \sim \triangle QAR \sim \triangle RCP$ with ratios of similarity $y : z : x$. Thus

$$\triangle PQR \text{ is similar to the triangle with sidelengths } y,\ z, \text{ and } x. \tag{1}$$

Next, notice that

$$\frac{RQ}{QP} = \frac{z}{y} = \frac{RA}{AF}$$

and, using directed angles between rays,

$$\begin{aligned}\measuredangle(RQ, QP) &= \measuredangle(RQ, QE) + \measuredangle(QE, QP) \\ &= \measuredangle(RQ, QE) + \measuredangle(RA, RQ) = \measuredangle(RA, QE) = \measuredangle(RA, AF).\end{aligned}$$

Thus $\triangle PQR \sim \triangle FAR$. Since $FA = y$ and $AR = z$, (1) then implies that $FR = x$. Similarly $FP = x$. Therefore $CRFP$ is a rhombus.

We conclude that CF is the perpendicular bisector of PR. Similarly, BE is the perpendicular bisector of PQ and AD is the perpendicular bisector of QR. It follows that AD, BE, and CF are concurrent at the circumcenter of PQR.

Solution 2. Let $X = CD \cap EF$, $Y = EF \cap AB$, $Z = AB \cap CD$, $X' = FA \cap BC$, $Y' = BC \cap DE$, and $Z' = DE \cap FA$. From $\angle A + \angle B + \angle C = 360^\circ + \theta/2$ we get $\angle A + \angle B > 180^\circ$ and $\angle B + \angle C > 180^\circ$, so Z and X' are respectively on the opposite sides of BC and AB from the hexagon. Similar conclusions hold for X, Y, Y', and Z'. Then

$$\angle YZX = \angle B + \angle C - 180^\circ = \angle E + \angle F - 180^\circ = \angle Y'Z'X',$$

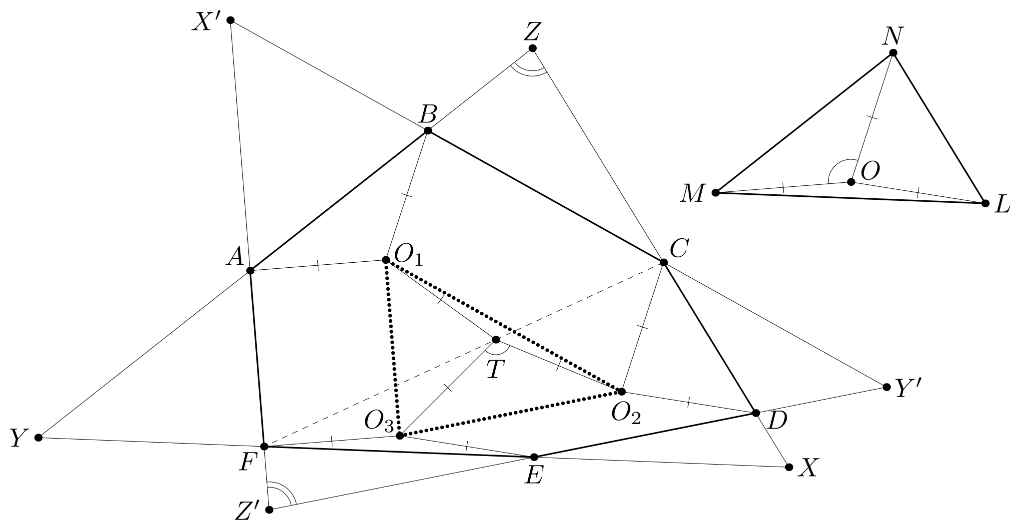
and similarly $\angle ZXY = \angle Z'X'Y'$ and $\angle XYZ = \angle X'Y'Z'$, so $\triangle XYZ \sim \triangle X'Y'Z'$. Thus there is a rotation R which sends $\triangle XYZ$ to a triangle with sides parallel to $\triangle X'Y'Z'$. Since $AB = DE$ we have $R(\overrightarrow{AB}) = \overrightarrow{DE}$. Similarly, $R(\overrightarrow{CD}) = \overrightarrow{FA}$ and $R(\overrightarrow{EF}) = \overrightarrow{BC}$. Therefore

$$\overrightarrow{0} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \left(\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{EF}\right) + R\left(\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{EF}\right).$$

If R is a rotation by 180°, then any two opposite sides of our hexagon are equal and parallel, so the three diagonals meet at their common midpoint. Otherwise, we must have

$$\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{EF} = \overrightarrow{0},$$

or else we would have two vectors with different directions whose sum is $\overrightarrow{0}$.



This allows us to consider a triangle LMN with $\overrightarrow{LM} = \overrightarrow{EF}$, $\overrightarrow{MN} = \overrightarrow{AB}$, and $\overrightarrow{NL} = \overrightarrow{CD}$. Let O be the circumcenter of $\triangle LMN$ and consider the points O_1, O_2, O_3 such that $\triangle AO_1B$, $\triangle CO_2D$, and $\triangle EO_3F$ are translations of $\triangle MON$, $\triangle NOL$, and $\triangle LOM$, respectively. Since FO_3 and AO_1 are translations of MO, quadrilateral AFO_3O_1 is a parallelogram and $O_3O_1 = FA = CD = NL$. Similarly, $O_1O_2 = LM$ and $O_2O_3 = MN$. Therefore $\triangle O_1O_2O_3 \cong \triangle LMN$. Moreover, by means of the rotation R one may check that these triangles have the same orientation.

Let T be the circumcenter of $\triangle O_1O_2O_3$. We claim that AD, BE, and CF meet at T. Let us show that C, T, and F are collinear. Notice that $CO_2 = O_2T = TO_3 = O_3F$ since they are all equal to the circumradius of $\triangle LMN$. Therefore $\triangle TO_3F$ and $\triangle CO_2T$ are isosceles. Using directed angles between rays again, we get

$$\measuredangle(TF, TO_3) = \measuredangle(FO_3, FT) \qquad \text{and} \qquad \measuredangle(TO_2, TC) = \measuredangle(CT, CO_2). \tag{2}$$

Also, T and O are the circumcenters of the congruent triangles $\triangle O_1O_2O_3$ and $\triangle LMN$ so we have $\measuredangle(TO_3, TO_2) = \measuredangle(ON, OM)$. Since CO_2 and FO_3 are translations of NO and MO respectively, this implies

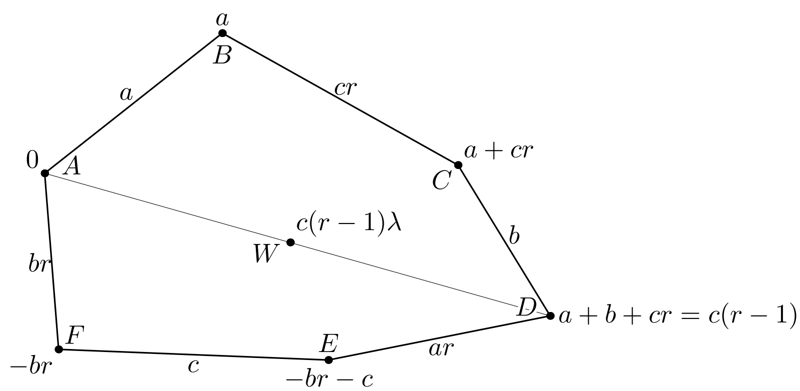
$$\measuredangle(TO_3, TO_2) = \measuredangle(CO_2, FO_3). \tag{3}$$

Adding the three equations in (2) and (3) gives

$$\measuredangle(TF, TC) = \measuredangle(CT, FT) = -\measuredangle(TF, TC)$$

which implies that T is on CF. Analogous arguments show that it is on AD and BE also. The desired result follows.

Solution 3. Place the hexagon on the complex plane, with A at the origin and vertices labelled clockwise. Now A, B, C, D, E, F represent the corresponding complex numbers. Also consider the complex numbers a, b, c, a', b', c' given by $B - A = a$, $D - C = b$, $F - E = c$, $E - D = a'$, $A - F = b'$, and $C - B = c'$. Let $k = |a|/|b|$. From $a/b' = -ke^{i\angle A}$ and $a'/b = -ke^{i\angle D}$ we get that $(a'/a)(b'/b) = e^{-i\theta}$ and similarly $(b'/b)(c'/c) = e^{-i\theta}$ and $(c'/c)(a'/a) = e^{-i\theta}$. It follows that $a' = ar$, $b' = br$, and $c' = cr$ for a complex number r with $|r| = 1$, as shown below.



We have

$$0 = a + cr + b + ar + c + br = (a + b + c)(1 + r).$$

If $r = -1$, then the hexagon is centrally symmetric and its diagonals intersect at its center of symmetry. Otherwise

$$a + b + c = 0.$$

Therefore

$$A = 0, \quad B = a, \quad C = a + cr, \quad D = c(r - 1), \quad E = -br - c, \quad F = -br.$$

Now consider a point W on AD given by the complex number $c(r-1)\lambda$, where λ is a real number with $0 < \lambda < 1$. Since $D \neq A$, we have $r \neq 1$, so we can define $s = 1/(r-1)$. From $r\overline{r} = |r|^2 = 1$ we get

$$1 + s = \frac{r}{r-1} = \frac{r}{r - r\overline{r}} = \frac{1}{1 - \overline{r}} = -\overline{s}.$$

Now,

$$\begin{aligned} W \text{ is on } BE &\iff c(r-1)\lambda - a \parallel a - (-br - c) = b(r-1) \iff c\lambda - as \parallel b \\ &\iff -a\lambda - b\lambda - as \parallel b \iff a(\lambda + s) \parallel b. \end{aligned}$$

One easily checks that $r \neq \pm 1$ implies that $\lambda + s \neq 0$ since s is not real. On the other hand,

$$\begin{aligned} W \text{ on } CF &\iff c(r-1)\lambda + br \parallel -br - (a + cr) = a(r-1) \iff c\lambda + b(1+s) \parallel a \\ &\iff -a\lambda - b\lambda - b\overline{s} \parallel a \iff b(\lambda + \overline{s}) \parallel a \iff b \parallel a(\lambda + s), \end{aligned}$$

where in the last step we use that $(\lambda + s)(\lambda + \overline{s}) = |\lambda + s|^2 \in \mathbb{R}_{>0}$. We conclude that $AD \cap BE = CF \cap BE$, and the desired result follows.

G6. Let the excircle of the triangle ABC lying opposite to A touch its side BC at the point A_1. Define the points B_1 and C_1 analogously. Suppose that the circumcentre of the triangle $A_1B_1C_1$ lies on the circumcircle of the triangle ABC. Prove that the triangle ABC is right-angled.

(Russia)

Solution 1. Denote the circumcircles of the triangles ABC and $A_1B_1C_1$ by Ω and Γ, respectively. Denote the midpoint of the arc CB of Ω containing A by A_0, and define B_0 as well as C_0 analogously. By our hypothesis the centre Q of Γ lies on Ω.

Lemma. One has $A_0B_1 = A_0C_1$. Moreover, the points A, A_0, B_1, and C_1 are concyclic. Finally, the points A and A_0 lie on the same side of B_1C_1. Similar statements hold for B and C.

Proof. Let us consider the case $A = A_0$ first. Then the triangle ABC is isosceles at A, which implies $AB_1 = AC_1$ while the remaining assertions of the Lemma are obvious. So let us suppose $A \neq A_0$ from now on.

By the definition of A_0, we have $A_0B = A_0C$. It is also well known and easy to show that $BC_1 = CB_1$. Next, we have $\angle C_1BA_0 = \angle ABA_0 = \angle ACA_0 = \angle B_1CA_0$. Hence the triangles A_0BC_1 and A_0CB_1 are congruent. This implies $A_0C_1 = A_0B_1$, establishing the first part of the Lemma. It also follows that $\angle A_0C_1A = \angle A_0B_1A$, as these are exterior angles at the corresponding vertices C_1 and B_1 of the congruent triangles A_0BC_1 and A_0CB_1. For that reason the points A, A_0, B_1, and C_1 are indeed the vertices of some cyclic quadrilateral two opposite sides of which are AA_0 and B_1C_1. $\square$

Now we turn to the solution. Evidently the points A_1, B_1, and C_1 lie interior to some semicircle arc of Γ, so the triangle $A_1B_1C_1$ is obtuse-angled. Without loss of generality, we will assume that its angle at B_1 is obtuse. Thus Q and B_1 lie on different sides of A_1C_1; obviously, the same holds for the points B and B_1. So, the points Q and B are on the same side of A_1C_1.

Notice that the perpendicular bisector of A_1C_1 intersects Ω at two points lying on different sides of A_1C_1. By the first statement from the Lemma, both points B_0 and Q are among these points of intersection; since they share the same side of A_1C_1, they coincide (see Figure 1).

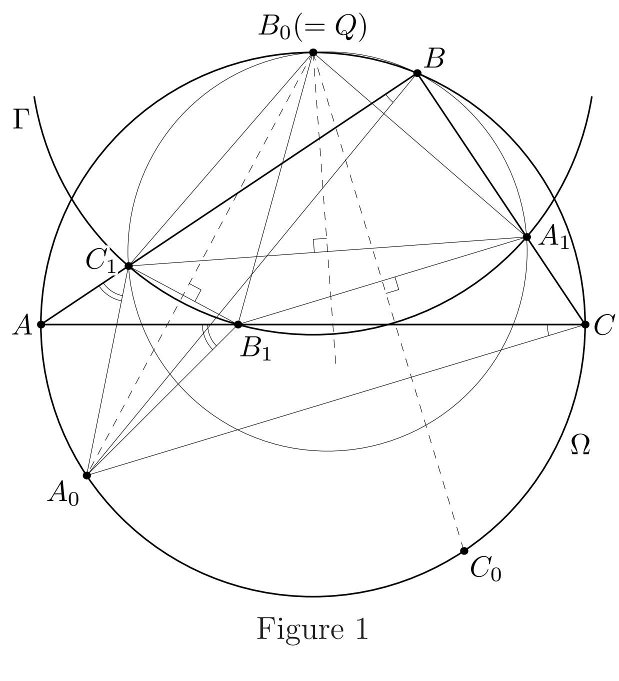


Figure 1

Now, by the first part of the Lemma again, the lines QA_0 and QC_0 are the perpendicular bisectors of B_1C_1 and A_1B_1, respectively. Thus

$$\angle C_1B_0A_1 = \angle C_1B_0B_1 + \angle B_1B_0A_1 = 2\angle A_0B_0B_1 + 2\angle B_1B_0C_0 = 2\angle A_0B_0C_0 = 180^\circ - \angle ABC,$$

recalling that A_0 and C_0 are the midpoints of the arcs CB and BA, respectively.

On the other hand, by the second part of the Lemma we have

$$\angle C_1B_0A_1 = \angle C_1BA_1 = \angle ABC.$$

From the last two equalities, we get $\angle ABC = 90^\circ$, whereby the problem is solved.

Solution 2. Let Q again denote the centre of the circumcircle of the triangle $A_1B_1C_1$, that lies on the circumcircle Ω of the triangle ABC. We first consider the case where Q coincides with one of the vertices of ABC, say $Q = B$. Then $BC_1 = BA_1$ and consequently the triangle ABC is isosceles at B. Moreover we have $BC_1 = B_1C$ in any triangle, and hence $BB_1 = BC_1 = B_1C$; similarly, $BB_1 = B_1A$. It follows that B_1 is the centre of Ω and that the triangle ABC has a right angle at B.

So from now on we may suppose $Q \notin \{A, B, C\}$. We start with the following well known fact.

Lemma. Let XYZ and $X'Y'Z'$ be two triangles with $XY = X'Y'$ and $YZ = Y'Z'$.

(i) If $XZ \neq X'Z'$ and $\angle YZX = \angle Y'Z'X'$, then $\angle ZXY + \angle Z'X'Y' = 180^\circ$.

(ii) If $\angle YZX + \angle X'Z'Y' = 180^\circ$, then $\angle ZXY = \angle Y'X'Z'$.

Proof. For both parts, we may move the triangle XYZ through the plane until $Y = Y'$ and $Z = Z'$. Possibly after reflecting one of the two triangles about YZ, we may also suppose that X and X' lie on the same side of YZ if we are in case (i) and on different sides if we are in case (ii). In both cases, the points X, Z, and X' are collinear due to the angle condition (see Fig. 2). Moreover we have $X \neq X'$, because in case (i) we assumed $XZ \neq X'Z'$ and in case (ii) these points even lie on different sides of YZ. Thus the triangle $XX'Y$ is isosceles at Y. The claim now follows by considering the equal angles at its base. □

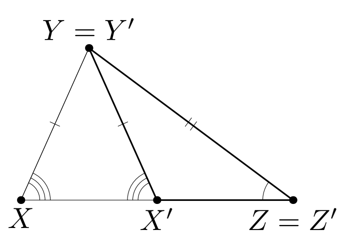


Figure 2(i)

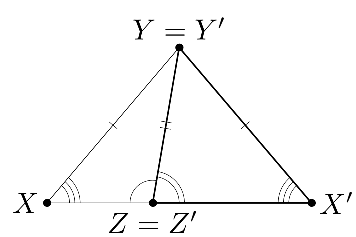


Figure 2(ii)

Relabeling the vertices of the triangle ABC if necessary we may suppose that Q lies in the interior of the arc AB of Ω not containing C. We will sometimes use tacitly that the six triangles QBA_1, QA_1C, QCB_1, QB_1A, QC_1A, and QBC_1 have the same orientation.

As Q cannot be the circumcentre of the triangle ABC, it is impossible that $QA = QB = QC$ and thus we may also suppose that $QC \neq QB$. Now the above Lemma (i) is applicable to the triangles QB_1C and QC_1B, since $QB_1 = QC_1$ and $B_1C = C_1B$, while $\angle B_1CQ = \angle C_1BQ$ holds as both angles appear over the same side of the chord QA in Ω (see Fig. 3). So we get

$$\angle CQB_1 + \angle BQC_1 = 180^\circ. \tag{1}$$

We claim that $QC = QA$. To see this, let us assume for the sake of a contradiction that $QC \neq QA$. Then arguing similarly as before but now with the triangles QA_1C and QC_1A we get

$$\angle A_1QC + \angle C_1QA = 180^\circ.$$

Adding this equation to (1), we get $\angle A_1QB_1 + \angle BQA = 360^\circ$, which is absurd as both summands lie in the interval $(0^\circ, 180^\circ)$.

This proves $QC = QA$; so the triangles QA_1C and QC_1A are congruent their sides being equal, which in turn yields

$$\angle A_1QC = \angle C_1QA. \tag{2}$$

Finally our Lemma (ii) is applicable to the triangles QA_1B and QB_1A. Indeed we have $QA_1 = QB_1$ and $A_1B = B_1A$ as usual, and the angle condition $\angle A_1BQ + \angle QAB_1 = 180^\circ$ holds as A and B lie on different sides of the chord QC in Ω. Consequently we have

$$\angle BQA_1 = \angle B_1QA. \tag{3}$$

From (1) and (3) we get

$$(\angle B_1QC + \angle B_1QA) + (\angle C_1QB - \angle BQA_1) = 180^\circ,$$

i.e. $\angle CQA + \angle A_1QC_1 = 180^\circ$. In light of (2) this may be rewritten as $2\angle CQA = 180^\circ$ and as Q lies on Ω this implies that the triangle ABC has a right angle at B.

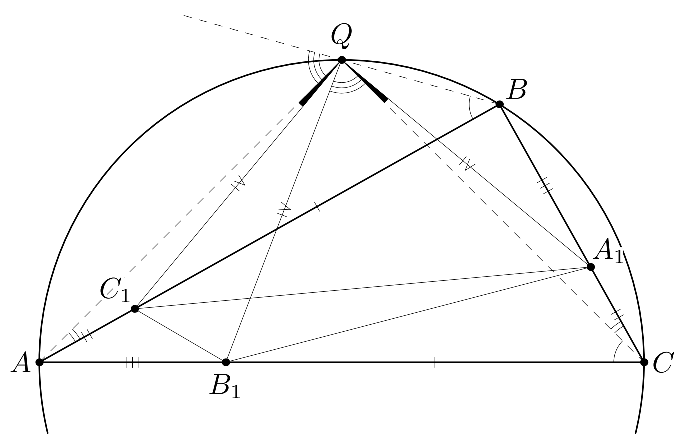


Figure 3

Comment 1. One may also check that Q is in the interior of Ω if and only if the triangle ABC is acute-angled.

Comment 2. The original proposal asked to prove the converse statement as well: if the triangle ABC is right-angled, then the point Q lies on its circumcircle. The Problem Selection Committee thinks that the above simplified version is more suitable for the competition.

Number Theory

N1. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

(Malaysia)

Answer. $f(n) = n$.

Solution 1. Setting $m = n = 2$ tells us that $4 + f(2) \mid 2f(2) + 2$. Since $2f(2) + 2 < 2(4 + f(2))$, we must have $2f(2) + 2 = 4 + f(2)$, so $f(2) = 2$. Plugging in $m = 2$ then tells us that $4 + f(n) \mid 4 + n$, which implies that $f(n) \leqslant n$ for all n.

Setting $m = n$ gives $n^2 + f(n) \mid nf(n) + n$, so $nf(n) + n \geqslant n^2 + f(n)$ which we rewrite as $(n-1)(f(n) - n) \geqslant 0$. Therefore $f(n) \geqslant n$ for all $n \geqslant 2$. This is trivially true for $n = 1$ also.

It follows that $f(n) = n$ for all n. This function obviously satisfies the desired property.

Solution 2. Setting $m = f(n)$ we get $f(n)(f(n)+1) \mid f(n)f(f(n)) + n$. This implies that $f(n) \mid n$ for all n.

Now let m be any positive integer, and let $p > 2m^2$ be a prime number. Note that $p > mf(m)$ also. Plugging in $n = p - mf(m)$ we learn that $m^2 + f(n)$ divides p. Since $m^2 + f(n)$ cannot equal 1, it must equal p. Therefore $p - m^2 = f(n) \mid n = p - mf(m)$. But $p - mf(m) < p < 2(p - m^2)$, so we must have $p - mf(m) = p - m^2$, i.e., $f(m) = m$.

Solution 3. Plugging $m = 1$ we obtain $1 + f(n) \leqslant f(1) + n$, so $f(n) \leqslant n + c$ for the constant $c = f(1) - 1$. Assume that $f(n) \neq n$ for some fixed n. When m is large enough (e.g. $m \geqslant \max(n, c+1)$) we have

$$mf(m) + n \leqslant m(m+c) + n \leqslant 2m^2 < 2(m^2 + f(n)),$$

so we must have $mf(m) + n = m^2 + f(n)$. This implies that

$$0 \neq f(n) - n = m(f(m) - m),$$

which is impossible for $m > |f(n) - n|$. It follows that f is the identity function.

N2. Prove that for any pair of positive integers k and n there exist k positive integers $m_1, m_2, \ldots, m_k$ such that

$$1+\frac{2^k-1}{n}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

(Japan)

Solution 1. We proceed by induction on k. For $k=1$ the statement is trivial. Assuming we have proved it for $k=j-1$, we now prove it for $k=j$.

Case 1. $n=2t-1$ for some positive integer t.

Observe that

$$1+\frac{2^j-1}{2t-1}=\frac{2(t+2^{j-1}-1)}{2t}\cdot\frac{2t}{2t-1}=\left(1+\frac{2^{j-1}-1}{t}\right)\left(1+\frac{1}{2t-1}\right).$$

By the induction hypothesis we can find $m_1, \ldots, m_{j-1}$ such that

$$1+\frac{2^{j-1}-1}{t}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_{j-1}}\right),$$

so setting $m_j=2t-1$ gives the desired expression.

Case 2. $n=2t$ for some positive integer t.

Now we have

$$1+\frac{2^j-1}{2t}=\frac{2t+2^j-1}{2t+2^j-2}\cdot\frac{2t+2^j-2}{2t}=\left(1+\frac{1}{2t+2^j-2}\right)\left(1+\frac{2^{j-1}-1}{t}\right),$$

noting that $2t+2^j-2>0$. Again, we use that

$$1+\frac{2^{j-1}-1}{t}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_{j-1}}\right).$$

Setting $m_j=2t+2^j-2$ then gives the desired expression.

Solution 2. Consider the base 2 expansions of the residues of $n-1$ and $-n$ modulo 2^k:

$$\begin{aligned} n-1 &\equiv 2^{a_1}+2^{a_2}+\cdots+2^{a_r} \pmod{2^k} &&\text{where } 0\leqslant a_1<a_2<\ldots<a_r\leqslant k-1,\\ -n &\equiv 2^{b_1}+2^{b_2}+\cdots+2^{b_s} \pmod{2^k} &&\text{where } 0\leqslant b_1<b_2<\ldots<b_s\leqslant k-1. \end{aligned}$$

Since $-1\equiv 2^0+2^1+\cdots+2^{k-1} \pmod{2^k}$, we have $\{a_1,\ldots,a_r\}\cup\{b_1\ldots,b_s\}=\{0,1,\ldots,k-1\}$ and $r+s=k$. Write

$$\begin{aligned} S_p &= 2^{a_p}+2^{a_{p+1}}+\cdots+2^{a_r} &&\text{for } 1\leqslant p\leqslant r,\\ T_q &= 2^{b_1}+2^{b_2}+\cdots+2^{b_q} &&\text{for } 1\leqslant q\leqslant s. \end{aligned}$$

Also set $S_{r+1} = T_0 = 0$. Notice that $S_1 + T_s = 2^k - 1$ and $n + T_s \equiv 0 \pmod{2^k}$. We have

$$\begin{aligned} 1 + \frac{2^k - 1}{n} = \frac{n + S_1 + T_s}{n} &= \frac{n + S_1 + T_s}{n + T_s} \cdot \frac{n + T_s}{n} \\ &= \prod_{p=1}^{r} \frac{n + S_p + T_s}{n + S_{p+1} + T_s} \cdot \prod_{q=1}^{s} \frac{n + T_q}{n + T_{q-1}} \\ &= \prod_{p=1}^{r} \left(1 + \frac{2^{a_p}}{n + S_{p+1} + T_s}\right) \cdot \prod_{q=1}^{s} \left(1 + \frac{2^{b_q}}{n + T_{q-1}}\right), \end{aligned}$$

so if we define

$$m_p = \frac{n + S_{p+1} + T_s}{2^{a_p}} \quad \text{for } 1 \leqslant p \leqslant r \quad \text{and} \quad m_{r+q} = \frac{n + T_{q-1}}{2^{b_q}} \quad \text{for } 1 \leqslant q \leqslant s,$$

the desired equality holds. It remains to check that every m_i is an integer. For $1 \leqslant p \leqslant r$ we have

$$n + S_{p+1} + T_s \equiv n + T_s \equiv 0 \pmod{2^{a_p}}$$

and for $1 \leqslant q \leqslant r$ we have

$$n + T_{q-1} \equiv n + T_s \equiv 0 \pmod{2^{b_q}}.$$

The desired result follows.

N3. Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

(Belgium)

Solution. Let p_n be the largest prime divisor of $n^4 + n^2 + 1$ and let q_n be the largest prime divisor of $n^2 + n + 1$. Then $p_n = q_{n^2}$, and from

$$n^4 + n^2 + 1 = (n^2+1)^2 - n^2 = (n^2 - n + 1)(n^2 + n + 1) = ((n-1)^2 + (n-1) + 1)(n^2 + n + 1)$$

it follows that $p_n = \max\{q_n, q_{n-1}\}$ for $n \geqslant 2$. Keeping in mind that $n^2 - n + 1$ is odd, we have

$$\gcd(n^2 + n + 1, n^2 - n + 1) = \gcd(2n, n^2 - n + 1) = \gcd(n, n^2 - n + 1) = 1.$$

Therefore $q_n \neq q_{n-1}$.

To prove the result, it suffices to show that the set

$$S = \{n \in \mathbb{Z}_{\geqslant 2} \mid q_n > q_{n-1} \text{ and } q_n > q_{n+1}\}$$

is infinite, since for each $n \in S$ one has

$$p_n = \max\{q_n, q_{n-1}\} = q_n = \max\{q_n, q_{n+1}\} = p_{n+1}.$$

Suppose on the contrary that S is finite. Since $q_2 = 7 < 13 = q_3$ and $q_3 = 13 > 7 = q_4$, the set S is non-empty. Since it is finite, we can consider its largest element, say m.

Note that it is impossible that $q_m > q_{m+1} > q_{m+2} > \ldots$ because all these numbers are positive integers, so there exists a $k \geqslant m$ such that $q_k < q_{k+1}$ (recall that $q_k \neq q_{k+1}$). Next observe that it is impossible to have $q_k < q_{k+1} < q_{k+2} < \ldots$, because $q_{(k+1)^2} = p_{k+1} = \max\{q_k, q_{k+1}\} = q_{k+1}$, so let us take the smallest $\ell \geqslant k+1$ such that $q_\ell > q_{\ell+1}$. By the minimality of ℓ we have $q_{\ell-1} < q_\ell$, so $\ell \in S$. Since $\ell \geqslant k+1 > k \geqslant m$, this contradicts the maximality of m, and hence S is indeed infinite.

Comment. Once the factorization of $n^4 + n^2 + 1$ is found and the set S is introduced, the problem is mainly about ruling out the case that

$$q_k < q_{k+1} < q_{k+2} < \ldots \tag{1}$$

might hold for some $k \in \mathbb{Z}_{>0}$. In the above solution, this is done by observing $q_{(k+1)^2} = \max(q_k, q_{k+1})$. Alternatively one may notice that (1) implies that $q_{j+2} - q_j \geqslant 6$ for $j \geqslant k+1$, since every prime greater than 3 is congruent to -1 or 1 modulo 6. Then there is some integer $C \geqslant 0$ such that $q_n \geqslant 3n - C$ for all $n \geqslant k$.

Now let the integer t be sufficiently large (e.g. $t = \max\{k+1, C+3\}$) and set $p = q_{t-1} \geqslant 2t$. Then $p \mid (t-1)^2 + (t-1) + 1$ implies that $p \mid (p-t)^2 + (p-t) + 1$, so p and q_{p-t} are prime divisors of $(p-t)^2 + (p-t) + 1$. But $p - t > t - 1 \geqslant k$, so $q_{p-t} > q_{t-1} = p$ and $p \cdot q_{p-t} > p^2 > (p-t)^2 + (p-t) + 1$, a contradiction.

N4. Determine whether there exists an infinite sequence of nonzero digits $a_1, a_2, a_3, \ldots$ and a positive integer N such that for every integer $k > N$, the number $\overline{a_k a_{k-1} \ldots a_1}$ is a perfect square.

(Iran)

Answer. No.

Solution. Assume that $a_1, a_2, a_3, \ldots$ is such a sequence. For each positive integer k, let $y_k = \overline{a_k a_{k-1} \ldots a_1}$. By the assumption, for each $k > N$ there exists a positive integer x_k such that $y_k = x_k^2$.

I. For every n, let 5^{γ_n} be the greatest power of 5 dividing x_n. Let us show first that $2\gamma_n \geqslant n$ for every positive integer $n > N$.

Assume, to the contrary, that there exists a positive integer $n > N$ such that $2\gamma_n < n$, which yields

$$y_{n+1} = \overline{a_{n+1} a_n \ldots a_1} = 10^n a_{n+1} + \overline{a_n a_{n-1} \ldots a_1} = 10^n a_{n+1} + y_n = 5^{2\gamma_n}\left(2^n 5^{n-2\gamma_n} a_{n+1} + \frac{y_n}{5^{2\gamma_n}}\right).$$

Since $5 \nmid y_n / 5^{2\gamma_n}$, we obtain $\gamma_{n+1} = \gamma_n < n < n+1$. By the same arguments we obtain that $\gamma_n = \gamma_{n+1} = \gamma_{n+2} = \ldots$. Denote this common value by γ.

Now, for each $k \geqslant n$ we have

$$(x_{k+1} - x_k)(x_{k+1} + x_k) = x_{k+1}^2 - x_k^2 = y_{k+1} - y_k = a_{k+1} \cdot 10^k.$$

One of the numbers $x_{k+1} - x_k$ and $x_{k+1} + x_k$ is not divisible by $5^{\gamma+1}$ since otherwise one would have $5^{\gamma+1} \mid \big((x_{k+1} - x_k) + (x_{k+1} + x_k)\big) = 2x_{k+1}$. On the other hand, we have $5^k \mid (x_{k+1} - x_k)(x_{k+1} + x_k)$, so $5^{k-\gamma}$ divides one of these two factors. Thus we get

$$5^{k-\gamma} \leqslant \max\{x_{k+1} - x_k, x_{k+1} + x_k\} < 2x_{k+1} = 2\sqrt{y_{k+1}} < 2 \cdot 10^{(k+1)/2},$$

which implies $5^{2k} < 4 \cdot 5^{2\gamma} \cdot 10^{k+1}$, or $(5/2)^k < 40 \cdot 5^{2\gamma}$. The last inequality is clearly false for sufficiently large values of k. This contradiction shows that $2\gamma_n \geqslant n$ for all $n > N$.

II. Consider now any integer $k > \max\{N/2, 2\}$. Since $2\gamma_{2k+1} \geqslant 2k+1$ and $2\gamma_{2k+2} \geqslant 2k+2$, we have $\gamma_{2k+1} \geqslant k+1$ and $\gamma_{2k+2} \geqslant k+1$. So, from $y_{2k+2} = a_{2k+2} \cdot 10^{2k+1} + y_{2k+1}$ we obtain $5^{2k+2} \mid y_{2k+2} - y_{2k+1} = a_{2k+2} \cdot 10^{2k+1}$ and thus $5 \mid a_{2k+2}$, which implies $a_{2k+2} = 5$. Therefore,

$$(x_{2k+2} - x_{2k+1})(x_{2k+2} + x_{2k+1}) = x_{2k+2}^2 - x_{2k+1}^2 = y_{2k+2} - y_{2k+1} = 5 \cdot 10^{2k+1} = 2^{2k+1} \cdot 5^{2k+2}.$$

Setting $A_k = x_{2k+2}/5^{k+1}$ and $B_k = x_{2k+1}/5^{k+1}$, which are integers, we obtain

$$(A_k - B_k)(A_k + B_k) = 2^{2k+1}. \tag{1}$$

Both A_k and B_k are odd, since otherwise y_{2k+2} or y_{2k+1} would be a multiple of 10 which is false by $a_1 \neq 0$; so one of the numbers $A_k - B_k$ and $A_k + B_k$ is not divisible by 4. Therefore (1) yields $A_k - B_k = 2$ and $A_k + B_k = 2^{2k}$, hence $A_k = 2^{2k-1} + 1$ and thus

$$x_{2k+2} = 5^{k+1} A_k = 10^{k+1} \cdot 2^{k-2} + 5^{k+1} > 10^{k+1},$$

since $k \geqslant 2$. This implies that $y_{2k+2} > 10^{2k+2}$ which contradicts the fact that y_{2k+2} contains $2k+2$ digits. The desired result follows.

Solution 2. Again, we assume that a sequence $a_1, a_2, a_3, \ldots$ satisfies the problem conditions, introduce the numbers x_k and y_k as in the previous solution, and notice that

$$y_{k+1} - y_k = (x_{k+1} - x_k)(x_{k+1} + x_k) = 10^k a_{k+1} \tag{2}$$

for all $k > N$. Consider any such k. Since $a_1 \neq 0$, the numbers x_k and x_{k+1} are not multiples of 10, and therefore the numbers $p_k = x_{k+1} - x_k$ and $q_k = x_{k+1} + x_k$ cannot be simultaneously multiples of 20, and hence one of them is not divisible either by 4 or by 5. In view of (2), this means that the other one is divisible by either 5^k or by 2^{k-1}. Notice also that p_k and q_k have the same parity, so both are even.

On the other hand, we have $x_{k+1}^2 = x_k^2 + 10^k a_{k+1} \geqslant x_k^2 + 10^k > 2x_k^2$, so $x_{k+1}/x_k > \sqrt{2}$, which implies that

$$1 < \frac{q_k}{p_k} = 1 + \frac{2}{x_{k+1}/x_k - 1} < 1 + \frac{2}{\sqrt{2} - 1} < 6. \tag{3}$$

Thus, if one of the numbers p_k and q_k is divisible by 5^k, then we have

$$10^{k+1} > 10^k a_{k+1} = p_k q_k \geqslant \frac{(5^k)^2}{6}$$

and hence $(5/2)^k < 60$ which is false for sufficiently large k. So, assuming that k is large, we get that 2^{k-1} divides one of the numbers p_k and q_k. Hence

$$\{p_k,\, q_k\} = \{2^{k-1} \cdot 5^{r_k} b_k,\, 2 \cdot 5^{k-r_k} c_k\} \quad \text{with nonnegative integers } b_k,\, c_k,\, r_k \text{ such that } b_k c_k = a_{k+1}.$$

Moreover, from (3) we get

$$6 > \frac{2^{k-1} \cdot 5^{r_k} b_k}{2 \cdot 5^{k-r_k} c_k} \geqslant \frac{1}{36} \cdot \left(\frac{2}{5}\right)^k \cdot 5^{2r_k} \quad \text{and} \quad 6 > \frac{2 \cdot 5^{k-r_k} c_k}{2^{k-1} \cdot 5^{r_k} b_k} \geqslant \frac{4}{9} \cdot \left(\frac{5}{2}\right)^k \cdot 5^{-2r_k},$$

so

$$\alpha k + c_1 < r_k < \alpha k + c_2 \qquad \text{for } \alpha = \tfrac{1}{2}\log_5\left(\tfrac{5}{2}\right) < 1 \text{ and some constants } c_2 > c_1. \tag{4}$$

Consequently, for $C = c_2 - c_1 + 1 - \alpha > 0$ we have

$$(k+1) - r_{k+1} \leqslant k - r_k + C. \tag{5}$$

Next, we will use the following easy lemma.

Lemma. Let s be a positive integer. Then $5^{s+2^s} \equiv 5^s \pmod{10^s}$.

Proof. Euler's theorem gives $5^{2^s} \equiv 1 \pmod{2^s}$, so $5^{s+2^s} - 5^s = 5^s(5^{2^s} - 1)$ is divisible by 2^s and 5^s.

Now, for every large k we have

$$x_{k+1} = \frac{p_k + q_k}{2} = 5^{r_k} \cdot 2^{k-2} b_k + 5^{k-r_k} c_k \equiv 5^{k-r_k} c_k \pmod{10^{r_k}} \tag{6}$$

since $r_k \leqslant k-2$ by (4); hence $y_{k+1} \equiv 5^{2(k-r_k)} c_k^2 \pmod{10^{r_k}}$. Let us consider some large integer s, and choose the minimal k such that $2(k - r_k) \geqslant s + 2^s$; it exists by (4). Set $d = 2(k-r_k) - (s+2^s)$. By (4) we have $2^s < 2(k - r_k) < \left(\frac{2}{\alpha} - 2\right) r_k - \frac{2c_1}{\alpha}$; if s is large this implies $r_k > s$, so (6) also holds modulo 10^s. Then (6) and the lemma give

$$y_{k+1} \equiv 5^{2(k-r_k)} c_k^2 = 5^{s+2^s} \cdot 5^d c_k^2 \equiv 5^s \cdot 5^d c_k^2 \pmod{10^s}. \tag{7}$$

By (5) and the minimality of k we have $d \leqslant 2C$, so $5^d c_k^2 \leqslant 5^{2C} \cdot 81 = D$. Using $5^4 < 10^3$ we obtain

$$5^s \cdot 5^d c_k^2 < 10^{3s/4} D < 10^{s-1}$$

for sufficiently large s. This, together with (7), shows that the sth digit from the right in y_{k+1}, which is a_s, is zero. This contradicts the problem condition.

N5. Fix an integer $k \geqslant 2$. Two players, called Ana and Banana, play the following *game of numbers*: Initially, some integer $n \geqslant k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leqslant m' < m$ that is coprime to m. The first player who cannot move anymore loses.

An integer $n \geqslant k$ is called *good* if Banana has a winning strategy when the initial number is n, and *bad* otherwise.

Consider two integers $n, n' \geqslant k$ with the property that each prime number $p \leqslant k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

(Italy)

Solution 1. Let us first observe that the number appearing on the blackboard decreases after every move; so the game necessarily ends after at most n steps, and consequently there always has to be some player possessing a winning strategy. So if some $n \geqslant k$ is bad, then Ana has a winning strategy in the game with starting number n.

More precisely, if $n \geqslant k$ is such that there is a good integer m with $n > m \geqslant k$ and $\gcd(m, n) = 1$, then n itself is bad, for Ana has the following winning strategy in the game with initial number n: She proceeds by first playing m and then using Banana's strategy for the game with starting number m.

Otherwise, if some integer $n \geqslant k$ has the property that every integer m with $n > m \geqslant k$ and $\gcd(m, n) = 1$ is bad, then n is good. Indeed, if Ana can make a first move at all in the game with initial number n, then she leaves it in a position where the first player has a winning strategy, so that Banana can defeat her.

In particular, this implies that any two good numbers have a non–trivial common divisor. Also, k itself is good.

For brevity, we say that $n \longrightarrow x$ is a *move* if n and x are two coprime integers with $n > x \geqslant k$.

Claim 1. If n is good and n' is a multiple of n, then n' is also good.

Proof. If n' were bad, there would have to be some move $n' \longrightarrow x$, where x is good. As n' is a multiple of n this implies that the two good numbers n and x are coprime, which is absurd. □

Claim 2. If r and s denote two positive integers for which $rs \geqslant k$ is bad, then $r^2 s$ is also bad.

Proof. Since rs is bad, there is a move $rs \longrightarrow x$ for some good x. Evidently x is coprime to $r^2 s$ as well, and hence the move $r^2 s \longrightarrow x$ shows that $r^2 s$ is indeed bad. □

Claim 3. If $p > k$ is prime and $n \geqslant k$ is bad, then np is also bad.

Proof. Otherwise we choose a counterexample with n being as small as possible. In particular, np is good. Since n is bad, there is a move $n \longrightarrow x$ for some good x. Now $np \longrightarrow x$ cannot be a valid move, which tells us that x has to be divisible by p. So we can write $x = p^r y$, where r and y denote some positive integers, the latter of which is not divisible by p.

Note that $y = 1$ is impossible, for then we would have $x = p^r$ and the move $x \longrightarrow k$ would establish that x is bad. In view of this, there is a least power y^α of y that is at least as large as k. Since the numbers np and y^α are coprime and the former is good, the latter has to be bad. Moreover, the minimality of α implies $y^\alpha < ky < py = \frac{x}{p^{r-1}} < \frac{n}{p^{r-1}}$. So $p^{r-1} \cdot y^\alpha < n$ and consequently all the numbers $y^\alpha, py^\alpha, \ldots, p^r \cdot y^\alpha = p(p^{r-1} \cdot y^\alpha)$ are bad due to the minimal choice of n. But now by Claim 1 the divisor x of $p^r \cdot y^\alpha$ cannot be good, whereby we have reached a contradiction that proves Claim 3. □

We now deduce the statement of the problem from these three claims. To this end, we call two integers $a, b \geqslant k$ *similar* if they are divisible by the same prime numbers not exceeding k. We are to prove that if a and b are similar, then either both of them are good or both are bad. As in this case the product ab is similar to both a and b, it suffices to show the following: if $c \geqslant k$ is similar to some of its multiples d, then either both c and d are good or both are bad.

Assuming that this is not true in general, we choose a counterexample (c_0, d_0) with d_0 being as small as possible. By Claim 1, c_0 is bad whilst d_0 is good. Plainly d_0 is strictly greater than c_0 and hence the quotient $\frac{d_0}{c_0}$ has some prime factor p. Clearly p divides d_0. If $p \leqslant k$, then p divides c_0 as well due to the similarity, and hence d_0 is actually divisible by p^2. So $\frac{d_0}{p}$ is good by the contrapositive of Claim 2. Since $c_0 \mid \frac{d_0}{p}$, the pair $(c_0, \frac{d_0}{p})$ contradicts the supposed minimality of d_0. This proves $p > k$, but now we get the same contradiction using Claim 3 instead of Claim 2. Thereby the problem is solved.

Solution 2. We use the same analysis of the game of numbers as in the first five paragraphs of the first solution. Let us call a prime number p *small* in case $p \leqslant k$ and *big* otherwise. We again call two integers *similar* if their sets of small prime factors coincide.

Claim 4. For each integer $b \geqslant k$ having some small prime factor, there exists an integer x similar to it with $b \geqslant x \geqslant k$ and having no big prime factors.

Proof. Unless b has a big prime factor we may simply choose $x = b$. Now let p and q denote a small and a big prime factor of b, respectively. Let a be the product of all small prime factors of b. Further define n to be the least non–negative integer for which the number $x = p^n a$ is at least as large as k. It suffices to show that $b > x$. This is clear in case $n = 0$, so let us assume $n > 0$ from now on. Then we have $x < pk$ due to the minimality of n, $p \leqslant a$ because p divides a by construction, and $k < q$. Therefore $x < aq$ and, as the right hand side is a product of distinct prime factors of b, this implies indeed $x < b$. □

Let us now assume that there is a pair (a, b) of similar numbers such that a is bad and b is good. Take such a pair with $\max(a, b)$ being as small as possible. Since a is bad, there exists a move $a \longrightarrow r$ for some good r. Since the numbers k and r are both good, they have a common prime factor, which necessarily has to be small. Thus Claim 4 is applicable to r, which yields an integer r' similar to r containing small prime factors only and satisfying $r \geqslant r' \geqslant k$. Since $\max(r, r') = r < a \leqslant \max(a, b)$ the number r' is also good. Now let p denote a common prime factor of the good numbers r' and b. By our construction of r', this prime is small and due to the similarities it consequently divides a and r, contrary to $a \longrightarrow r$ being a move. Thereby the problem is solved.

Comment 1. Having reached Claim 4 of Solution 2, there are various other ways to proceed. For instance, one may directly obtain the following fact, which seems to be interesting in its own right:

Claim 5. Any two good numbers have a common small prime factor.

Proof. Otherwise there exists a pair (b, b') of good numbers with $b' \geqslant b \geqslant k$ all of whose common prime factors are big. Choose such a pair with b' being as small as possible. Since b and k are both good, there has to be a common prime factor p of b and k. Evidently p is small and thus it cannot divide b', which in turn tells us $b' > b$. Applying Claim 4 to b we get an integer x with $b \geqslant x \geqslant k$ that is similar to b and has no big prime divisors at all. By our assumption, b' and x are coprime, and as b' is good this implies that x is bad. Consequently there has to be some move $x \longrightarrow b^*$ such that b^* is good. But now all the small prime factors of b also appear in x and thus they cannot divide b^*. Therefore the pair (b^*, b) contradicts the supposed minimality of b'. □

From that point, it is easy to complete the solution: assume that there are two similar integers a and b such that a is bad and b is good. Since a is bad, there is a move $a \longrightarrow b'$ for some good b'. By Claim 5, there is a small prime p dividing b and b'. Due to the similarity of a and b, the prime p has to divide a as well, but this contradicts the fact that $a \longrightarrow b'$ is a valid move. Thereby the problem is solved.

Comment 2. There are infinitely many good numbers, e.g. all multiples of k. The increasing sequence $b_0, b_1, \ldots,$ of all good numbers may be constructed recursively as follows:

- Start with $b_0 = k$.
- If b_n has just been defined for some $n \geqslant 0$, then b_{n+1} is the smallest number $b > b_n$ that is coprime to none of $b_0, \ldots, b_n$.

This construction can be used to determine the set of good numbers for any specific k as explained in the next comment. It is already clear that if $k = p^\alpha$ is a prime power, then a number $b \geqslant k$ is good if and only if it is divisible by p.

Comment 3. Let $P > 1$ denote the product of all small prime numbers. Then any two integers $a, b \geqslant k$ that are congruent modulo P are similar. Thus the infinite word $W_k = (X_k, X_{k+1}, \ldots)$ defined by

$$X_i = \begin{cases} A & \text{if } i \text{ is bad} \\ B & \text{if } i \text{ is good} \end{cases}$$

for all $i \geqslant k$ is periodic and the length of its period divides P. As the prime power example shows, the true period can sometimes be much smaller than P. On the other hand, there are cases where the period is rather large; e.g., if $k = 15$, the sequence of good numbers begins with $15, 18, 20, 24, 30, 36, 40, 42, 45$ and the period of W_{15} is 30.

Comment 4. The original proposal contained two questions about the game of numbers, namely (a) to show that if two numbers have the same prime factors then either both are good or both are bad, and (b) to show that the word W_k introduced in the previous comment is indeed periodic. The Problem Selection Committee thinks that the above version of the problem is somewhat easier, even though it demands to prove a stronger result.

N6. Determine all functions $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfying

$$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right) \tag{1}$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$. (Here, $\mathbb{Z}_{>0}$ denotes the set of positive integers.)

(Israel)

Answer. There are three kinds of such functions, which are: all constant functions, the floor function, and the ceiling function.

Solution 1. I. We start by verifying that these functions do indeed satisfy (1). This is clear for all constant functions. Now consider any triple $(x, a, b) \in \mathbb{Q} \times \mathbb{Z} \times \mathbb{Z}_{>0}$ and set

$$q = \left\lfloor \frac{x+a}{b} \right\rfloor.$$

This means that q is an integer and $bq \leqslant x + a < b(q+1)$. It follows that $bq \leqslant \lfloor x \rfloor + a < b(q+1)$ holds as well, and thus we have

$$\left\lfloor \frac{\lfloor x \rfloor + a}{b} \right\rfloor = \left\lfloor \frac{x+a}{b} \right\rfloor,$$

meaning that the floor function does indeed satisfy (1). One can check similarly that the ceiling function has the same property.

II. Let us now suppose conversely that the function $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfies (1) for all $(x, a, b) \in \mathbb{Q} \times \mathbb{Z} \times \mathbb{Z}_{>0}$. According to the behaviour of the restriction of f to the integers we distinguish two cases.

Case 1: There is some $m \in \mathbb{Z}$ such that $f(m) \neq m$.

Write $f(m) = C$ and let $\eta \in \{-1, +1\}$ and b denote the sign and absolute value of $f(m) - m$, respectively. Given any integer r, we may plug the triple $(m, rb - C, b)$ into (1), thus getting $f(r) = f(r - \eta)$. Starting with m and using induction in both directions, we deduce from this that the equation $f(r) = C$ holds for all integers r. Now any rational number y can be written in the form $y = \frac{p}{q}$ with $(p, q) \in \mathbb{Z} \times \mathbb{Z}_{>0}$, and substituting $(C - p, p - C, q)$ into (1) we get $f(y) = f(0) = C$. Thus f is the constant function whose value is always C.

Case 2: One has $f(m) = m$ for all integers m.

Note that now the special case $b = 1$ of (1) takes a particularly simple form, namely

$$f(x) + a = f(x + a) \qquad \text{for all } (x, a) \in \mathbb{Q} \times \mathbb{Z}. \tag{2}$$

Defining $f\left(\frac{1}{2}\right) = \omega$ we proceed in three steps.

Step A. We show that $\omega \in \{0, 1\}$.

If $\omega \leqslant 0$, we may plug $\left(\frac{1}{2}, -\omega, 1 - 2\omega\right)$ into (1), obtaining $0 = f(0) = f\left(\frac{1}{2}\right) = \omega$. In the contrary case $\omega \geqslant 1$ we argue similarly using the triple $\left(\frac{1}{2}, \omega - 1, 2\omega - 1\right)$.

Step B. We show that $f(x) = \omega$ for all rational numbers x with $0 < x < 1$.

Assume that this fails and pick some rational number $\frac{a}{b} \in (0, 1)$ with minimal b such that $f\left(\frac{a}{b}\right) \neq \omega$. Obviously, $\gcd(a, b) = 1$ and $b \geqslant 2$. If b is even, then a has to be odd and we can substitute $\left(\frac{1}{2}, \frac{a-1}{2}, \frac{b}{2}\right)$ into (1), which yields

$$f\left(\frac{\omega + (a-1)/2}{b/2}\right) = f\left(\frac{a}{b}\right) \neq \omega. \tag{3}$$

Recall that $0 \leqslant (a-1)/2 < b/2$. Thus, in both cases $\omega = 0$ and $\omega = 1$, the left-hand part of (3) equals ω either by the minimality of b, or by $f(\omega) = \omega$. A contradiction.

Thus b has to be odd, so $b = 2k+1$ for some $k \geqslant 1$. Applying (1) to $\left(\frac{1}{2}, k, b\right)$ we get

$$f\left(\frac{\omega+k}{b}\right) = f\left(\frac{1}{2}\right) = \omega. \tag{4}$$

Since a and b are coprime, there exist integers $r \in \{1, 2, \ldots, b\}$ and m such that $ra - mb = k + \omega$. Note that we actually have $1 \leqslant r < b$, since the right hand side is not a multiple of b. If m is negative, then we have $ra - mb > b \geqslant k + \omega$, which is absurd. Similarly, $m \geqslant r$ leads to $ra - mb < br - br = 0$, which is likewise impossible; so we must have $0 \leqslant m \leqslant r - 1$.

We finally substitute $\left(\frac{k+\omega}{b}, m, r\right)$ into (1) and use (4) to learn

$$f\left(\frac{\omega+m}{r}\right) = f\left(\frac{a}{b}\right) \neq \omega.$$

But as above one may see that the left hand side has to equal ω due to the minimality of b. This contradiction concludes our step B.

Step C. Now notice that if $\omega = 0$, then $f(x) = \lfloor x \rfloor$ holds for all rational x with $0 \leqslant x < 1$ and hence by (2) this even holds for all rational numbers x. Similarly, if $\omega = 1$, then $f(x) = \lceil x \rceil$ holds for all $x \in \mathbb{Q}$. Thereby the problem is solved.

Comment 1. An alternative treatment of Steps B and C from the second case, due to the proposer, proceeds as follows. Let square brackets indicate the floor function in case $\omega = 0$ and the ceiling function if $\omega = 1$. We are to prove that $f(x) = [x]$ holds for all $x \in \mathbb{Q}$, and because of Step A and (2) we already know this in case $2x \in \mathbb{Z}$. Applying (1) to $(2x, 0, 2)$ we get

$$f(x) = f\left(\frac{f(2x)}{2}\right),$$

and by the previous observation this yields

$$f(x) = \left[\frac{f(2x)}{2}\right] \qquad \text{for all } x \in \mathbb{Q}. \tag{5}$$

An easy induction now shows

$$f(x) = \left[\frac{f(2^n x)}{2^n}\right] \qquad \text{for all } (x, n) \in \mathbb{Q} \times \mathbb{Z}_{>0}. \tag{6}$$

Now suppose first that x is not an integer but can be written in the form $\frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$ both being odd. Let d denote the multiplicative order of 2 modulo q and let m be any large integer. Plugging $n = dm$ into (6) and using (2) we get

$$f(x) = \left[\frac{f(2^{dm}x)}{2^{dm}}\right] = \left[\frac{f(x) + (2^{dm}-1)x}{2^{dm}}\right] = \left[x + \frac{f(x) - x}{2^{dm}}\right].$$

Since x is not an integer, the square bracket function is continuous at x; hence as m tends to infinity the above fomula gives $f(x) = [x]$. To complete the argument we just need to observe that if some $y \in \mathbb{Q}$ satisfies $f(y) = [y]$, then (5) yields $f\left(\frac{y}{2}\right) = f\left(\frac{[y]}{2}\right) = \left[\frac{[y]}{2}\right] = \left[\frac{y}{2}\right]$.

Solution 2. Here we just give another argument for the second case of the above solution. Again we use equation (2). It follows that the set S of all zeros of f contains for each $x \in \mathbb{Q}$ exactly one term from the infinite sequence $\ldots, x-2, x-1, x, x+1, x+2, \ldots$.

Next we claim that

$$\text{if } (p,q) \in \mathbb{Z} \times \mathbb{Z}_{>0} \text{ and } \tfrac{p}{q} \in S, \text{ then } \tfrac{p}{q+1} \in S \text{ holds as well.} \tag{7}$$

To see this we just plug $\left(\frac{p}{q}, p, q+1\right)$ into (1), thus getting $f\left(\frac{p}{q+1}\right) = f\left(\frac{p}{q}\right) = 0$.

From this we get that

$$\text{if } x, y \in \mathbb{Q},\ x > y > 0, \text{ and } x \in S, \text{ then } y \in S. \tag{8}$$

Indeed, if we write $x = \frac{p}{q}$ and $y = \frac{r}{s}$ with $p, q, r, s \in \mathbb{Z}_{>0}$, then $ps > qr$ and (7) tells us

$$0 = f\left(\frac{p}{q}\right) = f\left(\frac{pr}{qr}\right) = f\left(\frac{pr}{qr+1}\right) = \ldots = f\left(\frac{pr}{ps}\right) = f\left(\frac{r}{s}\right).$$

Essentially the same argument also establishes that

$$\text{if } x, y \in \mathbb{Q},\ x < y < 0, \text{ and } x \in S, \text{ then } y \in S. \tag{9}$$

From (8) and (9) we get $0 \in S \subseteq (-1, +1)$ and hence the real number $\alpha = \sup(S)$ exists and satisfies $0 \leqslant \alpha \leqslant 1$.

Let us assume that we actually had $0 < \alpha < 1$. Note that $f(x) = 0$ if $x \in (0, \alpha) \cap \mathbb{Q}$ by (8), and $f(x) = 1$ if $x \in (\alpha, 1) \cap \mathbb{Q}$ by (9) and (2). Let K denote the unique positive integer satisfying $K\alpha < 1 \leqslant (K+1)\alpha$. The first of these two inequalities entails $\alpha < \frac{1+\alpha}{K+1}$, and thus there is a rational number $x \in \left(\alpha, \frac{1+\alpha}{K+1}\right)$. Setting $y = (K+1)x - 1$ and substituting $(y, 1, K+1)$ into (1) we learn

$$f\left(\frac{f(y)+1}{K+1}\right) = f\left(\frac{y+1}{K+1}\right) = f(x).$$

Since $\alpha < x < 1$ and $0 < y < \alpha$, this simplifies to

$$f\left(\frac{1}{K+1}\right) = 1.$$

But, as $0 < \frac{1}{K+1} \leqslant \alpha$, this is only possible if $\alpha = \frac{1}{K+1}$ and $f(\alpha) = 1$. From this, however, we get the contradiction

$$0 = f\left(\frac{1}{(K+1)^2}\right) = f\left(\frac{\alpha+0}{K+1}\right) = f\left(\frac{f(\alpha)+0}{K+1}\right) = f(\alpha) = 1.$$

Thus our assumption $0 < \alpha < 1$ has turned out to be wrong and it follows that $\alpha \in \{0, 1\}$. If $\alpha = 0$, then we have $S \subseteq (-1, 0]$, whence $S = (-1, 0] \cap \mathbb{Q}$, which in turn yields $f(x) = \lceil x \rceil$ for all $x \in \mathbb{Q}$ due to (2). Similarly, $\alpha = 1$ entails $S = [0, 1) \cap \mathbb{Q}$ and $f(x) = \lfloor x \rfloor$ for all $x \in \mathbb{Q}$. Thereby the solution is complete.

Comment 2. It seems that all solutions to this problems involve some case distinction separating the constant solutions from the unbounded ones, though the "descriptions" of the cases may be different depending on the work that has been done at the beginning of the solution. For instance, these two cases can also be "f is periodic on the integers" and "f is not periodic on the integers". The case leading to the unbounded solutions appears to be the harder one.

In most approaches, the cases leading to the two functions $x \longmapsto \lfloor x \rfloor$ and $x \longmapsto \lceil x \rceil$ can easily be treated parallelly, but sometimes it may be useful to know that there is some symmetry in the problem interchanging these two functions. Namely, if a function $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfies (1), then so does the function $g\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ defined by $g(x) = -f(-x)$ for all $x \in \mathbb{Q}$. For that reason, we could have restricted our attention to the case $\omega = 0$ in the first solution and, once $\alpha \in \{0, 1\}$ had been obtained, to the case $\alpha = 0$ in the second solution.

N7. Let ν be an irrational positive number, and let m be a positive integer. A pair (a, b) of positive integers is called *good* if

$$a\lceil b\nu\rceil - b\lfloor a\nu\rfloor = m. \qquad (*)$$

A good pair (a, b) is called *excellent* if neither of the pairs $(a-b, b)$ and $(a, b-a)$ is good. (As usual, by $\lfloor x\rfloor$ and $\lceil x\rceil$ we denote the integer numbers such that $x - 1 < \lfloor x\rfloor \leqslant x$ and $x \leqslant \lceil x\rceil < x + 1$.)

Prove that the number of excellent pairs is equal to the sum of the positive divisors of m.

(U.S.A.)

Solution. For positive integers a and b, let us denote

$$f(a, b) = a\lceil b\nu\rceil - b\lfloor a\nu\rfloor.$$

We will deal with various values of m; thus it is convenient to say that a pair (a, b) is m-*good* or m-*excellent* if the corresponding conditions are satisfied.

To start, let us investigate how the values $f(a + b, b)$ and $f(a, b + a)$ are related to $f(a, b)$. If $\{a\nu\} + \{b\nu\} < 1$, then we have $\lfloor(a + b)\nu\rfloor = \lfloor a\nu\rfloor + \lfloor b\nu\rfloor$ and $\lceil(a + b)\nu\rceil = \lceil a\nu\rceil + \lceil b\nu\rceil - 1$, so

$$f(a + b, b) = (a + b)\lceil b\nu\rceil - b(\lfloor a\nu\rfloor + \lfloor b\nu\rfloor) = f(a, b) + b(\lceil b\nu\rceil - \lfloor b\nu\rfloor) = f(a, b) + b$$

and

$$f(a, b + a) = a(\lceil b\nu\rceil + \lceil a\nu\rceil - 1) - (b + a)\lfloor a\nu\rfloor = f(a, b) + a(\lceil a\nu\rceil - 1 - \lfloor a\nu\rfloor) = f(a, b).$$

Similarly, if $\{a\nu\} + \{b\nu\} \geqslant 1$ then one obtains

$$f(a + b, b) = f(a, b) \quad \text{and} \quad f(a, b + a) = f(a, b) + a.$$

So, in both cases one of the numbers $f(a + b, a)$ and $f(a, b + a)$ is equal to $f(a, b)$ while the other is greater than $f(a, b)$ by one of a and b. Thus, exactly one of the pairs $(a + b, b)$ and $(a, b + a)$ is excellent (for an appropriate value of m).

Now let us say that the pairs $(a + b, b)$ and $(a, b + a)$ are the *children* of the pair (a, b), while this pair is their *parent*. Next, if a pair (c, d) can be obtained from (a, b) by several passings from a parent to a child, we will say that (c, d) is a *descendant* of (a, b), while (a, b) is an *ancestor* of (c, d) (a pair is neither an ancestor nor a descendant of itself). Thus each pair (a, b) has two children, it has a unique parent if $a \neq b$, and no parents otherwise. Therefore, each pair of distinct positive integers has a unique ancestor of the form (a, a); our aim is now to find how many m-excellent descendants each such pair has.

Notice now that if a pair (a, b) is m-excellent then $\min\{a, b\} \leqslant m$. Indeed, if $a = b$ then $f(a, a) = a = m$, so the statement is valid. Otherwise, the pair (a, b) is a child of some pair (a', b'). If $b = b'$ and $a = a' + b'$, then we should have $m = f(a, b) = f(a', b') + b'$, so $b = b' = m - f(a', b') < m$. Similarly, if $a = a'$ and $b = b' + a'$ then $a < m$.

Let us consider the set S_m of all pairs (a, b) such that $f(a, b) \leqslant m$ and $\min\{a, b\} \leqslant m$. Then all the ancestors of the elements in S_m are again in S_m, and each element in S_m either is of the form (a, a) with $a \leqslant m$, or has a unique ancestor of this form. From the arguments above we see that all m-excellent pairs lie in S_m.

We claim now that the set S_m is finite. Indeed, assume, for instance, that it contains infinitely many pairs (c, d) with $d > 2m$. Such a pair is necessarily a child of $(c, d-c)$, and thus a descendant of some pair (c, d') with $m < d' \leqslant 2m$. Therefore, one of the pairs $(a, b) \in S_m$ with $m < b \leqslant 2m$

has infinitely many descendants in S_m, and all these descendants have the form $(a, b+ka)$ with k a positive integer. Since $f(a, b+ka)$ does not decrease as k grows, it becomes constant for $k \geqslant k_0$, where k_0 is some positive integer. This means that $\{a\nu\} + \{(b+ka)\nu\} < 1$ for all $k \geqslant k_0$. But this yields $1 > \{(b+ka)\nu\} = \{(b+k_0a)\nu\} + (k-k_0)\{a\nu\}$ for all $k > k_0$, which is absurd.

Similarly, one can prove that S_m contains finitely many pairs (c, d) with $c > 2m$, thus finitely many elements at all.

We are now prepared for proving the following crucial lemma.

Lemma. Consider any pair (a, b) with $f(a, b) \neq m$. Then the number $g(a, b)$ of its m-excellent descendants is equal to the number $h(a, b)$ of ways to represent the number $t = m - f(a, b)$ as $t = ka + \ell b$ with k and ℓ being some nonnegative integers.

Proof. We proceed by induction on the number N of descendants of (a, b) in S_m. If $N = 0$ then clearly $g(a, b) = 0$. Assume that $h(a, b) > 0$; without loss of generality, we have $a \leqslant b$. Then, clearly, $m - f(a, b) \geqslant a$, so $f(a, b+a) \leqslant f(a, b) + a \leqslant m$ and $a \leqslant m$, hence $(a, b+a) \in S_m$ which is impossible. Thus in the base case we have $g(a, b) = h(a, b) = 0$, as desired.

Now let $N > 0$. Assume that $f(a+b, b) = f(a, b) + b$ and $f(a, b+a) = f(a, b)$ (the other case is similar). If $f(a, b) + b \neq m$, then by the induction hypothesis we have

$$g(a, b) = g(a+b, b) + g(a, b+a) = h(a+b, b) + h(a, b+a).$$

Notice that both pairs $(a+b, b)$ and $(a, b+a)$ are descendants of (a, b) and thus each of them has strictly less descendants in S_m than (a, b) does.

Next, each one of the $h(a+b, b)$ representations of $m - f(a+b, b) = m - b - f(a, b)$ as the sum $k'(a+b) + \ell' b$ provides the representation $m - f(a, b) = ka + \ell b$ with $k = k' < k' + \ell' + 1 = \ell$. Similarly, each one of the $h(a, b+a)$ representations of $m - f(a, b+a) = m - f(a, b)$ as the sum $k'a + \ell'(b+a)$ provides the representation $m - f(a, b) = ka + \ell b$ with $k = k' + \ell' \geqslant \ell' = \ell$. This correspondence is obviously bijective, so

$$h(a, b) = h(a+b, b) + h(a, b+a) = g(a, b),$$

as required.

Finally, if $f(a, b) + b = m$ then $(a+b, b)$ is m-excellent, so $g(a, b) = 1 + g(a, b+a) = 1 + h(a, b+a)$ by the induction hypothesis. On the other hand, the number $m - f(a, b) = b$ has a representation $0 \cdot a + 1 \cdot b$ and sometimes one more representation as $ka + 0 \cdot b$; this last representation exists simultaneously with the representation $m - f(a, b+a) = ka + 0 \cdot (b+a)$, so $h(a, b) = 1 + h(a, b+a)$ as well. Thus in this case the step is also proved. $\square$

Now it is easy to finish the solution. There exists a unique m-excellent pair of the form (a, a), and each other m-excellent pair (a, b) has a unique ancestor of the form (x, x) with $x < m$. By the lemma, for every $x < m$ the number of its m-excellent descendants is $h(x, x)$, which is the number of ways to represent $m - f(x, x) = m - x$ as $kx + \ell x$ (with nonnegative integer k and ℓ). This number is 0 if $x \nmid m$, and m/x otherwise. So the total number of excellent pairs is

$$1 + \sum_{x \mid m,\, x<m} \frac{m}{x} = 1 + \sum_{d \mid m,\, d>1} d = \sum_{d \mid m} d,$$

as required.

Comment. Let us present a sketch of an outline of a different solution. The plan is to check that the number of excellent pairs does not depend on the (irrational) number ν, and to find this number for some appropriate value of ν. For that, we first introduce some geometrical language. We deal only with the excellent pairs (a, b) with $a \neq b$.

Part I. Given an irrational positive ν, for every positive integer n we introduce two integral points $F_\nu(n) = (n, \lfloor n\nu \rfloor)$ and $C_\nu(n) = (n, \lceil n\nu \rceil)$ on the coordinate plane Oxy. Then $(*)$ reads as $[OF_\nu(a)C_\nu(b)] = m/2$; here $[\cdot]$ stands for the signed area. Next, we rewrite in these terms the condition on a pair (a, b) to be excellent. Let ℓ_ν, ℓ_ν^+, and ℓ_ν^- be the lines determined by the equations $y = \nu x$, $y = \nu x + 1$, and $y = \nu x - 1$, respectively.

a). Firstly, we deal with all excellent pairs (a, b) with $a < b$. Given some value of a, all the points C such that $[OF_\nu(a)C] = m/2$ lie on some line $f_\nu(a)$; if there exist any good pairs (a, b) at all, this line has to contain at least one integral point, which happens exactly when $\gcd(a, \lfloor a\nu \rfloor) \mid m$.

Let $P_\nu(a)$ be the point of intersection of ℓ_ν^+ and $f_\nu(a)$, and let $p_\nu(a)$ be its abscissa; notice that $p_\nu(a)$ is irrational if it is nonzero. Now, if (a, b) is good, then the point $C_\nu(b)$ lies on $f_\nu(a)$, which means that the point of $f_\nu(a)$ with abscissa b lies between ℓ_ν and ℓ_ν^+ and is integral. If in addition the pair $(a, b - a)$ is not good, then the point of $f_\nu(a)$ with abscissa $b - a$ lies above ℓ_ν^+ (see Fig. 1). Thus, the pair (a, b) with $b > a$ is excellent exactly when $p_\nu(a)$ lies between $b - a$ and b, and the point of $f_\nu(a)$ with abscissa b is integral (which means that this point is $C_\nu(b)$).

Notice now that, if $p_\nu(a) > a$, then the number of excellent pairs of the form (a, b) (with $b > a$) is $\gcd(a, \lfloor a\nu \rfloor)$.

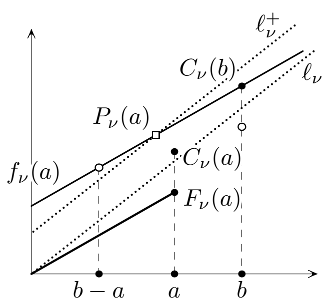


Figure 1

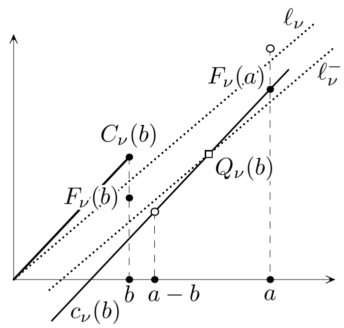


Figure 2

b). Analogously, considering the pairs (a, b) with $a > b$, we fix the value of b, introduce the line $c_\nu(b)$ containing all the points F with $[OFC_\nu(b)] = m/2$, assume that this line contains an integral point (which means $\gcd(b, \lceil b\nu \rceil) \mid m$), and denote the common point of $c_\nu(b)$ and ℓ_ν^- by $Q_\nu(b)$, its abscissa being $q_\nu(b)$. Similarly to the previous case, we obtain that the pair (a, b) is excellent exactly when $q_\nu(a)$ lies between $a - b$ and a, and the point of $c_\nu(b)$ with abscissa a is integral (see Fig. 2). Again, if $q_\nu(b) > b$, then the number of excellent pairs of the form (a, b) (with $a > b$) is $\gcd(b, \lceil b\nu \rceil)$.

Part II, sketchy. Having obtained such a description, one may check how the number of excellent pairs changes as ν grows. (Having done that, one may find this number for one appropriate value of ν; for instance, it is relatively easy to make this calculation for $\nu \in \left(1, 1 + \frac{1}{m}\right)$.)

Consider, for the initial value of ν, some excellent pair (a, t) with $a > t$. As ν grows, this pair eventually stops being excellent; this happens when the point $Q_\nu(t)$ passes through $F_\nu(a)$. At the same moment, the pair $(a + t, t)$ becomes excellent instead.

This process halts when the point $Q_\nu(t)$ eventually disappears, i.e. when ν passes through the ratio of the coordinates of the point $T = C_\nu(t)$. Hence, the point T afterwards is regarded as $F_\nu(t)$. Thus, all the old excellent pairs of the form (a, t) with $a > t$ disappear; on the other hand, the same number of excellent pairs with the first element being t just appear.

Similarly, if some pair (t, b) with $t < b$ is initially ν-excellent, then at some moment it stops being excellent when $P_\nu(t)$ passes through $C_\nu(b)$; at the same moment, the pair $(t, b-t)$ becomes excellent. This process eventually stops when $b - t < t$. At this moment, again the second element of the pair becomes fixed, and the first one starts to increase.

These ideas can be made precise enough to show that the number of excellent pairs remains unchanged, as required.

We should warn the reader that the rigorous elaboration of Part II is technically quite involved, mostly by the reason that the set of moments when the collection of excellent pairs changes is infinite. Especially much care should be applied to the limit points of this set, which are exactly the points when the line ℓ_ν passes through some point of the form $C_\nu(b)$.

The same ideas may be explained in an algebraic language instead of a geometrical one; the same technicalities remain in this way as well.

55th International Mathematical Olympiad

PROBLEMS SHORT LIST

WITH SOLUTIONS

IMO 2014

Cape Town - South Africa

Shortlisted Problems with Solutions

55^{th} International Mathematical Olympiad

Cape Town, South Africa, 2014

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2015.

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2014 thank the following 43 countries for contributing 141 problem proposals.

Australia, Austria, Belgium, Benin, Bulgaria, Colombia, Croatia, Cyprus, Czech Republic, Denmark, Ecuador, Estonia, Finland, France, Georgia, Germany, Greece, Hong Kong, Hungary, Iceland, India, Indonesia, Iran, Ireland, Japan, Lithuania, Luxembourg, Malaysia, Mongolia, Netherlands, Nigeria, Pakistan, Russia, Saudi Arabia, Serbia, Slovakia, Slovenia, South Korea, Thailand, Turkey, Ukraine, United Kingdom, U.S.A.

Problem Selection Committee

Johan Meyer
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Problems

Algebra

A1. Let $z_0 < z_1 < z_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geqslant 1$ such that

$$z_n < \frac{z_0 + z_1 + \cdots + z_n}{n} \leqslant z_{n+1}.$$

(Austria)

A2. Define the function $f\colon (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2}, \\ x^2 & \text{if } x \geqslant \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a$, $b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

(Denmark)

A3. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \leqslant i \leqslant n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i^{th} step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots + x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leqslant cD$.

(Georgia)

A4. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m) + n\big) + f(m) = f(n) + f(3m) + 2014$$

for all integers m and n.

(Netherlands)

A5. Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers x and y one has

$$|y^2 - P(x)| \leqslant 2|x| \quad \text{if and only if} \quad |x^2 - P(y)| \leqslant 2|y|.$$

Determine all possible values of $P(0)$.

(Belgium)

A6. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$n^2 + 4f(n) = f(f(n))^2$$

for all $n \in \mathbb{Z}$.

(United Kingdom)

Combinatorics

C1. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

(Serbia)

C2. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

(Iran)

C3. Let $n \geqslant 2$ be an integer. Consider an $n \times n$ chessboard divided into n^2 unit squares. We call a configuration of n rooks on this board *happy* if every row and every column contains exactly one rook. Find the greatest positive integer k such that for every happy configuration of rooks, we can find a $k \times k$ square without a rook on any of its k^2 unit squares.

(Croatia)

C4. Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively.

S-tetrominoes Z-tetrominoes

Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove than no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

(Hungary)

C5. Consider $n \geqslant 3$ lines in the plane such that no two lines are parallel and no three have a common point. These lines divide the plane into polygonal regions; let $\mathcal{F}$ be the set of regions having finite area. Prove that it is possible to colour $\left\lceil \sqrt{n/2} \right\rceil$ of the lines blue in such a way that no region in $\mathcal{F}$ has a completely blue boundary. (For a real number x, $\lceil x \rceil$ denotes the least integer which is not smaller than x.)

(Austria)

C6. We are given an infinite deck of cards, each with a real number on it. For every real number x, there is exactly one card in the deck that has x written on it. Now two players draw disjoint sets A and B of 100 cards each from this deck. We would like to define a rule that declares one of them a winner. This rule should satisfy the following conditions:

1. The winner only depends on the relative order of the 200 cards: if the cards are laid down in increasing order face down and we are told which card belongs to which player, but not what numbers are written on them, we can still decide the winner.

2. If we write the elements of both sets in increasing order as $A = \{a_1, a_2, \ldots, a_{100}\}$ and $B = \{b_1, b_2, \ldots, b_{100}\}$, and $a_i > b_i$ for all i, then A beats B.

3. If three players draw three disjoint sets A, B, C from the deck, A beats B and B beats C, then A also beats C.

How many ways are there to define such a rule? Here, we consider two rules as different if there exist two sets A and B such that A beats B according to one rule, but B beats A according to the other.

(Russia)

C7. Let M be a set of $n \geqslant 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

(Russia)

C8. A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.

Determine all possible first moves of the first player after which he has a winning strategy.

(Russia)

C9. There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*.

Suppose that Turbo's path entirely covers all circles. Prove that n must be odd.

(India)

Geometry

G1. The points P and Q are chosen on the side BC of an acute-angled triangle ABC so that $\angle PAB = \angle ACB$ and $\angle QAC = \angle CBA$. The points M and N are taken on the rays AP and AQ, respectively, so that $AP = PM$ and $AQ = QN$. Prove that the lines BM and CN intersect on the circumcircle of the triangle ABC.

(Georgia)

G2. Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

(Estonia)

G3. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$. (Here we always assume that an angle bisector is a ray.)

(Russia)

G4. Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

(United Kingdom)

G5. Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D = 90°$. Point H is the foot of the perpendicular from A to BD. The points S and T are chosen on the sides AB and AD, respectively, in such a way that H lies inside triangle SCT and

$$\angle SHC - \angle BSC = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ .$$

Prove that the circumcircle of triangle SHT is tangent to the line BD.

(Iran)

G6. Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF. Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T, respectively. We call the pair (E, F) *interesting*, if the quadrilateral $KSAT$ is cyclic.

Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC} .$$

(Iran)

G7. Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V, respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y. Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

(U.S.A.)

Number Theory

N1. Let $n \geqslant 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leqslant k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n.

(Serbia)

N2. Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1\,.$$

(U.S.A.)

N3. A coin is called a *Cape Town coin* if its value is $1/n$ for some positive integer n. Given a collection of Cape Town coins of total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into at most 100 groups each of total value at most 1.

(Luxembourg)

N4. Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k \geqslant 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

(Hong Kong)

N5. Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

(Belgium)

N6. Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geqslant n + 2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

(Serbia)

N7. Let $c \geqslant 1$ be an integer. Define a sequence of positive integers by $a_1 = c$ and

$$a_{n+1} = a_n^3 - 4c \cdot a_n^2 + 5c^2 \cdot a_n + c$$

for all $n \geqslant 1$. Prove that for each integer $n \geqslant 2$ there exists a prime number p dividing a_n but none of the numbers $a_1, \ldots, a_{n-1}$.

(Austria)

N8. For every real number x, let $\|x\|$ denote the distance between x and the nearest integer. Prove that for every pair (a, b) of positive integers there exist an odd prime p and a positive integer k satisfying

$$\left\| \frac{a}{p^k} \right\| + \left\| \frac{b}{p^k} \right\| + \left\| \frac{a+b}{p^k} \right\| = 1.$$

(Hungary)

Solutions

Algebra

A1. Let $z_0 < z_1 < z_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geqslant 1$ such that

$$z_n < \frac{z_0 + z_1 + \cdots + z_n}{n} \leqslant z_{n+1}. \tag{1}$$

(Austria)

Solution. For $n = 1, 2, \ldots$ define

$$d_n = (z_0 + z_1 + \cdots + z_n) - n z_n.$$

The sign of d_n indicates whether the first inequality in (1) holds; i.e., it is satisfied if and only if $d_n > 0$.

Notice that

$$n z_{n+1} - (z_0 + z_1 + \cdots + z_n) = (n+1) z_{n+1} - (z_0 + z_1 + \cdots + z_n + z_{n+1}) = -d_{n+1},$$

so the second inequality in (1) is equivalent to $d_{n+1} \leqslant 0$. Therefore, we have to prove that there is a unique index $n \geqslant 1$ that satisfies $d_n > 0 \geqslant d_{n+1}$.

By its definition the sequence $d_1, d_2, \ldots$ consists of integers and we have

$$d_1 = (z_0 + z_1) - 1 \cdot z_1 = z_0 > 0.$$

From

$$d_{n+1} - d_n = \big((z_0 + \cdots + z_n + z_{n+1}) - (n+1) z_{n+1}\big) - \big((z_0 + \cdots + z_n) - n z_n\big) = n(z_n - z_{n+1}) < 0$$

we can see that $d_{n+1} < d_n$ and thus the sequence strictly decreases.

Hence, we have a decreasing sequence $d_1 > d_2 > \ldots$ of integers such that its first element d_1 is positive. The sequence must drop below 0 at some point, and thus there is a unique index n, that is the index of the last positive term, satisfying $d_n > 0 \geqslant d_{n+1}$.

Comment. Omitting the assumption that $z_1, z_2, \ldots$ are integers allows the numbers d_n to be all positive. In such cases the desired n does not exist. This happens for example if $z_n = 2 - \frac{1}{2^n}$ for all integers $n \geqslant 0$.

A2. Define the function $f\colon (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2}, \\ x^2 & \text{if } x \geqslant \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a$, $b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

(Denmark)

Solution. Note that

$$f(x) - x = \tfrac{1}{2} > 0$$

if $x < \frac{1}{2}$ and

$$f(x) - x = x^2 - x < 0$$

if $x \geqslant \frac{1}{2}$. So if we consider $(0,1)$ as being divided into the two subintervals $I_1 = (0, \frac{1}{2})$ and $I_2 = [\frac{1}{2}, 1)$, the inequality

$$(a_n - a_{n-1})(b_n - b_{n-1}) = \big(f(a_{n-1}) - a_{n-1}\big)\big(f(b_{n-1}) - b_{n-1}\big) < 0$$

holds if and only if a_{n-1} and b_{n-1} lie in distinct subintervals.

Let us now assume, to the contrary, that a_k and b_k always lie in the same subinterval. Consider the distance $d_k = |a_k - b_k|$. If both a_k and b_k lie in I_1, then

$$d_{k+1} = |a_{k+1} - b_{k+1}| = \left|a_k + \tfrac{1}{2} - b_k - \tfrac{1}{2}\right| = d_k.$$

If, on the other hand, a_k and b_k both lie in I_2, then $\min(a_k, b_k) \geqslant \frac{1}{2}$ and $\max(a_k, b_k) = \min(a_k, b_k) + d_k \geqslant \frac{1}{2} + d_k$, which implies

$$d_{k+1} = |a_{k+1} - b_{k+1}| = \left|a_k^2 - b_k^2\right| = \left|(a_k - b_k)(a_k + b_k)\right| \geqslant |a_k - b_k|\big(\tfrac{1}{2} + \tfrac{1}{2} + d_k\big) = d_k(1 + d_k) \geqslant d_k.$$

This means that the difference d_k is non-decreasing, and in particular $d_k \geqslant d_0 > 0$ for all k.

We can even say more. If a_k and b_k lie in I_2, then

$$d_{k+2} \geqslant d_{k+1} \geqslant d_k(1 + d_k) \geqslant d_k(1 + d_0).$$

If a_k and b_k both lie in I_1, then a_{k+1} and b_{k+1} both lie in I_2, and so we have

$$d_{k+2} \geqslant d_{k+1}(1 + d_{k+1}) \geqslant d_{k+1}(1 + d_0) = d_k(1 + d_0).$$

In either case, $d_{k+2} \geqslant d_k(1 + d_0)$, and inductively we get

$$d_{2m} \geqslant d_0(1 + d_0)^m.$$

For sufficiently large m, the right-hand side is greater than 1, but since a_{2m}, b_{2m} both lie in $(0,1)$, we must have $d_{2m} < 1$, a contradiction.

Thus there must be a positive integer n such that a_{n-1} and b_{n-1} do not lie in the same subinterval, which proves the desired statement.

A3. For a sequence $x_1, x_2, \dots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \leqslant i \leqslant n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i^{th} step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots + x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leqslant cD$.

(Georgia)

Answer. $c = 2$.

Solution. If the initial numbers are 1, -1, 2, and -2, then Dave may arrange them as $1, -2, 2, -1$, while George may get the sequence $1, -1, 2, -2$, resulting in $D = 1$ and $G = 2$. So we obtain $c \geqslant 2$.

Therefore, it remains to prove that $G \leqslant 2D$. Let $x_1, x_2, \dots, x_n$ be the numbers Dave and George have at their disposal. Assume that Dave and George arrange them into sequences $d_1, d_2, \dots, d_n$ and $g_1, g_2, \dots, g_n$, respectively. Put

$$M = \max_{1 \leqslant i \leqslant n} |x_i|, \quad S = |x_1 + \cdots + x_n|, \quad \text{and} \quad N = \max\{M, S\}.$$

We claim that

$$D \geqslant S, \tag{1}$$

$$D \geqslant \frac{M}{2}, \qquad \text{and} \tag{2}$$

$$G \leqslant N = \max\{M, S\}. \tag{3}$$

These inequalities yield the desired estimate, as $G \leqslant \max\{M, S\} \leqslant \max\{M, 2S\} \leqslant 2D$.

The inequality (1) is a direct consequence of the definition of the price.

To prove (2), consider an index i with $|d_i| = M$. Then we have

$$M = |d_i| = \big|(d_1 + \cdots + d_i) - (d_1 + \cdots + d_{i-1})\big| \leqslant |d_1 + \cdots + d_i| + |d_1 + \cdots + d_{i-1}| \leqslant 2D,$$

as required.

It remains to establish (3). Put $h_i = g_1 + g_2 + \cdots + g_i$. We will prove by induction on i that $|h_i| \leqslant N$. The base case $i = 1$ holds, since $|h_1| = |g_1| \leqslant M \leqslant N$. Notice also that $|h_n| = S \leqslant N$.

For the induction step, assume that $|h_{i-1}| \leqslant N$. We distinguish two cases.

Case 1. Assume that no two of the numbers g_i, g_{i+1}, $\dots$, g_n have opposite signs.

Without loss of generality, we may assume that they are all nonnegative. Then one has $h_{i-1} \leqslant h_i \leqslant \cdots \leqslant h_n$, thus

$$|h_i| \leqslant \max\big\{|h_{i-1}|, |h_n|\big\} \leqslant N.$$

Case 2. Among the numbers $g_i, g_{i+1}, \dots, g_n$ there are positive and negative ones.

Then there exists some index $j \geqslant i$ such that $h_{i-1}g_j \leqslant 0$. By the definition of George's sequence we have

$$|h_i| = |h_{i-1} + g_i| \leqslant |h_{i-1} + g_j| \leqslant \max\bigl\{|h_{i-1}|, |g_j|\bigr\} \leqslant N.$$

Thus, the induction step is established.

Comment 1. One can establish the weaker inequalities $D \geqslant \frac{M}{2}$ and $G \leqslant D + \frac{M}{2}$ from which the result also follows.

Comment 2. One may ask a more specific question to find the maximal suitable c if the number n is fixed. For $n = 1$ or 2, the answer is $c = 1$. For $n = 3$, the answer is $c = \frac{3}{2}$, and it is reached e.g., for the collection $1, 2, -4$. Finally, for $n \geqslant 4$ the answer is $c = 2$. In this case the arguments from the solution above apply, and the answer is reached e.g., for the same collection $1, -1, 2, -2$, augmented by several zeroes.

A4. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m)+n\big)+f(m)=f(n)+f(3m)+2014 \tag{1}$$

for all integers m and n.

(Netherlands)

Answer. There is only one such function, namely $n \longmapsto 2n+1007$.

Solution. Let f be a function satisfying (1). Set $C=1007$ and define the function $g\colon \mathbb{Z} \to \mathbb{Z}$ by $g(m)=f(3m)-f(m)+2C$ for all $m \in \mathbb{Z}$; in particular, $g(0)=2C$. Now (1) rewrites as

$$f\big(f(m)+n\big)=g(m)+f(n)$$

for all $m, n \in \mathbb{Z}$. By induction in both directions it follows that

$$f\big(tf(m)+n\big)=tg(m)+f(n) \tag{2}$$

holds for all $m, n, t \in \mathbb{Z}$. Applying this, for any $r \in \mathbb{Z}$, to the triples $\big(r, 0, f(0)\big)$ and $\big(0, 0, f(r)\big)$ in place of (m, n, t) we obtain

$$f(0)g(r)=f\big(f(r)f(0)\big)-f(0)=f(r)g(0)\,.$$

Now if $f(0)$ vanished, then $g(0)=2C>0$ would entail that f vanishes identically, contrary to (1). Thus $f(0)\neq 0$ and the previous equation yields $g(r)=\alpha\, f(r)$, where $\alpha=\frac{g(0)}{f(0)}$ is some nonzero constant.

So the definition of g reveals $f(3m)=(1+\alpha)f(m)-2C$, i.e.,

$$f(3m)-\beta=(1+\alpha)\big(f(m)-\beta\big) \tag{3}$$

for all $m \in \mathbb{Z}$, where $\beta=\frac{2C}{\alpha}$. By induction on k this implies

$$f(3^k m)-\beta=(1+\alpha)^k\big(f(m)-\beta\big) \tag{4}$$

for all integers $k \geqslant 0$ and m.

Since $3 \nmid 2014$, there exists by (1) some value $d=f(a)$ attained by f that is not divisible by 3. Now by (2) we have $f(n+td)=f(n)+tg(a)=f(n)+\alpha\cdot tf(a)$, i.e.,

$$f(n+td)=f(n)+\alpha\cdot td \tag{5}$$

for all $n, t \in \mathbb{Z}$.

Let us fix any positive integer k with $d \mid (3^k-1)$, which is possible, since $\gcd(3,d)=1$. E.g., by the Euler–Fermat theorem, we may take $k=\varphi(|d|)$. Now for each $m \in \mathbb{Z}$ we get

$$f(3^k m)=f(m)+\alpha(3^k-1)m$$

from (5), which in view of (4) yields $\big((1+\alpha)^k-1\big)\big(f(m)-\beta\big)=\alpha(3^k-1)m$. Since $\alpha\neq 0$, the right hand side does not vanish for $m\neq 0$, wherefore the first factor on the left hand side cannot vanish either. It follows that

$$f(m)=\frac{\alpha(3^k-1)}{(1+\alpha)^k-1}\cdot m+\beta\,.$$

So f is a linear function, say $f(m) = Am + \beta$ for all $m \in \mathbb{Z}$ with some constant $A \in \mathbb{Q}$. Plugging this into (1) one obtains $(A^2 - 2A)m + (A\beta - 2C) = 0$ for all m, which is equivalent to the conjunction of

$$A^2 = 2A \qquad \text{and} \qquad A\beta = 2C\,. \tag{6}$$

The first equation is equivalent to $A \in \{0, 2\}$, and as $C \neq 0$ the second one gives

$$A = 2 \qquad \text{and} \qquad \beta = C\,. \tag{7}$$

This shows that f is indeed the function mentioned in the answer and as the numbers found in (7) do indeed satisfy the equations (6) this function is indeed as desired.

Comment 1. One may see that $\alpha = 2$. A more pedestrian version of the above solution starts with a direct proof of this fact, that can be obtained by substituting some special values into (1), e.g., as follows.

Set $D = f(0)$. Plugging $m = 0$ into (1) and simplifying, we get

$$f(n + D) = f(n) + 2C \tag{8}$$

for all $n \in \mathbb{Z}$. In particular, for $n = 0, D, 2D$ we obtain $f(D) = 2C + D$, $f(2D) = f(D) + 2C = 4C + D$, and $f(3D) = f(2D) + 2C = 6C + D$. So substituting $m = D$ and $n = r - D$ into (1) and applying (8) with $n = r - D$ afterwards we learn

$$f(r + 2C) + 2C + D = \big(f(r) - 2C\big) + (6C + D) + 2C\,,$$

i.e., $f(r + 2C) = f(r) + 4C$. By induction in both directions it follows that

$$f(n + 2Ct) = f(n) + 4Ct \tag{9}$$

holds for all $n, t \in \mathbb{Z}$.

Claim. If a and b denote two integers with the property that $f(n + a) = f(n) + b$ holds for all $n \in \mathbb{Z}$, then $b = 2a$.

Proof. Applying induction in both directions to the assumption we get $f(n + ta) = f(n) + tb$ for all $n, t \in \mathbb{Z}$. Plugging $(n, t) = (0, 2C)$ into this equation and $(n, t) = (0, a)$ into (9) we get $f(2aC) - f(0) = 2bC = 4aC$, and, as $C \neq 0$, the claim follows. □

Now by (1), for any $m \in \mathbb{Z}$, the numbers $a = f(m)$ and $b = f(3m) - f(m) + 2C$ have the property mentioned in the claim, whence we have

$$f(3m) - C = 3\big(f(m) - C\big)\,.$$

In view of (3) this tells us indeed that $\alpha = 2$.

Now the solution may be completed as above, but due to our knowledge of $\alpha = 2$ we get the desired formula $f(m) = 2m + C$ directly without having the need to go through all linear functions. Now it just remains to check that this function does indeed satisfy (1).

Comment 2. It is natural to wonder what happens if one replaces the number 2014 appearing in the statement of the problem by some arbitrary integer B.

If B is odd, there is no such function, as can be seen by using the same ideas as in the above solution.

If $B \neq 0$ is even, however, then the only such function is given by $n \longmapsto 2n + B/2$. In case $3 \nmid B$ this was essentially proved above, but for the general case one more idea seems to be necessary. Writing $B = 3^\nu \cdot k$ with some integers ν and k such that $3 \nmid k$ one can obtain $f(n) = 2n + B/2$ for all n that are divisible by 3^ν in the same manner as usual; then one may use the formula $f(3n) = 3f(n) - B$ to establish the remaining cases.

Finally, in case $B = 0$ there are more solutions than just the function $n \longmapsto 2n$. It can be shown that all these other functions are periodic; to mention just one kind of example, for any even integers r and s the function

$$f(n) = \begin{cases} r & \text{if } n \text{ is even,} \\ s & \text{if } n \text{ is odd,} \end{cases}$$

also has the property under discussion.

A5. Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers x and y one has

$$|y^2 - P(x)| \leqslant 2\,|x| \quad \text{if and only if} \quad |x^2 - P(y)| \leqslant 2\,|y|\,. \tag{1}$$

Determine all possible values of $P(0)$.

(Belgium)

Answer. The set of possible values of $P(0)$ is $(-\infty, 0) \cup \{1\}$.

Solution.

Part I. We begin by verifying that these numbers are indeed possible values of $P(0)$. To see that each negative real number $-C$ can be $P(0)$, it suffices to check that for every $C > 0$ the polynomial $P(x) = -\left(\frac{2x^2}{C} + C\right)$ has the property described in the statement of the problem. Due to symmetry it is enough for this purpose to prove $|y^2 - P(x)| > 2\,|x|$ for any two real numbers x and y. In fact we have

$$|y^2 - P(x)| = y^2 + \frac{x^2}{C} + \frac{(|x| - C)^2}{C} + 2\,|x| \geqslant \frac{x^2}{C} + 2\,|x| \geqslant 2\,|x|\,,$$

where in the first estimate equality can only hold if $|x| = C$, whilst in the second one it can only hold if $x = 0$. As these two conditions cannot be met at the same time, we have indeed $|y^2 - P(x)| > 2\,|x|$.

To show that $P(0) = 1$ is possible as well, we verify that the polynomial $P(x) = x^2 + 1$ satisfies (1). Notice that for all real numbers x and y we have

$$\begin{aligned} |y^2 - P(x)| \leqslant 2\,|x| &\iff (y^2 - x^2 - 1)^2 \leqslant 4x^2 \\ &\iff 0 \leqslant \big((y^2 - (x-1)^2\big)\big((x+1)^2 - y^2\big) \\ &\iff 0 \leqslant (y - x + 1)(y + x - 1)(x + 1 - y)(x + 1 + y) \\ &\iff 0 \leqslant \big((x+y)^2 - 1\big)\left(1 - (x-y)^2\right)\,. \end{aligned}$$

Since this inequality is symmetric in x and y, we are done.

Part II. Now we show that no values other than those mentioned in the answer are possible for $P(0)$. To reach this we let P denote any polynomial satisfying (1) and $P(0) \geqslant 0$; as we shall see, this implies $P(x) = x^2 + 1$ for all real x, which is actually more than what we want.

First step: We prove that P is even.

By (1) we have

$$|y^2 - P(x)| \leqslant 2\,|x| \iff |x^2 - P(y)| \leqslant 2\,|y| \iff |y^2 - P(-x)| \leqslant 2\,|x|$$

for all real numbers x and y. Considering just the equivalence of the first and third statement and taking into account that y^2 may vary through $\mathbb{R}_{\geqslant 0}$ we infer that

$$\big[P(x) - 2\,|x|, P(x) + 2\,|x|\big] \cap \mathbb{R}_{\geqslant 0} = \big[P(-x) - 2\,|x|, P(-x) + 2\,|x|\big] \cap \mathbb{R}_{\geqslant 0}$$

holds for all $x \in \mathbb{R}$. We claim that there are infinitely many real numbers x such that $P(x) + 2\,|x| \geqslant 0$. This holds in fact for any real polynomial with $P(0) \geqslant 0$; in order to see this, we may assume that the coefficient of P appearing in front of x is nonnegative. In this case the desired inequality holds for all sufficiently small positive real numbers.

For such numbers x satisfying $P(x) + 2\,|x| \geqslant 0$ we have $P(x) + 2\,|x| = P(-x) + 2\,|x|$ by the previous displayed formula, and hence also $P(x) = P(-x)$. Consequently the polynomial $P(x) - P(-x)$ has infinitely many zeros, wherefore it has to vanish identically. Thus P is indeed even.

Second step: We prove that $P(t) > 0$ for all $t \in \mathbb{R}$.

Let us assume for a moment that there exists a real number $t \neq 0$ with $P(t) = 0$. Then there is some open interval I around t such that $|P(y)| \leqslant 2|y|$ holds for all $y \in I$. Plugging $x = 0$ into (1) we learn that $y^2 = P(0)$ holds for all $y \in I$, which is clearly absurd. We have thus shown $P(t) \neq 0$ for all $t \neq 0$.

In combination with $P(0) \geqslant 0$ this informs us that our claim could only fail if $P(0) = 0$. In this case there is by our first step a polynomial $Q(x)$ such that $P(x) = x^2 Q(x)$. Applying (1) to $x = 0$ and an arbitrary $y \neq 0$ we get $|y\,Q(y)| > 2$, which is surely false when y is sufficiently small.

Third step: We prove that P is a quadratic polynomial.

Notice that P cannot be constant, for otherwise if $x = \sqrt{P(0)}$ and y is sufficiently large, the first part of (1) is false whilst the second part is true. So the degree n of P has to be at least 1. By our first step n has to be even as well, whence in particular $n \geqslant 2$.

Now assume that $n \geqslant 4$. Plugging $y = \sqrt{P(x)}$ into (1) we get $\left|x^2 - P\big(\sqrt{P(x)}\big)\right| \leqslant 2\sqrt{P(x)}$ and hence

$$P\big(\sqrt{P(x)}\big) \leqslant x^2 + 2\sqrt{P(x)}$$

for all real x. Choose positive real numbers x_0, a, and b such that if $x \in (x_0, \infty)$, then $ax^n < P(x) < bx^n$; this is indeed possible, for if $d > 0$ denotes the leading coefficient of P, then $\lim\limits_{x\to\infty} \frac{P(x)}{x^n} = d$, whence for instance the numbers $a = \frac{d}{2}$ and $b = 2d$ work provided that x_0 is chosen large enough.

Now for all sufficiently large real numbers x we have

$$a^{n/2+1} x^{n^2/2} < aP(x)^{n/2} < P\big(\sqrt{P(x)}\big) \leqslant x^2 + 2\sqrt{P(x)} < x^{n/2} + 2b^{1/2} x^{n/2},$$

i.e.

$$x^{(n^2-n)/2} < \frac{1 + 2b^{1/2}}{a^{n/2+1}},$$

which is surely absurd. Thus P is indeed a quadratic polynomial.

Fourth step: We prove that $P(x) = x^2 + 1$.

In the light of our first three steps there are two real numbers $a > 0$ and b such that $P(x) = ax^2 + b$. Now if x is large enough and $y = \sqrt{a}\,x$, the left part of (1) holds and the right part reads $|(1 - a^2)x^2 - b| \leqslant 2\sqrt{a}\,x$. In view of the fact that $a > 0$ this is only possible if $a = 1$. Finally, substituting $y = x + 1$ with $x > 0$ into (1) we get

$$|2x + 1 - b| \leqslant 2x \iff |2x + 1 + b| \leqslant 2x + 2,$$

i.e.,

$$b \in [1, 4x + 1] \iff b \in [-4x - 3, 1]$$

for all $x > 0$. Choosing x large enough, we can achieve that at least one of these two statements holds; then both hold, which is only possible if $b = 1$, as desired.

Comment 1. There are some issues with this problem in that its most natural solutions seem to use some basic facts from analysis, such as the continuity of polynomials or the intermediate value theorem. Yet these facts are intuitively obvious and implicitly clear to the students competing at this level of difficulty, so that the Problem Selection Committee still thinks that the problem is suitable for the IMO.

Comment 2. It seems that most solutions will in the main case, where $P(0)$ is nonnegative, contain an argument that is somewhat asymptotic in nature showing that P is quadratic, and some part narrowing that case down to $P(x) = x^2 + 1$.

Comment 3. It is also possible to skip the first step and start with the second step directly, but then one has to work a bit harder to rule out the case $P(0) = 0$. Let us sketch one possibility of doing this: Take the auxiliary polynomial $Q(x)$ such that $P(x) = xQ(x)$. Applying (1) to $x = 0$ and an arbitrary $y \neq 0$ we get $|Q(y)| > 2$. Hence we either have $Q(z) \geqslant 2$ for all real z or $Q(z) \leqslant -2$ for all real z. In particular there is some $\eta \in \{-1, +1\}$ such that $P(\eta) \geqslant 2$ and $P(-\eta) \leqslant -2$. Substituting $x = \pm\eta$ into (1) we learn

$$|y^2 - P(\eta)| \leqslant 2 \iff |1 - P(y)| \leqslant 2\,|y| \iff |y^2 - P(-\eta)| \leqslant 2\,.$$

But for $y = \sqrt{P(\eta)}$ the first statement is true, whilst the third one is false.

Also, if one has not obtained the evenness of P before embarking on the fourth step, one needs to work a bit harder there, but not in a way that is likely to cause major difficulties.

Comment 4. Truly curious people may wonder about the set of all polynomials having property (1). As explained in the solution above, $P(x) = x^2 + 1$ is the only one with $P(0) = 1$. On the other hand, it is not hard to notice that for negative $P(0)$ there are more possibilities than those mentioned above. E.g., as remarked by the proposer, if a and b denote two positive real numbers with $ab > 1$ and Q denotes a polynomial attaining nonnegative values only, then $P(x) = -\big(ax^2 + b + Q(x)\big)$ works.

More generally, it may be proved that if $P(x)$ satisfies (1) and $P(0) < 0$, then $-P(x) > 2\,|x|$ holds for all $x \in \mathbb{R}$ so that one just considers the equivalence of two false statements. One may generate all such polynomials P by going through all combinations of a solution of the polynomial equation

$$x = A(x)B(x) + C(x)D(x)$$

and a real $E > 0$, and setting

$$P(x) = -\big(A(x)^2 + B(x)^2 + C(x)^2 + D(x)^2 + E\big)$$

for each of them.

A6. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that

$$n^2+4f(n)=f(f(n))^2 \tag{1}$$

for all $n\in\mathbb{Z}$.

(United Kingdom)

Answer. The possibilities are:

- $f(n)=n+1$ for all n;
- or, for some $a\geqslant 1$, $\quad f(n)=\begin{cases} n+1, & n>-a, \\ -n+1, & n\leqslant -a; \end{cases}$
- or $f(n)=\begin{cases} n+1, & n>0, \\ 0, & n=0, \\ -n+1, & n<0. \end{cases}$

Solution 1.

Part I. Let us first check that each of the functions above really satisfies the given functional equation. If $f(n)=n+1$ for all n, then we have

$$n^2+4f(n)=n^2+4n+4=(n+2)^2=f(n+1)^2=f(f(n))^2.$$

If $f(n)=n+1$ for $n>-a$ and $f(n)=-n+1$ otherwise, then we have the same identity for $n>-a$ and

$$n^2+4f(n)=n^2-4n+4=(2-n)^2=f(1-n)^2=f(f(n))^2$$

otherwise. The same applies to the third solution (with $a=0$), where in addition one has

$$0^2+4f(0)=0=f(f(0))^2.$$

Part II. It remains to prove that these are really the only functions that satisfy our functional equation. We do so in three steps:

Step 1: We prove that $f(n)=n+1$ for $n>0$.

Consider the sequence (a_k) given by $a_k=f^k(1)$ for $k\geqslant 0$. Setting $n=a_k$ in (1), we get

$$a_k^2+4a_{k+1}=a_{k+2}^2.$$

Of course, $a_0=1$ by definition. Since $a_2^2=1+4a_1$ is odd, a_2 has to be odd as well, so we set $a_2=2r+1$ for some $r\in\mathbb{Z}$. Then $a_1=r^2+r$ and consequently

$$a_3^2=a_1^2+4a_2=(r^2+r)^2+8r+4.$$

Since $8r+4\neq 0$, $a_3^2\neq(r^2+r)^2$, so the difference between a_3^2 and $(r^2+r)^2$ is at least the distance from $(r^2+r)^2$ to the nearest even square (since $8r+4$ and r^2+r are both even). This implies that

$$|8r+4|=\left|a_3^2-(r^2+r)^2\right|\geqslant(r^2+r)^2-(r^2+r-2)^2=4(r^2+r-1),$$

(for $r=0$ and $r=-1$, the estimate is trivial, but this does not matter). Therefore, we ave

$$4r^2\leqslant|8r+4|-4r+4.$$

If $|r| \geqslant 4$, then

$$4r^2 \geqslant 16|r| \geqslant 12|r| + 16 > 8|r| + 4 + 4|r| + 4 \geqslant |8r+4| - 4r + 4,$$

a contradiction. Thus $|r| < 4$. Checking all possible remaining values of r, we find that $(r^2+r)^2 + 8r + 4$ is only a square in three cases: $r = -3$, $r = 0$ and $r = 1$. Let us now distinguish these three cases:

- $r = -3$, thus $a_1 = 6$ and $a_2 = -5$. For each $k \geqslant 1$, we have

  $$a_{k+2} = \pm\sqrt{a_k^2 + 4a_{k+1}},$$

  and the sign needs to be chosen in such a way that $a_{k+1}^2 + 4a_{k+2}$ is again a square. This yields $a_3 = -4$, $a_4 = -3$, $a_5 = -2$, $a_6 = -1$, $a_7 = 0$, $a_8 = 1$, $a_9 = 2$. At this point we have reached a contradiction, since $f(1) = f(a_0) = a_1 = 6$ and at the same time $f(1) = f(a_8) = a_9 = 2$.

- $r = 0$, thus $a_1 = 0$ and $a_2 = 1$. Then $a_3^2 = a_1^2 + 4a_2 = 4$, so $a_3 = \pm 2$. This, however, is a contradiction again, since it gives us $f(1) = f(a_0) = a_1 = 0$ and at the same time $f(1) = f(a_2) = a_3 = \pm 2$.

- $r = 1$, thus $a_1 = 2$ and $a_2 = 3$. We prove by induction that $a_k = k+1$ for all $k \geqslant 0$ in this case, which we already know for $k \leqslant 2$ now. For the induction step, assume that $a_{k-1} = k$ and $a_k = k+1$. Then

  $$a_{k+1}^2 = a_{k-1}^2 + 4a_k = k^2 + 4k + 4 = (k+2)^2,$$

  so $a_{k+1} = \pm(k+2)$. If $a_{k+1} = -(k+2)$, then

  $$a_{k+2}^2 = a_k^2 + 4a_{k+1} = (k+1)^2 - 4k - 8 = k^2 - 2k - 7 = (k-1)^2 - 8.$$

  The latter can only be a square if $k = 4$ (since 1 and 9 are the only two squares whose difference is 8). Then, however, $a_4 = 5$, $a_5 = -6$ and $a_6 = \pm 1$, so

  $$a_7^2 = a_5^2 + 4a_6 = 36 \pm 4,$$

  but neither 32 nor 40 is a perfect square. Thus $a_{k+1} = k+2$, which completes our induction. This also means that $f(n) = f(a_{n-1}) = a_n = n+1$ for all $n \geqslant 1$.

Step 2: We prove that either $f(0) = 1$, or $f(0) = 0$ and $f(n) \neq 0$ for $n \neq 0$.

Set $n = 0$ in (1) to get

$$4f(0) = f(f(0))^2.$$

This means that $f(0) \geqslant 0$. If $f(0) = 0$, then $f(n) \neq 0$ for all $n \neq 0$, since we would otherwise have

$$n^2 = n^2 + 4f(n) = f(f(n))^2 = f(0)^2 = 0.$$

If $f(0) > 0$, then we know that $f(f(0)) = f(0) + 1$ from the first step, so

$$4f(0) = \big(f(0)+1\big)^2,$$

which yields $f(0) = 1$.

Step 3: We discuss the values of $f(n)$ for $n < 0$.

Lemma. For every $n \geqslant 1$, we have $f(-n) = -n + 1$ or $f(-n) = n + 1$. Moreover, if $f(-n) = -n + 1$ for some $n \geqslant 1$, then also $f(-n+1) = -n + 2$.

Proof. We prove this statement by strong induction on n. For $n = 1$, we get

$$1 + 4f(-1) = f(f(-1))^2.$$

Thus $f(-1)$ needs to be nonnegative. If $f(-1) = 0$, then $f(f(-1)) = f(0) = \pm 1$, so $f(0) = 1$ (by our second step). Otherwise, we know that $f(f(-1)) = f(-1) + 1$, so

$$1 + 4f(-1) = \big(f(-1) + 1\big)^2,$$

which yields $f(-1) = 2$ and thus establishes the base case. For the induction step, we consider two cases:

- If $f(-n) \leqslant -n$, then

  $$f(f(-n))^2 = (-n)^2 + 4f(-n) \leqslant n^2 - 4n < (n-2)^2,$$

  so $|f(f(-n))| \leqslant n - 3$ (for $n = 2$, this case cannot even occur). If $f(f(-n)) \geqslant 0$, then we already know from the first two steps that $f(f(f(-n))) = f(f(-n)) + 1$, unless perhaps if $f(0) = 0$ and $f(f(-n)) = 0$. However, the latter would imply $f(-n) = 0$ (as shown in Step 2) and thus $n = 0$, which is impossible. If $f(f(-n)) < 0$, we can apply the induction hypothesis to $f(f(-n))$. In either case, $f(f(f(-n))) = \pm f(f(-n)) + 1$. Therefore,

  $$f(-n)^2 + 4f(f(-n)) = f(f(f(-n)))^2 = \big(\pm f(f(-n)) + 1\big)^2,$$

  which gives us

  $$\begin{aligned} n^2 \leqslant f(-n)^2 &= \big(\pm f(f(-n)) + 1\big)^2 - 4f(f(-n)) \leqslant f(f(-n))^2 + 6|f(f(-n))| + 1 \\ &\leqslant (n-3)^2 + 6(n-3) + 1 = n^2 - 8, \end{aligned}$$

  a contradiction.

- Thus, we are left with the case that $f(-n) > -n$. Now we argue as in the previous case: if $f(-n) \geqslant 0$, then $f(f(-n)) = f(-n) + 1$ by the first two steps, since $f(0) = 0$ and $f(-n) = 0$ would imply $n = 0$ (as seen in Step 2) and is thus impossible. If $f(-n) < 0$, we can apply the induction hypothesis, so in any case we can infer that $f(f(-n)) = \pm f(-n) + 1$. We obtain

  $$(-n)^2 + 4f(-n) = \big(\pm f(-n) + 1\big)^2,$$

  so either

  $$n^2 = f(-n)^2 - 2f(-n) + 1 = \big(f(-n) - 1\big)^2,$$

  which gives us $f(-n) = \pm n + 1$, or

  $$n^2 = f(-n)^2 - 6f(-n) + 1 = \big(f(-n) - 3\big)^2 - 8.$$

  Since 1 and 9 are the only perfect squares whose difference is 8, we must have $n = 1$, which we have already considered.

Finally, suppose that $f(-n) = -n + 1$ for some $n \geqslant 2$. Then

$$f(-n+1)^2 = f(f(-n))^2 = (-n)^2 + 4f(-n) = (n-2)^2,$$

so $f(-n+1) = \pm(n-2)$. However, we already know that $f(-n+1) = -n+2$ or $f(-n+1) = n$, so $f(-n+1) = -n+2$. □

Combining everything we know, we find the solutions as stated in the answer:

- One solution is given by $f(n) = n + 1$ for all n.
- If $f(n)$ is not always equal to $n + 1$, then there is a largest integer m (which cannot be positive) for which this is not the case. In view of the lemma that we proved, we must then have $f(n) = -n+1$ for any integer $n < m$. If $m = -a < 0$, we obtain $f(n) = -n+1$ for $n \leqslant -a$ (and $f(n) = n + 1$ otherwise). If $m = 0$, we have the additional possibility that $f(0) = 0$, $f(n) = -n + 1$ for negative n and $f(n) = n + 1$ for positive n.

Solution 2. Let us provide an alternative proof for Part II, which also proceeds in several steps.

Step 1. Let a be an arbitrary integer and $b = f(a)$. We first concentrate on the case where $|a|$ is sufficiently large.

1. If $b = 0$, then (1) applied to a yields $a^2 = f(f(a))^2$, thus
$$f(a) = 0 \quad \Rightarrow \quad a = \pm f(0). \tag{2}$$
From now on, we set $D = |f(0)|$. Throughout Step 1, we will assume that $a \notin \{-D, 0, D\}$, thus $b \neq 0$.

2. From (1), noticing that $f(f(a))$ and a have the same parity, we get
$$0 \neq 4|b| = \left|f(f(a))^2 - a^2\right| \geqslant a^2 - \big(|a| - 2\big)^2 = 4|a| - 4.$$
Hence we have
$$|b| = |f(a)| \geqslant |a| - 1 \qquad \text{for } a \notin \{-D, 0, D\}. \tag{3}$$
For the rest of Step 1, we also assume that $|a| \geqslant E = \max\{D + 2, 10\}$. Then by (3) we have $|b| \geqslant D + 1$ and thus $|f(b)| \geqslant D$.

3. Set $c = f(b)$; by (3), we have $|c| \geqslant |b| - 1$. Thus (1) yields
$$a^2 + 4b = c^2 \geqslant \big(|b| - 1\big)^2,$$
which implies
$$a^2 \geqslant \big(|b| - 1\big)^2 - 4|b| = \big(|b| - 3\big)^2 - 8 > \big(|b| - 4\big)^2$$
because $|b| \geqslant |a| - 1 \geqslant 9$. Thus (3) can be refined to
$$|a| + 3 \geqslant |f(a)| \geqslant |a| - 1 \qquad \text{for } |a| \geqslant E.$$
Now, from $c^2 = a^2 + 4b$ with $|b| \in \big[|a| - 1, |a| + 3\big]$ we get $c^2 = (a \pm 2)^2 + d$, where $d \in \{-16, -12, -8, -4, 0, 4, 8\}$. Since $|a \pm 2| \geqslant 8$, this can happen only if $c^2 = (a \pm 2)^2$, which in turn yields $b = \pm a + 1$. To summarise,
$$f(a) = 1 \pm a \qquad \text{for } |a| \geqslant E. \tag{4}$$

We have shown that, with at most finitely many exceptions, $f(a) = 1 \pm a$. Thus it will be convenient for our second step to introduce the sets
$$Z_+ = \big\{a \in \mathbb{Z}\colon f(a) = a + 1\big\}, \quad Z_- = \big\{a \in \mathbb{Z}\colon f(a) = 1 - a\big\}, \quad \text{and} \quad Z_0 = \mathbb{Z} \backslash \big(Z_+ \cup Z_-\big).$$

Step 2. Now we investigate the structure of the sets Z_+, Z_-, and Z_0.

4. Note that $f(E+1) = 1 \pm (E+1)$. If $f(E+1) = E+2$, then $E+1 \in Z_+$. Otherwise we have $f(1+E) = -E$; then the original equation (1) with $n = E+1$ gives us $(E-1)^2 = f(-E)^2$, so $f(-E) = \pm(E-1)$. By (4) this may happen only if $f(-E) = 1-E$, so in this case $-E \in Z_+$. In any case we find that $Z_+ \neq \varnothing$.

5. Now take any $a \in Z_+$. We claim that every integer $x \geqslant a$ also lies in Z_+. We proceed by induction on x, the base case $x = a$ being covered by our assumption. For the induction step, assume that $f(x-1) = x$ and plug $n = x-1$ into (1). We get $f(x)^2 = (x+1)^2$, so either $f(x) = x+1$ or $f(x) = -(x+1)$.

   Assume that $f(x) = -(x+1)$ and $x \neq -1$, since otherwise we already have $f(x) = x+1$. Plugging $n = x$ into (1), we obtain $f(-x-1)^2 = (x-2)^2 - 8$, which may happen only if $x-2 = \pm 3$ and $f(-x-1) = \pm 1$. Plugging $n = -x-1$ into (1), we get $f(\pm 1)^2 = (x+1)^2 \pm 4$, which in turn may happen only if $x+1 \in \{-2, 0, 2\}$.

   Thus $x \in \{-1, 5\}$ and at the same time $x \in \{-3, -1, 1\}$, which gives us $x = -1$. Since this has already been excluded, we must have $f(x) = x+1$, which completes our induction.

6. Now we know that either $Z_+ = \mathbb{Z}$ (if Z_+ is not bounded below), or $Z_+ = \{a \in \mathbb{Z} \colon a \geqslant a_0\}$, where a_0 is the smallest element of Z_+. In the former case, $f(n) = n+1$ for all $n \in \mathbb{Z}$, which is our first solution. So we assume in the following that Z_+ is bounded below and has a smallest element a_0.

   If $Z_0 = \varnothing$, then we have $f(x) = x+1$ for $x \geqslant a_0$ and $f(x) = 1-x$ for $x < a_0$. In particular, $f(0) = 1$ in any case, so $0 \in Z_+$ and thus $a_0 \leqslant 0$. Thus we end up with the second solution listed in the answer. It remains to consider the case where $Z_0 \neq \varnothing$.

7. Assume that there exists some $a \in Z_0$ with $b = f(a) \notin Z_0$, so that $f(b) = 1 \pm b$. Then we have $a^2 + 4b = (1 \pm b)^2$, so either $a^2 = (b-1)^2$ or $a^2 = (b-3)^2 - 8$. In the former case we have $b = 1 \pm a$, which is impossible by our choice of a. So we get $a^2 = (b-3)^2 - 8$, which implies $f(b) = 1-b$ and $|a| = 1$, $|b-3| = 3$.

   If $b = 0$, then we have $f(b) = 1$, so $b \in Z_+$ and therefore $a_0 \leqslant 0$; hence $a = -1$. But then $f(a) = 0 = a+1$, so $a \in Z_+$, which is impossible.

   If $b = 6$, then we have $f(6) = -5$, so $f(-5)^2 = 16$ and $f(-5) \in \{-4, 4\}$. Then $f(f(-5))^2 = 25 + 4f(-5) \in \{9, 41\}$, so $f(-5) = -4$ and $-5 \in Z_+$. This implies $a_0 \leqslant -5$, which contradicts our assumption that $\pm 1 = a \notin Z_+$.

8. Thus we have shown that $f(Z_0) \subseteq Z_0$, and Z_0 is finite. Take any element $c \in Z_0$, and consider the sequence defined by $c_i = f^i(c)$. All elements of the sequence (c_i) lie in Z_0, hence it is bounded. Choose an index k for which $|c_k|$ is maximal, so that in particular $|c_{k+1}| \leqslant |c_k|$ and $|c_{k+2}| \leqslant |c_k|$. Our functional equation (1) yields

$$(|c_k| - 2)^2 - 4 = |c_k|^2 - 4|c_k| \leqslant c_k^2 + 4c_{k+1} = c_{k+2}^2.$$

   Since c_k and c_{k+2} have the same parity and $|c_{k+2}| \leqslant |c_k|$, this leaves us with three possibilities: $|c_{k+2}| = |c_k|$, $|c_{k+2}| = |c_k| - 2$, and $|c_k| - 2 = \pm 2$, $c_{k+2} = 0$.

   If $|c_{k+2}| = |c_k| - 2$, then $f(c_k) = c_{k+1} = 1 - |c_k|$, which means that $c_k \in Z_-$ or $c_k \in Z_+$, and we reach a contradiction.

   If $|c_{k+2}| = |c_k|$, then $c_{k+1} = 0$, thus $c_{k+3}^2 = 4c_{k+2}$. So either $c_{k+3} \neq 0$ or (by maximality of $|c_{k+2}| = |c_k|$) $c_i = 0$ for all i. In the former case, we can repeat the entire argument with c_{k+2} in the place of c_k. Now $|c_{k+4}| = |c_{k+2}|$ is not possible any more since $c_{k+3} \neq 0$, leaving us with the only possibility $|c_k| - 2 = |c_{k+2}| - 2 = \pm 2$.

Thus we know now that either all c_i are equal to 0, or $|c_k| = 4$. If $c_k = \pm 4$, then either $c_{k+1} = 0$ and $|c_{k+2}| = |c_k| = 4$, or $c_{k+2} = 0$ and $c_{k+1} = -4$. From this point onwards, all elements of the sequence are either 0 or ± 4.

Let c_r be the last element of the sequence that is not equal to 0 or ± 4 (if such an element exists). Then $c_{r+1}, c_{r+2} \in \{-4, 0, 4\}$, so

$$c_r^2 = c_{r+2}^2 - 4c_{r+1} \in \{-16, 0, 16, 32\},$$

which gives us a contradiction. Thus all elements of the sequence are equal to 0 or ± 4, and since the choice of $c_0 = c$ was arbitrary, $Z_0 \subseteq \{-4, 0, 4\}$.

9. Finally, we show that $4 \notin Z_0$ and $-4 \notin Z_0$. Suppose that $4 \in Z_0$. Then in particular a_0 (the smallest element of Z_+) cannot be less than 4, since this would imply $4 \in Z_+$. So $-3 \in Z_-$, which means that $f(-3) = 4$. Then $25 = (-3)^2 + 4f(-3) = f(f(-3))^2 = f(4)^2$, so $f(4) = \pm 5 \notin Z_0$, and we reach a contradiction.

   Suppose that $-4 \in Z_0$. The only possible values for $f(-4)$ that are left are 0 and -4. Note that $4f(0) = f(f(0))^2$, so $f(0) \geqslant 0$. If $f(-4) = 0$, then we get $16 = (-4)^2 + 0 = f(0)^2$, thus $f(0) = 4$. But then $f(f(-4)) \notin Z_0$, which is impossible. Thus $f(-4) = -4$, which gives us $0 = (-4)^2 + 4f(-4) = f(f(-4))^2 = 16$, and this is clearly absurd.

   Now we are left with $Z_0 = \{0\}$ and $f(0) = 0$ as the only possibility. If $1 \in Z_-$, then $f(1) = 0$, so $1 = 1^2 + 4f(1) = f(f(1))^2 = f(0)^2 = 0$, which is another contradiction. Thus $1 \in Z_+$, meaning that $a_0 \leqslant 1$. On the other hand, $a_0 \leqslant 0$ would imply $0 \in Z_+$, so we can only have $a_0 = 1$. Thus Z_+ comprises all positive integers, and Z_- comprises all negative integers. This gives us the third solution.

Comment. All solutions known to the Problem Selection Committee are quite lengthy and technical, as the two solutions presented here show. It is possible to make the problem easier by imposing additional assumptions, such as $f(0) \neq 0$ or $f(n) \geqslant 1$ for all $n \geqslant 0$, to remove some of the technicalities.

Combinatorics

C1. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

(Serbia)

Solution 1. Let k be the number of rectangles in the dissection. The set of all points that are corners of one of the rectangles can be divided into three disjoint subsets:

- A, which consists of the four corners of the original rectangle R, each of which is the corner of exactly one of the smaller rectangles,
- B, which contains points where exactly two of the rectangles have a common corner (T-junctions, see the figure below),
- C, which contains points where four of the rectangles have a common corner (crossings, see the figure below).

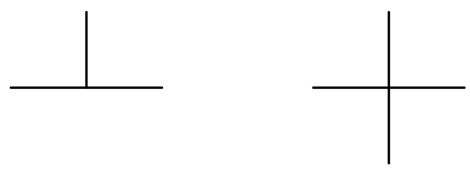


Figure 1: A T-junction and a crossing

We denote the number of points in B by b and the number of points in C by c. Since each of the k rectangles has exactly four corners, we get

$$4k = 4 + 2b + 4c.$$

It follows that $2b \leqslant 4k-4$, so $b \leqslant 2k-2$.

Each of the n given points has to lie on a side of one of the smaller rectangles (but not of the original rectangle R). If we extend this side as far as possible along borders between rectangles, we obtain a line segment whose ends are T-junctions. Note that every point in B can only be an endpoint of at most one such segment containing one of the given points, since it is stated that no two of them lie on a common line parallel to the sides of R. This means that

$$b \geqslant 2n.$$

Combining our two inequalities for b, we get

$$2k-2 \geqslant b \geqslant 2n,$$

thus $k \geqslant n+1$, which is what we wanted to prove.

Solution 2. Let k denote the number of rectangles. In the following, we refer to the directions of the sides of R as 'horizontal' and 'vertical' respectively. Our goal is to prove the inequality $k \geqslant n+1$ for fixed n. Equivalently, we can prove the inequality $n \leqslant k-1$ for each k, which will be done by induction on k. For $k=1$, the statement is trivial.

Now assume that $k>1$. If none of the line segments that form the borders between the rectangles is horizontal, then we have $k-1$ vertical segments dividing R into k rectangles. On each of them, there can only be one of the n points, so $n \leqslant k-1$, which is exactly what we want to prove.

Otherwise, consider the lowest horizontal line h that contains one or more of these line segments. Let R' be the rectangle that results when everything that lies below h is removed from R (see the example in the figure below).

The rectangles that lie entirely below h form blocks of rectangles separated by vertical line segments. Suppose there are r blocks and k_i rectangles in the i^{th} block. The left and right border of each block has to extend further upwards beyond h. Thus we can move any points that lie on these borders upwards, so that they now lie inside R'. This can be done without violating the conditions, one only needs to make sure that they do not get to lie on a common horizontal line with one of the other given points.

All other borders between rectangles in the i^{th} block have to lie entirely below h. There are k_i-1 such line segments, each of which can contain at most one of the given points. Finally, there can be one point that lies on h. All other points have to lie in R' (after moving some of them as explained in the previous paragraph).

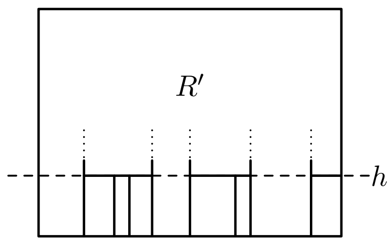


Figure 2: Illustration of the inductive argument

We see that R' is divided into $k-\sum_{i=1}^{r} k_i$ rectangles. Applying the induction hypothesis to R', we find that there are at most

$$\left(k-\sum_{i=1}^{r} k_i\right)-1+\sum_{i=1}^{r}(k_i-1)+1=k-r$$

points. Since $r \geqslant 1$, this means that $n \leqslant k-1$, which completes our induction.

C2. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

(Iran)

Solution. Let P_k be the product of the numbers on the sheets after k steps.

Suppose that in the $(k+1)^{\text{th}}$ step the numbers a and b are replaced by $a+b$. In the product, the number ab is replaced by $(a+b)^2$, and the other factors do not change. Since $(a+b)^2 \geqslant 4ab$, we see that $P_{k+1} \geqslant 4P_k$. Starting with $P_0 = 1$, a straightforward induction yields

$$P_k \geqslant 4^k$$

for all integers $k \geqslant 0$; in particular

$$P_{m\cdot 2^{m-1}} \geqslant 4^{m\cdot 2^{m-1}} = (2^m)^{2^m},$$

so by the AM–GM inequality, the sum of the numbers written on the sheets after $m2^{m-1}$ steps is at least

$$2^m \cdot \sqrt[2^m]{P_{m\cdot 2^{m-1}}} \geqslant 2^m \cdot 2^m = 4^m.$$

Comment 1. It is possible to achieve the sum 4^m in $m2^{m-1}$ steps. For example, starting from 2^m equal numbers on the sheets, in 2^{m-1} consecutive steps we can double all numbers. After m such doubling rounds we have the number 2^m on every sheet.

Comment 2. There are several versions of the solution above. E.g., one may try to assign to each positive integer n a *weight* w_n in such a way that the sum of the weights of the numbers written on the sheets increases, say, by at least 2 in each step. For this purpose, one needs the inequality

$$2w_{a+b} \geqslant w_a + w_b + 2 \tag{1}$$

to be satisfied for all positive integers a and b.

Starting from $w_1 = 1$ and trying to choose the weights as small as possible, one may find that these weights can be defined as follows: For every positive integer n, one chooses k to be the maximal integer such that $n \geqslant 2^k$, and puts

$$w_n = k + \frac{n}{2^k} = \min_{d\in\mathbb{Z}_{\geqslant 0}} \left(d + \frac{n}{2^d}\right). \tag{2}$$

Now, in order to prove that these weights satisfy (1), one may take arbitrary positive integers a and b, and choose an integer $d \geqslant 0$ such that $w_{a+b} = d + \frac{a+b}{2^d}$. Then one has

$$2w_{a+b} = 2d + 2\cdot\frac{a+b}{2^d} = \left((d-1) + \frac{a}{2^{d-1}}\right) + \left((d-1) + \frac{b}{2^{d-1}}\right) + 2 \geqslant w_a + w_b + 2.$$

Since the initial sum of the weights was 2^m, after $m2^{m-1}$ steps the sum is at least $(m+1)2^m$. To finish the solution, one may notice that by (2) for every positive integer a one has

$$w_a \leqslant m + \frac{a}{2^m}, \qquad \text{i.e.,} \quad a \geqslant 2^m(-m + w_a). \tag{3}$$

So the sum of the numbers $a_1, a_2, \ldots, a_{2^m}$ on the sheets can be estimated as

$$\sum_{i=1}^{2^m} a_i \geqslant \sum_{i=1}^{2^m} 2^m(-m + w_{a_i}) = -m2^m \cdot 2^m + 2^m \sum_{i=1}^{2^m} w_{a_i} \geqslant -m4^m + (m+1)4^m = 4^m,$$

as required.

For establishing the inequalities (1) and (3), one may also use the convexity argument, instead of the second definition of w_n in (2).

One may check that $\log_2 n \leqslant w_n \leqslant \log_2 n + 1$; thus, in some rough sense, this approach is obtained by "taking the logarithm" of the solution above.

Comment 3. An intuitive strategy to minimise the sum of numbers is that in every step we choose the two smallest numbers. We may call this the *greedy strategy.* In the following paragraphs we prove that the greedy strategy indeed provides the least possible sum of numbers.

Claim. Starting from any sequence $x_1, \ldots, x_N$ of positive real numbers on N sheets, for any number k of steps, the greedy strategy achieves the lowest possible sum of numbers.

Proof. We apply induction on k; for $k = 1$ the statement is obvious. Let $k \geqslant 2$, and assume that the claim is true for smaller values.

Every sequence of k steps can be encoded as $S = \big((i_1, j_1), \ldots, (i_k, j_k)\big)$, where, for $r = 1, 2, \ldots, k$, the numbers i_r and j_r are the indices of the two sheets that are chosen in the r^{th} step. The resulting final sum will be some linear combination of $x_1, \ldots, x_N$, say, $c_1 x_1 + \cdots + c_N x_N$ with positive integers $c_1, \ldots, c_N$ that depend on S only. Call the numbers $(c_1, \ldots, c_N)$ the *characteristic vector* of S.

Choose a sequence $S_0 = \big((i_1, j_1), \ldots, (i_k, j_k)\big)$ of steps that produces the minimal sum, starting from $x_1, \ldots, x_N$, and let $(c_1, \ldots, c_N)$ be the characteristic vector of S. We may assume that the sheets are indexed in such an order that $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_N$. If the sheets (and the numbers) are permuted by a permutation π of the indices $(1, 2, \ldots, N)$ and then the same steps are performed, we can obtain the sum $\sum\limits_{t=1}^{N} c_t x_{\pi(t)}$. By the rearrangement inequality, the smallest possible sum can be achieved when the numbers $(x_1, \ldots, x_N)$ are in non-decreasing order. So we can assume that also $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N$.

Let ℓ be the largest index with $c_1 = \cdots = c_\ell$, and let the r^{th} step be the first step for which $c_{i_r} = c_1$ or $c_{j_r} = c_1$. The role of i_r and j_r is symmetrical, so we can assume $c_{i_r} = c_1$ and thus $i_r \leqslant \ell$. We show that $c_{j_r} = c_1$ and $j_r \leqslant \ell$ hold, too.

Before the r^{th} step, on the ${i_r}^{\text{th}}$ sheet we had the number x_{i_r}. On the ${j_r}^{\text{th}}$ sheet there was a linear combination that contains the number x_{j_r} with a positive integer coefficient, and possibly some other terms. In the r^{th} step, the number x_{i_r} joins that linear combination. From this point, each sheet contains a linear combination of $x_1, \ldots, x_N$, with the coefficient of x_{j_r} being not smaller than the coefficient of x_{i_r}. This is preserved to the end of the procedure, so we have $c_{j_r} \geqslant c_{i_r}$. But $c_{i_r} = c_1$ is maximal among the coefficients, so we have $c_{j_r} = c_{i_r} = c_1$ and thus $j_r \leqslant \ell$.

Either from $c_{j_r} = c_{i_r} = c_1$ or from the arguments in the previous paragraph we can see that none of the ${i_r}^{\text{th}}$ and the ${j_r}^{\text{th}}$ sheets were used before step r. Therefore, the final linear combination of the numbers does not change if the step (i_r, j_r) is performed first: the sequence of steps

$$S_1 = \big((i_r, j_r), (i_1, j_1), \ldots, (i_{r-1}, j_{r-1}), (i_{r+1}, j_{r+1}), \ldots, (i_N, j_N)\big)$$

also produces the same minimal sum at the end. Therefore, we can replace S_0 by S_1 and we may assume that $r = 1$ and $c_{i_1} = c_{j_1} = c_1$.

As $i_1 \neq j_1$, we can see that $\ell \geqslant 2$ and $c_1 = c_2 = c_{i_1} = c_{j_1}$. Let π be such a permutation of the indices $(1, 2, \ldots, N)$ that exchanges $1, 2$ with i_r, j_r and does not change the remaining indices. Let

$$S_2 = \big((\pi(i_1), \pi(j_1)), \ldots, (\pi(i_N), \pi(j_N))\big).$$

Since $c_{\pi(i)} = c_i$ for all indices i, this sequence of steps produces the same, minimal sum. Moreover, in the first step we chose $x_{\pi(i_1)} = x_1$ and $x_{\pi(j_1)} = x_2$, the two smallest numbers.

Hence, it is possible to achieve the optimal sum if we follow the greedy strategy in the first step. By the induction hypothesis, following the greedy strategy in the remaining steps we achieve the optimal sum.

C3. Let $n \geqslant 2$ be an integer. Consider an $n \times n$ chessboard divided into n^2 unit squares. We call a configuration of n rooks on this board *happy* if every row and every column contains exactly one rook. Find the greatest positive integer k such that for every happy configuration of rooks, we can find a $k \times k$ square without a rook on any of its k^2 unit squares.

(Croatia)

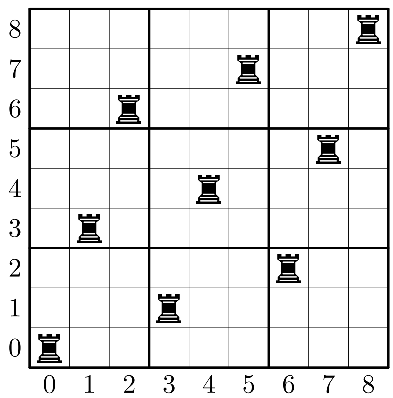
Answer. $\lfloor \sqrt{n-1} \rfloor$.

Solution. Let ℓ be a positive integer. We will show that **(i)** if $n > \ell^2$ then each happy configuration contains an empty $\ell \times \ell$ square, but **(ii)** if $n \leqslant \ell^2$ then there exists a happy configuration not containing such a square. These two statements together yield the answer.

(i). Assume that $n > \ell^2$. Consider any happy configuration. There exists a row R containing a rook in its leftmost square. Take ℓ consecutive rows with R being one of them. Their union U contains exactly ℓ rooks. Now remove the $n - \ell^2 \geqslant 1$ leftmost columns from U (thus at least one rook is also removed). The remaining part is an $\ell^2 \times \ell$ rectangle, so it can be split into ℓ squares of size $\ell \times \ell$, and this part contains at most $\ell - 1$ rooks. Thus one of these squares is empty.

(ii). Now we assume that $n \leqslant \ell^2$. Firstly, we will construct a happy configuration with no empty $\ell \times \ell$ square for the case $n = \ell^2$. After that we will modify it to work for smaller values of n.

Let us enumerate the rows from bottom to top as well as the columns from left to right by the numbers $0, 1, \ldots, \ell^2 - 1$. Every square will be denoted, as usual, by the pair (r, c) of its row and column numbers. Now we put the rooks on all squares of the form $(i\ell + j, j\ell + i)$ with $i, j = 0, 1, \ldots, \ell - 1$ (the picture below represents this arrangement for $\ell = 3$). Since each number from 0 to $\ell^2 - 1$ has a unique representation of the form $i\ell + j$ $(0 \leqslant i, j \leqslant \ell - 1)$, each row and each column contains exactly one rook.



Next, we show that each $\ell \times \ell$ square A on the board contains a rook. Consider such a square A, and consider ℓ consecutive rows the union of which contains A. Let the lowest of these rows have number $p\ell + q$ with $0 \leqslant p, q \leqslant \ell - 1$ (notice that $p\ell + q \leqslant \ell^2 - \ell$). Then the rooks in this union are placed in the columns with numbers $q\ell + p$, $(q+1)\ell + p$, $\ldots$, $(\ell - 1)\ell + p$, $p + 1$, $\ell + (p+1)$, $\ldots$, $(q-1)\ell + p + 1$, or, putting these numbers in increasing order,

$$p + 1,\ \ell + (p+1),\ \ldots,\ (q-1)\ell + (p+1),\ q\ell + p,\ (q+1)\ell + p,\ \ldots,\ (\ell - 1)\ell + p.$$

One readily checks that the first number in this list is at most $\ell - 1$ (if $p = \ell - 1$, then $q = 0$, and the first listed number is $q\ell + p = \ell - 1$), the last one is at least $(\ell - 1)\ell$, and the difference between any two consecutive numbers is at most ℓ. Thus, one of the ℓ consecutive columns intersecting A contains a number listed above, and the rook in this column is inside A, as required. The construction for $n = \ell^2$ is established.

It remains to construct a happy configuration of rooks not containing an empty $\ell \times \ell$ square for $n < \ell^2$. In order to achieve this, take the construction for an $\ell^2 \times \ell^2$ square described above and remove the $\ell^2 - n$ bottom rows together with the $\ell^2 - n$ rightmost columns. We will have a rook arrangement with no empty $\ell \times \ell$ square, but several rows and columns may happen to be empty. Clearly, the number of empty rows is equal to the number of empty columns, so one can find a bijection between them, and put a rook on any crossing of an empty row and an empty column corresponding to each other.

Comment. Part **(i)** allows several different proofs. E.g., in the last paragraph of the solution, it suffices to deal only with the case $n = \ell^2 + 1$. Notice now that among the four corner squares, at least one is empty. So the rooks in its row and in its column are distinct. Now, deleting this row and column we obtain an $\ell^2 \times \ell^2$ square with $\ell^2 - 1$ rooks in it. This square can be partitioned into ℓ^2 squares of size $\ell \times \ell$, so one of them is empty.

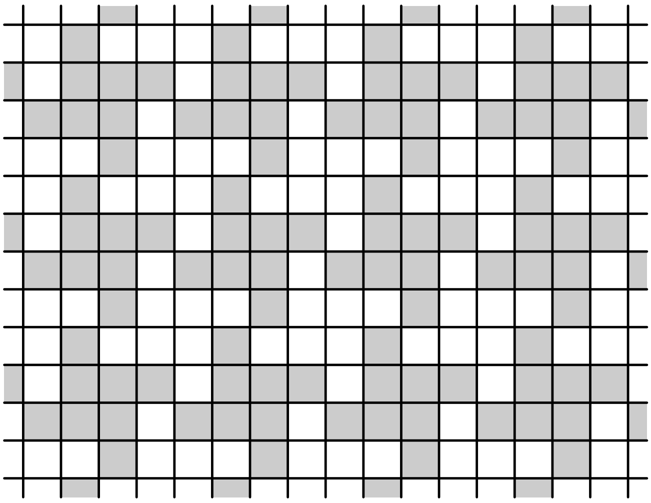
C4. Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively.

S-tetrominoes Z-tetrominoes

Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove than no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

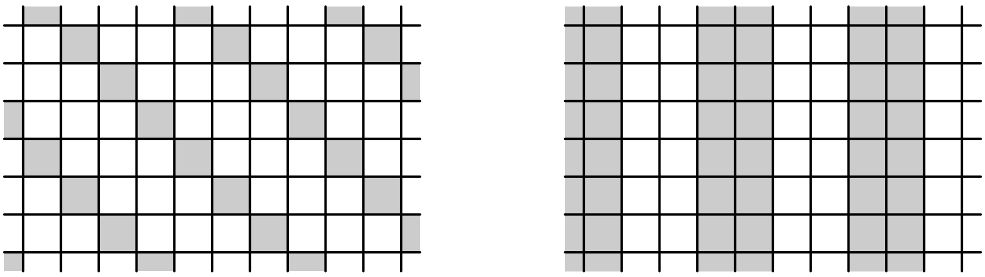
(Hungary)

Solution 1. We may assume that polygon P is the union of some squares of an infinite chessboard. Colour the squares of the chessboard with two colours as the figure below illustrates.



Observe that no matter how we tile P, any S-tetromino covers an even number of black squares, whereas any Z-tetromino covers an odd number of them. As P can be tiled exclusively by S-tetrominoes, it contains an even number of black squares. But if some S-tetrominoes and some Z-tetrominoes cover an even number of black squares, then the number of Z-tetrominoes must be even.

Comment. An alternative approach makes use of the following two colourings, which are perhaps somewhat more natural:



Let s_1 and s_2 be the number of S-tetrominoes of the first and second type (as shown in the figure above) respectively that are used in a tiling of P. Likewise, let z_1 and z_2 be the number of Z-tetrominoes of the first and second type respectively. The first colouring shows that $s_1 + z_2$ is invariant modulo 2, the second colouring shows that $s_1 + z_1$ is invariant modulo 2. Adding these two conditions, we find that $z_1 + z_2$ is invariant modulo 2, which is what we have to prove. Indeed, the sum of the two colourings (regarding white as 0 and black as 1 and adding modulo 2) is the colouring shown in the solution.

Solution 2. Let us assign coordinates to the squares of the infinite chessboard in such a way that the squares of P have nonnegative coordinates only, and that the first coordinate increases as one moves to the right, while the second coordinate increases as one moves upwards. Write the integer $3^i \cdot (-3)^j$ into the square with coordinates (i, j), as in the following figure:

$\vdots$					
81	$\vdots$				
−27	−81	$\vdots$			
9	27	81	$\cdots$		
−3	−9	−27	−81	$\cdots$	
1	3	9	27	81	$\cdots$

The sum of the numbers written into four squares that can be covered by an S-tetromino is either of the form

$$3^i \cdot (-3)^j \cdot \big(1 + 3 + 3 \cdot (-3) + 3^2 \cdot (-3)\big) = -32 \cdot 3^i \cdot (-3)^j$$

(for the first type of S-tetrominoes), or of the form

$$3^i \cdot (-3)^j \cdot \big(3 + 3 \cdot (-3) + (-3) + (-3)^2\big) = 0,$$

and thus divisible by 32. For this reason, the sum of the numbers written into the squares of P, and thus also the sum of the numbers covered by Z-tetrominoes in the second covering, is likewise divisible by 32. Now the sum of the entries of a Z-tetromino is either of the form

$$3^i \cdot (-3)^j \cdot \big(3 + 3^2 + (-3) + 3 \cdot (-3)\big) = 0$$

(for the first type of Z-tetrominoes), or of the form

$$3^i \cdot (-3)^j \cdot \big(1 + (-3) + 3 \cdot (-3) + 3 \cdot (-3)^2\big) = 16 \cdot 3^i \cdot (-3)^j,$$

i.e., 16 times an odd number. Thus in order to obtain a total that is divisible by 32, an even number of the latter kind of Z-tetrominoes needs to be used. Rotating everything by 90°, we find that the number of Z-tetrominoes of the first kind is even as well. So we have even proven slightly more than necessary.

Comment 1. In the second solution, 3 and -3 can be replaced by other combinations as well. For example, for any positive integer $a \equiv 3 \pmod 4$, we can write $a^i \cdot (-a)^j$ into the square with coordinates (i, j) and apply the same argument.

Comment 2. As the second solution shows, we even have the stronger result that the parity of the number of each of the four types of tetrominoes in a tiling of P by S- and Z-tetrominoes is an invariant of P. This also remains true if there is no tiling of P that uses only S-tetrominoes.

C5. Consider $n \geqslant 3$ lines in the plane such that no two lines are parallel and no three have a common point. These lines divide the plane into polygonal regions; let $\mathcal{F}$ be the set of regions having finite area. Prove that it is possible to colour $\lceil \sqrt{n/2} \rceil$ of the lines blue in such a way that no region in $\mathcal{F}$ has a completely blue boundary. (For a real number x, $\lceil x \rceil$ denotes the least integer which is not smaller than x.)

(Austria)

Solution. Let L be the given set of lines. Choose a maximal (by inclusion) subset $B \subseteq L$ such that when we colour the lines of B blue, no region in $\mathcal{F}$ has a completely blue boundary. Let $|B| = k$. We claim that $k \geqslant \lceil \sqrt{n/2} \rceil$.

Let us colour all the lines of $L \backslash B$ red. Call a point *blue* if it is the intersection of two blue lines. Then there are $\binom{k}{2}$ blue points.

Now consider any red line ℓ. By the maximality of B, there exists at least one region $A \in \mathcal{F}$ whose only red side lies on ℓ. Since A has at least three sides, it must have at least one blue vertex. Let us take one such vertex and associate it to ℓ.

Since each blue point belongs to four regions (some of which may be unbounded), it is associated to at most four red lines. Thus the total number of red lines is at most $4\binom{k}{2}$. On the other hand, this number is $n-k$, so

$$n - k \leqslant 2k(k-1), \quad \text{thus} \quad n \leqslant 2k^2 - k \leqslant 2k^2,$$

and finally $k \geqslant \lceil \sqrt{n/2} \rceil$, which gives the desired result.

Comment 1. The constant factor in the estimate can be improved in different ways; we sketch two of them below. On the other hand, the Problem Selection Committee is not aware of any results showing that it is sometimes impossible to colour k lines satisfying the desired condition for $k \gg \sqrt{n}$. In this situation we find it more suitable to keep the original formulation of the problem.

1. Firstly, we show that in the proof above one has in fact $k = |B| \geqslant \lceil \sqrt{2n/3} \rceil$.

Let us make weighted associations as follows. Let a region A whose only red side lies on ℓ have k vertices, so that $k-2$ of them are blue. We associate each of these blue vertices to ℓ, and put the weight $\frac{1}{k-2}$ on each such association. So the sum of the weights of all the associations is exactly $n-k$.

Now, one may check that among the four regions adjacent to a blue vertex v, at most two are triangles. This means that the sum of the weights of all associations involving v is at most $1+1+\frac{1}{2}+\frac{1}{2}=3$. This leads to the estimate

$$n - k \leqslant 3\binom{k}{2},$$

or

$$2n \leqslant 3k^2 - k < 3k^2,$$

which yields $k \geqslant \lceil \sqrt{2n/3} \rceil$.

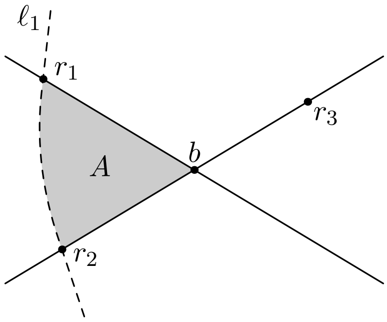
2. Next, we even show that $k = |B| \geqslant \lceil \sqrt{n} \rceil$. For this, we specify the process of associating points to red lines in one more different way.

Call a point *red* if it lies on a red line as well as on a blue line. Consider any red line ℓ, and take an arbitrary region $A \in \mathcal{F}$ whose only red side lies on ℓ. Let r', r, b_1, $\ldots$, b_k be its vertices in clockwise order with $r', r \in \ell$; then the points r', r are red, while all the points b_1, $\ldots$, b_k are blue. Let us associate to ℓ the red point r and the blue point b_1. One may notice that to each pair of a red point r and a blue point b, at most one red line can be associated, since there is at most one region A having r and b as two clockwise consecutive vertices.

We claim now that at most two red lines are associated to each blue point b; this leads to the desired bound

$$n - k \leqslant 2\binom{k}{2} \quad \Longleftrightarrow \quad n \leqslant k^2.$$

Assume, to the contrary, that three red lines ℓ_1, ℓ_2, and ℓ_3 are associated to the same blue point b. Let r_1, r_2, and r_3 respectively be the red points associated to these lines; all these points are distinct. The point b defines four blue rays, and each point r_i is the red point closest to b on one of these rays. So we may assume that the points r_2 and r_3 lie on one blue line passing through b, while r_1 lies on the other one.



Now consider the region A used to associate r_1 and b with ℓ_1. Three of its clockwise consecutive vertices are r_1, b, and either r_2 or r_3 (say, r_2). Since A has only one red side, it can only be the triangle r_1br_2; but then both ℓ_1 and ℓ_2 pass through r_2, as well as some blue line. This is impossible by the problem assumptions.

Comment 2. The condition that the lines be non-parallel is essentially not used in the solution, nor in the previous comment; thus it may be omitted.

C6. We are given an infinite deck of cards, each with a real number on it. For every real number x, there is exactly one card in the deck that has x written on it. Now two players draw disjoint sets A and B of 100 cards each from this deck. We would like to define a rule that declares one of them a winner. This rule should satisfy the following conditions:

1. The winner only depends on the relative order of the 200 cards: if the cards are laid down in increasing order face down and we are told which card belongs to which player, but not what numbers are written on them, we can still decide the winner.
2. If we write the elements of both sets in increasing order as $A = \{a_1, a_2, \ldots, a_{100}\}$ and $B = \{b_1, b_2, \ldots, b_{100}\}$, and $a_i > b_i$ for all i, then A beats B.
3. If three players draw three disjoint sets A, B, C from the deck, A beats B and B beats C, then A also beats C.

How many ways are there to define such a rule? Here, we consider two rules as different if there exist two sets A and B such that A beats B according to one rule, but B beats A according to the other.

(Russia)

Answer. 100.

Solution 1. We prove a more general statement for sets of cardinality n (the problem being the special case $n = 100$, then the answer is n). In the following, we write $A > B$ or $B < A$ for "A beats B".

Part I. Let us first define n different rules that satisfy the conditions. To this end, fix an index $k \in \{1, 2, \ldots, n\}$. We write both A and B in increasing order as $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ and say that A beats B if and only if $a_k > b_k$. This rule clearly satisfies all three conditions, and the rules corresponding to different k are all different. Thus there are at least n different rules.

Part II. Now we have to prove that there is no other way to define such a rule. Suppose that our rule satisfies the conditions, and let $k \in \{1, 2, \ldots, n\}$ be minimal with the property that

$$A_k = \{1, 2, \ldots, k, n+k+1, n+k+2, \ldots, 2n\} \prec B_k = \{k+1, k+2, \ldots, n+k\}.$$

Clearly, such a k exists, since this holds for $k = n$ by assumption. Now consider two disjoint sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, both in increasing order (i.e., $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$). We claim that $X \prec Y$ if (and only if – this follows automatically) $x_k < y_k$.

To prove this statement, pick arbitrary real numbers $u_i, v_i, w_i \notin X \cup Y$ such that

$$u_1 < u_2 < \cdots < u_{k-1} < \min(x_1, y_1), \quad \max(x_n, y_n) < v_{k+1} < v_{k+2} < \cdots < v_n,$$

and

$$x_k \; < \; v_1 < v_2 < \cdots < v_k \; < \; w_1 < w_2 < \cdots < w_n \; < \; u_k < u_{k+1} < \cdots < u_n \; < \; y_k,$$

and set

$$U = \{u_1, u_2, \ldots, u_n\}, \; V = \{v_1, v_2, \ldots, v_n\}, \; W = \{w_1, w_2, \ldots, w_n\}.$$

Then

- $u_i < y_i$ and $x_i < v_i$ for all i, so $U \prec Y$ and $X \prec V$ by the second condition.

- The elements of $U \cup W$ are ordered in the same way as those of $A_{k-1} \cup B_{k-1}$, and since $A_{k-1} \succ B_{k-1}$ by our choice of k, we also have $U \succ W$ (if $k = 1$, this is trivial).
- The elements of $V \cup W$ are ordered in the same way as those of $A_k \cup B_k$, and since $A_k \prec B_k$ by our choice of k, we also have $V \prec W$.

It follows that
$$X \prec V \prec W \prec U \prec Y,$$
so $X \prec Y$ by the third condition, which is what we wanted to prove.

Solution 2. Another possible approach to Part II of this problem is induction on n. For $n = 1$, there is trivially only one rule in view of the second condition.

In the following, we assume that our claim (namely, that there are no possible rules other than those given in Part I) holds for $n-1$ in place of n. We start with the following observation:

Claim. At least one of the two relations
$$\big(\{2\} \cup \{2i-1 \,|\, 2 \leqslant i \leqslant n\}\big) \prec \big(\{1\} \cup \{2i \,|\, 2 \leqslant i \leqslant n\}\big)$$
and
$$\big(\{2i-1 \,|\, 1 \leqslant i \leqslant n-1\} \cup \{2n\}\big) \prec \big(\{2i \,|\, 1 \leqslant i \leqslant n-1\} \cup \{2n-1\}\big)$$
holds.

Proof. Suppose that the first relation does not hold. Since our rule may only depend on the relative order, we must also have
$$\big(\{2\} \cup \{3i-2 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-2\}\big) \succ \big(\{1\} \cup \{3i-1 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n\}\big).$$
Likewise, if the second relation does not hold, then we must also have
$$\big(\{1\} \cup \{3i-1 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n\}\big) \succ \big(\{3\} \cup \{3i \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-1\}\big).$$
Now condition 3 implies that
$$\big(\{2\} \cup \{3i-2 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-2\}\big) \succ \big(\{3\} \cup \{3i \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-1\}\big),$$
which contradicts the second condition. □

Now we distinguish two cases, depending on which of the two relations actually holds:

First case: $\big(\{2\} \cup \{2i-1 \,|\, 2 \leqslant i \leqslant n\}\big) \prec \big(\{1\} \cup \{2i \,|\, 2 \leqslant i \leqslant n\}\big)$.

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two disjoint sets, both in increasing order. We claim that the winner can be decided only from the values of $a_2, \ldots, a_n$ and $b_2, \ldots, b_n$, while a_1 and b_1 are actually irrelevant. Suppose that this was not the case, and assume without loss of generality that $a_2 < b_2$. Then the relative order of $a_1, a_2, \ldots, a_n, b_2, \ldots, b_n$ is fixed, and the position of b_1 has to decide the winner. Suppose that for some value $b_1 = x$, B wins, while for some other value $b_1 = y$, A wins.

Write $B_x = \{x, b_2, \ldots, b_n\}$ and $B_y = \{y, b_2, \ldots, b_n\}$, and let $\varepsilon > 0$ be smaller than half the distance between any two of the numbers in $B_x \cup B_y \cup A$. For any set M, let $M \pm \varepsilon$ be the set obtained by adding/subtracting ε to all elements of M. By our choice of ε, the relative order of the elements of $(B_y + \varepsilon) \cup A$ is still the same as for $B_y \cup A$, while the relative order of the elements of $(B_x - \varepsilon) \cup A$ is still the same as for $B_x \cup A$. Thus $A \prec B_x - \varepsilon$, but $A \succ B_y + \varepsilon$. Moreover, if $y > x$, then $B_x - \varepsilon \prec B_y + \varepsilon$ by condition 2, while otherwise the relative order of the elements in $(B_x - \varepsilon) \cup (B_y + \varepsilon)$ is the same as for the two sets $\{2\} \cup \{2i-1 \mid 2 \leqslant i \leqslant n\}$ and $\{1\} \cup \{2i \mid 2 \leqslant i \leqslant n\}$, so that $B_x - \varepsilon \prec B_y + \varepsilon$. In either case, we obtain

$$A \prec B_x - \varepsilon \prec B_y + \varepsilon \prec A,$$

which contradicts condition 3.

So we know now that the winner does not depend on a_1, b_1. Therefore, we can define a new rule $\prec^*$ on sets of cardinality $n-1$ by saying that $A \prec^* B$ if and only if $A \cup \{a\} \prec B \cup \{b\}$ for some a, b (or equivalently, all a, b) such that $a < \min A$, $b < \min B$ and $A \cup \{a\}$ and $B \cup \{b\}$ are disjoint. The rule $\prec^*$ satisfies all conditions again, so by the induction hypothesis, there exists an index i such that $A \prec^* B$ if and only if the i^{th} smallest element of A is less than the i^{th} smallest element of B. This implies that $C \prec D$ if and only if the $(i+1)^{\text{th}}$ smallest element of C is less than the $(i+1)^{\text{th}}$ smallest element of D, which completes our induction.

Second case: $\big(\{2i-1 \mid 1 \leqslant i \leqslant n-1\} \cup \{2n\}\big) \prec \big(\{2i \mid 1 \leqslant i \leqslant n-1\} \cup \{2n-1\}\big)$.
Set $-A = \{-a \mid a \in A\}$ for any $A \subseteq \mathbb{R}$. For any two disjoint sets $A, B \subseteq \mathbb{R}$ of cardinality n, we write $A \prec^\circ B$ to mean $(-B) \prec (-A)$. It is easy to see that $\prec^\circ$ defines a rule to determine a winner that satisfies the three conditions of our problem as well as the relation of the first case. So it follows in the same way as in the first case that for some i, $A \prec^\circ B$ if and only if the i^{th} smallest element of A is less than the i^{th} smallest element of B, which is equivalent to the condition that the i^{th} largest element of $-A$ is greater than the i^{th} largest element of $-B$. This proves that the original rule $\prec$ also has the desired form.

Comment. The problem asks for all possible partial orders on the set of n-element subsets of $\mathbb{R}$ such that any two disjoint sets are comparable, the order relation only depends on the relative order of the elements, and $\{a_1, a_2, \ldots, a_n\} \prec \{b_1, b_2, \ldots, b_n\}$ whenever $a_i < b_i$ for all i.

As the proposer points out, one may also ask for all **total** orders on all n-element subsets of $\mathbb{R}$ (dropping the condition of disjointness and requiring that $\{a_1, a_2, \ldots, a_n\} \preceq \{b_1, b_2, \ldots, b_n\}$ whenever $a_i \leqslant b_i$ for all i). It turns out that the number of possibilities in this case is $n!$, and all possible total orders are obtained in the following way. Fix a permutation $\sigma \in S_n$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two subsets of $\mathbb{R}$ with $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$. Then we say that $A \succ_\sigma B$ if and only if $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ is lexicographically greater than $(b_{\sigma(1)}, \ldots, b_{\sigma(n)})$.

It seems, however, that this formulation adds rather more technicalities to the problem than additional ideas.

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C7. Let M be a set of $n \geqslant 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

(Russia)

Solution. A line is said to be *red* if it contains two points of M. As no three points of M are collinear, each red line determines a unique pair of points of M. Moreover, there are precisely $\binom{n}{2} < \frac{n^2}{2}$ red lines. By the *value of a segment* we mean the number of red lines intersecting it in its interior, and the *value of a set of segments* is defined to be the sum of the values of its elements. We will prove that (i) the value of the initial set of segments is smaller than $n^3/2$ and that (ii) each step decreases the value of the set of segments present by at least 2. Since such a value can never be negative, these two assertions imply the statement of the problem.

To show (i) we just need to observe that each segment has a value that is smaller than $n^2/2$. Thus the combined value of the n initial segments is indeed below $n \cdot n^2/2 = n^3/2$.

It remains to establish (ii). Suppose that at some moment we have two segments AB and CD sharing an interior point S, and that at the next moment we have the two segments AC and BD instead. Let X_{AB} denote the set of red lines intersecting the segment AB in its interior and let the sets X_{AC}, X_{BD}, and X_{CD} be defined similarly. We are to prove that $|X_{AC}| + |X_{BD}| + 2 \leqslant |X_{AB}| + |X_{CD}|$.

As a first step in this direction, we claim that

$$|X_{AC} \cup X_{BD}| + 2 \leqslant |X_{AB} \cup X_{CD}| . \tag{1}$$

Indeed, if g is a red line intersecting, e.g. the segment AC in its interior, then it has to intersect the triangle ACS once again, either in the interior of its side AS, or in the interior of its side CS, or at S, meaning that it belongs to X_{AB} or to X_{CD} (see Figure 1). Moreover, the red lines AB and CD contribute to $X_{AB} \cup X_{CD}$ but not to $X_{AC} \cup X_{BD}$. Thereby (1) is proved.

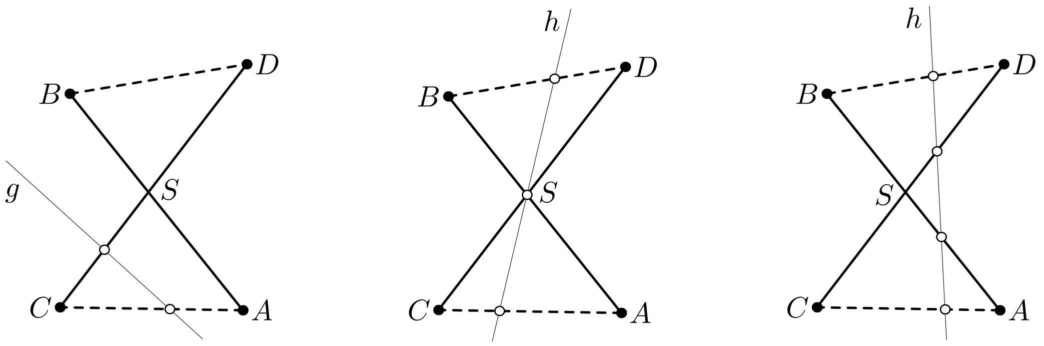


Figure 1 Figure 2 Figure 3

Similarly but more easily one obtains

$$|X_{AC} \cap X_{BD}| \leqslant |X_{AB} \cap X_{CD}| . \tag{2}$$

Indeed, a red line h appearing in $X_{AC} \cap X_{BD}$ belongs, for similar reasons as above, also to $X_{AB} \cap X_{CD}$. To make the argument precise, one may just distinguish the cases $S \in h$ (see Figure 2) and $S \notin h$ (see Figure 3). Thereby (2) is proved.

Adding (1) and (2) we obtain the desired conclusion, thus completing the solution of this problem.

Comment 1. There is a problem belonging to the folklore, in the solution of which one may use the same kind of operation:

Given n red and n green points in the plane, prove that one may draw n nonintersecting segments each of which connects a red point with a green point.

A standard approach to this problem consists in taking n arbitrary segments connecting the red points with the green points, and to perform the same operation as in the above proposal whenever an intersection occurs. Now each time one performs such a step, the total length of the segments that are present decreases due to the triangle inequality. So, as there are only finitely many possibilities for the set of segments present, the process must end at some stage.

In the above proposal, however, considering the sum of the Euclidean lengths of the segment that are present does not seem to help much, for even though it shows that the process must necessarily terminate after some finite number of steps, it does not seem to easily yield any upper bound on the number of these steps that grows polynomially with n.

One may regard the concept of the value of a segment introduced in the above solution as an appropriately discretised version of Euclidean length suitable for obtaining such a bound.

The Problem Selection Committee still believes the problem to be sufficiently original for the competition.

Comment 2. There are some other essentially equivalent ways of presenting the same solution. E.g., put $M = \{A_1, A_2, \ldots, A_n\}$, denote the set of segments present at any moment by $\{e_1, e_2, \ldots, e_n\}$, and called a triple (i, j, k) of indices with $i \neq j$ *intersecting*, if the line A_iA_j intersects the segment e_k. It may then be shown that the number S of intersecting triples satisfies $0 \leqslant S < n^3$ at the beginning and decreases by at least 4 in each step.

Comment 3. It is not difficult to construct an example where cn^2 moves are possible (for some absolute constant $c > 0$). It would be interesting to say more about the gap between cn^2 and cn^3.

C8. A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.

Determine all possible first moves of the first player after which he has a winning strategy.

(Russia)

Answer. All the moves except for taking the empty card.

Solution. Let us identify each card with the set of digits written on it. For any collection of cards $C_1, C_2, \ldots, C_k$ denote by their *sum* the set $C_1 \triangle C_2 \triangle \cdots \triangle C_k$ consisting of all elements belonging to an odd number of the C_i's. Denote the first and the second player by $\mathcal{F}$ and $\mathcal{S}$, respectively.

Since each digit is written on exactly 512 cards, the sum of all the cards is $\varnothing$. Therefore, at the end of the game the sum of all the cards of $\mathcal{F}$ will be the same as that of $\mathcal{S}$; denote this sum by C. Then the player who took C can throw it out and get the desired situation, while the other one cannot. Thus, the player getting card C wins, and no draw is possible.

Now, given a nonempty card B, one can easily see that all the cards can be split into 512 pairs of the form $(X, X \triangle B)$ because $(X \triangle B) \triangle B = X$. The following lemma shows a property of such a partition that is important for the solution.

Lemma. Let $B \neq \varnothing$ be some card. Let us choose 512 cards so that exactly one card is chosen from every pair $(X, X \triangle B)$. Then the sum of all chosen cards is either $\varnothing$ or B.

Proof. Let b be some element of B. Enumerate the pairs; let X_i be the card not containing b in the i^{th} pair, and let Y_i be the other card in this pair. Then the sets X_i are exactly all the sets not containing b, therefore each digit $a \neq b$ is written on exactly 256 of these cards, so $X_1 \triangle X_2 \triangle \cdots \triangle X_{512} = \varnothing$. Now, if we replace some summands in this sum by the other elements from their pairs, we will simply add B several times to this sum, thus the sum will either remain unchanged or change by B, as required. □

Now we consider two cases.

Case 1. Assume that $\mathcal{F}$ takes the card $\varnothing$ on his first move. In this case, we present a winning strategy for $\mathcal{S}$.

Let $\mathcal{S}$ take an arbitrary card A. Assume that $\mathcal{F}$ takes card B after that; then $\mathcal{S}$ takes $A \triangle B$. Split all 1024 cards into 512 pairs of the form $(X, X \triangle B)$; we call two cards in one pair *partners*. Then the four cards taken so far form two pairs $(\varnothing, B)$ and $(A, A \triangle B)$ belonging to $\mathcal{F}$ and $\mathcal{S}$, respectively. On each of the subsequent moves, when $\mathcal{F}$ takes some card, $\mathcal{S}$ should take the partner of this card in response.

Consider the situation at the end of the game. Let us for a moment replace card A belonging to $\mathcal{S}$ by $\varnothing$. Then he would have one card from each pair; by our lemma, the sum of all these cards would be either $\varnothing$ or B. Now, replacing $\varnothing$ back by A we get that the actual sum of the cards of $\mathcal{S}$ is either A or $A \triangle B$, and he has both these cards. Thus $\mathcal{S}$ wins.

Case 2. Now assume that $\mathcal{F}$ takes some card $A \neq \varnothing$ on his first move. Let us present a winning strategy for $\mathcal{F}$ in this case.

Assume that $\mathcal{S}$ takes some card $B \neq \varnothing$ on his first move; then $\mathcal{F}$ takes $A \triangle B$. Again, let us split all the cards into pairs of the form $(X, X \triangle B)$; then the cards which have not been taken yet form several complete pairs and one extra element (card $\varnothing$ has not been taken while its partner B has). Now, on each of the subsequent moves, if $\mathcal{S}$ takes some element from a complete pair, then $\mathcal{F}$ takes its partner. If $\mathcal{S}$ takes the extra element, then $\mathcal{F}$ takes an arbitrary card Y, and the partner of Y becomes the new extra element.

Thus, on his last move $\mathcal{S}$ is forced to take the extra element. After that player $\mathcal{F}$ has cards A and $A \triangle B$, player $\mathcal{S}$ has cards B and $\varnothing$, and $\mathcal{F}$ has exactly one element from every other pair. Thus the situation is the same as in the previous case with roles reversed, and $\mathcal{F}$ wins.

Finally, if $\mathcal{S}$ takes $\varnothing$ on his first move then $\mathcal{F}$ denotes any card which has not been taken yet by B and takes $A \triangle B$. After that, the same strategy as above is applicable.

Comment 1. If one wants to avoid the unusual question about the first move, one may change the formulation as follows. (The difficulty of the problem would decrease somewhat.)

A card deck consists of 1023 cards; on each card, a nonempty set of distinct decimal digits is written in such a way that no two of these sets coincide. Two players alternately take cards from the deck, one card per turn. When the deck is empty, each player checks if he can throw out one of his cards so that for each of the ten digits, he still holds an even number of cards with this digit. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.

Determine which of the players (if any) has a winning strategy.

The winner in this version is the first player. The analysis of the game from the first two paragraphs of the previous solution applies to this version as well, except for the case $C = \varnothing$ in which the result is a draw. Then the strategy for $\mathcal{S}$ in Case 1 works for $\mathcal{F}$ in this version: the sum of all his cards at the end is either A or $A \triangle B$, thus nonempty in both cases.

Comment 2. Notice that all the cards form a vector space over $\mathbb{F}_2$, with $\triangle$ the operation of addition. Due to the automorphisms of this space, all possibilities for $\mathcal{F}$'s first move except $\varnothing$ are equivalent. The same holds for the response by $\mathcal{S}$ if $\mathcal{F}$ takes the card $\varnothing$ on his first move.

Comment 3. It is not that hard to show that in the initial game, $\mathcal{F}$ has a winning move, by the idea of "strategy stealing".

Namely, assume that $\mathcal{S}$ has a winning strategy. Let us take two card decks and start two games, in which $\mathcal{S}$ will act by his strategy. In the first game, $\mathcal{F}$ takes an arbitrary card A_1; assume that $\mathcal{S}$ takes some B_1 in response. Then $\mathcal{F}$ takes the card B_1 at the second game; let the response by $\mathcal{S}$ be A_2. Then $\mathcal{F}$ takes A_2 in the first game and gets a response B_2, and so on.

This process stops at some moment when in the second game $\mathcal{S}$ takes $A_i = A_1$. At this moment the players hold the same sets of cards in both games, but with roles reversed. Now, if some cards remain in the decks, $\mathcal{F}$ takes an arbitrary card from the first deck starting a similar cycle.

At the end of the game, player $\mathcal{F}$'s cards in the first game are exactly player $\mathcal{S}$'s cards in the second game, and vice versa. Thus in one of the games $\mathcal{F}$ will win, which is impossible by our assumption.

One may notice that the strategy in Case 2 is constructed exactly in this way from the strategy in Case 1. This is possible since every response by $\mathcal{S}$ wins if $\mathcal{F}$ takes the card $\varnothing$ on his first move.

C9. There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*.

Suppose that Turbo's path entirely covers all circles. Prove that n must be odd.

(India)

Solution 1. Replace every cross (i.e. intersection of two circles) by two small circle arcs that indicate the direction in which the snail should leave the cross (see Figure 1.1). Notice that the placement of the small arcs does not depend on the direction of moving on the curves; no matter which direction the snail is moving on the circle arcs, he will follow the same curves (see Figure 1.2). In this way we have a set of curves, that are the possible paths of the snail. Call these curves *snail orbits* or just *orbits*. Every snail orbit is a simple closed curve that has no intersection with any other orbit.

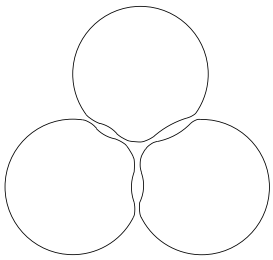


Figure 1.1

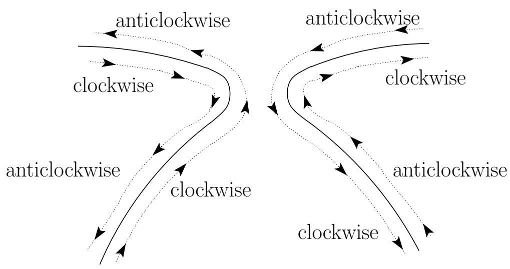


Figure 1.2

We prove the following, more general statement.

> $(*)$ *In any configuration of n circles such that no two of them are tangent, the number of snail orbits has the same parity as the number n.* (Note that it is not assumed that all circle pairs intersect.)

This immediately solves the problem.

Let us introduce the following operation that will be called *flipping a cross*. At a cross, remove the two small arcs of the orbits, and replace them by the other two arcs. Hence, when the snail arrives at a flipped cross, he will continue on the other circle as before, but he will preserve the orientation in which he goes along the circle arcs (see Figure 2).

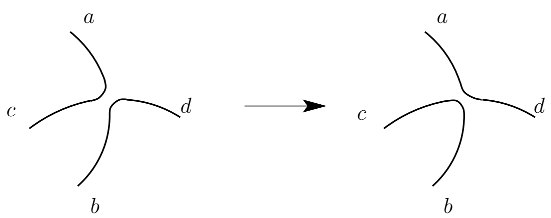


Figure 2

Consider what happens to the number of orbits when a cross is flipped. Denote by a, b, c, and d the four arcs that meet at the cross such that a and b belong to the same circle. Before the flipping a and b were connected to c and d, respectively, and after the flipping a and b are connected to d and c, respectively.

The orbits passing through the cross are closed curves, so each of the arcs a, b, c, and d is connected to another one by orbits outside the cross. We distinguish three cases.

Case 1: a is connected to b and c is connected to d by the orbits outside the cross (see Figure 3.1).

We show that this case is impossible. Remove the two small arcs at the cross, connect a to b, and connect c to d at the cross. Let γ be the new closed curve containing a and b, and let δ be the new curve that connects c and d. These two curves intersect at the cross. So one of c and d is inside γ and the other one is outside γ. Then the two closed curves have to meet at least one more time, but this is a contradiction, since no orbit can intersect itself.

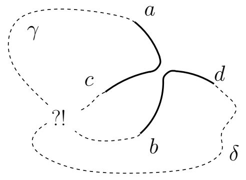


Figure 3.1

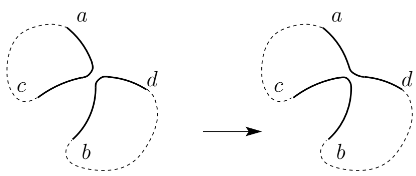


Figure 3.2

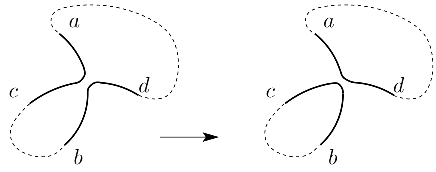


Figure 3.3

Case 2: a is connected to c and b is connected to d (see Figure 3.2).

Before the flipping a and c belong to one orbit and b and d belong to another orbit. Flipping the cross merges the two orbits into a single orbit. Hence, the number of orbits decreases by 1.

Case 3: a is connected to d and b is connected to c (see Figure 3.3).

Before the flipping the arcs a, b, c, and d belong to a single orbit. Flipping the cross splits that orbit in two. The number of orbits increases by 1.

As can be seen, every flipping decreases or increases the number of orbits by one, thus changes its parity.

Now flip every cross, one by one. Since every pair of circles has 0 or 2 intersections, the number of crosses is even. Therefore, when all crosses have been flipped, the original parity of the number of orbits is restored. So it is sufficient to prove $(*)$ for the new configuration, where all crosses are flipped. Of course also in this new configuration the (modified) orbits are simple closed curves not intersecting each other.

Orient the orbits in such a way that the snail always moves anticlockwise along the circle arcs. Figure 4 shows the same circles as in Figure 1 after flipping all crosses and adding orientation. (Note that this orientation may be different from the orientation of the orbit as a planar curve; the orientation of every orbit may be negative as well as positive, like the middle orbit in Figure 4.) If the snail moves around an orbit, the total angle change in his moving direction, the *total curvature*, is either $+2\pi$ or -2π, depending on the orientation of the orbit. Let P and N be the number of orbits with positive and negative orientation, respectively. Then the total curvature of all orbits is $(P - N) \cdot 2\pi$.

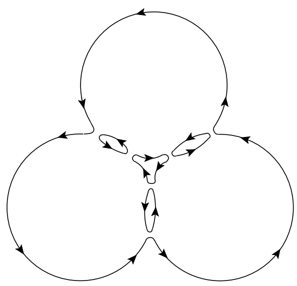


Figure 4

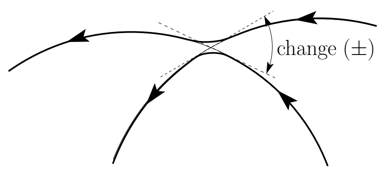


Figure 5

Double-count the total curvature of all orbits. Along every circle the total curvature is 2π. At every cross, the two turnings make two changes with some angles having the same absolute value but opposite signs, as depicted in Figure 5. So the changes in the direction at the crosses cancel out. Hence, the total curvature is $n \cdot 2\pi$.

Now we have $(P - N) \cdot 2\pi = n \cdot 2\pi$, so $P - N = n$. The number of (modified) orbits is $P + N$, that has a same parity as $P - N = n$.

Solution 2. We present a different proof of $(*)$.

We perform a sequence of small modification steps on the configuration of the circles in such a way that at the end they have no intersection at all (see Figure 6.1). We use two kinds of local changes to the structure of the orbits (see Figure 6.2):

- *Type-1 step:* An arc of a circle is moved over an arc of another circle; such a step creates or removes two intersections.

- *Type-2 step:* An arc of a circle is moved through the intersection of two other circles.

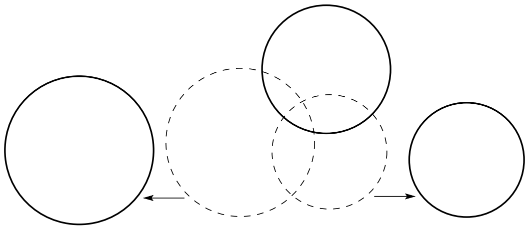


Figure 6.1

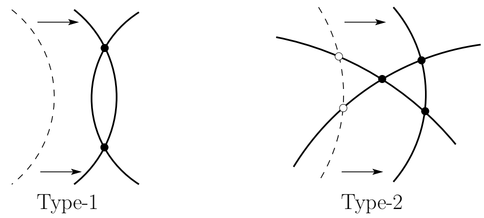


Figure 6.2

We assume that in every step only one circle is moved, and that this circle is moved over at most one arc or intersection point of other circles.

We will show that the parity of the number of orbits does not change in any step. As every circle becomes a separate orbit at the end of the procedure, this fact proves $(*)$.

Consider what happens to the number of orbits when a Type-1 step is performed. The two intersection points are created or removed in a small neighbourhood. Denote some points of the two circles where they enter or leave this neighbourhood by a, b, c, and d in this order around the neighbourhood; let a and b belong to one circle and let c and d belong to the other circle. The two circle arcs may have the same or opposite orientations. Moreover, the four end-points of the two arcs are connected by the other parts of the orbits. This can happen in two ways without intersection: either a is connected to d and b is connected to c, or a is connected to b and c is connected to d. Altogether we have four cases, as shown in Figure 7.

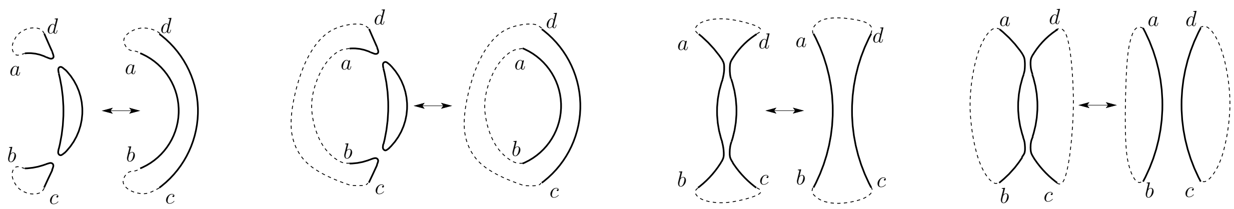


Figure 7

We can see that the number of orbits is changed by -2 or $+2$ in the leftmost case when the arcs have the same orientation, a is connected to d, and b is connected to c. In the other three cases the number of orbits is not changed. Hence, Type-1 steps do not change the parity of the number of orbits.

Now consider a Type-2 step. The three circles enclose a small, triangular region; by the step, this triangle is replaced by another triangle. Again, the modification of the orbits is done in some small neighbourhood; the structure does not change outside. Each side of the triangle shaped region can be convex or concave; the number of concave sides can be 0, 1, 2 or 3, so there are 4 possible arrangements of the orbits inside the neighbourhood, as shown in Figure 8.

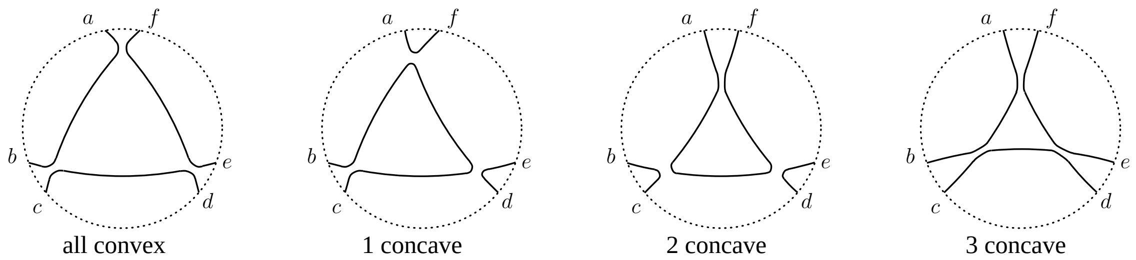


Figure 8

Denote the points where the three circles enter or leave the neighbourhood by a, b, c, d, e, and f in this order around the neighbourhood. As can be seen in Figure 8, there are only two essentially different cases; either a, c, e are connected to b, d, f, respectively, or a, c, e are connected to f, b, d, respectively. The step either preserves the set of connections or switches to the other arrangement. Obviously, in the earlier case the number of orbits is not changed; therefore we have to consider only the latter case.

The points a, b, c, d, e, and f are connected by the orbits outside, without intersection. If a was connected to c, say, then this orbit would isolate b, so this is impossible. Hence, each of a, b, c, d, e and f must be connected either to one of its neighbours or to the opposite point. If say a is connected to d, then this orbit separates b and c from e and f, therefore b must be connected to c and e must be connected to f. Altogether there are only two cases and their reverses: either each point is connected to one of its neighbours or two opposite points are connected and the the remaining neigh boring pairs are connected to each other. See Figure 9.

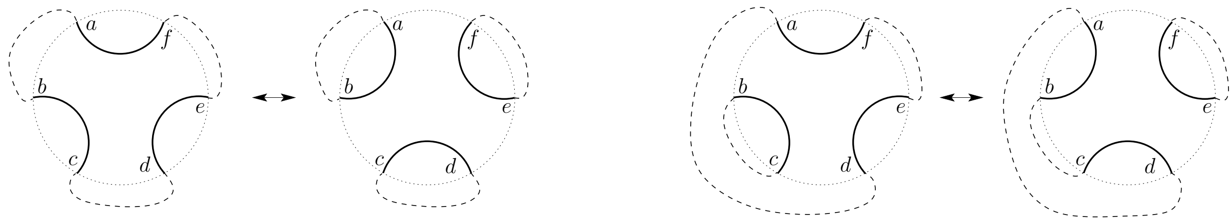


Figure 9

We can see that if only neighbouring points are connected, then the number of orbits is changed by $+2$ or -2. If two opposite points are connected (a and d in the figure), then the orbits are re-arranged, but their number is unchanged. Hence, Type-2 steps also preserve the parity. This completes the proof of $(*)$.

Solution 3. Like in the previous solutions, we do not need all circle pairs to intersect but we assume that the circles form a connected set. Denote by $\mathcal{C}$ and $\mathcal{P}$ the sets of circles and their intersection points, respectively.

The circles divide the plane into several simply connected, bounded regions and one unbounded region. Denote the set of these regions by $\mathcal{R}$. We say that an intersection point or a region is *odd* or *even* if it is contained inside an odd or even number of circles, respectively. Let $\mathcal{P}_{\text{odd}}$ and $\mathcal{R}_{\text{odd}}$ be the sets of odd intersection points and odd regions, respectively.

Claim.

$$|\mathcal{R}_{\text{odd}}| - |\mathcal{P}_{\text{odd}}| \equiv n \pmod 2. \tag{1}$$

Proof. For each circle $c \in \mathcal{C}$, denote by R_c, P_c, and X_c the number of regions inside c, the number of intersection points inside c, and the number of circles intersecting c, respectively. The circles divide each other into several arcs; denote by A_c the number of such arcs inside c. By double counting the regions and intersection points inside the circles we get

$$|\mathcal{R}_{\text{odd}}| \equiv \sum_{c\in\mathcal{C}} R_c \pmod 2 \quad \text{and} \quad |\mathcal{P}_{\text{odd}}| \equiv \sum_{c\in\mathcal{C}} P_c \pmod 2.$$

For each circle c, apply EULER's polyhedron theorem to the (simply connected) regions in c. There are $2X_c$ intersection points on c; they divide the circle into $2X_c$ arcs. The polyhedron theorem yields $(R_c+1)+(P_c+2X_c)=(A_c+2X_c)+2$, considering the exterior of c as a single region. Therefore,

$$R_c + P_c = A_c + 1. \tag{2}$$

Moreover, we have four arcs starting from every interior points inside c and a single arc starting into the interior from each intersection point on the circle. By double-counting the end-points of the interior arcs we get $2A_c = 4P_c + 2X_c$, so

$$A_c = 2P_c + X_c. \tag{3}$$

The relations (2) and (3) together yield

$$R_c - P_c = X_c + 1. \tag{4}$$

By summing up (4) for all circles we obtain

$$\sum_{c\in\mathcal{C}} R_c - \sum_{c\in\mathcal{C}} P_c = \sum_{c\in\mathcal{C}} X_c + |\mathcal{C}|,$$

which yields

$$|\mathcal{R}_{\text{odd}}| - |\mathcal{P}_{\text{odd}}| \equiv \sum_{c\in\mathcal{C}} X_c + n \pmod 2. \tag{5}$$

Notice that in $\sum_{c\in\mathcal{C}} X_c$ each intersecting circle pair is counted twice, i.e., for both circles in the pair, so

$$\sum_{c\in\mathcal{C}} X_c \equiv 0 \pmod 2,$$

which finishes the proof of the Claim. □

Now insert the same small arcs at the intersections as in the first solution, and suppose that there is a single snail orbit b.

First we show that the odd regions are inside the curve b, while the even regions are outside. Take a region $r \in \mathcal{R}$ and a point x in its interior, and draw a ray y, starting from x, that does not pass through any intersection point of the circles and is neither tangent to any of the circles. As is well-known, x is inside the curve b if and only if y intersects b an odd number of times (see Figure 10). Notice that if an arbitrary circle c contains x in its interior, then c intersects y at a single point; otherwise, if x is outside c, then c has 2 or 0 intersections with y. Therefore, y intersects b an odd number of times if and only if x is contained in an odd number of circles, so if and only if r is odd.

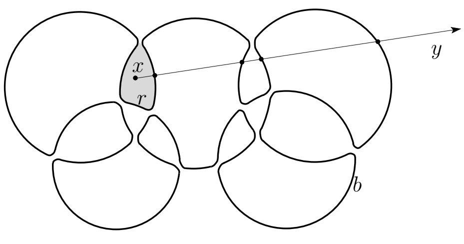


Figure 10

Now consider an intersection point p of two circles c_1 and c_2 and a small neighbourhood around p. Suppose that p is contained inside k circles.

We have four regions that meet at p. Let r_1 be the region that lies outside both c_1 and c_2, let r_2 be the region that lies inside both c_1 and c_2, and let r_3 and r_4 be the two remaining regions, each lying inside exactly one of c_1 and c_2. The region r_1 is contained inside the same k circles as p; the region r_2 is contained also by c_1 and c_2, so by $k+2$ circles in total; each of the regions r_3 and r_4 is contained inside $k+1$ circles. After the small arcs have been inserted at p, the regions r_1 and r_2 get connected, and the regions r_3 and r_4 remain separated at p (see Figure 11). If p is an odd point, then r_1 and r_2 are odd, so two odd regions are connected at p. Otherwise, if p is even, then we have two even regions connected at p.

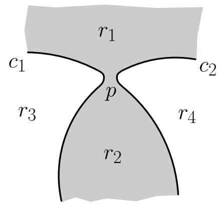


Figure 11

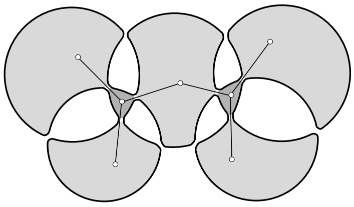


Figure 12

Consider the system of odd regions and their connections at the odd points as a graph. In this graph the odd regions are the vertices, and each odd point establishes an edge that connects two vertices (see Figure 12). As b is a single closed curve, this graph is connected and contains no cycle, so the graph is a tree. Then the number of vertices must be by one greater than the number of edges, so

$$|\mathcal{R}_{\text{odd}}| - |\mathcal{P}_{\text{odd}}| = 1. \tag{9}$$

The relations (1) and (9) together prove that n must be odd.

Comment. For every odd n there exists at least one configuration of n circles with a single snail orbit. Figure 13 shows a possible configuration with 5 circles. In general, if a circle is rotated by $k \cdot \frac{360^\circ}{n}$ $(k = 1, 2, \ldots, n-1)$ around an interior point other than the centre, the circle and its rotated copies together provide a single snail orbit.

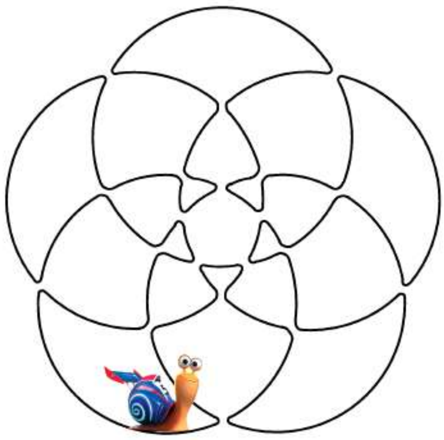


Figure 13

Geometry

G1. The points P and Q are chosen on the side BC of an acute-angled triangle ABC so that $\angle PAB = \angle ACB$ and $\angle QAC = \angle CBA$. The points M and N are taken on the rays AP and AQ, respectively, so that $AP = PM$ and $AQ = QN$. Prove that the lines BM and CN intersect on the circumcircle of the triangle ABC.

(Georgia)

Solution 1. Denote by S the intersection point of the lines BM and CN. Let moreover $\beta = \angle QAC = \angle CBA$ and $\gamma = \angle PAB = \angle ACB$. From these equalities it follows that the triangles ABP and CAQ are similar (see Figure 1). Therefore we obtain

$$\frac{BP}{PM} = \frac{BP}{PA} = \frac{AQ}{QC} = \frac{NQ}{QC}.$$

Moreover,

$$\angle BPM = \beta + \gamma = \angle CQN.$$

Hence the triangles BPM and NQC are similar. This gives $\angle BMP = \angle NCQ$, so the triangles BPM and BSC are also similar. Thus we get

$$\angle CSB = \angle BPM = \beta + \gamma = 180^\circ - \angle BAC,$$

which completes the solution.

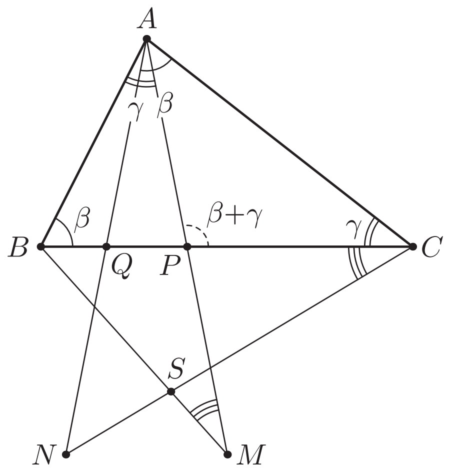


Figure 1

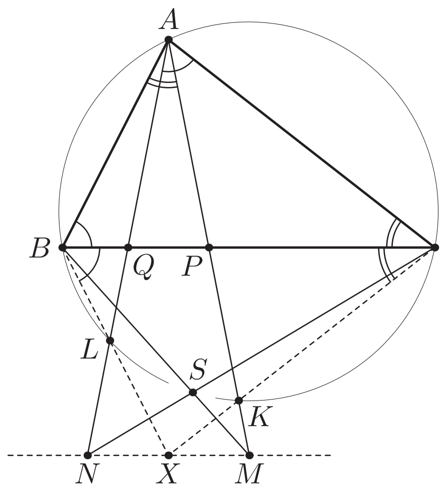


Figure 2

Solution 2. As in the previous solution, denote by S the intersection point of the lines BM and NC. Let moreover the circumcircle of the triangle ABC intersect the lines AP and AQ again at K and L, respectively (see Figure 2).

Note that $\angle LBC = \angle LAC = \angle CBA$ and similarly $\angle KCB = \angle KAB = \angle BCA$. It implies that the lines BL and CK meet at a point X, being symmetric to the point A with respect to the line BC. Since $AP = PM$ and $AQ = QN$, it follows that X lies on the line MN. Therefore, using PASCAL's theorem for the hexagon $ALBSCK$, we infer that S lies on the circumcircle of the triangle ABC, which finishes the proof.

Comment. Both solutions can be modified to obtain a more general result, with the equalities

$$AP = PM \quad \text{and} \quad AQ = QN$$

replaced by

$$\frac{AP}{PM} = \frac{QN}{AQ}.$$

G2. Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

(Estonia)

Solution. Denote

$$a = \frac{BK}{KC}, \qquad b = \frac{CL}{LA}, \qquad c = \frac{AM}{MB}.$$

By Ceva's theorem, $abc = 1$, so we may, without loss of generality, assume that $a \geqslant 1$. Then at least one of the numbers b or c is not greater than 1. Therefore at least one of the pairs (a, b), (b, c) has its first component not less than 1 and the second one not greater than 1. Without loss of generality, assume that $1 \leqslant a$ and $b \leqslant 1$.

Therefore, we obtain $bc \leqslant 1$ and $1 \leqslant ca$, or equivalently

$$\frac{AM}{MB} \leqslant \frac{LA}{CL} \qquad \text{and} \qquad \frac{MB}{AM} \leqslant \frac{BK}{KC}.$$

The first inequality implies that the line passing through M and parallel to BC intersects the *segment* AL at a point X (see Figure 1). Therefore the inradius of the triangle ALM is not less than the inradius r_1 of triangle AMX.

Similarly, the line passing through M and parallel to AC intersects the *segment* BK at a point Y, so the inradius of the triangle BMK is not less than the inradius r_2 of the triangle BMY. Thus, to complete our solution, it is enough to show that $r_1 + r_2 \geqslant r$, where r is the inradius of the triangle ABC. We prove that in fact $r_1 + r_2 = r$.

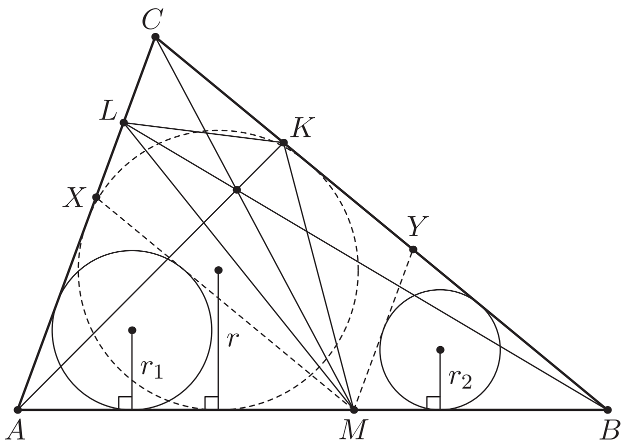


Figure 1

Since $MX \parallel BC$, the dilation with centre A that takes M to B takes the incircle of the triangle AMX to the incircle of the triangle ABC. Therefore

$$\frac{r_1}{r} = \frac{AM}{AB}, \qquad \text{and similarly} \quad \frac{r_2}{r} = \frac{MB}{AB}.$$

Adding these equalities gives $r_1 + r_2 = r$, as required.

Comment. Alternatively, one can use Desargues' theorem instead of Ceva's theorem, as follows: The lines AB, BC, CA dissect the plane into seven regions. One of them is bounded, and amongst the other six, three are two-sided and three are three-sided. Now define the points $P = BC \cap LM$, $Q = CA \cap MK$, and $R = AB \cap KL$ (in the projective plane). By Desargues' theorem, the points P, Q, R lie on a common line ℓ. This line intersects only unbounded regions. If we now assume (without loss of generality) that P, Q and R lie on ℓ in that order, then one of the segments PQ or QR lies inside a two-sided region. If, for example, this segment is PQ, then the triangles ALM and BMK will satisfy the statement of the problem for the same reason.

G3. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$. (Here we always assume that an angle bisector is a ray.)

(Russia)

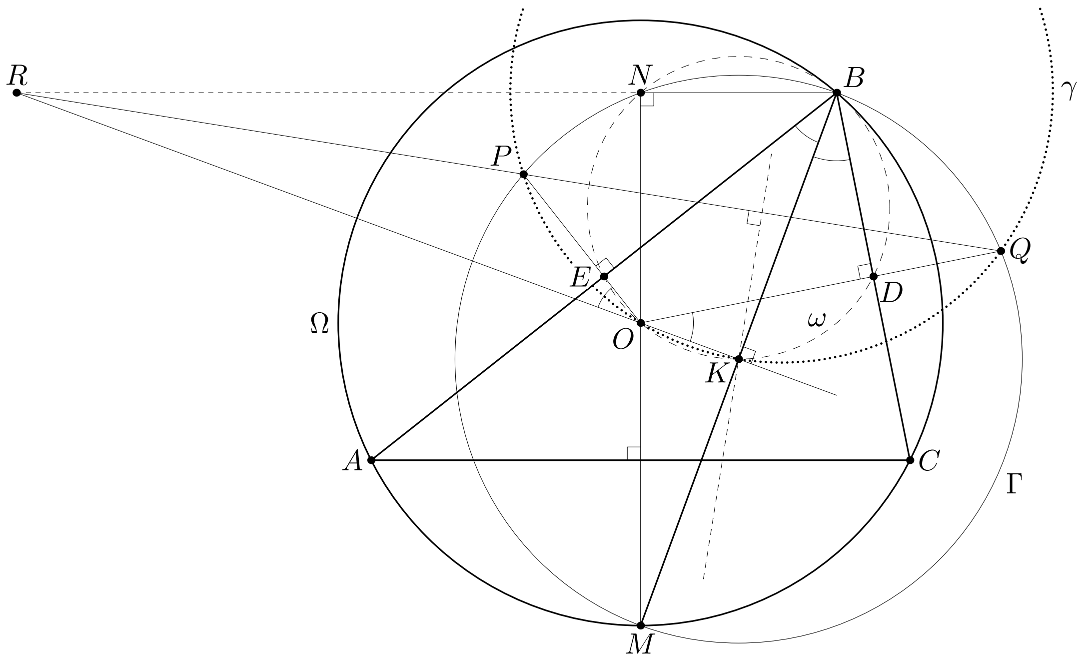
Solution. Let K be the midpoint of BM, i.e., the centre of Γ. Notice that $AB \neq BC$ implies $K \neq O$. Clearly, the lines OM and OK are the perpendicular bisectors of AC and BM, respectively. Therefore, R is the intersection point of PQ and OK.

Let N be the second point of intersection of Γ with the line OM. Since BM is a diameter of Γ, the lines BN and AC are both perpendicular to OM. Hence $BN \parallel AC$, and it suffices to prove that BN passes through R. Our plan for doing this is to interpret the lines BN, OK, and PQ as the radical axes of three appropriate circles.

Let ω be the circle with diameter BO. Since $\angle BNO = \angle BKO = 90^\circ$, the points N and K lie on ω.

Next we show that the points O, K, P, and Q are concyclic. To this end, let D and E be the midpoints of BC and AB, respectively. Clearly, D and E lie on the rays OQ and OP, respectively. By our assumptions about the triangle ABC, the points B, E, O, K, and D lie in this order on ω. It follows that $\angle EOR = \angle EBK = \angle KBD = \angle KOD$, so the line KO externally bisects the angle POQ. Since the point K is the centre of Γ, it also lies on the perpendicular bisector of PQ. So K coincides with the midpoint of the arc POQ of the circumcircle γ of triangle POQ.

Thus the lines OK, BN, and PQ are pairwise radical axes of the circles ω, γ, and Γ. Hence they are concurrent at R, as required.



G4. Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0,1)$. For a variable point $P \notin \{A,B,C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

(United Kingdom)

Solution 1. Throughout the solution, we denote by $\measuredangle(a,b)$ the directed angle between the lines a and b.

Let D be the point on the segment AB such that $BD = \lambda \cdot BA$. We will show that either $Q = D$, or $\measuredangle(DQ,QB) = \measuredangle(AB,BC)$; this would mean that the point Q varies over the constant circle through D tangent to BC at B, as required.

Denote the circumcircles of the triangles AMP and BMC by ω_A and ω_B, respectively. The lines AP, BC, and MQ are pairwise radical axes of the circles Γ, ω_A, and ω_B, thus either they are parallel, or they share a common point X.

Assume that these lines are parallel (see Figure 1). Then the segments AP, QM, and BC have a common perpendicular bisector; the reflection in this bisector maps the segment CP to BA, and maps M to Q. Therefore, in this case Q lies on AB, and $BQ/AB = CM/CP = BD/AB$; so we have $Q = D$.

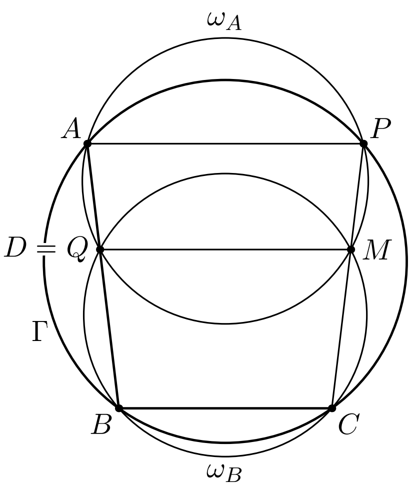


Figure 1

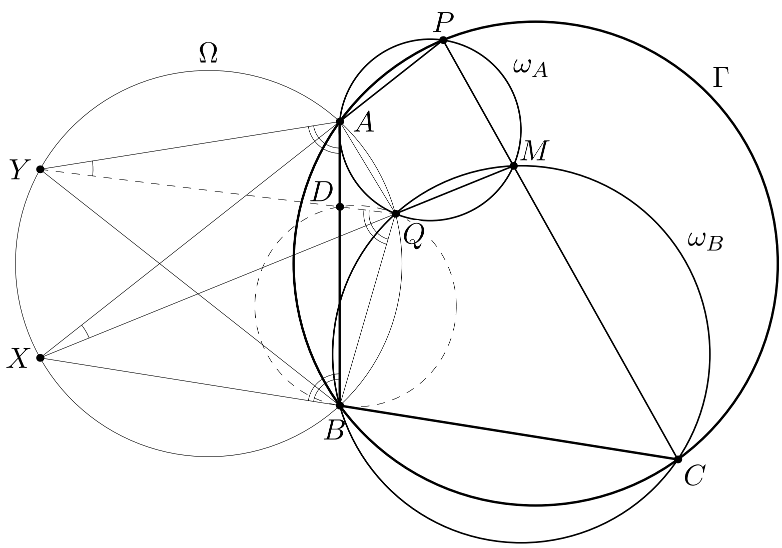


Figure 2

Now assume that the lines AP, QM, and BC are concurrent at some point X (see Figure 2). Notice that the points A, B, Q, and X lie on a common circle Ω by MIQUEL's theorem applied to the triangle XPC. Let us denote by Y the symmetric image of X about the perpendicular bisector of AB. Clearly, Y lies on Ω, and the triangles YAB and $\triangle XBA$ are congruent. Moreover, the triangle XPC is similar to the triangle XBA, so it is also similar to the triangle YAB.

Next, the points D and M correspond to each other in similar triangles YAB and XPC, since $BD/BA = CM/CP = \lambda$. Moreover, the triangles YAB and XPC are equi-oriented, so $\measuredangle(MX,XP) = \measuredangle(DY,YA)$. On the other hand, since the points A, Q, X, and Y lie on Ω, we have $\measuredangle(QY,YA) = \measuredangle(MX,XP)$. Therefore, $\measuredangle(QY,YA) = \measuredangle(DY,YA)$, so the points Y, D, and Q are collinear.

Finally, we have $\measuredangle(DQ,QB) = \measuredangle(YQ,QB) = \measuredangle(YA,AB) = \measuredangle(AB,BX) = \measuredangle(AB,BC)$, as desired.

Comment. In the original proposal, λ was supposed to be an arbitrary real number distinct from 0 and 1, and the point M was defined by $\overrightarrow{CM} = \lambda \cdot \overrightarrow{CP}$. The Problem Selection Committee decided to add the restriction $\lambda \in (0, 1)$ in order to avoid a large case distinction.

Solution 2. As in the previous solution, we introduce the radical centre $X = AP \cap BC \cap MQ$ of the circles ω_A, ω_B, and Γ. Next, we also notice that the points A, Q, B, and X lie on a common circle Ω.

If the point P lies on the arc BAC of Γ, then the point X is outside Γ, thus the point Q belongs to the ray XM, and therefore the points P, A, and Q lie on the same side of BC. Otherwise, if P lies on the arc BC not containing A, then X lies inside Γ, so M and Q lie on different sides of BC; thus again Q and A lie on the same side of BC. So, in each case the points Q and A lie on the same side of BC.

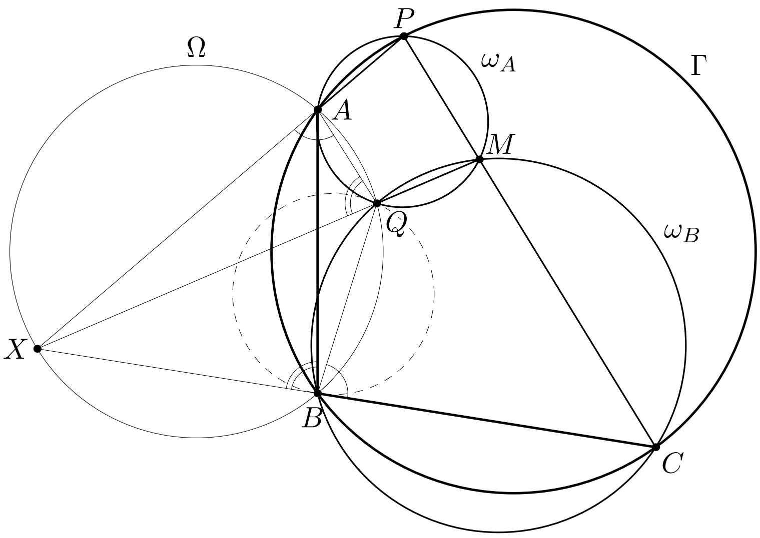


Figure 3

Now we prove that the ratio

$$\frac{QB}{\sin \angle QBC} = \frac{QB}{QX} \cdot \frac{QX}{\sin \angle QBX}$$

is constant. Since the points A, Q, B, and X are concyclic, we have

$$\frac{QX}{\sin \angle QBX} = \frac{AX}{\sin \angle ABC}.$$

Next, since the points B, Q, M, and C are concyclic, the triangles XBQ and XMC are similar, so

$$\frac{QB}{QX} = \frac{CM}{CX} = \lambda \cdot \frac{CP}{CX}.$$

Analogously, the triangles XCP and XAB are also similar, so

$$\frac{CP}{CX} = \frac{AB}{AX}.$$

Therefore, we obtain

$$\frac{QB}{\sin \angle QBC} = \lambda \cdot \frac{AB}{AX} \cdot \frac{AX}{\sin \angle ABC} = \lambda \cdot \frac{AB}{\sin \angle ABC},$$

so this ratio is indeed constant. Thus the circle passing through Q and tangent to BC at B is also constant, and Q varies over this fixed circle.

Comment. It is not hard to guess that the desired circle should be tangent to BC at B. Indeed, the second paragraph of this solution shows that this circle lies on one side of BC; on the other hand, in the limit case $P = B$, the point Q also coincides with B.

Solution 3. Let us perform an inversion centred at C. Denote by X' the image of a point X under this inversion.

The circle Γ maps to the line Γ' passing through the constant points A' and B', and containing the variable point P'. By the problem condition, the point M varies over the circle γ which is the homothetic image of Γ with centre C and coefficient λ. Thus M' varies over the constant line $\gamma' \parallel A'B'$ which is the homothetic image of $A'B'$ with centre C and coefficient $1/\lambda$, and $M = \gamma' \cap CP'$. Next, the circumcircles ω_A and ω_B of the triangles AMP and BMC map to the circumcircle ω'_A of the triangle $A'M'P'$ and to the line $B'M'$, respectively; the point Q thus maps to the second point of intersection of $B'M'$ with ω'_A (see Figure 4).

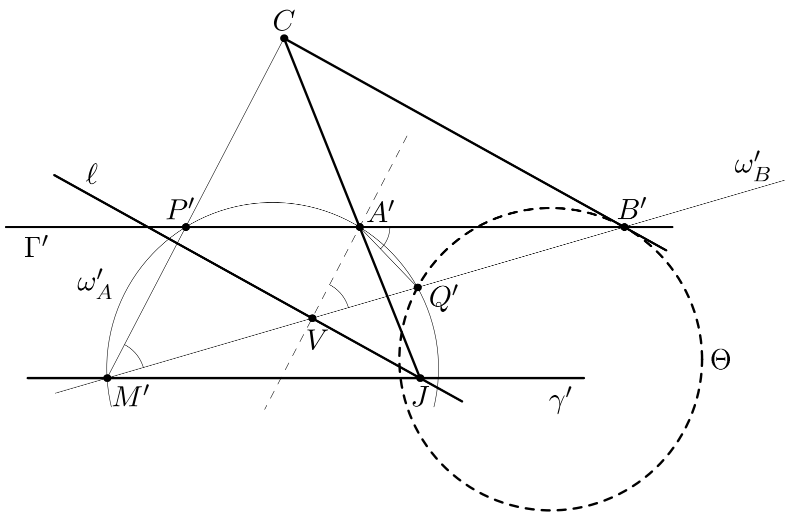


Figure 4

Let J be the (constant) common point of the lines γ' and CA', and let ℓ be the (constant) line through J parallel to CB'. Let V be the common point of the lines ℓ and $B'M'$. Applying Pappus' theorem to the triples (C, J, A') and (V, B', M') we get that the points $CB' \cap JV$, $JM' \cap A'B'$, and $CM' \cap A'V$ are collinear. The first two of these points are ideal, hence so is the third, which means that $CM' \parallel A'V$.

Now we have $\measuredangle(Q'A', A'P') = \measuredangle(Q'M', M'P') = \angle(VM', A'V)$, which means that the triangles $B'Q'A'$ and $B'A'V$ are similar, and $(B'A')^2 = B'Q' \cdot B'V$. Thus Q' is the image of V under the second (fixed) inversion with centre B' and radius $B'A'$. Since V varies over the constant line ℓ, Q' varies over some constant circle Θ. Thus, applying the first inversion back we get that Q also varies over some fixed circle.

One should notice that this last circle is not a line; otherwise Θ would contain C, and thus ℓ would contain the image of C under the second inversion. This is impossible, since $CB' \parallel \ell$.

G5. Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D = 90^\circ$. Point H is the foot of the perpendicular from A to BD. The points S and T are chosen on the sides AB and AD, respectively, in such a way that H lies inside triangle SCT and

$$\angle SHC - \angle BSC = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ .$$

Prove that the circumcircle of triangle SHT is tangent to the line BD.

(Iran)

Solution. Let the line passing through C and perpendicular to the line SC intersect the line AB at Q (see Figure 1). Then

$$\angle SQC = 90^\circ - \angle BSC = 180^\circ - \angle SHC ,$$

which implies that the points C, H, S, and Q lie on a common circle. Moreover, since SQ is a diameter of this circle, we infer that the circumcentre K of triangle SHC lies on the line AB. Similarly, we prove that the circumcentre L of triangle CHT lies on the line AD.

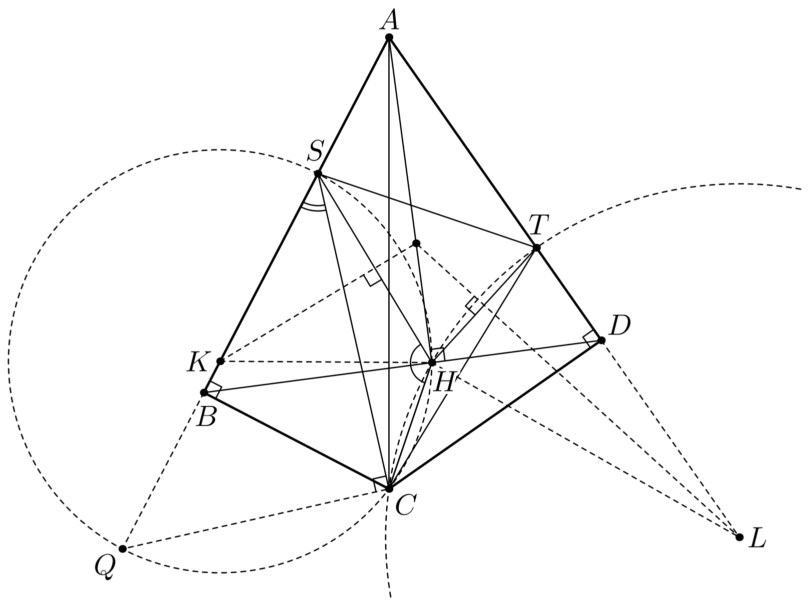


Figure 1

In order to prove that the circumcircle of triangle SHT is tangent to BD, it suffices to show that the perpendicular bisectors of HS and HT intersect on the line AH. However, these two perpendicular bisectors coincide with the angle bisectors of angles AKH and ALH. Therefore, in order to complete the solution, it is enough (by the bisector theorem) to show that

$$\frac{AK}{KH} = \frac{AL}{LH} . \tag{1}$$

We present two proofs of this equality.

First proof. Let the lines KL and HC intersect at M (see Figure 2). Since $KH = KC$ and $LH = LC$, the points H and C are symmetric to each other with respect to the line KL. Therefore M is the midpoint of HC. Denote by O the circumcentre of quadrilateral $ABCD$. Then O is the midpoint of AC. Therefore we have $OM \parallel AH$ and hence $OM \perp BD$. This together with the equality $OB = OD$ implies that OM is the perpendicular bisector of BD and therefore $BM = DM$.

Since $CM \perp KL$, the points B, C, M, and K lie on a common circle with diameter KC. Similarly, the points L, C, M, and D lie on a circle with diameter LC. Thus, using the sine law, we obtain

$$\frac{AK}{AL} = \frac{\sin\angle ALK}{\sin\angle AKL} = \frac{DM}{CL} \cdot \frac{CK}{BM} = \frac{CK}{CL} = \frac{KH}{LH} ,$$

which finishes the proof of (1).

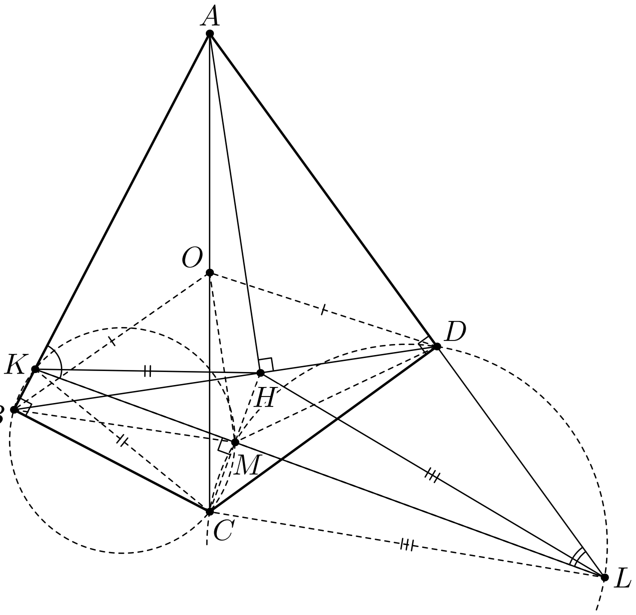


Figure 2

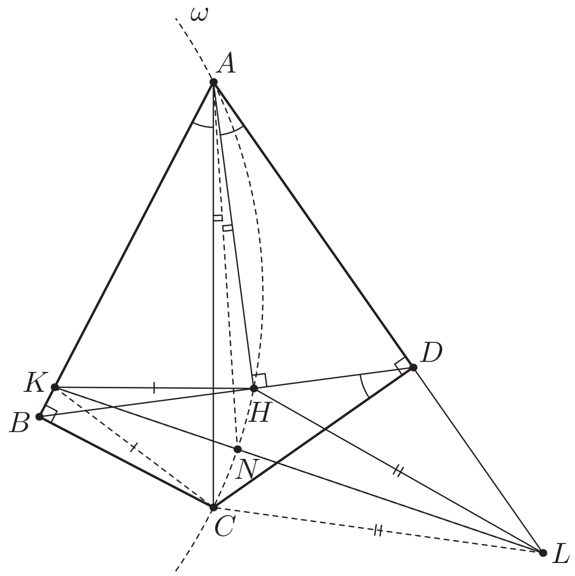


Figure 3

Second proof. If the points A, H, and C are collinear, then $AK = AL$ and $KH = LH$, so the equality (1) follows. Assume therefore that the points A, H, and C do not lie in a line and consider the circle ω passing through them (see Figure 3). Since the quadrilateral $ABCD$ is cyclic,

$$\angle BAC = \angle BDC = 90^\circ - \angle ADH = \angle HAD\,.$$

Let $N \neq A$ be the intersection point of the circle ω and the angle bisector of $\angle CAH$. Then AN is also the angle bisector of $\angle BAD$. Since H and C are symmetric to each other with respect to the line KL and $HN = NC$, it follows that both N and the centre of ω lie on the line KL. This means that the circle ω is an Apollonius circle of the points K and L. This immediately yields (1).

Comment. Either proof can be used to obtain the following generalised result:

Let $ABCD$ be a convex quadrilateral and let H be a point in its interior with $\angle BAC = \angle DAH$. The points S and T are chosen on the sides AB and AD, respectively, in such a way that H lies inside triangle SCT and

$$\angle SHC - \angle BSC = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ\,.$$

Then the circumcentre of triangle SHT lies on the line AH (and moreover the circumcentre of triangle SCT lies on AC).

G6. Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF. Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T, respectively. We call the pair (E, F) *interesting*, if the quadrilateral $KSAT$ is cyclic.

Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC}.$$

(Iran)

Solution 1. For any interesting pair (E, F), we will say that the corresponding triangle EFK is also *interesting*.

Let EFK be an interesting triangle. Firstly, we prove that $\angle KEF = \angle KFE = \angle A$, which also means that the circumcircle ω_1 of the triangle AEF is tangent to the lines KE and KF.

Denote by ω the circle passing through the points K, S, A, and T. Let the line AM intersect the line ST and the circle ω (for the second time) at N and L, respectively (see Figure 1).

Since $EF \parallel TS$ and M is the midpoint of EF, N is the midpoint of ST. Moreover, since K and M are symmetric to each other with respect to the line ST, we have $\angle KNS = \angle MNS = \angle LNT$. Thus the points K and L are symmetric to each other with respect to the perpendicular bisector of ST. Therefore $KL \parallel ST$.

Let G be the point symmetric to K with respect to N. Then G lies on the line EF, and we may assume that it lies on the ray MF. One has

$$\angle KGE = \angle KNS = \angle SNM = \angle KLA = 180^\circ - \angle KSA$$

(if $K = L$, then the angle KLA is understood to be the angle between AL and the tangent to ω at L). This means that the points K, G, E, and S are concyclic. Now, since $KSGT$ is a parallelogram, we obtain $\angle KEF = \angle KSG = 180^\circ - \angle TKS = \angle A$. Since $KE = KF$, we also have $\angle KFE = \angle KEF = \angle A$.

After having proved this fact, one may finish the solution by different methods.

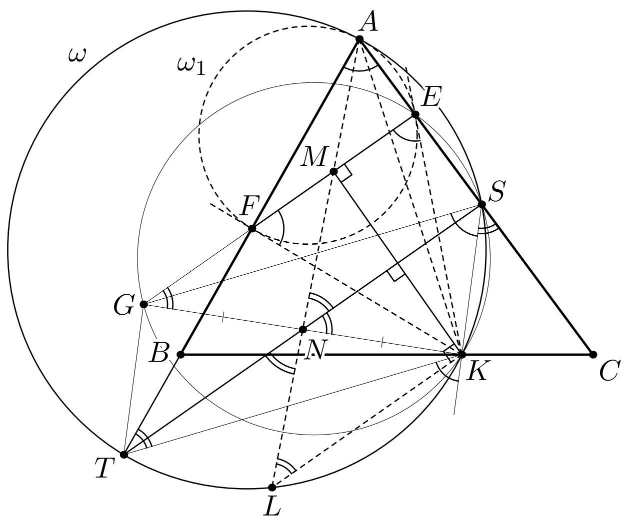


Figure 1

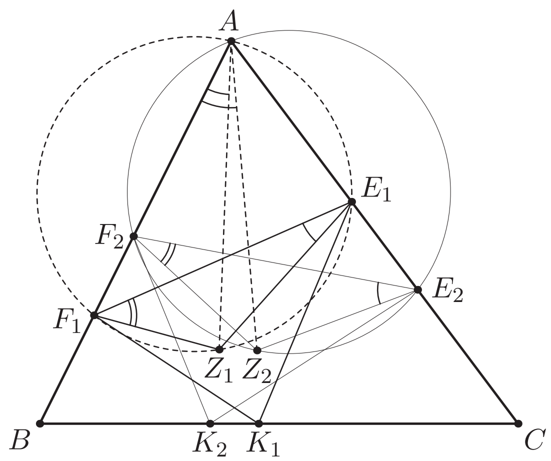


Figure 2

First method. We have just proved that all interesting triangles are similar to each other. This allows us to use the following lemma.

Lemma. Let ABC be an arbitrary triangle. Choose two points E_1 and E_2 on the side AC, two points F_1 and F_2 on the side AB, and two points K_1 and K_2 on the side BC, in a way that the triangles $E_1F_1K_1$ and $E_2F_2K_2$ are similar. Then the six circumcircles of the triangles AE_iF_i, BF_iK_i, and CE_iK_i $(i = 1, 2)$ meet at a common point Z. Moreover, Z is the centre of the spiral similarity that takes the triangle $E_1F_1K_1$ to the triangle $E_2F_2K_2$.

Proof. Firstly, notice that for each $i = 1, 2$, the circumcircles of the triangles AE_iF_i, BF_iK_i, and CK_iE_i have a common point Z_i by MIQUEL's theorem. Moreover, we have

$$\measuredangle(Z_iF_i, Z_iE_i) = \measuredangle(AB, CA), \quad \measuredangle(Z_iK_i, Z_iF_i) = \measuredangle(BC, AB), \quad \measuredangle(Z_iE_i, Z_iK_i) = \measuredangle(CA, BC).$$

This yields that the points Z_1 and Z_2 correspond to each other in similar triangles $E_1F_1K_1$ and $E_2F_2K_2$. Thus, if they coincide, then this common point is indeed the desired centre of a spiral similarity.

Finally, in order to show that $Z_1 = Z_2$, one may notice that $\measuredangle(AB, AZ_1) = \measuredangle(E_1F_1, E_1Z_1) = \measuredangle(E_2F_2, E_2Z_2) = \measuredangle(AB, AZ_2)$ (see Figure 2). Similarly, one has $\measuredangle(BC, BZ_1) = \measuredangle(BC, BZ_2)$ and $\measuredangle(CA, CZ_1) = \measuredangle(CA, CZ_2)$. This yields $Z_1 = Z_2$. $\square$

Now, let P and Q be the feet of the perpendiculars from B and C onto AC and AB, respectively, and let R be the midpoint of BC (see Figure 3). Then R is the circumcentre of the cyclic quadrilateral $BCPQ$. Thus we obtain $\angle APQ = \angle B$ and $\angle RPC = \angle C$, which yields $\angle QPR = \angle A$. Similarly, we show that $\angle PQR = \angle A$. Thus, all interesting triangles are similar to the triangle PQR.

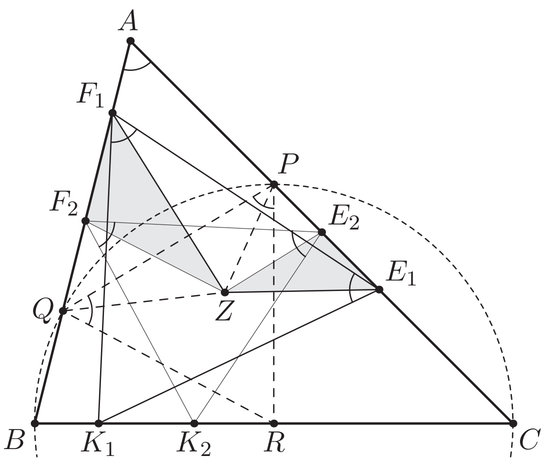


Figure 3

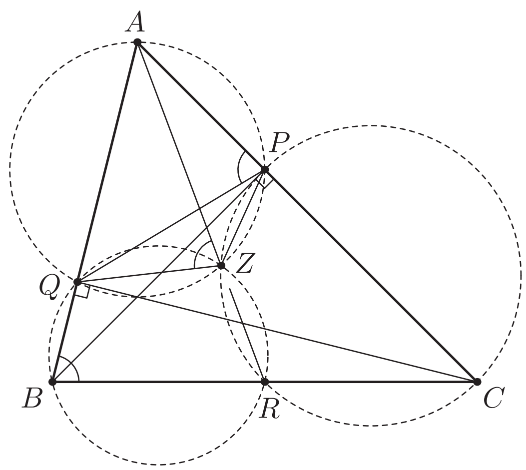


Figure 4

Denote now by Z the common point of the circumcircles of APQ, BQR, and CPR. Let $E_1F_1K_1$ and $E_2F_2K_2$ be two interesting triangles. By the lemma, Z is the centre of any spiral similarity taking one of the triangles $E_1F_1K_1$, $E_2F_2K_2$, and PQR to some other of them. Therefore the triangles ZE_1E_2 and ZF_1F_2 are similar, as well as the triangles ZE_1F_1 and ZPQ. Hence

$$\frac{E_1E_2}{F_1F_2} = \frac{ZE_1}{ZF_1} = \frac{ZP}{ZQ}.$$

Moreover, the equalities $\angle AZQ = \angle APQ = \angle ABC = 180^\circ - \angle QZR$ show that the point Z lies on the line AR (see Figure 4). Therefore the triangles AZP and ACR are similar, as well as the triangles AZQ and ABR. This yields

$$\frac{ZP}{ZQ} = \frac{ZP}{RC} \cdot \frac{RB}{ZQ} = \frac{AZ}{AC} \cdot \frac{AB}{AZ} = \frac{AB}{AC},$$

which completes the solution.

Second method. Now we will start from the fact that ω_1 is tangent to the lines KE and KF (see Figure 5). We prove that if (E, F) is an interesting pair, then

$$\frac{AE}{AB}+\frac{AF}{AC}=2\cos\angle A. \tag{1}$$

Let Y be the intersection point of the segments BE and CF. The points B, K, and C are collinear, hence applying Pascal's theorem to the degenerated hexagon $AFFYEE$, we infer that Y lies on the circle ω_1.

Denote by Z the second intersection point of the circumcircle of the triangle BFY with the line BC (see Figure 6). By Miquel's theorem, the points C, Z, Y, and E are concyclic. Therefore we obtain

$$BF\cdot AB+CE\cdot AC=BY\cdot BE+CY\cdot CF=BZ\cdot BC+CZ\cdot BC=BC^2\,.$$

On the other hand, $BC^2=AB^2+AC^2-2AB\cdot AC\cos\angle A$, by the cosine law. Hence

$$(AB-AF)\cdot AB+(AC-AE)\cdot AC=AB^2+AC^2-2AB\cdot AC\cos\angle A\,,$$

which simplifies to the desired equality (1).

Let now (E_1, F_1) and (E_2, F_2) be two interesting pairs of points. Then we get

$$\frac{AE_1}{AB}+\frac{AF_1}{AC}=\frac{AE_2}{AB}+\frac{AF_2}{AC}\,,$$

which gives the desired result.

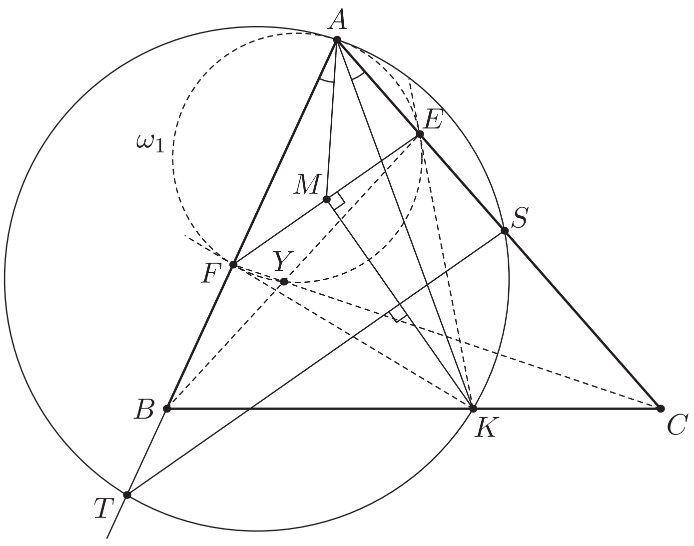


Figure 5

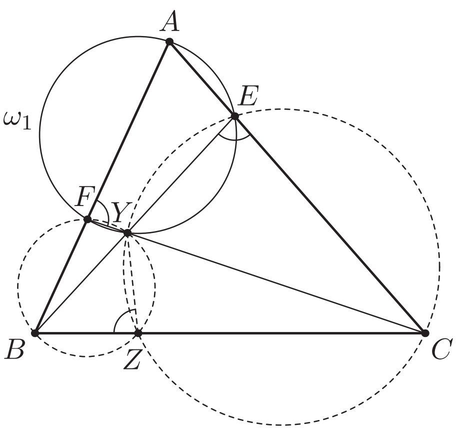


Figure 6

Third method. Again, we make use of the fact that all interesting triangles are similar (and equi-oriented). Let us put the picture onto a complex plane such that A is at the origin, and identify each point with the corresponding complex number.

Let EFK be any interesting triangle. The equalities $\angle KEF=\angle KFE=\angle A$ yield that the ratio $\nu=\frac{K-E}{F-E}$ is the same for all interesting triangles. This in turn means that the numbers E, F, and K satisfy the linear equation

$$K=\mu E+\nu F, \qquad \text{where} \quad \mu=1-\nu. \tag{2}$$

Now let us choose the points X and Y on the rays AB and AC, respectively, so that $\angle CXA = \angle AYB = \angle A = \angle KEF$ (see Figure 7). Then each of the triangles AXC and YAB is similar to any interesting triangle, which also means that

$$C = \mu A + \nu X = \nu X \quad \text{and} \quad B = \mu Y + \nu A = \mu Y. \tag{3}$$

Moreover, one has $X/Y = \overline{C/B}$.

Since the points E, F, and K lie on AC, AB, and BC, respectively, one gets

$$E = \rho Y, \quad F = \sigma X, \quad \text{and} \quad K = \lambda B + (1-\lambda)C$$

for some real ρ, σ, and λ. In view of (3), the equation (2) now reads $\lambda B + (1-\lambda)C = K = \mu E + \nu F = \rho B + \sigma C$, or

$$(\lambda - \rho)B = (\sigma + \lambda - 1)C.$$

Since the nonzero complex numbers B and C have different arguments, the coefficients in the brackets vanish, so $\rho = \lambda$ and $\sigma = 1 - \lambda$. Therefore,

$$\frac{E}{Y} + \frac{F}{X} = \rho + \sigma = 1. \tag{4}$$

Now, if (E_1, F_1) and (E_2, F_2) are two distinct interesting pairs, one may apply (4) to both pairs. Subtracting, we get

$$\frac{E_1 - E_2}{Y} = \frac{F_2 - F_1}{X}, \qquad \text{so} \quad \frac{E_1 - E_2}{F_2 - F_1} = \frac{Y}{X} = \frac{\overline{B}}{\overline{C}}.$$

Taking absolute values provides the required result.

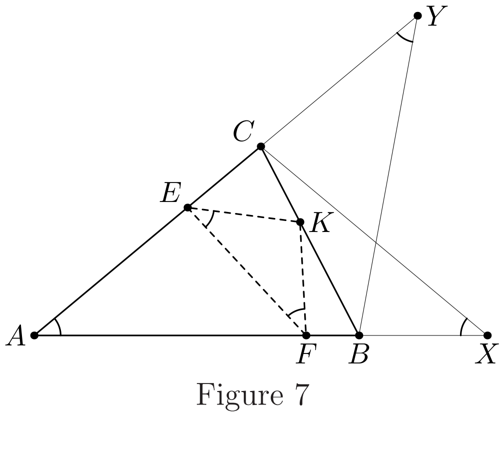


Figure 7

Comment 1. One may notice that the triangle PQR is also interesting.

Comment 2. In order to prove that $\angle KEF = \angle KFE = \angle A$, one may also use the following well-known fact:

Let AEF be a triangle with $AE \neq AF$, and let K be the common point of the symmedian taken from A and the perpendicular bisector of EF. Then the lines KE and KF are tangent to the circumcircle ω_1 of the triangle AEF.

In this case, however, one needs to deal with the case $AE = AF$ separately.

Solution 2. Let (E, F) be an interesting pair. This time we prove that

$$\frac{AM}{AK} = \cos\angle A\,. \tag{5}$$

As in Solution 1, we introduce the circle ω passing through the points K, S, A, and T, together with the points N and L at which the line AM intersect the line ST and the circle ω for the second time, respectively. Let moreover O be the centre of ω (see Figures 8 and 9). As in Solution 1, we note that N is the midpoint of ST and show that $KL \parallel ST$, which implies $\angle FAM = \angle EAK$.

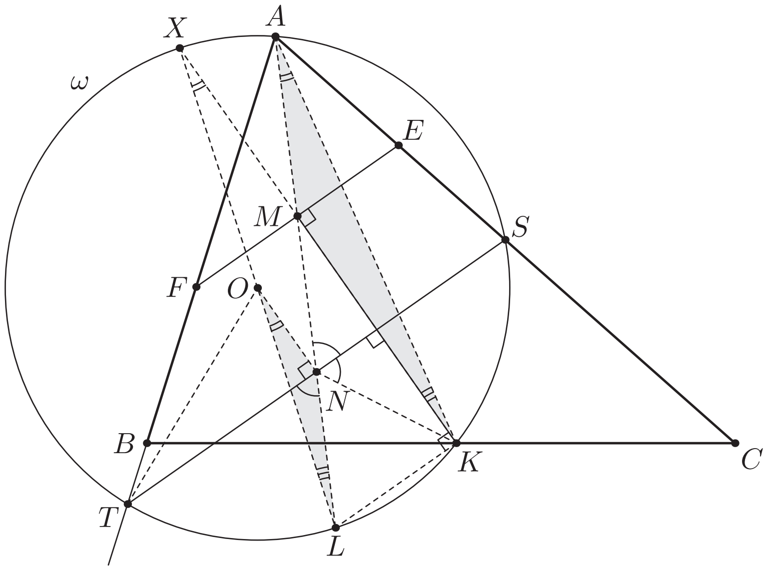


Figure 8

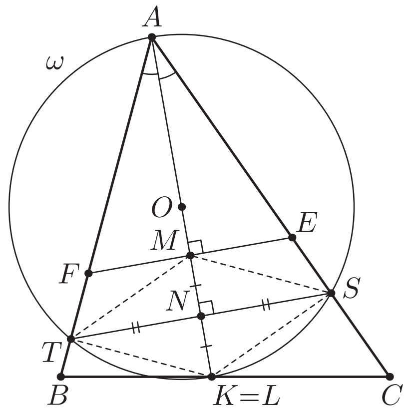


Figure 9

Suppose now that $K \neq L$ (see Figure 8). Then $KL \parallel ST$, and consequently the lines KM and KL are perpendicular. It implies that the lines LO and KM meet at a point X lying on the circle ω. Since the lines ON and XM are both perpendicular to the line ST, they are parallel to each other, and hence $\angle LON = \angle LXK = \angle MAK$. On the other hand, $\angle OLN = \angle MKA$, so we infer that triangles NOL and MAK are similar. This yields

$$\frac{AM}{AK} = \frac{ON}{OL} = \frac{ON}{OT} = \cos\angle TON = \cos\angle A\,.$$

If, on the other hand, $K = L$, then the points A, M, N, and K lie on a common line, and this line is the perpendicular bisector of ST (see Figure 9). This implies that AK is a diameter of ω, which yields $AM = 2OK - 2NK = 2ON$. So also in this case we obtain

$$\frac{AM}{AK} = \frac{2ON}{2OT} = \cos\angle TON = \cos\angle A\,.$$

Thus (5) is proved.

Let P and Q be the feet of the perpendiculars from B and C onto AC and AB, respectively (see Figure 10). We claim that the point M lies on the line PQ. Consider now the composition of the dilatation with factor $\cos\angle A$ and centre A, and the reflection with respect to the angle bisector of $\angle BAC$. This transformation is a similarity that takes B, C, and K to P, Q, and M, respectively. Since K lies on the line BC, the point M lies on the line PQ.

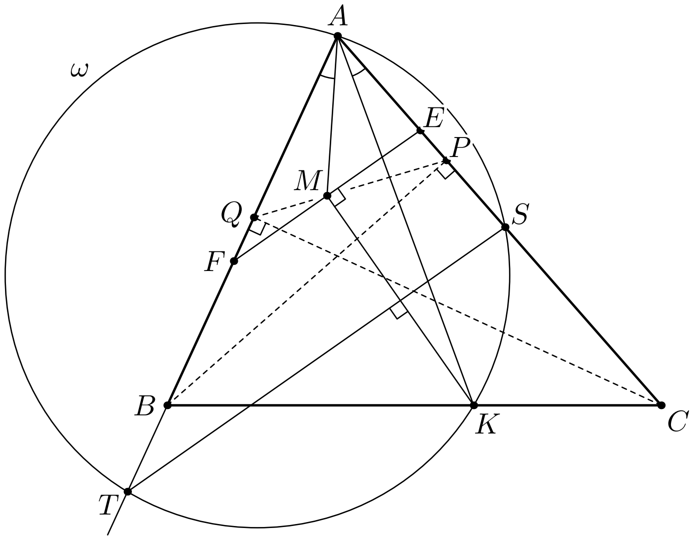


Figure 10

Suppose that $E \neq P$. Then also $F \neq Q$, and by MENELAUS' theorem, we obtain

$$\frac{AQ}{FQ} \cdot \frac{FM}{EM} \cdot \frac{EP}{AP} = 1\,.$$

Using the similarity of the triangles APQ and ABC, we infer that

$$\frac{EP}{FQ} = \frac{AP}{AQ} = \frac{AB}{AC}\,, \quad \text{and hence} \quad \frac{EP}{AB} = \frac{FQ}{AC}\,.$$

The last equality holds obviously also in case $E = P$, because then $F = Q$. Moreover, since the line PQ intersects the *segment* EF, we infer that the point E lies on the segment AP if and only if the point F lies outside of the segment AQ.

Let now (E_1, F_1) and (E_2, F_2) be two interesting pairs. Then we obtain

$$\frac{E_1P}{AB} = \frac{F_1Q}{AC} \qquad \text{and} \qquad \frac{E_2P}{AB} = \frac{F_2Q}{AC}\,.$$

If P lies between the points E_1 and E_2, we add the equalities above, otherwise we subtract them. In any case we obtain

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC}\,,$$

which completes the solution.

G7. Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V, respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y. Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

(U.S.A.)

Solution 1. We start with some general observations. Set $\alpha = \angle A/2$, $\beta = \angle B/2$, $\gamma = \angle C/2$. Then obviously $\alpha + \beta + \gamma = 90^\circ$. Since $\angle UIC = 90^\circ$, we obtain $\angle IUC = \alpha + \beta$. Therefore $\angle BIV = \angle IUC - \angle IBC = \alpha = \angle BAI = \angle BYV$, which implies that the points B, Y, I, and V lie on a common circle (see Figure 1).

Assume now that the points I, X and Y are collinear. We prove that $\angle YIA = 90^\circ$.

Let the line XU intersect AB at N. Since the lines AI, UX, and VY are parallel, we get

$$\frac{NX}{AI} = \frac{YN}{YA} = \frac{VU}{VI} = \frac{XU}{AI},$$

implying $NX = XU$. Moreover, $\angle BIU = \alpha = \angle BNU$. This implies that the quadrilateral $BUIN$ is cyclic, and since BI is the angle bisector of $\angle UBN$, we infer that $NI = UI$. Thus in the isosceles triangle NIU, the point X is the midpoint of the base NU. This gives $\angle IXN = 90^\circ$, i.e., $\angle YIA = 90^\circ$.

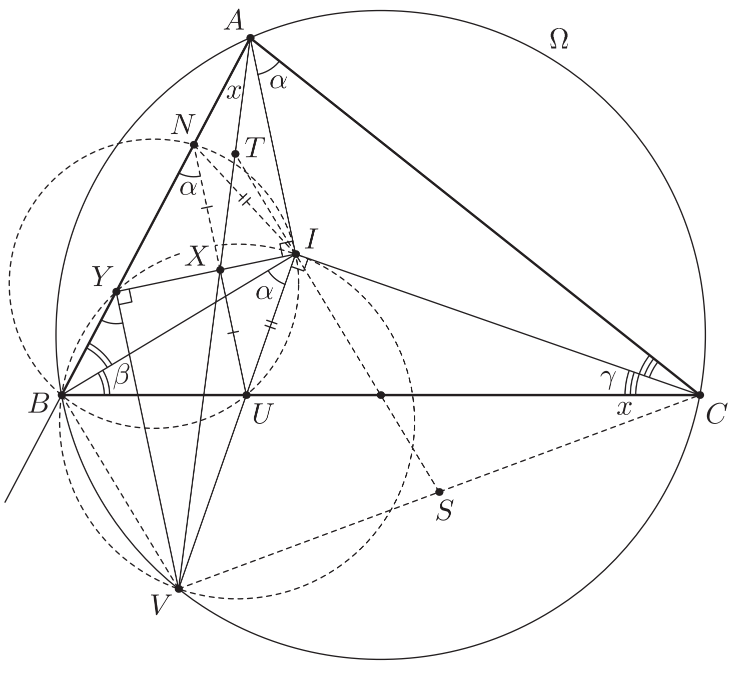


Figure 1

Let S be the midpoint of the segment VC. Let moreover T be the intersection point of the lines AX and SI, and set $x = \angle BAV = \angle BCV$. Since $\angle CIA = 90^\circ + \beta$ and $SI = SC$, we obtain

$$\angle TIA = 180^\circ - \angle AIS = 90^\circ - \beta - \angle CIS = 90^\circ - \beta - \gamma - x = \alpha - x = \angle TAI,$$

which implies that $TI = TA$. Therefore, since $\angle XIA = 90^\circ$, the point T is the midpoint of AX, i.e., $T = W$.

To complete our solution, it remains to show that the intersection point of the lines IS and BC coincide with the midpoint of the segment BC. But since S is the midpoint of the segment VC, it suffices to show that the lines BV and IS are parallel.

Since the quadrilateral $BYIV$ is cyclic, $\angle VBI = \angle VYI = \angle YIA = 90^\circ$. This implies that BV is the external angle bisector of the angle ABC, which yields $\angle VAC = \angle VCA$. Therefore $2\alpha - x = 2\gamma + x$, which gives $\alpha = \gamma + x$. Hence $\angle SCI = \alpha$, so $\angle VSI = 2\alpha$.

On the other hand, $\angle BVC = 180^\circ - \angle BAC = 180^\circ - 2\alpha$, which implies that the lines BV and IS are parallel. This completes the solution.

Solution 2. As in Solution 1, we first prove that the points B, Y, I, V lie on a common circle and $\angle YIA = 90^\circ$. The remaining part of the solution is based on the following lemma, which holds true for any triangle ABC, not necessarily with the property that I, X, Y are collinear.

Lemma. Let ABC be the triangle inscribed in a circle Γ and let I be its incentre. Assume that the line passing through I and perpendicular to the line AI intersects the side AB at the point Y. Let the circumcircle of the triangle BYI intersect the circle Γ for the second time at V, and let the excircle of the triangle ABC opposite to the vertex A be tangent to the side BC at E. Then

$$\angle BAV = \angle CAE\,.$$

Proof. Let ρ be the composition of the inversion with centre A and radius $\sqrt{AB \cdot AC}$, and the symmetry with respect to AI. Clearly, ρ interchanges B and C.

Let J be the excentre of the triangle ABC opposite to A (see Figure 2). Then we have $\angle JAC = \angle BAI$ and $\angle JCA = 90^\circ + \gamma = \angle BIA$, so the triangles ACJ and AIB are similar, and therefore $AB \cdot AC = AI \cdot AJ$. This means that ρ interchanges I and J. Moreover, since Y lies on AB and $\angle AIY = 90^\circ$, the point $Y' = \rho(Y)$ lies on AC, and $\angle JY'A = 90^\circ$. Thus ρ maps the circumcircle γ of the triangle BYI to a circle γ' with diameter JC.

Finally, since V lies on both Γ and γ, the point $V' = \rho(V)$ lies on the line $\rho(\Gamma) = AB$ as well as on γ', which in turn means that $V' = E$. This implies the desired result. $\square$

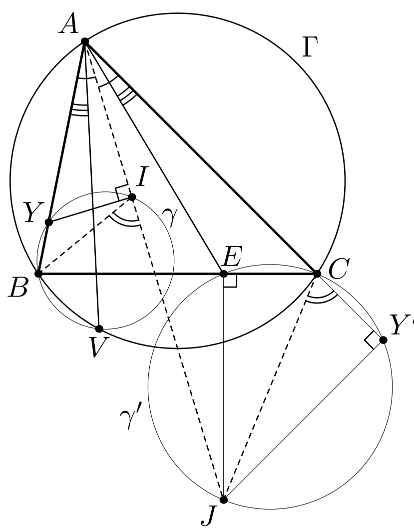


Figure 2

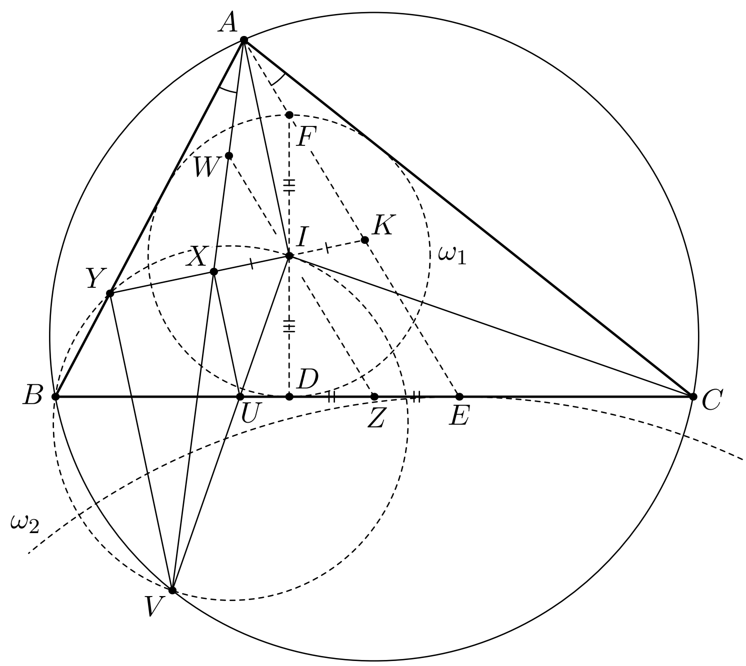


Figure 3

Now we turn to the solution of the problem.

Assume that the incircle ω_1 of the triangle ABC is tangent to BC at D, and let the excircle ω_2 of the triangle ABC opposite to the vertex A touch the side BC at E (see Figure 3). The homothety with centre A that takes ω_2 to ω_1 takes the point E to some point F, and the tangent to ω_1 at F is parallel to BC. Therefore DF is a diameter of ω_1. Moreover, Z is the midpoint of DE. This implies that the lines IZ and FE are parallel.

Let $K = YI \cap AE$. Since $\angle YIA = 90^\circ$, the lemma yields that I is the midpoint of XK. This implies that the segments IW and AK are parallel. Therefore, the points W, I and Z are collinear.

Comment 1. The properties $\angle YIA = 90^\circ$ and $VA = VC$ can be established in various ways. The main difficulty of the problem seems to find out how to use these properties in connection to the points W and Z.

In Solution 2 this principal part is more or less covered by the lemma, for which we have presented a direct proof. On the other hand, this lemma appears to be a combination of two well-known facts; let us formulate them in terms of the lemma statement.

Let the line IY intersect AC at P (see Figure 4). The first fact states that the circumcircle ω of the triangle VYP is tangent to the segments AB and AC, as well as to the circle Γ. The second fact states that for such a circle, the angles BAV and CAE are equal.

The awareness of this lemma may help a lot in solving this problem; so the Jury might also consider a variation of the proposed problem, for which the lemma does not seem to be useful; see Comment 3.

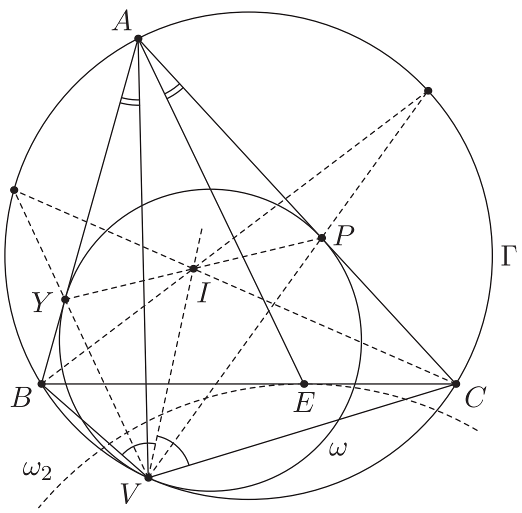


Figure 4

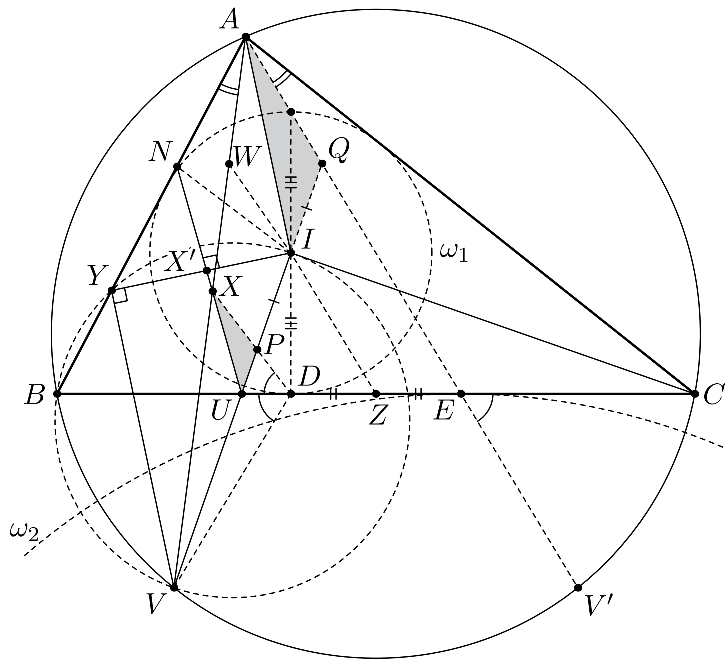


Figure 5

Comment 2. The proposed problem stated the equivalence: the point I lies on the line XY if and only if I lies on the line WZ. Here we sketch the proof of the "if" part (see Figure 5).

As in Solution 2, let BC touch the circles ω_1 and ω_2 at D and E, respectively. Since $IZ \parallel AE$ and W lies on IZ, the line DX is also parallel to AE. Therefore, the triangles XUP and AIQ are similar. Moreover, the line DX is symmetric to AE with respect to I, so $IP = IQ$, where $P = UV \cap XD$ and $Q = UV \cap AE$. Thus we obtain

$$\frac{UV}{VI} = \frac{UX}{IA} = \frac{UP}{IQ} = \frac{UP}{IP}.$$

So the pairs IU and PV are harmonic conjugates, and since $\angle UDI = 90^\circ$, we get $\angle VDB = \angle BDX = \angle BEA$. Therefore the point V' symmetric to V with respect to the perpendicular bisector of BC lies on the line AE. So we obtain $\angle BAV = \angle CAE$.

The rest can be obtained by simply reversing the arguments in Solution 2. The points B, V, I, and Y are concyclic. The lemma implies that $\angle YIA = 90^\circ$. Moreover, the points B, U, I, and N, where $N = UX \cap AB$, lie on a common circle, so $IN = IU$. Since $IY \perp UN$, the point $X' = IY \cap UN$ is the midpoint of UN. But in the trapezoid $AYVI$, the line XU is parallel to the sides AI and YV, so $NX = UX'$. This yields $X = X'$.

The reasoning presented in Solution 1 can also be reversed, but it requires a lot of technicalities. Therefore the Problem Selection Committee proposes to consider only the "only if" part of the original proposal, which is still challenging enough.

Comment 3. The Jury might also consider the following variation of the proposed problem.

Let ABC be a triangle with circumcircle Ω and incentre I. Let the line through I perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at U and V, respectively. Let the line through U parallel to AI intersect AV at X. Prove that if the lines XI and AI are perpendicular, then the midpoint of the segment AC lies on the line XI (see Figure 6).

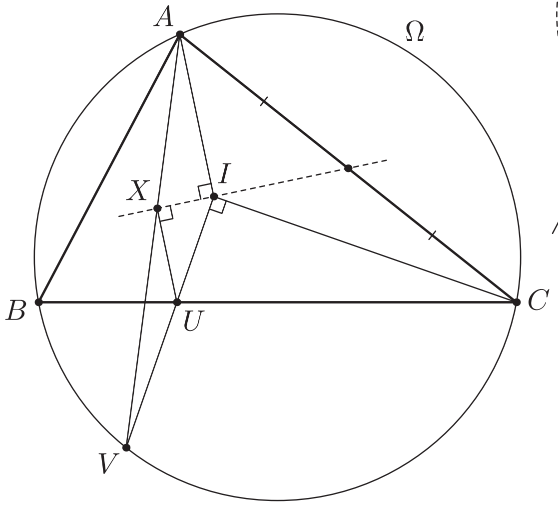


Figure 6

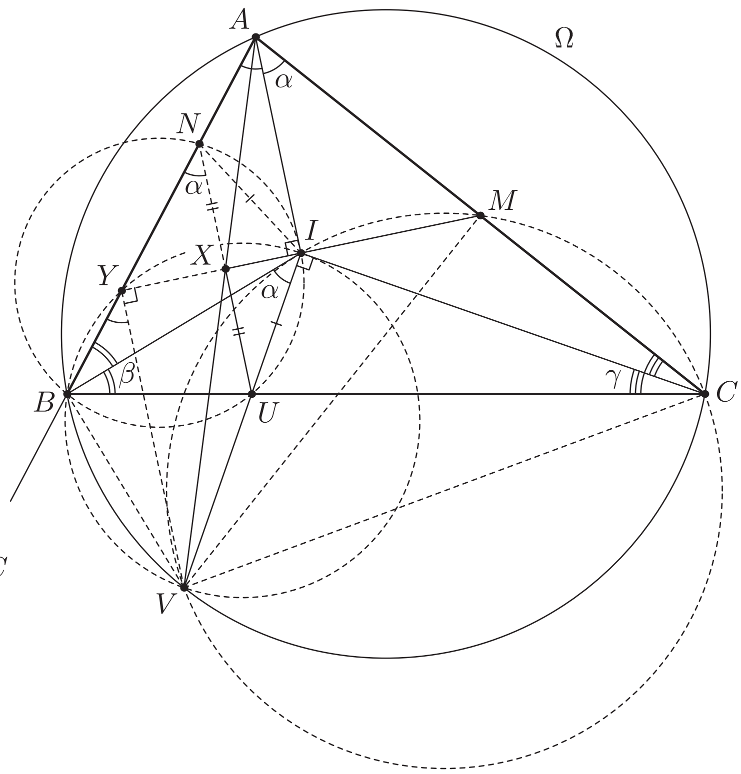


Figure 7

Since the solution contains the arguments used above, we only sketch it.

Let $N = XU \cap AB$ (see Figure 7). Then $\angle BNU = \angle BAI = \angle BIU$, so the points B, U, I, and N lie on a common circle. Therefore $IU = IN$, and since $IX \perp NU$, it follows that $NX = XU$.

Now set $Y = XI \cap AB$. The equality $NX = XU$ implies that

$$\frac{VX}{VA} = \frac{XU}{AI} = \frac{NX}{AI} = \frac{YX}{YI},$$

and therefore $YV \parallel AI$. Hence $\angle BYV = \angle BAI = \angle BIV$, so the points B, V, I, Y are concyclic. Next we have $IY \perp YV$, so $\angle IBV = 90^\circ$. This implies that BV is the external angle bisector of the angle ABC, which gives $\angle VAC = \angle VCA$.

So in order to show that $M = XI \cap AC$ is the midpoint of AC, it suffices to prove that $\angle VMC = 90^\circ$. But this follows immediately from the observation that the points V, C, M, and I are concyclic, as $\angle MIV = \angle YBV = 180^\circ - \angle ACV$.

The converse statement is also true, but its proof requires some technicalities as well.

Number Theory

N1. Let $n \geqslant 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leqslant k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n.

(Serbia)

Answer. $(n-2)2^n + 1$.

Solution 1.

Part I. First we show that every integer greater than $(n-2)2^n + 1$ can be represented as such a sum. This is achieved by induction on n.

For $n = 2$, the set A_n consists of the two elements 2 and 3. Every positive integer m except for 1 can be represented as the sum of elements of A_n in this case: as $m = 2 + 2 + \cdots + 2$ if m is even, and as $m = 3 + 2 + 2 + \cdots + 2$ if m is odd.

Now consider some $n > 2$, and take an integer $m > (n-2)2^n + 1$. If m is even, then consider

$$\frac{m}{2} \geqslant \frac{(n-2)2^n + 2}{2} = (n-2)2^{n-1} + 1 > (n-3)2^{n-1} + 1.$$

By the induction hypothesis, there is a representation of the form

$$\frac{m}{2} = (2^{n-1} - 2^{k_1}) + (2^{n-1} - 2^{k_2}) + \cdots + (2^{n-1} - 2^{k_r})$$

for some k_i with $0 \leqslant k_i < n-1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \cdots + (2^n - 2^{k_r+1}),$$

giving us the desired representation as a sum of elements of A_n. If m is odd, we consider

$$\frac{m - (2^n - 1)}{2} > \frac{(n-2)2^n + 1 - (2^n - 1)}{2} = (n-3)2^{n-1} + 1.$$

By the induction hypothesis, there is a representation of the form

$$\frac{m - (2^n - 1)}{2} = (2^{n-1} - 2^{k_1}) + (2^{n-1} - 2^{k_2}) + \cdots + (2^{n-1} - 2^{k_r})$$

for some k_i with $0 \leqslant k_i < n-1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \cdots + (2^n - 2^{k_r+1}) + (2^n - 1),$$

giving us the desired representation of m once again.

Part II. It remains to show that there is no representation for $(n-2)2^n + 1$. Let N be the smallest positive integer that satisfies $N \equiv 1 \pmod{2^n}$, and which can be represented as a sum of elements of A_n. Consider a representation of N, i.e.,

$$N = (2^n - 2^{k_1}) + (2^n - 2^{k_2}) + \cdots + (2^n - 2^{k_r}), \tag{1}$$

where $0 \leqslant k_1, k_2, \ldots, k_r < n$. Suppose first that two of the terms in the sum are the same, i.e., $k_i = k_j$ for some $i \neq j$. If $k_i = k_j = n-1$, then we can simply remove these two terms to get a representation for

$$N - 2(2^n - 2^{n-1}) = N - 2^n$$

as a sum of elements of A_n, which contradicts our choice of N. If $k_i = k_j = k < n-1$, replace the two terms by $2^n - 2^{k+1}$, which is also an element of A_n, to get a representation for

$$N - 2(2^n - 2^k) + 2^n - 2^{k+1} = N - 2^n.$$

This is a contradiction once again. Therefore, all k_i have to be distinct, which means that

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \leqslant 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

On the other hand, taking (1) modulo 2^n, we find

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \equiv -N \equiv -1 \pmod{2^n}.$$

Thus we must have $2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} = 2^n - 1$, which is only possible if each element of $\{0, 1, \ldots, n-1\}$ occurs as one of the k_i. This gives us

$$N = n2^n - (2^0 + 2^1 + \cdots + 2^{n-1}) = (n-1)2^n + 1.$$

In particular, this means that $(n-2)2^n + 1$ cannot be represented as a sum of elements of A_n.

Solution 2. The fact that $m = (n-2)2^n + 1$ cannot be represented as a sum of elements of A_n can also be shown in other ways. We prove the following statement by induction on n:

Claim. If a, b are integers with $a \geqslant 0$, $b \geqslant 1$, and $a + b < n$, then $a2^n + b$ cannot be written as a sum of elements of A_n.

Proof. The claim is clearly true for $n = 2$ (since $a = 0$, $b = 1$ is the only possibility). For $n > 2$, assume that there exist integers a, b with $a \geqslant 0$, $b \geqslant 1$ and $a + b < n$ as well as elements $m_1, m_2, \ldots, m_r$ of A_n such that

$$a2^n + b = m_1 + m_2 + \cdots + m_r.$$

We can suppose, without loss of generality, that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r$. Let ℓ be the largest index for which $m_\ell = 2^n - 1$ ($\ell = 0$ if $m_1 \neq 2^n - 1$). Clearly, ℓ and b must have the same parity. Now

$$(a - \ell)2^n + (b + \ell) = m_{\ell+1} + m_{\ell+2} + \cdots + m_r$$

and thus

$$(a - \ell)2^{n-1} + \frac{b + \ell}{2} = \frac{m_{\ell+1}}{2} + \frac{m_{\ell+2}}{2} + \cdots + \frac{m_r}{2}.$$

Note that $m_{\ell+1}/2, m_{\ell+2}/2, \ldots, m_r/2$ are elements of A_{n-1}. Moreover, $a - \ell$ and $(b + \ell)/2$ are integers, and $(b + \ell)/2 \geqslant 1$. If $a - \ell$ was negative, then we would have

$$a2^n + b \geqslant \ell(2^n - 1) \geqslant (a + 1)(2^n - 1) = a2^n + 2^n - a - 1,$$

thus $n \geqslant a + b + 1 \geqslant 2^n$, which is impossible. So $a - \ell \geqslant 0$. By the induction hypothesis, we must have $a - \ell + \frac{b+\ell}{2} \geqslant n - 1$, which gives us a contradiction, since

$$a - \ell + \frac{b + \ell}{2} \leqslant a - \ell + b + \ell - 1 = a + b - 1 < n - 1. \qquad \square$$

Considering the special case $a = n - 2$, $b = 1$ now completes the proof.

Solution 3. Denote by B_n the set of all positive integers that can be written as a sum of elements of A_n. In this solution, we explicitly describe all the numbers in B_n by an argument similar to the first solution.

For a positive integer n, we denote by $\sigma_2(n)$ the sum of its digits in the binary representation. Notice that every positive integer m has a unique representation of the form $m = s2^n - t$ with some positive integer s and $0 \leqslant t \leqslant 2^n - 1$.

Lemma. For any two integers $s \geqslant 1$ and $0 \leqslant t \leqslant 2^n - 1$, the number $m = s2^n - t$ belongs to B_n if and only if $s \geqslant \sigma_2(t)$.

Proof. For $t = 0$, the statement of the Lemma is obvious, since $m = 2s \cdot (2^n - 2^{n-1})$.

Now suppose that $t \geqslant 1$, and let

$$t = 2^{k_1} + \cdots + 2^{k_\sigma} \qquad (0 \leqslant k_1 < \cdots < k_\sigma \leqslant n-1, \quad \sigma = \sigma_2(t))$$

be its binary expansion. If $s \geqslant \sigma$, then $m \in B_n$ since

$$m = (s-\sigma)2^n + (\sigma 2^n - t) = 2(s-\sigma) \cdot (2^n - 2^{n-1}) + \sum_{i=1}^{\sigma}(2^n - 2^{k_i}).$$

Assume now that there exist integers s and t with $1 \leqslant s < \sigma_2(t)$ and $0 \leqslant t \leqslant 2^n - 1$ such that the number $m = s2^n - t$ belongs to B_n. Among all such instances, choose the one for which m is smallest, and let

$$m = \sum_{i=1}^{d}(2^n - 2^{\ell_i}) \qquad (0 \leqslant \ell_i \leqslant n-1)$$

be the corresponding representation. If all the ℓ_i's are distinct, then $\sum_{i=1}^{d} 2^{\ell_i} \leqslant \sum_{j=0}^{n-1} 2^j = 2^n - 1$, so one has $s = d$ and $t = \sum_{i=1}^{d} 2^{\ell_i}$, whence $s = d = \sigma_2(t)$; this is impossible. Therefore, two of the ℓ_i's must be equal, say $\ell_{d-1} = \ell_d$. Then $m \geqslant 2(2^n - 2^{\ell_d}) \geqslant 2^n$, so $s \geqslant 2$.

Now we claim that the number $m' = m - 2^n = (s-1)2^n - t$ also belongs to B_n, which contradicts the minimality assumption. Indeed, one has

$$(2^n - 2^{\ell_{d-1}}) + (2^n - 2^{\ell_d}) = 2(2^n - 2^{\ell_d}) = 2^n + (2^n - 2^{\ell_d+1}),$$

so

$$m' = \sum_{i=1}^{d-2}(2^n - 2^{\ell_i}) + (2^n - 2^{\ell_d+1})$$

is the desired representation of m' (if $\ell_d = n-1$, then the last summand is simply omitted). This contradiction finishes the proof. $\square$

By our lemma, the largest number M which does not belong to B_n must have the form

$$m_t = (\sigma_2(t) - 1)2^n - t$$

for some t with $1 \leqslant t \leqslant 2^n - 1$, so M is just the largest of these numbers. For $t_0 = 2^n - 1$ we have $m_{t_0} = (n-1)2^n - (2^n - 1) = (n-2)2^n + 1$; for every other value of t one has $\sigma_2(t) \leqslant n-1$, thus $m_t \leqslant (\sigma(t) - 1)2^n \leqslant (n-2)2^n < m_{t_0}$. This means that $M = m_{t_0} = (n-2)2^n + 1$.

N2. Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1\,. \tag{1}$$

(U.S.A.)

Answer. Either $(x, y) = (1, 1)$ or $\{x, y\} = \{m^3 + m^2 - 2m - 1, m^3 + 2m^2 - m - 1\}$ for some positive integer $m \geqslant 2$.

Solution. Let (x, y) be any pair of positive integers solving (1). We shall prove that it appears in the list displayed above. The converse assertion that all these pairs do actually satisfy (1) either may be checked directly by means of a somewhat laborious calculation, or it can be seen by going in reverse order through the displayed equations that follow.

In case $x = y$ the given equation reduces to $x^{2/3} = 1$, which is equivalent to $x = 1$, whereby he have found the first solution.

To find the solutions with $x \neq y$ we may assume $x > y$ due to symmetry. Then the integer $n = x - y$ is positive and (1) may be rewritten as

$$\sqrt[3]{7(y + n)^2 - 13(y + n)y + 7y^2} = n + 1\,.$$

Raising this to the third power and simplifying the result one obtains

$$y^2 + yn = n^3 - 4n^2 + 3n + 1\,.$$

To complete the square on the left hand side, we multiply by 4 and add n^2, thus getting

$$(2y + n)^2 = 4n^3 - 15n^2 + 12n + 4 = (n - 2)^2(4n + 1)\,.$$

This shows that the cases $n = 1$ and $n = 2$ are impossible, whence $n > 2$, and $4n + 1$ is the square of the rational number $\frac{2y+n}{n-2}$. Consequently, it has to be a perfect square, and, since it is odd as well, there has to exist some nonnegative integer m such that $4n + 1 = (2m + 1)^2$, i.e.

$$n = m^2 + m\,.$$

Notice that $n > 2$ entails $m \geqslant 2$. Substituting the value of n just found into the previous displayed equation we arrive at

$$(2y + m^2 + m)^2 = (m^2 + m - 2)^2(2m + 1)^2 = (2m^3 + 3m^2 - 3m - 2)^2\,.$$

Extracting square roots and taking $2m^3 + 3m^2 - 3m - 2 = (m - 1)(2m^2 + 5m + 2) > 0$ into account we derive $2y + m^2 + m = 2m^3 + 3m^2 - 3m - 2$, which in turn yields

$$y = m^3 + m^2 - 2m - 1\,.$$

Notice that $m \geqslant 2$ implies that $y = (m^3 - 1) + (m - 2)m$ is indeed positive, as it should be. In view of $x = y + n = y + m^2 + m$ it also follows that

$$x = m^3 + 2m^2 - m - 1\,,$$

and that this integer is positive as well.

Comment. Alternatively one could ask to find all pairs (x, y) of – not necessarily positive – integers solving (1). The answer to that question is a bit nicer than the answer above: the set of solutions are now described by

$$\{x, y\} = \{m^3 + m^2 - 2m - 1, m^3 + 2m^2 - m - 1\}\,,$$

where m varies through $\mathbb{Z}$. This may be shown using essentially the same arguments as above. We finally observe that the pair $(x, y) = (1, 1)$, that appears to be sporadic above, corresponds to $m = -1$.

N3. A coin is called a *Cape Town coin* if its value is $1/n$ for some positive integer n. Given a collection of Cape Town coins of total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into at most 100 groups each of total value at most 1.

(Luxembourg)

Solution. We will show that for every positive integer N any collection of Cape Town coins of total value at most $N - \frac{1}{2}$ can be split into N groups each of total value at most 1. The problem statement is a particular case for $N = 100$.

We start with some preparations. If several given coins together have a total value also of the form $\frac{1}{k}$ for a positive integer k, then we may merge them into one new coin. Clearly, if the resulting collection can be split in the required way then the initial collection can also be split.

After each such merging, the total number of coins decreases, thus at some moment we come to a situation when no more merging is possible. At this moment, for every even k there is at most one coin of value $\frac{1}{k}$ (otherwise two such coins may be merged), and for every odd $k > 1$ there are at most $k - 1$ coins of value $\frac{1}{k}$ (otherwise k such coins may also be merged).

Now, clearly, each coin of value 1 should form a single group; if there are d such coins then we may remove them from the collection and replace N by $N - d$. So from now on we may assume that there are no coins of value 1.

Finally, we may split all the coins in the following way. For each $k = 1, 2, \ldots, N$ we put all the coins of values $\frac{1}{2k-1}$ and $\frac{1}{2k}$ into a group G_k; the total value of G_k does not exceed

$$(2k-2) \cdot \frac{1}{2k-1} + \frac{1}{2k} < 1.$$

It remains to distribute the "small" coins of values which are less than $\frac{1}{2N}$; we will add them one by one. In each step, take any remaining small coin. The total value of coins in the groups at this moment is at most $N - \frac{1}{2}$, so there exists a group of total value at most $\frac{1}{N}\left(N - \frac{1}{2}\right) = 1 - \frac{1}{2N}$; thus it is possible to put our small coin into this group. Acting so, we will finally distribute all the coins.

Comment 1. The algorithm may be modified, at least the step where one distributes the coins of values $\geqslant \frac{1}{2N}$. One different way is to put into G_k all the coins of values $\frac{1}{(2k-1)2^s}$ for all integer $s \geqslant 0$. One may easily see that their total value also does not exceed 1.

Comment 2. The original proposal also contained another part, suggesting to show that a required splitting may be impossible if the total value of coins is at most 100. There are many examples of such a collection, e.g. one may take 98 coins of value 1, one coin of value $\frac{1}{2}$, two coins of value $\frac{1}{3}$, and four coins of value $\frac{1}{5}$.

The Problem Selection Committee thinks that this part is less suitable for the competition.

N4. Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geqslant 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

(Hong Kong)

Solution 1. If n is odd, let $k = n^m$ for $m = 1, 2, \ldots$. Then $a_k = n^{n^m - m}$, which is odd for each m.

Henceforth, assume that n is even, say $n = 2t$ for some integer $t \geqslant 1$. Then, for any $m \geqslant 2$, the integer $n^{2^m} - 2^m = 2^m(2^{2^m - m} \cdot t^{2^m} - 1)$ has an odd prime divisor p, since $2^m - m > 1$. Then, for $k = p \cdot 2^m$, we have

$$n^k = (n^{2^m})^p \equiv (2^m)^p = (2^p)^m \equiv 2^m,$$

where the congruences are taken modulo p (recall that $2^p \equiv 2 \pmod p$, by Fermat's little theorem). Also, from $n^k - 2^m < n^k < n^k + 2^m(p-1)$, we see that the fraction $\dfrac{n^k}{k}$ lies strictly between the consecutive integers $\dfrac{n^k - 2^m}{p \cdot 2^m}$ and $\dfrac{n^k + 2^m(p-1)}{p \cdot 2^m}$, which gives

$$\left\lfloor \frac{n^k}{k} \right\rfloor = \frac{n^k - 2^m}{p \cdot 2^m}.$$

We finally observe that $\dfrac{n^k - 2^m}{p \cdot 2^m} = \dfrac{\frac{n^k}{2^m} - 1}{p}$ is an odd integer, since the integer $\dfrac{n^k}{2^m} - 1$ is odd (recall that $k > m$). Note that for different values of m, we get different values of k, due to the different powers of 2 in the prime factorisation of k.

Solution 2. Treat the (trivial) case when n is odd as in Solution 1.

Now assume that n is even and $n > 2$. Let p be a prime divisor of $n - 1$.

Proceed by induction on i to prove that p^{i+1} is a divisor of $n^{p^i} - 1$ for every $i \geqslant 0$. The case $i = 0$ is true by the way in which p is chosen. Suppose the result is true for some $i \geqslant 0$. The factorisation

$$n^{p^{i+1}} - 1 = (n^{p^i} - 1)[n^{p^i(p-1)} + n^{p^i(p-2)} + \cdots + n^{p^i} + 1],$$

together with the fact that each of the p terms between the square brackets is congruent to 1 modulo p, implies that the result is also true for $i + 1$.

Hence $\left\lfloor \dfrac{n^{p^i}}{p^i} \right\rfloor = \dfrac{n^{p^i} - 1}{p^i}$, an odd integer for each $i \geqslant 1$.

Finally, we consider the case $n = 2$. We observe that $3 \cdot 4^i$ is a divisor of $2^{3 \cdot 4^i} - 4^i$ for every $i \geqslant 1$: Trivially, 4^i is a divisor of $2^{3\cdot 4^i} - 4^i$, since $3 \cdot 4^i > 2i$. Furthermore, since $2^{3\cdot 4^i}$ and 4^i are both congruent to 1 modulo 3, we have $3 \mid 2^{3\cdot 4^i} - 4^i$. Hence, $\left\lfloor \dfrac{2^{3\cdot 4^i}}{3 \cdot 4^i} \right\rfloor = \dfrac{2^{3\cdot 4^i} - 4^i}{3 \cdot 4^i} = \dfrac{2^{3\cdot 4^i - 2i} - 1}{3}$, which is odd for every $i \geqslant 1$.

Comment. The case n even and $n > 2$ can also be solved by recursively defining the sequence $(k_i)_{i\geqslant 1}$ by $k_1 = 1$ and $k_{i+1} = n^{k_i} - 1$ for $i \geqslant 1$. Then (k_i) is strictly increasing and it follows (by induction on i) that $k_i \mid n^{k_i} - 1$ for all $i \geqslant 1$, so the k_i are as desired.

The case $n = 2$ can also be solved as follows: Let $i \geqslant 2$. By Bertrand's postulate, there exists a prime number p such that $2^{2^i - 1} < p \cdot 2^i < 2^{2^i}$. This gives

$$p \cdot 2^i < 2^{2^i} < 2p \cdot 2^i. \tag{1}$$

Also, we have that $p \cdot 2^i$ is a divisor of $2^{p \cdot 2^i} - 2^{2^i}$, hence, using (1), we get that

$$\left\lfloor \frac{2^{p \cdot 2^i}}{p \cdot 2^i} \right\rfloor = \frac{2^{p \cdot 2^i} - 2^{2^i} + p \cdot 2^i}{p \cdot 2^i} = \frac{2^{p \cdot 2^i - i} - 2^{2^i - i} + p}{p},$$

which is an odd integer.

Solution 3. Treat the (trivial) case when n is odd as in Solution 1.

Let n be even, and let p be a prime divisor of $n+1$. Define the sequence $(a_i)_{i \geqslant 1}$ by

$$a_i = \min\left\{a \in \mathbb{Z}_{>0} \colon 2^i \text{ divides } ap + 1\right\}.$$

Recall that there exists a with $1 \leqslant a < 2^i$ such that $ap \equiv -1 \pmod{2^i}$, so each a_i satisfies $1 \leqslant a_i < 2^i$. This implies that $a_i p + 1 < p \cdot 2^i$. Also, $a_i \to \infty$ as $i \to \infty$, whence there are infinitely many i such that $a_i < a_{i+1}$. From now on, we restrict ourselves only to these i.

Notice that p is a divisor of $n^p + 1$, which, in turn, divides $n^{p \cdot 2^i} - 1$. It follows that $p \cdot 2^i$ is a divisor of $n^{p \cdot 2^i} - (a_i p + 1)$, and we consequently see that the integer $\left\lfloor \frac{n^{p \cdot 2^i}}{p \cdot 2^i} \right\rfloor = \frac{n^{p \cdot 2^i} - (a_i p + 1)}{p \cdot 2^i}$ is odd, since 2^{i+1} divides $n^{p \cdot 2^i}$, but not $a_i p + 1$.

N5. Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

(Belgium)

Answer. $(p, x, y) \in \{(3, 2, 5), (3, 5, 2)\} \cup \{(2, n, 2^k - n) \mid 0 < n < 2^k\}$.

Solution 1. For $p = 2$, clearly all pairs of two positive integers x and y whose sum is a power of 2 satisfy the condition. Thus we assume in the following that $p > 2$, and we let a and b be positive integers such that $x^{p-1} + y = p^a$ and $x + y^{p-1} = p^b$. Assume further, without loss of generality, that $x \leqslant y$, so that $p^a = x^{p-1} + y \leqslant x + y^{p-1} = p^b$, which means that $a \leqslant b$ (and thus $p^a \mid p^b$).

Now we have

$$p^b = y^{p-1} + x = (p^a - x^{p-1})^{p-1} + x.$$

We take this equation modulo p^a and take into account that $p - 1$ is even, which gives us

$$0 \equiv x^{(p-1)^2} + x \pmod{p^a}.$$

If $p \mid x$, then $p^a \mid x$, since $x^{(p-1)^2-1} + 1$ is not divisible by p in this case. However, this is impossible, since $x \leqslant x^{p-1} < p^a$. Thus we know that $p \nmid x$, which means that

$$p^a \mid x^{(p-1)^2-1} + 1 = x^{p(p-2)} + 1.$$

By FERMAT's little theorem, $x^{(p-1)^2} \equiv 1 \pmod{p}$, thus p divides $x+1$. Let p^r be the highest power of p that divides $x + 1$. By the binomial theorem, we have

$$x^{p(p-2)} = \sum_{k=0}^{p(p-2)} \binom{p(p-2)}{k} (-1)^{p(p-2)-k}(x+1)^k.$$

Except for the terms corresponding to $k = 0$, $k = 1$ and $k = 2$, all terms in the sum are clearly divisible by p^{3r} and thus by p^{r+2}. The remaining terms are

$$-\frac{p(p-2)(p^2-2p-1)}{2}(x+1)^2,$$

which is divisible by p^{2r+1} and thus also by p^{r+2},

$$p(p-2)(x+1),$$

which is divisible by p^{r+1}, but not p^{r+2} by our choice of r, and the final term -1 corresponding to $k = 0$. It follows that the highest power of p that divides $x^{p(p-2)} + 1$ is p^{r+1}.

On the other hand, we already know that p^a divides $x^{p(p-2)} + 1$, which means that $a \leqslant r + 1$. Moreover,

$$p^r \leqslant x + 1 \leqslant x^{p-1} + y = p^a.$$

Hence we either have $a = r$ or $a = r + 1$.

If $a = r$, then $x = y = 1$ needs to hold in the inequality above, which is impossible for $p > 2$. Thus $a = r + 1$. Now since $p^r \leqslant x + 1$, we get

$$x = \frac{x^2 + x}{x + 1} \leqslant \frac{x^{p-1} + y}{x + 1} = \frac{p^a}{x + 1} \leqslant \frac{p^a}{p^r} = p,$$

so we must have $x = p - 1$ for p to divide $x + 1$.

It follows that $r = 1$ and $a = 2$. If $p \geqslant 5$, we obtain

$$p^a = x^{p-1} + y > (p-1)^4 = (p^2 - 2p + 1)^2 > (3p)^2 > p^2 = p^a,$$

a contradiction. So the only case that remains is $p = 3$, and indeed $x = 2$ and $y = p^a - x^{p-1} = 5$ satisfy the conditions.

Comment 1. In this solution, we are implicitly using a special case of the following lemma known as "lifting the exponent":

Lemma. Let n be a positive integer, let p be an odd prime, and let $v_p(m)$ denote the exponent of the highest power of p that divides m.

If x and y are integers not divisible by p such that $p \mid x - y$, then we have

$$v_p(x^n - y^n) = v_p(x - y) + v_p(n).$$

Likewise, if x and y are integers not divisible by p such that $p \mid x + y$, then we have

$$v_p(x^n + y^n) = v_p(x + y) + v_p(n).$$

Comment 2. There exist various ways of solving the problem involving the "lifting the exponent" lemma. Let us sketch another one.

The cases $x = y$ and $p \mid x$ are ruled out easily, so we assume that $p > 2$, $x < y$, and $p \nmid x$. In this case we also have $p^a < p^b$ and $p \mid x + 1$.

Now one has

$$y^p - x^p \equiv y(y^{p-1} + x) - x(x^{p-1} + y) \equiv 0 \pmod{p^a},$$

so by the lemma mentioned above one has $p^{a-1} \mid y - x$ and hence $y = x + tp^{a-1}$ for some positive integer t. Thus one gets

$$x(x^{p-2} + 1) = x^{p-1} + x = (x^{p-1} + y) - (y - x) = p^{a-1}(p - t).$$

The factors on the left-hand side are coprime. So if $p \mid x$, then $x^{p-2} + 1 \mid p - t$, which is impossible since $x < x^{p-2} + 1$. Therefore, $p \nmid x$, and thus $x \mid p - t$. Since $p \mid x + 1$, the only remaining case is $x = p - 1$, $t = 1$, and $y = p^{a-1} + p - 1$. Now the solution can be completed in the same way as before.

Solution 2. Again, we can focus on the case that $p > 2$. If $p \mid x$, then also $p \mid y$. In this case, let p^k and p^ℓ be the highest powers of p that divide x and y respectively, and assume without loss of generality that $k \leqslant \ell$. Then p^k divides $x + y^{p-1}$ while p^{k+1} does not, but $p^k < x + y^{p-1}$, which yields a contradiction. So x and y are not divisible by p. FERMAT's little theorem yields $0 \equiv x^{p-1} + y \equiv 1 + y \pmod{p}$, so $y \equiv -1 \pmod{p}$ and for the same reason $x \equiv -1 \pmod{p}$.

In particular, $x, y \geqslant p - 1$ and thus $x^{p-1} + y \geqslant 2(p-1) > p$, so $x^{p-1} + y$ and $y^{p-1} + x$ are both at least equal to p^2. Now we have

$$x^{p-1} \equiv -y \pmod{p^2} \qquad \text{and} \qquad y^{p-1} \equiv -x \pmod{p^2}.$$

These two congruences, together with the EULER–FERMAT theorem, give us

$$1 \equiv x^{p(p-1)} \equiv (-y)^p \equiv -y^p \equiv xy \pmod{p^2}.$$

Since $x \equiv y \equiv -1 \pmod{p}$, $x - y$ is divisible by p, so $(x-y)^2$ is divisible by p^2. This means that

$$(x+y)^2 = (x-y)^2 + 4xy \equiv 4 \pmod{p^2},$$

so p^2 divides $(x+y-2)(x+y+2)$. We already know that $x + y \equiv -2 \pmod{p}$, so $x + y - 2 \equiv -4 \not\equiv 0 \pmod{p}$. This means that p^2 divides $x + y + 2$.

Using the same notation as in the first solution, we subtract the two original equations to obtain

$$p^b - p^a = y^{p-1} - x^{p-1} + x - y = (y - x)(y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1). \tag{1}$$

The second factor is symmetric in x and y, so it can be written as a polynomial of the elementary symmetric polynomials $x + y$ and xy with integer coefficients. In particular, its value modulo

p^2 is characterised by the two congruences $xy \equiv 1 \pmod{p^2}$ and $x + y \equiv -2 \pmod{p^2}$. Since both congruences are satisfied when $x = y = -1$, we must have

$$y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1 \equiv (-1)^{p-2} + (-1)^{p-3}(-1) + \cdots + (-1)^{p-2} - 1 \pmod{p^2},$$

which simplifies to $y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1 \equiv -p \pmod{p^2}$. Thus the second factor in (1) is divisible by p, but not p^2.

This means that p^{a-1} has to divide the other factor $y - x$. It follows that

$$0 \equiv x^{p-1} + y \equiv x^{p-1} + x \equiv x(x+1)(x^{p-3} - x^{p-4} + \cdots + 1) \pmod{p^{a-1}}.$$

Since $x \equiv -1 \pmod{p}$, the last factor is $x^{p-3} - x^{p-4} + \cdots + 1 \equiv p - 2 \pmod{p}$ and in particular not divisible by p. We infer that $p^{a-1} \mid x + 1$ and continue as in the first solution.

Comment. Instead of reasoning by means of elementary symmetric polynomials, it is possible to provide a more direct argument as well. For odd r, $(x+1)^2$ divides $(x^r + 1)^2$, and since p divides $x + 1$, we deduce that p^2 divides $(x^r + 1)^2$. Together with the fact that $xy \equiv 1 \pmod{p^2}$, we obtain

$$0 \equiv y^r(x^r + 1)^2 \equiv x^{2r}y^r + 2x^r y^r + y^r \equiv x^r + 2 + y^r \pmod{p^2}.$$

We apply this congruence with $r = p - 2 - 2k$ (where $0 \leqslant k < (p-2)/2$) to find that

$$x^k y^{p-2-k} + x^{p-2-k} y^k \equiv (xy)^k (x^{p-2-2k} + y^{p-2-2k}) \equiv 1^k \cdot (-2) \equiv -2 \pmod{p^2}.$$

Summing over all k yields

$$y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1 \equiv \tfrac{p-1}{2} \cdot (-2) - 1 \equiv -p \pmod{p^2}$$

once again.

N6. Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geqslant n+2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

(Serbia)

Solution 1. Let $A = a_1 \cdots a_n$. Throughout the solution, all intervals will be nonempty and have integer end-points. For any interval X, the length of X will be denoted by $|X|$.

Define the following two families of intervals:

$$\mathcal{S} = \big\{[x,y] \colon x < y \text{ are consecutive marked points}\big\}$$
$$\mathcal{T} = \big\{[x,y] \colon x < y \text{ are integers, } 0 \leqslant x \leqslant A-1, \text{ and no point is marked in } (x,y)\big\}$$

We are interested in computing $\sum\limits_{X \in \mathcal{S}} |X|^2$ modulo a_1.

Note that the number A is marked, so in the definition of $\mathcal{T}$ the condition $y \leqslant A$ is enforced without explicitly prescribing it.

Assign weights to the intervals in $\mathcal{T}$, depending only on their lengths. The weight of an arbitrary interval $Y \in \mathcal{T}$ will be $w\big(|Y|\big)$, where

$$w(k) = \begin{cases} 1 & \text{if } k = 1, \\ 2 & \text{if } k \geqslant 2\,. \end{cases}$$

Consider an arbitrary interval $X \in \mathcal{S}$ and its sub-intervals $Y \in \mathcal{T}$. Clearly, X has one sub-interval of length $|X|$, two sub-intervals of length $|X|-1$ and so on; in general X has $|X|-d+1$ sub-intervals of length d for every $d = 1, 2, \ldots, |X|$. The sum of the weights of the sub-intervals of X is

$$\sum_{Y \in \mathcal{T},\, Y \subseteq X} w\big(|Y|\big) = \sum_{d=1}^{|X|} (|X|-d+1) \cdot w(d) = |X| \cdot 1 + \big((|X|-1) + (|X|-2) + \cdots + 1\big) \cdot 2 = |X|^2.$$

Since the intervals in $\mathcal{S}$ are non-overlapping, every interval $Y \in \mathcal{T}$ is a sub-interval of a single interval $X \in \mathcal{S}$. Therefore,

$$\sum_{X \in \mathcal{S}} |X|^2 = \sum_{X \in \mathcal{S}} \left(\sum_{Y \in \mathcal{T},\, Y \subseteq X} w\big(|Y|\big) \right) = \sum_{Y \in \mathcal{T}} w\big(|Y|\big). \tag{1}$$

For every $d = 1, 2, \ldots, a_1$, we count how many intervals in $\mathcal{T}$ are of length d. Notice that the multiples of a_1 are all marked, so the lengths of the intervals in $\mathcal{S}$ and $\mathcal{T}$ cannot exceed a_1. Let x be an arbitrary integer with $0 \leqslant x \leqslant A-1$ and consider the interval $[x, x+d]$. Let r_1, $\ldots$, r_n be the remainders of x modulo a_1, $\ldots$, a_n, respectively. Since $a_1, \ldots, a_n$ are pairwise coprime, the number x is uniquely identified by the sequence $(r_1, \ldots, r_n)$, due to the Chinese remainder theorem.

For every $i = 1, \ldots, n$, the property that the interval $(x, x+d)$ does not contain any multiple of a_i is equivalent with $r_i + d \leqslant a_i$, i.e. $r_i \in \{0, 1, \ldots, a_i - d\}$, so there are $a_i - d + 1$ choices for the number r_i for each i. Therefore, the number of the remainder sequences $(r_1, \ldots, r_n)$ that satisfy $[x, x+d] \in \mathcal{T}$ is precisely $(a_1 + 1 - d) \cdots (a_n + 1 - d)$. Denote this product by $f(d)$.

Now we can group the last sum in (1) by length of the intervals. As we have seen, for every $d = 1, \dots, a_1$ there are $f(d)$ intervals $Y \in \mathcal{T}$ with $|Y| = d$. Therefore, (1) can be continued as

$$\sum_{X \in \mathcal{S}} |X|^2 = \sum_{Y \in \mathcal{T}} w\big(|Y|\big) = \sum_{d=1}^{a_1} f(d) \cdot w(d) = 2\sum_{d=1}^{a_1} f(d) - f(1). \tag{2}$$

Having the formula (2), the solution can be finished using the following well-known fact:

Lemma. If p is a prime, $F(x)$ is a polynomial with integer coefficients, and $\deg F \leqslant p-2$, then $\sum_{x=1}^{p} F(x)$ is divisible by p.

Proof. Obviously, it is sufficient to prove the lemma for monomials of the form x^k with $k \leqslant p-2$. Apply induction on k. If $k = 0$ then $F = 1$, and the statement is trivial.

Let $1 \leqslant k \leqslant p-2$, and assume that the lemma is proved for all lower degrees. Then

$$\begin{aligned} 0 \equiv p^{k+1} &= \sum_{x=1}^{p} \big(x^{k+1} - (x-1)^{k+1}\big) = \sum_{x=1}^{p} \left(\sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k+1}{\ell} x^{\ell} \right) \\ &= (k+1)\sum_{x=1}^{p} x^k + \sum_{\ell=0}^{k-1} (-1)^{k-\ell} \binom{k+1}{\ell} \sum_{x=1}^{p} x^{\ell} \equiv (k+1)\sum_{x=1}^{p} x^k \pmod{p}. \end{aligned}$$

Since $0 < k+1 < p$, this proves $\sum_{x=1}^{p} x^k \equiv 0 \pmod{p}$. □

In (2), by applying the lemma to the polynomial f and the prime a_1, we obtain that $\sum_{d=1}^{a_1} f(d)$ is divisible by a_1. The term $f(1) = a_1 \cdots a_n$ is also divisible by a_1; these two facts together prove that $\sum_{X \in \mathcal{S}} |X|^2$ is divisible by a_1.

Comment 1. With suitable sets of weights, the same method can be used to sum up other expressions on the lengths of the segments. For example, $w(1) = 1$ and $w(k) = 6(k-1)$ for $k \geqslant 2$ can be used to compute $\sum_{X \in \mathcal{S}} |X|^3$ and to prove that this sum is divisible by a_1 if a_1 is a prime with $a_1 \geqslant n+3$. See also Comment 2 after the second solution.

Solution 2. The conventions from the first paragraph of the first solution are still in force. We shall prove the following more general statement:

> ($\boxplus$) *Let p denote a prime number, let $p = a_1 < a_2 < \cdots < a_n$ be n pairwise coprime positive integers, and let d be an integer with $1 \leqslant d \leqslant p-n$. Mark all integers that are divisible by at least one of the numbers $a_1, \dots, a_n$ on the interval $I = [0, a_1 a_2 \cdots a_n]$ of the real line. These points split I into a number of smaller segments, say of lengths $b_1, \dots, b_k$. Then the sum $\sum_{i=1}^{k} \binom{b_i}{d}$ is divisible by p.*

Applying ($\boxplus$) to $d = 1$ and $d = 2$ and using the equation $x^2 = 2\binom{x}{2} + \binom{x}{1}$, one easily gets the statement of the problem.

To prove ($\boxplus$) itself, we argue by induction on n. The base case $n = 1$ follows from the known fact that the binomial coefficient $\binom{p}{d}$ is divisible by p whenever $1 \leqslant d \leqslant p-1$.

Let us now assume that $n \geqslant 2$, and that the statement is known whenever $n-1$ rather than n coprime integers are given together with some integer $d \in [1, p-n+1]$. Suppose that the numbers $p = a_1 < a_2 < \cdots < a_n$ and d are as above. Write $A' = \prod_{i=1}^{n-1} a_i$ and $A = A'\, a_n$. Mark the points on the real axis divisible by one of the numbers $a_1, \ldots, a_{n-1}$ *green* and those divisible by a_n *red*. The green points divide $[0, A']$ into certain sub-intervals, say J_1, J_2, …, and J_ℓ.

To translate intervals we use the notation $[a, b] + m = [a + m, b + m]$ whenever $a, b, m \in \mathbb{Z}$.

For each $i \in \{1, 2, \ldots, \ell\}$ let $\mathcal{F}_i$ be the family of intervals into which the red points partition the intervals J_i, $J_i + A'$, …, and $J_i + (a_n - 1)A'$. We are to prove that

$$\sum_{i=1}^{\ell} \sum_{X \in \mathcal{F}_i} \binom{|X|}{d}$$

is divisible by p.

Let us fix any index i with $1 \leqslant i \leqslant \ell$ for a while. Since the numbers A' and a_n are coprime by hypothesis, the numbers $0, A', \ldots, (a_n - 1)A'$ form a complete system of residues modulo a_n. Moreover, we have $|J_i| \leqslant p < a_n$, as in particular all multiples of p are green. So each of the intervals J_i, $J_i + A'$, …, and $J_i + (a_n - 1)A'$ contains at most one red point. More precisely, for each $j \in \{1, \ldots, |J_i| - 1\}$ there is exactly one amongst those intervals containing a red point splitting it into an interval of length j followed by an interval of length $|J_i| - j$, while the remaining $a_n - |J_i| + 1$ such intervals have no red points in their interiors. For these reasons

$$\begin{aligned}
\sum_{X \in \mathcal{F}_i} \binom{|X|}{d} &= 2\left(\binom{1}{d} + \cdots + \binom{|J_i| - 1}{d}\right) + (a_n - |J_i| + 1)\binom{|J_i|}{d} \\
&= 2\binom{|J_i|}{d+1} + (a_n - d + 1)\binom{|J_i|}{d} - (d+1)\binom{|J_i|}{d+1} \\
&= (1 - d)\binom{|J_i|}{d+1} + (a_n - d + 1)\binom{|J_i|}{d}.
\end{aligned}$$

So it remains to prove that

$$(1 - d)\sum_{i=1}^{\ell} \binom{|J_i|}{d+1} + (a_n - d + 1)\sum_{i=1}^{\ell} \binom{|J_i|}{d}$$

is divisible by p. By the induction hypothesis, however, it is even true that both summands are divisible by p, for $1 \leqslant d < d + 1 \leqslant p - (n - 1)$. This completes the proof of ($\boxplus$) and hence the solution of the problem.

Comment 2. The statement ($\boxplus$) can also be proved by the method of the first solution, using the weights $w(x) = \binom{x-2}{d-2}$.

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N7. Let $c \geqslant 1$ be an integer. Define a sequence of positive integers by $a_1 = c$ and

$$a_{n+1} = a_n^3 - 4c \cdot a_n^2 + 5c^2 \cdot a_n + c$$

for all $n \geqslant 1$. Prove that for each integer $n \geqslant 2$ there exists a prime number p dividing a_n but none of the numbers $a_1, \ldots, a_{n-1}$.

(Austria)

Solution. Let us define $x_0 = 0$ and $x_n = a_n / c$ for all integers $n \geqslant 1$. It is easy to see that the sequence (x_n) thus obtained obeys the recursive law

$$x_{n+1} = c^2(x_n^3 - 4x_n^2 + 5x_n) + 1 \tag{1}$$

for all integers $n \geqslant 0$. In particular, all of its terms are positive integers; notice that $x_1 = 1$ and $x_2 = 2c^2 + 1$. Since

$$x_{n+1} = c^2 x_n (x_n - 2)^2 + c^2 x_n + 1 > x_n \tag{2}$$

holds for all integers $n \geqslant 0$, it is also strictly increasing. Since x_{n+1} is by (1) coprime to c for any $n \geqslant 0$, it suffices to prove that for each $n \geqslant 2$ there exists a prime number p dividing x_n but none of the numbers $x_1, \ldots, x_{n-1}$. Let us begin by establishing three preliminary claims.

Claim 1. If $i \equiv j \pmod m$ holds for some integers $i, j \geqslant 0$ and $m \geqslant 1$, then $x_i \equiv x_j \pmod{x_m}$ holds as well.

Proof. Evidently, it suffices to show $x_{i+m} \equiv x_i \pmod{x_m}$ for all integers $i \geqslant 0$ and $m \geqslant 1$. For this purpose we may argue for fixed m by induction on i using $x_0 = 0$ in the base case $i = 0$. Now, if we have $x_{i+m} \equiv x_i \pmod{x_m}$ for some integer i, then the recursive equation (1) yields

$$x_{i+m+1} \equiv c^2(x_{i+m}^3 - 4x_{i+m}^2 + 5x_{i+m}) + 1 \equiv c^2(x_i^3 - 4x_i^2 + 5x_i) + 1 \equiv x_{i+1} \pmod{x_m},$$

which completes the induction. □

Claim 2. If the integers $i, j \geqslant 2$ and $m \geqslant 1$ satisfy $i \equiv j \pmod m$, then $x_i \equiv x_j \pmod{x_m^2}$ holds as well.

Proof. Again it suffices to prove $x_{i+m} \equiv x_i \pmod{x_m^2}$ for all integers $i \geqslant 2$ and $m \geqslant 1$. As above, we proceed for fixed m by induction on i. The induction step is again easy using (1), but this time the base case $i = 2$ requires some calculation. Set $L = 5c^2$. By (1) we have $x_{m+1} \equiv Lx_m + 1 \pmod{x_m^2}$, and hence

$$\begin{aligned} x_{m+1}^3 - 4x_{m+1}^2 + 5x_{m+1} &\equiv (Lx_m + 1)^3 - 4(Lx_m + 1)^2 + 5(Lx_m + 1) \\ &\equiv (3Lx_m + 1) - 4(2Lx_m + 1) + 5(Lx_m + 1) \equiv 2 \pmod{x_m^2}, \end{aligned}$$

which in turn gives indeed $x_{m+2} \equiv 2c^2 + 1 \equiv x_2 \pmod{x_m^2}$. □

Claim 3. For each integer $n \geqslant 2$, we have $x_n > x_1 \cdot x_2 \cdots x_{n-2}$.

Proof. The cases $n = 2$ and $n = 3$ are clear. Arguing inductively, we assume now that the claim holds for some $n \geqslant 3$. Recall that $x_2 \geqslant 3$, so by monotonicity and (2) we get $x_n \geqslant x_3 \geqslant x_2(x_2 - 2)^2 + x_2 + 1 \geqslant 7$. It follows that

$$x_{n+1} > x_n^3 - 4x_n^2 + 5x_n > 7x_n^2 - 4x_n^2 > x_n^2 > x_n x_{n-1},$$

which by the induction hypothesis yields $x_{n+1} > x_1 \cdot x_2 \cdots x_{n-1}$, as desired. □

Now we direct our attention to the problem itself: let any integer $n \geqslant 2$ be given. By Claim 3 there exists a prime number p appearing with a higher exponent in the prime factorisation of x_n than in the prime factorisation of $x_1 \cdots x_{n-2}$. In particular, $p \mid x_n$, and it suffices to prove that p divides none of $x_1, \ldots, x_{n-1}$.

Otherwise let $k \in \{1, \ldots, n-1\}$ be minimal such that p divides x_k. Since x_{n-1} and x_n are coprime by (1) and $x_1 = 1$, we actually have $2 \leqslant k \leqslant n-2$. Write $n = qk + r$ with some integers $q \geqslant 0$ and $0 \leqslant r < k$. By Claim 1 we have $x_n \equiv x_r \pmod{x_k}$, whence $p \mid x_r$. Due to the minimality of k this entails $r = 0$, i.e. $k \mid n$.

Thus from Claim 2 we infer

$$x_n \equiv x_k \pmod{x_k^2}.$$

Now let $\alpha \geqslant 1$ be maximal with the property $p^\alpha \mid x_k$. Then x_k^2 is divisible by $p^{\alpha+1}$ and by our choice of p so is x_n. So by the previous congruence x_k is a multiple of $p^{\alpha+1}$ as well, contrary to our choice of α. This is the final contradiction concluding the solution.

N8. For every real number x, let $\|x\|$ denote the distance between x and the nearest integer. Prove that for every pair (a, b) of positive integers there exist an odd prime p and a positive integer k satisfying

$$\left\|\frac{a}{p^k}\right\| + \left\|\frac{b}{p^k}\right\| + \left\|\frac{a+b}{p^k}\right\| = 1. \tag{1}$$

(Hungary)

Solution. Notice first that $\left\lfloor x + \frac{1}{2} \right\rfloor$ is an integer nearest to x, so $\|x\| = \left|\left\lfloor x + \frac{1}{2}\right\rfloor - x\right|$. Thus we have

$$\left\lfloor x + \frac{1}{2} \right\rfloor = x \pm \|x\|. \tag{2}$$

For every rational number r and every prime number p, denote by $v_p(r)$ the exponent of p in the prime factorisation of r. Recall the notation $(2n-1)!!$ for the product of all odd positive integers not exceeding $2n-1$, i.e., $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$.

Lemma. For every positive integer n and every odd prime p, we have

$$v_p\big((2n-1)!!\big) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor.$$

Proof. For every positive integer k, let us count the multiples of p^k among the factors $1, 3, \ldots, 2n-1$. If ℓ is an arbitrary integer, the number $(2\ell - 1)p^k$ is listed above if and only if

$$0 < (2\ell-1)p^k \leqslant 2n \quad \Longleftrightarrow \quad \frac{1}{2} < \ell \leqslant \frac{n}{p^k} + \frac{1}{2} \quad \Longleftrightarrow \quad 1 \leqslant \ell \leqslant \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor.$$

Hence, the number of multiples of p^k among the factors is precisely $m_k = \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor$. Thus we obtain

$$v_p\big((2n-1)!!\big) = \sum_{i=1}^{n} v_p(2i-1) = \sum_{i=1}^{n} \sum_{k=1}^{v_p(2i-1)} 1 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} 1 = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor. \qquad \square$$

In order to prove the problem statement, consider the rational number

$$N = \frac{(2a+2b-1)!!}{(2a-1)!! \cdot (2b-1)!!} = \frac{(2a+1)(2a+3)\cdots(2a+2b-1)}{1 \cdot 3 \cdots (2b-1)}.$$

Obviously, $N > 1$, so there exists a prime p with $v_p(N) > 0$. Since N is a fraction of two odd numbers, p is odd.

By our lemma,

$$0 < v_p(N) = \sum_{k=1}^{\infty} \left(\left\lfloor \frac{a+b}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{a}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{b}{p^k} + \frac{1}{2} \right\rfloor \right).$$

Therefore, there exists some positive integer k such that the integer number

$$d_k = \left\lfloor \frac{a+b}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{a}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{b}{p^k} + \frac{1}{2} \right\rfloor$$

is positive, so $d_k \geqslant 1$. By (2) we have

$$1 \leqslant d_k = \frac{a+b}{p^k} - \frac{a}{p^k} - \frac{b}{p^k} \pm \left\|\frac{a+b}{p^k}\right\| \pm \left\|\frac{a}{p^k}\right\| \pm \left\|\frac{b}{p^k}\right\| = \pm \left\|\frac{a+b}{p^k}\right\| \pm \left\|\frac{a}{p^k}\right\| \pm \left\|\frac{b}{p^k}\right\|. \tag{3}$$

Since $\|x\| < \frac{1}{2}$ for every rational x with odd denominator, the relation (3) can only be satisfied if all three signs on the right-hand side are positive and $d_k = 1$. Thus we get

$$\left\|\frac{a}{p^k}\right\| + \left\|\frac{b}{p^k}\right\| + \left\|\frac{a+b}{p^k}\right\| = d_k = 1,$$

as required.

Comment 1. There are various choices for the number N in the solution. Here we sketch such a version.

Let x and y be two rational numbers with odd denominators. It is easy to see that the condition $\|x\| + \|y\| + \|x+y\| = 1$ is satisfied if and only if

$$\text{either}\quad \{x\} < \tfrac{1}{2}, \quad \{y\} < \tfrac{1}{2}, \quad \{x+y\} > \tfrac{1}{2}, \qquad \text{or}\quad \{x\} > \tfrac{1}{2}, \quad \{y\} > \tfrac{1}{2}, \quad \{x+y\} < \tfrac{1}{2},$$

where $\{x\}$ denotes the fractional part of x.

In the context of our problem, the first condition seems easier to deal with. Also, one may notice that

$$\{x\} < \tfrac{1}{2} \iff \varkappa(x) = 0 \qquad \text{and} \qquad \{x\} \geqslant \tfrac{1}{2} \iff \varkappa(x) = 1, \tag{4}$$

where

$$\varkappa(x) = \lfloor 2x \rfloor - 2\lfloor x \rfloor.$$

Now it is natural to consider the number

$$M = \frac{\binom{2a+2b}{a+b}}{\binom{2a}{a}\binom{2b}{b}},$$

since

$$v_p(M) = \sum_{k=1}^{\infty}\left(\varkappa\left(\frac{2(a+b)}{p^k}\right) - \varkappa\left(\frac{2a}{p^k}\right) - \varkappa\left(\frac{2b}{p^k}\right)\right).$$

One may see that $M > 1$, and that $v_2(M) \leqslant 0$. Thus, there exist an odd prime p and a positive integer k with

$$\varkappa\left(\frac{2(a+b)}{p^k}\right) - \varkappa\left(\frac{2a}{p^k}\right) - \varkappa\left(\frac{2b}{p^k}\right) > 0.$$

In view of (4), the last inequality yields

$$\left\{\frac{a}{p^k}\right\} < \frac{1}{2}, \quad \left\{\frac{b}{p^k}\right\} < \frac{1}{2}, \quad \text{and} \quad \left\{\frac{a+b}{p^k}\right\} > \frac{1}{2}, \tag{5}$$

which is what we wanted to obtain.

Comment 2. Once one tries to prove the existence of suitable p and k satisfying (5), it seems somehow natural to suppose that $a \leqslant b$ and to add the restriction $p^k > a$. In this case the inequalities (5) can be rewritten as

$$2a < p^k, \quad 2mp^k < 2b < (2m+1)p^k, \quad \text{and} \quad (2m+1)p^k < 2(a+b) < (2m+2)p^k$$

for some positive integer m. This means exactly that one of the numbers $2a+1, 2a+3, \ldots, 2a+2b-1$ is divisible by some number of the form p^k which is greater than $2a$.

Using more advanced techniques, one can show that such a number p^k exists even with $k = 1$. This was shown in 2004 by Laishram and Shorey; the methods used for this proof are elementary but still quite involved. In fact, their result generalises a theorem by Sylvester which states that for every pair of integers (n, k) with $n \geqslant k \geqslant 1$, the product $(n+1)(n+2)\cdots(n+k)$ is divisible by some prime $p > k$. We would like to mention here that Sylvester's theorem itself does not seem to suffice for solving the problem.

Shortlisted Problems

with Solutions

56th International Mathematical Olympiad

IMO

THAILAND 2015

Shortlisted Problems with Solutions

56th International Mathematical Olympiad

Chiang Mai, Thailand, 4–16

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2016.

Contributing Countries

The Organizing Committee and the Problem Selection Committee of IMO 2015 thank the following 53 countries for contributing 155 problem proposals:

Albania, Algeria, Armenia, Australia, Austria, Brazil, Bulgaria, Canada, Costa Rica, Croatia, Cyprus, Denmark, El Salvador, Estonia, Finland, France, Georgia, Germany, Greece, Hong Kong, Hungary, India, Iran, Ireland, Israel, Italy, Japan, Kazakhstan, Lithuania, Luxembourg, Montenegro, Morocco, Netherlands, Pakistan, Poland, Romania, Russia, Saudi Arabia, Serbia, Singapore, Slovakia, Slovenia, South Africa, South Korea, Sweden, Turkey, Turkmenistan, Taiwan, Tanzania, Ukraine, United Kingdom, U.S.A., Uzbekistan

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Problems

Algebra

A1. Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geqslant \frac{k a_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \cdots + a_n \geqslant n$ for every $n \geqslant 2$.

(Serbia)

A2. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f\big(x - f(y)\big) = f\big(f(x)\big) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

(Croatia)

A3. Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leqslant r < s \leqslant 2n} (s - r - n) x_r x_s\,,$$

where $-1 \leqslant x_i \leqslant 1$ for all $i = 1, 2, \ldots, 2n$.

(Austria)

A4. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x + f(x+y)\big) + f(xy) = x + f(x+y) + y f(x)$$

for all real numbers x and y.

(Albania)

A5. Let $2\mathbb{Z}+1$ denote the set of odd integers. Find all functions $f\colon \mathbb{Z} \to 2\mathbb{Z}+1$ satisfying

$$f\big(x + f(x) + y\big) + f\big(x - f(x) - y\big) = f(x+y) + f(x-y)$$

for every $x, y \in \mathbb{Z}$.

(U.S.A.)

A6. Let n be a fixed integer with $n \geqslant 2$. We say that two polynomials P and Q with real coefficients are *block-similar* if for each $i \in \{1, 2, \ldots, n\}$ the sequences

$$\begin{aligned} &P(2015i), P(2015i-1), \ldots, P(2015i-2014) \quad \text{and} \\ &Q(2015i), Q(2015i-1), \ldots, Q(2015i-2014) \end{aligned}$$

are permutations of each other.

(a) Prove that there exist distinct block-similar polynomials of degree $n+1$.

(b) Prove that there do not exist distinct block-similar polynomials of degree n.

(Canada)

Combinatorics

C1. In Lineland there are $n \geqslant 1$ towns, arranged along a road running from left to right. Each town has a *left bulldozer* (put to the left of the town and facing left) and a *right bulldozer* (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a right and a left bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, the bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B being to the right of A. We say that town A can *sweep town B away* if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly, B can sweep A away if the left bulldozer of B can move to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town which cannot be swept away by any other one.

(Estonia)

C2. Let $\mathcal{V}$ be a finite set of points in the plane. We say that $\mathcal{V}$ is *balanced* if for any two distinct points $A, B \in \mathcal{V}$, there exists a point $C \in \mathcal{V}$ such that $AC = BC$. We say that $\mathcal{V}$ is *center-free* if for any distinct points $A, B, C \in \mathcal{V}$, there does not exist a point $P \in \mathcal{V}$ such that $PA = PB = PC$.

(a) Show that for all $n \geqslant 3$, there exists a balanced set consisting of n points.

(b) For which $n \geqslant 3$ does there exist a balanced, center-free set consisting of n points?

(Netherlands)

C3. For a finite set A of positive integers, we call a partition of A into two disjoint nonempty subsets A_1 and A_2 *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

(Ukraine)

C4. Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leqslant n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.

(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.

(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

(Finland)

C5. Consider an infinite sequence $a_1, a_2, \dots$ of positive integers with $a_i \leqslant 2015$ for all $i \geqslant 1$. Suppose that for any two distinct indices i and j we have $i + a_i \neq j + a_j$.

Prove that there exist two positive integers b and N such that

$$\left|\sum_{i=m+1}^{n} (a_i - b)\right| \leqslant 1007^2$$

whenever $n > m \geqslant N$.

(Australia)

C6. Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

(U.S.A.)

C7. In a company of people some pairs are enemies. A group of people is called *unsociable* if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

(Russia)

Geometry

G1. Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

(Australia)

G2. Let ABC be a triangle inscribed into a circle Ω with center O. A circle Γ with center A meets the side BC at points D and E such that D lies between B and E. Moreover, let F and G be the common points of Γ and Ω. We assume that F lies on the arc AB of Ω not containing C, and G lies on the arc AC of Ω not containing B. The circumcircles of the triangles BDF and CEG meet the sides AB and AC again at K and L, respectively. Suppose that the lines FK and GL are distinct and intersect at X. Prove that the points A, X, and O are collinear.

(Greece)

G3. Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

(Georgia)

G4. Let ABC be an acute triangle, and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC again at P and Q, respectively. Let T be the point such that the quadrilateral $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of the triangle ABC. Determine all possible values of BT/BM.

(Russia)

G5. Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC, respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

(El Salvador)

G6. Let ABC be an acute triangle with $AB > AC$, and let Γ be its circumcircle. Let H, M, and F be the orthocenter of the triangle, the midpoint of BC, and the foot of the altitude from A, respectively. Let Q and K be the two points on Γ that satisfy $\angle AQH = 90^\circ$ and $\angle QKH = 90^\circ$. Prove that the circumcircles of the triangles KQH and KFM are tangent to each other.

(Ukraine)

G7. Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segments PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

(Bulgaria)

G8. A *triangulation* of a convex polygon Π is a partitioning of Π into triangles by diagonals having no common points other than the vertices of the polygon. We say that a triangulation is a *Thaiangulation* if all triangles in it have the same area.

Prove that any two different Thaiangulations of a convex polygon Π differ by exactly two triangles. (In other words, prove that it is possible to replace one pair of triangles in the first Thaiangulation with a different pair of triangles so as to obtain the second Thaiangulation.)

(Bulgaria)

Number Theory

N1. Determine all positive integers M for which the sequence $a_0, a_1, a_2, \ldots$, defined by $a_0 = \frac{2M+1}{2}$ and $a_{k+1} = a_k\lfloor a_k \rfloor$ for $k = 0, 1, 2, \ldots$, contains at least one integer term.

(Luxembourg)

N2. Let a and b be positive integers such that $a!b!$ is a multiple of $a! + b!$. Prove that $3a \geqslant 2b + 2$.

(United Kingdom)

N3. Let m and n be positive integers such that $m > n$. Define $x_k = (m+k)/(n+k)$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \cdots x_{n+1} - 1$ is divisible by an odd prime.

(Austria)

N4. Suppose that $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are two sequences of positive integers satisfying $a_0, b_0 \geqslant 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1$$

for all $n \geqslant 0$. Prove that the sequence (a_n) is eventually periodic; in other words, there exist integers $N \geqslant 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geqslant N$.

(France)

N5. Determine all triples (a, b, c) of positive integers for which $ab - c$, $bc - a$, and $ca - b$ are powers of 2.

Explanation: A power of 2 is an integer of the form 2^n, where n denotes some nonnegative integer.

(Serbia)

N6. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:

(*i*) If $m, n \in \mathbb{Z}_{>0}$, then $\dfrac{f^n(m) - m}{n} \in \mathbb{Z}_{>0}$;

(*ii*) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

(Singapore)

N7. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called k-*good* if $\gcd\big(f(m) + n, f(n) + m\big) \leqslant k$ for all $m \neq n$. Find all k such that there exists a k-good function.

(Canada)

N8. For every positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, define

$$\mho(n) = \sum_{i\colon p_i > 10^{100}} \alpha_i.$$

That is, $\mho(n)$ is the number of prime factors of n greater than 10^{100}, counted with multiplicity.

Find all strictly increasing functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$\mho\big(f(a) - f(b)\big) \leqslant \mho(a - b) \qquad \text{for all integers } a \text{ and } b \text{ with } a > b.$$

(Brazil)

Solutions

Algebra

A1. Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geqslant \frac{ka_k}{a_k^2 + (k-1)} \tag{1}$$

for every positive integer k. Prove that $a_1 + a_2 + \cdots + a_n \geqslant n$ for every $n \geqslant 2$.

(Serbia)

Solution. From the constraint (1), it can be seen that

$$\frac{k}{a_{k+1}} \leqslant \frac{a_k^2 + (k-1)}{a_k} = a_k + \frac{k-1}{a_k},$$

and so

$$a_k \geqslant \frac{k}{a_{k+1}} - \frac{k-1}{a_k}.$$

Summing up the above inequality for $k = 1, \ldots, m$, we obtain

$$a_1 + a_2 + \cdots + a_m \geqslant \left(\frac{1}{a_2} - \frac{0}{a_1}\right) + \left(\frac{2}{a_3} - \frac{1}{a_2}\right) + \cdots + \left(\frac{m}{a_{m+1}} - \frac{m-1}{a_m}\right) = \frac{m}{a_{m+1}}. \tag{2}$$

Now we prove the problem statement by induction on n. The case $n = 2$ can be done by applying (1) to $k = 1$:

$$a_1 + a_2 \geqslant a_1 + \frac{1}{a_1} \geqslant 2.$$

For the induction step, assume that the statement is true for some $n \geqslant 2$. If $a_{n+1} \geqslant 1$, then the induction hypothesis yields

$$\bigl(a_1 + \cdots + a_n\bigr) + a_{n+1} \geqslant n + 1. \tag{3}$$

Otherwise, if $a_{n+1} < 1$ then apply (2) as

$$\bigl(a_1 + \cdots + a_n\bigr) + a_{n+1} \geqslant \frac{n}{a_{n+1}} + a_{n+1} = \frac{n-1}{a_{n+1}} + \left(\frac{1}{a_{n+1}} + a_{n+1}\right) > (n-1) + 2.$$

That completes the solution.

Comment 1. It can be seen easily that having equality in the statement requires $a_1 = a_2 = 1$ in the base case $n = 2$, and $a_{n+1} = 1$ in (3). So the equality $a_1 + \cdots + a_n = n$ is possible only in the trivial case $a_1 = \cdots = a_n = 1$.

Comment 2. After obtaining (2), there are many ways to complete the solution. We outline three such possibilities.

- With defining $s_n = a_1 + \cdots + a_n$, the induction step can be replaced by

$$s_{n+1} = s_n + a_{n+1} \geqslant s_n + \frac{n}{s_n} \geqslant n + 1,$$

because the function $x \mapsto x + \dfrac{n}{x}$ increases on $[n, \infty)$.

- By applying the AM–GM inequality to the numbers $a_1 + \cdots + a_k$ and ka_{k+1}, we can conclude

$$a_1 + \cdots + a_k + ka_{k+1} \geqslant 2k$$

  and sum it up for $k = 1, \ldots, n-1$.

- We can derive the symmetric estimate

$$\sum_{1 \leqslant i < j \leqslant n} a_i a_j = \sum_{j=2}^{n} (a_1 + \cdots + a_{j-1}) a_j \geqslant \sum_{j=2}^{n} (j-1) = \frac{n(n-1)}{2}$$

  and combine it with the AM–QM inequality.

A2. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f\big(x - f(y)\big) = f\big(f(x)\big) - f(y) - 1 \tag{1}$$

holds for all $x, y \in \mathbb{Z}$.

(Croatia)

Answer. There are two such functions, namely the constant function $x \mapsto -1$ and the successor function $x \mapsto x + 1$.

Solution 1. It is immediately checked that both functions mentioned in the answer are as desired.

Now let f denote any function satisfying (1) for all $x, y \in \mathbb{Z}$. Substituting $x = 0$ and $y = f(0)$ into (1) we learn that the number $z = -f\big(f(0)\big)$ satisfies $f(z) = -1$. So by plugging $y = z$ into (1) we deduce that

$$f(x + 1) = f\big(f(x)\big) \tag{2}$$

holds for all $x \in \mathbb{Z}$. Thereby (1) simplifies to

$$f\big(x - f(y)\big) = f(x + 1) - f(y) - 1\,. \tag{3}$$

We now work towards showing that f is linear by contemplating the difference $f(x+1) - f(x)$ for any $x \in \mathbb{Z}$. By applying (3) with $y = x$ and (2) in this order, we obtain

$$f(x + 1) - f(x) = f\big(x - f(x)\big) + 1 = f\big(f(x - 1 - f(x))\big) + 1\,.$$

Since (3) shows $f\big(x - 1 - f(x)\big) = f(x) - f(x) - 1 = -1$, this simplifies to

$$f(x + 1) = f(x) + A\,,$$

where $A = f(-1) + 1$ is some absolute constant.

Now a standard induction in both directions reveals that f is indeed linear and that in fact we have $f(x) = Ax + B$ for all $x \in \mathbb{Z}$, where $B = f(0)$. Substituting this into (2) we obtain that

$$Ax + (A + B) = A^2x + (AB + B)$$

holds for all $x \in \mathbb{Z}$; applying this to $x = 0$ and $x = 1$ we infer $A + B = AB + B$ and $A^2 = A$. The second equation leads to $A = 0$ or $A = 1$. In case $A = 1$, the first equation gives $B = 1$, meaning that f has to be the successor function. If $A = 0$, then f is constant and (1) shows that its constant value has to be -1. Thereby the solution is complete.

Comment. After (2) and (3) have been obtained, there are several other ways to combine them so as to obtain linearity properties of f. For instance, using (2) thrice in a row and then (3) with $x = f(y)$ one may deduce that

$$f(y + 2) = f\big(f(y + 1)\big) = f\big(f\big(f(y)\big)\big) = f\big(f(y) + 1\big) = f(y) + f(0) + 1$$

holds for all $y \in \mathbb{Z}$. It follows that f behaves linearly on the even numbers and on the odd numbers separately, and moreover that the slopes of these two linear functions coincide. From this point, one may complete the solution with some straightforward case analysis.

A different approach using the equations (2) and (3) will be presented in Solution 2. To show that it is also possible to start in a completely different way, we will also present a third solution that avoids these equations entirely.

Solution 2. We commence by deriving (2) and (3) as in the first solution. Now provided that f is injective, (2) tells us that f is the successor function. Thus we may assume from now on that f is not injective, i.e., that there are two integers $a > b$ with $f(a) = f(b)$. A straightforward induction using (2) in the induction step reveals that we have $f(a + n) = f(b + n)$ for all nonnegative integers n. Consequently, the sequence $\gamma_n = f(b + n)$ is periodic and thus in particular bounded, which means that the numbers

$$\varphi = \min_{n \geqslant 0} \gamma_n \quad \text{and} \quad \psi = \max_{n \geqslant 0} \gamma_n$$

exist.

Let us pick any integer y with $f(y) = \varphi$ and then an integer $x \geqslant a$ with $f\big(x - f(y)\big) = \varphi$. Due to the definition of φ and (3) we have

$$\varphi \leqslant f(x + 1) = f\big(x - f(y)\big) + f(y) + 1 = 2\varphi + 1\,,$$

whence $\varphi \geqslant -1$. The same reasoning applied to ψ yields $\psi \leqslant -1$. Since $\varphi \leqslant \psi$ holds trivially, it follows that $\varphi = \psi = -1$, or in other words that we have $f(t) = -1$ for all integers $t \geqslant a$.

Finally, if any integer y is given, we may find an integer x which is so large that $x + 1 \geqslant a$ and $x - f(y) \geqslant a$ hold. Due to (3) and the result from the previous paragraph we get

$$f(y) = f(x + 1) - f\big(x - f(y)\big) - 1 = (-1) - (-1) - 1 = -1\,.$$

Thereby the problem is solved.

Solution 3. Set $d = f(0)$. By plugging $x = f(y)$ into (1) we obtain

$$f^3(y) = f(y) + d + 1 \tag{4}$$

for all $y \in \mathbb{Z}$, where the left-hand side abbreviates $f\big(f(f(y))\big)$. When we replace x in (1) by $f(x)$ we obtain $f\big(f(x) - f(y)\big) = f^3(x) - f(y) - 1$ and as a consequence of (4) this simplifies to

$$f\big(f(x) - f(y)\big) = f(x) - f(y) + d\,. \tag{5}$$

Now we consider the set

$$E = \{f(x) - d \mid x \in \mathbb{Z}\}\,.$$

Given two integers a and b from E, we may pick some integers x and y with $f(x) = a + d$ and $f(y) = b + d$; now (5) tells us that $f(a - b) = (a - b) + d$, which means that $a - b$ itself exemplifies $a - b \in E$. Thus,

$$E \text{ is closed under taking differences.} \tag{6}$$

Also, the definitions of d and E yield $0 \in E$. If $E = \{0\}$, then f is a constant function and (1) implies that the only value attained by f is indeed -1.

So let us henceforth suppose that E contains some number besides zero. It is known that in this case (6) entails E to be the set of all integer multiples of some positive integer k. Indeed, this holds for

$$k = \min\big\{|x| \,\big|\, x \in E \text{ and } x \neq 0\big\}\,,$$

as one may verify by an argument based on division with remainder.

Thus we have

$$\{f(x) \mid x \in \mathbb{Z}\} = \{k \cdot t + d \mid t \in \mathbb{Z}\}\,. \tag{7}$$

Due to (5) and (7) we get

$$f(k \cdot t) = k \cdot t + d$$

for all $t \in \mathbb{Z}$, whence in particular $f(k) = k + d$. So by comparing the results of substituting $y = 0$ and $y = k$ into (1) we learn that

$$f(z + k) = f(z) + k \tag{8}$$

holds for all integers z. In plain English, this means that on any residue class modulo k the function f is linear with slope 1.

Now by (7) the set of all values attained by f is such a residue class. Hence, there exists an absolute constant c such that $f\big(f(x)\big) = f(x) + c$ holds for all $x \in \mathbb{Z}$. Thereby (1) simplifies to

$$f\big(x - f(y)\big) = f(x) - f(y) + c - 1\,. \tag{9}$$

On the other hand, considering (1) modulo k we obtain $d \equiv -1 \pmod{k}$ because of (7). So by (7) again, f attains the value -1.

Thus we may apply (9) to some integer y with $f(y) = -1$, which gives $f(x + 1) = f(x) + c$. So f is a linear function with slope c. Hence, (8) leads to $c = 1$, wherefore there is an absolute constant d' with $f(x) = x + d'$ for all $x \in \mathbb{Z}$. Using this for $x = 0$ we obtain $d' = d$ and finally (4) discloses $d = 1$, meaning that f is indeed the successor function.

A3. Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1\leqslant r<s\leqslant 2n}(s-r-n)x_rx_s\,,$$

where $-1\leqslant x_i\leqslant 1$ for all $i=1,2,\ldots,2n$.

(Austria)

Answer. $n(n-1)$.

Solution 1. Let Z be the expression to be maximized. Since this expression is linear in every variable x_i and $-1\leqslant x_i\leqslant 1$, the maximum of Z will be achieved when $x_i=-1$ or 1. Therefore, it suffices to consider only the case when $x_i\in\{-1,1\}$ for all $i=1,2,\ldots,2n$.

For $i=1,2,\ldots,2n$, we introduce auxiliary variables

$$y_i=\sum_{r=1}^{i}x_r-\sum_{r=i+1}^{2n}x_r\,.$$

Taking squares of both sides, we have

$$\begin{aligned}y_i^2&=\sum_{r=1}^{2n}x_r^2+\sum_{r<s\leqslant i}2x_rx_s+\sum_{i<r<s}2x_rx_s-\sum_{r\leqslant i<s}2x_rx_s\\&=2n+\sum_{r<s\leqslant i}2x_rx_s+\sum_{i<r<s}2x_rx_s-\sum_{r\leqslant i<s}2x_rx_s\,,\end{aligned}\tag{1}$$

where the last equality follows from the fact that $x_r\in\{-1,1\}$. Notice that for every $r<s$, the coefficient of x_rx_s in (1) is 2 for each $i=1,\ldots,r-1,s,\ldots,2n$, and this coefficient is -2 for each $i=r,\ldots,s-1$. This implies that the coefficient of x_rx_s in $\sum_{i=1}^{2n}y_i^2$ is $2(2n-s+r)-2(s-r)=4(n-s+r)$. Therefore, summing (1) for $i=1,2,\ldots,2n$ yields

$$\sum_{i=1}^{2n}y_i^2=4n^2+\sum_{1\leqslant r<s\leqslant 2n}4(n-s+r)x_rx_s=4n^2-4Z.\tag{2}$$

Hence, it suffices to find the minimum of the left-hand side.

Since $x_r\in\{-1,1\}$, we see that y_i is an even integer. In addition, $y_i-y_{i-1}=2x_i=\pm 2$, and so y_{i-1} and y_i are consecutive even integers for every $i=2,3,\ldots,2n$. It follows that $y_{i-1}^2+y_i^2\geqslant 4$, which implies

$$\sum_{i=1}^{2n}y_i^2=\sum_{j=1}^{n}\left(y_{2j-1}^2+y_{2j}^2\right)\geqslant 4n.\tag{3}$$

Combining (2) and (3), we get

$$4n\leqslant\sum_{i=1}^{2n}y_i^2=4n^2-4Z.\tag{4}$$

Hence, $Z\leqslant n(n-1)$.

If we set $x_i=1$ for odd indices i and $x_i=-1$ for even indices i, then we obtain equality in (3) (and thus in (4)). Therefore, the maximum possible value of Z is $n(n-1)$, as desired.

Comment 1. $Z=n(n-1)$ can be achieved by several other examples. In particular, x_i needs not be ± 1. For instance, setting $x_i=(-1)^i$ for all $2\leqslant i\leqslant 2n$, we find that the coefficient of x_1 in Z is 0. Therefore, x_1 can be chosen arbitrarily in the interval $[-1,1]$.

Nevertheless, if $x_i\in\{-1,1\}$ for all $i=1,2,\ldots,2n$, then the equality $Z=n(n-1)$ holds only when $(y_1,y_2,\ldots,y_{2n})=(0,\pm 2,0,\pm 2,\ldots,0,\pm 2)$ or $(\pm 2,0,\pm 2,0,\ldots,\pm 2,0)$. In each case, we can reconstruct x_i accordingly. The sum $\sum_{i=1}^{2n}x_i$ in the optimal cases needs not be 0, but it must equal 0 or ± 2.

Comment 2. Several variations in setting up the auxiliary variables are possible. For instance, one may let $x_{2n+i} = -x_i$ and $y'_i = x_i + x_{i+1} + \cdots + x_{i+n-1}$ for any $1 \leqslant i \leqslant 2n$. Similarly to Solution 1, we obtain $Y := y'^2_1 + y'^2_2 + \cdots + y'^2_{2n} = 2n^2 - 2Z$. Then, it suffices to show that $Y \geqslant 2n$. If n is odd, then each y'_i is odd, and so $y'^2_i \geqslant 1$. If n is even, then each y'_i is even. We can check that at least one of y'_i, y'_{i+1}, y'_{n+i}, and y'_{n+i+1} is nonzero, so that $y'^2_i + y'^2_{i+1} + y'^2_{n+i} + y'^2_{n+i+1} \geqslant 4$; summing these up for $i = 1, 3, \ldots, n-1$ yields $Y \geqslant 2n$.

Solution 2. We present a different method of obtaining the bound $Z \leqslant n(n-1)$. As in the previous solution, we reduce the problem to the case $x_i \in \{-1, 1\}$. For brevity, we use the notation $[2n] = \{1, 2, \ldots, 2n\}$.

Consider any $x_1, x_2, \ldots, x_{2n} \in \{-1, 1\}$. Let

$$A = \{i \in [2n] : x_i = 1\} \quad \text{and} \quad B = \{i \in [2n] : x_i = -1\}.$$

For any subsets X and Y of $[2n]$ we define

$$e(X, Y) = \sum_{r<s,\, r\in X,\, s\in Y} (s - r - n).$$

One may observe that

$$e(A,A)+e(A,B)+e(B,A)+e(B,B) = e([2n],[2n]) = \sum_{1\leqslant r<s\leqslant 2n} (s-r-n) = -\frac{(n-1)n(2n-1)}{3}.$$

Therefore, we have

$$Z = e(A,A) - e(A,B) - e(B,A) + e(B,B) = 2\big(e(A,A) + e(B,B)\big) + \frac{(n-1)n(2n-1)}{3}. \quad (5)$$

Thus, we need to maximize $e(A,A) + e(B,B)$, where A and B form a partition of $[2n]$.

Due to the symmetry, we may assume that $|A| = n - p$ and $|B| = n + p$, where $0 \leqslant p \leqslant n$. From now on, we fix the value of p and find an upper bound for Z in terms of n and p.

Let $a_1 < a_2 < \cdots < a_{n-p}$ and $b_1 < b_2 < \cdots < b_{n+p}$ list all elements of A and B, respectively. Then

$$e(A,A) = \sum_{1\leqslant i<j\leqslant n-p} (a_j - a_i - n) = \sum_{i=1}^{n-p} (2i - 1 - n + p)a_i - \binom{n-p}{2} \cdot n \quad (6)$$

and similarly

$$e(B,B) = \sum_{i=1}^{n+p} (2i - 1 - n - p)b_i - \binom{n+p}{2} \cdot n. \quad (7)$$

Thus, now it suffices to maximize the value of

$$M = \sum_{i=1}^{n-p} (2i - 1 - n + p)a_i + \sum_{i=1}^{n+p} (2i - 1 - n - p)b_i. \quad (8)$$

In order to get an upper bound, we will apply the rearrangement inequality to the sequence $a_1, a_2, \ldots, a_{n-p}, b_1, b_2, \ldots, b_{n+p}$ (which is a permutation of $1, 2, \ldots, 2n$), together with the sequence of coefficients of these numbers in (8). The coefficients of a_i form the sequence

$$n-p-1, n-p-3, \ldots, 1-n+p,$$

and those of b_i form the sequence

$$n+p-1, n+p-3, \ldots, 1-n-p.$$

Altogether, these coefficients are, in descending order:

- $n+p+1-2i$, for $i=1,2,\dots,p$;
- $n-p+1-2i$, counted twice, for $i=1,2,\dots,n-p$; and
- $-(n+p+1-2i)$, for $i=p,p-1,\dots,1$.

Thus, the rearrangement inequality yields

$$\begin{aligned} M \leqslant \sum_{i=1}^{p}(n+p+1-2i)(2n+1-i) \\ + \sum_{i=1}^{n-p}(n-p+1-2i)\big((2n+2-p-2i)+(2n+1-p-2i)\big) \\ - \sum_{i=1}^{p}(n+p+1-2i)i\,. \end{aligned} \tag{9}$$

Finally, combining the information from (5), (6), (7), and (9), we obtain

$$\begin{aligned} Z \leqslant \frac{(n-1)n(2n-1)}{3} - 2n\left(\binom{n-p}{2}+\binom{n+p}{2}\right) \\ + 2\sum_{i=1}^{p}(n+p+1-2i)(2n+1-2i) + 2\sum_{i=1}^{n-p}(n-p+1-2i)(4n-2p+3-4i)\,, \end{aligned}$$

which can be simplified to

$$Z \leqslant n(n-1) - \frac{2}{3}\,p(p-1)(p+1)\,.$$

Since p is a nonnegative integer, this yields $Z \leqslant n(n-1)$.

A4. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x + f(x+y)\big) + f(xy) = x + f(x+y) + yf(x) \tag{1}$$

for all real numbers x and y.

(Albania)

Answer. There are two such functions, namely the identity function and $x \mapsto 2 - x$.

Solution. Clearly, each of the functions $x \mapsto x$ and $x \mapsto 2 - x$ satisfies (1). It suffices now to show that they are the only solutions to the problem.

Suppose that f is any function satisfying (1). Then setting $y = 1$ in (1), we obtain

$$f\big(x + f(x+1)\big) = x + f(x+1); \tag{2}$$

in other words, $x + f(x+1)$ is a fixed point of f for every $x \in \mathbb{R}$.

We distinguish two cases regarding the value of $f(0)$.

Case 1. $f(0) \neq 0$.

By letting $x = 0$ in (1), we have

$$f\big(f(y)\big) + f(0) = f(y) + yf(0).$$

So, if y_0 is a fixed point of f, then substituting $y = y_0$ in the above equation we get $y_0 = 1$. Thus, it follows from (2) that $x + f(x+1) = 1$ for all $x \in \mathbb{R}$. That is, $f(x) = 2 - x$ for all $x \in \mathbb{R}$.

Case 2. $f(0) = 0$.

By letting $y = 0$ and replacing x by $x+1$ in (1), we obtain

$$f\big(x + f(x+1) + 1\big) = x + f(x+1) + 1. \tag{3}$$

From (1), the substitution $x = 1$ yields

$$f\big(1 + f(y+1)\big) + f(y) = 1 + f(y+1) + yf(1). \tag{4}$$

By plugging $x = -1$ into (2), we see that $f(-1) = -1$. We then plug $y = -1$ into (4) and deduce that $f(1) = 1$. Hence, (4) reduces to

$$f\big(1 + f(y+1)\big) + f(y) = 1 + f(y+1) + y. \tag{5}$$

Accordingly, if both y_0 and $y_0 + 1$ are fixed points of f, then so is $y_0 + 2$. Thus, it follows from (2) and (3) that $x + f(x+1) + 2$ is a fixed point of f for every $x \in \mathbb{R}$; i.e.,

$$f\big(x + f(x+1) + 2\big) = x + f(x+1) + 2.$$

Replacing x by $x - 2$ simplifies the above equation to

$$f\big(x + f(x-1)\big) = x + f(x-1).$$

On the other hand, we set $y = -1$ in (1) and get

$$f\big(x + f(x-1)\big) = x + f(x-1) - f(x) - f(-x).$$

Therefore, $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Finally, we substitute (x, y) by $(-1, -y)$ in (1) and use the fact that $f(-1) = -1$ to get

$$f\big(-1 + f(-y-1)\big) + f(y) = -1 + f(-y-1) + y.$$

Since f is an odd function, the above equation becomes

$$-f\big(1 + f(y+1)\big) + f(y) = -1 - f(y+1) + y.$$

By adding this equation to (5), we conclude that $f(y) = y$ for all $y \in \mathbb{R}$.

A5. Let $2\mathbb{Z}+1$ denote the set of odd integers. Find all functions $f\colon \mathbb{Z} \to 2\mathbb{Z}+1$ satisfying

$$f\big(x+f(x)+y\big)+f\big(x-f(x)-y\big)=f(x+y)+f(x-y) \tag{1}$$

for every $x, y \in \mathbb{Z}$.

(U.S.A.)

Answer. Fix an odd positive integer d, an integer k, and odd integers $\ell_0, \ell_1, \ldots, \ell_{d-1}$. Then the function defined as

$$f(md+i)=2kmd+\ell_i d \qquad (m \in \mathbb{Z}, \quad i=0,1,\ldots,d-1)$$

satisfies the problem requirements, and these are all such functions.

Solution. Throughout the solution, all functions are assumed to map integers to integers.

For any function g and any nonzero integer t, define

$$\Delta_t g(x)=g(x+t)-g(x).$$

For any nonzero integers a and b, notice that $\Delta_a\Delta_b g=\Delta_b\Delta_a g$. Moreover, if $\Delta_a g=0$ and $\Delta_b g=0$, then $\Delta_{a+b} g=0$ and $\Delta_{at} g=0$ for all nonzero integers t. We say that g is *t-quasi-periodic* if $\Delta_t g$ is a constant function (in other words, if $\Delta_1\Delta_t g=0$, or $\Delta_1 g$ is t-periodic). In this case, we call t a *quasi-period* of g. We say that g is *quasi-periodic* if it is t-quasi-periodic for some nonzero integer t.

Notice that a quasi-period of g is a period of $\Delta_1 g$. So if g is quasi-periodic, then its minimal positive quasi-period t divides all its quasi-periods.

We now assume that f satisfies (1). First, by setting $a=x+y$, the problem condition can be rewritten as

$$\Delta_{f(x)} f(a)=\Delta_{f(x)} f\big(2x-a-f(x)\big) \qquad \text{for all } x, a \in \mathbb{Z}. \tag{2}$$

Let b be an arbitrary integer and let k be an arbitrary positive integer. Applying (2) when a is substituted by $b, b+f(x), \ldots, b+(k-1)f(x)$ and summing up all these equations, we get

$$\Delta_{kf(x)} f(b)=\Delta_{kf(x)} f\big(2x-b-kf(x)\big).$$

Notice that a similar argument works when k is negative, so that

$$\Delta_M f(b)=\Delta_M f(2x-b-M) \qquad \text{for any nonzero integer } M \text{ such that } f(x) \mid M. \tag{3}$$

We now prove two lemmas.

Lemma 1. For any distinct integers x and y, the function $\Delta_{\operatorname{lcm}(f(x),f(y))} f$ is $2(y-x)$-periodic.

Proof. Denote $L=\operatorname{lcm}\big(f(x), f(y)\big)$. Applying (3) twice, we obtain

$$\Delta_L f(b)=\Delta_L f(2x-b-L)=\Delta_L f\big(2y-(b+2(y-x))-L\big)=\Delta_L f\big(b+2(y-x)\big).$$

Thus, the function $\Delta_L f$ is $2(y-x)$-periodic, as required. □

Lemma 2. Let g be a function. If t and s are nonzero integers such that $\Delta_{ts} g=0$ and $\Delta_t\Delta_t g=0$, then $\Delta_t g=0$.

Proof. Assume, without loss of generality, that s is positive. Let a be an arbitrary integer. Since $\Delta_t\Delta_t g=0$, we have

$$\Delta_t g(a)=\Delta_t g(a+t)=\cdots=\Delta_t g\big(a+(s-1)t\big).$$

The sum of these s equal numbers is $\Delta_{ts} g(a)=0$, so each of them is zero, as required. □

We now return to the solution.

Step 1. *We prove that f is quasi-periodic.*

Let $Q = \operatorname{lcm}\big(f(0), f(1)\big)$. Applying Lemma 1, we get that the function $g = \Delta_Q f$ is 2-periodic. In other words, the values of g are constant on even numbers and on odd numbers separately. Moreover, setting $M = Q$ and $x = b = 0$ in (3), we get $g(0) = g(-Q)$. Since 0 and $-Q$ have different parities, the value of g at even numbers is the same as that at odd numbers. Thus, g is constant, which means that Q is a quasi-period of f.

Step 2. *Denote the minimal positive quasi-period of f by T. We prove that $T \mid f(x)$ for all integers x.*

Since an odd number Q is a quasi-period of f, the number T is also odd. Now suppose, to the contrary, that there exist an odd prime p, a positive integer α, and an integer u such that $p^\alpha \mid T$ but $p^\alpha \nmid f(u)$. Setting $x = u$ and $y = 0$ in (1), we have $2f(u) = f\big(u+f(u)\big)+f\big(u-f(u)\big)$, so p^α does not divide the value of f at one of the points $u+f(u)$ or $u-f(u)$. Denote this point by v.

Let $L = \operatorname{lcm}\big(f(u), f(v)\big)$. Since $|u-v| = f(u)$, from Lemma 1 we get $\Delta_{2f(u)}\Delta_L f = 0$. Hence the function $\Delta_L f$ is $2f(u)$-periodic as well as T-periodic, so it is $\gcd\big(T, 2f(u)\big)$-periodic, or $\Delta_{\gcd(T,2f(u))}\Delta_L f = 0$. Similarly, observe that the function $\Delta_{\gcd(T,2f(u))} f$ is L-periodic as well as T-periodic, so we may conclude that $\Delta_{\gcd(T,L)}\Delta_{\gcd(T,2f(u))} f = 0$. Since $p^\alpha \nmid L$, both $\gcd\big(T, 2f(u)\big)$ and $\gcd(T, L)$ divide T/p. We thus obtain $\Delta_{T/p}\Delta_{T/p} f = 0$, which yields

$$\Delta_{T/p}\Delta_{T/p}\Delta_1 f = 0.$$

Since $\Delta_T\Delta_1 f = 0$, we can apply Lemma 2 to the function $\Delta_1 f$, obtaining $\Delta_{T/p}\Delta_1 f = 0$. However, this means that f is (T/p)-quasi-periodic, contradicting the minimality of T. Our claim is proved.

Step 3. *We describe all functions f.*

Let d be the greatest common divisor of all values of f. Then d is odd. By Step 2, d is a quasi-period of f, so that $\Delta_d f$ is constant. Since the value of $\Delta_d f$ is even and divisible by d, we may denote this constant by $2dk$, where k is an integer. Next, for all $i = 0, 1, \dots, d-1$, define $\ell_i = f(i)/d$; notice that ℓ_i is odd. Then

$$f(md+i) = \Delta_{md} f(i) + f(i) = 2kmd + \ell_i d \qquad \text{for all } m \in \mathbb{Z} \quad \text{and } i = 0, 1, \dots, d-1.$$

This shows that all functions satisfying (1) are listed in the answer.

It remains to check that all such functions indeed satisfy (1). This is equivalent to checking (2), which is true because for every integer x, the value of $f(x)$ is divisible by d, so that $\Delta_{f(x)} f$ is constant.

Comment. After obtaining Lemmas 1 and 2, it is possible to complete the steps in a different order. Here we sketch an alternative approach.

For any function g and any nonzero integer t, we say that g is *t-pseudo-periodic* if $\Delta_t\Delta_t g = 0$. In this case, we call t a *pseudo-period* of g, and we say that g is *pseudo-periodic*.

Let us first prove a basic property: if a function g is pseudo-periodic, then its minimal positive pseudo-period divides all its pseudo-periods. To establish this, it suffices to show that if t and s are pseudo-periods of g with $t \neq s$, then so is $t-s$. Indeed, suppose that $\Delta_t\Delta_t g = \Delta_s\Delta_s g = 0$. Then $\Delta_t\Delta_t\Delta_s g = \Delta_{ts}\Delta_s g = 0$, so that $\Delta_t\Delta_s g = 0$ by Lemma 2. Taking differences, we obtain $\Delta_t\Delta_{t-s} g = \Delta_s\Delta_{t-s} g = 0$, and thus $\Delta_{t-s}\Delta_{t-s} g = 0$.

Now let f satisfy the problem condition. We will show that f is pseudo-periodic. When this is done, we will let T' be the minimal pseudo-period of f, and show that T' divides $2f(x)$ for every integer x, using arguments similar to Step 2 of the solution. Then we will come back to Step 1 by showing that T' is also a quasi-period of f.

First, Lemma 1 yields that $\Delta_{2(y-x)}\Delta_{\operatorname{lcm}(f(x),f(y))}f = 0$ for every distinct integers x and y. Hence f is pseudo-periodic with pseudo-period $L_{x,y} = \operatorname{lcm}\big(2(y-x), f(x), f(y)\big)$.

We now show that $T' \mid 2f(x)$ for every integer x. Suppose, to the contrary, that there exists an integer u, a prime p, and a positive integer α such that $p^\alpha \mid T'$ and $p^\alpha \nmid 2f(u)$. Choose v as in Step 2 and employ Lemma 1 to obtain $\Delta_{2f(u)}\Delta_{\operatorname{lcm}(f(u),f(v))}f = 0$. However, this implies that $\Delta_{T'/p}\Delta_{T'/p}f = 0$, a contradiction with the minimality of T'.

We now claim that $\Delta_{T'}\Delta_2 f = 0$. Indeed, Lemma 1 implies that there exists an integer s such that $\Delta_s\Delta_2 f = 0$. Hence $\Delta_{T's}\Delta_2 f = \Delta_{T'}\Delta_{T'}\Delta_2 f = 0$, which allows us to conclude that $\Delta_{T'}\Delta_2 f = 0$ by Lemma 2. (The last two paragraphs are similar to Step 2 of the solution.)

Now, it is not difficult to finish the solution, though more work is needed to eliminate the factors of 2 from the subscripts of $\Delta_{T'}\Delta_2 f = 0$. Once this is done, we will obtain an odd quasi-period of f that divides $f(x)$ for all integers x. Then we can complete the solution as in Step 3.

A6. Let n be a fixed integer with $n \geqslant 2$. We say that two polynomials P and Q with real coefficients are *block-similar* if for each $i \in \{1, 2, \ldots, n\}$ the sequences

$$\begin{aligned} &P(2015i), P(2015i-1), \ldots, P(2015i-2014) \quad \text{and} \\ &Q(2015i), Q(2015i-1), \ldots, Q(2015i-2014) \end{aligned}$$

are permutations of each other.

(*a*) Prove that there exist distinct block-similar polynomials of degree $n+1$.

(*b*) Prove that there do not exist distinct block-similar polynomials of degree n.

(Canada)

Solution 1. For convenience, we set $k = 2015 = 2\ell + 1$.

Part (*a*). Consider the following polynomials of degree $n+1$:

$$P(x) = \prod_{i=0}^{n} (x - ik) \qquad \text{and} \qquad Q(x) = \prod_{i=0}^{n} (x - ik - 1).$$

Since $Q(x) = P(x-1)$ and $P(0) = P(k) = P(2k) = \cdots = P(nk)$, these polynomials are block-similar (and distinct).

Part (*b*). For every polynomial $F(x)$ and every nonnegative integer m, define $\Sigma_F(m) = \sum_{i=1}^{m} F(i)$; in particular, $\Sigma_F(0) = 0$. It is well-known that for every nonnegative integer d the sum $\sum_{i=1}^{m} i^d$ is a polynomial in m of degree $d+1$. Thus Σ_F may also be regarded as a real polynomial of degree $\deg F + 1$ (with the exception that if $F = 0$, then $\Sigma_F = 0$ as well). This allows us to consider the values of Σ_F at all real points (where the initial definition does not apply).

Assume for the sake of contradiction that there exist two distinct block-similar polynomials $P(x)$ and $Q(x)$ of degree n. Then both polynomials $\Sigma_{P-Q}(x)$ and $\Sigma_{P^2-Q^2}(x)$ have roots at the points $0, k, 2k, \ldots, nk$. This motivates the following lemma, where we use the special polynomial

$$T(x) = \prod_{i=0}^{n} (x - ik).$$

Lemma. Assume that $F(x)$ is a nonzero polynomial such that $0, k, 2k, \ldots, nk$ are among the roots of the polynomial $\Sigma_F(x)$. Then $\deg F \geqslant n$, and there exists a polynomial $G(x)$ such that $\deg G = \deg F - n$ and $F(x) = T(x)G(x) - T(x-1)G(x-1)$.

Proof. If $\deg F < n$, then $\Sigma_F(x)$ has at least $n+1$ roots, while its degree is less than $n+1$. Therefore, $\Sigma_F(x) = 0$ and hence $F(x) = 0$, which is impossible. Thus $\deg F \geqslant n$.

The lemma condition yields that $\Sigma_F(x) = T(x)G(x)$ for some polynomial $G(x)$ such that $\deg G = \deg \Sigma_F - (n+1) = \deg F - n$.

Now, let us define $F_1(x) = T(x)G(x) - T(x-1)G(x-1)$. Then for every positive integer n we have

$$\Sigma_{F_1}(n) = \sum_{i=1}^{n} \big(T(x)G(x) - T(x-1)G(x-1)\big) = T(n)G(n) - T(0)G(0) = T(n)G(n) = \Sigma_F(n),$$

so the polynomial $\Sigma_{F-F_1}(x) = \Sigma_F(x) - \Sigma_{F_1}(x)$ has infinitely many roots. This means that this polynomial is zero, which in turn yields $F(x) = F_1(x)$, as required. □

First, we apply the lemma to the nonzero polynomial $R_1(x) = P(x) - Q(x)$. Since the degree of $R_1(x)$ is at most n, we conclude that it is exactly n. Moreover, $R_1(x) = \alpha \cdot \big(T(x) - T(x-1)\big)$ for some nonzero constant α.

Our next aim is to prove that the polynomial $S(x) = P(x) + Q(x)$ is constant. Assume the contrary. Then, notice that the polynomial $R_2(x) = P(x)^2 - Q(x)^2 = R_1(x)S(x)$ is also nonzero and satisfies the lemma condition. Since $n < \deg R_1 + \deg S = \deg R_2 \leqslant 2n$, the lemma yields

$$R_2(x) = T(x)G(x) - T(x-1)G(x-1)$$

with some polynomial $G(x)$ with $0 < \deg G \leqslant n$.

Since the polynomial $R_1(x) = \alpha\big(T(x) - T(x-1)\big)$ divides the polynomial

$$R_2(x) = T(x)\big(G(x) - G(x-1)\big) + G(x-1)\big(T(x) - T(x-1)\big),$$

we get $R_1(x) \mid T(x)\big(G(x) - G(x-1)\big)$. On the other hand,

$$\gcd\big(T(x), R_1(x)\big) = \gcd\big(T(x), T(x) - T(x-1)\big) = \gcd\big(T(x), T(x-1)\big) = 1,$$

since both $T(x)$ and $T(x-1)$ are the products of linear polynomials, and their roots are distinct. Thus $R_1(x) \mid G(x) - G(x-1)$. However, this is impossible since $G(x) - G(x-1)$ is a nonzero polynomial of degree less than $n = \deg R_1$.

Thus, our assumption is wrong, and $S(x)$ is a constant polynomial, say $S(x) = \beta$. Notice that the polynomials $\big(2P(x) - \beta\big)/\alpha$ and $\big(2Q(x) - \beta\big)/\alpha$ are also block-similar and distinct. So we may replace the initial polynomials by these ones, thus obtaining two block-similar polynomials $P(x)$ and $Q(x)$ with $P(x) = -Q(x) = T(x) - T(x-1)$. It remains to show that this is impossible.

For every $i = 1, 2 \ldots, n$, the values $T(ik - k + 1)$ and $T(ik - 1)$ have the same sign. This means that the values $P(ik - k + 1) = T(ik - k + 1)$ and $P(ik) = -T(ik - 1)$ have opposite signs, so $P(x)$ has a root in each of the n segments $[ik - k + 1, ik]$. Since $\deg P = n$, it must have exactly one root in each of them.

Thus, the sequence $P(1), P(2), \ldots, P(k)$ should change sign exactly once. On the other hand, since $P(x)$ and $-P(x)$ are block-similar, this sequence must have as many positive terms as negative ones. Since $k = 2\ell + 1$ is odd, this shows that the middle term of the sequence above must be zero, so $P(\ell + 1) = 0$, or $T(\ell + 1) = T(\ell)$. However, this is not true since

$$|T(\ell + 1)| = |\ell + 1| \cdot |\ell| \cdot \prod_{i=2}^{n} |\ell + 1 - ik| < |\ell| \cdot |\ell + 1| \cdot \prod_{i=2}^{n} |\ell - ik| = |T(\ell)|,$$

where the strict inequality holds because $n \geqslant 2$. We come to the final contradiction.

Comment 1. In the solution above, we used the fact that $k > 1$ is odd. One can modify the arguments of the last part in order to work for every (not necessarily odd) sufficiently large value of k; namely, when k is even, one may show that the sequence $P(1), P(2), \ldots, P(k)$ has different numbers of positive and negative terms.

On the other hand, the problem statement with k replaced by 2 is false, since the polynomials $P(x) = T(x) - T(x-1)$ and $Q(x) = T(x-1) - T(x)$ *are* block-similar in this case, due to the fact that $P(2i - 1) = -P(2i) = Q(2i) = -Q(2i - 1) = T(2i - 1)$ for all $i = 1, 2, \ldots, n$. Thus, every complete solution should use the relation $k > 2$.

One may easily see that the condition $n \geqslant 2$ is also substantial, since the polynomials x and $k + 1 - x$ become block-similar if we set $n = 1$.

It is easily seen from the solution that the result still holds if we assume that the polynomials have degree *at most* n.

Solution 2. We provide an alternative argument for part (b).

Assume again that there exist two distinct block-similar polynomials $P(x)$ and $Q(x)$ of degree n. Let $R(x) = P(x) - Q(x)$ and $S(x) = P(x) + Q(x)$. For brevity, we also denote the segment $\big[(i-1)k+1, ik\big]$ by I_i, and the set $\big\{(i-1)k+1, (i-1)k+2, \ldots, ik\big\}$ of all integer points in I_i by Z_i.

Step 1. *We prove that $R(x)$ has exactly one root in each segment I_i, $i = 1, 2, \ldots, n$, and all these roots are simple.*

Indeed, take any $i \in \{1, 2, \ldots, n\}$ and choose some points $p^-, p^+ \in Z_i$ so that

$$P(p^-) = \min_{x \in Z_i} P(x) \quad \text{and} \quad P(p^+) = \max_{x \in Z_i} P(x).$$

Since the sequences of values of P and Q in Z_i are permutations of each other, we have $R(p^-) = P(p^-) - Q(p^-) \leqslant 0$ and $R(p^+) = P(p^+) - Q(p^+) \geqslant 0$. Since $R(x)$ is continuous, there exists at least one root of $R(x)$ between p^- and p^+ — thus in I_i.

So, $R(x)$ has at least one root in each of the n disjoint segments I_i with $i = 1, 2, \ldots, n$. Since $R(x)$ is nonzero and its degree does not exceed n, it should have exactly one root in each of these segments, and all these roots are simple, as required.

Step 2. *We prove that $S(x)$ is constant.*

We start with the following claim.

Claim. For every $i = 1, 2, \ldots, n$, the sequence of values $S\big((i-1)k+1\big)$, $S\big((i-1)k+2\big)$, $\ldots$, $S(ik)$ cannot be strictly increasing.

Proof. Fix any $i \in \{1, 2, \ldots, n\}$. Due to the symmetry, we may assume that $P\big(ik\big) \leqslant Q(ik)$. Choose now p^- and p^+ as in Step 1. If we had $P(p^+) = P(p^-)$, then P would be constant on Z_i, so all the elements of Z_i would be the roots of $R(x)$, which is not the case. In particular, we have $p^+ \neq p^-$. If $p^- > p^+$, then $S(p^-) = P(p^-) + Q(p^-) \leqslant Q(p^+) + P(p^+) = S(p^+)$, so our claim holds.

We now show that the remaining case $p^- < p^+$ is impossible. Assume first that $P(p^+) > Q(p^+)$. Then, like in Step 1, we have $R(p^-) \leqslant 0$, $R(p^+) > 0$, and $R(ik) \leqslant 0$, so $R(x)$ has a root in each of the intervals $[p^-, p^+)$ and $(p^+, ik]$. This contradicts the result of Step 1.

We are left only with the case $p^- < p^+$ and $P(p^+) = Q(p^+)$ (thus p^+ is the unique root of $R(x)$ in I_i). If $p^+ = ik$, then the values of $R(x)$ on $Z_i \setminus \{ik\}$ are all of the same sign, which is absurd since their sum is zero. Finally, if $p^- < p^+ < ik$, then $R(p^-)$ and $R(ik)$ are both negative. This means that $R(x)$ should have an even number of roots in $[p^-, ik]$, counted with multiplicity. This also contradicts the result of Step 1. $\square$

In a similar way, one may prove that for every $i = 1, 2, \ldots, n$, the sequence $S\big((i-1)k+1\big)$, $S\big((i-1)k+2\big)$, $\ldots$, $S(ik)$ cannot be strictly decreasing. This means that the polynomial $\Delta S(x) = S(x) - S(x-1)$ attains at least one nonnegative value, as well as at least one nonpositive value, on the set Z_i (and even on $Z_i \setminus \big\{(i-1)k+1\big\}$); so ΔS has a root in I_i.

Thus ΔS has at least n roots; however, its degree is less than n, so ΔS should be identically zero. This shows that $S(x)$ is a constant, say $S(x) \equiv \beta$.

Step 3. Notice that the polynomials $P(x) - \beta/2$ and $Q(x) - \beta/2$ are also block-similar and distinct. So we may replace the initial polynomials by these ones, thus reaching $P(x) = -Q(x)$.

Then $R(x) = 2P(x)$, so $P(x)$ has exactly one root in each of the segments I_i, $i = 1, 2, \ldots, n$. On the other hand, $P(x)$ and $-P(x)$ should attain the same number of positive values on Z_i. Since k is odd, this means that Z_i contains exactly one root of $P(x)$; moreover, this root should be at the center of Z_i, because $P(x)$ has the same number of positive and negative values on Z_i.

Thus we have found all n roots of $P(x)$, so

$$P(x) = c \prod_{i=1}^{n} (x - ik + \ell) \qquad \text{for some } c \in \mathbb{R} \setminus \{0\},$$

where $\ell = (k-1)/2$. It remains to notice that for every $t \in Z_1 \setminus \{1\}$ we have

$$|P(t)| = |c| \cdot |t - \ell - 1| \cdot \prod_{i=2}^{n} |t - ik + \ell| < |c| \cdot \ell \cdot \prod_{i=2}^{n} |1 - ik + \ell| = |P(1)|,$$

so $P(1) \neq -P(t)$ for all $t \in Z_1$. This shows that $P(x)$ is not block-similar to $-P(x)$. The final contradiction.

Comment 2. One may merge Steps 1 and 2 in the following manner. As above, we set $R(x) = P(x) - Q(x)$ and $S(x) = P(x) + Q(x)$.

We aim to prove that the polynomial $S(x) = 2P(x) - R(x) = 2Q(x) + R(x)$ is constant. Since the degrees of $R(x)$ and $S(x)$ do not exceed n, it suffices to show that the total number of roots of $R(x)$ and $\Delta S(x) = S(x) - S(x-1)$ is at least $2n$. For this purpose, we prove the following claim.

Claim. For every $i = 1, 2, \ldots, n$, either each of R and ΔS has a root in I_i, or R has at least two roots in I_i.

Proof. Fix any $i \in \{1, 2, \ldots, n\}$. Let $r \in Z_i$ be a point such that $|R(r)| = \max_{x \in Z_i} |R(x)|$; we may assume that $R(r) > 0$. Next, let $p^-, q^+ \in I_i$ be some points such that $P(p^-) = \min_{x \in Z_i} P(x)$ and $Q(q^+) = \max_{x \in Z_i} Q(x)$. Notice that $P(p^-) \leqslant Q(r) < P(r)$ and $Q(q^+) \geqslant P(r) > Q(r)$, so r is different from p^- and q^+.

Without loss of generality, we may assume that $p^- < r$. Then we have $R(p^-) = P(p^-) - Q(p^-) \leqslant 0 < R(r)$, so $R(x)$ has a root in $[p^-, r)$. If $q^+ > r$, then, similarly, $R(q^+) \leqslant 0 < R(r)$, and $R(x)$ also has a root in $(r, q^+]$; so $R(x)$ has two roots in I_i, as required.

In the remaining case we have $q^+ < r$; it suffices now to show that in this case ΔS has a root in I_i. Since $P(p^-) \leqslant Q(r)$ and $|R(p^-)| \leqslant R(r)$, we have $S(p^-) = 2P(p^-) - R(p^-) \leqslant 2Q(r) + R(r) = S(r)$. Similarly, we get $S(q^+) = 2Q(q^+) + R(q^+) \geqslant 2P(r) - R(r) = S(r)$. Therefore, the sequence of values of S on Z_i is neither strictly increasing nor strictly decreasing, which shows that ΔS has a root in I_i. □

Comment 3. After finding the relation $P(x) - Q(x) = \alpha\big(T(x) - T(x-1)\big)$ from Solution 1, one may also follow the approach presented in Solution 2. Knowledge of the difference of polynomials may simplify some steps; e.g., it is clear now that $P(x) - Q(x)$ has exactly one root in each of the segments I_i.

Combinatorics

C1. In Lineland there are $n \geqslant 1$ towns, arranged along a road running from left to right. Each town has a *left bulldozer* (put to the left of the town and facing left) and a *right bulldozer* (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a right and a left bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, the bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B being to the right of A. We say that town A can *sweep town B away* if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly, B can sweep A away if the left bulldozer of B can move to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town which cannot be swept away by any other one.

(Estonia)

Solution 1. Let $T_1, T_2, \ldots, T_n$ be the towns enumerated from left to right. Observe first that, if town T_i can sweep away town T_j, then T_i also can sweep away every town located between T_i and T_j.

We prove the problem statement by strong induction on n. The base case $n = 1$ is trivial.

For the induction step, we first observe that the left bulldozer in T_1 and the right bulldozer in T_n are completely useless, so we may forget them forever. Among the other $2n-2$ bulldozers, we choose the largest one. Without loss of generality, it is the right bulldozer of some town T_k with $k < n$.

Surely, with this large bulldozer T_k can sweep away all the towns to the right of it. Moreover, none of these towns can sweep T_k away; so they also cannot sweep away any town to the left of T_k. Thus, if we remove the towns $T_{k+1}, T_{k+2}, \ldots, T_n$, none of the remaining towns would change its status of being (un)sweepable away by the others.

Applying the induction hypothesis to the remaining towns, we find a unique town among $T_1, T_2, \ldots, T_k$ which cannot be swept away. By the above reasons, it is also the unique such town in the initial situation. Thus the induction step is established.

Solution 2. We start with the same enumeration and the same observation as in Solution 1. We also denote by ℓ_i and r_i the sizes of the left and the right bulldozers belonging to T_i, respectively. One may easily see that no two towns T_i and T_j with $i < j$ can sweep each other away, for this would yield $r_i > \ell_j > r_i$.

Clearly, there is no town which can sweep T_n away from the right. Then we may choose the leftmost town T_k which cannot be swept away from the right. One can observe now that no town T_i with $i > k$ may sweep away some town T_j with $j < k$, for otherwise T_i would be able to sweep T_k away as well.

Now we prove two claims, showing together that T_k is the unique town which cannot be swept away, and thus establishing the problem statement.

Claim 1. T_k also cannot be swept away from the left.

Proof. Let T_m be some town to the left of T_k. By the choice of T_k, town T_m can be swept away from the right by some town T_p with $p > m$. As we have already observed, p cannot be greater than k. On the other hand, T_m cannot sweep T_p away, so *a fortiori* it cannot sweep T_k away. $\square$

Claim 2. Any town T_m with $m \neq k$ can be swept away by some other town.

Proof. If $m < k$, then T_m can be swept away from the right due to the choice of T_k. In the remaining case we have $m > k$.

Let T_p be a town among $T_k, T_{k+1}, \dots, T_{m-1}$ having the largest right bulldozer. We claim that T_p can sweep T_m away. If this is not the case, then $r_p < \ell_q$ for some q with $p < q \leqslant m$. But this means that ℓ_q is greater than all the numbers r_i with $k \leqslant i \leqslant m-1$, so T_q can sweep T_k away. This contradicts the choice of T_k. □

Comment 1. One may employ the same ideas within the inductive approach. Here we sketch such a solution.

Assume that the problem statement holds for the collection of towns $T_1, T_2, \dots, T_{n-1}$, so that there is a unique town T_i among them which cannot be swept away by any other of them. Thus we need to prove that in the full collection $T_1, T_2, \dots, T_n$, exactly one of the towns T_i and T_n cannot be swept away.

If T_n cannot sweep T_i away, then it remains to prove that T_n can be swept away by some other town. This can be established as in the second paragraph of the proof of Claim 2.

If T_n can sweep T_i away, then it remains to show that T_n cannot be swept away by any other town. Since T_n can sweep T_i away, it also can sweep all the towns $T_i, T_{i+1}, \dots, T_{n-1}$ away, so T_n cannot be swept away by any of those. On the other hand, none of the remaining towns $T_1, T_2, \dots, T_{i-1}$ can sweep T_i away, so that they cannot sweep T_n away as well.

Comment 2. Here we sketch yet another inductive approach. Assume that $n > 1$. Firstly, we find a town which can be swept away by each of its neighbors (each town has two neighbors, except for the bordering ones each of which has one); we call such town a *loser*. Such a town exists, because there are $n-1$ pairs of neighboring towns, and in each of them there is only one which can sweep the other away; so there exists a town which is a winner in none of these pairs.

Notice that a loser can be swept away, but it cannot sweep any other town away (due to its neighbors' protection). Now we remove a loser, and *suggest* its left bulldozer to its *right* neighbor (if it exists), and its right bulldozer to a *left* one (if it exists). Surely, a town accepts a suggestion if a suggested bulldozer is larger than the town's one of the same orientation.

Notice that suggested bulldozers are useless in attack (by the definition of a loser), but may serve for defensive purposes. Moreover, each suggested bulldozer's protection works for the same pairs of remaining towns as before the removal.

By the induction hypothesis, the new configuration contains exactly one town which cannot be swept away. The arguments above show that the initial one also satisfies this property.

Solution 3. We separately prove that (i) there exists a town which cannot be swept away, and that (ii) there is at most one such town. We also make use of the two observations from the previous solutions.

To prove (i), assume contrariwise that every town can be swept away. Let t_1 be the leftmost town; next, for every $k = 1, 2, \dots$ we inductively choose t_{k+1} to be some town which can sweep t_k away. Now we claim that for every $k = 1, 2, \dots$, the town t_{k+1} is to the right of t_k; this leads to the contradiction, since the number of towns is finite.

Induction on k. The base case $k = 1$ is clear due to the choice of t_1. Assume now that for all j with $1 \leqslant j < k$, the town t_{j+1} is to the right of t_j. Suppose that t_{k+1} is situated to the left of t_k; then it lies between t_j and t_{j+1} (possibly coinciding with t_j) for some $j < k$. Therefore, t_{k+1} can be swept away by t_{j+1}, which shows that it cannot sweep t_{j+1} away — so t_{k+1} also cannot sweep t_k away. This contradiction proves the induction step.

To prove (ii), we also argue indirectly and choose two towns A and B neither of which can be swept away, with A being to the left of B. Consider the largest bulldozer b between them (taking into consideration the right bulldozer of A and the left bulldozer of B). Without loss of generality, b is a left bulldozer; then it is situated in some town to the right of A, and this town may sweep A away since nothing prevents it from doing that. A contradiction.

Comment 3. The Problem Selection Committee decided to reformulate this problem. The original formulation was as follows.

Let n be a positive integer. There are n cards in a deck, enumerated from bottom to top with numbers $1, 2, \ldots, n$. For each $i = 1, 2, \ldots, n$, an even number a_i is printed on the lower side and an odd number b_i is printed on the upper side of the i^{th} card. We say that the i^{th} card opens *the j^{th} card, if $i < j$ and $b_i < a_k$ for every $k = i+1, i+2, \ldots, j$. Similarly, we say that the i^{th} card* closes *the j^{th} card, if $i > j$ and $a_i < b_k$ for every $k = i-1, i-2, \ldots, j$. Prove that the deck contains exactly one card which is neither opened nor closed by any other card.*

C2. Let $\mathcal{V}$ be a finite set of points in the plane. We say that $\mathcal{V}$ is *balanced* if for any two distinct points $A, B \in \mathcal{V}$, there exists a point $C \in \mathcal{V}$ such that $AC = BC$. We say that $\mathcal{V}$ is *center-free* if for any distinct points $A, B, C \in \mathcal{V}$, there does not exist a point $P \in \mathcal{V}$ such that $PA = PB = PC$.

(*a*) Show that for all $n \geqslant 3$, there exists a balanced set consisting of n points.

(*b*) For which $n \geqslant 3$ does there exist a balanced, center-free set consisting of n points?

(Netherlands)

Answer for part (*b*). All odd integers $n \geqslant 3$.

Solution.

Part (*a*). Assume that n is odd. Consider a regular n-gon. Label the vertices of the n-gon as $A_1, A_2, \ldots, A_n$ in counter-clockwise order, and set $\mathcal{V} = \{A_1, \ldots, A_n\}$. We check that $\mathcal{V}$ is balanced. For any two distinct vertices A_i and A_j, let $k \in \{1, 2, \ldots, n\}$ be the solution of $2k \equiv i + j \pmod{n}$. Then, since $k - i \equiv j - k \pmod{n}$, we have $A_iA_k = A_jA_k$, as required.

Now assume that n is even. Consider a regular $(3n-6)$-gon, and let O be its circumcenter. Again, label its vertices as $A_1, \ldots, A_{3n-6}$ in counter-clockwise order, and choose $\mathcal{V} = \{O, A_1, A_2, \ldots, A_{n-1}\}$. We check that $\mathcal{V}$ is balanced. For any two distinct vertices A_i and A_j, we always have $OA_i = OA_j$. We now consider the vertices O and A_i. First note that the triangle $OA_iA_{n/2-1+i}$ is equilateral for all $i \leqslant \frac{n}{2}$. Hence, if $i \leqslant \frac{n}{2}$, then we have $OA_{n/2-1+i} = A_iA_{n/2-1+i}$; otherwise, if $i > \frac{n}{2}$, then we have $OA_{i-n/2+1} = A_iA_{i-n/2+1}$. This completes the proof.

An example of such a construction when $n = 10$ is shown in Figure 1.

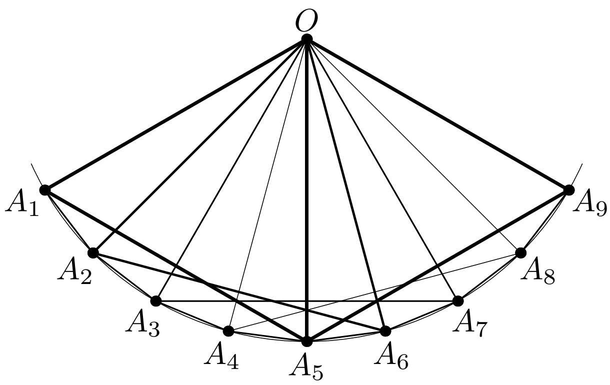

Figure 1

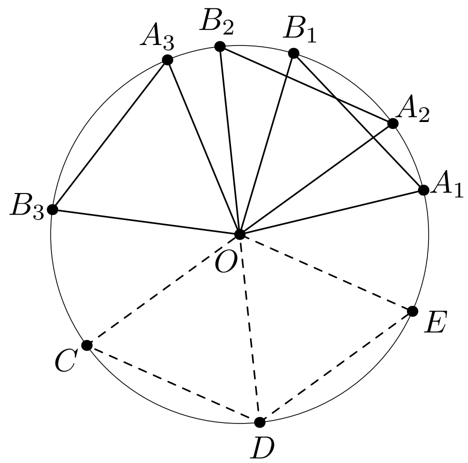

Figure 2

Comment (*a*). There are many ways to construct an example by placing equilateral triangles in a circle. Here we present one general method.

Let O be the center of a circle and let $A_1, B_1, \ldots, A_k, B_k$ be distinct points on the circle such that the triangle OA_iB_i is equilateral for each i. Then $\mathcal{V} = \{O, A_1, B_1, \ldots, A_k, B_k\}$ is balanced. To construct a set of even cardinality, put extra points C, D, E on the circle such that triangles OCD and ODE are equilateral (see Figure 2). Then $\mathcal{V} = \{O, A_1, B_1, \ldots, A_k, B_k, C, D, E\}$ is balanced.

Part (*b*). We now show that there exists a balanced, center-free set containing n points for all odd $n \geqslant 3$, and that one does not exist for any even $n \geqslant 3$.

If n is odd, then let $\mathcal{V}$ be the set of vertices of a regular n-gon. We have shown in part (*a*) that $\mathcal{V}$ is balanced. We claim that $\mathcal{V}$ is also center-free. Indeed, if P is a point such that

$PA = PB = PC$ for some three distinct vertices A, B and C, then P is the circumcenter of the n-gon, which is not contained in $\mathcal{V}$.

Now suppose that $\mathcal{V}$ is a balanced, center-free set of even cardinality n. We will derive a contradiction. For a pair of distinct points $A, B \in \mathcal{V}$, we say that a point $C \in \mathcal{V}$ is *associated* with the pair $\{A, B\}$ if $AC = BC$. Since there are $\frac{n(n-1)}{2}$ pairs of points, there exists a point $P \in \mathcal{V}$ which is associated with at least $\left\lceil \frac{n(n-1)}{2} / n \right\rceil = \frac{n}{2}$ pairs. Note that none of these $\frac{n}{2}$ pairs can contain P, so that the union of these $\frac{n}{2}$ pairs consists of at most $n-1$ points. Hence there exist two such pairs that share a point. Let these two pairs be $\{A, B\}$ and $\{A, C\}$. Then $PA = PB = PC$, which is a contradiction.

Comment (*b*). We can rephrase the argument in graph theoretic terms as follows. Let $\mathcal{V}$ be a balanced, center-free set consisting of n points. For any pair of distinct vertices $A, B \in \mathcal{V}$ and for any $C \in \mathcal{V}$ such that $AC = BC$, draw directed edges $A \to C$ and $B \to C$. Then all pairs of vertices generate altogether at least $n(n-1)$ directed edges; since the set is center-free, these edges are distinct. So we must obtain a graph in which any two vertices are connected in both directions. Now, each vertex has exactly $n-1$ incoming edges, which means that $n-1$ is even. Hence n is odd.

C3. For a finite set A of positive integers, we call a partition of A into two disjoint nonempty subsets A_1 and A_2 *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

(Ukraine)

Answer. 3024.

Solution. Let $A = \{a_1, a_2, \ldots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$. For a finite nonempty set B of positive integers, denote by $\operatorname{lcm} B$ and $\gcd B$ the least common multiple and the greatest common divisor of the elements in B, respectively.

Consider any good partition (A_1, A_2) of A. By definition, $\operatorname{lcm} A_1 = d = \gcd A_2$ for some positive integer d. For any $a_i \in A_1$ and $a_j \in A_2$, we have $a_i \leqslant d \leqslant a_j$. Therefore, we have $A_1 = \{a_1, a_2, \ldots, a_k\}$ and $A_2 = \{a_{k+1}, a_{k+2}, \ldots, a_n\}$ for some k with $1 \leqslant k < n$. Hence, each good partition is determined by an element a_k, where $1 \leqslant k < n$. We call such a_k *partitioning*.

It is convenient now to define $\ell_k = \operatorname{lcm}(a_1, a_2, \ldots, a_k)$ and $g_k = \gcd(a_{k+1}, a_{k+2}, \ldots, a_n)$ for $1 \leqslant k \leqslant n-1$. So a_k is partitioning exactly when $\ell_k = g_k$.

We proceed by proving some properties of partitioning elements, using the following claim.

Claim. If a_{k-1} and a_k are partitioning where $2 \leqslant k \leqslant n-1$, then $g_{k-1} = g_k = a_k$.

Proof. Assume that a_{k-1} and a_k are partitioning. Since $\ell_{k-1} = g_{k-1}$, we have $\ell_{k-1} \mid a_k$. Therefore, $g_k = \ell_k = \operatorname{lcm}(\ell_{k-1}, a_k) = a_k$, and $g_{k-1} = \gcd(a_k, g_k) = a_k$, as desired. □

Property 1. For every $k = 2, 3, \ldots, n-2$, at least one of a_{k-1}, a_k, and a_{k+1} is not partitioning.

Proof. Suppose, to the contrary, that all three numbers a_{k-1}, a_k, and a_{k+1} are partitioning. The claim yields that $a_{k+1} = g_k = a_k$, a contradiction. □

Property 2. The elements a_1 and a_2 cannot be simultaneously partitioning. Also, a_{n-2} and a_{n-1} cannot be simultaneously partitioning

Proof. Assume that a_1 and a_2 are partitioning. By the claim, it follows that $a_2 = g_1 = \ell_1 = \operatorname{lcm}(a_1) = a_1$, a contradiction.

Similarly, assume that a_{n-2} and a_{n-1} are partitioning. The claim yields that $a_{n-1} = g_{n-1} = \gcd(a_n) = a_n$, a contradiction. □

Now let A be an n-element set with exactly 2015 good partitions. Clearly, we have $n \geqslant 5$. Using Property 2, we find that there is at most one partitioning element in each of $\{a_1, a_2\}$ and $\{a_{n-2}, a_{n-1}\}$. By Property 1, there are at least $\left\lfloor \frac{n-5}{3} \right\rfloor$ non-partitioning elements in $\{a_3, a_4, \ldots, a_{n-3}\}$. Therefore, there are at most $(n-1) - 2 - \left\lfloor \frac{n-5}{3} \right\rfloor = \left\lceil \frac{2(n-2)}{3} \right\rceil$ partitioning elements in A. Thus, $\left\lceil \frac{2(n-2)}{3} \right\rceil \geqslant 2015$, which implies that $n \geqslant 3024$.

Finally, we show that there exists a set of 3024 positive integers with exactly 2015 partitioning elements. Indeed, in the set $A = \{2 \cdot 6^i, 3 \cdot 6^i, 6^{i+1} \mid 0 \leqslant i \leqslant 1007\}$, each element of the form $3 \cdot 6^i$ or 6^i, except 6^{1008}, is partitioning.

Therefore, the minimum possible value of n is 3024.

Comment. Here we will work out the general case when 2015 is replaced by an arbitrary positive integer m. Note that the bound $\left\lceil \frac{2(n-2)}{3} \right\rceil \geqslant m$ obtained in the solution is, in fact, true for any positive integers m and n. Using this bound, one can find that $n \geqslant \left\lceil \frac{3m}{2} \right\rceil + 1$.

To show that the bound is sharp, one constructs a set of $\left\lceil \frac{3m}{2} \right\rceil + 1$ elements with exactly m good partitions. Indeed, the minimum is attained on the set $\{6^i, 2 \cdot 6^i, 3 \cdot 6^i \mid 0 \leqslant i \leqslant t-1\} \cup \{6^t\}$ for every even $m = 2t$, and $\{2 \cdot 6^i, 3 \cdot 6^i, 6^{i+1} \mid 0 \leqslant i \leqslant t-1\}$ for every odd $m = 2t-1$.

C4. Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leqslant n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.

(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.

(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

(Finland)

Answer. The game ends in a draw when $n = 1, 2, 4, 6$; otherwise B wins.

Solution. For brevity, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

Firstly, we show that B wins whenever $n \neq 1, 2, 4, 6$. For this purpose, we provide a strategy which guarantees that B can always make a move after A's move, and also guarantees that the game does not end in a draw.

We begin with an important observation.

Lemma. Suppose that B's first pick is n and that A has made the k^{th} move where $k \geqslant 2$. Then B can also make the k^{th} move.

Proof. Let $\mathcal{S}$ be the set of the first k numbers chosen by A. Since $\mathcal{S}$ does not contain consecutive integers, we see that the set $[n] \setminus \mathcal{S}$ consists of k "contiguous components" if $1 \in \mathcal{S}$, and $k + 1$ components otherwise. Since B has chosen only $k - 1$ numbers, there is at least one component of $[n] \setminus \mathcal{S}$ consisting of numbers not yet picked by B. Hence, B can choose a number from this component. □

We will now describe a winning strategy for B, when $n \neq 1, 2, 4, 6$. By symmetry, we may assume that A's first choice is a number not exceeding $\frac{n+1}{2}$. So B can pick the number n in B's first turn. We now consider two cases.

Case 1. n is odd and $n \geqslant 3$. The only way the game ends in a draw is that A eventually picks all the odd numbers from the set $[n]$. However, B has already chosen n, so this cannot happen. Thus B can continue to apply the lemma until A cannot make a move.

Case 2. n is even and $n \geqslant 8$. Since B has picked n, the game is a draw only if A can eventually choose all the odd numbers from the set $[n - 1]$. So B picks a number from the set $\{1, 3, 5, \ldots, n - 3\}$ not already chosen by A, on B's second move. This is possible since the set consists of $\frac{n-2}{2} \geqslant 3$ numbers and A has chosen only 2 numbers. Hereafter B can apply the lemma until A cannot make a move.

Hence, in both cases A loses.

We are left with the cases $n = 1, 2, 4, 6$. The game is trivially a draw when $n = 1, 2$. When $n = 4$, A has to first pick 1 to avoid losing. Similarly, B has to choose 4 as well. It then follows that the game ends in a draw.

When $n = 6$, B gets at least a draw by the lemma or by using a mirror strategy. On the other hand, A may also get at least a draw in the following way. In the first turn, A chooses 1. After B's response by a number b, A finds a neighbor c of b which differs from 1 and 2, and reserves c for A's third move. Now, clearly A can make the second move by choosing a number different from $1, 2, c - 1, c, c + 1$. Therefore A will not lose.

Comment 1. We present some explicit winning strategies for B.

We start with the case n is odd and $n \geqslant 3$. B starts by picking n in the first turn. On the k^{th} move for $k \geqslant 2$, B chooses the number exactly 1 less than A's k^{th} pick. The only special case is when A's k^{th} choice is 1. In this situation, A's first pick was a number $a > 1$ and B can respond by choosing $a - 1$ on the k^{th} move instead.

We now give an alternative winning strategy in the case n is even and $n \geqslant 8$. We first present a winning strategy for the case when A's first pick is 1. We consider two cases depending on A's second move.

Case 1. *A's second pick is* 3. Then B chooses $n-3$ on the second move. On the k^{th} move, B chooses the number exactly 1 less than A's k^{th} pick except that B chooses 2 if A's k^{th} pick is $n-2$ or $n-1$.

Case 2. *A's second pick is $a > 3$.* Then B chooses $a - 2$ on the second move. Afterwards on the k^{th} move, B picks the number exactly 1 less than A's k^{th} pick.

One may easily see that this strategy guarantees B's victory, when A's first pick is 1.

The following claim shows how to extend the strategy to the general case.

Claim. Assume that B has an explicit strategy leading to a victory after A picks 1 on the first move. Then B also has an explicit strategy leading to a victory after any first moves of A.

Proof. Let S be an optimal strategy of B after A picks 1 on the first move. Assume that A picks some number $a > 1$ on this move; we show how B can make use of S in order to win in this case.

In parallel to the real play, B starts an *imaginary* play. The positions in these plays differ by flipping the segment $[1, a]$; so, if a player chooses some number x in the real play, then the same player chooses a number x or $a + 1 - x$ in the imaginary play, depending on whether $x > a$ or $x \leqslant a$. Thus A's first pick in the imaginary play is 1.

Clearly, a number is chosen in the real play exactly if the corresponding number is chosen in the imaginary one. Next, if an unchosen number is neighboring to one chosen by A in the imaginary play, then the corresponding number also has this property in the real play, so A also cannot choose it. One can easily see that a similar statement with real and imaginary plays interchanged holds for B instead of A.

Thus, when A makes some move in the real play, B may imagine the corresponding legal move in the imaginary one. Then B chooses the response according to S in the imaginary game and makes the corresponding legal move in the real one. Acting so, B wins the imaginary game, thus B will also win the real one. $\square$

Hence, B has a winning strategy for all even n greater or equal to 8.

Notice that the claim can also be used to simplify the argument when n is odd.

Comment 2. One may also employ symmetry when n is odd. In particular, B could use a mirror strategy. However, additional ideas are required to modify the strategy after A picks $\frac{n+1}{2}$.

C5. Consider an infinite sequence $a_1, a_2, \dots$ of positive integers with $a_i \leqslant 2015$ for all $i \geqslant 1$. Suppose that for any two distinct indices i and j we have $i + a_i \neq j + a_j$.

Prove that there exist two positive integers b and N such that

$$\left| \sum_{i=m+1}^{n} (a_i - b) \right| \leqslant 1007^2$$

whenever $n > m \geqslant N$.

(Australia)

Solution 1. We visualize the set of positive integers as a sequence of points. For each n we draw an arrow emerging from n that points to $n + a_n$; so the *length* of this arrow is a_n. Due to the condition that $m + a_m \neq n + a_n$ for $m \neq n$, each positive integer receives at most one arrow. There are some positive integers, such as 1, that receive no arrows; these will be referred to as *starting points* in the sequel. When one starts at any of the starting points and keeps following the arrows, one is led to an infinite path, called its *ray*, that visits a strictly increasing sequence of positive integers. Since the length of any arrow is at most 2015, such a ray, say with starting point s, meets every interval of the form $[n, n + 2014]$ with $n \geqslant s$ at least once.

Suppose for the sake of contradiction that there would be at least 2016 starting points. Then we could take an integer n that is larger than the first 2016 starting points. But now the interval $[n, n + 2014]$ must be met by at least 2016 rays in distinct points, which is absurd. We have thereby shown that the number b of starting points satisfies $1 \leqslant b \leqslant 2015$. Let N denote any integer that is larger than all starting points. We contend that b and N are as required.

To see this, let any two integers m and n with $n > m \geqslant N$ be given. The sum $\sum_{i=m+1}^{n} a_i$ gives the total length of the arrows emerging from $m + 1, \dots, n$. Taken together, these arrows form b subpaths of our rays, some of which may be empty. Now on each ray we look at the first number that is larger than m; let $x_1, \dots, x_b$ denote these numbers, and let $y_1, \dots, y_b$ enumerate in corresponding order the numbers defined similarly with respect to n. Then the list of differences $y_1 - x_1, \dots, y_b - x_b$ consists of the lengths of these paths and possibly some zeros corresponding to empty paths. Consequently, we obtain

$$\sum_{i=m+1}^{n} a_i = \sum_{j=1}^{b} (y_j - x_j),$$

whence

$$\sum_{i=m+1}^{n} (a_i - b) = \sum_{j=1}^{b} (y_j - n) - \sum_{j=1}^{b} (x_j - m).$$

Now each of the b rays meets the interval $[m + 1, m + 2015]$ at some point and thus $x_1 - m, \dots, x_b - m$ are b distinct members of the set $\{1, 2, \dots, 2015\}$. Moreover, since $m + 1$ is not a starting point, it must belong to some ray; so 1 has to appear among these numbers, wherefore

$$1 + \sum_{j=1}^{b-1} (j + 1) \leqslant \sum_{j=1}^{b} (x_j - m) \leqslant 1 + \sum_{j=1}^{b-1} (2016 - b + j).$$

The same argument applied to n and $y_1, \dots, y_b$ yields

$$1 + \sum_{j=1}^{b-1} (j + 1) \leqslant \sum_{j=1}^{b} (y_j - n) \leqslant 1 + \sum_{j=1}^{b-1} (2016 - b + j).$$

So altogether we get

$$\left|\sum_{i=m+1}^{n}(a_i-b)\right| \leqslant \sum_{j=1}^{b-1}\big((2016-b+j)-(j+1)\big) = (b-1)(2015-b)$$
$$\leqslant \left(\frac{(b-1)+(2015-b)}{2}\right)^2 = 1007^2\,,$$

as desired.

Solution 2. Set $s_n = n + a_n$ for all positive integers n. By our assumptions, we have

$$n+1 \leqslant s_n \leqslant n+2015$$

for all $n \in \mathbb{Z}_{>0}$. The members of the sequence $s_1, s_2, \ldots$ are distinct. We shall investigate the set

$$M = \mathbb{Z}_{>0} \setminus \{s_1, s_2, \ldots\}\,.$$

Claim. At most 2015 numbers belong to M.

Proof. Otherwise let $m_1 < m_2 < \cdots < m_{2016}$ be any 2016 distinct elements from M. For $n = m_{2016}$ we have

$$\{s_1, \ldots, s_n\} \cup \{m_1, \ldots, m_{2016}\} \subseteq \{1, 2, \ldots, n+2015\}\,,$$

where on the left-hand side we have a disjoint union containing altogether $n+2016$ elements. But the set on the right-hand side has only $n+2015$ elements. This contradiction proves our claim. □

Now we work towards proving that the positive integers $b = |M|$ and $N = \max(M)$ are as required. Recall that we have just shown $b \leqslant 2015$.

Let us consider any integer $r \geqslant N$. As in the proof of the above claim, we see that

$$B_r = M \cup \{s_1, \ldots, s_r\} \tag{1}$$

is a subset of $[1, r+2015] \cap \mathbb{Z}$ with precisely $b+r$ elements. Due to the definitions of M and N, we also know $[1, r+1] \cap \mathbb{Z} \subseteq B_r$. It follows that there is a set $C_r \subseteq \{1, 2, \ldots, 2014\}$ with $|C_r| = b-1$ and

$$B_r = \big([1, r+1] \cap \mathbb{Z}\big) \cup \big\{r+1+x \,\big|\, x \in C_r\big\}\,. \tag{2}$$

For any finite set of integers J we denote the sum of its elements by $\sum J$. Now the equations (1) and (2) give rise to two ways of computing $\sum B_r$ and the comparison of both methods leads to

$$\sum M + \sum_{i=1}^{r} s_i = \sum_{i=1}^{r} i + b(r+1) + \sum C_r\,,$$

or in other words to

$$\sum M + \sum_{i=1}^{r}(a_i - b) = b + \sum C_r\,. \tag{3}$$

After this preparation, we consider any two integers m and n with $n > m \geqslant N$. Plugging $r = n$ and $r = m$ into (3) and subtracting the estimates that result, we deduce

$$\sum_{i=m+1}^{n}(a_i - b) = \sum C_n - \sum C_m\,.$$

Since C_n and C_m are subsets of $\{1, 2, \ldots, 2014\}$ with $|C_n| = |C_m| = b - 1$, it is clear that the absolute value of the right-hand side of the above inequality attains its largest possible value if either $C_m = \{1, 2, \ldots, b-1\}$ and $C_n = \{2016 - b, \ldots, 2014\}$, or the other way around. In these two cases we have

$$\left|\sum C_n - \sum C_m\right| = (b-1)(2015-b)\,,$$

so in the general case we find

$$\left|\sum_{i=m+1}^{n} (a_i - b)\right| \leqslant (b-1)(2015-b) \leqslant \left(\frac{(b-1)+(2015-b)}{2}\right)^2 = 1007^2\,,$$

as desired.

Comment. The sets C_n may be visualized by means of the following process: Start with an empty blackboard. For $n \geqslant 1$, the following happens during the n^{th} step. The number a_n gets written on the blackboard, then all numbers currently on the blackboard are decreased by 1, and finally all zeros that have arisen get swept away.

It is not hard to see that the numbers present on the blackboard after n steps are distinct and form the set C_n. Moreover, it is possible to complete a solution based on this idea.

C6. Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

(U.S.A.)

Solution 1. Define an *odd* (respectively, *even*) *representation* of n to be a representation of n as a sum of an odd (respectively, even) number of distinct elements of S. Let $\mathbb{Z}_{>0}$ denote the set of all positive integers.

Suppose, to the contrary, that there exist only finitely many positive integers that are not clean. Therefore, there exists a positive integer N such that every integer $n > N$ has exactly one odd representation.

Clearly, S is infinite. We now claim the following properties of odd and even representations.

Property 1. Any positive integer n has at most one odd and at most one even representation.

Proof. We first show that every integer n has at most one even representation. Since S is infinite, there exists $x \in S$ such that $x > \max\{n, N\}$. Then, the number $n+x$ must be clean, and x does not appear in any even representation of n. If n has more than one even representation, then we obtain two distinct odd representations of $n + x$ by adding x to the even representations of n, which is impossible. Therefore, n can have at most one even representation.

Similarly, there exist two distinct elements $y, z \in S$ such that $y, z > \max\{n, N\}$. If n has more than one odd representation, then we obtain two distinct odd representations of $n+y+z$ by adding y and z to the odd representations of n. This is again a contradiction. $\square$

Property 2. Fix $s \in S$. Suppose that a number $n > N$ has no even representation. Then $n + 2as$ has an even representation containing s for all integers $a \geqslant 1$.

Proof. It is sufficient to prove the following statement: If n has no even representation without s, then $n+2s$ has an even representation containing s (and hence no even representation without s by Property 1).

Notice that the odd representation of $n + s$ does not contain s; otherwise, we have an even representation of n without s. Then, adding s to this odd representation of $n + s$, we get that $n + 2s$ has an even representation containing s, as desired. $\square$

Property 3. Every sufficiently large integer has an even representation.

Proof. Fix any $s \in S$, and let r be an arbitrary element in $\{1, 2, \dots, 2s\}$. Then, Property 2 implies that the set $Z_r = \{r + 2as\colon a \geqslant 0\}$ contains at most one number exceeding N with no even representation. Therefore, Z_r contains finitely many positive integers with no even representation, and so does $\mathbb{Z}_{>0} = \bigcup_{r=1}^{2s} Z_r$. $\square$

In view of Properties 1 and 3, we may assume that N is chosen such that every $n > N$ has exactly one odd and exactly one even representation. In particular, each element $s > N$ of S has an even representation.

Property 4. For any $s, t \in S$ with $N < s < t$, the even representation of t contains s.

Proof. Suppose the contrary. Then, $s+t$ has at least two odd representations: one obtained by adding s to the even representation of t and one obtained by adding t to the even representation of s. Since the latter does not contain s, these two odd representations of $s + t$ are distinct, a contradiction. $\square$

Let $s_1 < s_2 < \cdots$ be all the elements of S, and set $\sigma_n = \sum_{i=1}^{n} s_i$ for each nonnegative integer n. Fix an integer k such that $s_k > N$. Then, Property 4 implies that for every $i > k$ the even representation of s_i contains all the numbers $s_k, s_{k+1}, \dots, s_{i-1}$. Therefore,

$$s_i = s_k + s_{k+1} + \cdots + s_{i-1} + R_i = \sigma_{i-1} - \sigma_{k-1} + R_i, \tag{1}$$

where R_i is a sum of some of $s_1, \dots, s_{k-1}$. In particular, $0 \leqslant R_i \leqslant s_1 + \cdots + s_{k-1} = \sigma_{k-1}$.

Let j_0 be an integer satisfying $j_0 > k$ and $\sigma_{j_0} > 2\sigma_{k-1}$. Then (1) shows that, for every $j > j_0$,

$$s_{j+1} \geqslant \sigma_j - \sigma_{k-1} > \sigma_j/2. \tag{2}$$

Next, let $p > j_0$ be an index such that $R_p = \min_{i>j_0} R_i$. Then,

$$s_{p+1} = s_k + s_{k+1} + \cdots + s_p + R_{p+1} = (s_p - R_p) + s_p + R_{p+1} \geqslant 2s_p.$$

Therefore, there is no element of S larger than s_p but smaller than $2s_p$. It follows that the even representation τ of $2s_p$ does not contain any element larger than s_p. On the other hand, inequality (2) yields $2s_p > s_1 + \cdots + s_{p-1}$, so τ must contain a term larger than s_{p-1}. Thus, it must contain s_p. After removing s_p from τ, we have that s_p has an odd representation not containing s_p, which contradicts Property 1 since s_p itself also forms an odd representation of s_p.

Solution 2. We will also use Property 1 from Solution 1.

We first define some terminology and notations used in this solution. Let $\mathbb{Z}_{\geqslant 0}$ denote the set of all nonnegative integers. All sums mentioned are regarded as sums of distinct elements of S. Moreover, a sum is called *even* or *odd* depending on the parity of the number of terms in it. All closed or open intervals refer to sets of all integers inside them, e.g., $[a, b] = \{x \in \mathbb{Z}\colon a \leqslant x \leqslant b\}$.

Again, let $s_1 < s_2 < \cdots$ be all elements of S, and denote $\sigma_n = \sum_{i=1}^{n} s_i$ for each positive integer n. Let O_n (respectively, E_n) be the set of numbers representable as an odd (respectively, even) sum of elements of $\{s_1, \ldots, s_n\}$. Set $E = \bigcup_{n=1}^{\infty} E_n$ and $O = \bigcup_{n=1}^{\infty} O_n$. We assume that $0 \in E_n$ since 0 is representable as a sum of 0 terms.

We now proceed to our proof. Assume, to the contrary, that there exist only finitely many positive integers that are not clean and denote the number of non-clean positive integers by $m - 1$. Clearly, S is infinite. By Property 1 from Solution 1, every positive integer n has at most one odd and at most one even representation.

<u>*Step 1.*</u> We estimate s_{n+1} and σ_{n+1}.

Upper bounds: Property 1 yields $|O_n| = |E_n| = 2^{n-1}$, so $\left|[1, 2^{n-1} + m] \setminus O_n\right| \geqslant m$. Hence, there exists a clean integer $x_n \in [1, 2^{n-1} + m] \setminus O_n$. The definition of O_n then yields that the odd representation of x_n contains a term larger than s_n. Therefore, $s_{n+1} \leqslant x_n \leqslant 2^{n-1} + m$ for every positive integer n. Moreover, since s_1 is the smallest clean number, we get $\sigma_1 = s_1 \leqslant m$. Then,

$$\sigma_{n+1} = \sum_{i=2}^{n+1} s_i + s_1 \leqslant \sum_{i=2}^{n+1} (2^{i-2} + m) + m = 2^n - 1 + (n+1)m$$

for every positive integer n. Notice that this estimate also holds for $n = 0$.

Lower bounds: Since $O_{n+1} \subseteq [1, \sigma_{n+1}]$, we have $\sigma_{n+1} \geqslant |O_{n+1}| = 2^n$ for all positive integers n. Then,

$$s_{n+1} = \sigma_{n+1} - \sigma_n \geqslant 2^n - (2^{n-1} - 1 + nm) = 2^{n-1} + 1 - nm$$

for every positive integer n.

Combining the above inequalities, we have

$$2^{n-1} + 1 - nm \leqslant s_{n+1} \leqslant 2^{n-1} + m \qquad \text{and} \qquad 2^n \leqslant \sigma_{n+1} \leqslant 2^n - 1 + (n+1)m, \tag{3}$$

for every positive integer n.

<u>*Step 2.*</u> We prove Property 3 from Solution 1.

For every integer x and set of integers Y, define $x \pm Y = \{x \pm y\colon y \in Y\}$.

In view of Property 1, we get

$$E_{n+1} = E_n \sqcup (s_{n+1} + O_n) \qquad \text{and} \qquad O_{n+1} = O_n \sqcup (s_{n+1} + E_n),$$

where $\sqcup$ denotes the disjoint union operator. Notice also that $s_{n+2} \geqslant 2^n + 1 - (n+1)m > 2^{n-1} - 1 + nm \geqslant \sigma_n$ for every sufficiently large n. We now claim the following.

Claim 1. $(\sigma_n - s_{n+1}, s_{n+2} - s_{n+1}) \subseteq E_n$ for every sufficiently large n.

Proof. For sufficiently large n, all elements of (σ_n, s_{n+2}) are clean. Clearly, the elements of (σ_n, s_{n+2}) can be in neither O_n nor $O \setminus O_{n+1}$. So, $(\sigma_n, s_{n+2}) \subseteq O_{n+1} \setminus O_n = s_{n+1} + E_n$, which yields the claim. □

Now, Claim 1 together with inequalities (3) implies that, for all sufficiently large n,

$$E \supseteq E_n \supseteq (\sigma_n - s_{n+1}, s_{n+2} - s_{n+1}) \supseteq \big(2nm, 2^{n-1} - (n+2)m\big).$$

This easily yields that $\mathbb{Z}_{\geqslant 0} \setminus E$ is also finite. Since $\mathbb{Z}_{\geqslant 0} \setminus O$ is also finite, by Property 1, there exists a positive integer N such that every integer $n > N$ has exactly one even and one odd representation.

Step 3. We investigate the structures of E_n and O_n.

Suppose that $z \in E_{2n}$. Since z can be represented as an even sum using $\{s_1, s_2, \ldots, s_{2n}\}$, so can its complement $\sigma_{2n} - z$. Thus, we get $E_{2n} = \sigma_{2n} - E_{2n}$. Similarly, we have

$$E_{2n} = \sigma_{2n} - E_{2n}, \quad O_{2n} = \sigma_{2n} - O_{2n}, \qquad E_{2n+1} = \sigma_{2n+1} - O_{2n+1}, \quad O_{2n+1} = \sigma_{2n+1} - E_{2n+1}. \tag{4}$$

Claim 2. For every sufficiently large n, we have

$$[0, \sigma_n] \supseteq O_n \supseteq (N, \sigma_n - N) \qquad \text{and} \qquad [0, \sigma_n] \supseteq E_n \supseteq (N, \sigma_n - N).$$

Proof. Clearly $O_n, E_n \subseteq [0, \sigma_n]$ for every positive integer n. We now prove $O_n, E_n \supseteq (N, \sigma_n - N)$. Taking n sufficiently large, we may assume that $s_{n+1} \geqslant 2^{n-1} + 1 - nm > \frac{1}{2}(2^{n-1} - 1 + nm) \geqslant \sigma_n/2$. Therefore, the odd representation of every element of $(N, \sigma_n/2]$ cannot contain a term larger than s_n. Thus, $(N, \sigma_n/2] \subseteq O_n$. Similarly, since $s_{n+1} + s_1 > \sigma_n/2$, we also have $(N, \sigma_n/2] \subseteq E_n$. Equations (4) then yield that, for sufficiently large n, the interval $(N, \sigma_n - N)$ is a subset of both O_n and E_n, as desired. □

Step 4. We obtain a final contradiction.

Notice that $0 \in \mathbb{Z}_{\geqslant 0} \setminus O$ and $1 \in \mathbb{Z}_{\geqslant 0} \setminus E$. Therefore, the sets $\mathbb{Z}_{\geqslant 0} \setminus O$ and $\mathbb{Z}_{\geqslant 0} \setminus E$ are nonempty. Denote $o = \max(\mathbb{Z}_{\geqslant 0} \setminus O)$ and $e = \max(\mathbb{Z}_{\geqslant 0} \setminus E)$. Observe also that $e, o \leqslant N$.

Taking k sufficiently large, we may assume that $\sigma_{2k} > 2N$ and that Claim 2 holds for all $n \geqslant 2k$. Due to (4) and Claim 2, we have that $\sigma_{2k} - e$ is the minimal number greater than N which is not in E_{2k}, i.e., $\sigma_{2k} - e = s_{2k+1} + s_1$. Similarly,

$$\sigma_{2k} - o = s_{2k+1}, \qquad \sigma_{2k+1} - e = s_{2k+2}, \qquad \text{and} \qquad \sigma_{2k+1} - o = s_{2k+2} + s_1.$$

Therefore, we have

$$\begin{aligned} s_1 &= (s_{2k+1} + s_1) - s_{2k+1} = (\sigma_{2k} - e) - (\sigma_{2k} - o) = o - e \\ &= (\sigma_{2k+1} - e) - (\sigma_{2k+1} - o) = s_{2k+2} - (s_{2k+2} + s_1) = -s_1, \end{aligned}$$

which is impossible since $s_1 > 0$.

C7. In a company of people some pairs are enemies. A group of people is called *unsociable* if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

(Russia)

Solution 1. Let $G = (V, E)$ be a graph where V is the set of people in the company and E is the set of the enemy pairs — the edges of the graph. In this language, partitioning into 11 disjoint enemy-free subsets means properly coloring the vertices of this graph with 11 colors.

We will prove the following more general statement.

Claim. Let G be a graph with chromatic number $k \geqslant 3$. Then G contains at least $2^{k-1} - k$ unsociable groups.

Recall that the chromatic number of G is the least k such that a proper coloring

$$V = V_1 \sqcup \cdots \sqcup V_k \tag{1}$$

exists. In view of $2^{11} - 12 > 2015$, the claim implies the problem statement.

Let G be a graph with chromatic number k. We say that a proper coloring (1) of G is *leximinimal*, if the k-tuple $(|V_1|, |V_2|, \ldots, |V_k|)$ is lexicographically minimal; in other words, the following conditions are satisfied: the number $n_1 = |V_1|$ is minimal; the number $n_2 = |V_2|$ is minimal, subject to the previously chosen value of n_1; $\ldots$; the number $n_{k-1} = |V_{k-1}|$ is minimal, subject to the previously chosen values of $n_1, \ldots, n_{k-2}$.

The following lemma is the core of the proof.

Lemma 1. Suppose that $G = (V, E)$ is a graph with odd chromatic number $k \geqslant 3$, and let (1) be one of its leximinimal colorings. Then G contains an odd cycle which visits all color classes $V_1, V_2, \ldots, V_k$.

Proof of Lemma 1. Let us call a cycle *colorful* if it visits all color classes.

Due to the definition of the chromatic number, V_1 is nonempty. Choose an arbitrary vertex $v \in V_1$. We construct a colorful odd cycle that has only one vertex in V_1, and this vertex is v.

We draw a subgraph of G as follows. Place v in the center, and arrange the sets $V_2, V_3, \ldots, V_k$ in counterclockwise circular order around it. For convenience, let $V_{k+1} = V_2$. We will draw arrows to add direction to some edges of G, and mark the vertices these arrows point to. First we draw arrows from v to all its neighbors in V_2, and mark all those neighbors. If some vertex $u \in V_i$ with $i \in \{2, 3, \ldots, k\}$ is already marked, we draw arrows from u to all its neighbors in V_{i+1} which are not marked yet, and we mark all of them. We proceed doing this as long as it is possible. The process of marking is exemplified in Figure 1.

Notice that by the rules of our process, in the final state, marked vertices in V_i cannot have unmarked neighbors in V_{i+1}. Moreover, v is connected to all marked vertices by directed paths.

Now move each marked vertex to the next color class in circular order (see an example in Figure 3). In view of the arguments above, the obtained coloring $V_1 \sqcup W_2 \sqcup \cdots \sqcup W_k$ is proper. Notice that v has a neighbor $w \in W_2$, because otherwise

$$\big(V_1 \setminus \{v\}\big) \sqcup \big(W_2 \cup \{v\}\big) \sqcup W_3 \sqcup \cdots \sqcup W_k$$

would be a proper coloring lexicographically smaller than (1). If w was unmarked, i.e., w was an element of V_2, then it would be marked at the beginning of the process and thus moved to V_3, which did not happen. Therefore, w is marked and $w \in V_k$.

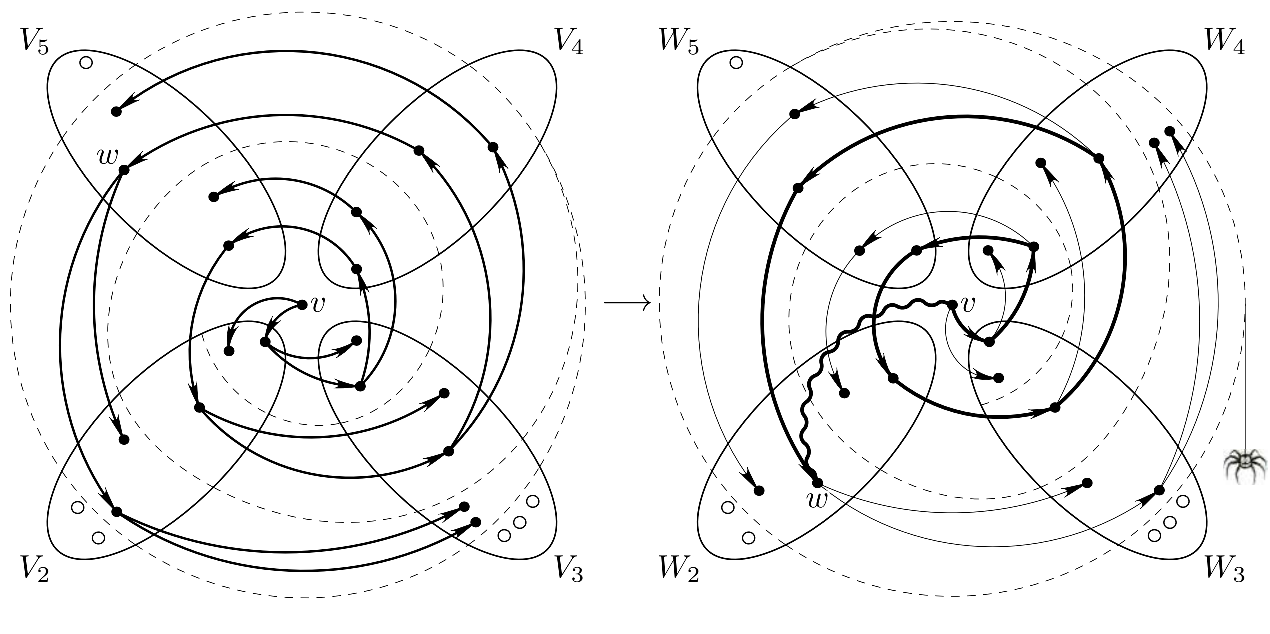

Figure 1 Figure 2

Since w is marked, there exists a directed path from v to w. This path moves through the sets $V_2, \ldots, V_k$ in circular order, so the number of edges in it is divisible by $k-1$ and thus even. Closing this path by the edge $w \to v$, we get a colorful odd cycle, as required. $\square$

Proof of the claim. Let us choose a leximinimal coloring (1) of G. For every set $C \subseteq \{1, 2, \ldots, k\}$ such that $|C|$ is odd and greater than 1, we will provide an odd cycle visiting exactly those color classes whose indices are listed in the set C. This property ensures that we have different cycles for different choices of C, and it proves the claim because there are $2^{k-1} - k$ choices for the set C.

Let $V_C = \bigcup_{c \in C} V_c$, and let G_C be the induced subgraph of G on the vertex set V_C. We also have the induced coloring of V_C with $|C|$ colors; this coloring is of course proper. Notice further that the induced coloring is leximinimal: if we had a lexicographically smaller coloring $(W_c)_{c \in C}$ of G_C, then these classes, together the original color classes V_i for $i \notin C$, would provide a proper coloring which is lexicographically smaller than (1). Hence Lemma 1, applied to the subgraph G_C and its leximinimal coloring $(V_c)_{c \in C}$, provides an odd cycle that visits exactly those color classes that are listed in the set C. $\square$

Solution 2. We provide a different proof of the claim from the previous solution.

We say that a graph is *critical* if deleting any vertex from the graph decreases the graph's chromatic number. Obviously every graph contains a critical induced subgraph with the same chromatic number.

Lemma 2. Suppose that $G = (V, E)$ is a critical graph with chromatic number $k \geqslant 3$. Then every vertex v of G is contained in at least $2^{k-2} - 1$ unsociable groups.

Proof. For every set $X \subseteq V$, denote by $n(X)$ the number of neighbors of v in the set X.

Since G is critical, there exists a proper coloring of $G \setminus \{v\}$ with $k-1$ colors, so there exists a proper coloring $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ of G such that $V_1 = \{v\}$. Among such colorings, take one for which the sequence $\big(n(V_2), n(V_3), \ldots, n(V_k)\big)$ is lexicographically minimal. Clearly, $n(V_i) > 0$ for every $i = 2, 3, \ldots, k$; otherwise $V_2 \sqcup \ldots \sqcup V_{i-1} \sqcup (V_i \cup V_1) \sqcup V_{i+1} \sqcup \ldots V_k$ would be a proper coloring of G with $k-1$ colors.

We claim that for every $C \subseteq \{2, 3, \ldots, k\}$ with $|C| \geqslant 2$ being even, G contains an unsociable group so that the set of its members' colors is precisely $C \cup \{1\}$. Since the number of such sets C is $2^{k-2} - 1$, this proves the lemma. Denote the elements of C by $c_1, \ldots, c_{2\ell}$ in increasing order. For brevity, let $U_i = V_{c_i}$. Denote by N_i the set of neighbors of v in U_i.

We show that for every $i = 1, \dots, 2\ell - 1$ and $x \in N_i$, the subgraph induced by $U_i \cup U_{i+1}$ contains a path that connects x with another point in N_{i+1}. For the sake of contradiction, suppose that no such path exists. Let S be the set of vertices that lie in the connected component of x in the subgraph induced by $U_i \cup U_{i+1}$, and let $P = U_i \cap S$, and $Q = U_{i+1} \cap S$ (see Figure 3). Since x is separated from N_{i+1}, the sets Q and N_{i+1} are disjoint. So, if we re-color G by replacing U_i and U_{i+1} by $(U_i \cup Q) \setminus P$ and $(U_{i+1} \cup P) \setminus Q$, respectively, we obtain a proper coloring such that $n(U_i) = n(V_{c_i})$ is decreased and only $n(U_{i+1}) = n(V_{c_{i+1}})$ is increased. That contradicts the lexicographical minimality of $\big(n(V_2), n(V_3), \dots, n(V_k)\big)$.

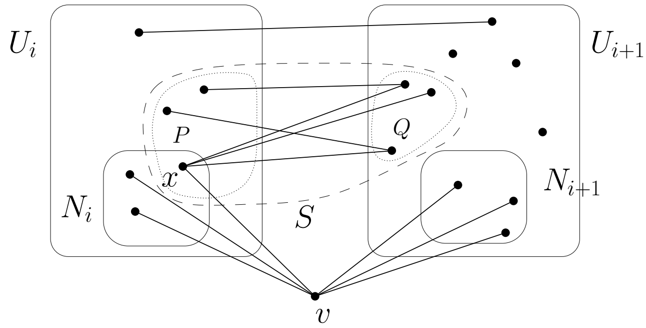

Figure 3

Next, we build a path through $U_1, U_2, \dots, U_{2\ell}$ as follows. Let the starting point of the path be an arbitrary vertex v_1 in the set N_1. For $i \leqslant 2\ell - 1$, if the vertex $v_i \in N_i$ is already defined, connect v_i to some vertex in N_{i+1} in the subgraph induced by $U_i \cup U_{i+1}$, and add these edges to the path. Denote the new endpoint of the path by v_{i+1}; by the construction we have $v_{i+1} \in N_{i+1}$ again, so the process can be continued. At the end we have a path that starts at $v_1 \in N_1$ and ends at some $v_{2\ell} \in N_{2\ell}$. Moreover, all edges in this path connect vertices in neighboring classes: if a vertex of the path lies in U_i, then the next vertex lies in U_{i+1} or U_{i-1}. Notice that the path is not necessary simple, so take a minimal subpath of it. The minimal subpath is simple and connects the same endpoints v_1 and $v_{2\ell}$. The property that every edge steps to a neighboring color class (i.e., from U_i to U_{i+1} or U_{i-1}) is preserved. So the resulting path also visits all of $U_1, \dots, U_{2\ell}$, and its length must be odd. Closing the path with the edges vv_1 and $v_{2\ell}v$ we obtain the desired odd cycle (see Figure 4). $\square$

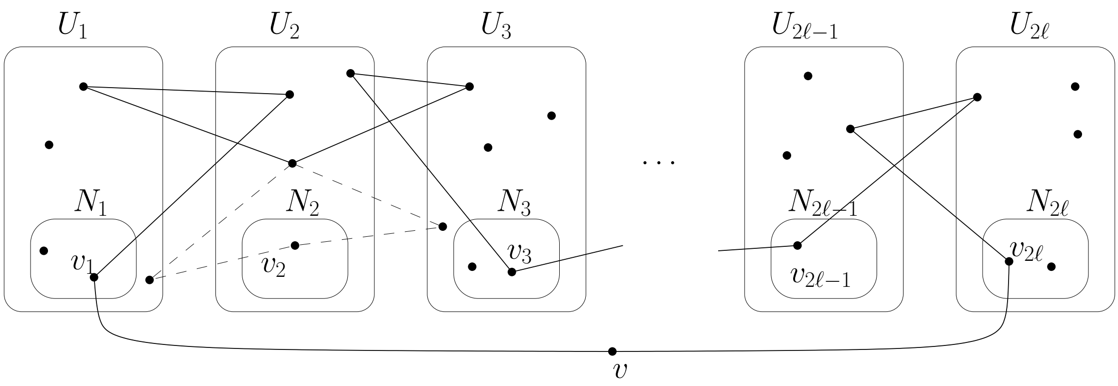

Figure 4

Now we prove the claim by induction on $k \geqslant 3$. The base case $k = 3$ holds by applying Lemma 2 to a critical subgraph. For the induction step, let G_0 be a critical k-chromatic subgraph of G, and let v be an arbitrary vertex of G_0. By Lemma 2, G_0 has at least $2^{k-2} - 1$ unsociable groups containing v. On the other hand, the graph $G_0 \setminus \{v\}$ has chromatic number $k - 1$, so it contains at least $2^{k-2} - (k-1)$ unsociable groups by the induction hypothesis. Altogether, this gives $2^{k-2} - 1 + 2^{k-2} - (k-1) = 2^{k-1} - k$ distinct unsociable groups in G_0 (and thus in G).

Comment 1. The claim we proved is sharp. The complete graph with k vertices has chromatic number k and contains exactly $2^{k-1} - k$ unsociable groups.

Comment 2. The proof of Lemma 2 works for odd values of $|C| \geqslant 3$ as well. Hence, the second solution shows the analogous statement that the number of even sized unsociable groups is at least $2^k - 1 - \binom{k}{2}$.

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Geometry

G1. Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

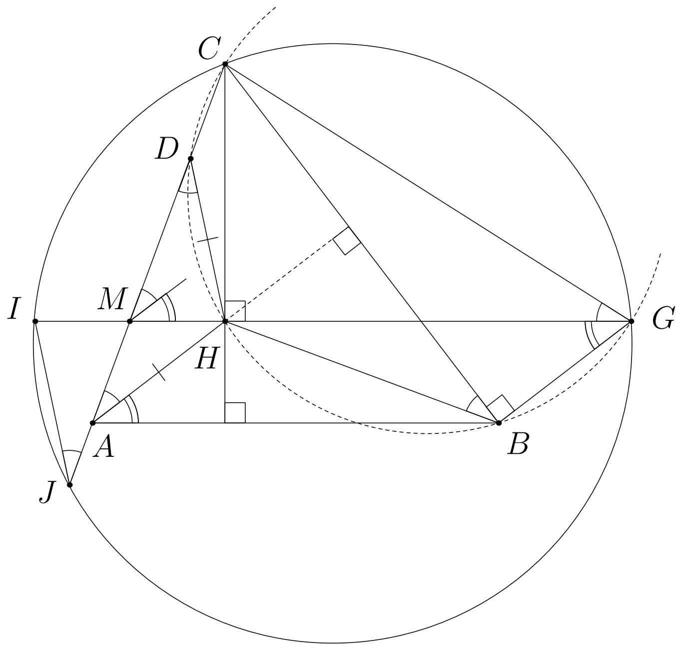
(Australia)

Solution 1. Since $HG \parallel AB$ and $BG \parallel AH$, we have $BG \perp BC$ and $CH \perp GH$. Therefore, the quadrilateral $BGCH$ is cyclic. Since H is the orthocenter of the triangle ABC, we have $\angle HAC = 90^\circ - \angle ACB = \angle CBH$. Using that $BGCH$ and $CGJI$ are cyclic quadrilaterals, we get

$$\angle CJI = \angle CGH = \angle CBH = \angle HAC.$$

Let M be the intersection of AC and GH, and let $D \neq A$ be the point on the line AC such that $AH = HD$. Then $\angle MJI = \angle HAC = \angle MDH$.

Since $\angle MJI = \angle MDH$, $\angle IMJ = \angle HMD$, and $IM = MH$, the triangles IMJ and HMD are congruent, and thus $IJ = HD = AH$.



Comment. Instead of introducing the point D, one can complete the solution by using the law of sines in the triangles IJM and AMH, yielding

$$\frac{IJ}{IM} = \frac{\sin \angle IMJ}{\sin \angle MJI} = \frac{\sin \angle AMH}{\sin \angle HAM} = \frac{AH}{MH} = \frac{AH}{IM}.$$

Solution 2. Obtain $\angle CGH = \angle HAC$ as in the previous solution. In the parallelogram $ABGH$ we have $\angle BAH = \angle HGB$. It follows that

$$\angle HMC = \angle BAC = \angle BAH + \angle HAC = \angle HGB + \angle CGH = \angle CGB.$$

So the right triangles CMH and CGB are similar. Also, in the circumcircle of triangle GCI we have similar triangles MIJ and MCG. Therefore,

$$\frac{IJ}{CG} = \frac{MI}{MC} = \frac{MH}{MC} = \frac{GB}{GC} = \frac{AH}{CG}.$$

Hence $IJ = AH$.

G2. Let ABC be a triangle inscribed into a circle Ω with center O. A circle Γ with center A meets the side BC at points D and E such that D lies between B and E. Moreover, let F and G be the common points of Γ and Ω. We assume that F lies on the arc AB of Ω not containing C, and G lies on the arc AC of Ω not containing B. The circumcircles of the triangles BDF and CEG meet the sides AB and AC again at K and L, respectively. Suppose that the lines FK and GL are distinct and intersect at X. Prove that the points A, X, and O are collinear.

(Greece)

Solution 1. It suffices to prove that the lines FK and GL are symmetric about AO. Now the segments AF and AG, being chords of Ω with the same length, are clearly symmetric with respect to AO. Hence it is enough to show

$$\angle KFA = \angle AGL\,. \tag{1}$$

Let us denote the circumcircles of BDF and CEG by ω_B and ω_C, respectively. To prove (1), we start from

$$\angle KFA = \angle DFG + \angle GFA - \angle DFK\,.$$

In view of the circles ω_B, Γ, and Ω, this may be rewritten as

$$\angle KFA = \angle CEG + \angle GBA - \angle DBK = \angle CEG - \angle CBG\,.$$

Due to the circles ω_C and Ω, we obtain $\angle KFA = \angle CLG - \angle CAG = \angle AGL$. Thereby the problem is solved.

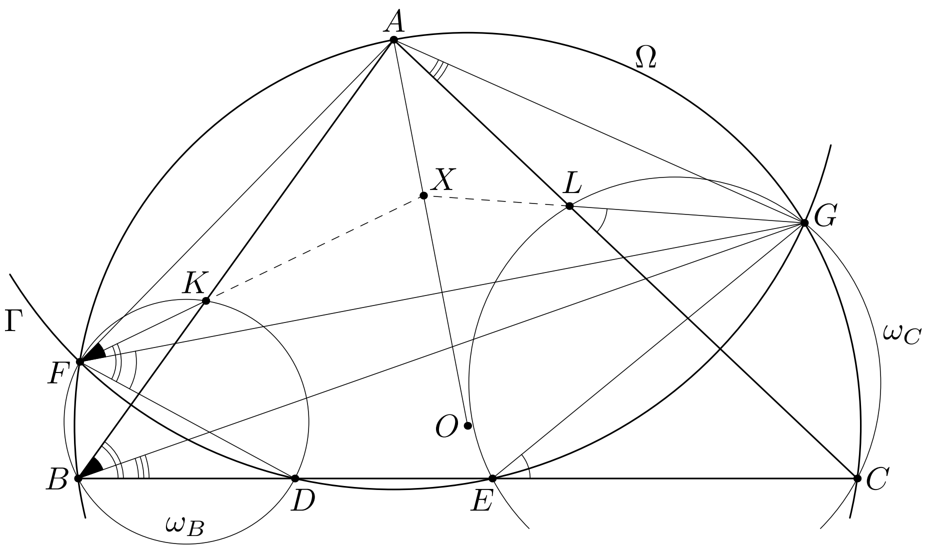


Figure 1

Solution 2. Again, we denote the circumcircle of $BDKF$ by ω_B. In addition, we set $\alpha = \angle BAC$, $\varphi = \angle ABF$, and $\psi = \angle EDA = \angle AED$ (see Figure 2). Notice that $AF = AG$ entails $\varphi = \angle GCA$, so all three of α, φ, and ψ respect the "symmetry" between B and C of our configuration. Again, we reduce our task to proving (1).

This time, we start from

$$2\angle KFA = 2(\angle DFA - \angle DFK)\,.$$

Since the triangle AFD is isosceles, we have

$$\angle DFA = \angle ADF = \angle EDF - \psi = \angle BFD + \angle EBF - \psi\,.$$

Moreover, because of the circle ω_B we have $\angle DFK = \angle CBA$. Altogether, this yields

$$2\angle KFA = \angle DFA + \big(\angle BFD + \angle EBF - \psi\big) - 2\angle CBA\,,$$

which simplifies to

$$2\angle KFA = \angle BFA + \varphi - \psi - \angle CBA\,.$$

Now the quadrilateral $AFBC$ is cyclic, so this entails $2\angle KFA = \alpha + \varphi - \psi$.

Due to the "symmetry" between B and C alluded to above, this argument also shows that $2\angle AGL = \alpha + \varphi - \psi$. This concludes the proof of (1).

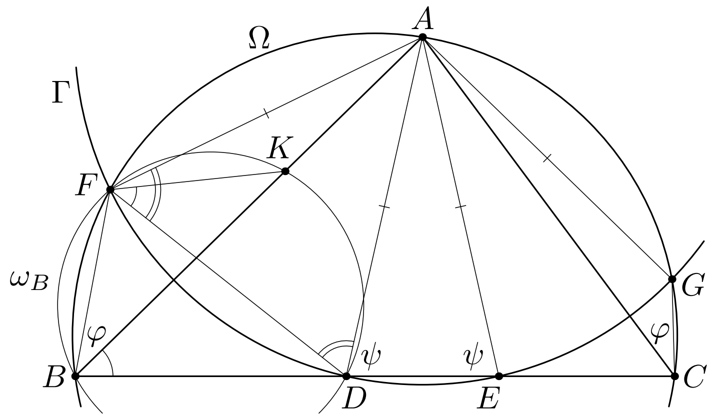


Figure 2

Comment 1. As the first solution shows, the assumption that A be the center of Γ may be weakened to the following one: The center of Γ lies on the line OA. The second solution may be modified to yield the same result.

Comment 2. It might be interesting to remark that $\angle GDK = 90^\circ$. To prove this, let G' denote the point on Γ diametrically opposite to G. Because of $\angle KDF = \angle KBF = \angle AGF = \angle G'DF$, the points D, K, and G' are collinear, which leads to the desired result. Notice that due to symmetry we also have $\angle LEF = 90^\circ$.

Moreover, a standard argument shows that the triangles AGL and BGE are similar. By symmetry again, also the triangles AFK and CDF are similar.

There are several ways to derive a solution from these facts. For instance, one may argue that

$$\begin{aligned}\angle KFA = \angle BFA - \angle BFK &= \angle BFA - \angle EDG' = (180^\circ - \angle AGB) - (180^\circ - \angle G'GE)\\ &= \angle AGE - \angle AGB = \angle BGE = \angle AGL\,.\end{aligned}$$

Comment 3. The original proposal did not contain the point X in the assumption and asked instead to prove that the lines FK, GL, and AO are concurrent. This differs from the version given above only insofar as it also requires to show that these lines cannot be parallel. The Problem Selection Committee removed this part from the problem intending to make it thus more suitable for the Olympiad.

For the sake of completeness, we would still like to sketch one possibility for proving $FK \nparallel AO$ here. As the points K and O lie in the angular region $\angle FAG$, it suffices to check $\angle KFA + \angle FAO < 180^\circ$. Multiplying by 2 and making use of the formulae from the second solution, we see that this is equivalent to $(\alpha + \varphi - \psi) + (180^\circ - 2\varphi) < 360^\circ$, which in turn is an easy consequence of $\alpha < 180^\circ$.

G3. Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

(Georgia)

Solution 1. Let K be the projection of D onto AB; then $AH = HK$ (see Figure 1). Since $PH \parallel DK$, we have
$$\frac{PD}{PB} = \frac{HK}{HB} = \frac{AH}{HB}. \tag{1}$$

Let L be the projection of Q onto DB. Since PQ is tangent to ω and $\angle DQB = \angle BLQ = 90^\circ$, we have $\angle PQD = \angle QBP = \angle DQL$. Therefore, QD and QB are respectively the internal and the external bisectors of $\angle PQL$. By the angle bisector theorem, we obtain
$$\frac{PD}{DL} = \frac{PQ}{QL} = \frac{PB}{BL}. \tag{2}$$

The relations (1) and (2) yield $\dfrac{AH}{HB} = \dfrac{PD}{PB} = \dfrac{DL}{LB}$. So, the spiral similarity τ centered at B and sending A to D maps H to L. Moreover, τ sends the semicircle with diameter AB passing through C to ω. Due to $CH \perp AB$ and $QL \perp DB$, it follows that $\tau(C) = Q$.

Hence, the triangles ABD and CBQ are similar, so $\angle ADB = \angle CQB$. This means that the lines AD and CQ meet at some point T, and this point satisfies $\angle BDT = \angle BQT$. Therefore, T lies on ω, as needed.

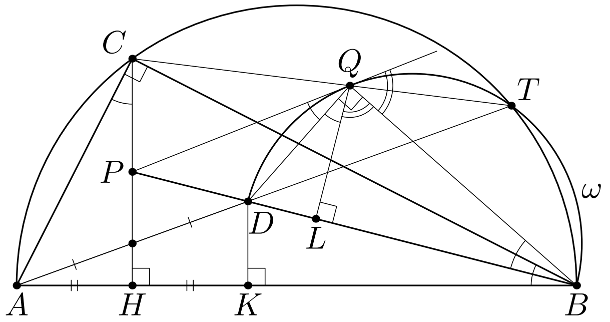


Figure 1

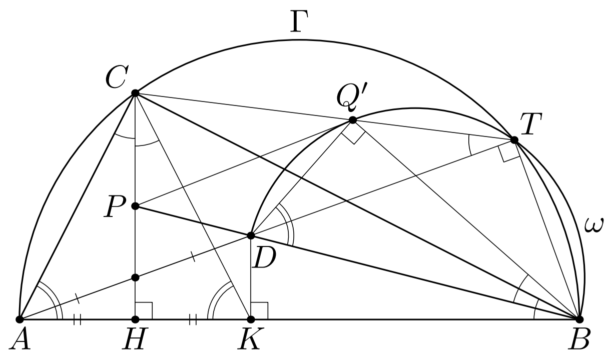


Figure 2

Comment 1. Since $\angle BAD = \angle BCQ$, the point T lies also on the circumcircle of the triangle ABC.

Solution 2. Let Γ be the circumcircle of ABC, and let AD meet ω at T. Then $\angle ATB = \angle ACB = 90^\circ$, so T lies on Γ as well. As in the previous solution, let K be the projection of D onto AB; then $AH = HK$ (see Figure 2).

Our goal now is to prove that the points C, Q, and T are collinear. Let CT meet ω again at Q'. Then, it suffices to show that PQ' is tangent to ω, or that $\angle PQ'D = \angle Q'BD$.

Since the quadrilateral $BDQ'T$ is cyclic and the triangles AHC and KHC are congruent, we have $\angle Q'BD = \angle Q'TD = \angle CTA = \angle CBA = \angle ACH = \angle HCK$. Hence, the right triangles CHK and $BQ'D$ are similar. This implies that $\dfrac{HK}{CK} = \dfrac{Q'D}{BD}$, and thus $HK \cdot BD = CK \cdot Q'D$.

Notice that $PH \parallel DK$; therefore, we have $\dfrac{PD}{BD} = \dfrac{HK}{BK}$, and so $PD \cdot BK = HK \cdot BD$. Consequently, $PD \cdot BK = HK \cdot BD = CK \cdot Q'D$, which yields $\dfrac{PD}{Q'D} = \dfrac{CK}{BK}$.

Since $\angle CKA = \angle KAC = \angle BDQ'$, the triangles CKB and PDQ' are similar, so $\angle PQ'D = \angle CBA = \angle Q'BD$, as required.

Comment 2. There exist several other ways to prove that PQ' is tangent to ω. For instance, one may compute $\dfrac{PD}{PB}$ and $\dfrac{PQ'}{PB}$ in terms of AH and HB to verify that $PQ'^2 = PD \cdot PB$, concluding that PQ' is tangent to ω.

Another possible approach is the following. As in Solution 2, we introduce the points T and Q' and mention that the triangles ABC and DBQ' are similar (see Figure 3).

Let M be the midpoint of AD, and let L be the projection of Q' onto AB. Construct E on the line AB so that EP is parallel to AD. Projecting from P, we get $(A, B; H, E) = (A, D; M, \infty) = -1$.

Since $\dfrac{EA}{AB} = \dfrac{PD}{DB}$, the point P is the image of E under the similarity transform mapping ABC to DBQ'. Therefore, we have $(D, B; L, P) = (A, B; H, E) = -1$, which means that $Q'D$ and $Q'B$ are respectively the internal and the external bisectors of $\angle PQ'L$. This implies that PQ' is tangent to ω, as required.

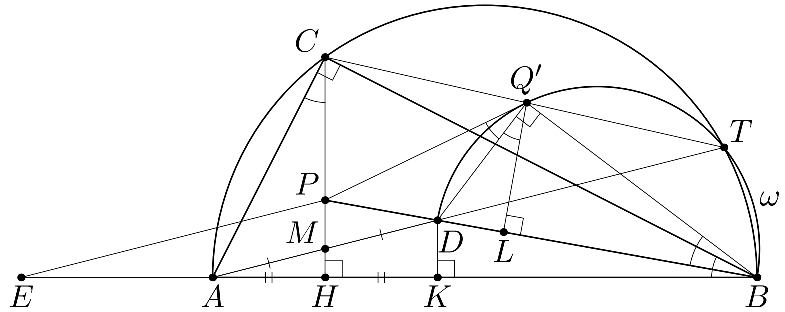


Figure 3

Solution 3. Introduce the points T and Q' as in the previous solution. Note that T lies on the circumcircle of ABC. Here we present yet another proof that PQ' is tangent to ω.

Let Ω be the circle completing the semicircle ω. Construct a point F symmetric to C with respect to AB. Let $S \neq T$ be the second intersection point of FT and Ω (see Figure 4).

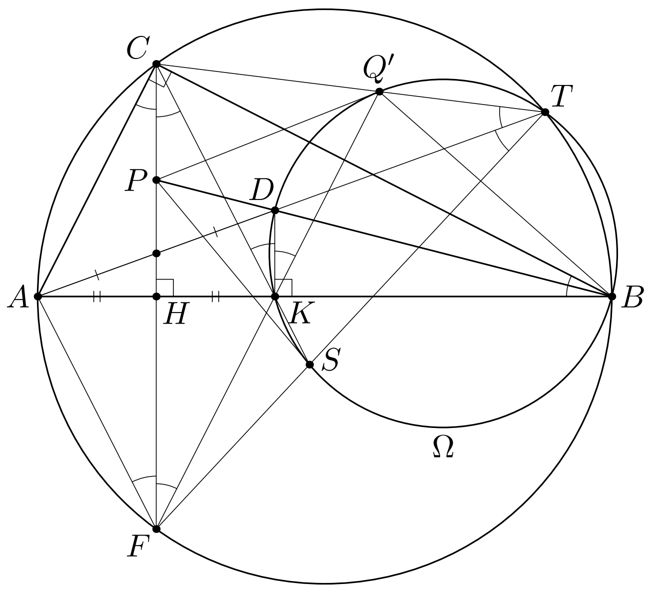


Figure 4

Since $AC = AF$, we have $\angle DKC = \angle HCK = \angle CBA = \angle CTA = \angle DTS = 180^\circ - \angle SKD$. Thus, the points C, K, and S are collinear. Notice also that $\angle Q'KD = \angle Q'TD = \angle HCK = \angle KFH = 180^\circ - \angle DKF$. This implies that the points F, K, and Q' are collinear.

Applying Pascal's theorem to the degenerate hexagon $KQ'Q'TSS$, we get that the tangents to Ω passing through Q' and S intersect on CF. The relation $\angle Q'TD = \angle DTS$ yields that Q' and S are symmetric with respect to BD. Therefore, the two tangents also intersect on BD. Thus, the two tangents pass through P. Hence, PQ' is tangent to ω, as needed.

G4. Let ABC be an acute triangle, and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC again at P and Q, respectively. Let T be the point such that the quadrilateral $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of the triangle ABC. Determine all possible values of BT/BM.

(Russia)

Answer. $\sqrt{2}$.

Solution 1. Let S be the center of the parallelogram $BPTQ$, and let $B' \neq B$ be the point on the ray BM such that $BM = MB'$ (see Figure 1). It follows that $ABCB'$ is a parallelogram. Then, $\angle ABB' = \angle PQM$ and $\angle BB'A = \angle B'BC = \angle MPQ$, and so the triangles ABB' and MQP are similar. It follows that AM and MS are corresponding medians in these triangles. Hence,

$$\angle SMP = \angle B'AM = \angle BCA = \angle BTA. \tag{1}$$

Since $\angle ACT = \angle PBT$ and $\angle TAC = \angle TBC = \angle BTP$, the triangles TCA and PBT are similar. Again, as TM and PS are corresponding medians in these triangles, we have

$$\angle MTA = \angle TPS = \angle BQP = \angle BMP. \tag{2}$$

Now we deal separately with two cases.

Case 1. S does not lie on BM. Since the configuration is symmetric between A and C, we may assume that S and A lie on the same side with respect to the line BM.

Applying (1) and (2), we get

$$\angle BMS = \angle BMP - \angle SMP = \angle MTA - \angle BTA = \angle MTB,$$

and so the triangles BSM and BMT are similar. We now have $BM^2 = BS \cdot BT = BT^2/2$, so $BT = \sqrt{2}BM$.

Case 2. S lies on BM. It follows from (2) that $\angle BCA = \angle MTA = \angle BQP = \angle BMP$ (see Figure 2). Thus, $PQ \parallel AC$ and $PM \parallel AT$. Hence, $BS/BM = BP/BA = BM/BT$, so $BT^2 = 2BM^2$ and $BT = \sqrt{2}BM$.

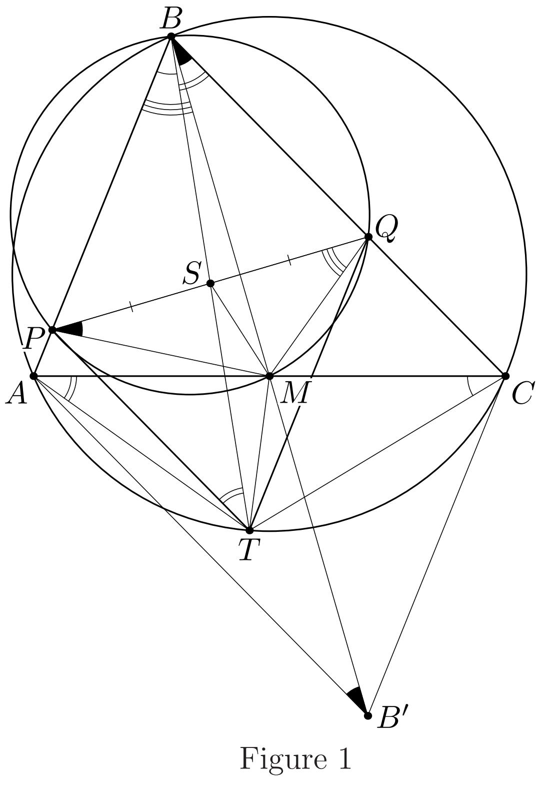


Figure 1

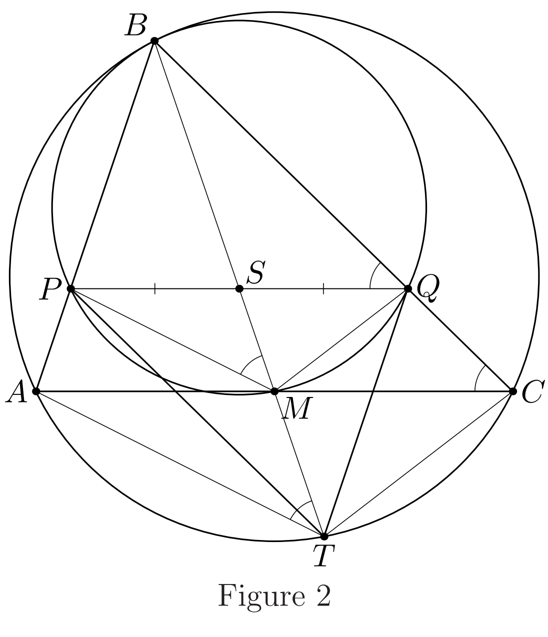


Figure 2

Comment 1. Here is another way to show that the triangles BSM and BMT are similar. Denote by Ω the circumcircle of the triangle ABC. Let R be the second point of intersection of ω and Ω, and let τ be the spiral similarity centered at R mapping ω to Ω. Then, one may show that τ maps each point X on ω to a point Y on Ω such that B, X, and Y are collinear (see Figure 3). If we let K and L be the second points of intersection of BM with Ω and of BT with ω, respectively, then it follows that the triangle MKT is the image of SML under τ. We now obtain $\angle BSM = \angle TMB$, which implies the desired result.

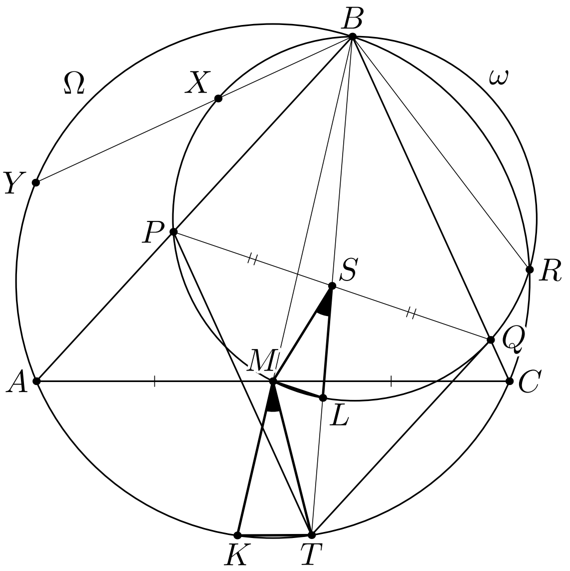


Figure 3

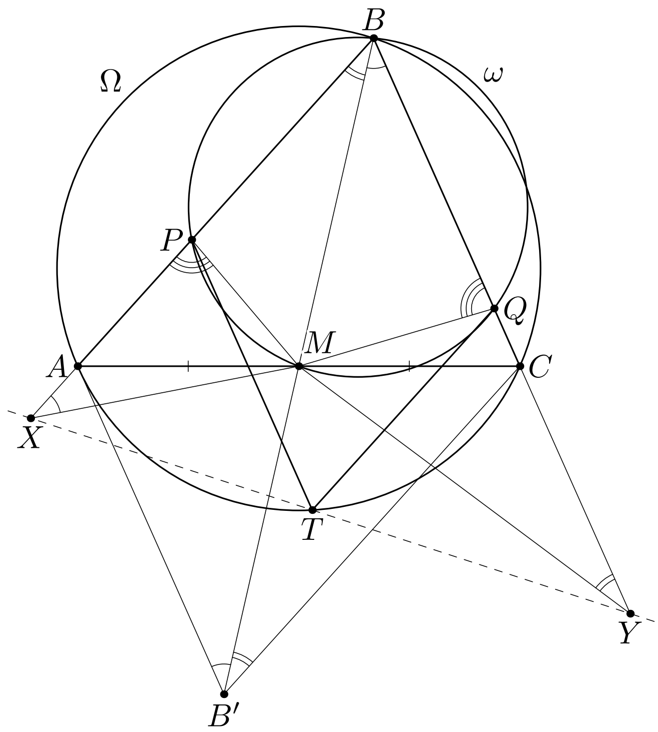


Figure 4

Solution 2. Again, we denote by Ω the circumcircle of the triangle ABC.

Choose the points X and Y on the rays BA and BC respectively, so that $\angle MXB = \angle MBC$ and $\angle BYM = \angle ABM$ (see Figure 4). Then the triangles BMX and YMB are similar. Since $\angle XPM = \angle BQM$, the points P and Q correspond to each other in these triangles. So, if $\overrightarrow{BP} = \mu \cdot \overrightarrow{BX}$, then $\overrightarrow{BQ} = (1-\mu) \cdot \overrightarrow{BY}$. Thus

$$\overrightarrow{BT} = \overrightarrow{BP} + \overrightarrow{BQ} = \overrightarrow{BY} + \mu \cdot (\overrightarrow{BX} - \overrightarrow{BY}) = \overrightarrow{BY} + \mu \cdot \overrightarrow{YX},$$

which means that T lies on the line XY.

Let $B' \neq B$ be the point on the ray BM such that $BM = MB'$. Then $\angle MB'A = \angle MBC = \angle MXB$ and $\angle CB'M = \angle ABM = \angle BYM$. This means that the triangles BMX, BAB', YMB, and $B'CB$ are all similar; hence $BA \cdot BX = BM \cdot BB' = BC \cdot BY$. Thus there exists an inversion centered at B which swaps A with X, M with B', and C with Y. This inversion then swaps Ω with the line XY, and hence it preserves T. Therefore, we have $BT^2 = BM \cdot BB' = 2BM^2$, and $BT = \sqrt{2}BM$.

Solution 3. We begin with the following lemma.

Lemma. Let $ABCT$ be a cyclic quadrilateral. Let P and Q be points on the sides BA and BC respectively, such that $BPTQ$ is a parallelogram. Then $BP \cdot BA + BQ \cdot BC = BT^2$.

Proof. Let the circumcircle of the triangle QTC meet the line BT again at J (see Figure 5). The power of B with respect to this circle yields

$$BQ \cdot BC = BJ \cdot BT. \tag{3}$$

We also have $\angle TJQ = 180^\circ - \angle QCT = \angle TAB$ and $\angle QTJ = \angle ABT$, and so the triangles TJQ and BAT are similar. We now have $TJ/TQ = BA/BT$. Therefore,

$$TJ \cdot BT = TQ \cdot BA = BP \cdot BA. \tag{4}$$

Combining (3) and (4) now yields the desired result. $\square$

Let X and Y be the midpoints of BA and BC respectively (see Figure 6). Applying the lemma to the cyclic quadrilaterals $PBQM$ and $ABCT$, we obtain

$$BX \cdot BP + BY \cdot BQ = BM^2$$

and

$$BP \cdot BA + BQ \cdot BC = BT^2.$$

Since $BA = 2BX$ and $BC = 2BY$, we have $BT^2 = 2BM^2$, and so $BT = \sqrt{2}BM$.

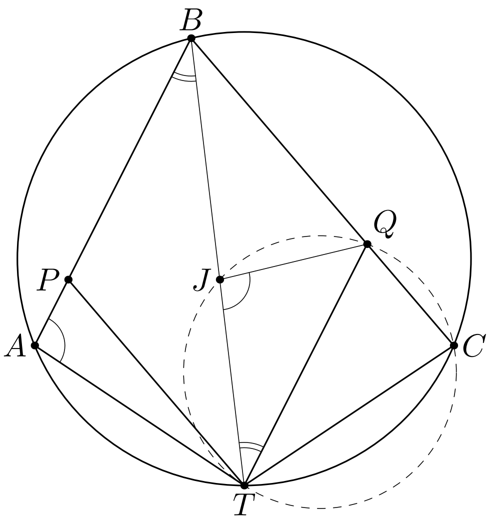


Figure 5

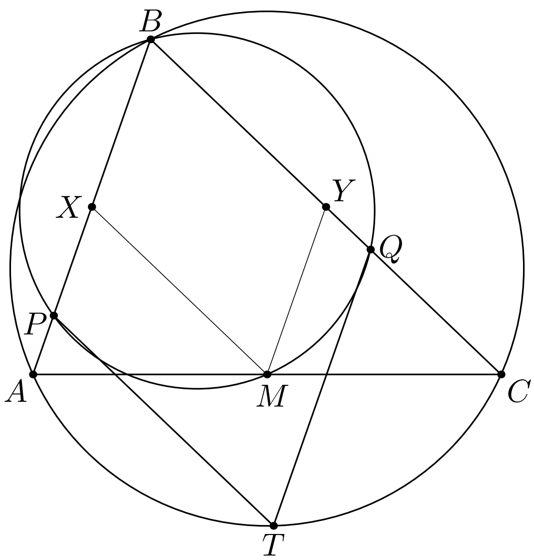


Figure 6

Comment 2. Here we give another proof of the lemma using Ptolemy's theorem. We readily have

$$TC \cdot BA + TA \cdot BC = AC \cdot BT.$$

The lemma now follows from

$$\frac{BP}{TC} = \frac{BQ}{TA} = \frac{BT}{AC} = \frac{\sin \angle BCT}{\sin \angle ABC}.$$

G5. Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC, respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

(El Salvador)

Solution 1. We may assume that $CA > CB$. Observe that H' and I' lie inside the segments CF and CG, respectively. Therefore, M lies outside $\triangle ABC$ (see Figure 1).

Due to the powers of points A and B with respect to the circle Γ, we have

$$CH' \cdot CA = AH \cdot AC = AD^2 = BD^2 = BI \cdot BC = CI' \cdot CB.$$

Therefore, $CH' \cdot CF = CI' \cdot CG$. Hence, the quadrilateral $H'I'GF$ is cyclic, and so $\angle I'H'C = \angle CGF$.

Let DF and DG meet Γ again at R and S, respectively. We claim that the points R and S lie on the line $H'I'$.

Observe that $FH' \cdot FA = FH \cdot FC = FR \cdot FD$. Thus, the quadrilateral $ADH'R$ is cyclic, and hence $\angle RH'F = \angle FDA = \angle CGF = \angle I'H'C$. Therefore, the points R, H', and I' are collinear. Similarly, the points S, H', and I' are also collinear, and so all the points R, H', Q, I', S, and M are all collinear.

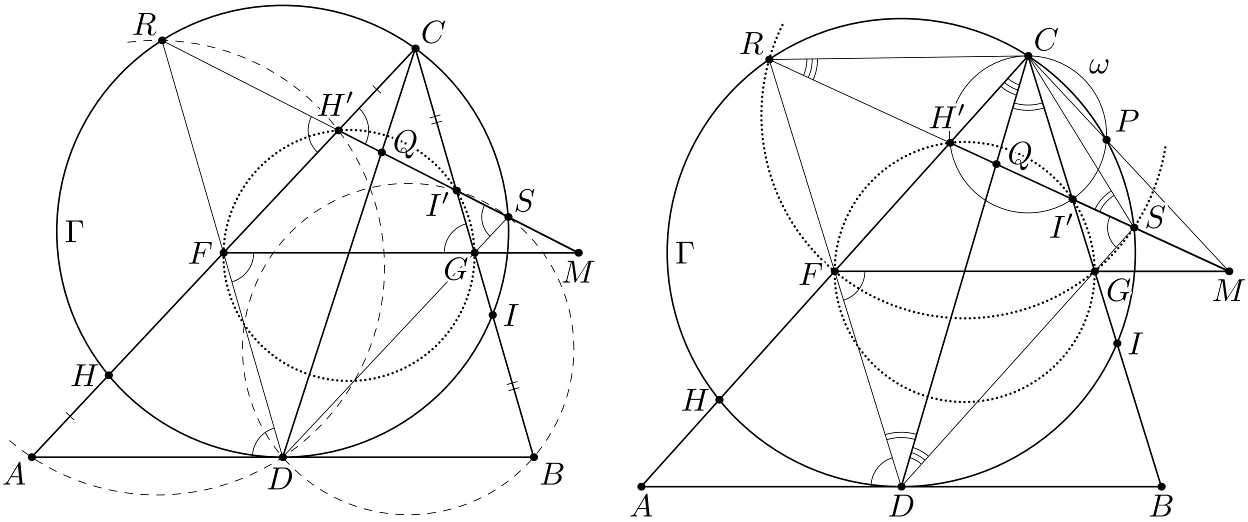


Figure 1 Figure 2

Then, $\angle RSD = \angle RDA = \angle DFG$. Hence, the quadrilateral $RSGF$ is cyclic (see Figure 2). Therefore, $MH' \cdot MI' = MF \cdot MG = MR \cdot MS = MP \cdot MC$. Thus, the quadrilateral $CPI'H'$ is also cyclic. Let ω be its circumcircle.

Notice that $\angle H'CQ = \angle SDC = \angle SRC$ and $\angle QCI' = \angle CDR = \angle CSR$. Hence, $\triangle CH'Q \sim \triangle RCQ$ and $\triangle CI'Q \sim \triangle SCQ$, and therefore $QH' \cdot QR = QC^2 = QI' \cdot QS$.

We apply the inversion with center Q and radius QC. Observe that the points R, C, and S are mapped to H', C, and I', respectively. Therefore, the circumcircle Γ of $\triangle RCS$ is mapped to the circumcircle ω of $\triangle H'CI'$. Since P and C belong to both circles and the point C is preserved by the inversion, we have that P is also mapped to itself. We then get $QP^2 = QC^2$. Hence, $QP = QC$.

Comment 1. The problem statement still holds when Γ intersects the sides CA and CB outside segments AF and BG, respectively.

Solution 2. Let $X = HI \cap AB$, and let the tangent to Γ at C meet AB at Y. Let XC meet Γ again at X' (see Figure 3). Projecting from C, X, and C again, we have $(X, A; D, B) = (X', H; D, I) = (C, I; D, H) = (Y, B; D, A)$. Since A and B are symmetric about D, it follows that X and Y are also symmetric about D.

Now, Menelaus' theorem applied to $\triangle ABC$ with the line HIX yields

$$1 = \frac{CH}{HA} \cdot \frac{BI}{IC} \cdot \frac{AX}{XB} = \frac{AH'}{H'C} \cdot \frac{CI'}{I'B} \cdot \frac{BY}{YA}.$$

By the converse of Menelaus' theorem applied to $\triangle ABC$ with points H', I', Y, we get that the points H', I', Y are collinear.

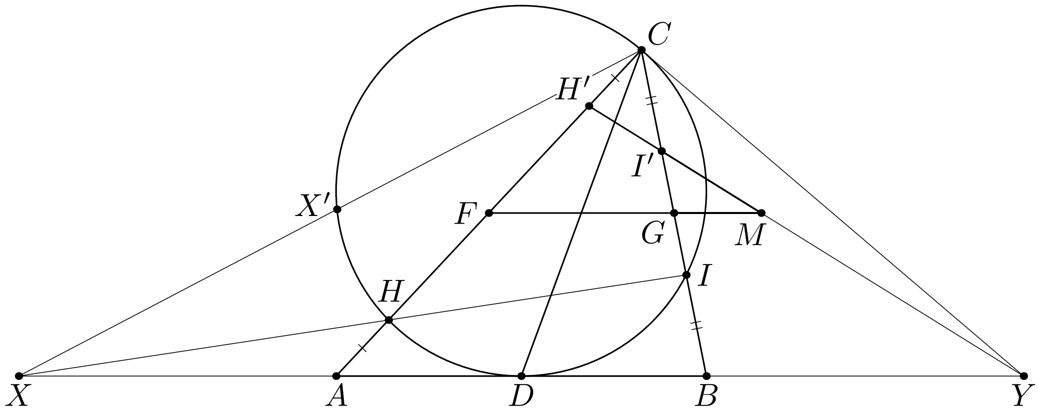


Figure 3

Let T be the midpoint of CD, and let O be the center of Γ. Let CM meet TY at N. To avoid confusion, we clean some superfluous details out of the picture (see Figure 4).

Let $V = MT \cap CY$. Since $MT \parallel YD$ and $DT = TC$, we get $CV = VY$. Then Ceva's theorem applied to $\triangle CTY$ with the point M yields

$$1 = \frac{TQ}{QC} \cdot \frac{CV}{VY} \cdot \frac{YN}{NT} = \frac{TQ}{QC} \cdot \frac{YN}{NT}.$$

Therefore, $\frac{TQ}{QC} = \frac{TN}{NY}$. So, $NQ \parallel CY$, and thus $NQ \perp OC$.

Note that the points O, N, T, and Y are collinear. Therefore, $CQ \perp ON$. So, Q is the orthocenter of $\triangle OCN$, and hence $OQ \perp CP$. Thus, Q lies on the perpendicular bisector of CP, and therefore $CQ = QP$, as required.

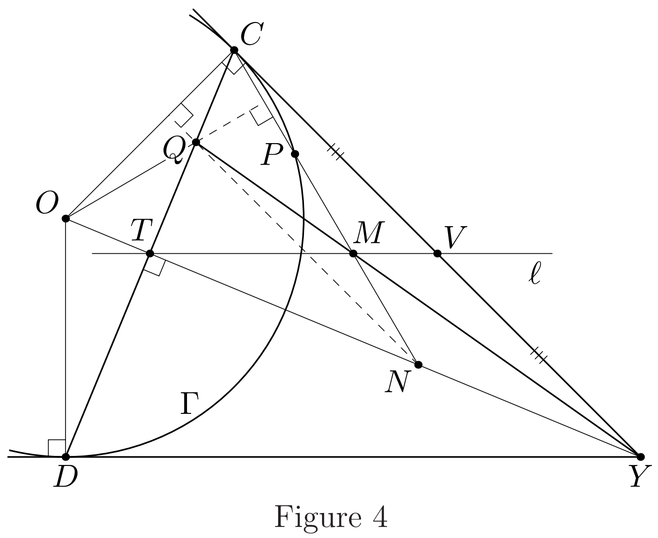


Figure 4

Comment 2. The second part of Solution 2 provides a proof of the following more general statement, which does not involve a specific choice of Q on CD.

Let YC and YD be two tangents to a circle Γ with center O (see Figure 4). Let ℓ be the midline of $\triangle YCD$ parallel to YD. Let Q and M be two points on CD and ℓ, respectively, such that the line QM passes through Y. Then $OQ \perp CM$.

G6. Let ABC be an acute triangle with $AB > AC$, and let Γ be its circumcircle. Let H, M, and F be the orthocenter of the triangle, the midpoint of BC, and the foot of the altitude from A, respectively. Let Q and K be the two points on Γ that satisfy $\angle AQH = 90^\circ$ and $\angle QKH = 90^\circ$. Prove that the circumcircles of the triangles KQH and KFM are tangent to each other.

(Ukraine)

Solution 1. Let A' be the point diametrically opposite to A on Γ. Since $\angle AQA' = 90^\circ$ and $\angle AQH = 90^\circ$, the points Q, H, and A' are collinear. Similarly, if Q' denotes the point on Γ diametrically opposite to Q, then K, H, and Q' are collinear. Let the line AHF intersect Γ again at E; it is known that M is the midpoint of the segment HA' and that F is the midpoint of HE. Let J be the midpoint of HQ'.

Consider any point T such that TK is tangent to the circle KQH at K with Q and T lying on different sides of KH (see Figure 1). Then $\angle HKT = \angle HQK$ and we are to prove that $\angle MKT = \angle CFK$. Thus it remains to show that $\angle HQK = \angle CFK + \angle HKM$. Due to $\angle HQK = 90^\circ - \angle Q'HA'$ and $\angle CFK = 90^\circ - \angle KFA$, this means the same as $\angle Q'HA' = \angle KFA - \angle HKM$. Now, since the triangles KHE and AHQ' are similar with F and J being the midpoints of corresponding sides, we have $\angle KFA = \angle HJA$, and analogously one may obtain $\angle HKM = \angle JQH$. Thereby our task is reduced to verifying

$$\angle Q'HA' = \angle HJA - \angle JQH\,.$$

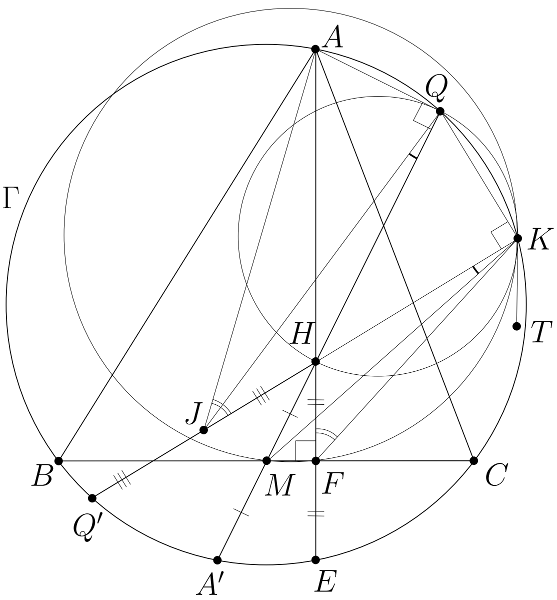


Figure 1

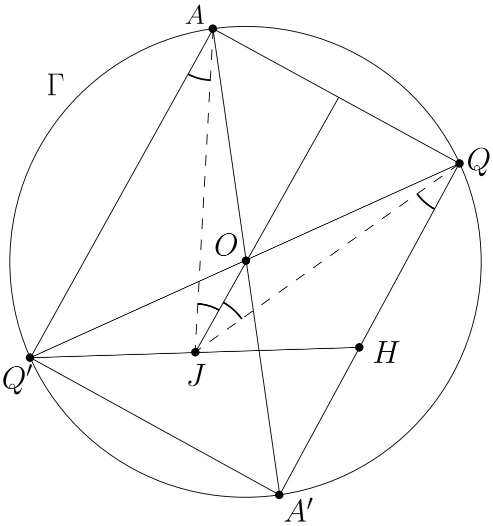


Figure 2

To avoid confusion, let us draw a new picture at this moment (see Figure 2). Owing to $\angle Q'HA' = \angle JQH + \angle HJQ$ and $\angle HJA = \angle QJA + \angle HJQ$, we just have to show that $2\angle JQH = \angle QJA$. To this end, it suffices to remark that $AQA'Q'$ is a rectangle and that J, being defined to be the midpoint of HQ', has to lie on the mid parallel of QA' and $Q'A$.

Solution 2. We define the points A' and E and prove that the ray MH passes through Q in the same way as in the first solution. Notice that the points A' and E can play analogous roles to the points Q and K, respectively: point A' is the second intersection of the line MH with Γ, and E is the point on Γ with the property $\angle HEA' = 90^\circ$ (see Figure 3).

In the circles KQH and $EA'H$, the line segments HQ and HA' are diameters, respectively; so, these circles have a common tangent t at H, perpendicular to MH. Let R be the radical center of the circles ABC, KQH and $EA'H$. Their pairwise radical axes are the lines QK, $A'E$ and the line t; they all pass through R. Let S be the midpoint of HR; by $\angle QKH =$

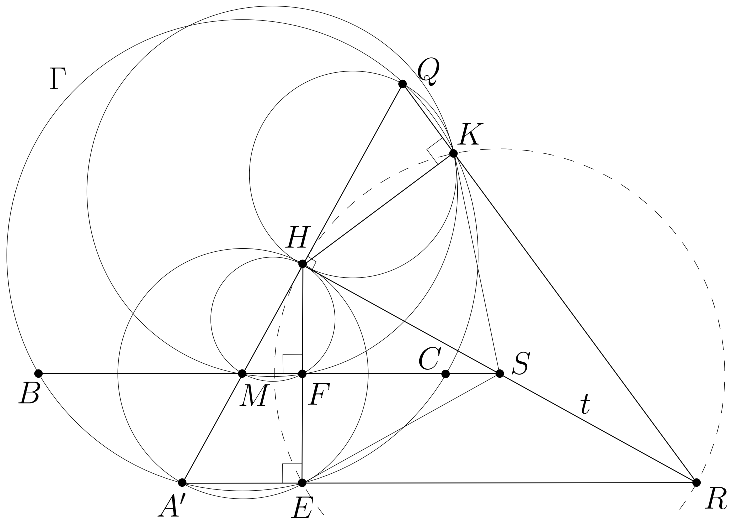


Figure 3

$\angle HEA' = 90^\circ$, the quadrilateral $HERK$ is cyclic and its circumcenter is S; hence we have $SK = SE = SH$. The line BC, being the perpendicular bisector of HE, passes through S.

The circle HMF also is tangent to t at H; from the power of S with respect to the circle HMF we have

$$SM \cdot SF = SH^2 = SK^2.$$

So, the power of S with respect to the circles KQH and KFM is SK^2. Therefore, the line segment SK is tangent to both circles at K.

G7. Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segments PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

(Bulgaria)

Solution 1. Denote by γ_A, γ_B, γ_C, and γ_D the incircles of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$, respectively.

We start with proving that the quadrilateral $ABCD$ also has an incircle which will be referred to as Ω. Denote the points of tangency as in Figure 1. It is well-known that $QQ_1 = OO_1$ (if $BC \parallel PR$, this is obvious; otherwise, one may regard the two circles involved as the incircle and an excircle of the triangle formed by the lines OQ, PR, and BC). Similarly, $OO_1 = PP_1$. Hence we have $QQ_1 = PP_1$. The other equalities of segment lengths marked in Figure 1 can be proved analogously. These equalities, together with $AP_1 = AS_1$ and similar ones, yield $AB + CD = AD + BC$, as required.

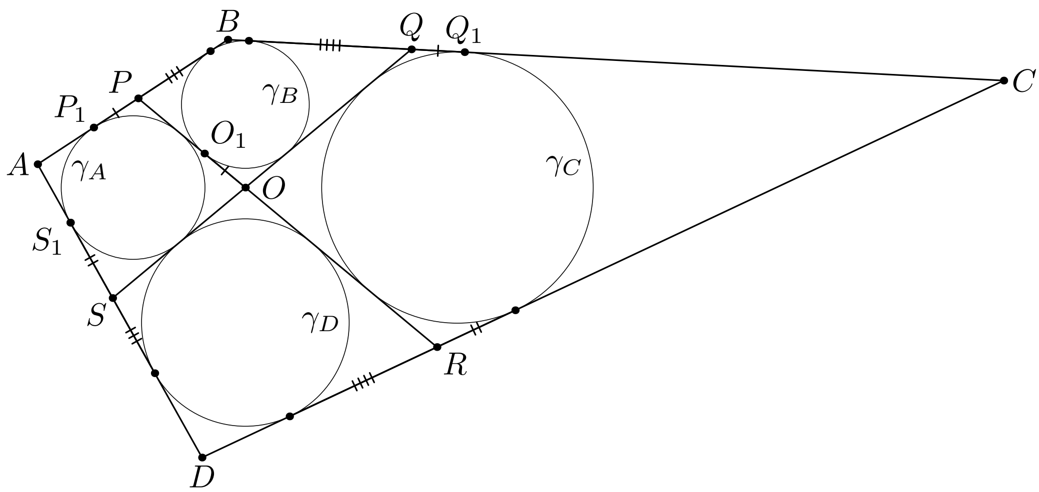


Figure 1

Next, let us draw the lines parallel to QS through P and R, and also draw the lines parallel to PR through Q and S. These lines form a parallelogram; denote its vertices by A', B', C', and D' as shown in Figure 2.

Since the quadrilateral $APOS$ has an incircle, we have $AP - AS = OP - OS = A'S - A'P$. It is well-known that in this case there also exists a circle ω_A tangent to the four rays AP, AS, $A'P$, and $A'S$. It is worth mentioning here that in case when, say, the lines AB and $A'B'$ coincide, the circle ω_A is just tangent to AB at P. We introduce the circles ω_B, ω_C, and ω_D in a similar manner.

Assume that the radii of the circles ω_A and ω_C are different. Let X be the center of the homothety having a positive scale factor and mapping ω_A to ω_C.

Now, MONGE's theorem applied to the circles ω_A, Ω, and ω_C shows that the points A, C, and X are collinear. Applying the same theorem to the circles ω_A, ω_B, and ω_C, we see that the points P, Q, and X are also collinear. Similarly, the points R, S, and X are collinear, as required.

If the radii of ω_A and ω_C are equal but these circles do not coincide, then the degenerate version of the same theorem yields that the three lines AC, PQ, and RS are parallel to the line of centers of ω_A and ω_C.

Finally, we need to say a few words about the case when ω_A and ω_C coincide (and thus they also coincide with Ω, ω_B, and ω_D). It may be regarded as the limit case in the following manner.

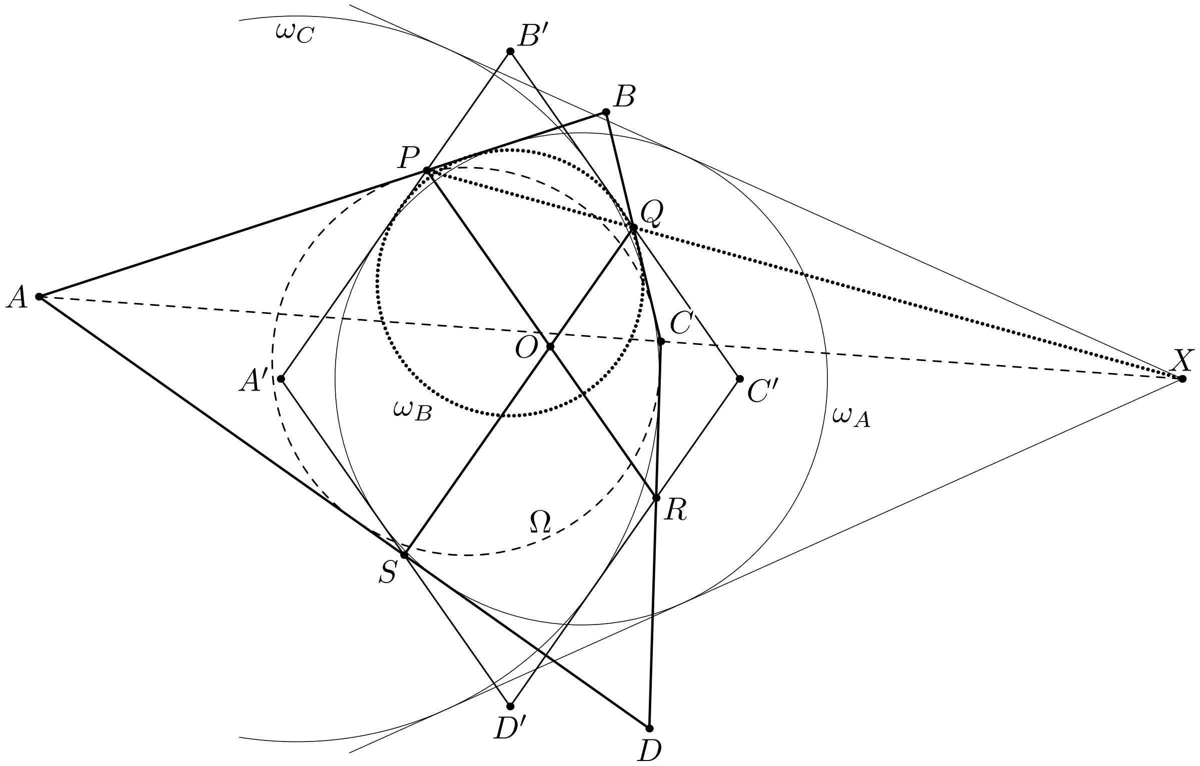


Figure 2

Let us fix the positions of A, P, O, and S (thus we also fix the circles ω_A, γ_A, γ_B, and γ_D). Now we vary the circle γ_C inscribed into $\angle QOR$; for each of its positions, one may reconstruct the lines BC and CD as the external common tangents to γ_B, γ_C and γ_C, γ_D different from PR and QS, respectively. After such variation, the circle Ω changes, so the result obtained above may be applied.

Solution 2. Applying Menelaus' theorem to $\triangle ABC$ with the line PQ and to $\triangle ACD$ with the line RS, we see that the line AC meets PQ and RS at the same point (possibly at infinity) if and only if

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1. \tag{1}$$

So, it suffices to prove (1).

We start with the following result.

Lemma 1. Let $EFGH$ be a circumscribed quadrilateral, and let M be its incenter. Then

$$\frac{EF \cdot FG}{GH \cdot HE} = \frac{FM^2}{HM^2}.$$

Proof. Notice that $\angle EMH + \angle GMF = \angle FME + \angle HMG = 180^\circ$, $\angle FGM = \angle MGH$, and $\angle HEM = \angle MEF$ (see Figure 3). By the law of sines, we get

$$\frac{EF}{FM} \cdot \frac{FG}{FM} = \frac{\sin\angle FME \cdot \sin\angle GMF}{\sin\angle MEF \cdot \sin\angle FGM} = \frac{\sin\angle HMG \cdot \sin\angle EMH}{\sin\angle MGH \cdot \sin\angle HEM} = \frac{GH}{HM} \cdot \frac{HE}{HM}. \qquad \square$$

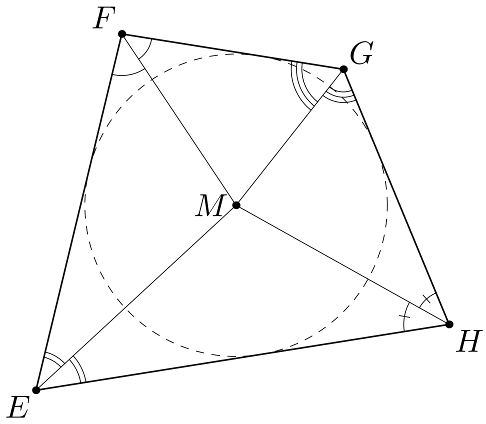


Figure 3

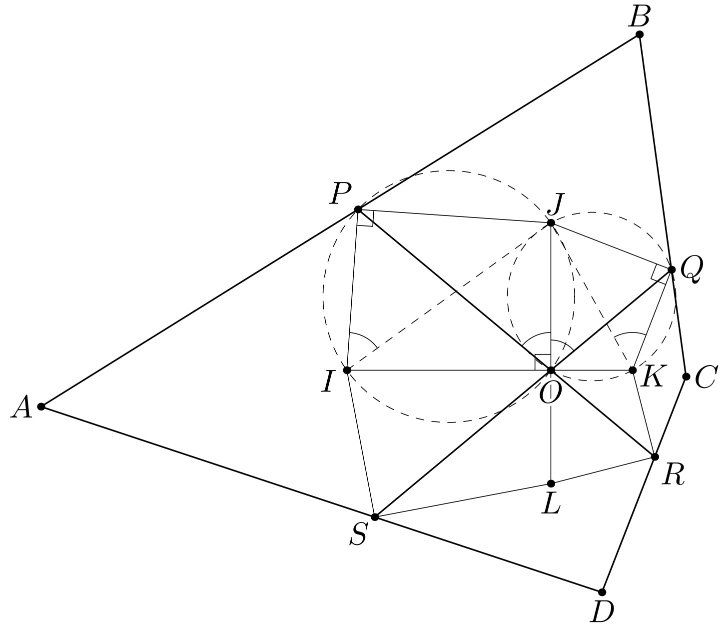


Figure 4

We denote by I, J, K, and L the incenters of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$, respectively. Applying Lemma 1 to these four quadrilaterals we get

$$\frac{AP \cdot PO}{OS \cdot SA} \cdot \frac{BQ \cdot QO}{OP \cdot PB} \cdot \frac{CR \cdot RO}{OQ \cdot QC} \cdot \frac{DS \cdot SO}{OR \cdot RD} = \frac{PI^2}{SI^2} \cdot \frac{QJ^2}{PJ^2} \cdot \frac{RK^2}{QK^2} \cdot \frac{SL^2}{RL^2},$$

which reduces to

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = \frac{PI^2}{PJ^2} \cdot \frac{QJ^2}{QK^2} \cdot \frac{RK^2}{RL^2} \cdot \frac{SL^2}{SI^2}. \tag{2}$$

Next, we have $\angle IPJ = \angle JOI = 90^\circ$, and the line OP separates I and J (see Figure 4). This means that the quadrilateral $IPJO$ is cyclic. Similarly, we get that the quadrilateral $JQKO$ is cyclic with $\angle JQK = 90^\circ$. Thus, $\angle QKJ = \angle QOJ = \angle JOP = \angle JIP$. Hence, the right triangles IPJ and KQJ are similar. Therefore, $\dfrac{PI}{PJ} = \dfrac{QK}{QJ}$. Likewise, we obtain $\dfrac{RK}{RL} = \dfrac{SI}{SL}$. These two equations together with (2) yield (1).

Comment. Instead of using the sine law, one may prove Lemma 1 by the following approach.

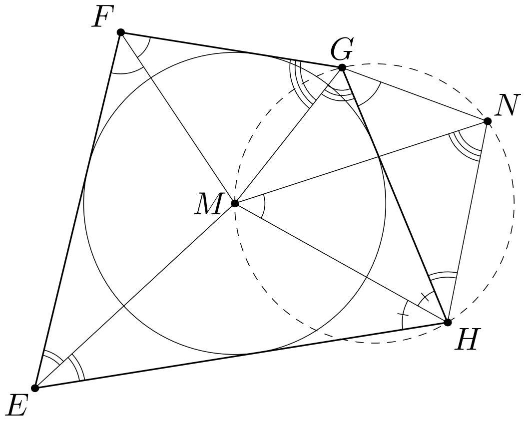


Figure 5

Let N be the point such that $\triangle NHG \sim \triangle MEF$ and such that N and M lie on different sides of the line GH, as shown in Figure 5. Then $\angle GNH + \angle HMG = \angle FME + \angle HMG = 180^\circ$. So, the quadrilateral $GNHM$ is cyclic. Thus, $\angle MNH = \angle MGH = \angle FGM$ and $\angle HMN = \angle HGN = \angle EFM = \angle MFG$. Hence, $\triangle HMN \sim \triangle MFG$. Therefore, $\dfrac{HM}{HG} = \dfrac{HM}{HN} \cdot \dfrac{HN}{HG} = \dfrac{MF}{MG} \cdot \dfrac{EM}{EF}$. Similarly, we obtain $\dfrac{HM}{HE} = \dfrac{MF}{ME} \cdot \dfrac{GM}{GF}$. By multiplying these two equations, we complete the proof.

Solution 3. We present another approach for showing (1) from Solution 2.

Lemma 2. Let $EFGH$ and $E'F'G'H'$ be circumscribed quadrilaterals such that $\angle E + \angle E' = \angle F + \angle F' = \angle G + \angle G' = \angle H + \angle H' = 180^\circ$. Then

$$\frac{EF \cdot GH}{FG \cdot HE} = \frac{E'F' \cdot G'H'}{F'G' \cdot H'E'}.$$

Proof. Let M and M' be the incenters of $EFGH$ and $E'F'G'H'$, respectively. We use the notation $[XYZ]$ for the area of a triangle XYZ.

Taking into account the relation $\angle FME + \angle F'M'E' = 180^\circ$ together with the analogous ones, we get

$$\frac{EF \cdot GH}{FG \cdot HE} = \frac{[MEF] \cdot [MGH]}{[MFG] \cdot [MHE]} = \frac{ME \cdot MF \cdot \sin\angle FME \cdot MG \cdot MH \cdot \sin\angle HMG}{MF \cdot MG \cdot \sin\angle GMF \cdot MH \cdot ME \cdot \sin\angle EMH}$$
$$= \frac{M'E' \cdot M'F' \cdot \sin\angle F'M'E' \cdot M'G' \cdot M'H' \cdot \sin\angle H'M'G'}{M'F' \cdot M'G' \cdot \sin\angle G'M'F' \cdot M'H' \cdot M'E' \cdot \sin\angle E'M'H'} = \frac{E'F' \cdot G'H'}{F'G' \cdot H'E'}. \qquad \square$$

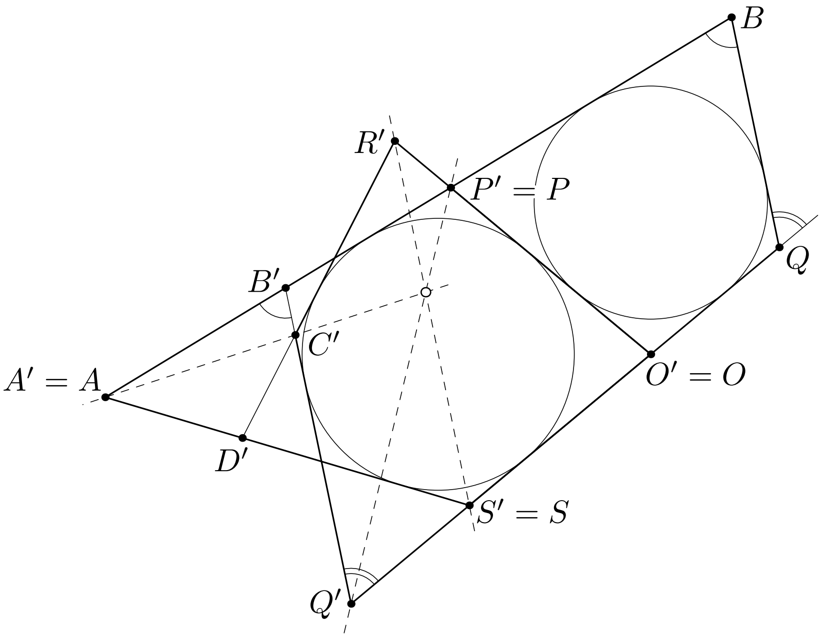


Figure 6

Denote by h the homothety centered at O that maps the incircle of $CROQ$ to the incircle of $APOS$. Let $Q' = h(Q)$, $C' = h(C)$, $R' = h(R)$, $O' = O$, $S' = S$, $A' = A$, and $P' = P$. Furthermore, define $B' = A'P' \cap C'Q'$ and $D' = A'S' \cap C'R'$ as shown in Figure 6. Then

$$\frac{AP \cdot OS}{PO \cdot SA} = \frac{A'P' \cdot O'S'}{P'O' \cdot S'A'}$$

holds trivially. We also have

$$\frac{CR \cdot OQ}{RO \cdot QC} = \frac{C'R' \cdot O'Q'}{R'O' \cdot Q'C'}$$

by the similarity of the quadrilaterals $CROQ$ and $C'R'O'Q'$.

Next, consider the circumscribed quadrilaterals $BQOP$ and $B'Q'O'P'$ whose incenters lie on different sides of the quadrilaterals' shared side line $OP = O'P'$. Observe that $BQ \parallel B'Q'$ and that B' and Q' lie on the lines BP and QO, respectively. It is now easy to see that the two quadrilaterals satisfy the hypotheses of Lemma 2. Thus, we deduce

$$\frac{BQ \cdot OP}{QO \cdot PB} = \frac{B'Q' \cdot O'P'}{Q'O' \cdot P'B'}.$$

Similarly, we get

$$\frac{DS \cdot OR}{SO \cdot RD} = \frac{D'S' \cdot O'R'}{S'O' \cdot R'D'}.$$

Multiplying these four equations, we obtain

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = \frac{A'P'}{P'B'} \cdot \frac{B'Q'}{Q'C'} \cdot \frac{C'R'}{R'D'} \cdot \frac{D'S'}{S'A'}. \tag{3}$$

Finally, we apply Brianchon's theorem to the circumscribed hexagon $A'P'R'C'Q'S'$ and deduce that the lines $A'C'$, $P'Q'$, and $R'S'$ are either concurrent or parallel to each other. So, by Menelaus' theorem, we obtain

$$\frac{A'P'}{P'B'} \cdot \frac{B'Q'}{Q'C'} \cdot \frac{C'R'}{R'D'} \cdot \frac{D'S'}{S'A'} = 1.$$

This equation together with (3) yield (1).

G8. A *triangulation* of a convex polygon Π is a partitioning of Π into triangles by diagonals having no common points other than the vertices of the polygon. We say that a triangulation is a *Thaiangulation* if all triangles in it have the same area.

Prove that any two different Thaiangulations of a convex polygon Π differ by exactly two triangles. (In other words, prove that it is possible to replace one pair of triangles in the first Thaiangulation with a different pair of triangles so as to obtain the second Thaiangulation.)

(Bulgaria)

Solution 1. We denote by $[S]$ the area of a polygon S.

Recall that each triangulation of a convex n-gon has exactly $n-2$ triangles. This means that all triangles in any two Thaiangulations of a convex polygon Π have the same area.

Let $\mathcal{T}$ be a triangulation of a convex polygon Π. If four vertices A, B, C, and D of Π form a parallelogram, and $\mathcal{T}$ contains two triangles whose union is this parallelogram, then we say that $\mathcal{T}$ *contains* parallelogram $ABCD$. Notice here that if two Thaiangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of Π differ by two triangles, then the union of these triangles is a quadrilateral each of whose diagonals bisects its area, i.e., a parallelogram.

We start with proving two properties of triangulations.

Lemma 1. A triangulation of a convex polygon Π cannot contain two parallelograms.

Proof. Arguing indirectly, assume that P_1 and P_2 are two parallelograms contained in some triangulation $\mathcal{T}$. If they have a common triangle in $\mathcal{T}$, then we may assume that P_1 consists of triangles ABC and ADC of $\mathcal{T}$, while P_2 consists of triangles ADC and CDE (see Figure 1). But then $BC \parallel AD \parallel CE$, so the three vertices B, C, and E of Π are collinear, which is absurd.

Assume now that P_1 and P_2 contain no common triangle. Let $P_1 = ABCD$. The sides AB, BC, CD, and DA partition Π into several parts, and P_2 is contained in one of them; we may assume that this part is cut off from P_1 by AD. Then one may label the vertices of P_2 by X, Y, Z, and T so that the polygon $ABCDXYZT$ is convex (see Figure 2; it may happen that $D = X$ and/or $T = A$, but still this polygon has at least six vertices). But the sum of the external angles of this polygon at B, C, Y, and Z is already 360°, which is impossible. A final contradiction. □

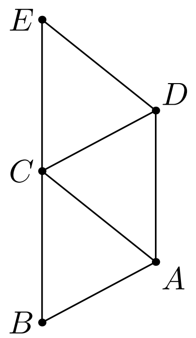


Figure 1

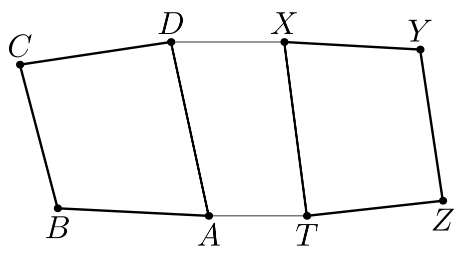


Figure 2

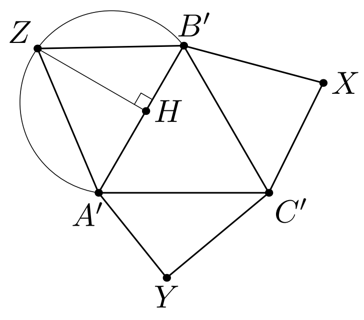


Figure 3

Lemma 2. Every triangle in a Thaiangulation $\mathcal{T}$ of Π contains a side of Π.

Proof. Let ABC be a triangle in $\mathcal{T}$. Apply an affine transform such that ABC maps to an equilateral triangle; let $A'B'C'$ be the image of this triangle, and Π' be the image of Π. Clearly, $\mathcal{T}$ maps into a Thaiangulation $\mathcal{T}'$ of Π'.

Assume that none of the sides of $\triangle A'B'C'$ is a side of Π'. Then $\mathcal{T}'$ contains some other triangles with these sides, say, $A'B'Z$, $C'A'Y$, and $B'C'X$; notice that $A'ZB'XC'Y$ is a convex hexagon (see Figure 3). The sum of its external angles at X, Y, and Z is less than 360°. So one of these angles (say, at Z) is less than 120°, hence $\angle A'ZB' > 60^\circ$. Then Z lies on a circular arc subtended by $A'B'$ and having angular measure less than 240°; consequently, the altitude ZH of $\triangle A'B'Z$ is less than $\sqrt{3}\,A'B'/2$. Thus $[A'B'Z] < [A'B'C']$, and $\mathcal{T}'$ is not a Thaiangulation. A contradiction. □

Now we pass to the solution. We say that a triangle in a triangulation of Π is an *ear* if it contains two sides of Π. Note that each triangulation of a polygon contains some ear.

Arguing indirectly, we choose a convex polygon Π with the least possible number of sides such that some two Thaiangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of Π violate the problem statement (thus Π has at least five sides). Consider now any ear ABC in $\mathcal{T}_1$, with AC being a diagonal of Π. If $\mathcal{T}_2$ also contains $\triangle ABC$, then one may cut $\triangle ABC$ off from Π, getting a polygon with a smaller number of sides which also violates the problem statement. This is impossible; thus $\mathcal{T}_2$ does not contain $\triangle ABC$.

Next, $\mathcal{T}_1$ contains also another triangle with side AC, say $\triangle ACD$. By Lemma 2, this triangle contains a side of Π, so D is adjacent to either A or C on the boundary of Π. We may assume that D is adjacent to C.

Assume that $\mathcal{T}_2$ does not contain the triangle BCD. Then it contains two different triangles BCX and CDY (possibly, with $X = Y$); since these triangles have no common interior points, the polygon $ABCDYX$ is convex (see Figure 4). But, since $[ABC] = [BCX] = [ACD] = [CDY]$, we get $AX \parallel BC$ and $AY \parallel CD$ which is impossible. Thus $\mathcal{T}_2$ contains $\triangle BCD$.

Therefore, $[ABD] = [ABC] + [ACD] - [BCD] = [ABC]$, and $ABCD$ is a parallelogram contained in $\mathcal{T}_1$. Let $\mathcal{T}'$ be the Thaiangulation of Π obtained from $\mathcal{T}_1$ by replacing the diagonal AC with BD; then $\mathcal{T}'$ is distinct from $\mathcal{T}_2$ (otherwise $\mathcal{T}_1$ and $\mathcal{T}_2$ would differ by two triangles). Moreover, $\mathcal{T}'$ shares a common ear BCD with $\mathcal{T}_2$. As above, cutting this ear away we obtain that $\mathcal{T}_2$ and $\mathcal{T}'$ differ by two triangles forming a parallelogram different from $ABCD$. Thus $\mathcal{T}'$ contains two parallelograms, which contradicts Lemma 1.

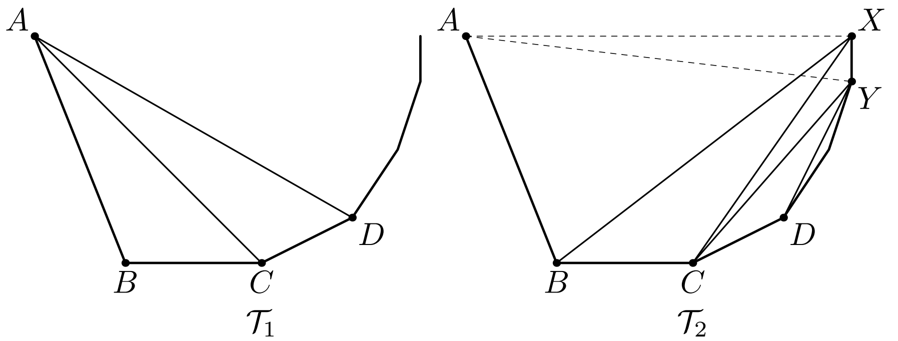


Figure 4

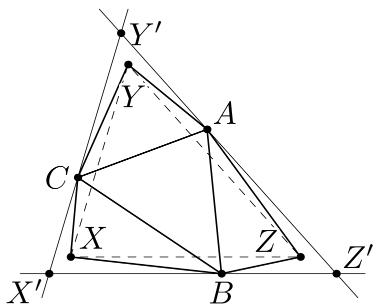


Figure 5

Comment 1. Lemma 2 is equivalent to the well-known Erdős–Debrunner inequality stating that for any triangle PQR and any points A, B, C lying on the sides QR, RP, and PQ, respectively, we have

$$[ABC] \geqslant \min\big\{[ABR], [BCP], [CAQ]\big\}. \tag{1}$$

To derive this inequality from Lemma 2, one may assume that (1) does not hold, and choose some points X, Y, and Z inside the triangles BCP, CAQ, and ABR, respectively, so that $[ABC] = [ABZ] = [BCX] = [CAY]$. Then a convex hexagon $AZBXCY$ has a Thaiangulation containing $\triangle ABC$, which contradicts Lemma 2.

Conversely, assume that a Thaiangulation $\mathcal{T}$ of Π contains a triangle ABC none of whose sides is a side of Π, and let ABZ, AYC, and XBC be other triangles in $\mathcal{T}$ containing the corresponding sides. Then $AZBXCY$ is a convex hexagon.

Consider the lines through A, B, and C parallel to YZ, ZX, and XY, respectively. They form a triangle $X'Y'Z'$ similar to $\triangle XYZ$ (see Figure 5). By (1) we have

$$[ABC] \geqslant \min\big\{[ABZ'], [BCX'], [CAY']\big\} > \min\big\{[ABZ], [BCX], [CAY]\big\},$$

so $\mathcal{T}$ is not a Thaiangulation.

Solution 2. We will make use of the preliminary observations from Solution 1, together with Lemma 1.

Arguing indirectly, we choose a convex polygon Π with the least possible number of sides such that some two Thaiangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of Π violate the statement (thus Π has at least five sides). Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ share a diagonal d splitting Π into two smaller polygons Π_1 and Π_2. Since the problem statement holds for any of them, the induced Thaiangulations of each of Π_i differ by two triangles forming a parallelogram (the Thaiangulations induced on Π_i by $\mathcal{T}_1$ and T_2 may not coincide, otherwise $\mathcal{T}_1$ and $\mathcal{T}_2$ would differ by at most two triangles). But both these parallelograms are contained in $\mathcal{T}_1$; this contradicts Lemma 1. Therefore, $\mathcal{T}_1$ and $\mathcal{T}_2$ share no diagonal. Hence they also share no triangle.

We consider two cases.

Case 1. Assume that some vertex B of Π is an endpoint of some diagonal in $\mathcal{T}_1$, as well as an endpoint of some diagonal in $\mathcal{T}_2$.

Let A and C be the vertices of Π adjacent to B. Then $\mathcal{T}_1$ contains some triangles ABX and BCY, while $\mathcal{T}_2$ contains some triangles ABX' and BCY'. Here, some of the points X, X', Y, and Y' may coincide; however, in view of our assumption together with the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ share no triangle, all four triangles ABX, BCY, ABX', and BCY' are distinct.

Since $[ABX] = [BCY] = [ABX'] = [BCY']$, we have $XX' \parallel AB$ and $YY' \parallel BC$. Now, if $X = Y$, then X' and Y' lie on different lines passing through X and are distinct from that point, so that $X' \neq Y'$. In this case, we may switch the two Thaiangulations. So, hereafter we assume that $X \neq Y$.

In the convex pentagon $ABCYX$ we have either $\angle BAX + \angle AXY > 180^\circ$ or $\angle XYC + \angle YCB > 180^\circ$ (or both); due to the symmetry, we may assume that the first inequality holds. Let r be the ray emerging from X and co-directed with $\overrightarrow{AB}$; our inequality shows that r points to the interior of the pentagon (and thus to the interior of Π). Therefore, the ray opposite to r points outside Π, so X' lies on r; moreover, X' lies on the "arc" CY of Π not containing X. So the segments XX' and YB intersect (see Figure 6).

Let O be the intersection point of the rays r and BC. Since the triangles ABX' and BCY' have no common interior points, Y' must lie on the "arc" CX' which is situated inside the triangle XBO. Therefore, the line YY' meets two sides of $\triangle XBO$, none of which may be XB (otherwise the diagonals XB and YY' would share a common point). Thus YY' intersects BO, which contradicts $YY' \parallel BC$.

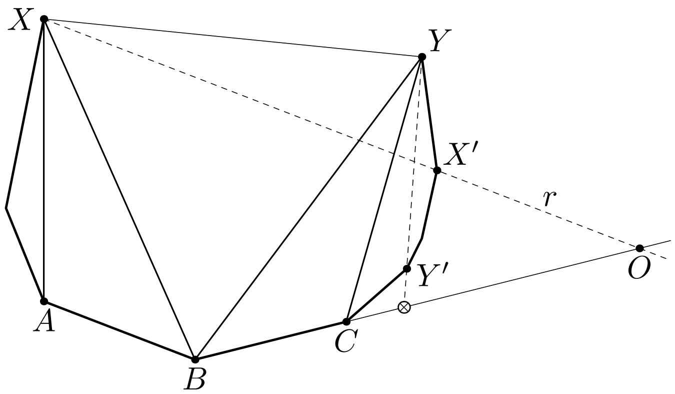


Figure 6

Case 2. In the remaining case, each vertex of Π is an endpoint of a diagonal in at most one of $\mathcal{T}_1$ and $\mathcal{T}_2$. On the other hand, a triangulation cannot contain two consecutive vertices with no diagonals from each. Therefore, the vertices of Π alternatingly emerge diagonals in $\mathcal{T}_1$ and in $\mathcal{T}_2$. In particular, Π has an even number of sides.

Next, we may choose five consecutive vertices A, B, C, D, and E of Π in such a way that

$$\angle ABC + \angle BCD > 180^\circ \quad \text{and} \quad \angle BCD + \angle CDE > 180^\circ. \tag{2}$$

In order to do this, it suffices to choose three consecutive vertices B, C, and D of Π such that the sum of their external angles is at most 180°. This is possible, since Π has at least six sides.

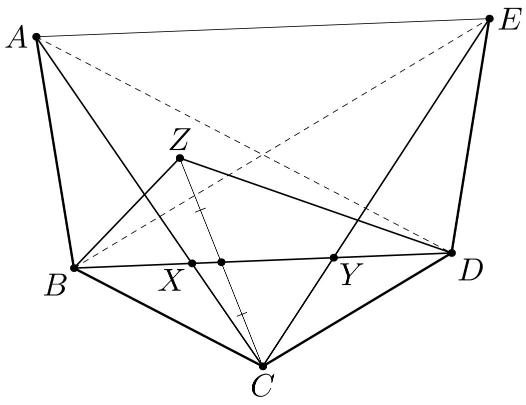


Figure 7

We may assume that $\mathcal{T}_1$ has no diagonals from B and D (and thus contains the triangles ABC and CDE), while $\mathcal{T}_2$ has no diagonals from A, C, and E (and thus contains the triangle BCD). Now, since $[ABC] = [BCD] = [CDE]$, we have $AD \parallel BC$ and $BE \parallel CD$ (see Figure 7). By (2) this yields that $AD > BC$ and $BE > CD$. Let $X = AC \cap BD$ and $Y = CE \cap BD$; then the inequalities above imply that $AX > CX$ and $EY > CY$.

Finally, $\mathcal{T}_2$ must also contain some triangle BDZ with $Z \neq C$; then the ray CZ lies in the angle ACE. Since $[BCD] = [BDZ]$, the diagonal BD bisects CZ. Together with the inequalities above, this yields that Z lies inside the triangle ACE (but Z is distinct from A and E), which is impossible. The final contradiction.

Comment 2. Case 2 may also be accomplished with the use of Lemma 2. Indeed, since each triangulation of an n-gon contains $n-2$ triangles neither of which may contain three sides of Π, Lemma 2 yields that each Thaiangulation contains exactly two ears. But each vertex of Π is a vertex of an ear either in $\mathcal{T}_1$ or in $\mathcal{T}_2$, so Π cannot have more than four vertices.

Number Theory

N1. Determine all positive integers M for which the sequence $a_0, a_1, a_2, \ldots$, defined by $a_0 = \frac{2M+1}{2}$ and $a_{k+1} = a_k\lfloor a_k\rfloor$ for $k = 0, 1, 2, \ldots$, contains at least one integer term.

(Luxembourg)

Answer. All integers $M \geqslant 2$.

Solution 1. Define $b_k = 2a_k$ for all $k \geqslant 0$. Then

$$b_{k+1} = 2a_{k+1} = 2a_k\lfloor a_k\rfloor = b_k\left\lfloor \frac{b_k}{2}\right\rfloor.$$

Since b_0 is an integer, it follows that b_k is an integer for all $k \geqslant 0$.

Suppose that the sequence $a_0, a_1, a_2, \ldots$ does not contain any integer term. Then b_k must be an odd integer for all $k \geqslant 0$, so that

$$b_{k+1} = b_k\left\lfloor \frac{b_k}{2}\right\rfloor = \frac{b_k(b_k-1)}{2}. \tag{1}$$

Hence

$$b_{k+1} - 3 = \frac{b_k(b_k-1)}{2} - 3 = \frac{(b_k-3)(b_k+2)}{2} \tag{2}$$

for all $k \geqslant 0$.

Suppose that $b_0 - 3 > 0$. Then equation (2) yields $b_k - 3 > 0$ for all $k \geqslant 0$. For each $k \geqslant 0$, define c_k to be the highest power of 2 that divides $b_k - 3$. Since $b_k - 3$ is even for all $k \geqslant 0$, the number c_k is positive for every $k \geqslant 0$.

Note that $b_k + 2$ is an odd integer. Therefore, from equation (2), we have that $c_{k+1} = c_k - 1$. Thus, the sequence $c_0, c_1, c_2, \ldots$ of positive integers is strictly decreasing, a contradiction. So, $b_0 - 3 \leqslant 0$, which implies $M = 1$.

For $M = 1$, we can check that the sequence is constant with $a_k = \frac{3}{2}$ for all $k \geqslant 0$. Therefore, the answer is $M \geqslant 2$.

Solution 2. We provide an alternative way to show $M = 1$ once equation (1) has been reached. We claim that $b_k \equiv 3 \pmod{2^m}$ for all $k \geqslant 0$ and $m \geqslant 1$. If this is true, then we would have $b_k = 3$ for all $k \geqslant 0$ and hence $M = 1$.

To establish our claim, we proceed by induction on m. The base case $b_k \equiv 3 \pmod 2$ is true for all $k \geqslant 0$ since b_k is odd. Now suppose that $b_k \equiv 3 \pmod{2^m}$ for all $k \geqslant 0$. Hence $b_k = 2^m d_k + 3$ for some integer d_k. We have

$$3 \equiv b_{k+1} \equiv (2^m d_k + 3)(2^{m-1}d_k + 1) \equiv 3 \cdot 2^{m-1}d_k + 3 \pmod{2^m},$$

so that d_k must be even. This implies that $b_k \equiv 3 \pmod{2^{m+1}}$, as required.

Comment. The reason the number 3 which appears in both solutions is important, is that it is a nontrivial fixed point of the recurrence relation for b_k.

N2. Let a and b be positive integers such that $a!b!$ is a multiple of $a! + b!$. Prove that $3a \geqslant 2b + 2$.

(United Kingdom)

Solution 1. If $a > b$, we immediately get $3a \geqslant 2b + 2$. In the case $a = b$, the required inequality is equivalent to $a \geqslant 2$, which can be checked easily since $(a, b) = (1, 1)$ does not satisfy $a! + b! \mid a!b!$. We now assume $a < b$ and denote $c = b - a$. The required inequality becomes $a \geqslant 2c + 2$.

Suppose, to the contrary, that $a \leqslant 2c + 1$. Define $M = \frac{b!}{a!} = (a + 1)(a + 2) \cdots (a + c)$. Since $a! + b! \mid a!b!$ implies $1 + M \mid a!M$, we obtain $1 + M \mid a!$. Note that we must have $c < a$; otherwise $1 + M > a!$, which is impossible. We observe that $c! \mid M$ since M is a product of c consecutive integers. Thus $\gcd(1 + M, c!) = 1$, which implies

$$1 + M \,\Big|\, \frac{a!}{c!} = (c + 1)(c + 2) \cdots a. \tag{1}$$

If $a \leqslant 2c$, then $\frac{a!}{c!}$ is a product of $a - c \leqslant c$ integers not exceeding a whereas M is a product of c integers exceeding a. Therefore, $1 + M > \frac{a!}{c!}$, which is a contradiction.

It remains to exclude the case $a = 2c + 1$. Since $a + 1 = 2(c + 1)$, we have $c + 1 \mid M$. Hence, we can deduce from (1) that $1 + M \mid (c + 2)(c + 3) \cdots a$. Now $(c + 2)(c + 3) \cdots a$ is a product of $a - c - 1 = c$ integers not exceeding a; thus it is smaller than $1 + M$. Again, we arrive at a contradiction.

Comment 1. One may derive a weaker version of (1) and finish the problem as follows. After assuming $a \leqslant 2c + 1$, we have $\left\lfloor \frac{a}{2} \right\rfloor \leqslant c$, so $\left\lfloor \frac{a}{2} \right\rfloor! \mid M$. Therefore,

$$1 + M \,\Big|\, \left(\left\lfloor \frac{a}{2} \right\rfloor + 1\right) \left(\left\lfloor \frac{a}{2} \right\rfloor + 2\right) \cdots a.$$

Observe that $\left(\left\lfloor \frac{a}{2} \right\rfloor + 1\right) \left(\left\lfloor \frac{a}{2} \right\rfloor + 2\right) \cdots a$ is a product of $\left\lceil \frac{a}{2} \right\rceil$ integers not exceeding a. This leads to a contradiction when a is even since $\left\lceil \frac{a}{2} \right\rceil = \frac{a}{2} \leqslant c$ and M is a product of c integers exceeding a.

When a is odd, we can further deduce that $1 + M \mid \left(\frac{a+3}{2}\right) \left(\frac{a+5}{2}\right) \cdots a$ since $\left\lfloor \frac{a}{2} \right\rfloor + 1 = \frac{a+1}{2} \mid a + 1$. Now $\left(\frac{a+3}{2}\right) \left(\frac{a+5}{2}\right) \cdots a$ is a product of $\frac{a-1}{2} \leqslant c$ numbers not exceeding a, and we get a contradiction.

Solution 2. As in Solution 1, we may assume that $a < b$ and let $c = b - a$. Suppose, to the contrary, that $a \leqslant 2c + 1$. From $a! + b! \mid a!b!$, we have

$$N = 1 + (a + 1)(a + 2) \cdots (a + c) \,\big|\, (a + c)!,$$

which implies that all prime factors of N are at most $a + c$.

Let p be a prime factor of N. If $p \leqslant c$ or $p \geqslant a + 1$, then p divides one of $a + 1, \ldots, a + c$ which is impossible. Hence $a \geqslant p \geqslant c + 1$. Furthermore, we must have $2p > a + c$; otherwise, $a + 1 \leqslant 2c + 2 \leqslant 2p \leqslant a + c$ so $p \mid N - 1$, again impossible. Thus, we have $p \in \left(\frac{a+c}{2}, a\right]$, and $p^2 \nmid (a + c)!$ since $2p > a + c$. Therefore, $p^2 \nmid N$ as well.

If $a \leqslant c + 2$, then the interval $\left(\frac{a+c}{2}, a\right]$ contains at most one integer and hence at most one prime number, which has to be a. Since $p^2 \nmid N$, we must have $N = p = a$ or $N = 1$, which is absurd since $N > a \geqslant 1$. Thus, we have $a \geqslant c + 3$, and so $\frac{a+c+1}{2} \geqslant c + 2$. It follows that p lies in the interval $[c + 2, a]$.

Thus, every prime appearing in the prime factorization of N lies in the interval $[c+2, a]$, and its exponent is exactly 1. So we must have $N \mid (c+2)(c+3) \cdots a$. However, $(c+2)(c+3) \cdots a$ is a product of $a - c - 1 \leqslant c$ numbers not exceeding a, so it is less than N. This is a contradiction.

Comment 2. The original problem statement also asks to determine when the equality $3a = 2b + 2$ holds. It can be checked that the answer is $(a, b) = (2, 2), (4, 5)$.

N3. Let m and n be positive integers such that $m > n$. Define $x_k = (m+k)/(n+k)$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \cdots x_{n+1} - 1$ is divisible by an odd prime.

(Austria)

Solution. Assume that $x_1, x_2, \ldots, x_{n+1}$ are integers. Define the integers

$$a_k = x_k - 1 = \frac{m+k}{n+k} - 1 = \frac{m-n}{n+k} > 0$$

for $k = 1, 2, \ldots, n+1$.

Let $P = x_1 x_2 \cdots x_{n+1} - 1$. We need to prove that P is divisible by an odd prime, or in other words, that P is not a power of 2. To this end, we investigate the powers of 2 dividing the numbers a_k.

Let 2^d be the largest power of 2 dividing $m-n$, and let 2^c be the largest power of 2 not exceeding $2n+1$. Then $2n+1 \leqslant 2^{c+1}-1$, and so $n+1 \leqslant 2^c$. We conclude that 2^c is one of the numbers $n+1, n+2, \ldots, 2n+1$, and that it is the only multiple of 2^c appearing among these numbers. Let ℓ be such that $n+\ell = 2^c$. Since $\frac{m-n}{n+\ell}$ is an integer, we have $d \geqslant c$. Therefore, $2^{d-c+1} \nmid a_\ell = \frac{m-n}{n+\ell}$, while $2^{d-c+1} \mid a_k$ for all $k \in \{1, \ldots, n+1\} \setminus \{\ell\}$.

Computing modulo 2^{d-c+1}, we get

$$P = (a_1+1)(a_2+1)\cdots(a_{n+1}+1) - 1 \equiv (a_\ell + 1) \cdot 1^n - 1 \equiv a_\ell \not\equiv 0 \pmod{2^{d-c+1}}.$$

Therefore, $2^{d-c+1} \nmid P$.

On the other hand, for any $k \in \{1, \ldots, n+1\} \setminus \{\ell\}$, we have $2^{d-c+1} \mid a_k$. So $P \geqslant a_k \geqslant 2^{d-c+1}$, and it follows that P is not a power of 2.

Comment. Instead of attempting to show that P is not a power of 2, one may try to find an odd factor of P (greater than 1) as follows:

From $a_k = \frac{m-n}{n+k} \in \mathbb{Z}_{>0}$, we get that $m-n$ is divisible by $n+1, n+2, \ldots, 2n+1$, and thus it is also divisible by their least common multiple L. So $m-n = qL$ for some positive integer q; hence $x_k = q \cdot \frac{L}{n+k} + 1$.

Then, since $n+1 \leqslant 2^c = n+\ell \leqslant 2n+1 \leqslant 2^{c+1}-1$, we have $2^c \mid L$, but $2^{c+1} \nmid L$. So $\frac{L}{n+\ell}$ is odd, while $\frac{L}{n+k}$ is even for $k \neq \ell$. Computing modulo $2q$ yields

$$x_1 x_2 \cdots x_{n+1} - 1 \equiv (q+1) \cdot 1^n - 1 \equiv q \pmod{2q}.$$

Thus, $x_1 x_2 \cdots x_{n+1} - 1 = 2qr + q = q(2r+1)$ for some integer r.

Since $x_1 x_2 \cdots x_{n+1} - 1 \geqslant x_1 x_2 - 1 \geqslant (q+1)^2 - 1 > q$, we have $r \geqslant 1$. This implies that $x_1 x_2 \cdots x_{n+1} - 1$ is divisible by an odd prime.

N4. Suppose that $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are two sequences of positive integers satisfying $a_0, b_0 \geqslant 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1$$

for all $n \geqslant 0$. Prove that the sequence (a_n) is eventually periodic; in other words, there exist integers $N \geqslant 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geqslant N$.

(France)

Solution 1. Let $s_n = a_n + b_n$. Notice that if $a_n \mid b_n$, then $a_{n+1} = a_n + 1$, $b_{n+1} = b_n - 1$ and $s_{n+1} = s_n$. So, a_n increases by 1 and s_n does not change until the first index is reached with $a_n \nmid s_n$. Define

$$W_n = \left\{m \in \mathbb{Z}_{>0} \colon m \geqslant a_n \text{ and } m \nmid s_n\right\} \quad \text{and} \quad w_n = \min W_n.$$

Claim 1. The sequence (w_n) is non-increasing.

Proof. If $a_n \mid b_n$ then $a_{n+1} = a_n + 1$. Due to $a_n \mid s_n$, we have $a_n \notin W_n$. Moreover $s_{n+1} = s_n$; therefore, $W_{n+1} = W_n$ and $w_{n+1} = w_n$.

Otherwise, if $a_n \nmid b_n$, then $a_n \nmid s_n$, so $a_n \in W_n$ and thus $w_n = a_n$. We show that $a_n \in W_{n+1}$; this implies $w_{n+1} \leqslant a_n = w_n$. By the definition of W_{n+1}, we need that $a_n \geqslant a_{n+1}$ and $a_n \nmid s_{n+1}$. The first relation holds because of $\gcd(a_n, b_n) < a_n$. For the second relation, observe that in $s_{n+1} = \gcd(a_n, b_n) + \operatorname{lcm}(a_n, b_n)$, the second term is divisible by a_n, but the first term is not. So $a_n \nmid s_{n+1}$; that completes the proof of the claim. □

Let $w = \min\limits_n w_n$ and let N be an index with $w = w_N$. Due to Claim 1, we have $w_n = w$ for all $n \geqslant N$.

Let $g_n = \gcd(w, s_n)$. As we have seen, starting from an arbitrary index $n \geqslant N$, the sequence $a_n, a_{n+1}, \ldots$ increases by 1 until it reaches w, which is the first value not dividing s_n; then it drops to $\gcd(w, s_n) + 1 = g_n + 1$.

Claim 2. The sequence (g_n) is constant for $n \geqslant N$.

Proof. If $a_n \mid b_n$, then $s_{n+1} = s_n$ and hence $g_{n+1} = g_n$. Otherwise we have $a_n = w$,

$$\begin{aligned} \gcd(a_n, b_n) &= \gcd(a_n, s_n) = \gcd(w, s_n) = g_n, \\ s_{n+1} &= \gcd(a_n, b_n) + \operatorname{lcm}(a_n, b_n) = g_n + \frac{a_n b_n}{g_n} = g_n + \frac{w(s_n - w)}{g_n}, \qquad (1) \\ \text{and} \quad g_{n+1} &= \gcd(w, s_{n+1}) = \gcd\left(w,\ g_n + \frac{s_n - w}{g_n} w\right) = \gcd(w, g_n) = g_n. \end{aligned}$$

□

Let $g = g_N$. We have proved that the sequence (a_n) eventually repeats the following cycle:

$$g + 1 \mapsto g + 2 \mapsto \ldots \mapsto w \mapsto g + 1.$$

Solution 2. By Claim 1 in the first solution, we have $a_n \leqslant w_n \leqslant w_0$, so the sequence (a_n) is bounded, and hence it has only finitely many values.

Let $M = \operatorname{lcm}(a_1, a_2, \ldots)$, and consider the sequence b_n modulo M. Let r_n be the remainder of b_n, divided by M. For every index n, since $a_n \mid M \mid b_n - r_n$, we have $\gcd(a_n, b_n) = \gcd(a_n, r_n)$, and therefore

$$a_{n+1} = \gcd(a_n, r_n) + 1.$$

Moreover,

$$\begin{aligned} r_{n+1} \equiv b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1 &= \frac{a_n}{\gcd(a_n, b_n)} b_n - 1 \\ &= \frac{a_n}{\gcd(a_n, r_n)} b_n - 1 \equiv \frac{a_n}{\gcd(a_n, r_n)} r_n - 1 \pmod{M}. \end{aligned}$$

Hence, the pair (a_n, r_n) uniquely determines the pair (a_{n+1}, r_{n+1}). Since there are finitely many possible pairs, the sequence of pairs (a_n, r_n) is eventually periodic; in particular, the sequence (a_n) is eventually periodic.

Comment. We show that there are only four possibilities for g and w (as defined in Solution 1), namely

$$(w, g) \in \{(2,1), (3,1), (4,2), (5,1)\}. \tag{2}$$

This means that the sequence (a_n) eventually repeats one of the following cycles:

$$(2), \quad (2,3), \quad (3,4), \quad \text{or} \quad (2,3,4,5). \tag{3}$$

Using the notation of Solution 1, for $n \geqslant N$ the sequence (a_n) has a cycle $(g+1, g+2, \ldots, w)$ such that $g = \gcd(w, s_n)$. By the observations in the proof of Claim 2, the numbers $g+1, \ldots, w-1$ all divide s_n; so the number $L = \operatorname{lcm}(g+1, g+2, \ldots, w-1)$ also divides s_n. Moreover, g also divides w.

Now choose any $n \geqslant N$ such that $a_n = w$. By (1), we have

$$s_{n+1} = g + \frac{w(s_n - w)}{g} = s_n \cdot \frac{w}{g} - \frac{w^2 - g^2}{g}.$$

Since L divides both s_n and s_{n+1}, it also divides the number $T = \frac{w^2-g^2}{g}$.

Suppose first that $w \geqslant 6$, which yields $g + 1 \leqslant \frac{w}{2} + 1 \leqslant w - 2$. Then $(w-2)(w-1) \mid L \mid T$, so we have either $w^2 - g^2 \geqslant 2(w-1)(w-2)$, or $g = 1$ and $w^2 - g^2 = (w-1)(w-2)$. In the former case we get $(w-1)(w-5) + (g^2 - 1) \leqslant 0$ which is false by our assumption. The latter equation rewrites as $3w = 3$, so $w = 1$, which is also impossible.

Now we are left with the cases when $w \leqslant 5$ and $g \mid w$. The case $(w, g) = (4, 1)$ violates the condition $L \mid \frac{w^2-g^2}{g}$; all other such pairs are listed in (2).

In the table below, for each pair (w, g), we provide possible sequences (a_n) and (b_n). That shows that the cycles shown in (3) are indeed possible.

$w = 2$	$g = 1$	$a_n = 2$	$b_n = 2 \cdot 2^n + 1$
$w = 3$	$g = 1$	$(a_{2k}, a_{2k+1}) = (2, 3)$	$(b_{2k}, b_{2k+1}) = (6 \cdot 3^k + 2, 6 \cdot 3^k + 1)$
$w = 4$	$g = 2$	$(a_{2k}, a_{2k+1}) = (3, 4)$	$(b_{2k}, b_{2k+1}) = (12 \cdot 2^k + 3, 12 \cdot 2^k + 2)$
$w = 5$	$g = 1$	$(a_{4k}, \ldots, a_{4k+3}) = (2, 3, 4, 5)$	$(b_{4k}, \ldots, b_{4k+3}) = (6 \cdot 5^k + 4, \ldots, 6 \cdot 5^k + 1)$

N5. Determine all triples (a, b, c) of positive integers for which $ab - c$, $bc - a$, and $ca - b$ are powers of 2.

Explanation: A power of 2 is an integer of the form 2^n, where n denotes some nonnegative integer.

(Serbia)

Answer. There are sixteen such triples, namely $(2, 2, 2)$, the three permutations of $(2, 2, 3)$, and the six permutations of each of $(2, 6, 11)$ and $(3, 5, 7)$.

Solution 1. It can easily be verified that these sixteen triples are as required. Now let (a, b, c) be any triple with the desired property. If we would have $a = 1$, then both $b - c$ and $c - b$ were powers of 2, which is impossible since their sum is zero; because of symmetry, this argument shows $a, b, c \geqslant 2$.

Case 1. Among a, b, and c there are at least two equal numbers.

Without loss of generality we may suppose that $a = b$. Then $a^2 - c$ and $a(c - 1)$ are powers of 2. The latter tells us that actually a and $c - 1$ are powers of 2. So there are nonnegative integers α and γ with $a = 2^\alpha$ and $c = 2^\gamma + 1$. Since $a^2 - c = 2^{2\alpha} - 2^\gamma - 1$ is a power of 2 and thus incongruent to -1 modulo 4, we must have $\gamma \leqslant 1$. Moreover, each of the terms $2^{2\alpha} - 2$ and $2^{2\alpha} - 3$ can only be a power of 2 if $\alpha = 1$. It follows that the triple (a, b, c) is either $(2, 2, 2)$ or $(2, 2, 3)$.

Case 2. The numbers a, b, and c are distinct.

Due to symmetry we may suppose that

$$2 \leqslant a < b < c\,. \tag{1}$$

We are to prove that the triple (a, b, c) is either $(2, 6, 11)$ or $(3, 5, 7)$. By our hypothesis, there exist three nonnegative integers α, β, and γ such that

$$bc - a = 2^\alpha\,, \tag{2}$$

$$ac - b = 2^\beta\,, \tag{3}$$

$$\text{and} \qquad ab - c = 2^\gamma\,. \tag{4}$$

Evidently we have

$$\alpha > \beta > \gamma\,. \tag{5}$$

Depending on how large a is, we divide the argument into two further cases.

Case 2.1. $a = 2$.

We first prove that $\gamma = 0$. Assume for the sake of contradiction that $\gamma > 0$. Then c is even by (4) and, similarly, b is even by (5) and (3). So the left-hand side of (2) is congruent to 2 modulo 4, which is only possible if $bc = 4$. As this contradicts (1), we have thereby shown that $\gamma = 0$, i.e., that $c = 2b - 1$.

Now (3) yields $3b - 2 = 2^\beta$. Due to $b > 2$ this is only possible if $\beta \geqslant 4$. If $\beta = 4$, then we get $b = 6$ and $c = 2 \cdot 6 - 1 = 11$, which is a solution. It remains to deal with the case $\beta \geqslant 5$. Now (2) implies

$$9 \cdot 2^\alpha = 9b(2b - 1) - 18 = (3b - 2)(6b + 1) - 16 = 2^\beta(2^{\beta+1} + 5) - 16\,,$$

and by $\beta \geqslant 5$ the right-hand side is not divisible by 32. Thus $\alpha \leqslant 4$ and we get a contradiction to (5).

Case 2.2. $a \geqslant 3$.

Pick an integer $\vartheta \in \{-1, +1\}$ such that $c - \vartheta$ is not divisible by 4. Now

$$2^\alpha + \vartheta \cdot 2^\beta = (bc - a\vartheta^2) + \vartheta(ca - b) = (b + a\vartheta)(c - \vartheta)$$

is divisible by 2^β and, consequently, $b + a\vartheta$ is divisible by $2^{\beta-1}$. On the other hand, $2^\beta = ac - b > (a-1)c \geqslant 2c$ implies in view of (1) that a and b are smaller than $2^{\beta-1}$. All this is only possible if $\vartheta = 1$ and $a + b = 2^{\beta-1}$. Now (3) yields

$$ac - b = 2(a + b)\,, \tag{6}$$

whence $4b > a + 3b = a(c-1) \geqslant ab$, which in turn yields $a = 3$.

So (6) simplifies to $c = b + 2$ and (2) tells us that $b(b+2) - 3 = (b-1)(b+3)$ is a power of 2. Consequently, the factors $b - 1$ and $b + 3$ are powers of 2 themselves. Since their difference is 4, this is only possible if $b = 5$ and thus $c = 7$. Thereby the solution is complete.

Solution 2. As in the beginning of the first solution, we observe that $a, b, c \geqslant 2$. Depending on the parities of a, b, and c we distinguish three cases.

Case 1. The numbers a, b, and c are even.

Let 2^A, 2^B, and 2^C be the largest powers of 2 dividing a, b, and c respectively. We may assume without loss of generality that $1 \leqslant A \leqslant B \leqslant C$. Now 2^B is the highest power of 2 dividing $ac - b$, whence $ac - b = 2^B \leqslant b$. Similarly, we deduce $bc - a = 2^A \leqslant a$. Adding both estimates we get $(a + b)c \leqslant 2(a + b)$, whence $c \leqslant 2$. So $c = 2$ and thus $A = B = C = 1$; moreover, we must have had equality throughout, i.e., $a = 2^A = 2$ and $b = 2^B = 2$. We have thereby found the solution $(a, b, c) = (2, 2, 2)$.

Case 2. The numbers a, b, and c are odd.

If any two of these numbers are equal, say $a = b$, then $ac - b = a(c - 1)$ has a nontrivial odd divisor and cannot be a power of 2. Hence a, b, and c are distinct. So we may assume without loss of generality that $a < b < c$.

Let α and β denote the nonnegative integers for which $bc - a = 2^\alpha$ and $ac - b = 2^\beta$ hold. Clearly, we have $\alpha > \beta$, and thus 2^β divides

$$a \cdot 2^\alpha - b \cdot 2^\beta = a(bc - a) - b(ac - b) = b^2 - a^2 = (b + a)(b - a)\,.$$

Since a is odd, it is not possible that both factors $b+a$ and $b-a$ are divisible by 4. Consequently, one of them has to be a multiple of $2^{\beta-1}$. Hence one of the numbers $2(b + a)$ and $2(b - a)$ is divisible by 2^β and in either case we have

$$ac - b = 2^\beta \leqslant 2(a + b)\,. \tag{7}$$

This in turn yields $(a - 1)b < ac - b < 4b$ and thus $a = 3$ (recall that a is odd and larger than 1). Substituting this back into (7) we learn $c \leqslant b + 2$. But due to the parity $b < c$ entails that $b + 2 \leqslant c$ holds as well. So we get $c = b + 2$ and from $bc - a = (b - 1)(b + 3)$ being a power of 2 it follows that $b = 5$ and $c = 7$.

Case 3. Among a, b, and c both parities occur.

Without loss of generality, we suppose that c is odd and that $a \leqslant b$. We are to show that (a, b, c) is either $(2, 2, 3)$ or $(2, 6, 11)$. As at least one of a and b is even, the expression $ab - c$ is odd; since it is also a power of 2, we obtain

$$ab - c = 1\,. \tag{8}$$

If $a = b$, then $c = a^2 - 1$, and from $ac - b = a(a^2 - 2)$ being a power of 2 it follows that both a and $a^2 - 2$ are powers of 2, whence $a = 2$. This gives rise to the solution $(2, 2, 3)$.

We may suppose $a < b$ from now on. As usual, we let $\alpha > \beta$ denote the integers satisfying

$$2^\alpha = bc - a \quad \text{and} \quad 2^\beta = ac - b\,. \tag{9}$$

If $\beta = 0$ it would follow that $ac - b = ab - c = 1$ and hence that $b = c = 1$, which is absurd. So β and α are positive and consequently a and b are even. Substituting $c = ab - 1$ into (9) we obtain

$$2^\alpha = ab^2 - (a+b)\,, \tag{10}$$

$$\text{and} \quad 2^\beta = a^2 b - (a+b)\,. \tag{11}$$

The addition of both equation yields $2^\alpha + 2^\beta = (ab-2)(a+b)$. Now $ab - 2$ is even but not divisible by 4, so the highest power of 2 dividing $a + b$ is $2^{\beta-1}$. For this reason, the equations (10) and (11) show that the highest powers of 2 dividing either of the numbers ab^2 and a^2b is likewise $2^{\beta-1}$. Thus there is an integer $\tau \geqslant 1$ together with odd integers A, B, and C such that $a = 2^\tau A$, $b = 2^\tau B$, $a + b = 2^{3\tau} C$, and $\beta = 1 + 3\tau$.

Notice that $A + B = 2^{2\tau} C \geqslant 4C$. Moreover, (11) entails $A^2 B - C = 2$. Thus $8 = 4A^2B - 4C \geqslant 4A^2B - A - B \geqslant A^2(3B - 1)$. Since A and B are odd with $A < B$, this is only possible if $A = 1$ and $B = 3$. Finally, one may conclude $C = 1$, $\tau = 1$, $a = 2$, $b = 6$, and $c = 11$. We have thereby found the triple $(2, 6, 11)$. This completes the discussion of the third case, and hence the solution.

Comment. In both solutions, there are many alternative ways to proceed in each of its cases. Here we present a different treatment of the part "$a < b$" of Case 3 in Solution 2, assuming that (8) and (9) have already been written down:

Put $d = \gcd(a, b)$ and define the integers p and q by $a = dp$ and $b = dq$; notice that $p < q$ and $\gcd(p, q) = 1$. Now (8) implies $c = d^2pq - 1$ and thus we have

$$2^\alpha = d(d^2pq^2 - p - q)$$

$$\text{and} \quad 2^\beta = d(d^2p^2q - p - q)\,. \tag{12}$$

Now 2^β divides $2^\alpha - 2^\beta = d^3pq(q - p)$ and, as p and q are easily seen to be coprime to $d^2p^2q - p - q$, it follows that

$$(d^2p^2q - p - q) \mid d^2(q - p)\,. \tag{13}$$

In particular, we have $d^2p^2q - p - q \leqslant d^2(q - p)$, i.e., $d^2(p^2q + p - q) \leqslant p + q$. As $p^2q + p - q > 0$, this may be weakened to $p^2q + p - q \leqslant p + q$. Hence $p^2q \leqslant 2q$, which is only possible if $p = 1$.

Going back to (13), we get

$$(d^2q - q - 1) \mid d^2(q - 1)\,. \tag{14}$$

Now $2(d^2q - q - 1) \leqslant d^2(q - 1)$ would entail $d^2(q + 1) \leqslant 2(q + 1)$ and thus $d = 1$. But this would tell us that $a = dp = 1$, which is absurd. This argument proves $2(d^2q - q - 1) > d^2(q - 1)$ and in the light of (14) it follows that $d^2q - q - 1 = d^2(q - 1)$, i.e., $q = d^2 - 1$. Plugging this together with $p = 1$ into (12) we infer $2^\beta = d^3(d^2 - 2)$. Hence d and $d^2 - 2$ are powers of 2. Consequently, $d = 2$, $q = 3$, $a = 2$, $b = 6$, and $c = 11$, as desired.

N6. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:

(*i*) If $m, n \in \mathbb{Z}_{>0}$, then $\dfrac{f^n(m) - m}{n} \in \mathbb{Z}_{>0}$;

(*ii*) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

(Singapore)

Solution. We split the solution into three steps. In the first of them, we show that the function f is injective and explain how this leads to a useful visualization of f. Then comes the second step, in which most of the work happens: its goal is to show that for any $n \in \mathbb{Z}_{>0}$ the sequence $n, f(n), f^2(n), \ldots$ is an arithmetic progression. Finally, in the third step we put everything together, thus solving the problem.

Step 1. We commence by checking that f is injective. For this purpose, we consider any $m, k \in \mathbb{Z}_{>0}$ with $f(m) = f(k)$. By (*i*), every positive integer n has the property that

$$\frac{k-m}{n} = \frac{f^n(m) - m}{n} - \frac{f^n(k) - k}{n}$$

is a difference of two integers and thus integral as well. But for $n = |k - m| + 1$ this is only possible if $k = m$. Thereby, the injectivity of f is established.

Now recall that due to condition (*ii*) there are finitely many positive integers $a_1, \ldots, a_k$ such that $\mathbb{Z}_{>0}$ is the disjoint union of $\{a_1, \ldots, a_k\}$ and $\{f(n) \mid n \in \mathbb{Z}_{>0}\}$. Notice that by plugging $n = 1$ into condition (*i*) we get $f(m) > m$ for all $m \in \mathbb{Z}_{>0}$.

We contend that every positive integer n may be expressed uniquely in the form $n = f^j(a_i)$ for some $j \geqslant 0$ and $i \in \{1, \ldots, k\}$. The uniqueness follows from the injectivity of f. The existence can be proved by induction on n in the following way. If $n \in \{a_1, \ldots, a_k\}$, then we may take $j = 0$; otherwise there is some $n' < n$ with $f(n') = n$ to which the induction hypothesis may be applied.

The result of the previous paragraph means that every positive integer appears exactly once in the following infinite picture, henceforth referred to as "the Table":

a_1	$f(a_1)$	$f^2(a_1)$	$f^3(a_1)$	$\ldots$
a_2	$f(a_2)$	$f^2(a_2)$	$f^3(a_2)$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
a_k	$f(a_k)$	$f^2(a_k)$	$f^3(a_k)$	$\ldots$

The Table

Step 2. Our next goal is to prove that each row of the Table is an arithmetic progression. Assume contrariwise that the number t of rows which are arithmetic progressions would satisfy $0 \leqslant t < k$. By permuting the rows if necessary we may suppose that precisely the first t rows are arithmetic progressions, say with steps $T_1, \ldots, T_t$. Our plan is to find a further row that is "not too sparse" in an asymptotic sense, and then to prove that such a row has to be an arithmetic progression as well.

Let us write $T = \operatorname{lcm}(T_1, T_2, \ldots, T_t)$ and $A = \max\{a_1, a_2, \ldots, a_t\}$ if $t > 0$; and $T = 1$ and $A = 0$ if $t = 0$. For every integer $n \geqslant A$, the interval $\Delta_n = [n+1, n+T]$ contains exactly T/T_i

elements of the i^{th} row $(1 \leqslant i \leqslant t)$. Therefore, the number of elements from the last $(k-t)$ rows of the Table contained in Δ_n does not depend on $n \geqslant A$. It is not possible that none of these intervals Δ_n contains an element from the $k-t$ last rows, because infinitely many numbers appear in these rows. It follows that for each $n \geqslant A$ the interval Δ_n contains at least one member from these rows.

This yields that for every positive integer d, the interval $\big[A+1, A+(d+1)(k-t)T\big]$ contains at least $(d+1)(k-t)$ elements from the last $k-t$ rows; therefore, there exists an index x with $t+1 \leqslant x \leqslant k$, possibly depending on d, such that our interval contains at least $d+1$ elements from the x^{th} row. In this situation we have

$$f^d(a_x) \leqslant A + (d+1)(k-t)T\,.$$

Finally, since there are finitely many possibilities for x, there exists an index $x \geqslant t+1$ such that the set

$$X = \big\{d \in \mathbb{Z}_{>0} \,\big|\, f^d(a_x) \leqslant A + (d+1)(k-t)T\big\}$$

is infinite. Thereby we have found the "dense row" promised above.

By assumption (i), for every $d \in X$ the number

$$\beta_d = \frac{f^d(a_x) - a_x}{d}$$

is a positive integer not exceeding

$$\frac{A + (d+1)(k-t)T}{d} \leqslant \frac{Ad + 2d(k-t)T}{d} = A + 2(k-t)T\,.$$

This leaves us with finitely many choices for β_d, which means that there exists a number T_x such that the set

$$Y = \big\{d \in X \,\big|\, \beta_d = T_x\big\}$$

is infinite. Notice that we have $f^d(a_x) = a_x + d \cdot T_x$ for all $d \in Y$.

Now we are prepared to prove that the numbers in the x^{th} row form an arithmetic progression, thus coming to a contradiction with our assumption. Let us fix any positive integer j. Since the set Y is infinite, we can choose a number $y \in Y$ such that $y - j > \big|f^j(a_x) - (a_x + jT_x)\big|$. Notice that both numbers

$$f^y(a_x) - f^j(a_x) = f^{y-j}\big(f^j(a_x)\big) - f^j(a_x) \quad \text{and} \quad f^y(a_x) - (a_x + jT_x) = (y-j)T_x$$

are divisible by $y-j$. Thus, the difference between these numbers is also divisible by $y-j$. Since the absolute value of this difference is less than $y-j$, it has to vanish, so we get $f^j(a_x) = a_x + j \cdot T_x$.

Hence, it is indeed true that all rows of the Table are arithmetic progressions.

Step 3. Keeping the above notation in force, we denote the step of the i^{th} row of the table by T_i. Now we claim that we have $f(n) - n = f(n+T) - (n+T)$ for all $n \in \mathbb{Z}_{>0}$, where

$$T = \operatorname{lcm}(T_1, \ldots, T_k)\,.$$

To see this, let any $n \in \mathbb{Z}_{>0}$ be given and denote the index of the row in which it appears in the Table by i. Then we have $f^j(n) = n + j \cdot T_i$ for all $j \in \mathbb{Z}_{>0}$, and thus indeed

$$f(n+T) - f(n) = f^{1+T/T_i}(n) - f(n) = (n + T + T_i) - (n + T_i) = T\,.$$

This concludes the solution.

Comment 1. There are some alternative ways to complete the second part once the index x corresponding to a "dense row" is found. For instance, one may show that for some integer T_x^* the set

$$Y^* = \{ j \in \mathbb{Z}_{>0} \mid f^{j+1}(a_x) - f^j(a_x) = T_x^* \}$$

is infinite, and then one may conclude with a similar divisibility argument.

Comment 2. It may be checked that, conversely, any way to fill out the Table with finitely many arithmetic progressions so that each positive integer appears exactly once, gives rise to a function f satisfying the two conditions mentioned in the problem. For example, we may arrange the positive integers as follows:

2	4	6	8	10	...
1	5	9	13	17	...
3	7	11	15	19	...

This corresponds to the function

$$f(n) = \begin{cases} n+2 & \text{if } n \text{ is even;} \\ n+4 & \text{if } n \text{ is odd.} \end{cases}$$

As this example shows, it is not true that the function $n \mapsto f(n) - n$ has to be constant.

N7. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called *k-good* if $\gcd\big(f(m)+n, f(n)+m\big) \leqslant k$ for all $m \neq n$. Find all k such that there exists a k-good function.

(Canada)

Answer. $k \geqslant 2$.

Solution 1. For any function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, let $G_f(m,n) = \gcd\big(f(m)+n, f(n)+m\big)$. Note that a k-good function is also $(k+1)$-good for any positive integer k. Hence, it suffices to show that there does not exist a 1-good function and that there exists a 2-good function.

We first show that there is no 1-good function. Suppose that there exists a function f such that $G_f(m,n) = 1$ for all $m \neq n$. Now, if there are two distinct even numbers m and n such that $f(m)$ and $f(n)$ are both even, then $2 \mid G_f(m,n)$, a contradiction. A similar argument holds if there are two distinct odd numbers m and n such that $f(m)$ and $f(n)$ are both odd. Hence we can choose an even m and an odd n such that $f(m)$ is odd and $f(n)$ is even. This also implies that $2 \mid G_f(m,n)$, a contradiction.

We now construct a 2-good function. Define $f(n) = 2^{g(n)+1} - n - 1$, where g is defined recursively by $g(1) = 1$ and $g(n+1) = (2^{g(n)+1})!$.

For any positive integers $m > n$, set

$$A = f(m) + n = 2^{g(m)+1} - m + n - 1, \qquad B = f(n) + m = 2^{g(n)+1} - n + m - 1.$$

We need to show that $\gcd(A,B) \leqslant 2$. First, note that $A + B = 2^{g(m)+1} + 2^{g(n)+1} - 2$ is not divisible by 4, so that $4 \nmid \gcd(A,B)$. Now we suppose that there is an odd prime p for which $p \mid \gcd(A,B)$ and derive a contradiction.

We first claim that $2^{g(m-1)+1} \geqslant B$. This is a rather weak bound; one way to prove it is as follows. Observe that $g(k+1) > g(k)$ and hence $2^{g(k+1)+1} \geqslant 2^{g(k)+1} + 1$ for every positive integer k. By repeatedly applying this inequality, we obtain $2^{g(m-1)+1} \geqslant 2^{g(n)+1} + (m-1) - n = B$.

Now, since $p \mid B$, we have $p - 1 < B \leqslant 2^{g(m-1)+1}$, so that $p - 1 \mid (2^{g(m-1)+1})! = g(m)$. Hence $2^{g(m)} \equiv 1 \pmod{p}$, which yields $A + B \equiv 2^{g(n)+1} \pmod{p}$. However, since $p \mid A + B$, this implies that $p = 2$, a contradiction.

Solution 2. We provide an alternative construction of a 2-good function f.

Let $\mathcal{P}$ be the set consisting of 4 and all odd primes. For every $p \in \mathcal{P}$, we say that a number $a \in \{0, 1, \ldots, p-1\}$ is *p-useful* if $a \not\equiv -a \pmod{p}$. Note that a residue modulo p which is neither 0 nor 2 is p-useful (the latter is needed only when $p = 4$).

We will construct f recursively; in some steps, we will also define a p-useful number a_p. After the m^{th} step, the construction will satisfy the following conditions:

(i) The values of $f(n)$ have already been defined for all $n \leqslant m$, and p-useful numbers a_p have already been defined for all $p \leqslant m+2$;

(ii) If $n \leqslant m$ and $p \leqslant m+2$, then $f(n) + n \not\equiv a_p \pmod{p}$;

(iii) $\gcd\big(f(n_1) + n_2, f(n_2) + n_1\big) \leqslant 2$ for all $n_1 < n_2 \leqslant m$.

If these conditions are satisfied, then f will be a 2-good function.

Step 1. Set $f(1) = 1$ and $a_3 = 1$. Clearly, all the conditions are satisfied.

Step m, for $m \geqslant 2$. We need to determine $f(m)$ and, if $m + 2 \in \mathcal{P}$, the number a_{m+2}.

Defining $f(m)$. Let $X_m = \{p \in \mathcal{P} \colon p \mid f(n) + m \text{ for some } n < m\}$. We will determine $f(m) \bmod p$ for all $p \in X_m$ and then choose $f(m)$ using the Chinese Remainder Theorem.

Take any $p \in X_m$. If $p \leqslant m+1$, then we define $f(m) \equiv -a_p - m \pmod{p}$. Otherwise, if $p \geqslant m+2$, then we define $f(m) \equiv 0 \pmod{p}$.

Defining a_{m+2}. Now let $p = m+2$ and suppose that $p \in \mathcal{P}$. We choose a_p to be a residue modulo p that is not congruent to 0, 2, or $f(n)+n$ for any $n \leqslant m$. Since $f(1)+1 = 2$, there are at most $m+1 < p$ residues to avoid, so we can always choose a remaining residue.

We first check that (ii) is satisfied. We only need to check it if $p = m+2$ or $n = m$. In the former case, we have $f(n)+n \not\equiv a_p \pmod{p}$ by construction. In the latter case, if $n = m$ and $p \leqslant m+1$, then we have $f(m)+m \equiv -a_p \not\equiv a_p \pmod{p}$, where we make use of the fact that a_p is p-useful.

Now we check that (iii) holds. Suppose, to the contrary, that $p \mid \gcd\big(f(n)+m, f(m)+n\big)$ for some $n < m$. Then $p \in X_m$ and $p \mid f(m)+n$. If $p \geqslant m+2$, then $0 \equiv f(m)+n \equiv n \pmod{p}$, which is impossible since $n < m < p$.

Otherwise, if $p \leqslant m+1$, then

$$0 \equiv \big(f(m)+n\big) + \big(f(n)+m\big) \equiv \big(f(n)+n\big) + \big(f(m)+m\big) \equiv \big(f(n)+n\big) - a_p \pmod{p}.$$

This implies that $f(n)+n \equiv a_p \pmod{p}$, a contradiction with (ii).

Comment 1. For any $p \in \mathcal{P}$, we may also define a_p at step m for an arbitrary $m \leqslant p-2$. The construction will work as long as we define a finite number of a_p at each step.

Comment 2. When attempting to construct a 2-good function f recursively, the following way seems natural. Start with setting $f(1) = 1$. Next, for each integer $m > 1$, introduce the set X_m like in Solution 2 and define $f(m)$ so as to satisfy

$$\begin{aligned} f(m) &\equiv f(m-p) \pmod{p} &&\text{for all } p \in X_m \text{ with } p < m, \quad \text{and} \\ f(m) &\equiv 0 \pmod{p} &&\text{for all } p \in X_m \text{ with } p \geqslant m. \end{aligned}$$

This construction might seem to work. Indeed, consider a fixed $p \in \mathcal{P}$, and suppose that p divides $\gcd\big(f(n)+m, f(m)+n\big)$ for some $n < m$. Choose such m and n so that $\max(m,n)$ is minimal. Then $p \in X_m$. We can check that $p < m$, so that the construction implies that p divides $\gcd\big(f(n)+(m-p), f(m-p)+n\big)$. Since $\max(n, m-p) < \max(m,n)$, this *almost* leads to a contradiction—the only trouble is the possibility that $n = m-p$. However, this flaw may happen to be not so easy to fix.

We will present one possible way to repair this argument in the next comment.

Comment 3. There are many recursive constructions for a 2-good function f. Here we sketch one general approach which may be specified in different ways. For convenience, we denote by $\mathbb{Z}_p$ the set of residues modulo p; all operations on elements of $\mathbb{Z}_p$ are also performed modulo p.

The general structure is the same as in Solution 2, i.e. using the Chinese Remainder Theorem to successively determine $f(m)$. But instead of designating a common "safe" residue a_p for future steps, we act as follows.

For every $p \in \mathcal{P}$, in some step of the process we define p subsets $B_p^{(1)}, B_p^{(2)}, \ldots, B_p^{(p)} \subset \mathbb{Z}_p$. The meaning of these sets is that

$$f(m)+m \text{ should be congruent to some element in } B_p^{(i)} \text{ whenever } m \equiv i \pmod{p} \text{ for } i \in \mathbb{Z}_p. \tag{1}$$

Moreover, in every such subset we specify a *safe* element $b_p^{(i)} \in B_p^{(i)}$. The meaning now is that in future steps, it is safe to set $f(m)+m \equiv b_p^{(i)} \pmod{p}$ whenever $m \equiv i \pmod{p}$. In view of (1), this safety will follow from the condition that $p \nmid \gcd\big(b_p^{(i)} + (j-i), c^{(j)} - (j-i)\big)$ for all $j \in \mathbb{Z}_p$ and all $c^{(j)} \in B_p^{(j)}$. In turn, this condition can be rewritten as

$$-b_p^{(i)} \notin B_p^{(j)}, \qquad \text{where} \quad j \equiv i - b_p^{(i)} \pmod{p}. \tag{2}$$

The construction in Solution 2 is equivalent to setting $b_p^{(i)} = -a_p$ and $B_p^{(i)} = \mathbb{Z}_p \setminus \{a_p\}$ for all i. However, there are different, more technical specifications of our approach.

One may view the (incomplete) construction in Comment 2 as defining $B_p^{(i)}$ and $b_p^{(i)}$ at step $p-1$ by setting $B_p^{(0)} = \{b_p^{(0)}\} = \{0\}$ and $B_p^{(i)} = \{b_p^{(i)}\} = \{f(i) + i \mod p\}$ for every $i = 1, 2, \ldots, p-1$. However, this construction violates (2) as soon as some number of the form $f(i) + i$ is divisible by some p with $i + 2 \leqslant p \in \mathcal{P}$, since then $-b_p^{(i)} = b_p^{(i)} \in B_p^{(i)}$.

Here is one possible way to repair this construction. For all $p \in \mathcal{P}$, we define the sets $B_p^{(i)}$ and the elements $b_p^{(i)}$ at step $(p-2)$ as follows. Set $B_p^{(1)} = \{b_p^{(1)}\} = \{2\}$ and $B_p^{(-1)} = B_p^{(0)} = \{b_p^{(-1)}\} = \{b_p^{(0)}\} = \{-1\}$. Next, for all $i = 2, \ldots, p-2$, define $B_p^{(i)} = \{i, f(i) + i \mod p\}$ and $b_p^{(i)} = i$. One may see that these definitions agree with both (1) and (2).

N8. For every positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, define

$$\mho(n) = \sum_{i\colon p_i > 10^{100}} \alpha_i.$$

That is, $\mho(n)$ is the number of prime factors of n greater than 10^{100}, counted with multiplicity.

Find all strictly increasing functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$\mho\big(f(a) - f(b)\big) \leqslant \mho(a - b) \qquad \text{for all integers } a \text{ and } b \text{ with } a > b. \tag{1}$$

(Brazil)

Answer. $f(x) = ax + b$, where b is an arbitrary integer, and a is an arbitrary positive integer with $\mho(a) = 0$.

Solution. A straightforward check shows that all the functions listed in the answer satisfy the problem condition. It remains to show the converse.

Assume that f is a function satisfying the problem condition. Notice that the function $g(x) = f(x) - f(0)$ also satisfies this condition. Replacing f by g, we assume from now on that $f(0) = 0$; then $f(n) > 0$ for any positive integer n. Thus, we aim to prove that there exists a positive integer a with $\mho(a) = 0$ such that $f(n) = an$ for all $n \in \mathbb{Z}$.

We start by introducing some notation. Set $N = 10^{100}$. We say that a prime p is *large* if $p > N$, and p is *small* otherwise; let $\mathcal{S}$ be the set of all small primes. Next, we say that a positive integer is *large* or *small* if all its prime factors are such (thus, the number 1 is the unique number which is both large and small). For a positive integer k, we denote the greatest large divisor of k and the greatest small divisor of k by $L(k)$ and $S(k)$, respectively; thus, $k = L(k)S(k)$.

We split the proof into three steps.

Step 1. *We prove that for every large k, we have $k \mid f(a) - f(b) \iff k \mid a - b$. In other words, $L\big(f(a) - f(b)\big) = L(a - b)$ for all integers a and b with $a > b$.*

We use induction on k. The base case $k = 1$ is trivial. For the induction step, assume that k_0 is a large number, and that the statement holds for all large numbers k with $k < k_0$.

Claim 1. For any integers x and y with $0 < x - y < k_0$, the number k_0 does not divide $f(x) - f(y)$.

Proof. Assume, to the contrary, that $k_0 \mid f(x) - f(y)$. Let $\ell = L(x - y)$; then $\ell \leqslant x - y < k_0$. By the induction hypothesis, $\ell \mid f(x) - f(y)$, and thus $\operatorname{lcm}(k_0, \ell) \mid f(x) - f(y)$. Notice that $\operatorname{lcm}(k_0, \ell)$ is large, and $\operatorname{lcm}(k_0, \ell) \geqslant k_0 > \ell$. But then

$$\mho\big(f(x) - f(y)\big) \geqslant \mho\big(\operatorname{lcm}(k_0, \ell)\big) > \mho(\ell) = \mho(x - y),$$

which is impossible. $\square$

Now we complete the induction step. By Claim 1, for every integer a each of the sequences

$$f(a), f(a+1), \ldots, f(a + k_0 - 1) \quad \text{and} \quad f(a+1), f(a+2), \ldots, f(a + k_0)$$

forms a complete residue system modulo k_0. This yields $f(a) \equiv f(a + k_0) \pmod{k_0}$. Thus, $f(a) \equiv f(b) \pmod{k_0}$ whenever $a \equiv b \pmod{k_0}$.

Finally, if $a \not\equiv b \pmod{k_0}$ then there exists an integer b' such that $b' \equiv b \pmod{k_0}$ and $|a - b'| < k_0$. Then $f(b) \equiv f(b') \not\equiv f(a) \pmod{k_0}$. The induction step is proved.

Step 2. *We prove that for some small integer a there exist infinitely many integers n such that $f(n) = an$. In other words, f is linear on some infinite set.*

We start with the following general statement.

Claim 2. There exists a constant c such that $f(t) < ct$ for every positive integer $t > N$.

Proof. Let d be the product of all small primes, and let α be a positive integer such that $2^\alpha > f(N)$. Then, for every $p \in \mathcal{S}$ the numbers $f(0), f(1), \ldots, f(N)$ are distinct modulo p^α. Set $P = d^\alpha$ and $c = P + f(N)$.

Choose any integer $t > N$. Due to the choice of α, for every $p \in \mathcal{S}$ there exists at most one nonnegative integer $i \leqslant N$ with $p^\alpha \mid f(t) - f(i)$. Since $|\mathcal{S}| < N$, we can choose a nonnegative integer $j \leqslant N$ such that $p^\alpha \nmid f(t) - f(j)$ for all $p \in \mathcal{S}$. Therefore, $S\big(f(t) - f(j)\big) < P$.

On the other hand, Step 1 shows that $L\big(f(t) - f(j)\big) = L(t - j) \leqslant t - j$. Since $0 \leqslant j \leqslant N$, this yields

$$f(t) = f(j) + L\big(f(t) - f(j)\big) \cdot S\big(f(t) - f(j)\big) < f(N) + (t - j)P \leqslant \big(P + f(N)\big)t = ct. \quad \square$$

Now let $\mathcal{T}$ be the set of large primes. For every $t \in \mathcal{T}$, Step 1 implies $L\big(f(t)\big) = t$, so the ratio $f(t)/t$ is an integer. Now Claim 2 leaves us with only finitely many choices for this ratio, which means that there exists an infinite subset $\mathcal{T}' \subseteq \mathcal{T}$ and a positive integer a such that $f(t) = at$ for all $t \in \mathcal{T}'$, as required.

Since $L(t) = L\big(f(t)\big) = L(a)L(t)$ for all $t \in \mathcal{T}'$, we get $L(a) = 1$, so the number a is small.

Step 3. *We show that $f(x) = ax$ for all $x \in \mathbb{Z}$.*

Let $R_i = \big\{x \in \mathbb{Z} \colon x \equiv i \pmod{N!}\big\}$ denote the residue class of i modulo $N!$.

Claim 3. Assume that for some r, there are infinitely many $n \in R_r$ such that $f(n) = an$. Then $f(x) = ax$ for all $x \in R_{r+1}$.

Proof. Choose any $x \in R_{r+1}$. By our assumption, we can select $n \in R_r$ such that $f(n) = an$ and $|n - x| > \big|f(x) - ax\big|$. Since $n - x \equiv r - (r + 1) = -1 \pmod{N!}$, the number $|n - x|$ is large. Therefore, by Step 1 we have $f(x) \equiv f(n) = an \equiv ax \pmod{n - x}$, so $n - x \mid f(x) - ax$. Due to the choice of n, this yields $f(x) = ax$. $\square$

To complete Step 3, notice that the set $\mathcal{T}'$ found in Step 2 contains infinitely many elements of some residue class R_i. Applying Claim 3, we successively obtain that $f(x) = ax$ for all $x \in R_{i+1}, R_{i+2}, \ldots, R_{i+N!} = R_i$. This finishes the solution.

Comment 1. As the proposer also mentions, one may also consider the version of the problem where the condition (1) is replaced by the condition that $L\big(f(a) - f(b)\big) = L(a - b)$ for all integers a and b with $a > b$. This allows to remove of Step 1 from the solution.

Comment 2. Step 2 is the main step of the solution. We sketch several different approaches allowing to perform this step using statements which are weaker than Claim 2.

Approach 1. Let us again denote the product of all small primes by d. We focus on the values $f(d^i)$, $i \geqslant 0$. In view of Step 1, we have $L\big(f(d^i) - f(d^k)\big) = L(d^i - d^k) = d^{i-k} - 1$ for all $i > k \geqslant 0$.

Acting similarly to the beginning of the proof of Claim 2, one may choose a number $\alpha \geqslant 0$ such that the residues of the numbers $f(d^i)$, $i = 0, 1, \ldots, N$, are distinct modulo p^α for each $p \in \mathcal{S}$. Then, for every $i > N$, there exists an exponent $k = k(i) \leqslant N$ such that $S\big(f(d^i) - f(d^k)\big) < P = d^\alpha$.

Since there are only finitely many options for $k(i)$, as well as for the corresponding numbers $S\big(f(d^i) - f(d^k)\big)$, there exists an infinite set I of exponents $i > N$ such that $k(i)$ attains the same value k_0 for all $i \in I$, and such that, moreover, $S\big(f(d^i) - f(d^{k_0})\big)$ attains the same value s_0 for all $i \in I$. Therefore, for all such i we have

$$f(d^i) = f(d^{k_0}) + L\big(f(d^i) - f(d^{k_0})\big) \cdot S\big(f(d^i) - f(d^{k_0})\big) = f(d^{k_0}) + \big(d^{i-k_0} - 1\big)s_0,$$

which means that f is linear on the infinite set $\{d^i : i \in I\}$ (although with rational coefficients).

Finally, one may implement the relation $f(d^i) \equiv f(1) \pmod{d^i - 1}$ in order to establish that in fact $f(d^i)/d^i$ is a (small and fixed) integer for all $i \in I$.

Approach 2. Alternatively, one may start with the following lemma.

Lemma. There exists a positive constant c such that

$$L\left(\prod_{i=1}^{3N}\big(f(k)-f(i)\big)\right)=\prod_{i=1}^{3N}L\big(f(k)-f(i)\big)\geqslant c\big(f(k)\big)^{2N}$$

for all $k>3N$.

Proof. Let k be an integer with $k>3N$. Set $\Pi=\prod_{i=1}^{3N}\big(f(k)-f(i)\big)$.

Notice that for every prime $p\in\mathcal{S}$, at most one of the numbers in the set

$$\mathcal{H}=\big\{f(k)-f(i)\colon 1\leqslant i\leqslant 3N\big\}$$

is divisible by a power of p which is greater than $f(3N)$; we say that such elements of $\mathcal{H}$ are *bad*. Now, for each element $h\in\mathcal{H}$ which is not bad we have $S(h)\leqslant f(3N)^N$, while the bad elements do not exceed $f(k)$. Moreover, there are less than N bad elements in $\mathcal{H}$. Therefore,

$$S(\Pi)=\prod_{h\in\mathcal{H}}S(h)\leqslant\big(f(3N)\big)^{3N^2}\cdot\big(f(k)\big)^N.$$

This easily yields the lemma statement in view of the fact that $L(\Pi)S(\Pi)=\Pi\geqslant\mu\big(f(k)\big)^{3N}$ for some absolute constant μ. □

As a corollary of the lemma, one may get a weaker version of Claim 2 stating that there exists a positive constant C such that $f(k)\leqslant Ck^{3/2}$ for all $k>3N$. Indeed, from Step 1 we have

$$k^{3N}\geqslant\prod_{i=1}^{3N}L(k-i)=\prod_{i=1}^{3N}L\big(f(k)-f(i)\big)\geqslant c\big(f(k)\big)^{2N},$$

so $f(k)\leqslant c^{-1/(2N)}k^{3/2}$.

To complete Step 2 now, set $a=f(1)$. Due to the estimates above, we may choose a positive integer n_0 such that $\big|f(n)-an\big|<\frac{n(n-1)}{2}$ for all $n\geqslant n_0$.

Take any $n\geqslant n_0$ with $n\equiv 2\pmod{N!}$. Then $L\big(f(n)-f(0)\big)=L(n)=n/2$ and $L\big(f(n)-f(1)\big)=L(n-1)=n-1$; these relations yield $f(n)\equiv f(0)=0\equiv an\pmod{n/2}$ and $f(n)\equiv f(1)=a\equiv an\pmod{n-1}$, respectively. Thus, $\frac{n(n-1)}{2}\mid f(n)-an$, which shows that $f(n)=an$ in view of the estimate above.

Comment 3. In order to perform Step 3, it suffices to establish the equality $f(n)=an$ for any infinite set of values of n. However, if this set has some good structure, then one may find easier ways to complete this step.

For instance, after showing, as in Approach 2, that $f(n)=an$ for all $n\geqslant n_0$ with $n\equiv 2\pmod{N!}$, one may proceed as follows. Pick an arbitrary integer x and take any large prime p which is greater than $|f(x)-ax|$. By the Chinese Remainder Theorem, there exists a positive integer $n>\max(x,n_0)$ such that $n\equiv 2\pmod{N!}$ and $n\equiv x\pmod{p}$. By Step 1, we have $f(x)\equiv f(n)=an\equiv ax\pmod{p}$. Due to the choice of p, this is possible only if $f(x)=ax$.

CHIANG MAI, THAILAND

4 - 16 JULY 2015

Shortlisted Problems with Solutions

57^{th} International Mathematical Olympiad
Hong Kong, 2016

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2017.

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2016 thank the following 40 countries for contributing 121 problem proposals:

Albania, Algeria, Armenia, Australia, Austria, Belarus, Belgium, Bulgaria, Colombia, Cyprus, Czech Republic, Denmark, Estonia, France, Georgia, Greece, Iceland, India, Iran, Ireland, Israel, Japan, Latvia, Luxembourg, Malaysia, Mexico, Mongolia, Netherlands, Philippines, Russia, Serbia, Slovakia, Slovenia, South Africa, Taiwan, Tanzania, Thailand, Trinidad and Tobago, Turkey, Ukraine.

Problem Selection Committee

Front row from left: Yong-Gao Chen, Andy Liu, Tat Wing Leung (Chairman).
Back row from left: Yi-Jun Yao, Yun-Hao Fu, Yi-Jie He, Zhongtao Wu, Heung Wing Joseph Lee, Chi Hong Chow, Ka Ho Law, Tak Wing Ching.

Problems

Algebra

A1. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geqslant 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leqslant \left(\frac{a+b+c}{3}\right)^2 + 1.$$

A2. Find the smallest real constant C such that for any positive real numbers a_1, a_2, a_3, a_4 and a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k and l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leqslant C.$$

A3. Find all integers $n \geqslant 3$ with the following property: for all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ satisfying $|a_k| + |b_k| = 1$ for $1 \leqslant k \leqslant n$, there exist $x_1, x_2, \ldots, x_n$, each of which is either -1 or 1, such that

$$\left|\sum_{k=1}^{n} x_k a_k\right| + \left|\sum_{k=1}^{n} x_k b_k\right| \leqslant 1.$$

A4. Denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$x f(x^2) f(f(y)) + f(y f(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right)$$

for all positive real numbers x and y.

A5.

(a) Prove that for every positive integer n, there exists a fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n} + 1$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

(b) Prove that there are infinitely many positive integers n such that there is no fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n}$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

A6. The equation

$$(x-1)(x-2)\cdots(x-2016)=(x-1)(x-2)\cdots(x-2016)$$

is written on the board. One tries to erase some linear factors from both sides so that each side still has at least one factor, and the resulting equation has no real roots. Find the least number of linear factors one needs to erase to achieve this.

A7. Denote by $\mathbb{R}$ the set of all real numbers. Find all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that $f(0)\neq 0$ and

$$f(x+y)^2=2f(x)f(y)+\max\left\{f(x^2)+f(y^2),f(x^2+y^2)\right\}$$

for all real numbers x and y.

A8. Determine the largest real number a such that for all $n\geqslant 1$ and for all real numbers $x_0,x_1,\ldots,x_n$ satisfying $0=x_0<x_1<x_2<\cdots<x_n$, we have

$$\frac{1}{x_1-x_0}+\frac{1}{x_2-x_1}+\cdots+\frac{1}{x_n-x_{n-1}}\geqslant a\left(\frac{2}{x_1}+\frac{3}{x_2}+\cdots+\frac{n+1}{x_n}\right).$$

Combinatorics

C1. The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down 001, 111 and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

C2. Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:

- each cell contains a distinct divisor;
- the sums of all rows are equal; and
- the sums of all columns are equal.

C3. Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

C4. Find all positive integers n for which we can fill in the entries of an $n \times n$ table with the following properties:

- each entry can be one of I, M and O;
- in each row and each column, the letters I, M and O occur the same number of times; and
- in any diagonal whose number of entries is a multiple of three, the letters I, M and O occur the same number of times.

C5. Let $n \geqslant 3$ be a positive integer. Find the maximum number of diagonals of a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

C6. There are $n \geqslant 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

After each year, the ferry company will close a ferry route between some two islands X and Y. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected by a ferry route to exactly one of X and Y, a new route between this island and the other of X and Y is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

C7. Let $n \geqslant 2$ be an integer. In the plane, there are n segments given in such a way that any two segments have an intersection point in the interior, and no three segments intersect at a single point. Jeff places a snail at one of the endpoints of each of the segments and claps his hands $n-1$ times. Each time when he claps his hands, all the snails move along their own segments and stay at the next intersection points until the next clap. Since there are $n-1$ intersection points on each segment, all snails will reach the furthest intersection points from their starting points after $n-1$ claps.

(a) Prove that if n is odd then Jeff can always place the snails so that no two of them ever occupy the same intersection point.

(b) Prove that if n is even then there must be a moment when some two snails occupy the same intersection point no matter how Jeff places the snails.

C8. Let n be a positive integer. Determine the smallest positive integer k with the following property: it is possible to mark k cells on a $2n \times 2n$ board so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contains two marked cells.

Geometry

G1. In a convex pentagon $ABCDE$, let F be a point on AC such that $\angle FBC = 90^\circ$. Suppose triangles ABF, ACD and ADE are similar isosceles triangles with

$$\angle FAB = \angle FBA = \angle DAC = \angle DCA = \angle EAD = \angle EDA.$$

Let M be the midpoint of CF. Point X is chosen such that $AMXE$ is a parallelogram. Show that BD, EM and FX are concurrent.

G2. Let ABC be a triangle with circumcircle Γ and incentre I. Let M be the midpoint of side BC. Denote by D the foot of perpendicular from I to side BC. The line through I perpendicular to AI meets sides AB and AC at F and E respectively. Suppose the circumcircle of triangle AEF intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

G3. Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be *nice* if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geqslant 0, y \geqslant 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

G4. Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

G5. Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.

G6. Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

G7. Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120^\circ$.

G8. Let A_1, B_1 and C_1 be points on sides BC, CA and AB of an acute triangle ABC respectively, such that AA_1, BB_1 and CC_1 are the internal angle bisectors of triangle ABC. Let I be the incentre of triangle ABC, and H be the orthocentre of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geqslant AI + BI + CI.$$

Number Theory

N1. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geqslant 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

N2. Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all possible integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

N3. Define $P(n) = n^2 + n + 1$. For any positive integers a and b, the set

$$\{P(a), P(a+1), P(a+2), \ldots, P(a+b)\}$$

is said to be *fragrant* if none of its elements is relatively prime to the product of the other elements. Determine the smallest size of a fragrant set.

N4. Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

- $m = 1$ and $l = 2k$; or
- $l | k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

N5. Let a be a positive integer which is not a square number. Denote by A the set of all positive integers k such that

$$k = \frac{x^2 - a}{x^2 - y^2} \tag{1}$$

for some integers x and y with $x > \sqrt{a}$. Denote by B the set of all positive integers k such that (1) is satisfied for some integers x and y with $0 \leqslant x < \sqrt{a}$. Prove that $A = B$.

N6. Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

N7. Let n be an odd positive integer. In the Cartesian plane, a cyclic polygon P with area S is chosen. All its vertices have integral coordinates, and all squares of its side lengths are divisible by n. Prove that $2S$ is an integer divisible by n.

N8. Find all polynomials $P(x)$ of odd degree d and with integer coefficients satisfying the following property: for each positive integer n, there exist n positive integers $x_1, x_2, \ldots, x_n$ such that $\frac{1}{2} < \frac{P(x_i)}{P(x_j)} < 2$ and $\frac{P(x_i)}{P(x_j)}$ is the d-th power of a rational number for every pair of indices i and j with $1 \leqslant i, j \leqslant n$.

Solutions

Algebra

A1. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geqslant 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leqslant \left(\frac{a+b+c}{3}\right)^2 + 1. \tag{1}$$

Solution 1. We first show the following.

- **Claim.** For any positive real numbers x, y with $xy \geqslant 1$, we have

$$(x^2+1)(y^2+1) \leqslant \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2. \tag{2}$$

Proof. Note that $xy \geqslant 1$ implies $(\frac{x+y}{2})^2 - 1 \geqslant xy - 1 \geqslant 0$. We find that

$$(x^2+1)(y^2+1) = (xy-1)^2 + (x+y)^2 \leqslant \left(\left(\frac{x+y}{2}\right)^2 - 1\right)^2 + (x+y)^2 = \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2.$$

$\square$

Without loss of generality, assume $a \geqslant b \geqslant c$. This implies $a \geqslant 1$. Let $d = \frac{a+b+c}{3}$. Note that

$$ad = \frac{a(a+b+c)}{3} \geqslant \frac{1+1+1}{3} = 1.$$

Then we can apply (2) to the pair (a, d) and the pair (b, c). We get

$$(a^2+1)(d^2+1)(b^2+1)(c^2+1) \leqslant \left(\left(\frac{a+d}{2}\right)^2 + 1\right)^2 \left(\left(\frac{b+c}{2}\right)^2 + 1\right)^2. \tag{3}$$

Next, from

$$\frac{a+d}{2} \cdot \frac{b+c}{2} \geqslant \sqrt{ad} \cdot \sqrt{bc} \geqslant 1,$$

we can apply (2) again to the pair $(\frac{a+d}{2}, \frac{b+c}{2})$. Together with (3), we have

$$(a^2+1)(d^2+1)(b^2+1)(c^2+1) \leqslant \left(\left(\frac{a+b+c+d}{4}\right)^2 + 1\right)^4 = (d^2+1)^4.$$

Therefore, $(a^2+1)(b^2+1)(c^2+1) \leqslant (d^2+1)^3$, and (1) follows by taking cube root of both sides.

Comment. After justifying the Claim, one may also obtain (1) by mixing variables. Indeed, the function involved is clearly continuous, and hence it suffices to check that the condition $xy \geqslant 1$ is preserved under each mixing step. This is true since whenever $ab, bc, ca \geqslant 1$, we have

$$\frac{a+b}{2} \cdot \frac{a+b}{2} \geqslant ab \geqslant 1 \quad \text{and} \quad \frac{a+b}{2} \cdot c \geqslant \frac{1+1}{2} = 1.$$

Solution 2. Let $f(x) = \ln(1+x^2)$. Then the inequality (1) to be shown is equivalent to

$$\frac{f(a)+f(b)+f(c)}{3} \leqslant f\left(\frac{a+b+c}{3}\right),$$

while (2) becomes

$$\frac{f(x)+f(y)}{2} \leqslant f\left(\frac{x+y}{2}\right)$$

for $xy \geqslant 1$.

Without loss of generality, assume $a \geqslant b \geqslant c$. From the Claim in Solution 1, we have

$$\frac{f(a)+f(b)+f(c)}{3} \leqslant \frac{f(a)+2f(\frac{b+c}{2})}{3}.$$

Note that $a \geqslant 1$ and $\frac{b+c}{2} \geqslant \sqrt{bc} \geqslant 1$. Since

$$f''(x) = \frac{2(1-x^2)}{(1+x^2)^2},$$

we know that f is concave on $[1, \infty)$. Then we can apply Jensen's Theorem to get

$$\frac{f(a)+2f(\frac{b+c}{2})}{3} \leqslant f\left(\frac{a+2\cdot\frac{b+c}{2}}{3}\right) = f\left(\frac{a+b+c}{3}\right).$$

This completes the proof.

A2. Find the smallest real constant C such that for any positive real numbers a_1, a_2, a_3, a_4 and a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k and l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leqslant C. \tag{1}$$

Answer. The smallest C is $\frac{1}{2}$.

Solution. We first show that $C \leqslant \frac{1}{2}$. For any positive real numbers $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4 \leqslant a_5$, consider the five fractions

$$\frac{a_1}{a_2}, \frac{a_3}{a_4}, \frac{a_1}{a_5}, \frac{a_2}{a_3}, \frac{a_4}{a_5}. \tag{2}$$

Each of them lies in the interval $(0, 1]$. Therefore, by the Pigeonhole Principle, at least three of them must lie in $(0, \frac{1}{2}]$ or lie in $(\frac{1}{2}, 1]$ simultaneously. In particular, there must be two consecutive terms in (2) which belong to an interval of length $\frac{1}{2}$ (here, we regard $\frac{a_1}{a_2}$ and $\frac{a_4}{a_5}$ as consecutive). In other words, the difference of these two fractions is less than $\frac{1}{2}$. As the indices involved in these two fractions are distinct, we can choose them to be i, j, k, l and conclude that $C \leqslant \frac{1}{2}$.

Next, we show that $C = \frac{1}{2}$ is best possible. Consider the numbers $1, 2, 2, 2, n$ where n is a large real number. The fractions formed by two of these numbers in ascending order are $\frac{1}{n}, \frac{2}{n}, \frac{1}{2}, \frac{2}{2}, \frac{2}{1}, \frac{n}{2}, \frac{n}{1}$. Since the indices i, j, k, l are distinct, $\frac{1}{n}$ and $\frac{2}{n}$ cannot be chosen simultaneously. Therefore the minimum value of the left-hand side of (1) is $\frac{1}{2} - \frac{2}{n}$. When n tends to infinity, this value approaches $\frac{1}{2}$, and so C cannot be less than $\frac{1}{2}$.

These conclude that $C = \frac{1}{2}$ is the smallest possible choice.

Comment. The conclusion still holds if $a_1, a_2, \ldots, a_5$ are pairwise distinct, since in the construction, we may replace the 2's by real numbers sufficiently close to 2.

There are two possible simplifications for this problem:

(i) the answer $C = \frac{1}{2}$ is given to the contestants; or

(ii) simply ask the contestants to prove the inequality (1) for $C = \frac{1}{2}$.

A3. Find all integers $n \geqslant 3$ with the following property: for all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ satisfying $|a_k| + |b_k| = 1$ for $1 \leqslant k \leqslant n$, there exist $x_1, x_2, \ldots, x_n$, each of which is either -1 or 1, such that

$$\left|\sum_{k=1}^{n} x_k a_k\right| + \left|\sum_{k=1}^{n} x_k b_k\right| \leqslant 1. \tag{1}$$

Answer. n can be any odd integer greater than or equal to 3.

Solution 1. For any even integer $n \geqslant 4$, we consider the case

$$a_1 = a_2 = \cdots = a_{n-1} = b_n = 0 \quad \text{and} \quad b_1 = b_2 = \cdots = b_{n-1} = a_n = 1.$$

The condition $|a_k| + |b_k| = 1$ is satisfied for each $1 \leqslant k \leqslant n$. No matter how we choose each x_k, both sums $\sum_{k=1}^{n} x_k a_k$ and $\sum_{k=1}^{n} x_k b_k$ are odd integers. This implies $|\sum_{k=1}^{n} x_k a_k| \geqslant 1$ and $|\sum_{k=1}^{n} x_k b_k| \geqslant 1$, which shows (1) cannot hold.

For any odd integer $n \geqslant 3$, we may assume without loss of generality $b_k \geqslant 0$ for $1 \leqslant k \leqslant n$ (this can be done by flipping the pair (a_k, b_k) to $(-a_k, -b_k)$ and x_k to $-x_k$ if necessary) and $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m \geqslant 0 > a_{m+1} \geqslant \cdots \geqslant a_n$. We claim that the choice $x_k = (-1)^{k+1}$ for $1 \leqslant k \leqslant n$ will work. Define

$$s = \sum_{k=1}^{m} x_k a_k \quad \text{and} \quad t = -\sum_{k=m+1}^{n} x_k a_k.$$

Note that

$$s = (a_1 - a_2) + (a_3 - a_4) + \cdots \geqslant 0$$

by the assumption $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m$ (when m is odd, there is a single term a_m at the end, which is also positive). Next, we have

$$s = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots \leqslant a_1 \leqslant 1.$$

Similarly,

$$t = (-a_n + a_{n-1}) + (-a_{n-2} + a_{n-3}) + \cdots \geqslant 0$$

and

$$t = -a_n + (a_{n-1} - a_{n-2}) + (a_{n-3} - a_{n-4}) + \cdots \leqslant -a_n \leqslant 1.$$

From the condition, we have $a_k + b_k = 1$ for $1 \leqslant k \leqslant m$ and $-a_k + b_k = 1$ for $m+1 \leqslant k \leqslant n$. It follows that $\sum_{k=1}^{n} x_k a_k = s - t$ and $\sum_{k=1}^{n} x_k b_k = 1 - s - t$. Hence it remains to prove

$$|s - t| + |1 - s - t| \leqslant 1$$

under the constraint $0 \leqslant s, t \leqslant 1$. By symmetry, we may assume $s \geqslant t$. If $1 - s - t \geqslant 0$, then we have

$$|s - t| + |1 - s - t| = s - t + 1 - s - t = 1 - 2t \leqslant 1.$$

If $1 - s - t \leqslant 0$, then we have

$$|s - t| + |1 - s - t| = s - t - 1 + s + t = 2s - 1 \leqslant 1.$$

Hence, the inequality is true in both cases.

These show n can be any odd integer greater than or equal to 3.

Solution 2. The even case can be handled in the same way as Solution 1. For the odd case, we prove by induction on n.

Firstly, for $n = 3$, we may assume without loss of generality $a_1 \geqslant a_2 \geqslant a_3 \geqslant 0$ and $b_1 = a_1 - 1$ (if $b_1 = 1 - a_1$, we may replace each b_k by $-b_k$).

- **Case 1.** $b_2 = a_2 - 1$ and $b_3 = a_3 - 1$, in which case we take $(x_1, x_2, x_3) = (1, -1, 1)$.

Let $c = a_1 - a_2 + a_3$ so that $0 \leqslant c \leqslant 1$. Then $|b_1 - b_2 + b_3| = |a_1 - a_2 + a_3 - 1| = 1 - c$ and hence $|c| + |b_1 - b_2 + b_3| = 1$.

- **Case 2.** $b_2 = 1 - a_2$ and $b_3 = 1 - a_3$, in which case we take $(x_1, x_2, x_3) = (1, -1, 1)$.

Let $c = a_1 - a_2 + a_3$ so that $0 \leqslant c \leqslant 1$. Since $a_3 \leqslant a_2$ and $a_1 \leqslant 1$, we have

$$c - 1 \leqslant b_1 - b_2 + b_3 = a_1 + a_2 - a_3 - 1 \leqslant 1 - c.$$

This gives $|b_1 - b_2 + b_3| \leqslant 1 - c$ and hence $|c| + |b_1 - b_2 + b_3| \leqslant 1$.

- **Case 3.** $b_2 = a_2 - 1$ and $b_3 = 1 - a_3$, in which case we take $(x_1, x_2, x_3) = (-1, 1, 1)$.

Let $c = -a_1 + a_2 + a_3$. If $c \geqslant 0$, then $a_3 \leqslant 1$ and $a_2 \leqslant a_1$ imply

$$c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 + a_2 - a_3 + 1 \leqslant 1 - c.$$

If $c < 0$, then $a_1 \leqslant a_2 + 1$ and $a_3 \geqslant 0$ imply

$$-c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 + a_2 - a_3 + 1 \leqslant 1 + c.$$

In both cases, we get $|-b_1 + b_2 + b_3| \leqslant 1 - |c|$ and hence $|c| + |-b_1 + b_2 + b_3| \leqslant 1$.

- **Case 4.** $b_2 = 1 - a_2$ and $b_3 = a_3 - 1$, in which case we take $(x_1, x_2, x_3) = (-1, 1, 1)$.

Let $c = -a_1 + a_2 + a_3$. If $c \geqslant 0$, then $a_2 \leqslant 1$ and $a_3 \leqslant a_1$ imply

$$c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 - a_2 + a_3 + 1 \leqslant 1 - c.$$

If $c < 0$, then $a_1 \leqslant a_3 + 1$ and $a_2 \geqslant 0$ imply

$$-c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 - a_2 + a_3 + 1 \leqslant 1 + c.$$

In both cases, we get $|-b_1 + b_2 + b_3| \leqslant 1 - |c|$ and hence $|c| + |-b_1 + b_2 + b_3| \leqslant 1$.

We have found x_1, x_2, x_3 satisfying (1) in each case for $n = 3$.

Now, let $n \geqslant 5$ be odd and suppose the result holds for any smaller odd cases. Again we may assume $a_k \geqslant 0$ for each $1 \leqslant k \leqslant n$. By the Pigeonhole Principle, there are at least three indices k for which $b_k = a_k - 1$ or $b_k = 1 - a_k$. Without loss of generality, suppose $b_k = a_k - 1$ for $k = 1, 2, 3$. Again by the Pigeonhole Principle, as a_1, a_2, a_3 lies between 0 and 1, the difference of two of them is at most $\frac{1}{2}$. By changing indices if necessary, we may assume $0 \leqslant d = a_1 - a_2 \leqslant \frac{1}{2}$.

By the inductive hypothesis, we can choose $x_3, x_4, \ldots, x_n$ such that $a' = \sum_{k=3}^{n} x_k a_k$ and $b' = \sum_{k=3}^{n} x_k b_k$ satisfy $|a'| + |b'| \leqslant 1$. We may further assume $a' \geqslant 0$.

- **Case 1.** $b' \geqslant 0$, in which case we take $(x_1, x_2) = (-1, 1)$.

  We have $|-a_1 + a_2 + a'| + |-(a_1 - 1) + (a_2 - 1) + b'| = |-d + a'| + |-d + b'| \leqslant \max\{a' + b' - 2d, a' - b', b' - a', 2d - a' - b'\} \leqslant 1$ since $0 \leqslant a', b', a' + b' \leqslant 1$ and $0 \leqslant d \leqslant \frac{1}{2}$.

- **Case 2.** $0 > b' \geqslant -a'$, in which case we take $(x_1, x_2) = (-1, 1)$.

  We have $|-a_1 + a_2 + a'| + |-(a_1 - 1) + (a_2 - 1) + b'| = |-d + a'| + |-d + b'|$. If $-d + a' \geqslant 0$, this equals $a' - b' = |a'| + |b'| \leqslant 1$. If $-d + a' < 0$, this equals $2d - a' - b' \leqslant 2d \leqslant 1$.

- **Case 3.** $b' < -a'$, in which case we take $(x_1, x_2) = (1, -1)$.

  We have $|a_1 - a_2 + a'| + |(a_1 - 1) - (a_2 - 1) + b'| = |d + a'| + |d + b'|$. If $d + b' \geqslant 0$, this equals $2d + a' + b' < 2d \leqslant 1$. If $d + b' < 0$, this equals $a' - b' = |a'| + |b'| \leqslant 1$.

Therefore, we have found $x_1, x_2, \ldots, x_n$ satisfying (1) in each case. By induction, the property holds for all odd integers $n \geqslant 3$.

A4. Denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right) \tag{1}$$

for all positive real numbers x and y.

Answer. $f(x) = \frac{1}{x}$ for any $x \in \mathbb{R}^+$.

Solution 1. Taking $x = y = 1$ in (1), we get $f(1)f(f(1)) + f(f(1)) = 2f(1)f(f(1))$ and hence $f(1) = 1$. Swapping x and y in (1) and comparing with (1) again, we find

$$xf(x^2)f(f(y)) + f(yf(x)) = yf(y^2)f(f(x)) + f(xf(y)). \tag{2}$$

Taking $y = 1$ in (2), we have $xf(x^2) + f(f(x)) = f(f(x)) + f(x)$, that is,

$$f(x^2) = \frac{f(x)}{x}. \tag{3}$$

Take $y = 1$ in (1) and apply (3) to $xf(x^2)$. We get $f(x) + f(f(x)) = f(x)(f(f(x^2)) + 1)$, which implies

$$f(f(x^2)) = \frac{f(f(x))}{f(x)}. \tag{4}$$

For any $x \in \mathbb{R}^+$, we find that

$$f(f(x)^2) \overset{(3)}{=} \frac{f(f(x))}{f(x)} \overset{(4)}{=} f(f(x^2)) \overset{(3)}{=} f\left(\frac{f(x)}{x}\right). \tag{5}$$

It remains to show the following key step.

• **Claim.** The function f is injective.

Proof. Using (3) and (4), we rewrite (1) as

$$f(x)f(f(y)) + f(yf(x)) = f(xy)\left(\frac{f(f(x))}{f(x)} + \frac{f(f(y))}{f(y)}\right). \tag{6}$$

Take $x = y$ in (6) and apply (3). This gives $f(x)f(f(x)) + f(xf(x)) = 2\frac{f(f(x))}{x}$, which means

$$f(xf(x)) = f(f(x))\left(\frac{2}{x} - f(x)\right). \tag{7}$$

Using (3), equation (2) can be rewritten as

$$f(x)f(f(y)) + f(yf(x)) = f(y)f(f(x)) + f(xf(y)). \tag{8}$$

Suppose $f(x) = f(y)$ for some $x, y \in \mathbb{R}^+$. Then (8) implies

$$f(yf(y)) = f(yf(x)) = f(xf(y)) = f(xf(x)).$$

Using (7), this gives

$$f(f(y))\left(\frac{2}{y} - f(y)\right) = f(f(x))\left(\frac{2}{x} - f(x)\right).$$

Noting $f(x) = f(y)$, we find $x = y$. This establishes the injectivity. $\square$

By the Claim and (5), we get the only possible solution $f(x)=\frac{1}{x}$. It suffices to check that this is a solution. Indeed, the left-hand side of (1) becomes

$$x \cdot \frac{1}{x^2} \cdot y + \frac{x}{y} = \frac{y}{x} + \frac{x}{y},$$

while the right-hand side becomes

$$\frac{1}{xy}(x^2+y^2) = \frac{x}{y} + \frac{y}{x}.$$

The two sides agree with each other.

Solution 2. Taking $x=y=1$ in (1), we get $f(1)f(f(1))+f(f(1))=2f(1)f(f(1))$ and hence $f(1)=1$. Putting $x=1$ in (1), we have $f(f(y))+f(y)=f(y)(1+f(f(y^2)))$ so that

$$f(f(y)) = f(y)f(f(y^2)). \tag{9}$$

Putting $y=1$ in (1), we get $xf(x^2)+f(f(x))=f(x)(f(f(x^2))+1)$. Using (9), this gives

$$xf(x^2)=f(x). \tag{10}$$

Replace y by $\frac{1}{x}$ in (1). Then we have

$$xf(x^2)f\left(f\left(\frac{1}{x}\right)\right)+f\left(\frac{f(x)}{x}\right)=f(f(x^2))+f\left(f\left(\frac{1}{x^2}\right)\right).$$

The relation (10) shows $f(\frac{f(x)}{x})=f(f(x^2))$. Also, using (9) with $y=\frac{1}{x}$ and using (10) again, the last equation reduces to

$$f(x)f\left(\frac{1}{x}\right)=1. \tag{11}$$

Replace x by $\frac{1}{x}$ and y by $\frac{1}{y}$ in (1) and apply (11). We get

$$\frac{1}{xf(x^2)f(f(y))}+\frac{1}{f(yf(x))}=\frac{1}{f(xy)}\left(\frac{1}{f(f(x^2))}+\frac{1}{f(f(y^2))}\right).$$

Clearing denominators, we can use (1) to simplify the numerators and obtain

$$f(xy)^2f(f(x^2))f(f(y^2))=xf(x^2)f(f(y))f(yf(x)).$$

Using (9) and (10), this is the same as

$$f(xy)^2f(f(x))=f(x)^2f(y)f(yf(x)). \tag{12}$$

Substitute $y=f(x)$ in (12) and apply (10) (with x replaced by $f(x)$). We have

$$f(xf(x))^2=f(x)f(f(x)). \tag{13}$$

Taking $y=x$ in (12), squaring both sides, and using (10) and (13), we find that

$$f(f(x))=x^4f(x)^3. \tag{14}$$

Finally, we combine (9), (10) and (14) to get

$$y^4f(y)^3 \overset{(14)}{=} f(f(y)) \overset{(9)}{=} f(y)f(f(y^2)) \overset{(14)}{=} f(y)y^8f(y^2)^3 \overset{(10)}{=} y^5f(y)^4,$$

which implies $f(y)=\frac{1}{y}$. This is a solution by the checking in Solution 1.

A5.

(a) Prove that for every positive integer n, there exists a fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n} + 1$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

(b) Prove that there are infinitely many positive integers n such that there is no fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n}$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

Solution.

(a) Let r be the unique positive integer for which $r^2 \leqslant n < (r+1)^2$. Write $n = r^2 + s$. Then we have $0 \leqslant s \leqslant 2r$. We discuss in two cases according to the parity of s.

- **Case 1.** s is even.

  Consider the number $(r + \frac{s}{2r})^2 = r^2 + s + (\frac{s}{2r})^2$. We find that

$$n = r^2 + s \leqslant r^2 + s + \left(\frac{s}{2r}\right)^2 \leqslant r^2 + s + 1 = n + 1.$$

  It follows that

$$\sqrt{n} \leqslant r + \frac{s}{2r} \leqslant \sqrt{n+1}.$$

  Since s is even, we can choose the fraction $r + \frac{s}{2r} = \frac{r^2+(s/2)}{r}$ since $r \leqslant \sqrt{n}$.

- **Case 2.** s is odd.

  Consider the number $(r + 1 - \frac{2r+1-s}{2(r+1)})^2 = (r+1)^2 - (2r+1-s) + (\frac{2r+1-s}{2(r+1)})^2$. We find that

$$\begin{aligned} n = r^2 + s = (r+1)^2 - (2r+1-s) &\leqslant (r+1)^2 - (2r+1-s) + \left(\frac{2r+1-s}{2(r+1)}\right)^2 \\ &\leqslant (r+1)^2 - (2r+1-s) + 1 = n+1. \end{aligned}$$

  It follows that

$$\sqrt{n} \leqslant r + 1 - \frac{2r+1-s}{2(r+1)} \leqslant \sqrt{n+1}.$$

  Since s is odd, we can choose the fraction $(r+1) - \frac{2r+1-s}{2(r+1)} = \frac{(r+1)^2 - r + ((s-1)/2)}{r+1}$ since $r + 1 \leqslant \sqrt{n} + 1$.

(b) We show that for every positive integer r, there is no fraction $\frac{a}{b}$ with $b \leqslant \sqrt{r^2+1}$ such that $\sqrt{r^2+1} \leqslant \frac{a}{b} \leqslant \sqrt{r^2+2}$. Suppose on the contrary that such a fraction exists. Since $b \leqslant \sqrt{r^2+1} < r + 1$ and b is an integer, we have $b \leqslant r$. Hence,

$$(br)^2 < b^2(r^2+1) \leqslant a^2 \leqslant b^2(r^2+2) \leqslant b^2r^2 + 2br < (br+1)^2.$$

This shows the square number a^2 is strictly bounded between the two consecutive squares $(br)^2$ and $(br+1)^2$, which is impossible. Hence, we have found infinitely many $n = r^2 + 1$ for which there is no fraction of the desired form.

A6. The equation

$$(x-1)(x-2)\cdots(x-2016)=(x-1)(x-2)\cdots(x-2016)$$

is written on the board. One tries to erase some linear factors from both sides so that each side still has at least one factor, and the resulting equation has no real roots. Find the least number of linear factors one needs to erase to achieve this.

Answer. 2016.

Solution. Since there are 2016 common linear factors on both sides, we need to erase at least 2016 factors. We claim that the equation has no real roots if we erase all factors $(x-k)$ on the left-hand side with $k \equiv 2,3 \pmod 4$, and all factors $(x-m)$ on the right-hand side with $m \equiv 0,1 \pmod 4$. Therefore, it suffices to show that no real number x satisfies

$$\prod_{j=0}^{503}(x-4j-1)(x-4j-4)=\prod_{j=0}^{503}(x-4j-2)(x-4j-3). \tag{1}$$

- **Case 1.** $x=1,2,\ldots,2016$.
  In this case, one side of (1) is zero while the other side is not. This shows x cannot satisfy (1).
- **Case 2.** $4k+1<x<4k+2$ or $4k+3<x<4k+4$ for some $k=0,1,\ldots,503$.
  For $j=0,1,\ldots,503$ with $j\neq k$, the product $(x-4j-1)(x-4j-4)$ is positive. For $j=k$, the product $(x-4k-1)(x-4k-4)$ is negative. This shows the left-hand side of (1) is negative. On the other hand, each product $(x-4j-2)(x-4j-3)$ on the right-hand side of (1) is positive. This yields a contradiction.
- **Case 3.** $x<1$ or $x>2016$ or $4k<x<4k+1$ for some $k=1,2,\ldots,503$.
  The equation (1) can be rewritten as

$$1=\prod_{j=0}^{503}\frac{(x-4j-1)(x-4j-4)}{(x-4j-2)(x-4j-3)}=\prod_{j=0}^{503}\left(1-\frac{2}{(x-4j-2)(x-4j-3)}\right).$$

Note that $(x-4j-2)(x-4j-3)>2$ for $0\leqslant j\leqslant 503$ in this case. So each term in the product lies strictly between 0 and 1, and the whole product must be less than 1, which is impossible.

- **Case 4.** $4k+2<x<4k+3$ for some $k=0,1,\ldots,503$.
  This time we rewrite (1) as

$$\begin{aligned}1&=\frac{x-1}{x-2}\cdot\frac{x-2016}{x-2015}\prod_{j=1}^{503}\frac{(x-4j)(x-4j-1)}{(x-4j+1)(x-4j-2)}\\&=\frac{x-1}{x-2}\cdot\frac{x-2016}{x-2015}\prod_{j=1}^{503}\left(1+\frac{2}{(x-4j+1)(x-4j-2)}\right).\end{aligned}$$

Clearly, $\frac{x-1}{x-2}$ and $\frac{x-2016}{x-2015}$ are both greater than 1. For the range of x in this case, each term in the product is also greater than 1. Then the right-hand side must be greater than 1 and hence a contradiction arises.

From the four cases, we conclude that (1) has no real roots. Hence, the minimum number of linear factors to be erased is 2016.

Comment. We discuss the general case when 2016 is replaced by a positive integer n. The above solution works equally well when n is divisible by 4.

If $n \equiv 2 \pmod 4$, one may leave $l(x) = (x-1)(x-2)\cdots(x-\frac{n}{2})$ on the left-hand side and $r(x) = (x-\frac{n}{2}-1)(x-\frac{n}{2}-2)\cdots(x-n)$ on the right-hand side. One checks that for $x < \frac{n+1}{2}$, we have $|l(x)| < |r(x)|$, while for $x > \frac{n+1}{2}$, we have $|l(x)| > |r(x)|$.

If $n \equiv 3 \pmod 4$, one may leave $l(x) = (x-1)(x-2)\cdots(x-\frac{n+1}{2})$ on the left-hand side and $r(x) = (x-\frac{n+3}{2})(x-\frac{x+5}{2})\cdots(x-n)$ on the right-hand side. For $x < 1$ or $\frac{n+1}{2} < x < \frac{n+3}{2}$, we have $l(x) > 0 > r(x)$. For $1 < x < \frac{n+1}{2}$, we have $|l(x)| < |r(x)|$. For $x > \frac{n+3}{2}$, we have $|l(x)| > |r(x)|$.

If $n \equiv 1 \pmod 4$, as the proposer mentioned, the situation is a bit more out of control. Since the construction for $n-1 \equiv 0 \pmod 4$ works, the answer can be either n or $n-1$. For $n = 5$, we can leave the products $(x-1)(x-2)(x-3)(x-4)$ and $(x-5)$. For $n = 9$, the only example that works is $l(x) = (x-1)(x-2)(x-9)$ and $r(x) = (x-3)(x-4)\cdots(x-8)$, while there seems to be no such partition for $n = 13$.

A7. Denote by $\mathbb{R}$ the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \neq 0$ and

$$f(x+y)^2 = 2f(x)f(y) + \max\left\{f(x^2) + f(y^2), f(x^2+y^2)\right\} \tag{1}$$

for all real numbers x and y.

Answer.

- $f(x) = -1$ for any $x \in \mathbb{R}$; or
- $f(x) = x - 1$ for any $x \in \mathbb{R}$.

Solution 1. Taking $x = y = 0$ in (1), we get $f(0)^2 = 2f(0)^2 + \max\{2f(0), f(0)\}$. If $f(0) > 0$, then $f(0)^2 + 2f(0) = 0$ gives no positive solution. If $f(0) < 0$, then $f(0)^2 + f(0) = 0$ gives $f(0) = -1$. Putting $y = 0$ in (1), we have $f(x)^2 = -2f(x) + f(x^2)$, which is the same as $(f(x)+1)^2 = f(x^2) + 1$. Let $g(x) = f(x) + 1$. Then for any $x \in \mathbb{R}$, we have

$$g(x^2) = g(x)^2 \geqslant 0. \tag{2}$$

From (1), we find that $f(x+y)^2 \geqslant 2f(x)f(y) + f(x^2) + f(y^2)$. In terms of g, this becomes $(g(x+y)-1)^2 \geqslant 2(g(x)-1)(g(y)-1) + g(x^2) + g(y^2) - 2$. Using (2), this means

$$(g(x+y)-1)^2 \geqslant (g(x)+g(y)-1)^2 - 1. \tag{3}$$

Putting $x = 1$ in (2), we get $g(1) = 0$ or 1. The two cases are handled separately.

- **Case 1.** $g(1) = 0$, which is the same as $f(1) = -1$.

We put $x = -1$ and $y = 0$ in (1). This gives $f(-1)^2 = -2f(-1) - 1$, which forces $f(-1) = -1$. Next, we take $x = -1$ and $y = 1$ in (1) to get $1 = 2 + \max\{-2, f(2)\}$. This clearly implies $1 = 2 + f(2)$ and hence $f(2) = -1$, that is, $g(2) = 0$. From (2), we can prove inductively that $g(2^{2^n}) = g(2)^{2^n} = 0$ for any $n \in \mathbb{N}$. Substitute $y = 2^{2^n} - x$ in (3). We obtain

$$(g(x) + g(2^{2^n} - x) - 1)^2 \leqslant (g(2^{2^n}) - 1)^2 + 1 = 2.$$

For any fixed $x \geqslant 0$, we consider n to be sufficiently large so that $2^{2^n} - x > 0$. From (2), this implies $g(2^{2^n} - x) \geqslant 0$ so that $g(x) \leqslant 1 + \sqrt{2}$. Using (2) again, we get

$$g(x)^{2^n} = g(x^{2^n}) \leqslant 1 + \sqrt{2}$$

for any $n \in \mathbb{N}$. Therefore, $|g(x)| \leqslant 1$ for any $x \geqslant 0$.

If there exists $a \in \mathbb{R}$ for which $g(a) \neq 0$, then for sufficiently large n we must have $g((a^2)^{\frac{1}{2^n}}) = g(a^2)^{\frac{1}{2^n}} > \frac{1}{2}$. By taking $x = -y = -(a^2)^{\frac{1}{2^n}}$ in (1), we obtain

$$\begin{aligned}1 &= 2f(x)f(-x) + \max\left\{2f(x^2), f(2x^2)\right\} \\ &= 2(g(x)-1)(g(-x)-1) + \max\left\{2(g(x^2)-1), g(2x^2)-1\right\} \\ &\leqslant 2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + 0 = \frac{1}{2}\end{aligned}$$

since $|g(-x)| = |g(x)| \in (\frac{1}{2}, 1]$ by (2) and the choice of x, and since $g(z) \leqslant 1$ for $z \geqslant 0$. This yields a contradiction and hence $g(x) = 0$ must hold for any x. This means $f(x) = -1$ for any $x \in \mathbb{R}$, which clearly satisfies (1).

• **Case 2.** $g(1)=1$, which is the same as $f(1)=0$.

We put $x=-1$ and $y=1$ in (1) to get $1=\max\{0,f(2)\}$. This clearly implies $f(2)=1$ and hence $g(2)=2$. Setting $x=2n$ and $y=2$ in (3), we have

$$(g(2n+2)-1)^2 \geqslant (g(2n)+1)^2-1.$$

By induction on n, it is easy to prove that $g(2n)\geqslant n+1$ for all $n\in\mathbb{N}$. For any real number $a>1$, we choose a large $n\in\mathbb{N}$ and take k to be the positive integer such that $2k\leqslant a^{2^n}<2k+2$. From (2) and (3), we have

$$(g(a)^{2^n}-1)^2+1=(g(a^{2^n})-1)^2+1\geqslant (g(2k)+g(a^{2^n}-2k)-1)^2\geqslant k^2>\frac{1}{4}(a^{2^n}-2)^2$$

since $g(a^{2^n}-2k)\geqslant 0$. For large n, this clearly implies $g(a)^{2^n}>1$. Thus,

$$(g(a)^{2^n})^2>(g(a)^{2^n}-1)^2+1>\frac{1}{4}(a^{2^n}-2)^2.$$

This yields

$$g(a)^{2^n}>\frac{1}{2}(a^{2^n}-2). \tag{4}$$

Note that

$$\frac{a^{2^n}}{a^{2^n}-2}=1+\frac{2}{a^{2^n}-2}\leqslant\left(1+\frac{2}{2^n(a^{2^n}-2)}\right)^{2^n}$$

by binomial expansion. This can be rewritten as

$$(a^{2^n}-2)^{\frac{1}{2^n}}\geqslant\frac{a}{1+\frac{2}{2^n(a^{2^n}-2)}}.$$

Together with (4), we conclude $g(a)\geqslant a$ by taking n sufficiently large.

Consider $x=na$ and $y=a>1$ in (3). This gives $(g((n+1)a)-1)^2\geqslant(g(na)+g(a)-1)^2-1$. By induction on n, it is easy to show $g(na)\geqslant(n-1)(g(a)-1)+a$ for any $n\in\mathbb{N}$. We choose a large $n\in\mathbb{N}$ and take k to be the positive integer such that $ka\leqslant 2^{2^n}<(k+1)a$. Using (2) and (3), we have

$$2^{2^{n+1}}>(2^{2^n}-1)^2+1=(g(2^{2^n})-1)^2+1\geqslant(g(2^{2^n}-ka)+g(ka)-1)^2\geqslant((k-1)(g(a)-1)+a-1)^2,$$

from which it follows that

$$2^{2^n}\geqslant(k-1)(g(a)-1)+a-1>\frac{2^{2^n}}{a}(g(a)-1)-2(g(a)-1)+a-1$$

holds for sufficiently large n. Hence, we must have $\frac{g(a)-1}{a}\leqslant 1$, which implies $g(a)\leqslant a+1$ for any $a>1$. Then for large $n\in\mathbb{N}$, from (3) and (2) we have

$$4a^{2^{n+1}}=(2a^{2^n})^2\geqslant(g(2a^{2^n})-1)^2\geqslant(2g(a^{2^n})-1)^2-1=(2g(a)^{2^n}-1)^2-1.$$

This implies

$$2a^{2^n} > \frac{1}{2}(1+\sqrt{4a^{2^{n+1}}+1}) \geqslant g(a)^{2^n}.$$

When n tends to infinity, this forces $g(a) \leqslant a$. Together with $g(a) \geqslant a$, we get $g(a) = a$ for all real numbers $a > 1$, that is, $f(a) = a - 1$ for all $a > 1$.

Finally, for any $x \in \mathbb{R}$, we choose y sufficiently large in (1) so that $y, x+y > 1$. This gives $(x+y-1)^2 = 2f(x)(y-1) + \max\{f(x^2)+y^2-1, x^2+y^2-1\}$, which can be rewritten as

$$2(x-1-f(x))y = -x^2+2x-2-2f(x)+\max\{f(x^2), x^2\}.$$

As the right-hand side is fixed, this can only hold for all large y when $f(x) = x-1$. We now check that this function satisfies (1). Indeed, we have

$$\begin{aligned} f(x+y)^2 &= (x+y-1)^2 = 2(x-1)(y-1) + (x^2+y^2-1) \\ &= 2f(x)f(y) + \max\{f(x^2)+f(y^2), f(x^2+y^2)\}. \end{aligned}$$

Solution 2. Taking $x = y = 0$ in (1), we get $f(0)^2 = 2f(0)^2 + \max\{2f(0), f(0)\}$. If $f(0) > 0$, then $f(0)^2 + 2f(0) = 0$ gives no positive solution. If $f(0) < 0$, then $f(0)^2 + f(0) = 0$ gives $f(0) = -1$. Putting $y = 0$ in (1), we have

$$f(x)^2 = -2f(x) + f(x^2). \tag{5}$$

Replace x by $-x$ in (5) and compare with (5) again. We get $f(x)^2+2f(x) = f(-x)^2+2f(-x)$, which implies

$$f(x) = f(-x) \quad \text{or} \quad f(x)+f(-x) = -2. \tag{6}$$

Taking $x = y$ and $x = -y$ respectively in (1) and comparing the two equations obtained, we have

$$f(2x)^2 - 2f(x)^2 = 1 - 2f(x)f(-x). \tag{7}$$

Combining (6) and (7) to eliminate $f(-x)$, we find that $f(2x)$ can be ± 1 (when $f(x) = f(-x)$) or $\pm(2f(x)+1)$ (when $f(x)+f(-x) = -2$).

We prove the following.

• **Claim.** $f(x) + f(-x) = -2$ for any $x \in \mathbb{R}$.

Proof. Suppose there exists $a \in \mathbb{R}$ such that $f(a) + f(-a) \neq -2$. Then $f(a) = f(-a) \neq -1$ and we may assume $a > 0$. We first show that $f(a) \neq 1$. Suppose $f(a) = 1$. Consider $y = a$ in (7). We get $f(2a)^2 = 1$. Taking $x = y = a$ in (1), we have $1 = 2 + \max\{2f(a^2), f(2a^2)\}$. From (5), $f(a^2) = 3$ so that $1 \geqslant 2 + 6$. This is impossible, and thus $f(a) \neq 1$.

As $f(a) \neq \pm 1$, we have $f(a) = \pm(2f(\frac{a}{2})+1)$. Similarly, $f(-a) = \pm(2f(-\frac{a}{2})+1)$. These two expressions are equal since $f(a) = f(-a)$. If $f(\frac{a}{2}) = f(-\frac{a}{2})$, then the above argument works when we replace a by $\frac{a}{2}$. In particular, we have $f(a)^2 = f(2 \cdot \frac{a}{2})^2 = 1$, which is a contradiction. Therefore, (6) forces $f(\frac{a}{2}) + f(-\frac{a}{2}) = -2$. Then we get

$$\pm\left(2f\left(\frac{a}{2}\right)+1\right) = \pm\left(-2f\left(\frac{a}{2}\right)-3\right).$$

For any choices of the two signs, we either get a contradiction or $f(\frac{a}{2}) = -1$, in which case $f(\frac{a}{2}) = f(-\frac{a}{2})$ and hence $f(a) = \pm 1$ again. Therefore, there is no such real number a and the Claim follows. $\square$

Replace x and y by $-x$ and $-y$ in (1) respectively and compare with (1). We get

$$f(x+y)^2 - 2f(x)f(y) = f(-x-y)^2 - 2f(-x)f(-y).$$

Using the Claim, this simplifies to $f(x+y) = f(x)+f(y)+1$. In addition, (5) can be rewritten as $(f(x)+1)^2 = f(x^2)+1$. Therefore, the function g defined by $g(x) = f(x)+1$ satisfies $g(x+y) = g(x)+g(y)$ and $g(x)^2 = g(x^2)$. The latter relation shows $g(y)$ is nonnegative for $y \geqslant 0$. For such a function satisfying the Cauchy Equation $g(x+y) = g(x)+g(y)$, it must be monotonic increasing and hence $g(x) = cx$ for some constant c.

From $(cx)^2 = g(x)^2 = g(x^2) = cx^2$, we get $c = 0$ or 1, which corresponds to the two functions $f(x) = -1$ and $f(x) = x-1$ respectively, both of which are solutions to (1) as checked in Solution 1.

Solution 3. As in Solution 2, we find that $f(0) = -1$,

$$(f(x)+1)^2 = f(x^2)+1 \tag{8}$$

and

$$f(x) = f(-x) \quad \text{or} \quad f(x)+f(-x) = -2 \tag{9}$$

for any $x \in \mathbb{R}$. We shall show that one of the statements in (9) holds for all $x \in \mathbb{R}$. Suppose $f(a) = f(-a)$ but $f(a)+f(-a) \neq -2$, while $f(b) \neq f(-b)$ but $f(b)+f(-b) = -2$. Clearly, $a, b \neq 0$ and $f(a), f(b) \neq -1$.

Taking $y = a$ and $y = -a$ in (1) respectively and comparing the two equations obtained, we have $f(x+a)^2 = f(x-a)^2$, that is, $f(x+a) = \pm f(x-a)$. This implies $f(x+2a) = \pm f(x)$ for all $x \in \mathbb{R}$. Putting $x = b$ and $x = -2a-b$ respectively, we find $f(2a+b) = \pm f(b)$ and $f(-2a-b) = \pm f(-b) = \pm(-2-f(b))$. Since $f(b) \neq -1$, the term $\pm(-2-f(b))$ is distinct from $\pm f(b)$ in any case. So $f(2a+b) \neq f(-2a-b)$. From (9), we must have $f(2a+b)+f(-2a-b) = -2$. Note that we also have $f(b)+f(-b) = -2$ where $|f(b)|, |f(-b)|$ are equal to $|f(2a+b)|, |f(-2a-b)|$ respectively. The only possible case is $f(2a+b) = f(b)$ and $f(-2a-b) = f(-b)$.

Applying the argument to $-a$ instead of a and using induction, we have $f(2ka+b) = f(b)$ and $f(2ka-b) = f(-b)$ for any integer k. Note that $f(b)+f(-b) = -2$ and $f(b) \neq -1$ imply one of $f(b), f(-b)$ is less than -1. Without loss of generality, assume $f(b) < -1$. We consider $x = \sqrt{2ka+b}$ in (8) for sufficiently large k so that

$$(f(x)+1)^2 = f(2ka+b)+1 = f(b)+1 < 0$$

yields a contradiction. Therefore, one of the statements in (9) must hold for all $x \in \mathbb{R}$.

- **Case 1.** $f(x) = f(-x)$ for any $x \in \mathbb{R}$.

  For any $a \in \mathbb{R}$, setting $x = y = \frac{a}{2}$ and $x = -y = \frac{a}{2}$ in (1) respectively and comparing these, we obtain $f(a)^2 = f(0)^2 = 1$, which means $f(a) = \pm 1$ for all $a \in \mathbb{R}$. If $f(a) = 1$ for some a, we may assume $a > 0$ since $f(a) = f(-a)$. Taking $x = y = \sqrt{a}$ in (1), we get

$$f(2\sqrt{a})^2 = 2f(\sqrt{a})^2 + \max\{2, f(2a)\} = 2f(\sqrt{a})^2 + 2.$$

Note that the left-hand side is ± 1 while the right-hand side is an even integer. This is a contradiction. Therefore, $f(x) = -1$ for all $x \in \mathbb{R}$, which is clearly a solution.

- **Case 2.** $f(x) + f(-x) = -2$ for any $x \in \mathbb{R}$.

  This case can be handled in the same way as in Solution 2, which yields another solution $f(x) = x - 1$.

A8. Determine the largest real number a such that for all $n \geqslant 1$ and for all real numbers $x_0, x_1, \ldots, x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$, we have

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geqslant a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right). \tag{1}$$

Answer. The largest a is $\frac{4}{9}$.

Solution 1. We first show that $a = \frac{4}{9}$ is admissible. For each $2 \leqslant k \leqslant n$, by the Cauchy-Schwarz Inequality, we have

$$(x_{k-1} + (x_k - x_{k-1}))\left(\frac{(k-1)^2}{x_{k-1}} + \frac{3^2}{x_k - x_{k-1}}\right) \geqslant (k-1+3)^2,$$

which can be rewritten as

$$\frac{9}{x_k - x_{k-1}} \geqslant \frac{(k+2)^2}{x_k} - \frac{(k-1)^2}{x_{k-1}}. \tag{2}$$

Summing (2) over $k = 2, 3, \ldots, n$ and adding $\frac{9}{x_1}$ to both sides, we have

$$9\sum_{k=1}^{n} \frac{1}{x_k - x_{k-1}} \geqslant 4\sum_{k=1}^{n} \frac{k+1}{x_k} + \frac{n^2}{x_n} > 4\sum_{k=1}^{n} \frac{k+1}{x_k}.$$

This shows (1) holds for $a = \frac{4}{9}$.

Next, we show that $a = \frac{4}{9}$ is the optimal choice. Consider the sequence defined by $x_0 = 0$ and $x_k = x_{k-1} + k(k+1)$ for $k \geqslant 1$, that is, $x_k = \frac{1}{3}k(k+1)(k+2)$. Then the left-hand side of (1) equals

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1},$$

while the right-hand side equals

$$a\sum_{k=1}^{n} \frac{k+1}{x_k} = 3a\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{3}{2}a\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{3}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)a.$$

When n tends to infinity, the left-hand side tends to 1 while the right-hand side tends to $\frac{9}{4}a$. Therefore a has to be at most $\frac{4}{9}$.

Hence the largest value of a is $\frac{4}{9}$.

Solution 2. We shall give an alternative method to establish (1) with $a = \frac{4}{9}$. We define $y_k = x_k - x_{k-1} > 0$ for $1 \leqslant k \leqslant n$. By the Cauchy-Schwarz Inequality, for $1 \leqslant k \leqslant n$, we have

$$(y_1 + y_2 + \cdots + y_k)\left(\sum_{j=1}^{k} \frac{1}{y_j}\binom{j+1}{2}^2\right) \geqslant \left(\binom{2}{2} + \binom{3}{2} + \cdots + \binom{k+1}{2}\right)^2 = \binom{k+2}{3}^2.$$

This can be rewritten as

$$\frac{k+1}{y_1+y_2+\cdots+y_k} \leqslant \frac{36}{k^2(k+1)(k+2)^2}\left(\sum_{j=1}^{k}\frac{1}{y_j}\binom{j+1}{2}^2\right). \tag{3}$$

Summing (3) over $k=1,2,\ldots,n$, we get

$$\frac{2}{y_1}+\frac{3}{y_1+y_2}+\cdots+\frac{n+1}{y_1+y_2+\cdots+y_n} \leqslant \frac{c_1}{y_1}+\frac{c_2}{y_2}+\cdots+\frac{c_n}{y_n} \tag{4}$$

where for $1 \leqslant m \leqslant n$,

$$\begin{aligned} c_m &= 36\binom{m+1}{2}^2 \sum_{k=m}^{n} \frac{1}{k^2(k+1)(k+2)^2} \\ &= \frac{9m^2(m+1)^2}{4}\sum_{k=m}^{n}\left(\frac{1}{k^2(k+1)^2}-\frac{1}{(k+1)^2(k+2)^2}\right) \\ &= \frac{9m^2(m+1)^2}{4}\left(\frac{1}{m^2(m+1)^2}-\frac{1}{(n+1)^2(n+2)^2}\right) < \frac{9}{4}. \end{aligned}$$

From (4), the inequality (1) holds for $a=\frac{4}{9}$. This is also the upper bound as can be verified in the same way as Solution 1.

Combinatorics

C1. The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down 001, 111 and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

Answer. The minimum number of guesses is 2 if $n = 2k$ and 1 if $n \neq 2k$.

Solution 1. Let X be the binary string chosen by the leader and let X' be the binary string of length n every digit of which is different from that of X. The strings written by the deputy leader are the same as those in the case when the leader's string is X' and k is changed to $n - k$. In view of this, we may assume $k \geqslant \frac{n}{2}$. Also, for the particular case $k = \frac{n}{2}$, this argument shows that the strings X and X' cannot be distinguished, and hence in that case the contestant has to guess at least twice.

It remains to show that the number of guesses claimed suffices. Consider any string Y which differs from X in m digits where $0 < m < 2k$. Without loss of generality, assume the first m digits of X and Y are distinct. Let Z be the binary string obtained from X by changing its first k digits. Then Z is written by the deputy leader. Note that Z differs from Y by $|m - k|$ digits where $|m - k| < k$ since $0 < m < 2k$. From this observation, the contestant must know that Y is not the desired string.

As we have assumed $k \geqslant \frac{n}{2}$, when $n < 2k$, every string $Y \neq X$ differs from X in fewer than $2k$ digits. When $n = 2k$, every string except X and X' differs from X in fewer than $2k$ digits. Hence, the answer is as claimed.

Solution 2. Firstly, assume $n \neq 2k$. Without loss of generality suppose the first digit of the leader's string is 1. Then among the $\binom{n}{k}$ strings written by the deputy leader, $\binom{n-1}{k}$ will begin with 1 and $\binom{n-1}{k-1}$ will begin with 0. Since $n \neq 2k$, we have $k + (k - 1) \neq n - 1$ and so $\binom{n-1}{k} \neq \binom{n-1}{k-1}$. Thus, by counting the number of strings written by the deputy leader that start with 0 and 1, the contestant can tell the first digit of the leader's string. The same can be done on the other digits, so 1 guess suffices when $n \neq 2k$.

Secondly, for the case $n = 2$ and $k = 1$, the answer is clearly 2. For the remaining cases where $n = 2k > 2$, the deputy leader would write down the same strings if the leader's string X is replaced by X' obtained by changing each digit of X. This shows at least 2 guesses are needed. We shall show that 2 guesses suffice in this case. Suppose the first two digits of the leader's string are the same. Then among the strings written by the deputy leader, the prefices 01 and 10 will occur $\binom{2k-2}{k-1}$ times each, while the prefices 00 and 11 will occur $\binom{2k-2}{k}$ times each. The two numbers are interchanged if the first two digits of the leader's string are different. Since $\binom{2k-2}{k-1} \neq \binom{2k-2}{k}$, the contestant can tell whether the first two digits of the leader's string are the same or not. He can work out the relation of the first digit and the other digits in the same way and reduce the leader's string to only 2 possibilities. The proof is complete.

C2. Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:

- each cell contains a distinct divisor;
- the sums of all rows are equal; and
- the sums of all columns are equal.

Answer. 1.

Solution 1. Suppose all positive divisors of n can be arranged into a rectangular table of size $k \times l$ where the number of rows k does not exceed the number of columns l. Let the sum of numbers in each column be s. Since n belongs to one of the columns, we have $s \geqslant n$, where equality holds only when $n = 1$.

For $j = 1, 2, \ldots, l$, let d_j be the largest number in the j-th column. Without loss of generality, assume $d_1 > d_2 > \cdots > d_l$. Since these are divisors of n, we have

$$d_l \leqslant \frac{n}{l}. \tag{1}$$

As d_l is the maximum entry of the l-th column, we must have

$$d_l \geqslant \frac{s}{k} \geqslant \frac{n}{k}. \tag{2}$$

The relations (1) and (2) combine to give $\frac{n}{l} \geqslant \frac{n}{k}$, that is, $k \geqslant l$. Together with $k \leqslant l$, we conclude that $k = l$. Then all inequalities in (1) and (2) are equalities. In particular, $s = n$ and so $n = 1$, in which case the conditions are clearly satisfied.

Solution 2. Clearly $n = 1$ works. Then we assume $n > 1$ and let its prime factorization be $n = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$. Suppose the table has k rows and l columns with $1 < k \leqslant l$. Note that kl is the number of positive divisors of n and the sum of all entries is the sum of positive divisors of n, which we denote by $\sigma(n)$. Consider the column containing n. Since the column sum is $\frac{\sigma(n)}{l}$, we must have $\frac{\sigma(n)}{l} > n$. Therefore, we have

$$\begin{aligned}(r_1+1)(r_2+1)\cdots(r_t+1) = kl \leqslant l^2 &< \left(\frac{\sigma(n)}{n}\right)^2 \\ &= \left(1 + \frac{1}{p_1} + \cdots + \frac{1}{p_1^{r_1}}\right)^2 \cdots \left(1 + \frac{1}{p_t} + \cdots + \frac{1}{p_t^{r_t}}\right)^2.\end{aligned}$$

This can be rewritten as

$$f(p_1, r_1) f(p_2, r_2) \cdots f(p_t, r_t) < 1 \tag{3}$$

where

$$f(p, r) = \frac{r+1}{\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^r}\right)^2} = \frac{(r+1)\left(1 - \frac{1}{p}\right)^2}{\left(1 - \frac{1}{p^{r+1}}\right)^2}.$$

Direct computation yields

$$f(2,1)=\frac{8}{9},\quad f(2,2)=\frac{48}{49},\quad f(3,1)=\frac{9}{8}.$$

Also, we find that

$$\begin{aligned}
f(2,r) &\geqslant \left(1-\frac{1}{2^{r+1}}\right)^{-2}>1 \quad \text{for } r\geqslant 3,\\
f(3,r) &\geqslant \frac{4}{3}\left(1-\frac{1}{3^{r+1}}\right)^{-2}>\frac{4}{3}>\frac{9}{8} \quad \text{for } r\geqslant 2, \text{ and}\\
f(p,r) &\geqslant \frac{32}{25}\left(1-\frac{1}{p^{r+1}}\right)^{-2}>\frac{32}{25}>\frac{9}{8} \quad \text{for } p\geqslant 5.
\end{aligned}$$

From these values and bounds, it is clear that (3) holds only when $n=2$ or 4. In both cases, it is easy to see that the conditions are not satisfied. Hence, the only possible n is 1.

C3. Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

Solution. For $k = 1, 2, 3$, let a_k be the number of isosceles triangles whose vertices contain exactly k colours. Suppose on the contrary that $a_3 = 0$. Let b, c, d be the number of vertices of the three different colours respectively. We now count the number of pairs $(\triangle, E)$ where $\triangle$ is an isosceles triangle and E is a side of $\triangle$ whose endpoints are of different colours.

On the one hand, since we have assumed $a_3 = 0$, each triangle in the pair must contain exactly two colours, and hence each triangle contributes twice. Thus the number of pairs is $2a_2$.

On the other hand, if we pick any two vertices A, B of distinct colours, then there are three isosceles triangles having these as vertices, two when AB is not the base and one when AB is the base since n is odd. Note that the three triangles are all distinct as $(n, 3) = 1$. In this way, we count the number of pairs to be $3(bc + cd + db)$. However, note that $2a_2$ is even while $3(bc + cd + db)$ is odd, as each of b, c, d is. This yields a contradiction and hence $a_3 \geqslant 1$.

Comment. A slightly stronger version of this problem is to replace the condition $(n, 6) = 1$ by n being odd (where equilateral triangles are regarded as isosceles triangles). In that case, the only difference in the proof is that by fixing any two vertices A, B, one can find exactly one or three isosceles triangles having these as vertices. But since only parity is concerned in the solution, the proof goes the same way.

The condition that there is an odd number of vertices of each colour is necessary, as can be seen from the following example. Consider $n = 25$ and we label the vertices $A_0, A_1, \ldots, A_{24}$. Suppose colour 1 is used for A_0, colour 2 is used for $A_5, A_{10}, A_{15}, A_{20}$, while colour 3 is used for the remaining vertices. Then any isosceles triangle having colours 1 and 2 must contain A_0 and one of $A_5, A_{10}, A_{15}, A_{20}$. Clearly, the third vertex must have index which is a multiple of 5 so it is not of colour 3.

C4. Find all positive integers n for which we can fill in the entries of an $n \times n$ table with the following properties:

- each entry can be one of I, M and O;
- in each row and each column, the letters I, M and O occur the same number of times; and
- in any diagonal whose number of entries is a multiple of three, the letters I, M and O occur the same number of times.

Answer. n can be any multiple of 9.

Solution. We first show that such a table exists when n is a multiple of 9. Consider the following 9×9 table.

$$\begin{pmatrix} I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \\ I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \\ I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \end{pmatrix} \tag{1}$$

It is a direct checking that the table (1) satisfies the requirements. For $n = 9k$ where k is a positive integer, we form an $n \times n$ table using $k \times k$ copies of (1). For each row and each column of the table of size n, since there are three I's, three M's and three O's for any nine consecutive entries, the numbers of I, M and O are equal. In addition, every diagonal of the large table whose number of entries is divisible by 3 intersects each copy of (1) at a diagonal with number of entries divisible by 3 (possibly zero). Therefore, every such diagonal also contains the same number of I, M and O.

Next, consider any $n \times n$ table for which the requirements can be met. As the number of entries of each row should be a multiple of 3, we let $n = 3k$ where k is a positive integer. We divide the whole table into $k \times k$ copies of 3×3 blocks. We call the entry at the centre of such a 3×3 square a *vital entry*. We also call any row, column or diagonal that contains at least one vital entry a *vital line*. We compute the number of pairs (l, c) where l is a vital line and c is an entry belonging to l that contains the letter M. We let this number be N.

On the one hand, since each vital line contains the same number of I, M and O, it is obvious that each vital row and each vital column contain k occurrences of M. For vital diagonals in either direction, we count there are exactly

$$1 + 2 + \cdots + (k-1) + k + (k-1) + \cdots + 2 + 1 = k^2$$

occurrences of M. Therefore, we have $N = 4k^2$.

On the other hand, there are $3k^2$ occurrences of M in the whole table. Note that each entry belongs to exactly 1 or 4 vital lines. Therefore, N must be congruent to $3k^2$ mod 3.

From the double counting, we get $4k^2 \equiv 3k^2 \pmod 3$, which forces k to be a multiple of 3. Therefore, n has to be a multiple of 9 and the proof is complete.

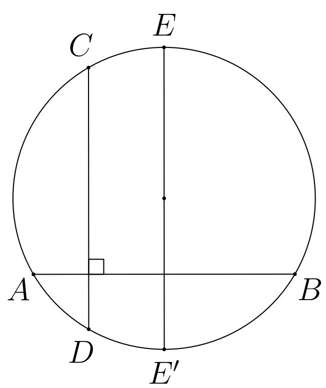
C5. Let $n \geqslant 3$ be a positive integer. Find the maximum number of diagonals of a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

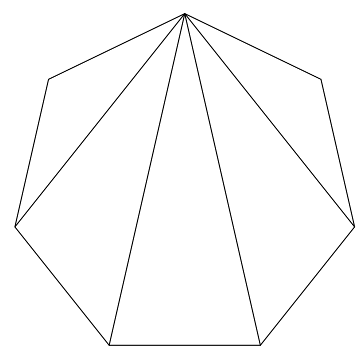
Answer. $n-2$ if n is even and $n-3$ if n is odd.

Solution 1. We consider two cases according to the parity of n.

• **Case 1.** n is odd.

We first claim that no pair of diagonals is perpendicular. Suppose A, B, C, D are vertices where AB and CD are perpendicular, and let E be the vertex lying on the perpendicular bisector of AB. Let E' be the opposite point of E on the circumcircle of the regular polygon. Since $EC = E'D$ and C, D, E are vertices of the regular polygon, E' should also belong to the polygon. This contradicts the fact that a regular polygon with an odd number of vertices does not contain opposite points on the circumcircle.



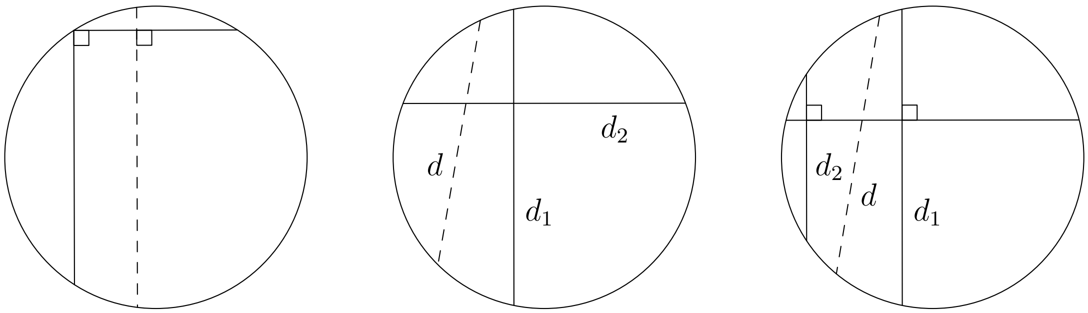


Therefore in the odd case we can only select diagonals which do not intersect. In the maximal case these diagonals should divide the regular n-gon into $n-2$ triangles, so we can select at most $n-3$ diagonals. This can be done, for example, by selecting all diagonals emanated from a particular vertex.

• **Case 2.** n is even.

If there is no intersection, then the proof in the odd case works. Suppose there are two perpendicular diagonals selected. We consider the set S of all selected diagonals parallel to one of them which intersect with some selected diagonals. Suppose S contains k diagonals and the number of distinct endpoints of the k diagonals is l.

Firstly, consider the longest diagonal in one of the two directions in S. No other diagonal in S can start from either endpoint of that diagonal, since otherwise it has to meet another longer diagonal in S. The same holds true for the other direction. Ignoring these two longest diagonals and their four endpoints, the remaining $k-2$ diagonals share $l-4$ endpoints where each endpoint can belong to at most two diagonals. This gives $2(l-4) \geqslant 2(k-2)$, so that $k \leqslant l-2$.

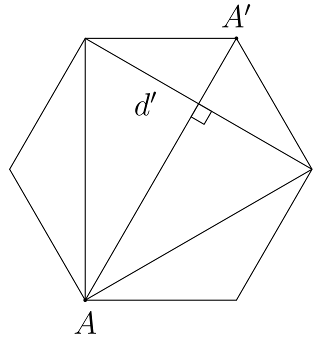


Consider a group of consecutive vertices of the regular n-gon so that each of the two outermost vertices is an endpoint of a diagonal in S, while the interior points are not. There are l such groups. We label these groups $P_1, P_2, \ldots, P_l$ in this order. We claim that each selected diagonal outside S must connect vertices of the same group P_i. Consider any diagonal d joining vertices from distinct groups P_i and P_j. Let d_1 and d_2 be two diagonals in S each having one of the outermost points of P_i as endpoint. Then d must meet either d_1, d_2 or a diagonal in S which is perpendicular to both d_1 and d_2. In any case d should belong to S by definition, which is a contradiction.

Within the same group P_i, there are no perpendicular diagonals since the vertices belong to the same side of a diameter of the circumcircle. Hence there can be at most $|P_i|-2$ selected diagonals within P_i, including the one joining the two outermost points of P_i when $|P_i| > 2$. Therefore, the maximum number of diagonals selected is

$$\sum_{i=1}^{l}(|P_i|-2)+k=\sum_{i=1}^{l}|P_i|-2l+k=(n+l)-2l+k=n-l+k\leqslant n-2.$$

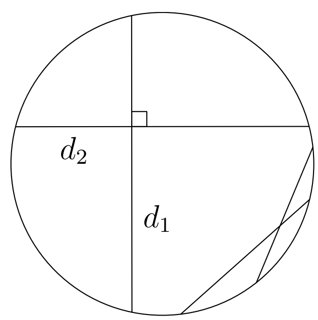
This upper bound can be attained as follows. We take any vertex A and let A' be the vertex for which AA' is a diameter of the circumcircle. If we select all diagonals emanated from A together with the diagonal d' joining the two neighbouring vertices of A', then the only pair of diagonals that meet each other is AA' and d', which are perpendicular to each other. In total we can take $n-2$ diagonals.

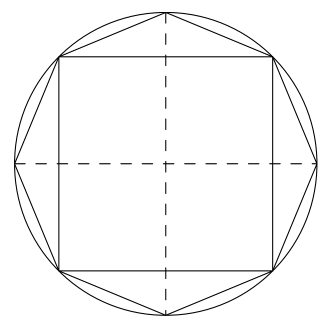


Solution 2. The constructions and the odd case are the same as Solution 1. Instead of dealing separately with the case where n is even, we shall prove by induction more generally that we can select at most $n-2$ diagonals for any cyclic n-gon with circumcircle Γ.

The base case $n=3$ is trivial since there is no diagonal at all. Suppose the upper bound holds for any cyclic polygon with fewer than n sides. For a cyclic n-gon, if there is a selected diagonal which does not intersect any other selected diagonal, then this diagonal divides the n-gon into an m-gon and an l-gon (with $m+l=n+2$) so that each selected diagonal belongs to one of them. Without loss of generality, we may assume the m-gon lies on the same side of a diameter of Γ. Then no two selected diagonals of the m-gon can intersect, and hence we can select at most $m-3$ diagonals. Also, we can apply the inductive hypothesis to the l-gon. This shows the maximum number of selected diagonals is $(m-3)+(l-2)+1=n-2$.

It remains to consider the case when all selected diagonals meet at least one other selected diagonal. Consider a pair of selected perpendicular diagonals d_1, d_2. They divide the circumference of Γ into four arcs, each of which lies on the same side of a diameter of Γ. If there are two selected diagonals intersecting each other and neither is parallel to d_1 or d_2, then their endpoints must belong to the same arc determined by d_1, d_2, and hence they cannot be perpendicular. This violates the condition, and hence all selected diagonals must have the same direction as one of d_1, d_2.





Take the longest selected diagonal in one of the two directions. We argue as in Solution 1 that its endpoints do not belong to any other selected diagonal. The same holds true for the longest diagonal in the other direction. Apart from these four endpoints, each of the remaining $n-4$ vertices can belong to at most two selected diagonals. Thus we can select at most $\frac{1}{2}(2(n-4)+4)=n-2$ diagonals. Then the proof follows by induction.

C6. There are $n \geqslant 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

After each year, the ferry company will close a ferry route between some two islands X and Y. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected by a ferry route to exactly one of X and Y, a new route between this island and the other of X and Y is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

Solution. Initially, we pick any pair of islands A and B which are connected by a ferry route and put A in set $\mathcal{A}$ and B in set $\mathcal{B}$. From the condition, without loss of generality there must be another island which is connected to A. We put such an island C in set $\mathcal{B}$. We say that two sets of islands form a *network* if each island in one set is connected to each island in the other set.

Next, we shall included all islands to $\mathcal{A} \cup \mathcal{B}$ one by one. Suppose we have two sets $\mathcal{A}$ and $\mathcal{B}$ which form a network where $3 \leqslant |\mathcal{A} \cup \mathcal{B}| < n$. This relation no longer holds only when a ferry route between islands $A \in \mathcal{A}$ and $B \in \mathcal{B}$ is closed. In that case, we define $\mathcal{A}' = \{A, B\}$, and $\mathcal{B}' = (\mathcal{A} \cup \mathcal{B}) - \{A, B\}$. Note that $\mathcal{B}'$ is nonempty. Consider any island $C \in \mathcal{A} - \{A\}$. From the relation of $\mathcal{A}$ and $\mathcal{B}$, we know that C is connected to B. If C was not connected to A before the route between A and B closes, then there will be a route added between C and A afterwards. Hence, C must now be connected to both A and B. The same holds true for any island in $\mathcal{B} - \{B\}$. Therefore, $\mathcal{A}'$ and $\mathcal{B}'$ form a network, and $\mathcal{A}' \cup \mathcal{B}' = \mathcal{A} \cup \mathcal{B}$. Hence these islands can always be partitioned into sets $\mathcal{A}$ and $\mathcal{B}$ which form a network.

As $|\mathcal{A} \cup \mathcal{B}| < n$, there are some islands which are not included in $\mathcal{A} \cup \mathcal{B}$. From the condition, after some years there must be a ferry route between an island A in $\mathcal{A} \cup \mathcal{B}$ and an island D outside $\mathcal{A} \cup \mathcal{B}$ which closes. Without loss of generality assume $A \in \mathcal{A}$. Then each island in $\mathcal{B}$ must then be connected to D, no matter it was or not before. Hence, we can put D in set $\mathcal{A}$ so that the new sets $\mathcal{A}$ and $\mathcal{B}$ still form a network and the size of $\mathcal{A} \cup \mathcal{B}$ is increased by 1. The same process can be done to increase the size of $\mathcal{A} \cup \mathcal{B}$. Eventually, all islands are included in this way so we may now assume $|\mathcal{A} \cup \mathcal{B}| = n$.

Suppose a ferry route between $A \in \mathcal{A}$ and $B \in \mathcal{B}$ is closed after some years. We put A and B in set $\mathcal{A}'$ and all remaining islands in set $\mathcal{B}'$. Then $\mathcal{A}'$ and $\mathcal{B}'$ form a network. This relation no longer holds only when a route between A, without loss of generality, and $C \in \mathcal{B}'$ is closed. Since this must eventually occur, at that time island B will be connected to all other islands and the result follows.

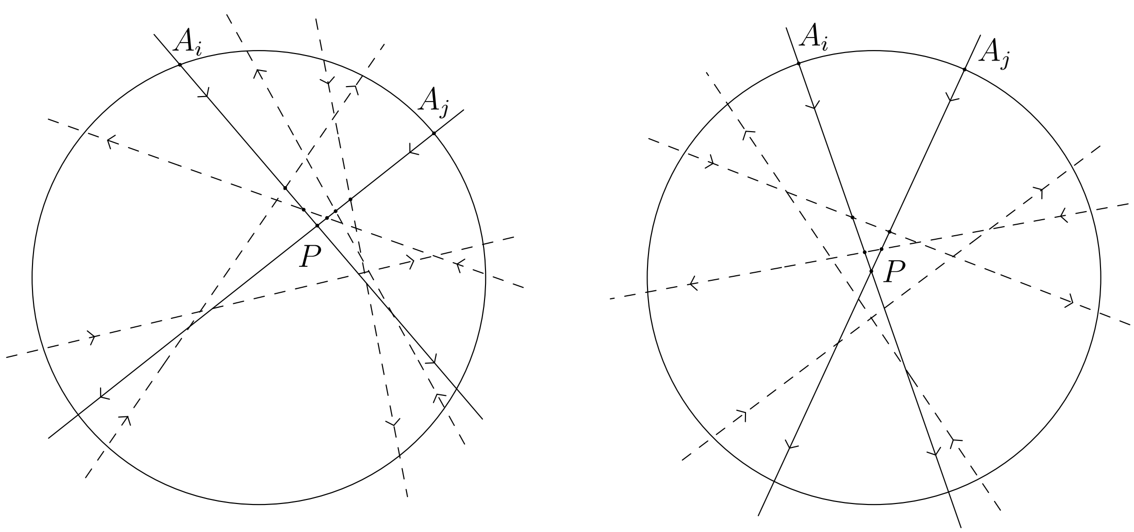
C7. Let $n \geqslant 2$ be an integer. In the plane, there are n segments given in such a way that any two segments have an intersection point in the interior, and no three segments intersect at a single point. Jeff places a snail at one of the endpoints of each of the segments and claps his hands $n-1$ times. Each time when he claps his hands, all the snails move along their own segments and stay at the next intersection points until the next clap. Since there are $n-1$ intersection points on each segment, all snails will reach the furthest intersection points from their starting points after $n-1$ claps.

(a) Prove that if n is odd then Jeff can always place the snails so that no two of them ever occupy the same intersection point.

(b) Prove that if n is even then there must be a moment when some two snails occupy the same intersection point no matter how Jeff places the snails.

Solution. We consider a big disk which contains all the segments. We extend each segment to a line l_i so that each of them cuts the disk at two distinct points A_i, B_i.

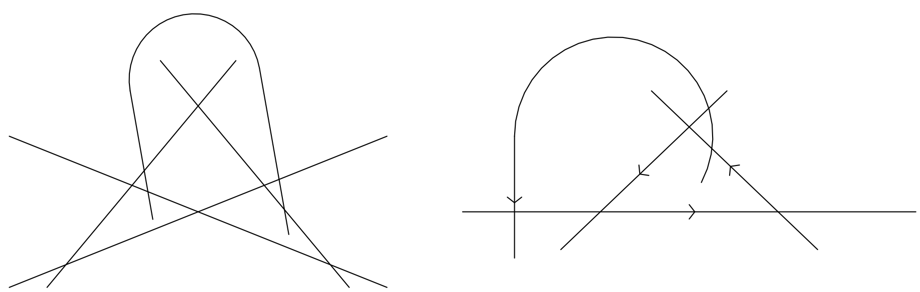
(a) For odd n, we travel along the circumference of the disk and mark each of the points A_i or B_i 'in' and 'out' alternately. Since every pair of lines intersect in the disk, there are exactly $n-1$ points between A_i and B_i for any fixed $1 \leqslant i \leqslant n$. As n is odd, this means one of A_i and B_i is marked 'in' and the other is marked 'out'. Then Jeff can put a snail on the endpoint of each segment which is closer to the 'in' side of the corresponding line. We claim that the snails on l_i and l_j do not meet for any pairs i, j, hence proving part (a).



Without loss of generality, we may assume the snails start at A_i and A_j respectively. Let l_i intersect l_j at P. Note that there is an odd number of points between arc A_iA_j. Each of these points belongs to a line l_k. Such a line l_k must intersect exactly one of the segments A_iP and A_jP, making an odd number of intersections. For the other lines, they may intersect both segments A_iP and A_jP, or meet none of them. Therefore, the total number of intersection points on segments A_iP and A_jP (not counting P) is odd. However, if the snails arrive at P at the same time, then there should be the same number of intersections on A_iP and A_jP, which gives an even number of intersections. This is a contradiction so the snails do not meet each other.

(b) For even n, we consider any way that Jeff places the snails and mark each of the points A_i or B_i 'in' and 'out' according to the directions travelled by the snails. In this case there must be two neighbouring points A_i and A_j both of which are marked 'in'. Let P be the intersection of the segments A_iB_i and A_jB_j. Then any other segment meeting one of the segments A_iP and A_jP must also meet the other one, and so the number of intersections on A_iP and A_jP are the same. This shows the snails starting from A_i and A_j will meet at P.

Comment. The conclusions do not hold for pseudosegments, as can be seen from the following examples.

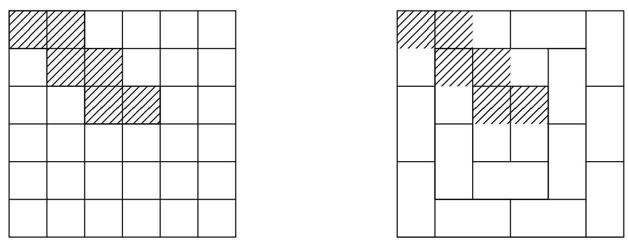


C8. Let n be a positive integer. Determine the smallest positive integer k with the following property: it is possible to mark k cells on a $2n \times 2n$ board so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contains two marked cells.

Answer. $2n$.

Solution. We first construct an example of marking $2n$ cells satisfying the requirement. Label the rows and columns $1, 2, \ldots, 2n$ and label the cell in the i-th row and the j-th column (i, j).

For $i = 1, 2, \ldots, n$, we mark the cells (i, i) and $(i, i+1)$. We claim that the required partition exists and is unique. The two diagonals of the board divide the board into four regions. Note that the domino covering cell $(1,1)$ must be vertical. This in turn shows that each domino covering $(2,2), (3,3), \ldots, (n,n)$ is vertical. By induction, the dominoes in the left region are all vertical. By rotational symmetry, the dominoes in the bottom region are horizontal, and so on. This shows that the partition exists and is unique.



It remains to show that this value of k is the smallest possible. Assume that only $k < 2n$ cells are marked, and there exists a partition P satisfying the requirement. It suffices to show there exists another desirable partition distinct from P. Let d be the main diagonal of the board.

Construct the following graph with edges of two colours. Its vertices are the cells of the board. Connect two vertices with a red edge if they belong to the same domino of P. Connect two vertices with a blue edge if their reflections in d are connected by a red edge. It is possible that two vertices are connected by edges of both colours. Clearly, each vertex has both red and blue degrees equal to 1. Thus the graph splits into cycles where the colours of edges in each cycle alternate (a cycle may have length 2).

Consider any cell c lying on the diagonal d. Its two edges are symmetrical with respect to d. Thus they connect c to different cells. This shows c belongs to a cycle $C(c)$ of length at least 4. Consider a part of this cycle $c_0, c_1, \ldots, c_m$ where $c_0 = c$ and m is the least positive integer such that c_m lies on d. Clearly, c_m is distinct from c. From the construction, the path symmetrical to this with respect to d also lies in the graph, and so these paths together form $C(c)$. Hence, $C(c)$ contains exactly two cells from d. Then all $2n$ cells in d belong to n cycles $C_1, C_2, \ldots, C_n$, each has length at least 4.

By the Pigeonhole Principle, there exists a cycle C_i containing at most one of the k marked cells. We modify P as follows. We remove all dominoes containing the vertices of C_i, which correspond to the red edges of C_i. Then we put the dominoes corresponding to the blue edges of C_i. Since C_i has at least 4 vertices, the resultant partition P' is different from P. Clearly, no domino in P' contains two marked cells as C_i contains at most one marked cell. This shows the partition is not unique and hence k cannot be less than $2n$.

Geometry

G1. In a convex pentagon $ABCDE$, let F be a point on AC such that $\angle FBC = 90^\circ$. Suppose triangles ABF, ACD and ADE are similar isosceles triangles with

$$\angle FAB = \angle FBA = \angle DAC = \angle DCA = \angle EAD = \angle EDA. \tag{1}$$

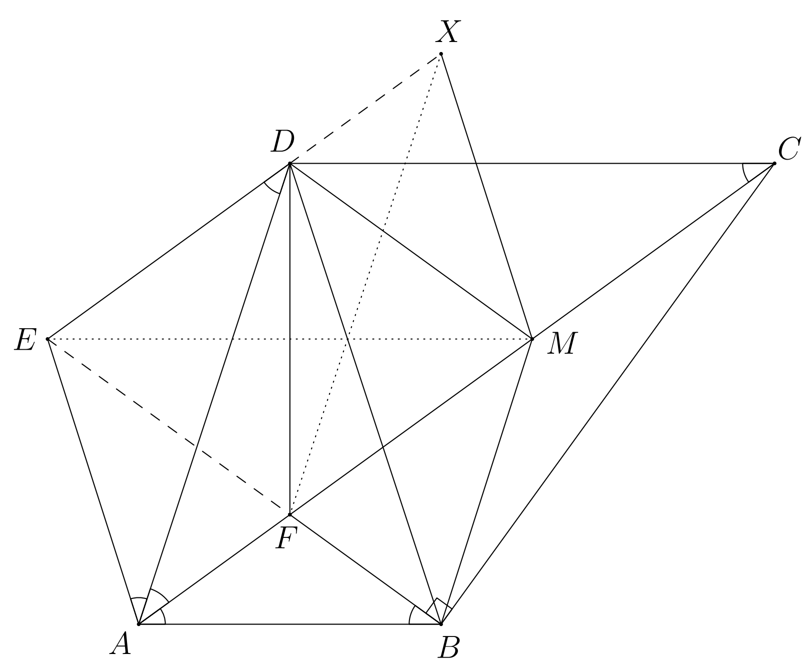
Let M be the midpoint of CF. Point X is chosen such that $AMXE$ is a parallelogram. Show that BD, EM and FX are concurrent.

Solution 1. Denote the common angle in (1) by θ. As $\triangle ABF \sim \triangle ACD$, we have $\frac{AB}{AC} = \frac{AF}{AD}$ so that $\triangle ABC \sim \triangle AFD$. From $EA = ED$, we get

$$\angle AFD = \angle ABC = 90^\circ + \theta = 180^\circ - \frac{1}{2}\angle AED.$$

Hence, F lies on the circle with centre E and radius EA. In particular, $EF = EA = ED$. As $\angle EFA = \angle EAF = 2\theta = \angle BFC$, points B, F, E are collinear.

As $\angle EDA = \angle MAD$, we have $ED // AM$ and hence E, D, X are collinear. As M is the midpoint of CF and $\angle CBF = 90^\circ$, we get $MF = MB$. In the isosceles triangles EFA and MFB, we have $\angle EFA = \angle MFB$ and $AF = BF$. Therefore, they are congruent to each other. Then we have $BM = AE = XM$ and $BE = BF + FE = AF + FM = AM = EX$. This shows $\triangle EMB \cong \triangle EMX$. As F and D lie on EB and EX respectively and $EF = ED$, we know that lines BD and XF are symmetric with respect to EM. It follows that the three lines are concurrent.



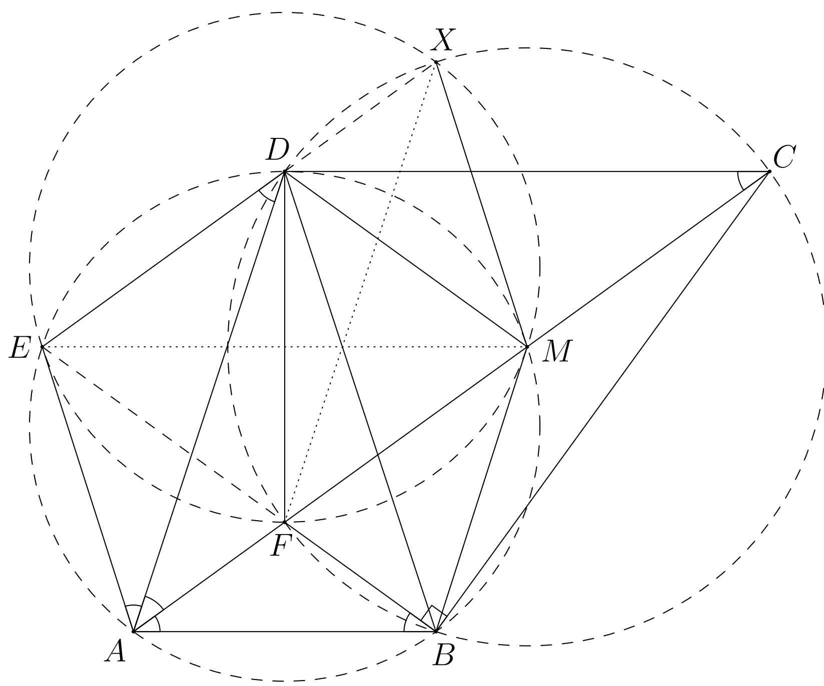
<u>Solution 2</u>. From $\angle CAD = \angle EDA$, we have $AC//ED$. Together with $AC//EX$, we know that E, D, X are collinear. Denote the common angle in (1) by θ. From $\triangle ABF \sim \triangle ACD$, we get $\frac{AB}{AC} = \frac{AF}{AD}$ so that $\triangle ABC \sim \triangle AFD$. This yields $\angle AFD = \angle ABC = 90^\circ + \theta$ and hence $\angle FDC = 90^\circ$, implying that $BCDF$ is cyclic. Let Γ_1 be its circumcircle.

Next, from $\triangle ABF \sim \triangle ADE$, we have $\frac{AB}{AD} = \frac{AF}{AE}$ so that $\triangle ABD \sim \triangle AFE$. Therefore,

$$\angle AFE = \angle ABD = \theta + \angle FBD = \theta + \angle FCD = 2\theta = 180^\circ - \angle BFA.$$

This implies B, F, E are collinear. Note that F is the incentre of triangle DAB. Point E lies on the internal angle bisector of $\angle DBA$ and lies on the perpendicular bisector of AD. It follows that E lies on the circumcircle Γ_2 of triangle ABD, and $EA = EF = ED$.

Also, since CF is a diameter of Γ_1 and M is the midpoint of CF, M is the centre of Γ_1 and hence $\angle AMD = 2\theta = \angle ABD$. This shows M lies on Γ_2. Next, $\angle MDX = \angle MAE = \angle DXM$ since $AMXE$ is a parallelogram. Hence $MD = MX$ and X lies on Γ_1.



We now have two ways to complete the solution.

- **Method 1.** From $EF = EA = XM$ and $EX//FM$, $EFMX$ is an isosceles trapezoid and is cyclic. Denote its circumcircle by Γ_3. Since BD, EM, FX are the three radical axes of $\Gamma_1, \Gamma_2, \Gamma_3$, they must be concurrent.

- **Method 2.** As $\angle DMF = 2\theta = \angle BFM$, we have $DM//EB$. Also,

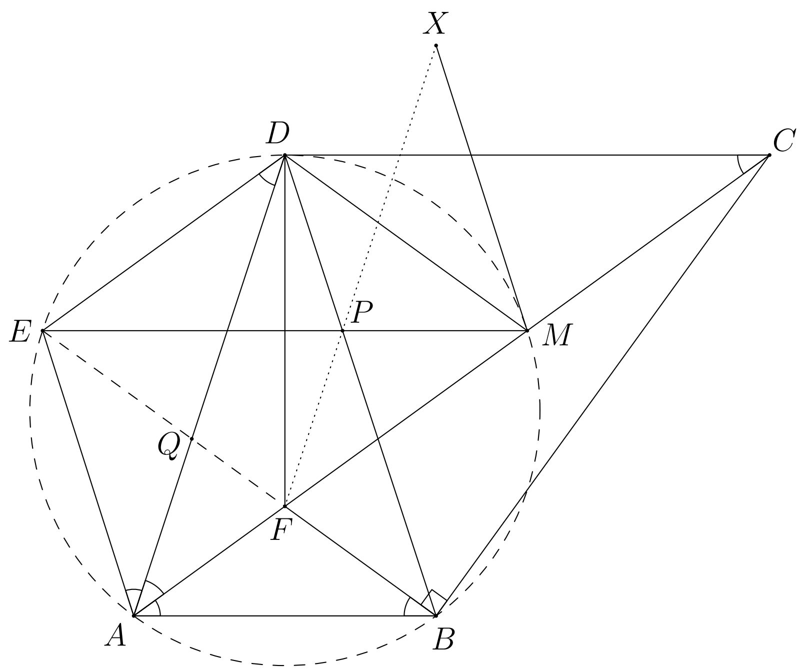
$$\angle BFD + \angle XBF = \angle BFC + \angle CFD + 90^\circ - \angle CBX = 2\theta + (90^\circ - \theta) + 90^\circ - \theta = 180^\circ$$

implies $DF//XB$. These show the corresponding sides of triangles DMF and BEX are parallel. By Desargues' Theorem, the two triangles are perspective and hence DB, ME, FX meet at a point.

Comment. In Solution 2, both the Radical Axis Theorem and Desargues' Theorem could imply DB, ME, FX are parallel. However, this is impossible as can be seen from the configuration. For example, it is obvious that DB and ME meet each other.

Solution 3. Let the common angle in (1) be θ. From $\triangle ABF \sim \triangle ACD$, we have $\frac{AB}{AC} = \frac{AF}{AD}$ so that $\triangle ABC \sim \triangle AFD$. Then $\angle ADF = \angle ACB = 90° - 2\theta = 90° - \angle BAD$ and hence $DF \perp AB$. As $FA = FB$, this implies $\triangle DAB$ is isosceles with $DA = DB$. Then F is the incentre of $\triangle DAB$.

Next, from $\angle AED = 180° - 2\theta = 180° - \angle DBA$, points A, B, D, E are concyclic. Since we also have $EA = ED$, this shows E, F, B are collinear and $EA = EF = ED$.



Note that C lies on the internal angle bisector of $\angle BAD$ and lies on the external angle bisector of $\angle DBA$. It follows that it is the A-excentre of triangle DAB. As M is the midpoint of CF, M lies on the circumcircle of triangle DAB and it is the centre of the circle passing through D, F, B, C. By symmetry, $DEFM$ is a rhombus. Then the midpoints of AX, EM and DF coincide, and it follows that $DAFX$ is a parallelogram.

Let P be the intersection of BD and EM, and Q be the intersection of AD and BE. From $\angle BAC = \angle DCA$, we know that DC, AB, EM are parallel. Thus we have $\frac{DP}{PB} = \frac{CM}{MA}$. This is further equal to $\frac{AE}{BE}$ since $CM = DM = DE = AE$ and $MA = BE$. From $\triangle AEQ \sim \triangle BEA$, we find that

$$\frac{DP}{PB} = \frac{AE}{BE} = \frac{AQ}{BA} = \frac{QF}{FB}$$

by the Angle Bisector Theorem. This implies $QD /\!/ FP$ and hence F, P, X are collinear, as desired.

G2. Let ABC be a triangle with circumcircle Γ and incentre I. Let M be the midpoint of side BC. Denote by D the foot of perpendicular from I to side BC. The line through I perpendicular to AI meets sides AB and AC at F and E respectively. Suppose the circumcircle of triangle AEF intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Solution 1. Let AM meet Γ again at Y and XY meet BC at D'. It suffices to show $D' = D$. We shall apply the following fact.

• **Claim.** For any cyclic quadrilateral $PQRS$ whose diagonals meet at T, we have

$$\frac{QT}{TS} = \frac{PQ \cdot QR}{PS \cdot SR}.$$

Proof. We use $[W_1W_2W_3]$ to denote the area of $W_1W_2W_3$. Then

$$\frac{QT}{TS} = \frac{[PQR]}{[PSR]} = \frac{\frac{1}{2}PQ \cdot QR \sin\angle PQR}{\frac{1}{2}PS \cdot SR \sin\angle PSR} = \frac{PQ \cdot QR}{PS \cdot SR}.$$

□

Applying the Claim to $ABYC$ and $XBYC$ respectively, we have $1 = \frac{BM}{MC} = \frac{AB \cdot BY}{AC \cdot CY}$ and $\frac{BD'}{D'C} = \frac{XB \cdot BY}{XC \cdot CY}$. These combine to give

$$\frac{BD'}{CD'} = \frac{XB}{XC} \cdot \frac{BY}{CY} = \frac{XB}{XC} \cdot \frac{AC}{AB}. \tag{1}$$

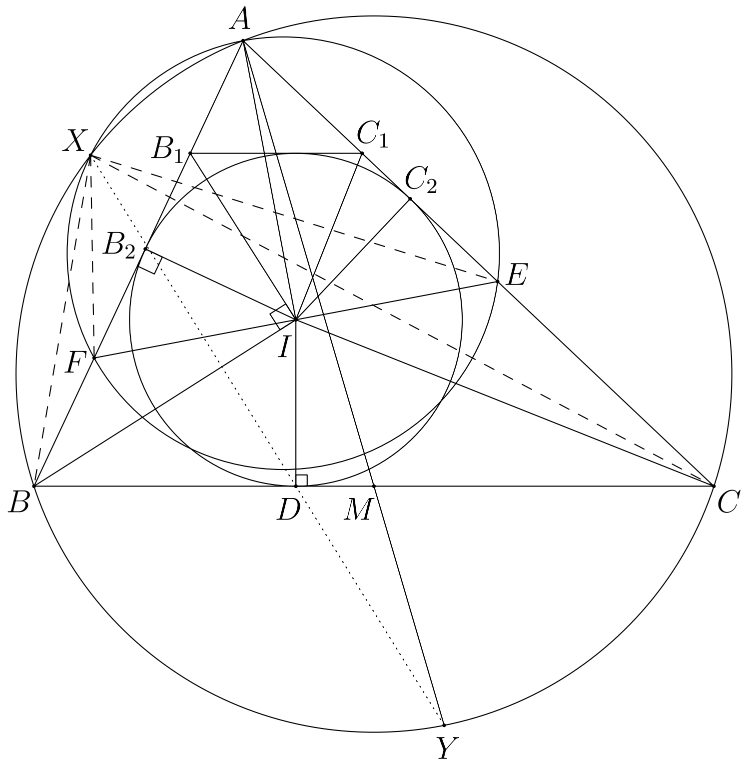
Next, we use directed angles to find that $\measuredangle XBF = \measuredangle XBA = \measuredangle XCA = \measuredangle XCE$ and $\measuredangle XFB = \measuredangle XFA = \measuredangle XEA = \measuredangle XEC$. This shows triangles XBF and XCE are directly similar. In particular, we have

$$\frac{XB}{XC} = \frac{BF}{CE}. \tag{2}$$

In the following, we give two ways to continue the proof.

• **Method 1.** Here is a geometrical method. As $\angle FIB = \angle AIB - 90^\circ = \frac{1}{2}\angle ACB = \angle ICB$ and $\angle FBI = \angle IBC$, the triangles FBI and IBC are similar. Analogously, triangles EIC and IBC are also similar. Hence, we get

$$\frac{FB}{IB} = \frac{BI}{BC} \quad \text{and} \quad \frac{EC}{IC} = \frac{IC}{BC}. \tag{3}$$



Next, construct a line parallel to BC and tangent to the incircle. Suppose it meets sides AB and AC at B_1 and C_1 respectively. Let the incircle touch AB and AC at B_2 and C_2 respectively. By homothety, the line B_1I is parallel to the external angle bisector of $\angle ABC$, and hence $\angle B_1IB = 90°$. Since $\angle BB_2I = 90°$, we get $BB_2 \cdot BB_1 = BI^2$, and similarly $CC_2 \cdot CC_1 = CI^2$. Hence,

$$\frac{BI^2}{CI^2} = \frac{BB_2 \cdot BB_1}{CC_2 \cdot CC_1} = \frac{BB_1}{CC_1} \cdot \frac{BD}{CD} = \frac{AB}{AC} \cdot \frac{BD}{CD}. \tag{4}$$

Combining (1), (2), (3) and (4), we conclude

$$\frac{BD'}{CD'} = \frac{XB}{XC} \cdot \frac{AC}{AB} = \frac{BF}{CE} \cdot \frac{AC}{AB} = \frac{BI^2}{CI^2} \cdot \frac{AC}{AB} = \frac{BD}{CD}$$

so that $D' = D$. The result then follows.

• **Method 2.** We continue the proof of Solution 1 using trigonometry. Let $\beta = \frac{1}{2}\angle ABC$ and $\gamma = \frac{1}{2}\angle ACB$. Observe that $\angle FIB = \angle AIB - 90° = \gamma$. Hence, $\frac{BF}{FI} = \frac{\sin \angle FIB}{\sin \angle IBF} = \frac{\sin\gamma}{\sin\beta}$. Similarly, $\frac{CE}{EI} = \frac{\sin\beta}{\sin\gamma}$. As $FI = EI$, we get

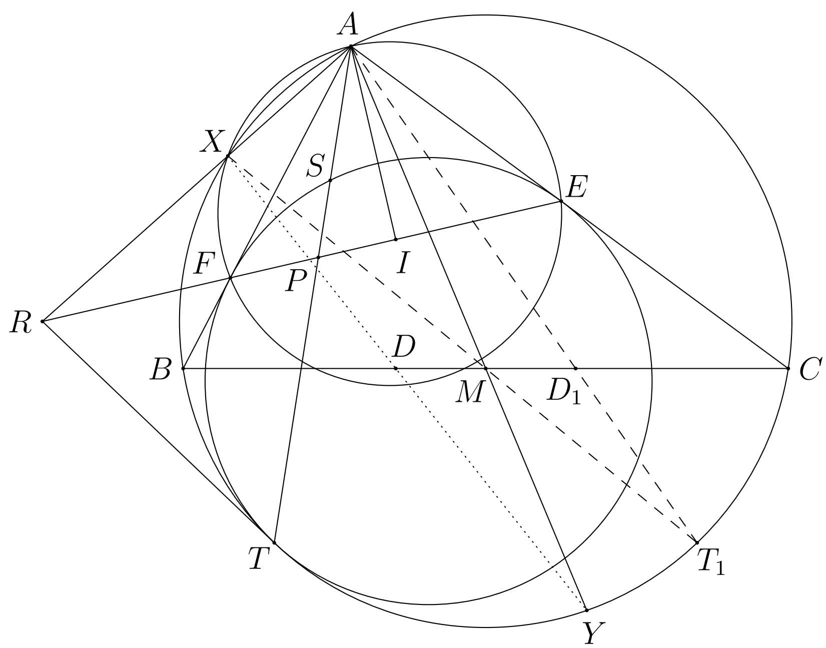
$$\frac{BF}{CE} = \frac{BF}{FI} \cdot \frac{EI}{CE} = \left(\frac{\sin\gamma}{\sin\beta}\right)^2. \tag{5}$$

Together with (1) and (2), we find that

$$\frac{BD'}{CD'} = \frac{AC}{AB} \cdot \left(\frac{\sin\gamma}{\sin\beta}\right)^2 = \frac{\sin 2\beta}{\sin 2\gamma} \cdot \left(\frac{\sin\gamma}{\sin\beta}\right)^2 = \frac{\tan\gamma}{\tan\beta} = \frac{ID/CD}{ID/BD} = \frac{BD}{CD}.$$

This shows $D' = D$ and the result follows.

Solution 2. Let ω_A be the A-mixtilinear incircle of triangle ABC. From the properties of mixtilinear incircles, ω_A touches sides AB and AC at F and E respectively. Suppose ω_A is tangent to Γ at T. Let AM meet Γ again at Y, and let D_1, T_1 be the reflections of D and T with respect to the perpendicular bisector of BC respectively. It is well-known that $\angle BAT = \angle D_1AC$ so that A, D_1, T_1 are collinear.



We then show that X, M, T_1 are collinear. Let R be the radical centre of ω_A, Γ and the circumcircle of triangle AEF. Then R lies on AX, EF and the tangent at T to Γ. Let AT meet ω_A again at S and meet EF at P. Obviously, $SFTE$ is a harmonic quadrilateral. Projecting from T, the pencil $(R, P; F, E)$ is harmonic. We further project the pencil onto Γ from A, so that $XBTC$ is a harmonic quadrilateral. As $TT_1 // BC$, the projection from T_1 onto BC maps T to a point at infinity, and hence maps X to the midpoint of BC, which is M. This shows X, M, T_1 are collinear.

We have two ways to finish the proof.

• **Method 1.** Note that both AY and XT_1 are chords of Γ passing through the midpoint M of the chord BC. By the Butterfly Theorem, XY and AT_1 cut BC at a pair of symmetric points with respect to M, and hence X, D, Y are collinear. The proof is thus complete.

• **Method 2.** Here, we finish the proof without using the Butterfly Theorem. As DTT_1D_1 is an isosceles trapezoid, we have

$$\measuredangle YTD = \measuredangle YTT_1 + \measuredangle T_1TD = \measuredangle YAT_1 + \measuredangle AD_1D = \measuredangle YMD$$

so that D, T, Y, M are concyclic. As X, M, T_1 are collinear, we have

$$\measuredangle AYD = \measuredangle MTD = \measuredangle D_1T_1M = \measuredangle AT_1X = \measuredangle AYX.$$

This shows X, D, Y are collinear.

G3. Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be *nice* if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

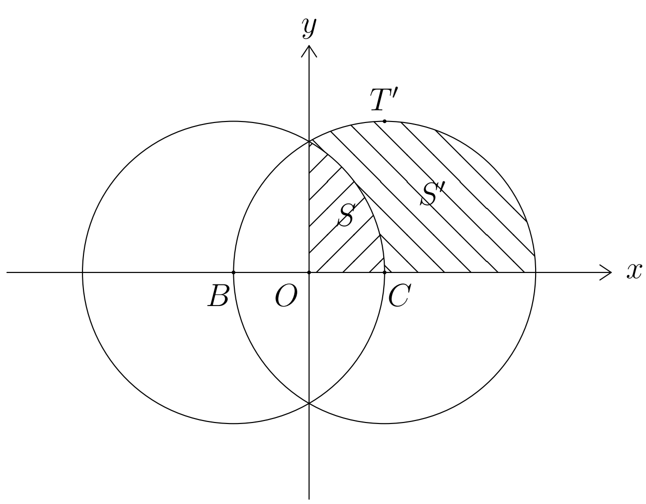
(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geqslant 0, y \geqslant 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

Solution. If in the similarity of $\triangle ABC$ and $\triangle P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$, BC corresponds to the longest side of $\triangle P_1P_2P_3$, then we have $BC \geqslant AB \geqslant AC$. The condition $BC \geqslant AB$ is equivalent to $(x+1)^2 + y^2 \leqslant 4$, while $AB \geqslant AC$ is trivially satisfied for any point in the first quadrant. Then we first define

$$S = \{(x, y) : (x+1)^2 + y^2 \leqslant 4, x \geqslant 0, y \geqslant 0\}.$$

Note that S is the intersection of a disk and the first quadrant, so it is bounded and convex, and we can choose any $T \in S$ to meet condition (i). For any point A in S, the relation $BC \geqslant AB \geqslant AC$ always holds. Therefore, the point A in (ii) is uniquely determined, while its existence is guaranteed by the above construction.



Next, if in the similarity of $\triangle A'BC$ and $\triangle P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$, BC corresponds to the second longest side of $\triangle P_1P_2P_3$, then we have $A'B \geqslant BC \geqslant A'C$. The two inequalities are equivalent to $(x+1)^2 + y^2 \geqslant 4$ and $(x-1)^2 + y^2 \leqslant 4$ respectively. Then we define

$$S' = \{(x, y) : (x+1)^2 + y^2 \geqslant 4, (x-1)^2 + y^2 \leqslant 4, x \geqslant 0, y \geqslant 0\}.$$

The boundedness condition is satisfied while (ii) can be argued as in the previous case. For (i), note that S' contains points inside the disk $(x-1)^2+y^2 \leqslant 4$ and outside the disk $(x+1)^2+y^2 \geqslant 4$. This shows we can take $T'=(1,2)$ in (i), which is the topmost point of the circle $(x-1)^2+y^2=4$.

It remains to check that the product $BA \cdot BA'$ is a constant. Suppose we are given a triangle $P_1P_2P_3$ with $P_1P_2 \geqslant P_2P_3 \geqslant P_3P_1$. By the similarity, we have

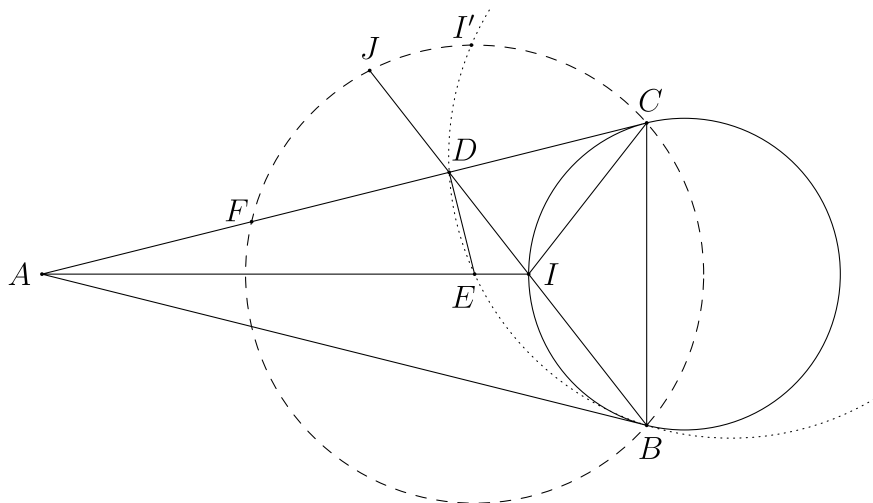
$$BA = BC \cdot \frac{P_2P_3}{P_1P_2} \quad \text{and} \quad BA' = BC \cdot \frac{P_1P_2}{P_2P_3}.$$

Thus $BA \cdot BA' = BC^2 = 4$, which is certainly independent of the triangle $P_1P_2P_3$.

Comment. The original version of this problem includes the condition that the interiors of S and S' are disjoint. We remove this condition since it is hard to define the interior of a point set rigorously.

G4. Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

Solution 1.



Let Γ be the circle with centre E passing through B and C. Since $ED \perp AC$, the point F symmetric to C with respect to D lies on Γ. From $\angle DCI = \angle ICB = \angle CBI$, the line DC is a tangent to the circumcircle of triangle IBC. Let J be the symmetric point of I with respect to D. Using directed lengths, from

$$DC \cdot DF = -DC^2 = -DI \cdot DB = DJ \cdot DB,$$

the point J also lies on Γ. Let I' be the reflection of I in AC. Since IJ and CF bisect each other, $CJFI$ is a parallelogram. From $\angle FI'C = \angle CIF = \angle FJC$, we find that I' lies on Γ. This gives $EI' = EB$.

Note that AC is the internal angle bisector of $\angle BDI'$. This shows DE is the external angle bisector of $\angle BDI'$ as $DE \perp AC$. Together with $EI' = EB$, it is well-known that E lies on the circumcircle of triangle BDI'.

Solution 2. Let I' be the reflection of I in AC and let S be the intersection of $I'C$ and AI. Using directed angles, we let $\theta = \measuredangle ACI = \measuredangle ICB = \measuredangle CBI$. We have

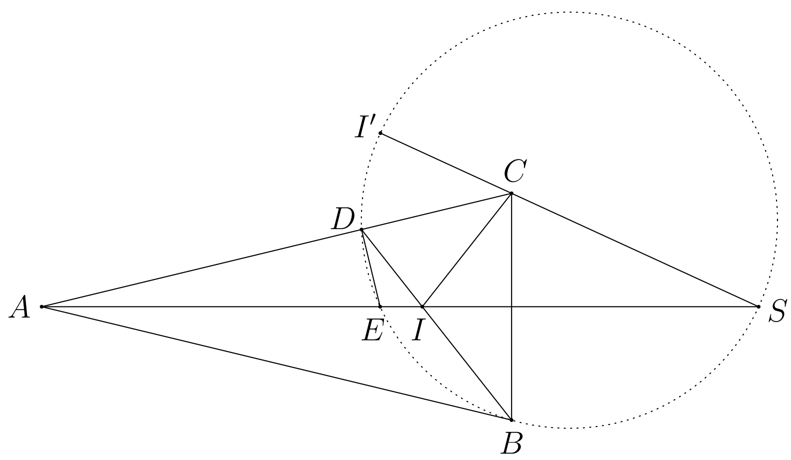
$$\measuredangle I'SE = \measuredangle I'CA + \measuredangle CAI = \theta + \left(\frac{\pi}{2} + 2\theta\right) = 3\theta + \frac{\pi}{2}$$

and

$$\measuredangle I'DE = \measuredangle I'DC + \frac{\pi}{2} = \measuredangle CDI + \frac{\pi}{2} = \measuredangle DCB + \measuredangle CBD + \frac{\pi}{2} = 3\theta + \frac{\pi}{2}.$$

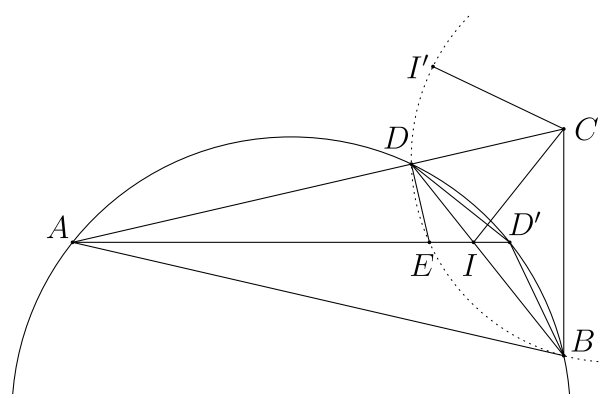
This shows I', D, E, S are concyclic.

Next, we find $\measuredangle I'SB = 2\measuredangle I'SE = 6\theta$ and $\measuredangle I'DB = 2\measuredangle CDI = 6\theta$. Therefore, I', D, B, S are concyclic so that I', D, E, B, S lie on the same circle. The result then follows.



Comment. The point S constructed in Solution 2 may lie on the same side as A of BC. Also, since S lies on the circumcircle of the non-degenerate triangle BDE, we implicitly know that S is not an ideal point. Indeed, one can verify that $I'C$ and AI are parallel if and only if triangle ABC is equilateral.

Solution 3. Let I' be the reflection of I in AC, and let D' be the second intersection of AI and the circumcircle of triangle ABD. Since AD' bisects $\angle BAD$, point D' is the midpoint of the arc BD and $DD' = BD' = CD'$. Obviously, A, E, D' lie on AI in this order.



We find that $\angle ED'D = \angle AD'D = \angle ABD = \angle IBC = \angle ICB$. Next, since D' is the circumcentre of triangle BCD, we have $\angle EDD' = 90^\circ - \angle D'DC = \angle CBD = \angle IBC$. The two relations show that triangles $ED'D$ and ICB are similar. Therefore, we have

$$\frac{BC}{CI'} = \frac{BC}{CI} = \frac{DD'}{D'E} = \frac{BD'}{D'E}.$$

Also, we get

$$\angle BCI' = \angle BCA + \angle ACI' = \angle BCA + \angle ICA = \angle BCA + \angle DBC = \angle BDA = \angle BD'E.$$

These show triangles BCI' and $BD'E$ are similar, and hence triangles BCD' and $BI'E$ are similar. As BCD' is isosceles, we obtain $BE = I'E$.

As DE is the external angle bisector of $\angle BDI'$ and $EI' = EB$, we know that E lies on the circumcircle of triangle BDI'.

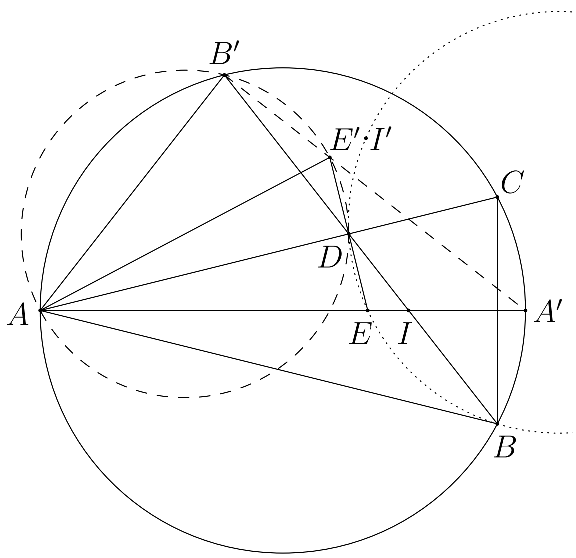
<u>Solution 4</u>. Let AI and BI meet the circumcircle of triangle ABC again at A' and B' respectively, and let E' be the reflection of E in AC. From

$$\begin{aligned}\angle B'AE' = \angle B'AD - \angle E'AD = \frac{\angle ABC}{2} - \frac{\angle BAC}{2} = 90^\circ - \angle BAC - \frac{\angle ABC}{2} \\ = 90^\circ - \angle B'DA = \angle B'DE',\end{aligned}$$

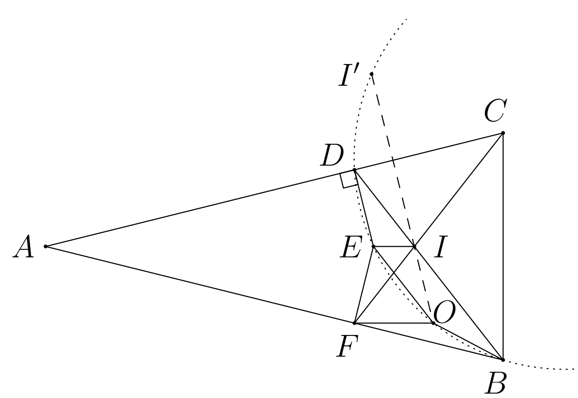
points B', A, D, E' are concyclic. Then

$$\angle DB'E' = \angle DAE' = \frac{\angle BAC}{2} = \angle BAA' = \angle DB'A'$$

and hence B', E', A' are collinear. It is well-known that $A'B'$ is the perpendicular bisector of CI, so that $CE' = IE'$. Let I' be the reflection of I in AC. This implies $BE = CE = I'E$. As DE is the external angle bisector of $\angle BDI'$ and $EI' = EB$, we know that E lies on the circumcircle of triangle BDI'.



<u>Solution 5</u>. Let F be the intersection of CI and AB. Clearly, F and D are symmetric with respect to AI. Let O be the circumcentre of triangle BIF, and let I' be the reflection of I in AC.



From $\angle BFO = 90^\circ - \angle FIB = \frac{1}{2}\angle BAC = \angle BAI$, we get $EI//FO$. Also, from the relation $\angle OIB = 90^\circ - \angle BFI = 90^\circ - \angle CDI = \angle I'ID$, we know that O, I, I' are collinear.

Note that $ED//OI$ since both are perpendicular to AC. Then $\angle FEI = \angle DEI = \angle OIE$. Together with $EI//FO$, the quadrilateral $EFOI$ is an isosceles trapezoid. Therefore, we find that $\angle DIE = \angle FIE = \angle OEI$ so $OE//ID$. Then $DEOI$ is a parallelogram. Hence, we have $DI' = DI = EO$, which shows $DEOI'$ is an isosceles trapezoid. In addition, $ED = OI = OB$ and $OE//BD$ imply $EOBD$ is another isosceles trapezoid. In particular, both $DEOI'$ and $EOBD$ are cyclic. This shows B, D, E, I' are concyclic.

<u>Solution 6</u>. Let I' be the reflection of I in AC. Denote by T and M the projections from I to sides AB and BC respectively. Since BI is the perpendicular bisector of TM, we have

$$DT = DM. \tag{1}$$

Since $\angle ADE = \angle ATI = 90^\circ$ and $\angle DAE = \angle TAI$, we have $\triangle ADE \sim \triangle ATI$. This shows $\frac{AD}{AE} = \frac{AT}{AI} = \frac{AT}{AI'}$. Together with $\angle DAT = 2\angle DAE = \angle EAI'$, this yields $\triangle DAT \sim \triangle EAI'$. In particular, we have

$$\frac{DT}{EI'} = \frac{AT}{AI'} = \frac{AT}{AI}. \tag{2}$$

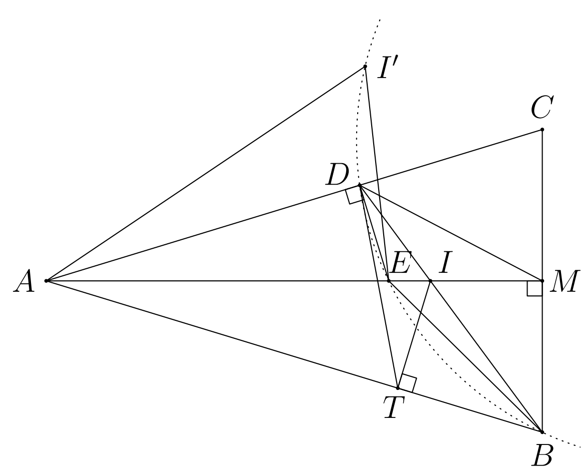
Obviously, the right-angled triangles AMB and ATI are similar. Then we get

$$\frac{AM}{AB} = \frac{AT}{AI}. \tag{3}$$

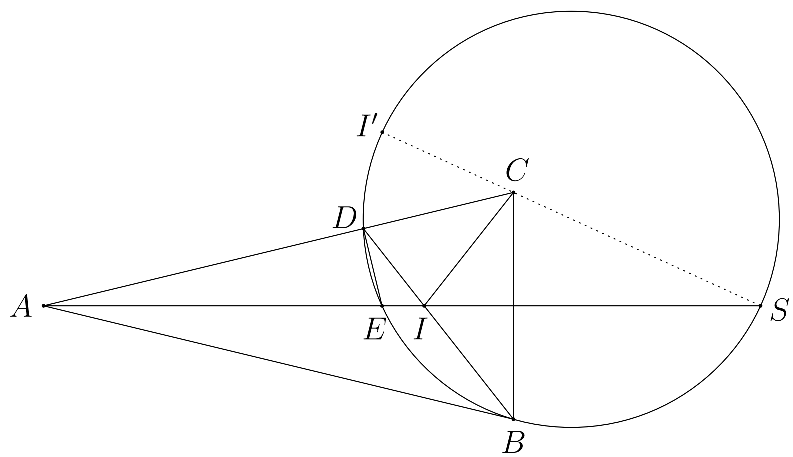
Next, from $\triangle AMB \sim \triangle ATI \sim \triangle ADE$, we have $\frac{AM}{AB} = \frac{AD}{AE}$ so that $\triangle ADM \sim \triangle AEB$. It follows that

$$\frac{DM}{EB} = \frac{AM}{AB}. \tag{4}$$

Combining (1), (2), (3) and (4), we get $EB = EI'$. As DE is the external angle bisector of $\angle BDI'$, we know that E lies on the circumcircle of triangle BDI'.



Comment. A stronger version of this problem is to ask the contestants to prove the reflection of I in AC lies on the circumcircle of triangle BDE if and only if $AB = AC$. Some of the above solutions can be modified to prove the converse statement to the original problem. For example, we borrow some ideas from Solution 2 to establish the converse as follows.



Let I' be the reflection of I in AC and suppose B, E, D, I' lie on a circle Γ. Let AI meet Γ again at S. As DE is the external angle bisector of $\angle BDI'$, we have $EB = EI'$. Using directed angles, we get

$$\measuredangle CI'S = \measuredangle CI'D + \measuredangle DI'S = \measuredangle DIC + \measuredangle DES = \measuredangle DIC + \measuredangle DEA = \measuredangle DIC + \measuredangle DCB = 0.$$

This means I', C, S are collinear. From this we get $\measuredangle BSE = \measuredangle ESI' = \measuredangle ESC$ and hence AS bisects both $\angle BAC$ and $\angle BSC$. Clearly, S and A are distinct points. It follows that $\triangle BAS \cong \triangle CAS$ and thus $AB = AC$.

As in some of the above solutions, an obvious way to prove the stronger version is to establish the following equivalence: $BE = EI'$ if and only if $AB = AC$. In addition to the ideas used in those solutions, one can use trigonometry to express the lengths of BE and EI' in terms of the side lengths of triangle ABC and to establish the equivalence.

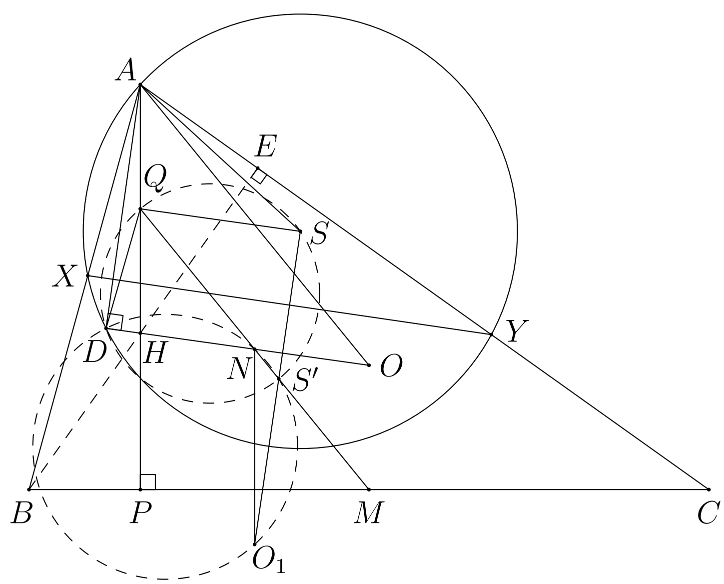
G5. Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.

Solution 1. Denote the orthocentre and circumcentre of triangle ABC by H and O respectively. Let Q be the midpoint of AH and N be the nine-point centre of triangle ABC. It is known that Q lies on the nine-point circle of triangle ABC, N is the midpoint of QM and that QM is parallel to AO.

Let the perpendicular from S to XY meet line QM at S'. Let E be the foot of altitude from B to side AC. Since Q and S lie on the perpendicular bisector of AD, using directed angles, we have

$$\begin{aligned}\measuredangle SDQ &= \measuredangle QAS = \measuredangle XAS - \measuredangle XAQ = \left(\frac{\pi}{2} - \measuredangle AYX\right) - \measuredangle BAP = \measuredangle CBA - \measuredangle AYX \\ &= (\measuredangle CBA - \measuredangle ACB) - \measuredangle BCA - \measuredangle AYX = \measuredangle PEM - (\measuredangle BCA + \measuredangle AYX) \\ &= \measuredangle PQM - \angle(BC, XY) = \frac{\pi}{2} - \angle(S'Q, BC) - \angle(BC, XY) = \measuredangle SS'Q.\end{aligned}$$

This shows D, S', S, Q are concyclic.



Let the perpendicular from N to BC intersect line SS' at O_1. (Note that the two lines coincide when S is the midpoint of AO, in which case the result is true since the circumcentre of triangle XSY must lie on this line.) It suffices to show that O_1 is the circumcentre of triangle XSY since N lies on the perpendicular bisector of PM. From

$$\measuredangle DS'O_1 = \measuredangle DQS = \measuredangle SQA = \angle(SQ, QA) = \angle(OD, O_1N) = \measuredangle DNO_1$$

since $SQ//OD$ and $QA//O_1N$, we know that D, O_1, S', N are concyclic. Therefore, we get

$$\measuredangle SDS' = \measuredangle SQS' = \angle(SQ, QS') = \angle(ND, NS') = \measuredangle DNS',$$

so that SD is a tangent to the circle through D, O_1, S', N. Then we have

$$SS' \cdot SO_1 = SD^2 = SX^2. \tag{1}$$

Next, we show that S and S' are symmetric with respect to XY. By the Sine Law, we have

$$\frac{SS'}{\sin\angle SQS'} = \frac{SQ}{\sin\angle SS'Q} = \frac{SQ}{\sin\angle SDQ} = \frac{SQ}{\sin\angle SAQ} = \frac{SA}{\sin\angle SQA}.$$

It follows that

$$SS' = SA \cdot \frac{\sin\angle SQS'}{\sin\angle SQA} = SA \cdot \frac{\sin\angle HOA}{\sin\angle OHA} = SA \cdot \frac{AH}{AO} = SA \cdot 2\cos A,$$

which is twice the distance from S to XY. Note that S and C lie on the same side of the perpendicular bisector of PM if and only if $\angle SAC < \angle OAC$ if and only if $\angle YXA > \angle CBA$. This shows S and O_1 lie on different sides of XY. As S' lies on ray SO_1, it follows that S and S' cannot lie on the same side of XY. Therefore, S and S' are symmetric with respect to XY.

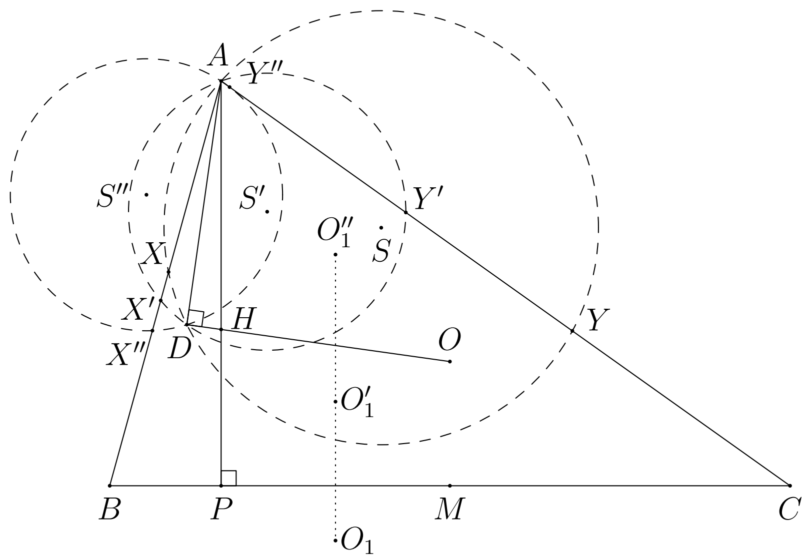
Let d be the diameter of the circumcircle of triangle XSY. As SS' is twice the distance from S to XY and $SX = SY$, we have $SS' = 2\frac{SX^2}{d}$. It follows from (1) that $d = 2SO_1$. As SO_1 is the perpendicular bisector of XY, point O_1 is the circumcentre of triangle XSY.

Solution 2. Denote the orthocentre and circumcentre of triangle ABC by H and O respectively. Let O_1 be the circumcentre of triangle XSY. Consider two other possible positions of S. We name them S' and S'' and define the analogous points $X', Y', O_1', X'', Y''O_1''$ accordingly. Note that S, S', S'' lie on the perpendicular bisector of AD.

As XX' and YY' meet at A and the circumcircles of triangles AXY and $AX'Y'$ meet at D, there is a spiral similarity with centre D mapping XY to $X'Y'$. We find that

$$\measuredangle SXY = \frac{\pi}{2} - \measuredangle YAX = \frac{\pi}{2} - \measuredangle Y'AX' = \measuredangle S'X'Y'$$

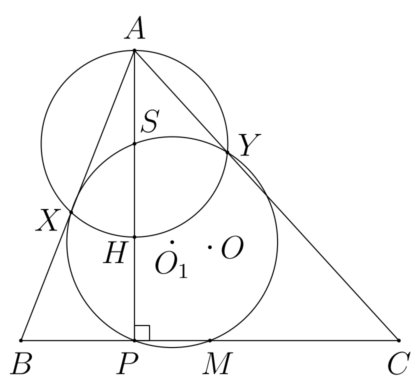
and similarly $\measuredangle SYX = \measuredangle S'Y'X'$. This shows triangles SXY and $S'X'Y'$ are directly similar. Then the spiral similarity with centre D takes points S, X, Y, O_1 to S', X', Y', O_1'. Similarly, there is a spiral similarity with centre D mapping S, X, Y, O_1 to S'', X'', Y'', O_1''. From these, we see that there is a spiral similarity taking the corresponding points S, S', S'' to points O_1, O_1', O_1''. In particular, O_1, O_1', O_1'' are collinear.

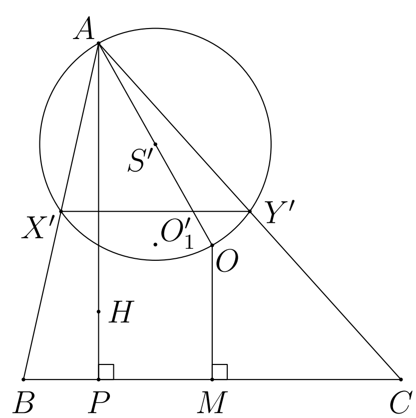


It now suffices to show that O_1 lies on the perpendicular bisector of PM for two special cases.

Firstly, we take S to be the midpoint of AH. Then X and Y are the feet of altitudes from C and B respectively. It is well-known that the circumcircle of triangle XSY is the nine-point circle of triangle ABC. Then O_1 is the nine-point centre and $O_1P = O_1M$. Indeed, P and M also lies on the nine-point circle.

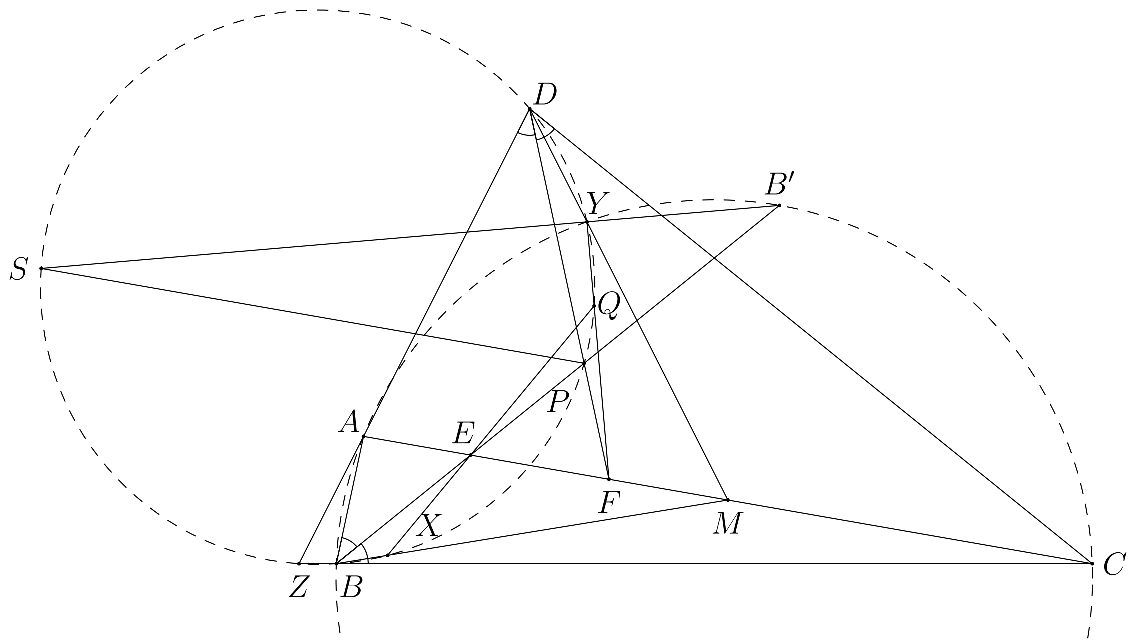
Secondly, we take S' to be the midpoint of AO. Then X' and Y' are the midpoints of AB and AC respectively. Then $X'Y'//BC$. Clearly, S' lies on the perpendicular bisector of PM. This shows the perpendicular bisectors of $X'Y'$ and PM coincide. Hence, we must have $O_1'P = O_1'M$.





G6. Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

Solution 1.



Let ω_1 be the circumcircle of triangle ABC. We first prove that Y lies on ω_1. Let Y' be the point on ray MD such that $MY' \cdot MD = MA^2$. Then triangles MAY' and MDA are oppositely similar. Since $MC^2 = MA^2 = MY' \cdot MD$, triangles MCY' and MDC are also oppositely similar. Therefore, using directed angles, we have

$$\measuredangle AY'C = \measuredangle AY'M + \measuredangle MY'C = \measuredangle MAD + \measuredangle DCM = \measuredangle CDA = \measuredangle ABC$$

so that Y' lies on ω_1.

Let Z be the intersection point of lines BC and AD. Since $\measuredangle PDZ = \measuredangle PBC = \measuredangle PBZ$, point Z lies on ω. In addition, from $\measuredangle Y'BZ = \measuredangle Y'BC = \measuredangle Y'AC = \measuredangle Y'AM = \measuredangle Y'DZ$, we also know that Y' lies on ω. Note that $\angle ADC$ is acute implies $MA \neq MD$ so $MY' \neq MD$. Therefore, Y' is the second intersection of DM and ω. Then $Y' = Y$ and hence Y lies on ω_1.

Next, by the Angle Bisector Theorem and the similar triangles, we have

$$\frac{FA}{FC} = \frac{AD}{CD} = \frac{AD}{AM} \cdot \frac{CM}{CD} = \frac{YA}{YM} \cdot \frac{YM}{YC} = \frac{YA}{YC}.$$

Hence, FY is the internal angle bisector of $\angle AYC$.

Let B' be the second intersection of the internal angle bisector of $\angle CBA$ and ω_1. Then B' is the midpoint of arc AC not containing B. Therefore, YB' is the external angle bisector of $\angle AYC$, so that $B'Y \perp FY$.

Denote by l the line through P parallel to AC. Suppose l meets line $B'Y$ at S. From

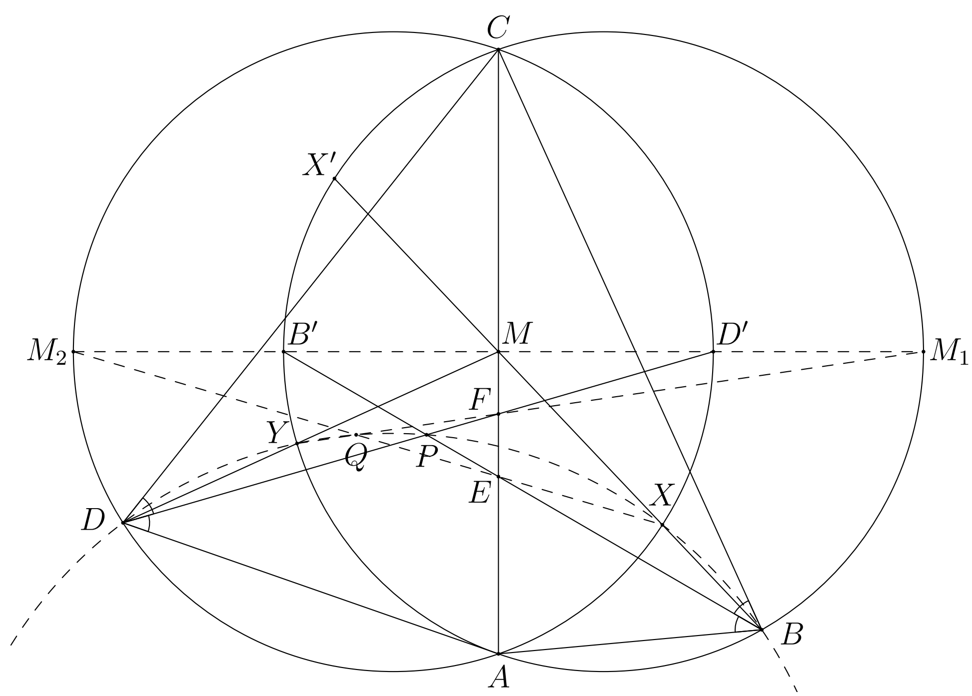
$$\begin{aligned}\measuredangle PSY &= \angle(AC, B'Y) = \measuredangle ACY + \measuredangle CYB' = \measuredangle ACY + \measuredangle CAB' = \measuredangle ACY + \measuredangle B'CA \\ &= \measuredangle B'CY = \measuredangle B'BY = \measuredangle PBY,\end{aligned}$$

the point S lies on ω. Similarly, the line through X perpendicular to XE also passes through the second intersection of l and ω, which is the point S. From $QY \perp YS$ and $QX \perp XS$, point Q lies on ω and QS is a diameter of ω. Therefore, $PQ \perp PS$ so that $PQ \perp AC$.

Solution 2. Denote by ω_1 and ω_2 the circumcircles of triangles ABC and ADC respectively. Since $\angle ABC = \angle ADC$, we know that ω_1 and ω_2 are symmetric with respect to the midpoint M of AC.

Firstly, we show that X lies on ω_2. Let X_1 be the second intersection of ray MB and ω_2 and X' be its symmetric point with respect to M. Then X' lies on ω_1 and $X'AX_1C$ is a parallelogram. Hence, we have

$$\begin{aligned}\measuredangle DX_1B &= \measuredangle DX_1A + \measuredangle AX_1B = \measuredangle DCA + \measuredangle AX_1X' = \measuredangle DCA + \measuredangle CX'X_1 \\ &= \measuredangle DCA + \measuredangle CAB = \angle(CD, AB).\end{aligned}$$



Also, we have

$$\measuredangle DPB = \measuredangle PDC + \angle(CD, AB) + \measuredangle ABP = \angle(CD, AB).$$

These yield $\measuredangle DX_1B = \measuredangle DPB$ and hence X_1 lies on ω. It follows that $X_1 = X$ and X lies on ω_2. Similarly, Y lies on ω_1.

Next, we prove that Q lies on ω. Suppose the perpendicular bisector of AC meet ω_1 at B' and M_1 and meet ω_2 at D' and M_2, so that B, M_1 and D' lie on the same side of AC. Note that B' lies on the angle bisector of $\angle ABC$ and similarly D' lies on DP.

If we denote the area of $W_1W_2W_3$ by $[W_1W_2W_3]$, then

$$\frac{BA \cdot X'A}{BC \cdot X'C} = \frac{\frac{1}{2}BA \cdot X'A \sin \angle BAX'}{\frac{1}{2}BC \cdot X'C \sin \angle BCX'} = \frac{[BAX']}{[BCX']} = \frac{MA}{MC} = 1.$$

As BE is the angle bisector of $\angle ABC$, we have

$$\frac{EA}{EC} = \frac{BA}{BC} = \frac{X'C}{X'A} = \frac{XA}{XC}.$$

Therefore, XE is the angle bisector of $\angle AXC$, so that M_2 lies on the line joining X, E, Q. Analogously, M_1, F, Q, Y are collinear. Thus,

$$\begin{aligned}\measuredangle XQY &= \measuredangle M_2QM_1 = \measuredangle QM_2M_1 + \measuredangle M_2M_1Q = \measuredangle XM_2D' + \measuredangle B'M_1Y \\ &= \measuredangle XDD' + \measuredangle B'BY = \measuredangle XDP + \measuredangle PBY = \measuredangle XBP + \measuredangle PBY = \measuredangle XBY,\end{aligned}$$

which implies Q lies on ω.

Finally, as M_1 and M_2 are symmetric with respect to M, the quadrilateral $X'M_2XM_1$ is a parallelogram. Consequently,

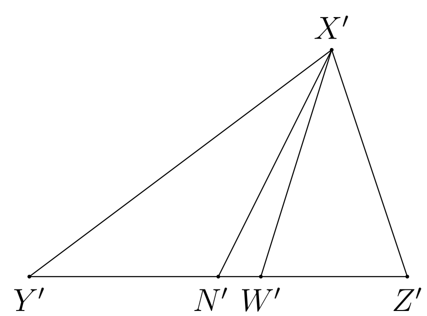
$$\measuredangle XQP = \measuredangle XBP = \measuredangle X'BB' = \measuredangle X'M_1B' = \measuredangle XM_2M_1.$$

This shows $QP//M_2M_1$. As $M_2M_1 \perp AC$, we get $QP \perp AC$.

<u>Solution 3</u>. We first state two results which will be needed in our proof.

• **Claim 1.** In $\triangle X'Y'Z'$ with $X'Y' \neq X'Z'$, let N' be the midpoint of $Y'Z'$ and W' be the foot of internal angle bisector from X'. Then $\tan^2 \measuredangle W'X'Z' = \tan \measuredangle N'X'W' \tan \measuredangle Z'W'X'$.

Proof.



Without loss of generality, assume $X'Y' > X'Z'$. Then W' lies between N' and Z'. The signs of both sides agree so it suffices to establish the relation for ordinary angles. Let $\angle W'X'Z' = \alpha$, $\angle N'X'W' = \beta$ and $\angle Z'W'X' = \gamma$. We have

$$\frac{\sin(\gamma - \alpha)}{\sin(\alpha - \beta)} = \frac{N'X'}{N'Y'} = \frac{N'X'}{N'Z'} = \frac{\sin(\gamma + \alpha)}{\sin(\alpha + \beta)}.$$

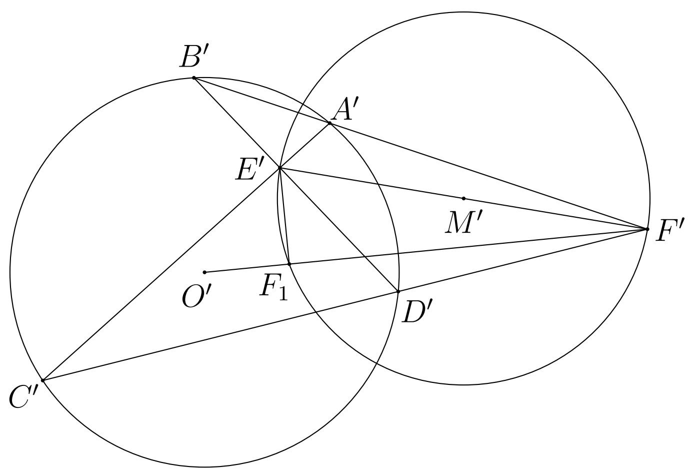
This implies

$$\frac{\tan\gamma-\tan\alpha}{\tan\gamma+\tan\alpha}=\frac{\sin\gamma\cos\alpha-\cos\gamma\sin\alpha}{\sin\gamma\cos\alpha+\cos\gamma\sin\alpha}=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}=\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}.$$

Expanding and simplifying, we get the desired result $\tan^2\alpha=\tan\beta\tan\gamma$. $\square$

• **Claim 2.** Let $A'B'C'D'$ be a quadrilateral inscribed in circle Γ. Let diagonals $A'C'$ and $B'D'$ meet at E', and F' be the intersection of lines $A'B'$ and $C'D'$. Let M' be the midpoint of $E'F'$. Then the power of M' with respect to Γ is equal to $(M'E')^2$.

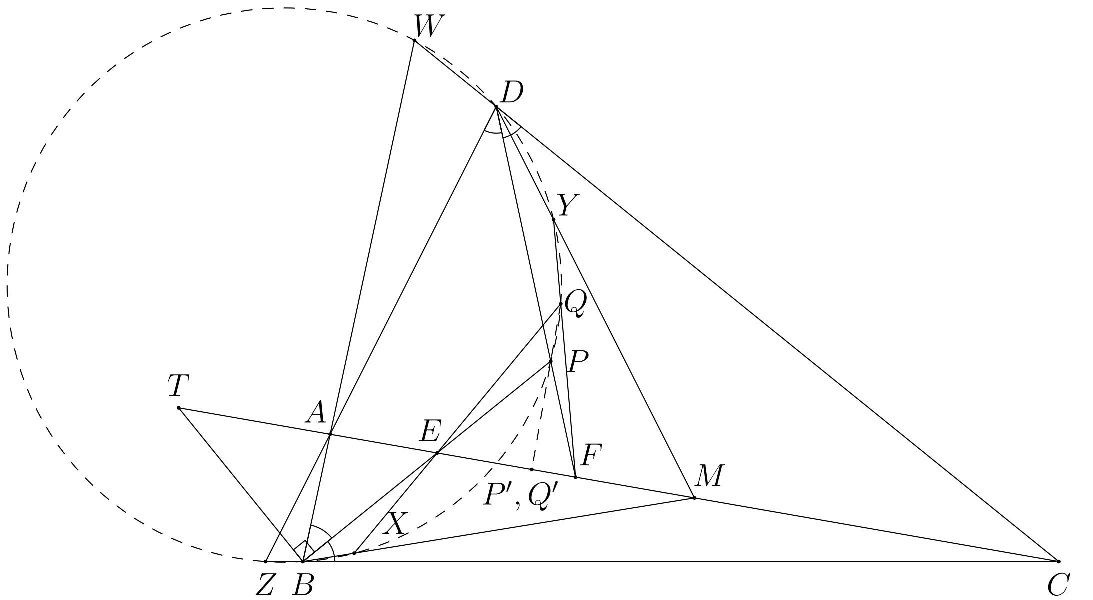
Proof.



Let O' be the centre of Γ and let Γ' be the circle with centre M' passing through E'. Let F_1 be the inversion image of F' with respect to Γ. It is well-known that E' lies on the polar of F' with respect to Γ. This shows $E'F_1\perp O'F'$ and hence F_1 lies on Γ'. It follows that the inversion image of Γ' with respect to Γ is Γ' itself. This shows Γ' is orthogonal to Γ, and thus the power of M' with respect to Γ is the square of radius of Γ', which is $(M'E')^2$. $\square$

We return to the main problem. Let Z be the intersection of lines AD and BC, and W be the intersection of lines AB and CD. Since $\measuredangle PDZ=\measuredangle PBC=\measuredangle PBZ$, point Z lies on ω. Similarly, W lies on ω. Applying Claim 2 to the cyclic quadrilateral $ZBDW$, we know that the power of M with respect to ω is MA^2. Hence, $MX\cdot MB=MA^2$.

Suppose the line through B perpendicular to BE meets line AC at T. Then BE and BT are the angle bisectors of $\angle CBA$. This shows $(T,E;A,C)$ is harmonic. Thus, we have $ME\cdot MT=MA^2=MX\cdot MB$. It follows that E,T,B,X are concyclic.



The result is trivial for the special case $AD = CD$ since P, Q lie on the perpendicular bisector of AC in that case. Similarly, the case $AB = CB$ is trivial. It remains to consider the general cases where we can apply Claim 1 in the latter part of the proof.

Let the projections from P and Q to AC be P' and Q' respectively. Then $PQ \perp AC$ if and only if $P' = Q'$ if and only if $\frac{EP'}{FP'} = \frac{EQ'}{FQ'}$ in terms of directed lengths. Note that

$$\frac{EP'}{FP'} = \frac{\tan \measuredangle EFP}{\tan \measuredangle FEP} = \frac{\tan \measuredangle AFD}{\tan \measuredangle AEB}.$$

Next, we have $\frac{EQ'}{FQ'} = \frac{\tan \measuredangle EFQ}{\tan \measuredangle FEQ}$ where $\measuredangle FEQ = \measuredangle TEX = \measuredangle TBX = \frac{\pi}{2} + \measuredangle EBM$ and by symmetry $\measuredangle EFQ = \frac{\pi}{2} + \measuredangle FDM$. Combining all these, it suffices to show

$$\frac{\tan \measuredangle AFD}{\tan \measuredangle AEB} = \frac{\tan \measuredangle MBE}{\tan \measuredangle MDF}.$$

We now apply Claim 1 twice to get

$$\tan \measuredangle AFD \tan \measuredangle MDF = \tan^2 \measuredangle FDC = \tan^2 \measuredangle EBA = \tan \measuredangle MBE \tan \measuredangle AEB.$$

The result then follows.

G7. Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

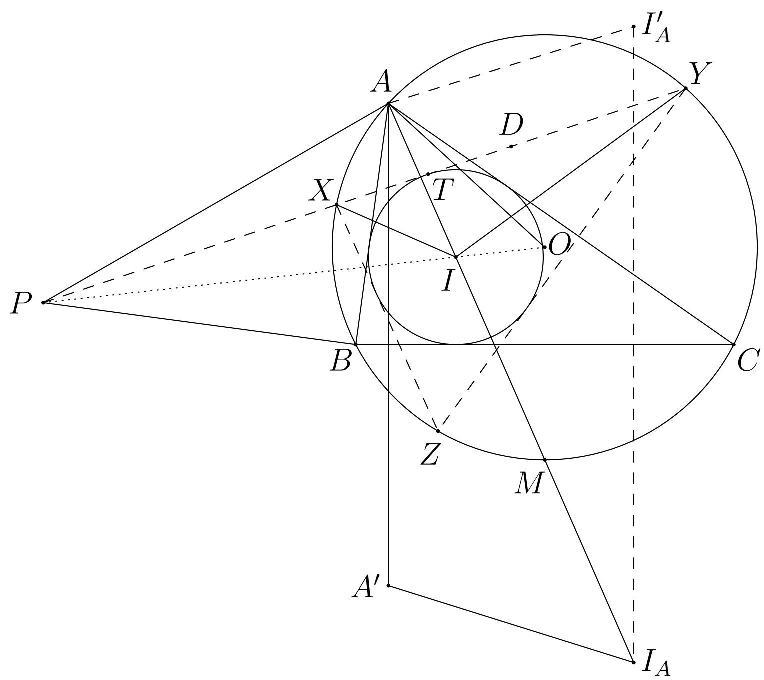
(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120^\circ$.

Solution 1.

(a) Let A' be the reflection of A in BC and let M be the second intersection of line AI and the circumcircle Γ of triangle ABC. As triangles ABA' and AOC are isosceles with $\angle ABA' = 2\angle ABC = \angle AOC$, they are similar to each other. Also, triangles ABI_A and AIC are similar. Therefore we have

$$\frac{AA'}{AI_A} = \frac{AA'}{AB} \cdot \frac{AB}{AI_A} = \frac{AC}{AO} \cdot \frac{AI}{AC} = \frac{AI}{AO}.$$

Together with $\angle A'AI_A = \angle IAO$, we find that triangles $AA'I_A$ and AIO are similar.



Denote by P' the intersection of line AP and line OI. Using directed angles, we have

$$\begin{aligned}\measuredangle MAP' &= \measuredangle I'_A A I_A = \measuredangle I'_A AA' - \measuredangle I_A AA' = \measuredangle AA'I_A - \angle(AM, OM) \\ &= \measuredangle AIO - \measuredangle AMO = \measuredangle MOP'.\end{aligned}$$

This shows M, O, A, P' are concyclic.

Denote by R and r the circumradius and inradius of triangle ABC. Then

$$IP' = \frac{IA \cdot IM}{IO} = \frac{IO^2 - R^2}{IO}$$

is independent of A. Hence, BP also meets line OI at the same point P' so that $P' = P$, and P lies on OI.

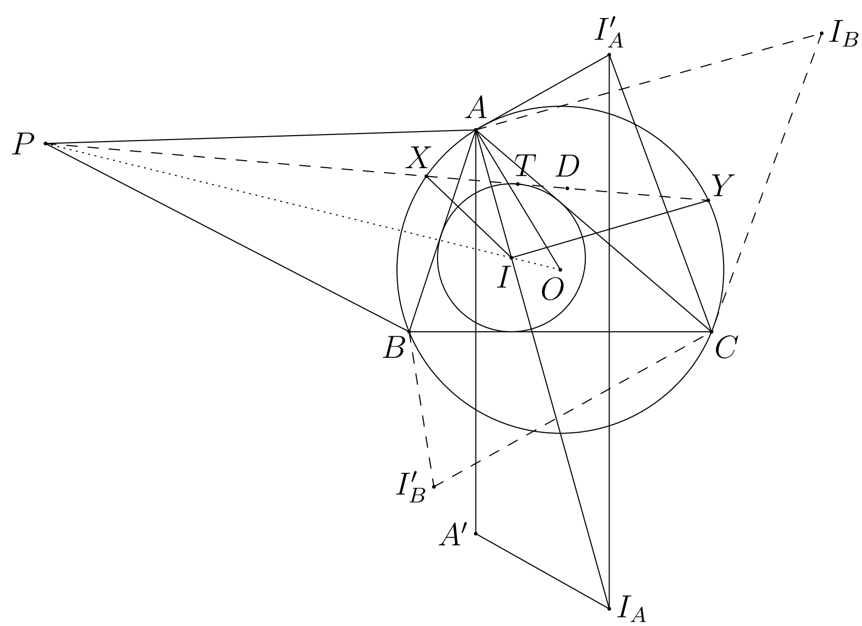
(b) By Poncelet's Porism, the other tangents to the incircle of triangle ABC from X and Y meet at a point Z on Γ. Let T be the touching point of the incircle to XY, and let D be the midpoint of XY. We have

$$\begin{aligned} OD &= IT \cdot \frac{OP}{IP} = r\left(1 + \frac{OI}{IP}\right) = r\left(1 + \frac{OI^2}{OI \cdot IP}\right) = r\left(1 + \frac{R^2 - 2Rr}{R^2 - IO^2}\right) \\ &= r\left(1 + \frac{R^2 - 2Rr}{2Rr}\right) = \frac{R}{2} = \frac{OX}{2}. \end{aligned}$$

This shows $\angle XZY = 60^\circ$ and hence $\angle XIY = 120^\circ$.

Solution 2.

(a) Note that triangles AI_BC and I_ABC are similar since their corresponding interior angles are equal. Therefore, the four triangles AI'_BC, AI_BC, I_ABC and I'_ABC are all similar. From $\triangle AI'_BC \sim \triangle I'_ABC$, we get $\triangle AI'_AC \sim \triangle I'_BBC$. From $\measuredangle ABP = \measuredangle I'_BBC = \measuredangle AI'_AC$ and $\measuredangle BAP = \measuredangle I'_AAC$, the triangles ABP and AI'_AC are directly similar.



Consider the inversion with centre A and radius $\sqrt{AB \cdot AC}$ followed by the reflection in AI. Then B and C are mapped to each other, and I and I_A are mapped to each other.

From the similar triangles obtained, we have $AP \cdot AI'_A = AB \cdot AC$ so that P is mapped to I'_A under the transformation. In addition, line AO is mapped to the altitude from A, and hence O is mapped to the reflection of A in BC, which we call point A'. Note that $AA'I_AI'_A$ is an isosceles trapezoid, which shows it is inscribed in a circle. The preimage of this circle is a straight line, meaning that O, I, P are collinear.

(b) Denote by R and r the circumradius and inradius of triangle ABC. Note that by the above transformation, we have $\triangle APO \sim \triangle AA'I'_A$ and $\triangle AA'I_A \sim \triangle AIO$. Therefore, we find that

$$PO = A'I'_A \cdot \frac{AO}{AI'_A} = AI_A \cdot \frac{AO}{A'I_A} = \frac{AI_A}{A'I_A} \cdot AO = \frac{AO}{IO} \cdot AO.$$

This shows $PO \cdot IO = R^2$, and it follows that P and I are mapped to each other under the inversion with respect to the circumcircle Γ of triangle ABC. Then $PX \cdot PY$, which is the power of P with respect to Γ, equals $PI \cdot PO$. This yields X, I, O, Y are concyclic.

Let T be the touching point of the incircle to XY, and let D be the midpoint of XY. Then

$$OD = IT \cdot \frac{PO}{PI} = r \cdot \frac{PO}{PO - IO} = r \cdot \frac{R^2}{R^2 - IO^2} = r \cdot \frac{R^2}{2Rr} = \frac{R}{2}.$$

This shows $\angle DOX = 60^\circ$ and hence $\angle XIY = \angle XOY = 120^\circ$.

Comment. A simplification of this problem is to ask part (a) only. Note that the question in part (b) implicitly requires P to lie on OI, or otherwise the angle is not uniquely determined as we can find another tangent from P to the incircle.

G8. Let A_1, B_1 and C_1 be points on sides BC, CA and AB of an acute triangle ABC respectively, such that AA_1, BB_1 and CC_1 are the internal angle bisectors of triangle ABC. Let I be the incentre of triangle ABC, and H be the orthocentre of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geqslant AI + BI + CI.$$

Solution. Without loss of generality, assume $\alpha = \angle BAC \leqslant \beta = \angle CBA \leqslant \gamma = \angle ACB$. Denote by a, b, c the lengths of BC, CA, AB respectively. We first show that triangle $A_1B_1C_1$ is acute.

Choose points D and E on side BC such that $B_1D // AB$ and B_1E is the internal angle bisector of $\angle BB_1C$. As $\angle B_1DB = 180^\circ - \beta$ is obtuse, we have $BB_1 > B_1D$. Thus,

$$\frac{BE}{EC} = \frac{BB_1}{B_1C} > \frac{DB_1}{B_1C} = \frac{BA}{AC} = \frac{BA_1}{A_1C}.$$

Therefore, $BE > BA_1$ and $\frac{1}{2}\angle BB_1C = \angle BB_1E > \angle BB_1A_1$. Similarly, $\frac{1}{2}\angle BB_1A > \angle BB_1C_1$. It follows that

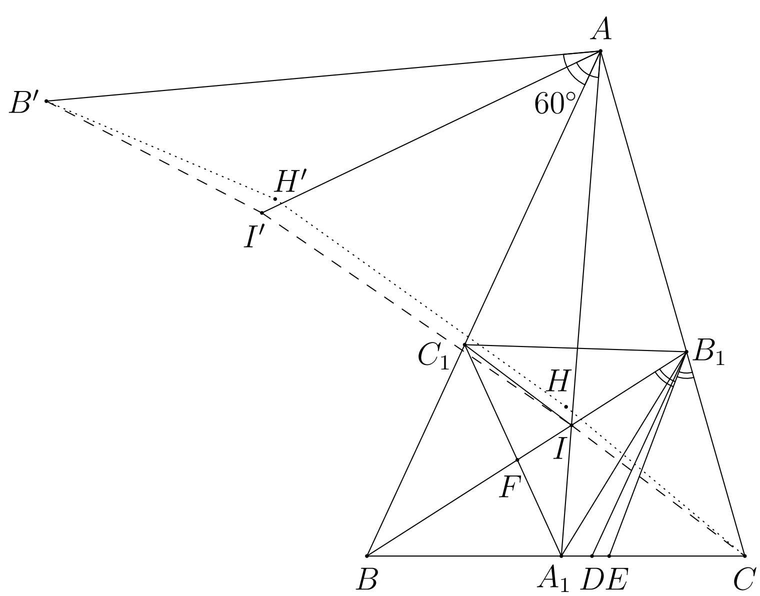
$$\angle A_1B_1C_1 = \angle BB_1A_1 + \angle BB_1C_1 < \frac{1}{2}(\angle BB_1C + \angle BB_1A) = 90^\circ$$

is acute. By symmetry, triangle $A_1B_1C_1$ is acute.

Let BB_1 meet A_1C_1 at F. From $\alpha \leqslant \gamma$, we get $a \leqslant c$, which implies

$$BA_1 = \frac{ca}{b+c} \leqslant \frac{ac}{a+b} = BC_1$$

and hence $\angle BC_1A_1 \leqslant \angle BA_1C_1$. As BF is the internal angle bisector of $\angle A_1BC_1$, this shows $\angle B_1FC_1 = \angle BFA_1 \leqslant 90^\circ$. Hence, H lies on the same side of BB_1 as C_1. This shows H lies inside triangle BB_1C_1. Similarly, from $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$, we know that H lies inside triangles CC_1B_1 and AA_1C_1.



As $\alpha \leqslant \beta \leqslant \gamma$, we have $\alpha \leqslant 60^\circ \leqslant \gamma$. Then $\angle BIC \leqslant 120^\circ \leqslant \angle AIB$. Firstly, suppose $\angle AIC \geqslant 120^\circ$.

Rotate points B, I, H through 60° about A to B', I', H' so that B' and C lie on different sides of AB. Since triangle $AI'I$ is equilateral, we have

$$AI + BI + CI = I'I + B'I' + IC = B'I' + I'I + IC. \tag{1}$$

Similarly,

$$AH + BH + CH = H'H + B'H' + HC = B'H' + H'H + HC. \tag{2}$$

As $\angle AII' = \angle AI'I = 60^\circ$, $\angle AI'B' = \angle AIB \geqslant 120^\circ$ and $\angle AIC \geqslant 120^\circ$, the quadrilateral $B'I'IC$ is convex and lies on the same side of $B'C$ as A.

Next, since H lies inside triangle ACC_1, H lies outside $B'I'IC$. Also, H lying inside triangle ABI implies H' lies inside triangle $AB'I'$. This shows H' lies outside $B'I'IC$ and hence the convex quadrilateral $B'I'IC$ is contained inside the quadrilateral $B'H'HC$. It follows that the perimeter of $B'I'IC$ cannot exceed the perimeter of $B'H'HC$. From (1) and (2), we conclude that

$$AH + BH + CH \geqslant AI + BI + CI.$$

For the case $\angle AIC < 120^\circ$, we can rotate B, I, H through 60° about C to B', I', H' so that B' and A lie on different sides of BC. The proof is analogous to the previous case and we still get the desired inequality.

Number Theory

N1. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geqslant 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)). \tag{1}$$

Answer.

- $P(x) = c$ where $1 \leqslant c \leqslant 9$ is an integer; or
- $P(x) = x$.

Solution 1. We consider three cases according to the degree of P.

- **Case 1.** $P(x)$ is a constant polynomial.

Let $P(x) = c$ where c is an integer constant. Then (1) becomes $S(c) = c$. This holds if and only if $1 \leqslant c \leqslant 9$.

- **Case 2.** $\deg P = 1$.

We have the following observation. For any positive integers m, n, we have

$$S(m+n) \leqslant S(m) + S(n), \tag{2}$$

and equality holds if and only if there is no carry in the addition $m + n$.

Let $P(x) = ax + b$ for some integers a, b where $a \neq 0$. As $P(n)$ is positive for large n, we must have $a \geqslant 1$. The condition (1) becomes $S(an + b) = aS(n) + b$ for all $n \geqslant 2016$. Setting $n = 2025$ and $n = 2020$ respectively, we get

$$S(2025a + b) - S(2020a + b) = (aS(2025) + b) - (aS(2020) + b) = 5a.$$

On the other hand, (2) implies

$$S(2025a + b) = S((2020a + b) + 5a) \leqslant S(2020a + b) + S(5a).$$

These give $5a \leqslant S(5a)$. As $a \geqslant 1$, this holds only when $a = 1$, in which case (1) reduces to $S(n + b) = S(n) + b$ for all $n \geqslant 2016$. Then we find that

$$S(n+1+b) - S(n+b) = (S(n+1) + b) - (S(n) + b) = S(n+1) - S(n). \tag{3}$$

If $b > 0$, we choose n such that $n + 1 + b = 10^k$ for some sufficiently large k. Note that all the digits of $n + b$ are 9's, so that the left-hand side of (3) equals $1 - 9k$. As n is a positive integer less than $10^k - 1$, we have $S(n) < 9k$. Therefore, the right-hand side of (3) is at least $1 - (9k - 1) = 2 - 9k$, which is a contradiction.

The case $b < 0$ can be handled similarly by considering $n + 1$ to be a large power of 10. Therefore, we conclude that $P(x) = x$, in which case (1) is trivially satisfied.

- **Case 3.** $\deg P \geqslant 2$.

Suppose the leading term of P is $a_d n^d$ where $a_d \neq 0$. Clearly, we have $a_d > 0$. Consider $n = 10^k - 1$ in (1). We get $S(P(n)) = P(9k)$. Note that $P(n)$ grows asymptotically as fast as n^d, so $S(P(n))$ grows asymptotically as no faster than a constant multiple of k. On the other hand, $P(9k)$ grows asymptotically as fast as k^d. This shows the two sides of the last equation cannot be equal for sufficiently large k since $d \geqslant 2$.

Therefore, we conclude that $P(x) = c$ where $1 \leqslant c \leqslant 9$ is an integer, or $P(x) = x$.

Solution 2. Let $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$. Clearly $a_d > 0$. There exists an integer $m \geqslant 1$ such that $|a_i| < 10^m$ for all $0 \leqslant i \leqslant d$. Consider $n = 9 \times 10^k$ for a sufficiently large integer k in (1). If there exists an index $0 \leqslant i \leqslant d-1$ such that $a_i < 0$, then all digits of $P(n)$ in positions from 10^{ik+m+1} to $10^{(i+1)k-1}$ are all 9's. Hence, we have $S(P(n)) \geqslant 9(k-m-1)$. On the other hand, $P(S(n)) = P(9)$ is a fixed constant. Therefore, (1) cannot hold for large k. This shows $a_i \geqslant 0$ for all $0 \leqslant i \leqslant d-1$.

Hence, $P(n)$ is an integer formed by the nonnegative integers $a_d \times 9^d, a_{d-1} \times 9^{d-1}, \ldots, a_0$ by inserting some zeros in between. This yields

$$S(P(n)) = S(a_d \times 9^d) + S(a_{d-1} \times 9^{d-1}) + \cdots + S(a_0).$$

Combining with (1), we have

$$S(a_d \times 9^d) + S(a_{d-1} \times 9^{d-1}) + \cdots + S(a_0) = P(9) = a_d \times 9^d + a_{d-1} \times 9^{d-1} + \cdots + a_0.$$

As $S(m) \leqslant m$ for any positive integer m, with equality when $1 \leqslant m \leqslant 9$, this forces each $a_i \times 9^i$ to be a positive integer between 1 and 9. In particular, this shows $a_i = 0$ for $i \geqslant 2$ and hence $d \leqslant 1$. Also, we have $a_1 \leqslant 1$ and $a_0 \leqslant 9$. If $a_1 = 1$ and $1 \leqslant a_0 \leqslant 9$, we take $n = 10^k + (10 - a_0)$ for sufficiently large k in (1). This yields a contradiction since

$$S(P(n)) = S(10^k + 10) = 2 \neq 11 = P(11 - a_0) = P(S(n)).$$

The zero polynomial is also rejected since $P(n)$ is positive for large n. The remaining candidates are $P(x) = x$ or $P(x) = a_0$ where $1 \leqslant a_0 \leqslant 9$, all of which satisfy (1), and hence are the only solutions.

N2. Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all possible integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

Answer. All composite numbers together with 2.

Solution. In this solution, we always use p_i to denote primes congruent to 1 mod 3, and use q_j to denote primes congruent to 2 mod 3. When we express a positive integer m using its prime factorization, we also include the special case $m = 1$ by allowing the exponents to be zeros. We first compute $\tau_1(m)$ for a positive integer m.

- **Claim.** Let $m = 3^x p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}$ be the prime factorization of m. Then

$$\tau_1(m) = \prod_{i=1}^{s} (a_i + 1) \left\lceil \frac{1}{2} \prod_{j=1}^{t} (b_j + 1) \right\rceil . \tag{1}$$

Proof. To choose a divisor of m congruent to 1 mod 3, it cannot have the prime divisor 3, while there is no restriction on choosing prime factors congruent to 1 mod 3. Also, we have to choose an even number of prime factors (counted with multiplicity) congruent to 2 mod 3.

If $\prod_{j=1}^{t} (b_j + 1)$ is even, then we may assume without loss of generality $b_1 + 1$ is even. We can choose the prime factors $q_2, q_3, \ldots, q_t$ freely in $\prod_{j=2}^{t} (b_j + 1)$ ways. Then the parity of the number of q_1 is uniquely determined, and hence there are $\frac{1}{2}(b_1 + 1)$ ways to choose the exponent of q_1. Hence (1) is verified in this case.

If $\prod_{j=1}^{t} (b_j + 1)$ is odd, we use induction on t to count the number of choices. When $t = 1$, there are $\lceil \frac{b_1+1}{2} \rceil$ choices for which the exponent is even and $\lfloor \frac{b_1+1}{2} \rfloor$ choices for which the exponent is odd. For the inductive step, we find that there are

$$\left\lceil \frac{1}{2} \prod_{j=1}^{t-1} (b_j + 1) \right\rceil \cdot \left\lceil \frac{b_t + 1}{2} \right\rceil + \left\lfloor \frac{1}{2} \prod_{j=1}^{t-1} (b_j + 1) \right\rfloor \cdot \left\lfloor \frac{b_t + 1}{2} \right\rfloor = \left\lceil \frac{1}{2} \prod_{j=1}^{t} (b_j + 1) \right\rceil$$

choices with an even number of prime factors and hence $\lfloor \frac{1}{2} \prod_{j=1}^{t} (b_j + 1) \rfloor$ choices with an odd number of prime factors. Hence (1) is also true in this case. $\square$

Let $n = 3^x 2^y 5^z p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}$. Using the well-known formula for computing the divisor function, we get

$$\tau(10n) = (x + 1)(y + 2)(z + 2) \prod_{i=1}^{s} (a_i + 1) \prod_{j=1}^{t} (b_j + 1). \tag{2}$$

By the Claim, we have

$$\tau_1(10n) = \prod_{i=1}^{s} (a_i + 1) \left\lceil \frac{1}{2}(y + 2)(z + 2) \prod_{j=1}^{t} (b_j + 1) \right\rceil . \tag{3}$$

If $c = (y+2)(z+2)\prod_{j=1}^{t}(b_j+1)$ is even, then (2) and (3) imply

$$\frac{\tau(10n)}{\tau_1(10n)} = 2(x+1).$$

In this case $\frac{\tau(10n)}{\tau_1(10n)}$ can be any even positive integer as x runs through all nonnegative integers.

If c is odd, which means y, z are odd and each b_j is even, then (2) and (3) imply

$$\frac{\tau(10n)}{\tau_1(10n)} = \frac{2(x+1)c}{c+1}. \tag{4}$$

For this to be an integer, we need $c+1$ divides $2(x+1)$ since c and $c+1$ are relatively prime. Let $2(x+1) = k(c+1)$. Then (4) reduces to

$$\frac{\tau(10n)}{\tau_1(10n)} = kc = k(y+2)(z+2)\prod_{j=1}^{t}(b_j+1). \tag{5}$$

Noting that y, z are odd, the integers $y+2$ and $z+2$ are at least 3. This shows the integer in this case must be composite. On the other hand, for any odd composite number ab with $a, b \geqslant 3$, we may simply take $n = 3^{\frac{ab-1}{2}} \cdot 2^{a-2} \cdot 5^{b-2}$ so that $\frac{\tau(10n)}{\tau_1(10n)} = ab$ from (5).

We conclude that the fraction can be any even integer or any odd composite number. Equivalently, it can be 2 or any composite number.

N3. Define $P(n)=n^2+n+1$. For any positive integers a and b, the set

$$\{P(a), P(a+1), P(a+2), \ldots, P(a+b)\}$$

is said to be *fragrant* if none of its elements is relatively prime to the product of the other elements. Determine the smallest size of a fragrant set.

Answer. 6.

Solution. We have the following observations.

(i) $(P(n), P(n+1))=1$ for any n.

We have $(P(n), P(n+1))=(n^2+n+1, n^2+3n+3)=(n^2+n+1, 2n+2)$. Noting that n^2+n+1 is odd and $(n^2+n+1, n+1)=(1, n+1)=1$, the claim follows.

(ii) $(P(n), P(n+2))=1$ for $n \not\equiv 2 \pmod 7$ and $(P(n), P(n+2))=7$ for $n \equiv 2 \pmod 7$.

From $(2n+7)P(n)-(2n-1)P(n+2)=14$ and the fact that $P(n)$ is odd, $(P(n), P(n+2))$ must be a divisor of 7. The claim follows by checking $n \equiv 0, 1, \ldots, 6 \pmod 7$ directly.

(iii) $(P(n), P(n+3))=1$ for $n \not\equiv 1 \pmod 3$ and $3 \mid (P(n), P(n+3))$ for $n \equiv 1 \pmod 3$.

From $(n+5)P(n)-(n-1)P(n+3)=18$ and the fact that $P(n)$ is odd, $(P(n), P(n+3))$ must be a divisor of 9. The claim follows by checking $n \equiv 0, 1, 2 \pmod 3$ directly.

Suppose there exists a fragrant set with at most 5 elements. We may assume it contains exactly 5 elements $P(a), P(a+1), \ldots, P(a+4)$ since the following argument also works with fewer elements. Consider $P(a+2)$. From (i), it is relatively prime to $P(a+1)$ and $P(a+3)$. Without loss of generality, assume $(P(a), P(a+2))>1$. From (ii), we have $a \equiv 2 \pmod 7$. The same observation implies $(P(a+1), P(a+3))=1$. In order that the set is fragrant, $(P(a), P(a+3))$ and $(P(a+1), P(a+4))$ must both be greater than 1. From (iii), this holds only when both a and $a+1$ are congruent to 1 mod 3, which is a contradiction.

It now suffices to construct a fragrant set of size 6. By the Chinese Remainder Theorem, we can take a positive integer a such that

$$a \equiv 7 \pmod{19}, \quad a+1 \equiv 2 \pmod 7, \quad a+2 \equiv 1 \pmod 3.$$

For example, we may take $a=197$. From (ii), both $P(a+1)$ and $P(a+3)$ are divisible by 7. From (iii), both $P(a+2)$ and $P(a+5)$ are divisible by 3. One also checks from $19 \mid P(7)=57$ and $19 \mid P(11)=133$ that $P(a)$ and $P(a+4)$ are divisible by 19. Therefore, the set $\{P(a), P(a+1), \ldots, P(a+5)\}$ is fragrant.

Therefore, the smallest size of a fragrant set is 6.

Comment. "Fragrant Harbour" is the English translation of "Hong Kong".

A stronger version of this problem is to show that there exists a fragrant set of size k for any $k \geqslant 6$. We present a proof here.

For each even positive integer m which is not divisible by 3, since $m^2+3 \equiv 3 \pmod 4$, we can find a prime $p_m \equiv 3 \pmod 4$ such that $p_m \mid m^2+3$. Clearly, $p_m>3$.

If $b = 2t \geqslant 6$, we choose a such that $3|2(a+t)+1$ and $p_m|2(a+t)+1$ for each $1 \leqslant m \leqslant b$ with $m \equiv 2, 4 \pmod 6$. For $0 \leqslant r \leqslant t$ and $3|r$, we have $a+t \pm r \equiv 1 \pmod 3$ so that $3|P(a+t \pm r)$. For $0 \leqslant r \leqslant t$ and $(r, 3) = 1$, we have

$$4P(a+t \pm r) \equiv (-1 \pm 2r)^2 + 2(-1 \pm 2r) + 4 = 4r^2 + 3 \equiv 0 \pmod{p_{2r}}.$$

Hence, $\{P(a), P(a+1), \ldots, P(a+b)\}$ is fragrant.

If $b = 2t+1 \geqslant 7$ (the case $b = 5$ has been done in the original problem), we choose a such that $3|2(a+t)+1$ and $p_m|2(a+t)+1$ for $1 \leqslant m \leqslant b$ with $m \equiv 2, 4 \pmod 6$, and that $a + b \equiv 9 \pmod{13}$. Note that a exists by the Chinese Remainder Theorem since $p_m \neq 13$ for all m. The even case shows that $\{P(a), P(a+1), \ldots, P(a+b-1)\}$ is fragrant. Also, one checks from $13|P(9) = 91$ and $13|P(3) = 13$ that $P(a+b)$ and $P(a+b-6)$ are divisible by 13. The proof is thus complete.

N4. Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

- $m = 1$ and $l = 2k$; or
- $l|k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

Solution 1. It is given that

$$n^k + mn^l + 1 | n^{k+l} - 1. \tag{1}$$

This implies

$$n^k + mn^l + 1 | (n^{k+l} - 1) + (n^k + mn^l + 1) = n^{k+l} + n^k + mn^l. \tag{2}$$

We have two cases to discuss.

- **Case 1.** $l \geqslant k$.
  Since $(n^k + mn^l + 1, n) = 1$, (2) yields

$$n^k + mn^l + 1 | n^l + mn^{l-k} + 1.$$

In particular, we get $n^k + mn^l + 1 \leqslant n^l + mn^{l-k} + 1$. As $n \geqslant 2$ and $k \geqslant 1$, $(m-1)n^l$ is at least $2(m-1)n^{l-k}$. It follows that the inequality cannot hold when $m \geqslant 2$. For $m = 1$, the above divisibility becomes

$$n^k + n^l + 1 | n^l + n^{l-k} + 1.$$

Note that $n^l + n^{l-k} + 1 < n^l + n^l + 1 < 2(n^k + n^l + 1)$. Thus we must have $n^l + n^{l-k} + 1 = n^k + n^l + 1$ so that $l = 2k$, which gives the first result.

- **Case 2.** $l < k$.
  This time (2) yields

$$n^k + mn^l + 1 | n^k + n^{k-l} + m.$$

In particular, we get $n^k + mn^l + 1 \leqslant n^k + n^{k-l} + m$, which implies

$$m \leqslant \frac{n^{k-l} - 1}{n^l - 1}. \tag{3}$$

On the other hand, from (1) we may let $n^{k+l} - 1 = (n^k + mn^l + 1)t$ for some positive integer t. Obviously, t is less than n^l, which means $t \leqslant n^l - 1$ as it is an integer. Then we have $n^{k+l} - 1 \leqslant (n^k + mn^l + 1)(n^l - 1)$, which is the same as

$$m \geqslant \frac{n^{k-l} - 1}{n^l - 1}. \tag{4}$$

Equations (3) and (4) combine to give $m = \frac{n^{k-l}-1}{n^l-1}$. As this is an integer, we have $l | k - l$. This means $l|k$ and it corresponds to the second result.

Solution 2. As in Solution 1, we begin with equation (2).

- **Case 1.** $l \geqslant k$.
  Then (2) yields

$$n^k + mn^l + 1 | n^l + mn^{l-k} + 1.$$

Since $2(n^k+mn^l+1) > 2mn^l+1 > n^l+mn^{l-k}+1$, it follows that $n^k+mn^l+1 = n^l+mn^{l-k}+1$, that is,

$$m(n^l - n^{l-k}) = n^l - n^k.$$

If $m \geqslant 2$, then $m(n^l - n^{l-k}) \geqslant 2n^l - 2n^{l-k} \geqslant 2n^l - n^l > n^l - n^k$ gives a contradiction. Hence $m = 1$ and $l - k = k$, which means $m = 1$ and $l = 2k$.

- **Case 2.** $l < k$.
  Then (2) yields

$$n^k + mn^l + 1 | n^k + n^{k-l} + m.$$

Since $2(n^k+mn^l+1) > 2n^k+m > n^k+n^{k-l}+m$, it follows that $n^k+mn^l+1 = n^k+n^{k-l}+m$. This gives $m = \frac{n^{k-l}-1}{n^l-1}$. Note that $n^l - 1 | n^{k-l} - 1$ implies $l | k - l$ and hence $l | k$. The proof is thus complete.

Comment. Another version of this problem is as follows: let n, m, k and l be positive integers with $n \neq 1$ such that k and l do not divide each other. Show that $n^k + mn^l + 1$ does not divide $n^{k+l} - 1$.

N5. Let a be a positive integer which is not a square number. Denote by A the set of all positive integers k such that

$$k = \frac{x^2 - a}{x^2 - y^2} \tag{1}$$

for some integers x and y with $x > \sqrt{a}$. Denote by B the set of all positive integers k such that (1) is satisfied for some integers x and y with $0 \leqslant x < \sqrt{a}$. Prove that $A = B$.

Solution 1. We first prove the following preliminary result.

• **Claim.** For fixed k, let x, y be integers satisfying (1). Then the numbers x_1, y_1 defined by

$$x_1 = \frac{1}{2}\left(x - y + \frac{(x-y)^2 - 4a}{x+y}\right), \quad y_1 = \frac{1}{2}\left(x - y - \frac{(x-y)^2 - 4a}{x+y}\right)$$

are integers and satisfy (1) (with x, y replaced by x_1, y_1 respectively).

Proof. Since $x_1 + y_1 = x - y$ and

$$x_1 = \frac{x^2 - xy - 2a}{x+y} = -x + \frac{2(x^2 - a)}{x+y} = -x + 2k(x-y),$$

both x_1 and y_1 are integers. Let $u = x + y$ and $v = x - y$. The relation (1) can be rewritten as

$$u^2 - (4k-2)uv + (v^2 - 4a) = 0.$$

By Vieta's Theorem, the number $z = \frac{v^2 - 4a}{u}$ satisfies

$$v^2 - (4k-2)vz + (z^2 - 4a) = 0.$$

Since x_1 and y_1 are defined so that $v = x_1 + y_1$ and $z = x_1 - y_1$, we can reverse the process and verify (1) for x_1, y_1. $\square$

We first show that $B \subset A$. Take any $k \in B$ so that (1) is satisfied for some integers x, y with $0 \leqslant x < \sqrt{a}$. Clearly, $y \neq 0$ and we may assume y is positive. Since a is not a square, we have $k > 1$. Hence, we get $0 \leqslant x < y < \sqrt{a}$. Define

$$x_1 = \frac{1}{2}\left|x - y + \frac{(x-y)^2 - 4a}{x+y}\right|, \quad y_1 = \frac{1}{2}\left(x - y - \frac{(x-y)^2 - 4a}{x+y}\right).$$

By the Claim, x_1, y_1 are integers satisfying (1). Also, we have

$$x_1 \geqslant -\frac{1}{2}\left(x - y + \frac{(x-y)^2 - 4a}{x+y}\right) = \frac{2a + x(y-x)}{x+y} \geqslant \frac{2a}{x+y} > \sqrt{a}.$$

This implies $k \in A$ and hence $B \subset A$.

Next, we shall show that $A \subset B$. Take any $k \in A$ so that (1) is satisfied for some integers x, y with $x > \sqrt{a}$. Again, we may assume y is positive. Among all such representations of k, we choose the one with smallest $x + y$. Define

$$x_1 = \frac{1}{2}\left|x - y + \frac{(x-y)^2 - 4a}{x+y}\right|, \quad y_1 = \frac{1}{2}\left(x - y - \frac{(x-y)^2 - 4a}{x+y}\right).$$

By the Claim, x_1, y_1 are integers satisfying (1). Since $k > 1$, we get $x > y > \sqrt{a}$. Therefore, we have $y_1 > \frac{4a}{x+y} > 0$ and $\frac{4a}{x+y} < x + y$. It follows that

$$x_1 + y_1 \leqslant \max\left\{x - y, \frac{4a - (x-y)^2}{x+y}\right\} < x + y.$$

If $x_1 > \sqrt{a}$, we get a contradiction due to the minimality of $x + y$. Therefore, we must have $0 \leqslant x_1 < \sqrt{a}$, which means $k \in B$ so that $A \subset B$.

The two subset relations combine to give $A = B$.

Solution 2. The relation (1) is equivalent to

$$ky^2 - (k-1)x^2 = a. \tag{2}$$

Motivated by Pell's Equation, we prove the following, which is essentially the same as the Claim in Solution 1.

• **Claim.** If (x_0, y_0) is a solution to (2), then $((2k-1)x_0 \pm 2ky_0, (2k-1)y_0 \pm 2(k-1)x_0)$ is also a solution to (2).

Proof. We check directly that

$$\begin{aligned}
&k((2k-1)y_0 \pm 2(k-1)x_0)^2 - (k-1)((2k-1)x_0 \pm 2ky_0)^2 \\
&= (k(2k-1)^2 - (k-1)(2k)^2)y_0^2 + (k(2(k-1))^2 - (k-1)(2k-1)^2)x_0^2 \\
&= ky_0^2 - (k-1)x_0^2 = a.
\end{aligned}$$

□

If (2) is satisfied for some $0 \leqslant x < \sqrt{a}$ and nonnegative integer y, then clearly (1) implies $y > x$. Also, we have $k > 1$ since a is not a square number. By the Claim, consider another solution to (2) defined by

$$x_1 = (2k-1)x + 2ky, \quad y_1 = (2k-1)y + 2(k-1)x.$$

It satisfies $x_1 \geqslant (2k-1)x + 2k(x+1) = (4k-1)x + 2k > x$. Then we can replace the old solution by a new one which has a larger value in x. After a finite number of replacements, we must get a solution with $x > \sqrt{a}$. This shows $B \subset A$.

If (2) is satisfied for some $x > \sqrt{a}$ and nonnegative integer y, by the Claim we consider another solution to (2) defined by

$$x_1 = |(2k-1)x - 2ky|, \quad y_1 = (2k-1)y - 2(k-1)x.$$

From (2), we get $\sqrt{k}y > \sqrt{k-1}x$. This implies $ky > \sqrt{k(k-1)}x > (k-1)x$ and hence $(2k-1)x - 2ky < x$. On the other hand, the relation (1) implies $x > y$. Then it is clear that $(2k-1)x - 2ky > -x$. These combine to give $x_1 < x$, which means we have found a solution to (2) with x having a smaller absolute value. After a finite number of steps, we shall obtain a solution with $0 \leqslant x < \sqrt{a}$. This shows $A \subset B$.

The desired result follows from $B \subset A$ and $A \subset B$.

Solution 3. It suffices to show $A \cup B$ is a subset of $A \cap B$. We take any $k \in A \cup B$, which means there exist integers x, y satisfying (1). Since a is not a square, it follows that $k \neq 1$. As in Solution 2, the result follows readily once we have proved the existence of a solution (x_1, y_1) to (1) with $|x_1| > |x|$, and, in case of $x > \sqrt{a}$, another solution (x_2, y_2) with $|x_2| < |x|$.

Without loss of generality, assume $x, y \geqslant 0$. Let $u = x + y$ and $v = x - y$. Then $u \geqslant v$ and (1) becomes

$$k = \frac{(u+v)^2 - 4a}{4uv}. \tag{3}$$

This is the same as

$$v^2 + (2u - 4ku)v + u^2 - 4a = 0.$$

Let $v_1 = 4ku - 2u - v$. Then $u + v_1 = 4ku - u - v \geqslant 8u - u - v > u + v$. By Vieta's Theorem, v_1 satisfies

$$v_1^2 + (2u - 4ku)v_1 + u^2 - 4a = 0.$$

This gives $k = \frac{(u+v_1)^2 - 4a}{4uv_1}$. As k is an integer, $u + v_1$ must be even. Therefore, $x_1 = \frac{u+v_1}{2}$ and $y_1 = \frac{v_1 - u}{2}$ are integers. By reversing the process, we can see that (x_1, y_1) is a solution to (1), with $x_1 = \frac{u+v_1}{2} > \frac{u+v}{2} = x \geqslant 0$. This completes the first half of the proof.

Suppose $x > \sqrt{a}$. Then $u + v > 2\sqrt{a}$ and (3) can be rewritten as

$$u^2 + (2v - 4kv)u + v^2 - 4a = 0.$$

Let $u_2 = 4kv - 2v - u$. By Vieta's Theorem, we have $uu_2 = v^2 - 4a$ and

$$u_2^2 + (2v - 4kv)u_2 + v^2 - 4a = 0. \tag{4}$$

By $u > 0$, $u + v > 2\sqrt{a}$ and (3), we have $v > 0$. If $u_2 \geqslant 0$, then $vu_2 \leqslant uu_2 = v^2 - 4a < v^2$. This shows $u_2 < v \leqslant u$ and $0 < u_2 + v < u + v$. If $u_2 < 0$, then $(u_2 + v) + (u + v) = 4kv > 0$ and $u_2 + v < u + v$ imply $|u_2 + v| < u + v$. In any case, since $u_2 + v$ is even from (4), we can define $x_2 = \frac{u_2+v}{2}$ and $y_2 = \frac{u_2-v}{2}$ so that (1) is satisfied with $|x_2| < x$, as desired. The proof is thus complete.

N6. Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

Answer. $f(n) = n^2$ for any $n \in \mathbb{N}$.

Solution. It is given that

$$f(m) + f(n) - mn \mid mf(m) + nf(n). \tag{1}$$

Taking $m = n = 1$ in (1), we have $2f(1) - 1 \mid 2f(1)$. Then $2f(1) - 1 \mid 2f(1) - (2f(1) - 1) = 1$ and hence $f(1) = 1$.

Let $p \geqslant 7$ be a prime. Taking $m = p$ and $n = 1$ in (1), we have $f(p) - p + 1 \mid pf(p) + 1$ and hence

$$f(p) - p + 1 \mid pf(p) + 1 - p(f(p) - p + 1) = p^2 - p + 1.$$

If $f(p) - p + 1 = p^2 - p + 1$, then $f(p) = p^2$. If $f(p) - p + 1 \neq p^2 - p + 1$, as $p^2 - p + 1$ is an odd positive integer, we have $p^2 - p + 1 \geqslant 3(f(p) - p + 1)$, that is,

$$f(p) \leqslant \frac{1}{3}(p^2 + 2p - 2). \tag{2}$$

Taking $m = n = p$ in (1), we have $2f(p) - p^2 \mid 2pf(p)$. This implies

$$2f(p) - p^2 \mid 2pf(p) - p(2f(p) - p^2) = p^3.$$

By (2) and $f(p) \geqslant 1$, we get

$$-p^2 < 2f(p) - p^2 \leqslant \frac{2}{3}(p^2 + 2p - 2) - p^2 < -p$$

since $p \geqslant 7$. This contradicts the fact that $2f(p) - p^2$ is a factor of p^3. Thus we have proved that $f(p) = p^2$ for all primes $p \geqslant 7$.

Let n be a fixed positive integer. Choose a sufficiently large prime p. Consider $m = p$ in (1). We obtain

$$f(p) + f(n) - pn \mid pf(p) + nf(n) - n(f(p) + f(n) - pn) = pf(p) - nf(p) + pn^2.$$

As $f(p) = p^2$, this implies $p^2 - pn + f(n) \mid p(p^2 - pn + n^2)$. As p is sufficiently large and n is fixed, p cannot divide $f(n)$, and so $(p, p^2 - pn + f(n)) = 1$. It follows that $p^2 - pn + f(n) \mid p^2 - pn + n^2$ and hence

$$p^2 - pn + f(n) \mid (p^2 - pn + n^2) - (p^2 - pn + f(n)) = n^2 - f(n).$$

Note that $n^2 - f(n)$ is fixed while $p^2 - pn + f(n)$ is chosen to be sufficiently large. Therefore, we must have $n^2 - f(n) = 0$ so that $f(n) = n^2$ for any positive integer n.

Finally, we check that when $f(n) = n^2$ for any positive integer n, then

$$f(m) + f(n) - mn = m^2 + n^2 - mn$$

and

$$mf(m) + nf(n) = m^3 + n^3 = (m + n)(m^2 + n^2 - mn).$$

The latter expression is divisible by the former for any positive integers m, n. This shows $f(n) = n^2$ is the only solution.

N7. Let n be an odd positive integer. In the Cartesian plane, a cyclic polygon P with area S is chosen. All its vertices have integral coordinates, and the squares of its side lengths are all divisible by n. Prove that $2S$ is an integer divisible by n.

Solution. Let $P = A_1A_2\dots A_k$ and let $A_{k+i} = A_i$ for $i \geqslant 1$. By the Shoelace Formula, the area of any convex polygon with integral coordinates is half an integer. Therefore, $2S$ is an integer. We shall prove by induction on $k \geqslant 3$ that $2S$ is divisible by n. Clearly, it suffices to consider $n = p^t$ where p is an odd prime and $t \geqslant 1$.

For the base case $k = 3$, let the side lengths of P be $\sqrt{na}, \sqrt{nb}, \sqrt{nc}$ where a, b, c are positive integers. By Heron's Formula,

$$16S^2 = n^2(2ab + 2bc + 2ca - a^2 - b^2 - c^2).$$

This shows $16S^2$ is divisible by n^2. Since n is odd, $2S$ is divisible by n.

Assume $k \geqslant 4$. If the square of length of one of the diagonals is divisible by n, then that diagonal divides P into two smaller polygons, to which the induction hypothesis applies. Hence we may assume that none of the squares of diagonal lengths is divisible by n. As usual, we denote by $\nu_p(r)$ the exponent of p in the prime decomposition of r. We claim the following.

- **Claim.** $\nu_p(A_1A_m^2) > \nu_p(A_1A_{m+1}^2)$ for $2 \leqslant m \leqslant k-1$.

Proof. The case $m = 2$ is obvious since $\nu_p(A_1A_2^2) \geqslant p^t > \nu_p(A_1A_3^2)$ by the condition and the above assumption.

Suppose $\nu_p(A_1A_2^2) > \nu_p(A_1A_3^2) > \cdots > \nu_p(A_1A_m^2)$ where $3 \leqslant m \leqslant k-1$. For the induction step, we apply Ptolemy's Theorem to the cyclic quadrilateral $A_1A_{m-1}A_mA_{m+1}$ to get

$$A_1A_{m+1} \times A_{m-1}A_m + A_1A_{m-1} \times A_mA_{m+1} = A_1A_m \times A_{m-1}A_{m+1},$$

which can be rewritten as

$$\begin{aligned} A_1A_{m+1}^2 \times A_{m-1}A_m^2 = {} & A_1A_{m-1}^2 \times A_mA_{m+1}^2 + A_1A_m^2 \times A_{m-1}A_{m+1}^2 \\ & - 2A_1A_{m-1} \times A_mA_{m+1} \times A_1A_m \times A_{m-1}A_{m+1}. \end{aligned} \tag{1}$$

From this, $2A_1A_{m-1} \times A_mA_{m+1} \times A_1A_m \times A_{m-1}A_{m+1}$ is an integer. We consider the component of p of each term in (1). By the inductive hypothesis, we have $\nu_p(A_1A_{m-1}^2) > \nu_p(A_1A_m^2)$. Also, we have $\nu_p(A_mA_{m+1}^2) \geqslant p^t > \nu_p(A_{m-1}A_{m+1}^2)$. These give

$$\nu_p(A_1A_{m-1}^2 \times A_mA_{m+1}^2) > \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2). \tag{2}$$

Next, we have $\nu_p(4A_1A_{m-1}^2 \times A_mA_{m+1}^2 \times A_1A_m^2 \times A_{m-1}A_{m+1}^2) = \nu_p(A_1A_{m-1}^2 \times A_mA_{m+1}^2) + \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2) > 2\nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2)$ from (2). This implies

$$\nu_p(2A_1A_{m-1} \times A_mA_{m+1} \times A_1A_m \times A_{m-1}A_{m+1}) > \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2). \tag{3}$$

Combining (1), (2) and (3), we conclude that

$$\nu_p(A_1A_{m+1}^2 \times A_{m-1}A_m^2) = \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2).$$

By $\nu_p(A_{m-1}A_m^2) \geqslant p^t > \nu_p(A_{m-1}A_{m+1}^2)$, we get $\nu_p(A_1A_{m+1}^2) < \nu_p(A_1A_m^2)$. The Claim follows by induction. $\square$

From the Claim, we get a chain of inequalities

$$p^t > \nu_p(A_1A_3^2) > \nu_p(A_1A_4^2) > \cdots > \nu_p(A_1A_k^2) \geqslant p^t,$$

which yields a contradiction. Therefore, we can show by induction that $2S$ is divisible by n.

Comment. The condition that P is cyclic is crucial. As a counterexample, consider the rhombus with vertices $(0,3),(4,0),(0,-3),(-4,0)$. Each of its squares of side lengths is divisible by 5, while $2S = 48$ is not.

The proposer also gives a proof for the case n is even. One just needs an extra technical step for the case $p = 2$.

N8. Find all polynomials $P(x)$ of odd degree d and with integer coefficients satisfying the following property: for each positive integer n, there exist n positive integers $x_1, x_2, \dots, x_n$ such that $\frac{1}{2} < \frac{P(x_i)}{P(x_j)} < 2$ and $\frac{P(x_i)}{P(x_j)}$ is the d-th power of a rational number for every pair of indices i and j with $1 \leqslant i, j \leqslant n$.

Answer. $P(x) = a(rx+s)^d$ where a, r, s are integers with $a \neq 0$, $r \geqslant 1$ and $(r, s) = 1$.

Solution. Let $P(x) = a_d x^d + a_{d-1}x^{d-1} + \dots + a_0$. Consider the substitution $y = da_d x + a_{d-1}$. By defining $Q(y) = P(x)$, we find that Q is a polynomial with rational coefficients without the term y^{d-1}. Let $Q(y) = b_d y^d + b_{d-2}y^{d-2} + b_{d-3}y^{d-3} + \dots + b_0$ and $B = \max_{0 \leqslant i \leqslant d}\{|b_i|\}$ (where $b_{d-1} = 0$).

The condition shows that for each $n \geqslant 1$, there exist integers $y_1, y_2, \dots, y_n$ such that $\frac{1}{2} < \frac{Q(y_i)}{Q(y_j)} < 2$ and $\frac{Q(y_i)}{Q(y_j)}$ is the d-th power of a rational number for $1 \leqslant i, j \leqslant n$. Since n can be arbitrarily large, we may assume all x_i's and hence y_i's are integers larger than some absolute constant in the following.

By Dirichlet's Theorem, since d is odd, we can find a sufficiently large prime p such that $p \equiv 2 \pmod{d}$. In particular, we have $(p-1, d) = 1$. For this fixed p, we choose n to be sufficiently large. Then by the Pigeonhole Principle, there must be $d+1$ of $y_1, y_2, \dots, y_n$ which are congruent mod p. Without loss of generality, assume $y_i \equiv y_j \pmod{p}$ for $1 \leqslant i, j \leqslant d+1$. We shall establish the following.

- **Claim.** $\frac{Q(y_i)}{Q(y_1)} = \frac{y_i^d}{y_1^d}$ for $2 \leqslant i \leqslant d+1$.

Proof. Let $\frac{Q(y_i)}{Q(y_1)} = \frac{l^d}{m^d}$ where $(l, m) = 1$ and $l, m > 0$. This can be rewritten in the expanded form

$$b_d(m^d y_i^d - l^d y_1^d) = -\sum_{j=0}^{d-2} b_j(m^d y_i^j - l^d y_1^j). \tag{1}$$

Let c be the common denominator of Q, so that $cQ(k)$ is an integer for any integer k. Note that c depends only on P and so we may assume $(p, c) = 1$. Then $y_1 \equiv y_i \pmod{p}$ implies $cQ(y_1) \equiv cQ(y_i) \pmod{p}$.

- **Case 1.** $p | cQ(y_1)$.

In this case, there is a cancellation of p in the numerator and denominator of $\frac{cQ(y_i)}{cQ(y_1)}$, so that $m^d \leqslant p^{-1}|cQ(y_1)|$. Noting $|Q(y_1)| < 2By_1^d$ as y_1 is large, we get

$$m \leqslant p^{-\frac{1}{d}}(2cB)^{\frac{1}{d}} y_1. \tag{2}$$

For large y_1 and y_i, the relation $\frac{1}{2} < \frac{Q(y_i)}{Q(y_1)} < 2$ implies

$$\frac{1}{3} < \frac{y_i^d}{y_1^d} < 3. \tag{3}$$

We also have

$$\frac{1}{2} < \frac{l^d}{m^d} < 2. \tag{4}$$

Now, the left-hand side of (1) is

$$b_d(my_i - ly_1)(m^{d-1}y_i^{d-1} + m^{d-2}y_i^{d-2}ly_1 + \cdots + l^{d-1}y_1^{d-1}).$$

Suppose on the contrary that $my_i - ly_1 \neq 0$. Then the absolute value of the above expression is at least $|b_d|m^{d-1}y_i^{d-1}$. On the other hand, the absolute value of the right-hand side of (1) is at most

$$\begin{aligned}\sum_{j=0}^{d-2} B(m^d y_i^j + l^d y_1^j) &\leqslant (d-1)B(m^d y_i^{d-2} + l^d y_1^{d-2}) \\ &\leqslant (d-1)B(7m^d y_i^{d-2}) \\ &\leqslant 7(d-1)B(p^{-\frac{1}{d}}(2cB)^{\frac{1}{d}}y_1)m^{d-1}y_i^{d-2} \\ &\leqslant 21(d-1)Bp^{-\frac{1}{d}}(2cB)^{\frac{1}{d}}m^{d-1}y_i^{d-1}\end{aligned}$$

by using successively (3), (4), (2) and again (3). This shows

$$|b_d|m^{d-1}y_i^{d-1} \leqslant 21(d-1)Bp^{-\frac{1}{d}}(2cB)^{\frac{1}{d}}m^{d-1}y_i^{d-1},$$

which is a contradiction for large p as b_d, B, c, d depend only on the polynomial P. Therefore, we have $my_i - ly_1 = 0$ in this case.

- **Case 2.** $(p, cQ(y_1)) = 1$.

From $cQ(y_1) \equiv cQ(y_i) \pmod p$, we have $l^d \equiv m^d \pmod p$. Since $(p-1, d) = 1$, we use Fermat Little Theorem to conclude $l \equiv m \pmod p$. Then $p | my_i - ly_1$. Suppose on the contrary that $my_i - ly_1 \neq 0$. Then the left-hand side of (1) has absolute value at least $|b_d|pm^{d-1}y_i^{d-1}$. Similar to Case 1, the right-hand side of (1) has absolute value at most

$$21(d-1)B(2cB)^{\frac{1}{d}}m^{d-1}y_i^{d-1},$$

which must be smaller than $|b_d|pm^{d-1}y_i^{d-1}$ for large p. Again this yields a contradiction and hence $my_i - ly_1 = 0$.

In both cases, we find that $\frac{Q(y_i)}{Q(y_1)} = \frac{l^d}{m^d} = \frac{y_i^d}{y_1^d}$. $\square$

From the Claim, the polynomial $Q(y_1)y^d - y_1^d Q(y)$ has roots $y = y_1, y_2, \ldots, y_{d+1}$. Since its degree is at most d, this must be the zero polynomial. Hence, $Q(y) = b_d y^d$. This implies $P(x) = a_d(x + \frac{a_{d-1}}{da_d})^d$. Let $\frac{a_{d-1}}{da_d} = \frac{s}{r}$ with integers r, s where $r \geqslant 1$ and $(r, s) = 1$. Since P has integer coefficients, we need $r^d | a_d$. Let $a_d = r^d a$. Then $P(x) = a(rx+s)^d$. It is obvious that such a polynomial satisfies the conditions.

Comment. In the proof, the use of prime and Dirichlet's Theorem can be avoided. One can easily show that each $P(x_i)$ can be expressed in the form uv_i^d where u, v_i are integers and u cannot be divisible by the d-th power of a prime (note that u depends only on P). By fixing a large integer q and by choosing a large n, we can apply the Pigeonhole Principle and assume

$x_1 \equiv x_2 \equiv \cdots \equiv x_{d+1} \pmod{q}$ and $v_1 \equiv v_2 \equiv \cdots \equiv v_{d+1} \pmod{q}$. Then the remaining proof is similar to Case 2 of the Solution.

Alternatively, we give another modification of the proof as follows.

We take a sufficiently large n and consider the corresponding positive integers $y_1, y_2, \dots, y_n$. For each $2 \leqslant i \leqslant n$, let $\frac{Q(y_i)}{Q(y_1)} = \frac{l_i^d}{m_i^d}$.

As in Case 1, if there are d indices i such that the integers $\frac{c|Q(y_1)|}{m_i^d}$ are bounded below by a constant depending only on P, we can establish the Claim using those y_i's and complete the proof. Similarly, as in Case 2, if there are d indices i such that the integers $|m_i y_i - l_i y_1|$ are bounded below, then the proof goes the same. So it suffices to consider the case where $\frac{c|Q(y_1)|}{m_i^d} \leqslant M$ and $|m_i y_i - l_i y_1| \leqslant N$ for all $2 \leqslant i \leqslant n'$ where M, N are fixed constants and n' is large. Since there are only finitely many choices for m_i and $m_i y_i - l_i y_1$, by the Pigeonhole Principle, we can assume without loss of generality $m_i = m$ and $m_i y_i - l_i y_1 = t$ for $2 \leqslant i \leqslant d+2$. Then

$$\frac{Q(y_i)}{Q(y_1)} = \frac{l_i^d}{m^d} = \frac{(my_i - t)^d}{m^d y_1^d}$$

so that $Q(y_1)(my-t)^d - m^d y_1^d Q(y)$ has roots $y = y_2, y_3, \dots, y_{d+2}$. Its degree is at most d and hence it is the zero polynomial. Therefore, $Q(y) = \frac{b_d}{m^d}(my-t)^d$. Indeed, Q does not have the term y^{d-1}, which means t should be 0. This gives the corresponding $P(x)$ of the desired form.

The two modifications of the Solution work equally well when the degree d is even.

IMO 2017
RIO DE JANEIRO - BRAZIL
58th International Mathematical Olympiad

Shortlisted Problems

(with solutions)

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58th International Mathematical Olympiad

Rio de Janeiro, 12–23 July 2017

Note of Confidentiality

> **The Shortlist has to be kept strictly confidential**
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> **International Mathematical Olympiad.**
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Contributing Countries

The Organizing Committee and the Problem Selection Committee of IMO 2017 thank the following 51 countries for contributing 150 problem proposals:

Albania, Algeria, Armenia, Australia, Austria, Azerbaijan, Belarus, Belgium, Bulgaria, Cuba, Cyprus, Czech Republic, Denmark, Estonia, France, Georgia, Germany, Greece, Hong Kong, India, Iran, Ireland, Israel, Italy, Japan, Kazakhstan, Latvia, Lithuania, Luxembourg, Mexico, Montenegro, Morocco, Netherlands, Romania, Russia, Serbia, Singapore, Slovakia, Slovenia, South Africa, Sweden, Switzerland, Taiwan, Tajikistan, Tanzania, Thailand, Trinidad and Tobago, Turkey, Ukraine, United Kingdom, U.S.A.

Problem Selection Committee

Carlos Gustavo Tamm de Araújo Moreira (Gugu) (chairman),
Luciano Monteiro de Castro, Ilya I. Bogdanov, Géza Kós, Carlos Yuzo Shine,
Zhuo Qun (Alex) Song, Ralph Costa Teixeira, Eduardo Tengan

Problems

Algebra

A1. Let $a_1, a_2, \ldots, a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \ldots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

(Trinidad and Tobago)

A2. Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:

- In the first line, Gugu writes down every number of the form $a-b$, where a and b are two (not necessarily distinct) numbers on his napkin.
- In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the *first line*.
- In the third line, Gugu writes down every number of the form $a^2+b^2-c^2-d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the *first line*.

Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

(Austria)

A3. Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of S under f. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for every g in $\mathcal{A}$ with $g \neq f$. Show that $f(T) = T$.

(India)

A4. A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n}(a_i + a_j) \qquad \text{for all } n > 2017.$$

Prove that this sequence is bounded, i.e., there is a constant M such that $|a_n| \leqslant M$ for all positive integers n.

(Russia)

A5. An integer $n \geqslant 3$ is given. We call an n-tuple of real numbers $(x_1, x_2, \dots, x_n)$ *Shiny* if for each permutation $y_1, y_2, \dots, y_n$ of these numbers we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1 y_2 + y_2 y_3 + y_3 y_4 + \cdots + y_{n-1} y_n \geqslant -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leqslant i < j \leqslant n} x_i x_j \geqslant K$$

holds for every Shiny n-tuple $(x_1, x_2, \dots, x_n)$.

(Serbia)

A6. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)f(y)) + f(x+y) = f(xy)$$

for all $x, y \in \mathbb{R}$.

(Albania)

A7. Let $a_0, a_1, a_2, \dots$ be a sequence of integers and $b_0, b_1, b_2, \dots$ be a sequence of *positive* integers such that $a_0 = 0, a_1 = 1$, and

$$a_{n+1} = \begin{cases} a_n b_n + a_{n-1}, & \text{if } b_{n-1} = 1 \\ a_n b_n - a_{n-1}, & \text{if } b_{n-1} > 1 \end{cases} \qquad \text{for } n = 1, 2, \dots.$$

Prove that at least one of the two numbers a_{2017} and a_{2018} must be greater than or equal to 2017.

(Australia)

A8. Assume that a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following condition:

For every $x, y \in \mathbb{R}$ such that $\big(f(x)+y\big)\big(f(y)+x\big) > 0$, we have $f(x)+y = f(y)+x$.

Prove that $f(x) + y \leqslant f(y) + x$ whenever $x > y$.

(Netherlands)

Combinatorics

C1. A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

(Singapore)

C2. Let n be a positive integer. Define a *chameleon* to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters a, b, and c. Define a *swap* to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X, there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

(Australia)

C3. Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:

(1) Choose any number of the form 2^j, where j is a non-negative integer, and put it into an empty cell.

(2) Choose two (not necessarily adjacent) cells with the same number in them; denote that number by 2^j. Replace the number in one of the cells with 2^{j+1} and erase the number in the other cell.

At the end of the game, one cell contains the number 2^n, where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

(Thailand)

C4. Let $N \geqslant 2$ be an integer. $N(N+1)$ soccer players, no two of the same height, stand in a row in some order. Coach Ralph wants to remove $N(N-1)$ people from this row so that in the remaining row of $2N$ players, no one stands between the two tallest ones, no one stands between the third and the fourth tallest ones, ..., and finally no one stands between the two shortest ones. Show that this is always possible.

(Russia)

C5. A hunter and an invisible rabbit play a game in the Euclidean plane. The hunter's starting point H_0 coincides with the rabbit's starting point R_0. In the n^{th} round of the game ($n \geqslant 1$), the following happens.

(1) First the invisible rabbit moves secretly and unobserved from its current point R_{n-1} to some new point R_n with $R_{n-1}R_n = 1$.

(2) The hunter has a tracking device (e.g. dog) that returns an approximate position R'_n of the rabbit, so that $R_nR'_n \leqslant 1$.

(3) The hunter then visibly moves from point H_{n-1} to a new point H_n with $H_{n-1}H_n = 1$.

Is there a strategy for the hunter that guarantees that after 10^9 such rounds the distance between the hunter and the rabbit is below 100?

(Austria)

C6. Let $n > 1$ be an integer. An $n \times n \times n$ cube is composed of n^3 unit cubes. Each unit cube is painted with one color. For each $n \times n \times 1$ box consisting of n^2 unit cubes (of any of the three possible orientations), we consider the set of the colors present in that box (each color is listed only once). This way, we get $3n$ sets of colors, split into three groups according to the orientation. It happens that for every set in any group, the same set appears in both of the other groups. Determine, in terms of n, the maximal possible number of colors that are present.

(Russia)

C7. For any finite sets X and Y of positive integers, denote by $f_X(k)$ the k^{th} smallest positive integer not in X, and let

$$X * Y = X \cup \{f_X(y) \colon y \in Y\}.$$

Let A be a set of $a > 0$ positive integers, and let B be a set of $b > 0$ positive integers. Prove that if $A * B = B * A$, then

$$\underbrace{A * (A * \cdots * (A * (A * A)) \ldots)}_{A \text{ appears } b \text{ times}} = \underbrace{B * (B * \cdots * (B * (B * B)) \ldots)}_{B \text{ appears } a \text{ times}}.$$

(U.S.A.)

C8. Let n be a given positive integer. In the Cartesian plane, each lattice point with nonnegative coordinates initially contains a butterfly, and there are no other butterflies. The *neighborhood* of a lattice point c consists of all lattice points within the axis-aligned $(2n+1) \times (2n+1)$ square centered at c, apart from c itself. We call a butterfly *lonely*, *crowded*, or *comfortable*, depending on whether the number of butterflies in its neighborhood N is respectively less than, greater than, or equal to half of the number of lattice points in N.

Every minute, all lonely butterflies fly away simultaneously. This process goes on for as long as there are any lonely butterflies. Assuming that the process eventually stops, determine the number of comfortable butterflies at the final state.

(Bulgaria)

Geometry

G1. Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

(Italy)

G2. Let R and S be distinct points on circle Ω, and let t denote the tangent line to Ω at R. Point R' is the reflection of R with respect to S. A point I is chosen on the smaller arc RS of Ω so that the circumcircle Γ of triangle ISR' intersects t at two different points. Denote by A the common point of Γ and t that is closest to R. Line AI meets Ω again at J. Show that JR' is tangent to Γ.

(Luxembourg)

G3. Let O be the circumcenter of an acute scalene triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

(Ukraine)

G4. In triangle ABC, let ω be the excircle opposite A. Let D, E, and F be the points where ω is tangent to lines BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

(Denmark)

G5. Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

(Ukraine)

G6. Let $n \geqslant 3$ be an integer. Two regular n-gons $\mathcal{A}$ and $\mathcal{B}$ are given in the plane. Prove that the vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary are consecutive.

(That is, prove that there exists a line separating those vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary from the other vertices of $\mathcal{A}$.)

(Czech Republic)

G7. A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c, and I_d be the incenters of the triangles DAB, ABC, BCD, and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90^\circ$.

(Kazakhstan)

G8. There are 2017 mutually external circles drawn on a blackboard, such that no two are tangent and no three share a common tangent. A *tangent segment* is a line segment that is a common tangent to two circles, starting at one tangent point and ending at the other one. Luciano is drawing tangent segments on the blackboard, one at a time, so that no tangent segment intersects any other circles or previously drawn tangent segments. Luciano keeps drawing tangent segments until no more can be drawn. Find all possible numbers of tangent segments when he stops drawing.

(Australia)

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Number Theory

N1. The sequence $a_0, a_1, a_2, \ldots$ of positive integers satisfies

$$a_{n+1} = \begin{cases} \sqrt{a_n}, & \text{if } \sqrt{a_n} \text{ is an integer} \\ a_n + 3, & \text{otherwise} \end{cases} \qquad \text{for every } n \geqslant 0.$$

Determine all values of $a_0 > 1$ for which there is at least one number a such that $a_n = a$ for infinitely many values of n.

(South Africa)

N2. Let $p \geqslant 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices $i \in \{0, 1, \ldots, p-1\}$ have been chosen. Then the following number is computed:

$$M = a_0 + 10 \cdot a_1 + \cdots + 10^{p-1} \cdot a_{p-1} = \sum_{j=0}^{p-1} a_j \cdot 10^j.$$

The goal of Eduardo is to make the number M divisible by p, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

(Morocco)

N3. Determine all integers $n \geqslant 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leqslant i \leqslant n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \cdots + a_{i+n-1}$$

is divisible by n. (We let $a_i = a_{i-n}$ when $i > n$.)

(Thailand)

N4. Call a rational number *short* if it has finitely many digits in its decimal expansion. For a positive integer m, we say that a positive integer t is m-*tastic* if there exists a number $c \in \{1, 2, 3, \ldots, 2017\}$ such that $\dfrac{10^t - 1}{c \cdot m}$ is short, and such that $\dfrac{10^k - 1}{c \cdot m}$ is not short for any $1 \leqslant k < t$. Let $S(m)$ be the set of m-tastic numbers. Consider $S(m)$ for $m = 1, 2, \ldots$. What is the maximum number of elements in $S(m)$?

(Turkey)

N5. Find all pairs (p, q) of prime numbers with $p > q$ for which the number

$$\frac{(p+q)^{p+q}(p-q)^{p-q} - 1}{(p+q)^{p-q}(p-q)^{p+q} - 1}$$

is an integer.

(Japan)

N6. Find the smallest positive integer n, or show that no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \qquad \text{and} \qquad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

are integers.

(Singapore)

N7. Say that an ordered pair (x, y) of integers is an *irreducible lattice point* if x and y are relatively prime. For any finite set S of irreducible lattice points, show that there is a homogenous polynomial in two variables, $f(x, y)$, with integer coefficients, of degree at least 1, such that $f(x, y) = 1$ for each (x, y) in the set S.

Note: A homogenous polynomial of degree n is any nonzero polynomial of the form

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_{n-1} x y^{n-1} + a_n y^n.$$

(U.S.A.)

N8. Let p be an odd prime number and $\mathbb{Z}_{>0}$ be the set of positive integers. Suppose that a function $f\colon \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \{0, 1\}$ satisfies the following properties:

- $f(1, 1) = 0$;
- $f(a, b) + f(b, a) = 1$ for any pair of relatively prime positive integers (a, b) not both equal to 1;
- $f(a + b, b) = f(a, b)$ for any pair of relatively prime positive integers (a, b).

Prove that

$$\sum_{n=1}^{p-1} f(n^2, p) \geqslant \sqrt{2p} - 2.$$

(Italy)

Solutions

Algebra

A1. Let $a_1, a_2, \ldots, a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \ldots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

(Trinidad and Tobago)

Solution 1. We first prove that, for $x > 0$,

$$a_i(x+1)^{1/a_i} \leqslant x + a_i, \tag{1}$$

with equality if and only if $a_i = 1$. It is clear that equality occurs if $a_i = 1$.

If $a_i > 1$, the AM–GM inequality applied to a single copy of $x+1$ and $a_i - 1$ copies of 1 yields

$$\frac{(x+1) + \overbrace{1+1+\cdots+1}^{a_i-1 \text{ ones}}}{a_i} \geqslant \sqrt[a_i]{(x+1)\cdot 1^{a_i-1}} \implies a_i(x+1)^{1/a_i} \leqslant x + a_i.$$

Since $x+1 > 1$, the inequality is strict for $a_i > 1$.

Multiplying the inequalities (1) for $i = 1, 2, \ldots, n$ yields

$$\prod_{i=1}^{n} a_i(x+1)^{1/a_i} \leqslant \prod_{i=1}^{n}(x+a_i) \iff M(x+1)^{\sum_{i=1}^{n} 1/a_i} - \prod_{i=1}^{n}(x+a_i) \leqslant 0 \iff P(x) \leqslant 0$$

with equality iff $a_i = 1$ for all $i \in \{1, 2, \ldots, n\}$. But this implies $M = 1$, which is not possible. Hence $P(x) < 0$ for all $x \in \mathbb{R}^+$, and P has no positive roots.

Comment 1. Inequality (1) can be obtained in several ways. For instance, we may also use the binomial theorem: since $a_i \geqslant 1$,

$$\left(1 + \frac{x}{a_i}\right)^{a_i} = \sum_{j=0}^{a_i} \binom{a_i}{j}\left(\frac{x}{a_i}\right)^j \geqslant \binom{a_i}{0} + \binom{a_i}{1}\cdot\frac{x}{a_i} = 1 + x.$$

Both proofs of (1) mimic proofs to Bernoulli's inequality for a positive integer exponent a_i; we can use this inequality directly:

$$\left(1 + \frac{x}{a_i}\right)^{a_i} \geqslant 1 + a_i \cdot \frac{x}{a_i} = 1 + x,$$

and so

$$x + a_i = a_i\left(1 + \frac{x}{a_i}\right) \geqslant a_i(1+x)^{1/a_i},$$

or its (reversed) formulation, with exponent $1/a_i \leqslant 1$:

$$(1+x)^{1/a_i} \leqslant 1 + \frac{1}{a_i}\cdot x = \frac{x+a_i}{a_i} \implies a_i(1+x)^{1/a_i} \leqslant x + a_i.$$

Solution 2. We will prove that, in fact, all coefficients of the polynomial $P(x)$ are non-positive, and at least one of them is negative, which implies that $P(x) < 0$ for $x > 0$.

Indeed, since $a_j \geqslant 1$ for all j and $a_j > 1$ for some j (since $a_1 a_2 \dots a_n = M > 1$), we have $k = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < n$, so the coefficient of x^n in $P(x)$ is $-1 < 0$. Moreover, the coefficient of x^r in $P(x)$ is negative for $k < r \leqslant n = \deg(P)$.

For $0 \leqslant r \leqslant k$, the coefficient of x^r in $P(x)$ is

$$M \cdot \binom{k}{r} - \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{n-r} \leqslant n} a_{i_1} a_{i_2} \cdots a_{i_{n-r}} = a_1 a_2 \cdots a_n \cdot \binom{k}{r} - \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{n-r} \leqslant n} a_{i_1} a_{i_2} \cdots a_{i_{n-r}},$$

which is non-positive iff

$$\binom{k}{r} \leqslant \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}}. \tag{2}$$

We will prove (2) by induction on r. For $r = 0$ it is an equality because the constant term of $P(x)$ is $P(0) = 0$, and if $r = 1$, (2) becomes $k = \sum_{i=1}^{n} \frac{1}{a_i}$. For $r > 1$, if (2) is true for a given $r < k$, we have

$$\binom{k}{r+1} = \frac{k-r}{r+1} \cdot \binom{k}{r} \leqslant \frac{k-r}{r+1} \cdot \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}},$$

and it suffices to prove that

$$\frac{k-r}{r+1} \cdot \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}} \leqslant \sum_{1 \leqslant j_1 < \cdots < j_r < j_{r+1} \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r} a_{j_{r+1}}},$$

which is equivalent to

$$\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - r \right) \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}} \leqslant (r+1) \sum_{1 \leqslant j_1 < \cdots < j_r < j_{r+1} \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r} a_{j_{r+1}}}.$$

Since there are $r+1$ ways to choose a fraction $\frac{1}{a_{j_i}}$ from $\frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r} a_{j_{r+1}}}$ to factor out, every term $\frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r} a_{j_{r+1}}}$ in the right hand side appears exactly $r+1$ times in the product

$$\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}}.$$

Hence all terms in the right hand side cancel out.

The remaining terms in the left hand side can be grouped in sums of the type

$$\frac{1}{a_{j_1}^2 a_{j_2} \cdots a_{j_r}} + \frac{1}{a_{j_1} a_{j_2}^2 \cdots a_{j_r}} + \cdots + \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}^2} - \frac{r}{a_{j_1} a_{j_2} \cdots a_{j_r}}$$
$$= \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}} \left(\frac{1}{a_{j_1}} + \frac{1}{a_{j_2}} + \cdots + \frac{1}{a_{j_r}} - r \right),$$

which are all non-positive because $a_i \geqslant 1 \implies \frac{1}{a_i} \leqslant 1$, $i = 1, 2, \dots, n$.

Comment 2. The result is valid for any real numbers a_i, $i = 1, 2, \dots, n$ with $a_i \geqslant 1$ and product M greater than 1. A variation of Solution 1, namely using *weighted* AM–GM (or the Bernoulli inequality for real exponents), actually proves that $P(x) < 0$ for $x > -1$ and $x \neq 0$.

A2. Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:

- In the first line, Gugu writes down every number of the form $a-b$, where a and b are two (not necessarily distinct) numbers on his napkin.
- In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the *first line*.
- In the third line, Gugu writes down every number of the form $a^2+b^2-c^2-d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the *first line*.

Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

(Austria)

Answer: $-2, 0, 2$.

Solution 1. Call a number q *good* if every number in the second line appears in the third line unconditionally. We first show that the numbers 0 and ± 2 are good. The third line necessarily contains 0, so 0 is good. For any two numbers a, b in the first line, write $a = x-y$ and $b = u-v$, where x, y, u, v are (not necessarily distinct) numbers on the napkin. We may now write

$$2ab = 2(x-y)(u-v) = (x-v)^2 + (y-u)^2 - (x-u)^2 - (y-v)^2,$$

which shows that 2 is good. By negating both sides of the above equation, we also see that -2 is good.

We now show that $-2, 0$, and 2 are the only good numbers. Assume for sake of contradiction that q is a good number, where $q \notin \{-2, 0, 2\}$. We now consider some particular choices of numbers on Gugu's napkin to arrive at a contradiction.

Assume that the napkin contains the integers $1, 2, \ldots, 10$. Then, the first line contains the integers $-9, -8, \ldots, 9$. The second line then contains q and $81q$, so the third line must also contain both of them. But the third line only contains integers, so q must be an integer. Furthermore, the third line contains no number greater than $162 = 9^2+9^2-0^2-0^2$ or less than -162, so we must have $-162 \leqslant 81q \leqslant 162$. This shows that the only possibilities for q are ± 1.

Now assume that $q = \pm 1$. Let the napkin contain $0, 1, 4, 8, 12, 16, 20, 24, 28, 32$. The first line contains ± 1 and ± 4, so the second line contains ± 4. However, for every number a in the first line, $a \not\equiv 2 \pmod 4$, so we may conclude that $a^2 \equiv 0, 1 \pmod 8$. Consequently, every number in the third line must be congruent to $-2, -1, 0, 1, 2 \pmod 8$; in particular, ± 4 cannot be in the third line, which is a contradiction.

Solution 2. Let q be a good number, as defined in the first solution, and define the polynomial $P(x_1, \ldots, x_{10})$ as

$$\prod_{i<j}(x_i - x_j) \prod_{a_i \in S}\left(q(x_1-x_2)(x_3-x_4) - (a_1-a_2)^2 - (a_3-a_4)^2 + (a_5-a_6)^2 + (a_7-a_8)^2\right),$$

where $S = \{x_1, \ldots, x_{10}\}$.

We claim that $P(x_1, \ldots, x_{10}) = 0$ for every choice of real numbers $(x_1, \ldots, x_{10})$. If any two of the x_i are equal, then $P(x_1, \ldots, x_{10}) = 0$ trivially. If no two are equal, assume that Gugu has those ten numbers $x_1, \ldots, x_{10}$ on his napkin. Then, the number $q(x_1-x_2)(x_3-x_4)$ is in the second line, so we must have some $a_1, \ldots, a_8$ so that

$$q(x_1-x_2)(x_3-x_4) - (a_1-a_2)^2 - (a_3-a_4)^2 + (a_5-a_6)^2 + (a_7-a_8)^2 = 0,$$

and hence $P(x_1, \ldots, x_{10}) = 0$.

Since every polynomial that evaluates to zero everywhere is the zero polynomial, and the product of two nonzero polynomials is necessarily nonzero, we may define F such that

$$F(x_1, \ldots, x_{10}) \equiv q(x_1 - x_2)(x_3 - x_4) - (a_1 - a_2)^2 - (a_3 - a_4)^2 + (a_5 - a_6)^2 + (a_7 - a_8)^2 \equiv 0 \quad (1)$$

for some particular choice $a_i \in S$.

Each of the sets $\{a_1, a_2\}$, $\{a_3, a_4\}$, $\{a_5, a_6\}$, and $\{a_7, a_8\}$ is equal to at most one of the four sets $\{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_4\}$, and $\{x_2, x_4\}$. Thus, without loss of generality, we may assume that at most one of the sets $\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}$, and $\{a_7, a_8\}$ is equal to $\{x_1, x_3\}$. Let u_1, u_3, u_5, u_7 be the indicator functions for this equality of sets: that is, $u_i = 1$ if and only if $\{a_i, a_{i+1}\} = \{x_1, x_3\}$. By assumption, at least three of the u_i are equal to 0.

We now compute the coefficient of $x_1 x_3$ in F. It is equal to $q + 2(u_1 + u_3 - u_5 - u_7) = 0$, and since at least three of the u_i are zero, we must have that $q \in \{-2, 0, 2\}$, as desired.

A3. Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of S under f. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for every g in $\mathcal{A}$ with $g \neq f$. Show that $f(T) = T$.

(India)

Solution. For $n \geqslant 1$, denote the n-th composition of f with itself by

$$f^n \stackrel{\text{def}}{=} \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

By hypothesis, if $g \in \mathcal{A}$ satisfies $f \circ g \circ f = g \circ f \circ g$, then $g = f$. A natural idea is to try to plug in $g = f^n$ for some n in the expression $f \circ g \circ f = g \circ f \circ g$ in order to get $f^n = f$, which solves the problem:

Claim. If there exists $n \geqslant 3$ such that $f^{n+2} = f^{2n+1}$, then the restriction $f \colon T \to T$ of f to T is a bijection.

Proof. Indeed, by hypothesis, $f^{n+2} = f^{2n+1} \iff f \circ f^n \circ f = f^n \circ f \circ f^n \implies f^n = f$. Since $n - 2 \geqslant 1$, the image of f^{n-2} is contained in $T = f(S)$, hence f^{n-2} restricts to a function $f^{n-2} \colon T \to T$. This is the inverse of $f \colon T \to T$. In fact, given $t \in T$, say $t = f(s)$ with $s \in S$, we have

$$t = f(s) = f^n(s) = f^{n-2}(f(t)) = f(f^{n-2}(t)), \qquad \text{i.e.,} \qquad f^{n-2} \circ f = f \circ f^{n-2} = \text{id on } T$$

(here id stands for the identity function). Hence, the restriction $f \colon T \to T$ of f to T is bijective with inverse given by $f^{n-2} \colon T \to T$. $\square$

It remains to show that n as in the claim exists. For that, define

$$S_m \stackrel{\text{def}}{=} f^m(S) \qquad (S_m \text{ is image of } f^m)$$

Clearly the image of f^{m+1} is contained in the image of f^m, i.e., there is a descending chain of subsets of S

$$S \supseteq S_1 \supseteq S_2 \supseteq S_3 \supseteq S_4 \supseteq \cdots,$$

which must eventually stabilise since S is finite, i.e., there is a $k \geqslant 1$ such that

$$S_k = S_{k+1} = S_{k+2} = S_{k+3} = \cdots \stackrel{\text{def}}{=} S_\infty.$$

Hence f restricts to a *surjective* function $f \colon S_\infty \to S_\infty$, which is also *bijective* since $S_\infty \subseteq S$ is finite. To sum up, $f \colon S_\infty \to S_\infty$ is a permutation of the elements of the finite set S_∞, hence there exists an integer $r \geqslant 1$ such that $f^r = \text{id}$ on S_∞ (for example, we may choose $r = |S_\infty|!$). In other words,

$$f^{m+r} = f^m \text{ on } S \text{ for all } m \geqslant k. \qquad (*)$$

Clearly, $(*)$ also implies that $f^{m+tr} = f^m$ for all integers $t \geqslant 1$ and $m \geqslant k$. So, to find n as in the claim and finish the problem, it is enough to choose m and t in order to ensure that there exists $n \geqslant 3$ satisfying

$$\begin{cases} 2n + 1 = m + tr \\ n + 2 = m \end{cases} \iff \begin{cases} m = 3 + tr \\ n = m - 2. \end{cases}$$

This can be clearly done by choosing m large enough with $m \equiv 3 \pmod{r}$. For instance, we may take $n = 2kr + 1$, so that

$$f^{n+2} = f^{2kr+3} = f^{4kr+3} = f^{2n+1}$$

where the middle equality follows by $(*)$ since $2kr + 3 \geqslant k$.

A4. A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n}(a_i + a_j) \qquad \text{for all } n > 2017.$$

Prove that this sequence is bounded, i.e., there is a constant M such that $|a_n| \leqslant M$ for all positive integers n.

(Russia)

Solution 1. Set $D = 2017$. Denote

$$M_n = \max_{k<n} a_k \qquad \text{and} \qquad m_n = -\min_{k<n} a_k = \max_{k<n}(-a_k).$$

Clearly, the sequences (m_n) and (M_n) are nondecreasing. We need to prove that both are bounded.

Consider an arbitrary $n > D$; our first aim is to bound a_n in terms of m_n and M_n.

(i) There exist indices p and q such that $a_n = -(a_p + a_q)$ and $p + q = n$. Since $a_p, a_q \leqslant M_n$, we have $a_n \geqslant -2M_n$.

(ii) On the other hand, choose an index $k < n$ such that $a_k = M_n$. Then, we have

$$a_n = -\max_{\ell<n}(a_{n-\ell} + a_\ell) \leqslant -(a_{n-k} + a_k) = -a_{n-k} - M_n \leqslant m_n - M_n.$$

Summarizing (i) and (ii), we get

$$-2M_n \leqslant a_n \leqslant m_n - M_n,$$

whence

$$m_n \leqslant m_{n+1} \leqslant \max\{m_n, 2M_n\} \quad \text{and} \quad M_n \leqslant M_{n+1} \leqslant \max\{M_n, m_n - M_n\}. \tag{1}$$

Now, say that an index $n > D$ is *lucky* if $m_n \leqslant 2M_n$. Two cases are possible.

Case 1. Assume that there exists a lucky index n. In this case, (1) yields $m_{n+1} \leqslant 2M_n$ and $M_n \leqslant M_{n+1} \leqslant M_n$. Therefore, $M_{n+1} = M_n$ and $m_{n+1} \leqslant 2M_n = 2M_{n+1}$. So, the index $n+1$ is also lucky, and $M_{n+1} = M_n$. Applying the same arguments repeatedly, we obtain that all indices $k > n$ are lucky (i.e., $m_k \leqslant 2M_k$ for all these indices), and $M_k = M_n$ for all such indices. Thus, all of the m_k and M_k are bounded by $2M_n$.

Case 2. Assume now that there is no lucky index, i.e., $2M_n < m_n$ for all $n > D$. Then (1) shows that for all $n > D$ we have $m_n \leqslant m_{n+1} \leqslant m_n$, so $m_n = m_{D+1}$ for all $n > D$. Since $M_n < m_n/2$ for all such indices, all of the m_n and M_n are bounded by m_{D+1}.

Thus, in both cases the sequences (m_n) and (M_n) are bounded, as desired.

Solution 2. As in the previous solution, let $D = 2017$. If the sequence is bounded above, say, by Q, then we have that $a_n \geqslant \min\{a_1, \ldots, a_D, -2Q\}$ for all n, so the sequence is bounded. Assume for sake of contradiction that the sequence is not bounded above. Let $\ell = \min\{a_1, \ldots, a_D\}$, and $L = \max\{a_1, \ldots, a_D\}$. Call an index n *good* if the following criteria hold:

$$a_n > a_i \text{ for each } i < n, \quad a_n > -2\ell, \quad \text{and} \quad n > D \tag{2}$$

We first show that there must be some good index n. By assumption, we may take an index N such that $a_N > \max\{L, -2\ell\}$. Choose n minimally such that $a_n = \max\{a_1, a_2, \ldots, a_N\}$. Now, the first condition in (2) is satisfied because of the minimality of n, and the second and third conditions are satisfied because $a_n \geqslant a_N > L, -2\ell$, and $L \geqslant a_i$ for every i such that $1 \leqslant i \leqslant D$.

Let n be a good index. We derive a contradiction. We have that

$$a_n + a_u + a_v \leqslant 0, \tag{3}$$

whenever $u + v = n$.

We define the index u to maximize a_u over $1 \leqslant u \leqslant n-1$, and let $v = n-u$. Then, we note that $a_u \geqslant a_v$ by the maximality of a_u.

Assume first that $v \leqslant D$. Then, we have that

$$a_N + 2\ell \leqslant 0,$$

because $a_u \geqslant a_v \geqslant \ell$. But this contradicts our assumption that $a_n > -2\ell$ in the second criteria of (2).

Now assume that $v > D$. Then, there exist some indices w_1, w_2 summing up to v such that

$$a_v + a_{w_1} + a_{w_2} = 0.$$

But combining this with (3), we have

$$a_n + a_u \leqslant a_{w_1} + a_{w_2}.$$

Because $a_n > a_u$, we have that $\max\{a_{w_1}, a_{w_2}\} > a_u$. But since each of the w_i is less than v, this contradicts the maximality of a_u.

Comment 1. We present two harder versions of this problem below.

Version 1. Let $a_1, a_2, \ldots$ be a sequence of numbers that satisfies the relation

$$a_n = -\max_{i+j+k=n}(a_i + a_j + a_k) \qquad \text{for all } n > 2017.$$

Then, this sequence is bounded.

Proof. Set $D = 2017$. Denote

$$M_n = \max_{k<n} a_k \qquad \text{and} \qquad m_n = -\min_{k<n} a_k = \max_{k<n}(-a_k).$$

Clearly, the sequences (m_n) and (M_n) are nondecreasing. We need to prove that both are bounded.

Consider an arbitrary $n > 2D$; our first aim is to bound a_n in terms of m_i and M_i. Set $k = \lfloor n/2 \rfloor$.

(i) Choose indices p, q, and r such that $a_n = -(a_p + a_q + a_r)$ and $p + q + r = n$. Without loss of generality, $p \geqslant q \geqslant r$.

Assume that $p \geqslant k + 1 (> D)$; then $p > q + r$. Hence

$$-a_p = \max_{i_1+i_2+i_3=p}(a_{i_1} + a_{i_2} + a_{i_3}) \geqslant a_q + a_r + a_{p-q-r},$$

and therefore $a_n = -(a_p + a_q + a_r) \geqslant (a_q + a_r + a_{p-q-r}) - a_q - a_r = a_{p-q-r} \geqslant -m_n$.

Otherwise, we have $k \geqslant p \geqslant q \geqslant r$. Since $n < 3k$, we have $r < k$. Then $a_p, a_q \leqslant M_{k+1}$ and $a_r \leqslant M_k$, whence $a_n \geqslant -2M_{k+1} - M_k$.

Thus, in any case $a_n \geqslant -\max\{m_n, 2M_{k+1} + M_k\}$.

(ii) On the other hand, choose $p \leqslant k$ and $q \leqslant k-1$ such that $a_p = M_{k+1}$ and $a_q = M_k$. Then $p+q < n$, so $a_n \leqslant -(a_p + a_q + a_{n-p-q}) = -a_{n-p-q} - M_{k+1} - M_k \leqslant m_n - M_{k+1} - M_k$.

To summarize,

$$-\max\{m_n, 2M_{k+1} + M_k\} \leqslant a_n \leqslant m_n - M_{k+1} - M_k,$$

whence

$$m_n \leqslant m_{n+1} \leqslant \max\{m_n, 2M_{k+1} + M_k\} \quad \text{and} \quad M_n \leqslant M_{n+1} \leqslant \max\{M_n, m_n - M_{k+1} - M_k\}. \tag{4}$$

Now, say that an index $n > 2D$ is *lucky* if $m_n \leqslant 2M_{\lfloor n/2\rfloor+1} + M_{\lfloor n/2\rfloor}$. Two cases are possible.

Case 1. Assume that there exists a lucky index n; set $k = \lfloor n/2\rfloor$. In this case, (4) yields $m_{n+1} \leqslant 2M_{k+1} + M_k$ and $M_n \leqslant M_{n+1} \leqslant M_n$ (the last relation holds, since $m_n - M_{k+1} - M_k \leqslant (2M_{k+1} + M_k) - M_{k+1} - M_k = M_{k+1} \leqslant M_n$). Therefore, $M_{n+1} = M_n$ and $m_{n+1} \leqslant 2M_{k+1} + M_k$; the last relation shows that the index $n+1$ is also lucky.

Thus, all indices $N > n$ are lucky, and $M_N = M_n \geqslant m_N/3$, whence all the m_N and M_N are bounded by $3M_n$.

Case 2. Conversely, assume that there is no lucky index, i.e., $2M_{\lfloor n/2\rfloor+1} + M_{\lfloor n/2\rfloor} < m_n$ for all $n > 2D$. Then (4) shows that for all $n > 2D$ we have $m_n \leqslant m_{n+1} \leqslant m_n$, i.e., $m_N = m_{2D+1}$ for all $N > 2D$. Since $M_N < m_{2N+1}/3$ for all such indices, all the m_N and M_N are bounded by m_{2D+1}.

Thus, in both cases the sequences (m_n) and (M_n) are bounded, as desired. □

Version 2. Let $a_1, a_2, \dots$ be a sequence of numbers that satisfies the relation

$$a_n = -\max_{i_1+\cdots+i_k=n}(a_{i_1} + \cdots + a_{i_k}) \qquad \text{for all } n > 2017.$$

Then, this sequence is bounded.

Proof. As in the solutions above, let $D = 2017$. If the sequence is bounded above, say, by Q, then we have that $a_n \geqslant \min\{a_1, \dots, a_D, -kQ\}$ for all n, so the sequence is bounded. Assume for sake of contradiction that the sequence is not bounded above. Let $\ell = \min\{a_1, \dots, a_D\}$, and $L = \max\{a_1, \dots, a_D\}$. Call an index n *good* if the following criteria hold:

$$a_n > a_i \text{ for each } i < n, \quad a_n > -k\ell, \quad \text{and} \quad n > D \tag{5}$$

We first show that there must be some good index n. By assumption, we may take an index N such that $a_N > \max\{L, -k\ell\}$. Choose n minimally such that $a_n = \max\{a_1, a_2, \dots, a_N\}$. Now, the first condition is satisfied because of the minimality of n, and the second and third conditions are satisfied because $a_n \geqslant a_N > L, -k\ell$, and $L \geqslant a_i$ for every i such that $1 \leqslant i \leqslant D$.

Let n be a good index. We derive a contradiction. We have that

$$a_n + a_{v_1} + \cdots + a_{v_k} \leqslant 0, \tag{6}$$

whenever $v_1 + \cdots + v_k = n$.

We define the sequence of indices $v_1, \dots, v_{k-1}$ to greedily maximize a_{v_1}, then a_{v_2}, and so forth, selecting only from indices such that the equation $v_1 + \cdots + v_k = n$ can be satisfied by positive integers $v_1, \dots, v_k$. More formally, we define them inductively so that the following criteria are satisfied by the v_i:

1. $1 \leqslant v_i \leqslant n - (k-i) - (v_1 + \cdots + v_{i-1})$.
2. a_{v_i} is maximal among all choices of v_i from the first criteria.

First of all, we note that for each i, the first criteria is always satisfiable by some v_i, because we are guaranteed that

$$v_{i-1} \leqslant n - (k-(i-1)) - (v_1 + \cdots + v_{i-2}),$$

which implies

$$1 \leqslant n - (k-i) - (v_1 + \cdots + v_{i-1}).$$

Secondly, the sum $v_1 + \cdots + v_{k-1}$ is at most $n-1$. Define $v_k = n - (v_1 + \cdots + v_{k-1})$. Then, (6) is satisfied by the v_i. We also note that $a_{v_i} \geqslant a_{v_j}$ for all $i < j$; otherwise, in the definition of v_i, we could have selected v_j instead.

Assume first that $v_k \leqslant D$. Then, from (6), we have that

$$a_n + k\ell \leqslant 0,$$

by using that $a_{v_1} \geqslant \cdots \geqslant a_{v_k} \geqslant \ell$. But this contradicts our assumption that $a_n > -k\ell$ in the second criteria of (5).

Now assume that $v_k > D$, and then we must have some indices $w_1, \dots, w_k$ summing up to v_k such that

$$a_{v_k} + a_{w_1} + \cdots + a_{w_k} = 0.$$

But combining this with (6), we have

$$a_n + a_{v_1} + \cdots + a_{v_{k-1}} \leqslant a_{w_1} + \cdots + a_{w_k}.$$

Because $a_n > a_{v_1} \geqslant \cdots \geqslant a_{v_{k-1}}$, we have that $\max\{a_{w_1}, \dots, a_{w_k}\} > a_{v_{k-1}}$. But since each of the w_i is less than v_k, in the definition of the v_{k-1} we could have chosen one of the w_i instead, which is a contradiction. $\square$

Comment 2. It seems that each sequence satisfying the condition in Version 2 is eventually periodic, at least when its terms are integers.

However, up to this moment, the Problem Selection Committee is not aware of a proof for this fact (even in the case $k = 2$).

A5. An integer $n \geqslant 3$ is given. We call an n-tuple of real numbers $(x_1, x_2, \dots, x_n)$ *Shiny* if for each permutation $y_1, y_2, \dots, y_n$ of these numbers we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1y_2 + y_2y_3 + y_3y_4 + \cdots + y_{n-1}y_n \geqslant -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leqslant i<j \leqslant n} x_i x_j \geqslant K$$

holds for every Shiny n-tuple $(x_1, x_2, \dots, x_n)$.

(Serbia)

Answer: $K = -(n-1)/2$.

Solution 1. First of all, we show that we may not take a larger constant K. Let t be a positive number, and take $x_2 = x_3 = \cdots = t$ and $x_1 = -1/(2t)$. Then, every product x_ix_j $(i \neq j)$ is equal to either t^2 or $-1/2$. Hence, for every permutation y_i of the x_i, we have

$$y_1y_2 + \cdots + y_{n-1}y_n \geqslant (n-3)t^2 - 1 \geqslant -1.$$

This justifies that the n-tuple $(x_1, \dots, x_n)$ is Shiny. Now, we have

$$\sum_{i<j} x_i x_j = -\frac{n-1}{2} + \frac{(n-1)(n-2)}{2}t^2.$$

Thus, as t approaches 0 from above, $\sum_{i<j} x_ix_j$ gets arbitrarily close to $-(n-1)/2$. This shows that we may not take K any larger than $-(n-1)/2$. It remains to show that $\sum_{i<j} x_ix_j \geqslant -(n-1)/2$ for any Shiny choice of the x_i.

From now onward, assume that $(x_1, \dots, x_n)$ is a Shiny n-tuple. Let the z_i $(1 \leqslant i \leqslant n)$ be some permutation of the x_i to be chosen later. The indices for z_i will always be taken modulo n. We will first split up the sum $\sum_{i<j} x_ix_j = \sum_{i<j} z_iz_j$ into $\lfloor (n-1)/2 \rfloor$ expressions, each of the form $y_1y_2 + \cdots + y_{n-1}y_n$ for some permutation y_i of the z_i, and some leftover terms. More specifically, write

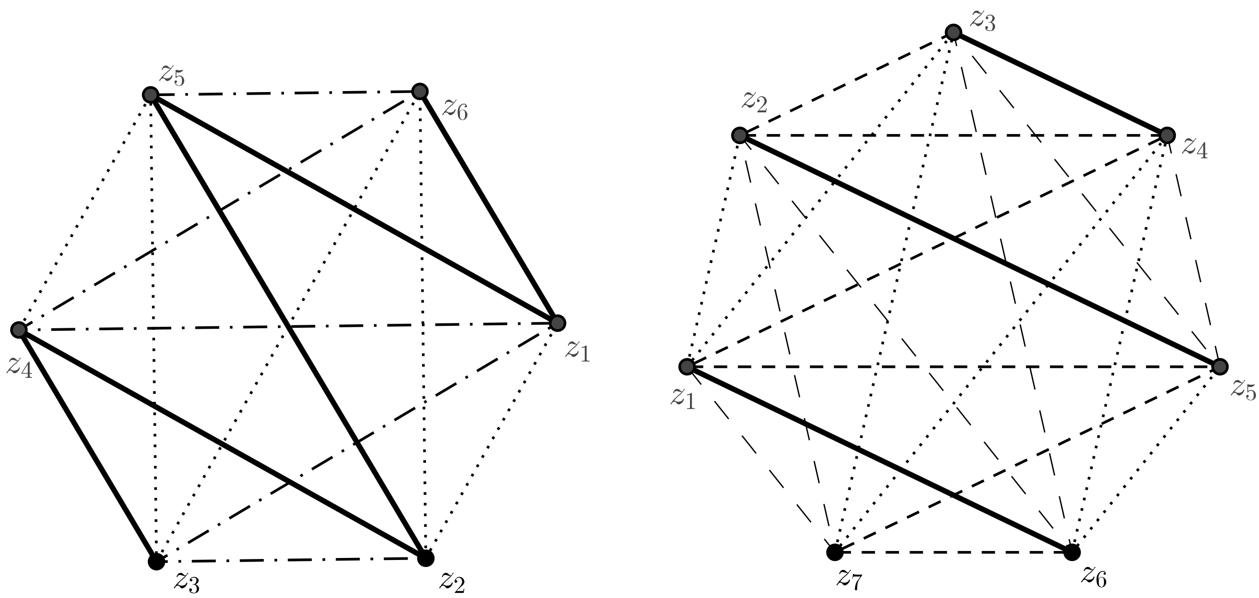
$$\sum_{i<j} z_iz_j = \sum_{q=0}^{n-1} \sum_{\substack{i+j \equiv q \pmod n \\ i \not\equiv j \pmod n}} z_iz_j = \sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\substack{i+j \equiv 2p-1, 2p \pmod n \\ i \not\equiv j \pmod n}} z_iz_j + L, \tag{1}$$

where $L = z_1z_{-1} + z_2z_{-2} + \cdots + z_{(n-1)/2}z_{-(n-1)/2}$ if n is odd, and $L = z_1z_{-1} + z_1z_{-2} + z_2z_{-2} + \cdots + z_{(n-2)/2}z_{-n/2}$ if n is even. We note that for each $p = 1, 2, \dots, \lfloor (n-1)/2 \rfloor$, there is some permutation y_i of the z_i such that

$$\sum_{\substack{i+j \equiv 2p-1, 2p \pmod n \\ i \not\equiv j \pmod n}} z_iz_j = \sum_{k=1}^{n-1} y_ky_{k+1},$$

because we may choose $y_{2i-1} = z_{i+p-1}$ for $1 \leqslant i \leqslant (n+1)/2$ and $y_{2i} = z_{p-i}$ for $1 \leqslant i \leqslant n/2$.

We show (1) graphically for $n = 6, 7$ in the diagrams below. The edges of the graphs each represent a product z_iz_j, and the dashed and dotted series of lines represents the sum of the edges, which is of the form $y_1y_2 + \cdots + y_{n-1}y_n$ for some permutation y_i of the z_i precisely when the series of lines is a Hamiltonian path. The filled edges represent the summands of L.



Now, because the z_i are Shiny, we have that (1) yields the following bound:

$$\sum_{i<j} z_i z_j \geqslant -\left\lfloor \frac{n-1}{2} \right\rfloor + L.$$

It remains to show that, for each n, there exists some permutation z_i of the x_i such that $L \geqslant 0$ when n is odd, and $L \geqslant -1/2$ when n is even. We now split into cases based on the parity of n and provide constructions of the permutations z_i.

Since we have not made any assumptions yet about the x_i, we may now assume without loss of generality that

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k \leqslant 0 \leqslant x_{k+1} \leqslant \cdots \leqslant x_n. \tag{2}$$

Case 1: n is odd.

Without loss of generality, assume that k (from (2)) is even, because we may negate all the x_i if k is odd. We then have $x_1x_2, x_3x_4, \ldots, x_{n-2}x_{n-1} \geqslant 0$ because the factors are of the same sign. Let $L = x_1x_2 + x_3x_4 + \cdots + x_{n-2}x_{n-1} \geqslant 0$. We choose our z_i so that this definition of L agrees with the sum of the leftover terms in (1). Relabel the x_i as z_i such that

$$\{z_1, z_{n-1}\}, \{z_2, z_{n-2}\}, \ldots, \{z_{(n-1)/2}, z_{(n+1)/2}\}$$

are some permutation of

$$\{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_{n-2}, x_{n-1}\},$$

and $z_n = x_n$. Then, we have $L = z_1z_{n-1} + \cdots + z_{(n-1)/2}z_{(n+1)/2}$, as desired.

Case 2: n is even.

Let $L = x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n$. Assume without loss of generality $k \neq 1$. Now, we have

$$2L = (x_1x_2 + \cdots + x_{n-1}x_n) + (x_1x_2 + \cdots + x_{n-1}x_n) \geqslant (x_2x_3 + \cdots + x_{n-1}x_n) + x_kx_{k+1}$$

$$\geqslant x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \geqslant -1,$$

where the first inequality holds because the only negative term in L is x_kx_{k+1}, the second inequality holds because $x_1 \leqslant x_k \leqslant 0 \leqslant x_{k+1} \leqslant x_n$, and the third inequality holds because the x_i are assumed to be Shiny. We thus have that $L \geqslant -1/2$. We now choose a suitable z_i such that the definition of L matches the leftover terms in (1).

Relabel the x_i with z_i in the following manner: $x_{2i-1} = z_{-i}, x_{2i} = z_i$ (again taking indices modulo n). We have that

$$L = \sum_{\substack{i+j \equiv 0,-1 \pmod n \\ i \not\equiv j \pmod n}} z_i z_j,$$

as desired.

Solution 2. We present another proof that $\sum_{i<j} x_i x_j \geqslant -(n-1)/2$ for any Shiny n-tuple $(x_1, \ldots, x_n)$. Assume an ordering of the x_i as in (2), and let $\ell = n - k$. Assume without loss of generality that $k \geqslant \ell$. Also assume $k \neq n$, (as otherwise, all of the x_i are nonpositive, and so the inequality is trivial). Define the sets of indices $S = \{1, 2, \ldots, k\}$ and $T = \{k+1, \ldots, n\}$. Define the following sums:

$$K = \sum_{\substack{i<j \\ i,j \in S}} x_i x_j, \quad M = \sum_{\substack{i \in S \\ j \in T}} x_i x_j, \quad \text{and} \quad L = \sum_{\substack{i<j \\ i,j \in T}} x_i x_j$$

By definition, $K, L \geqslant 0$ and $M \leqslant 0$. We aim to show that $K + L + M \geqslant -(n-1)/2$.

We split into cases based on whether $k = \ell$ or $k > \ell$.

Case 1: $k > \ell$.

Consider all permutations $\phi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $\phi^{-1}(T) = \{2, 4, \ldots, 2\ell\}$. Note that there are $k!\ell!$ such permutations ϕ. Define

$$f(\phi) = \sum_{i=1}^{n-1} x_{\phi(i)} x_{\phi(i+1)}.$$

We know that $f(\phi) \geqslant -1$ for every permutation ϕ with the above property. Averaging $f(\phi)$ over all ϕ gives

$$-1 \leqslant \frac{1}{k!\ell!} \sum_{\phi} f(\phi) = \frac{2\ell}{k\ell} M + \frac{2(k-\ell-1)}{k(k-1)} K,$$

where the equality holds because there are $k\ell$ products in M, of which 2ℓ are selected for each ϕ, and there are $k(k-1)/2$ products in K, of which $k - \ell - 1$ are selected for each ϕ. We now have

$$K + L + M \geqslant K + L + \left(-\frac{k}{2} - \frac{k-\ell-1}{k-1} K\right) = -\frac{k}{2} + \frac{\ell}{k-1} K + L.$$

Since $k \leqslant n - 1$ and $K, L \geqslant 0$, we get the desired inequality.

Case 2: $k = \ell = n/2$.

We do a similar approach, considering all $\phi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $\phi^{-1}(T) = \{2, 4, \ldots, 2\ell\}$, and defining f the same way. Analogously to Case 1, we have

$$-1 \leqslant \frac{1}{k!\ell!} \sum_{\phi} f(\phi) = \frac{2\ell - 1}{k\ell} M,$$

because there are $k\ell$ products in M, of which $2\ell - 1$ are selected for each ϕ. Now, we have that

$$K + L + M \geqslant M \geqslant -\frac{n^2}{4(n-1)} \geqslant -\frac{n-1}{2},$$

where the last inequality holds because $n \geqslant 4$.

A6. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)f(y)) + f(x+y) = f(xy) \qquad (*)$$

for all $x, y \in \mathbb{R}$.

(Albania)

Answer: There are 3 solutions:

$$x \mapsto 0 \qquad \text{or} \qquad x \mapsto x-1 \qquad \text{or} \qquad x \mapsto 1-x \qquad (x \in \mathbb{R}).$$

Solution. An easy check shows that all the 3 above mentioned functions indeed satisfy the original equation $(*)$.

In order to show that these are the only solutions, first observe that if $f(x)$ is a solution then $-f(x)$ is also a solution. Hence, without loss of generality we may (and will) assume that $f(0) \leqslant 0$ from now on. We have to show that either f is identically zero or $f(x) = x-1$ $(\forall x \in \mathbb{R})$.

Observe that, for a fixed $x \neq 1$, we may choose $y \in \mathbb{R}$ so that $x + y = xy \iff y = \frac{x}{x-1}$, and therefore from the original equation $(*)$ we have

$$f\Big(f(x) \cdot f\Big(\frac{x}{x-1}\Big)\Big) = 0 \qquad (x \neq 1). \tag{1}$$

In particular, plugging in $x = 0$ in (1), we conclude that f has at least one zero, namely $(f(0))^2$:

$$f\big((f(0))^2\big) = 0. \tag{2}$$

We analyze two cases (recall that $f(0) \leqslant 0$):

Case 1: $f(0) = 0$.

Setting $y = 0$ in the original equation we get the identically zero solution:

$$f(f(x)f(0)) + f(x) = f(0) \implies f(x) = 0 \text{ for all } x \in \mathbb{R}.$$

From now on, we work on the main

Case 2: $f(0) < 0$.

We begin with the following

Claim 1.

$$f(1) = 0, \qquad f(a) = 0 \implies a = 1, \qquad \text{and} \qquad f(0) = -1. \tag{3}$$

Proof. We need to show that 1 is the *unique* zero of f. First, observe that f has at least one zero a by (2); if $a \neq 1$ then setting $x = a$ in (1) we get $f(0) = 0$, a contradiction. Hence from (2) we get $(f(0))^2 = 1$. Since we are assuming $f(0) < 0$, we conclude that $f(0) = -1$. □

Setting $y = 1$ in the original equation $(*)$ we get

$$f(f(x)f(1)) + f(x+1) = f(x) \iff f(0) + f(x+1) = f(x) \iff f(x+1) = f(x) + 1 \qquad (x \in \mathbb{R}).$$

An easy induction shows that

$$f(x+n) = f(x) + n \qquad (x \in \mathbb{R},\ n \in \mathbb{Z}). \tag{4}$$

Now we make the following

Claim 2. f is injective.

Proof. Suppose that $f(a) = f(b)$ with $a \neq b$. Then by (4), for all $N \in \mathbb{Z}$,

$$f(a + N + 1) = f(b + N) + 1.$$

Choose any integer $N < -b$; then there exist $x_0, y_0 \in \mathbb{R}$ with $x_0 + y_0 = a + N + 1$, $x_0 y_0 = b + N$. Since $a \neq b$, we have $x_0 \neq 1$ and $y_0 \neq 1$. Plugging in x_0 and y_0 in the original equation $(*)$ we get

$$\begin{aligned} f(f(x_0)f(y_0)) + f(a + N + 1) = f(b + N) &\iff f(f(x_0)f(y_0)) + 1 = 0 \\ &\iff f(f(x_0)f(y_0) + 1) = 0 \qquad \text{by (4)} \\ &\iff f(x_0)f(y_0) = 0 \qquad \text{by (3)}. \end{aligned}$$

However, by Claim 1 we have $f(x_0) \neq 0$ and $f(y_0) \neq 0$ since $x_0 \neq 1$ and $y_0 \neq 1$, a contradiction. □

Now the end is near. For any $t \in \mathbb{R}$, plug in $(x, y) = (t, -t)$ in the original equation $(*)$ to get

$$\begin{aligned} f(f(t)f(-t)) + f(0) = f(-t^2) &\iff f(f(t)f(-t)) = f(-t^2) + 1 \qquad \text{by (3)} \\ &\iff f(f(t)f(-t)) = f(-t^2 + 1) \qquad \text{by (4)} \\ &\iff f(t)f(-t) = -t^2 + 1 \qquad \text{by injectivity of } f. \end{aligned}$$

Similarly, plugging in $(x, y) = (t, 1 - t)$ in $(*)$ we get

$$\begin{aligned} f(f(t)f(1-t)) + f(1) = f(t(1-t)) &\iff f(f(t)f(1-t)) = f(t(1-t)) \quad \text{by (3)} \\ &\iff f(t)f(1-t) = t(1-t) \qquad \text{by injectivity of } f. \end{aligned}$$

But since $f(1 - t) = 1 + f(-t)$ by (4), we get

$$\begin{aligned} f(t)f(1-t) = t(1-t) &\iff f(t)(1 + f(-t)) = t(1-t) \iff f(t) + (-t^2 + 1) = t(1-t) \\ &\iff f(t) = t - 1, \end{aligned}$$

as desired.

Comment. Other approaches are possible. For instance, after Claim 1, we may define

$$g(x) \stackrel{\text{def}}{=} f(x) + 1.$$

Replacing $x + 1$ and $y + 1$ in place of x and y in the original equation $(*)$, we get

$$f(f(x+1)f(y+1)) + f(x + y + 2) = f(xy + x + y + 1) \qquad (x, y \in \mathbb{R}),$$

and therefore, using (4) (so that in particular $g(x) = f(x + 1)$), we may rewrite $(*)$ as

$$g(g(x)g(y)) + g(x + y) = g(xy + x + y) \qquad (x, y \in \mathbb{R}). \tag{$**$}$$

We are now to show that $g(x) = x$ for all $x \in \mathbb{R}$ under the assumption (Claim 1) that 0 is the unique zero of g.

Claim 3. Let $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

(a) $g(x + n) = x + n$, and the conditions $g(x) = n$ and $x = n$ are equivalent.

(b) $g(nx) = ng(x)$.

Proof. For part (a), just note that $g(x+n) = x+n$ is just a reformulation of (4). Then $g(x) = n \iff g(x-n) = 0 \iff x - n = 0$ since 0 is the unique zero of g. For part (b), we may assume that $x \neq 0$ since the result is obvious when $x = 0$. Plug in $y = n/x$ in $(**)$ and use part (a) to get

$$g\left(g(x)g\left(\frac{n}{x}\right)\right) + g\left(x + \frac{n}{x}\right) = g\left(n + x + \frac{n}{x}\right) \iff g\left(g(x)g\left(\frac{n}{x}\right)\right) = n \iff g(x)g\left(\frac{n}{x}\right) = n.$$

In other words, for $x \neq 0$ we have

$$g(x) = \frac{n}{g\big(n/x\big)}.$$

In particular, for $n = 1$, we get $g(1/x) = 1/g(x)$, and therefore replacing $x \leftarrow nx$ in the last equation we finally get

$$g(nx) = \frac{n}{g\big(1/x\big)} = ng(x),$$

as required.

Claim 4. The function g is *additive*, i.e., $g(a+b) = g(a) + g(b)$ for all $a, b \in \mathbb{R}$.

Proof. Set $x \leftarrow -x$ and $y \leftarrow -y$ in $(**)$; since g is an odd function (by Claim 3(b) with $n = -1$), we get

$$g(g(x)g(y)) - g(x+y) = -g(-xy + x + y).$$

Subtracting the last relation from $(**)$ we have

$$2g(x+y) = g(xy + x + y) + g(-xy + x + y)$$

and since by Claim 3(b) we have $2g(x+y) = g(2(x+y))$, we may rewrite the last equation as

$$g(\alpha + \beta) = g(\alpha) + g(\beta) \qquad \text{where} \qquad \begin{cases} \alpha = xy + x + y \\ \beta = -xy + x + y. \end{cases}$$

In other words, we have additivity for all $\alpha, \beta \in \mathbb{R}$ for which there are real numbers x and y satisfying

$$x + y = \frac{\alpha + \beta}{2} \qquad \text{and} \qquad xy = \frac{\alpha - \beta}{2},$$

i.e., for all $\alpha, \beta \in \mathbb{R}$ such that $(\frac{\alpha+\beta}{2})^2 - 4 \cdot \frac{\alpha-\beta}{2} \geqslant 0$. Therefore, given any $a, b \in \mathbb{R}$, we may choose $n \in \mathbb{Z}$ large enough so that we have additivity for $\alpha = na$ and $\beta = nb$, i.e.,

$$g(na) + g(nb) = g(na + nb) \iff ng(a) + ng(b) = ng(a+b)$$

by Claim 3(b). Cancelling n, we get the desired result. (Alternatively, setting either $(\alpha, \beta) = (a, b)$ or $(\alpha, \beta) = (-a, -b)$ will ensure that $(\frac{\alpha+\beta}{2})^2 - 4 \cdot \frac{\alpha-\beta}{2} \geqslant 0$). □

Now we may finish the solution. Set $y = 1$ in $(**)$, and use Claim 3 to get

$$g(g(x)g(1)) + g(x+1) = g(2x+1) \iff g(g(x)) + g(x) + 1 = 2g(x) + 1 \iff g(g(x)) = g(x).$$

By additivity, this is equivalent to $g(g(x) - x) = 0$. Since 0 is the unique zero of g by assumption, we finally get $g(x) - x = 0 \iff g(x) = x$ for all $x \in \mathbb{R}$.

A7. Let $a_0, a_1, a_2, \ldots$ be a sequence of integers and $b_0, b_1, b_2, \ldots$ be a sequence of *positive* integers such that $a_0 = 0, a_1 = 1$, and

$$a_{n+1} = \begin{cases} a_n b_n + a_{n-1}, & \text{if } b_{n-1} = 1 \\ a_n b_n - a_{n-1}, & \text{if } b_{n-1} > 1 \end{cases} \qquad \text{for } n = 1, 2, \ldots.$$

Prove that at least one of the two numbers a_{2017} and a_{2018} must be greater than or equal to 2017.

(Australia)

Solution 1. The value of b_0 is irrelevant since $a_0 = 0$, so we may assume that $b_0 = 1$.

Lemma. We have $a_n \geqslant 1$ for all $n \geqslant 1$.

Proof. Let us suppose otherwise in order to obtain a contradiction. Let

$$n \geqslant 1 \text{ be the smallest integer with } a_n \leqslant 0. \tag{1}$$

Note that $n \geqslant 2$. It follows that $a_{n-1} \geqslant 1$ and $a_{n-2} \geqslant 0$. Thus we cannot have $a_n = a_{n-1}b_{n-1} + a_{n-2}$, so we must have $a_n = a_{n-1}b_{n-1} - a_{n-2}$. Since $a_n \leqslant 0$, we have $a_{n-1} \leqslant a_{n-2}$. Thus we have $a_{n-2} \geqslant a_{n-1} \geqslant a_n$.

Let

$$r \text{ be the smallest index with } a_r \geqslant a_{r+1} \geqslant a_{r+2}. \tag{2}$$

Then $r \leqslant n-2$ by the above, but also $r \geqslant 2$: if $b_1 = 1$, then $a_2 = a_1 = 1$ and $a_3 = a_2 b_2 + a_1 > a_2$; if $b_1 > 1$, then $a_2 = b_1 > 1 = a_1$.

By the minimal choice (2) of r, it follows that $a_{r-1} < a_r$. And since $2 \leqslant r \leqslant n-2$, by the minimal choice (1) of n we have $a_{r-1}, a_r, a_{r+1} > 0$. In order to have $a_{r+1} \geqslant a_{r+2}$, we must have $a_{r+2} = a_{r+1}b_{r+1} - a_r$ so that $b_r \geqslant 2$. Putting everything together, we conclude that

$$a_{r+1} = a_r b_r \pm a_{r-1} \geqslant 2a_r - a_{r-1} = a_r + (a_r - a_{r-1}) > a_r,$$

which contradicts (2). $\square$

To complete the problem, we prove that $\max\{a_n, a_{n+1}\} \geqslant n$ by induction. The cases $n = 0, 1$ are given. Assume it is true for all non-negative integers strictly less than n, where $n \geqslant 2$. There are two cases:

Case 1: $b_{n-1} = 1$.

Then $a_{n+1} = a_n b_n + a_{n-1}$. By the inductive assumption one of a_{n-1}, a_n is at least $n-1$ and the other, by the lemma, is at least 1. Hence

$$a_{n+1} = a_n b_n + a_{n-1} \geqslant a_n + a_{n-1} \geqslant (n-1) + 1 = n.$$

Thus $\max\{a_n, a_{n+1}\} \geqslant n$, as desired.

Case 2: $b_{n-1} > 1$.

Since we defined $b_0 = 1$ there is an index r with $1 \leqslant r \leqslant n-1$ such that

$$b_{n-1}, b_{n-2}, \ldots, b_r \geqslant 2 \qquad \text{and} \qquad b_{r-1} = 1.$$

We have $a_{r+1} = a_r b_r + a_{r-1} \geqslant 2a_r + a_{r-1}$. Thus $a_{r+1} - a_r \geqslant a_r + a_{r-1}$.

Now we claim that $a_r + a_{r-1} \geqslant r$. Indeed, this holds by inspection for $r = 1$; for $r \geqslant 2$, one of a_r, a_{r-1} is at least $r-1$ by the inductive assumption, while the other, by the lemma, is at least 1. Hence $a_r + a_{r-1} \geqslant r$, as claimed, and therefore $a_{r+1} - a_r \geqslant r$ by the last inequality in the previous paragraph.

Since $r \geqslant 1$ and, by the lemma, $a_r \geqslant 1$, from $a_{r+1} - a_r \geqslant r$ we get the following two inequalities:

$$a_{r+1} \geqslant r + 1 \qquad \text{and} \qquad a_{r+1} > a_r.$$

Now observe that

$$a_m > a_{m-1} \implies a_{m+1} > a_m \text{ for } m = r+1, r+2, \ldots, n-1,$$

since $a_{m+1} = a_m b_m - a_{m-1} \geqslant 2a_m - a_{m-1} = a_m + (a_m - a_{m-1}) > a_m$. Thus

$$a_n > a_{n-1} > \cdots > a_{r+1} \geqslant r+1 \implies a_n \geqslant n.$$

So $\max\{a_n, a_{n+1}\} \geqslant n$, as desired.

Solution 2. We say that an index $n > 1$ is *bad* if $b_{n-1} = 1$ and $b_{n-2} > 1$; otherwise n is *good.* The value of b_0 is irrelevant to the definition of (a_n) since $a_0 = 0$; so we assume that $b_0 > 1$.

Lemma 1. (a) $a_n \geqslant 1$ for all $n > 0$.

(b) If $n > 1$ is good, then $a_n > a_{n-1}$.

Proof. Induction on n. In the base cases $n = 1, 2$ we have $a_1 = 1 \geqslant 1$, $a_2 = b_1 a_1 \geqslant 1$, and finally $a_2 > a_1$ if 2 is good, since in this case $b_1 > 1$.

Now we assume that the lemma statement is proved for $n = 1, 2, \ldots, k$ with $k \geqslant 2$, and prove it for $n = k+1$. Recall that a_k and a_{k-1} are positive by the induction hypothesis.

Case 1: k is bad.

We have $b_{k-1} = 1$, so $a_{k+1} = b_k a_k + a_{k-1} \geqslant a_k + a_{k-1} > a_k \geqslant 1$, as required.

Case 2: k is good.

We already have $a_k > a_{k-1} \geqslant 1$ by the induction hypothesis. We consider three easy subcases.

Subcase 2.1: $b_k > 1$.

Then $a_{k+1} \geqslant b_k a_k - a_{k-1} \geqslant a_k + (a_k - a_{k-1}) > a_k \geqslant 1$.

Subcase 2.2: $b_k = b_{k-1} = 1$.

Then $a_{k+1} = a_k + a_{k-1} > a_k \geqslant 1$.

Subcase 2.3: $b_k = 1$ but $b_{k-1} > 1$.

Then $k+1$ is bad, and we need to prove only (a), which is trivial: $a_{k+1} = a_k - a_{k-1} \geqslant 1$.

So, in all three subcases we have verified the required relations. □

Lemma 2. Assume that $n > 1$ is bad. Then there exists a $j \in \{1, 2, 3\}$ such that $a_{n+j} \geqslant a_{n-1} + j + 1$, and $a_{n+i} \geqslant a_{n-1} + i$ for all $1 \leqslant i < j$.

Proof. Recall that $b_{n-1} = 1$. Set

$$m = \inf\{i > 0 \colon b_{n+i-1} > 1\}$$

(possibly $m = +\infty$). We claim that $j = \min\{m, 3\}$ works. Again, we distinguish several cases, according to the value of m; in each of them we use Lemma 1 without reference.

Case 1: $m = 1$, so $b_n > 1$.

Then $a_{n+1} \geqslant 2a_n + a_{n-1} \geqslant a_{n-1} + 2$, as required.

Case 2: $m = 2$, so $b_n = 1$ and $b_{n+1} > 1$.

Then we successively get

$$a_{n+1} = a_n + a_{n-1} \geqslant a_{n-1} + 1,$$
$$a_{n+2} \geqslant 2a_{n+1} + a_n \geqslant 2(a_{n-1} + 1) + a_n = a_{n-1} + (a_{n-1} + a_n + 2) \geqslant a_{n-1} + 4,$$

which is even better than we need.

Case 3: $m > 2$, so $b_n = b_{n+1} = 1$.

Then we successively get

$$a_{n+1} = a_n + a_{n-1} \geqslant a_{n-1} + 1, \quad a_{n+2} = a_{n+1} + a_n \geqslant a_{n-1} + 1 + a_n \geqslant a_{n-1} + 2,$$
$$a_{n+3} \geqslant a_{n+2} + a_{n+1} \geqslant (a_{n-1} + 1) + (a_{n-1} + 2) \geqslant a_{n-1} + 4,$$

as required. $\square$

Lemmas 1(b) and 2 provide enough information to prove that $\max\{a_n, a_{n+1}\} \geqslant n$ for all n and, moreover, that $a_n \geqslant n$ often enough. Indeed, assume that we have found some n with $a_{n-1} \geqslant n-1$. If n is good, then by Lemma 1(b) we have $a_n \geqslant n$ as well. If n is bad, then Lemma 2 yields $\max\{a_{n+i}, a_{n+i+1}\} \geqslant a_{n-1} + i + 1 \geqslant n + i$ for all $0 \leqslant i < j$ and $a_{n+j} \geqslant a_{n-1} + j + 1 \geqslant n + j$; so $n + j$ is the next index to start with.

A8. Assume that a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following condition:

For every $x, y \in \mathbb{R}$ such that $\big(f(x)+y\big)\big(f(y)+x\big) > 0$, we have $f(x)+y = f(y)+x$.

Prove that $f(x) + y \leqslant f(y) + x$ whenever $x > y$.

(Netherlands)

Solution 1. Define $g(x) = x - f(x)$. The condition on f then rewrites as follows:

For every $x, y \in \mathbb{R}$ such that $\big((x+y) - g(x)\big)\big((x+y) - g(y)\big) > 0$, we have $g(x) = g(y)$.

This condition may in turn be rewritten in the following form:

If $g(x) \neq g(y)$, then the number $x+y$ lies (non-strictly) between $g(x)$ and $g(y)$. $\qquad (*)$

Notice here that the function $g_1(x) = -g(-x)$ also satisfies $(*)$, since

$$\begin{aligned} g_1(x) \neq g_1(y) \quad \Longrightarrow \quad g(-x) \neq g(-y) \quad &\Longrightarrow \quad -(x+y) \text{ lies between } g(-x) \text{ and } g(-y) \\ &\Longrightarrow \quad x+y \text{ lies between } g_1(x) \text{ and } g_1(y). \end{aligned}$$

On the other hand, the relation we need to prove reads now as

$$g(x) \leqslant g(y) \qquad \text{whenever } x < y. \tag{1}$$

Again, this condition is equivalent to the same one with g replaced by g_1.

If $g(x) = 2x$ for all $x \in \mathbb{R}$, then $(*)$ is obvious; so in what follows we consider the other case. We split the solution into a sequence of lemmas, strengthening one another. We always consider some value of x with $g(x) \neq 2x$ and denote $X = g(x)$.

Lemma 1. Assume that $X < 2x$. Then on the interval $(X - x; x]$ the function g attains at most two values — namely, X and, possibly, some $Y > X$. Similarly, if $X > 2x$, then g attains at most two values on $[x; X - x)$ — namely, X and, possibly, some $Y < X$.

Proof. We start with the first claim of the lemma. Notice that $X - x < x$, so the considered interval is nonempty.

Take any $a \in (X - x; x)$ with $g(a) \neq X$ (if it exists). If $g(a) < X$, then $(*)$ yields $g(a) \leqslant a + x \leqslant g(x) = X$, so $a \leqslant X - x$ which is impossible. Thus, $g(a) > X$ and hence by $(*)$ we get $X \leqslant a + x \leqslant g(a)$.

Now, for any $b \in (X - x; x)$ with $g(b) \neq X$ we similarly get $b + x \leqslant g(b)$. Therefore, the number $a + b$ (which is smaller than each of $a + x$ and $b + x$) cannot lie between $g(a)$ and $g(b)$, which by $(*)$ implies that $g(a) = g(b)$. Hence g may attain only two values on $(X - x; x]$, namely X and $g(a) > X$.

To prove the second claim, notice that $g_1(-x) = -X < 2 \cdot (-x)$, so g_1 attains at most two values on $(-X + x, -x]$, i.e., $-X$ and, possibly, some $-Y > -X$. Passing back to g, we get what we need. $\square$

Lemma 2. If $X < 2x$, then g is constant on $(X - x; x)$. Similarly, if $X > 2x$, then g is constant on $(x; X - x)$.

Proof. Again, it suffices to prove the first claim only. Assume, for the sake of contradiction, that there exist $a, b \in (X - x; x)$ with $g(a) \neq g(b)$; by Lemma 1, we may assume that $g(a) = X$ and $Y = g(b) > X$.

Notice that $\min\{X - a, X - b\} > X - x$, so there exists a $u \in (X - x; x)$ such that $u < \min\{X - a, X - b\}$. By Lemma 1, we have either $g(u) = X$ or $g(u) = Y$. In the former case, by $(*)$ we have $X \leqslant u + b \leqslant Y$ which contradicts $u < X - b$. In the second case, by $(*)$ we have $X \leqslant u + a \leqslant Y$ which contradicts $u < X - a$. Thus the lemma is proved. $\square$

Lemma 3. If $X < 2x$, then $g(a) = X$ for all $a \in (X-x; x)$. Similarly, if $X > 2x$, then $g(a) = X$ for all $a \in (x; X - x)$.

Proof. Again, we only prove the first claim.

By Lemmas 1 and 2, this claim may be violated only if g takes on a constant value $Y > X$ on $(X - x, x)$. Choose any $a, b \in (X - x; x)$ with $a < b$. By $(*)$, we have

$$Y \geqslant b + x \geqslant X. \tag{2}$$

In particular, we have $Y \geqslant b + x > 2a$. Applying Lemma 2 to a in place of x, we obtain that g is constant on $(a, Y - a)$. By (2) again, we have $x \leqslant Y - b < Y - a$; so $x, b \in (a; Y - a)$. But $X = g(x) \neq g(b) = Y$, which is a contradiction. □

Now we are able to finish the solution. Assume that $g(x) > g(y)$ for some $x < y$. Denote $X = g(x)$ and $Y = g(y)$; by $(*)$, we have $X \geqslant x + y \geqslant Y$, so $Y - y \leqslant x < y \leqslant X - x$, and hence $(Y - y; y) \cap (x; X - x) = (x, y) \neq \varnothing$. On the other hand, since $Y - y < y$ and $x < X-x$, Lemma 3 shows that g should attain a constant value X on $(x; X-x)$ and a constant value $Y \neq X$ on $(Y - y; y)$. Since these intervals overlap, we get the final contradiction.

Solution 2. As in the previous solution, we pass to the function g satisfying $(*)$ and notice that we need to prove the condition (1). We will also make use of the function g_1.

If g is constant, then (1) is clearly satisfied. So, in the sequel we assume that g takes on at least two different values. Now we collect some information about the function g.

Claim 1. For any $c \in \mathbb{R}$, all the solutions of $g(x) = c$ are bounded.

Proof. Fix any $y \in \mathbb{R}$ with $g(y) \neq c$. Assume first that $g(y) > c$. Now, for any x with $g(x) = c$, by $(*)$ we have $c \leqslant x + y \leqslant g(y)$, or $c - y \leqslant x \leqslant g(y) - y$. Since c and y are constant, we get what we need.

If $g(y) < c$, we may switch to the function g_1 for which we have $g_1(-y) > -c$. By the above arguments, we obtain that all the solutions of $g_1(-x) = -c$ are bounded, which is equivalent to what we need. □

As an immediate consequence, the function g takes on infinitely many values, which shows that the next claim is indeed widely applicable.

Claim 2. If $g(x) < g(y) < g(z)$, then $x < z$.

Proof. By $(*)$, we have $g(x) \leqslant x + y \leqslant g(y) \leqslant z + y \leqslant g(z)$, so $x + y \leqslant z + y$, as required. □

Claim 3. Assume that $g(x) > g(y)$ for some $x < y$. Then $g(a) \in \{g(x), g(y)\}$ for all $a \in [x; y]$.

Proof. If $g(y) < g(a) < g(x)$, then the triple (y, a, x) violates Claim 2. If $g(a) < g(y) < g(x)$, then the triple (a, y, x) violates Claim 2. If $g(y) < g(x) < g(a)$, then the triple (y, x, a) violates Claim 2. The only possible cases left are $g(a) \in \{g(x), g(y)\}$. □

In view of Claim 3, we say that an interval I (which may be open, closed, or semi-open) is a *Dirichlet interval** if the function g takes on just two values on I.

Assume now, for the sake of contradiction, that (1) is violated by some $x < y$. By Claim 3, $[x; y]$ is a Dirichlet interval. Set

$$r = \inf\{a \colon (a; y] \text{ is a Dirichlet interval}\} \quad \text{and} \quad s = \sup\{b \colon [x; b) \text{ is a Dirichlet interval}\}.$$

Clearly, $r \leqslant x < y \leqslant s$. By Claim 1, r and s are finite. Denote $X = g(x)$, $Y = g(y)$, and $\Delta = (y - x)/2$.

Suppose first that there exists a $t \in (r; r + \Delta)$ with $f(t) = Y$. By the definition of r, the interval $(r - \Delta; y]$ is not Dirichlet, so there exists an $r' \in (r - \Delta; r]$ such that $g(r') \notin \{X, Y\}$.

*The name *Dirichlet interval* is chosen for the reason that g theoretically might act similarly to the Dirichlet function on this interval.

The function g attains at least three distinct values on $[r'; y]$, namely $g(r')$, $g(x)$, and $g(y)$. Claim 3 now yields $g(r') \leqslant g(y)$; the equality is impossible by the choice of r', so in fact $g(r') < Y$. Applying $(*)$ to the pairs (r', y) and (t, x) we obtain $r' + y \leqslant Y \leqslant t + x$, whence $r - \Delta + y < r' + y \leqslant t + x < r + \Delta + x$, or $y - x < 2\Delta$. This is a contradiction.

Thus, $g(t) = X$ for all $t \in (r; r + \Delta)$. Applying the same argument to g_1, we get $g(t) = Y$ for all $t \in (s - \Delta; s)$.

Finally, choose some $s_1, s_2 \in (s - \Delta; s)$ with $s_1 < s_2$ and denote $\delta = (s_2 - s_1)/2$. As before, we choose $r' \in (r - \delta; r)$ with $g(r') \notin \{X, Y\}$ and obtain $g(r') < Y$. Choose any $t \in (r; r + \delta)$; by the above arguments, we have $g(t) = X$ and $g(s_1) = g(s_2) = Y$. As before, we apply $(*)$ to the pairs (r', s_2) and (t, s_1) obtaining $r - \delta + s_2 < r' + s_2 \leqslant Y \leqslant t + s_1 < r + \delta + s_1$, or $s_2 - s_1 < 2\delta$. This is a final contradiction.

Comment 1. The original submission discussed the same functions f, but the question was different — namely, the following one:

Prove that the equation $f(x) = 2017x$ has at most one solution, and the equation $f(x) = -2017x$ has at least one solution.

The Problem Selection Committee decided that the question we are proposing is more natural, since it provides more natural information about the function g (which is indeed the main character in this story). On the other hand, the new problem statement is strong enough in order to imply the original one easily.

Namely, we will deduce from the new problem statement (along with the facts used in the solutions) that (i) for every $N > 0$ the equation $g(x) = -Nx$ has at most one solution, and (ii) for every $N > 1$ the equation $g(x) = Nx$ has at least one solution.

Claim (i) is now trivial. Indeed, g is proven to be non-decreasing, so $g(x) + Nx$ is strictly increasing and thus has at most one zero.

We proceed on claim (ii). If $g(0) = 0$, then the required root has been already found. Otherwise, we may assume that $g(0) > 0$ and denote $c = g(0)$. We intend to prove that $x = c/N$ is the required root. Indeed, by monotonicity we have $g(c/N) \geqslant g(0) = c$; if we had $g(c/N) > c$, then $(*)$ would yield $c \leqslant 0 + c/N \leqslant g(c/N)$ which is false. Thus, $g(x) = c = Nx$.

Comment 2. There are plenty of functions g satisfying $(*)$ (and hence of functions f satisfying the problem conditions). One simple example is $g_0(x) = 2x$. Next, for any increasing sequence $A = (\dots, a_{-1}, a_0, a_1, \dots)$ which is unbounded in both directions (i.e., for every N this sequence contains terms greater than N, as well as terms smaller than $-N$), the function g_A defined by

$$g_A(x) = a_i + a_{i+1} \qquad \text{whenever } x \in [a_i; a_{i+1})$$

satisfies $(*)$. Indeed, pick any $x < y$ with $g(x) \neq g(y)$; this means that $x \in [a_i; a_{i+1})$ and $y \in [a_j; a_{j+1})$ for some $i < j$. Then we have $g(x) = a_i + a_{i+1} \leqslant x + y < a_j + a_{j+1} = g(y)$, as required.

There also exist examples of the mixed behavior; e.g., for an arbitrary sequence A as above and an arbitrary subset $I \subseteq \mathbb{Z}$ the function

$$g_{A,I}(x) = \begin{cases} g_0(x), & x \in [a_i; a_{i+1}) \quad \text{with } i \in I; \\ g_A(x), & x \in [a_i; a_{i+1}) \quad \text{with } i \notin I \end{cases}$$

also satisfies $(*)$.

Finally, it is even possible to provide a complete description of all functions g satisfying $(*)$ (and hence of all functions f satisfying the problem conditions); however, it seems to be far out of scope for the IMO. This description looks as follows.

Let A be any closed subset of $\mathbb{R}$ which is unbounded in both directions. Define the functions i_A, s_A, and g_A as follows:

$$i_A(x) = \inf\{a \in A \colon a \geqslant x\}, \quad s_A(x) = \sup\{a \in A \colon a \leqslant x\}, \quad g_A(x) = i_A(x) + s_A(x).$$

It is easy to see that for different sets A and B the functions g_A and g_B are also different (since, e.g., for any $a \in A \setminus B$ the function g_B is constant in a small neighborhood of a, but the function g_A is not). One may check, similarly to the arguments above, that each such function satisfies $(*)$.

Finally, one more modification is possible. Namely, for any $x \in A$ one may redefine $g_A(x)$ (which is $2x$) to be any of the numbers

$$g_{A+}(x) = i_{A+}(x) + x \quad \text{or} \quad g_{A-}(x) = x + s_{A-}(x),$$
$$\text{where} \qquad i_{A+}(x) = \inf\{a \in A\colon a > x\} \quad \text{and} \quad s_{A-}(x) = \sup\{a \in A\colon a < x\}.$$

This really changes the value if x has some right (respectively, left) semi-neighborhood disjoint from A, so there are at most countably many possible changes; all of them can be performed independently.

With some effort, one may show that the construction above provides all functions g satisfying $(*)$.

Combinatorics

C1. A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

(Singapore)

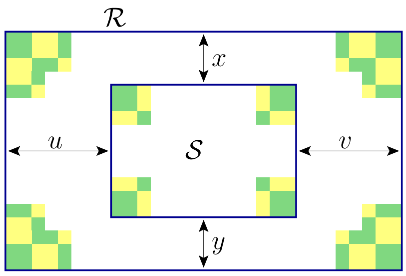
Solution. Let the width and height of $\mathcal{R}$ be odd numbers a and b. Divide $\mathcal{R}$ into ab unit squares and color them green and yellow in a checkered pattern. Since the side lengths of a and b are odd, the corner squares of $\mathcal{R}$ will all have the same color, say green.

Call a rectangle (either $\mathcal{R}$ or a small rectangle) *green* if its corners are all green; call it *yellow* if the corners are all yellow, and call it *mixed* if it has both green and yellow corners. In particular, $\mathcal{R}$ is a green rectangle.

We will use the following trivial observations.

- Every mixed rectangle contains the same number of green and yellow squares;
- Every green rectangle contains one more green square than yellow square;
- Every yellow rectangle contains one more yellow square than green square.

The rectangle $\mathcal{R}$ is green, so it contains more green unit squares than yellow unit squares. Therefore, among the small rectangles, at least one is green. Let $\mathcal{S}$ be such a small green rectangle, and let its distances from the sides of $\mathcal{R}$ be x, y, u and v, as shown in the picture. The top-left corner of $\mathcal{R}$ and the top-left corner of $\mathcal{S}$ have the same color, which happen if and only if x and u have the same parity. Similarly, the other three green corners of $\mathcal{S}$ indicate that x and v have the same parity, y and u have the same parity, i.e. x, y, u and v are all odd or all even.



C2. Let n be a positive integer. Define a *chameleon* to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters a, b, and c. Define a *swap* to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X, there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

(Australia)

Solution 1. To start, notice that the swap of two identical letters does not change a chameleon, so we may assume there are no such swaps.

For any two chameleons X and Y, define their *distance* $d(X, Y)$ to be the minimal number of swaps needed to transform X into Y (or vice versa). Clearly, $d(X, Y) + d(Y, Z) \geqslant d(X, Z)$ for any three chameleons X, Y, and Z.

Lemma. Consider two chameleons

$$P = \underbrace{aa\ldots a}_{n}\underbrace{bb\ldots b}_{n}\underbrace{cc\ldots c}_{n} \quad \text{and} \quad Q = \underbrace{cc\ldots c}_{n}\underbrace{bb\ldots b}_{n}\underbrace{aa\ldots a}_{n}.$$

Then $d(P, Q) \geqslant 3n^2$.

Proof. For any chameleon X and any pair of distinct letters $u, v \in \{a, b, c\}$, we define $f_{u,v}(X)$ to be the number of pairs of positions in X such that the left one is occupied by u, and the right one is occupied by v. Define $f(X) = f_{a,b}(X) + f_{a,c}(X) + f_{b,c}(X)$. Notice that $f_{a,b}(P) = f_{a,c}(P) = f_{b,c}(P) = n^2$ and $f_{a,b}(Q) = f_{a,c}(Q) = f_{b,c}(Q) = 0$, so $f(P) = 3n^2$ and $f(Q) = 0$.

Now consider some swap changing a chameleon X to X'; say, the letters a and b are swapped. Then $f_{a,b}(X)$ and $f_{a,b}(X')$ differ by exactly 1, while $f_{a,c}(X) = f_{a,c}(X')$ and $f_{b,c}(X) = f_{b,c}(X')$. This yields $|f(X) - f(X')| = 1$, i.e., on any swap the value of f changes by 1. Hence $d(X, Y) \geqslant |f(X) - f(Y)|$ for any two chameleons X and Y. In particular, $d(P, Q) \geqslant |f(P) - f(Q)| = 3n^2$, as desired. □

Back to the problem, take any chameleon X and notice that $d(X, P) + d(X, Q) \geqslant d(P, Q) \geqslant 3n^2$ by the lemma. Consequently, $\max\{d(X, P), d(X, Q)\} \geqslant \frac{3n^2}{2}$, which establishes the problem statement.

Comment 1. The problem may be reformulated in a graph language. Construct a graph G with the chameleons as vertices, two vertices being connected with an edge if and only if these chameleons differ by a single swap. Then $d(X, Y)$ is the usual distance between the vertices X and Y in this graph. Recall that the *radius* of a connected graph G is defined as

$$r(G) = \min_{v \in V} \max_{u \in V} d(u, v).$$

So we need to prove that the radius of the constructed graph is at least $3n^2/2$.

It is well-known that the radius of any connected graph is at least the half of its diameter (which is simply $\max_{u,v \in V} d(u, v)$). Exactly this fact has been used above in order to finish the solution.

Solution 2. We use the notion of distance from Solution 1, but provide a different lower bound for it.

In any chameleon X, we enumerate the positions in it from left to right by $1, 2, \ldots, 3n$. Define $s_c(X)$ as the sum of positions occupied by c. The value of s_c changes by at most 1 on each swap, but this fact alone does not suffice to solve the problem; so we need an improvement.

For every chameleon X, denote by $X_{\overline{c}}$ the sequence obtained from X by removing all n letters c. Enumerate the positions in $X_{\overline{c}}$ from left to right by $1, 2, \ldots, 2n$, and define $s_{\overline{c},b}(X)$ as the sum of positions in $X_{\overline{c}}$ occupied by b. (In other words, here we consider the positions of the b's relatively to the a's only.) Finally, denote

$$d'(X, Y) := |s_c(X) - s_c(Y)| + |s_{\overline{c},b}(X) - s_{\overline{c},b}(Y)|.$$

Now consider any swap changing a chameleon X to X'. If no letter c is involved into this swap, then $s_c(X) = s_c(X')$; on the other hand, exactly one letter b changes its position in $X_{\bar{c}}$, so $|s_{\bar{c},b}(X) - s_{\bar{c},b}(X')| = 1$. If a letter c is involved into a swap, then $X_{\bar{c}} = X'_{\bar{c}}$, so $s_{\bar{c},b}(X) = s_{\bar{c},b}(X')$ and $|s_c(X) - s_c(X')| = 1$. Thus, in all cases we have $d'(X, X') = 1$.

As in the previous solution, this means that $d(X, Y) \geqslant d'(X, Y)$ for any two chameleons X and Y. Now, for any chameleon X we will indicate a chameleon Y with $d'(X, Y) \geqslant 3n^2/2$, thus finishing the solution.

The function s_c attains all integer values from $1 + \cdots + n = \frac{n(n+1)}{2}$ to $(2n+1) + \cdots + 3n = 2n^2 + \frac{n(n+1)}{2}$. If $s_c(X) \leqslant n^2 + \frac{n(n+1)}{2}$, then we put the letter c into the last n positions in Y; otherwise we put the letter c into the first n positions in Y. In either case we already have $|s_c(X) - s_c(Y)| \geqslant n^2$.

Similarly, $s_{\bar{c},b}$ ranges from $\frac{n(n+1)}{2}$ to $n^2 + \frac{n(n+1)}{2}$. So, if $s_{\bar{c},b}(X) \leqslant \frac{n^2}{2} + \frac{n(n+1)}{2}$, then we put the letter b into the last n positions in Y which are still free; otherwise, we put the letter b into the first n such positions. The remaining positions are occupied by a. In any case, we have $|s_{\bar{c},b}(X) - s_{\bar{c},b}(Y)| \geqslant \frac{n^2}{2}$, thus $d'(X, Y) \geqslant n^2 + \frac{n^2}{2} = \frac{3n^2}{2}$, as desired.

Comment 2. The two solutions above used two lower bounds $|f(X) - f(Y)|$ and $d'(X, Y)$ for the number $d(X, Y)$. One may see that these bounds are closely related to each other, as

$$f_{a,c}(X) + f_{b,c}(X) = s_c(X) - \frac{n(n+1)}{2} \quad \text{and} \quad f_{a,b}(X) = s_{\bar{c},b}(X) - \frac{n(n+1)}{2}.$$

One can see that, e.g., the bound $d'(X, Y)$ could as well be used in the proof of the lemma in Solution 1.

Let us describe here an even sharper bound which also can be used in different versions of the solutions above.

In each chameleon X, enumerate the occurrences of a from the left to the right as $a_1, a_2, \ldots, a_n$. Since we got rid of swaps of identical letters, the relative order of these letters remains the same during the swaps. Perform the same operation with the other letters, obtaining new letters $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$. Denote by A the set of the $3n$ obtained letters.

Since all $3n$ letters became different, for any chameleon X and any $s \in A$ we may define the position $N_s(X)$ of s in X (thus $1 \leqslant N_s(X) \leqslant 3n$). Now, for any two chameleons X and Y we say that a pair of letters $(s, t) \in A \times A$ is an (X, Y)-*inversion* if $N_s(X) < N_t(X)$ but $N_s(Y) > N_t(Y)$, and define $d^*(X, Y)$ to be the number of (X, Y)-inversions. Then for any two chameleons Y and Y' differing by a single swap, we have $|d^*(X, Y) - d^*(X, Y')| = 1$. Since $d^*(X, X) = 0$, this yields $d(X, Y) \geqslant d^*(X, Y)$ for any pair of chameleons X and Y. The bound d^* may also be used in both Solution 1 and Solution 2.

Comment 3. In fact, one may prove that the distance d^* defined in the previous comment *coincides* with d. Indeed, if $X \neq Y$, then there exist an (X, Y)-inversion (s, t). One can show that such s and t may be chosen to occupy consecutive positions in Y. Clearly, s and t correspond to different letters among $\{a, b, c\}$. So, swapping them in Y we get another chameleon Y' with $d^*(X, Y') = d^*(X, Y) - 1$. Proceeding in this manner, we may change Y to X in $d^*(X, Y)$ steps.

Using this fact, one can show that the estimate in the problem statement is sharp for all $n \geqslant 2$. (For $n = 1$ it is not sharp, since any permutation of three letters can be changed to an opposite one in no less than three swaps.) We outline the proof below.

For any $k \geqslant 0$, define

$$X_{2k} = \underbrace{abc\,abc\ldots abc}_{3k \text{ letters}}\underbrace{cba\,cba\ldots cba}_{3k \text{ letters}} \quad \text{and} \quad X_{2k+3} = \underbrace{abc\,abc\ldots abc}_{3k \text{ letters}}\,abc\,bca\,cab\,\underbrace{cba\,cba\ldots cba}_{3k \text{ letters}}.$$

We claim that for every $n \geqslant 2$ and every chameleon Y, we have $d^*(X_n, Y) \leqslant \lceil 3n^2/2 \rceil$. This will mean that for every $n \geqslant 2$ the number $3n^2/2$ in the problem statement cannot be changed by any number larger than $\lceil 3n^2/2 \rceil$.

For any distinct letters $u, v \in \{a, b, c\}$ and any two chameleons X and Y, we define $d^*_{u,v}(X, Y)$ as the number of (X, Y)-inversions (s, t) such that s and t are instances of u and v (in any of the two possible orders). Then $d^*(X, Y) = d^*_{a,b}(X, Y) + d^*_{b,c}(X, Y) + d^*_{c,a}(X, Y)$.

We start with the case when $n = 2k$ is even; denote $X = X_{2k}$. We show that $d^*_{a,b}(X, Y) \leqslant 2k^2$ for any chameleon Y; this yields the required estimate. Proceed by the induction on k with the trivial base case $k = 0$. To perform the induction step, notice that $d^*_{a,b}(X, Y)$ is indeed the minimal number of swaps needed to change $Y_{\overline{c}}$ into $X_{\overline{c}}$. One may show that moving a_1 and a_{2k} in Y onto the first and the last positions in Y, respectively, takes at most $2k$ swaps, and that subsequent moving b_1 and b_{2k} onto the second and the second last positions takes at most $2k - 2$ swaps. After performing that, one may delete these letters from both $X_{\overline{c}}$ and $Y_{\overline{c}}$ and apply the induction hypothesis; so $X_{\overline{c}}$ can be obtained from $Y_{\overline{c}}$ using at most $2(k-1)^2 + 2k + (2k-2) = 2k^2$ swaps, as required.

If $n = 2k + 3$ is odd, the proof is similar but more technically involved. Namely, we claim that $d^*_{a,b}(X_{2k+3}, Y) \leqslant 2k^2 + 6k + 5$ for any chameleon Y, and that the equality is achieved only if $Y_{\overline{c}} = bb \dots b\,aa \dots a$. The proof proceeds by a similar induction, with some care taken of the base case, as well as of extracting the equality case. Similar estimates hold for $d^*_{b,c}$ and $d^*_{c,a}$. Summing three such estimates, we obtain

$$d^*(X_{2k+3}, Y) \leqslant 3(2k^2 + 6k + 5) = \left\lceil \frac{3n^2}{2} \right\rceil + 1,$$

which is by 1 more than we need. But the equality could be achieved only if $Y_{\overline{c}} = bb \dots b\,aa \dots a$ and, similarly, $Y_{\overline{b}} = aa \dots a\,cc \dots c$ and $Y_{\overline{a}} = cc \dots c\,bb \dots b$. Since these three equalities cannot hold simultaneously, the proof is finished.

C3. Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:

(1) Choose any number of the form 2^j, where j is a non-negative integer, and put it into an empty cell.

(2) Choose two (not necessarily adjacent) cells with the same number in them; denote that number by 2^j. Replace the number in one of the cells with 2^{j+1} and erase the number in the other cell.

At the end of the game, one cell contains the number 2^n, where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

(Thailand)

Answer: $2\sum_{j=0}^{8}\binom{n}{j}-1$.

Solution 1. We will solve a more general problem, replacing the row of 9 cells with a row of k cells, where k is a positive integer. Denote by $m(n,k)$ the maximum possible number of moves Sir Alex can make starting with a row of k empty cells, and ending with one cell containing the number 2^n and all the other $k-1$ cells empty. Call an operation of type (1) an *insertion*, and an operation of type (2) a *merge*.

Only one move is possible when $k=1$, so we have $m(n,1)=1$. From now on we consider $k\geqslant 2$, and we may assume Sir Alex's last move was a merge. Then, just before the last move, there were exactly two cells with the number 2^{n-1}, and the other $k-2$ cells were empty.

Paint one of those numbers 2^{n-1} blue, and the other one red. Now trace back Sir Alex's moves, always painting the numbers blue or red following this rule: *if a and b merge into c, paint a and b with the same color as c.* Notice that in this backward process new numbers are produced only by reversing merges, since reversing an insertion simply means deleting one of the numbers. Therefore, all numbers appearing in the whole process will receive one of the two colors.

Sir Alex's first move is an insertion. Without loss of generality, assume this first number inserted is blue. Then, from this point on, until the last move, there is always at least one cell with a blue number.

Besides the last move, there is no move involving a blue and a red number, since all merges involves numbers with the same color, and insertions involve only one number. Call an insertion of a blue number or merge of two blue numbers a *blue move*, and define a *red move* analogously.

The whole sequence of blue moves could be repeated on another row of k cells to produce one cell with the number 2^{n-1} and all the others empty, so there are at most $m(n-1,k)$ blue moves.

Now we look at the red moves. Since every time we perform a red move there is at least one cell occupied with a blue number, the whole sequence of red moves could be repeated on a row of $k-1$ cells to produce one cell with the number 2^{n-1} and all the others empty, so there are at most $m(n-1,k-1)$ red moves. This proves that

$$m(n,k)\leqslant m(n-1,k)+m(n-1,k-1)+1.$$

On the other hand, we can start with an empty row of k cells and perform $m(n-1,k)$ moves to produce one cell with the number 2^{n-1} and all the others empty, and after that perform $m(n-1,k-1)$ moves on those $k-1$ empty cells to produce the number 2^{n-1} in one of them, leaving $k-2$ empty. With one more merge we get one cell with 2^n and the others empty, proving that

$$m(n,k)\geqslant m(n-1,k)+m(n-1,k-1)+1.$$

It follows that

$$m(n,k) = m(n-1,k) + m(n-1,k-1) + 1, \tag{1}$$

for $n \geqslant 1$ and $k \geqslant 2$.

If $k = 1$ or $n = 0$, we must insert 2^n on our first move and immediately get the final configuration, so $m(0,k) = 1$ and $m(n,1) = 1$, for $n \geqslant 0$ and $k \geqslant 1$. These initial values, together with the recurrence relation (1), determine $m(n,k)$ uniquely.

Finally, we show that

$$m(n,k) = 2\sum_{j=0}^{k-1}\binom{n}{j} - 1, \tag{2}$$

for all integers $n \geqslant 0$ and $k \geqslant 1$.

We use induction on n. Since $m(0,k) = 1$ for $k \geqslant 1$, (2) is true for the base case. We make the induction hypothesis that (2) is true for some fixed positive integer n and all $k \geqslant 1$. We have $m(n+1,1) = 1 = 2\binom{n+1}{0} - 1$, and for $k \geqslant 2$ the recurrence relation (1) and the induction hypothesis give us

$$m(n+1,k) = m(n,k) + m(n,k-1) + 1 = 2\sum_{j=0}^{k-1}\binom{n}{j} - 1 + 2\sum_{j=0}^{k-2}\binom{n}{j} - 1 + 1$$
$$= 2\sum_{j=0}^{k-1}\binom{n}{j} + 2\sum_{j=0}^{k-1}\binom{n}{j-1} - 1 = 2\sum_{j=0}^{k-1}\left(\binom{n}{j} + \binom{n}{j-1}\right) - 1 = 2\sum_{j=0}^{k-1}\binom{n+1}{j} - 1,$$

which completes the proof.

Comment 1. After deducing the recurrence relation (1), it may be convenient to homogenize the recurrence relation by defining $h(n,k) = m(n,k) + 1$. We get the new relation

$$h(n,k) = h(n-1,k) + h(n-1,k), \tag{3}$$

for $n \geqslant 1$ and $k \geqslant 2$, with initial values $h(0,k) = h(n,1) = 2$, for $n \geqslant 0$ and $k \geqslant 1$.

This may help one to guess the answer, and also with other approaches like the one we develop next.

Comment 2. We can use a generating function to find the answer without guessing. We work with the homogenized recurrence relation (3). Define $h(n,0) = 0$ so that (3) is valid for $k = 1$ as well. Now we set up the generating function $f(x,y) = \sum_{n,k\geqslant 0} h(n,k)x^n y^k$. Multiplying the recurrence relation (3) by $x^n y^k$ and summing over $n,k \geqslant 1$, we get

$$\sum_{n,k\geqslant 1} h(n,k)x^n y^k = x\sum_{n,k\geqslant 1} h(n-1,k)x^{n-1}y^k + xy\sum_{n,k\geqslant 1} h(n-1,k-1)x^{n-1}y^{k-1}.$$

Completing the missing terms leads to the following equation on $f(x,y)$:

$$f(x,y) - \sum_{n\geqslant 0} h(n,0)x^n - \sum_{k\geqslant 1} h(0,k)y^k = xf(x,y) - x\sum_{n\geqslant 0} h(n,0)x^n + xyf(x,y).$$

Substituting the initial values, we obtain

$$f(x,y) = \frac{2y}{1-y}\cdot\frac{1}{1-x(1+y)}.$$

Developing as a power series, we get

$$f(x,y) = 2\sum_{j\geqslant 1} y^j \cdot \sum_{n\geqslant 0}(1+y)^n x^n.$$

The coefficient of x^n in this power series is

$$2\sum_{j\geqslant 1} y^j \cdot (1+y)^n = 2\sum_{j\geqslant 1} y^j \cdot \sum_{i\geqslant 0}\binom{n}{i} y^i,$$

and extracting the coefficient of y^k in this last expression we finally obtain the value for $h(n,k)$,

$$h(n,k) = 2\sum_{j=0}^{k-1}\binom{n}{j}.$$

This proves that

$$m(n,k) = 2\sum_{j=0}^{k-1}\binom{n}{j} - 1.$$

The generating function approach also works if applied to the non-homogeneous recurrence relation (1), but the computations are less straightforward.

Solution 2. Define merges and insertions as in Solution 1. After each move made by Sir Alex we compute the number N of empty cells, and the sum S of all the numbers written in the cells. Insertions always increase S by some power of 2, and increase N exactly by 1. Merges do not change S and decrease N exactly by 1. Since the initial value of N is 0 and its final value is 1, the total number of insertions exceeds that of merges by exactly one. So, to maximize the number of moves, we need to maximize the number of insertions.

We will need the following lemma.

Lemma. If the binary representation of a positive integer A has d nonzero digits, then A cannot be represented as a sum of fewer than d powers of 2. Moreover, any representation of A as a sum of d powers of 2 must coincide with its binary representation.

Proof. Let s be the minimum number of summands in all possible representations of A as sum of powers of 2. Suppose there is such a representation with s summands, where two of the summands are equal to each other. Then, replacing those two summands with the result of their sum, we obtain a representation with fewer than s summands, which is a contradiction. We deduce that in any representation with s summands, the summands are all distinct, so any such representation must coincide with the unique binary representation of A, and $s = d$. □

Now we split the solution into a sequence of claims.

Claim 1. After every move, the number S is the sum of at most $k-1$ distinct powers of 2.

Proof. If S is the sum of k (or more) distinct powers of 2, the Lemma implies that the k cells are filled with these numbers. This is a contradiction since no more merges or insertions can be made. □

Let $A(n,k-1)$ denote the set of all positive integers not exceeding 2^n with at most $k-1$ nonzero digits in its base 2 representation. Since every insertion increases the value of S, by Claim 1, the total number of insertions is at most $|A(n,k-1)|$. We proceed to prove that it is possible to achieve this number of insertions.

Claim 2. Let $A(n,k-1) = \{a_1, a_2, \ldots, a_m\}$, with $a_1 < a_2 < \cdots < a_m$. If after some of Sir Alex's moves the value of S is a_j, with $j \in \{1,2,\ldots,m-1\}$, then there is a sequence of moves after which the value of S is exactly a_{j+1}.

Proof. Suppose $S = a_j$. Performing all possible merges, we eventually get different powers of 2 in all nonempty cells. After that, by Claim 1 there will be at least one empty cell, in which we want to insert $a_{j+1} - a_j$. It remains to show that $a_{j+1} - a_j$ is a power of 2.

For this purpose, we notice that if a_j has less than $k-1$ nonzero digits in base 2 then $a_{j+1} = a_j + 1$. Otherwise, we have $a_j = 2^{b_{k-1}} + \cdots + 2^{b_2} + 2^{b_1}$ with $b_1 < b_2 < \cdots < b_{k-1}$. Then, adding any number less than 2^{b_1} to a_j will result in a number with more than $k-1$ nonzero binary digits. On the other hand, $a_j + 2^{b_1}$ is a sum of k powers of 2, not all distinct, so by the Lemma it will be a sum of less then k distinct powers of 2. This means that $a_{j+1} - a_j = 2^{b_1}$, completing the proof. □

Claims 1 and 2 prove that the maximum number of insertions is $|A(n, k-1)|$. We now compute this number.

Claim 3. $|A(n, k-1)| = \sum_{j=0}^{k-1} \binom{n}{j}$.

Proof. The number 2^n is the only element of $A(n, k-1)$ with $n+1$ binary digits. Any other element has at most n binary digits, at least one and at most $k-1$ of them are nonzero (so they are ones). For each $j \in \{1, 2, \ldots, k-1\}$, there are $\binom{n}{j}$ such elements with exactly j binary digits equal to one. We conclude that $|A(n, k-1)| = 1 + \sum_{j=1}^{k-1} \binom{n}{j} = \sum_{j=0}^{k-1} \binom{n}{j}$. □

Recalling that the number of insertions exceeds that of merges by exactly 1, we deduce that the maximum number of moves is $2 \sum_{j=0}^{k-1} \binom{n}{j} - 1$.

C4. Let $N \geqslant 2$ be an integer. $N(N+1)$ soccer players, no two of the same height, stand in a row in some order. Coach Ralph wants to remove $N(N-1)$ people from this row so that in the remaining row of $2N$ players, no one stands between the two tallest ones, no one stands between the third and the fourth tallest ones, $\ldots$, and finally no one stands between the two shortest ones. Show that this is always possible.

(Russia)

Solution 1. Split the row into N blocks with $N+1$ consecutive people each. We will show how to remove $N-1$ people from each block in order to satisfy the coach's wish.

First, construct a $(N+1) \times N$ matrix where $x_{i,j}$ is the height of the i^{th} tallest person of the j^{th} block—in other words, each column lists the heights within a single block, sorted in decreasing order from top to bottom.

We will reorder this matrix by repeatedly swapping whole columns. First, by column permutation, make sure that $x_{2,1} = \max\{x_{2,i} \colon i = 1, 2, \ldots, N\}$ (the first column contains the largest height of the *second* row). With the first column fixed, permute the other ones so that $x_{3,2} = \max\{x_{3,i} \colon i = 2, \ldots, N\}$ (the second column contains the tallest person of the *third* row, first column excluded). In short, at step k $(k = 1, 2, \ldots, N-1)$, we permute the columns from k to N so that $x_{k+1,k} = \max\{x_{i,k} \colon i = k, k+1, \ldots, N\}$, and end up with an array like this:

$$
\begin{array}{ccccccccc}
\boldsymbol{x_{1,1}} & & x_{1,2} & & x_{1,3} & \cdots & x_{1,N-1} & & x_{1,N} \\
\boldsymbol{\vee} & & \vee & & \vee & & \vee & & \vee \\
\boldsymbol{x_{2,1}} & \boldsymbol{>} & \boldsymbol{x_{2,2}} & & x_{2,3} & \cdots & x_{2,N-1} & & x_{2,N} \\
\vee & & \boldsymbol{\vee} & & \vee & & \vee & & \vee \\
x_{3,1} & & \boldsymbol{x_{3,2}} & \boldsymbol{>} & \boldsymbol{x_{3,3}} & \cdots & x_{3,N-1} & & x_{3,N} \\
\vee & & \vee & & \boldsymbol{\vee} & & \vee & & \vee \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots \\
\vee & & \vee & & \vee & & \boldsymbol{\vee} & & \vee \\
x_{N,1} & & x_{N,2} & & x_{N,3} & \cdots & \boldsymbol{x_{N,N-1}} & \boldsymbol{>} & \boldsymbol{x_{N,N}} \\
\vee & & \vee & & \vee & & \vee & & \boldsymbol{\vee} \\
x_{N+1,1} & & x_{N+1,2} & & x_{N+1,3} & \cdots & x_{N+1,N-1} & & \boldsymbol{x_{N+1,N}}
\end{array}
$$

Now we make the bold choice: from the original row of people, remove everyone but those with heights

$$x_{1,1} > x_{2,1} > x_{2,2} > x_{3,2} > \cdots > x_{N,N-1} > x_{N,N} > x_{N+1,N} \tag{$*$}$$

Of course this *height* order $(*)$ is not necessarily their *spatial* order in the new row. We now need to convince ourselves that each pair $(x_{k,k}; x_{k+1,k})$ remains spatially together in this new row. But $x_{k,k}$ and $x_{k+1,k}$ belong to the same column/block of consecutive $N+1$ people; the only people that could possibly stand between them were also in this block, and they are all gone.

Solution 2. Split the people into N groups *by height*: group G_1 has the $N+1$ tallest ones, group G_2 has the next $N+1$ tallest, and so on, up to group G_N with the $N+1$ shortest people.

Now scan the original row from left to right, stopping as soon as you have scanned two people (consecutively or not) from the same group, say, G_i. Since we have N groups, this must happen before or at the $(N+1)^{\text{th}}$ person of the row. Choose this pair of people, removing all the other people from the same group G_i and also all people that have been scanned so far. The only people that could separate this pair's heights were in group G_i (and they are gone); the only people that could separate this pair's positions were already scanned (and they are gone too).

We are now left with $N-1$ groups (all except G_i). Since each of them lost at most one person, each one has at least N unscanned people left in the row. Repeat the scanning process from left to right, choosing the next two people from the same group, removing this group and everyone scanned up to that point. Once again we end up with two people who are next to each other in the remaining row and whose heights cannot be separated by anyone else who remains (since the rest of their group is gone). After picking these 2 pairs, we still have $N-2$ groups with at least $N-1$ people each.

If we repeat the scanning process a total of N times, it is easy to check that we will end up with 2 people from each group, for a total of $2N$ people remaining. The height order is guaranteed by the grouping, and the scanning construction from left to right guarantees that each pair from a group stand next to each other in the final row. We are done.

Solution 3. This is essentially the same as solution 1, but presented inductively. The essence of the argument is the following lemma.

Lemma. Assume that we have N disjoint groups of at least $N+1$ people in each, all people have distinct heights. Then one can choose two people from each group so that among the chosen people, the two tallest ones are in one group, the third and the fourth tallest ones are in one group, $\ldots$, and the two shortest ones are in one group.

Proof. Induction on $N \geqslant 1$; for $N=1$, the statement is trivial.

Consider now N groups $G_1, \ldots, G_N$ with at least $N+1$ people in each for $N \geqslant 2$. Enumerate the people by $1, 2, \ldots, N(N+1)$ according to their height, say, from tallest to shortest. Find the least s such that two people among $1, 2, \ldots, s$ are in one group (without loss of generality, say this group is G_N). By the minimality of s, the two mentioned people in G_N are s and some $i < s$.

Now we choose people i and s in G_N, forget about this group, and remove the people $1, 2, \ldots, s$ from $G_1, \ldots, G_{N-1}$. Due to minimality of s again, each of the obtained groups $G_1', \ldots, G_{N-1}'$ contains at least N people. By the induction hypothesis, one can choose a pair of people from each of $G_1', \ldots, G_{N-1}'$ so as to satisfy the required conditions. Since all these people have numbers greater than s, addition of the pair (s, i) from G_N does not violate these requirements. $\square$

To solve the problem, it suffices now to split the row into N contiguous groups with $N+1$ people in each and apply the Lemma to those groups.

Comment 1. One can identify each person with a pair of indices (p, h) $(p, h \in \{1, 2, \ldots, N(N+1)\})$ so that the p^{th} person in the row (say, from left to right) is the h^{th} tallest person in the group. Say that (a, b) *separates* (x_1, y_1) and (x_2, y_2) whenever a is strictly between x_1 and y_1, *or* b is strictly between x_2 and y_2. So the coach wants to pick $2N$ people $(p_i, h_i)(i = 1, 2, \ldots, 2N)$ such that no chosen person separates (p_1, h_1) from (p_2, h_2), no chosen person separates (p_3, h_3) and (p_4, h_4), and so on. This formulation reveals a duality between positions and heights. In that sense, solutions 1 and 2 are dual of each other.

Comment 2. The number $N(N+1)$ is sharp for $N=2$ and $N=3$, due to arrangements $1, 5, 3, 4, 2$ and $1, 10, 6, 4, 3, 9, 5, 8, 7, 2, 11$.

C5. A hunter and an invisible rabbit play a game in the Euclidean plane. The hunter's starting point H_0 coincides with the rabbit's starting point R_0. In the n^{th} round of the game $(n \geqslant 1)$, the following happens.

(1) First the invisible rabbit moves secretly and unobserved from its current point R_{n-1} to some new point R_n with $R_{n-1}R_n = 1$.

(2) The hunter has a tracking device (e.g. dog) that returns an approximate position R'_n of the rabbit, so that $R_nR'_n \leqslant 1$.

(3) The hunter then visibly moves from point H_{n-1} to a new point H_n with $H_{n-1}H_n = 1$.

Is there a strategy for the hunter that guarantees that after 10^9 such rounds the distance between the hunter and the rabbit is below 100?

(Austria)

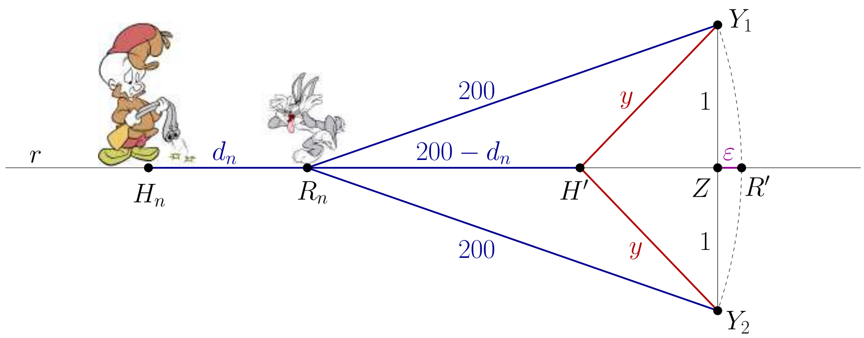
Answer: There is no such strategy for the hunter. The rabbit "wins".

Solution. If the answer were "yes", the hunter would have a strategy that would "work", no matter how the rabbit moved or where the radar pings R'_n appeared. We will show the opposite: with bad luck from the radar pings, there is no strategy for the hunter that guarantees that the distance stays below 100 in 10^9 rounds.

So, let d_n be the distance between the hunter and the rabbit after n rounds. Of course, if $d_n \geqslant 100$ for any $n < 10^9$, the rabbit has won — it just needs to move straight away from the hunter, and the distance will be kept at or above 100 thereon.

We will now show that, while $d_n < 100$, whatever given strategy the hunter follows, the rabbit has a way of increasing d_n^2 by at least $\frac{1}{2}$ every 200 rounds (as long as the radar pings are lucky enough for the rabbit). This way, d_n^2 will reach 10^4 in less than $2 \cdot 10^4 \cdot 200 = 4 \cdot 10^6 < 10^9$ rounds, and the rabbit wins.

Suppose the hunter is at H_n and the rabbit is at R_n. Suppose even that the rabbit *reveals* its position at this moment to the hunter (this allows us to ignore all information from previous radar pings). Let r be the line H_nR_n, and Y_1 and Y_2 be points which are 1 unit away from r and 200 units away from R_n, as in the figure below.



The rabbit's plan is simply to choose one of the points Y_1 or Y_2 and hop 200 rounds straight towards it. Since all hops stay within 1 distance unit from r, it is possible that all radar pings stay on r. In particular, in this case, the hunter has no way of knowing whether the rabbit chose Y_1 or Y_2.

Looking at such pings, what is the hunter going to do? If the hunter's strategy tells him to go 200 rounds straight to the right, he ends up at point H' in the figure. Note that the hunter does not have a better alternative! Indeed, after these 200 rounds he will always end up at a point to the left of H'. If his strategy took him to a point above r, he would end up even further from Y_2; and if his strategy took him below r, he would end up even further from Y_1. In other words, no matter what strategy the hunter follows, he can never be sure his distance to the rabbit will be less than $y \stackrel{\text{def}}{=} H'Y_1 = H'Y_2$ after these 200 rounds.

To estimate y^2, we take Z as the midpoint of segment Y_1Y_2, we take R' as a point 200 units to the right of R_n and we define $\varepsilon = ZR'$ (note that $H'R' = d_n$). Then

$$y^2 = 1 + (H'Z)^2 = 1 + (d_n - \varepsilon)^2$$

where

$$\varepsilon = 200 - R_n Z = 200 - \sqrt{200^2 - 1} = \frac{1}{200 + \sqrt{200^2 - 1}} > \frac{1}{400}.$$

In particular, $\varepsilon^2 + 1 = 400\varepsilon$, so

$$y^2 = d_n^2 - 2\varepsilon d_n + \varepsilon^2 + 1 = d_n^2 + \varepsilon(400 - 2d_n).$$

Since $\varepsilon > \frac{1}{400}$ and we assumed $d_n < 100$, this shows that $y^2 > d_n^2 + \frac{1}{2}$. So, as we claimed, with this list of radar pings, no matter what the hunter does, the rabbit might achieve $d_{n+200}^2 > d_n^2 + \frac{1}{2}$. The wabbit wins.

Comment 1. Many different versions of the solution above can be found by replacing 200 with some other number N for the number of hops the rabbit takes between reveals. If this is done, we have:

$$\varepsilon = N - \sqrt{N^2 - 1} > \frac{1}{N + \sqrt{N^2 - 1}} > \frac{1}{2N}$$

and

$$\varepsilon^2 + 1 = 2N\varepsilon,$$

so, as long as $N > d_n$, we would find

$$y^2 = d_n^2 + \varepsilon(2N - 2d_n) > d_n^2 + \frac{N - d_n}{N}.$$

For example, taking $N = 101$ is already enough—the squared distance increases by at least $\frac{1}{101}$ every 101 rounds, and $101^2 \cdot 10^4 = 1.0201 \cdot 10^8 < 10^9$ rounds are enough for the rabbit. If the statement is made sharper, some such versions might not work any longer.

Comment 2. The original statement asked whether the distance could be kept under 10^{10} in 10^{100} rounds.

C6. Let $n > 1$ be an integer. An $n \times n \times n$ cube is composed of n^3 unit cubes. Each unit cube is painted with one color. For each $n \times n \times 1$ box consisting of n^2 unit cubes (of any of the three possible orientations), we consider the set of the colors present in that box (each color is listed only once). This way, we get $3n$ sets of colors, split into three groups according to the orientation. It happens that for every set in any group, the same set appears in both of the other groups. Determine, in terms of n, the maximal possible number of colors that are present.

(Russia)

Answer: The maximal number is $\frac{n(n+1)(2n+1)}{6}$.

Solution 1. Call a $n \times n \times 1$ box an *x-box*, a *y-box*, or a *z-box*, according to the direction of its short side. Let C be the number of colors in a valid configuration. We start with the upper bound for C.

Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ be the sets of colors which appear in the big cube exactly once, exactly twice, and at least thrice, respectively. Let M_i be the set of unit cubes whose colors are in $\mathcal{C}_i$, and denote $n_i = |M_i|$.

Consider any x-box X, and let Y and Z be a y- and a z-box containing the same set of colors as X does.

Claim. $4|X \cap M_1| + |X \cap M_2| \leqslant 3n + 1$.

Proof. We distinguish two cases.

Case 1: $X \cap M_1 \neq \varnothing$.

A cube from $X \cap M_1$ should appear in all three boxes X, Y, and Z, so it should lie in $X \cap Y \cap Z$. Thus $X \cap M_1 = X \cap Y \cap Z$ and $|X \cap M_1| = 1$.

Consider now the cubes in $X \cap M_2$. There are at most $2(n-1)$ of them lying in $X \cap Y$ or $X \cap Z$ (because the cube from $X \cap Y \cap Z$ is in M_1). Let a be some *other* cube from $X \cap M_2$. Recall that there is just one other cube a' sharing a color with a. But both Y and Z should contain such cube, so $a' \in Y \cap Z$ (but $a' \notin X \cap Y \cap Z$). The map $a \mapsto a'$ is clearly injective, so the number of cubes a we are interested in does not exceed $|(Y \cap Z) \setminus X| = n - 1$. Thus $|X \cap M_2| \leqslant 2(n-1) + (n-1) = 3(n-1)$, and hence $4|X \cap M_1| + |X \cap M_2| \leqslant 4 + 3(n-1) = 3n+1$.

Case 2: $X \cap M_1 = \varnothing$.

In this case, the same argument applies with several changes. Indeed, $X \cap M_2$ contains at most $2n - 1$ cubes from $X \cap Y$ or $X \cap Z$. Any other cube a in $X \cap M_2$ corresponds to some $a' \in Y \cap Z$ (possibly with $a' \in X$), so there are at most n of them. All this results in $|X \cap M_2| \leqslant (2n-1) + n = 3n - 1$, which is even better than we need (by the assumptions of our case). □

Summing up the inequalities from the Claim over all x-boxes X, we obtain

$$4n_1 + n_2 \leqslant n(3n+1).$$

Obviously, we also have $n_1 + n_2 + n_3 = n^3$.

Now we are prepared to estimate C. Due to the definition of the M_i, we have $n_i \geqslant i|\mathcal{C}_i|$, so

$$C \leqslant n_1 + \frac{n_2}{2} + \frac{n_3}{3} = \frac{n_1 + n_2 + n_3}{3} + \frac{4n_1 + n_2}{6} \leqslant \frac{n^3}{3} + \frac{3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}.$$

It remains to present an example of an appropriate coloring in the above-mentioned number of colors. For each color, we present the set of all cubes of this color. These sets are:

1. n singletons of the form $S_i = \{(i,i,i)\}$ (with $1 \leqslant i \leqslant n$);
2. $3\binom{n}{2}$ doubletons of the forms $D^1_{i,j} = \{(i,j,j),(j,i,i)\}$, $D^2_{i,j} = \{(j,i,j),(i,j,i)\}$, and $D^3_{i,j} = \{(j,j,i),(i,i,j)\}$ (with $1 \leqslant i < j \leqslant n$);

3. $2\binom{n}{3}$ triplets of the form $T_{i,j,k} = \{(i,j,k),(j,k,i),(k,i,j)\}$ (with $1 \leqslant i < j < k \leqslant n$ or $1 \leqslant i < k < j \leqslant n$).

One may easily see that the i^{th} boxes of each orientation contain the same set of colors, and that

$$n + \frac{3n(n-1)}{2} + \frac{n(n-1)(n-2)}{3} = \frac{n(n+1)(2n+1)}{6}$$

colors are used, as required.

Solution 2. We will approach a new version of the original problem. In this new version, each cube may have a color, or be *invisible* (not both). Now we make sets of colors for each $n \times n \times 1$ box as before (where "invisible" is not considered a color) and group them by orientation, also as before. Finally, we require that, for every *non-empty* set in any group, the same set must appear in the other 2 groups. What is the maximum number of colors present with these new requirements?

Let us call *strange* a big $n \times n \times n$ cube whose painting scheme satisfies the new requirements, and let D be the number of colors in a strange cube. Note that any cube that satisfies the original requirements is also strange, so $\max(D)$ is an upper bound for the original answer.

Claim. $D \leqslant \frac{n(n+1)(2n+1)}{6}$.

Proof. The proof is by induction on n. If $n = 1$, we must paint the cube with at most 1 color.

Now, pick a $n \times n \times n$ strange cube A, where $n \geqslant 2$. If A is completely invisible, $D = 0$ and we are done. Otherwise, pick a non-empty set of colors $\mathcal{S}$ which corresponds to, say, the boxes X, Y and Z of different orientations.

Now find all cubes in A whose colors are in $\mathcal{S}$ and make them invisible. Since X, Y and Z are now completely invisible, we can throw them away and focus on the remaining $(n-1) \times (n-1) \times (n-1)$ cube B. The sets of colors in all the groups for B are the same as the sets for A, removing exactly the colors in $\mathcal{S}$, and no others! Therefore, every nonempty set that appears in one group for B still shows up in all possible orientations (it is possible that an empty set of colors in B only matched X, Y or Z before these were thrown away, but remember we do not require empty sets to match anyway). In summary, B is also strange.

By the induction hypothesis, we may assume that B has at most $\frac{(n-1)n(2n-1)}{6}$ colors. Since there were at most n^2 different colors in $\mathcal{S}$, we have that A has at most $\frac{(n-1)n(2n-1)}{6} + n^2 = \frac{n(n+1)(2n+1)}{6}$ colors. $\square$

Finally, the construction in the previous solution shows a painting scheme (with no invisible cubes) that reaches this maximum, so we are done.

C7. For any finite sets X and Y of positive integers, denote by $f_X(k)$ the k^{th} smallest positive integer not in X, and let

$$X * Y = X \cup \{f_X(y) \colon y \in Y\}.$$

Let A be a set of $a > 0$ positive integers, and let B be a set of $b > 0$ positive integers. Prove that if $A * B = B * A$, then

$$\underbrace{A * (A * \cdots * (A * (A * A)) \ldots)}_{A \text{ appears } b \text{ times}} = \underbrace{B * (B * \cdots * (B * (B * B)) \ldots)}_{B \text{ appears } a \text{ times}}.$$

(U.S.A.)

Solution 1. For any function $g \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ and any subset $X \subset \mathbb{Z}_{>0}$, we define $g(X) = \{g(x) \colon x \in X\}$. We have that the image of f_X is $f_X(\mathbb{Z}_{>0}) = \mathbb{Z}_{>0} \setminus X$. We now show a general lemma about the operation $*$, with the goal of showing that $*$ is associative.

Lemma 1. Let X and Y be finite sets of positive integers. The functions f_{X*Y} and $f_X \circ f_Y$ are equal.

Proof. We have

$$f_{X*Y}(\mathbb{Z}_{>0}) = \mathbb{Z}_{>0} \backslash (X*Y) = (\mathbb{Z}_{>0} \backslash X) \backslash f_X(Y) = f_X(\mathbb{Z}_{>0}) \backslash f_X(Y) = f_X(\mathbb{Z}_{>0} \backslash Y) = f_X(f_Y(\mathbb{Z}_{>0})).$$

Thus, the functions f_{X*Y} and $f_X \circ f_Y$ are strictly increasing functions with the same range. Because a strictly function is uniquely defined by its range, we have $f_{X*Y} = f_X \circ f_Y$. $\square$

Lemma 1 implies that $*$ is associative, in the sense that $(A * B) * C = A * (B * C)$ for any finite sets A, B, and C of positive integers. We prove the associativity by noting

$$\mathbb{Z}_{>0} \setminus ((A * B) * C) = f_{(A*B)*C}(\mathbb{Z}_{>0}) = f_{A*B}(f_C(\mathbb{Z}_{>0})) = f_A(f_B(f_C(\mathbb{Z}_{>0})))$$

$$= f_A(f_{B*C}(\mathbb{Z}_{>0}) = f_{A*(B*C)}(\mathbb{Z}_{>0}) = \mathbb{Z}_{>0} \setminus (A * (B * C)).$$

In light of the associativity of $*$, we may drop the parentheses when we write expressions like $A * (B * C)$. We also introduce the notation

$$X^{*k} = \underbrace{X * (X * \cdots * (X * (X * X)) \ldots)}_{X \text{ appears } k \text{ times}}.$$

Our goal is then to show that $A * B = B * A$ implies $A^{*b} = B^{*a}$. We will do so via the following general lemma.

Lemma 2. Suppose that X and Y are finite sets of positive integers satisfying $X * Y = Y * X$ and $|X| = |Y|$. Then, we must have $X = Y$.

Proof. Assume that X and Y are not equal. Let s be the largest number in exactly one of X and Y. Without loss of generality, say that $s \in X \setminus Y$. The number $f_X(s)$ counts the s^{th} number not in X, which implies that

$$f_X(s) = s + \big|X \cap \{1, 2, \ldots, f_X(s)\}\big|. \tag{1}$$

Since $f_X(s) \geqslant s$, we have that

$$\big\{f_X(s)+1, f_X(s)+2, \ldots\big\} \cap X = \big\{f_X(s)+1, f_X(s)+2, \ldots\big\} \cap Y,$$

which, together with the assumption that $|X| = |Y|$, gives

$$\big|X \cap \{1, 2, \ldots, f_X(s)\}\big| = \big|Y \cap \{1, 2, \ldots, f_X(s)\}\big|. \tag{2}$$

Now consider the equation

$$t - \left|Y \cap \{1, 2, \ldots, t\}\right| = s.$$

This equation is satisfied only when $t \in \left[f_Y(s), f_Y(s+1)\right)$, because the left hand side counts the number of elements up to t that are not in Y. We have that the value $t = f_X(s)$ satisfies the above equation because of (1) and (2). Furthermore, since $f_X(s) \notin X$ and $f_X(s) \geqslant s$, we have that $f_X(s) \notin Y$ due to the maximality of s. Thus, by the above discussion, we must have $f_X(s) = f_Y(s)$.

Finally, we arrive at a contradiction. The value $f_X(s)$ is neither in X nor in $f_X(Y)$, because s is not in Y by assumption. Thus, $f_X(s) \notin X * Y$. However, since $s \in X$, we have $f_Y(s) \in Y * X$, a contradiction. $\square$

We are now ready to finish the proof. Note first of all that $|A^{*b}| = ab = |B^{*a}|$. Moreover, since $A * B = B * A$, and $*$ is associative, it follows that $A^{*b} * B^{*a} = B^{*a} * A^{*b}$. Thus, by Lemma 2, we have $A^{*b} = B^{*a}$, as desired.

Comment 1. Taking $A = X^{*k}$ and $B = X^{*l}$ generates many non-trivial examples where $A*B = B*A$. There are also other examples not of this form. For example, if $A = \{1, 2, 4\}$ and $B = \{1, 3\}$, then $A * B = \{1, 2, 3, 4, 6\} = B * A$.

Solution 2. We will use Lemma 1 from Solution 1. Additionally, let X^{*k} be defined as in Solution 1. If X and Y are finite sets, then

$$f_X = f_Y \iff f_X(\mathbb{Z}_{>0}) = f_Y(\mathbb{Z}_{>0}) \iff (\mathbb{Z}_{>0} \setminus X) = (\mathbb{Z}_{>0} \setminus Y) \iff X = Y, \tag{3}$$

where the first equivalence is because f_X and f_Y are strictly increasing functions, and the second equivalence is because $f_X(\mathbb{Z}_{>0}) = \mathbb{Z}_{>0} \setminus X$ and $f_Y(\mathbb{Z}_{>0}) = \mathbb{Z}_{>0} \setminus Y$.

Denote $g = f_A$ and $h = f_B$. The given relation $A * B = B * A$ is equivalent to $f_{A*B} = f_{B*A}$ because of (3), and by Lemma 1 of the first solution, this is equivalent to $g \circ h = h \circ g$. Similarly, the required relation $A^{*b} = B^{*a}$ is equivalent to $g^b = h^a$. We will show that

$$g^b(n) = h^a(n) \tag{4}$$

for all $n \in \mathbb{Z}_{>0}$, which suffices to solve the problem.

To start, we claim that (4) holds for all sufficiently large n. Indeed, let p and q be the maximal elements of A and B, respectively; we may assume that $p \geqslant q$. Then, for every $n \geqslant p$ we have $g(n) = n + a$ and $h(n) = n + b$, whence $g^b(n) = n + ab = h^a(n)$, as was claimed.

In view of this claim, if (4) is not identically true, then there exists a maximal s with $g^b(s) \neq h^a(s)$. Without loss of generality, we may assume that $g(s) \neq s$, for if we had $g(s) = h(s) = s$, then s would satisfy (4). As g is increasing, we then have $g(s) > s$, so (4) holds for $n = g(s)$. But then we have

$$g(g^b(s)) = g^{b+1}(s) = g^b(n) = h^a(n) = h^a(g(s)) = g(h^a(s)),$$

where the last equality holds in view of $g \circ h = h \circ g$. By the injectivity of g, the above equality yields $g^b(s) = h^a(s)$, which contradicts the choice of s. Thus, we have proved that (4) is identically true on $\mathbb{Z}_{>0}$, as desired.

Comment 2. We present another proof of Lemma 2 of the first solution.

Let $x = |X| = |Y|$. Say that u is the smallest number in X and v is the smallest number in Y; assume without loss of generality that $u \leqslant v$.

Let T be any finite set of positive integers, and define $t = |T|$. Enumerate the elements of X as $x_1 < x_2 < \cdots < x_n$. Define $S_m = f_{(T*X^{*(m-1)})}(X)$, and enumerate its elements $s_{m,1} < s_{m,2} < \cdots < s_{m,n}$. Note that the S_m are pairwise disjoint; indeed, if we have $m < m'$, then

$$S_m \subset T * X^{*m} \subset T * X^{*(m'-1)} \quad \text{and} \quad S_{m'} = (T * X^{*m'}) \setminus (T * X^{*(m'-1)})$$

We claim the following statement, which essentially says that the S_m are eventually linear translates of each other:

Claim. For every i, there exists some m_i and c_i such that for all $m > m_i$, we have that $s_{m,i} = t+mn-c_i$. Furthermore, the c_i do not depend on the choice of T.

First, we show that this claim implies Lemma 2. We may choose $T = X$ and $T = Y$. Then, there is some m' such that for all $m \geqslant m'$, we have

$$f_{X^{*m}}(X) = f_{(Y*X^{*(m-1)})}(X). \tag{5}$$

Because u is the minimum element of X, v is the minimum element of Y, and $u \leqslant v$, we have that

$$\left(\bigcup_{m=m'}^{\infty} f_{X^{*m}}(X)\right) \cup X^{*m'} = \left(\bigcup_{m=m'}^{\infty} f_{(Y*X^{*(m-1)})}(X)\right) \cup \left(Y * X^{*(m'-1)}\right) = \{u, u+1, \dots\},$$

and in both the first and second expressions, the unions are of pairwise distinct sets. By (5), we obtain $X^{*m'} = Y * X^{*(m'-1)}$. Now, because X and Y commute, we get $X^{*m'} = X^{*(m'-1)} * Y$, and so $X = Y$.

We now prove the claim.

Proof of the claim. We induct downwards on i, first proving the statement for $i = n$, and so on.

Assume that m is chosen so that all elements of S_m are greater than all elements of T (which is possible because T is finite). For $i = n$, we have that $s_{m,n} > s_{k,n}$ for every $k < m$. Thus, all $(m-1)n$ numbers of the form $s_{k,u}$ for $k < m$ and $1 \leqslant u \leqslant n$ are less than $s_{m,n}$. We then have that $s_{m,n}$ is the $((m-1)n + x_n)^{th}$ number not in T, which is equal to $t + (m-1)n + x_n$. So we may choose $c_n = x_n - n$, which does not depend on T, which proves the base case for the induction.

For $i < n$, we have again that all elements $s_{m,j}$ for $j < i$ and $s_{p,i}$ for $p < m$ are less than $s_{m,i}$, so $s_{m,i}$ is the $((m-1)i + x_i)^{th}$ element not in T or of the form $s_{p,j}$ for $j > i$ and $p < m$. But by the inductive hypothesis, each of the sequences $s_{p,j}$ is eventually periodic with period n, and thus the sequence $s_{m,i}$ such must be as well. Since each of the sequences $s_{p,j} - t$ with $j > i$ eventually do not depend on T, the sequence $s_{m,i} - t$ eventually does not depend on T either, so the inductive step is complete. This proves the claim and thus Lemma 2. $\square$

C8. Let n be a given positive integer. In the Cartesian plane, each lattice point with nonnegative coordinates initially contains a butterfly, and there are no other butterflies. The *neighborhood* of a lattice point c consists of all lattice points within the axis-aligned $(2n+1) \times (2n+1)$ square centered at c, apart from c itself. We call a butterfly *lonely*, *crowded*, or *comfortable*, depending on whether the number of butterflies in its neighborhood N is respectively less than, greater than, or equal to half of the number of lattice points in N.

Every minute, all lonely butterflies fly away simultaneously. This process goes on for as long as there are any lonely butterflies. Assuming that the process eventually stops, determine the number of comfortable butterflies at the final state.

(Bulgaria)

Answer: $n^2 + 1$.

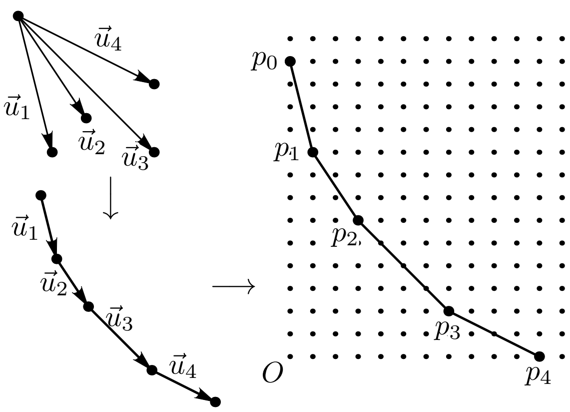
Solution. We always identify a butterfly with the lattice point it is situated at. For two points p and q, we write $p \geqslant q$ if each coordinate of p is at least the corresponding coordinate of q. Let O be the origin, and let $\mathcal{Q}$ be the set of initially occupied points, i.e., of all lattice points with nonnegative coordinates. Let $\mathcal{R}_{\mathrm{H}} = \{(x,0) : x \geqslant 0\}$ and $\mathcal{R}_{\mathrm{V}} = \{(0,y) : y \geqslant 0\}$ be the sets of the lattice points lying on the horizontal and vertical boundary rays of $\mathcal{Q}$. Denote by $N(a)$ the neighborhood of a lattice point a.

1. Initial observations. We call a set of lattice points *up-right closed* if its points stay in the set after being shifted by any lattice vector (i,j) with $i, j \geqslant 0$. Whenever the butterflies form a up-right closed set $\mathcal{S}$, we have $|N(p) \cap \mathcal{S}| \geqslant |N(q) \cap \mathcal{S}|$ for any two points $p, q \in \mathcal{S}$ with $p \geqslant q$. So, since $\mathcal{Q}$ is up-right closed, the set of butterflies at any moment also preserves this property. We assume all forthcoming sets of lattice points to be up-right closed.

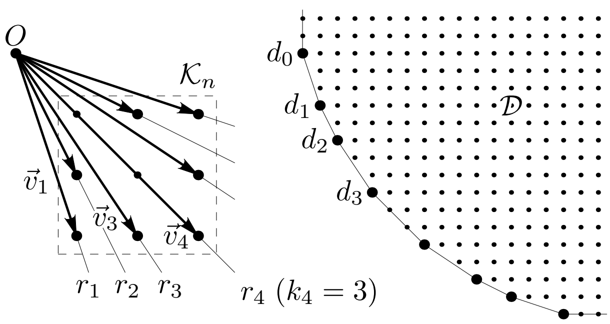
When speaking of some set $\mathcal{S}$ of lattice points, we call its points *lonely*, *comfortable*, or *crowded* with respect to this set (i.e., as if the butterflies were exactly at all points of $\mathcal{S}$). We call a set $\mathcal{S} \subset \mathcal{Q}$ *stable* if it contains no lonely points. In what follows, we are interested only in those stable sets whose complements in $\mathcal{Q}$ are finite, because one can easily see that only a finite number of butterflies can fly away on each minute.

If the initial set $\mathcal{Q}$ of butterflies contains some stable set $\mathcal{S}$, then, clearly no butterfly of this set will fly away. On the other hand, the set $\mathcal{F}$ of all butterflies in the end of the process is stable. This means that $\mathcal{F}$ is the largest (with respect to inclusion) stable set within $\mathcal{Q}$, and we are about to describe this set.

2. A description of a final set. The following notion will be useful. Let $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d\}$ be a set of d pairwise non-parallel lattice vectors, each having a positive x- and a negative y-coordinate. Assume that they are numbered in increasing order according to slope. We now define a *$\mathcal{U}$-curve* to be the broken line $p_0 p_1 \ldots p_d$ such that $p_0 \in \mathcal{R}_{\mathrm{V}}$, $p_d \in \mathcal{R}_{\mathrm{H}}$, and $\overrightarrow{p_{i-1}p_i} = \vec{u}_i$ for all $i = 1, 2, \ldots, m$ (see the Figure below to the left).



Construction of $\mathcal{U}$-curve



Construction of $\mathcal{D}$

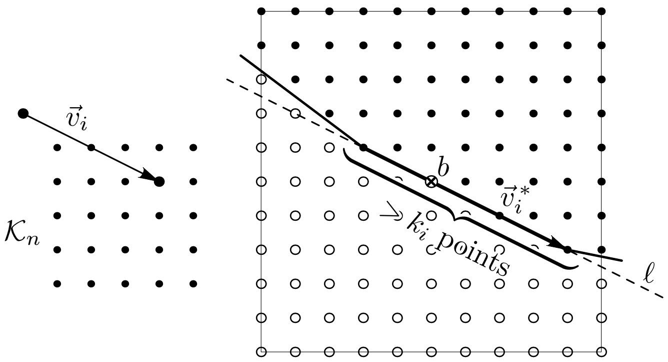
Now, let $\mathcal{K}_n = \{(i,j)\colon 1 \leqslant i \leqslant n,\ -n \leqslant j \leqslant -1\}$. Consider all the rays emerging at O and passing through a point from $\mathcal{K}_n$; number them as $r_1, \ldots, r_m$ in increasing order according to slope. Let A_i be the farthest from O lattice point in $r_i \cap \mathcal{K}_n$, set $k_i = |r_i \cap \mathcal{K}_n|$, let $\vec{v}_i = \overrightarrow{OA_i}$, and finally denote $\mathcal{V} = \{\vec{v}_i\colon 1 \leqslant i \leqslant m\}$; see the Figure above to the right. We will concentrate on the $\mathcal{V}$-curve $d_0d_1\ldots d_m$; let $\mathcal{D}$ be the set of all lattice points p such that $p \geqslant p'$ for some (not necessarily lattice) point p' on the $\mathcal{V}$-curve. In fact, we will show that $\mathcal{D} = \mathcal{F}$.

Clearly, the $\mathcal{V}$-curve is symmetric in the line $y = x$. Denote by D the convex hull of $\mathcal{D}$.

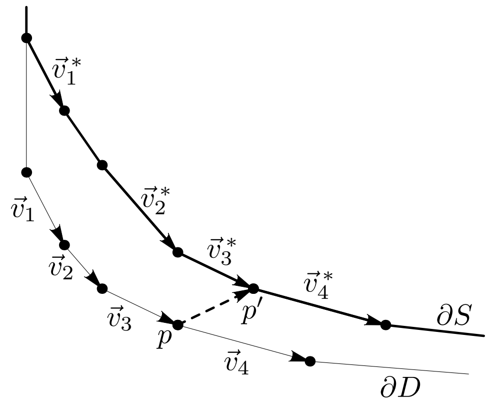
3. We prove that the set $\mathcal{D}$ contains all stable sets. Let $\mathcal{S} \subset \mathcal{Q}$ be a stable set (recall that it is assumed to be up-right closed and to have a finite complement in $\mathcal{Q}$). Denote by S its convex hull; clearly, the vertices of S are lattice points. The boundary of S consists of two rays (horizontal and vertical ones) along with some $\mathcal{V}_*$-curve for some set of lattice vectors $\mathcal{V}_*$.

Claim 1. For every $\vec{v}_i \in \mathcal{V}$, there is a $\vec{v}_i^{\,*} \in \mathcal{V}_*$ co-directed with $\vec{v}$ with $|\vec{v}_i^{\,*}| \geqslant |\vec{v}|$.

Proof. Let ℓ be the supporting line of S parallel to $\vec{v}_i$ (i.e., ℓ contains some point of S, and the set S lies on one side of ℓ). Take any point $b \in \ell \cap \mathcal{S}$ and consider $N(b)$. The line ℓ splits the set $N(b) \setminus \ell$ into two congruent parts, one having an empty intersection with $\mathcal{S}$. Hence, in order for b not to be lonely, at least half of the set $\ell \cap N(b)$ (which contains $2k_i$ points) should lie in S. Thus, the boundary of S contains a segment $\ell \cap S$ with at least $k_i + 1$ lattice points (including b) on it; this segment corresponds to the required vector $\vec{v}_i^{\,*} \in \mathcal{V}_*$. $\square$



Proof of Claim 1



Proof of Claim 2

Claim 2. Each stable set $\mathcal{S} \subseteq \mathcal{Q}$ lies in $\mathcal{D}$.

Proof. To show this, it suffices to prove that the $\mathcal{V}_*$-curve lies in D, i.e., that all its vertices do so. Let p' be an arbitrary vertex of the $\mathcal{V}_*$-curve; p' partitions this curve into two parts, $\mathcal{X}$ (being down-right of p) and $\mathcal{Y}$ (being up-left of p). The set $\mathcal{V}$ is split now into two parts: $\mathcal{V}_{\mathcal{X}}$ consisting of those $\vec{v}_i \in \mathcal{V}$ for which $\vec{v}_i^{\,*}$ corresponds to a segment in $\mathcal{X}$, and a similar part $\mathcal{V}_{\mathcal{Y}}$. Notice that the $\mathcal{V}$-curve consists of several segments corresponding to $\mathcal{V}_{\mathcal{X}}$, followed by those corresponding to $\mathcal{V}_{\mathcal{Y}}$. Hence there is a vertex p of the $\mathcal{V}$-curve separating $\mathcal{V}_{\mathcal{X}}$ from $\mathcal{V}_{\mathcal{Y}}$. Claim 1 now yields that $p' \geqslant p$, so $p' \in \mathcal{D}$, as required. $\square$

Claim 2 implies that the final set $\mathcal{F}$ is contained in $\mathcal{D}$.

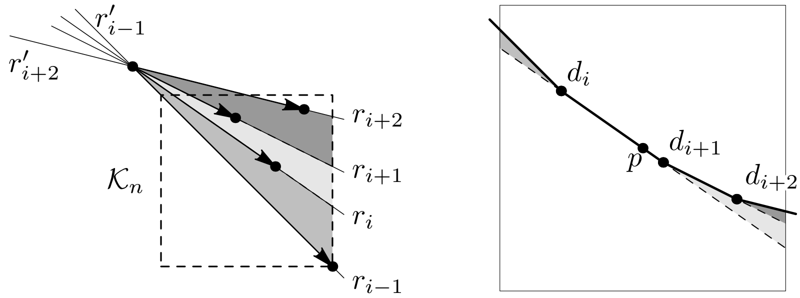
4. $\mathcal{D}$ is stable, and its comfortable points are known. Recall the definitions of r_i; let r_i' be the ray complementary to r_i. By our definitions, the set $N(O)$ contains no points between the rays r_i and r_{i+1}, as well as between r_i' and r_{i+1}'.

Claim 3. In the set $\mathcal{D}$, all lattice points of the $\mathcal{V}$-curve are comfortable.

Proof. Let p be any lattice point of the $\mathcal{V}$-curve, belonging to some segment d_id_{i+1}. Draw the line ℓ containing this segment. Then $\ell \cap \mathcal{D}$ contains exactly $k_i + 1$ lattice points, all of which lie in $N(p)$ except for p. Thus, exactly half of the points in $N(p) \cap \ell$ lie in $\mathcal{D}$. It remains to show that all points of $N(p)$ above ℓ lie in $\mathcal{D}$ (recall that all the points below ℓ lack this property).

Notice that each vector in $\mathcal{V}$ has one coordinate greater than $n/2$; thus the neighborhood of p contains parts of at most two segments of the $\mathcal{V}$-curve succeeding $d_i d_{i+1}$, as well as at most two of those preceding it.

The angles formed by these consecutive segments are obtained from those formed by r_j and r'_{j-1} (with $i-1 \leqslant j \leqslant i+2$) by shifts; see the Figure below. All the points in $N(p)$ above ℓ which could lie outside $\mathcal{D}$ lie in shifted angles between r_j, r_{j+1} or r'_j, r'_{j-1}. But those angles, restricted to $N(p)$, have no lattice points due to the above remark. The claim is proved. $\square$

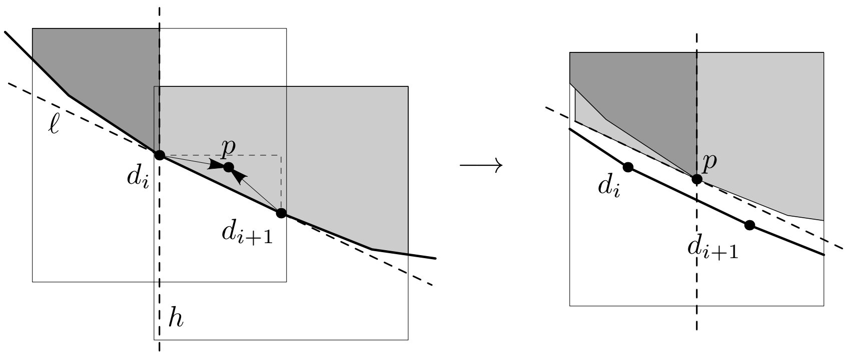


Proof of Claim 3

Claim 4. All the points of $\mathcal{D}$ which are not on the boundary of D are crowded.

Proof. Let $p \in \mathcal{D}$ be such a point. If it is to the up-right of some point p' on the curve, then the claim is easy: the shift of $N(p') \cap \mathcal{D}$ by $\overrightarrow{p'p}$ is still in $\mathcal{D}$, and $N(p)$ contains at least one more point of $\mathcal{D}$ — either below or to the left of p. So, we may assume that p lies in a right triangle constructed on some hypothenuse $d_i d_{i+1}$. Notice here that $d_i, d_{i+1} \in N(p)$.

Draw a line $\ell \parallel d_i d_{i+1}$ through p, and draw a vertical line h through d_i; see Figure below. Let $\mathcal{D}_{\mathrm{L}}$ and $\mathcal{D}_{\mathrm{R}}$ be the parts of $\mathcal{D}$ lying to the left and to the right of h, respectively (points of $\mathcal{D} \cap h$ lie in both parts).



Proof of Claim 4

Notice that the vectors $\overrightarrow{d_i p}$, $\overrightarrow{d_{i+1}d_{i+2}}$, $\overrightarrow{d_i d_{i+1}}$, $\overrightarrow{d_{i-1}d_i}$, and $\overrightarrow{pd_{i+1}}$ are arranged in non-increasing order by slope. This means that $\mathcal{D}_{\mathrm{L}}$ shifted by $\overrightarrow{d_i p}$ still lies in $\mathcal{D}$, as well as $\mathcal{D}_{\mathrm{R}}$ shifted by $\overrightarrow{d_{i+1}p}$. As we have seen in the proof of Claim 3, these two shifts cover all points of $N(p)$ above ℓ, along with those on ℓ to the left of p. Since $N(p)$ contains also d_i and d_{i+1}, the point p is crowded. $\square$

Thus, we have proved that $\mathcal{D} = \mathcal{F}$, and have shown that the lattice points on the $\mathcal{V}$-curve are exactly the comfortable points of $\mathcal{D}$. It remains to find their number.

Recall the definition of $\mathcal{K}_n$ (see Figure on the first page of the solution). Each segment $d_i d_{i+1}$ contains k_i lattice points different from d_i. Taken over all i, these points exhaust all the lattice points in the $\mathcal{V}$-curve, except for d_1, and thus the number of lattice points on the $\mathcal{V}$-curve is $1 + \sum_{i=1}^{m} k_i$. On the other hand, $\sum_{i=1}^{m} k_i$ is just the number of points in $\mathcal{K}_n$, so it equals n^2. Hence the answer to the problem is $n^2 + 1$.

Comment 1. The assumption that the process eventually stops is unnecessary for the problem, as one can see that, in fact, the process stops for every $n \geqslant 1$. Indeed, the proof of Claims 3 and 4 do not rely essentially on this assumption, and they together yield that the set $\mathcal{D}$ is stable. So, only butterflies that are not in $\mathcal{D}$ may fly away, and this takes only a finite time.

This assumption has been inserted into the problem statement in order to avoid several technical details regarding finiteness issues. It may also simplify several other arguments.

Comment 2. The description of the final set $\mathcal{F}(= \mathcal{D})$ seems to be crucial for the solution; the Problem Selection Committee is not aware of any solution that completely avoids such a description.

On the other hand, after the set $\mathcal{D}$ has been defined, the further steps may be performed in several ways. For example, in order to prove that all butterflies outside $\mathcal{D}$ will fly away, one may argue as follows. (Here we will also make use of the assumption that the process eventually stops.)

First of all, notice that the process can be modified in the following manner: *Each minute, exactly one of the lonely butterflies flies away, until there are no more lonely butterflies.* The modified process necessarily stops at the same state as the initial one. Indeed, one may observe, as in solution above, that the (unique) largest stable set is still the final set for the modified process.

Thus, in order to prove our claim, it suffices to indicate an order in which the butterflies should fly away in the new process; if we are able to exhaust the whole set $\mathcal{Q} \setminus \mathcal{D}$, we are done.

Let $\mathcal{C}_0 = d_0 d_1 \dots d_m$ be the $\mathcal{V}$-curve. Take its copy $\mathcal{C}$ and shift it downwards so that d_0 comes to some point below the origin O. Now we start moving $\mathcal{C}$ upwards continuously, until it comes back to its initial position $\mathcal{C}_0$. At each moment when $\mathcal{C}$ meets some lattice points, we convince all the butterflies at those points to fly away in a certain order. We will now show that we always have enough arguments for butterflies to do so, which will finish our argument for the claim..

Let $\mathcal{C}' = d'_0 d'_1 \dots d'_m$ be a position of $\mathcal{C}$ when it meets some butterflies. We assume that all butterflies under this current position of $\mathcal{C}$ were already convinced enough and flied away. Consider the lowest butterfly b on $\mathcal{C}'$. Let $d'_i d'_{i+1}$ be the segment it lies on; we choose i so that $b \neq d'_{i+1}$ (this is possible because $\mathcal{C}$ as not yet reached $\mathcal{C}_0$).

Draw a line ℓ containing the segment $d'_i d'_{i+1}$. Then all the butterflies in $N(b)$ are situated on or above ℓ; moreover, those on ℓ all lie on the segment $d_i d_{i+1}$. But this segment now contains at most k_i butterflies (including b), since otherwise some butterfly had to occupy d'_{i+1} which is impossible by the choice of b. Thus, b is lonely and hence may be convinced to fly away.

After b has flied away, we switch to the lowest of the remaining butterflies on $\mathcal{C}'$, and so on.

Claims 3 and 4 also allow some different proofs which are not presented here.

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Geometry

G1. Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

(Italy)

Solution 1. Throughout the solution, we refer to $\angle A$, $\angle B$, $\angle C$, $\angle D$, and $\angle E$ as internal angles of the pentagon $ABCDE$. Let the perpendicular bisectors of AC and BD, which pass respectively through B and C, meet at point I. Then $BD \perp CI$ and, similarly, $AC \perp BI$. Hence AC and BD meet at the orthocenter H of the triangle BIC, and $IH \perp BC$. It remains to prove that E lies on the line IH or, equivalently, $EI \perp BC$.

Lines IB and IC bisect $\angle B$ and $\angle C$, respectively. Since $IA = IC$, $IB = ID$, and $AB = BC = CD$, the triangles IAB, ICB and ICD are congruent. Hence $\angle IAB = \angle ICB = \angle C/2 = \angle A/2$, so the line IA bisects $\angle A$. Similarly, the line ID bisects $\angle D$. Finally, the line IE bisects $\angle E$ because I lies on all the other four internal bisectors of the angles of the pentagon.

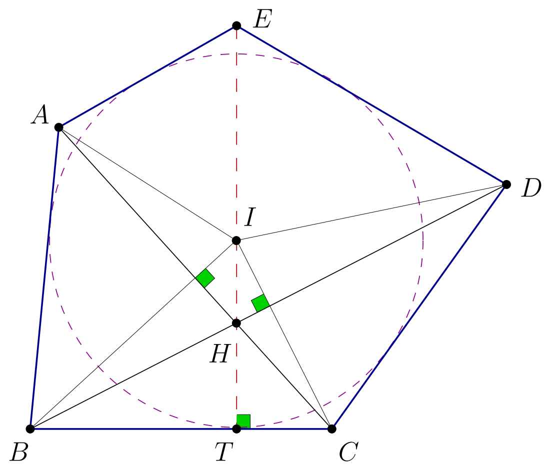
The sum of the internal angles in a pentagon is 540°, so

$$\angle E = 540^\circ - 2\angle A + 2\angle B.$$

In quadrilateral $ABIE$,

$$\begin{aligned}\angle BIE &= 360^\circ - \angle EAB - \angle ABI - \angle AEI = 360^\circ - \angle A - \frac{1}{2}\angle B - \frac{1}{2}\angle E \\ &= 360^\circ - \angle A - \frac{1}{2}\angle B - (270^\circ - \angle A - \angle B) \\ &= 90^\circ + \frac{1}{2}\angle B = 90^\circ + \angle IBC,\end{aligned}$$

which means that $EI \perp BC$, completing the proof.



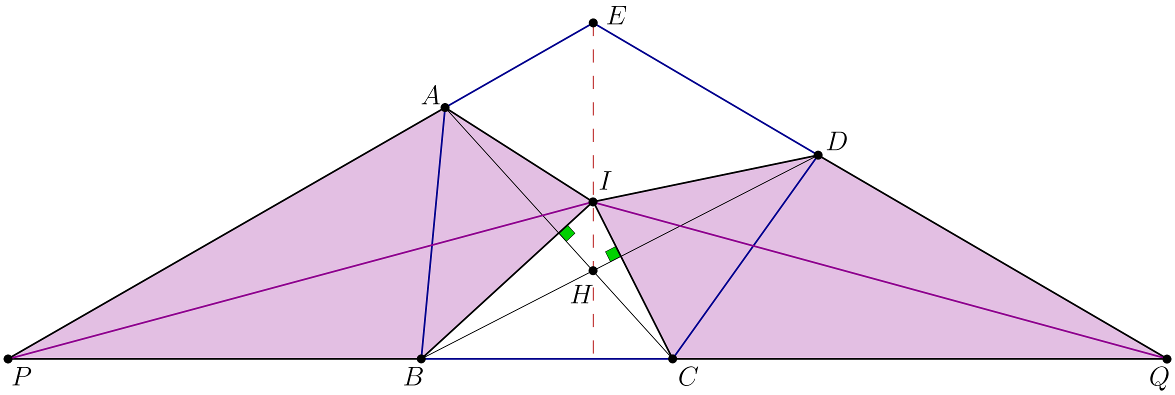
Solution 2. We present another proof of the fact that E lies on line IH. Since all five internal bisectors of $ABCDE$ meet at I, this pentagon has an inscribed circle with center I. Let this circle touch side BC at T.

Applying Brianchon's theorem to the (degenerate) hexagon $ABTCDE$ we conclude that AC, BD and ET are concurrent, so point E also lies on line IHT, completing the proof.

Solution 3. We present yet another proof that $EI \perp BC$. In pentagon $ABCDE$, $\angle E < 180^\circ \iff \angle A + \angle B + \angle C + \angle D > 360^\circ$. Then $\angle A + \angle B = \angle C + \angle D > 180^\circ$, so rays EA and CB meet at a point P, and rays BC and ED meet at a point Q. Now,

$$\angle PBA = 180^\circ - \angle B = 180^\circ - \angle D = \angle QDC$$

and, similarly, $\angle PAB = \angle QCD$. Since $AB = CD$, the triangles PAB and QCD are congruent with the same orientation. Moreover, PQE is isosceles with $EP = EQ$.



In Solution 1 we have proved that triangles IAB and ICD are also congruent with the same orientation. Then we conclude that quadrilaterals $PBIA$ and $QDIC$ are congruent, which implies $IP = IQ$. Then EI is the perpendicular bisector of PQ and, therefore, $EI \perp PQ \iff EI \perp BC$.

Comment. Even though all three solutions used the point I, there are solutions that do not need it. We present an outline of such a solution: if J is the incenter of $\triangle QCD$ (with P and Q as defined in Solution 3), then a simple angle chasing shows that triangles CJD and BHC are congruent. Then if S is the projection of J onto side CD and T is the orthogonal projection of H onto side BC, one can verify that

$$QT = QC + CT = QC + DS = QC + \frac{CD + DQ - QC}{2} = \frac{PB + BC + QC}{2} = \frac{PQ}{2},$$

so T is the midpoint of PQ, and E, H and T all lie on the perpendicular bisector of PQ.

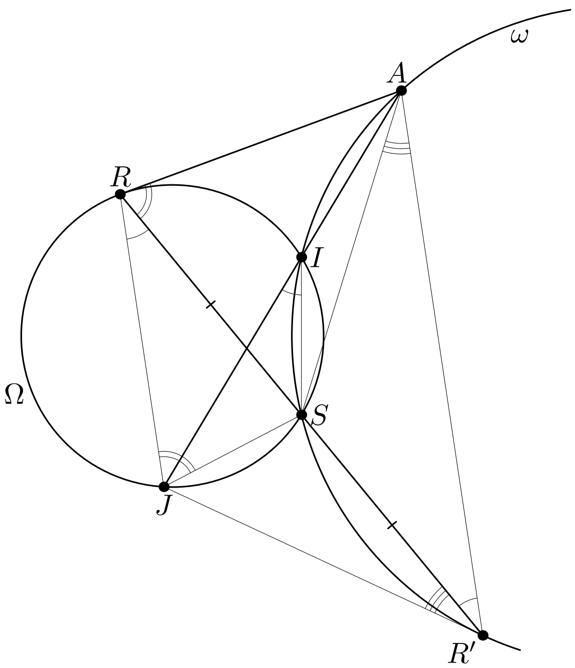
G2. Let R and S be distinct points on circle Ω, and let t denote the tangent line to Ω at R. Point R' is the reflection of R with respect to S. A point I is chosen on the smaller arc RS of Ω so that the circumcircle Γ of triangle ISR' intersects t at two different points. Denote by A the common point of Γ and t that is closest to R. Line AI meets Ω again at J. Show that JR' is tangent to Γ.

(Luxembourg)

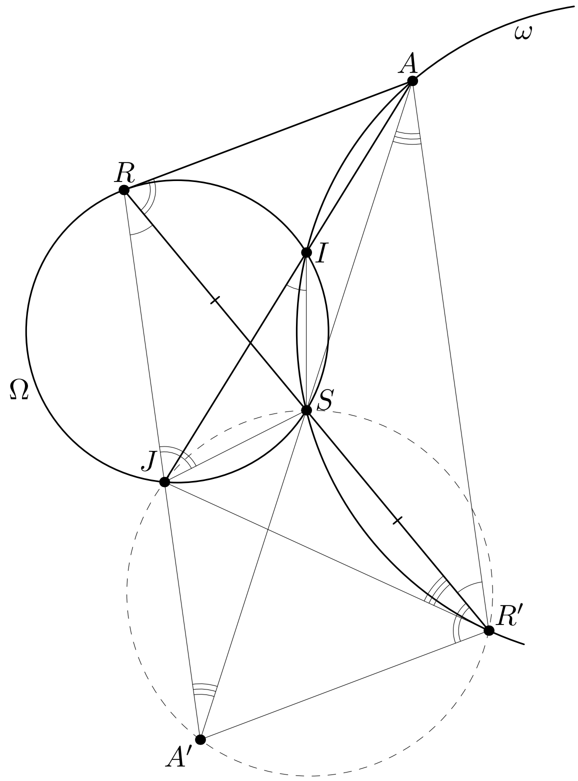
Solution 1. In the circles Ω and Γ we have $\angle JRS = \angle JIS = \angle AR'S$. On the other hand, since RA is tangent to Ω, we get $\angle SJR = \angle SRA$. So the triangles ARR' and SJR are similar, and

$$\frac{R'R}{RJ} = \frac{AR'}{SR} = \frac{AR'}{SR'}.$$

The last relation, together with $\angle AR'S = \angle JRR'$, yields $\triangle ASR' \sim \triangle R'JR$, hence $\angle SAR' = \angle RR'J$. It follows that JR' is tangent to Γ at R'.



Solution 1



Solution 2

Solution 2. As in Solution 1, we notice that $\angle JRS = \angle JIS = \angle AR'S$, so we have $RJ \parallel AR'$.

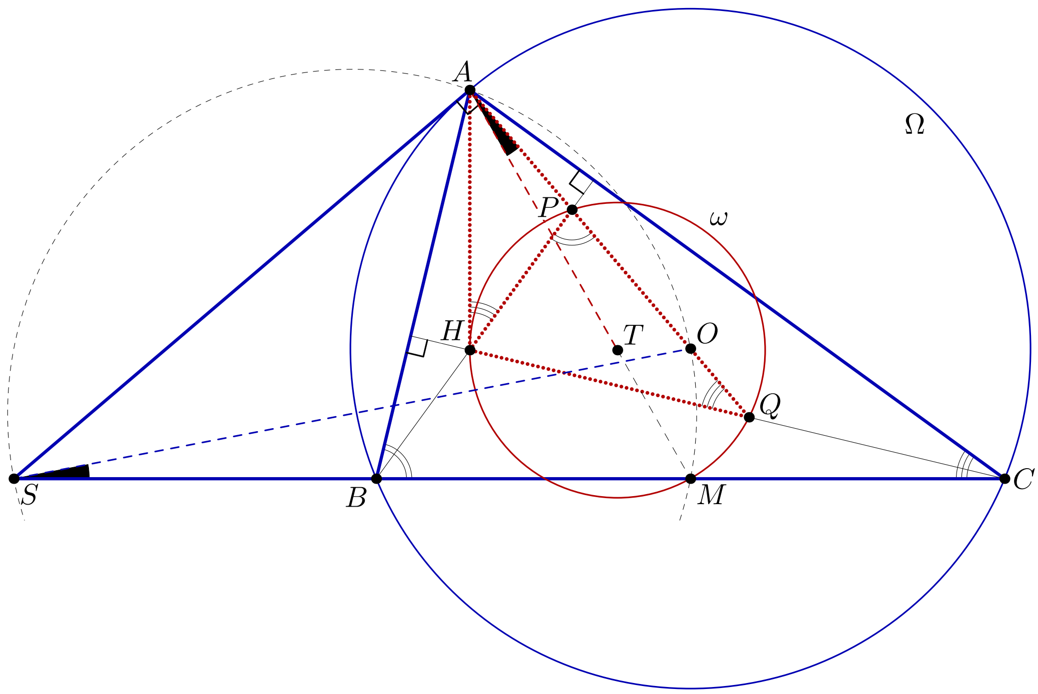
Let A' be the reflection of A about S; then $ARA'R'$ is a parallelogram with center S, and hence the point J lies on the line RA'.

From $\angle SR'A' = \angle SRA = \angle SJR$ we get that the points S, J, A', R' are concyclic. This proves that $\angle SR'J = \angle SA'J = \angle SA'R = \angle SAR'$, so JR' is tangent to Γ at R'.

G3. Let O be the circumcenter of an acute scalene triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

(Ukraine)

Solution. Suppose, without loss of generality, that $AB < AC$. We have $\angle PQH = 90^\circ - \angle QAB = 90^\circ - \angle OAB = \frac{1}{2}\angle AOB = \angle ACB$, and similarly $\angle QPH = \angle ABC$. Thus triangles ABC and HPQ are similar. Let Ω and ω be the circumcircles of ABC and HPQ, respectively. Since $\angle AHP = 90^\circ - \angle HAC = \angle ACB = \angle HQP$, line AH is tangent to ω.



Let T be the center of ω and let lines AT and BC meet at M. We will take advantage of the similarity between ABC and HPQ and the fact that AH is tangent to ω at H, with A on line PQ. Consider the corresponding tangent AS to Ω, with $S \in BC$. Then S and A correspond to each other in $\triangle ABC \sim \triangle HPQ$, and therefore $\angle OSM = \angle OAT = \angle OAM$. Hence quadrilateral $SAOM$ is cyclic, and since the tangent line AS is perpendicular to AO, $\angle OMS = 180^\circ - \angle OAS = 90^\circ$. This means that M is the orthogonal projection of O onto BC, which is its midpoint. So T lies on median AM of triangle ABC.

G4. In triangle ABC, let ω be the excircle opposite A. Let D, E, and F be the points where ω is tangent to lines BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

(Denmark)

Solution 1. Denote by Ω the circle $AEFPQ$, and denote by γ the circle PQM. Let the line AD meet ω again at $T \neq D$. We will show that γ is tangent to ω at T.

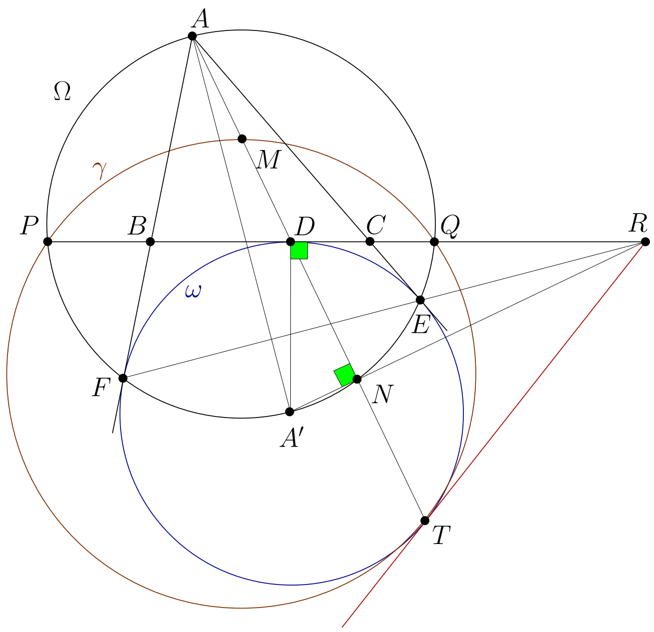
We first prove that points P, Q, M, T are concyclic. Let A' be the center of ω. Since $A'E \perp AE$ and $A'F \perp AF$, AA' is a diameter in Ω. Let N be the midpoint of DT; from $A'D = A'T$ we can see that $\angle A'NA = 90^\circ$ and therefore N also lies on the circle Ω. Now, from the power of D with respect to the circles γ and Ω we get

$$DP \cdot DQ = DA \cdot DN = 2DM \cdot \frac{DT}{2} = DM \cdot DT,$$

so P, Q, M, T are concyclic.

If $EF \parallel BC$, then ABC is isosceles and the problem is now immediate by symmetry. Otherwise, let the tangent line to ω at T meet line BC at point R. The tangent line segments RD and RT have the same length, so $A'R$ is the perpendicular bisector of DT; since $ND = NT$, N lies on this perpendicular bisector.

In right triangle $A'RD$, $RD^2 = RN \cdot RA' = RP \cdot RQ$, in which the last equality was obtained from the power of R with respect to Ω. Hence $RT^2 = RP \cdot RQ$, which implies that RT is also tangent to γ. Because RT is a common tangent to ω and γ, these two circles are tangent at T.



Solution 2. After proving that P, Q, M, T are concyclic, we finish the problem in a different fashion. We only consider the case in which EF and BC are not parallel. Let lines PQ and EF meet at point R. Since PQ and EF are radical axes of Ω, γ and ω, γ, respectively, R is the radical center of these three circles.

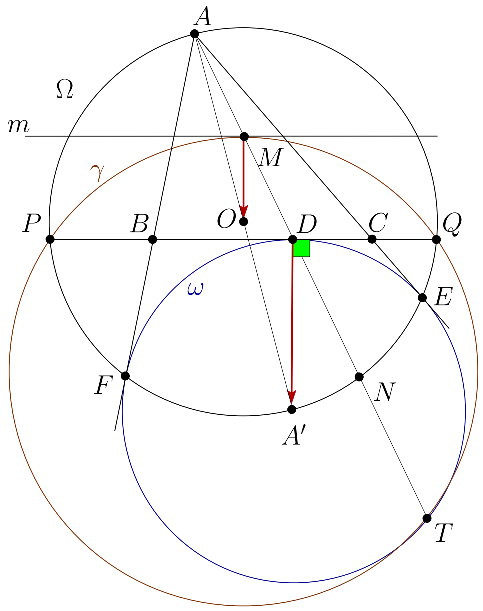
With respect to the circle ω, the line DR is the polar of D, and the line EF is the polar of A. So the pole of line ADT is $DR \cap EF = R$, and therefore RT is tangent to ω.

Finally, since T belongs to γ and ω and R is the radical center of γ, ω and Ω, line RT is the radical axis of γ and ω, and since it is tangent to ω, it is also tangent to γ. Because RT is a common tangent to ω and γ, these two circles are tangent at T.

Comment. In Solution 2 we defined the point R from Solution 1 in a different way.

Solution 3. We give an alternative proof that the circles are tangent at the common point T. Again, we start from the fact that P, Q, M, T are concyclic. Let point O be the midpoint of diameter AA'. Then MO is the midline of triangle ADA', so $MO \parallel A'D$. Since $A'D \perp PQ$, MO is perpendicular to PQ as well.

Looking at circle Ω, which has center O, $MO \perp PQ$ implies that MO is the perpendicular bisector of the chord PQ. Thus M is the midpoint of arc $\widehat{PQ}$ from γ, and the tangent line m to γ at M is parallel to PQ.



Consider the homothety with center T and ratio $\frac{TD}{TM}$. It takes D to M, and the line PQ to the line m. Since the circle that is tangent to a line at a given point and that goes through another given point is unique, this homothety also takes ω (tangent to PQ and going through T) to γ (tangent to m and going through T). We conclude that ω and γ are tangent at T.

G5. Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

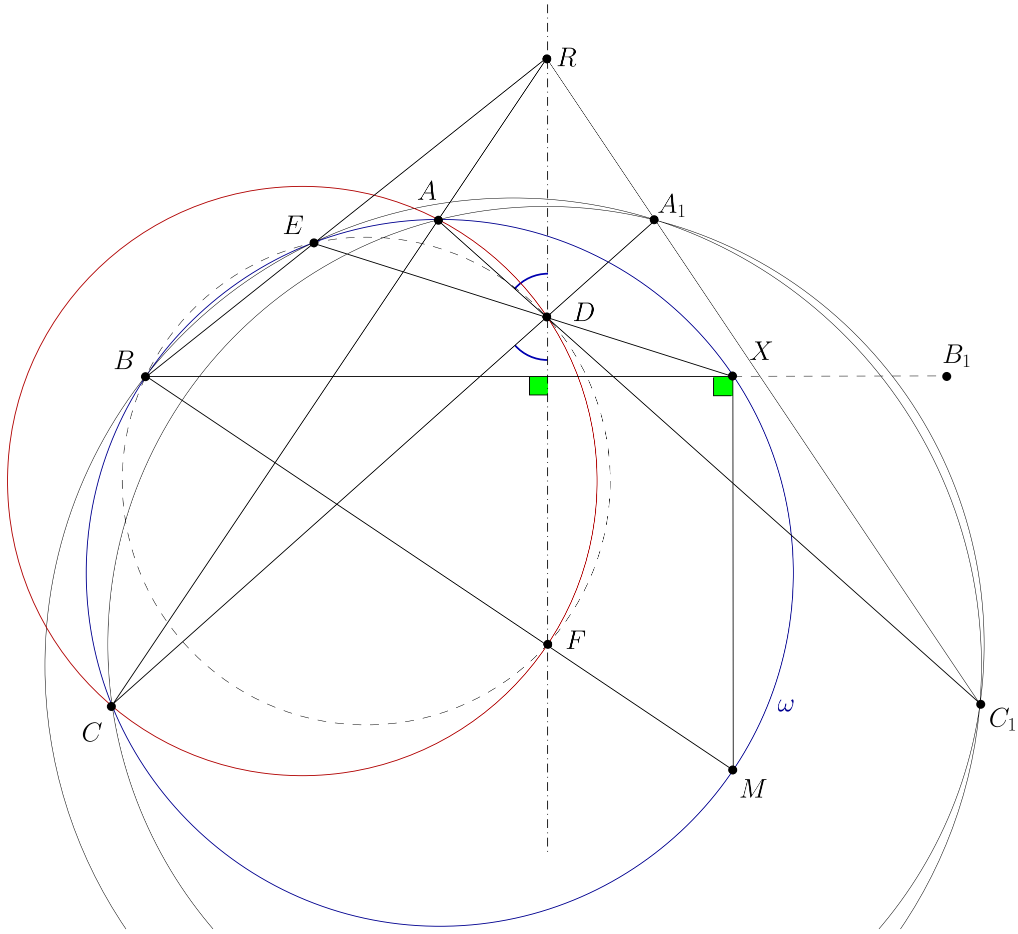
(Ukraine)

Solution 1. If $AA_1 = CC_1$, then the hexagon is symmetric about the line BB_1; in particular the circles ABC and A_1BC_1 are tangent to each other. So AA_1 and CC_1 must be different. Since the points A and A_1 can be interchanged with C and C_1, respectively, we may assume $AA_1 < CC_1$.

Let R be the radical center of the circles $AEBC$ and A_1EBC_1, and the circumcircle of the symmetric trapezoid ACC_1A_1; that is the common point of the pairwise radical axes AC, A_1C_1, and BE. By the symmetry of AC and A_1C_1, the point R lies on the common perpendicular bisector of AA_1 and CC_1, which is the external bisector of $\angle ADC$.

Let F be the second intersection of the line DR and the circle ACD. From the power of R with respect to the circles ω and $ACFD$ we have $RB \cdot RE = RA \cdot RC = RD \cdot DF$, so the points B, E, D and F are concyclic.

The line RDF is the external bisector of $\angle ADC$, so the point F bisects the arc $\widehat{CDA}$. By $AB = BC$, on circle ω, the point B is the midpoint of arc $\widehat{AEC}$; let M be the point diametrically opposite to B, that is the midpoint of the opposite arc $\widehat{CA}$ of ω. Notice that the points B, F and M lie on the perpendicular bisector of AC, so they are collinear.



Finally, let X be the second intersection point of ω and the line DE. Since BM is a diameter in ω, we have $\angle BXM = 90°$. Moreover,

$$\angle EXM = 180° - \angle MBE = 180° - \angle FBE = \angle EDF,$$

so MX and FD are parallel. Since BX is perpendicular to MX and BB_1 is perpendicular to FD, this shows that X lies on line BB_1.

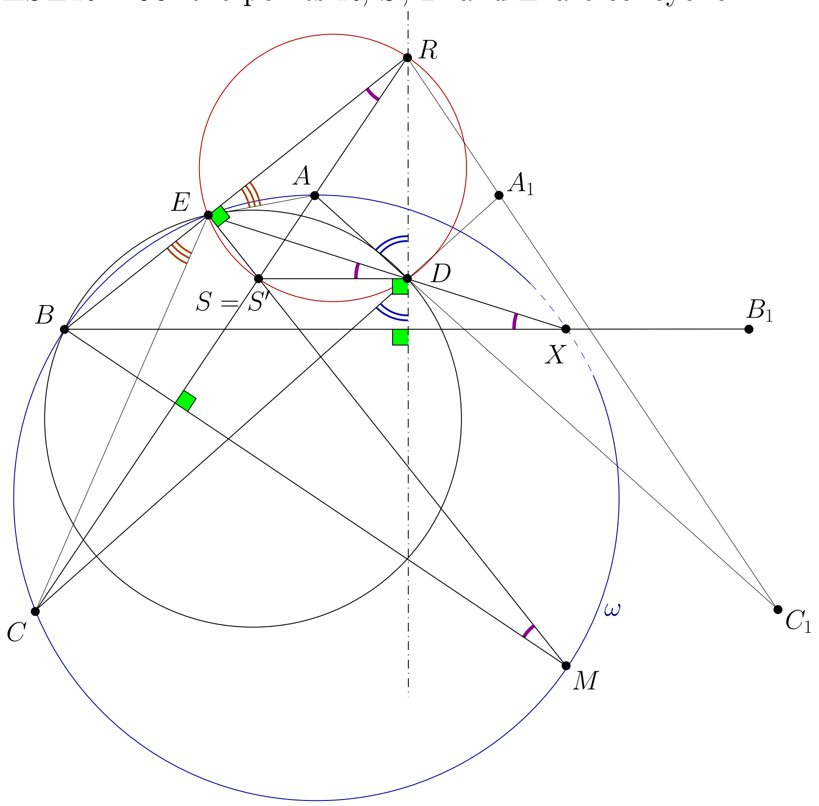
Solution 2. Define point M as the point opposite to B on circle ω, and point R as the intersection of lines AC, A_1C_1 and BE, and show that R lies on the external bisector of $\angle ADC$, like in the first solution.

Since B is the midpoint of the arc $\widehat{AEC}$, the line BER is the external bisector of $\angle CEA$. Now we show that the internal angle bisectors of $\angle ADC$ and $\angle CEA$ meet on the segment AC. Let the angle bisector of $\angle ADC$ meet AC at S, and let the angle bisector of $\angle CEA$, which is line EM, meet AC at S'. By applying the angle bisector theorem to both internal and external bisectors of $\angle ADC$ and $\angle CEA$,

$$AS : CS = AD : CD = AR : CR = AE : CE = AS' : CS',$$

so indeed $S = S'$.

By $\angle RDS = \angle SER = 90°$ the points R, S, D and E are concyclic.



Now let the lines BB_1 and DE meet at point X. Notice that $\angle EXB = \angle EDS$ because both BB_1 and DS are perpendicular to the line DR, we have that $\angle EDS = \angle ERS$ in circle $SRDE$, and $\angle ERS = \angle EMB$ because $SR \perp BM$ and $ER \perp ME$. Therefore, $\angle EXB = \angle EMB$, so indeed, the point X lies on ω.

G6. Let $n \geqslant 3$ be an integer. Two regular n-gons $\mathcal{A}$ and $\mathcal{B}$ are given in the plane. Prove that the vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary are consecutive.

(That is, prove that there exists a line separating those vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary from the other vertices of $\mathcal{A}$.)

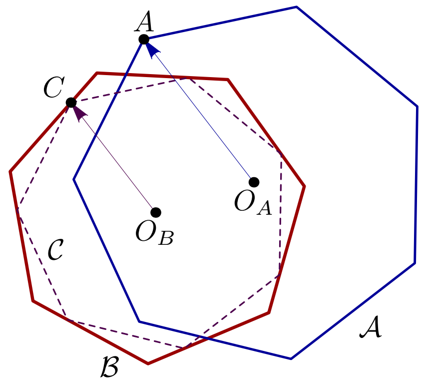
(Czech Republic)

Solution 1. In both solutions, by a *polygon* we always mean its interior together with its boundary.

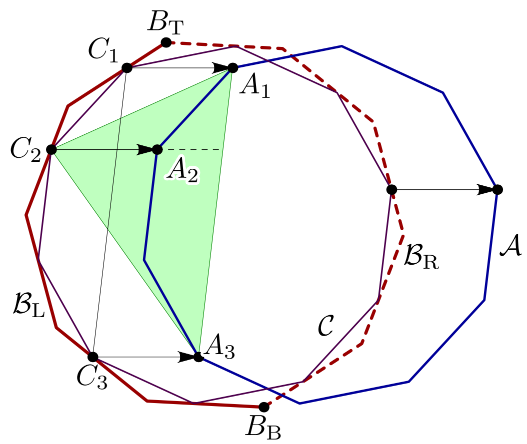
We start with finding a regular n-gon $\mathcal{C}$ which (i) is inscribed into $\mathcal{B}$ (that is, all vertices of $\mathcal{C}$ lie on the perimeter of $\mathcal{B}$); and (ii) is either a translation of $\mathcal{A}$, or a homothetic image of $\mathcal{A}$ with a positive factor.

Such a polygon may be constructed as follows. Let O_A and O_B be the centers of $\mathcal{A}$ and $\mathcal{B}$, respectively, and let A be an arbitrary vertex of $\mathcal{A}$. Let $\overrightarrow{O_BC}$ be the vector co-directional to $\overrightarrow{O_AA}$, with C lying on the perimeter of $\mathcal{B}$. The rotations of C around O_B by multiples of $2\pi/n$ form the required polygon. Indeed, it is regular, inscribed into $\mathcal{B}$ (due to the rotational symmetry of $\mathcal{B}$), and finally the translation/homothety mapping $\overrightarrow{O_AA}$ to $\overrightarrow{O_BC}$ maps $\mathcal{A}$ to $\mathcal{C}$.

Now we separate two cases.



Construction of $\mathcal{C}$



Case 1: Translation

Case 1: $\mathcal{C}$ is a translation of $\mathcal{A}$ by a vector $\vec{v}$.

Denote by t the translation transform by vector $\vec{v}$. We need to prove that the vertices of $\mathcal{C}$ which stay in $\mathcal{B}$ under t are consecutive. To visualize the argument, we refer the plane to Cartesian coordinates so that the x-axis is co-directional with $\vec{v}$. This way, the notions of right/left and top/bottom are also introduced, according to the x- and y-coordinates, respectively.

Let B_{T} and B_{B} be the top and the bottom vertices of $\mathcal{B}$ (if several vertices are extremal, we take the rightmost of them). They split the perimeter of $\mathcal{B}$ into the *right part* $\mathcal{B}_{\mathrm{R}}$ and the *left part* $\mathcal{B}_{\mathrm{L}}$ (the vertices B_{T} and B_{B} are assumed to lie in both parts); each part forms a connected subset of the perimeter of $\mathcal{B}$. So the vertices of $\mathcal{C}$ are also split into two parts $\mathcal{C}_{\mathrm{L}} \subset \mathcal{B}_{\mathrm{L}}$ and $\mathcal{C}_{\mathrm{R}} \subset \mathcal{B}_{\mathrm{R}}$, each of which consists of consecutive vertices.

Now, all the points in $\mathcal{B}_{\mathrm{R}}$ (and hence in $\mathcal{C}_{\mathrm{R}}$) move out from $\mathcal{B}$ under t, since they are the rightmost points of $\mathcal{B}$ on the corresponding horizontal lines. It remains to prove that the vertices of $\mathcal{C}_{\mathrm{L}}$ which stay in $\mathcal{B}$ under t are consecutive.

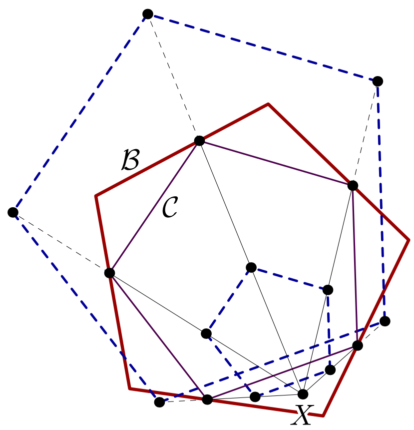
For this purpose, let C_1, C_2, and C_3 be three vertices in $\mathcal{C}_{\mathrm{L}}$ such that C_2 is between C_1 and C_3, and $t(C_1)$ and $t(C_3)$ lie in $\mathcal{B}$; we need to prove that $t(C_2) \in \mathcal{B}$ as well. Let $A_i = t(C_i)$. The line through C_2 parallel to $\vec{v}$ crosses the segment C_1C_3 to the right of C_2; this means that this line crosses A_1A_3 to the right of A_2, so A_2 lies inside the triangle $A_1C_2A_3$ which is contained in $\mathcal{B}$. This yields the desired result.

Case 2: $\mathcal{C}$ is a homothetic image of $\mathcal{A}$ centered at X with factor $k > 0$.

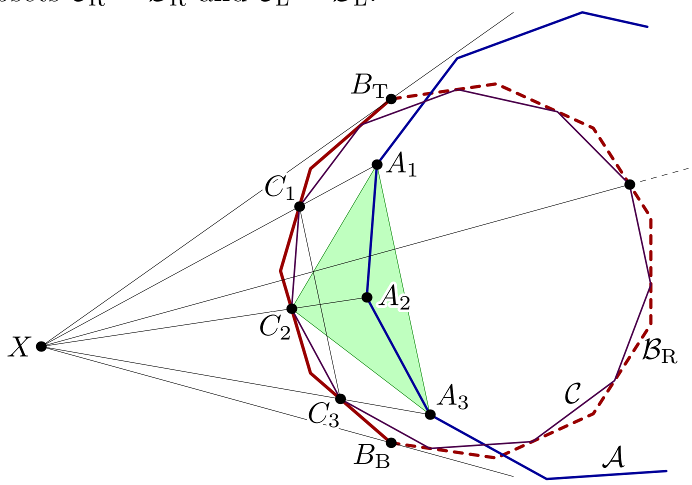
Denote by h the homothety mapping $\mathcal{C}$ to $\mathcal{A}$. We need now to prove that the vertices of $\mathcal{C}$ which stay in $\mathcal{B}$ after applying h are consecutive. If $X \in \mathcal{B}$, the claim is easy. Indeed, if $k < 1$, then the vertices of $\mathcal{A}$ lie on the segments of the form XC (C being a vertex of $\mathcal{C}$) which lie in $\mathcal{B}$. If $k > 1$, then the vertices of $\mathcal{A}$ lie on the extensions of such segments XC beyond C, and almost all these extensions lie outside $\mathcal{B}$. The exceptions may occur only in case when X lies on the boundary of $\mathcal{B}$, and they may cause one or two vertices of $\mathcal{A}$ stay on the boundary of $\mathcal{B}$. But even in this case those vertices are still consecutive.

So, from now on we assume that $X \notin \mathcal{B}$.

Now, there are two vertices B_{T} and $\mathcal{B}_{\mathrm{B}}$ of $\mathcal{B}$ such that $\mathcal{B}$ is contained in the angle $\angle B_{\mathrm{T}}XB_{\mathrm{B}}$; if there are several options, say, for B_{T}, then we choose the farthest one from X if $k > 1$, and the nearest one if $k < 1$. For the visualization purposes, we refer the plane to Cartesian coordinates so that the y-axis is co-directional with $\overrightarrow{B_{\mathrm{B}}B_{\mathrm{T}}}$, and X lies to the left of the line $B_{\mathrm{T}}B_{\mathrm{B}}$. Again, the perimeter of $\mathcal{B}$ is split by B_{T} and B_{B} into the *right part* $\mathcal{B}_{\mathrm{R}}$ and the *left part* $\mathcal{B}_{\mathrm{L}}$, and the set of vertices of $\mathcal{C}$ is split into two subsets $\mathcal{C}_{\mathrm{R}} \subset \mathcal{B}_{\mathrm{R}}$ and $\mathcal{C}_{\mathrm{L}} \subset \mathcal{B}_{\mathrm{L}}$.



Case 2, X inside $\mathcal{B}$

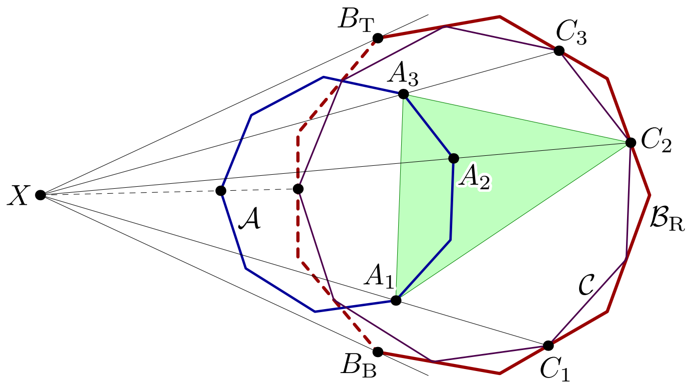


Subcase 2.1: $k > 1$

Subcase 2.1: $k > 1$.

In this subcase, all points from $\mathcal{B}_{\mathrm{R}}$ (and hence from $\mathcal{C}_{\mathrm{R}}$) move out from $\mathcal{B}$ under h, because they are the farthest points of $\mathcal{B}$ on the corresponding rays emanated from X. It remains to prove that the vertices of $\mathcal{C}_{\mathrm{L}}$ which stay in $\mathcal{B}$ under h are consecutive.

Again, let C_1, C_2, C_3 be three vertices in $\mathcal{C}_{\mathrm{L}}$ such that C_2 is between C_1 and C_3, and $h(C_1)$ and $h(C_3)$ lie in $\mathcal{B}$. Let $A_i = h(C_i)$. Then the ray XC_2 crosses the segment C_1C_3 beyond C_2, so this ray crosses A_1A_3 beyond A_2; this implies that A_2 lies in the triangle $A_1C_2A_3$, which is contained in $\mathcal{B}$.



Subcase 2.2: $k < 1$

Subcase 2.2: $k < 1$.

This case is completely similar to the previous one. All points from $\mathcal{B}_{\mathrm{L}}$ (and hence from $\mathcal{C}_{\mathrm{L}}$ move out from $\mathcal{B}$ under h, because they are the nearest points of $\mathcal{B}$ on the corresponding rays emanated from X. Assume that C_1, C_2, and C_3 are three vertices in $\mathcal{C}_{\mathrm{R}}$ such that C_2 lies between C_1 and C_3, and $h(C_1)$ and $h(C_3)$ lie in $\mathcal{B}$; let $A_i = h(C_i)$. Then A_2 lies on the segment XC_2, and the segments XA_2 and A_1A_3 cross each other. Thus A_2 lies in the triangle $A_1C_2A_3$, which is contained in $\mathcal{B}$.

Comment 1. In fact, Case 1 can be reduced to Case 2 via the following argument.

Assume that $\mathcal{A}$ and $\mathcal{C}$ are congruent. Apply to $\mathcal{A}$ a homothety centered at O_B with a factor slightly smaller than 1 to obtain a polygon $\mathcal{A}'$. With appropriately chosen factor, the vertices of $\mathcal{A}$ which were outside/inside $\mathcal{B}$ stay outside/inside it, so it suffices to prove our claim for $\mathcal{A}'$ instead of $\mathcal{A}$. And now, the polygon $\mathcal{A}'$ is a homothetic image of $\mathcal{C}$, so the arguments from Case 2 apply.

Comment 2. After the polygon $\mathcal{C}$ has been found, the rest of the solution uses only the convexity of the polygons, instead of regularity. Thus, it proves a more general statement:

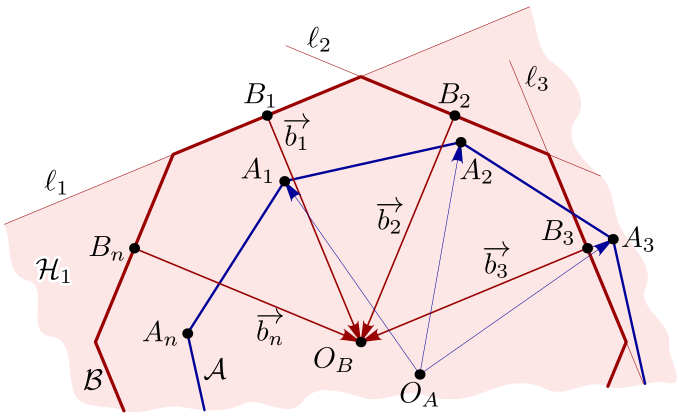
Assume that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are three convex polygons in the plane such that $\mathcal{C}$ is inscribed into $\mathcal{B}$, and $\mathcal{A}$ can be obtained from it via either translation or positive homothety. Then the vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary are consecutive.

Solution 2. Let O_A and O_B be the centers of $\mathcal{A}$ and $\mathcal{B}$, respectively. Denote $[n] = \{1, 2, \ldots, n\}$.

We start with introducing appropriate enumerations and notations. Enumerate the *sidelines* of $\mathcal{B}$ clockwise as $\ell_1, \ell_2, \ldots, \ell_n$. Denote by $\mathcal{H}_i$ the half-plane of ℓ_i that contains $\mathcal{B}$ ($\mathcal{H}_i$ is assumed to contain ℓ_i); by B_i the *midpoint* of the side belonging to ℓ_i; and finally denote $\overrightarrow{b_i} = \overrightarrow{B_iO_B}$. (As usual, the numbering is cyclic modulo n, so $\ell_{n+i} = \ell_i$ etc.)

Now, choose a vertex A_1 of $\mathcal{A}$ such that the vector $\overrightarrow{O_AA_1}$ points "mostly outside $\mathcal{H}_1$"; strictly speaking, this means that the scalar product $\langle\overrightarrow{O_AA_1}, \overrightarrow{b_1}\rangle$ is minimal. Starting from A_1, enumerate the vertices of $\mathcal{A}$ clockwise as $A_1, A_2, \ldots, A_n$; by the rotational symmetry, the choice of A_1 yields that the vector $\overrightarrow{O_AA_i}$ points "mostly outside $\mathcal{H}_i$", i.e.,

$$\langle\overrightarrow{O_AA_i}, \overrightarrow{b_i}\rangle = \min_{j\in[n]}\langle\overrightarrow{O_AA_j}, \overrightarrow{b_i}\rangle. \tag{1}$$



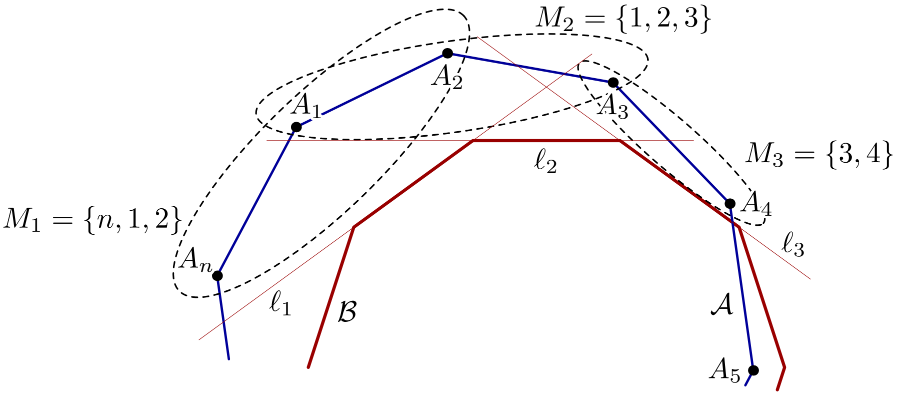
Enumerations and notations

We intend to reformulate the problem in more combinatorial terms, for which purpose we introduce the following notion. Say that a subset $I \subseteq [n]$ is *connected* if the elements of this set are consecutive in the cyclic order (in other words, if we join each i with $i+1 \mod n$ by an edge, this subset is connected in the usual graph sense). Clearly, the union of two connected subsets sharing at least one element is connected too. Next, for any half-plane $\mathcal{H}$ the indices of vertices of, say, $\mathcal{A}$ that lie in $\mathcal{H}$ form a connected set.

To access the problem, we denote

$$M = \{j \in [n] \colon A_j \notin \mathcal{B}\}, \qquad M_i = \{j \in [n] \colon A_j \notin \mathcal{H}_i\} \quad \text{for } i \in [n].$$

We need to prove that $[n] \setminus M$ is connected, which is equivalent to M being connected. On the other hand, since $\mathcal{B} = \bigcap_{i\in[n]} \mathcal{H}_i$, we have $M = \bigcup_{i\in[n]} M_i$, where the sets M_i are easier to investigate. We will utilize the following properties of these sets; the first one holds by the definition of M_i, along with the above remark.



The sets M_i

Property 1: Each set M_i is connected. □

Property 2: If M_i is nonempty, then $i \in M_i$.

Proof. Indeed, we have

$$j \in M_i \iff A_j \notin \mathcal{H}_i \iff \langle \overrightarrow{B_iA_j}, \overrightarrow{b_i} \rangle < 0 \iff \langle \overrightarrow{O_AA_j}, \overrightarrow{b_i} \rangle < \langle \overrightarrow{O_AB_i}, \overrightarrow{b_i} \rangle. \tag{2}$$

The right-hand part of the last inequality does not depend on j. Therefore, if some j lies in M_i, then by (1) so does i. □

In view of Property 2, it is useful to define the set

$$M' = \{i \in [n] : i \in M_i\} = \{i \in [n] : M_i \neq \varnothing\}.$$

Property 3: The set M' is connected.

Proof. To prove this property, we proceed on with the investigation started in (2) to write

$$i \in M' \iff A_i \in M_i \iff \langle \overrightarrow{B_iA_i}, \overrightarrow{b_i} \rangle < 0 \iff \langle \overrightarrow{O_BO_A}, \overrightarrow{b_i} \rangle < \langle \overrightarrow{O_BB_i}, \overrightarrow{b_i} \rangle + \langle \overrightarrow{A_iO_A}, \overrightarrow{b_i} \rangle.$$

The right-hand part of the obtained inequality does not depend on i, due to the rotational symmetry; denote its constant value by μ. Thus, $i \in M'$ if and only if $\langle \overrightarrow{O_BO_A}, \overrightarrow{b_i} \rangle < \mu$. This condition is in turn equivalent to the fact that B_i lies in a certain (open) half-plane whose boundary line is orthogonal to O_BO_A; thus, it defines a connected set. □

Now we can finish the solution. Since $M' \subseteq M$, we have

$$M = \bigcup_{i \in [n]} M_i = M' \cup \bigcup_{i \in [n]} M_i,$$

so M can be obtained from M' by adding all the sets M_i one by one. All these sets are connected, and each nonempty M_i contains an element of M' (namely, i). Thus their union is also connected.

Comment 3. Here we present a way in which one can come up with a solution like the one above.

Assume, for sake of simplicity, that O_A lies inside $\mathcal{B}$. Let us first put onto the plane a very small regular n-gon $\mathcal{A}'$ centered at O_A and aligned with $\mathcal{A}$; all its vertices lie inside $\mathcal{B}$. Now we start blowing it up, looking at the order in which the vertices leave $\mathcal{B}$. To go out of $\mathcal{B}$, a vertex should cross a certain side of $\mathcal{B}$ (which is hard to describe), or, equivalently, to cross at least one *sideline* of $\mathcal{B}$ — and this event is easier to describe. Indeed, the first vertex of $\mathcal{A}'$ to cross ℓ_i is the vertex A'_i (corresponding to A_i in $\mathcal{A}$); more generally, the vertices A'_j cross ℓ_i in such an order that the scalar product $\langle \overrightarrow{O_AA_j}, \overrightarrow{b_i} \rangle$ does not increase. For different indices i, these orders are just cyclic shifts of each other; and this provides some intuition for the notions and claims from Solution 2.

G7. A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c, and I_d be the incenters of the triangles DAB, ABC, BCD, and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90°$.

(Kazakhstan)

Solution. Denote by ω_a, ω_b, ω_c and ω_d the circles AI_bI_d, BI_aI_c, CI_bI_d, and DI_aI_c, let their centers be O_a, O_b, O_c and O_d, and let their radii be r_a, r_b, r_c and r_d, respectively.

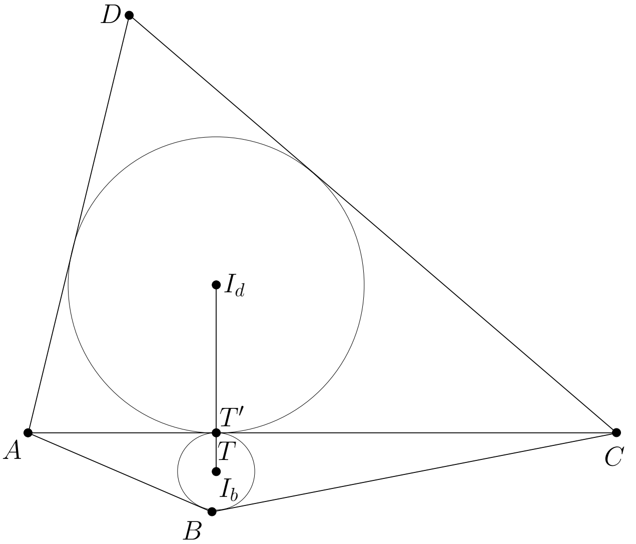
Claim 1. $I_bI_d \perp AC$ and $I_aI_c \perp BD$.

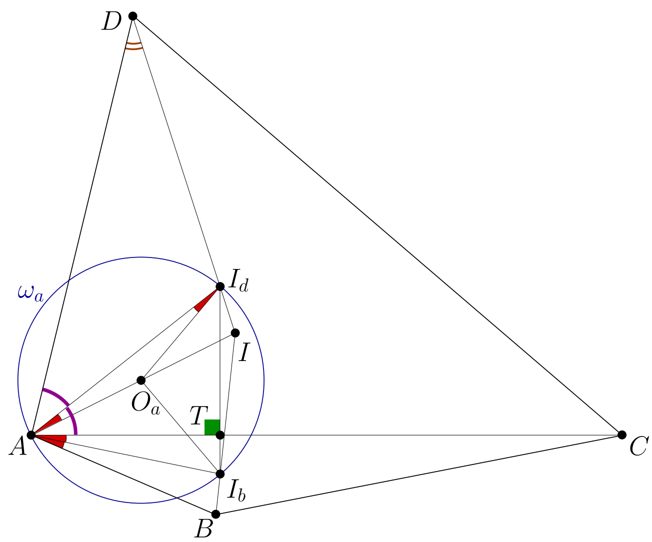
Proof. Let the incircles of triangles ABC and ACD be tangent to the line AC at T and T', respectively. (See the figure to the left.) We have $AT = \frac{AB+AC-BC}{2}$ in triangle ABC, $AT' = \frac{AD+AC-CD}{2}$ in triangle ACD, and $AB - BC = AD - CD$ in quadrilateral $ABCD$, so

$$AT = \frac{AC + AB - BC}{2} = \frac{AC + AD - CD}{2} = AT'.$$

This shows $T = T'$. As an immediate consequence, $I_bI_d \perp AC$.

The second statement can be shown analogously. □





Claim 2. The points O_a, O_b, O_c and O_d lie on the lines AI, BI, CI and DI, respectively.

Proof. By symmetry it suffices to prove the claim for O_a. (See the figure to the right above.)

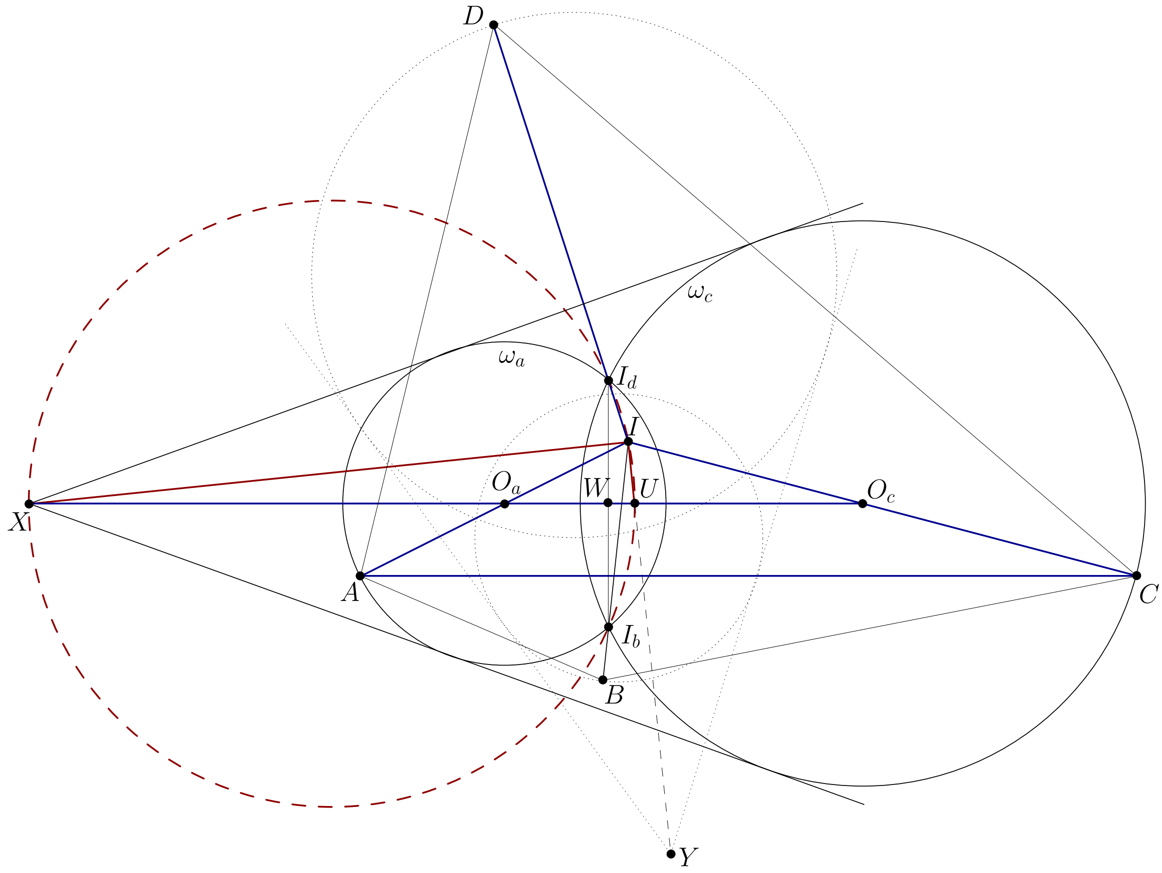
Notice first that the incircles of triangles ABC and ACD can be obtained from the incircle of the quadrilateral $ABCD$ with homothety centers B and D, respectively, and homothety factors less than 1, therefore the points I_b and I_d lie on the line segments BI and DI, respectively.

As is well-known, in every triangle the altitude and the diameter of the circumcircle starting from the same vertex are symmetric about the angle bisector. By Claim 1, in triangle AI_dI_b, the segment AT is the altitude starting from A. Since the foot T lies inside the segment I_bI_d, the circumcenter O_a of triangle AI_dI_b lies in the angle domain I_bAI_d in such a way that $\angle I_bAT = \angle O_aAI_d$. The points I_b and I_d are the incenters of triangles ABC and ACD, so the lines AI_b and AI_d bisect the angles $\angle BAC$ and $\angle CAD$, respectively. Then

$$\angle O_aAD = \angle O_aAI_d + \angle I_dAD = \angle I_bAT + \angle I_dAD = \tfrac{1}{2}\angle BAC + \tfrac{1}{2}\angle CAD = \tfrac{1}{2}\angle BAD,$$

so O_a lies on the angle bisector of $\angle BAD$, that is, on line AI. □

The point X is the external similitude center of ω_a and ω_c; let U be their internal similitude center. The points O_a and O_c lie on the perpendicular bisector of the common chord I_bI_d of ω_a and ω_c, and the two similitude centers X and U lie on the same line; by Claim 2, that line is parallel to AC.



From the similarity of the circles ω_a and ω_c, from $O_aI_b = O_aI_d = O_aA = r_a$ and $O_cI_b = O_cI_d = O_cC = r_c$, and from $AC \parallel O_aO_c$ we can see that

$$\frac{O_aX}{O_cX} = \frac{O_aU}{O_cU} = \frac{r_a}{r_c} = \frac{O_aI_b}{O_cI_b} = \frac{O_aI_d}{O_cI_d} = \frac{O_aA}{O_cC} = \frac{O_aI}{O_cI}.$$

So the points X, U, I_b, I_d, I lie on the Apollonius circle of the points O_a, O_c with ratio $r_a : r_c$. In this Apollonius circle XU is a diameter, and the lines IU and IX are respectively the internal and external bisectors of $\angle O_aIO_c = \angle AIC$, according to the angle bisector theorem. Moreover, in the Apollonius circle the diameter UX is the perpendicular bisector of I_bI_d, so the lines IX and IU are the internal and external bisectors of $\angle I_bII_d = \angle BID$, respectively.

Repeating the same argument for the points B, D instead of A, C, we get that the line IY is the internal bisector of $\angle AIC$ and the external bisector of $\angle BID$. Therefore, the lines IX and IY respectively are the internal and external bisectors of $\angle BID$, so they are perpendicular.

Comment. In fact the points O_a, O_b, O_c and O_d lie on the *line segments* AI, BI, CI and DI, respectively. For the point O_a this can be shown for example by $\angle I_dO_aA + \angle AO_aI_b = (180^\circ - 2\angle O_aAI_d) + (180^\circ - 2\angle I_bAO_a) = 360^\circ - \angle BAD = \angle ADI + \angle DIA + \angle AIB + \angle IBA > \angle I_dIA + \angle AII_b$.

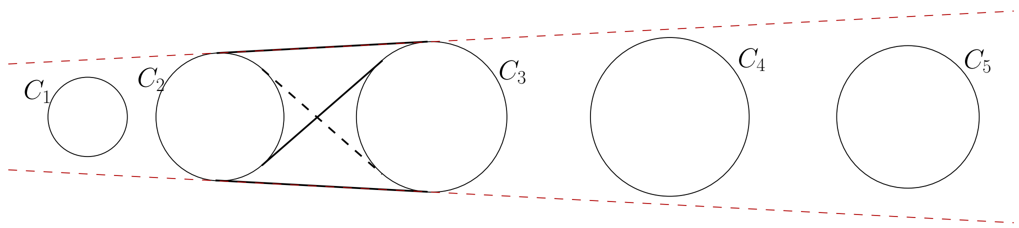
The solution also shows that the line IY passes through the point U, and analogously, IX passes through the internal similitude center of ω_b and ω_d.

G8. There are 2017 mutually external circles drawn on a blackboard, such that no two are tangent and no three share a common tangent. A *tangent segment* is a line segment that is a common tangent to two circles, starting at one tangent point and ending at the other one. Luciano is drawing tangent segments on the blackboard, one at a time, so that no tangent segment intersects any other circles or previously drawn tangent segments. Luciano keeps drawing tangent segments until no more can be drawn. Find all possible numbers of tangent segments when he stops drawing.

(Australia)

Answer: If there were n circles, there would always be exactly $3(n-1)$ segments; so the only possible answer is $3 \cdot 2017 - 3 = 6048$.

Solution 1. First, consider a particular arrangement of circles $C_1, C_2, \ldots, C_n$ where all the centers are aligned and each C_i is eclipsed from the other circles by its neighbors – for example, taking C_i with center $(i^2, 0)$ and radius $i/2$ works. Then the only tangent segments that can be drawn are between adjacent circles C_i and C_{i+1}, and exactly three segments can be drawn for each pair. So Luciano will draw exactly $3(n-1)$ segments in this case.



For the general case, start from a *final* configuration (that is, an arrangement of circles and segments in which no further segments can be drawn). The idea of the solution is to continuously resize and move the circles around the plane, one by one (in particular, making sure we never have 4 circles with a common tangent line), and show that the number of segments drawn remains constant as the picture changes. This way, we can reduce any circle/segment configuration to the particular one mentioned above, and the final number of segments must remain at $3n-3$.

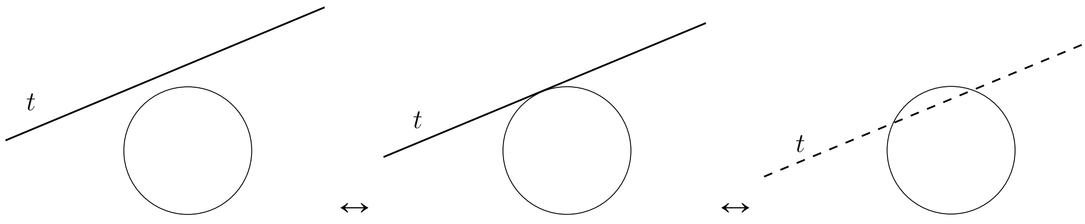
Some preliminary considerations: look at all possible tangent segments joining any two circles. A segment that is tangent to a circle A can do so in two possible orientations — it may come out of A in clockwise or counterclockwise orientation. Two segments touching the same circle with the same orientation will never intersect each other. Each pair (A, B) of circles has 4 choices of tangent segments, which can be identified by their orientations — for example, $(A+, B-)$ would be the segment which comes out of A in clockwise orientation and comes out of B in counterclockwise orientation. In total, we have $2n(n-1)$ possible segments, disregarding intersections.

Now we pick a circle C and start to continuously move and resize it, maintaining all existing tangent segments according to their identifications, including those involving C. We can keep our choice of tangent segments until the configuration reaches a *transition*. We lose nothing if we assume that C is kept at least ε units away from any other circle, where ε is a positive, fixed constant; therefore at a transition either: (1) a currently drawn tangent segment t suddenly becomes obstructed; or (2) a currently absent tangent segment t suddenly becomes unobstructed and available.

Claim. A transition can only occur when three circles C_1, C_2, C_3 are tangent to a common line ℓ containing t, in a way such that the three tangent segments lying on ℓ (joining the three circles pairwise) are not obstructed by any other circles or tangent segments (other than C_1, C_2, C_3).

Proof. Since (2) is effectively the reverse of (1), it suffices to prove the claim for (1). Suppose t has suddenly become obstructed, and let us consider two cases.

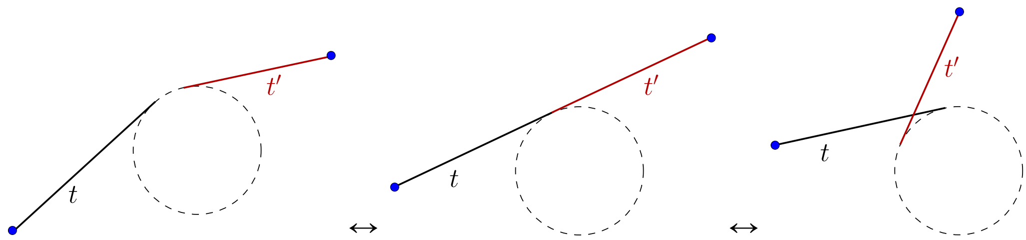
Case 1: t becomes obstructed by a circle



Then the new circle becomes the third circle tangent to ℓ, and no other circles or tangent segments are obstructing t.

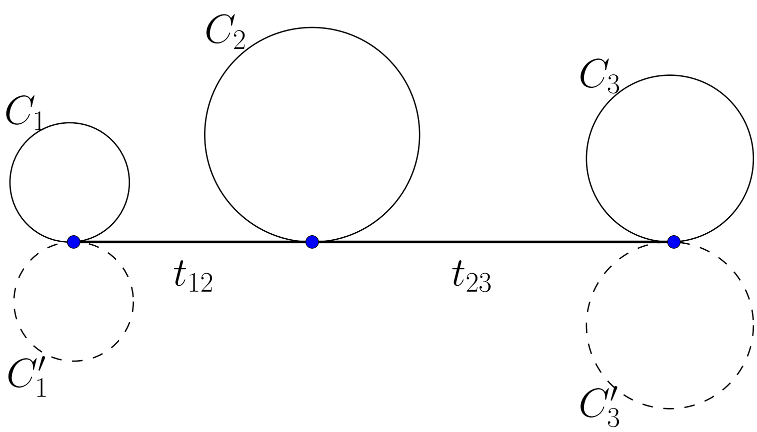
Case 2: t becomes obstructed by another tangent segment t'

When two segments t and t' first intersect each other, they must do so at a vertex of one of them. But if a vertex of t' first crossed an interior point of t, the circle associated to this vertex was already blocking t (absurd), or is about to (we already took care of this in case 1). So we only have to analyze the possibility of t and t' suddenly having a common vertex. However, if that happens, this vertex must belong to a single circle (remember we are keeping different circles at least ε units apart from each other throughout the moving/resizing process), and therefore they must have different orientations with respect to that circle.



Thus, at the transition moment, both t and t' are tangent to the same circle at a common point, that is, they must be on the same line ℓ and hence we again have three circles simultaneously tangent to ℓ. Also no other circles or tangent segments are obstructing t or t' (otherwise, they would have disappeared before this transition). □

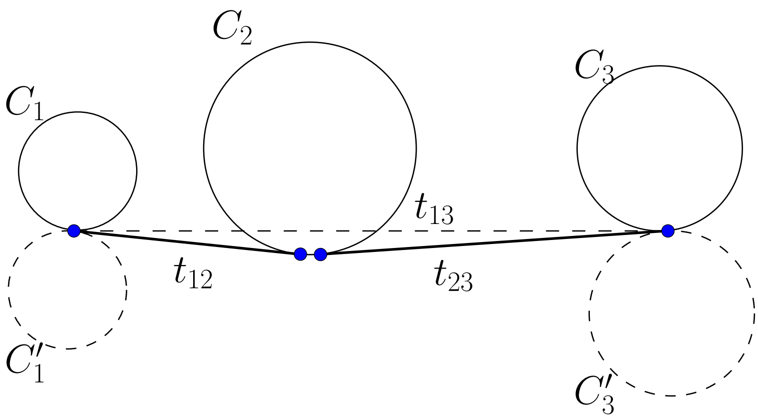
Next, we focus on the maximality of a configuration immediately before and after a transition, where three circles share a common tangent line ℓ. Let the three circles be C_1, C_2, C_3, ordered by their tangent points. The only possibly affected segments are the ones lying on ℓ, namely t_{12}, t_{23} and t_{13}. Since C_2 is in the middle, t_{12} and t_{23} must have different orientations with respect to C_2. For C_1, t_{12} and t_{13} must have the same orientation, while for C_3, t_{13} and t_{23} must have the same orientation. The figure below summarizes the situation, showing alternative positions for C_1 (namely, C_1 and C_1') and for C_3 (C_3 and C_3').



Now perturb the diagram slightly so the three circles no longer have a common tangent, while preserving the definition of t_{12}, t_{23} and t_{13} according to their identifications. First note that no other circles or tangent segments can obstruct any of these segments. Also recall that tangent segments joining the same circle at the same orientation will never obstruct each other.

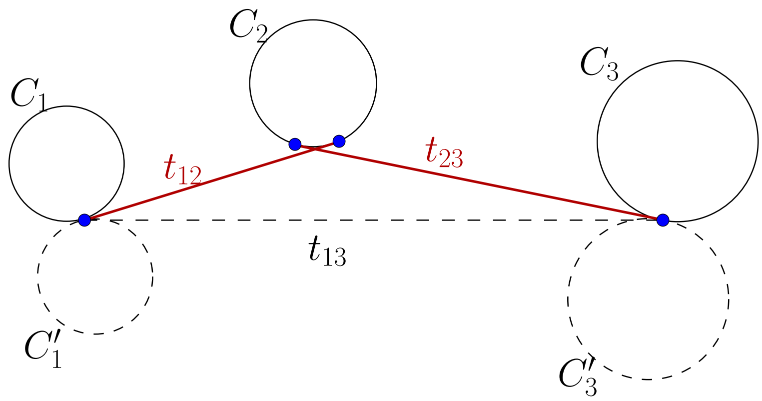
The availability of the tangent segments can now be checked using simple diagrams.

Case 1: t_{13} passes through C_2



In this case, t_{13} is not available, but both t_{12} and t_{23} are.

Case 2: t_{13} does not pass through C_2

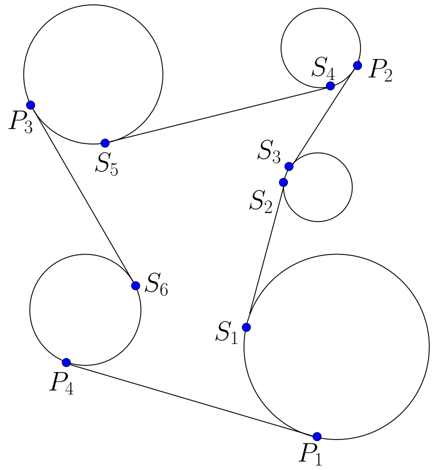


Now t_{13} is available, but t_{12} and t_{23} obstruct each other, so only one can be drawn.

In any case, exactly 2 out of these 3 segments can be drawn. Thus the maximal number of segments remains constant as we move or resize the circles, and we are done.

Solution 2. First note that all tangent segments lying on the boundary of the convex hull of the circles are always drawn since they do not intersect anything else. Now in the final picture, aside from the n circles, the blackboard is divided into regions. We can consider the picture as a plane (multi-)graph G in which the circles are the vertices and the tangent segments are the edges. The idea of this solution is to find a relation between the number of edges and the number of regions in G; then, once we prove that G is connected, we can use Euler's formula to finish the problem.

The boundary of each region consists of 1 or more (for now) simple closed curves, each made of arcs and tangent segments. The segment and the arc might meet smoothly (as in S_i, $i = 1, 2, \dots, 6$ in the figure below) or not (as in P_1, P_2, P_3, P_4; call such points *sharp corners* of the boundary). In other words, if a person walks along the border, her direction would suddenly turn an angle of π at a sharp corner.



Claim 1. The outer boundary B_1 of any internal region has at least 3 sharp corners.

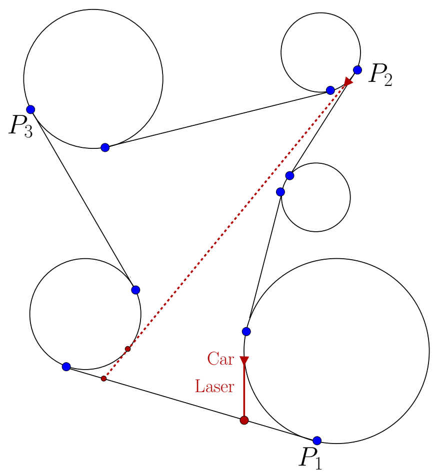
Proof. Let a person walk one lap along B_1 in the counterclockwise orientation. As she does so, she will turn *clockwise* as she moves along the circle arcs, and not turn at all when moving along the lines. On the other hand, her total rotation after one lap is 2π in the *counterclockwise* direction! Where could she be turning counterclockwise? She can only do so at sharp corners, and, even then, she turns only an angle of π there. But two sharp corners are not enough, since at least one arc must be present—so she must have gone through at least 3 sharp corners. □

Claim 2. Each internal region is simply connected, that is, has only one boundary curve.

Proof. Suppose, by contradiction, that some region has an outer boundary B_1 and inner boundaries $B_2, B_3, \ldots, B_m$ ($m \geqslant 2$). Let P_1 be one of the sharp corners of B_1.

Now consider a car starting at P_1 and traveling counterclockwise along B_1. It starts in reverse, i.e., it is initially facing the corner P_1. Due to the tangent conditions, the car may travel in a way so that its orientation only changes when it is moving along an arc. In particular, this means the car will sometimes travel forward. For example, if the car approaches a sharp corner when driving in reverse, it would continue travel forward after the corner, instead of making an immediate half-turn. This way, the orientation of the car only changes in a clockwise direction since the car always travels clockwise around each arc.

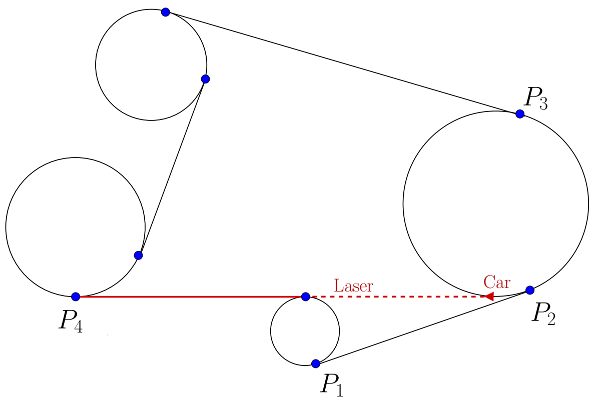
Now imagine there is a laser pointer at the front of the car, pointing directly ahead. Initially, the laser endpoint hits P_1, but, as soon as the car hits an arc, the endpoint moves clockwise around B_1. In fact, the laser endpoint must move continuously along B_1! Indeed, if the endpoint ever jumped (within B_1, or from B_1 to one of the inner boundaries), at the moment of the jump the interrupted laser would be a drawable tangent segment that Luciano missed (see figure below for an example).



Now, let P_2 and P_3 be the next two sharp corners the car goes through, after P_1 (the previous lemma assures their existence). At P_2 the car starts moving forward, and at P_3 it will start to move in reverse again. So, at P_3, the laser endpoint is at P_3 itself. So while the car moved counterclockwise between P_1 and P_3, the laser endpoint moved clockwise between P_1 and P_3. That means the laser beam itself scanned the whole region within B_1, and it should have crossed some of the inner boundaries. $\square$

Claim 3. Each region has exactly 3 sharp corners.

Proof. Consider again the car of the previous claim, with its laser still firmly attached to its front, traveling the same way as before and going through the same *consecutive* sharp corners P_1, P_2 and P_3. As we have seen, as the car goes counterclockwise from P_1 to P_3, the laser endpoint goes clockwise from P_1 to P_3, so together they cover the whole boundary. If there were a fourth sharp corner P_4, at some moment the laser endpoint would pass through it. But, since P_4 is a sharp corner, this means the car must be on the extension of a tangent segment going through P_4. Since the car is not on that segment itself (the car never goes through P_4), we would have 3 circles with a common tangent line, which is not allowed.



$\square$

We are now ready to finish the solution. Let r be the number of internal regions, and s be the number of tangent segments. Since each tangent segment contributes exactly 2 sharp corners to the diagram, and each region has exactly 3 sharp corners, we must have $2s = 3r$. Since the graph corresponding to the diagram is connected, we can use Euler's formula $n - s + r = 1$ and find $s = 3n - 3$ and $r = 2n - 2$.

Number Theory

N1. The sequence $a_0, a_1, a_2, \dots$ of positive integers satisfies

$$a_{n+1} = \begin{cases} \sqrt{a_n}, & \text{if } \sqrt{a_n} \text{ is an integer} \\ a_n + 3, & \text{otherwise} \end{cases} \qquad \text{for every } n \geqslant 0.$$

Determine all values of $a_0 > 1$ for which there is at least one number a such that $a_n = a$ for infinitely many values of n.

(South Africa)

Answer: All positive multiples of 3.

Solution. Since the value of a_{n+1} only depends on the value of a_n, if $a_n = a_m$ for two different indices n and m, then the sequence is eventually periodic. So we look for the values of a_0 for which the sequence is eventually periodic.

Claim 1. If $a_n \equiv -1 \pmod 3$, then, for all $m > n$, a_m is not a perfect square. It follows that the sequence is eventually strictly increasing, so it is not eventually periodic.

Proof. A square cannot be congruent to -1 modulo 3, so $a_n \equiv -1 \pmod 3$ implies that a_n is not a square, therefore $a_{n+1} = a_n + 3 > a_n$. As a consequence, $a_{n+1} \equiv a_n \equiv -1 \pmod 3$, so a_{n+1} is not a square either. By repeating the argument, we prove that, from a_n on, all terms of the sequence are not perfect squares and are greater than their predecessors, which completes the proof. $\square$

Claim 2. If $a_n \not\equiv -1 \pmod 3$ and $a_n > 9$ then there is an index $m > n$ such that $a_m < a_n$.

Proof. Let t^2 be the largest perfect square which is less than a_n. Since $a_n > 9$, t is at least 3. The first square in the sequence $a_n, a_n + 3, a_n + 6, \dots$ will be $(t+1)^2$, $(t+2)^2$ or $(t+3)^2$, therefore there is an index $m > n$ such that $a_m \leqslant t + 3 < t^2 < a_n$, as claimed. $\square$

Claim 3. If $a_n \equiv 0 \pmod 3$, then there is an index $m > n$ such that $a_m = 3$.

Proof. First we notice that, by the definition of the sequence, a multiple of 3 is always followed by another multiple of 3. If $a_n \in \{3, 6, 9\}$ the sequence will eventually follow the periodic pattern $3, 6, 9, 3, 6, 9, \dots$. If $a_n > 9$, let j be an index such that a_j is equal to the minimum value of the set $\{a_{n+1}, a_{n+2}, \dots\}$. We must have $a_j \leqslant 9$, otherwise we could apply Claim 2 to a_j and get a contradiction on the minimality hypothesis. It follows that $a_j \in \{3, 6, 9\}$, and the proof is complete. $\square$

Claim 4. If $a_n \equiv 1 \pmod 3$, then there is an index $m > n$ such that $a_m \equiv -1 \pmod 3$.

Proof. In the sequence, 4 is always followed by $2 \equiv -1 \pmod 3$, so the claim is true for $a_n = 4$. If $a_n = 7$, the next terms will be $10, 13, 16, 4, 2, \dots$ and the claim is also true. For $a_n \geqslant 10$, we again take an index $j > n$ such that a_j is equal to the minimum value of the set $\{a_{n+1}, a_{n+2}, \dots\}$, which by the definition of the sequence consists of non-multiples of 3. Suppose $a_j \equiv 1 \pmod 3$. Then we must have $a_j \leqslant 9$ by Claim 2 and the minimality of a_j. It follows that $a_j \in \{4, 7\}$, so $a_m = 2 < a_j$ for some $m > j$, contradicting the minimality of a_j. Therefore, we must have $a_j \equiv -1 \pmod 3$. $\square$

It follows from the previous claims that if a_0 is a multiple of 3 the sequence will eventually reach the periodic pattern $3, 6, 9, 3, 6, 9, \dots$; if $a_0 \equiv -1 \pmod 3$ the sequence will be strictly increasing; and if $a_0 \equiv 1 \pmod 3$ the sequence will be eventually strictly increasing.

So the sequence will be eventually periodic if, and only if, a_0 is a multiple of 3.

N2. Let $p \geqslant 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices $i \in \{0, 1, \ldots, p-1\}$ have been chosen. Then the following number is computed:

$$M = a_0 + 10 \cdot a_1 + \cdots + 10^{p-1} \cdot a_{p-1} = \sum_{j=0}^{p-1} a_j \cdot 10^j.$$

The goal of Eduardo is to make the number M divisible by p, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

(Morocco)

Solution. We say that a player makes the move (i, a_i) if he chooses the index i and then the element a_i of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in this move.

If $p = 2$ or $p = 5$ then Eduardo chooses $i = 0$ and $a_0 = 0$ in the first move, and wins, since, independently of the next moves, M will be a multiple of 10.

Now assume that the prime number p does not belong to $\{2, 5\}$. Eduardo chooses $i = p - 1$ and $a_{p-1} = 0$ in the first move. By Fermat's Little Theorem, $(10^{(p-1)/2})^2 = 10^{p-1} \equiv 1 \pmod{p}$, so $p \mid (10^{(p-1)/2})^2 - 1 = (10^{(p-1)/2} + 1)(10^{(p-1)/2} - 1)$. Since p is prime, either $p \mid 10^{(p-1)/2} + 1$ or $p \mid 10^{(p-1)/2} - 1$. Thus we have two cases:

Case a: $10^{(p-1)/2} \equiv -1 \pmod{p}$

In this case, for each move (i, a_i) of Fernando, Eduardo immediately makes the move $(j, a_j) = \left(i + \frac{p-1}{2}, a_i\right)$, if $0 \leqslant i \leqslant \frac{p-3}{2}$, or $(j, a_j) = \left(i - \frac{p-1}{2}, a_i\right)$, if $\frac{p-1}{2} \leqslant i \leqslant p-2$. We will have $10^j \equiv -10^i \pmod{p}$, and so $a_j \cdot 10^j = a_i \cdot 10^j \equiv -a_i \cdot 10^i \pmod{p}$. Notice that this move by Eduardo is always possible. Indeed, immediately before a move by Fernando, for any set of the type $\{r, r + (p-1)/2\}$ with $0 \leqslant r \leqslant (p-3)/2$, either no element of this set was chosen as an index by the players in the previous moves or else both elements of this set were chosen as indices by the players in the previous moves. Therefore, after each of his moves, Eduardo always makes the sum of the numbers $a_k \cdot 10^k$ corresponding to the already chosen pairs (k, a_k) divisible by p, and thus wins the game.

Case b: $10^{(p-1)/2} \equiv 1 \pmod{p}$

In this case, for each move (i, a_i) of Fernando, Eduardo immediately makes the move $(j, a_j) = \left(i + \frac{p-1}{2}, 9 - a_i\right)$, if $0 \leqslant i \leqslant \frac{p-3}{2}$, or $(j, a_j) = \left(i - \frac{p-1}{2}, 9 - a_i\right)$, if $\frac{p-1}{2} \leqslant i \leqslant p-2$. The same argument as above shows that Eduardo can always make such move. We will have $10^j \equiv 10^i \pmod{p}$, and so $a_j \cdot 10^j + a_i \cdot 10^i \equiv (a_i + a_j) \cdot 10^i = 9 \cdot 10^i \pmod{p}$. Therefore, at the end of the game, the sum of all terms $a_k \cdot 10^k$ will be congruent to

$$\sum_{i=0}^{\frac{p-3}{2}} 9 \cdot 10^i = 10^{(p-1)/2} - 1 \equiv 0 \pmod{p},$$

and Eduardo wins the game.

N3. Determine all integers $n \geqslant 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leqslant i \leqslant n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \cdots + a_{i+n-1}$$

is divisible by n. (We let $a_i = a_{i-n}$ when $i > n$.)

(Thailand)

Answer: These integers are exactly the prime numbers.

Solution. Let us first show that, if $n = ab$, with $a, b \geqslant 2$ integers, then the property in the statement of the problem does not hold. Indeed, in this case, let $a_k = a$ for $1 \leqslant k \leqslant n-1$ and $a_n = 0$. The sum $a_1 + a_2 + \cdots + a_n = a \cdot (n-1)$ is not divisible by n. Let i with $1 \leqslant i \leqslant n$ be an arbitrary index. Taking $j = b$ if $1 \leqslant i \leqslant n-b$, and $j = b+1$ if $n - b < i \leqslant n$, we have

$$a_i + a_{i+1} + \cdots + a_{i+j-1} = a \cdot b = n \equiv 0 \pmod{n}.$$

It follows that the given example is indeed a counterexample to the property of the statement.

Now let n be a prime number. Suppose by contradiction that the property in the statement of the problem does not hold. Then there are integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n such that for each i, $1 \leqslant i \leqslant n$, there is j, $1 \leqslant j \leqslant n$, for which the number $a_i + a_{i+1} + \cdots + a_{i+j-1}$ is divisible by n. Notice that, in any such case, we should have $1 \leqslant j \leqslant n-1$, since $a_1 + a_2 + \cdots + a_n$ is not divisible by n. So we may construct recursively a finite sequence of integers $0 = i_0 < i_1 < i_2 < \cdots < i_n$ with $i_{s+1} - i_s \leqslant n-1$ for $0 \leqslant s \leqslant n-1$ such that, for $0 \leqslant s \leqslant n-1$,

$$a_{i_s+1} + a_{i_s+2} + \cdots + a_{i_{s+1}} \equiv 0 \pmod{n}$$

(where we take indices modulo n). Indeed, for $0 \leqslant s < n$, we apply the previous observation to $i = i_s + 1$ in order to define $i_{s+1} = i_s + j$.

In the sequence of $n+1$ indices $i_0, i_1, i_2, \ldots, i_n$, by the pigeonhole principle, we have two distinct elements which are congruent modulo n. So there are indices r, s with $0 \leqslant r < s \leqslant n$ such that $i_s \equiv i_r \pmod{n}$ and

$$a_{i_r+1} + a_{i_r+2} + \cdots + a_{i_s} = \sum_{j=r}^{s-1} (a_{i_j+1} + a_{i_j+2} + \cdots + a_{i_{j+1}}) \equiv 0 \pmod{n}.$$

Since $i_s \equiv i_r \pmod{n}$, we have $i_s - i_r = k \cdot n$ for some positive integer k, and, since $i_{j+1} - i_j \leqslant n-1$ for $0 \leqslant j \leqslant n-1$, we have $i_s - i_r \leqslant (n-1) \cdot n$, so $k \leqslant n-1$. But in this case

$$a_{i_r+1} + a_{i_r+2} + \cdots + a_{i_s} = k \cdot (a_1 + a_2 + \cdots + a_n)$$

cannot be a multiple of n, since n is prime and neither k nor $a_1 + a_2 + \cdots + a_n$ is a multiple of n. A contradiction.

N4. Call a rational number *short* if it has finitely many digits in its decimal expansion. For a positive integer m, we say that a positive integer t is *m-tastic* if there exists a number $c \in \{1, 2, 3, \ldots, 2017\}$ such that $\dfrac{10^t - 1}{c \cdot m}$ is short, and such that $\dfrac{10^k - 1}{c \cdot m}$ is not short for any $1 \leqslant k < t$. Let $S(m)$ be the set of m-tastic numbers. Consider $S(m)$ for $m = 1, 2, \ldots$. What is the maximum number of elements in $S(m)$?

(Turkey)

Answer: 807.

Solution. First notice that $x \in \mathbb{Q}$ is short if and only if there are exponents $a, b \geqslant 0$ such that $2^a \cdot 5^b \cdot x \in \mathbb{Z}$. In fact, if x is short, then $x = \frac{n}{10^k}$ for some k and we can take $a = b = k$; on the other hand, if $2^a \cdot 5^b \cdot x = q \in \mathbb{Z}$ then $x = \frac{2^b \cdot 5^a q}{10^{a+b}}$, so x is short.

If $m = 2^a \cdot 5^b \cdot s$, with $\gcd(s, 10) = 1$, then $\frac{10^t - 1}{m}$ is short if and only if s divides $10^t - 1$. So we may (and will) suppose without loss of generality that $\gcd(m, 10) = 1$. Define

$$C = \{1 \leqslant c \leqslant 2017\colon \gcd(c, 10) = 1\}.$$

The m-tastic numbers are then precisely the smallest exponents $t > 0$ such that $10^t \equiv 1 \pmod{cm}$ for some integer $c \in C$, that is, the set of *orders of* 10 *modulo* cm. In other words,

$$S(m) = \{\operatorname{ord}_{cm}(10)\colon c \in C\}.$$

Since there are $4 \cdot 201 + 3 = 807$ numbers c with $1 \leqslant c \leqslant 2017$ and $\gcd(c, 10) = 1$, namely those such that $c \equiv 1, 3, 7, 9 \pmod{10}$,

$$|S(m)| \leqslant |C| = 807.$$

Now we find m such that $|S(m)| = 807$. Let

$$P = \{1 < p \leqslant 2017\colon p \text{ is prime}, p \neq 2, 5\}$$

and choose a positive integer α such that every $p \in P$ divides $10^\alpha - 1$ (e.g. $\alpha = \varphi(T)$, T being the product of all primes in P), and let $m = 10^\alpha - 1$.

Claim. For every $c \in C$, we have

$$\operatorname{ord}_{cm}(10) = c\alpha.$$

As an immediate consequence, this implies $|S(m)| = |C| = 807$, finishing the problem.

Proof. Obviously $\operatorname{ord}_m(10) = \alpha$. Let $t = \operatorname{ord}_{cm}(10)$. Then

$$cm \mid 10^t - 1 \quad \Longrightarrow \quad m \mid 10^t - 1 \quad \Longrightarrow \quad \alpha \mid t.$$

Hence $t = k\alpha$ for some $k \in \mathbb{Z}_{>0}$. We will show that $k = c$.

Denote by $\nu_p(n)$ the number of prime factors p in n, that is, the maximum exponent β for which $p^\beta \mid n$. For every $\ell \geqslant 1$ and $p \in P$, the Lifting the Exponent Lemma provides

$$\nu_p(10^{\ell\alpha} - 1) = \nu_p((10^\alpha)^\ell - 1) = \nu_p(10^\alpha - 1) + \nu_p(\ell) = \nu_p(m) + \nu_p(\ell),$$

so

$$\begin{aligned} cm \mid 10^{k\alpha} - 1 &\iff \forall p \in P;\ \nu_p(cm) \leqslant \nu_p(10^{k\alpha} - 1) \\ &\iff \forall p \in P;\ \nu_p(m) + \nu_p(c) \leqslant \nu_p(m) + \nu_p(k) \\ &\iff \forall p \in P;\ \nu_p(c) \leqslant \nu_p(k) \\ &\iff c \mid k. \end{aligned}$$

The first such k is $k = c$, so $\operatorname{ord}_{cm}(10) = c\alpha$. □

Comment. The Lifting the Exponent Lemma states that, for any odd prime p, any integers a, b coprime with p such that $p \mid a - b$, and any positive integer exponent n,

$$\nu_p(a^n - b^n) = \nu_p(a - b) + \nu_p(n),$$

and, for $p = 2$,

$$\nu_2(a^n - b^n) = \nu_2(a^2 - b^2) + \nu_p(n) - 1.$$

Both claims can be proved by induction on n.

N5. Find all pairs (p, q) of prime numbers with $p > q$ for which the number

$$\frac{(p+q)^{p+q}(p-q)^{p-q}-1}{(p+q)^{p-q}(p-q)^{p+q}-1}$$

is an integer.

(Japan)

Answer: The only such pair is $(3, 2)$.

Solution. Let $M = (p+q)^{p-q}(p-q)^{p+q} - 1$, which is relatively prime with both $p+q$ and $p-q$. Denote by $(p-q)^{-1}$ the multiplicative inverse of $(p-q)$ modulo M.

By eliminating the term -1 in the numerator,

$$(p+q)^{p+q}(p-q)^{p-q} - 1 \equiv (p+q)^{p-q}(p-q)^{p+q} - 1 \pmod M$$
$$(p+q)^{2q} \equiv (p-q)^{2q} \pmod M \tag{1}$$
$$\left((p+q)\cdot(p-q)^{-1}\right)^{2q} \equiv 1 \pmod M. \tag{2}$$

Case 1: $q \geqslant 5$.

Consider an arbitrary prime divisor r of M. Notice that M is odd, so $r \geqslant 3$. By (2), the multiplicative order of $\left((p+q)\cdot(p-q)^{-1}\right)$ modulo r is a divisor of the exponent $2q$ in (2), so it can be 1, 2, q or $2q$.

By Fermat's theorem, the order divides $r-1$. So, if the order is q or $2q$ then $r \equiv 1 \pmod q$. If the order is 1 or 2 then $r \mid (p+q)^2 - (p-q)^2 = 4pq$, so $r = p$ or $r = q$. The case $r = p$ is not possible, because, by applying Fermat's theorem,

$$M = (p+q)^{p-q}(p-q)^{p+q} - 1 \equiv q^{p-q}(-q)^{p+q} - 1 = \left(q^2\right)^p - 1 \equiv q^2 - 1 = (q+1)(q-1) \pmod p$$

and the last factors $q-1$ and $q+1$ are less than p and thus $p \nmid M$. Hence, all prime divisors of M are either q or of the form $kq+1$; it follows that all positive divisors of M are congruent to 0 or 1 modulo q.

Now notice that

$$M = \left((p+q)^{\frac{p-q}{2}}(p-q)^{\frac{p+q}{2}} - 1\right)\left((p+q)^{\frac{p-q}{2}}(p-q)^{\frac{p+q}{2}} + 1\right)$$

is the product of two consecutive positive odd numbers; both should be congruent to 0 or 1 modulo q. But this is impossible by the assumption $q \geqslant 5$. So, there is no solution in Case 1.

Case 2: $q = 2$.

By (1), we have $M \mid (p+q)^{2q} - (p-q)^{2q} = (p+2)^4 - (p-2)^4$, so

$$(p+2)^{p-2}(p-2)^{p+2} - 1 = M \leqslant (p+2)^4 - (p-2)^4 \leqslant (p+2)^4 - 1,$$
$$(p+2)^{p-6}(p-2)^{p+2} \leqslant 1.$$

If $p \geqslant 7$ then the left-hand side is obviously greater than 1. For $p = 5$ we have $(p+2)^{p-6}(p-2)^{p+2} = 7^{-1}\cdot 3^7$ which is also too large.

There remains only one candidate, $p = 3$, which provides a solution:

$$\frac{(p+q)^{p+q}(p-q)^{p-q}-1}{(p+q)^{p-q}(p-q)^{p+q}-1} = \frac{5^5\cdot 1^1 - 1}{5^1\cdot 1^5 - 1} = \frac{3124}{4} = 781.$$

So in Case 2 the only solution is $(p, q) = (3, 2)$.

Case 3: $q = 3$.

Similarly to Case 2, we have

$$M \mid (p+q)^{2q} - (p-q)^{2q} = 64 \cdot \left(\left(\frac{p+3}{2}\right)^6 - \left(\frac{p-3}{2}\right)^6 \right).$$

Since M is odd, we conclude that

$$M \mid \left(\frac{p+3}{2}\right)^6 - \left(\frac{p-3}{2}\right)^6$$

and

$$(p+3)^{p-3}(p-3)^{p+3} - 1 = M \leqslant \left(\frac{p+3}{2}\right)^6 - \left(\frac{p-3}{2}\right)^6 \leqslant \left(\frac{p+3}{2}\right)^6 - 1,$$
$$64(p+3)^{p-9}(p-3)^{p+3} \leqslant 1.$$

If $p \geqslant 11$ then the left-hand side is obviously greater than 1. If $p = 7$ then the left-hand side is $64 \cdot 10^{-2} \cdot 4^{10} > 1$. If $p = 5$ then the left-hand side is $64 \cdot 8^{-4} \cdot 2^8 = 2^2 > 1$. Therefore, there is no solution in Case 3.

N6. Find the smallest positive integer n, or show that no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \qquad \text{and} \qquad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

are integers.

(Singapore)

Answer: $n = 3$.

Solution 1. For $n = 1$, $a_1 \in \mathbb{Z}_{>0}$ and $\frac{1}{a_1} \in \mathbb{Z}_{>0}$ if and only if $a_1 = 1$. Next we show that

(i) *There are finitely many $(x, y) \in \mathbb{Q}^2_{>0}$ satisfying $x + y \in \mathbb{Z}$ and $\frac{1}{x} + \frac{1}{y} \in \mathbb{Z}$*

Write $x = \frac{a}{b}$ and $y = \frac{c}{d}$ with $a, b, c, d \in \mathbb{Z}_{>0}$ and $\gcd(a, b) = \gcd(c, d) = 1$. Then $x + y \in \mathbb{Z}$ and $\frac{1}{x} + \frac{1}{y} \in \mathbb{Z}$ is equivalent to the two divisibility conditions

$$bd \mid ad + bc \quad (1) \qquad \text{and} \qquad ac \mid ad + bc \quad (2)$$

Condition (1) implies that $d \mid ad + bc \iff d \mid bc \iff d \mid b$ since $\gcd(c, d) = 1$. Still from (1) we get $b \mid ad + bc \iff b \mid ad \iff b \mid d$ since $\gcd(a, b) = 1$. From $b \mid d$ and $d \mid b$ we have $b = d$.

An analogous reasoning with condition (2) shows that $a = c$. Hence $x = \frac{a}{b} = \frac{c}{d} = y$, i.e., the problem amounts to finding all $x \in \mathbb{Q}_{>0}$ such that $2x \in \mathbb{Z}_{>0}$ and $\frac{2}{x} \in \mathbb{Z}_{>0}$. Letting $n = 2x \in \mathbb{Z}_{>0}$, we have that $\frac{2}{x} \in \mathbb{Z}_{>0} \iff \frac{4}{n} \in \mathbb{Z}_{>0} \iff n = 1$, 2 or 4, and there are finitely many solutions, namely $(x, y) = (\frac{1}{2}, \frac{1}{2})$, $(1, 1)$ or $(2, 2)$.

(ii) *There are infinitely many triples $(x, y, z) \in \mathbb{Q}^2_{>0}$ such that $x + y + z \in \mathbb{Z}$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \in \mathbb{Z}$.*
We will look for triples such that $x + y + z = 1$, so we may write them in the form

$$(x, y, z) = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right) \qquad \text{with } a, b, c \in \mathbb{Z}_{>0}$$

We want these to satisfy

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a+b+c}{a} + \frac{a+b+c}{b} + \frac{a+b+c}{c} \in \mathbb{Z} \iff \frac{b+c}{a} + \frac{a+c}{b} + \frac{a+b}{c} \in \mathbb{Z}$$

Fixing $a = 1$, it suffices to find infinitely many pairs $(b, c) \in \mathbb{Z}^2_{>0}$ such that

$$\frac{1}{b} + \frac{1}{c} + \frac{c}{b} + \frac{b}{c} = 3 \iff b^2 + c^2 - 3bc + b + c = 0 \qquad (*)$$

To show that equation $(*)$ has infinitely many solutions, we use *Vieta jumping* (also known as *root flipping*): starting with $b = 2$, $c = 3$, the following algorithm generates infinitely many solutions. Let $c \geqslant b$, and view $(*)$ as a quadratic equation in b for c fixed:

$$b^2 - (3c - 1) \cdot b + (c^2 + c) = 0 \qquad (**)$$

Then there exists another root $b_0 \in \mathbb{Z}$ of $(**)$ which satisfies $b + b_0 = 3c - 1$ and $b \cdot b_0 = c^2 + c$. Since $c \geqslant b$ by assumption,

$$b_0 = \frac{c^2 + c}{b} \geqslant \frac{c^2 + c}{c} > c$$

Hence from the solution (b, c) we obtain another one (c, b_0) with $b_0 > c$, and we can then "jump" again, this time with c as the "variable" in the quadratic $(*)$. This algorithm will generate an infinite sequence of distinct solutions, whose first terms are

$(2, 3)$, $(3, 6)$, $(6, 14)$, $(14, 35)$, $(35, 90)$, $(90, 234)$, $(234, 611)$, $(611, 1598)$, $(1598, 4182)$, ...

Comment. Although not needed for solving this problem, we may also explicitly solve the recursion given by the Vieta jumping. Define the sequence (x_n) as follows:

$$x_0 = 2, \quad x_1 = 3 \qquad \text{and} \qquad x_{n+2} = 3x_{n+1} - x_n - 1 \text{ for } n \geqslant 0$$

Then the triple

$$(x, y, z) = \left(\frac{1}{1 + x_n + x_{n+1}}, \frac{x_n}{1 + x_n + x_{n+1}}, \frac{x_{n+1}}{1 + x_n + x_{n+1}}\right)$$

satisfies the problem conditions for all $n \in \mathbb{N}$. It is easy to show that $x_n = F_{2n+1} + 1$, where F_n denotes the n-th term of the Fibonacci sequence ($F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geqslant 0$).

Solution 2. Call the n-tuples $(a_1, a_2, \ldots, a_n) \in \mathbb{Q}_{>0}^n$ satisfying the conditions of the problem statement *good*, and those for which

$$f(a_1, \ldots, a_n) \overset{\text{def}}{=} (a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

is an integer *pretty*. Then good n-tuples are pretty, and if $(b_1, \ldots, b_n)$ is pretty then

$$\left(\frac{b_1}{b_1 + b_2 + \cdots + b_n}, \frac{b_2}{b_1 + b_2 + \cdots + b_n}, \ldots, \frac{b_n}{b_1 + b_2 + \cdots + b_n}\right)$$

is good since the sum of its components is 1, and the sum of the reciprocals of its components equals $f(b_1, \ldots, b_n)$. We declare pretty n-tuples proportional to each other *equivalent* since they are precisely those which give rise to the same good n-tuple. Clearly, each such equivalence class contains exactly one n-tuple of positive integers having no common prime divisors. Call such n-tuple a *primitive* pretty tuple. Our task is to find infinitely many primitive pretty n-tuples.

For $n = 1$, there is clearly a single primitive 1-tuple. For $n = 2$, we have $f(a, b) = \frac{(a+b)^2}{ab}$, which can be integral (for coprime $a, b \in \mathbb{Z}_{>0}$) only if $a = b = 1$ (see for instance (i) in the first solution).

Now we construct infinitely many primitive pretty triples for $n = 3$. Fix $b, c, k \in \mathbb{Z}_{>0}$; we will try to find sufficient conditions for the existence of an $a \in \mathbb{Q}_{>0}$ such that $f(a, b, c) = k$. Write $\sigma = b + c$, $\tau = bc$. From $f(a, b, c) = k$, we have that a should satisfy the quadratic equation

$$a^2 \cdot \sigma + a \cdot (\sigma^2 - (k-1)\tau) + \sigma\tau = 0 \tag{1}$$

whose discriminant is

$$\Delta = (\sigma^2 - (k-1)\tau)^2 - 4\sigma^2\tau = ((k+1)\tau - \sigma^2)^2 - 4k\tau^2.$$

We need it to be a square of an integer, say, $\Delta = M^2$ for some $M \in \mathbb{Z}$, i.e., we want

$$((k+1)\tau - \sigma^2)^2 - M^2 = 2k \cdot 2\tau^2$$

so that it suffices to set

$$(k+1)\tau - \sigma^2 = \tau^2 + k, \qquad M = \tau^2 - k.$$

The first relation reads $\sigma^2 = (\tau - 1)(k - \tau)$, so if b and c satisfy

$$\tau - 1 \mid \sigma^2 \qquad \text{i.e.} \qquad bc - 1 \mid (b+c)^2 \tag{2}$$

then $k = \frac{\sigma^2}{\tau - 1} + \tau$ will be integral, and we find rational solutions to (1), namely

$$a = \frac{\sigma}{\tau - 1} = \frac{b + c}{bc - 1} \qquad \text{or} \qquad a = \frac{\tau^2 - \tau}{\sigma} = \frac{bc \cdot (bc - 1)}{b + c}$$

We can now find infinitely many pairs (b, c) satisfying (2) by Vieta jumping. For example, if we impose

$$(b+c)^2 = 5 \cdot (bc-1)$$

then all pairs $(b, c) = (v_i, v_{i+1})$ satisfy the above condition, where

$$v_1 = 2,\ v_2 = 3, \qquad v_{i+2} = 3v_{i+1} - v_i \quad \text{for } i \geqslant 0$$

For $(b, c) = (v_i, v_{i+1})$, one of the solutions to (1) will be $a = (b+c)/(bc-1) = 5/(b+c) = 5/(v_i + v_{i+1})$. Then the pretty triple (a, b, c) will be equivalent to the integral pretty triple

$$(5, v_i(v_i + v_{i+1}), v_{i+1}(v_i + v_{i+1}))$$

After possibly dividing by 5, we obtain infinitely many primitive pretty triples, as required.

Comment. There are many other infinite series of $(b, c) = (v_i, v_{i+1})$ with $bc - 1 \mid (b+c)^2$. Some of them are:

$$\begin{array}{llll} v_1 = 1, & v_2 = 3, & v_{i+1} = 6v_i - v_{i-1}, & (v_i + v_{i+1})^2 = 8 \cdot (v_i v_{i+1} - 1); \\ v_1 = 1, & v_2 = 2, & v_{i+1} = 7v_i - v_{i-1}, & (v_i + v_{i+1})^2 = 9 \cdot (v_i v_{i+1} - 1); \\ v_1 = 1, & v_2 = 5, & v_{i+1} = 7v_i - v_{i-1}, & (v_i + v_{i+1})^2 = 9 \cdot (v_i v_{i+1} - 1) \end{array}$$

(the last two are in fact one sequence prolonged in two possible directions).

N7. Say that an ordered pair (x, y) of integers is an *irreducible lattice point* if x and y are relatively prime. For any finite set S of irreducible lattice points, show that there is a homogenous polynomial in two variables, $f(x, y)$, with integer coefficients, of degree at least 1, such that $f(x, y) = 1$ for each (x, y) in the set S.

Note: A homogenous polynomial of degree n is any nonzero polynomial of the form

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n.$$

(U.S.A.)

Solution 1. First of all, we note that finding a homogenous polynomial $f(x, y)$ such that $f(x, y) = \pm 1$ is enough, because we then have $f^2(x, y) = 1$. Label the irreducible lattice points (x_1, y_1) through (x_n, y_n). If any two of these lattice points (x_i, y_i) and (x_j, y_j) lie on the same line through the origin, then $(x_j, y_j) = (-x_i, -y_i)$ because both of the points are irreducible. We then have $f(x_j, y_j) = \pm f(x_i, y_i)$ whenever f is homogenous, so we can assume that no two of the lattice points are collinear with the origin by ignoring the extra lattice points.

Consider the homogenous polynomials $\ell_i(x, y) = y_ix - x_iy$ and define

$$g_i(x, y) = \prod_{j \neq i} \ell_j(x, y).$$

Then $\ell_i(x_j, y_j) = 0$ if and only if $j = i$, because there is only one lattice point on each line through the origin. Thus, $g_i(x_j, y_j) = 0$ for all $j \neq i$. Define $a_i = g_i(x_i, y_i)$, and note that $a_i \neq 0$.

Note that $g_i(x, y)$ is a degree $n - 1$ polynomial with the following two properties:

1. $g_i(x_j, y_j) = 0$ if $j \neq i$.
2. $g_i(x_i, y_i) = a_i$.

For any $N \geqslant n - 1$, there also exists a polynomial of degree N with the same two properties. Specifically, let $I_i(x, y)$ be a degree 1 homogenous polynomial such that $I_i(x_i, y_i) = 1$, which exists since (x_i, y_i) is irreducible. Then $I_i(x, y)^{N-(n-1)}g_i(x, y)$ satisfies both of the above properties and has degree N.

We may now reduce the problem to the following claim:

Claim: For each positive integer a, there is a homogenous polynomial $f_a(x, y)$, with integer coefficients, of degree at least 1, such that $f_a(x, y) \equiv 1 \pmod{a}$ for all relatively prime (x, y).

To see that this claim solves the problem, take a to be the least common multiple of the numbers a_i $(1 \leqslant i \leqslant n)$. Take f_a given by the claim, choose some power $f_a(x, y)^k$ that has degree at least $n - 1$, and subtract appropriate multiples of the g_i constructed above to obtain the desired polynomial.

We prove the claim by factoring a. First, if a is a power of a prime $(a = p^k)$, then we may choose either:

- $f_a(x, y) = (x^{p-1} + y^{p-1})^{\phi(a)}$ if p is odd;
- $f_a(x, y) = (x^2 + xy + y^2)^{\phi(a)}$ if $p = 2$.

Now suppose a is any positive integer, and let $a = q_1q_2 \cdots q_k$, where the q_i are prime powers, pairwise relatively prime. Let f_{q_i} be the polynomials just constructed, and let F_{q_i} be powers of these that all have the same degree. Note that

$$\frac{a}{q_i}F_{q_i}(x, y) \equiv \frac{a}{q_i} \pmod{a}$$

for any relatively prime x, y. By Bézout's lemma, there is an integer linear combination of the $\frac{a}{q_i}$ that equals 1. Thus, there is a linear combination of the F_{q_i} such that $F_{q_i}(x, y) \equiv 1 \pmod{a}$ for any relatively prime (x, y); and this polynomial is homogenous because all the F_{q_i} have the same degree.

Solution 2. As in the previous solution, label the irreducible lattice points $(x_1, y_1), \ldots, (x_n, y_n)$ and assume without loss of generality that no two of the points are collinear with the origin. We induct on n to construct a homogenous polynomial $f(x, y)$ such that $f(x_i, y_i) = 1$ for all $1 \leqslant i \leqslant n$.

If $n = 1$: Since x_1 and y_1 are relatively prime, there exist some integers c, d such that $cx_1 + dy_1 = 1$. Then $f(x, y) = cx + dy$ is suitable.

If $n \geqslant 2$: By the induction hypothesis we already have a homogeneous polynomial $g(x, y)$ with $g(x_1, y_1) = \ldots = g(x_{n-1}, y_{n-1}) = 1$. Let $j = \deg g$,

$$g_n(x, y) = \prod_{k=1}^{n-1} (y_k x - x_k y),$$

and $a_n = g_n(x_n, y_n)$. By assumption, $a_n \neq 0$. Take some integers c, d such that $cx_n + dy_n = 1$. We will construct $f(x, y)$ in the form

$$f(x, y) = g(x, y)^K - C \cdot g_n(x, y) \cdot (cx + dy)^L,$$

where K and L are some positive integers and C is some integer. We assume that $L = Kj - n + 1$ so that f is homogenous.

Due to $g(x_1, y_1) = \ldots = g(x_{n-1}, y_{n-1}) = 1$ and $g_n(x_1, y_1) = \ldots = g_n(x_{n-1}, y_{n-1}) = 0$, the property $f(x_1, y_1) = \ldots = f(x_{n-1}, y_{n-1}) = 1$ is automatically satisfied with any choice of K, L, and C.

Furthermore,

$$f(x_n, y_n) = g(x_n, y_n)^K - C \cdot g_n(x_n, y_n) \cdot (cx_n + dy_n)^L = g(x_n, y_n)^K - Ca_n.$$

If we have an exponent K such that $g(x_n, y_n)^K \equiv 1 \pmod{a_n}$, then we may choose C such that $f(x_n, y_n) = 1$. We now choose such a K.

Consider an arbitrary prime divisor p of a_n. By

$$p \mid a_n = g_n(x_n, y_n) = \prod_{k=1}^{n-1} (y_k x_n - x_k y_n),$$

there is some $1 \leqslant k < n$ such that $x_k y_n \equiv x_n y_k \pmod{p}$. We first show that $x_k x_n$ or $y_k y_n$ is relatively prime with p. This is trivial in the case $x_k y_n \equiv x_n y_k \not\equiv 0 \pmod{p}$. In the other case, we have $x_k y_n \equiv x_n y_k \equiv 0 \pmod{p}$, If, say $p \mid x_k$, then $p \nmid y_k$ because (x_k, y_k) is irreducible, so $p \mid x_n$; then $p \nmid y_n$ because (x_k, y_k) is irreducible. In summary, $p \mid x_k$ implies $p \nmid y_k y_n$. Similarly, $p \mid y_n$ implies $p \nmid x_k x_n$.

By the homogeneity of g we have the congruences

$$x_k^d \cdot g(x_n, y_n) = g(x_k x_n, x_k y_n) \equiv g(x_k x_n, y_k x_n) = x_n^d \cdot g(x_k, y_k) = x_n^d \pmod{p} \tag{1.1}$$

and

$$y_k^d \cdot g(x_n, y_n) = g(y_k x_n, y_k y_n) \equiv g(x_k y_n, y_k y_n) = y_n^d \cdot g(x_k, y_k) = y_n^d \pmod{p}. \tag{1.2}$$

If $p \nmid x_k x_n$, then take the $(p-1)^{st}$ power of (1.1); otherwise take the $(p-1)^{st}$ power of (1.2); by Fermat's theorem, in both cases we get

$$g(x_n, y_n)^{p-1} \equiv 1 \pmod{p}.$$

If $p^\alpha \mid m$, then we have

$$g(x_n, y_n)^{p^{\alpha-1}(p-1)} \equiv 1 \pmod{p^\alpha},$$

which implies that the exponent $K = n \cdot \varphi(a_n)$, which is a multiple of all $p^{\alpha-1}(p-1)$, is a suitable choice. (The factor n is added only so that $K \geqslant n$ and so $L > 0$.)

Comment. It is possible to show that there is no constant C for which, given any two irreducible lattice points, there is some homogenous polynomial f of degree at most C with integer coefficients that takes the value 1 on the two points. Indeed, if one of the points is $(1,0)$ and the other is (a,b), the polynomial $f(x,y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$ should satisfy $a_0 = 1$, and so $a^n \equiv 1 \pmod{b}$. If $a = 3$ and $b = 2^k$ with $k \geqslant 3$, then $n \geqslant 2^{k-2}$. If we choose $2^{k-2} > C$, this gives a contradiction.

N8. Let p be an odd prime number and $\mathbb{Z}_{>0}$ be the set of positive integers. Suppose that a function $f\colon \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \{0,1\}$ satisfies the following properties:

- $f(1,1) = 0$;
- $f(a,b) + f(b,a) = 1$ for any pair of relatively prime positive integers (a,b) not both equal to 1;
- $f(a+b,b) = f(a,b)$ for any pair of relatively prime positive integers (a,b).

Prove that

$$\sum_{n=1}^{p-1} f(n^2,p) \geqslant \sqrt{2p} - 2.$$

(Italy)

Solution 1. Denote by $\mathbb{A}$ the set of all pairs of coprime positive integers. Notice that for every $(a,b) \in \mathbb{A}$ there exists a pair $(u,v) \in \mathbb{Z}^2$ with $ua + vb = 1$. Moreover, if (u_0, v_0) is one such pair, then all such pairs are of the form $(u,v) = (u_0 + kb, v_0 - ka)$, where $k \in \mathbb{Z}$. So there exists a unique such pair (u,v) with $-b/2 < u \leqslant b/2$; we denote this pair by $(u,v) = g(a,b)$.

Lemma. Let $(a,b) \in \mathbb{A}$ and $(u,v) = g(a,b)$. Then $f(a,b) = 1 \iff u > 0$.

Proof. We induct on $a+b$. The base case is $a+b=2$. In this case, we have that $a=b=1$, $g(a,b) = g(1,1) = (0,1)$ and $f(1,1) = 0$, so the claim holds.

Assume now that $a+b>2$, and so $a \neq b$, since a and b are coprime. Two cases are possible.

Case 1: $a > b$.

Notice that $g(a-b,b) = (u, v+u)$, since $u(a-b) + (v+u)b = 1$ and $u \in (-b/2, b/2]$. Thus $f(a,b) = 1 \iff f(a-b,b) = 1 \iff u > 0$ by the induction hypothesis.

Case 2: $a < b$. (Then, clearly, $b \geqslant 2$.)

Now we estimate v. Since $vb = 1 - ua$, we have

$$1 + \frac{ab}{2} > vb \geqslant 1 - \frac{ab}{2}, \quad \text{so} \quad \frac{1+a}{2} \geqslant \frac{1}{b} + \frac{a}{2} > v \geqslant \frac{1}{b} - \frac{a}{2} > -\frac{a}{2}.$$

Thus $1 + a > 2v > -a$, so $a \geqslant 2v > -a$, hence $a/2 \geqslant v > -a/2$, and thus $g(b,a) = (v,u)$.

Observe that $f(a,b) = 1 \iff f(b,a) = 0 \iff f(b-a,a) = 0$. We know from Case 1 that $g(b-a,a) = (v, u+v)$. We have $f(b-a,a) = 0 \iff v \leqslant 0$ by the inductive hypothesis. Then, since $b > a \geqslant 1$ and $ua + vb = 1$, we have $v \leqslant 0 \iff u > 0$, and we are done. $\square$

The Lemma proves that, for all $(a,b) \in \mathbb{A}$, $f(a,b) = 1$ if and only if the inverse of a modulo b, taken in $\{1, 2, \ldots, b-1\}$, is at most $b/2$. Then, for any odd prime p and integer n such that $n \not\equiv 0 \pmod{p}$, $f(n^2,p) = 1$ iff the inverse of $n^2 \bmod p$ is less than $p/2$. Since $\{n^2 \bmod p \colon 1 \leqslant n \leqslant p-1\} = \{n^{-2} \bmod p \colon 1 \leqslant n \leqslant p-1\}$, including multiplicities (two for each quadratic residue in each set), we conclude that the desired sum is twice the number of quadratic residues that are less than $p/2$, i.e.,

$$\sum_{n=1}^{p-1} f(n^2,p) = 2\left|\left\{k \colon 1 \leqslant k \leqslant \frac{p-1}{2} \text{ and } k^2 \bmod p < \frac{p}{2}\right\}\right|. \tag{1}$$

Since the number of perfect squares in the interval $[1, p/2)$ is $\lfloor\sqrt{p/2}\rfloor > \sqrt{p/2} - 1$, we conclude that

$$\sum_{n=1}^{p-1} f(n^2,p) > 2\left(\sqrt{\frac{p}{2}} - 1\right) = \sqrt{2p} - 2.$$

Solution 2. We provide a different proof for the Lemma. For this purpose, we use continued fractions to find $g(a,b)=(u,v)$ explicitly.

The function f is completely determined on $\mathbb{A}$ by the following

Claim. Represent a/b as a continued fraction; that is, let a_0 be an integer and $a_1,\dots,a_k$ be positive integers such that $a_k \geqslant 2$ and

$$\frac{a}{b}=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\cdots+\cfrac{1}{a_k}}}}=[a_0;a_1,a_2,\dots,a_k].$$

Then $f(a,b)=0 \iff k$ is even.

Proof. We induct on b. If $b=1$, then $a/b=[a]$ and $k=0$. Then, for $a\geqslant 1$, an easy induction shows that $f(a,1)=f(1,1)=0$.

Now consider the case $b>1$. Perform the Euclidean division $a=qb+r$, with $0\leqslant r<b$. We have $r\neq 0$ because $\gcd(a,b)=1$. Hence

$$f(a,b)=f(r,b)=1-f(b,r),\quad \frac{a}{b}=[q;a_1,\dots,a_k],\quad\text{and}\quad \frac{b}{r}=[a_1;a_2,\dots,a_k].$$

Then the number of terms in the continued fraction representations of a/b and b/r differ by one. Since $r<b$, the inductive hypothesis yields

$$f(b,r)=0 \iff k-1 \text{ is even},$$

and thus

$$f(a,b)=0 \iff f(b,r)=1 \iff k-1 \text{ is odd} \iff k \text{ is even}. \qquad \square$$

Now we use the following well-known properties of continued fractions to prove the Lemma:

Let p_i and q_i be coprime positive integers with $[a_0;a_1,a_2,\dots,a_i]=p_i/q_i$, with the notation borrowed from the Claim. In particular, $a/b=[a_0;a_1,a_2,\dots,a_k]=p_k/q_k$. Assume that $k>0$ and define $q_{-1}=0$ if necessary. Then

- $q_k=a_kq_{k-1}+q_{k-2}$, and
- $aq_{k-1}-bp_{k-1}=p_kq_{k-1}-q_kp_{k-1}=(-1)^{k-1}$.

Assume that $k>0$. Then $a_k\geqslant 2$, and

$$b=q_k=a_kq_{k-1}+q_{k-2}\geqslant a_kq_{k-1}\geqslant 2q_{k-1} \implies q_{k-1}\leqslant\frac{b}{2},$$

with strict inequality for $k>1$, and

$$(-1)^{k-1}q_{k-1}a+(-1)^kp_{k-1}b=1.$$

Now we finish the proof of the Lemma. It is immediate for $k=0$. If $k=1$, then $(-1)^{k-1}=1$, so

$$-b/2<0\leqslant(-1)^{k-1}q_{k-1}\leqslant b/2.$$

If $k>1$, we have $q_{k-1}<b/2$, so

$$-b/2<(-1)^{k-1}q_{k-1}<b/2.$$

Thus, for any $k>0$, we find that $g(a,b)=((-1)^{k-1}q_{k-1},(-1)^kp_{k-1})$, and so

$$f(a,b)=1 \iff k \text{ is odd} \iff u=(-1)^{k-1}q_{k-1}>0.$$

Comment 1. The Lemma can also be established by observing that f is uniquely defined on $\mathbb{A}$, defining $f_1(a,b) = 1$ if $u > 0$ in $g(a,b) = (u,v)$ and $f_1(a,b) = 0$ otherwise, and verifying that f_1 satisfies all the conditions from the statement.

It seems that the main difficulty of the problem is in conjecturing the Lemma.

Comment 2. The case $p \equiv 1 \pmod 4$ is, in fact, easier than the original problem. We have, in general, for $1 \leqslant a \leqslant p-1$,

$$f(a,p) = 1-f(p,a) = 1-f(p-a,a) = f(a,p-a) = f(a+(p-a),p-a) = f(p,p-a) = 1-f(p-a,p).$$

If $p \equiv 1 \pmod 4$, then a is a quadratic residue modulo p if and only if $p-a$ is a quadratic residue modulo p. Therefore, denoting by r_k (with $1 \leqslant r_k \leqslant p-1$) the remainder of the division of k^2 by p, we get

$$\sum_{n=1}^{p-1} f(n^2,p) = \sum_{n=1}^{p-1} f(r_n,p) = \frac{1}{2}\sum_{n=1}^{p-1}(f(r_n,p)+f(p-r_n,p)) = \frac{p-1}{2}.$$

Comment 3. The estimate for the sum $\sum_{n=1}^{p} f(n^2,p)$ can be improved by refining the final argument in Solution 1. In fact, one can prove that

$$\sum_{n=1}^{p-1} f(n^2,p) \geqslant \frac{p-1}{16}.$$

By counting the number of perfect squares in the intervals $[kp,(k+1/2)p)$, we find that

$$\sum_{n=1}^{p-1} f(n^2,p) = \sum_{k=0}^{p-1}\left(\left\lfloor \sqrt{\left(k+\frac{1}{2}\right)p}\right\rfloor - \left\lfloor\sqrt{kp}\right\rfloor\right). \tag{2}$$

Each summand of (2) is non-negative. We now estimate the number of positive summands. Suppose that a summand is zero, i.e.,

$$\left\lfloor \sqrt{\left(k+\frac{1}{2}\right)p}\right\rfloor = \left\lfloor\sqrt{kp}\right\rfloor =: q.$$

Then both of the numbers kp and $kp+p/2$ lie within the interval $[q^2,(q+1)^2)$. Hence

$$\frac{p}{2} < (q+1)^2 - q^2,$$

which implies

$$q \geqslant \frac{p-1}{4}.$$

Since $q \leqslant \sqrt{kp}$, if the k^{th} summand of (2) is zero, then

$$k \geqslant \frac{q^2}{p} \geqslant \frac{(p-1)^2}{16p} > \frac{p-2}{16} \implies k \geqslant \frac{p-1}{16}.$$

So at least the first $\lceil \frac{p-1}{16} \rceil$ summands (from $k=0$ to $k = \lceil \frac{p-1}{16} \rceil - 1$) are positive, and the result follows.

Comment 4. The bound can be further improved by using different methods. In fact, we prove that

$$\sum_{n=1}^{p-1} f(n^2,p) \geqslant \frac{p-3}{4}.$$

To that end, we use the Legendre symbol

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } a \text{ is a nonzero quadratic residue mod } p \\ -1 & \text{otherwise.} \end{cases}$$

We start with the following Claim, which tells us that there are not too many consecutive quadratic residues or consecutive quadratic non-residues.

Claim. $\sum_{n=1}^{p-1}\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1$.

Proof. We have $\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = \left(\frac{n(n+1)}{p}\right)$. For $1 \leqslant n \leqslant p-1$, we get that $n(n+1) \equiv n^2(1+n^{-1}) \pmod p$, hence $\left(\frac{n(n+1)}{p}\right) = \left(\frac{1+n^{-1}}{p}\right)$. Since $\{1+n^{-1} \bmod p\colon 1 \leqslant n \leqslant p-1\} = \{0,2,3,\ldots,p-1 \bmod p\}$, we find

$$\sum_{n=1}^{p-1}\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = \sum_{n=1}^{p-1}\left(\frac{1+n^{-1}}{p}\right) = \sum_{n=1}^{p-1}\left(\frac{n}{p}\right) - 1 = -1,$$

because $\sum_{n=1}^{p}\left(\frac{n}{p}\right) = 0$. $\square$

Observe that (1) becomes

$$\sum_{n=1}^{p-1} f(n^2,p) = 2\,|S|, \quad S = \left\{r\colon 1 \leqslant r \leqslant \frac{p-1}{2} \text{ and } \left(\frac{r}{p}\right) = 1\right\}.$$

We connect S with the sum from the claim by pairing quadratic residues and quadratic non-residues. To that end, define

$$S' = \left\{r\colon 1 \leqslant r \leqslant \frac{p-1}{2} \text{ and } \left(\frac{r}{p}\right) = -1\right\}$$
$$T = \left\{r\colon \frac{p+1}{2} \leqslant r \leqslant p-1 \text{ and } \left(\frac{r}{p}\right) = 1\right\}$$
$$T' = \left\{r\colon \frac{p+1}{2} \leqslant r \leqslant p-1 \text{ and } \left(\frac{r}{p}\right) = -1\right\}$$

Since there are exactly $(p-1)/2$ nonzero quadratic residues modulo p, $|S| + |T| = (p-1)/2$. Also we obviously have $|T| + |T'| = (p-1)/2$. Then $|S| = |T'|$.

For the sake of brevity, define $t = |S| = |T'|$. If $\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1$, then exactly of one the numbers $\left(\frac{n}{p}\right)$ and $\left(\frac{n+1}{p}\right)$ is equal to 1, so

$$\left|\left\{n\colon 1 \leqslant n \leqslant \frac{p-3}{2} \text{ and } \left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1\right\}\right| \leqslant |S| + |S-1| = 2t.$$

On the other hand, if $\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1$, then exactly one of $\left(\frac{n}{p}\right)$ and $\left(\frac{n+1}{p}\right)$ is equal to -1, and

$$\left|\left\{n\colon \frac{p+1}{2} \leqslant n \leqslant p-2 \text{ and } \left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1\right\}\right| \leqslant |T'| + |T'-1| = 2t.$$

Thus, taking into account that the middle term $\left(\frac{(p-1)/2}{p}\right)\left(\frac{(p+1)/2}{p}\right)$ may happen to be -1,

$$\left|\left\{n\colon 1 \leqslant n \leqslant p-2 \text{ and } \left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1\right\}\right| \leqslant 4t+1.$$

This implies that

$$\left|\left\{n\colon 1 \leqslant n \leqslant p-2 \text{ and } \left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = 1\right\}\right| \geqslant (p-2) - (4t+1) = p-4t-3,$$

and so

$$-1 = \sum_{n=1}^{p-1}\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) \geqslant p-4t-3-(4t+1) = p-8t-4,$$

which implies $8t \geqslant p-3$, and thus

$$\sum_{n=1}^{p-1} f(n^2,p) = 2t \geqslant \frac{p-3}{4}.$$

Comment 5. It is possible to prove that

$$\sum_{n=1}^{p-1} f(n^2, p) \geqslant \frac{p-1}{2}.$$

The case $p \equiv 1 \pmod 4$ was already mentioned, and it is the equality case. If $p \equiv 3 \pmod 4$, then, by a theorem of Dirichlet, we have

$$\left|\left\{r : 1 \leqslant r \leqslant \frac{p-1}{2} \text{ and } \left(\frac{r}{p}\right) = 1\right\}\right| > \frac{p-1}{4},$$

which implies the result.

See https://en.wikipedia.org/wiki/Quadratic_residue#Dirichlet.27s_formulas for the full statement of the theorem. It seems that no elementary proof of it is known; a proof using complex analysis is available, for instance, in Chapter 7 of the book *Quadratic Residues and Non-Residues: Selected Topics*, by Steve Wright, available in https://arxiv.org/abs/1408.0235.

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SHORTLISTED PROBLEMS

WITH SOLUTIONS

59th International Mathematical Olympiad
July 3rd - July 14th, Cluj-Napoca, Romania

Shortlisted Problems

(with solutions)

59th International Mathematical Olympiad

Cluj-Napoca — Romania, 3–14 July 2018

Note of Confidentiality

The Shortlist has to be kept strictly confidential until the conclusion of the following International Mathematical Olympiad.

IMO General Regulations §6.6

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2018 thank the following 49 countries for contributing 168 problem proposals:

Armenia, Australia, Austria, Azerbaijan, Belarus, Belgium, Bosnia and Herzegovina, Brazil, Bulgaria, Canada, China, Croatia, Cyprus, Czech Republic, Denmark, Estonia, Germany, Greece, Hong Kong, Iceland, India, Indonesia, Iran, Ireland, Israel, Japan, Kosovo, Luxembourg, Mexico, Moldova, Mongolia, Netherlands, Nicaragua, Poland, Russia, Serbia, Singapore, Slovakia, Slovenia, South Africa, South Korea, Switzerland, Taiwan, Tanzania, Thailand, Turkey, Ukraine, United Kingdom, U.S.A.

Problem Selection Committee

Calin Popescu, Radu Gologan, Marian Andronache, Mihail Baluna,
Nicolae Beli, Ilya Bogdanov, Pavel Kozhevnikov, Géza Kós, Sever Moldoveanu

Problems

Algebra

A1. Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f\left(x^2 f(y)^2\right) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$.

(Switzerland)

A2. Find all positive integers $n \geqslant 3$ for which there exist real numbers $a_1, a_2, \ldots, a_n$, $a_{n+1} = a_1$, $a_{n+2} = a_2$ such that

$$a_i a_{i+1} + 1 = a_{i+2}$$

for all $i = 1, 2, \ldots, n$.

(Slovakia)

A3. Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x\in F} 1/x = \sum_{x\in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

(Luxembourg)

A4. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers such that $a_0 = 0$, $a_1 = 1$, and for every $n \geqslant 2$ there exists $1 \leqslant k \leqslant n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximal possible value of $a_{2018} - a_{2017}$.

(Belgium)

A5. Determine all functions $f : (0, \infty) \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

(South Korea)

A6. Let $m, n \geqslant 2$ be integers. Let $f(x_1, \ldots, x_n)$ be a polynomial with real coefficients such that

$$f(x_1, \ldots, x_n) = \left\lfloor \frac{x_1 + \ldots + x_n}{m} \right\rfloor \quad \text{for every } x_1, \ldots, x_n \in \{0, 1, \ldots, m-1\}.$$

Prove that the total degree of f is at least n.

(Brazil)

A7. Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a + b + c + d = 100$.

(Taiwan)

Combinatorics

C1. Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, \ldots, n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

(Iceland)

C2. Queenie and Horst play a game on a 20×20 chessboard. In the beginning the board is empty. In every turn, Horst places a black knight on an empty square in such a way that his new knight does not attack any previous knights. Then Queenie places a white queen on an empty square. The game gets finished when somebody cannot move.

Find the maximal positive K such that, regardless of the strategy of Queenie, Horst can put at least K knights on the board.

(Armenia)

C3. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n + 1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of those stones and moves it to the right by at most k squares (the stone should stay within the board). Sisyphus' aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

(Netherlands)

C4. An *anti-Pascal pyramid* is a finite set of numbers, placed in a triangle-shaped array so that the first row of the array contains one number, the second row contains two numbers, the third row contains three numbers and so on; and, except for the numbers in the bottom row, each number equals the absolute value of the difference of the two numbers below it. For instance, the triangle below is an anti-Pascal pyramid with four rows, in which every integer from 1 to $1 + 2 + 3 + 4 = 10$ occurs exactly once:

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9\,. \end{array}$$

Is it possible to form an anti-Pascal pyramid with 2018 rows, using every integer from 1 to $1 + 2 + \cdots + 2018$ exactly once?

(Iran)

C5. Let k be a positive integer. The organising committee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

(Russia)

C6. Let a and b be distinct positive integers. The following infinite process takes place on an initially empty board.

> (i) If there is at least a pair of equal numbers on the board, we choose such a pair and increase one of its components by a and the other by b.
>
> (ii) If no such pair exists, we write down two times the number 0.

Prove that, no matter how we make the choices in (i), operation (ii) will be performed only finitely many times.

(Serbia)

C7. Consider 2018 pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular *edges* that meet at *vertices*. Notice that there are an even number of vertices on each circle. Given the circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coloured twice — once for each of the two circles that cross at that point. If the two colourings agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least 2061 yellow points, then the vertices of some region are all yellow.

(India)

Geometry

G1. Let ABC be an acute-angled triangle with circumcircle Γ. Let D and E be points on the segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of the segments BD and CE intersect the small arcs $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ at points F and G respectively. Prove that $DE \parallel FG$.

(Greece)

G2. Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

(Australia)

G3. A circle ω of radius 1 is given. A collection T of triangles is called *good*, if the following conditions hold:

(i) each triangle from T is inscribed in ω;

(ii) no two triangles from T have a common interior point.

Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.

(South Africa)

G4. A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

(Mongolia)

G5. Let ABC be a triangle with circumcircle ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to ω.

(Denmark)

G6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. A point X is chosen inside the quadrilateral so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle AXB + \angle CXD = 180°$.

(Poland)

G7. Let O be the circumcentre, and Ω be the circumcircle of an acute-angled triangle ABC. Let P be an arbitrary point on Ω, distinct from A, B, C, and their antipodes in Ω. Denote the circumcentres of the triangles AOP, BOP, and COP by O_A, O_B, and O_C, respectively. The lines ℓ_A, ℓ_B, and ℓ_C perpendicular to BC, CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of the triangle formed by ℓ_A, ℓ_B, and ℓ_C is tangent to the line OP.

(Russia)

Number Theory

N1. Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the numbers of divisors of sn and of sk are equal.

(Ukraine)

N2. Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied:

(i) Each number in the table is congruent to 1 modulo n;

(ii) The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2.

Let R_i be the product of the numbers in the i^{th} row, and C_j be the product of the numbers in the j^{th} column. Prove that the sums $R_1 + \cdots + R_n$ and $C_1 + \cdots + C_n$ are congruent modulo n^4.

(Indonesia)

N3. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

(Serbia)

N4. Let a_1, a_2, $\ldots$, a_n, $\ldots$ be a sequence of positive integers such that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer for all $n \geqslant k$, where k is some positive integer. Prove that there exists a positive integer m such that $a_n = a_{n+1}$ for all $n \geqslant m$.

(Mongolia)

N5. Four positive integers x, y, z, and t satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both xy and zt are perfect squares?

(Russia)

N6. Let $f\colon \{1, 2, 3, \ldots\} \to \{2, 3, \ldots\}$ be a function such that $f(m+n) \mid f(m) + f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.

(Mexico)

N7. Let $n \geqslant 2018$ be an integer, and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be pairwise distinct positive integers not exceeding $5n$. Suppose that the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$$

forms an arithmetic progression. Prove that the terms of the sequence are equal.

(Thailand)

Solutions

Algebra

A1. Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f\left(x^2 f(y)^2\right) = f(x)^2 f(y) \tag{$*$}$$

for all $x, y \in \mathbb{Q}_{>0}$.

(Switzerland)

Answer: $f(x) = 1$ for all $x \in \mathbb{Q}_{>0}$.

Solution. Take any $a, b \in \mathbb{Q}_{>0}$. By substituting $x = f(a)$, $y = b$ and $x = f(b)$, $y = a$ into $(*)$ we get

$$f\big(f(a)\big)^2 f(b) = f\big(f(a)^2 f(b)^2\big) = f\big(f(b)\big)^2 f(a),$$

which yields

$$\frac{f\big(f(a)\big)^2}{f(a)} = \frac{f\big(f(b)\big)^2}{f(b)} \qquad \text{for all } a, b \in \mathbb{Q}_{>0}.$$

In other words, this shows that there exists a constant $C \in \mathbb{Q}_{>0}$ such that $f\big(f(a)\big)^2 = Cf(a)$, or

$$\left(\frac{f\big(f(a)\big)}{C}\right)^2 = \frac{f(a)}{C} \qquad \text{for all } a \in \mathbb{Q}_{>0}. \tag{1}$$

Denote by $f^n(x) = \underbrace{f(f(\ldots(f}_{n}(x))\ldots))$ the n^{th} iteration of f. Equality (1) yields

$$\frac{f(a)}{C} = \left(\frac{f^2(a)}{C}\right)^2 = \left(\frac{f^3(a)}{C}\right)^4 = \cdots = \left(\frac{f^{n+1}(a)}{C}\right)^{2^n}$$

for all positive integer n. So, $f(a)/C$ is the 2^n-th power of a rational number for all positive integer n. This is impossible unless $f(a)/C = 1$, since otherwise the exponent of some prime in the prime decomposition of $f(a)/C$ is not divisible by sufficiently large powers of 2. Therefore, $f(a) = C$ for all $a \in \mathbb{Q}_{>0}$.

Finally, after substituting $f \equiv C$ into $(*)$ we get $C = C^3$, whence $C = 1$. So $f(x) \equiv 1$ is the unique function satisfying $(*)$.

Comment 1. There are several variations of the solution above. For instance, one may start with finding $f(1) = 1$. To do this, let $d = f(1)$. By substituting $x = y = 1$ and $x = d^2$, $y = 1$ into $(*)$ we get $f(d^2) = d^3$ and $f(d^6) = f(d^2)^2 \cdot d = d^7$. By substituting now $x = 1$, $y = d^2$ we obtain $f(d^6) = d^2 \cdot d^3 = d^5$. Therefore, $d^7 = f(d^6) = d^5$, whence $d = 1$.

After that, the rest of the solution simplifies a bit, since we already know that $C = \frac{f(f(1))^2}{f(1)} = 1$. Hence equation (1) becomes merely $f(f(a))^2 = f(a)$, which yields $f(a) = 1$ in a similar manner.

Comment 2. There exist nonconstant functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(*)$ for all real $x, y > 0$ — e.g., $f(x) = \sqrt{x}$.

A2. Find all positive integers $n \geqslant 3$ for which there exist real numbers $a_1, a_2, \ldots, a_n$, $a_{n+1} = a_1$, $a_{n+2} = a_2$ such that
$$a_i a_{i+1} + 1 = a_{i+2}$$
for all $i = 1, 2, \ldots, n$.

(Slovakia)

Answer: n can be any multiple of 3.

Solution 1. For the sake of convenience, extend the sequence $a_1, \ldots, a_{n+2}$ to an infinite periodic sequence with period n. (n is not necessarily the shortest period.)

If n is divisible by 3, then $(a_1, a_2, \ldots) = (-1, -1, 2, -1, -1, 2, \ldots)$ is an obvious solution.

We will show that in every periodic sequence satisfying the recurrence, each positive term is followed by two negative values, and after them the next number is positive again. From this, it follows that n is divisible by 3.

If the sequence contains two consecutive positive numbers a_i, a_{i+1}, then $a_{i+2} = a_i a_{i+1} + 1 > 1$, so the next value is positive as well; by induction, all numbers are positive and greater than 1. But then $a_{i+2} = a_i a_{i+1} + 1 \geqslant 1 \cdot a_{i+1} + 1 > a_{i+1}$ for every index i, which is impossible: our sequence is periodic, so it cannot increase everywhere.

If the number 0 occurs in the sequence, $a_i = 0$ for some index i, then it follows that $a_{i+1} = a_{i-1}a_i + 1$ and $a_{i+2} = a_i a_{i+1} + 1$ are two consecutive positive elements in the sequences and we get the same contradiction again.

Notice that after any two consecutive negative numbers the next one must be positive: if $a_i < 0$ and $a_{i+1} < 0$, then $a_{i+2} = a_1 a_{i+1} + 1 > 1 > 0$. Hence, the positive and negative numbers follow each other in such a way that each positive term is followed by one or two negative values and then comes the next positive term.

Consider the case when the positive and negative values alternate. So, if a_i is a negative value then a_{i+1} is positive, a_{i+2} is negative and a_{i+3} is positive again.

Notice that $a_i a_{i+1} + 1 = a_{i+2} < 0 < a_{i+3} = a_{i+1}a_{i+2} + 1$; by $a_{i+1} > 0$ we conclude $a_i < a_{i+2}$. Hence, the negative values form an infinite increasing subsequence, $a_i < a_{i+2} < a_{i+4} < \ldots$, which is not possible, because the sequence is periodic.

The only case left is when there are consecutive negative numbers in the sequence. Suppose that a_i and a_{i+1} are negative; then $a_{i+2} = a_i a_{i+1} + 1 > 1$. The number a_{i+3} must be negative. We show that a_{i+4} also must be negative.

Notice that a_{i+3} is negative and $a_{i+4} = a_{i+2}a_{i+3} + 1 < 1 < a_i a_{i+1} + 1 = a_{i+2}$, so
$$a_{i+5} - a_{i+4} = (a_{i+3}a_{i+4} + 1) - (a_{i+2}a_{i+3} + 1) = a_{i+3}(a_{i+4} - a_{i+2}) > 0,$$
therefore $a_{i+5} > a_{i+4}$. Since at most one of a_{i+4} and a_{i+5} can be positive, that means that a_{i+4} must be negative.

Now a_{i+3} and a_{i+4} are negative and a_{i+5} is positive; so after two negative and a positive terms, the next three terms repeat the same pattern. That completes the solution.

Solution 2. We prove that the shortest period of the sequence must be 3. Then it follows that n must be divisible by 3.

Notice that the equation $x^2 + 1 = x$ has no real root, so the numbers $a_1, \ldots, a_n$ cannot be all equal, hence the shortest period of the sequence cannot be 1.

By applying the recurrence relation for i and $i + 1$,
$$(a_{i+2} - 1)a_{i+2} = a_i a_{i+1} a_{i+2} = a_i(a_{i+3} - 1), \quad \text{so}$$
$$a_{i+2}^2 - a_i a_{i+3} = a_{i+2} - a_i.$$

By summing over $i = 1, 2, \ldots, n$, we get

$$\sum_{i=1}^{n} (a_i - a_{i+3})^2 = 0.$$

That proves that $a_i = a_{i+3}$ for every index i, so the sequence $a_1, a_2, \ldots$ is indeed periodic with period 3. The shortest period cannot be 1, so it must be 3; therefore, n is divisible by 3.

Comment. By solving the system of equations $ab + 1 = c, \quad bc + 1 = a, \quad ca + 1 = b$, it can be seen that the pattern $(-1, -1, 2)$ is repeated in all sequences satisfying the problem conditions.

A3. Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x\in F} 1/x = \sum_{x\in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

(Luxembourg)

Solution 1. Argue indirectly. Agree, as usual, that the empty sum is 0 to consider rationals in $[0,1)$; adjoining 0 causes no harm, since $\sum_{x\in F} 1/x = 0$ for no nonempty finite subset F of S. For every rational r in $[0,1)$, let F_r be the unique finite subset of S such that $\sum_{x\in F_r} 1/x = r$. The argument hinges on the lemma below.

Lemma. If x is a member of S and q and r are rationals in $[0,1)$ such that $q - r = 1/x$, then x is a member of F_q if and only if it is not one of F_r.

Proof. If x is a member of F_q, then

$$\sum_{y\in F_q\smallsetminus\{x\}} \frac{1}{y} = \sum_{y\in F_q} \frac{1}{y} - \frac{1}{x} = q - \frac{1}{x} = r = \sum_{y\in F_r} \frac{1}{y},$$

so $F_r = F_q \smallsetminus \{x\}$, and x is not a member of F_r. Conversely, if x is not a member of F_r, then

$$\sum_{y\in F_r\cup\{x\}} \frac{1}{y} = \sum_{y\in F_r} \frac{1}{y} + \frac{1}{x} = r + \frac{1}{x} = q = \sum_{y\in F_q} \frac{1}{y},$$

so $F_q = F_r \cup \{x\}$, and x is a member of F_q. $\square$

Consider now an element x of S and a positive rational $r < 1$. Let $n = \lfloor rx \rfloor$ and consider the sets $F_{r-k/x}$, $k = 0, \ldots, n$. Since $0 \leqslant r - n/x < 1/x$, the set $F_{r-n/x}$ does not contain x, and a repeated application of the lemma shows that the $F_{r-(n-2k)/x}$ do not contain x, whereas the $F_{r-(n-2k-1)/x}$ do. Consequently, x is a member of F_r if and only if n is odd.

Finally, consider $F_{2/3}$. By the preceding, $\lfloor 2x/3 \rfloor$ is odd for each x in $F_{2/3}$, so $2x/3$ is not integral. Since $F_{2/3}$ is finite, there exists a positive rational ε such that $\lfloor (2/3 - \varepsilon)x \rfloor = \lfloor 2x/3 \rfloor$ for all x in $F_{2/3}$. This implies that $F_{2/3}$ is a subset of $F_{2/3-\varepsilon}$ which is impossible.

Comment. The solution above can be adapted to show that the problem statement still holds, if the condition $r < 1$ in **(2)** is replaced with $r < \delta$, for an arbitrary positive δ. This yields that, if S does not satisfy **(1)**, then there exist *infinitely many* positive rational numbers $r < 1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

Solution 2. A finite S clearly satisfies **(2)**, so let S be infinite. If S fails both conditions, so does $S \smallsetminus \{1\}$. We may and will therefore assume that S consists of integers greater than 1. Label the elements of S increasingly $x_1 < x_2 < \cdots$, where $x_1 \geqslant 2$.

We first show that S satisfies **(2)** if $x_{n+1} \geqslant 2x_n$ for all n. In this case, $x_n \geqslant 2^{n-1}x_1$ for all n, so

$$s = \sum_{n\geqslant 1} \frac{1}{x_n} \leqslant \sum_{n\geqslant 1} \frac{1}{2^{n-1}x_1} = \frac{2}{x_1}.$$

If $x_1 \geqslant 3$, or $x_1 = 2$ and $x_{n+1} > 2x_n$ for some n, then $\sum_{x\in F} 1/x < s < 1$ for every finite subset F of S, so S satisfies **(2)**; and if $x_1 = 2$ and $x_{n+1} = 2x_n$ for all n, that is, $x_n = 2^n$ for all n, then every finite subset F of S consists of powers of 2, so $\sum_{x\in F} 1/x \neq 1/3$ and again S satisfies **(2)**.

Finally, we deal with the case where $x_{n+1} < 2x_n$ for some n. Consider the positive rational $r = 1/x_n - 1/x_{n+1} < 1/x_{n+1}$. If $r = \sum_{x\in F} 1/x$ for no finite subset F of S, then S satisfies **(2)**.

We now assume that $r = \sum_{x \in F_0} 1/x$ for some finite subset F_0 of S, and show that S satisfies **(1)**. Since $\sum_{x \in F_0} 1/x = r < 1/x_{n+1}$, it follows that x_{n+1} is not a member of F_0, so

$$\sum_{x \in F_0 \cup \{x_{n+1}\}} \frac{1}{x} = \sum_{x \in F_0} \frac{1}{x} + \frac{1}{x_{n+1}} = r + \frac{1}{x_{n+1}} = \frac{1}{x_n}.$$

Consequently, $F = F_0 \cup \{x_{n+1}\}$ and $G = \{x_n\}$ are distinct finite subsets of S such that $\sum_{x \in F} 1/x = \sum_{x \in G} 1/x$, and S satisfies **(1)**.

A4. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers such that $a_0 = 0$, $a_1 = 1$, and for every $n \geqslant 2$ there exists $1 \leqslant k \leqslant n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximal possible value of $a_{2018} - a_{2017}$.

(Belgium)

Answer: The maximal value is $\frac{2016}{2017^2}$.

Solution 1. The claimed maximal value is achieved at

$$a_1 = a_2 = \cdots = a_{2016} = 1, \quad a_{2017} = \frac{a_{2016} + \cdots + a_0}{2017} = 1 - \frac{1}{2017},$$
$$a_{2018} = \frac{a_{2017} + \cdots + a_1}{2017} = 1 - \frac{1}{2017^2}.$$

Now we need to show that this value is optimal. For brevity, we use the notation

$$S(n,k) = a_{n-1} + a_{n-2} + \cdots + a_{n-k} \quad \text{for nonnegative integers } k \leqslant n.$$

In particular, $S(n,0) = 0$ and $S(n,1) = a_{n-1}$. In these terms, for every integer $n \geqslant 2$ there exists a positive integer $k \leqslant n$ such that $a_n = S(n,k)/k$.

For every integer $n \geqslant 1$ we define

$$M_n = \max_{1 \leqslant k \leqslant n} \frac{S(n,k)}{k}, \qquad m_n = \min_{1 \leqslant k \leqslant n} \frac{S(n,k)}{k}, \quad \text{and} \quad \Delta_n = M_n - m_n \geqslant 0.$$

By definition, $a_n \in [m_n, M_n]$ for all $n \geqslant 2$; on the other hand, $a_{n-1} = S(n,1)/1 \in [m_n, M_n]$. Therefore,

$$a_{2018} - a_{2017} \leqslant M_{2018} - m_{2018} = \Delta_{2018},$$

and we are interested in an upper bound for Δ_{2018}.

Also by definition, for any $0 < k \leqslant n$ we have $km_n \leqslant S(n,k) \leqslant kM_n$; notice that these inequalities are also valid for $k = 0$.

Claim 1. For every $n > 2$, we have $\Delta_n \leqslant \frac{n-1}{n}\Delta_{n-1}$.

Proof. Choose positive integers $k, \ell \leqslant n$ such that $M_n = S(n,k)/k$ and $m_n = S(n,\ell)/\ell$. We have $S(n,k) = a_{n-1} + S(n-1,k-1)$, so

$$k(M_n - a_{n-1}) = S(n,k) - ka_{n-1} = S(n-1,k-1) - (k-1)a_{n-1} \leqslant (k-1)(M_{n-1} - a_{n-1}),$$

since $S(n-1,k-1) \leqslant (k-1)M_{n-1}$. Similarly, we get

$$\ell(a_{n-1} - m_n) = (\ell-1)a_{n-1} - S(n-1,\ell-1) \leqslant (\ell-1)(a_{n-1} - m_{n-1}).$$

Since $m_{n-1} \leqslant a_{n-1} \leqslant M_{n-1}$ and $k, \ell \leqslant n$, the obtained inequalities yield

$$M_n - a_{n-1} \leqslant \frac{k-1}{k}(M_{n-1} - a_{n-1}) \leqslant \frac{n-1}{n}(M_{n-1} - a_{n-1}) \quad \text{and}$$
$$a_{n-1} - m_n \leqslant \frac{\ell-1}{\ell}(a_{n-1} - m_{n-1}) \leqslant \frac{n-1}{n}(a_{n-1} - m_{n-1}).$$

Therefore,

$$\Delta_n = (M_n - a_{n-1}) + (a_{n-1} - m_n) \leqslant \frac{n-1}{n}\big((M_{n-1} - a_{n-1}) + (a_{n-1} - m_{n-1})\big) = \frac{n-1}{n}\Delta_{n-1}. \qquad \square$$

Back to the problem, if $a_n = 1$ for all $n \leqslant 2017$, then $a_{2018} \leqslant 1$ and hence $a_{2018} - a_{2017} \leqslant 0$. Otherwise, let $2 \leqslant q \leqslant 2017$ be the minimal index with $a_q < 1$. We have $S(q,i) = i$ for all $i = 1, 2, \ldots, q-1$, while $S(q,q) = q-1$. Therefore, $a_q < 1$ yields $a_q = S(q,q)/q = 1 - \frac{1}{q}$.

Now we have $S(q+1,i) = i - \frac{1}{q}$ for $i = 1, 2, \ldots, q$, and $S(q+1,q+1) = q - \frac{1}{q}$. This gives us

$$m_{q+1} = \frac{S(q+1,1)}{1} = \frac{S(q+1,q+1)}{q+1} = \frac{q-1}{q} \quad \text{and} \quad M_{q+1} = \frac{S(q+1,q)}{q} = \frac{q^2-1}{q^2},$$

so $\Delta_{q+1} = M_{q+1} - m_{q+1} = (q-1)/q^2$. Denoting $N = 2017 \geqslant q$ and using Claim 1 for $n = q+2, q+3, \ldots, N+1$ we finally obtain

$$\Delta_{N+1} \leqslant \frac{q-1}{q^2} \cdot \frac{q+1}{q+2} \cdot \frac{q+2}{q+3} \cdots \frac{N}{N+1} = \frac{1}{N+1}\left(1 - \frac{1}{q^2}\right) \leqslant \frac{1}{N+1}\left(1 - \frac{1}{N^2}\right) = \frac{N-1}{N^2},$$

as required.

Comment 1. One may check that the maximal value of $a_{2018} - a_{2017}$ is attained at the unique sequence, which is presented in the solution above.

Comment 2. An easier question would be to determine the maximal value of $|a_{2018} - a_{2017}|$. In this version, the answer $\frac{1}{2018}$ is achieved at

$$a_1 = a_2 = \cdots = a_{2017} = 1, \quad a_{2018} = \frac{a_{2017} + \cdots + a_0}{2018} = 1 - \frac{1}{2018}.$$

To prove that this value is optimal, it suffices to notice that $\Delta_2 = \frac{1}{2}$ and to apply Claim 1 obtaining

$$|a_{2018} - a_{2017}| \leqslant \Delta_{2018} \leqslant \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{2017}{2018} = \frac{1}{2018}.$$

Solution 2. We present a different proof of the estimate $a_{2018} - a_{2017} \leqslant \frac{2016}{2017^2}$. We keep the same notations of $S(n,k)$, m_n and M_n from the previous solution.

Notice that $S(n,n) = S(n,n-1)$, as $a_0 = 0$. Also notice that for $0 \leqslant k \leqslant \ell \leqslant n$ we have $S(n,\ell) = S(n,k) + S(n-k,\ell-k)$.

Claim 2. For every positive integer n, we have $m_n \leqslant m_{n+1}$ and $M_{n+1} \leqslant M_n$, so the segment $[m_{n+1}, M_{n+1}]$ is contained in $[m_n, M_n]$.

Proof. Choose a positive integer $k \leqslant n+1$ such that $m_{n+1} = S(n+1,k)/k$. Then we have

$$km_{n+1} = S(n+1,k) = a_n + S(n,k-1) \geqslant m_n + (k-1)m_n = km_n,$$

which establishes the first inequality in the Claim. The proof of the second inequality is similar. □

Claim 3. For every positive integers $k \geqslant n$, we have $m_n \leqslant a_k \leqslant M_n$.

Proof. By Claim 2, we have $[m_k, M_k] \subseteq [m_{k-1}, M_{k-1}] \subseteq \cdots \subseteq [m_n, M_n]$. Since $a_k \in [m_k, M_k]$, the claim follows. □

Claim 4. For every integer $n \geqslant 2$, we have $M_n = S(n, n-1)/(n-1)$ and $m_n = S(n,n)/n$.

Proof. We use induction on n. The base case $n = 2$ is routine. To perform the induction step, we need to prove the inequalities

$$\frac{S(n,n)}{n} \leqslant \frac{S(n,k)}{k} \quad \text{and} \quad \frac{S(n,k)}{k} \leqslant \frac{S(n,n-1)}{n-1} \tag{1}$$

for every positive integer $k \leqslant n$. Clearly, these inequalities hold for $k = n$ and $k = n-1$, as $S(n,n) = S(n,n-1) > 0$. In the sequel, we assume that $k < n-1$.

Now the first inequality in (1) rewrites as $nS(n,k) \geqslant kS(n,n) = k\big(S(n,k)+S(n-k,n-k)\big)$, or, cancelling the terms occurring on both parts, as

$$(n-k)S(n,k) \geqslant kS(n-k,n-k) \iff S(n,k) \geqslant k \cdot \frac{S(n-k,n-k)}{n-k}.$$

By the induction hypothesis, we have $S(n-k,n-k)/(n-k) = m_{n-k}$. By Claim 3, we get $a_{n-i} \geqslant m_{n-k}$ for all $i = 1, 2, \ldots, k$. Summing these k inequalities we obtain

$$S(n,k) \geqslant km_{n-k} = k \cdot \frac{S(n-k,n-k)}{n-k},$$

as required.

The second inequality in (1) is proved similarly. Indeed, this inequality is equivalent to

$$\begin{aligned}(n-1)S(n,k) \leqslant kS(n,n-1) &\iff (n-k-1)S(n,k) \leqslant kS(n-k,n-k-1)\\ &\iff S(n,k) \leqslant k \cdot \frac{S(n-k,n-k-1)}{n-k-1} = kM_{n-k};\end{aligned}$$

the last inequality follows again from Claim 3, as each term in $S(n,k)$ is at most M_{n-k}. □

Now we can prove the required estimate for $a_{2018} - a_{2017}$. Set $N = 2017$. By Claim 4,

$$\begin{aligned}a_{N+1} - a_N \leqslant M_{N+1} - a_N = \frac{S(N+1,N)}{N} - a_N &= \frac{a_N + S(N,N-1)}{N} - a_N\\ &= \frac{S(N,N-1)}{N} - \frac{N-1}{N} \cdot a_N.\end{aligned}$$

On the other hand, the same Claim yields

$$a_N \geqslant m_N = \frac{S(N,N)}{N} = \frac{S(N,N-1)}{N}.$$

Noticing that each term in $S(N,N-1)$ is at most 1, so $S(N,N-1) \leqslant N-1$, we finally obtain

$$a_{N+1} - a_N \leqslant \frac{S(N,N-1)}{N} - \frac{N-1}{N} \cdot \frac{S(N,N-1)}{N} = \frac{S(N,N-1)}{N^2} \leqslant \frac{N-1}{N^2}.$$

Comment 1. Claim 1 in Solution 1 can be deduced from Claims 2 and 4 in Solution 2.

By Claim 4 we have $M_n = \frac{S(n,n-1)}{n-1}$ and $m_n = \frac{S(n,n)}{n} = \frac{S(n,n-1)}{n}$. It follows that $\Delta_n = M_n - m_n = \frac{S(n,n-1)}{(n-1)n}$ and so $M_n = n\Delta_n$ and $m_n = (n-1)\Delta_n$

Similarly, $M_{n-1} = (n-1)\Delta_{n-1}$ and $m_{n-1} = (n-2)\Delta_{n-1}$. Then the inequalities $m_{n-1} \leqslant m_n$ and $M_n \leqslant M_{n-1}$ from Claim 2 write as $(n-2)\Delta_{n-1} \leqslant (n-1)\Delta_n$ and $n\Delta_n \leqslant (n-1)\Delta_{n-1}$. Hence we have the double inequality

$$\frac{n-2}{n-1}\Delta_{n-1} \leqslant \Delta_n \leqslant \frac{n-1}{n}\Delta_{n-1}.$$

Comment 2. Both solutions above discuss the properties of an arbitrary sequence satisfying the problem conditions. Instead, one may investigate only an *optimal* sequence which maximises the value of $a_{2018} - a_{2017}$. Here we present an observation which allows to simplify such investigation — for instance, the proofs of Claim 1 in Solution 1 and Claim 4 in Solution 2.

The sequence (a_n) is uniquely determined by choosing, for every $n \geqslant 2$, a positive integer $k(n) \leqslant n$ such that $a_n = S(n, k(n))/k(n)$. Take an arbitrary $2 \leqslant n_0 \leqslant 2018$, and assume that all such integers $k(n)$, for $n \neq n_0$, are fixed. Then, for every n, the value of a_n is a linear function in a_{n_0} (whose possible values constitute some discrete subset of $[m_{n_0}, M_{n_0}]$ containing both endpoints). Hence, $a_{2018} - a_{2017}$ is also a linear function in a_{n_0}, so it attains its maximal value at one of the endpoints of the segment $[m_{n_0}, M_{n_0}]$.

This shows that, while dealing with an optimal sequence, we may assume $a_n \in \{m_n, M_n\}$ for all $2 \leqslant n \leqslant 2018$. Now one can easily see that, if $a_n = m_n$, then $m_{n+1} = m_n$ and $M_{n+1} \leqslant \frac{m_n + nM_n}{n+1}$; similar estimates hold in the case $a_n = M_n$. This already establishes Claim 1, and simplifies the inductive proof of Claim 4, both applied to an optimal sequence.

A5. Determine all functions $f:(0,\infty)\to\mathbb{R}$ satisfying

$$\left(x+\frac{1}{x}\right)f(y)=f(xy)+f\left(\frac{y}{x}\right) \tag{1}$$

for all $x,y>0$.

(South Korea)

Answer: $f(x)=C_1x+\dfrac{C_2}{x}$ with arbitrary constants C_1 and C_2.

Solution 1. Fix a real number $a>1$, and take a new variable t. For the values $f(t)$, $f(t^2)$, $f(at)$ and $f(a^2t^2)$, the relation (1) provides a system of linear equations:

$$\begin{array}{llcll}
x=y=t: & \left(t+\frac{1}{t}\right)f(t) & = & f(t^2)+f(1) & \text{(2a)}\\
x=\frac{t}{a},\ y=at: & \left(\frac{t}{a}+\frac{a}{t}\right)f(at) & = & f(t^2)+f(a^2) & \text{(2b)}\\
x=a^2t,\ y=t: & \left(a^2t+\frac{1}{a^2t}\right)f(t) & = & f(a^2t^2)+f\left(\frac{1}{a^2}\right) & \text{(2c)}\\
x=y=at: & \left(at+\frac{1}{at}\right)f(at) & = & f(a^2t^2)+f(1) & \text{(2d)}
\end{array}$$

In order to eliminate $f(t^2)$, take the difference of (2a) and (2b); from (2c) and (2d) eliminate $f(a^2t^2)$; then by taking a linear combination, eliminate $f(at)$ as well:

$$\left(t+\frac{1}{t}\right)f(t)-\left(\frac{t}{a}+\frac{a}{t}\right)f(at)=f(1)-f(a^2)\quad\text{and}$$

$$\left(a^2t+\frac{1}{a^2t}\right)f(t)-\left(at+\frac{1}{at}\right)f(at)=f(1/a^2)-f(1),\quad\text{so}$$

$$\left(\left(at+\frac{1}{at}\right)\left(t+\frac{1}{t}\right)-\left(\frac{t}{a}+\frac{a}{t}\right)\left(a^2t+\frac{1}{a^2t}\right)\right)f(t)$$
$$=\left(at+\frac{1}{at}\right)\left(f(1)-f(a^2)\right)-\left(\frac{t}{a}+\frac{a}{t}\right)\left(f(1/a^2)-f(1)\right).$$

Notice that on the left-hand side, the coefficient of $f(t)$ is nonzero and does not depend on t:

$$\left(at+\frac{1}{at}\right)\left(t+\frac{1}{t}\right)-\left(\frac{t}{a}+\frac{a}{t}\right)\left(a^2t+\frac{1}{a^2t}\right)=a+\frac{1}{a}-\left(a^3+\frac{1}{a^3}\right)<0.$$

After dividing by this fixed number, we get

$$f(t)=C_1t+\frac{C_2}{t} \tag{3}$$

where the numbers C_1 and C_2 are expressed in terms of a, $f(1)$, $f(a^2)$ and $f(1/a^2)$, and they do not depend on t.

The functions of the form (3) satisfy the equation:

$$\left(x+\frac{1}{x}\right)f(y)=\left(x+\frac{1}{x}\right)\left(C_1y+\frac{C_2}{y}\right)=\left(C_1xy+\frac{C_2}{xy}\right)+\left(C_1\frac{y}{x}+C_2\frac{x}{y}\right)=f(xy)+f\left(\frac{y}{x}\right).$$

Solution 2. We start with an observation. If we substitute $x = a \neq 1$ and $y = a^n$ in (1), we obtain

$$f(a^{n+1}) - \left(a + \frac{1}{a}\right) f(a^n) + f(a^{n-1}) = 0.$$

For the sequence $z_n = a^n$, this is a homogeneous linear recurrence of the second order, and its characteristic polynomial is $t^2 - \left(a + \frac{1}{a}\right) t + 1 = (t-a)(t - \frac{1}{a})$ with two distinct nonzero roots, namely a and $1/a$. As is well-known, the general solution is $z_n = C_1 a^n + C_2 (1/a)^n$ where the index n can be as well positive as negative. Of course, the numbers C_1 and C_2 may depend of the choice of a, so in fact we have two functions, C_1 and C_2, such that

$$f(a^n) = C_1(a) \cdot a^n + \frac{C_2(a)}{a^n} \quad \text{for every } a \neq 1 \text{ and every integer } n. \tag{4}$$

The relation (4) can be easily extended to rational values of n, so we may conjecture that C_1 and C_2 are constants, and whence $f(t) = C_1 t + \frac{C_2}{t}$. As it was seen in the previous solution, such functions indeed satisfy (1).

The equation (1) is linear in f; so if some functions f_1 and f_2 satisfy (1) and c_1, c_2 are real numbers, then $c_1 f_1(x) + c_2 f_2(x)$ is also a solution of (1). In order to make our formulas simpler, define

$$f_0(x) = f(x) - f(1) \cdot x.$$

This function is another one satisfying (1) and the extra constraint $f_0(1) = 0$. Repeating the same argument on linear recurrences, we can write $f_0(a) = K(a)a^n + \frac{L(a)}{a^n}$ with some functions K and L. By substituting $n = 0$, we can see that $K(a) + L(a) = f_0(1) = 0$ for every a. Hence,

$$f_0(a^n) = K(a)\left(a^n - \frac{1}{a^n}\right).$$

Now take two numbers $a > b > 1$ arbitrarily and substitute $x = (a/b)^n$ and $y = (ab)^n$ in (1):

$$\left(\frac{a^n}{b^n} + \frac{b^n}{a^n}\right) f_0\big((ab)^n\big) = f_0\big(a^{2n}\big) + f_0\big(b^{2n}\big), \quad \text{so}$$

$$\left(\frac{a^n}{b^n} + \frac{b^n}{a^n}\right) K(ab)\left((ab)^n - \frac{1}{(ab)^n}\right) = K(a)\left(a^{2n} - \frac{1}{a^{2n}}\right) + K(b)\left(b^{2n} - \frac{1}{b^{2n}}\right), \quad \text{or equivalently}$$

$$K(ab)\left(a^{2n} - \frac{1}{a^{2n}} + b^{2n} - \frac{1}{b^{2n}}\right) = K(a)\left(a^{2n} - \frac{1}{a^{2n}}\right) + K(b)\left(b^{2n} - \frac{1}{b^{2n}}\right). \tag{5}$$

By dividing (5) by a^{2n} and then taking limit with $n \to +\infty$ we get $K(ab) = K(a)$. Then (5) reduces to $K(a) = K(b)$. Hence, $K(a) = K(b)$ for all $a > b > 1$.

Fix $a > 1$. For every $x > 0$ there is some b and an integer n such that $1 < b < a$ and $x = b^n$. Then

$$f_0(x) = f_0(b^n) = K(b)\left(b^n - \frac{1}{b^n}\right) = K(a)\left(x - \frac{1}{x}\right).$$

Hence, we have $f(x) = f_0(x) + f(1)x = C_1 x + \frac{C_2}{x}$ with $C_1 = K(a) + f(1)$ and $C_2 = -K(a)$.

Comment. After establishing (5), there are several variants of finishing the solution. For example, instead of taking a limit, we can obtain a system of linear equations for $K(a)$, $K(b)$ and $K(ab)$ by substituting two positive integers n in (5), say $n = 1$ and $n = 2$. This approach leads to a similar ending as in the first solution.

Optionally, we define another function $f_1(x) = f_0(x) - C\left(x - \frac{1}{x}\right)$ and prescribe $K(c) = 0$ for another fixed c. Then we can choose $ab = c$ and decrease the number of terms in (5).

A6. Let $m, n \geqslant 2$ be integers. Let $f(x_1, \ldots, x_n)$ be a polynomial with real coefficients such that

$$f(x_1, \ldots, x_n) = \left\lfloor \frac{x_1 + \ldots + x_n}{m} \right\rfloor \quad \text{for every } x_1, \ldots, x_n \in \{0, 1, \ldots, m-1\}.$$

Prove that the total degree of f is at least n.

(Brazil)

Solution. We transform the problem to a single variable question by the following

Lemma. Let $a_1, \ldots, a_n$ be nonnegative integers and let $G(x)$ be a nonzero polynomial with $\deg G \leqslant a_1 + \ldots + a_n$. Suppose that some polynomial $F(x_1, \ldots, x_n)$ satisfies

$$F(x_1, \ldots, x_n) = G(x_1 + \ldots + x_n) \quad \text{for } (x_1, \ldots, x_n) \in \{0, 1, \ldots, a_1\} \times \ldots \times \{0, 1, \ldots, a_n\}.$$

Then F cannot be the zero polynomial, and $\deg F \geqslant \deg G$.

For proving the lemma, we will use *forward differences* of polynomials. If $p(x)$ is a polynomial with a single variable, then define $(\Delta p)(x) = p(x+1) - p(x)$. It is well-known that if p is a nonconstant polynomial then $\deg \Delta p = \deg p - 1$.

If $p(x_1, \ldots, x_n)$ is a polynomial with n variables and $1 \leqslant k \leqslant n$ then let

$$(\Delta_k p)(x_1, \ldots, x_n) = p(x_1, \ldots, x_{k-1}, x_k + 1, x_{k+1}, \ldots, x_n) - p(x_1, \ldots, x_n).$$

It is also well-known that either $\Delta_k p$ is the zero polynomial or $\deg(\Delta_k p) \leqslant \deg p - 1$.

Proof of the lemma. We apply induction on the degree of G. If G is a constant polynomial then we have $F(0, \ldots, 0) = G(0) \neq 0$, so F cannot be the zero polynomial.

Suppose that $\deg G \geqslant 1$ and the lemma holds true for lower degrees. Since $a_1 + \ldots + a_n \geqslant \deg G > 0$, at least one of $a_1, \ldots, a_n$ is positive; without loss of generality suppose $a_1 \geqslant 1$.

Consider the polynomials $F_1 = \Delta_1 F$ and $G_1 = \Delta G$. On the grid $\{0, \ldots, a_1 - 1\} \times \{0, \ldots, a_2\} \times \ldots \times \{0, \ldots, a_n\}$ we have

$$\begin{aligned} F_1(x_1, \ldots, x_n) &= F(x_1 + 1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n) = \\ &= G(x_1 + \ldots + x_n + 1) - G(x_1 + \ldots + x_n) = G_1(x_1 + \ldots + x_n). \end{aligned}$$

Since G is nonconstant, we have $\deg G_1 = \deg G - 1 \leqslant (a_1 - 1) + a_2 + \ldots + a_n$. Therefore we can apply the induction hypothesis to F_1 and G_1 and conclude that F_1 is not the zero polynomial and $\deg F_1 \geqslant \deg G_1$. Hence, $\deg F \geqslant \deg F_1 + 1 \geqslant \deg G_1 + 1 = \deg G$. That finishes the proof. $\square$

To prove the problem statement, take the unique polynomial $g(x)$ so that $g(x) = \left\lfloor \frac{x}{m} \right\rfloor$ for $x \in \{0, 1, \ldots, n(m-1)\}$ and $\deg g \leqslant n(m-1)$. Notice that precisely $n(m-1)+1$ values of g are prescribed, so $g(x)$ indeed exists and is unique. Notice further that the constraints $g(0) = g(1) = 0$ and $g(m) = 1$ together enforce $\deg g \geqslant 2$.

By applying the lemma to $a_1 = \ldots = a_n = m - 1$ and the polynomials f and g, we achieve $\deg f \geqslant \deg g$. Hence we just need a suitable lower bound on $\deg g$.

Consider the polynomial $h(x) = g(x+m) - g(x) - 1$. The degree of $g(x+m) - g(x)$ is $\deg g - 1 \geqslant 1$, so $\deg h = \deg g - 1 \geqslant 1$, and therefore h cannot be the zero polynomial. On the other hand, h vanishes at the points $0, 1, \ldots, n(m-1) - m$, so h has at least $(n-1)(m-1)$ roots. Hence,

$$\deg f \geqslant \deg g = \deg h + 1 \geqslant (n-1)(m-1) + 1 \geqslant n.$$

Comment 1. In the lemma we have equality for the choice $F(x_1, \ldots, x_n) = G(x_1 + \ldots + x_n)$, so it indeed transforms the problem to an equivalent single-variable question.

Comment 2. If $m \geqslant 3$, the polynomial $h(x)$ can be replaced by Δg. Notice that

$$(\Delta g)(x) = \begin{cases} 1 & \text{if } x \equiv -1 \pmod m \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } x = 0, 1, \ldots, n(m-1) - 1.$$

Hence, Δg vanishes at all integers x with $0 \leqslant x < n(m-1)$ and $x \not\equiv -1 \pmod m$. This leads to $\deg g \geqslant \frac{(m-1)^2 n}{m} + 1$.

If m is even then this lower bound can be improved to $n(m-1)$. For $0 \leqslant N < n(m-1)$, the $(N+1)^{\text{st}}$ forward difference at $x = 0$ is

$$(\Delta^{N+1})g(0) = \sum_{k=0}^{N} (-1)^{N-k} \binom{N}{k} (\Delta g)(k) = \sum_{\substack{0 \leqslant k \leqslant N \\ k \equiv -1 \pmod m}} (-1)^{N-k} \binom{N}{k}. \tag{$*$}$$

Since m is even, all signs in the last sum are equal; with $N = n(m-1)-1$ this proves $\Delta^{n(m-1)} g(0) \neq 0$, indicating that $\deg g \geqslant n(m-1)$.

However, there are infinitely many cases when all terms in $(*)$ cancel out, for example if m is an odd divisor of $n+1$. In such cases, $\deg f$ can be less than $n(m-1)$.

Comment 3. The lemma is closely related to the so-called

> **Alon–Füredi bound.** *Let $S_1, \ldots, S_n$ be nonempty finite sets in a field and suppose that the polynomial $P(x_1, \ldots, x_n)$ vanishes at the points of the grid $S_1 \times \ldots \times S_n$, except for a single point. Then* $\deg P \geqslant \sum_{i=1}^{n} \big(|S_i| - 1\big)$.
>
> (A well-known application of the Alon–Füredi bound was the former IMO problem 2007/6. Since then, this result became popular among the students and is part of the IMO training for many IMO teams.)

The proof of the lemma can be replaced by an application of the Alon–Füredi bound as follows. Let $d = \deg G$, and let G_0 be the unique polynomial such that $G_0(x) = G(x)$ for $x \in \{0, 1, \ldots, d-1\}$ but $\deg G_0 < d$. The polynomials G_0 and G are different because they have different degrees, and they attain the same values at $0, 1, \ldots, d-1$; that enforces $G_0(d) \neq G(d)$.

Choose some nonnegative integers $b_1, \ldots, b_n$ so that $b_1 \leqslant a_1, \ldots, b_n \leqslant a_n$, and $b_1 + \ldots + b_n = d$, and consider the polynomial

$$H(x_1, \ldots, x_n) = F(x_1, \ldots, x_n) - G_0(x_1 + \ldots + x_n)$$

on the grid $\{0, 1, \ldots, b_1\} \times \ldots \times \{0, 1, \ldots, b_n\}$.

At the point $(b_1, \ldots, b_n)$ we have $H(b_1, \ldots, b_n) = G(d) - G_0(d) \neq 0$. At all other points of the grid we have $F = G$ and therefore $H = G - G_0 = 0$. So, by the Alon–Füredi bound, $\deg H \geqslant b_1 + \ldots + b_n = d$. Since $\deg G_0 < d$, this implies $\deg F = \deg(H + G_0) = \deg H \geqslant d = \deg G$. $\square$

A7. Find the maximal value of

$$S=\sqrt[3]{\frac{a}{b+7}}+\sqrt[3]{\frac{b}{c+7}}+\sqrt[3]{\frac{c}{d+7}}+\sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a+b+c+d=100$.

(Taiwan)

Answer: $\frac{8}{\sqrt[3]{7}}$, reached when (a, b, c, d) is a cyclic permutation of $(1,49,1,49)$.

Solution 1. Since the value $8/\sqrt[3]{7}$ is reached, it suffices to prove that $S \leqslant 8/\sqrt[3]{7}$.

Assume that x, y, z, t is a permutation of the variables, with $x \leqslant y \leqslant z \leqslant t$. Then, by the rearrangement inequality,

$$S \leqslant\left(\sqrt[3]{\frac{x}{t+7}}+\sqrt[3]{\frac{t}{x+7}}\right)+\left(\sqrt[3]{\frac{y}{z+7}}+\sqrt[3]{\frac{z}{y+7}}\right).$$

Claim. The first bracket above does not exceed $\sqrt[3]{\frac{x+t+14}{7}}$.

Proof. Since

$$X^{3}+Y^{3}+3 X Y Z-Z^{3}=\frac{1}{2}(X+Y-Z)\left((X-Y)^{2}+(X+Z)^{2}+(Y+Z)^{2}\right),$$

the inequality $X+Y \leqslant Z$ is equivalent (when $X, Y, Z \geqslant 0$) to $X^{3}+Y^{3}+3 X Y Z \leqslant Z^{3}$. Therefore, the claim is equivalent to

$$\frac{x}{t+7}+\frac{t}{x+7}+3 \sqrt[3]{\frac{x t(x+t+14)}{7(x+7)(t+7)}} \leqslant \frac{x+t+14}{7}.$$

Notice that

$$\begin{aligned}
3 \sqrt[3]{\frac{x t(x+t+14)}{7(x+7)(t+7)}} &=3 \sqrt[3]{\frac{t(x+7)}{7(t+7)} \cdot \frac{x(t+7)}{7(x+7)} \cdot \frac{7(x+t+14)}{(t+7)(x+7)}} \\
&\leqslant \frac{t(x+7)}{7(t+7)}+\frac{x(t+7)}{7(x+7)}+\frac{7(x+t+14)}{(t+7)(x+7)}
\end{aligned}$$

by the AM–GM inequality, so it suffices to prove

$$\frac{x}{t+7}+\frac{t}{x+7}+\frac{t(x+7)}{7(t+7)}+\frac{x(t+7)}{7(x+7)}+\frac{7(x+t+14)}{(t+7)(x+7)} \leqslant \frac{x+t+14}{7}.$$

A straightforward check verifies that the last inequality is in fact an equality. $\square$

The claim leads now to

$$S \leqslant \sqrt[3]{\frac{x+t+14}{7}}+\sqrt[3]{\frac{y+z+14}{7}} \leqslant 2 \sqrt[3]{\frac{x+y+z+t+28}{14}}=\frac{8}{\sqrt[3]{7}},$$

the last inequality being due to the AM–CM inequality (or to the fact that $\sqrt[3]{\ }$ is concave on $[0, \infty)$).

Solution 2. We present a different proof for the estimate $S \leqslant 8/\sqrt[3]{7}$.

Start by using Hölder's inequality:

$$S^3 = \left(\sum_{\text{cyc}} \frac{\sqrt[6]{a} \cdot \sqrt[6]{a}}{\sqrt[3]{b+7}}\right)^3 \leqslant \sum_{\text{cyc}} \left(\sqrt[6]{a}\right)^3 \cdot \sum_{\text{cyc}} \left(\sqrt[6]{a}\right)^3 \cdot \sum_{\text{cyc}} \left(\frac{1}{\sqrt[3]{b+7}}\right)^3 = \left(\sum_{\text{cyc}} \sqrt{a}\right)^2 \sum_{\text{cyc}} \frac{1}{b+7}.$$

Notice that

$$\frac{(x-1)^2(x-7)^2}{x^2+7} \geqslant 0 \iff x^2 - 16x + 71 \geqslant \frac{448}{x^2+7}$$

yields

$$\sum \frac{1}{b+7} \leqslant \frac{1}{448} \sum \left(b - 16\sqrt{b} + 71\right) = \frac{1}{448}\left(384 - 16\sum \sqrt{b}\right) = \frac{48 - 2\sum\sqrt{b}}{56}.$$

Finally,

$$S^3 \leqslant \frac{1}{56}\left(\sum \sqrt{a}\right)^2 \left(48 - 2\sum \sqrt{a}\right) \leqslant \frac{1}{56}\left(\frac{\sum\sqrt{a} + \sum\sqrt{a} + \left(48 - 2\sum\sqrt{a}\right)}{3}\right)^3 = \frac{512}{7}$$

by the AM–GM inequality. The conclusion follows.

Comment. All the above works if we replace 7 and 100 with $k > 0$ and $2(k^2+1)$, respectively; in this case, the answer becomes

$$2\sqrt[3]{\frac{(k+1)^2}{k}}.$$

Even further, a linear substitution allows to extend the solutions to a version with 7 and 100 being replaced with arbitrary positive real numbers p and q satisfying $q \geqslant 4p$.

Combinatorics

C1. Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, \ldots, n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

(Iceland)

Solution. We show that one of possible examples is the set

$$S = \{1 \cdot 3^k,\ 2 \cdot 3^k \colon k = 1, 2, \ldots, n-1\} \cup \left\{1,\ \frac{3^n + 9}{2} - 1\right\}.$$

It is readily verified that all the numbers listed above are distinct (notice that the last two are not divisible by 3).

The sum of elements in S is

$$\Sigma = 1 + \left(\frac{3^n + 9}{2} - 1\right) + \sum_{k=1}^{n-1}(1 \cdot 3^k + 2 \cdot 3^k) = \frac{3^n + 9}{2} + \sum_{k=1}^{n-1} 3^{k+1} = \frac{3^n + 9}{2} + \frac{3^{n+1} - 9}{2} = 2 \cdot 3^n.$$

Hence, in order to show that this set satisfies the problem requirements, it suffices to present, for every $m = 2, 3, \ldots, n$, an m-element subset $A_m \subset S$ whose sum of elements equals 3^n.

Such a subset is

$$A_m = \{2 \cdot 3^k \colon k = n-m+1, n-m+2, \ldots, n-1\} \cup \{1 \cdot 3^{n-m+1}\}.$$

Clearly, $|A_m| = m$. The sum of elements in A_m is

$$3^{n-m+1} + \sum_{k=n-m+1}^{n-1} 2 \cdot 3^k = 3^{n-m+1} + \frac{2 \cdot 3^n - 2 \cdot 3^{n-m+1}}{2} = 3^n,$$

as required.

Comment. Let us present a more general construction. Let $s_1, s_2, \ldots, s_{2n-1}$ be a sequence of pairwise distinct positive integers satisfying $s_{2i+1} = s_{2i} + s_{2i-1}$ for all $i = 2, 3, \ldots, n-1$. Set $s_{2n} = s_1 + s_2 + \cdots + s_{2n-4}$.

Assume that s_{2n} is distinct from the other terms of the sequence. Then the set $S = \{s_1, s_2, \ldots, s_{2n}\}$ satisfies the problem requirements. Indeed, the sum of its elements is

$$\Sigma = \sum_{i=1}^{2n-4} s_i + (s_{2n-3} + s_{2n-2}) + s_{2n-1} + s_{2n} = s_{2n} + s_{2n-1} + s_{2n-1} + s_{2n} = 2s_{2n} + 2s_{2n-1}.$$

Therefore, we have

$$\frac{\Sigma}{2} = s_{2n} + s_{2n-1} = s_{2n} + s_{2n-2} + s_{2n-3} = s_{2n} + s_{2n-2} + s_{2n-4} + s_{2n-5} = \ldots,$$

which shows that the required sets A_m can be chosen as

$$A_m = \{s_{2n}, s_{2n-2}, \ldots, s_{2n-2m+4}, s_{2n-2m+3}\}.$$

So, the only condition to be satisfied is $s_{2n} \notin \{s_1, s_2, \ldots, s_{2n-1}\}$, which can be achieved in many different ways (e.g., by choosing properly the number s_1 after specifying $s_2, s_3, \ldots, s_{2n-1}$).

The solution above is an instance of this general construction. Another instance, for $n > 3$, is the set

$$\{F_1, F_2, \ldots, F_{2n-1}, F_1 + \cdots + F_{2n-4}\},$$

where $F_1 = 1$, $F_2 = 2$, $F_{n+1} = F_n + F_{n-1}$ is the usual Fibonacci sequence.

C2. Queenie and Horst play a game on a 20×20 chessboard. In the beginning the board is empty. In every turn, Horst places a black knight on an empty square in such a way that his new knight does not attack any previous knights. Then Queenie places a white queen on an empty square. The game gets finished when somebody cannot move.

Find the maximal positive K such that, regardless of the strategy of Queenie, Horst can put at least K knights on the board.

(Armenia)

Answer: $K = 20^2/4 = 100$. In case of a $4N \times 4M$ board, the answer is $K = 4NM$.

Solution. We show two strategies, one for Horst to place at least 100 knights, and another strategy for Queenie that prevents Horst from putting more than 100 knights on the board.

A strategy for Horst: *Put knights only on black squares, until all black squares get occupied.*

Colour the squares of the board black and white in the usual way, such that the white and black squares alternate, and let Horst put his knights on black squares as long as it is possible. Two knights on squares of the same colour never attack each other. The number of black squares is $20^2/2 = 200$. The two players occupy the squares in turn, so Horst will surely find empty black squares in his first 100 steps.

A strategy for Queenie: *Group the squares into cycles of length 4, and after each step of Horst, occupy the opposite square in the same cycle.*

Consider the squares of the board as vertices of a graph; let two squares be connected if two knights on those squares would attack each other. Notice that in a 4×4 board the squares can be grouped into 4 cycles of length 4, as shown in Figure 1. Divide the board into parts of size 4×4, and perform the same grouping in every part; this way we arrange the 400 squares of the board into 100 cycles (Figure 2).

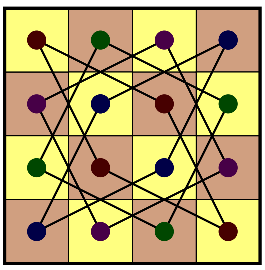

Figure 1

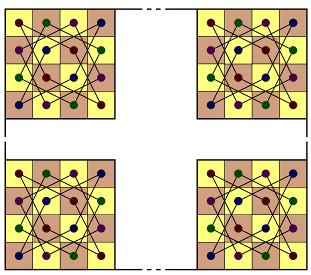

Figure 2

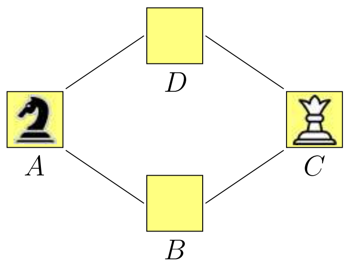

Figure 3

The strategy of Queenie can be as follows: Whenever Horst puts a new knight to a certain square A, which is part of some cycle $A - B - C - D - A$, let Queenie put her queen on the opposite square C in that cycle (Figure 3). From this point, Horst cannot put any knight on A or C because those squares are already occupied, neither on B or D because those squares are attacked by the knight standing on A. Hence, Horst can put at most one knight on each cycle, that is at most 100 knights in total.

Comment 1. Queenie's strategy can be prescribed by a simple rule: divide the board into 4×4 parts; whenever Horst puts a knight in a part P, Queenie reflects that square about the centre of P and puts her queen on the reflected square.

Comment 2. The result remains the same if Queenie moves first. In the first turn, she may put her first queen arbitrarily. Later, if she has to put her next queen on a square that already contains a queen, she may move arbitrarily again.

C3. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of those stones and moves it to the right by at most k squares (the stone should stay within the board). Sisyphus' aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

(Netherlands)

Solution. The stones are indistinguishable, and all have the same origin and the same final position. So, at any turn we can prescribe which stone from the chosen square to move. We do it in the following manner. Number the stones from 1 to n. At any turn, after choosing a square, Sisyphus moves the stone with the largest number from this square.

This way, when stone k is moved from some square, that square contains not more than k stones (since all their numbers are at most k). Therefore, stone k is moved by at most k squares at each turn. Since the total shift of the stone is exactly n, at least $\lceil n/k \rceil$ moves of stone k should have been made, for every $k = 1, 2, \ldots, n$.

By summing up over all $k = 1, 2, \ldots, n$, we get the required estimate.

Comment. The original submission contained the second part, asking for which values of n the equality can be achieved. The answer is $n = 1, 2, 3, 4, 5, 7$. The Problem Selection Committee considered this part to be less suitable for the competition, due to technicalities.

C4. An *anti-Pascal pyramid* is a finite set of numbers, placed in a triangle-shaped array so that the first row of the array contains one number, the second row contains two numbers, the third row contains three numbers and so on; and, except for the numbers in the bottom row, each number equals the absolute value of the difference of the two numbers below it. For instance, the triangle below is an anti-Pascal pyramid with four rows, in which every integer from 1 to $1+2+3+4=10$ occurs exactly once:

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9\,. \end{array}$$

Is it possible to form an anti-Pascal pyramid with 2018 rows, using every integer from 1 to $1+2+\cdots+2018$ exactly once?

(Iran)

Answer: No, it is not possible.

Solution. Let T be an anti-Pascal pyramid with n rows, containing every integer from 1 to $1+2+\cdots+n$, and let a_1 be the topmost number in T (Figure 1). The two numbers below a_1 are some a_2 and $b_2=a_1+a_2$, the two numbers below b_2 are some a_3 and $b_3=a_1+a_2+a_3$, and so on and so forth all the way down to the bottom row, where some a_n and $b_n=a_1+a_2+\cdots+a_n$ are the two neighbours below $b_{n-1}=a_1+a_2+\cdots+a_{n-1}$. Since the a_k are n pairwise distinct positive integers whose sum does not exceed the largest number in T, which is $1+2+\cdots+n$, it follows that they form a permutation of $1,2,\ldots,n$.

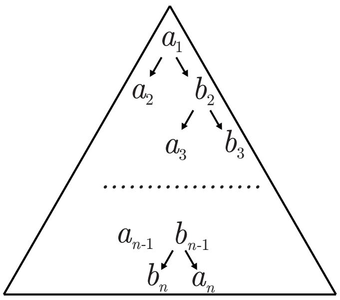

Figure 1

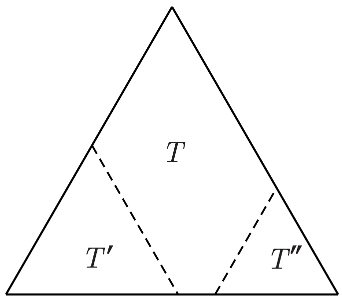

Figure 2

Consider now (Figure 2) the two 'equilateral' subtriangles of T whose bottom rows contain the numbers to the left, respectively right, of the pair a_n, b_n. (One of these subtriangles may very well be empty.) At least one of these subtriangles, say T', has side length $\ell \geqslant \lceil (n-2)/2 \rceil$. Since T' obeys the anti-Pascal rule, it contains ℓ pairwise distinct positive integers $a'_1, a'_2, \ldots, a'_\ell$, where a'_1 is at the apex, and a'_k and $b'_k = a'_1+a'_2+\cdots+a'_k$ are the two neighbours below b'_{k-1} for each $k=2,3\ldots,\ell$. Since the a_k all lie outside T', and they form a permutation of $1,2,\ldots,n$, the a'_k are all greater than n. Consequently,

$$\begin{aligned} b'_\ell &\geqslant (n+1)+(n+2)+\cdots+(n+\ell) = \frac{\ell(2n+\ell+1)}{2} \\ &\geqslant \frac{1}{2}\cdot\frac{n-2}{2}\left(2n+\frac{n-2}{2}+1\right) = \frac{5n(n-2)}{8}, \end{aligned}$$

which is greater than $1+2+\cdots+n = n(n+1)/2$ for $n=2018$. A contradiction.

Comment. The above estimate may be slightly improved by noticing that $b'_\ell \neq b_n$. This implies $n(n+1)/2 = b_n > b'_\ell \geqslant \lceil (n-2)/2 \rceil \left(2n + \lceil (n-2)/2 \rceil + 1\right)/2$, so $n \leqslant 7$ if n is odd, and $n \leqslant 12$ if n is even. It seems that the largest anti-Pascal pyramid whose entries are a permutation of the integers from 1 to $1+2+\cdots+n$ has 5 rows.

C5. Let k be a positive integer. The organising committee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

(Russia)

Answer: The required minimum is $k(4k^2 + k - 1)/2$.

Solution 1. Enumerate the days of the tournament $1, 2, \ldots, \binom{2k}{2}$. Let $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_{2k}$ be the days the players arrive to the tournament, arranged in *nondecreasing* order; similarly, let $e_1 \geqslant \cdots \geqslant e_{2k}$ be the days they depart arranged in *nonincreasing* order (it may happen that a player arrives on day b_i and departs on day e_j, where $i \neq j$). If a player arrives on day b and departs on day e, then his stay cost is $e - b + 1$. Therefore, the total stay cost is

$$\Sigma = \sum_{i=1}^{2k} e_i - \sum_{i=1}^{2k} b_i + n = \sum_{i=1}^{2k} (e_i - b_i + 1).$$

Bounding the total cost from below. To this end, estimate $e_{i+1} - b_{i+1} + 1$. Before day b_{i+1}, only i players were present, so at most $\binom{i}{2}$ matches could be played. Therefore, $b_{i+1} \leqslant \binom{i}{2} + 1$. Similarly, at most $\binom{i}{2}$ matches could be played after day e_{i+1}, so $e_i \geqslant \binom{2k}{2} - \binom{i}{2}$. Thus,

$$e_{i+1} - b_{i+1} + 1 \geqslant \binom{2k}{2} - 2\binom{i}{2} = k(2k-1) - i(i-1).$$

This lower bound can be improved for $i > k$: List the i players who arrived first, and the i players who departed last; at least $2i - 2k$ players appear in both lists. The matches between these players were counted twice, though the players in each pair have played only once. Therefore, if $i > k$, then

$$e_{i+1} - b_{i+1} + 1 \geqslant \binom{2k}{2} - 2\binom{i}{2} + \binom{2i-2k}{2} = (2k-i)^2.$$

An optimal tournament, We now describe a schedule in which the lower bounds above are all achieved simultaneously. Split players into two groups X and Y, each of cardinality k. Next, partition the schedule into three parts. During the first part, the players from X arrive one by one, and each newly arrived player immediately plays with everyone already present. During the third part (after all players from X have already departed) the players from Y depart one by one, each playing with everyone still present just before departing.

In the middle part, everyone from X should play with everyone from Y. Let $S_1, S_2, \ldots, S_k$ be the players in X, and let $T_1, T_2, \ldots, T_k$ be the players in Y. Let $T_1, T_2, \ldots, T_k$ arrive in this order; after T_j arrives, he immediately plays with all the S_i, $i > j$. Afterwards, players S_k, $S_{k-1}, \ldots, S_1$ depart in this order; each S_i plays with all the T_j, $i \leqslant j$, just before his departure, and S_k departs the day T_k arrives. For $0 \leqslant s \leqslant k-1$, the number of matches played between T_{k-s}'s arrival and S_{k-s}'s departure is

$$\sum_{j=k-s}^{k-1} (k-j) + 1 + \sum_{j=k-s}^{k-1} (k-j+1) = \frac{1}{2}s(s+1) + 1 + \frac{1}{2}s(s+3) = (s+1)^2.$$

Thus, if $i > k$, then the number of matches that have been played between T_{i-k+1}'s arrival, which is b_{i+1}, and S_{i-k+1}'s departure, which is e_{i+1}, is $(2k-i)^2$; that is, $e_{i+1} - b_{i+1} + 1 = (2k-i)^2$, showing the second lower bound achieved for all $i > k$.

If $i \leqslant k$, then the matches between the i players present before b_{i+1} all fall in the first part of the schedule, so there are $\binom{i}{2}$ such, and $b_{i+1} = \binom{i}{2} + 1$. Similarly, after e_{i+1}, there are i players left, all $\binom{i}{2}$ matches now fall in the third part of the schedule, and $e_{i+1} = \binom{2k}{2} - \binom{i}{2}$. The first lower bound is therefore also achieved for all $i \leqslant k$.

Consequently, all lower bounds are achieved simultaneously, and the schedule is indeed optimal.

Evaluation. Finally, evaluate the total cost for the optimal schedule:

$$\Sigma = \sum_{i=0}^{k}\bigl(k(2k-1) - i(i-1)\bigr) + \sum_{i=k+1}^{2k-1}(2k-i)^2 = (k+1)k(2k-1) - \sum_{i=0}^{k} i(i-1) + \sum_{j=1}^{k-1} j^2$$
$$= k(k+1)(2k-1) - k^2 + \frac{1}{2}k(k+1) = \frac{1}{2}k(4k^2+k-1).$$

Solution 2. Consider any tournament schedule. Label players $P_1, P_2, \ldots, P_{2k}$ in order of their arrival, and label them again $Q_{2k}, Q_{2k-1}, \ldots, Q_1$ in order of their departure, to define a permutation $a_1, a_2, \ldots, a_{2k}$ of $1, 2, \ldots, 2k$ by $P_i = Q_{a_i}$.

We first describe an optimal tournament for any given permutation $a_1, a_2, \ldots, a_{2k}$ of the indices $1, 2, \ldots, 2k$. Next, we find an optimal permutation and an optimal tournament.

Optimisation for a fixed $a_1, \ldots, a_{2k}$. We say that the *cost* of the match between P_i and P_j is the number of players present at the tournament when this match is played. Clearly, the Committee pays for each day the cost of the match of that day. Hence, we are to minimise the total cost of all matches.

Notice that Q_{2k}'s departure does not precede P_{2k}'s arrival. Hence, the number of players at the tournament monotonically increases (non-strictly) until it reaches $2k$, and then monotonically decreases (non-strictly). So, the best time to schedule the match between P_i and P_j is either when $P_{\max(i,j)}$ arrives, or when $Q_{\max(a_i,a_j)}$ departs, in which case the cost is $\min\bigl(\max(i,j), \max(a_i,a_j)\bigr)$.

Conversely, assuming that $i > j$, if this match is scheduled between the arrivals of P_i and P_{i+1}, then its cost will be exactly $i = \max(i,j)$. Similarly, one can make it cost $\max(a_i,a_j)$. Obviously, these conditions can all be simultaneously satisfied, so the minimal cost for a fixed sequence $a_1, a_2, \ldots, a_{2k}$ is

$$\Sigma(a_1, \ldots, a_{2k}) = \sum_{1 \leqslant i < j \leqslant 2k} \min\bigl(\max(i,j), \max(a_i,a_j)\bigr). \tag{1}$$

Optimising the sequence (a_i). Optimisation hinges on the lemma below.

Lemma. If $a \leqslant b$ and $c \leqslant d$, then

$$\min\bigl(\max(a,x), \max(c,y)\bigr) + \min\bigl(\max(b,x), \max(d,y)\bigr)$$
$$\geqslant \min\bigl(\max(a,x), \max(d,y)\bigr) + \min\bigl(\max(b,x), \max(c,y)\bigr).$$

Proof. Write $a' = \max(a,x) \leqslant \max(b,x) = b'$ and $c' = \max(c,y) \leqslant \max(d,y) = d'$ and check that $\min(a',c') + \min(b',d') \geqslant \min(a',d') + \min(b',c')$. $\square$

Consider a permutation $a_1, a_2, \ldots, a_{2k}$ such that $a_i < a_j$ for some $i < j$. Swapping a_i and a_j does not change the (i,j)th summand in (1), and for $\ell \notin \{i,j\}$ the sum of the (i,ℓ)th and the (j,ℓ)th summands does not increase by the Lemma. Hence the optimal value does not increase, but the number of disorders in the permutation increases. This process stops when $a_i = 2k+1-i$ for all i, so the required minimum is

$$S(2k, 2k-1, \ldots, 1) = \sum_{1 \leqslant i < j \leqslant 2k} \min\bigl(\max(i,j), \max(2k+1-i, 2k+1-j)\bigr)$$
$$= \sum_{1 \leqslant i < j \leqslant 2k} \min(j, 2k+1-i).$$

The latter sum is fairly tractable and yields the stated result; we omit the details.

Comment. If the number of players is odd, say, $2k-1$, the required minimum is $k(k-1)(4k-1)/2$. In this case, $|X| = k$, $|Y| = k-1$, the argument goes along the same lines, but some additional technicalities are to be taken care of.

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C6. Let a and b be distinct positive integers. The following infinite process takes place on an initially empty board.

(i) If there is at least a pair of equal numbers on the board, we choose such a pair and increase one of its components by a and the other by b.

(ii) If no such pair exists, we write down two times the number 0.

Prove that, no matter how we make the choices in (i), operation (ii) will be performed only finitely many times.

(Serbia)

Solution 1. We may assume $\gcd(a,b)=1$; otherwise we work in the same way with multiples of $d=\gcd(a,b)$.

Suppose that after N moves of type (ii) and some moves of type (i) we have to add two new zeros. For each integer k, denote by $f(k)$ the number of times that the number k appeared on the board up to this moment. Then $f(0)=2N$ and $f(k)=0$ for $k<0$. Since the board contains at most one $k-a$, every second occurrence of $k-a$ on the board produced, at some moment, an occurrence of k; the same stands for $k-b$. Therefore,

$$f(k)=\left\lfloor \frac{f(k-a)}{2}\right\rfloor+\left\lfloor \frac{f(k-b)}{2}\right\rfloor, \tag{1}$$

yielding

$$f(k)\geqslant \frac{f(k-a)+f(k-b)}{2}-1. \tag{2}$$

Since $\gcd(a,b)=1$, every integer $x>ab-a-b$ is expressible in the form $x=sa+tb$, with integer $s,t\geqslant 0$.

We will prove by induction on $s+t$ that if $x=sa+bt$, with s,t nonnegative integers, then

$$f(x)>\frac{f(0)}{2^{s+t}}-2. \tag{3}$$

The base case $s+t=0$ is trivial. Assume now that (3) is true for $s+t=v$. Then, if $s+t=v+1$ and $x=sa+tb$, at least one of the numbers s and t – say s – is positive, hence by (2),

$$f(x)=f(sa+tb)\geqslant \frac{f\big((s-1)a+tb\big)}{2}-1>\frac{1}{2}\left(\frac{f(0)}{2^{s+t-1}}-2\right)-1=\frac{f(0)}{2^{s+t}}-2.$$

Assume now that we must perform moves of type (ii) *ad infinitum*. Take $n=ab-a-b$ and suppose $b>a$. Since each of the numbers $n+1,n+2,\ldots,n+b$ can be expressed in the form $sa+tb$, with $0\leqslant s\leqslant b$ and $0\leqslant t\leqslant a$, after moves of type (ii) have been performed 2^{a+b+1} times and we have to add a new pair of zeros, each $f(n+k)$, $k=1,2,\ldots,b$, is at least 2. In this case (1) yields inductively $f(n+k)\geqslant 2$ for all $k\geqslant 1$. But this is absurd: after a finite number of moves, f cannot attain nonzero values at infinitely many points.

Solution 2. We start by showing that the result of the process in the problem does not depend on the way the operations are performed. For that purpose, it is convenient to modify the process a bit.

Claim 1. Suppose that the board initially contains a finite number of nonnegative integers, and one starts performing type (i) moves only. Assume that one had applied k moves which led to a *final* arrangement where no more type (i) moves are possible. Then, if one starts from the same initial arrangement, performing type (i) moves in an arbitrary fashion, then the process will necessarily stop at the same final arrangement

Proof. Throughout this proof, all moves are supposed to be of type (i).

Induct on k; the base case $k = 0$ is trivial, since no moves are possible. Assume now that $k \geqslant 1$. Fix some *canonical* process, consisting of k moves $M_1, M_2, \ldots, M_k$, and reaching the final arrangement A. Consider any *sample* process $m_1, m_2, \ldots$ starting with the same initial arrangement and proceeding as long as possible; clearly, it contains at least one move. We need to show that this process stops at A.

Let move m_1 consist in replacing two copies of x with $x + a$ and $x + b$. If move M_1 does the same, we may apply the induction hypothesis to the arrangement appearing after m_1. Otherwise, the canonical process should still contain at least one move consisting in replacing $(x, x) \mapsto (x + a, x + b)$, because the initial arrangement contains at least two copies of x, while the final one contains at most one such.

Let M_i be the first such move. Since the copies of x are indistinguishable and no other copy of x disappeared before M_i in the canonical process, the moves in this process can be permuted as $M_i, M_1, \ldots, M_{i-1}, M_{i+1}, \ldots, M_k$, without affecting the final arrangement. Now it suffices to perform the move $m_1 = M_i$ and apply the induction hypothesis as above. □

Claim 2. Consider any process starting from the empty board, which involved exactly n moves of type (ii) and led to a final arrangement where all the numbers are distinct. Assume that one starts with the board containing $2n$ zeroes (as if n moves of type (ii) were made in the beginning), applying type (i) moves in an arbitrary way. Then this process will reach the same final arrangement.

Proof. Starting with the board with $2n$ zeros, one may indeed model the first process mentioned in the statement of the claim, omitting the type (ii) moves. This way, one reaches the same final arrangement. Now, Claim 1 yields that this final arrangement will be obtained when type (i) moves are applied arbitrarily. □

Claim 2 allows now to reformulate the problem statement as follows: *There exists an integer n such that, starting from $2n$ zeroes, one may apply type (i) moves indefinitely.*

In order to prove this, we start with an obvious induction on $s + t = k \geqslant 1$ to show that if we start with 2^{s+t} zeros, then we can get simultaneously on the board, at some point, each of the numbers $sa + tb$, with $s + t = k$.

Suppose now that $a < b$. Then, an appropriate use of separate groups of zeros allows us to get two copies of each of the numbers $sa + tb$, with $1 \leqslant s, t \leqslant b$.

Define $N = ab - a - b$, and notice that after representing each of numbers $N + k$, $1 \leqslant k \leqslant b$, in the form $sa + tb$, $1 \leqslant s, t \leqslant b$ we can get, using enough zeros, the numbers $N + 1, N + 2, \ldots, N + a$ and the numbers $N + 1, N + 2, \ldots, N + b$.

From now on we can perform only moves of type (i). Indeed, if $n \geqslant N$, the occurrence of the numbers $n + 1, n + 2, \ldots, n + a$ and $n + 1, n + 2, \ldots, n + b$ and the replacement $(n + 1, n + 1) \mapsto (n + b + 1, n + a + 1)$ leads to the occurrence of the numbers $n + 2, n + 3, \ldots, n + a + 1$ and $n + 2, n + 3, \ldots, n + b + 1$.

Comment. The proofs of Claims 1 and 2 may be extended in order to show that in fact the number of moves in the canonical process is the same as in an arbitrary sample one.

C7. Consider 2018 pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular *edges* that meet at *vertices*. Notice that there are an even number of vertices on each circle. Given the circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coloured twice — once for each of the two circles that cross at that point. If the two colourings agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least 2061 yellow points, then the vertices of some region are all yellow.

(India)

Solution 1. Letting $n = 2018$, we will show that, if every region has at least one non-yellow vertex, then every circle contains at most $n + \lfloor\sqrt{n-2}\rfloor - 2$ yellow points. In the case at hand, the latter equals $2018 + 44 - 2 = 2060$, contradicting the hypothesis.

Consider the natural geometric graph G associated with the configuration of n circles. Fix any circle C in the configuration, let k be the number of yellow points on C, and find a suitable lower bound for the total number of yellow vertices of G in terms of k and n. It turns out that k is even, and G has at least

$$k + 2\binom{k/2}{2} + 2\binom{n-k/2-1}{2} = \frac{k^2}{2} - (n-2)k + (n-2)(n-1) \tag{$*$}$$

yellow vertices. The proof hinges on the two lemmata below.

Lemma 1. Let two circles in the configuration cross at x and y. Then x and y are either both yellow or both non-yellow.

Proof. This is because the numbers of interior vertices on the four arcs x and y determine on the two circles have like parities. □

In particular, each circle in the configuration contains an even number of yellow vertices.

Lemma 2. If $\widehat{xy}$, $\widehat{yz}$, and $\widehat{zx}$ are circular arcs of three pairwise distinct circles in the configuration, then the number of yellow vertices in the set $\{x, y, z\}$ is odd.

Proof. Let C_1, C_2, C_3 be the three circles under consideration. Assume, without loss of generality, that C_2 and C_3 cross at x, C_3 and C_1 cross at y, and C_1 and C_2 cross at z. Let k_1, k_2, k_3 be the numbers of interior vertices on the three circular arcs under consideration. Since each circle in the configuration, different from the C_i, crosses the cycle $\widehat{xy} \cup \widehat{yz} \cup \widehat{zx}$ at an even number of points (recall that no three circles are concurrent), and self-crossings are counted twice, the sum $k_1 + k_2 + k_3$ is even.

Let Z_1 be the colour z gets from C_1 and define the other colours similarly. By the preceding, the number of bichromatic pairs in the list (Z_1, Y_1), (X_2, Z_2), (Y_3, X_3) is odd. Since the total number of colour changes in a cycle Z_1–Y_1–Y_3–X_3–X_2–Z_2–Z_1 is even, the number of bichromatic pairs in the list (X_2, X_3), (Y_1, Y_3), (Z_1, Z_2) is odd, and the lemma follows. □

We are now in a position to prove that $(*)$ bounds the total number of yellow vertices from below. Refer to Lemma 1 to infer that the k yellow vertices on C pair off to form the pairs of points where C is crossed by $k/2$ circles in the configuration. By Lemma 2, these circles cross pairwise to account for another $2\binom{k/2}{2}$ yellow vertices. Finally, the remaining $n - k/2 - 1$ circles in the configuration cross C at non-yellow vertices, by Lemma 1, and Lemma 2 applies again to show that these circles cross pairwise to account for yet another $2\binom{n-k/2-1}{2}$ yellow vertices. Consequently, there are at least $(*)$ yellow vertices.

Next, notice that G is a plane graph on $n(n-1)$ degree 4 vertices, having exactly $2n(n-1)$ edges and exactly $n(n-1)+2$ faces (regions), the outer face inclusive (by Euler's formula for planar graphs).

Lemma 3. Each face of G has equally many red and blue vertices. In particular, each face has an even number of non-yellow vertices.

Proof. Trace the boundary of a face once in circular order, and consider the colours each vertex is assigned in the colouring of the two circles that cross at that vertex, to infer that colours of non-yellow vertices alternate. □

Consequently, if each region has at least one non-yellow vertex, then it has at least two such. Since each vertex of G has degree 4, consideration of vertex-face incidences shows that G has at least $n(n-1)/2+1$ non-yellow vertices, and hence at most $n(n-1)/2-1$ yellow vertices. (In fact, Lemma 3 shows that there are at least $n(n-1)/4+1/2$ red, respectively blue, vertices.)

Finally, recall the lower bound $(*)$ for the total number of yellow vertices in G, to write $n(n-1)/2-1 \geqslant k^2/2-(n-2)k+(n-2)(n-1)$, and conclude that $k \leqslant n+\lfloor\sqrt{n-2}\rfloor-2$, as claimed in the first paragraph.

Solution 2. The first two lemmata in Solution 1 show that the circles in the configuration split into two classes: Consider any circle C along with all circles that cross C at yellow points to form one class; the remaining circles then form the other class. Lemma 2 shows that any pair of circles in the same class cross at yellow points; otherwise, they cross at non-yellow points.

Call the circles from the two classes *white* and *black*, respectively. Call a region *yellow* if its vertices are all yellow. Let w and b be the numbers of white and black circles, respectively; clearly, $w+b=n$. Assume that $w \geqslant b$, and that there is no yellow region. Clearly, $b \geqslant 1$, otherwise each region is yellow. The white circles subdivide the plane into $w(w-1)+2$ larger regions — call them *white*. The white regions (or rather their boundaries) subdivide each black circle into *black arcs*. Since there are no yellow regions, each white region contains at least one black arc.

Consider any white region; let it contain $t \geqslant 1$ black arcs. We claim that the number of points at which these t arcs cross does not exceed $t-1$. To prove this, consider a multigraph whose vertices are these black arcs, two vertices being joined by an edge for each point at which the corresponding arcs cross. If this graph had more than $t-1$ edges, it would contain a cycle, since it has t vertices; this cycle would correspond to a closed contour formed by black sub-arcs, lying inside the region under consideration. This contour would, in turn, define at least one yellow region, which is impossible.

Let t_i be the number of black arcs inside the i^{th} white region. The total number of black arcs is $\sum_i t_i = 2wb$, and they cross at $2\binom{b}{2}=b(b-1)$ points. By the preceding,

$$b(b-1) \leqslant \sum_{i=1}^{w^2-w+2}(t_i-1) = \sum_{i=1}^{w^2-w+2} t_i-(w^2-w+2) = 2wb-(w^2-w+2),$$

or, equivalently, $(w-b)^2 \leqslant w+b-2=n-2$, which is the case if and only if $w-b \leqslant \lfloor\sqrt{n-2}\rfloor$. Consequently, $b \leqslant w \leqslant \left(n+\lfloor\sqrt{n-2}\rfloor\right)/2$, so there are at most $2(w-1) \leqslant n+\lfloor\sqrt{n-2}\rfloor-2$ yellow vertices on each circle — a contradiction.

Geometry

G1. Let ABC be an acute-angled triangle with circumcircle Γ. Let D and E be points on the segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of the segments BD and CE intersect the small arcs $\widehat{AB}$ and $\widehat{AC}$ at points F and G respectively. Prove that $DE \parallel FG$.

(Greece)

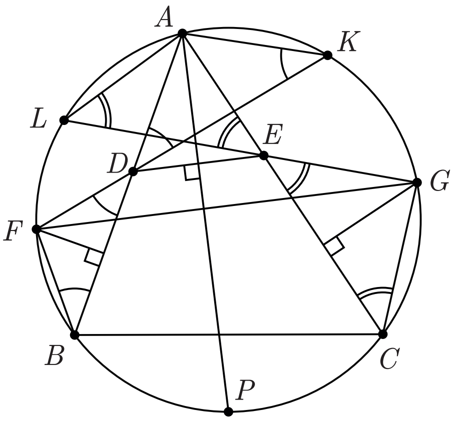
Solution 1. In the sequel, all the considered arcs are small arcs.

Let P be the midpoint of the arc $\widehat{BC}$. Then AP is the bisector of $\angle BAC$, hence, in the isosceles triangle ADE, $AP \perp DE$. So, the statement of the problem is equivalent to $AP \perp FG$.

In order to prove this, let K be the second intersection of Γ with FD. Then the triangle FBD is isosceles, therefore

$$\angle AKF = \angle ABF = \angle FDB = \angle ADK,$$

yielding $AK = AD$. In the same way, denoting by L the second intersection of Γ with GE, we get $AL = AE$. This shows that $AK = AL$.



Now $\angle FBD = \angle FDB$ gives $\widehat{AF} = \widehat{BF} + \widehat{AK} = \widehat{BF} + \widehat{AL}$, hence $\widehat{BF} = \widehat{LF}$. In a similar way, we get $\widehat{CG} = \widehat{GK}$. This yields

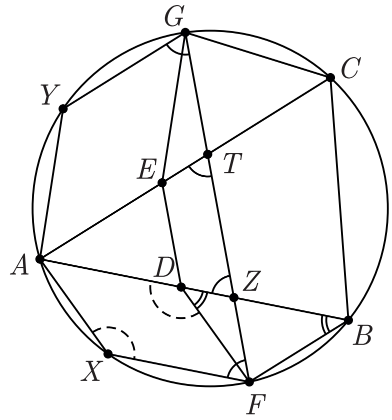
$$\angle(AP, FG) = \frac{\widehat{AF} + \widehat{PG}}{2} = \frac{\widehat{AL} + \widehat{LF} + \widehat{PC} + \widehat{CG}}{2} = \frac{\widehat{KL} + \widehat{LB} + \widehat{BC} + \widehat{CK}}{4} = 90^\circ.$$

Solution 2. Let $Z = AB \cap FG$, $T = AC \cap FG$. It suffices to prove that $\angle ATZ = \angle AZT$.

Let X be the point for which $FXAD$ is a parallelogram. Then

$$\angle FXA = \angle FDA = 180^\circ - \angle FDB = 180^\circ - \angle FBD,$$

where in the last equality we used that $FD = FB$. It follows that the quadrilateral $BFXA$ is cyclic, so X lies on Γ.

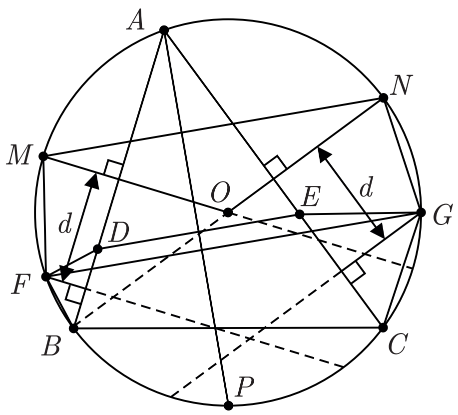


Analogously, if Y is the point for which $GYAE$ is a parallelogram, then Y lies on Γ. So the quadrilateral $XFGY$ is cyclic and $FX = AD = AE = GY$, hence $XFGY$ is an isosceles trapezoid.

Now, by $XF \parallel AZ$ and $YG \parallel AT$, it follows that $\angle ATZ = \angle YGF = \angle XFG = \angle AZT$.

Solution 3. As in the first solution, we prove that $FG \perp AP$, where P is the midpoint of the small arc $\overparen{BC}$.

Let O be the circumcentre of the triangle ABC, and let M and N be the midpoints of the small arcs $\overparen{AB}$ and $\overparen{AC}$, respectively. Then OM and ON are the perpendicular bisectors of AB and AC, respectively.



The distance d between OM and the perpendicular bisector of BD is $\frac{1}{2}AB - \frac{1}{2}BD = \frac{1}{2}AD$, hence it is equal to the distance between ON and the perpendicular bisector of CE.

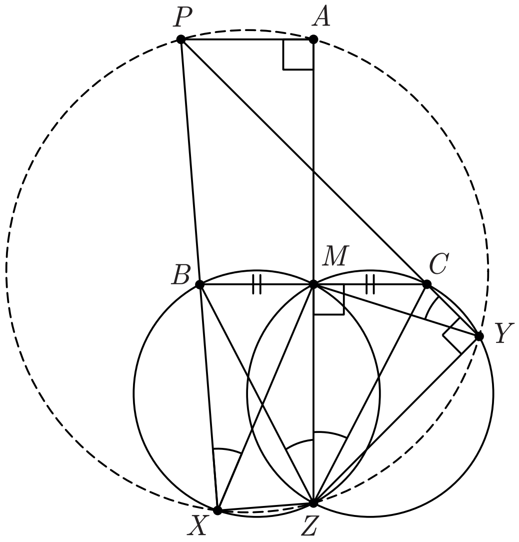
This shows that the isosceles trapezoid determined by the diameter δ of Γ through M and the chord parallel to δ through F is congruent to the isosceles trapezoid determined by the diameter δ' of Γ through N and the chord parallel to δ' through G. Therefore $MF = NG$, yielding $MN \parallel FG$.

Now
$$\angle(MN, AP) = \frac{1}{2}\left(\overparen{AM} + \overparen{PC} + \overparen{CN}\right) = \frac{1}{4}\left(\overparen{AB} + \overparen{BC} + \overparen{CA}\right) = 90^\circ,$$
hence $MN \perp AP$, and the conclusion follows.

G2. Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

(Australia)

Solution. Since $AB = AC$, AM is the perpendicular bisector of BC, hence $\angle PAM = \angle AMC = 90^\circ$.



Now let Z be the common point of AM and the perpendicular through Y to PC (notice that Z lies on to the ray AM beyond M). We have $\angle PAZ = \angle PYZ = 90^\circ$. Thus the points P, A, Y, and Z are concyclic.

Since $\angle CMZ = \angle CYZ = 90^\circ$, the quadrilateral $CYZM$ is cyclic, hence $\angle CZM = \angle CYM$. By the condition in the statement, $\angle CYM = \angle BXM$, and, by symmetry in ZM, $\angle CZM = \angle BZM$. Therefore, $\angle BXM = \angle BZM$. It follows that the points B, X, Z, and M are concyclic, hence $\angle BXZ = 180^\circ - \angle BMZ = 90^\circ$.

Finally, we have $\angle PXZ = \angle PYZ = \angle PAZ = 90^\circ$, hence the five points P, A, X, Y, Z are concyclic. In particular, the quadrilateral $APXY$ is cyclic, as required.

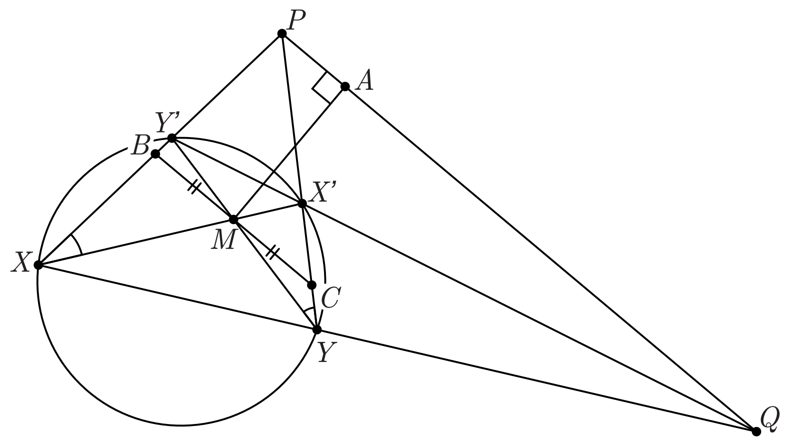
Comment 1. Clearly, the key point Z from the solution above can be introduced in several different ways, e.g., as the second meeting point of the circle CMY and the line AM, or as the second meeting point of the circles CMY and BMX, etc.

For some of definitions of Z its location is not obvious. For instance, if Z is defined as a common point of AM and the perpendicular through X to PX, it is not clear that Z lies on the ray AM beyond M. To avoid such slippery details some more restrictions on the construction may be required.

Comment 2. Let us discuss a connection to the Miquel point of a cyclic quadrilateral. Set $X' = MX \cap PC$, $Y' = MY \cap PB$, and $Q = XY \cap X'Y'$ (see the figure below).

We claim that $BC \parallel PQ$. (One way of proving this is the following. Notice that the quadruple of lines PX, PM, PY, PQ is harmonic, hence the quadruple B, M, C, $PQ \cap BC$ of their intersection points with BC is harmonic. Since M is the midpoint of BC, $PQ \cap BC$ is an ideal point, i.e., $PQ \parallel BC$.)

It follows from the given equality $\angle PXM = \angle PYM$ that the quadrilateral $XYX'Y'$ is cyclic. Note that A is the projection of M onto PQ. By a known description, A is the Miquel point for the sidelines $XY, XY', X'Y, X'Y'$. In particular, the circle PXY passes through A.



Comment 3. An alternative approach is the following. One can note that the (oriented) lengths of the segments CY and BX are both linear functions of a parameter $t = \cot \angle PXM$. As t varies, the intersection point S of the perpendicular bisectors of PX and PY traces a fixed line, thus the family of circles PXY has a fixed common point (other than P). By checking particular cases, one can show that this fixed point is A.

Comment 4. The problem states that $\angle PXM = \angle PYM$ implies that $APXY$ is cyclic. The original submission claims that these two conditions are in fact equivalent. The Problem Selection Committee omitted the converse part, since it follows easily from the direct one, by reversing arguments.

G3. A circle ω of radius 1 is given. A collection T of triangles is called *good*, if the following conditions hold:

(i) each triangle from T is inscribed in ω;

(ii) no two triangles from T have a common interior point.

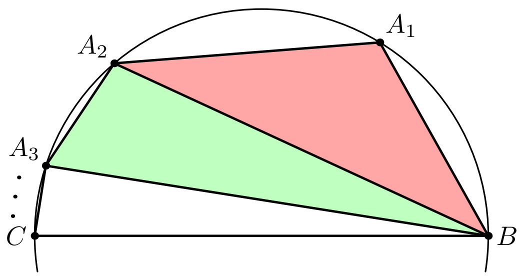
Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.

(South Africa)

Answer: $t \in (0, 4]$.

Solution. First, we show how to construct a good collection of n triangles, each of perimeter greater than 4. This will show that all $t \leqslant 4$ satisfy the required conditions.

Construct inductively an $(n+2)$-gon $BA_1A_2 \ldots A_nC$ inscribed in ω such that BC is a diameter, and BA_1A_2, BA_2A_3, $\ldots$, $BA_{n-1}A_n$, BA_nC is a good collection of n triangles. For $n = 1$, take any triangle BA_1C inscribed in ω such that BC is a diameter; its perimeter is greater than $2BC = 4$. To perform the inductive step, assume that the $(n+2)$-gon $BA_1A_2 \ldots A_nC$ is already constructed. Since $A_nB + A_nC + BC > 4$, one can choose a point A_{n+1} on the small arc $\widehat{CA_n}$, close enough to C, so that $A_nB + A_nA_{n+1} + BA_{n+1}$ is still greater than 4. Thus each of these new triangles BA_nA_{n+1} and $BA_{n+1}C$ has perimeter greater than 4, which completes the induction step.



We proceed by showing that no $t > 4$ satisfies the conditions of the problem. To this end, we assume that there exists a good collection T of n triangles, each of perimeter greater than t, and then bound n from above.

Take $\varepsilon > 0$ such that $t = 4 + 2\varepsilon$.

Claim. There exists a positive constant $\sigma = \sigma(\varepsilon)$ such that any triangle Δ with perimeter $2s \geqslant 4 + 2\varepsilon$, inscribed in ω, has area $S(\Delta)$ at least σ.

Proof. Let a, b, c be the side lengths of Δ. Since Δ is inscribed in ω, each side has length at most 2. Therefore, $s - a \geqslant (2 + \varepsilon) - 2 = \varepsilon$. Similarly, $s - b \geqslant \varepsilon$ and $s - c \geqslant \varepsilon$. By Heron's formula, $S(\Delta) = \sqrt{s(s-a)(s-b)(s-c)} \geqslant \sqrt{(2+\varepsilon)\varepsilon^3}$. Thus we can set $\sigma(\varepsilon) = \sqrt{(2+\varepsilon)\varepsilon^3}$. $\square$

Now we see that the total area S of all triangles from T is at least $n\sigma(\varepsilon)$. On the other hand, S does not exceed the area of the disk bounded by ω. Thus $n\sigma(\varepsilon) \leqslant \pi$, which means that n is bounded from above.

Comment 1. One may prove the Claim using the formula $S = \dfrac{abc}{4R}$ instead of Heron's formula.

Comment 2. In the statement of the problem condition (i) could be replaced by a weaker one: each triangle from T lies within ω. This does not affect the solution above, but reduces the number of ways to prove the Claim.

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G4. A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

(Mongolia)

Solution. By $\measuredangle(\ell, n)$ we always mean the directed angle of the lines ℓ and n, taken modulo 180°.

Let CC_2 meet Ω again at K (as usual, if CC_2 is tangent to Ω, we set $T = C_2$). We show that the line BB_2 contains K; similarly, AA_2 will also pass through K. For this purpose, it suffices to prove that

$$\measuredangle(C_2C, C_2A_1) = \measuredangle(B_2B, B_2A_1). \tag{1}$$

By the problem condition, CB and CA are the perpendicular bisectors of TA_1 and TB_1, respectively. Hence, C is the circumcentre of the triangle A_1TB_1. Therefore,

$$\measuredangle(CA_1, CB) = \measuredangle(CB, CT) = \measuredangle(B_1A_1, B_1T) = \measuredangle(B_1A_1, B_1B_2).$$

In circle Ω we have $\measuredangle(B_1A_1, B_1B_2) = \measuredangle(C_2A_1, C_2B_2)$. Thus,

$$\measuredangle(CA_1, CB) = \measuredangle(B_1A_1, B_1B_2) = \measuredangle(C_2A_1, C_2B_2). \tag{2}$$

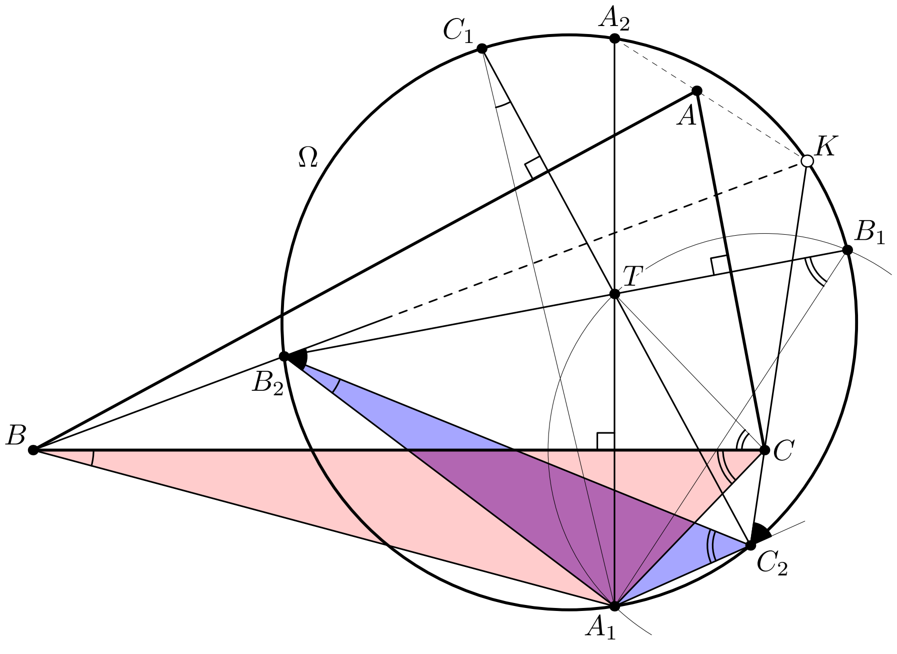
Similarly, we get

$$\measuredangle(BA_1, BC) = \measuredangle(C_1A_1, C_1C_2) = \measuredangle(B_2A_1, B_2C_2). \tag{3}$$

The two obtained relations yield that the triangles A_1BC and $A_1B_2C_2$ are similar and equioriented, hence

$$\frac{A_1B_2}{A_1B} = \frac{A_1C_2}{A_1C} \quad \text{and} \quad \measuredangle(A_1B, A_1C) = \measuredangle(A_1B_2, A_1C_2).$$

The second equality may be rewritten as $\measuredangle(A_1B, A_1B_2) = \measuredangle(A_1C, A_1C_2)$, so the triangles A_1BB_2 and A_1CC_2 are also similar and equioriented. This establishes (1).



Comment 1. In fact, the triangle A_1BC is an image of $A_1B_2C_2$ under a spiral similarity centred at A_1; in this case, the triangles ABB_2 and ACC_2 are also spirally similar with the same centre.

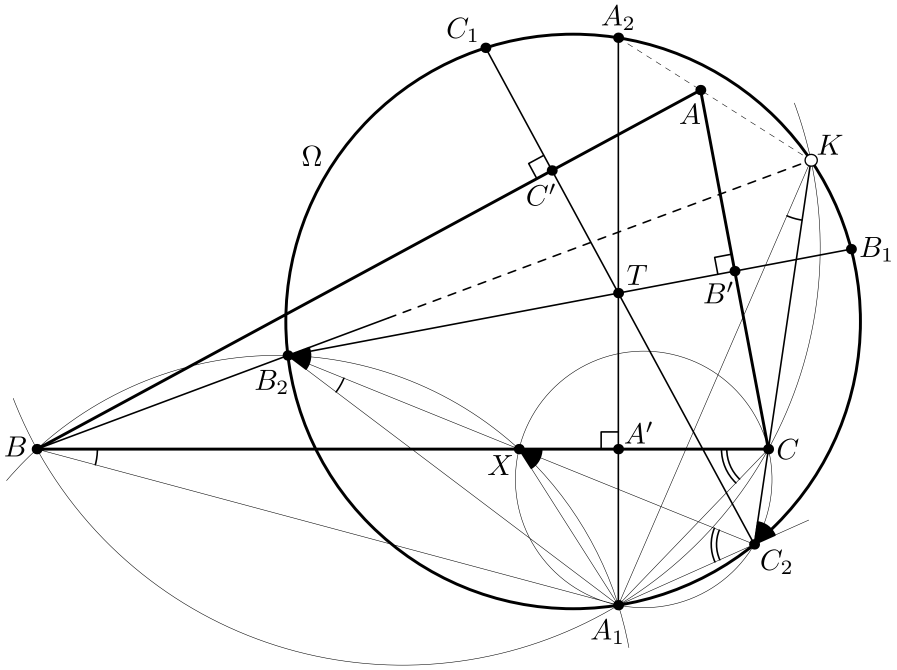
Comment 2. After obtaining (2) and (3), one can finish the solution in different ways.

For instance, introducing the point $X = BC \cap B_2C_2$, one gets from these relations that the 4-tuples (A_1, B, B_2, X) and (A_1, C, C_2, X) are both cyclic. Therefore, K is the Miquel point of the lines BB_2, CC_2, BC, and B_2C_2; this yields that the meeting point of BB_2 and CC_2 lies on Ω.

Yet another way is to show that the points A_1, B, C, and K are concyclic, as

$$\measuredangle(KC, KA_1) = \measuredangle(B_2C_2, B_2A_1) = \measuredangle(BC, BA_1).$$

By symmetry, the second point K' of intersection of BB_2 with Ω is also concyclic to A_1, B, and C, hence $K' = K$.



Comment 3. The requirement that the common point of the lines AA_2, BB_2, and CC_2 should lie on Ω may seem to make the problem easier, since it suggests some approaches. On the other hand, there are also different ways of showing that the lines AA_2, BB_2, and CC_2 are just concurrent.

In particular, the problem conditions yield that the lines A_2T, B_2T, and C_2T are perpendicular to the corresponding sides of the triangle ABC. One may show that the lines AT, BT, and CT are also perpendicular to the corresponding sides of the triangle $A_2B_2C_2$, i.e., the triangles ABC and $A_2B_2C_2$ are *orthologic*, and their orthology centres coincide. It is known that such triangles are also *perspective*, i.e. the lines AA_2, BB_2, and CC_2 are concurrent (in projective sense).

To show this mutual orthology, one may again apply angle chasing, but there are also other methods. Let A', B', and C' be the projections of T onto the sides of the triangle ABC. Then $A_2T \cdot TA' = B_2T \cdot TB' = C_2T \cdot TC'$, since all three products equal (minus) half the power of T with respect to Ω. This means that A_2, B_2, and C_2 are the poles of the sidelines of the triangle ABC with respect to some circle centred at T and having *pure imaginary* radius (in other words, the reflections of A_2, B_2, and C_2 in T are the poles of those sidelines with respect to some regular circle centred at T). Hence, dually, the vertices of the triangle ABC are also the poles of the sidelines of the triangle $A_2B_2C_2$.

G5. Let ABC be a triangle with circumcircle ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to ω.

(Denmark)

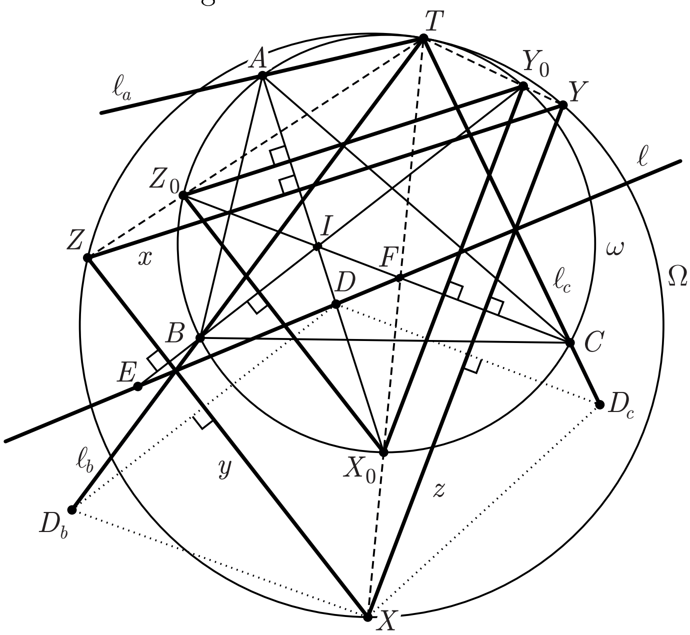
Preamble. Let $X = y \cap z$, $Y = x \cap z$, $Z = x \cap y$ and let Ω denote the circumcircle of the triangle XYZ. Denote by X_0, Y_0, and Z_0 the second intersection points of AI, BI and CI, respectively, with ω. It is known that Y_0Z_0 is the perpendicular bisector of AI, Z_0X_0 is the perpendicular bisector of BI, and X_0Y_0 is the perpendicular bisector of CI. In particular, the triangles XYZ and $X_0Y_0Z_0$ are homothetic, because their corresponding sides are parallel.

The solutions below mostly exploit the following approach. Consider the triangles XYZ and $X_0Y_0Z_0$, or some other pair of homothetic triangles Δ and δ inscribed into Ω and ω, respectively. In order to prove that Ω and ω are tangent, it suffices to show that the centre T of the homothety taking Δ to δ lies on ω (or Ω), or, in other words, to show that Δ and δ are perspective (i.e., the lines joining corresponding vertices are concurrent), with their perspector lying on ω (or Ω).

We use directed angles throughout all the solutions.

Solution 1.

Claim 1. The reflections ℓ_a, ℓ_b and ℓ_c of the line ℓ in the lines x, y, and z, respectively, are concurrent at a point T which belongs to ω.



Proof. Notice that $\measuredangle(\ell_b, \ell_c) = \measuredangle(\ell_b, \ell) + \measuredangle(\ell, \ell_c) = 2\measuredangle(y, \ell) + 2\measuredangle(\ell, z) = 2\measuredangle(y, z)$. But $y \perp BI$ and $z \perp CI$ implies $\measuredangle(y, z) = \measuredangle(BI, IC)$, so, since $2\measuredangle(BI, IC) = \measuredangle(BA, AC)$, we obtain

$$\measuredangle(\ell_b, \ell_c) = \measuredangle(BA, AC). \tag{1}$$

Since A is the reflection of D in x, A belongs to ℓ_a; similarly, B belongs to ℓ_b. Then (1) shows that the common point T' of ℓ_a and ℓ_b lies on ω; similarly, the common point T'' of ℓ_c and ℓ_b lies on ω.

If $B \notin \ell_a$ and $B \notin \ell_c$, then T' and T'' are the second point of intersection of ℓ_b and ω, hence they coincide. Otherwise, if, say, $B \in \ell_c$, then $\ell_c = BC$, so $\measuredangle(BA, AC) = \measuredangle(\ell_b, \ell_c) = \measuredangle(\ell_b, BC)$, which shows that ℓ_b is tangent at B to ω and $T' = T'' = B$. So T' and T'' coincide in all the cases, and the conclusion of the claim follows. $\square$

Now we prove that X, X_0, T are collinear. Denote by D_b and D_c the reflections of the point D in the lines y and z, respectively. Then D_b lies on ℓ_b, D_c lies on ℓ_c, and

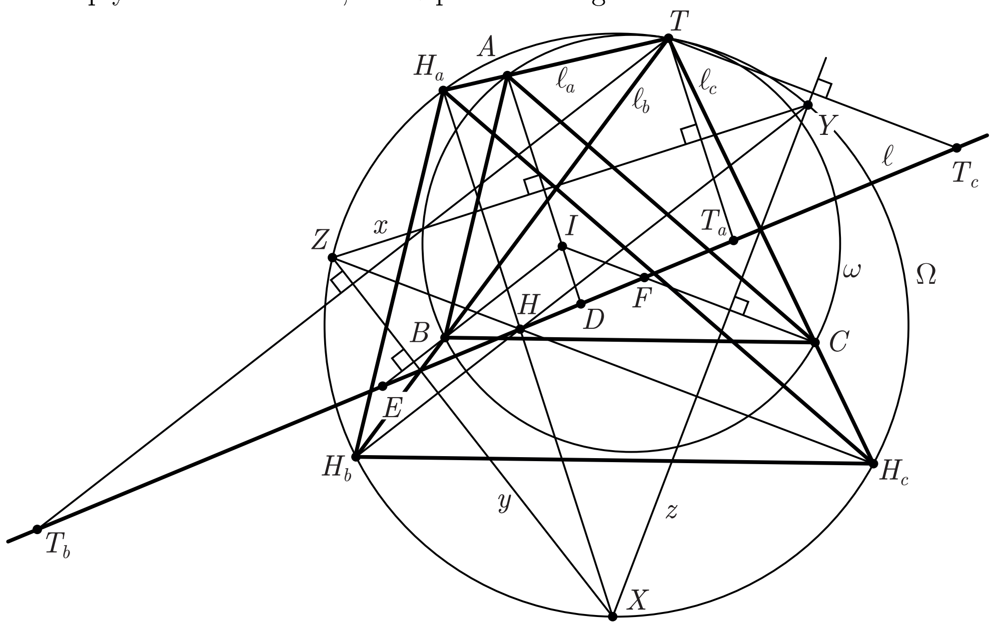
$$\begin{aligned}\measuredangle(D_bX, XD_c) &= \measuredangle(D_bX, DX) + \measuredangle(DX, XD_c) = 2\measuredangle(y, DX) + 2\measuredangle(DX, z) = 2\measuredangle(y, z)\\ &= \measuredangle(BA, AC) = \measuredangle(BT, TC),\end{aligned}$$

hence the quadrilateral XD_bTD_c is cyclic. Notice also that since $XD_b = XD = XD_c$, the points D, D_b, D_c lie on a circle with centre X. Using in this circle the diameter $D_cD'_c$ yields $\measuredangle(D_bD_c, D_cX) = 90^\circ + \measuredangle(D_bD'_c, D'_cX) = 90^\circ + \measuredangle(D_bD, DD_c)$. Therefore,

$$\begin{aligned}\measuredangle(\ell_b, XT) &= \measuredangle(D_bT, XT) = \measuredangle(D_bD_c, D_cX) = 90^\circ + \measuredangle(D_bD, DD_c)\\ &= 90^\circ + \measuredangle(BI, IC) = \measuredangle(BA, AI) = \measuredangle(BA, AX_0) = \measuredangle(BT, TX_0) = \measuredangle(\ell_b, X_0T),\end{aligned}$$

so the points X, X_0, T are collinear. By a similar argument, Y, Y_0, T and Z, Z_0, T are collinear. As mentioned in the preamble, the statement of the problem follows.

Comment 1. After proving Claim 1 one may proceed in another way. As it was shown, the reflections of ℓ in the sidelines of XYZ are concurrent at T. Thus ℓ is the Steiner line of T with respect to ΔXYZ (that is the line containing the reflections T_a, T_b, T_c of T in the sidelines of XYZ). The properties of the Steiner line imply that T lies on Ω, and ℓ passes through the orthocentre H of the triangle XYZ.



Let H_a, H_b, and H_c be the reflections of the point H in the lines x, y, and z, respectively. Then the triangle $H_aH_bH_c$ is inscribed in Ω and homothetic to ABC (by an easy angle chasing). Since $H_a \in \ell_a$, $H_b \in \ell_b$, and $H_c \in \ell_c$, the triangles $H_aH_bH_c$ and ABC form a required pair of triangles Δ and δ mentioned in the preamble.

Comment 2. The following observation shows how one may guess the description of the tangency point T from Solution 1.

Let us fix a direction and move the line ℓ parallel to this direction with constant speed.

Then the points D, E, and F are moving with constant speeds along the lines AI, BI, and CI, respectively. In this case x, y, and z are moving with constant speeds, defining a family of homothetic triangles XYZ with a common centre of homothety T. Notice that the triangle $X_0Y_0Z_0$ belongs to this family (for ℓ passing through I). We may specify the location of T considering the degenerate case when x, y, and z are concurrent. In this degenerate case all the lines x, y, z, ℓ, ℓ_a, ℓ_b, ℓ_c have a common point. Note that the lines ℓ_a, ℓ_b, ℓ_c remain constant as ℓ is moving (keeping its direction). Thus T should be the common point of ℓ_a, ℓ_b, and ℓ_c, lying on ω.

Solution 2. As mentioned in the preamble, it is sufficient to prove that the centre T of the homothety taking XYZ to $X_0Y_0Z_0$ belongs to ω. Thus, it suffices to prove that $\measuredangle(TX_0, TY_0) = \measuredangle(Z_0X_0, Z_0Y_0)$, or, equivalently, $\measuredangle(XX_0, YY_0) = \measuredangle(Z_0X_0, Z_0Y_0)$.

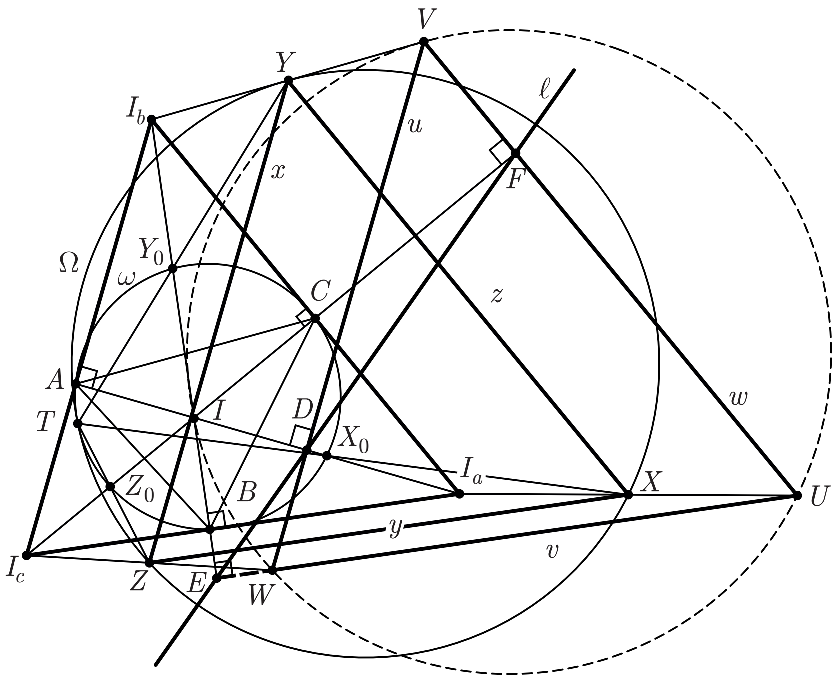
Recall that YZ and Y_0Z_0 are the perpendicular bisectors of AD and AI, respectively. Then, the vector $\overrightarrow{x}$ perpendicular to YZ and shifting the line Y_0Z_0 to YZ is equal to $\frac{1}{2}\overrightarrow{ID}$. Define the shifting vectors $\overrightarrow{y} = \frac{1}{2}\overrightarrow{IE}$, $\overrightarrow{z} = \frac{1}{2}\overrightarrow{IF}$ similarly. Consider now the triangle UVW formed by the perpendiculars to AI, BI, and CI through D, E, and F, respectively (see figure below). This is another triangle whose sides are parallel to the corresponding sides of XYZ.

Claim 2. $\overrightarrow{IU} = 2\overrightarrow{X_0X}$, $\overrightarrow{IV} = 2\overrightarrow{Y_0Y}$, $\overrightarrow{IW} = 2\overrightarrow{Z_0Z}$.

Proof. We prove one of the relations, the other proofs being similar. To prove the equality of two vectors it suffices to project them onto two non-parallel axes and check that their projections are equal.

The projection of $\overrightarrow{X_0X}$ onto IB equals $\vec{y}$, while the projection of $\overrightarrow{IU}$ onto IB is $\overrightarrow{IE} = 2\vec{y}$. The projections onto the other axis IC are $\vec{z}$ and $\overrightarrow{IF} = 2\vec{z}$. Then $\overrightarrow{IU} = 2\overrightarrow{X_0X}$ follows. $\square$

Notice that the line ℓ is the Simson line of the point I with respect to the triangle UVW; thus U, V, W, and I are concyclic. It follows from Claim 2 that $\measuredangle(XX_0, YY_0) = \measuredangle(IU, IV) = \measuredangle(WU, WV) = \measuredangle(Z_0X_0, Z_0Y_0)$, and we are done.



Solution 3. Let I_a, I_b, and I_c be the excentres of triangle ABC corresponding to A, B, and C, respectively. Also, let u, v, and w be the lines through D, E, and F which are perpendicular to AI, BI, and CI, respectively, and let UVW be the triangle determined by these lines, where $u = VW$, $v = UW$ and $w = UV$ (see figure above).

Notice that the line u is the reflection of I_bI_c in the line x, because u, x, and I_bI_c are perpendicular to AD and x is the perpendicular bisector of AD. Likewise, v and I_aI_c are reflections of each other in y, while w and I_aI_b are reflections of each other in z. It follows that X, Y, and Z are the midpoints of UI_a, VI_b and WI_c, respectively, and that the triangles UVW, XYZ and $I_aI_bI_c$ are either translates of each other or homothetic with a common homothety centre.

Construct the points T and S such that the quadrilaterals $UVIW$, $XYTZ$ and $I_aI_bSI_c$ are homothetic. Then T is the midpoint of IS. Moreover, note that ℓ is the Simson line of the point I with respect to the triangle UVW, hence I belongs to the circumcircle of the triangle UVW, therefore T belongs to Ω.

Consider now the homothety or translation h_1 that maps $XYZT$ to $I_aI_bI_cS$ and the homothety h_2 with centre I and factor $\frac{1}{2}$. Furthermore, let $h = h_2 \circ h_1$. The transform h can be a homothety or a translation, and

$$h(T) = h_2(h_1(T)) = h_2(S) = T,$$

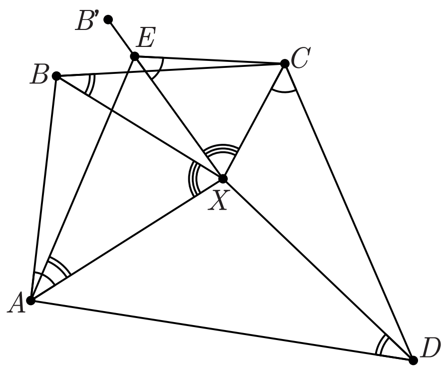
hence T is a fixed point of h. So, h is a homothety with centre T. Note that h_2 maps the excentres I_a, I_b, I_c to X_0, Y_0, Z_0 defined in the preamble. Thus the centre T of the homothety taking XYZ to $X_0Y_0Z_0$ belongs to Ω, and this completes the proof.

G6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. A point X is chosen inside the quadrilateral so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle AXB + \angle CXD = 180°$.

(Poland)

Solution 1. Let B' be the reflection of B in the internal angle bisector of $\angle AXC$, so that $\angle AXB' = \angle CXB$ and $\angle CXB' = \angle AXB$. If X, D, and B' are collinear, then we are done. Now assume the contrary.

On the ray XB' take a point E such that $XE \cdot XB = XA \cdot XC$, so that $\triangle AXE \sim \triangle BXC$ and $\triangle CXE \sim \triangle BXA$. We have $\angle XCE + \angle XCD = \angle XBA + \angle XAB < 180°$ and $\angle XAE + \angle XAD = \angle XDA + \angle XAD < 180°$, which proves that X lies inside the angles $\angle ECD$ and $\angle EAD$ of the quadrilateral $EADC$. Moreover, X lies in the interior of exactly one of the two triangles EAD, ECD (and in the exterior of the other).



The similarities mentioned above imply $XA \cdot BC = XB \cdot AE$ and $XB \cdot CE = XC \cdot AB$. Multiplying these equalities with the given equality $AB \cdot CD = BC \cdot DA$, we obtain $XA \cdot CD \cdot CE = XC \cdot AD \cdot AE$, or, equivalently,

$$\frac{XA \cdot DE}{AD \cdot AE} = \frac{XC \cdot DE}{CD \cdot CE}. \tag{$*$}$$

Lemma. Let PQR be a triangle, and let X be a point in the interior of the angle QPR such that $\angle QPX = \angle PRX$. Then $\dfrac{PX \cdot QR}{PQ \cdot PR} < 1$ if and only if X lies in the interior of the triangle PQR.

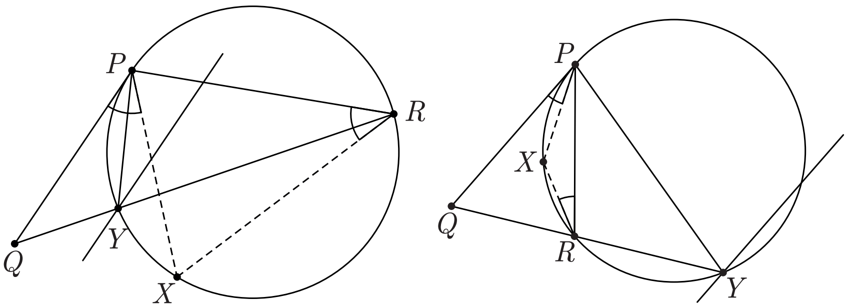
Proof. The locus of points X with $\angle QPX = \angle PRX$ lying inside the angle QPR is an arc α of the circle γ through R tangent to PQ at P. Let γ intersect the line QR again at Y (if γ is tangent to QR, then set $Y = R$). The similarity $\triangle QPY \sim \triangle QRP$ yields $PY = \dfrac{PQ \cdot PR}{QR}$. Now it suffices to show that $PX < PY$ if and only if X lies in the interior of the triangle PQR. Let m be a line through Y parallel to PQ. Notice that the points Z of γ satisfying $PZ < PY$ are exactly those between the lines m and PQ.

Case 1: Y lies in the segment QR (see the left figure below).

In this case Y splits α into two arcs $\overparen{PY}$ and $\overparen{YR}$. The arc $\overparen{PY}$ lies inside the triangle PQR, and $\overparen{PY}$ lies between m and PQ, hence $PX < PY$ for points $X \in \overparen{PY}$. The other arc $\overparen{YR}$ lies outside triangle PQR, and $\overparen{YR}$ is on the opposite side of m than P, hence $PX > PY$ for $X \in \overparen{YR}$.

Case 2: Y lies on the ray QR beyond R (see the right figure below).

In this case the whole arc α lies inside triangle PQR, and between m and PQ, thus $PX < PY$ for all $X \in \alpha$. $\square$



Applying the Lemma (to $\triangle EAD$ with the point X, and to $\triangle ECD$ with the point X), we obtain that exactly one of two expressions $\dfrac{XA \cdot DE}{AD \cdot AE}$ and $\dfrac{XC \cdot DE}{CD \cdot CE}$ is less than 1, which contradicts $(*)$.

Comment 1. One may show that $AB \cdot CD = XA \cdot XC + XB \cdot XD$. We know that D, X, E are collinear and $\angle DCE = \angle CXD = 180° - \angle AXB$. Therefore,

$$AB \cdot CD = XB \cdot \frac{\sin \angle AXB}{\sin \angle BAX} \cdot DE \cdot \frac{\sin \angle CED}{\sin \angle DCE} = XB \cdot DE.$$

Furthermore, $XB \cdot DE = XB \cdot (XD + XE) = XB \cdot XD + XB \cdot XE = XB \cdot XD + XA \cdot XC$.

Comment 2. For a convex quadrilateral $ABCD$ with $AB \cdot CD = BC \cdot DA$, it is known that $\angle DAC + \angle ABD + \angle BCA + \angle CDB = 180°$ (among other, it was used as a problem on the Regional round of All-Russian olympiad in 2012), but it seems that there is no essential connection between this fact and the original problem.

Solution 2. The solution consists of two parts. In Part 1 we show that it suffices to prove that

$$\frac{XB}{XD} = \frac{AB}{CD} \tag{1}$$

and

$$\frac{XA}{XC} = \frac{DA}{BC}. \tag{2}$$

In Part 2 we establish these equalities.

Part 1. Using the sine law and applying (1) we obtain

$$\frac{\sin \angle AXB}{\sin \angle XAB} = \frac{AB}{XB} = \frac{CD}{XD} = \frac{\sin \angle CXD}{\sin \angle XCD},$$

so $\sin \angle AXB = \sin \angle CXD$ by the problem conditions. Similarly, (2) yields $\sin \angle DXA = \sin \angle BXC$. If at least one of the pairs $(\angle AXB, \angle CXD)$ and $(\angle BXC, \angle DXA)$ consists of supplementary angles, then we are done. Otherwise, $\angle AXB = \angle CXD$ and $\angle DXA = \angle BXC$. In this case $X = AC \cap BD$, and the problem conditions yield that $ABCD$ is a parallelogram and hence a rhombus. In this last case the claim also holds.

Part 2. To prove the desired equality (1), invert $ABCD$ at centre X with unit radius; the images of points are denoted by primes.

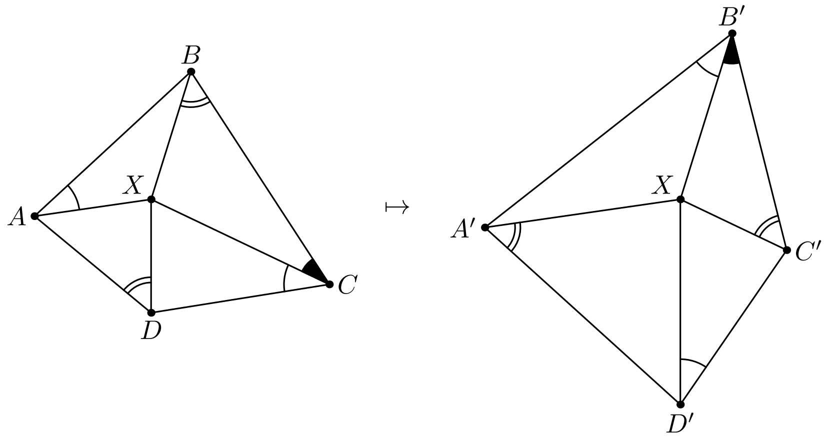
We have

$$\angle A'B'C' = \angle XB'A' + \angle XB'C' = \angle XAB + \angle XCB = \angle XCD + \angle XCB = \angle BCD.$$

Similarly, the corresponding angles of quadrilaterals $ABCD$ and $D'A'B'C'$ are equal.

Moreover, we have

$$A'B' \cdot C'D' = \frac{AB}{XA \cdot XB} \cdot \frac{CD}{XC \cdot XD} = \frac{BC}{XB \cdot XC} \cdot \frac{DA}{XD \cdot DA} = B'C' \cdot D'A'.$$



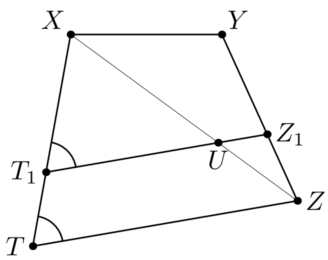
Now we need the following Lemma.

Lemma. Assume that the corresponding angles of convex quadrilaterals $XYZT$ and $X'Y'Z'T'$ are equal, and that $XY \cdot ZT = YZ \cdot TX$ and $X'Y' \cdot Z'T' = Y'Z' \cdot T'X'$. Then the two quadrilaterals are similar.

Proof. Take the quadrilateral XYZ_1T_1 similar to $X'Y'Z'T'$ and sharing the side XY with $XYZT$, such that Z_1 and T_1 lie on the rays YZ and XT, respectively, and $Z_1T_1 \parallel ZT$. We need to prove that $Z_1 = Z$ and $T_1 = T$. Assume the contrary. Without loss of generality, $TX > XT_1$. Let segments XZ and Z_1T_1 intersect at U. We have

$$\frac{T_1X}{T_1Z_1} < \frac{T_1X}{T_1U} = \frac{TX}{ZT} = \frac{XY}{YZ} < \frac{XY}{YZ_1},$$

thus $T_1X \cdot YZ_1 < T_1Z_1 \cdot XY$. A contradiction. □



It follows from the Lemma that the quadrilaterals $ABCD$ and $D'A'B'C'$ are similar, hence

$$\frac{BC}{AB} = \frac{A'B'}{D'A'} = \frac{AB}{XA \cdot XB} \cdot \frac{XD \cdot XA}{DA} = \frac{AB}{AD} \cdot \frac{XD}{XB},$$

and therefore

$$\frac{XB}{XD} = \frac{AB^2}{BC \cdot AD} = \frac{AB^2}{AB \cdot CD} = \frac{AB}{CD}.$$

We obtain (1), as desired; (2) is proved similarly.

Comment. Part 1 is an easy one, while part 2 seems to be crucial. On the other hand, after the proof of the similarity $D'A'B'C' \sim ABCD$ one may finish the solution in different ways, e.g., as follows. The similarity taking $D'A'B'C'$ to $ABCD$ maps X to the point X' *isogonally conjugate* of X with respect to $ABCD$ (i.e. to the point X' inside $ABCD$ such that $\angle BAX = \angle DAX'$, $\angle CBX = \angle ABX'$, $\angle DCX = \angle BCX'$, $\angle ADX = \angle CDX'$). It is known that the required equality $\angle AXB + \angle CXD = 180^\circ$ is one of known conditions on a point X inside $ABCD$ equivalent to the existence of its isogonal conjugate.

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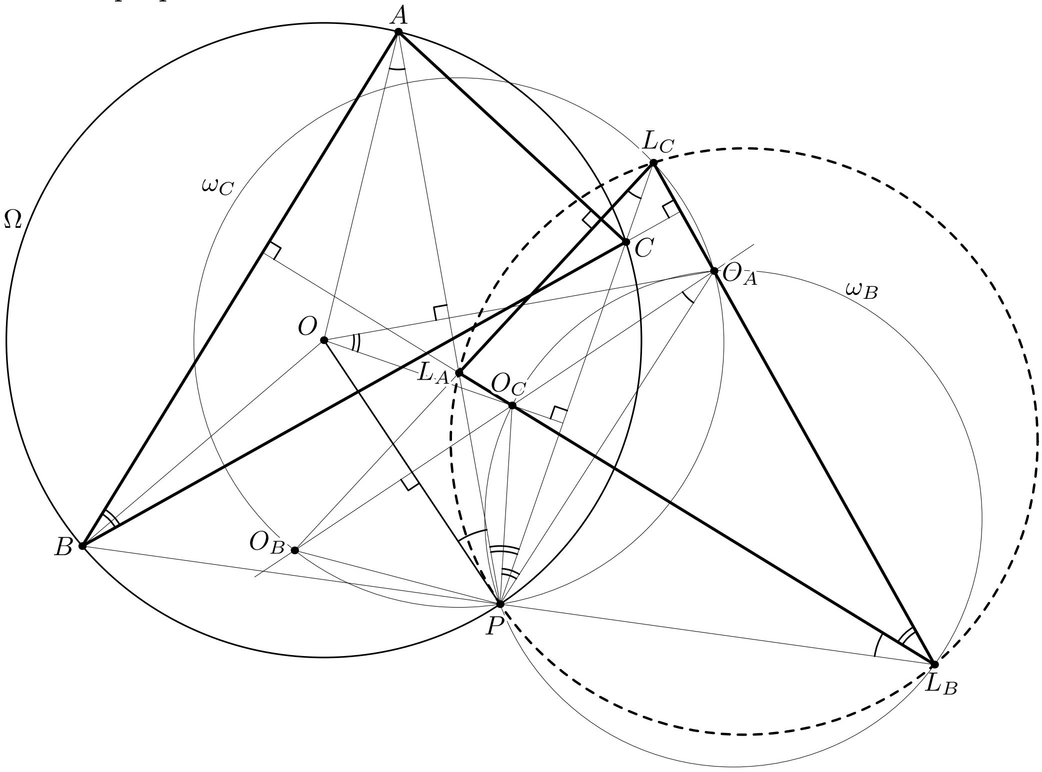
G7. Let O be the circumcentre, and Ω be the circumcircle of an acute-angled triangle ABC. Let P be an arbitrary point on Ω, distinct from A, B, C, and their antipodes in Ω. Denote the circumcentres of the triangles AOP, BOP, and COP by O_A, O_B, and O_C, respectively. The lines ℓ_A, ℓ_B, and ℓ_C perpendicular to BC, CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of the triangle formed by ℓ_A, ℓ_B, and ℓ_C is tangent to the line OP.

(Russia)

Solution. As usual, we denote the directed angle between the lines a and b by $\measuredangle(a, b)$. We frequently use the fact that $a_1 \perp a_2$ and $b_1 \perp b_2$ yield $\measuredangle(a_1, b_1) = \measuredangle(a_2, b_2)$.

Let the lines ℓ_B and ℓ_C meet at L_A; define the points L_B and L_C similarly. Note that the sidelines of the triangle $L_A L_B L_C$ are perpendicular to the corresponding sidelines of ABC. Points O_A, O_B, O_C are located on the corresponding sidelines of $L_A L_B L_C$; moreover, O_A, O_B, O_C all lie on the perpendicular bisector of OP.



Claim 1. The points L_B, P, O_A, and O_C are concyclic.

Proof. Since O is symmetric to P in $O_A O_C$, we have

$$\measuredangle(O_A P, O_C P) = \measuredangle(O_C O, O_A O) = \measuredangle(CP, AP) = \measuredangle(CB, AB) = \measuredangle(O_A L_B, O_C L_B). \qquad \square$$

Denote the circle through L_B, P, O_A, and O_C by ω_B. Define the circles ω_A and ω_C similarly.

Claim 2. The circumcircle of the triangle $L_A L_B L_C$ passes through P.

Proof. From cyclic quadruples of points in the circles ω_B and ω_C, we have

$$\begin{aligned}\measuredangle(L_C L_A, L_C P) &= \measuredangle(L_C O_B, L_C P) = \measuredangle(O_A O_B, O_A P) \\ &= \measuredangle(O_A O_C, O_A P) = \measuredangle(L_B O_C, L_B P) = \measuredangle(L_B L_A, L_B P). \qquad \square\end{aligned}$$

Claim 3. The points P, L_C, and C are collinear.

Proof. We have $\measuredangle(PL_C, L_C L_A) = \measuredangle(PL_C, L_C O_B) = \measuredangle(PO_A, O_A O_B)$. Further, since O_A is the centre of the circle AOP, $\measuredangle(PO_A, O_A O_B) = \measuredangle(PA, AO)$. As O is the circumcentre of the triangle PCA, $\measuredangle(PA, AO) = \pi/2 - \measuredangle(CA, CP) = \measuredangle(CP, L_C L_A)$. We obtain $\measuredangle(PL_C, L_C L_A) = \measuredangle(CP, L_C L_A)$, which shows that $P \in CL_C$. $\square$

Similarly, the points P, L_A, A are collinear, and the points P, L_B, B are also collinear.

Finally, the computation above also shows that

$$\measuredangle(OP, PL_A) = \measuredangle(PA, AO) = \measuredangle(PL_C, L_CL_A),$$

which means that OP is tangent to the circle $PL_AL_BL_C$.

Comment 1. The proof of Claim 2 may be replaced by the following remark: since P belongs to the circles ω_A and ω_C, P is the Miquel point of the four lines ℓ_A, ℓ_B, ℓ_C, and $O_AO_BO_C$.

Comment 2. Claims 2 and 3 can be proved in several different ways and, in particular, in the reverse order.

Claim 3 implies that the triangles ABC and $L_AL_BL_C$ are perspective with perspector P. Claim 2 can be derived from this observation using spiral similarity. Consider the centre Q of the spiral similarity that maps ABC to $L_AL_BL_C$. From known spiral similarity properties, the points L_A, L_B, P, Q are concyclic, and so are L_A, L_C, P, Q.

Comment 3. The final conclusion can also be proved it terms of spiral similarity: the spiral similarity with centre Q located on the circle ABC maps the circle ABC to the circle $PL_AL_BL_C$. Thus these circles are orthogonal.

Comment 4. Notice that the homothety with centre O and ratio 2 takes O_A to A' that is the common point of tangents to Ω at A and P. Similarly, let this homothety take O_B to B' and O_C to C'. Let the tangents to Ω at B and C meet at A'', and define the points B'' and C'' similarly. Now, replacing labels O with I, Ω with ω, and swapping labels $A \leftrightarrow A''$, $B \leftrightarrow B''$, $C \leftrightarrow C''$ we obtain the following

Reformulation. Let ω be the incircle, and let I be the incentre of a triangle ABC. Let P be a point of ω (other than the points of contact of ω with the sides of ABC). The tangent to ω at P meets the lines AB, BC, and CA at A', B', and C', respectively. Line ℓ_A parallel to the internal angle bisector of $\angle BAC$ passes through A'; define lines ℓ_B and ℓ_C similarly. Prove that the line IP is tangent to the circumcircle of the triangle formed by ℓ_A, ℓ_B, and ℓ_C.

Though this formulation is equivalent to the original one, it seems more challenging, since the point of contact is now "hidden".

Number Theory

N1. Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the numbers of divisors of sn and of sk are equal.

(Ukraine)

Answer: All pairs (n, k) such that $n \nmid k$ and $k \nmid n$.

Solution. As usual, the number of divisors of a positive integer n is denoted by $d(n)$. If $n = \prod_i p_i^{\alpha_i}$ is the prime factorisation of n, then $d(n) = \prod_i (\alpha_i + 1)$.

We start by showing that one cannot find any suitable number s if $k \mid n$ or $n \mid k$ (and $k \neq n$). Suppose that $n \mid k$, and choose any positive integer s. Then the set of divisors of sn is a proper subset of that of sk, hence $d(sn) < d(sk)$. Therefore, the pair (n, k) does not satisfy the problem requirements. The case $k \mid n$ is similar.

Now assume that $n \nmid k$ and $k \nmid n$. Let $p_1, \ldots, p_t$ be all primes dividing nk, and consider the prime factorisations

$$n = \prod_{i=1}^{t} p_i^{\alpha_i} \quad \text{and} \quad k = \prod_{i=1}^{t} p_i^{\beta_i}.$$

It is reasonable to search for the number s having the form

$$s = \prod_{i=1}^{t} p_i^{\gamma_i}.$$

The (nonnegative integer) exponents γ_i should be chosen so as to satisfy

$$\frac{d(sn)}{d(sk)} = \prod_{i=1}^{t} \frac{\alpha_i + \gamma_i + 1}{\beta_i + \gamma_i + 1} = 1. \tag{1}$$

First of all, if $\alpha_i = \beta_i$ for some i, then, regardless of the value of γ_i, the corresponding factor in (1) equals 1 and does not affect the product. So we may assume that there is no such index i. For the other factors in (1), the following lemma is useful.

Lemma. Let $\alpha > \beta$ be nonnegative integers. Then, for every integer $M \geqslant \beta + 1$, there exists a nonnegative integer γ such that

$$\frac{\alpha + \gamma + 1}{\beta + \gamma + 1} = 1 + \frac{1}{M} = \frac{M + 1}{M}.$$

Proof.

$$\frac{\alpha + \gamma + 1}{\beta + \gamma + 1} = 1 + \frac{1}{M} \iff \frac{\alpha - \beta}{\beta + \gamma + 1} = \frac{1}{M} \iff \gamma = M(\alpha - \beta) - (\beta + 1) \geqslant 0. \qquad \square$$

Now we can finish the solution. Without loss of generality, there exists an index u such that $\alpha_i > \beta_i$ for $i = 1, 2, \ldots, u$, and $\alpha_i < \beta_i$ for $i = u + 1, \ldots, t$. The conditions $n \nmid k$ and $k \nmid n$ mean that $1 \leqslant u \leqslant t - 1$.

Choose an integer X greater than all the α_i and β_i. By the lemma, we can define the numbers γ_i so as to satisfy

$$\frac{\alpha_i + \gamma_i + 1}{\beta_i + \gamma_i + 1} = \frac{uX + i}{uX + i - 1} \qquad \text{for } i = 1, 2, \ldots, u, \text{ and}$$

$$\frac{\beta_{u+i} + \gamma_{u+i} + 1}{\alpha_{u+i} + \gamma_{u+i} + 1} = \frac{(t - u)X + i}{(t - u)X + i - 1} \qquad \text{for } i = 1, 2, \ldots, t - u.$$

Then we will have

$$\frac{d(sn)}{d(sk)} = \prod_{i=1}^{u} \frac{uX+i}{uX+i-1} \cdot \prod_{i=1}^{t-u} \frac{(t-u)X+i-1}{(t-u)X+i} = \frac{u(X+1)}{uX} \cdot \frac{(t-u)X}{(t-u)(X+1)} = 1,$$

as required.

Comment. The lemma can be used in various ways, in order to provide a suitable value of s. In particular, one may apply induction on the number t of prime factors, using identities like

$$\frac{n}{n-1} = \frac{n^2}{n^2-1} \cdot \frac{n+1}{n}.$$

N2. Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied:

(i) Each number in the table is congruent to 1 modulo n;

(ii) The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2.

Let R_i be the product of the numbers in the i^{th} row, and C_j be the product of the numbers in the j^{th} column. Prove that the sums $R_1 + \cdots + R_n$ and $C_1 + \cdots + C_n$ are congruent modulo n^4.

(Indonesia)

Solution 1. Let $A_{i,j}$ be the entry in the i^{th} row and the j^{th} column; let P be the product of all n^2 entries. For convenience, denote $a_{i,j} = A_{i,j} - 1$ and $r_i = R_i - 1$. We show that

$$\sum_{i=1}^{n} R_i \equiv (n-1) + P \pmod{n^4}. \tag{1}$$

Due to symmetry of the problem conditions, the sum of all the C_j is also congruent to $(n-1)+P$ modulo n^4, whence the conclusion.

By condition (i), the number n divides $a_{i,j}$ for all i and j. So, every product of at least two of the $a_{i,j}$ is divisible by n^2, hence

$$R_i = \prod_{j=1}^{n}(1+a_{i,j}) = 1+\sum_{j=1}^{n} a_{i,j} + \sum_{1 \leqslant j_1 < j_2 \leqslant n} a_{i,j_1}a_{i,j_2} + \cdots \equiv 1+\sum_{j=1}^{n} a_{i,j} \equiv 1-n+\sum_{j=1}^{n} A_{i,j} \pmod{n^2}$$

for every index i. Using condition (ii), we obtain $R_i \equiv 1 \pmod{n^2}$, and so $n^2 \mid r_i$.

Therefore, every product of at least two of the r_i is divisible by n^4. Repeating the same argument, we obtain

$$P = \prod_{i=1}^{n} R_i = \prod_{i=1}^{n}(1+r_i) \equiv 1 + \sum_{i=1}^{n} r_i \pmod{n^4},$$

whence

$$\sum_{i=1}^{n} R_i = n + \sum_{i=1}^{n} r_i \equiv n + (P-1) \pmod{n^4},$$

as desired.

Comment. The original version of the problem statement contained also the condition

(iii) The product of all the numbers in the table is congruent to 1 modulo n^4.

This condition appears to be superfluous, so it was omitted.

Solution 2. We present a more straightforward (though lengthier) way to establish (1). We also use the notation of $a_{i,j}$.

By condition (i), all the $a_{i,j}$ are divisible by n. Therefore, we have

$$P = \prod_{i=1}^{n}\prod_{j=1}^{n}(1+a_{i,j}) \equiv 1 + \sum_{(i,j)} a_{i,j} + \sum_{(i_1,j_1),\,(i_2,j_2)} a_{i_1,j_1}a_{i_2,j_2}$$
$$+ \sum_{(i_1,j_1),\,(i_2,j_2),\,(i_3,j_3)} a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3} \pmod{n^4},$$

where the last two sums are taken over all unordered pairs/triples of *pairwise different* pairs (i,j); such conventions are applied throughout the solution.

Similarly,

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n}\prod_{j=1}^{n}(1+a_{i,j}) \equiv n + \sum_i\sum_j a_{i,j} + \sum_i \sum_{j_1,j_2} a_{i,j_1}a_{i,j_2} + \sum_i \sum_{j_1,j_2,j_3} a_{i,j_1}a_{i,j_2}a_{i,j_3} \pmod{n^4}.$$

Therefore,

$$\begin{aligned} P + (n-1) - \sum_i R_i \equiv \sum_{\substack{(i_1,j_1),(i_2,j_2)\\ i_1\neq i_2}} a_{i_1,j_1}a_{i_2,j_2} &+ \sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3)\\ i_1\neq i_2\neq i_3\neq i_1}} a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3} \\ &+ \sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3)\\ i_1\neq i_2 = i_3}} a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3} \pmod{n^4}. \end{aligned}$$

We show that in fact each of the three sums appearing in the right-hand part of this congruence is divisible by n^4; this yields (1). Denote those three sums by Σ_1, Σ_2, and Σ_3 in order of appearance. Recall that by condition (ii) we have

$$\sum_j a_{i,j} \equiv 0 \pmod{n^2} \qquad \text{for all indices } i.$$

For every two indices $i_1 < i_2$ we have

$$\sum_{j_1}\sum_{j_2} a_{i_1,j_1}a_{i_2,j_2} = \left(\sum_{j_1} a_{i_1,j_1}\right)\cdot\left(\sum_{j_2} a_{i_2,j_2}\right) \equiv 0 \pmod{n^4},$$

since each of the two factors is divisible by n^2. Summing over all pairs (i_1,i_2) we obtain $n^4 \mid \Sigma_1$.

Similarly, for every three indices $i_1 < i_2 < i_3$ we have

$$\sum_{j_1}\sum_{j_2}\sum_{j_3} a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3} = \left(\sum_{j_1} a_{i_1,j_1}\right)\cdot\left(\sum_{j_2} a_{i_2,j_2}\right)\cdot\left(\sum_{j_3} a_{i_3,j_3}\right)$$

which is divisible even by n^6. Hence $n^4 \mid \Sigma_2$.

Finally, for every indices $i_1 \neq i_2 = i_3$ and $j_2 < j_3$ we have

$$a_{i_2,j_2}\cdot a_{i_2,j_3}\cdot \sum_{j_1} a_{i_1,j_1} \equiv 0 \pmod{n^4},$$

since the three factors are divisible by n, n, and n^2, respectively. Summing over all 4-tuples of indices (i_1,i_2,j_2,j_3) we get $n^4 \mid \Sigma_3$.

N3. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

(Serbia)

Solution 1. Call a nonnegative integer *representable* if it equals the sum of several (possibly 0 or 1) distinct terms of the sequence. We say that two nonnegative integers b and c are *equivalent* (written as $b \sim c$) if they are either both representable or both non-representable.

One can easily compute

$$S_{n-1} := a_0 + \cdots + a_{n-1} = 2^n + 2^{\lceil n/2 \rceil} + 2^{\lfloor n/2 \rfloor} - 3.$$

Indeed, we have $S_n - S_{n-1} = 2^n + 2^{\lfloor n/2 \rfloor} = a_n$ so we can use the induction. In particular, $S_{2k-1} = 2^{2k} + 2^{k+1} - 3$.

Note that, if $n \geqslant 3$, then $2^{\lceil n/2 \rceil} \geqslant 2^2 > 3$, so

$$S_{n-1} = 2^n + 2^{\lceil n/2 \rceil} + 2^{\lfloor n/2 \rfloor} - 3 > 2^n + 2^{\lfloor n/2 \rfloor} = a_n.$$

Also notice that $S_{n-1} - a_n = 2^{\lceil n/2 \rceil} - 3 < a_n$.

The main tool of the solution is the following claim.

Claim 1. Assume that b is a positive integer such that $S_{n-1} - a_n < b < a_n$ for some $n \geqslant 3$. Then $b \sim S_{n-1} - b$.

Proof. As seen above, we have $S_{n-1} > a_n$. Denote $c = S_{n-1} - b$; then $S_{n-1} - a_n < c < a_n$, so the roles of b and c are symmetrical.

Assume that b is representable. The representation cannot contain a_i with $i \geqslant n$, since $b < a_n$. So b is the sum of some subset of $\{a_0, a_1, \ldots, a_{n-1}\}$; then c is the sum of the complement. The converse is obtained by swapping b and c. $\square$

We also need the following version of this claim.

Claim 2. For any $n \geqslant 3$, the number a_n can be represented as a sum of two or more distinct terms of the sequence if and only if $S_{n-1} - a_n = 2^{\lceil n/2 \rceil} - 3$ is representable.

Proof. Denote $c = S_{n-1} - a_n < a_n$. If a_n satisfies the required condition, then it is the sum of some subset of $\{a_0, a_1, \ldots, a_{n-1}\}$; then c is the sum of the complement. Conversely, if c is representable, then its representation consists only of the numbers from $\{a_0, \ldots, a_{n-1}\}$, so a_n is the sum of the complement. $\square$

By Claim 2, in order to prove the problem statement, it suffices to find infinitely many representable numbers of the form $2^t - 3$, as well as infinitely many non-representable ones.

Claim 3. For every $t \geqslant 3$, we have $2^t - 3 \sim 2^{4t-6} - 3$, and $2^{4t-6} - 3 > 2^t - 3$.

Proof. The inequality follows from $t \geqslant 3$. In order to prove the equivalence, we apply Claim 1 twice in the following manner.

First, since $S_{2t-3} - a_{2t-2} = 2^{t-1} - 3 < 2^t - 3 < 2^{2t-2} + 2^{t-1} = a_{2t-2}$, by Claim 1 we have $2^t - 3 \sim S_{2t-3} - (2^t - 3) = 2^{2t-2}$.

Second, since $S_{4t-7} - a_{4t-6} = 2^{2t-3} - 3 < 2^{2t-2} < 2^{4t-6} + 2^{2t-3} = a_{4t-6}$, by Claim 1 we have $2^{2t-2} \sim S_{4t-7} - 2^{2t-2} = 2^{4t-6} - 3$.

Therefore, $2^t - 3 \sim 2^{2t-2} \sim 2^{4t-6} - 3$, as required. $\square$

Now it is easy to find the required numbers. Indeed, the number $2^3 - 3 = 5 = a_0 + a_1$ is representable, so Claim 3 provides an infinite sequence of representable numbers

$$2^3 - 3 \sim 2^6 - 3 \sim 2^{18} - 3 \sim \cdots \sim 2^t - 3 \sim 2^{4t-6} - 3 \sim \cdots.$$

On the other hand, the number $2^7 - 3 = 125$ is non-representable (since by Claim 1 we have $125 \sim S_6 - 125 = 24 \sim S_4 - 24 = 17 \sim S_3 - 17 = 4$ which is clearly non-representable). So Claim 3 provides an infinite sequence of non-representable numbers

$$2^7 - 3 \sim 2^{22} - 3 \sim 2^{82} - 3 \sim \cdots \sim 2^t - 3 \sim 2^{4t-6} - 3 \sim \cdots.$$

Solution 2. We keep the notion of representability and the notation S_n from the previous solution. We say that an index n is *good* if a_n writes as a sum of smaller terms from the sequence $a_0, a_1, \dots$. Otherwise we say it is *bad*. We must prove that there are infinitely many good indices, as well as infinitely many bad ones.

Lemma 1. If $m \geqslant 0$ is an integer, then 4^m is representable if and only if either of $2m+1$ and $2m+2$ is good.

Proof. The case $m = 0$ is obvious, so we may assume that $m \geqslant 1$. Let $n = 2m+1$ or $2m+2$. Then $n \geqslant 3$. We notice that

$$S_{n-1} < a_{n-2} + a_n.$$

The inequality writes as $2^n + 2^{\lceil n/2 \rceil} + 2^{\lfloor n/2 \rfloor} - 3 < 2^n + 2^{\lfloor n/2 \rfloor} + 2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}$, i.e. as $2^{\lceil n/2 \rceil} < 2^{n-2} + 2^{\lfloor n/2 \rfloor - 1} + 3$. If $n \geqslant 4$, then $n/2 \leqslant n-2$, so $\lceil n/2 \rceil \leqslant n-2$ and $2^{\lceil n/2 \rceil} \leqslant 2^{n-2}$. For $n = 3$ the inequality verifies separately.

If n is good, then a_n writes as $a_n = a_{i_1} + \dots + a_{i_r}$, where $r \geqslant 2$ and $i_1 < \dots < i_r < n$. Then $i_r = n-1$ and $i_{r-1} = n-2$, for if $n-1$ or $n-2$ is missing from the sequence $i_1, \dots, i_r$, then $a_{i_1} + \dots + a_{i_r} \leqslant a_0 + \dots + a_{n-3} + a_{n-1} = S_{n-1} - a_{n-2} < a_n$. Thus, if n is good, then both $a_n - a_{n-1}$ and $a_n - a_{n-1} - a_{n-2}$ are representable.

We now consider the cases $n = 2m+1$ and $n = 2m+2$ separately.

If $n = 2m+1$, then $a_n - a_{n-1} = a_{2m+1} - a_{2m} = (2^{2m+1} + 2^m) - (2^{2m} + 2^m) = 2^{2m}$. So we proved that, if $2m+1$ is good, then 2^{2m} is representable. Conversely, if 2^{2m} is representable, then $2^{2m} < a_{2m}$, so 2^{2m} is a sum of some distinct terms a_i with $i < 2m$. It follows that $a_{2m+1} = a_{2m} + 2^{2m}$ writes as a_{2m} plus a sum of some distinct terms a_i with $i < 2m$. Hence $2m+1$ is good.

If $n = 2m+2$, then $a_n - a_{n-1} - a_{n-2} = a_{2m+2} - a_{2m+1} - a_{2m} = (2^{2m+2} + 2^{m+1}) - (2^{2m+1} + 2^m) - (2^{2m} + 2^m) = 2^{2m}$. So we proved that, if $2m+2$ is good, then 2^{2m} is representable. Conversely, if 2^{2m} is representable, then, as seen in the previous case, it writes as a sum of some distinct terms a_i with $i < 2m$. Hence $a_{2m+2} = a_{2m+1} + a_{2m} + 2^{2m}$ writes as $a_{2m+1} + a_{2m}$ plus a sum of some distinct terms a_i with $i < 2m$. Thus $2m+2$ is good. $\square$

Lemma 2. If $k \geqslant 2$, then 2^{4k-2} is representable if and only if 2^{k+1} is representable.

In particular, if $s \geqslant 2$, then 4^s is representable if and only if 4^{4s-3} is representable. Also, $4^{4s-3} > 4^s$.

Proof. We have $2^{4k-2} < a_{4k-2}$, so in a representation of 2^{4k-2} we can have only terms a_i with $i \leqslant 4k-3$. Notice that

$$a_0 + \dots + a_{4k-3} = 2^{4k-2} + 2^{2k} - 3 < 2^{4k-2} + 2^{2k} + 2^k = 2^{4k-2} + a_{2k}.$$

Hence, any representation of 2^{4k-2} must contain all terms from a_{2k} to a_{4k-3}. (If any of these terms is missing, then the sum of the remaining ones is $\leqslant (a_0 + \dots + a_{4k-3}) - a_{2k} < 2^{4k-2}$.) Hence, if 2^{4k-2} is representable, then $2^{4k-2} - \sum_{i=2k}^{4k-3} a_i$ is representable. But

$$2^{4k-2} - \sum_{i=2k}^{4k-3} a_i = 2^{4k-2} - (S_{4k-3} - S_{2k-1}) = 2^{4k-2} - (2^{4k-2} + 2^{2k} - 3) + (2^{2k} + 2^{k+1} - 3) = 2^{k+1}.$$

So, if 2^{4k-2} is representable, then 2^{k+1} is representable. Conversely, if 2^{k+1} is representable, then $2^{k+1} < 2^{2k} + 2^k = a_{2k}$, so 2^{k+1} writes as a sum of some distinct terms a_i with $i < 2k$. It follows that $2^{4k-2} = \sum_{i=2k}^{4k-3} a_i + 2^{k+1}$ writes as $a_{4k-3} + a_{4k-4} + \dots + a_{2k}$ plus the sum of some distinct terms a_i with $i < 2k$. Hence 2^{4k-2} is representable.

For the second statement, if $s \geqslant 2$, then we just take $k = 2s-1$ and we notice that $2^{k+1} = 4^s$ and $2^{4k-2} = 4^{4s-3}$. Also, $s \geqslant 2$ implies that $4s-3 > s$. $\square$

Now $4^2 = a_2 + a_3$ is representable, whereas $4^6 = 4096$ is not. Indeed, note that $4^6 = 2^{12} < a_{12}$, so the only available terms for a representation are $a_0, \ldots, a_{11}$, i.e., 2, 3, 6, 10, 20, 36, 72, 136, 272, 528, 1056, 2080. Their sum is $S_{11} = 4221$, which exceeds 4096 by 125. Then any representation of 4096 must contain all the terms from $a_0, \ldots, a_{11}$ that are greater that 125, i.e., 136, 272, 528, 1056, 2080. Their sum is 4072. Since $4096 - 4072 = 24$ and 24 is clearly not representable, 4096 is non-representable as well.

Starting with these values of m, by using Lemma 2, we can obtain infinitely many representable powers of 4, as well as infinitely many non-representable ones. By Lemma 1, this solves our problem.

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N4. Let a_1, a_2, …, a_n, … be a sequence of positive integers such that

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}$$

is an integer for all $n \geqslant k$, where k is some positive integer. Prove that there exists a positive integer m such that $a_n = a_{n+1}$ for all $n \geqslant m$.

(Mongolia)

Solution 1. The argument hinges on the following two facts: Let a, b, c be positive integers such that $N = b/c + (c-b)/a$ is an integer.

(1) If $\gcd(a,c)=1$, then c divides b; and
(2) If $\gcd(a,b,c)=1$, then $\gcd(a,b)=1$.

To prove **(1)**, write $ab = c(aN+b-c)$. Since $\gcd(a,c)=1$, it follows that c divides b. To prove **(2)**, write $c^2-bc = a(cN-b)$ to infer that a divides c^2-bc. Letting $d=\gcd(a,b)$, it follows that d divides c^2, and since the two are relatively prime by hypothesis, $d=1$.

Now, let $s_n = a_1/a_2 + a_2/a_3 + \cdots + a_{n-1}/a_n + a_n/a_1$, let $\delta_n = \gcd(a_1, a_n, a_{n+1})$ and write

$$s_{n+1}-s_n=\frac{a_n}{a_{n+1}}+\frac{a_{n+1}-a_n}{a_1}=\frac{a_n/\delta_n}{a_{n+1}/\delta_n}+\frac{a_{n+1}/\delta_n-a_n/\delta_n}{a_1/\delta_n}.$$

Let $n \geqslant k$. Since $\gcd(a_1/\delta_n, a_n/\delta_n, a_{n+1}/\delta_n)=1$, it follows by **(2)** that $\gcd(a_1/\delta_n, a_n/\delta_n)=1$. Let $d_n = \gcd(a_1,a_n)$. Then $d_n = \delta_n \cdot \gcd(a_1/\delta_n, a_n/\delta_n) = \delta_n$, so d_n divides a_{n+1}, and therefore d_n divides d_{n+1}.

Consequently, from some rank on, the d_n form a nondecreasing sequence of integers not exceeding a_1, so $d_n = d$ for all $n \geqslant \ell$, where ℓ is some positive integer.

Finally, since $\gcd(a_1/d, a_{n+1}/d)=1$, it follows by **(1)** that a_{n+1}/d divides a_n/d, so $a_n \geqslant a_{n+1}$ for all $n \geqslant \ell$. The conclusion follows.

Solution 2. We use the same notation s_n. This time, we explore the exponents of primes in the prime factorizations of the a_n for $n \geqslant k$.

To start, for every $n \geqslant k$, we know that the number

$$s_{n+1}-s_n=\frac{a_n}{a_{n+1}}+\frac{a_{n+1}}{a_1}-\frac{a_n}{a_1} \tag{$*$}$$

is integer. Multiplying it by a_1 we obtain that a_1a_n/a_{n+1} is integer as well, so that $a_{n+1} \mid a_1a_n$. This means that $a_n \mid a_1^{n-k}a_k$, so all prime divisors of a_n are among those of a_1a_k. There are finitely many such primes; therefore, it suffices to prove that the exponent of each of them in the prime factorization of a_n is eventually constant.

Choose any prime $p \mid a_1a_k$. Recall that $v_p(q)$ is the standard notation for the exponent of p in the prime factorization of a nonzero rational number q. Say that an index $n \geqslant k$ is *large* if $v_p(a_n) \geqslant v_p(a_1)$. We separate two cases.

Case 1: There exists a large index n.

If $v_p(a_{n+1}) < v_p(a_1)$, then $v_p(a_n/a_{n+1})$ and $v_p(a_n/a_1)$ are nonnegative, while $v_p(a_{n+1}/a_1)<0$; hence $(*)$ cannot be an integer. This contradiction shows that index $n+1$ is also large.

On the other hand, if $v_p(a_{n+1}) > v_p(a_n)$, then $v_p(a_n/a_{n+1})<0$, while $v_p\big((a_{n+1}-a_n)/a_1\big) \geqslant 0$, so $(*)$ is not integer again. Thus, $v_p(a_1) \leqslant v_p(a_{n+1}) \leqslant v_p(a_n)$.

The above arguments can now be applied successively to indices $n+1$, $n+2$, …, showing that all the indices greater than n are large, and the sequence $v_p(a_n), v_p(a_{n+1}), v_p(a_{n+2}), \ldots$ is nonincreasing — hence eventually constant.

Case 2: There is no large index.

We have $v_p(a_1) > v_p(a_n)$ for all $n \geqslant k$. If we had $v_p(a_{n+1}) < v_p(a_n)$ for some $n \geqslant k$, then $v_p(a_{n+1}/a_1) < v_p(a_n/a_1) < 0 < v_p(a_n/a_{n+1})$ which would also yield that $(*)$ is not integer. Therefore, in this case the sequence $v_p(a_k), v_p(a_{k+1}), v_p(a_{k+2}), \ldots$ is nondecreasing and bounded by $v_p(a_1)$ from above; hence it is also eventually constant.

Comment. Given any positive odd integer m, consider the m-tuple $(2, 2^2, \ldots, 2^{m-1}, 2^m)$. Appending an infinite string of 1's to this m-tuple yields an eventually constant sequence of integers satisfying the condition in the statement, and shows that the rank from which the sequence stabilises may be arbitrarily large.

There are more sophisticated examples. The solution to part **(b)** of **10532**, *Amer. Math. Monthly*, Vol. 105 No. 8 (Oct. 1998), 775–777 (available at `https://www.jstor.org/stable/2589009`), shows that, for every integer $m \geqslant 5$, there exists an m-tuple $(a_1, a_2, \ldots, a_m)$ of pairwise distinct positive integers such that $\gcd(a_1, a_2) = \gcd(a_2, a_3) = \cdots = \gcd(a_{m-1}, a_m) = \gcd(a_m, a_1) = 1$, and the sum $a_1/a_2 + a_2/a_3 + \cdots + a_{m-1}/a_m + a_m/a_1$ is an integer. Letting $a_{m+k} = a_1$, $k = 1, 2, \ldots$, extends such an m-tuple to an eventually constant sequence of positive integers satisfying the condition in the statement of the problem at hand.

Here is the example given by the proposers of **10532**. Let $b_1 = 2$, let $b_{k+1} = 1 + b_1 \cdots b_k = 1 + b_k(b_k - 1)$, $k \geqslant 1$, and set $B_m = b_1 \cdots b_{m-4} = b_{m-3} - 1$. The m-tuple $(a_1, a_2, \ldots, a_m)$ defined below satisfies the required conditions:

$$a_1 = 1, \quad a_2 = (8B_m + 1)B_m + 8, \quad a_3 = 8B_m + 1, \quad a_k = b_{m-k} \quad \text{for } 4 \leqslant k \leqslant m - 1,$$
$$a_m = \frac{a_2}{2} \cdot a_3 \cdot \frac{B_m}{2} = \left(\frac{1}{2}(8B_m + 1)B_m + 4\right) \cdot (8B_m + 1) \cdot \frac{B_m}{2}.$$

It is readily checked that $a_1 < a_{m-1} < a_{m-2} < \cdots < a_3 < a_2 < a_m$. For further details we refer to the solution mentioned above. Acquaintance with this example (or more elaborated examples derived from) offers no advantage in tackling the problem.

N5. Four positive integers x, y, z, and t satisfy the relations

$$xy - zt = x + y = z + t. \qquad (*)$$

Is it possible that both xy and zt are perfect squares?

(Russia)

Answer: No.

Solution 1. Arguing indirectly, assume that $xy = a^2$ and $zt = c^2$ with $a, c > 0$.

Suppose that the number $x + y = z + t$ is odd. Then x and y have opposite parity, as well as z and t. This means that both xy and zt are even, as well as $xy - zt = x + y$; a contradiction. Thus, $x + y$ is even, so the number $s = \frac{x+y}{2} = \frac{z+t}{2}$ is a positive integer.

Next, we set $b = \frac{|x-y|}{2}$, $d = \frac{|z-t|}{2}$. Now the problem conditions yield

$$s^2 = a^2 + b^2 = c^2 + d^2 \qquad (1)$$

and

$$2s = a^2 - c^2 = d^2 - b^2 \qquad (2)$$

(the last equality in (2) follows from (1)). We readily get from (2) that $a, d > 0$.

In the sequel we will use only the relations (1) and (2), along with the fact that a, d, s are positive integers, while b and c are nonnegative integers, at most one of which may be zero. Since both relations are symmetric with respect to the simultaneous swappings $a \leftrightarrow d$ and $b \leftrightarrow c$, we assume, without loss of generality, that $b \geqslant c$ (and hence $b > 0$). Therefore, $d^2 = 2s + b^2 > c^2$, whence

$$d^2 > \frac{c^2 + d^2}{2} = \frac{s^2}{2}. \qquad (3)$$

On the other hand, since $d^2 - b^2$ is even by (2), the numbers b and d have the same parity, so $0 < b \leqslant d - 2$. Therefore,

$$2s = d^2 - b^2 \geqslant d^2 - (d-2)^2 = 4(d-1), \qquad \text{i.e.,} \qquad d \leqslant \frac{s}{2} + 1. \qquad (4)$$

Combining (3) and (4) we obtain

$$2s^2 < 4d^2 \leqslant 4\left(\frac{s}{2} + 1\right)^2, \qquad \text{or} \qquad (s-2)^2 < 8,$$

which yields $s \leqslant 4$.

Finally, an easy check shows that each number of the form s^2 with $1 \leqslant s \leqslant 4$ has a unique representation as a sum of two squares, namely $s^2 = s^2 + 0^2$. Thus, (1) along with $a, d > 0$ imply $b = c = 0$, which is impossible.

Solution 2. We start with a complete description of all 4-tuples (x, y, z, t) of positive integers satisfying $(*)$. As in the solution above, we notice that the numbers

$$s = \frac{x+y}{2} = \frac{z+t}{2}, \qquad p = \frac{x-y}{2}, \qquad \text{and} \qquad q = \frac{z-t}{2}$$

are integers (we may, and will, assume that $p, q \geqslant 0$). We have

$$2s = xy - zt = (s+p)(s-p) - (s+q)(s-q) = q^2 - p^2,$$

so p and q have the same parity, and $q > p$.

Set now $k = \frac{q-p}{2}$, $\ell = \frac{q+p}{2}$. Then we have $s = \frac{q^2-p^2}{2} = 2k\ell$ and hence

$$\begin{aligned} x &= s+p = 2k\ell - k + \ell, & y &= s-p = 2k\ell + k - \ell, \\ z &= s+q = 2k\ell + k + \ell, & t &= s-q = 2k\ell - k - \ell. \end{aligned} \tag{5}$$

Recall here that $\ell \geqslant k > 0$ and, moreover, $(k, \ell) \neq (1,1)$, since otherwise $t = 0$.

Assume now that both xy and zt are squares. Then $xyzt$ is also a square. On the other hand, we have

$$\begin{aligned} xyzt &= (2k\ell - k + \ell)(2k\ell + k - \ell)(2k\ell + k + \ell)(2k\ell - k - \ell) \\ &= \big(4k^2\ell^2 - (k-\ell)^2\big)\big(4k^2\ell^2 - (k+\ell)^2\big) = (4k^2\ell^2 - k^2 - \ell^2)^2 - 4k^2\ell^2. \end{aligned} \tag{6}$$

Denote $D = 4k^2\ell^2 - k^2 - \ell^2 > 0$. From (6) we get $D^2 > xyzt$. On the other hand,

$$\begin{aligned} (D-1)^2 = D^2 - 2(4k^2\ell^2 - k^2 - \ell^2) + 1 &= (D^2 - 4k^2\ell^2) - (2k^2 - 1)(2\ell^2 - 1) + 2 \\ &= xyzt - (2k^2-1)(2\ell^2-1) + 2 < xyzt, \end{aligned}$$

since $\ell \geqslant 2$ and $k \geqslant 1$. Thus $(D-1)^2 < xyzt < D^2$, and $xyzt$ cannot be a perfect square; a contradiction.

Comment. The first part of Solution 2 shows that all 4-tuples of positive integers $x \geqslant y$, $z \geqslant t$ satisfying $(*)$ have the form (5), where $\ell \geqslant k > 0$ and $\ell \geqslant 2$. The converse is also true: every pair of positive integers $\ell \geqslant k > 0$, except for the pair $k = \ell = 1$, generates via (5) a 4-tuple of positive integers satisfying $(*)$.

N6. Let $f\colon \{1,2,3,\ldots\} \to \{2,3,\ldots\}$ be a function such that $f(m+n) \mid f(m)+f(n)$ for all pairs m,n of positive integers. Prove that there exists a positive integer $c>1$ which divides all values of f.

(Mexico)

Solution 1. For every positive integer m, define $S_m = \{n\colon m \mid f(n)\}$.

Lemma. If the set S_m is infinite, then $S_m = \{d, 2d, 3d, \ldots\} = d \cdot \mathbb{Z}_{>0}$ for some positive integer d.

Proof. Let $d = \min S_m$; the definition of S_m yields $m \mid f(d)$.

Whenever $n \in S_m$ and $n > d$, we have $m \mid f(n) \mid f(n-d) + f(d)$, so $m \mid f(n-d)$ and therefore $n - d \in S_m$. Let $r \leqslant d$ be the least positive integer with $n \equiv r \pmod{d}$; repeating the same step, we can see that $n-d, n-2d, \ldots, r \in S_m$. By the minimality of d, this shows $r = d$ and therefore $d \mid n$.

Starting from an arbitrarily large element of S_m, the process above reaches all multiples of d; so they all are elements of S_m. $\square$

The solution for the problem will be split into two cases.

Case 1: The function f is bounded.

Call a prime p *frequent* if the set S_p is infinite, i.e., if p divides $f(n)$ for infinitely many positive integers n; otherwise call p *sporadic*. Since the function f is bounded, there are only a finite number of primes that divide at least one $f(n)$; so altogether there are finitely many numbers n such that $f(n)$ has a sporadic prime divisor. Let N be a positive integer, greater than all those numbers n.

Let $p_1, \ldots, p_k$ be the frequent primes. By the lemma we have $S_{p_i} = d_i \cdot \mathbb{Z}_{>0}$ for some d_i. Consider the number

$$n = N d_1 d_2 \cdots d_k + 1.$$

Due to $n > N$, all prime divisors of $f(n)$ are frequent primes. Let p_i be any frequent prime divisor of $f(n)$. Then $n \in S_{p_i}$, and therefore $d_i \mid n$. But $n \equiv 1 \pmod{d_i}$, which means $d_i = 1$. Hence $S_{p_i} = 1 \cdot \mathbb{Z}_{>0} = \mathbb{Z}_{>0}$ and therefore p_i is a common divisor of all values $f(n)$.

Case 2: f is unbounded.

We prove that $f(1)$ divides all $f(n)$.

Let $a = f(1)$. Since $1 \in S_a$, by the lemma it suffices to prove that S_a is an infinite set.

Call a positive integer p a *peak* if $f(p) > \max\bigl(f(1), \ldots, f(p-1)\bigr)$. Since f is not bounded, there are infinitely many peaks. Let $1 = p_1 < p_2 < \ldots$ be the sequence of all peaks, and let $h_k = f(p_k)$. Notice that for any peak p_i and for any $k < p_i$, we have $f(p_i) \mid f(k) + f(p_i - k) < 2f(p_i)$, hence

$$f(k) + f(p_i - k) = f(p_i) = h_i. \tag{1}$$

By the pigeonhole principle, among the numbers $h_1, h_2, \ldots$ there are infinitely many that are congruent modulo a. Let $k_0 < k_1 < k_2 < \ldots$ be an infinite sequence of positive integers such that $h_{k_0} \equiv h_{k_1} \equiv \ldots \pmod{a}$. Notice that

$$f(p_{k_i} - p_{k_0}) = f(p_{k_i}) - f(p_{k_0}) = h_{k_i} - h_{k_0} \equiv 0 \pmod{a},$$

so $p_{k_i} - p_{k_0} \in S_a$ for all $i = 1, 2, \ldots$. This provides infinitely many elements in S_a.

Hence, S_a is an infinite set, and therefore $f(1) = a$ divides $f(n)$ for every n.

Comment. As an extension of the solution above, it can be proven that if f is not bounded then $f(n) = an$ with $a = f(1)$.

Take an arbitrary positive integer n; we will show that $f(n+1) = f(n) + a$. Then it follows by induction that $f(n) = an$.

Take a peak p such that $p > n+2$ and $h = f(p) > f(n) + 2a$. By (1) we have $f(p-1) = f(p) - f(1) = h - a$ and $f(n+1) = f(p) - f(p-n-1) = h - f(p-n-1)$. From $h - a = f(p-1) \mid f(n) + f(p-n-1) < f(n) + h < 2(h-a)$ we get $f(n) + f(p-n-1) = h - a$. Then

$$f(n+1) - f(n) = \big(h - f(p-n-1)\big) - \big(h - a - f(p-n-1)\big) = a.$$

On the other hand, there exists a wide family of bounded functions satisfying the required properties. Here we present a few examples:

$$f(n) = c; \quad f(n) = \begin{cases} 2c & \text{if } n \text{ is even} \\ c & \text{if } n \text{ is odd;} \end{cases} \quad f(n) = \begin{cases} 2018c & \text{if } n \leqslant 2018 \\ c & \text{if } n > 2018. \end{cases}$$

Solution 2. Let $d_n = \gcd\big(f(n), f(1)\big)$. From $d_{n+1} \mid f(1)$ and $d_{n+1} \mid f(n+1) \mid f(n) + f(1)$, we can see that $d_{n+1} \mid f(n)$; then $d_{n+1} \mid \gcd\big(f(n), f(1)\big) = d_n$. So the sequence $d_1, d_2, \ldots$ is nonincreasing in the sense that every element is a divisor of the previous elements. Let $d = \min(d_1, d_2, \ldots) = \gcd(d_1.d_2, \ldots) = \gcd\big(f(1), f(2), \ldots\big)$; we have to prove $d \geqslant 2$.

For the sake of contradiction, suppose that the statement is wrong, so $d = 1$; that means there is some index n_0 such that $d_n = 1$ for every $n \geqslant n_0$, i.e., $f(n)$ is coprime with $f(1)$.

Claim 1. If $2^k \geqslant n_0$ then $f(2^k) \leqslant 2^k$.

Proof. By the condition, $f(2n) \mid 2f(n)$; a trivial induction yields $f(2^k) \mid 2^k f(1)$. If $2^k \geqslant n_0$ then $f(2^k)$ is coprime with $f(1)$, so $f(2^k)$ is a divisor of 2^k. □

Claim 2. There is a constant C such that $f(n) < n + C$ for every n.

Proof. Take the first power of 2 which is greater than or equal to n_0: let $K = 2^k \geqslant n_0$. By Claim 1, we have $f(K) \leqslant K$. Notice that $f(n+K) \mid f(n) + f(K)$ implies $f(n+K) \leqslant f(n) + f(K) \leqslant f(n) + K$. If $n = tK + r$ for some $t \geqslant 0$ and $1 \leqslant r \leqslant K$, then we conclude

$$f(n) \leqslant K + f(n-K) \leqslant 2K + f(n-2K) \leqslant \ldots \leqslant tK + f(r) < n + \max\big(f(1), f(2), \ldots, f(K)\big),$$

so the claim is true with $C = \max\big(f(1), \ldots, f(K)\big)$. □

Claim 3. If $a, b \in \mathbb{Z}_{>0}$ are coprime then $\gcd\big(f(a), f(b)\big) \mid f(1)$. In particular, if $a, b \geqslant n_0$ are coprime then $f(a)$ and $f(b)$ are coprime.

Proof. Let $d = \gcd\big(f(a), f(b)\big)$. We can replicate Euclid's algorithm. Formally, apply induction on $a + b$. If $a = 1$ or $b = 1$ then we already have $d \mid f(1)$.

Without loss of generality, suppose $1 < a < b$. Then $d \mid f(a)$ and $d \mid f(b) \mid f(a) + f(b-a)$, so $d \mid f(b-a)$. Therefore d divides $\gcd\big(f(a), f(b-a)\big)$ which is a divisor of $f(1)$ by the induction hypothesis. □

Let $p_1 < p_2 < \ldots$ be the sequence of all prime numbers; for every k, let q_k be the lowest power of p_k with $q_k \geqslant n_0$. (Notice that there are only finitely many positive integers with $q_k \neq p_k$.)

Take a positive integer N, and consider the numbers

$$f(1), f(q_1), f(q_2), \ldots, f(q_N).$$

Here we have $N + 1$ numbers, each being greater than 1, and they are pairwise coprime by Claim 3. Therefore, they have at least $N+1$ different prime divisors in total, and their greatest prime divisor is at least p_{N+1}. Hence, $\max(f(1), f(q_1), \ldots, f(q_N)) \geqslant p_{N+1}$.

Choose N such that $\max(q_1, \ldots, q_N) = p_N$ (this is achieved if N is sufficiently large), and $p_{N+1} - p_N > C$ (that is possible, because there are arbitrarily long gaps between the primes). Then we establish a contradiction

$$p_{N+1} \leqslant \max(f(1), f(q_1), \ldots, f(q_N)) < \max(1 + C, q_1 + C, \ldots, q_N + C) = p_N + C < p_{N+1}$$

which proves the statement.

N7. Let $n \geqslant 2018$ be an integer, and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be pairwise distinct positive integers not exceeding $5n$. Suppose that the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n} \tag{1}$$

forms an arithmetic progression. Prove that the terms of the sequence are equal.

(Thailand)

Solution. Suppose that (1) is an arithmetic progression with nonzero difference. Let the difference be $\Delta = \frac{c}{d}$, where $d > 0$ and c, d are coprime.

We will show that too many denominators b_i should be divisible by d. To this end, for any $1 \leqslant i \leqslant n$ and any prime divisor p of d, say that *the index i is p-wrong*, if $v_p(b_i) < v_p(d)$. ($v_p(x)$ stands for the exponent of p in the prime factorisation of x.)

Claim 1. For any prime p, all p-wrong indices are congruent modulo p. In other words, the p-wrong indices (if they exist) are included in an arithmetic progression with difference p.

Proof. Let $\alpha = v_p(d)$. For the sake of contradiction, suppose that i and j are p-wrong indices (i.e., none of b_i and b_j is divisible by p^α) such that $i \not\equiv j \pmod p$. Then the least common denominator of $\frac{a_i}{b_i}$ and $\frac{a_j}{b_j}$ is not divisible by p^α. But this is impossible because in their difference, $(i-j)\Delta = \frac{(i-j)c}{d}$, the numerator is coprime to p, but p^α divides the denominator d. □

Claim 2. d has no prime divisors greater than 5.

Proof. Suppose that $p \geqslant 7$ is a prime divisor of d. Among the indices $1, 2, \ldots, n$, at most $\left\lceil \frac{n}{p} \right\rceil < \frac{n}{p} + 1$ are p-wrong, so p divides at least $\frac{p-1}{p}n - 1$ of $b_1, \ldots, b_n$. Since these denominators are distinct,

$$5n \geqslant \max\{b_i : p \mid b_i\} \geqslant \left(\frac{p-1}{p}n - 1\right)p = (p-1)(n-1) - 1 \geqslant 6(n-1) - 1 > 5n,$$

a contradiction. □

Claim 3. For every $0 \leqslant k \leqslant n - 30$, among the denominators $b_{k+1}, b_{k+2}, \ldots, b_{k+30}$, at least $\varphi(30) = 8$ are divisible by d.

Proof. By Claim 1, the 2-wrong, 3-wrong and 5-wrong indices can be covered by three arithmetic progressions with differences 2, 3 and 5. By a simple inclusion-exclusion, $(2-1)\cdot(3-1)\cdot(5-1) = 8$ indices are not covered; by Claim 2, we have $d \mid b_i$ for every uncovered index i. □

Claim 4. $|\Delta| < \frac{20}{n-2}$ and $d > \frac{n-2}{20}$.

Proof. From the sequence (1), remove all fractions with $b_n < \frac{n}{2}$, There remain at least $\frac{n}{2}$ fractions, and they cannot exceed $\frac{5n}{n/2} = 10$. So we have at least $\frac{n}{2}$ elements of the arithmetic progression (1) in the interval $(0, 10]$, hence the difference must be below $\frac{10}{n/2-1} = \frac{20}{n-2}$.

The second inequality follows from $\frac{1}{d} \leqslant \frac{|c|}{d} = |\Delta|$. □

Now we have everything to get the final contradiction. By Claim 3, we have $d \mid b_i$ for at least $\left\lfloor \frac{n}{30} \right\rfloor \cdot 8$ indices i. By Claim 4, we have $d \geqslant \frac{n-2}{20}$. Therefore,

$$5n \geqslant \max\{b_i : d \mid b_i\} \geqslant \left(\left\lfloor \frac{n}{30} \right\rfloor \cdot 8\right) \cdot d > \left(\frac{n}{30} - 1\right) \cdot 8 \cdot \frac{n-2}{20} > 5n.$$

Comment 1. It is possible that all terms in (1) are equal, for example with $a_i = 2i - 1$ and $b_i = 4i - 2$ we have $\frac{a_i}{b_i} = \frac{1}{2}$.

Comment 2. The bound $5n$ in the statement is far from sharp; the solution above can be modified to work for $9n$. For large n, the bound $5n$ can be replaced by $n^{\frac{3}{2}-\varepsilon}$.

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Problem Selection
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Martes, 16 de julio de 2019

Problema 1. Sea $\mathbb{Z}$ el conjunto de los números enteros. Determinar todas las funciones $f\colon \mathbb{Z} \to \mathbb{Z}$ tales que, para todos los enteros a y b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

Problema 2. En el triángulo ABC, el punto A_1 está en el lado BC y el punto B_1 está en el lado AC. Sean P y Q puntos en los segmentos AA_1 y BB_1, respectivamente, tales que PQ es paralelo a AB. Sea P_1 un punto en la recta PB_1 distinto de B_1, con B_1 entre P y P_1, y $\angle PP_1C = \angle BAC$. Análogamente, sea Q_1 un punto en la recta QA_1 distinto de A_1, con A_1 entre Q y Q_1, y $\angle CQ_1Q = \angle CBA$.

Demostrar que los puntos P, Q, P_1, y Q_1 son concíclicos.

Problema 3. Una red social tiene 2019 usuarios, algunos de los cuales son amigos. Siempre que el usuario A es amigo del usuario B, el usuario B también es amigo del usuario A. Eventos del siguiente tipo pueden ocurrir repetidamente, uno a la vez:

> Tres usuarios A, B, y C tales que A es amigo de B y de C, pero B y C no son amigos, cambian su estado de amistad de modo que B y C ahora son amigos, pero A ya no es amigo ni de B ni de C. Las otras relaciones de amistad no cambian.

Inicialmente, hay 1010 usuarios que tienen 1009 amigos cada uno, y hay 1009 usuarios que tienen 1010 amigos cada uno. Demostrar que hay una sucesión de este tipo de eventos después de la cual cada usuario es amigo como máximo de uno de los otros usuarios.

Tiempo: 4 horas y 30 minutos
Cada problema vale 7 puntos

Miércoles, 17 de julio de 2019

Problema 4. Encontrar todos los pares (k, n) de enteros positivos tales que

$$k! = (2^n - 1)(2^n - 2)(2^n - 4)\cdots(2^n - 2^{n-1}).$$

Problema 5. El Banco de Bath emite monedas con una H en una cara y una T en la otra. Harry tiene n monedas de este tipo alineadas de izquierda a derecha. Él realiza repetidamente la siguiente operación: si hay exactamente $k > 0$ monedas con la H hacia arriba, Harry voltea la k-ésima moneda contando desde la izquierda; en caso contrario, todas las monedas tienen la T hacia arriba y él se detiene. Por ejemplo, si $n = 3$ y la configuración inicial es THT, el proceso sería $THT \to HHT \to HTT \to TTT$, que se detiene después de tres operaciones.

(a) Demostrar que para cualquier configuración inicial que tenga Harry, el proceso se detiene después de un número finito de operaciones.

(b) Para cada configuración inicial C, sea $L(C)$ el número de operaciones que se realizan hasta que Harry se detiene. Por ejemplo, $L(THT) = 3$ y $L(TTT) = 0$. Determinar el valor promedio de $L(C)$ sobre todas las 2^n posibles configuraciones iniciales de C.

Problema 6. Sea I el incentro del triángulo acutángulo ABC con $AB \neq AC$. La circunferencia inscrita (o incírculo) ω de ABC es tangente a los lados BC, CA y AB en D, E y F, respectivamente. La recta que pasa por D y es perpendicular a EF corta a ω nuevamente en R. La recta AR corta a ω nuevamente en P. Las circunferencias circunscritas (o circuncírculos) de los triángulos PCE y PBF se cortan nuevamente en Q.

Demostrar que las rectas DI y PQ se cortan en la recta que pasa por A y es perpendicular a AI.

Problems

(with solutions)

60th International Mathematical Olympiad

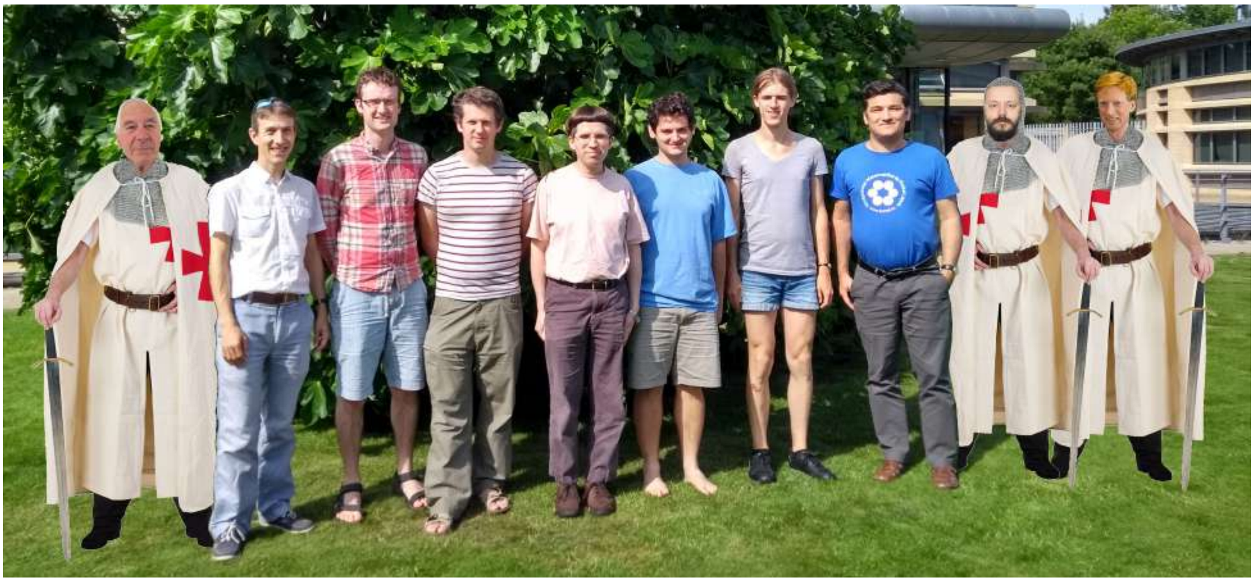
Bath — UK, 11th–22nd July 2019

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2019 thank the following 58 countries for contributing 204 problem proposals:

Albania, Armenia, Australia, Austria, Belarus, Belgium, Brazil, Bulgaria, Canada, China, Croatia, Cuba, Cyprus, Czech Republic, Denmark, Ecuador, El Salvador, Estonia, Finland, France, Georgia, Germany, Greece, Hong Kong, Hungary, India, Indonesia, Iran, Ireland, Israel, Italy, Japan, Kazakhstan, Kosovo, Luxembourg, Mexico, Netherlands, New Zealand, Nicaragua, Nigeria, North Macedonia, Philippines, Poland, Russia, Serbia, Singapore, Slovakia, Slovenia, South Africa, South Korea, Sweden, Switzerland, Taiwan, Tanzania, Thailand, Ukraine, USA, Vietnam.

Problem Selection Committee



Tony Gardiner, Edward Crane, Alexander Betts, James Cranch, Joseph Myers (chair), James Aaronson, Andrew Carlotti, Géza Kós, Ilya I. Bogdanov, Jack Shotton

Problems

Day 1

Problem 1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

(South Africa)

Problem 2. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

(Ukraine)

Problem 3. A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

> Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

(Croatia)

Day 2

Problem 4. Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

(El Salvador)

Problem 5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

(USA)

Problem 6. Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

(India)

Solutions

Day 1

Problem 1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)). \tag{1}$$

(South Africa)

Answer: The solutions are $f(n) = 0$ and $f(n) = 2n + K$ for any constant $K \in \mathbb{Z}$.

Common remarks. Most solutions to this problem first prove that f must be linear, before determining all linear functions satisfying (1).

Solution 1. Substituting $a = 0, b = n + 1$ gives $f(f(n+1)) = f(0) + 2f(n+1)$. Substituting $a = 1, b = n$ gives $f(f(n+1)) = f(2) + 2f(n)$.

In particular, $f(0) + 2f(n+1) = f(2) + 2f(n)$, and so $f(n+1) - f(n) = \frac{1}{2}(f(2) - f(0))$. Thus $f(n+1) - f(n)$ must be constant. Since f is defined only on $\mathbb{Z}$, this tells us that f must be a linear function; write $f(n) = Mn + K$ for arbitrary constants M and K, and we need only determine which choices of M and K work.

Now, (1) becomes

$$2Ma + K + 2(Mb + K) = M(M(a+b) + K) + K$$

which we may rearrange to form

$$(M - 2)\big(M(a+b) + K\big) = 0.$$

Thus, either $M = 2$, or $M(a+b) + K = 0$ for all values of $a + b$. In particular, the only possible solutions are $f(n) = 0$ and $f(n) = 2n + K$ for any constant $K \in \mathbb{Z}$, and these are easily seen to work.

Solution 2. Let $K = f(0)$.

First, put $a = 0$ in (1); this gives

$$f(f(b)) = 2f(b) + K \tag{2}$$

for all $b \in \mathbb{Z}$.

Now put $b = 0$ in (1); this gives

$$f(2a) + 2K = f(f(a)) = 2f(a) + K,$$

where the second equality follows from (2). Consequently,

$$f(2a) = 2f(a) - K \tag{3}$$

for all $a \in \mathbb{Z}$.

Substituting (2) and (3) into (1), we obtain

$$\begin{aligned} f(2a) + 2f(b) &= f(f(a+b)) \\ 2f(a) - K + 2f(b) &= 2f(a+b) + K \\ f(a) + f(b) &= f(a+b) + K. \end{aligned}$$

Thus, if we set $g(n) = f(n) - K$ we see that g satisfies the Cauchy equation $g(a + b) = g(a) + g(b)$. The solution to the Cauchy equation over $\mathbb{Z}$ is well-known; indeed, it may be proven by an easy induction that $g(n) = Mn$ for each $n \in \mathbb{Z}$, where $M = g(1)$ is a constant.

Therefore, $f(n) = Mn + K$, and we may proceed as in Solution 1.

Comment 1. Instead of deriving (3) by substituting $b = 0$ into (1), we could instead have observed that the right hand side of (1) is symmetric in a and b, and thus

$$f(2a) + 2f(b) = f(2b) + 2f(a).$$

Thus, $f(2a) - 2f(a) = f(2b) - 2f(b)$ for any $a, b \in \mathbb{Z}$, and in particular $f(2a) - 2f(a)$ is constant. Setting $a = 0$ shows that this constant is equal to $-K$, and so we obtain (3).

Comment 2. Some solutions initially prove that $f(f(n))$ is linear (sometimes via proving that $f(f(n)) - 3K$ satisfies the Cauchy equation). However, one can immediately prove that f is linear by substituting something of the form $f(f(n)) = M'n + K'$ into (2).

Problem 2. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

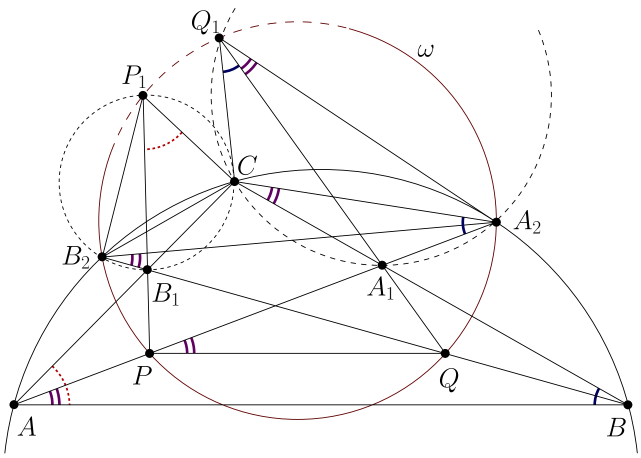
(Ukraine)

Solution 1. Throughout the solution we use oriented angles.

Let rays AA_1 and BB_1 intersect the circumcircle of $\triangle ACB$ at A_2 and B_2, respectively. By

$$\angle QPA_2 = \angle BAA_2 = \angle BB_2A_2 = \angle QB_2A_2,$$

points P, Q, A_2, B_2 are concyclic; denote the circle passing through these points by ω. We shall prove that P_1 and Q_1 also lie on ω.



By

$$\angle CA_2A_1 = \angle CA_2A = \angle CBA = \angle CQ_1Q = \angle CQ_1A_1,$$

points C, Q_1, A_2, A_1 are also concyclic. From that we get

$$\angle QQ_1A_2 = \angle A_1Q_1A_2 = \angle A_1CA_2 = \angle BCA_2 = \angle BAA_2 = \angle QPA_2,$$

so Q_1 lies on ω.

It follows similarly that P_1 lies on ω.

Solution 2. First consider the case when lines PP_1 and QQ_1 intersect each other at some point R.

Let line PQ meet the sides AC and BC at E and F, respectively. Then

$$\angle PP_1C = \angle BAC = \angle PEC,$$

so points C, E, P, P_1 lie on a circle; denote that circle by ω_P. It follows analogously that points C, F, Q, Q_1 lie on another circle; denote it by ω_Q.

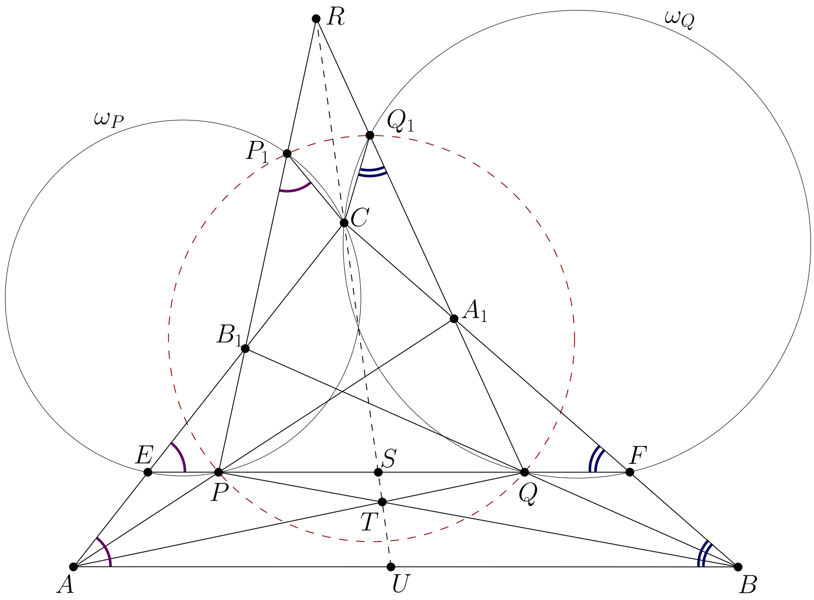
Let AQ and BP intersect at T. Applying Pappus' theorem to the lines AA_1P and BB_1Q provides that points $C = AB_1 \cap BA_1$, $R = A_1Q \cap B_1P$ and $T = AQ \cap BP$ are collinear.

Let line RCT meet PQ and AB at S and U, respectively. From $AB \parallel PQ$ we obtain

$$\frac{SP}{SQ} = \frac{UB}{UA} = \frac{SF}{SE},$$

so

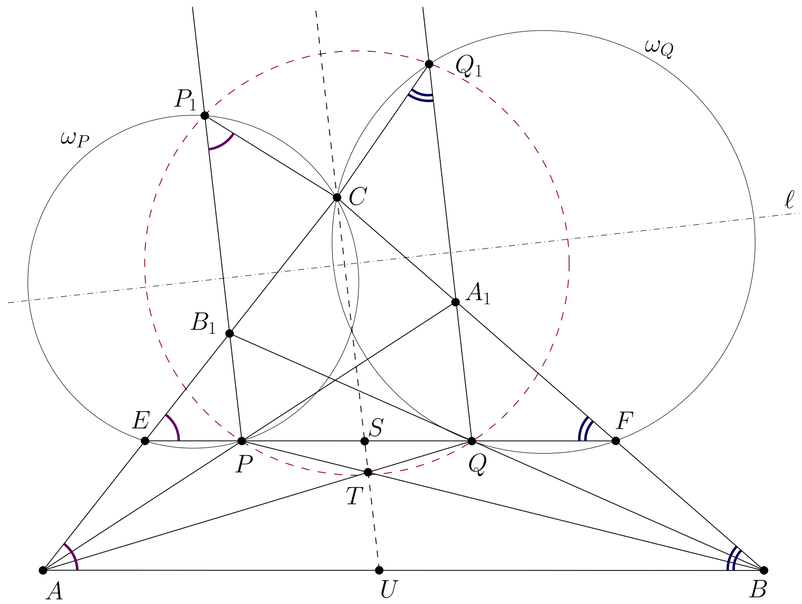
$$SP \cdot SE = SQ \cdot SF.$$



So, point S has equal powers with respect to ω_P and ω_Q, hence line RCS is their radical axis; then R also has equal powers to the circles, so $RP \cdot RP_1 = RQ \cdot RQ_1$, proving that points P, P_1, Q, Q_1 are indeed concyclic.

Now consider the case when PP_1 and QQ_1 are parallel. Like in the previous case, let AQ and BP intersect at T. Applying Pappus' theorem again to the lines AA_1P and BB_1Q, in this limit case it shows that line CT is parallel to PP_1 and QQ_1.

Let line CT meet PQ and AB at S and U, as before. The same calculation as in the previous case shows that $SP \cdot SE = SQ \cdot SF$, so S lies on the radical axis between ω_P and ω_Q.



Line CST, that is the radical axis between ω_P and ω_Q, is perpendicular to the line ℓ of centres of ω_P and ω_Q. Hence, the chords PP_1 and QQ_1 are perpendicular to ℓ. So the quadrilateral PP_1Q_1Q is an isosceles trapezium with symmetry axis ℓ, and hence is cyclic.

Comment. There are several ways of solving the problem involving Pappus' theorem. For example, one may consider the points $K = PB_1 \cap BC$ and $L = QA_1 \cap AC$. Applying Pappus' theorem to the lines AA_1P and QB_1B we get that K, L, and $PQ \cap AB$ are collinear, i.e. that $KL \parallel AB$. Therefore, cyclicity of P, Q, P_1, and Q_1 is equivalent to that of K, L, P_1, and Q_1. The latter is easy after noticing that C also lies on that circle. Indeed, e.g. $\angle(LK, LC) = \angle(AB, AC) = \angle(P_1K, P_1C)$ shows that K lies on circle KLC.

This approach also has some possible degeneracy, as the points K and L may happen to be ideal.

Problem 3. A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

> Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

(Croatia)

Common remarks. The problem has an obvious rephrasing in terms of graph theory. One is given a graph G with 2019 vertices, 1010 of which have degree 1009 and 1009 of which have degree 1010. One is allowed to perform operations on G of the following kind:

> Suppose that vertex A is adjacent to two distinct vertices B and C which are not adjacent to each other. Then one may remove the edges AB and AC from G and add the edge BC into G.

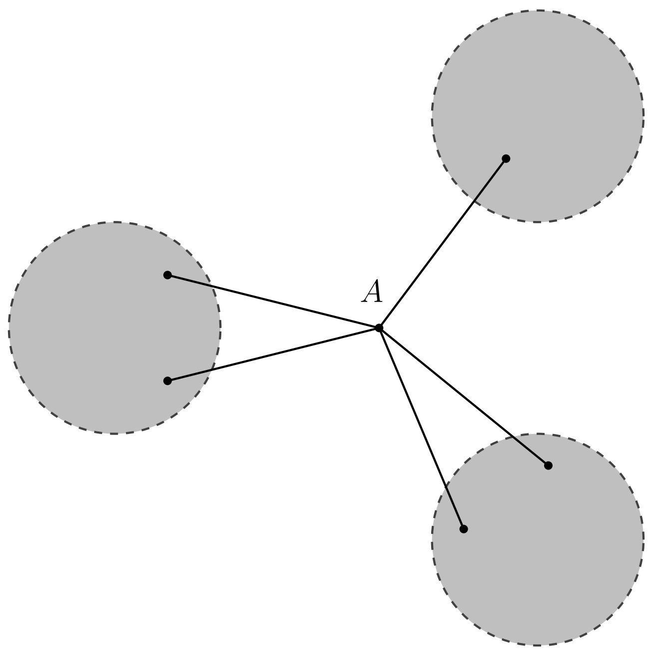
Call such an operation a *refriending*. One wants to prove that, via a sequence of such refriendings, one can reach a graph which is a disjoint union of single edges and vertices.

All of the solutions presented below will use this reformulation.

Solution 1. Note that the given graph is connected, since the total degree of any two vertices is at least 2018 and hence they are either adjacent or have at least one neighbour in common. Hence the given graph satisfies the following condition:

> Every connected component of G with at least three vertices is not complete and has a vertex of odd degree. (1)

We will show that if a graph G satisfies condition (1) and has a vertex of degree at least 2, then there is a refriending on G that preserves condition (1). Since refriendings decrease the total number of edges of G, by using a sequence of such refriendings, we must reach a graph G with maximal degree at most 1, so we are done.



Pick a vertex A of degree at least 2 in a connected component G' of G. Since no component of G with at least three vertices is complete we may assume that not all of the neighbours of A are adjacent to one another. (For example, pick a maximal complete subgraph K of G'. Some vertex A of K has a neighbour outside K, and this neighbour is not adjacent to every vertex of K by maximality.) Removing A from G splits G' into smaller connected components $G_1, \dots, G_k$ (possibly with $k = 1$), to each of which A is connected by at least one edge. We divide into several cases.

Case 1: $k \geqslant 2$ and A is connected to some G_i by at least two edges.

Choose a vertex B of G_i adjacent to A, and a vertex C in another component G_j adjacent to A. The vertices B and C are not adjacent, and hence removing edges AB and AC and adding in edge BC does not disconnect G'. It is easy to see that this preserves the condition, since the refriending does not change the parity of the degrees of vertices.

Case 2: $k \geqslant 2$ and A is connected to each G_i by exactly one edge.

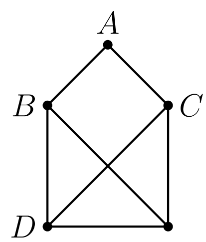
Consider the induced subgraph on any G_i and the vertex A. The vertex A has degree 1 in this subgraph; since the number of odd-degree vertices of a graph is always even, we see that G_i has a vertex of odd degree (in G). Thus if we let B and C be any distinct neighbours of A, then removing edges AB and AC and adding in edge BC preserves the above condition: the refriending creates two new components, and if either of these components has at least three vertices, then it cannot be complete and must contain a vertex of odd degree (since each G_i does).

Case 3: $k = 1$ and A is connected to G_1 by at least three edges.

By assumption, A has two neighbours B and C which are not adjacent to one another. Removing edges AB and AC and adding in edge BC does not disconnect G'. We are then done as in Case 1.

Case 4: $k = 1$ and A is connected to G_1 by exactly two edges.

Let B and C be the two neighbours of A, which are not adjacent. Removing edges AB and AC and adding in edge BC results in two new components: one consisting of a single vertex; and the other containing a vertex of odd degree. We are done unless this second component would be a complete graph on at least 3 vertices. But in this case, G_1 would be a complete graph minus the single edge BC, and hence has at least 4 vertices since G' is not a 4-cycle. If we let D be a third vertex of G_1, then removing edges BA and BD and adding in edge AD does not disconnect G'. We are then done as in Case 1.

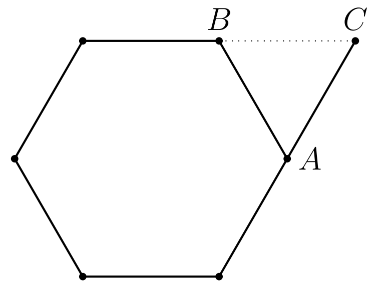


Comment. In fact, condition 1 above *precisely* characterises those graphs which can be reduced to a graph of maximal degree $\leqslant 1$ by a sequence of refriendings.

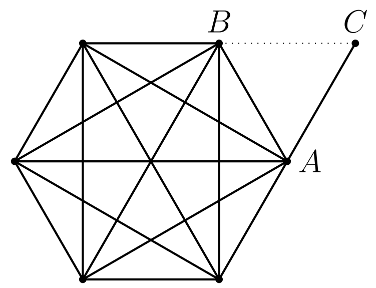
Solution 2. As in the previous solution, note that a refriending preserves the property that a graph has a vertex of odd degree and (trivially) the property that it is not complete; note also that our initial graph is connected. We describe an algorithm to reduce our initial graph to a graph of maximal degree at most 1, proceeding in two steps.

Step 1: There exists a sequence of refriendings reducing the graph to a tree.

Proof. Since the number of edges decreases with each refriending, it suffices to prove the following: as long as the graph contains a cycle, there exists a refriending such that the resulting graph is still connected. We will show that the graph in fact contains a cycle Z and vertices A, B, C such that A and B are adjacent in the cycle Z, C is not in Z, and is adjacent to A but not B. Removing edges AB and AC and adding in edge BC keeps the graph connected, so we are done.



To find this cycle Z and vertices A, B, C, we pursue one of two strategies. If the graph contains a triangle, we consider a largest complete subgraph K, which thus contains at least three vertices. Since the graph itself is not complete, there is a vertex C not in K connected to a vertex A of K. By maximality of K, there is a vertex B of K not connected to C, and hence we are done by choosing a cycle Z in K through the edge AB.



If the graph is triangle-free, we consider instead a smallest cycle Z. This cycle cannot be Hamiltonian (i.e. it cannot pass through every vertex of the graph), since otherwise by minimality the graph would then have no other edges, and hence would have even degree at every vertex. We may thus choose a vertex C not in Z adjacent to a vertex A of Z. Since the graph is triangle-free, it is not adjacent to any neighbour B of A in Z, and we are done. $\square$

Step 2: Any tree may be reduced to a disjoint union of single edges and vertices by a sequence of refriendings.

Proof. The refriending preserves the property of being acyclic. Hence, after applying a sequence of refriendings, we arrive at an acyclic graph in which it is impossible to perform any further refriendings. The maximal degree of any such graph is 1: if it had a vertex A with two neighbours B, C, then B and C would necessarily be nonadjacent since the graph is cycle-free, and so a refriending would be possible. Thus we reach a graph with maximal degree at most 1 as desired. $\square$

Day 2

Problem 4.

Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}). \tag{1}$$

(El Salvador)

Answer: The only such pairs are $(1, 1)$ and $(3, 2)$.

Common remarks. In all solutions, for any prime p and positive integer N, we will denote by $v_p(N)$ the exponent of the largest power of p that divides N. The right-hand side of (1) will be denoted by L_n; that is, $L_n = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1})$.

Solution 1. We will get an upper bound on n from the speed at which $v_2(L_n)$ grows.

From

$$L_n = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}) = 2^{1+2+\cdots+(n-1)}(2^n - 1)(2^{n-1} - 1) \cdots (2^1 - 1)$$

we read

$$v_2(L_n) = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}.$$

On the other hand, $v_2(k!)$ is expressed by the Legendre formula as

$$v_2(k!) = \sum_{i=1}^{\infty} \left\lfloor \frac{k}{2^i} \right\rfloor.$$

As usual, by omitting the floor functions,

$$v_2(k!) < \sum_{i=1}^{\infty} \frac{k}{2^i} = k.$$

Thus, $k! = L_n$ implies the inequality

$$\frac{n(n-1)}{2} < k. \tag{2}$$

In order to obtain an opposite estimate, observe that

$$L_n = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}) < (2^n)^n = 2^{n^2}.$$

We claim that

$$2^{n^2} < \left(\frac{n(n-1)}{2}\right)! \quad \text{for } n \geqslant 6. \tag{3}$$

For $n = 6$ the estimate (3) is true because $2^{6^2} < 6.9 \cdot 10^{10}$ and $\left(\frac{n(n-1)}{2}\right)! = 15! > 1.3 \cdot 10^{12}$.

For $n \geqslant 7$ we prove (3) by the following inequalities:

$$\begin{aligned}\left(\frac{n(n-1)}{2}\right)! &= 15! \cdot 16 \cdot 17 \cdots \frac{n(n-1)}{2} > 2^{36} \cdot 16^{\frac{n(n-1)}{2} - 15} \\ &= 2^{2n(n-1)-24} = 2^{n^2} \cdot 2^{n(n-2)-24} > 2^{n^2}.\end{aligned}$$

Putting together (2) and (3), for $n \geqslant 6$ we get a contradiction, since

$$L_n < 2^{n^2} < \left(\frac{n(n-1)}{2}\right)! < k! = L_n.$$

Hence $n \geqslant 6$ is not possible.

Checking manually the cases $n \leqslant 5$ we find

$$L_1 = 1 = 1!, \qquad L_2 = 6 = 3!, \qquad 5! < L_3 = 168 < 6!,$$
$$7! < L_4 = 20\,160 < 8! \qquad \text{and} \qquad 10! < L_5 = 9\,999\,360 < 11!.$$

So, there are two solutions:

$$(k,n) \in \{(1,1),(3,2)\}.$$

Solution 2. Like in the previous solution, the cases $n = 1,2,3,4$ are checked manually. We will exclude $n \geqslant 5$ by considering the exponents of 3 and 31 in (1).

For odd primes p and distinct integers a,b, coprime to p, with $p \mid a-b$, the Lifting The Exponent lemma asserts that

$$v_p(a^j - b^j) = v_p(a-b) + v_p(j).$$

Notice that 3 divides $2^j - 1$ if only if j is even; moreover, by the Lifting The Exponent lemma we have

$$v_3(2^{2j}-1) = v_3(4^j - 1) = 1 + v_3(j) = v_3(3j).$$

Hence,

$$v_3(L_n) = \sum_{2j\leqslant n} v_3(4^j - 1) = \sum_{j \leqslant \lfloor \frac{n}{2} \rfloor} v_3(3j).$$

Notice that the last expression is precisely the exponent of 3 in the prime factorisation of $\left(3\lfloor \frac{n}{2} \rfloor\right)!$. Therefore

$$v_3(k!) = v_3(L_n) = v_3\left(\left(3\lfloor \tfrac{n}{2} \rfloor\right)!\right)$$
$$3\left\lfloor \frac{n}{2} \right\rfloor \leqslant k \leqslant 3\left\lfloor \frac{n}{2} \right\rfloor + 2. \tag{4}$$

Suppose that $n \geqslant 5$. Note that every fifth factor in L_n is divisible by $31 = 2^5 - 1$, and hence we have $v_{31}(L_n) \geqslant \lfloor \frac{n}{5} \rfloor$. Then

$$\frac{n}{10} \leqslant \left\lfloor \frac{n}{5} \right\rfloor \leqslant v_{31}(L_n) = v_{31}(k!) = \sum_{j=1}^{\infty} \left\lfloor \frac{k}{31^j} \right\rfloor < \sum_{j=1}^{\infty} \frac{k}{31^j} = \frac{k}{30}. \tag{5}$$

By combining (4) and (5),

$$3n < k \leqslant \frac{3n}{2} + 2$$

so $n < \frac{4}{3}$ which is inconsistent with the inequality $n \geqslant 5$.

Comment 1. There are many combinations of the ideas above; for example combining (2) and (4) also provides $n < 5$. Obviously, considering the exponents of any two primes in (1), or considering one prime and the magnitude orders lead to an upper bound on n and k.

Comment 2. This problem has a connection to group theory. Indeed, the right-hand side is the order of the group $GL_n(\mathbb{F}_2)$ of invertible n-by-n matrices with entries modulo 2, while the left-hand side is the order of the symmetric group S_k on k elements. The result thus shows that the only possible isomorphisms between these groups are $GL_1(\mathbb{F}_2) \cong S_1$ and $GL_2(\mathbb{F}_2) \cong S_3$, and there are in fact isomorphisms in both cases. In general, $GL_n(\mathbb{F}_2)$ is a simple group for $n \geqslant 3$, as it is isomorphic to $PSL_n(\mathbb{F}_2)$.

There is also a near-solution of interest: the right-hand side for $n = 4$ is *half* of the left-hand side when $k = 8$; this turns out to correspond to an isomorphism $GL_4(\mathbb{F}_2) \cong A_8$ with the alternating group on eight elements.

However, while this indicates that the problem is a useful one, knowing group theory is of no use in solving it!

Problem 5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

(USA)

Answer: The average is $\frac{1}{4}n(n+1)$.

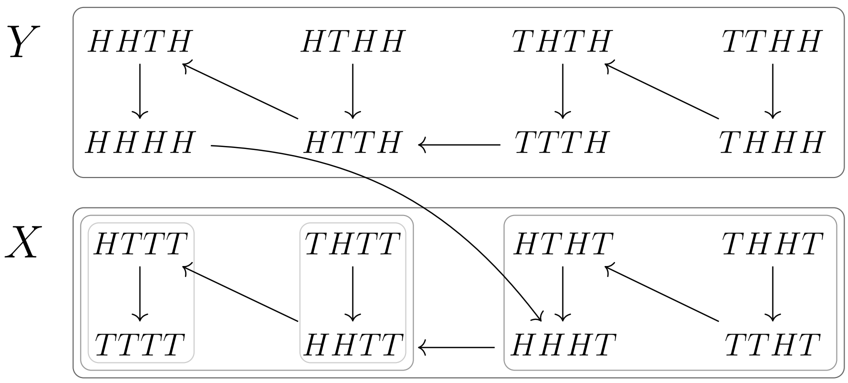
Common remarks. Throughout all these solutions, we let $E(n)$ denote the desired average value.

Solution 1. We represent the problem using a directed graph G_n whose vertices are the length-n strings of H's and T's. The graph features an edge from each string to its successor (except for $TT\cdots TT$, which has no successor). We will also write $\bar{H} = T$ and $\bar{T} = H$.

The graph G_0 consists of a single vertex: the empty string. The main claim is that G_n can be described explicitly in terms of G_{n-1}:

- We take two copies, X and Y, of G_{n-1}.
- In X, we take each string of $n-1$ coins and just append a T to it. In symbols, we replace $s_1 \cdots s_{n-1}$ with $s_1 \cdots s_{n-1}T$.
- In Y, we take each string of $n-1$ coins, flip every coin, reverse the order, and append an H to it. In symbols, we replace $s_1 \cdots s_{n-1}$ with $\bar{s}_{n-1}\bar{s}_{n-2} \cdots \bar{s}_1 H$.
- Finally, we add one new edge from Y to X, namely $HH\cdots HHH \to HH\cdots HHT$.

We depict G_4 below, in a way which indicates this recursive construction:



We prove the claim inductively. Firstly, X is correct as a subgraph of G_n, as the operation on coins is unchanged by an extra T at the end: if $s_1 \cdots s_{n-1}$ is sent to $t_1 \cdots t_{n-1}$, then $s_1 \cdots s_{n-1}T$ is sent to $t_1 \cdots t_{n-1}T$.

Next, Y is also correct as a subgraph of G_n, as if $s_1 \cdots s_{n-1}$ has k occurrences of H, then $\bar{s}_{n-1} \cdots \bar{s}_1 H$ has $(n-1-k)+1 = n-k$ occurrences of H, and thus (provided that $k > 0$), if $s_1 \cdots s_{n-1}$ is sent to $t_1 \cdots t_{n-1}$, then $\bar{s}_{n-1} \cdots \bar{s}_1 H$ is sent to $\bar{t}_{n-1} \cdots \bar{t}_1 H$.

Finally, the one edge from Y to X is correct, as the operation does send $HH\cdots HHH$ to $HH\cdots HHT$.

To finish, note that the sequences in X take an average of $E(n-1)$ steps to terminate, whereas the sequences in Y take an average of $E(n-1)$ steps to reach $HH\cdots H$ and then an additional n steps to terminate. Therefore, we have

$$E(n) = \frac{1}{2}\left(E(n-1) + (E(n-1)+n)\right) = E(n-1) + \frac{n}{2}.$$

We have $E(0) = 0$ from our description of G_0. Thus, by induction, we have $E(n) = \frac{1}{2}(1+\cdots+n) = \frac{1}{4}n(n+1)$, which in particular is finite.

Solution 2. We consider what happens with configurations depending on the coins they start and end with.

- If a configuration starts with H, the last $n-1$ coins follow the given rules, as if they were all the coins, until they are all T, then the first coin is turned over.
- If a configuration ends with T, the last coin will never be turned over, and the first $n-1$ coins follow the given rules, as if they were all the coins.
- If a configuration starts with T and ends with H, the middle $n-2$ coins follow the given rules, as if they were all the coins, until they are all T. After that, there are $2n-1$ more steps: first coins 1, 2, …, $n-1$ are turned over in that order, then coins n, $n-1$, …, 1 are turned over in that order.

As this covers all configurations, and the number of steps is clearly finite for 0 or 1 coins, it follows by induction on n that the number of steps is always finite.

We define $E_{AB}(n)$, where A and B are each one of H, T or $*$, to be the average number of steps over configurations of length n restricted to those that start with A, if A is not $*$, and that end with B, if B is not $*$ (so $*$ represents "either H or T"). The above observations tell us that, for $n \geqslant 2$:

- $E_{H*}(n) = E(n-1) + 1$.
- $E_{*T}(n) = E(n-1)$.
- $E_{HT}(n) = E(n-2) + 1$ (by using both the observations for $H*$ and for $*T$).
- $E_{TH}(n) = E(n-2) + 2n - 1$.

Now $E_{H*}(n) = \frac{1}{2}(E_{HH}(n) + E_{HT}(n))$, so $E_{HH}(n) = 2E(n-1) - E(n-2) + 1$. Similarly, $E_{TT}(n) = 2E(n-1) - E(n-2) - 1$. So

$$E(n) = \frac{1}{4}(E_{HT}(n) + E_{HH}(n) + E_{TT}(n) + E_{TH}(n)) = E(n-1) + \frac{n}{2}.$$

We have $E(0) = 0$ and $E(1) = \frac{1}{2}$, so by induction on n we have $E(n) = \frac{1}{4}n(n+1)$.

Solution 3. Let H_i be the number of H's in positions 1 to i inclusive (so H_n is the total number of H's), and let I_i be 1 if the i^{th} coin is an H, 0 otherwise. Consider the function

$$t(i) = I_i + 2(\min\{i, H_n\} - H_i).$$

We claim that $t(i)$ is the total number of times coin i is turned over (which implies that the process terminates). Certainly $t(i) = 0$ when all coins are T's, and $t(i)$ is always a nonnegative integer, so it suffices to show that when the k^{th} coin is turned over (where $k = H_n$), $t(k)$ goes down by 1 and all the other $t(i)$ are unchanged. We show this by splitting into cases:

- If $i < k$, I_i and H_i are unchanged, and $\min\{i, H_n\} = i$ both before and after the coin flip, so $t(i)$ is unchanged.
- If $i > k$, $\min\{i, H_n\} = H_n$ both before and after the coin flip, and both H_n and H_i change by the same amount, so $t(i)$ is unchanged.
- If $i = k$ and the coin is H, I_i goes down by 1, as do both $\min\{i, H_n\} = H_n$ and H_i; so $t(i)$ goes down by 1.
- If $i = k$ and the coin is T, I_i goes up by 1, $\min\{i, H_n\} = i$ is unchanged and H_i goes up by 1; so $t(i)$ goes down by 1.

We now need to compute the average value of

$$\sum_{i=1}^{n} t(i) = \sum_{i=1}^{n} I_i + 2\sum_{i=1}^{n} \min\{i, H_n\} - 2\sum_{i=1}^{n} H_i.$$

The average value of the first term is $\frac{1}{2}n$, and that of the third term is $-\frac{1}{2}n(n+1)$. To compute the second term, we sum over choices for the total number of H's, and then over the possible values of i, getting

$$2^{1-n}\sum_{j=0}^{n}\binom{n}{j}\sum_{i=1}^{n}\min\{i,j\} = 2^{1-n}\sum_{j=0}^{n}\binom{n}{j}\left(nj - \binom{j}{2}\right).$$

Now, in terms of trinomial coefficients,

$$\sum_{j=0}^{n} j\binom{n}{j} = \sum_{j=1}^{n}\binom{n}{n-j, j-1, 1} = n\sum_{j=0}^{n-1}\binom{n-1}{j} = 2^{n-1}n$$

and

$$\sum_{j=0}^{n}\binom{j}{2}\binom{n}{j} = \sum_{j=2}^{n}\binom{n}{n-j, j-2, 2} = \binom{n}{2}\sum_{j=0}^{n-2}\binom{n-2}{j} = 2^{n-2}\binom{n}{2}.$$

So the second term above is

$$2^{1-n}\left(2^{n-1}n^2 - 2^{n-2}\binom{n}{2}\right) = n^2 - \frac{n(n-1)}{4},$$

and the required average is

$$E(n) = \frac{1}{2}n + n^2 - \frac{n(n-1)}{4} - \frac{1}{2}n(n+1) = \frac{n(n+1)}{4}.$$

Solution 4. Harry has built a Turing machine to flip the coins for him. The machine is initially positioned at the k^{th} coin, where there are k coins showing H (and the position before the first coin is considered to be the 0^{th} coin). The machine then moves according to the following rules, stopping when it reaches the position before the first coin: if the coin at its current position is H, it flips the coin and moves to the previous coin, while if the coin at its current position is T, it flips the coin and moves to the next position.

Consider the maximal sequences of consecutive moves in the same direction. Suppose the machine has a consecutive moves to the next coin, before a move to the previous coin. After those a moves, the a coins flipped in those moves are all H's, as is the coin the machine is now at, so at least the next $a + 1$ moves will all be moves to the previous coin. Similarly, a consecutive moves to the previous coin are followed by at least $a + 1$ consecutive moves to the next coin. There cannot be more than n consecutive moves in the same direction, so this proves that the process terminates (with a move from the first coin to the position before the first coin).

Thus we have a (possibly empty) sequence $a_1 < \cdots < a_t \leqslant n$ giving the lengths of maximal sequences of consecutive moves in the same direction, where the final a_t moves must be moves to the previous coin, ending before the first coin. We claim there is a bijection between initial configurations of the coins and such sequences. This gives

$$E(n) = \frac{1}{2}(1 + 2 + \cdots + n) = \frac{n(n+1)}{4}$$

as required, since each i with $1 \leqslant i \leqslant n$ will appear in half of the sequences, and will contribute i to the number of moves when it does.

To see the bijection, consider following the sequence of moves backwards, starting with the machine just before the first coin and all coins showing T. This certainly determines a unique configuration of coins that could possibly correspond to the given sequence. Furthermore, every coin flipped as part of the a_j consecutive moves is also flipped as part of all subsequent sequences of a_k consecutive moves, for all $k > j$, meaning that, as we follow the moves backwards, each coin is always in the correct state when flipped to result in a move in the required direction. (Alternatively, since there are 2^n possible configurations of coins and 2^n possible such ascending sequences, the fact that the sequence of moves determines at most one configuration of coins, and thus that there is an injection from configurations of coins to such ascending sequences, is sufficient for it to be a bijection, without needing to show that coins are in the right state as we move backwards.)

Solution 5. We explicitly describe what happens with an arbitrary sequence C of n coins. Suppose that C contain k coins showing H at positions $1 \leqslant c_1 < c_2 < \cdots < c_k \leqslant n$.

Let i be the minimal index such that $c_i \geqslant k$. Then the first few steps will consist of turning over the k^{th}, $(k+1)^{\text{th}}$, $\ldots$, $c_i{}^{\text{th}}$, $(c_i - 1)^{\text{th}}$, $(c_i - 2)^{\text{th}}$, $\ldots$, k^{th} coins in this order. After that we get a configuration with $k-1$ coins showing H at the same positions as in the initial one, except for c_i. This part of the process takes $2(c_i - k) + 1$ steps.

After that, the process acts similarly; by induction on the number of H's we deduce that the process ends. Moreover, if the c_i disappear in order $c_{i_1}, \ldots, c_{i_k}$, the whole process takes

$$L(C) = \sum_{j=1}^{k} \bigl(2(c_{i_j} - (k+1-j)) + 1\bigr) = 2\sum_{j=1}^{k} c_j - 2\sum_{j=1}^{k}(k+1-j) + k = 2\sum_{j=1}^{k} c_j - k^2$$

steps.

Now let us find the total value S_k of $L(C)$ over all $\binom{n}{k}$ configurations with exactly k coins showing H. To sum up the above expression over those, notice that each number $1 \leqslant i \leqslant n$ appears as c_j exactly $\binom{n-1}{k-1}$ times. Thus

$$\begin{aligned} S_k &= 2\binom{n-1}{k-1}\sum_{i=1}^{n} i - \binom{n}{k}k^2 = 2\frac{(n-1)\cdots(n-k+1)}{(k-1)!} \cdot \frac{n(n+1)}{2} - \frac{n \cdots (n-k+1)}{k!}k^2 \\ &= \frac{n(n-1)\cdots(n-k+1)}{(k-1)!}\bigl((n+1) - k\bigr) = n(n-1)\binom{n-2}{k-1} + n\binom{n-1}{k-1}. \end{aligned}$$

Therefore, the total value of $L(C)$ over all configurations is

$$\sum_{k=1}^{n} S_k = n(n-1)\sum_{k=1}^{n}\binom{n-2}{k-1} + n\sum_{k=1}^{n}\binom{n-1}{k-1} = n(n-1)2^{n-2} + n2^{n-1} = 2^n\frac{n(n+1)}{4}.$$

Hence the required average is $E(n) = \frac{n(n+1)}{4}$.

Problem 6. Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

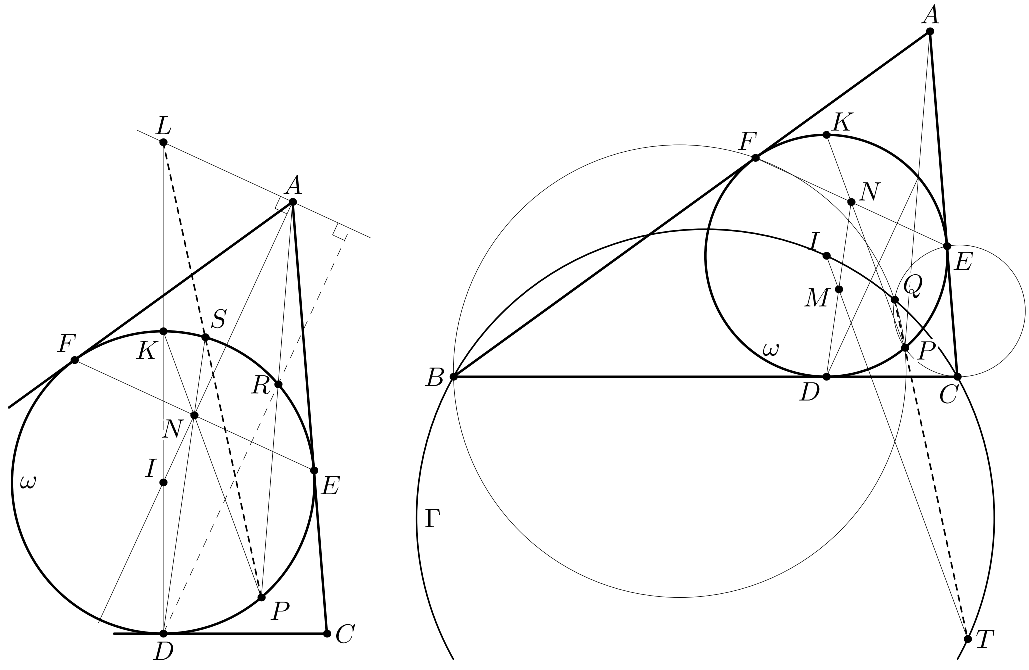
(India)

Common remarks. Throughout the solution, $\angle(a, b)$ denotes the directed angle between lines a and b, measured modulo π.

Solution 1.

Step 1. The external bisector of $\angle BAC$ is the line through A perpendicular to IA. Let DI meet this line at L and let DI meet ω at K. Let N be the midpoint of EF, which lies on IA and is the pole of line AL with respect to ω. Since $AN \cdot AI = AE^2 = AR \cdot AP$, the points R, N, I, and P are concyclic. As $IR = IP$, the line NI is the external bisector of $\angle PNR$, so PN meets ω again at the point symmetric to R with respect to AN – i.e. at K.

Let DN cross ω again at S. Opposite sides of any quadrilateral inscribed in the circle ω meet on the polar line of the intersection of the diagonals with respect to ω. Since L lies on the polar line AL of N with respect to ω, the line PS must pass through L. Thus it suffices to prove that the points S, Q, and P are collinear.

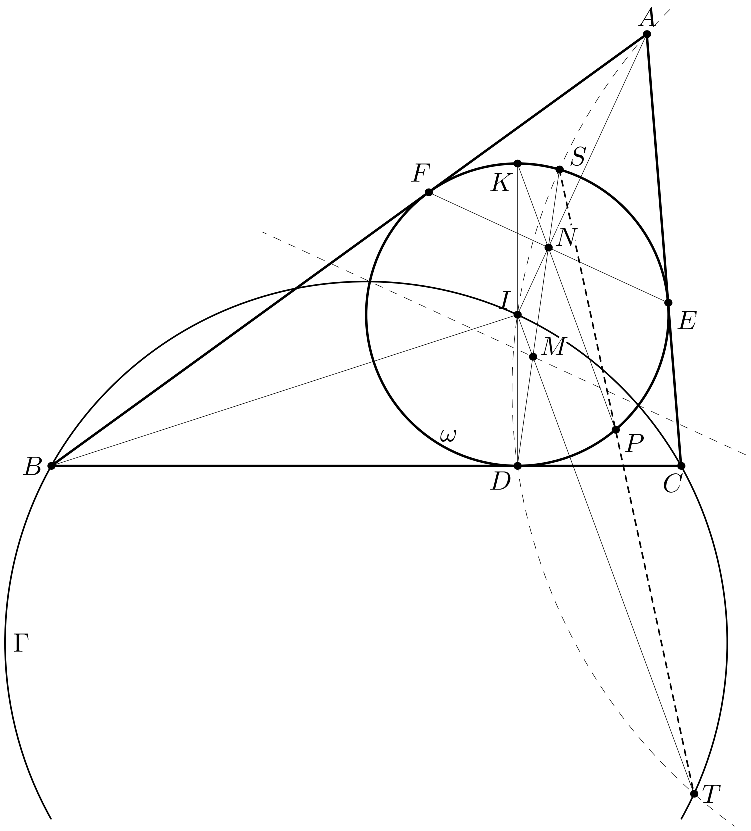


Step 2. Let Γ be the circumcircle of $\triangle BIC$. Notice that

$$\begin{aligned}\angle(BQ, QC) = \angle(BQ, QP) + \angle(PQ, QC) &= \angle(BF, FP) + \angle(PE, EC)\\ &= \angle(EF, EP) + \angle(FP, FE) = \angle(FP, EP) = \angle(DF, DE) = \angle(BI, IC),\end{aligned}$$

so Q lies on Γ. Let QP meet Γ again at T. It will now suffice to prove that S, P, and T are collinear. Notice that $\angle(BI, IT) = \angle(BQ, QT) = \angle(BF, FP) = \angle(FK, KP)$. Note $FD \perp FK$ and $FD \perp BI$ so $FK \parallel BI$ and hence IT is parallel to the line KNP. Since $DI = IK$, the line IT crosses DN at its midpoint M.

Step 3. Let F' and E' be the midpoints of DE and DF, respectively. Since $DE' \cdot E'F = DE'^2 = BE' \cdot E'I$, the point E' lies on the radical axis of ω and Γ; the same holds for F'. Therefore, this radical axis is $E'F'$, and it passes through M. Thus $IM \cdot MT = DM \cdot MS$, so S, I, D, and T are concyclic. This shows $\angle(DS, ST) = \angle(DI, IT) = \angle(DK, KP) = \angle(DS, SP)$, whence the points S, P, and T are collinear, as desired.



Comment. Here is a longer alternative proof in step 1 that P, S, and L are collinear, using a circular inversion instead of the fact that opposite sides of a quadrilateral inscribed in a circle ω meet on the polar line with respect to ω of the intersection of the diagonals. Let G be the foot of the altitude from N to the line $DIKL$. Observe that N, G, K, S are concyclic (opposite right angles) so

$$\angle DIP = 2\angle DKP = \angle GKN + \angle DSP = \angle GSN + \angle NSP = \angle GSP\,,$$

hence I, G, S, P are concyclic. We have $IG \cdot IL = IN \cdot IA = r^2$ since $\triangle IGN \sim \triangle IAL$. Inverting the circle $IGSP$ in circle ω, points P and S are fixed and G is taken to L so we find that P, S, and L are collinear.

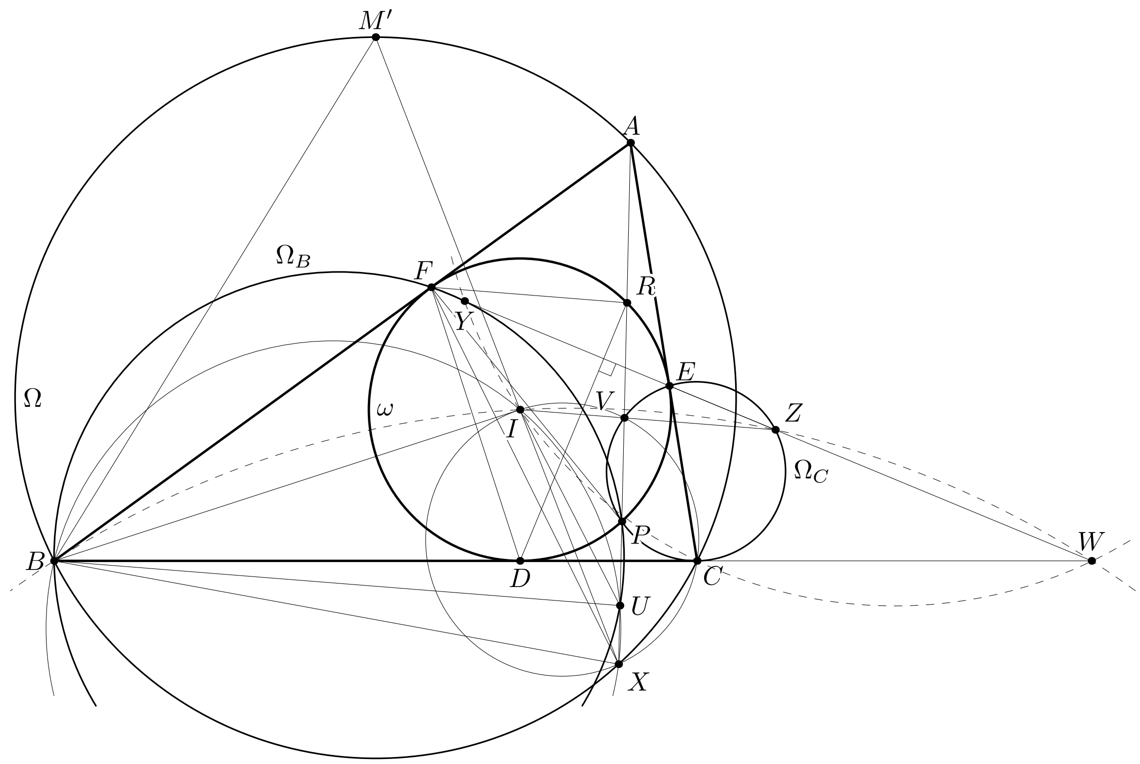
Solution 2. We start as in Solution 1. Namely, we introduce the same points K, L, N, and S, and show that the triples (P, N, K) and (P, S, L) are collinear. We conclude that K and R are symmetric in AI, and reduce the problem statement to showing that P, Q, and S are collinear.

Step 1. Let AR meet the circumcircle Ω of ABC again at X. The lines AR and AK are isogonal in the angle BAC; it is well known that in this case X is the tangency point of Ω with the A-mixtilinear circle. It is also well known that for this point X, the line XI crosses Ω again at the midpoint M' of arc BAC.

Step 2. Denote the circles BFP and CEP by Ω_B and Ω_C, respectively. Let Ω_B cross AR and EF again at U and Y, respectively. We have

$$\angle(UB, BF) = \angle(UP, PF) = \angle(RP, PF) = \angle(RF, FA),$$

so $UB \parallel RF$.



Next, we show that the points B, I, U, and X are concyclic. Since

$$\angle(UB, UX) = \angle(RF, RX) = \angle(AF, AR) + \angle(FR, FA) = \angle(M'B, M'X) + \angle(DR, DF),$$

it suffices to prove $\angle(IB, IX) = \angle(M'B, M'X) + \angle(DR, DF)$, or $\angle(IB, M'B) = \angle(DR, DF)$. But both angles equal $\angle(CI, CB)$, as desired. (This is where we used the fact that M' is the midpoint of arc BAC of Ω.)

It follows now from circles $BUIX$ and $BPUFY$ that

$$\angle(IU, UB) = \angle(IX, BX) = \angle(M'X, BX) = \frac{\pi - \angle A}{2}$$
$$= \angle(EF, AF) = \angle(YF, BF) = \angle(YU, BU),$$

so the points Y, U, and I are collinear.

Let EF meet BC at W. We have

$$\angle(IY, YW) = \angle(UY, FY) = \angle(UB, FB) = \angle(RF, AF) = \angle(CI, CW),$$

so the points W, Y, I, and C are concyclic.

Similarly, if V and Z are the second meeting points of Ω_C with AR and EF, we get that the 4-tuples (C, V, I, X) and (B, I, Z, W) are both concyclic.

Step 3. Let $Q' = CY \cap BZ$. We will show that $Q' = Q$.

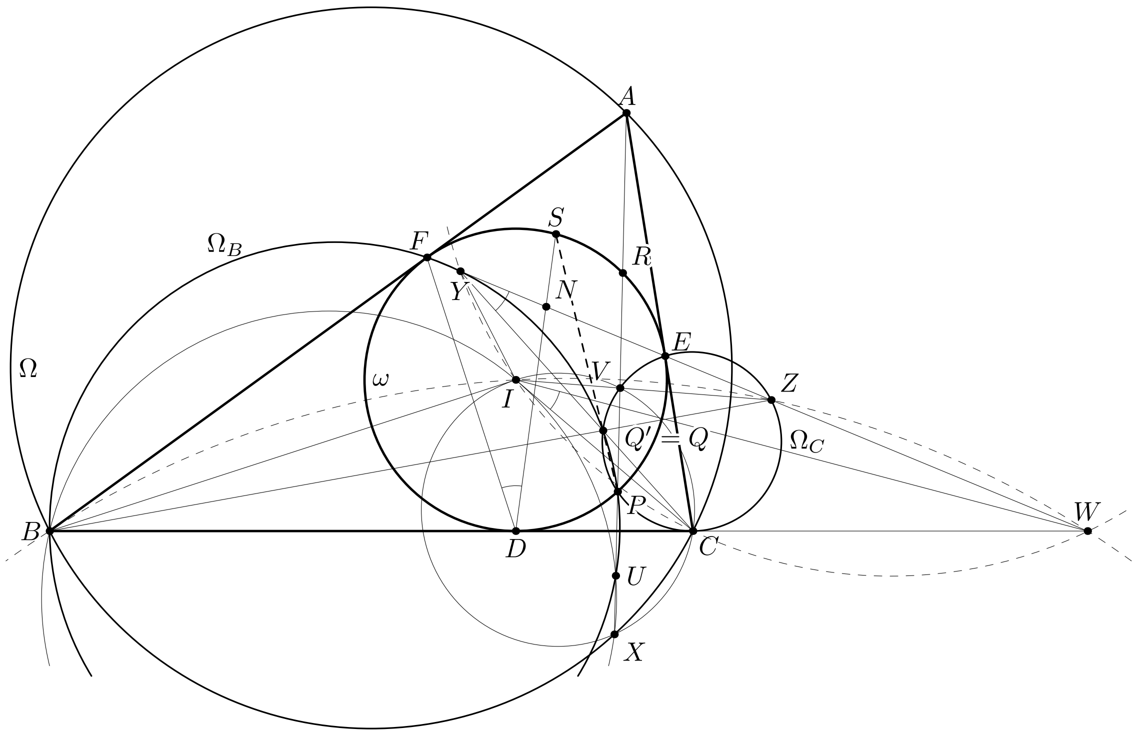
First of all, we have

$$\begin{aligned}\angle(Q'Y, Q'B) &= \angle(CY, ZB) = \angle(CY, ZY) + \angle(ZY, BZ)\\ &= \angle(CI, IW) + \angle(IW, IB) = \angle(CI, IB) = \frac{\pi - \angle A}{2} = \angle(FY, FB),\end{aligned}$$

so $Q' \in \Omega_B$. Similarly, $Q' \in \Omega_C$. Thus $Q' \in \Omega_B \cap \Omega_C = \{P, Q\}$ and it remains to prove that $Q' \neq P$. If we had $Q' = P$, we would have $\angle(PY, PZ) = \angle(Q'Y, Q'Z) = \angle(IC, IB)$. This would imply

$$\angle(PY, YF) + \angle(EZ, ZP) = \angle(PY, PZ) = \angle(IC, IB) = \angle(PE, PF),$$

so circles Ω_B and Ω_C would be tangent at P. That is excluded in the problem conditions, so $Q' = Q$.



Step 4. Now we are ready to show that P, Q, and S are collinear.

Notice that A and D are the poles of EW and DW with respect to ω, so W is the pole of AD. Hence, $WI \perp AD$. Since $CI \perp DE$, this yields $\angle(IC, WI) = \angle(DE, DA)$. On the other hand, DA is a symmedian in $\triangle DEF$, so $\angle(DE, DA) = \angle(DN, DF) = \angle(DS, DF)$. Therefore,

$$\begin{aligned}\angle(PS, PF) &= \angle(DS, DF) = \angle(DE, DA) = \angle(IC, IW)\\ &= \angle(YC, YW) = \angle(YQ, YF) = \angle(PQ, PF),\end{aligned}$$

which yields the desired collinearity.

IMO 2019 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2019 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Solve over $\mathbb{Z}$ the functional equation $f(2a) + 2f(b) = f(f(a+b))$.

2. In triangle ABC point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that $\overline{PQ} \parallel \overline{AB}$. Point P_1 is chosen on ray PB_1 beyond B_1 such that $\angle PP_1C = \angle BAC$. Point Q_1 is chosen on ray QA_1 beyond A_1 such that $\angle CQ_1Q = \angle CBA$. Prove that points P_1, Q_1, P, Q are cyclic.

3. A social network has 2019 users, some pairs of which are friends (friendship is symmetric). If A, B, C are three users such that AB are friends and AC are friends but BC is not, then the administrator may perform the following operation: change the friendships such that BC are friends, but AB and AC are no longer friends.

   Initially, 1009 users have 1010 friends and 1010 users have 1009 friends. Prove that the administrator can make a sequence of operations such that all users have at most 1 friend.

4. Solve over positive integers the equation
$$k! = \prod_{i=0}^{n-1}(2^n - 2^i) = (2^n-1)(2^n-2)(2^n-4)\dots(2^n-2^{n-1}).$$

5. Let n be a positive integer. Harry has n coins lined up on his desk, which can show either heads or tails. He does the following operation: if there are k coins which show heads and $k > 0$, then he flips the kth coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \to HHT \to HTT \to TTT$, which takes three steps.)

   Prove the process will always terminate, and determine the average number of steps this takes over all 2^n configurations.

6. Let ABC be a triangle with incenter I and incircle ω. Let D, E, F denote the tangency points of ω with $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. The line through D perpendicular to $\overline{EF}$ meets ω again at R (other than D), and line AR meets ω again at P (other than R). Suppose the circumcircles of $\triangle PCE$ and $\triangle PBF$ meet again at Q (other than P). Prove that lines DI and PQ meet on the external $\angle A$-bisector.

§1 IMO 2019/1, proposed by Liam Baker (SAF)

Solve over $\mathbb{Z}$ the functional equation $f(2a) + 2f(b) = f(f(a+b))$.

Notice that $f(x) \equiv 0$ or $f(x) \equiv 2x + k$ work and are clearly the only linear solutions. We now prove all solutions are linear.

Let $P(a, b)$ be the assertion.

Claim — For each $x \in \mathbb{Z}$ we have $f(2x) = 2f(x) - f(0)$.

Proof. Compare $P(0, x)$ and $P(x, 0)$. □

Now, $P(a, b)$ and $P(0, a+b)$ give

$$\begin{aligned} f(f(a+b)) &= f(2a) + 2f(b) = f(0) + 2f(a+b) \\ \implies [2f(a) - f(0)] + 2f(b) &= f(0) + 2f(a+b) \\ \implies (f(a) - f(0)) + (f(b) - f(0)) &= (f(a+b) - f(0)). \end{aligned}$$

Thus the map $x \mapsto f(x) - f(0)$ is additive, therefore linear.

Remark. The same proof works on the functional equation

$$f(2a) + 2f(b) = g(a+b)$$

where g is an arbitrary function (it implies that f is linear).

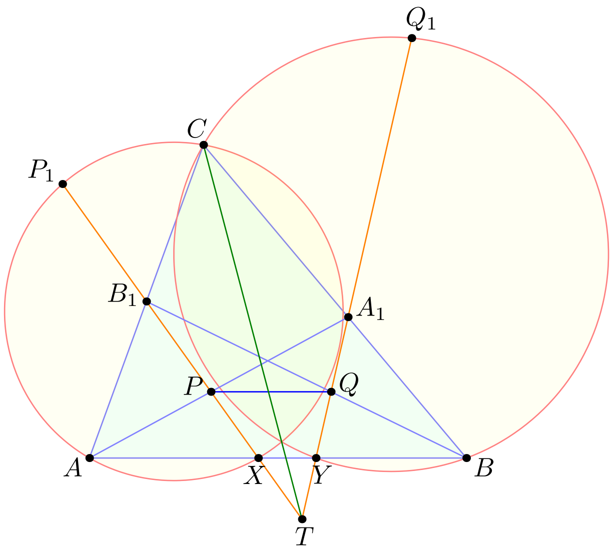
§2 IMO 2019/2, proposed by Anton Trygub (UKR)

In triangle ABC point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that $\overline{PQ} \parallel \overline{AB}$. Point P_1 is chosen on ray PB_1 beyond B_1 such that $\angle PP_1C = \angle BAC$. Point Q_1 is chosen on ray QA_1 beyond A_1 such that $\angle CQ_1Q = \angle CBA$. Prove that points P_1, Q_1, P, Q are cyclic.

We present two solutions.

First solution by bary (Evan Chen) Let PB_1 and QA_1 meet line AB at X and Y. Since $\overline{XY} \parallel \overline{PQ}$ it is equivalent to show P_1XYQ_1 is cyclic (Reim's theorem) Note that P_1CXA and Q_1CYB are cyclic.

Letting $T = \overline{PX} \cap \overline{QY}$ (possibly at infinity), it suffices to show that the radical axis of $\triangle CXA$ and $\triangle CYB$ passes through T, because that would imply P_1XYQ_1 is cyclic (by power of a point when T is Euclidean, and because it is an isosceles trapezoid if T is at infinity).



To this end we use barycentric coordinates on $\triangle ABC$. We begin by writing

$$P = (u+t : s : r), \quad Q = (t : u+s : r)$$

from which it follows that $A_1 = (0 : s : r)$ and $B_1 = (t : 0 : r)$.

Next, compute $X = \left(\det\left[\begin{smallmatrix} u+t & r \\ t & r \end{smallmatrix}\right] : \det\left[\begin{smallmatrix} s & r \\ 0 & r \end{smallmatrix}\right] : 0\right) = (u : s : 0)$. Similarly, $Y = (t : u : 0)$. So we have computed all points.

Claim — Line B_1X has equation $-rs \cdot x + ru \cdot y + st \cdot z = 0$, while line C_1Y has equation $ru \cdot x - rt \cdot y + st \cdot z = 0$.

Proof. Line B_1X is $0 = \det(B_1, X, -) = \det \left[\begin{smallmatrix} t & 0 & r \\ u & s & 0 \\ x & y & z \end{smallmatrix}\right]$. Line C_1Y is analogous. □

Claim — The radical axis $(u+t)y - (u+s)x = 0$.

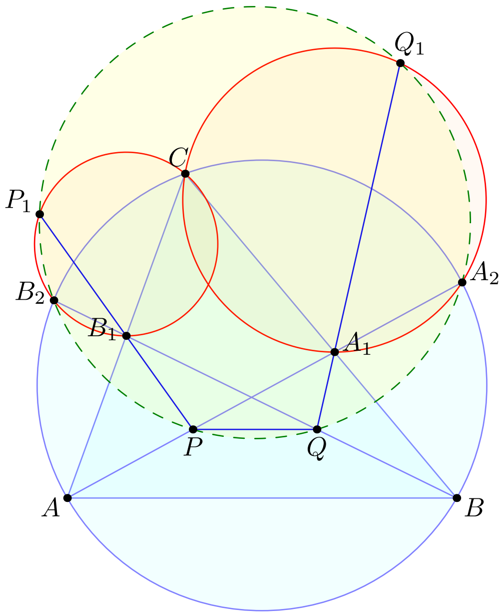
Proof. Circle (AXC) is given by $-a^2yz - b^2zx - c^2xy + (x+y+z) \cdot \frac{c^2 \cdot u}{u+s} y = 0$. Similarly, circle (BYC) has equation $-a^2yz - b^2zx - c^2xy + (x+y+z) \cdot \frac{c^2 \cdot u}{u+t} x = 0$. Subtracting gives the radical axis. $\square$

Finally, to see these three lines are concurrent, we now compute

$$\begin{aligned} \det \begin{bmatrix} -rs & ru & st \\ ru & -rt & st \\ -(u+s) & u+t & 0 \end{bmatrix} &= rst\left[[u(u+t) - t(u+s)] + [s(u+t) - u(u+s)]\right] \\ &= rst\left[(u^2 - st) + (st - u^2)\right] = 0. \end{aligned}$$

This completes the proof.

Second official solution by tricky angle chasing Let lines AA_1 and BB_1 meet at the circumcircle of $\triangle ABC$ again at points A_2 and B_2. By Reim's theorem, PQA_2B_2 are cyclic.



Claim — The points P, Q, A_2, Q_1 are cyclic. Similarly the points P, Q, B_2, P_1 are cyclic.

Proof. Note that $CA_1A_2Q_1$ is cyclic since $\measuredangle CQ_1A_1 = \measuredangle CQ_1Q = \measuredangle CBA = \measuredangle CA_2A = \measuredangle CA_2A_1$. Then $\measuredangle QQ_1A_2 = \measuredangle A_1Q_1A_2 = \measuredangle A_1CA_2 = \measuredangle BCA_2 = \measuredangle BAA_2 = \measuredangle QPA_2$. $\square$

This claim obviously solves the problem.

§3 IMO 2019/3, proposed by Adrian Beker (HRV)

A social network has 2019 users, some pairs of which are friends (friendship is symmetric). If A, B, C are three users such that AB are friends and AC are friends but BC is not, then the administrator may perform the following operation: change the friendships such that BC are friends, but AB and AC are no longer friends.

Initially, 1009 users have 1010 friends and 1010 users have 1009 friends. Prove that the administrator can make a sequence of operations such that all users have at most 1 friend.

We take the obvious graph formulation and call the move a *toggle*.

Claim — Let G be a connected graph. Then one can toggle G without disconnecting the graph, unless G is a clique, a cycle, or a tree.

Proof. Assume G is connected and not a tree, so it has a cycle. Take the smallest cycle C; by hypothesis $C \neq G$.

If C is not a triangle (equivalently, G is triangle-free), then let $b \notin C$ be a vertex adjacent to C, say at a. Take a vertex c of the cycle adjacent to a (hence not to b). Then we can toggle abc.

Now assume there exists a triangle; let K be the maximal clique. By hypothesis, $K \neq G$. We take an edge $e = ab$ dangling off the clique, with $a \in K$ and $b \notin K$. Note some vertex c of K is not adjacent to b; now toggle abc. □

Back to the original problem; let G_{imo} be the given graph. The point is that we can apply toggles (by the claim) repeatedly, without disconnecting the graph, until we get a tree. This is because

- G_{imo} is connected, since any two vertices which are not adjacent have a common neighbor by pigeonhole ($1009 + 1009 + 2 > 2019$).
- G_{imo} cannot become a cycle, because it initially has an odd-degree vertex, and toggles preserve parity of degree!
- G_{imo} is obviously not a clique initially (and hence not afterwards).

So, we can eventually get G_{imo} to be a tree.

Once G_{imo} is a tree the problem follows by repeatedly applying toggles arbitrarily until no more are possible; the graph (although now disconnected) remains acyclic (in particular having no triangles) and therefore can only terminate in the desired situation.

Remark. Assume G_{imo} is connected. Then we have shown more strongly that if G_{imo} is not a clique, and has any vertex of odd degree, Noting that toggles preserve parity of degree, these conditions are actually necessary too. So this characterizes all connected graphs (and thus all graphs) for which the goal is possible.

§4 IMO 2019/4, proposed by Gabriel Chicas Reyes (SLV)

Solve over positive integers the equation

$$k! = \prod_{i=0}^{n-1}(2^n - 2^i) = (2^n-1)(2^n-2)(2^n-4)\dots(2^n-2^{n-1}).$$

The answer is $(n,k)=(1,1)$ and $(n,k)=(2,3)$ which work.

Let $A=\prod_i(2^n-2^k)$, and assume $A=k!$ for some $k\ge 3$. Recall by exponent lifting that

$$\nu_3(2^t-1) = \begin{cases} 0 & t \text{ odd} \\ 1+\nu_3(t) & t \text{ even.} \end{cases}$$

Consequently, we can compute

$$\begin{aligned} k > \nu_2(k!) = \nu_2(A) &= 1+2+\dots+(n-1) = \frac{n(n-1)}{2} \\ \frac{k}{3} \le \nu_3(k!) = \nu_3(A) &= \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor + \dots < \frac{3}{4}n. \end{aligned}$$

where the very first inequality can be justified say by Legendre's formula $\nu_2(k!) = k - s_2(k)$.

In this way, we get

$$\frac{9}{4}n > k > \frac{n(n-1)}{2}$$

which means $n \le \frac{11}{2}$; a manual check then shows the solutions we claimed earlier are the only ones.

> **Remark.** An amusing corollary of the problem pointed out in the Shortlist is that the symmetric group S_k cannot be isomorphic to the group $\mathrm{GL}_n(\mathbb{F}_2)$ unless $(n,k)=(1,1)$ or $(n,k)=(2,3)$, which indeed produce isomorphisms.

§5 IMO 2019/5, proposed by David Altizio (USA)

Let n be a positive integer. Harry has n coins lined up on his desk, which can show either heads or tails. He does the following operation: if there are k coins which show heads and $k > 0$, then he flips the kth coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \to HHT \to HTT \to TTT$, which takes three steps.)

Prove the process will always terminate, and determine the average number of steps this takes over all 2^n configurations.

The answer is

$$E_n = \frac{1}{2}(1 + \cdots + n) = \frac{1}{4}n(n+1)$$

which is finite.

We'll represent the operation by a directed graph G_n on vertices $\{0,1\}^n$ (each string points to its successor) with 1 corresponding to heads and 0 corresponding to tails. For $b \in \{0,1\}$ we let $\overline{b} = 1 - b$, and denote binary strings as a sequence of n symbols.

The main claim is that G_n can be described explicitly in terms of G_{n-1}:

- We take two copies X and Y of G_{n-1}.
- In X, we take each string of length $n-1$ and just append a 0 to it. In symbols, we replace $s_1 \dots s_{n-1} \mapsto s_1 \dots s_{n-1}0$.
- In Y, we toggle every bit, then reverse the order, and then append a 1 to it. In symbols, we replace $s_1 \dots s_{n-1} \mapsto \overline{s}_{n-1}\overline{s}_{n-2} \dots \overline{s}_1 1$.
- Finally, we add one new edge from Y to X by $11\dots1 \to 11\dots110$.

An illustration of G_4 is given below.

$$\begin{tikzcd}
0001 \arrow[d] & 0101 \arrow[l] & 0111 \arrow[l] & 0011 \arrow[l] \\
1001 \arrow[d] & 1011 \arrow[l] & & \\
1101 \arrow[d] & & & \\
1111 \arrow[d, Rightarrow] & & & \\
1110 \arrow[d] & 1010 \arrow[l] & 0010 \arrow[l] & 0110 \arrow[l] \\
1100 \arrow[d] & 0100 \arrow[l] & & \\
1000 \arrow[d] & & & \\
0000 & & &
\end{tikzcd}$$

To prove this claim, we need only show the arrows of this directed graph remain valid. The graph X is correct as a subgraph of G_n, since the extra 0 makes no difference. As for Y, note that if $s = s_1 \dots s_{n-1}$ had k ones, then the modified string has $(n-1-k)+1 = n-k$ ones, ergo $\overline{s}_{n-1} \dots \overline{s}_1 1 \mapsto \overline{s}_{n-1} \dots \overline{s}_{k+1} s_k \overline{s}_{k-1} \dots \overline{s}_1 1$ which is what we wanted. Finally, the one edge from Y to X is obviously correct.

To finish, let E_n denote the desired expected value. Since $1 \dots 1$ takes n steps to finish we have

$$E_n = \frac{1}{2}\left[E_{n-1} + (E_{n-1} + n)\right]$$

based on cases on whether the chosen string is in X or Y or not. By induction, we have $E_n = \frac{1}{2}(1 + \cdots + n) = \frac{1}{4}n(n+1)$, as desired.

Remark. Actually, the following is true: if the indices of the 1's are $1 \le i_1 < \dots < i_\ell \le 1$, then the number of operations required is

$$2(i_1 + \dots + i_\ell) - \ell^2.$$

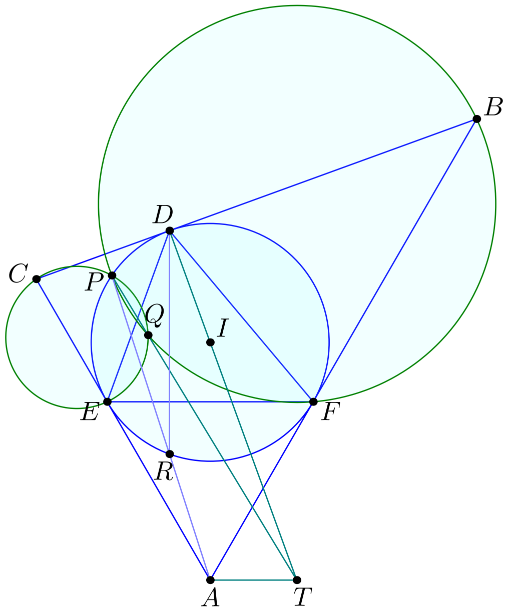
This problem also has an interpretation as a Turing machine: the head starts at a position on the tape (the binary string). If it sees a 1, it changes the cell to a 0 and moves left; if it sees a 0, it changes the cell to a 1 and moves right.

§6 IMO 2019/6, proposed by Anant Mudgal (IND)

Let ABC be a triangle with incenter I and incircle ω. Let D, E, F denote the tangency points of ω with $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. The line through D perpendicular to $\overline{EF}$ meets ω again at R (other than D), and line AR meets ω again at P (other than R). Suppose the circumcircles of $\triangle PCE$ and $\triangle PBF$ meet again at Q (other than P). Prove that lines DI and PQ meet on the external $\angle A$-bisector.

We present three solutions.

First solution by complex numbers (Evan Chen, with Yang Liu) We use complex numbers with $D = x$, $E = y$, $F = z$.



Then $A = \frac{2yz}{y+z}$, $R = \frac{-yz}{x}$ and so

$$P = \frac{A - R}{1 - R\overline{A}} = \frac{\frac{2yz}{y+z} + \frac{yz}{x}}{1 + \frac{yz}{x} \cdot \frac{2}{y+z}} = \frac{yz(2x + y + z)}{2yz + x(y+z)}.$$

We now compute

$$\begin{aligned} O_B &= \det \begin{bmatrix} P & P\overline{P} & 1 \\ F & F\overline{F} & 1 \\ B & B\overline{B} & 1 \end{bmatrix} \div \det \begin{bmatrix} P & \overline{P} & 1 \\ F & \overline{F} & 1 \\ B & \overline{B} & 1 \end{bmatrix} = \det \begin{bmatrix} P & 1 & 1 \\ z & 1 & 1 \\ \frac{2xz}{x+z} & \frac{4xz}{(x+z)^2} & 1 \end{bmatrix} \div \det \begin{bmatrix} P & 1/P & 1 \\ z & 1/z & 1 \\ \frac{2xz}{x+z} & \frac{2}{x+z} & 1 \end{bmatrix} \\
&= \frac{1}{x+z} \det \begin{bmatrix} P & 0 & 1 \\ z & 0 & 1 \\ 2xz(x+z) & -(x-z)^2 & (x+z)^2 \end{bmatrix} \div \det \begin{bmatrix} P & 1/P & 1 \\ z & 1/z & 1 \\ 2xz & 2 & x+z \end{bmatrix} \\
&= \frac{(x-z)^2}{x+z} \cdot \frac{P-z}{(x+z)(P/z - z/P) + 2z - 2x + \frac{2xz}{P} - 2P} \\
&= \frac{(x-z)^2}{x+z} \cdot \frac{P-z}{(\frac{x}{z}-1)P - 2(x-z) + (xz-z^2)\frac{1}{P}} \\
&= \frac{x-z}{x+z} \cdot \frac{P-z}{P/z + z/P - 2} = \frac{x-z}{x+z} \cdot \frac{P-z}{\frac{(P-z)^2}{Pz}} = \frac{x-z}{x+z} \cdot \frac{1}{\frac{1}{z} - \frac{1}{P}} \\
&= \frac{x-z}{x+z} \cdot \frac{y(2x+y+z)}{y(2x+y+z) - (2yz+xy+xz)} = \frac{x-z}{x+z} \cdot \frac{yz(2x+y+z)}{xy+y^2-yz-xz} \\
&= \frac{x-z}{x+z} \cdot \frac{yz(2x+y+z)}{(y-z)(x+y)}. \end{aligned}$$

Similarly

$$O_C = \frac{x-y}{x+y} \cdot \frac{yz(2x+y+z)}{(z-y)(x+z)}.$$

Therefore, subtraction gives

$$O_B - O_C = \frac{yz(2x+y+z)}{(x+y)(x+z)(y-z)} \left[(x-z) + (x-y) \right] = \frac{yz(2x+y+z)(2x-y-z)}{(x+y)(x+z)(z-y)}.$$

It remains to compute T. Since $T \in \overline{ID}$ we have $t/x \in \mathbb{R}$ so $\overline{t} = t/x^2$. Also,

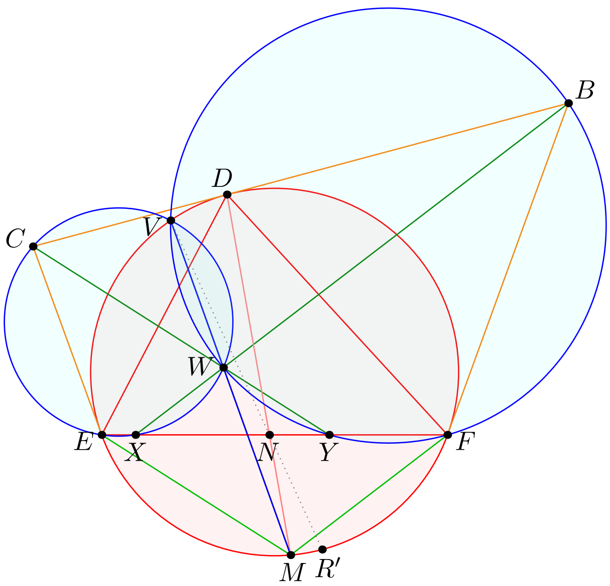
$$\begin{aligned} \frac{t - \frac{2yz}{y+z}}{y+z} \in i\mathbb{R} \implies 0 &= \frac{t - \frac{2yz}{y+z}}{y+z} + \frac{\frac{t}{x^2} - \frac{2}{y+z}}{\frac{1}{y} + \frac{1}{z}} \\
&= \frac{1 + \frac{yz}{x^2}}{y+z} t - \frac{2yz}{(y+z)^2} - \frac{2yz}{(y+z)^2} \\
\implies t &= \frac{x^2}{x^2+yz} \cdot \frac{4yz}{y+z} \end{aligned}$$

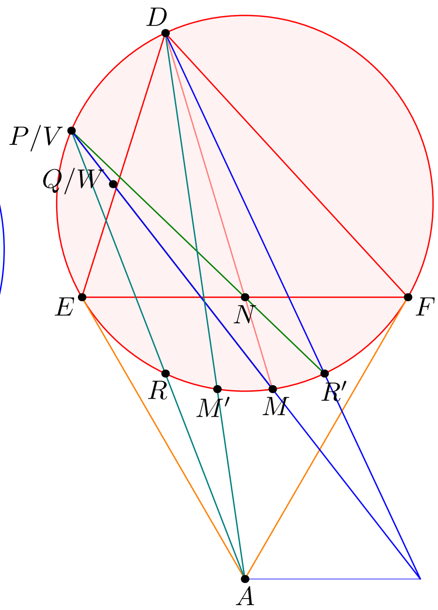
Thus

$$\begin{aligned} P - T &= \frac{yz(2x+y+z)}{2yz + x(y+z)} - \frac{4x^2yz}{(x^2+yz)(y+z)} \\
&= yz \cdot \frac{(2x+y+z)(x^2+yz)(y+z) - 4x^2(2yz+xy+xz)}{(y+z)(x^2+yz)(2yz+xy+xz)} \\
&= -yz \cdot \frac{(2x-y-z)(x^2y + x^2z + 4xyz + y^2z + yz^2)}{(y+z)(x^2+yz)(2yz+xy+xz)}. \end{aligned}$$

This gives $\overline{PT} \perp \overline{O_BO_C}$ as needed.

Second solution by tethered moving points, with optimization (Evan Chen) Fix $\triangle DEF$ and ω, with $B = \overline{DD} \cap \overline{FF}$ and $C = \overline{DD} \cap \overline{EE}$. We consider a variable point M on ω and let X, Y be on $\overline{EF}$ with $\overline{CY} \cap \parallel \overline{ME}$, $\overline{BX} \cap \parallel \overline{MF}$. We define $W = \overline{CY} \cap \overline{BX}$. Also, let line MW meet ω again at V.





Claim (Angle chasing) — Pentagons $CVWXE$ and $BVWYF$ are cyclic.

Proof. By $\measuredangle EVW = \measuredangle EVM = \measuredangle EFM = \measuredangle CEM = \measuredangle ECW$ and $\measuredangle EXW = \measuredangle EFM = \measuredangle CEM = \measuredangle ECW$. □

Let $N = \overline{DM} \cap \overline{EF}$ and R' be the D-antipode on ω.

Claim (Black magic) — The points V, N, R' are collinear.

Proof. We use tethered moving points with $\triangle DEF$ fixed.

Obviously the map $\omega \mapsto \overline{EF} \mapsto \omega$ by $M \mapsto N \mapsto \overline{R'N} \cap \omega$ is projective. Also, the map $\omega \mapsto \overline{EF} \mapsto \omega$ by $M \mapsto X \mapsto V$ is also projective (the first by projection to the line at infinity at back; the second say by inversion at E).

So it suffices to check for three points. When $M = E$ we get $N = E$ so $\overline{R'N} \cap \omega = E$, while $W = E$ and thus $V = E$. The case $M = F$ is similar. Finally, if $M = R'$, then W is the center of ω and so $V = \overline{R'N} \cap \overline{EF} = D$. □

We now address the original problem by specializing M: choose it so that N is the midpoint of $\overline{EF}$. Let $M' = \overline{DA} \cap (DEF)$.

Claim — After this specialization, $V = P$ and $W = Q$.

Proof. Thus $\overline{RR'}$ and $\overline{MM'}$ are parallel to $\overline{EF}$. From $(EF;PR) = -1 = (EF;N\infty) \stackrel{R'}{=} (EF;NV)$, we derive that $P = V$ and $Q = R$, proving (i). □

Finally, the concurrence requested follows by Pascal theorem on $M'MDR'PR$.

Third solution by power of a point linearity (Luke Robitaille) Let us define

$$f(\bullet) = \operatorname{Pow}(\bullet, (CPE)) - \operatorname{Pow}(\bullet, (BPF))$$

which is a linear function from the plane to $\mathbb{R}$.

Define $W = \overline{BA} \cap \overline{PE}$, $V = \overline{AC} \cap \overline{PF}$. Also, let $W_1 = \overline{ER} \cap \overline{AB}$, $V_1 = \overline{FR} \cap \overline{AB}$. Note that

$$-1 = (PR; EF) \overset{E}{=} (WA; W_1F)$$

and similarly $(VA; V_1E) = -1$.

Claim — We have

$$\begin{aligned} f(F) &= \frac{|EF| \cdot (s-c) \sin C/2}{\sin B/2} \\ f(E) &= -\frac{|EF| \cdot (s-b) \sin B/2}{\sin C/2}. \end{aligned}$$

Proof. We have

$$f(W) = WF^2 - WB \cdot WF = WF \cdot BF$$

where lengths are directed. Next,

$$\begin{aligned} f(F) &= \frac{AF \cdot f(W) + FW \cdot f(A)}{AW} \\ &= \frac{AF \cdot WF \cdot BF + FW \cdot (AE \cdot AC - AF \cdot AB)}{AW} \\ &= \frac{WF(AF \cdot BF + AF \cdot AB) + FW \cdot AE \cdot AC}{AW} \\ &= \frac{WF \cdot AF^2 - WF \cdot AE \cdot AC}{AW} = \frac{WF}{AW} \cdot (AE^2 - AE \cdot AC) \\ &= \frac{WF}{AW} \cdot AE \cdot CE = -\frac{W_1F}{AW_1} \cdot AE \cdot CE. \end{aligned}$$

Since $\triangle DEF$ is acute, the point R lies inside $\triangle AEF$. Thus W_1 lies inside segment $\overline{AF}$ and the ratio $\frac{W_1F}{AW_1}$ is positive. We now determine its value: by the ratio lemma

$$\begin{aligned} \frac{|W_1F|}{|AW_1|} &= \frac{|EF| \sin \angle W_1EF}{|AE| \sin \angle AEW_1} \\ &= \frac{|EF| \sin \angle REF}{|AE| \sin \angle AER} \\ &= \frac{|EF| \sin \angle RDF}{|AE| \sin \angle EDR} \\ &= \frac{|EF| \sin C/2}{|AE| \sin B/2}. \end{aligned}$$

Also, we have $AE \cdot CE < 0$ since E lies inside $\overline{AC}$. Hence

$$f(F) = -\frac{|EF| \sin C/2}{|AE| \sin B/2} . \cdot AE \cdot CE = |EF| \cdot \frac{|CE| \sin B/2}{\sin C/2} = |EF| \cdot \frac{(s-c) \sin B/2}{\sin C/2}.$$

The calculation for $f(E)$ is similar, (noting the sign flips since f is anti-symmetric in terms of B and C). $\square$

Let $Z \in \overline{DI}$ with $\angle ZAI = 90^\circ$ be the point requested in the problem now. Our goal is to show $f(Z) = 0$. We assume WLOG that $AB < AC$, so $\frac{ZA}{EF} > 0$. Then

$$\begin{aligned}
|ZA| &= |AI| \cdot \tan \angle AIZ \\
&= |AI| \cdot \tan \angle(\overline{AI}, \overline{DI}) \\
&= \frac{s-a}{\cos A/2} \cdot \tan(\overline{BC}, \overline{EF}) \\
&= \frac{s-a}{\cos A/2} \tan(B/2 - C/2).
\end{aligned}$$

To this end we compute

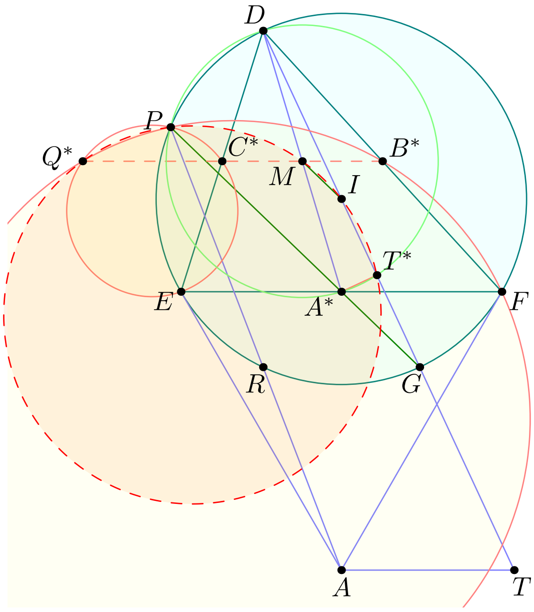
$$\begin{aligned}
f(Z) &= f(A) + [f(Z) - f(A)] = f(A) + \frac{ZA}{EF}\left[f(E) - f(F)\right] \\
&= f(A) - \frac{ZA}{EF}\left[\frac{|EF| \cdot (s-b)\sin B/2}{\sin C/2} + \frac{|EF| \cdot (s-c)\sin C/2}{\sin B/2}\right] \\
&= f(A) - |ZA|\left[\frac{(s-b)\sin B/2}{\sin C/2} + \frac{(s-c)\sin C/2}{\sin B/2}\right] \\
&= [b(s-a) - c(s-a)] - |ZA|\left[\frac{(s-b)\sin B/2}{\sin C/2} + \frac{(s-c)\sin C/2}{\sin B/2}\right] \\
&= (b-c)(s-a) - \frac{s-a}{\cos A/2}\tan(B/2 - C/2)\left[\frac{(s-b)\sin B/2}{\sin C/2} + \frac{(s-c)\sin C/2}{\sin B/2}\right].
\end{aligned}$$

Dividing out,

$$\begin{aligned}
\frac{f(Z)}{s-a} &= (b-c) - \frac{1}{\cos A/2}\tan(B/2 - C/2)\left[\frac{r\cos B/2}{\sin C/2} + \frac{r\cos C/2}{\sin B/2}\right] \\
&= (b-c) - \frac{r\tan(B/2 - C/2)}{\cos A/2} \cdot \frac{\cos B/2 \sin B/2 + \cos C/2 \sin C/2}{\sin C/2 \sin B/2} \\
&= (b-c) - \frac{r\tan(B/2 - C/2)}{\cos A/2} \cdot \frac{\sin B + \sin C}{2\sin C/2 \sin B/2} \\
&= (b-c) - \frac{r\tan(B/2 - C/2)}{\cos A/2} \cdot \frac{\sin(B/2 + C/2)\cos(B/2 - C/2)}{\sin C/2 \sin B/2} \\
&= (b-c) - r\frac{\sin(B/2 - C/2)}{\sin B/2 \sin C/2} \\
&= (b-c) - r(\cot C/2 - \cot B/2) = (b-c) - ((s-c) - (s-b)) = 0.
\end{aligned}$$

Fourth solution by incircle inversion (USA IMO live stream, led by Andrew Gu) Let T be the intersection of line DI and the external $\angle A$-bisector. Also, let G be the antipode of D on ω.

We perform inversion around ω, using $\bullet^*$ for the inverse. Then $\triangle A^*B^*C^*$ is the medial triangle of $\triangle DEF$, and T^* is the foot from A^* on to $\overline{DI}$. If we denote Q^* as the second intersection of (PC^*E) and (PB^*F), then the goal it show that Q^* lies on (PIT^*).



Claim — Points Q^*, B^*, C^* are collinear.

Proof. $\measuredangle PQ^*C^* = \measuredangle PEC^* = \measuredangle PED = \measuredangle PFD = \measuredangle PFB^* = \measuredangle PQ^*B^*$. □

Claim (cf Brazil 2011/5) — Points P, A^*, G are collinear.

Proof. Project harmonic quadrilateral $PERF$ through G, noting $\overline{GR} \parallel \overline{EF}$. □

Denote by M the center of parallelogram $DC^*A^*B^*$. Note that it is the center of the circle with diameter $\overline{DA^*}$, which passes through P and T^*. Also, $\overline{MI} \parallel \overline{PA^*G}$.

Claim — Points P, M, I, T^* are cyclic.

Proof. $\measuredangle IT^*P = \measuredangle DT^*P = \measuredangle DA^*P = \measuredangle MA^*P = \measuredangle A^*PM = \measuredangle PMI$. □

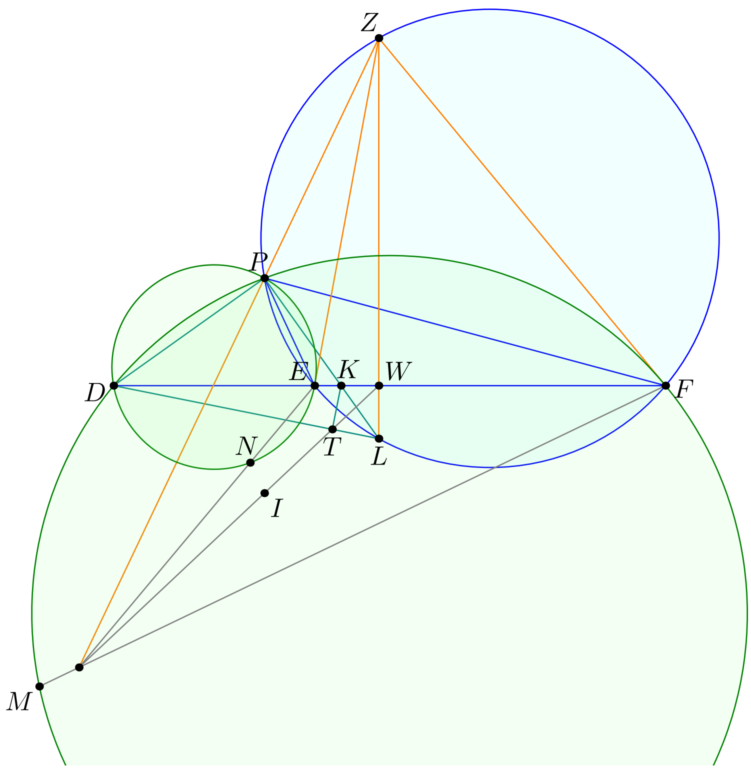
Claim — Points P, M, I, Q^* are cyclic.

Proof. $\measuredangle MQ^*P = \measuredangle C^*Q^*P = \measuredangle C^*EP = \measuredangle DEP = \measuredangle DGP = \measuredangle GPI = \measuredangle MIP$. □

Fifth solution by double inversion (Brandon Wang, Luke Robitaille, Michael Ren, Evan Chen) We outline one final approach. After inverting about ω as in the previous approach, we then apply another inversion around P. Dropping the apostrophes/stars/etc now one can check that the problem we arrive at becomes the following.

Proposition (Doubly inverted problem)
In $\triangle PEF$, the P-symmedian meets $\overline{EF}$ and (PEF) at K, L. Let $D \in \overline{EF}$ with $\angle DPK = 90^\circ$, and let T be the foot from K to $\overline{DL}$. Denote by I the reflection of P about $\overline{EF}$. Finally, let $PDNE$ and $PDMF$ be cyclic harmonic quadrilaterals. Then lines EN, MF, TI, are concurrent.

The proof proceeds in three steps. Suppose the line through L perpendicular to $\overline{EF}$ meets $\overline{EF}$ at W and (PEF) at Z.



1. Since $\measuredangle ZEP = \measuredangle WLP = \measuredangle WDP$, it follows $\overline{ZE}$ is tangent to $(PDNE)$. Similarly, $\overline{ZF}$ is tangent to $(PDMF)$.

2. $\triangle WTP$ is the orthic triangle of $\triangle DKL$, so $\overline{WD}$ bisects $\angle PWT$ and $\overline{WTI}$ collinear.

3. $-1 = E(PN; DZ) = F(PM; DZ) = W(PI; DZ)$, so $\overline{EN}$, $\overline{FM}$, $\overline{WI}$ meet on $\overline{PZ}$.

IMO 2019

60TH INTERNATIONAL MATHEMATICAL OLYMPIAD

July 11th - July 22nd,
Bath, United Kingdom

SHORTLISTED PROBLEMS

WITH SOLUTIONS

Shortlisted Problems

(with solutions)

60th International Mathematical Olympiad

Bath — UK, 11th–22nd July 2019

Note of Confidentiality

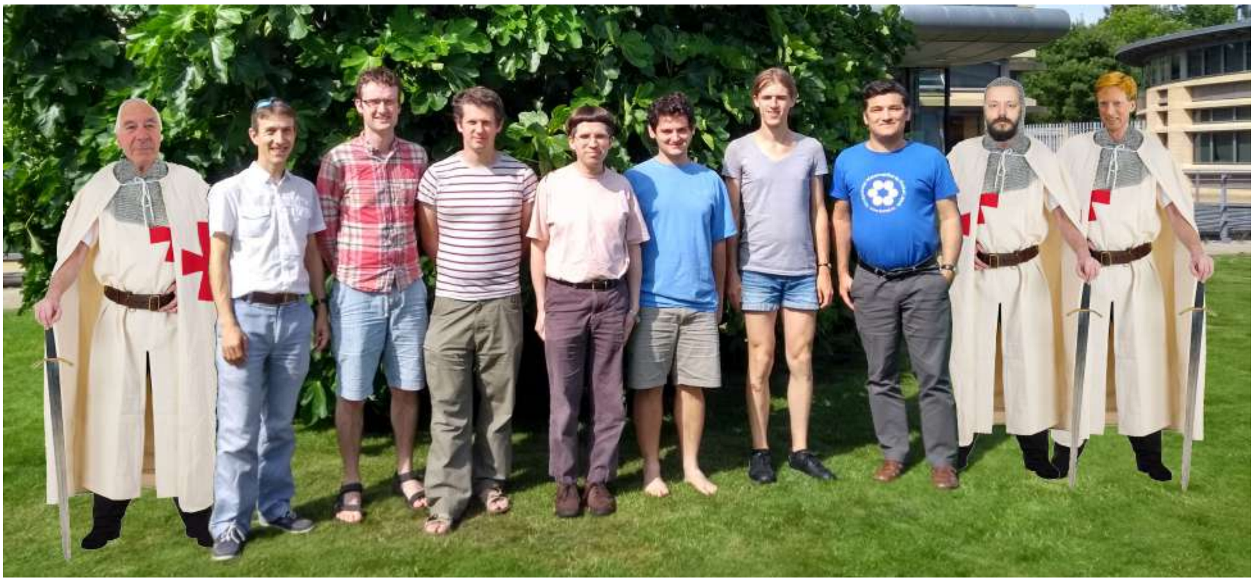
> **The Shortlist has to be kept strictly confidential**
> **until the conclusion of the following**
> **International Mathematical Olympiad.**
>
> **IMO General Regulations §6.6**

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2019 thank the following 58 countries for contributing 204 problem proposals:

Albania, Armenia, Australia, Austria, Belarus, Belgium, Brazil, Bulgaria, Canada, China, Croatia, Cuba, Cyprus, Czech Republic, Denmark, Ecuador, El Salvador, Estonia, Finland, France, Georgia, Germany, Greece, Hong Kong, Hungary, India, Indonesia, Iran, Ireland, Israel, Italy, Japan, Kazakhstan, Kosovo, Luxembourg, Mexico, Netherlands, New Zealand, Nicaragua, Nigeria, North Macedonia, Philippines, Poland, Russia, Serbia, Singapore, Slovakia, Slovenia, South Africa, South Korea, Sweden, Switzerland, Taiwan, Tanzania, Thailand, Ukraine, USA, Vietnam.

Problem Selection Committee



Tony Gardiner, Edward Crane, Alexander Betts, James Cranch, Joseph Myers (chair), James Aaronson, Andrew Carlotti, Géza Kós, Ilya I. Bogdanov, Jack Shotton

Problems

Algebra

A1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

(South Africa)

A2. Let u_1, u_2, $\ldots$, u_{2019} be real numbers satisfying

$$u_1 + u_2 + \cdots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \ldots, u_{2019})$ and $b = \max(u_1, u_2, \ldots, u_{2019})$. Prove that

$$ab \leqslant -\frac{1}{2019}.$$

(Germany)

A3. Let $n \geqslant 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be a strictly increasing sequence of n positive real numbers with sum equal to 2. Let X be a subset of $\{1, 2, \ldots, n\}$ such that the value of

$$\left| 1 - \sum_{i \in X} a_i \right|$$

is minimised. Prove that there exists a strictly increasing sequence of n positive real numbers $(b_1, b_2, \ldots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

(New Zealand)

A4. Let $n \geqslant 2$ be a positive integer and a_1, a_2, $\ldots$, a_n be real numbers such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Define the set A by

$$A = \left\{ (i, j) \,\middle|\, 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1 \right\}.$$

Prove that, if A is not empty, then

$$\sum_{(i,j) \in A} a_i a_j < 0.$$

(China)

A5. Let x_1, x_2, $\ldots$, x_n be different real numbers. Prove that

$$\sum_{1 \leqslant i \leqslant n} \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

(Kazakhstan)

A6. A polynomial $P(x, y, z)$ in three variables with real coefficients satisfies the identities

$$P(x, y, z) = P(x, y, xy - z) = P(x, zx - y, z) = P(yz - x, y, z).$$

Prove that there exists a polynomial $F(t)$ in one variable such that

$$P(x, y, z) = F(x^2 + y^2 + z^2 - xyz).$$

(Russia)

A7. Let $\mathbb{Z}$ be the set of integers. We consider functions $f \colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(x + y) + y\big) = f\big(f(x) + y\big)$$

for all integers x and y. For such a function, we say that an integer v is f-*rare* if the set

$$X_v = \{x \in \mathbb{Z} \colon f(x) = v\}$$

is finite and nonempty.

(a) Prove that there exists such a function f for which there is an f-rare integer.

(b) Prove that no such function f can have more than one f-rare integer.

(Netherlands)

Combinatorics

C1. The infinite sequence $a_0, a_1, a_2, \ldots$ of (not necessarily different) integers has the following properties: $0 \leqslant a_i \leqslant i$ for all integers $i \geqslant 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geqslant 0$.

Prove that all integers $N \geqslant 0$ occur in the sequence (that is, for all $N \geqslant 0$, there exists $i \geqslant 0$ with $a_i = N$).

(Netherlands)

C2. You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leqslant r \leqslant 2n - 2$ you can choose a subset of the blocks whose total weight is at least r but at most $r + 2$.

(Thailand)

C3. Let n be a positive integer. Harry has n coins lined up on his desk, each showing heads or tails. He repeatedly does the following operation: if there are k coins showing heads and $k > 0$, then he flips the k^{th} coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \to HHT \to HTT \to TTT$, which takes three steps.)

Letting C denote the initial configuration (a sequence of n H's and T's), write $\ell(C)$ for the number of steps needed before all coins show T. Show that this number $\ell(C)$ is finite, and determine its average value over all 2^n possible initial configurations C.

(USA)

C4. On a flat plane in Camelot, King Arthur builds a labyrinth $\mathfrak{L}$ consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.

After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

Let $k(\mathfrak{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth $\mathfrak{L}$, Morgana can always place at least k knights such that no two of them can ever meet. For each n, what are all possible values for $k(\mathfrak{L})$, where $\mathfrak{L}$ is a labyrinth with n walls?

(Canada)

C5. On a certain social network, there are 2019 users, some pairs of which are friends, where friendship is a symmetric relation. Initially, there are 1010 people with 1009 friends each and 1009 people with 1010 friends each. However, the friendships are rather unstable, so events of the following kind may happen repeatedly, one at a time:

> Let A, B, and C be people such that A is friends with both B and C, but B and C are not friends; then B and C become friends, but A is no longer friends with them.

Prove that, regardless of the initial friendships, there exists a sequence of such events after which each user is friends with at most one other user.

(Croatia)

C6. Let $n > 1$ be an integer. Suppose we are given $2n$ points in a plane such that no three of them are collinear. The points are to be labelled $A_1, A_2, \ldots, A_{2n}$ in some order. We then consider the $2n$ angles $\angle A_1A_2A_3$, $\angle A_2A_3A_4$, $\ldots$, $\angle A_{2n-2}A_{2n-1}A_{2n}$, $\angle A_{2n-1}A_{2n}A_1$, $\angle A_{2n}A_1A_2$. We measure each angle in the way that gives the smallest positive value (i.e. between 0° and 180°). Prove that there exists an ordering of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

(USA)

C7. There are 60 empty boxes $B_1, \ldots, B_{60}$ in a row on a table and an unlimited supply of pebbles. Given a positive integer n, Alice and Bob play the following game.

In the first round, Alice takes n pebbles and distributes them into the 60 boxes as she wishes. Each subsequent round consists of two steps:

(a) Bob chooses an integer k with $1 \leqslant k \leqslant 59$ and splits the boxes into the two groups $B_1, \ldots, B_k$ and $B_{k+1}, \ldots, B_{60}$.

(b) Alice picks one of these two groups, adds one pebble to each box in that group, and removes one pebble from each box in the other group.

Bob wins if, at the end of any round, some box contains no pebbles. Find the smallest n such that Alice can prevent Bob from winning.

(Czech Republic)

C8. Alice has a map of Wonderland, a country consisting of $n \geqslant 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

Comment. This problem could be posed with an explicit statement about points being awarded for weaker bounds cn for some $c > 4$, in the style of IMO 2014 Problem 6.

(Thailand)

C9. For any two different real numbers x and y, we define $D(x, y)$ to be the unique integer d satisfying $2^d \leqslant |x - y| < 2^{d+1}$. Given a set of reals $\mathcal{F}$, and an element $x \in \mathcal{F}$, we say that the *scales* of x in $\mathcal{F}$ are the values of $D(x, y)$ for $y \in \mathcal{F}$ with $x \neq y$.

Let k be a given positive integer. Suppose that each member x of $\mathcal{F}$ has at most k different scales in $\mathcal{F}$ (note that these scales may depend on x). What is the maximum possible size of $\mathcal{F}$?

(Italy)

Geometry

G1. Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

(Nigeria)

G2. Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

(Vietnam)

G3. In triangle ABC, let A_1 and B_1 be two points on sides BC and AC, and let P and Q be two points on segments AA_1 and BB_1, respectively, so that line PQ is parallel to AB. On ray PB_1, beyond B_1, let P_1 be a point so that $\angle PP_1C = \angle BAC$. Similarly, on ray QA_1, beyond A_1, let Q_1 be a point so that $\angle CQ_1Q = \angle CBA$. Show that points P, Q, P_1, and Q_1 are concyclic.

(Ukraine)

G4. Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

(Australia)

G5. Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \neq 2 \cdot \angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if $\operatorname{area}(BCD) + \operatorname{area}(ADE) = \operatorname{area}(ABD) + \operatorname{area}(ABP)$.

(Hungary)

G6. Let I be the incentre of acute-angled triangle ABC. Let the incircle meet BC, CA, and AB at D, E, and F, respectively. Let line EF intersect the circumcircle of the triangle at P and Q, such that F lies between E and P. Prove that $\angle DPA + \angle AQD = \angle QIP$.

(Slovakia)

G7. The incircle ω of acute-angled scalene triangle ABC has centre I and meets sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at $Q \neq P$. Prove that lines DI and PQ meet on the external bisector of angle BAC.

(India)

G8. Let $\mathcal{L}$ be the set of all lines in the plane and let f be a function that assigns to each line $\ell \in \mathcal{L}$ a point $f(\ell)$ on ℓ. Suppose that for any point X, and for any three lines ℓ_1, ℓ_2, ℓ_3 passing through X, the points $f(\ell_1)$, $f(\ell_2)$, $f(\ell_3)$ and X lie on a circle.

Prove that there is a unique point P such that $f(\ell) = P$ for any line ℓ passing through P.

(Australia)

Number Theory

N1. Find all pairs (m, n) of positive integers satisfying the equation

$$(2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}) = m!$$

(El Salvador)

N2. Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

(Nigeria)

N3. We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \ldots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

(Czech Republic)

N4. Let $\mathbb{Z}_{>0}$ be the set of positive integers. A positive integer constant C is given. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that, for all positive integers a and b satisfying $a + b > C$,

$$a + f(b) \mid a^2 + b\,f(a).$$

(Croatia)

N5. Let a be a positive integer. We say that a positive integer b is *a-good* if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geqslant b$. Suppose b is a positive integer such that b is a-good, but $b + 2$ is not a-good. Prove that $b + 1$ is prime.

(Netherlands)

N6. Let $H = \left\{ \lfloor i\sqrt{2} \rfloor : i \in \mathbb{Z}_{>0} \right\} = \{1, 2, 4, 5, 7, \ldots\}$, and let n be a positive integer. Prove that there exists a constant C such that, if $A \subset \{1, 2, \ldots, n\}$ satisfies $|A| \geqslant C\sqrt{n}$, then there exist $a, b \in A$ such that $a - b \in H$. (Here $\mathbb{Z}_{>0}$ is the set of positive integers, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

(Brazil)

N7. Prove that there is a constant $c > 0$ and infinitely many positive integers n with the following property: there are infinitely many positive integers that cannot be expressed as the sum of fewer than $cn \log(n)$ pairwise coprime n^{th} powers.

(Canada)

N8. Let a and b be two positive integers. Prove that the integer

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square. (Here $\lceil z \rceil$ denotes the least integer greater than or equal to z.)

(Russia)

Solutions

Algebra

A1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)). \tag{1}$$

(South Africa)

Answer: The solutions are $f(n) = 0$ and $f(n) = 2n + K$ for any constant $K \in \mathbb{Z}$.

Common remarks. Most solutions to this problem first prove that f must be linear, before determining all linear functions satisfying (1).

Solution 1. Substituting $a = 0, b = n+1$ gives $f(f(n+1)) = f(0) + 2f(n+1)$. Substituting $a = 1, b = n$ gives $f(f(n+1)) = f(2) + 2f(n)$.

In particular, $f(0) + 2f(n+1) = f(2) + 2f(n)$, and so $f(n+1) - f(n) = \frac{1}{2}(f(2) - f(0))$. Thus $f(n+1) - f(n)$ must be constant. Since f is defined only on $\mathbb{Z}$, this tells us that f must be a linear function; write $f(n) = Mn + K$ for arbitrary constants M and K, and we need only determine which choices of M and K work.

Now, (1) becomes

$$2Ma + K + 2(Mb + K) = M(M(a+b) + K) + K$$

which we may rearrange to form

$$(M-2)\big(M(a+b) + K\big) = 0.$$

Thus, either $M = 2$, or $M(a+b) + K = 0$ for all values of $a + b$. In particular, the only possible solutions are $f(n) = 0$ and $f(n) = 2n + K$ for any constant $K \in \mathbb{Z}$, and these are easily seen to work.

Solution 2. Let $K = f(0)$.

First, put $a = 0$ in (1); this gives

$$f(f(b)) = 2f(b) + K \tag{2}$$

for all $b \in \mathbb{Z}$.

Now put $b = 0$ in (1); this gives

$$f(2a) + 2K = f(f(a)) = 2f(a) + K,$$

where the second equality follows from (2). Consequently,

$$f(2a) = 2f(a) - K \tag{3}$$

for all $a \in \mathbb{Z}$.

Substituting (2) and (3) into (1), we obtain

$$\begin{aligned} f(2a) + 2f(b) &= f(f(a+b)) \\ 2f(a) - K + 2f(b) &= 2f(a+b) + K \\ f(a) + f(b) &= f(a+b) + K. \end{aligned}$$

Thus, if we set $g(n) = f(n) - K$ we see that g satisfies the Cauchy equation $g(a+b) = g(a)+g(b)$. The solution to the Cauchy equation over $\mathbb{Z}$ is well-known; indeed, it may be proven by an easy induction that $g(n) = Mn$ for each $n \in \mathbb{Z}$, where $M = g(1)$ is a constant.

Therefore, $f(n) = Mn + K$, and we may proceed as in Solution 1.

Comment 1. Instead of deriving (3) by substituting $b = 0$ into (1), we could instead have observed that the right hand side of (1) is symmetric in a and b, and thus

$$f(2a) + 2f(b) = f(2b) + 2f(a).$$

Thus, $f(2a) - 2f(a) = f(2b) - 2f(b)$ for any $a, b \in \mathbb{Z}$, and in particular $f(2a) - 2f(a)$ is constant. Setting $a = 0$ shows that this constant is equal to $-K$, and so we obtain (3).

Comment 2. Some solutions initially prove that $f(f(n))$ is linear (sometimes via proving that $f(f(n)) - 3K$ satisfies the Cauchy equation). However, one can immediately prove that f is linear by substituting something of the form $f(f(n)) = M'n + K'$ into (2).

A2. Let $u_1, u_2, \ldots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \cdots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \ldots, u_{2019})$ and $b = \max(u_1, u_2, \ldots, u_{2019})$. Prove that

$$ab \leqslant -\frac{1}{2019}.$$

(Germany)

Solution 1. Notice first that $b > 0$ and $a < 0$. Indeed, since $\sum_{i=1}^{2019} u_i^2 = 1$, the variables u_i cannot be all zero, and, since $\sum_{i=1}^{2019} u_i = 0$, the nonzero elements cannot be all positive or all negative.

Let $P = \{i : u_i > 0\}$ and $N = \{i : u_i \leqslant 0\}$ be the indices of positive and nonpositive elements in the sequence, and let $p = |P|$ and $n = |N|$ be the sizes of these sets; then $p + n = 2019$. By the condition $\sum_{i=1}^{2019} u_i = 0$ we have $0 = \sum_{i=1}^{2019} u_i = \sum_{i \in P} u_i - \sum_{i \in N} |u_i|$, so

$$\sum_{i \in P} u_i = \sum_{i \in N} |u_i|. \tag{1}$$

After this preparation, estimate the sum of squares of the positive and nonpositive elements as follows:

$$\sum_{i \in P} u_i^2 \leqslant \sum_{i \in P} b u_i = b \sum_{i \in P} u_i = b \sum_{i \in N} |u_i| \leqslant b \sum_{i \in N} |a| = -nab; \tag{2}$$

$$\sum_{i \in N} u_i^2 \leqslant \sum_{i \in N} |a| \cdot |u_i| = |a| \sum_{i \in N} |u_i| = |a| \sum_{i \in P} u_i \leqslant |a| \sum_{i \in P} b = -pab. \tag{3}$$

The sum of these estimates is

$$1 = \sum_{i=1}^{2019} u_i^2 = \sum_{i \in P} u_i^2 + \sum_{i \in N} u_i^2 \leqslant -(p+n)ab = -2019ab;$$

that proves $ab \leqslant \frac{-1}{2019}$.

Comment 1. After observing $\sum_{i \in P} u_i^2 \leqslant b \sum_{i \in P} u_i$ and $\sum_{i \in N} u_i^2 \leqslant |a| \sum_{i \in P} |u_i|$, instead of $(2, 3)$ an alternative continuation is

$$|ab| \geqslant \frac{\sum_{i \in P} u_i^2}{\sum_{i \in P} u_i} \cdot \frac{\sum_{i \in N} u_i^2}{\sum_{i \in N} |u_i|} = \frac{\sum_{i \in P} u_i^2}{\left(\sum_{i \in P} u_i\right)^2} \sum_{i \in N} u_i^2 \geqslant \frac{1}{p} \sum_{i \in N} u_i^2$$

(by the AM-QM or the Cauchy–Schwarz inequality) and similarly $|ab| \geqslant \frac{1}{n} \sum_{i \in P} u_i^2$.

Solution 2. As in the previous solution we conclude that $a < 0$ and $b > 0$.

For every index i, the number u_i is a convex combination of a and b, so

$$u_i = x_i a + y_i b \quad \text{with some weights } 0 \leqslant x_i, y_i \leqslant 1, \text{ with } x_i + y_i = 1.$$

Let $X = \sum_{i=1}^{2019} x_i$ and $Y = \sum_{i=1}^{2019} y_i$. From $0 = \sum_{i=1}^{2019} u_i = \sum_{i=1}^{2019} (x_i a + y_i b) = -|a| X + bY$, we get

$$|a| X = bY. \tag{4}$$

From $\sum_{i=1}^{2019}(x_i + y_i) = 2019$ we have

$$X + Y = 2019. \tag{5}$$

The system of linear equations $(4, 5)$ has a unique solution:

$$X = \frac{2019b}{|a| + b}, \quad Y = \frac{2019|a|}{|a| + b}.$$

Now apply the following estimate to every u_i^2 in their sum:

$$u_i^2 = x_i^2 a^2 + 2x_i y_i ab + y_i^2 b^2 \leqslant x_i a^2 + y_i b^2;$$

we obtain that

$$1 = \sum_{i=1}^{2019} u_i^2 \leqslant \sum_{i=1}^{2019}(x_i a^2 + y_i b^2) = Xa^2 + Yb^2 = \frac{2019b}{|a| + b}|a|^2 + \frac{2019|a|}{|a| + b}b^2 = 2019|a|b = -2019ab.$$

Hence, $ab \leqslant \dfrac{-1}{2019}$.

Comment 2. The idea behind Solution 2 is the following thought. Suppose we fix $a < 0$ and $b > 0$, fix $\sum u_i = 0$ and vary the u_i to achieve the maximum value of $\sum u_i^2$. Considering varying any two of the u_i while preserving their sum: the maximum value of $\sum u_i^2$ is achieved when those two are as far apart as possible, so all but at most one of the u_i are equal to a or b. Considering a weighted version of the problem, we see the maximum (with fractional numbers of u_i having each value) is achieved when $\dfrac{2019b}{|a| + b}$ of them are a and $\dfrac{2019|a|}{|a| + b}$ are b.

In fact, this happens in the solution: the number u_i is replaced by x_i copies of a and y_i copies of b.

A3. Let $n \geqslant 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be a strictly increasing sequence of n positive real numbers with sum equal to 2. Let X be a subset of $\{1, 2, \ldots, n\}$ such that the value of

$$\left|1 - \sum_{i \in X} a_i\right|$$

is minimised. Prove that there exists a strictly increasing sequence of n positive real numbers $(b_1, b_2, \ldots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

(New Zealand)

Common remarks. In all solutions, we say an index set X is (a_i)-*minimising* if it has the property in the problem for the given sequence (a_i). Write X^c for the complement of X, and $[a, b]$ for the interval of integers k such that $a \leqslant k \leqslant b$. Note that

$$\left|1 - \sum_{i \in X} a_i\right| = \left|1 - \sum_{i \in X^c} a_i\right|,$$

so we may exchange X and X^c where convenient. Let

$$\Delta = \sum_{i \in X^c} a_i - \sum_{i \in X} a_i$$

and note that X is (a_i)-minimising if and only if it minimises $|\Delta|$, and that $\sum_{i \in X} a_i = 1$ if and only if $\Delta = 0$.

In some solutions, a scaling process is used. If we have a strictly increasing sequence of positive real numbers c_i (typically obtained by perturbing the a_i in some way) such that

$$\sum_{i \in X} c_i = \sum_{i \in X^c} c_i,$$

then we may put $b_i = 2c_i / \sum_{j=1}^{n} c_j$. So it suffices to construct such a sequence without needing its sum to be 2.

The solutions below show various possible approaches to the problem. Solutions 1 and 2 perturb a few of the a_i to form the b_i (with scaling in the case of Solution 1, without scaling in the case of Solution 2). Solutions 3 and 4 look at properties of the index set X. Solution 3 then perturbs many of the a_i to form the b_i, together with scaling. Rather than using such perturbations, Solution 4 constructs a sequence (b_i) directly from the set X with the required properties. Solution 4 can be used to give a complete description of sets X that are (a_i)-minimising for some (a_i).

Solution 1. Without loss of generality, assume $\sum_{i \in X} a_i \leqslant 1$, and we may assume strict inequality as otherwise $b_i = a_i$ works. Also, X clearly cannot be empty.

If $n \in X$, add Δ to a_n, producing a sequence of c_i with $\sum_{i \in X} c_i = \sum_{i \in X^c} c_i$, and then scale as described above to make the sum equal to 2. Otherwise, there is some k with $k \in X$ and $k + 1 \in X^c$. Let $\delta = a_{k+1} - a_k$.

- If $\delta > \Delta$, add Δ to a_k and then scale.
- If $\delta < \Delta$, then considering $X \cup \{k+1\} \setminus \{k\}$ contradicts X being (a_i)-minimising.
- If $\delta = \Delta$, choose any $j \neq k, k+1$ (possible since $n \geqslant 3$), and any ϵ less than the least of a_1 and all the differences $a_{i+1} - a_i$. If $j \in X$ then add $\Delta - \epsilon$ to a_k and ϵ to a_j, then scale; otherwise, add Δ to a_k and $\epsilon/2$ to a_{k+1}, and subtract $\epsilon/2$ from a_j, then scale.

Solution 2. This is similar to Solution 1, but without scaling. As in that solution, without loss of generality, assume $\sum_{i\in X} a_i < 1$.

Suppose there exists $1 \leqslant j \leqslant n-1$ such that $j \in X$ but $j+1 \in X^c$. Then $a_{j+1} - a_j \geqslant \Delta$, because otherwise considering $X \cup \{j+1\} \setminus \{j\}$ contradicts X being (a_i)-minimising.

If $a_{j+1} - a_j > \Delta$, put

$$b_i = \begin{cases} a_j + \Delta/2, & \text{if } i = j; \\ a_{j+1} - \Delta/2, & \text{if } i = j+1; \\ a_i, & \text{otherwise.} \end{cases}$$

If $a_{j+1} - a_j = \Delta$, choose any ϵ less than the least of $\Delta/2$, a_1 and all the differences $a_{i+1} - a_i$. If $|X| \geqslant 2$, choose $k \in X$ with $k \neq j$, and put

$$b_i = \begin{cases} a_j + \Delta/2 - \epsilon, & \text{if } i = j; \\ a_{j+1} - \Delta/2, & \text{if } i = j+1; \\ a_k + \epsilon, & \text{if } i = k; \\ a_i, & \text{otherwise.} \end{cases}$$

Otherwise, $|X^c| \geqslant 2$, so choose $k \in X^c$ with $k \neq j+1$, and put

$$b_i = \begin{cases} a_j + \Delta/2, & \text{if } i = j; \\ a_{j+1} - \Delta/2 + \epsilon, & \text{if } i = j+1; \\ a_k - \epsilon, & \text{if } i = k; \\ a_i, & \text{otherwise.} \end{cases}$$

If there is no $1 \leqslant j \leqslant n$ such that $j \in X$ but $j+1 \in X^c$, there must be some $1 < k \leqslant n$ such that $X = [k, n]$ (certainly X cannot be empty). We must have $a_1 > \Delta$, as otherwise considering $X \cup \{1\}$ contradicts X being (a_i)-minimising. Now put

$$b_i = \begin{cases} a_1 - \Delta/2, & \text{if } i = 1; \\ a_n + \Delta/2, & \text{if } i = n; \\ a_i, & \text{otherwise.} \end{cases}$$

Solution 3. Without loss of generality, assume $\sum_{i\in X} a_i \leqslant 1$, so $\Delta \geqslant 0$. If $\Delta = 0$ we can take $b_i = a_i$, so now assume that $\Delta > 0$.

Suppose that there is some $k \leqslant n$ such that $|X \cap [k,n]| > |X^c \cap [k,n]|$. If we choose the largest such k then $|X \cap [k,n]| - |X^c \cap [k,n]| = 1$. We can now find the required sequence (b_i) by starting with $c_i = a_i$ for $i < k$ and $c_i = a_i + \Delta$ for $i \geqslant k$, and then scaling as described above.

If no such k exists, we will derive a contradiction. For each $i \in X$ we can choose $i < j_i \leqslant n$ in such a way that $j_i \in X^c$ and all the j_i are different. (For instance, note that necessarily $n \in X^c$ and now just work downwards; each time an $i \in X$ is considered, let j_i be the least element of X^c greater than i and not yet used.) Let Y be the (possibly empty) subset of $[1,n]$ consisting of those elements in X^c that are also not one of the j_i. In any case

$$\Delta = \sum_{i\in X}(a_{j_i} - a_i) + \sum_{j\in Y} a_j$$

where each term in the sums is positive. Since $n \geqslant 3$ the total number of terms above is at least two. Take a least such term and its corresponding index i and consider the set Z which we form from X by removing i and adding j_i (if it is a term of the first type) or just by adding j if it is a term of the second type. The corresponding expression of Δ for Z has the sign of its least term changed, meaning that the sum is still nonnegative but strictly less than Δ, which contradicts X being (a_i)-minimising.

Solution 4. This uses some similar ideas to Solution 3, but describes properties of the index sets X that are sufficient to describe a corresponding sequence (b_i) that is not derived from (a_i).

Note that, for two subsets X, Y of $[1,n]$, the following are equivalent:

- $|X \cap [i,n]| \leqslant |Y \cap [i,n]|$ for all $1 \leqslant i \leqslant n$;
- Y is at least as large as X, and for all $1 \leqslant j \leqslant |Y|$, the j^{th} largest element of Y is at least as big as the j^{th} largest element of X;
- there is an injective function $f : X \to Y$ such that $f(i) \geqslant i$ for all $i \in X$.

If these equivalent conditions are satisfied, we write $X \preceq Y$. We write $X \prec Y$ if $X \preceq Y$ and $X \neq Y$.

Note that if $X \prec Y$, then $\sum_{i\in X} a_i < \sum_{i\in Y} a_i$ (the second description above makes this clear).

We claim first that, if $n \geqslant 3$ and $X \prec X^c$, then there exists Y with $X \prec Y \prec X^c$. Indeed, as $|X| \leqslant |X^c|$, we have $|X^c| \geqslant 2$. Define Y to consist of the largest element of X^c, together with all but the largest element of X; it is clear both that Y is distinct from X and X^c, and that $X \preceq Y \preceq X^c$, which is what we need.

But, in this situation, we have

$$\sum_{i\in X} a_i < \sum_{i\in Y} a_i < \sum_{i\in X^c} a_i \qquad \text{and} \qquad 1 - \sum_{i\in X} a_i = -\left(1 - \sum_{i\in X^c} a_i\right),$$

so $|1 - \sum_{i\in Y} a_i| < |1 - \sum_{i\in X} a_i|$.

Hence if X is (a_i)-minimising, we do not have $X \prec X^c$, and similarly we do not have $X^c \prec X$.

Considering the first description above, this immediately implies the following Claim.

Claim. There exist $1 \leqslant k, \ell \leqslant n$ such that $|X \cap [k,n]| > \frac{n-k+1}{2}$ and $|X \cap [\ell,n]| < \frac{n-\ell+1}{2}$.

We now construct our sequence (b_i) using this claim. Let k and ℓ be the greatest values satisfying the claim, and without loss of generality suppose $k = n$ and $\ell < n$ (otherwise replace X by its complement). As ℓ is maximal, $n - \ell$ is even and $|X \cap [\ell,n]| = \frac{n-\ell}{2}$. For sufficiently small positive ϵ, we take

$$b_i = i\epsilon + \begin{cases} 0, & \text{if } i < \ell; \\ \delta, & \text{if } \ell \leqslant i \leqslant n-1; \\ \gamma, & \text{if } i = n. \end{cases}$$

Let $M = \sum_{i\in X} i$. So we require

$$M\epsilon + \left(\frac{n-\ell}{2} - 1\right)\delta + \gamma = 1$$

and

$$\frac{n(n+1)}{2}\epsilon + (n-\ell)\delta + \gamma = 2.$$

These give

$$\gamma = 2\delta + \left(\frac{n(n+1)}{2} - 2M\right)\epsilon$$

and for sufficiently small positive ϵ, solving for γ and δ gives $0 < \delta < \gamma$ (since $\epsilon = 0$ gives $\delta = 1/(\frac{n-\ell}{2} + 1)$ and $\gamma = 2\delta$), so the sequence is strictly increasing and has positive values.

Comment. This solution also shows that the claim gives a complete description of sets X that are (a_i)-minimising for some (a_i).

Another approach to proving the claim is as follows. We prove the existence of ℓ with the claimed property; the existence of k follows by considering the complement of X.

Suppose, for a contradiction, that for all $1 \leqslant \ell \leqslant n$ we have $|X \cap [\ell, n]| \geqslant \left\lceil \frac{n-\ell+1}{2} \right\rceil$. If we ever have strict inequality, consider the set $Y = \{n, n-2, n-4, \ldots\}$. This set may be obtained from X by possibly removing some elements and reducing the values of others. (To see this, consider the largest $k \in X \setminus Y$, if any; remove it, and replace it by the greatest $j \in X^c$ with $j < k$, if any. Such steps preserve the given inequality, and are possible until we reach the set Y.) So if we had strict inequality, and so $X \neq Y$, we have

$$\sum_{i \in X} a_i > \sum_{i \in Y} a_i > 1,$$

contradicting X being (a_i)-minimising. Otherwise, we always have equality, meaning that $X = Y$. But now consider $Z = Y \cup \{n-1\} \setminus \{n\}$. Since $n \geqslant 3$, we have

$$\sum_{i \in Y} a_i > \sum_{i \in Z} a_i > \sum_{i \in Y^c} a_i = 2 - \sum_{i \in Y} a_i,$$

and so Z contradicts X being (a_i)-minimising.

A4. Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Define the set A by

$$A = \{(i, j) \mid 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1\}.$$

Prove that, if A is not empty, then

$$\sum_{(i,j)\in A} a_i a_j < 0.$$

(China)

Solution 1. Define sets B and C by

$$B = \{(i, j) \mid 1 \leqslant i, j \leqslant n, |a_i - a_j| \geqslant 1\},$$
$$C = \{(i, j) \mid 1 \leqslant i, j \leqslant n, |a_i - a_j| < 1\}.$$

We have

$$\sum_{(i,j)\in A} a_i a_j = \frac{1}{2} \sum_{(i,j)\in B} a_i a_j$$
$$\sum_{(i,j)\in B} a_i a_j = \sum_{1\leqslant i,j\leqslant n} a_i a_j - \sum_{(i,j)\notin B} a_i a_j = 0 - \sum_{(i,j)\in C} a_i a_j.$$

So it suffices to show that if A (and hence B) are nonempty, then

$$\sum_{(i,j)\in C} a_i a_j > 0.$$

Partition the indices into sets P, Q, R, and S such that

$$P = \{i \mid a_i \leqslant -1\} \qquad R = \{i \mid 0 < a_i < 1\}$$
$$Q = \{i \mid -1 < a_i \leqslant 0\} \qquad S = \{i \mid 1 \leqslant a_i\}.$$

Then

$$\sum_{(i,j)\in C} a_i a_j \geqslant \sum_{i\in P\cup S} a_i^2 + \sum_{i,j\in Q\cup R} a_i a_j = \sum_{i\in P\cup S} a_i^2 + \left(\sum_{i\in Q\cup R} a_i\right)^2 \geqslant 0.$$

The first inequality holds because all of the positive terms in the RHS are also in the LHS, and all of the negative terms in the LHS are also in the RHS. The first inequality attains equality only if both sides have the same negative terms, which implies $|a_i - a_j| < 1$ whenever $i, j \in Q \cup R$; the second inequality attains equality only if $P = S = \varnothing$. But then we would have $A = \varnothing$. So A nonempty implies that the inequality holds strictly, as required.

Solution 2. Consider P, Q, R, S as in Solution 1, set

$$p = \sum_{i\in P} a_i, \quad q = \sum_{i\in Q} a_i, \quad r = \sum_{i\in R} a_i, \quad s = \sum_{i\in S} a_i,$$

and let

$$t_+ = \sum_{(i,j)\in A,\ a_i a_j \geqslant 0} a_i a_j, \quad t_- = \sum_{(i,j)\in A,\ a_i a_j \leqslant 0} a_i a_j.$$

We know that $p + q + r + s = 0$, and we need to prove that $t_+ + t_- < 0$.

Notice that $t_+ \leqslant p^2/2 + pq + rs + s^2/2$ (with equality only if $p = s = 0$), and $t_- \leqslant pr + ps + qs$ (with equality only if there do not exist $i \in Q$ and $j \in R$ with $a_j - a_i > 1$). Therefore,

$$t_+ + t_- \leqslant \frac{p^2 + s^2}{2} + pq + rs + pr + ps + qs = \frac{(p + q + r + s)^2}{2} - \frac{(q + r)^2}{2} = -\frac{(q + r)^2}{2} \leqslant 0.$$

If A is not empty and $p = s = 0$, then there must exist $i \in Q, j \in R$ with $|a_i - a_j| > 1$, and hence the earlier equality conditions cannot both occur.

Comment. The RHS of the original inequality cannot be replaced with any constant $c < 0$ (independent of n). Indeed, take

$$a_1 = -\frac{n}{n+2}, a_2 = \cdots = a_{n-1} = \frac{1}{n+2}, a_n = \frac{2}{n+2}.$$

Then $\displaystyle\sum_{(i,j)\in A} a_i a_j = -\frac{2n}{(n+2)^2}$, which converges to zero as $n \to \infty$.

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A5. Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1\leqslant i\leqslant n}\prod_{j\neq i}\frac{1-x_ix_j}{x_i-x_j}=\begin{cases}0, & \text{if } n \text{ is even;}\\ 1, & \text{if } n \text{ is odd.}\end{cases}$$

(Kazakhstan)

Common remarks. Let $G(x_1, x_2, \ldots, x_n)$ be the function of the n variables $x_1, x_2, \ldots, x_n$ on the LHS of the required identity.

Solution 1 (Lagrange interpolation). Since both sides of the identity are rational functions, it suffices to prove it when all $x_i \notin \{\pm 1\}$. Define

$$f(t)=\prod_{i=1}^{n}(1-x_it)\,,$$

and note that

$$f(x_i)=(1-x_i^2)\prod_{j\neq i}1-x_ix_j\,.$$

Using the nodes $+1, -1, x_1, \ldots, x_n$, the Lagrange interpolation formula gives us the following expression for f:

$$\sum_{i=1}^{n}f(x_i)\frac{(x-1)(x+1)}{(x_i-1)(x_i+1)}\prod_{j\neq i}\frac{x-x_j}{x_i-x_j}\;+\;f(1)\frac{x+1}{1+1}\prod_{1\leqslant i\leqslant n}\frac{x-x_i}{1-x_i}\;+\;f(-1)\frac{x-1}{-1-1}\prod_{1\leqslant i\leqslant n}\frac{x-x_i}{1-x_i}\,.$$

The coefficient of t^{n+1} in $f(t)$ is 0, since f has degree n. The coefficient of t^{n+1} in the above expression of f is

$$\begin{aligned}0&=\sum_{1\leqslant i\leqslant n}\frac{f(x_i)}{\prod_{j\neq i}(x_i-x_j)\cdot(x_i-1)(x_i+1)}+\frac{f(1)}{\prod_{1\leqslant j\leqslant n}(1-x_j)\cdot(1+1)}+\frac{f(-1)}{\prod_{1\leqslant j\leqslant n}(-1-x_j)\cdot(-1-1)}\\&=-G(x_1,\ldots,x_n)+\frac{1}{2}+\frac{(-1)^{n+1}}{2}.\end{aligned}$$

Comment. The main difficulty is to think of including the two extra nodes ± 1 and evaluating the coefficient t^{n+1} in f when $n+1$ is higher than the degree of f.

It is possible to solve the problem using Lagrange interpolation on the nodes $x_1, \ldots, x_n$, but the definition of the polynomial being interpolated should depend on the parity of n. For n even, consider the polynomial

$$P(x)=\prod_i(1-xx_i)-\prod_i(x-x_i).$$

Lagrange interpolation shows that G is the coefficient of x^{n-1} in the polynomial $P(x)/(1-x^2)$, i.e. 0. For n odd, consider the polynomial

$$P(x)=\prod_i(1-xx_i)-x\prod_i(x-x_i).$$

Now G is the coefficient of x^{n-1} in $P(x)/(1-x^2)$, which is 1.

Solution 2 (using symmetries). Observe that G is symmetric in the variables $x_1, \ldots, x_n$. Define $V = \prod_{i<j}(x_j - x_i)$ and let $F = G \cdot V$, which is a polynomial in $x_1, \ldots, x_n$. Since V is alternating, F is also alternating (meaning that, if we exchange any two variables, then F changes sign). Every alternating polynomial in n variables $x_1, \ldots, x_n$ vanishes when any two variables x_i, x_j $(i \neq j)$ are equal, and is therefore divisible by $x_i - x_j$ for each pair $i \neq j$. Since these linear factors are pairwise coprime, V divides F exactly as a polynomial. Thus G is in fact a symmetric polynomial in $x_1, \ldots, x_n$.

Now observe that if all x_i are nonzero and we set $y_i = 1/x_i$ for $i = 1, \ldots, n$, then we have

$$\frac{1 - y_i y_j}{y_i - y_j} = \frac{1 - x_i x_j}{x_i - x_j},$$

so that

$$G\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) = G(x_1, \ldots, x_n).$$

By continuity this is an identity of rational functions. Since G is a polynomial, it implies that G is constant. (If G were not constant, we could choose a point $(c_1, \ldots, c_n)$ with all $c_i \neq 0$, such that $G(c_1, \ldots, c_n) \neq G(0, \ldots, 0)$; then $g(x) := G(c_1 x, \ldots, c_n x)$ would be a nonconstant polynomial in the variable x, so $|g(x)| \to \infty$ as $x \to \infty$, hence $\left|G\left(\frac{y}{c_1}, \ldots, \frac{y}{c_n}\right)\right| \to \infty$ as $y \to 0$, which is impossible since G is a polynomial.)

We may identify the constant by substituting $x_i = \zeta^i$, where ζ is a primitive n^{th} root of unity in $\mathbb{C}$. In the i^{th} term in the sum in the original expression we have a factor $1 - \zeta^i \zeta^{n-i} = 0$, unless $i = n$ or $2i = n$. In the case where n is odd, the only exceptional term is $i = n$, which gives the value $\prod_{j \neq n} \frac{1-\zeta^j}{1-\zeta^j} = 1$. When n is even, we also have the term $\prod_{j \neq \frac{n}{2}} \frac{1+\zeta^j}{-1-\zeta^j} = (-1)^{n-1} = -1$, so the sum is 0.

Comment. If we write out an explicit expression for F,

$$F = \sum_{1 \leqslant i \leqslant n} (-1)^{n-i} \prod_{\substack{j<k \\ j,k \neq i}} (x_k - x_j) \prod_{j \neq i} (1 - x_i x_j)$$

then to prove directly that F vanishes when $x_i = x_j$ for some $i \neq j$, but no other pair of variables coincide, we have to check carefully that the two nonzero terms in this sum cancel.

A different and slightly less convenient way to identify the constant is to substitute $x_i = 1 + \epsilon \zeta^i$, and throw away terms that are $O(\epsilon)$ as $\epsilon \to 0$.

Solution 3 (breaking symmetry). Consider G as a rational function in x_n with coefficients that are rational functions in the other variables. We can write

$$G(x_1, \ldots, x_n) = \frac{P(x_n)}{\prod_{j \neq n} (x_n - x_j)}$$

where $P(x_n)$ is a polynomial in x_n whose coefficients are rational functions in the other variables. We then have

$$P(x_n) = \left(\prod_{j \neq n} (1 - x_n x_j)\right) + \sum_{1 \leqslant i \leqslant n-1} (x_i x_n - 1) \left(\prod_{j \neq i,n} (x_n - x_j)\right) \left(\prod_{j \neq i,n} \frac{1 - x_i x_j}{x_i - x_j}\right).$$

For any $k \neq n$, substituting $x_n = x_k$ (which is valid when manipulating the numerator $P(x_n)$

on its own), we have (noting that $x_n - x_j$ vanishes when $j = k$)

$$\begin{aligned}
P(x_k) &= \left(\prod_{j\neq n}(1-x_kx_j)\right) + \sum_{1\leqslant i\leqslant n-1}(x_ix_k-1)\left(\prod_{j\neq i,n}(x_k-x_j)\right)\left(\prod_{j\neq i,n}\frac{1-x_ix_j}{x_i-x_j}\right)\\
&= \left(\prod_{j\neq n}(1-x_kx_j)\right) + \left(x_k^2-1\right)\left(\prod_{j\neq k,n}(x_k-x_j)\right)\left(\prod_{j\neq k,n}\frac{1-x_kx_j}{x_k-x_j}\right)\\
&= \left(\prod_{j\neq n}(1-x_kx_j)\right) + \left(x_k^2-1\right)\left(\prod_{j\neq k,n}(1-x_kx_j)\right)\\
&= 0.
\end{aligned}$$

Note that P is a polynomial in x_n of degree $n-1$. For any choice of distinct real numbers $x_1, \ldots, x_{n-1}$, P has those real numbers as its roots, and the denominator has the same degree and the same roots. This shows that G is constant in x_n, for any fixed choice of distinct $x_1, \ldots, x_{n-1}$. Now, G is symmetric in all n variables, so it must be also be constant in each of the other variables. G is therefore a constant that depends only on n. The constant may be identified as in the previous solution.

Comment. There is also a solution in which we recognise the expression for F in the comment after Solution 2 as the final column expansion of a certain matrix obtained by modifying the final column of the Vandermonde matrix. The task is then to show that the matrix can be modified by column operations either to make the final column identically zero (in the case where n even) or to recover the Vandermonde matrix (in the case where n odd). The polynomial $P/(1-x^2)$ is helpful for this task, where P is the parity-dependent polynomial defined in the comment after Solution 1.

A6. A polynomial $P(x, y, z)$ in three variables with real coefficients satisfies the identities

$$P(x, y, z) = P(x, y, xy - z) = P(x, zx - y, z) = P(yz - x, y, z). \qquad (*)$$

Prove that there exists a polynomial $F(t)$ in one variable such that

$$P(x, y, z) = F(x^2 + y^2 + z^2 - xyz).$$

(Russia)

Common remarks. The polynomial $x^2 + y^2 + z^2 - xyz$ satisfies the condition $(*)$, so every polynomial of the form $F(x^2 + y^2 + z^2 - xyz)$ does satisfy $(*)$. We will use without comment the fact that two polynomials have the same coefficients if and only if they are equal as functions.

Solution 1. In the first two steps, we deal with any polynomial $P(x, y, z)$ satisfying $P(x, y, z) = P(x, y, xy - z)$. Call such a polynomial *weakly symmetric*, and call a polynomial satisfying the full conditions in the problem *symmetric*.

Step 1. We start with the description of weakly symmetric polynomials. We claim that they are exactly the polynomials in x, y, and $z(xy - z)$. Clearly, all such polynomials are weakly symmetric. For the converse statement, consider $P_1(x, y, z) := P(x, y, z + \frac{1}{2}xy)$, which satisfies $P_1(x, y, z) = P_1(x, y, -z)$ and is therefore a polynomial in x, y, and z^2. This means that P is a polynomial in x, y, and $(z - \frac{1}{2}xy)^2 = -z(xy - z) + \frac{1}{4}x^2y^2$, and therefore a polynomial in x, y, and $z(xy - z)$.

Step 2. Suppose that P is weakly symmetric. Consider the monomials in $P(x, y, z)$ of highest total degree. Our aim is to show that in each such monomial $\mu x^a y^b z^c$ we have $a, b \geqslant c$. Consider the expansion

$$P(x, y, z) = \sum_{i,j,k} \mu_{ijk} x^i y^j \big(z(xy - z)\big)^k. \qquad (1.1)$$

The maximal total degree of a summand in (1.1) is $m = \max_{i,j,k\colon \mu_{ijk} \neq 0}(i + j + 3k)$. Now, for any i, j, k satisfying $i + j + 3k = m$ the summand $\mu_{i,j,k} x^i y^j \big(z(xy - z)\big)^k$ has leading term of the form $\mu x^{i+k} y^{j+k} z^k$. No other nonzero summand in (1.1) may have a term of this form in its expansion, hence this term does not cancel in the whole sum. Therefore, $\deg P = m$, and the leading component of P is exactly

$$\sum_{i+j+3k=m} \mu_{i,j,k} x^{i+k} y^{j+k} z^k,$$

and each summand in this sum satisfies the condition claimed above.

Step 3. We now prove the problem statement by induction on $m = \deg P$. For $m = 0$ the claim is trivial. Consider now a symmetric polynomial P with $\deg P > 0$. By Step 2, each of its monomials $\mu x^a y^b z^c$ of the highest total degree satisfies $a, b \geqslant c$. Applying other weak symmetries, we obtain $a, c \geqslant b$ and $b, c \geqslant a$; therefore, P has a unique leading monomial of the form $\mu(xyz)^c$. The polynomial $P_0(x, y, z) = P(x, y, z) - \mu\big(xyz - x^2 - y^2 - z^2\big)^c$ has smaller total degree. Since P_0 is symmetric, it is representable as a polynomial function of $xyz - x^2 - y^2 - z^2$. Then P is also of this form, completing the inductive step.

Comment. We could alternatively carry out Step 1 by an induction on $n = \deg_z P$, in a manner similar to Step 3. If $n = 0$, the statement holds. Assume that $n > 0$ and check the leading component of P with respect to z:

$$P(x, y, z) = Q_n(x, y) z^n + R(x, y, z),$$

where $\deg_z R < n$. After the change $z \mapsto xy - z$, the leading component becomes $Q_n(x,y)(-z)^n$; on the other hand, it should remain the same. Hence n is even. Now consider the polynomial

$$P_0(x,y,z) = P(x,y,z) - Q_n(x,y) \cdot \big(z(z-xy)\big)^{n/2}.$$

It is also weakly symmetric, and $\deg_z P_0 < n$. By the inductive hypothesis, it has the form $P_0(x,y,z) = S\big(x,y,z(z-xy)\big)$. Hence the polynomial

$$P(x,y,z) = S\big(x,y,z(xy-z)\big) + Q_n(x,y)\big(z(z-xy)\big)^{n/2}$$

also has this form. This completes the inductive step.

Solution 2. We will rely on the well-known identity

$$\cos^2 u + \cos^2 v + \cos^2 w - 2\cos u \cos v \cos w - 1 = 0 \quad \text{whenever } u+v+w=0. \tag{2.1}$$

Claim 1. The polynomial $P(x,y,z)$ is constant on the surface

$$\mathfrak{S} = \big\{(2\cos u, 2\cos v, 2\cos w) : u+v+w=0\big\}.$$

Proof. Notice that for $x = 2\cos u$, $y = 2\cos v$, $z = 2\cos w$, the Vieta jumps $x \mapsto yz - x$, $y \mapsto zx - y$, $z \mapsto xy - z$ in $(*)$ replace (u,v,w) by $(v-w,-v,w)$, $(u,w-u,-w)$ and $(-u,v,u-v)$, respectively. For example, for the first type of jump we have

$$yz - x = 4\cos v\cos w - 2\cos u = 2\cos(v+w) + 2\cos(v-w) - 2\cos u = 2\cos(v-w).$$

Define $G(u,v,w) = P(2\cos u, 2\cos v, 2\cos w)$. For $u+v+w=0$, the jumps give

$$\begin{aligned} G(u,v,w) &= G(v-w,-v,w) = G(w-v,-v,(v-w)-(-v)) = G(-u-2v,-v,2v-w) \\ &= G(u+2v,v,w-2v)\,. \end{aligned}$$

By induction,

$$G(u,v,w) = G\big(u+2kv, v, w-2kv\big) \quad (k \in \mathbb{Z}). \tag{2.2}$$

Similarly,

$$G(u,v,w) = G\big(u, v-2\ell u, w+2\ell u\big) \quad (\ell \in \mathbb{Z}). \tag{2.3}$$

And, of course, we have

$$G(u,v,w) = G\big(u+2p\pi, v+2q\pi, w-2(p+q)\pi\big) \quad (p,q \in \mathbb{Z}). \tag{2.4}$$

Take two nonzero real numbers u, v such that u, v and π are linearly independent over $\mathbb{Q}$. By combining (2.2–2.4), we can see that G is constant on a dense subset of the plane $u+v+w=0$. By continuity, G is constant on the entire plane and therefore P is constant on $\mathfrak{S}$. □

Claim 2. The polynomial $T(x,y,z) = x^2+y^2+z^2-xyz-4$ divides $P(x,y,z) - P(2,2,2)$.

Proof. By dividing P by T with remainders, there exist some polynomials $R(x,y,z)$, $A(y,z)$ and $B(y,z)$ such that

$$P(x,y,z) - P(2,2,2) = T(x,y,z) \cdot R(x,y,z) + A(y,z)x + B(y,z). \tag{2.5}$$

On the surface $\mathfrak{S}$ the LHS of (2.5) is zero by Claim 1 (since $(2,2,2) \in \mathfrak{S}$) and $T = 0$ by (2.1). Hence, $A(y,z)x + B(y,z)$ vanishes on $\mathfrak{S}$.

Notice that for every $y = 2\cos v$ and $z = 2\cos w$ with $\frac{\pi}{3} < v, w < \frac{2\pi}{3}$, there are two distinct values of x such that $(x,y,z) \in \mathfrak{S}$, namely $x_1 = 2\cos(v+w)$ (which is negative), and $x_2 = 2\cos(v-w)$ (which is positive). This can happen only if $A(y,z) = B(y,z) = 0$. Hence, $A(y,z) = B(y,z) = 0$ for $|y| < 1$, $|z| < 1$. The polynomials A and B vanish on an open set, so A and B are both the zero polynomial. □

The quotient $(P(x, y, z) - P(2, 2, 2))/T(x, y, z)$ is a polynomial of lower degree than P and it also satisfies $(*)$. The problem statement can now be proven by induction on the degree of P.

Comment. In the proof of (2.2) and (2.3) we used two consecutive Vieta jumps; in fact from $(*)$ we used only $P(x, y, xy - z) = P(x, zx - y, z) = P(yz - x, y, z)$.

Solution 3 (using algebraic geometry, just for interest). Let $Q = x^2 + y^2 + z^2 - xyz$ and let $t \in \mathbb{C}$. Checking where $Q - t$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ and $\frac{\partial Q}{\partial z}$ vanish simultaneously, we find that the surface $Q = t$ is smooth except for the cases $t = 0$, when the only singular point is $(0, 0, 0)$, and $t = 4$, when the four points $(\pm 2, \pm 2, \pm 2)$ that satisfy $xyz = 8$ are the only singular points. The singular points are the fixed points of the group Γ of polynomial automorphisms of $\mathbb{C}^3$ generated by the three *Vieta involutions*

$$\iota_1 : (x, y, z) \mapsto (x, y, xy - z), \quad \iota_2 : (x, y, z) \mapsto (x, xz - y, z), \quad \iota_3 : (x, y, z) \mapsto (yz - x, y, z)\,.$$

Γ acts on each surface $\mathcal{V}_t : Q - t = 0$. If $Q - t$ were reducible then the surface $Q = t$ would contain a curve of singular points. Therefore $Q - t$ is irreducible in $\mathbb{C}[x, y, z]$. (One can also prove algebraically that $Q - t$ is irreducible, for example by checking that its discriminant as a quadratic polynomial in x is not a square in $\mathbb{C}[y, z]$, and likewise for the other two variables.) In the following solution we will only use the algebraic surface $\mathcal{V}_0$.

Let U be the Γ-orbit of $(3, 3, 3)$. Consider $\iota_3 \circ \iota_2$, which leaves z invariant. For each fixed value of z, $\iota_3 \circ \iota_2$ acts linearly on (x, y) by the matrix

$$M_z := \begin{pmatrix} z^2 - 1 & -z \\ z & -1 \end{pmatrix}.$$

The reverse composition $\iota_2 \circ \iota_3$ acts by $M_z^{-1} = M_z^{adj}$. Note $\det M_z = 1$ and $\mathrm{tr} M_z = z^2 - 2$. When z does not lie in the real interval $[-2, 2]$, the eigenvalues of M_z do not have absolute value 1, so every orbit of the group generated by M_z on $\mathbb{C}^2 \setminus \{(0, 0)\}$ is unbounded. For example, fixing $z = 3$ we find $(3F_{2k+1}, 3F_{2k-1}, 3) \in U$ for every $k \in \mathbb{Z}$, where $(F_n)_{n \in \mathbb{Z}}$ is the Fibonacci sequence with $F_0 = 0$, $F_1 = 1$.

Now we may start at any point $(3F_{2k+1}, 3F_{2k-1}, 3)$ and iteratively apply $\iota_1 \circ \iota_2$ to generate another infinite sequence of distinct points of U, Zariski dense in the hyperbola cut out of $\mathcal{V}_0$ by the plane $x - 3F_{2k+1} = 0$. (The plane $x = a$ cuts out an irreducible conic when $a \notin \{-2, 0, 2\}$.) Thus the Zariski closure $\overline{U}$ of U contains infinitely many distinct algebraic curves in $\mathcal{V}_0$. Since $\mathcal{V}_0$ is an irreducible surface this implies that $\overline{U} = \mathcal{V}_0$.

For any polynomial P satisfying $(*)$, we have $P - P(3, 3, 3) = 0$ at each point of U. Since $\overline{U} = \mathcal{V}_0$, $P - P(3, 3, 3)$ vanishes on $\mathcal{V}_0$. Then Hilbert's Nullstellensatz and the irreducibility of Q imply that $P - P(3, 3, 3)$ is divisible by Q. Now $(P - P(3, 3, 3))/Q$ is a polynomial also satisfying $(*)$, so we may complete the proof by an induction on the total degree, as in the other solutions.

Comment. We remark that Solution 2 used a trigonometric parametrisation of a real component of $\mathcal{V}_4$; in contrast $\mathcal{V}_0$ is birationally equivalent to the projective space $\mathbb{P}^2$ under the maps

$$(x, y, z) \to (x : y : z), \quad (a : b : c) \to \left(\frac{a^2 + b^2 + c^2}{bc}, \frac{a^2 + b^2 + c^2}{ac}, \frac{a^2 + b^2 + c^2}{ab} \right).$$

The set U in Solution 3 is contained in $\mathbb{Z}^3$ so it is nowhere dense in $\mathcal{V}_0$ in the classical topology.

Comment (background to the problem). A triple $(a, b, c) \in \mathbb{Z}^3$ is called a *Markov triple* if $a^2 + b^2 + c^2 = 3abc$, and an integer that occurs as a coordinate of some Markov triple is called a *Markov number*. (The spelling *Markoff* is also frequent.) Markov triples arose in A. Markov's work in the 1870s on the reduction theory of indefinite binary quadratic forms. For every Markov triple,

$(3a, 3b, 3c)$ lies on $Q = 0$. It is well known that all nonzero Markov triples can be generated from $(1, 1, 1)$ by sequences of Vieta involutions, which are the substitutions described in equation $(*)$ in the problem statement. There has been recent work by number theorists about the properties of Markov numbers (see for example Jean Bourgain, Alex Gamburd and Peter Sarnak, *Markoff triples and strong approximation*, Comptes Rendus Math. **345**, no. 2, 131–135 (2016), arXiv:1505.06411). Each Markov number occurs in infinitely many triples, but a famous old open problem is the unicity conjecture, which asserts that each Markov number occurs in only one Markov triple (up to permutations and sign changes) as the largest coordinate in absolute value in that triple. It is a standard fact in the modern literature on Markov numbers that the Markov triples are Zariski dense in the Markov surface. Proving this is the main work of Solution 3. Algebraic geometry is definitely off-syllabus for the IMO, and one still has to work a bit to prove the Zariski density. On the other hand the approaches of Solutions 1 and 2 are elementary and only use tools expected to be known by IMO contestants. Therefore we do not think that the existence of a solution using algebraic geometry necessarily makes this problem unsuitable for the IMO.

A7. Let $\mathbb{Z}$ be the set of integers. We consider functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(x+y)+y\big) = f\big(f(x)+y\big)$$

for all integers x and y. For such a function, we say that an integer v is *f-rare* if the set

$$X_v = \{x \in \mathbb{Z}\colon f(x) = v\}$$

is finite and nonempty.

(a) Prove that there exists such a function f for which there is an f-rare integer.

(b) Prove that no such function f can have more than one f-rare integer.

(Netherlands)

Solution 1. a) Let f be the function where $f(0) = 0$ and $f(x)$ is the largest power of 2 dividing $2x$ for $x \neq 0$. The integer 0 is evidently f-rare, so it remains to verify the functional equation.

Since $f(2x) = 2f(x)$ for all x, it suffices to verify the functional equation when at least one of x and y is odd (the case $x = y = 0$ being trivial). If y is odd, then we have

$$f(f(x+y)+y) = 2 = f(f(x)+y)$$

since all the values attained by f are even. If, on the other hand, x is odd and y is even, then we already have

$$f(x+y) = 2 = f(x)$$

from which the functional equation follows immediately.

b) An easy inductive argument (substituting $x + ky$ for x) shows that

$$f(f(x+ky)+y) = f(f(x)+y) \tag{$*$}$$

for all integers x, y and k. If v is an f-rare integer and a is the least element of X_v, then by substituting $y = a - f(x)$ in the above, we see that

$$f(x + k\cdot(a - f(x))) - f(x) + a \in X_v$$

for all integers x and k, so that in particular

$$f(x + k\cdot(a - f(x))) \geqslant f(x)$$

for all integers x and k, by assumption on a. This says that on the (possibly degenerate) arithmetic progression through x with common difference $a - f(x)$, the function f attains its minimal value at x.

Repeating the same argument with a replaced by the greatest element b of X_v shows that

$$f(x + k\cdot(b - f(x)) \leqslant f(x)$$

for all integers x and k. Combined with the above inequality, we therefore have

$$f(x + k\cdot(a - f(x))\cdot(b - f(x))) = f(x) \tag{$\dagger$}$$

for all integers x and k.

Thus if $f(x) \neq a, b$, then the set $X_{f(x)}$ contains a nondegenerate arithmetic progression, so is infinite. So the only possible f-rare integers are a and b.

In particular, the f-rare integer v we started with must be one of a or b, so that $f(v) = f(a) = f(b) = v$. This means that there cannot be any other f-rare integers v', as they would on the one hand have to be either a or b, and on the other would have to satisfy $f(v') = v'$. Thus v is the unique f-rare integer.

Comment 1. If f is a solution to the functional equation, then so too is any conjugate of f by a translation, i.e. any function $x \mapsto f(x+n)-n$ for an integer n. Thus in proving part (b), one is free to consider only functions f for which 0 is f-rare, as in the following solution.

Solution 2, part (b) only. Suppose v is f-rare, and let a and b be the least and greatest elements of X_v, respectively. Substituting $x=v$ and $y=a-v$ into the equation shows that

$$f(v)-v+a \in X_v$$

and in particular $f(v) \geqslant v$. Repeating the same argument with $x=v$ and $y=b-v$ shows that $f(v) \leqslant v$, and hence $f(v)=v$.

Suppose now that v' is a second f-rare integer. We may assume that $v=0$ (see Comment 1). We've seen that $f(v')=v'$; we claim that in fact $f(kv')=v'$ for all positive integers k. This gives a contradiction unless $v'=v=0$.

This claim is proved by induction on k. Supposing it to be true for k, we substitute $y=kv'$ and $x=0$ into the functional equation to yield

$$f((k+1)v')=f(f(0)+kv')=f(kv')=v'$$

using that $f(0)=0$. This completes the induction, and hence the proof.

Comment 2. There are many functions f satisfying the functional equation for which there is an f-rare integer. For instance, one may generalise the construction in part (a) of Solution 1 by taking a sequence $1=a_0, a_1, a_2, \ldots$ of positive integers with each a_i a proper divisor of a_{i+1} and choosing arbitrary functions $f_i\colon (\mathbb{Z}/a_i\mathbb{Z}) \setminus \{0\} \to a_i\mathbb{Z} \setminus \{0\}$ from the nonzero residue classes modulo a_i to the nonzero multiples of a_i. One then defines a function $f\colon \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) := \begin{cases} f_{i+1}(x \bmod a_{i+1}), & \text{if } a_i \mid x \text{ but } a_{i+1} \nmid x; \\ 0, & \text{if } x=0. \end{cases}$$

If one writes $v(x)$ for the largest i such that $a_i \mid x$ (with $v(0)=\infty$), then it is easy to verify the functional equation for f separately in the two cases $v(y)>v(x)$ and $v(x) \geqslant v(y)$. Hence this f satisfies the functional equation and 0 is an f-rare integer.

Comment 3. In fact, if v is an f-rare integer for an f satisfying the functional equation, then its fibre $X_v=\{v\}$ must be a singleton. We may assume without loss of generality that $v=0$. We've already seen in Solution 1 that 0 is either the greatest or least element of X_0; replacing f with the function $x \mapsto -f(-x)$ if necessary, we may assume that 0 is the least element of X_0. We write b for the largest element of X_0, supposing for contradiction that $b>0$, and write $N=(2b)!$.

It now follows from $(*)$ that we have

$$f(f(Nb)+b)=f(f(0)+b)=f(b)=0,$$

from which we see that $f(Nb)+b \in X_0 \subseteq [0,b]$. It follows that $f(Nb) \in [-b,0)$, since by construction $Nb \notin X_v$. Now it follows that $(f(Nb)-0)\cdot(f(Nb)-b)$ is a divisor of N, so from $(\dagger)$ we see that $f(Nb)=f(0)=0$. This yields the desired contradiction.

Combinatorics

C1. The infinite sequence a_0, a_1, a_2, $\ldots$ of (not necessarily different) integers has the following properties: $0 \leqslant a_i \leqslant i$ for all integers $i \geqslant 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geqslant 0$.

Prove that all integers $N \geqslant 0$ occur in the sequence (that is, for all $N \geqslant 0$, there exists $i \geqslant 0$ with $a_i = N$).

(Netherlands)

Solution. We prove by induction on k that every initial segment of the sequence, $a_0, a_1, \ldots, a_k$, consists of the following elements (counted with multiplicity, and not necessarily in order), for some $\ell \geqslant 0$ with $2\ell \leqslant k+1$:

$$0, 1, \ldots, \ell - 1, \quad 0, 1, \ldots, k - \ell.$$

For $k = 0$ we have $a_0 = 0$, which is of this form. Now suppose that for $k = m$ the elements $a_0, a_1, \ldots, a_m$ are $0, 0, 1, 1, 2, 2, \ldots, \ell - 1, \ell - 1, \ell, \ell + 1, \ldots, m - \ell - 1, m - \ell$ for some ℓ with $0 \leqslant 2\ell \leqslant m + 1$. It is given that

$$\binom{m+1}{a_0} + \binom{m+1}{a_1} + \cdots + \binom{m+1}{a_m} + \binom{m+1}{a_{m+1}} = 2^{m+1},$$

which becomes

$$\left(\binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{\ell-1}\right) + \left(\binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{m-\ell}\right) + \binom{m+1}{a_{m+1}} = 2^{m+1},$$

or, using $\binom{m+1}{i} = \binom{m+1}{m+1-i}$, that

$$\left(\binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{\ell-1}\right) + \left(\binom{m+1}{m+1} + \binom{m+1}{m} + \cdots + \binom{m+1}{\ell+1}\right) + \binom{m+1}{a_{m+1}} = 2^{m+1}.$$

On the other hand, it is well known that

$$\binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{m+1} = 2^{m+1},$$

and so, by subtracting, we get

$$\binom{m+1}{a_{m+1}} = \binom{m+1}{\ell}.$$

From this, using the fact that the binomial coefficients $\binom{m+1}{i}$ are increasing for $i \leqslant \frac{m+1}{2}$ and decreasing for $i \geqslant \frac{m+1}{2}$, we conclude that either $a_{m+1} = \ell$ or $a_{m+1} = m + 1 - \ell$. In either case, $a_0, a_1, \ldots, a_{m+1}$ is again of the claimed form, which concludes the induction.

As a result of this description, any integer $N \geqslant 0$ appears as a term of the sequence a_i for some $0 \leqslant i \leqslant 2N$.

C2. You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leqslant r \leqslant 2n-2$ you can choose a subset of the blocks whose total weight is at least r but at most $r+2$.

(Thailand)

Solution 1. We prove the following more general statement by induction on n.

Claim. Suppose that you have n blocks, each of weight at least 1, and of total weight $s \leqslant 2n$. Then for every r with $-2 \leqslant r \leqslant s$, you can choose some of the blocks whose total weight is at least r but at most $r+2$.

Proof. The base case $n=1$ is trivial. To prove the inductive step, let x be the largest block weight. Clearly, $x \geqslant s/n$, so $s-x \leqslant \frac{n-1}{n}s \leqslant 2(n-1)$. Hence, if we exclude a block of weight x, we can apply the inductive hypothesis to show the claim holds (for this smaller set) for any $-2 \leqslant r \leqslant s-x$. Adding the excluded block to each of those combinations, we see that the claim also holds when $x-2 \leqslant r \leqslant s$. So if $x-2 \leqslant s-x$, then we have covered the whole interval $[-2, s]$. But each block weight is at least 1, so we have $x-2 \leqslant (s-(n-1))-2 = s-(2n-(n-1)) \leqslant s-(s-(n-1)) \leqslant s-x$, as desired. $\square$

Comment. Instead of inducting on sets of blocks with total weight $s \leqslant 2n$, we could instead prove the result only for $s=2n$. We would then need to modify the inductive step to scale up the block weights before applying the induction hypothesis.

Solution 2. Let $x_1, \ldots, x_n$ be the weights of the blocks in weakly increasing order. Consider the set S of sums of the form $\sum_{j \in J} x_j$ for a subset $J \subseteq \{1, 2, \ldots, n\}$. We want to prove that the *mesh* of S – i.e. the largest distance between two adjacent elements – is at most 2.

For $0 \leqslant k \leqslant n$, let S_k denote the set of sums of the form $\sum_{i \in J} x_i$ for a subset $J \subseteq \{1, 2, \ldots, k\}$. We will show by induction on k that the mesh of S_k is at most 2.

The base case $k=0$ is trivial (as $S_0 = \{0\}$). For $k>0$ we have

$$S_k = S_{k-1} \cup (x_k + S_{k-1})$$

(where $(x_k + S_{k-1})$ denotes $\{x_k + s : s \in S_{k-1}\}$), so it suffices to prove that $x_k \leqslant \sum_{j<k} x_j + 2$. But if this were not the case, we would have $x_l > \sum_{j<k} x_j + 2 \geqslant k+1$ for all $l \geqslant k$, and hence

$$2n = \sum_{j=1}^{n} x_j > (n+1-k)(k+1) + k - 1.$$

This rearranges to $n > k(n+1-k)$, which is false for $1 \leqslant k \leqslant n$, giving the desired contradiction.

C3. Let n be a positive integer. Harry has n coins lined up on his desk, each showing heads or tails. He repeatedly does the following operation: if there are k coins showing heads and $k > 0$, then he flips the k^{th} coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \to HHT \to HTT \to TTT$, which takes three steps.)

Letting C denote the initial configuration (a sequence of n H's and T's), write $\ell(C)$ for the number of steps needed before all coins show T. Show that this number $\ell(C)$ is finite, and determine its average value over all 2^n possible initial configurations C.

(USA)

Answer: The average is $\frac{1}{4}n(n+1)$.

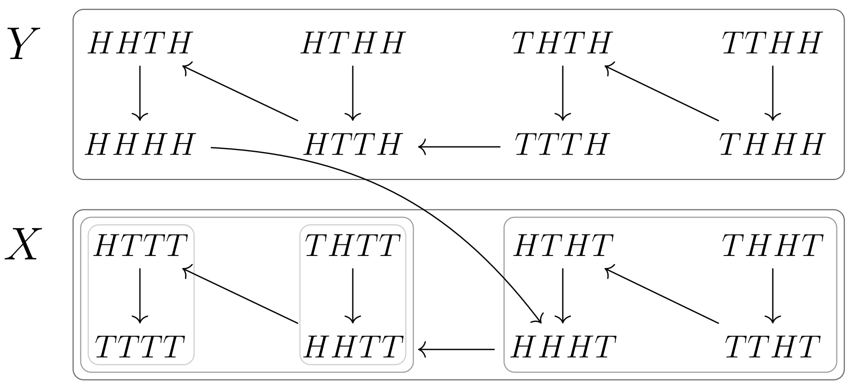
Common remarks. Throughout all these solutions, we let $E(n)$ denote the desired average value.

Solution 1. We represent the problem using a directed graph G_n whose vertices are the length-n strings of H's and T's. The graph features an edge from each string to its successor (except for $TT\cdots TT$, which has no successor). We will also write $\bar{H} = T$ and $\bar{T} = H$.

The graph G_0 consists of a single vertex: the empty string. The main claim is that G_n can be described explicitly in terms of G_{n-1}:

- We take two copies, X and Y, of G_{n-1}.
- In X, we take each string of $n-1$ coins and just append a T to it. In symbols, we replace $s_1 \cdots s_{n-1}$ with $s_1 \cdots s_{n-1}T$.
- In Y, we take each string of $n-1$ coins, flip every coin, reverse the order, and append an H to it. In symbols, we replace $s_1 \cdots s_{n-1}$ with $\bar{s}_{n-1}\bar{s}_{n-2} \cdots \bar{s}_1 H$.
- Finally, we add one new edge from Y to X, namely $HH \cdots HHH \to HH \cdots HHT$.

We depict G_4 below, in a way which indicates this recursive construction:



We prove the claim inductively. Firstly, X is correct as a subgraph of G_n, as the operation on coins is unchanged by an extra T at the end: if $s_1 \cdots s_{n-1}$ is sent to $t_1 \cdots t_{n-1}$, then $s_1 \cdots s_{n-1}T$ is sent to $t_1 \cdots t_{n-1}T$.

Next, Y is also correct as a subgraph of G_n, as if $s_1 \cdots s_{n-1}$ has k occurrences of H, then $\bar{s}_{n-1} \cdots \bar{s}_1 H$ has $(n-1-k)+1 = n-k$ occurrences of H, and thus (provided that $k > 0$), if $s_1 \cdots s_{n-1}$ is sent to $t_1 \cdots t_{n-1}$, then $\bar{s}_{n-1} \cdots \bar{s}_1 H$ is sent to $\bar{t}_{n-1} \cdots \bar{t}_1 H$.

Finally, the one edge from Y to X is correct, as the operation does send $HH \cdots HHH$ to $HH \cdots HHT$.

To finish, note that the sequences in X take an average of $E(n-1)$ steps to terminate, whereas the sequences in Y take an average of $E(n-1)$ steps to reach $HH\cdots H$ and then an additional n steps to terminate. Therefore, we have

$$E(n) = \frac{1}{2}\left(E(n-1) + (E(n-1)+n)\right) = E(n-1) + \frac{n}{2}.$$

We have $E(0) = 0$ from our description of G_0. Thus, by induction, we have $E(n) = \frac{1}{2}(1+\cdots+n) = \frac{1}{4}n(n+1)$, which in particular is finite.

Solution 2. We consider what happens with configurations depending on the coins they start and end with.

- If a configuration starts with H, the last $n-1$ coins follow the given rules, as if they were all the coins, until they are all T, then the first coin is turned over.
- If a configuration ends with T, the last coin will never be turned over, and the first $n-1$ coins follow the given rules, as if they were all the coins.
- If a configuration starts with T and ends with H, the middle $n-2$ coins follow the given rules, as if they were all the coins, until they are all T. After that, there are $2n-1$ more steps: first coins $1, 2, \ldots, n-1$ are turned over in that order, then coins $n, n-1, \ldots, 1$ are turned over in that order.

As this covers all configurations, and the number of steps is clearly finite for 0 or 1 coins, it follows by induction on n that the number of steps is always finite.

We define $E_{AB}(n)$, where A and B are each one of H, T or $*$, to be the average number of steps over configurations of length n restricted to those that start with A, if A is not $*$, and that end with B, if B is not $*$ (so $*$ represents "either H or T"). The above observations tell us that, for $n \geqslant 2$:

- $E_{H*}(n) = E(n-1) + 1$.
- $E_{*T}(n) = E(n-1)$.
- $E_{HT}(n) = E(n-2) + 1$ (by using both the observations for $H*$ and for $*T$).
- $E_{TH}(n) = E(n-2) + 2n - 1$.

Now $E_{H*}(n) = \frac{1}{2}(E_{HH}(n) + E_{HT}(n))$, so $E_{HH}(n) = 2E(n-1) - E(n-2) + 1$. Similarly, $E_{TT}(n) = 2E(n-1) - E(n-2) - 1$. So

$$E(n) = \frac{1}{4}(E_{HT}(n) + E_{HH}(n) + E_{TT}(n) + E_{TH}(n)) = E(n-1) + \frac{n}{2}.$$

We have $E(0) = 0$ and $E(1) = \frac{1}{2}$, so by induction on n we have $E(n) = \frac{1}{4}n(n+1)$.

Solution 3. Let H_i be the number of heads in positions 1 to i inclusive (so H_n is the total number of heads), and let I_i be 1 if the i^{th} coin is a head, 0 otherwise. Consider the function

$$t(i) = I_i + 2(\min\{i, H_n\} - H_i).$$

We claim that $t(i)$ is the total number of times coin i is turned over (which implies that the process terminates). Certainly $t(i) = 0$ when all coins are tails, and $t(i)$ is always a nonnegative integer, so it suffices to show that when the k^{th} coin is turned over (where $k = H_n$), $t(k)$ goes down by 1 and all the other $t(i)$ are unchanged. We show this by splitting into cases:

- If $i < k$, I_i and H_i are unchanged, and $\min\{i, H_n\} = i$ both before and after the coin flip, so $t(i)$ is unchanged.
- If $i > k$, $\min\{i, H_n\} = H_n$ both before and after the coin flip, and both H_n and H_i change by the same amount, so $t(i)$ is unchanged.
- If $i = k$ and the coin is heads, I_i goes down by 1, as do both $\min\{i, H_n\} = H_n$ and H_i; so $t(i)$ goes down by 1.
- If $i = k$ and the coin is tails, I_i goes up by 1, $\min\{i, H_n\} = i$ is unchanged and H_i goes up by 1; so $t(i)$ goes down by 1.

We now need to compute the average value of

$$\sum_{i=1}^{n} t(i) = \sum_{i=1}^{n} I_i + 2\sum_{i=1}^{n} \min\{i, H_n\} - 2\sum_{i=1}^{n} H_i.$$

The average value of the first term is $\frac{1}{2}n$, and that of the third term is $-\frac{1}{2}n(n+1)$. To compute the second term, we sum over choices for the total number of heads, and then over the possible values of i, getting

$$2^{1-n}\sum_{j=0}^{n}\binom{n}{j}\sum_{i=1}^{n}\min\{i,j\} = 2^{1-n}\sum_{j=0}^{n}\binom{n}{j}\left(nj - \binom{j}{2}\right).$$

Now, in terms of trinomial coefficients,

$$\sum_{j=0}^{n} j\binom{n}{j} = \sum_{j=1}^{n}\binom{n}{n-j, j-1, 1} = n\sum_{j=0}^{n-1}\binom{n-1}{j} = 2^{n-1}n$$

and

$$\sum_{j=0}^{n}\binom{j}{2}\binom{n}{j} = \sum_{j=2}^{n}\binom{n}{n-j, j-2, 2} = \binom{n}{2}\sum_{j=0}^{n-2}\binom{n-2}{j} = 2^{n-2}\binom{n}{2}.$$

So the second term above is

$$2^{1-n}\left(2^{n-1}n^2 - 2^{n-2}\binom{n}{2}\right) = n^2 - \frac{n(n-1)}{4},$$

and the required average is

$$E(n) = \frac{1}{2}n + n^2 - \frac{n(n-1)}{4} - \frac{1}{2}n(n+1) = \frac{n(n+1)}{4}.$$

Solution 4. Harry has built a Turing machine to flip the coins for him. The machine is initially positioned at the k^{th} coin, where there are k heads (and the position before the first coin is considered to be the 0^{th} coin). The machine then moves according to the following rules, stopping when it reaches the position before the first coin: if the coin at its current position is H, it flips the coin and moves to the previous coin, while if the coin at its current position is T, it flips the coin and moves to the next position.

Consider the maximal sequences of consecutive moves in the same direction. Suppose the machine has a consecutive moves to the next coin, before a move to the previous coin. After those a moves, the a coins flipped in those moves are all heads, as is the coin the machine is now at, so at least the next $a + 1$ moves will all be moves to the previous coin. Similarly, a consecutive moves to the previous coin are followed by at least $a + 1$ consecutive moves to the next coin. There cannot be more than n consecutive moves in the same direction, so this proves that the process terminates (with a move from the first coin to the position before the first coin).

Thus we have a (possibly empty) sequence $a_1 < \cdots < a_t \leqslant n$ giving the lengths of maximal sequences of consecutive moves in the same direction, where the final a_t moves must be moves to the previous coin, ending before the first coin. We claim there is a bijection between initial configurations of the coins and such sequences. This gives

$$E(n) = \frac{1}{2}(1 + 2 + \cdots + n) = \frac{n(n+1)}{4}$$

as required, since each i with $1 \leqslant i \leqslant n$ will appear in half of the sequences, and will contribute i to the number of moves when it does.

To see the bijection, consider following the sequence of moves backwards, starting with the machine just before the first coin and all coins showing tails. This certainly determines a unique configuration of coins that could possibly correspond to the given sequence. Furthermore, every coin flipped as part of the a_j consecutive moves is also flipped as part of all subsequent sequences of a_k consecutive moves, for all $k > j$, meaning that, as we follow the moves backwards, each coin is always in the correct state when flipped to result in a move in the required direction. (Alternatively, since there are 2^n possible configurations of coins and 2^n possible such ascending sequences, the fact that the sequence of moves determines at most one configuration of coins, and thus that there is an injection from configurations of coins to such ascending sequences, is sufficient for it to be a bijection, without needing to show that coins are in the right state as we move backwards.)

Solution 5. We explicitly describe what happens with an arbitrary sequence C of n coins. Suppose that C contain k heads at positions $1 \leqslant c_1 < c_2 < \cdots < c_k \leqslant n$.

Let i be the minimal index such that $c_i \geqslant k$. Then the first few steps will consist of turning over the k^{th}, $(k+1)^{\text{th}}$, …, $c_i{}^{\text{th}}$, $(c_i - 1)^{\text{th}}$, $(c_i - 2)^{\text{th}}$, …, k^{th} coins in this order. After that we get a configuration with $k-1$ heads at the same positions as in the initial one, except for c_i. This part of the process takes $2(c_i - k) + 1$ steps.

After that, the process acts similarly; by induction on the number of heads we deduce that the process ends. Moreover, if the c_i disappear in order $c_{i_1}, \ldots, c_{i_k}$, the whole process takes

$$\ell(C) = \sum_{j=1}^{k} \bigl(2(c_{i_j} - (k+1-j)) + 1\bigr) = 2\sum_{j=1}^{k} c_j - 2\sum_{j=1}^{k}(k+1-j) + k = 2\sum_{j=1}^{k} c_j - k^2$$

steps.

Now let us find the total value S_k of $\ell(C)$ over all $\binom{n}{k}$ configurations with exactly k heads. To sum up the above expression over those, notice that each number $1 \leqslant i \leqslant n$ appears as c_j exactly $\binom{n-1}{k-1}$ times. Thus

$$\begin{aligned} S_k &= 2\binom{n-1}{k-1}\sum_{i=1}^{n} i - \binom{n}{k}k^2 = 2\frac{(n-1)\cdots(n-k+1)}{(k-1)!} \cdot \frac{n(n+1)}{2} - \frac{n\cdots(n-k+1)}{k!}k^2 \\ &= \frac{n(n-1)\cdots(n-k+1)}{(k-1)!}\bigl((n+1)-k\bigr) = n(n-1)\binom{n-2}{k-1} + n\binom{n-1}{k-1}. \end{aligned}$$

Therefore, the total value of $\ell(C)$ over all configurations is

$$\sum_{k=1}^{n} S_k = n(n-1)\sum_{k=1}^{n}\binom{n-2}{k-1} + n\sum_{k=1}^{n}\binom{n-1}{k-1} = n(n-1)2^{n-2} + n2^{n-1} = 2^n\frac{n(n+1)}{4}.$$

Hence the required average is $E(n) = \frac{n(n+1)}{4}$.

C4. On a flat plane in Camelot, King Arthur builds a labyrinth $\mathfrak{L}$ consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.

After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

Let $k(\mathfrak{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth $\mathfrak{L}$, Morgana can always place at least k knights such that no two of them can ever meet. For each n, what are all possible values for $k(\mathfrak{L})$, where $\mathfrak{L}$ is a labyrinth with n walls?

(Canada)

Answer: The only possible value of k is $k = n + 1$, no matter what shape the labyrinth is.

Solution 1. First we show by induction that the n walls divide the plane into $\binom{n+1}{2}+1$ regions. The claim is true for $n = 0$ as, when there are no walls, the plane forms a single region. When placing the n^{th} wall, it intersects each of the $n-1$ other walls exactly once and hence splits each of n of the regions formed by those other walls into two regions. By the induction hypothesis, this yields $\left(\binom{n}{2} + 1\right) + n = \binom{n+1}{2} + 1$ regions, proving the claim.

Now let G be the graph with vertices given by the $\binom{n+1}{2} + 1$ regions, and with two regions connected by an edge if there is a door between them.

We now show that no matter how Merlin paints the n walls, Morgana can place at least $n + 1$ knights. No matter how the walls are painted, there are exactly $\binom{n}{2}$ intersection points, each of which corresponds to a single edge in G. Consider adding the edges of G sequentially and note that each edge reduces the number of connected components by at most one. Therefore the number of connected components of G is at least $\binom{n+1}{2}+1-\binom{n}{2} = n+1$. If Morgana places a knight in regions corresponding to different connected components of G, then no two knights can ever meet.

Now we give a construction showing that, no matter what shape the labyrinth is, Merlin can colour it such that there are exactly $n + 1$ connected components, allowing Morgana to place at most $n + 1$ knights.

First, we choose a coordinate system on the labyrinth so that none of the walls run due north-south, or due east-west. We then have Merlin paint the west face of each wall red, and the east face of each wall blue. We label the regions according to how many walls the region is on the east side of: the labels are integers between 0 and n.

We claim that, for each i, the regions labelled i are connected by doors. First, we note that for each i with $0 \leqslant i \leqslant n$ there is a unique region labelled i which is unbounded to the north.

Now, consider a knight placed in some region with label i, and ask them to walk north (moving east or west by following the walls on the northern sides of regions, as needed). This knight will never get stuck: each region is convex, and so, if it is bounded to the north, it has a single northernmost vertex with a door northwards to another region with label i.

Eventually it will reach a region which is unbounded to the north, which will be the unique such region with label i. Hence every region with label i is connected to this particular region, and so all regions with label i are connected to each other.

As a result, there are exactly $n + 1$ connected components, and Morgana can place at most $n + 1$ knights.

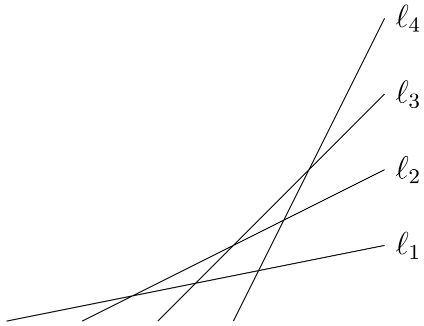
Comment. Variations on this argument exist: some of them capture more information, and some of them capture less information, about the connected components according to this system of numbering.

For example, it can be shown that the unbounded regions are numbered $0, 1, \ldots, n-1, n, n-1, \ldots, 1$ as one cycles around them, that the regions labelled 0 and n are the only regions in their connected components, and that each other connected component forms a single chain running between the two unbounded ones. It is also possible to argue that the regions are acyclic without revealing much about their structure.

Solution 2. We give another description of a strategy for Merlin to paint the walls so that Morgana can place no more than $n+1$ knights.

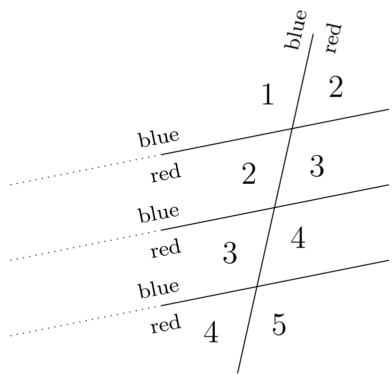
Merlin starts by building a labyrinth of n walls of his own design. He places walls in turn with increasing positive gradients, placing each so far to the right that all intersection points of previously-placed lines lie to the left of it. He paints each in such a way that blue is on the left and red is on the right.

For example, here is a possible sequence of four such lines $\ell_1, \ell_2, \ell_3, \ell_4$:



We say that a region is "on the right" if it has x-coordinate unbounded above (note that if we only have one wall, then both regions are on the right). We claim inductively that, after placing n lines, there are $n+1$ connected components in the resulting labyrinth, each of which contains exactly one region on the right. This is certainly true after placing 0 lines, as then there is only one region (and hence one connected component) and it is on the right.

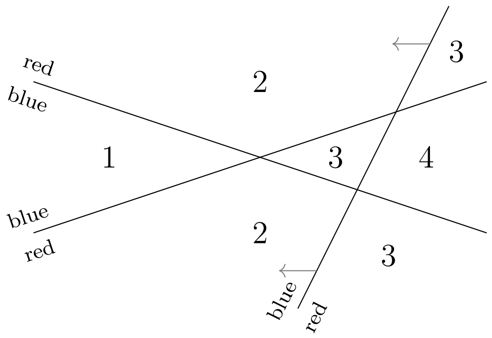
When placing the n^{th} line, it then cuts every one of the $n-1$ previously placed lines, and since it is to the right of all intersection points, the regions it cuts are exactly the n regions on the right.

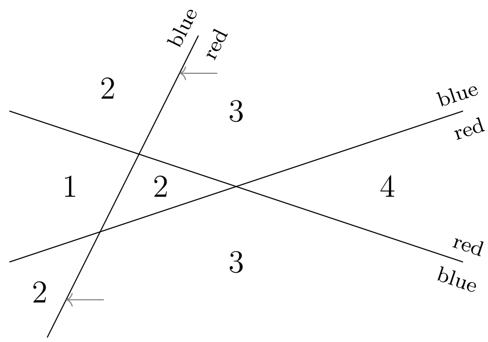


The addition of this line leaves all previous connected components with exactly one region on the right, and creates a new connected component containing exactly one region, and that region is also on the right. As a result, by induction, this particular labyrinth will have $n+1$ connected components.

Having built this labyrinth, Merlin then moves the walls one-by-one (by a sequence of continuous translations and rotations of lines) into the proper position of the given labyrinth, in such a way that no two lines ever become parallel.

The only time the configuration is changed is when one wall is moved through an intersection point of two others:





Note that all moves really do switch between two configurations like this: all sets of three lines have this colour configuration initially, and the rules on rotations mean they are preserved (in particular, we cannot create three lines creating a triangle with three red edges inwards, or three blue edges inwards).

However, as can be seen, such a move preserves the number of connected components, so in the painting this provides for Arthur's actual labyrinth, Morgana can still only place at most $n + 1$ knights.

Comment. While these constructions are superficially distinct, they in fact result in the same colourings for any particular labyrinth. In fact, using the methods of Solution 2, it is possible to show that these are the only colourings that result in exactly $n + 1$ connected components.

C5. On a certain social network, there are 2019 users, some pairs of which are friends, where friendship is a symmetric relation. Initially, there are 1010 people with 1009 friends each and 1009 people with 1010 friends each. However, the friendships are rather unstable, so events of the following kind may happen repeatedly, one at a time:

> Let A, B, and C be people such that A is friends with both B and C, but B and C are not friends; then B and C become friends, but A is no longer friends with them.

Prove that, regardless of the initial friendships, there exists a sequence of such events after which each user is friends with at most one other user.

(Croatia)

Common remarks. The problem has an obvious rephrasing in terms of graph theory. One is given a graph G with 2019 vertices, 1010 of which have degree 1009 and 1009 of which have degree 1010. One is allowed to perform operations on G of the following kind:

> Suppose that vertex A is adjacent to two distinct vertices B and C which are not adjacent to each other. Then one may remove the edges AB and AC from G and add the edge BC into G.

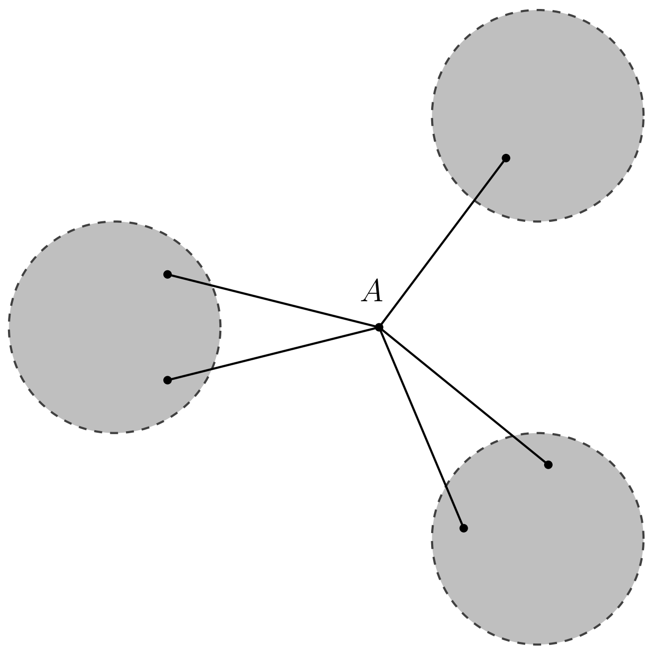
Call such an operation a *refriending*. One wants to prove that, via a sequence of such refriendings, one can reach a graph which is a disjoint union of single edges and vertices.

All of the solutions presented below will use this reformulation.

Solution 1. Note that the given graph is connected, since the total degree of any two vertices is at least 2018 and hence they are either adjacent or have at least one neighbour in common. Hence the given graph satisfies the following condition:

> Every connected component of G with at least three vertices is not complete and has a vertex of odd degree. (1)

We will show that if a graph G satisfies condition (1) and has a vertex of degree at least 2, then there is a refriending on G that preserves condition (1). Since refriendings decrease the total number of edges of G, by using a sequence of such refriendings, we must reach a graph G with maximal degree at most 1, so we are done.



Pick a vertex A of degree at least 2 in a connected component G' of G. Since no component of G with at least three vertices is complete we may assume that not all of the neighbours of A are adjacent to one another. (For example, pick a maximal complete subgraph K of G'. Some vertex A of K has a neighbour outside K, and this neighbour is not adjacent to every vertex of K by maximality.) Removing A from G splits G' into smaller connected components $G_1, \ldots, G_k$ (possibly with $k = 1$), to each of which A is connected by at least one edge. We divide into several cases.

Case 1: $k \geqslant 2$ and A is connected to some G_i by at least two edges.

Choose a vertex B of G_i adjacent to A, and a vertex C in another component G_j adjacent to A. The vertices B and C are not adjacent, and hence removing edges AB and AC and adding in edge BC does not disconnect G'. It is easy to see that this preserves the condition, since the refriending does not change the parity of the degrees of vertices.

Case 2: $k \geqslant 2$ and A is connected to each G_i by exactly one edge.

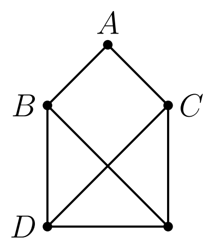
Consider the induced subgraph on any G_i and the vertex A. The vertex A has degree 1 in this subgraph; since the number of odd-degree vertices of a graph is always even, we see that G_i has a vertex of odd degree (in G). Thus if we let B and C be any distinct neighbours of A, then removing edges AB and AC and adding in edge BC preserves the above condition: the refriending creates two new components, and if either of these components has at least three vertices, then it cannot be complete and must contain a vertex of odd degree (since each G_i does).

Case 3: $k = 1$ and A is connected to G_1 by at least three edges.

By assumption, A has two neighbours B and C which are not adjacent to one another. Removing edges AB and AC and adding in edge BC does not disconnect G'. We are then done as in Case 1.

Case 4: $k = 1$ and A is connected to G_1 by exactly two edges.

Let B and C be the two neighbours of A, which are not adjacent. Removing edges AB and AC and adding in edge BC results in two new components: one consisting of a single vertex; and the other containing a vertex of odd degree. We are done unless this second component would be a complete graph on at least 3 vertices. But in this case, G_1 would be a complete graph minus the single edge BC, and hence has at least 4 vertices since G' is not a 4-cycle. If we let D be a third vertex of G_1, then removing edges BA and BD and adding in edge AD does not disconnect G'. We are then done as in Case 1.

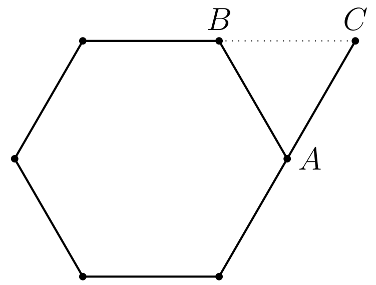


Comment. In fact, condition 1 above *precisely* characterises those graphs which can be reduced to a graph of maximal degree $\leqslant 1$ by a sequence of refriendings.

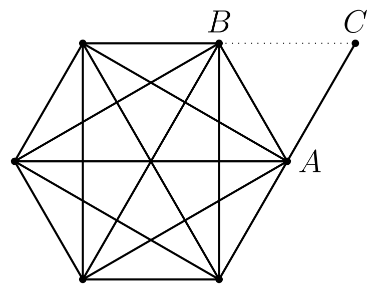
Solution 2. As in the previous solution, note that a refriending preserves the property that a graph has a vertex of odd degree and (trivially) the property that it is not complete; note also that our initial graph is connected. We describe an algorithm to reduce our initial graph to a graph of maximal degree at most 1, proceeding in two steps.

Step 1: There exists a sequence of refriendings reducing the graph to a tree.

Proof. Since the number of edges decreases with each refriending, it suffices to prove the following: as long as the graph contains a cycle, there exists a refriending such that the resulting graph is still connected. We will show that the graph in fact contains a cycle Z and vertices A, B, C such that A and B are adjacent in the cycle Z, C is not in Z, and is adjacent to A but not B. Removing edges AB and AC and adding in edge BC keeps the graph connected, so we are done.



To find this cycle Z and vertices A, B, C, we pursue one of two strategies. If the graph contains a triangle, we consider a largest complete subgraph K, which thus contains at least three vertices. Since the graph itself is not complete, there is a vertex C not in K connected to a vertex A of K. By maximality of K, there is a vertex B of K not connected to C, and hence we are done by choosing a cycle Z in K through the edge AB.



If the graph is triangle-free, we consider instead a smallest cycle Z. This cycle cannot be Hamiltonian (i.e. it cannot pass through every vertex of the graph), since otherwise by minimality the graph would then have no other edges, and hence would have even degree at every vertex. We may thus choose a vertex C not in Z adjacent to a vertex A of Z. Since the graph is triangle-free, it is not adjacent to any neighbour B of A in Z, and we are done. □

Step 2: Any tree may be reduced to a disjoint union of single edges and vertices by a sequence of refriendings.

Proof. The refriending preserves the property of being acyclic. Hence, after applying a sequence of refriendings, we arrive at an acyclic graph in which it is impossible to perform any further refriendings. The maximal degree of any such graph is 1: if it had a vertex A with two neighbours B, C, then B and C would necessarily be nonadjacent since the graph is cycle-free, and so a refriending would be possible. Thus we reach a graph with maximal degree at most 1 as desired. □

C6. Let $n > 1$ be an integer. Suppose we are given $2n$ points in a plane such that no three of them are collinear. The points are to be labelled $A_1, A_2, \ldots, A_{2n}$ in some order. We then consider the $2n$ angles $\angle A_1A_2A_3, \angle A_2A_3A_4, \ldots, \angle A_{2n-2}A_{2n-1}A_{2n}, \angle A_{2n-1}A_{2n}A_1, \angle A_{2n}A_1A_2$. We measure each angle in the way that gives the smallest positive value (i.e. between 0° and 180°). Prove that there exists an ordering of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

(USA)

Comment. The first three solutions all use the same construction involving a line separating the points into groups of n points each, but give different proofs that this construction works. Although Solution 1 is very short, the Problem Selection Committee does not believe any of the solutions is easy to find and thus rates this as a problem of medium difficulty.

Solution 1. Let ℓ be a line separating the points into two groups (L and R) with n points in each. Label the points $A_1, A_2, \ldots, A_{2n}$ so that $L = \{A_1, A_3, \ldots, A_{2n-1}\}$. We claim that this labelling works.

Take a line $s = A_{2n}A_1$.

(a) Rotate s around A_1 until it passes through A_2; the rotation is performed in a direction such that s is never parallel to ℓ.

(b) Then rotate the new s around A_2 until it passes through A_3 in a similar manner.

(c) Perform $2n - 2$ more such steps, after which s returns to its initial position.

The total (directed) rotation angle Θ of s is clearly a multiple of 180°. On the other hand, s was never parallel to ℓ, which is possible only if $\Theta = 0$. Now it remains to partition all the $2n$ angles into those where s is rotated anticlockwise, and the others.

Solution 2. When tracing a cyclic path through the A_i in order, with straight line segments between consecutive points, let θ_i be the exterior angle at A_i, with a sign convention that it is positive if the path turns left and negative if the path turns right. Then $\sum_{i=1}^{2n} \theta_i = 360k^\circ$ for some integer k. Let $\phi_i = \angle A_{i-1}A_iA_{i+1}$ (indices mod $2n$), defined as in the problem; thus $\phi_i = 180^\circ - |\theta_i|$.

Let L be the set of i for which the path turns left at A_i and let R be the set for which it turns right. Then $S = \sum_{i\in L} \phi_i - \sum_{i\in R} \phi_i = (180(|L| - |R|) - 360k)^\circ$, which is a multiple of 360° since the number of points is even. We will show that the points can be labelled such that $S = 0$, in which case L and R satisfy the required condition of the problem.

Note that the value of S is defined for a slightly larger class of configurations: it is OK for two points to coincide, as long as they are not consecutive, and OK for three points to be collinear, as long as A_i, A_{i+1} and A_{i+2} do not appear on a line *in that order*. In what follows it will be convenient, although not strictly necessary, to consider such configurations.

Consider how S changes if a single one of the A_i is moved along some straight-line path (not passing through any A_j and not lying on any line A_jA_k, but possibly crossing such lines). Because S is a multiple of 360°, and the angles change continuously, S can only change when a point moves between R and L. Furthermore, if $\phi_j = 0$ when A_j moves between R and L, S is unchanged; it only changes if $\phi_j = 180^\circ$ when A_j moves between those sets.

For any starting choice of points, we will now construct a new configuration, with labels such that $S = 0$, that can be perturbed into the original one without any ϕ_i passing through 180°, so that $S = 0$ for the original configuration with those labels as well.

Take some line such that there are n points on each side of that line. The new configuration has n copies of a single point on each side of the line, and a path that alternates between sides of the line; all angles are 0, so this configuration has $S = 0$. Perturbing the points into their original positions, while keeping each point on its side of the line, no angle ϕ_i can pass through $180°$, because no straight line can go from one side of the line to the other and back. So the perturbation process leaves $S = 0$.

Comment. More complicated variants of this solution are also possible; for example, a path defined using four quadrants of the plane rather than just two half-planes.

Solution 3. First, let ℓ be a line in the plane such that there are n points on one side and the other n points on the other side. For convenience, assume ℓ is horizontal (otherwise, we can rotate the plane). Then we can use the terms "above", "below", "left" and "right" in the usual way. We denote the n points above the line in an arbitrary order as $P_1, P_2, \ldots, P_n$, and the n points below the line as $Q_1, Q_2, \ldots, Q_n$.

If we connect P_i and Q_j with a line segment, the line segment will intersect with the line ℓ. Denote the intersection as I_{ij}. If P_i is connected to Q_j and Q_k, where $j < k$, then I_{ij} and I_{ik} are two different points, because P_i, Q_j and Q_k are not collinear.

Now we define a "sign" for each angle $\angle Q_j P_i Q_k$. Assume $j < k$. We specify that the sign is positive for the following two cases:

- if i is odd and I_{ij} is to the left of I_{ik},
- if i is even and I_{ij} is to the right of I_{ik}.

Otherwise the sign of the angle is negative. If $j > k$, then the sign of $\angle Q_j P_i Q_k$ is taken to be the same as for $\angle Q_k P_i Q_j$.

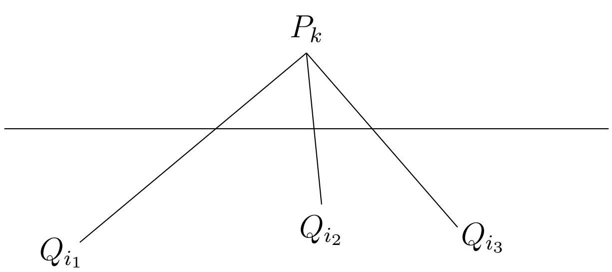
Similarly, we can define the sign of $\angle P_j Q_i P_k$ with $j < k$ (or equivalently $\angle P_k Q_i P_j$). For example, it is positive when i is odd and I_{ji} is to the left of I_{ki}.

Henceforth, whenever we use the notation $\angle Q_j P_i Q_k$ or $\angle P_j Q_i P_k$ for a numerical quantity, it is understood to denote either the (geometric) measure of the angle or the negative of this measure, depending on the sign as specified above.

We now have the following important fact for signed angle measures:

$$\angle Q_{i_1} P_k Q_{i_3} = \angle Q_{i_1} P_k Q_{i_2} + \angle Q_{i_2} P_k Q_{i_3} \tag{1}$$

for all points P_k, Q_{i_1}, Q_{i_2} and Q_{i_3} with $i_1 < i_2 < i_3$. The following figure shows a "natural" arrangement of the points. Equation (1) still holds for any other arrangement, as can be easily verified.



Similarly, we have

$$\angle P_{i_1} Q_k P_{i_3} = \angle P_{i_1} Q_k P_{i_2} + \angle P_{i_2} Q_k P_{i_3}, \tag{2}$$

for all points Q_k, P_{i_1}, P_{i_2} and P_{i_3}, with $i_1 < i_2 < i_3$.

We are now ready to specify the desired ordering $A_1, \ldots, A_{2n}$ of the points:

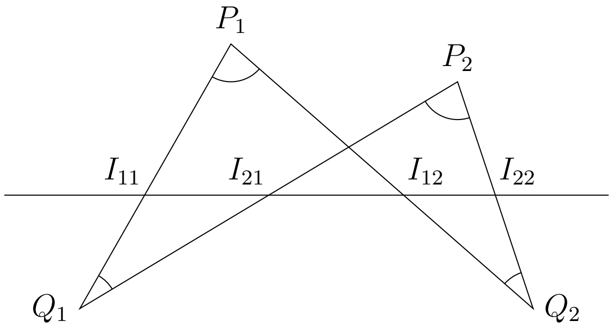
- if $i \leqslant n$ is odd, put $A_i = P_i$ and $A_{2n+1-i} = Q_i$;
- if $i \leqslant n$ is even, put $A_i = Q_i$ and $A_{2n+1-i} = P_i$.

For example, for $n = 3$ this ordering is $P_1, Q_2, P_3, Q_3, P_2, Q_1$. This sequence alternates between P's and Q's, so the above conventions specify a sign for each of the angles $A_{i-1}A_iA_{i+1}$. We claim that the sum of these $2n$ signed angles equals 0. If we can show this, it would complete the proof.

We prove the claim by induction. For brevity, we use the notation $\angle P_i$ to denote whichever of the $2n$ angles has its vertex at P_i, and $\angle Q_i$ similarly.

First let $n = 2$. If the four points can be arranged to form a convex quadrilateral, then the four line segments P_1Q_1, P_1Q_2, P_2Q_1 and P_2Q_2 constitute a self-intersecting quadrilateral. We use several figures to illustrate the possible cases.

The following figure is one possible arrangement of the points.



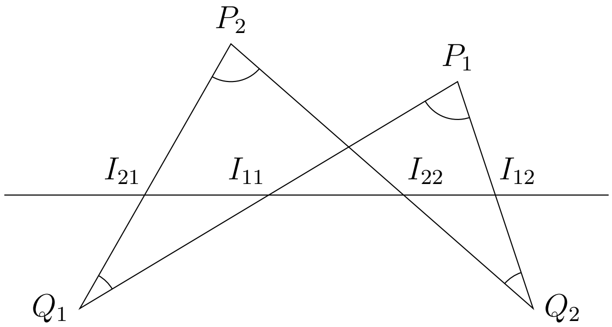
Then $\angle P_1$ and $\angle Q_1$ are positive, $\angle P_2$ and $\angle Q_2$ are negative, and we have

$$|\angle P_1| + |\angle Q_1| = |\angle P_2| + |\angle Q_2|.$$

With signed measures, we have

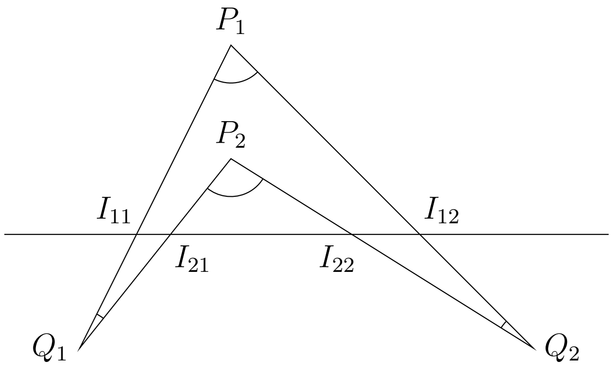
$$\angle P_1 + \angle Q_1 + \angle P_2 + \angle Q_2 = 0. \tag{3}$$

If we switch the labels of P_1 and P_2, we have the following picture:



Switching labels P_1 and P_2 has the effect of flipping the sign of all four angles (as well as swapping the magnitudes on the relabelled points); that is, the new values of $(\angle P_1, \angle P_2, \angle Q_1, \angle Q_2)$ equal the old values of $(-\angle P_2, -\angle P_1, -\angle Q_1, -\angle Q_2)$. Consequently, equation (3) still holds. Similarly, when switching the labels of Q_1 and Q_2, or both the P's and the Q's, equation (3) still holds.

The remaining subcase of $n = 2$ is that one point lies inside the triangle formed by the other three. We have the following picture.



We have

$$|\angle P_1| + |\angle Q_1| + |\angle Q_2| = |\angle P_2|.$$

and equation (3) holds.

Again, switching the labels for P's or the Q's will not affect the validity of equation (3). Also, if the point lying inside the triangle of the other three is one of the Q's rather than the P's, the result still holds, since our sign convention is preserved when we relabel Q's as P's and vice-versa and reflect across ℓ.

We have completed the proof of the claim for $n = 2$.

Assume the claim holds for $n = k$, and we wish to prove it for $n = k + 1$. Suppose we are given our $2(k + 1)$ points. First ignore P_{k+1} and Q_{k+1}, and form $2k$ angles from $P_1, \ldots, P_k$, $Q_1, \ldots, Q_k$ as in the $n = k$ case. By the induction hypothesis we have

$$\sum_{i=1}^{k} (\angle P_i + \angle Q_i) = 0.$$

When we add in the two points P_{k+1} and Q_{k+1}, this changes our angles as follows:

- the angle at P_k changes from $\angle Q_{k-1}P_kQ_k$ to $\angle Q_{k-1}P_kQ_{k+1}$;
- the angle at Q_k changes from $\angle P_{k-1}Q_kP_k$ to $\angle P_{k-1}Q_kP_{k+1}$;
- two new angles $\angle Q_kP_{k+1}Q_{k+1}$ and $\angle P_kQ_{k+1}P_{k+1}$ are added.

We need to prove the changes have no impact on the total sum. In other words, we need to prove

$$(\angle Q_{k-1}P_kQ_{k+1} - \angle Q_{k-1}P_kQ_k) + (\angle P_{k-1}Q_kP_{k+1} - \angle P_{k-1}Q_kP_k) + (\angle P_{k+1} + \angle Q_{k+1}) = 0. \quad (4)$$

In fact, from equations (1) and (2), we have

$$\angle Q_{k-1}P_kQ_{k+1} - \angle Q_{k-1}P_kQ_k = \angle Q_kP_kQ_{k+1},$$

and

$$\angle P_{k-1}Q_kP_{k+1} - \angle P_{k-1}Q_kP_k = \angle P_kQ_kP_{k+1}.$$

Therefore, the left hand side of equation (4) becomes $\angle Q_kP_kQ_{k+1} + \angle P_kQ_kP_{k+1} + \angle Q_kP_{k+1}Q_{k+1} + \angle P_kQ_{k+1}P_{k+1}$, which equals 0, simply by applying the $n = 2$ case of the claim. This completes the induction.

Solution 4. We shall think instead of the problem as asking us to assign a weight ± 1 to each angle, such that the weighted sum of all the angles is zero.

Given an ordering $A_1, \ldots, A_{2n}$ of the points, we shall assign weights according to the following recipe: walk in order from point to point, and assign the left turns $+1$ and the right turns -1. This is the same weighting as in Solution 3, and as in that solution, the weighted sum is a multiple of $360°$.

We now aim to show the following:

Lemma. Transposing any two consecutive points in the ordering changes the weighted sum by $\pm 360°$ or 0.

Knowing that, we can conclude quickly: if the ordering $A_1, \ldots, A_{2n}$ has weighted angle sum $360k°$, then the ordering $A_{2n}, \ldots, A_1$ has weighted angle sum $-360k°$ (since the angles are the same, but left turns and right turns are exchanged). We can reverse the ordering of $A_1, \ldots, A_{2n}$ by a sequence of transpositions of consecutive points, and in doing so the weighted angle sum must become zero somewhere along the way.

We now prove that lemma:

Proof. Transposing two points amounts to taking a section $A_kA_{k+1}A_{k+2}A_{k+3}$ as depicted, reversing the central line segment $A_{k+1}A_{k+2}$, and replacing its two neighbours with the dotted lines.

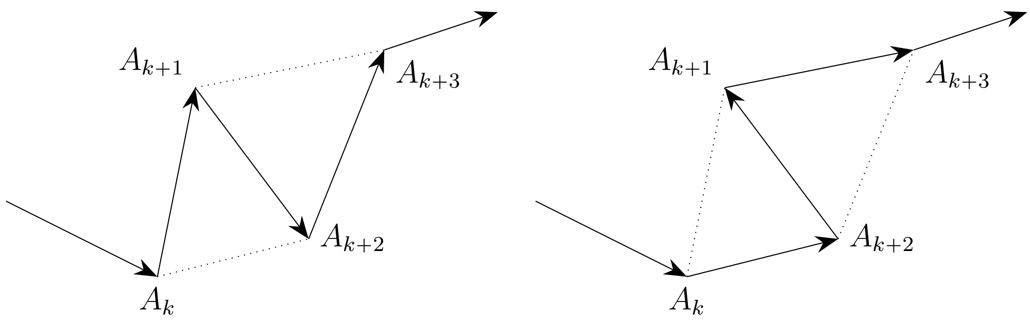


Figure 1: Transposing two consecutive vertices: before (left) and afterwards (right)

In each triangle, we alter the sum by $\pm 180°$. Indeed, using (anticlockwise) directed angles modulo $360°$, we either add or subtract all three angles of each triangle.

Hence both triangles together alter the sum by $\pm 180 \pm 180°$, which is $\pm 360°$ or 0. $\square$

C7. There are 60 empty boxes $B_1, \ldots, B_{60}$ in a row on a table and an unlimited supply of pebbles. Given a positive integer n, Alice and Bob play the following game.

In the first round, Alice takes n pebbles and distributes them into the 60 boxes as she wishes. Each subsequent round consists of two steps:

(a) Bob chooses an integer k with $1 \leqslant k \leqslant 59$ and splits the boxes into the two groups $B_1, \ldots, B_k$ and $B_{k+1}, \ldots, B_{60}$.

(b) Alice picks one of these two groups, adds one pebble to each box in that group, and removes one pebble from each box in the other group.

Bob wins if, at the end of any round, some box contains no pebbles. Find the smallest n such that Alice can prevent Bob from winning.

(Czech Republic)

Answer: $n = 960$. In general, if there are $N > 1$ boxes, the answer is $n = \left\lfloor \frac{N}{2} + 1 \right\rfloor \left\lceil \frac{N}{2} + 1 \right\rceil - 1$.

Common remarks. We present solutions for the general case of $N > 1$ boxes, and write $M = \left\lfloor \frac{N}{2} + 1 \right\rfloor \left\lceil \frac{N}{2} + 1 \right\rceil - 1$ for the claimed answer. For $1 \leqslant k < N$, say that Bob makes a *k-move* if he splits the boxes into a left group $\{B_1, \ldots, B_k\}$ and a right group $\{B_{k+1}, \ldots, B_N\}$. Say that one configuration *dominates* another if it has at least as many pebbles in each box, and say that it *strictly dominates* the other configuration if it also has more pebbles in at least one box. (Thus, if Bob wins in some configuration, he also wins in every configuration that it dominates.)

It is often convenient to consider 'V-shaped' configurations; for $1 \leqslant i \leqslant N$, let V_i be the configuration where B_j contains $1 + |j - i|$ pebbles (i.e. where the i^{th} box has a single pebble and the numbers increase by one in both directions, so the first box has i pebbles and the last box has $N + 1 - i$ pebbles). Note that V_i contains $\frac{1}{2}i(i+1) + \frac{1}{2}(N+1-i)(N+2-i) - 1$ pebbles. If $i = \left\lceil \frac{N}{2} \right\rceil$, this number equals M.

Solutions split naturally into a strategy for Alice (starting with M pebbles and showing she can prevent Bob from winning) and a strategy for Bob (showing he can win for any starting configuration with at most $M - 1$ pebbles). The following observation is also useful to simplify the analysis of strategies for Bob.

Observation A. Consider two consecutive rounds. Suppose that in the first round Bob made a k-move and Alice picked the left group, and then in the second round Bob makes an ℓ-move, with $\ell > k$. We may then assume, without loss of generality, that Alice again picks the left group.

Proof. Suppose Alice picks the right group in the second round. Then the combined effect of the two rounds is that each of the boxes $B_{k+1}, \ldots, B_\ell$ lost two pebbles (and the other boxes are unchanged). Hence this configuration is strictly dominated by that before the first round, and it suffices to consider only Alice's other response. $\square$

Solution 1 (Alice). Alice initially distributes pebbles according to $V_{\left\lceil \frac{N}{2} \right\rceil}$. Suppose the current configuration of pebbles dominates V_i. If Bob makes a k-move with $k \geqslant i$ then Alice picks the left group, which results in a configuration that dominates V_{i+1}. Likewise, if Bob makes a k-move with $k < i$ then Alice picks the right group, which results in a configuration that dominates V_{i-1}. Since none of $V_1, \ldots, V_N$ contains an empty box, Alice can prevent Bob from ever winning.

Solution 1 (Bob). The key idea in this solution is the following claim.

Claim. If there exist a positive integer k such that there are at least $2k$ boxes that have at most k pebbles each then Bob can force a win.

Proof. We ignore the other boxes. First, Bob makes a k-move (splits the $2k$ boxes into two groups of k boxes each). Without loss of generality, Alice picks the left group. Then Bob makes a $(k+1)$-move, ..., a $(2k-1)$-move. By Observation A, we may suppose Alice always picks the left group. After Bob's $(2k-1)$-move, the rightmost box becomes empty and Bob wins. $\square$

Now, we claim that if $n < M$ then either there already exists an empty box, or there exist a positive integer k and $2k$ boxes with at most k pebbles each (and thus Bob can force a win). Otherwise, assume each box contains at least 1 pebble, and for each $1 \leqslant k \leqslant \lfloor \frac{N}{2} \rfloor$, at least $N-(2k-1) = N+1-2k$ boxes contain at least $k+1$ pebbles. Summing, there are at least as many pebbles in total as in $V_{\lceil \frac{N}{2} \rceil}$; that is, at least M pebbles, as desired.

Solution 2 (Alice). Let $K = \lfloor \frac{N}{2} + 1 \rfloor$. Alice starts with the boxes in the configuration V_K. For each of Bob's $N-1$ possible choices, consider the subset of rounds in which he makes that choice. In that subset of rounds, Alice alternates between picking the left group and picking the right group; the first time Bob makes that choice, Alice picks the group containing the K^{th} box. Thus, at any time during the game, the number of pebbles in each box depends only on which choices Bob has made an odd number of times. This means that the number of pebbles in a box could decrease by at most the number of choices for which Alice would have started by removing a pebble from the group containing that box. These numbers are, for each box,

$$\left\lfloor \tfrac{N}{2} \right\rfloor, \left\lfloor \tfrac{N}{2} - 1 \right\rfloor, \ldots, 1, 0, 1, \ldots, \left\lceil \tfrac{N}{2} - 1 \right\rceil.$$

These are pointwise less than the numbers of pebbles the boxes started with, meaning that no box ever becomes empty with this strategy.

Solution 2 (Bob). Let $K = \lfloor \frac{N}{2} + 1 \rfloor$. For Bob's strategy, we consider a configuration X with at most $M-1$ pebbles, and we make use of Observation A. Consider two configurations with M pebbles: V_K and V_{N+1-K} (if n is odd, they are the same configuration; if n is even, one is the reverse of the other). The configuration X has fewer pebbles than V_K in at least one box, and fewer pebbles than V_{N+1-K} in at least one box.

Suppose first that, with respect to one of those configurations (without loss of generality V_K), X has fewer pebbles in one of the boxes in the half where they have $1, 2, \ldots, \lceil \frac{N}{2} \rceil$ pebbles (the right half in V_K if N is even; if N is odd, we can take it to be the right half, without loss of generality, as the configuration is symmetric). Note that the number cannot be fewer in the box with 1 pebble in V_K, because then it would have 0 pebbles. Bob then does a K-move. If Alice picks the right group, the total number of pebbles goes down and we restart Bob's strategy with a smaller number of pebbles. If Alice picks the left group, Bob follows with a $(K+1)$-move, a $(K+2)$-move, and so on; by Observation A we may assume Alice always picks the left group. But whichever box in the right half had fewer pebbles in X than in V_K ends up with 0 pebbles at some point in this sequence of moves.

Otherwise, N is even, and for both of those configurations, there are fewer pebbles in X only on the $2, 3, \ldots, \frac{N}{2}+1$ side. That is, the numbers of pebbles in X are at least

$$\frac{N}{2}, \frac{N}{2}-1, \ldots, 1, 1, \ldots, \frac{N}{2} \tag{C}$$

with equality occurring at least once on each side. Bob does an $\frac{N}{2}$-move. Whichever group Alice chooses, the total number of pebbles is unchanged, and the side from which pebbles are removed now has a box with fewer pebbles than in (C), so the previous case of Bob's strategy can now be applied.

Solution 3 (Bob). For any configuration C, define $L(C)$ to be the greatest integer such that, for all $0 \leqslant i \leqslant N-1$, the box B_{i+1} contains at least $L(C)-i$ pebbles. Similarly, define $R(C)$ to be greatest integer such that, for all $0 \leqslant i \leqslant N-1$, the box B_{N-i} contains at least $R(C)-i$ pebbles. (Thus, C dominates the 'left half' of $V_{L(C)}$ and the 'right half' of $V_{N+1-R(C)}$.) Then C dominates a 'V-shaped' configuration if and only if $L(C)+R(C) \geqslant N+1$. Note that if C dominates a V-shaped configuration, it has at least M pebbles.

Now suppose that there are fewer than M pebbles, so we have $L(C)+R(C) \leqslant N$. Then Bob makes an $L(C)$-move (or more generally any move with at least $L(C)$ boxes on the left and $R(C)$ boxes on the right). Let C' be the new configuration, and suppose that no box becomes empty (otherwise Bob has won). If Alice picks the left group, we have $L(C')=L(C)+1$ and $R(C')=R(C)-1$. Otherwise, we have $L(C')=L(C)-1$ and $R(C')=R(C)+1$. In either case, we have $L(C')+R(C') \leqslant N$.

Bob then repeats this strategy, until one of the boxes becomes empty. Since the condition in Observation A holds, we may assume that Alice picks a group on the same side each time. Then one of L and R is strictly decreasing; without loss of generality assume that L strictly decreases. At some point we reach $L=1$. If B_2 is still nonempty, then B_1 must contain a single pebble. Bob makes a 1-move, and by Observation A, Alice must (eventually) pick the right group, making this box empty.

C8. Alice has a map of Wonderland, a country consisting of $n \geqslant 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

Comment. This problem could be posed with an explicit statement about points being awarded for weaker bounds cn for some $c > 4$, in the style of IMO 2014 Problem 6.

(Thailand)

Solution. We will show Alice needs to ask at most $4n - 7$ questions. Her strategy has the following phases. In what follows, S is the set of towns that Alice, so far, does not know to have more than one outgoing road (so initially $|S| = n$).

Phase 1. Alice chooses any two towns, say A and B. Without loss of generality, suppose that the King of Hearts' answer is that the road goes from A to B.

At the end of this phase, Alice has asked 1 question.

Phase 2. During this phase there is a single (variable) town T that is known to have at least one incoming road but not yet known to have any outgoing roads. Initially, T is B. Alice does the following $n - 2$ times: she picks a town X she has not asked about before, and asks the direction of the road between T and X. If it is from X to T, T is unchanged; if it is from T to X, X becomes the new choice of town T, as the previous T is now known to have an outgoing road.

At the end of this phase, Alice has asked a total of $n - 1$ questions. The final town T is not yet known to have any outgoing roads, while every other town has exactly one outgoing road known. The undirected graph of roads whose directions are known is a tree.

Phase 3. During this phase, Alice asks about the directions of all roads between T and another town she has not previously asked about, stopping if she finds two outgoing roads from T. This phase involves at most $n - 2$ questions. If she does not find two outgoing roads from T, she has answered her original question with at most $2n - 3 \leqslant 4n - 7$ questions, so in what follows we suppose that she does find two outgoing roads, asking a total of k questions in this phase, where $2 \leqslant k \leqslant n - 2$ (and thus $n \geqslant 4$ for what follows).

For every question where the road goes towards T, the town at the other end is removed from S (as it already had one outgoing road known), while the last question resulted in T being removed from S. So at the end of this phase, $|S| = n - k + 1$, while a total of $n + k - 1$ questions have been asked. Furthermore, the undirected graph of roads within S whose directions are known contains no cycles (as T is no longer a member of S, all questions asked in this phase involved T and the graph was a tree before this phase started). Every town in S has exactly one outgoing road known (not necessarily to another town in S).

Phase 4. During this phase, Alice repeatedly picks any pair of towns in S for which she does not know the direction of the road between them. Because every town in S has exactly one outgoing road known, this always results in the removal of one of those two towns from S. Because there are no cycles in the graph of roads of known direction within S, this can continue until there are at most 2 towns left in S.

If it ends with t towns left, $n - k + 1 - t$ questions were asked in this phase, so a total of $2n - t$ questions have been asked.

Phase 5. During this phase, Alice asks about all the roads from the remaining towns in S that she has not previously asked about. She has definitely already asked about any road between those towns (if $t = 2$). She must also have asked in one of the first two phases about at least one other road involving one of those towns (as those phases resulted in a tree with $n > 2$ vertices). So she asks at most $t(n-t)-1$ questions in this phase.

At the end of this phase, Alice knows whether any town has at most one outgoing road. If $t = 1$, at most $3n-3 \leqslant 4n-7$ questions were needed in total, while if $t = 2$, at most $4n-7$ questions were needed in total.

Comment 1. The version of this problem originally submitted asked only for an upper bound of $5n$, which is much simpler to prove. The Problem Selection Committee preferred a version with an asymptotically optimal constant. In the following comment, we will show that the constant is optimal.

Comment 2. We will show that Alice *cannot* always find out by asking at most $4n - 3(\log_2 n) - 15$ questions, if $n \geqslant 8$.

To show this, we suppose the King of Hearts is choosing the directions as he goes along, only picking the direction of a road when Alice asks about it for the first time. We provide a strategy for the King of Hearts that ensures that, after the given number of questions, the map is still consistent both with the existence of a town with at most one outgoing road, and with the nonexistence of such a town. His strategy has the following phases. When describing how the King of Hearts' answer to a question is determined below, we always assume he is being asked about a road for the first time (otherwise, he just repeats his previous answer for that road). This strategy is described throughout in graph-theoretic terms (vertices and edges rather than towns and roads).

Phase 1. In this phase, we consider the undirected graph formed by edges whose directions are known. The phase terminates when there are exactly 8 connected components whose undirected graphs are trees. The following invariant is maintained: in a component with k vertices whose undirected graph is a tree, every vertex has at most $\lfloor \log_2 k \rfloor$ edges into it.

- If the King of Hearts is asked about an edge between two vertices in the same component, or about an edge between two components at least one of which is not a tree, he chooses any direction for that edge arbitrarily.
- If he is asked about an edge between a vertex in component A that has a vertices and is a tree and a vertex in component B that has b vertices and is a tree, suppose without loss of generality that $a \geqslant b$. He then chooses the edge to go from A to B. In this case, the new number of edges into any vertex is at most $\max\{\lfloor \log_2 a \rfloor, \lfloor \log_2 b \rfloor + 1\} \leqslant \lfloor \log_2(a+b) \rfloor$.

In all cases, the invariant is preserved, and the number of tree components either remains unchanged or goes down by 1. Assuming Alice does not repeat questions, the process must eventually terminate with 8 tree components, and at least $n-8$ questions having been asked.

Note that each tree component contains at least one vertex with no outgoing edges. Colour one such vertex in each tree component red.

Phase 2. Let V_1, V_2 and V_3 be the three of the red vertices whose components are smallest (so their components together have at most $\left\lfloor \frac{3}{8}n \right\rfloor$ vertices, with each component having at most $\left\lfloor \frac{3}{8}n - 2 \right\rfloor$ vertices). Let sets C_1, C_2, $\ldots$ be the connected components after removing the V_j. By construction, there are no edges with known direction between C_i and C_j for $i \neq j$, and there are at least five such components.

If at any point during this phase, the King of Hearts is asked about an edge within one of the C_i, he chooses an arbitrary direction. If he is asked about an edge between C_i and C_j for $i \neq j$, he answers so that all edges go from C_i to C_{i+1} and C_{i+2}, with indices taken modulo the number of components, and chooses arbitrarily for other pairs. This ensures that all vertices other than the V_j will have more than one outgoing edge.

For edges involving one of the V_j he answers as follows, so as to remain consistent for as long as possible with both possibilities for whether one of those vertices has at most one outgoing edge. Note that as they were red vertices, they have no outgoing edges at the start of this phase. For edges between two of the V_j, he answers that the edges go from V_1 to V_2, from V_2 to V_3 and from V_3 to V_1. For edges between V_j and some other vertex, he always answers that the edge goes into V_j, except for the last such edge for which he is asked the question for any given V_j, for which he answers that the edge goes out of V_j. Thus, as long as at least one of the V_j has not had the question answered for all the vertices that are not among the V_j, his answers are still compatible both with all vertices having more than one outgoing edge, and with that V_j having only one outgoing edge.

At the start of this phase, each of the V_j has at most $\lfloor \log_2 \lfloor \frac{3}{8}n - 2 \rfloor \rfloor < (\log_2 n) - 1$ incoming edges. Thus, Alice cannot determine whether some vertex has only one outgoing edge within $3(n - 3 - ((\log_2 n) - 1)) - 1$ questions in this phase; that is, $4n - 3(\log_2 n) - 15$ questions total.

Comment 3. We can also improve the upper bound slightly, to $4n - 2(\log_2 n) + 1$. (We do not know where the precise minimum number of questions lies between $4n - 3(\log_2 n) + O(1)$ and $4n - 2(\log_2 n) + O(1)$.) Suppose $n \geqslant 5$ (otherwise no questions are required at all).

To do this, we replace Phases 1 and 2 of the given solution with a different strategy that also results in a spanning tree where one vertex V is not known to have any outgoing edges, and all other vertices have exactly one outgoing edge known, but where there is more control over the numbers of incoming edges. In Phases 3 and 4 we then take more care about the order in which pairs of towns are chosen, to ensure that each of the remaining towns has already had a question asked about at least $\log_2 n + O(1)$ edges.

Define trees T_m with 2^m vertices, exactly one of which (the *root*) has no outgoing edges and the rest of which have exactly one outgoing edge, as follows: T_0 is a single vertex, while T_m is constructed by joining the roots of two copies of T_{m-1} with an edge in either direction. If $n = 2^m$ we can readily ask $n-1$ questions, resulting in a tree T_m for the edges with known direction: first ask about 2^{m-1} disjoint pairs of vertices, then about 2^{m-2} disjoint pairs of the roots of the resulting T_1 trees, and so on. For the general case, where n is not a power of 2, after k stages of this process we have $\lfloor n/2^k \rfloor$ trees, each of which is like T_k but may have some extra vertices (but, however, a unique root). If there are an even number of trees, then ask about pairs of their roots. If there are an odd number (greater than 1) of trees, when a single T_k is left over, ask about its root together with that of one of the T_{k+1} trees.

Say $m = \lfloor \log_2 n \rfloor$. The result of that process is a single T_m tree, possibly with some extra vertices but still a unique root V. That root has at least m incoming edges, and we may list vertices V_0, …, V_{m-1} with edges to V, such that, for all $0 \leqslant i < m$, vertex V_i itself has at least i incoming edges.

Now divide the vertices other than V into two parts: A has all vertices at an odd distance from V and B has all the vertices at an even distance from B. Both A and B are nonempty; A contains the V_i, while B contains a sequence of vertices with at least 0, 1, …, $m - 2$ incoming edges respectively, similar to the V_i. There are no edges with known direction within A or within B.

In Phase 3, then ask about edges between V and other vertices: first those in B, in order of increasing number of incoming edges to the other vertex, then those in A, again in order of increasing number of incoming edges, which involves asking at most $n - 1 - m$ questions in this phase. If two outgoing edges are not found from V, at most $2n - 2 - m \leqslant 4n - 2(\log_2 n) + 1$ questions needed to be asked in total, so we suppose that two outgoing edges were found, with k questions asked in this phase, where $2 \leqslant k \leqslant n - 1 - m$. The state of S is as described in the solution above, with the additional property that, since S must still contain all vertices with edges to V, it contains the vertices V_i described above.

In Phase 4, consider the vertices left in B, in increasing order of number of edges incoming to a vertex. If s is the least number of incoming edges to such a vertex, then, for any $s \leqslant t \leqslant m - 2$, there are at least $m - t - 2$ vertices with more than t incoming edges. Repeatedly asking about the pair of vertices left in B with the least numbers of incoming edges results in a single vertex left over (if any were in B at all at the start of this phase) with at least $m - 2$ incoming edges. Doing the same with A (which must be nonempty) leaves a vertex with at least $m - 1$ incoming edges.

Thus if only A is nonempty we ask at most $n - m$ questions in Phase 5, so in total at most $3n - m - 1$ questions, while if both are nonempty we ask at most $2n - 2m + 1$ questions in Phase 5, so in total at most $4n - 2m - 1 < 4n - 2(\log_2 n) + 1$ questions.

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C9. For any two different real numbers x and y, we define $D(x, y)$ to be the unique integer d satisfying $2^d \leqslant |x - y| < 2^{d+1}$. Given a set of reals $\mathcal{F}$, and an element $x \in \mathcal{F}$, we say that the *scales* of x in $\mathcal{F}$ are the values of $D(x, y)$ for $y \in \mathcal{F}$ with $x \neq y$.

Let k be a given positive integer. Suppose that each member x of $\mathcal{F}$ has at most k different scales in $\mathcal{F}$ (note that these scales may depend on x). What is the maximum possible size of $\mathcal{F}$?

(Italy)

Answer: The maximum possible size of $\mathcal{F}$ is 2^k.

Common remarks. For convenience, we extend the use of the word *scale*: we say that the scale between two reals x and y is $D(x, y)$.

Solution. We first construct a set $\mathcal{F}$ with 2^k members, each member having at most k different scales in $\mathcal{F}$. Take $\mathcal{F} = \{0, 1, 2, \ldots, 2^k - 1\}$. The scale between any two members of $\mathcal{F}$ is in the set $\{0, 1, \ldots, k - 1\}$.

We now show that 2^k is an upper bound on the size of $\mathcal{F}$. For every finite set $\mathcal{S}$ of real numbers, and every real x, let $r_{\mathcal{S}}(x)$ denote the number of different scales of x in $\mathcal{S}$. That is, $r_{\mathcal{S}}(x) = |\{D(x, y) : x \neq y \in \mathcal{S}\}|$. Thus, for every element x of the set $\mathcal{F}$ in the problem statement, we have $r_{\mathcal{F}}(x) \leqslant k$. The condition $|\mathcal{F}| \leqslant 2^k$ is an immediate consequence of the following lemma.

Lemma. Let $\mathcal{S}$ be a finite set of real numbers, and define

$$w(\mathcal{S}) = \sum_{x \in \mathcal{S}} 2^{-r_{\mathcal{S}}(x)}.$$

Then $w(\mathcal{S}) \leqslant 1$.

Proof. Induction on $n = |\mathcal{S}|$. If $\mathcal{S} = \{x\}$, then $r_{\mathcal{S}}(x) = 0$, so $w(\mathcal{S}) = 1$.

Assume now $n \geqslant 2$, and let $x_1 < \cdots < x_n$ list the members of $\mathcal{S}$. Let d be the minimal scale between two distinct elements of $\mathcal{S}$; then there exist neighbours x_t and x_{t+1} with $D(x_t, x_{t+1}) = d$. Notice that for any two indices i and j with $j - i > 1$ we have $D(x_i, x_j) > d$, since

$$|x_i - x_j| = |x_{i+1} - x_i| + |x_j - x_{i+1}| \geqslant 2^d + 2^d = 2^{d+1}.$$

Now choose the minimal $i \leqslant t$ and the maximal $j \geqslant t + 1$ such that $D(x_i, x_{i+1}) = D(x_{i+1}, x_{i+2}) = \cdots = D(x_{j-1}, x_j) = d$.

Let E be the set of all the x_s with even indices $i \leqslant s \leqslant j$, O be the set of those with odd indices $i \leqslant s \leqslant j$, and R be the rest of the elements (so that $\mathcal{S}$ is the disjoint union of E, O and R). Set $\mathcal{S}_O = R \cup O$ and $\mathcal{S}_E = R \cup E$; we have $|\mathcal{S}_O| < |\mathcal{S}|$ and $|\mathcal{S}_E| < |\mathcal{S}|$, so $w(\mathcal{S}_O), w(\mathcal{S}_E) \leqslant 1$ by the inductive hypothesis.

Clearly, $r_{\mathcal{S}_O}(x) \leqslant r_{\mathcal{S}}(x)$ and $r_{\mathcal{S}_E}(x) \leqslant r_{\mathcal{S}}(x)$ for any $x \in R$, and thus

$$\begin{aligned}\sum_{x \in R} 2^{-r_{\mathcal{S}}(x)} &= \frac{1}{2} \sum_{x \in R} \left(2^{-r_{\mathcal{S}}(x)} + 2^{-r_{\mathcal{S}}(x)}\right) \\ &\leqslant \frac{1}{2} \sum_{x \in R} \left(2^{-r_{\mathcal{S}_O}(x)} + 2^{-r_{\mathcal{S}_E}(x)}\right).\end{aligned}$$

On the other hand, for every $x \in O$, there is no $y \in \mathcal{S}_O$ such that $D_{\mathcal{S}_O}(x, y) = d$ (as all candidates from $\mathcal{S}$ were in E). Hence, we have $r_{\mathcal{S}_O}(x) \leqslant r_{\mathcal{S}}(x) - 1$, and thus

$$\sum_{x \in O} 2^{-r_{\mathcal{S}}(x)} \leqslant \frac{1}{2} \sum_{x \in O} 2^{-r_{\mathcal{S}_O}(x)}.$$

Similarly, for every $x \in E$, we have

$$\sum_{x\in E} 2^{-r_S(x)} \leqslant \frac{1}{2}\sum_{x\in E} 2^{-r_{S_E}(x)}.$$

We can then combine these to give

$$\begin{aligned}
w(S) &= \sum_{x\in R} 2^{-r_S(x)} + \sum_{x\in O} 2^{-r_S(x)} + \sum_{x\in E} 2^{-r_S(x)} \\
&\leqslant \frac{1}{2}\sum_{x\in R}\left(2^{-r_{S_O}(x)} + 2^{-r_{S_E}(x)}\right) + \frac{1}{2}\sum_{x\in O} 2^{-r_{S_O}(x)} + \frac{1}{2}\sum_{x\in E} 2^{-r_{S_E}(x)} \\
&= \frac{1}{2}\left(\sum_{x\in \mathcal{S}_O} 2^{-r_{S_O}(x)} + \sum_{x\in \mathcal{S}_E} 2^{-r_{S_E}(x)}\right) \qquad (\text{since } \mathcal{S}_O = O \cup R \text{ and } \mathcal{S}_E = E \cup R) \\
&= \frac{1}{2}\left(w(\mathcal{S}_O) + w(\mathcal{S}_E)\right)) \qquad (\text{by definition of } w(\cdot)) \\
&\leqslant 1 \qquad (\text{by the inductive hypothesis})
\end{aligned}$$

which completes the induction. □

Comment 1. The sets O and E above are not the only ones we could have chosen. Indeed, we could instead have used the following definitions:

Let d be the maximal scale between two distinct elements of $\mathcal{S}$; that is, $d = D(x_1, x_n)$. Let $O = \{x \in \mathcal{S} : D(x, x_n) = d\}$ (a 'left' part of the set) and let $E = \{x \in \mathcal{S} : D(x_1, x) = d\}$ (a 'right' part of the set). Note that these two sets are disjoint, and nonempty (since they contain x_1 and x_n respectively). The rest of the proof is then the same as in Solution 1.

Comment 2. Another possible set $\mathcal{F}$ containing 2^k members could arise from considering a binary tree of height k, allocating a real number to each leaf, and trying to make the scale between the values of two leaves dependent only on the (graph) distance between them. The following construction makes this more precise.

We build up sets $\mathcal{F}_k$ recursively. Let $\mathcal{F}_0 = \{0\}$, and then let $\mathcal{F}_{k+1} = \mathcal{F}_k \cup \{x + 3 \cdot 4^k : x \in \mathcal{F}_k\}$ (i.e. each half of $\mathcal{F}_{k+1}$ is a copy of F_k). We have that $\mathcal{F}_k$ is contained in the interval $[0, 4^{k+1})$, and so it follows by induction on k that every member of F_{k+1} has k different scales in its own half of F_{k+1} (by the inductive hypothesis), and only the single scale $2k+1$ in the other half of F_{k+1}.

Both of the constructions presented here have the property that every member of $\mathcal{F}$ has exactly k different scales in $\mathcal{F}$. Indeed, it can be seen that this must hold (up to a slight perturbation) for any such maximal set. Suppose there were some element x with only $k-1$ different scales in $\mathcal{F}$ (and every other element had at most k different scales). Then we take some positive real ϵ, and construct a new set $\mathcal{F}' = \{y : y \in \mathcal{F}, y \leqslant x\} \cup \{y + \epsilon : y \in \mathcal{F}, y \geqslant x\}$. We have $|\mathcal{F}'| = |\mathcal{F}| + 1$, and if ϵ is sufficiently small then $\mathcal{F}'$ will also satisfy the property that no member has more than k different scales in $\mathcal{F}'$.

This observation might be used to motivate the idea of weighting members of an arbitrary set $\mathcal{S}$ of reals according to how many different scales they have in $\mathcal{S}$.

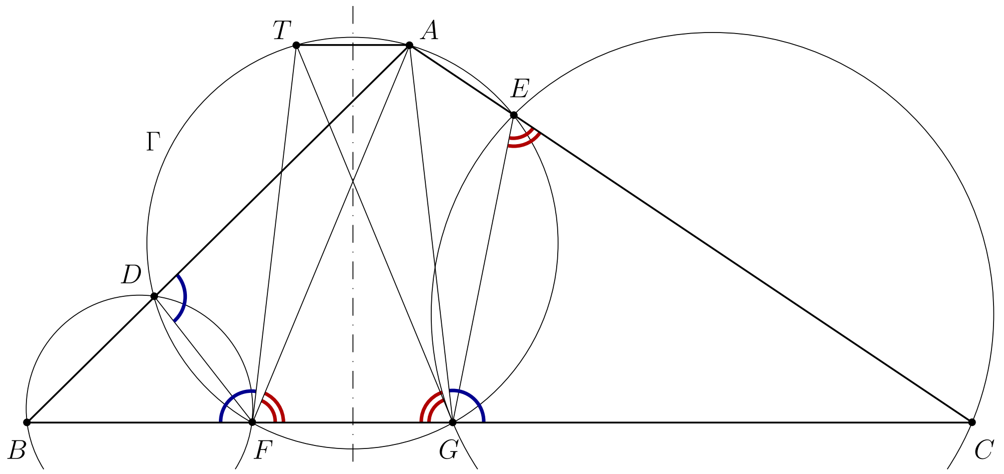
Geometry

G1. Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

(Nigeria)

Solution. Notice that $\angle TFB = \angle FDA$ because FT is tangent to circle BDF, and moreover $\angle FDA = \angle CGA$ because quadrilateral $ADFG$ is cyclic. Similarly, $\angle TGB = \angle GEC$ because GT is tangent to circle CEG, and $\angle GEC = \angle CFA$. Hence,

$$\angle TFB = \angle CGA \quad \text{and} \quad \angle TGB = \angle CFA. \tag{1}$$



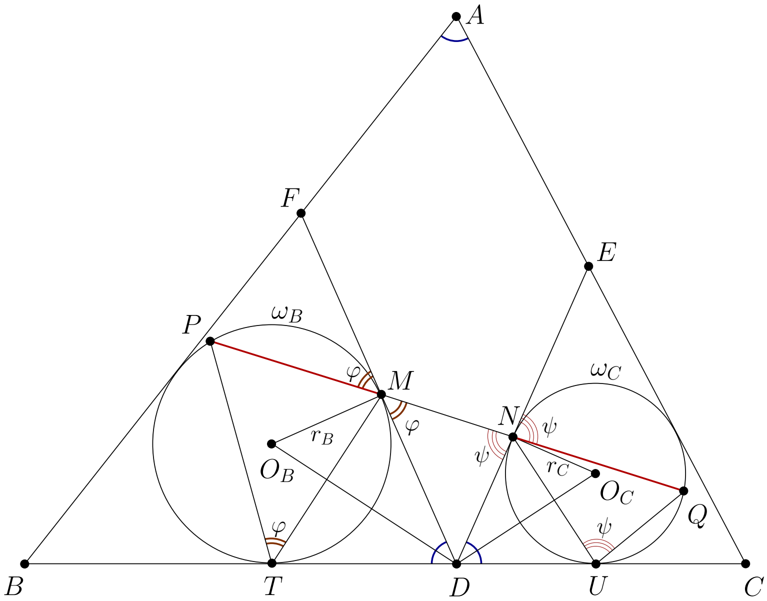
Triangles FGA and GFT have a common side FG, and by (1) their angles at F, G are the same. So, these triangles are congruent. So, their altitudes starting from A and T, respectively, are equal and hence AT is parallel to line $BFGC$.

Comment. Alternatively, we can prove first that T lies on Γ. For example, this can be done by showing that $\angle AFT = \angle AGT$ using (1). Then the statement follows as $\angle TAF = \angle TGF = \angle GFA$.

G2. Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

(Vietnam)

Solution. Denote the centres of ω_B and ω_C by O_B and O_C, let their radii be r_B and r_C, and let BC be tangent to the two circles at T and U, respectively.



From the cyclic quadrilaterals $AFDC$ and $ABDE$ we have

$$\angle MDO_B = \frac{1}{2}\angle FDB = \frac{1}{2}\angle BAC = \frac{1}{2}\angle CDE = \angle O_C DN,$$

so the right-angled triangles DMO_B and DNO_C are similar. The ratio of similarity between the two triangles is

$$\frac{DN}{DM} = \frac{O_C N}{O_B M} = \frac{r_C}{r_B}.$$

Let $\varphi = \angle DMN$ and $\psi = \angle MND$. The lines FM and EN are tangent to ω_B and ω_C, respectively, so

$$\angle MTP = \angle FMP = \angle DMN = \varphi \quad \text{and} \quad \angle QUN = \angle QNE = \angle MND = \psi.$$

(It is possible that P or Q coincides with T or U, or lie inside triangles DMT or DUN, respectively. To reduce case-sensitivity, we may use directed angles or simply ignore angles MTP and QUN.)

In the circles ω_B and ω_C the lengths of chords MP and NQ are

$$MP = 2r_B \cdot \sin\angle MTP = 2r_B \cdot \sin\varphi \quad \text{and} \quad NQ = 2r_C \cdot \sin\angle QUN = 2r_C \cdot \sin\psi.$$

By applying the sine rule to triangle DNM we get

$$\frac{DN}{DM} = \frac{\sin\angle DMN}{\sin\angle MND} = \frac{\sin\varphi}{\sin\psi}.$$

Finally, putting the above observations together, we get

$$\frac{MP}{NQ} = \frac{2r_B\sin\varphi}{2r_C\sin\psi} = \frac{r_B}{r_C}\cdot\frac{\sin\varphi}{\sin\psi} = \frac{DM}{DN}\cdot\frac{\sin\varphi}{\sin\psi} = \frac{\sin\psi}{\sin\varphi}\cdot\frac{\sin\varphi}{\sin\psi} = 1,$$

so $MP = NQ$ as required.

G3. In triangle ABC, let A_1 and B_1 be two points on sides BC and AC, and let P and Q be two points on segments AA_1 and BB_1, respectively, so that line PQ is parallel to AB. On ray PB_1, beyond B_1, let P_1 be a point so that $\angle PP_1C = \angle BAC$. Similarly, on ray QA_1, beyond A_1, let Q_1 be a point so that $\angle CQ_1Q = \angle CBA$. Show that points P, Q, P_1, and Q_1 are concyclic.

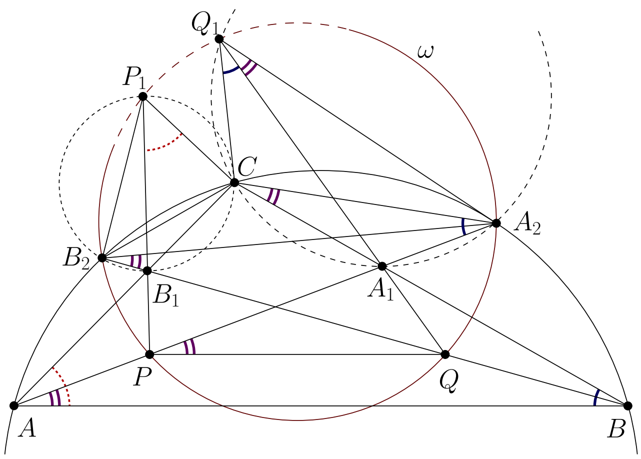
(Ukraine)

Solution 1. Throughout the solution we use oriented angles.

Let rays AA_1 and BB_1 intersect the circumcircle of $\triangle ACB$ at A_2 and B_2, respectively. By

$$\angle QPA_2 = \angle BAA_2 = \angle BB_2A_2 = \angle QB_2A_2,$$

points P, Q, A_2, B_2 are concyclic; denote the circle passing through these points by ω. We shall prove that P_1 and Q_1 also lie on ω.



By

$$\angle CA_2A_1 = \angle CA_2A = \angle CBA = \angle CQ_1Q = \angle CQ_1A_1,$$

points C, Q_1, A_2, A_1 are also concyclic. From that we get

$$\angle QQ_1A_2 = \angle A_1Q_1A_2 = \angle A_1CA_2 = \angle BCA_2 = \angle BAA_2 = \angle QPA_2,$$

so Q_1 lies on ω.

It follows similarly that P_1 lies on ω.

Solution 2. First consider the case when lines PP_1 and QQ_1 intersect each other at some point R.

Let line PQ meet the sides AC and BC at E and F, respectively. Then

$$\angle PP_1C = \angle BAC = \angle PEC,$$

so points C, E, P, P_1 lie on a circle; denote that circle by ω_P. It follows analogously that points C, F, Q, Q_1 lie on another circle; denote it by ω_Q.

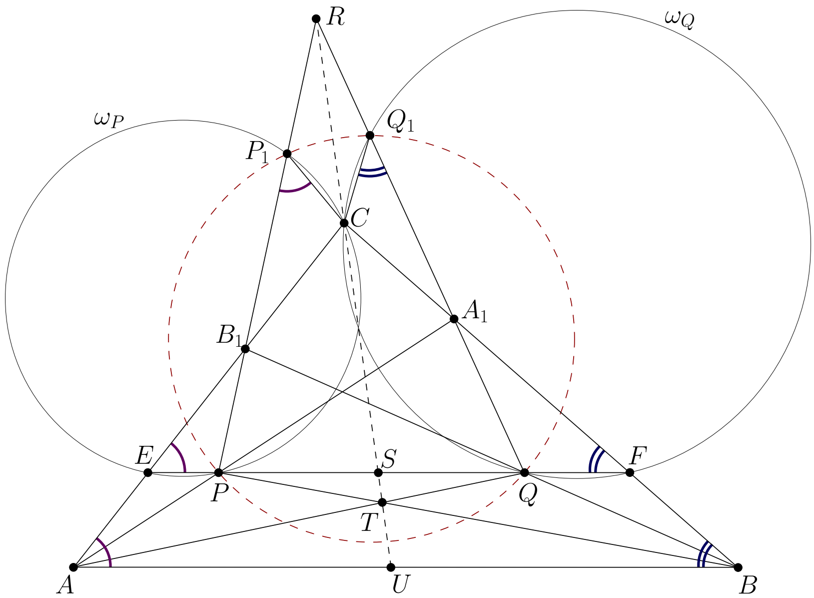
Let AQ and BP intersect at T. Applying Pappus' theorem to the lines AA_1P and BB_1Q provides that points $C = AB_1 \cap BA_1$, $R = A_1Q \cap B_1P$ and $T = AQ \cap BP$ are collinear.

Let line RCT meet PQ and AB at S and U, respectively. From $AB \parallel PQ$ we obtain

$$\frac{SP}{SQ} = \frac{UB}{UA} = \frac{SF}{SE},$$

so

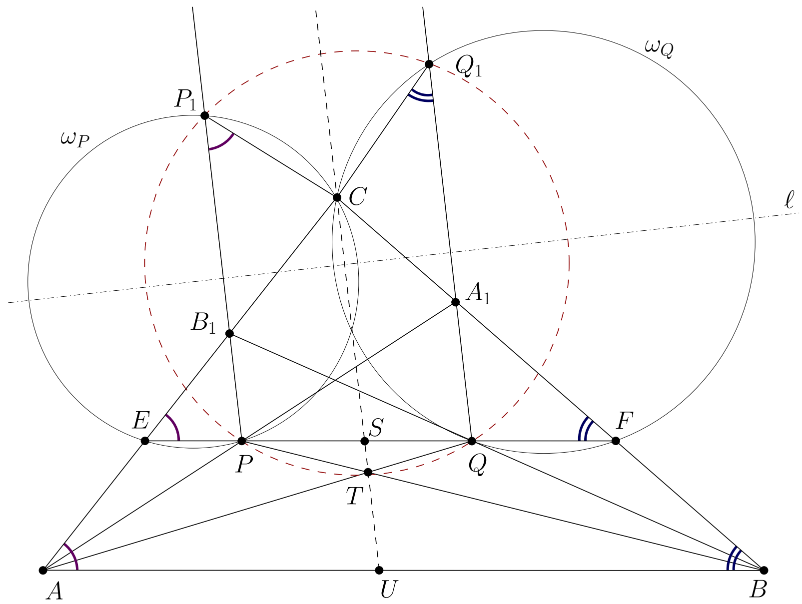
$$SP \cdot SE = SQ \cdot SF.$$



So, point S has equal powers with respect to ω_P and ω_Q, hence line RCS is their radical axis; then R also has equal powers to the circles, so $RP \cdot RP_1 = RQ \cdot RQ_1$, proving that points P, P_1, Q, Q_1 are indeed concyclic.

Now consider the case when PP_1 and QQ_1 are parallel. Like in the previous case, let AQ and BP intersect at T. Applying Pappus' theorem again to the lines AA_1P and BB_1Q, in this limit case it shows that line CT is parallel to PP_1 and QQ_1.

Let line CT meet PQ and AB at S and U, as before. The same calculation as in the previous case shows that $SP \cdot SE = SQ \cdot SF$, so S lies on the radical axis between ω_P and ω_Q.



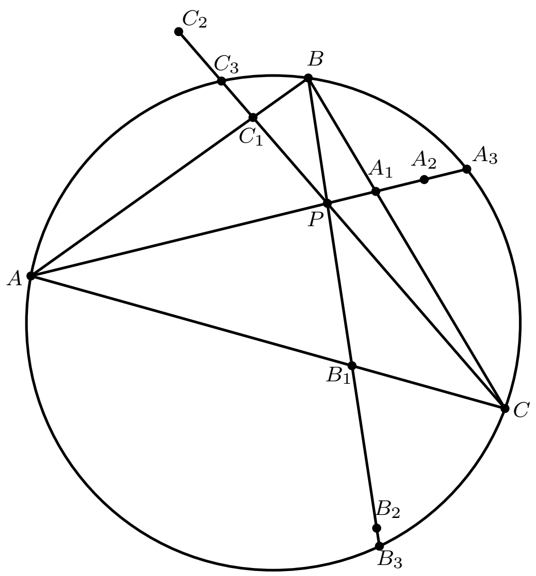
Line CST, that is the radical axis between ω_P and ω_Q, is perpendicular to the line ℓ of centres of ω_P and ω_Q. Hence, the chords PP_1 and QQ_1 are perpendicular to ℓ. So the quadrilateral PP_1Q_1Q is an isosceles trapezium with symmetry axis ℓ, and hence is cyclic.

Comment. There are several ways of solving the problem involving Pappus' theorem. For example, one may consider the points $K = PB_1 \cap BC$ and $L = QA_1 \cap AC$. Applying Pappus' theorem to the lines AA_1P and QB_1B we get that K, L, and $PQ \cap AB$ are collinear, i.e. that $KL \parallel AB$. Therefore, cyclicity of P, Q, P_1, and Q_1 is equivalent to that of K, L, P_1, and Q_1. The latter is easy after noticing that C also lies on that circle. Indeed, e.g. $\angle(LK, LC) = \angle(AB, AC) = \angle(P_1K, P_1C)$ shows that K lies on circle KLC.

This approach also has some possible degeneracy, as the points K and L may happen to be ideal.

G4. Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

(Australia)



Solution 1. Since

$$\angle APB + \angle BPC + \angle CPA = 2\pi = (\pi - \angle ACB) + (\pi - \angle BAC) + (\pi - \angle CBA),$$

at least one of the following inequalities holds:

$$\angle APB \geqslant \pi - \angle ACB, \quad \angle BPC \geqslant \pi - \angle BAC, \quad \angle CPA \geqslant \pi - \angle CBA.$$

Without loss of generality, we assume that $\angle BPC \geqslant \pi - \angle BAC$. We have $\angle BPC > \angle BAC$ because P is inside $\triangle ABC$. So $\angle BPC \geqslant \max(\angle BAC, \pi - \angle BAC)$ and hence

$$\sin \angle BPC \leqslant \sin \angle BAC. \tag{$*$}$$

Let the rays AP, BP, and CP cross the circumcircle Ω again at A_3, B_3, and C_3, respectively. We will prove that at least one of the ratios $\frac{PB_1}{B_1B_3}$ and $\frac{PC_1}{C_1C_3}$ is at least 1, which yields that one of the points B_2 and C_2 does not lie strictly inside Ω.

Because A, B, C, B_3 lie on a circle, the triangles CB_1B_3 and BB_1A are similar, so

$$\frac{CB_1}{B_1B_3} = \frac{BB_1}{B_1A}.$$

Applying the sine rule we obtain

$$\frac{PB_1}{B_1B_3} = \frac{PB_1}{CB_1} \cdot \frac{CB_1}{B_1B_3} = \frac{PB_1}{CB_1} \cdot \frac{BB_1}{B_1A} = \frac{\sin \angle ACP}{\sin \angle BPC} \cdot \frac{\sin \angle BAC}{\sin \angle PBA}.$$

Similarly,

$$\frac{PC_1}{C_1C_3} = \frac{\sin \angle PBA}{\sin \angle BPC} \cdot \frac{\sin \angle BAC}{\sin \angle ACP}.$$

Multiplying these two equations we get

$$\frac{PB_1}{B_1B_3} \cdot \frac{PC_1}{C_1C_3} = \frac{\sin^2 \angle BAC}{\sin^2 \angle BPC} \geqslant 1$$

using $(*)$, which yields the desired conclusion.

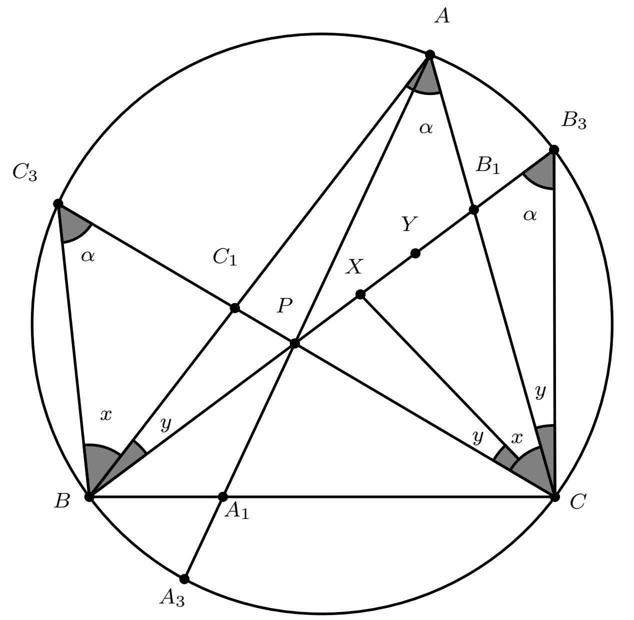
Comment. It also cannot happen that all three points A_2, B_2, and C_2 lie strictly *outside* Ω. The same proof works almost literally, starting by assuming without loss of generality that $\angle BPC \leqslant \pi - \angle BAC$ and using $\angle BPC > \angle BAC$ to deduce that $\sin \angle BPC \geqslant \sin \angle BAC$. It is possible for A_2, B_2, and C_2 all to lie on the circumcircle; from the above solution we may derive that this happens if and only if P is the orthocentre of the triangle ABC, (which lies strictly inside ABC if and only if ABC is acute).

Solution 2. Define points A_3, B_3, and C_3 as in Solution 1. Assume for the sake of contradiction that A_2, B_2, and C_2 all lie strictly inside circle ABC. It follows that $PA_1 < A_1A_3$, $PB_1 < B_1B_3$, and $PC_1 < C_1C_3$.

Observe that $\triangle PBC_3 \sim \triangle PCB_3$. Let X be the point on side PB_3 that corresponds to point C_1 on side PC_3 under this similarity. In other words, X lies on segment PB_3 and satisfies $PX : XB_3 = PC_1 : C_1C_3$. It follows that

$$\angle XCP = \angle PBC_1 = \angle B_3BA = \angle B_3CB_1 .$$

Hence lines CX and CB_1 are isogonal conjugates in $\triangle PCB_3$.



Let Y be the foot of the bisector of $\angle B_3CP$ in $\triangle PCB_3$. Since $PC_1 < C_1C_3$, we have $PX < XB_3$. Also, we have $PY < YB_3$ because $PB_1 < B_1B_3$ and Y lies between X and B_1. By the angle bisector theorem in $\triangle PCB_3$, we have $PY : YB_3 = PC : CB_3$. So $PC < CB_3$ and it follows that $\angle PB_3C < \angle CPB_3$. Now since $\angle PB_3C = \angle BB_3C = \angle BAC$, we have

$$\angle BAC < \angle CPB_3 .$$

Similarly, we have

$$\angle CBA < \angle APC_3 \quad \text{and} \quad \angle ACB < \angle BPA_3 = \angle B_3PA .$$

Adding these three inequalities yields $\pi < \pi$, and this contradiction concludes the proof.

Solution 3. Choose coordinates such that the circumcentre of $\triangle ABC$ is at the origin and the circumradius is 1. Then we may think of A, B, and C as vectors in $\mathbb{R}^2$ such that

$$|A|^2 = |B|^2 = |C|^2 = 1\,.$$

P may be represented as a convex combination $\alpha A+\beta B+\gamma C$ where $\alpha,\beta,\gamma>0$ and $\alpha+\beta+\gamma=1$. Then

$$A_1 = \frac{\beta B+\gamma C}{\beta+\gamma} = \frac{1}{1-\alpha}P - \frac{\alpha}{1-\alpha}A,$$

so

$$A_2 = 2A_1 - P = \frac{1+\alpha}{1-\alpha}P - \frac{2\alpha}{1-\alpha}A\,.$$

Hence

$$|A_2|^2 = \left(\frac{1+\alpha}{1-\alpha}\right)^2|P|^2 + \left(\frac{2\alpha}{1-\alpha}\right)^2|A|^2 - \frac{4\alpha(1+\alpha)}{(1-\alpha)^2}\,A\cdot P\,.$$

Using $|A|^2=1$ we obtain

$$\frac{(1-\alpha)^2}{2(1+\alpha)}|A_2|^2 = \frac{1+\alpha}{2}|P|^2 + \frac{2\alpha^2}{1+\alpha} - 2\alpha A\cdot P. \tag{1}$$

Likewise

$$\frac{(1-\beta)^2}{2(1+\beta)}|B_2|^2 = \frac{1+\beta}{2}|P|^2 + \frac{2\beta^2}{1+\beta} - 2\beta B\cdot P \tag{2}$$

and

$$\frac{(1-\gamma)^2}{2(1+\gamma)}|C_2|^2 = \frac{1+\gamma}{2}|P|^2 + \frac{2\gamma^2}{1+\gamma} - 2\gamma C\cdot P. \tag{3}$$

Summing (1), (2) and (3) we obtain on the LHS the positive linear combination

$$\text{LHS} = \frac{(1-\alpha)^2}{2(1+\alpha)}|A_2|^2 + \frac{(1-\beta)^2}{2(1+\beta)}|B_2|^2 + \frac{(1-\gamma)^2}{2(1+\gamma)}|C_2|^2$$

and on the RHS the quantity

$$\left(\frac{1+\alpha}{2}+\frac{1+\beta}{2}+\frac{1+\gamma}{2}\right)|P|^2 + \left(\frac{2\alpha^2}{1+\alpha}+\frac{2\beta^2}{1+\beta}+\frac{2\gamma^2}{1+\gamma}\right) - 2(\alpha A\cdot P+\beta B\cdot P+\gamma C\cdot P)\,.$$

The first term is $2|P|^2$ and the last term is $-2P\cdot P$, so

$$\begin{aligned}\text{RHS} &= \left(\frac{2\alpha^2}{1+\alpha}+\frac{2\beta^2}{1+\beta}+\frac{2\gamma^2}{1+\gamma}\right)\\ &= \frac{3\alpha-1}{2}+\frac{(1-\alpha)^2}{2(1+\alpha)}+\frac{3\beta-1}{2}+\frac{(1-\beta)^2}{2(1+\beta)}+\frac{3\gamma-1}{2}+\frac{(1-\gamma)^2}{2(1+\gamma)}\\ &= \frac{(1-\alpha)^2}{2(1+\alpha)}+\frac{(1-\beta)^2}{2(1+\beta)}+\frac{(1-\gamma)^2}{2(1+\gamma)}\,.\end{aligned}$$

Here we used the fact that

$$\frac{3\alpha-1}{2}+\frac{3\beta-1}{2}+\frac{3\gamma-1}{2}=0\,.$$

We have shown that a linear combination of $|A_1|^2$, $|B_1|^2$, and $|C_1|^2$ with positive coefficients is equal to the sum of the coefficients. Therefore at least one of $|A_1|^2$, $|B_1|^2$, and $|C_1|^2$ must be at least 1, as required.

Comment. This proof also works when P is any point for which $\alpha,\beta,\gamma>-1$, $\alpha+\beta+\gamma=1$, and $\alpha,\beta,\gamma\neq 1$. (In any cases where $\alpha=1$ or $\beta=1$ or $\gamma=1$, some points in the construction are not defined.)

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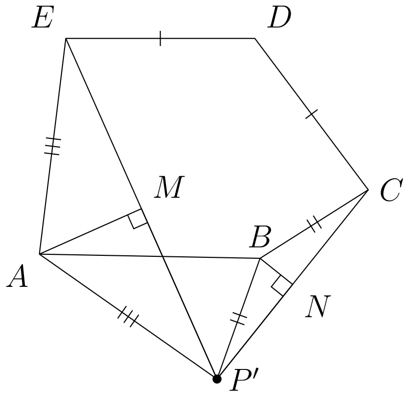
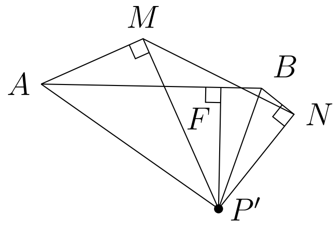
G5. Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \neq 2 \cdot \angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if $\operatorname{area}(BCD) + \operatorname{area}(ADE) = \operatorname{area}(ABD) + \operatorname{area}(ABP)$.

(Hungary)

Solution 1. Let P' be the reflection of P across line AB, and let M and N be the midpoints of $P'E$ and $P'C$ respectively. Convexity ensures that P' is distinct from both E and C, and hence from both M and N. We claim that both the area condition and the collinearity condition in the problem are equivalent to the condition that the (possibly degenerate) right-angled triangles $AP'M$ and $BP'N$ are directly similar (equivalently, $AP'E$ and $BP'C$ are directly similar).



For the equivalence with the collinearity condition, let F denote the foot of the perpendicular from P' to AB, so that F is the midpoint of PP'. We have that P lies on CE if and only if F lies on MN, which occurs if and only if we have the equality $\angle AFM = \angle BFN$ of signed angles modulo π. By concyclicity of $AP'FM$ and $BFP'N$, this is equivalent to $\angle AP'M = \angle BP'N$, which occurs if and only if $AP'M$ and $BP'N$ are directly similar.



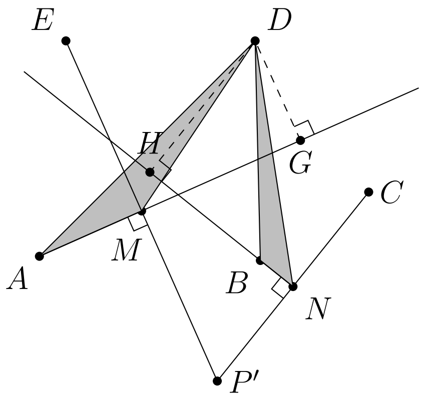
For the other equivalence with the area condition, we have the equality of signed areas $\operatorname{area}(ABD) + \operatorname{area}(ABP) = \operatorname{area}(AP'BD) = \operatorname{area}(AP'D) + \operatorname{area}(BDP')$. Using the identity $\operatorname{area}(ADE) - \operatorname{area}(AP'D) = \operatorname{area}(ADE) + \operatorname{area}(ADP') = 2\operatorname{area}(ADM)$, and similarly for B, we find that the area condition is equivalent to the equality

$$\operatorname{area}(DAM) = \operatorname{area}(DBN).$$

Now note that A and B lie on the perpendicular bisectors of $P'E$ and $P'C$, respectively. If we write G and H for the feet of the perpendiculars from D to these perpendicular bisectors respectively, then this area condition can be rewritten as

$$MA \cdot GD = NB \cdot HD.$$

(In this condition, we interpret all lengths as signed lengths according to suitable conventions: for instance, we orient $P'E$ from P' to E, orient the parallel line DH in the same direction, and orient the perpendicular bisector of $P'E$ at an angle $\pi/2$ clockwise from the oriented segment $P'E$ – we adopt the analogous conventions at B.)



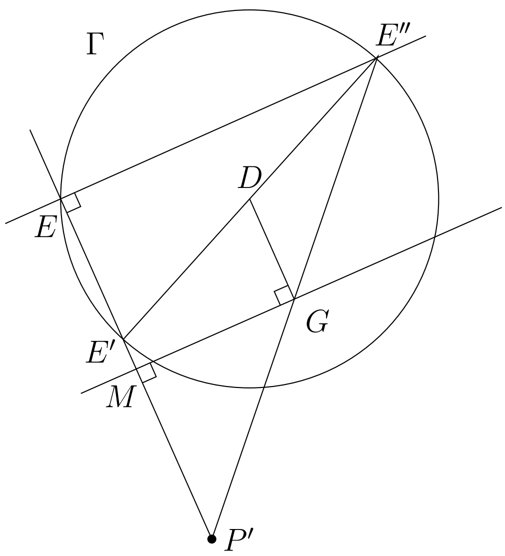
To relate the signed lengths GD and HD to the triangles $AP'M$ and $BP'N$, we use the following calculation.

Claim. Let Γ denote the circle centred on D with both E and C on the circumference, and h the power of P' with respect to Γ. Then we have the equality

$$GD \cdot P'M = HD \cdot P'N = \frac{1}{4}h \neq 0.$$

Proof. Firstly, we have $h \neq 0$, since otherwise P' would lie on Γ, and hence the internal angle bisectors of $\angle EDP'$ and $\angle P'DC$ would pass through A and B respectively. This would violate the angle inequality $\angle EDC \neq 2 \cdot \angle ADB$ given in the question.

Next, let E' denote the second point of intersection of $P'E$ with Γ, and let E'' denote the point on Γ diametrically opposite E', so that $E''E$ is perpendicular to $P'E$. The point G lies on the perpendicular bisectors of the sides $P'E$ and EE'' of the right-angled triangle $P'EE''$; it follows that G is the midpoint of $P'E''$. Since D is the midpoint of $E'E''$, we have that $GD = \frac{1}{2}P'E'$. Since $P'M = \frac{1}{2}P'E$, we have $GD \cdot P'M = \frac{1}{4}P'E' \cdot P'E = \frac{1}{4}h$. The other equality $HD \cdot P'N$ follows by exactly the same argument.



□

From this claim, we see that the area condition is equivalent to the equality

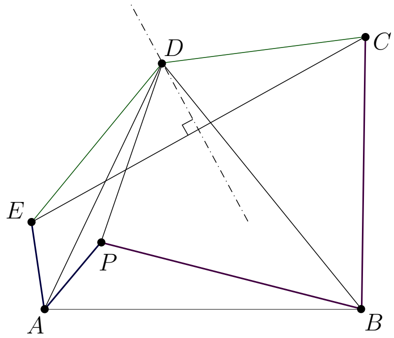
$$(MA : P'M) = (NB : P'N)$$

of ratios of signed lengths, which is equivalent to direct similarity of $AP'M$ and $BP'N$, as desired.

Solution 2. Along the perpendicular bisector of CE, define the linear function

$$f(X) = \text{area}(BCX) + \text{area}(AXE) - \text{area}(ABX) - \text{area}(ABP),$$

where, from now on, we always use signed areas. Thus, we want to show that C, P, E are collinear if and only if $f(D) = 0$.



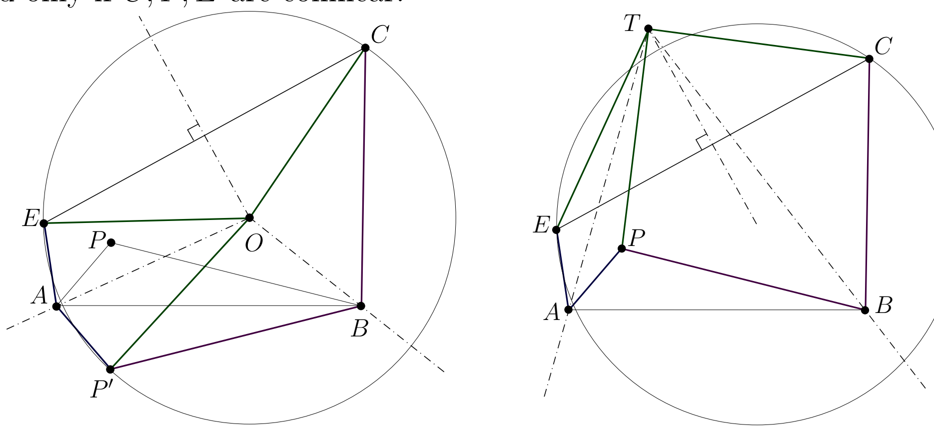
Let P' be the reflection of P across line AB. The point P' does not lie on the line CE. To see this, we let A'' and B'' be the points obtained from A and B by dilating with scale factor 2 about P', so that P is the orthogonal projection of P' onto $A''B''$. Since A lies on the perpendicular bisector of $P'E$, the triangle $A''EP'$ is right-angled at E (and $B''CP'$ similarly). If P' were to lie on CE, then the lines $A''E$ and $B''C$ would be perpendicular to CE and A'' and B'' would lie on the opposite side of CE to D. It follows that the line $A''B''$ does not meet triangle CDE, and hence point P does not lie inside CDE. But then P must lie inside $ABCE$, and it is clear that such a point cannot reflect to a point P' on CE.

We thus let O be the centre of the circle CEP'. The lines AO and BO are the perpendicular bisectors of EP' and CP', respectively, so

$$\begin{aligned}\text{area}(BCO) + \text{area}(AOE) &= \text{area}(OP'B) + \text{area}(P'OA) = \text{area}(P'BOA)\\ &= \text{area}(ABO) + \text{area}(BAP') = \text{area}(ABO) + \text{area}(ABP),\end{aligned}$$

and hence $f(O) = 0$.

Notice that if point O coincides with D then points A, B lie in angle domain CDE and $\angle EOC = 2 \cdot \angle AOB$, which is not allowed. So, O and D must be distinct. Since f is linear and vanishes at O, it follows that $f(D) = 0$ if and only if f is constant zero – we want to show this occurs if and only if C, P, E are collinear.



In the one direction, suppose firstly that C, P, E are not collinear, and let T be the centre of the circle CEP. The same calculation as above provides

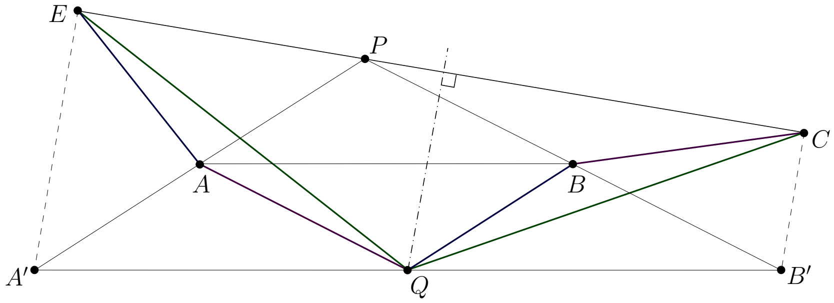
$$\text{area}(BCT) + \text{area}(ATE) = \text{area}(PBTA) = \text{area}(ABT) - \text{area}(ABP)$$

so

$$f(T) = -2\,\text{area}(ABP) \neq 0.$$

Hence, the linear function f is nonconstant with its zero is at O, so that $f(D) \neq 0$.

In the other direction, suppose that the points C, P, E are collinear. We will show that f is constant zero by finding a second point (other than O) at which it vanishes.



Let Q be the reflection of P across the midpoint of AB, so $PAQB$ is a parallelogram. It is easy to see that Q is on the perpendicular bisector of CE; for instance if A' and B' are the points produced from A and B by dilating about P with scale factor 2, then the projection of Q to CE is the midpoint of the projections of A' and B', which are E and C respectively. The triangles BCQ and AQE are indirectly congruent, so

$$f(Q) = \big(\text{area}(BCQ) + \text{area}(AQE)\big) - \big(\text{area}(ABQ) - \text{area}(BAP)\big) = 0 - 0 = 0.$$

The points O and Q are distinct. To see this, consider the circle ω centred on Q with P' on the circumference; since triangle $PP'Q$ is right-angled at P', it follows that P lies outside ω. On the other hand, P lies between C and E on the line CPE. It follows that C and E cannot both lie on ω, so that ω is not the circle CEP' and $Q \neq O$.

Since O and Q are distinct zeroes of the linear function f, we have $f(D) = 0$ as desired.

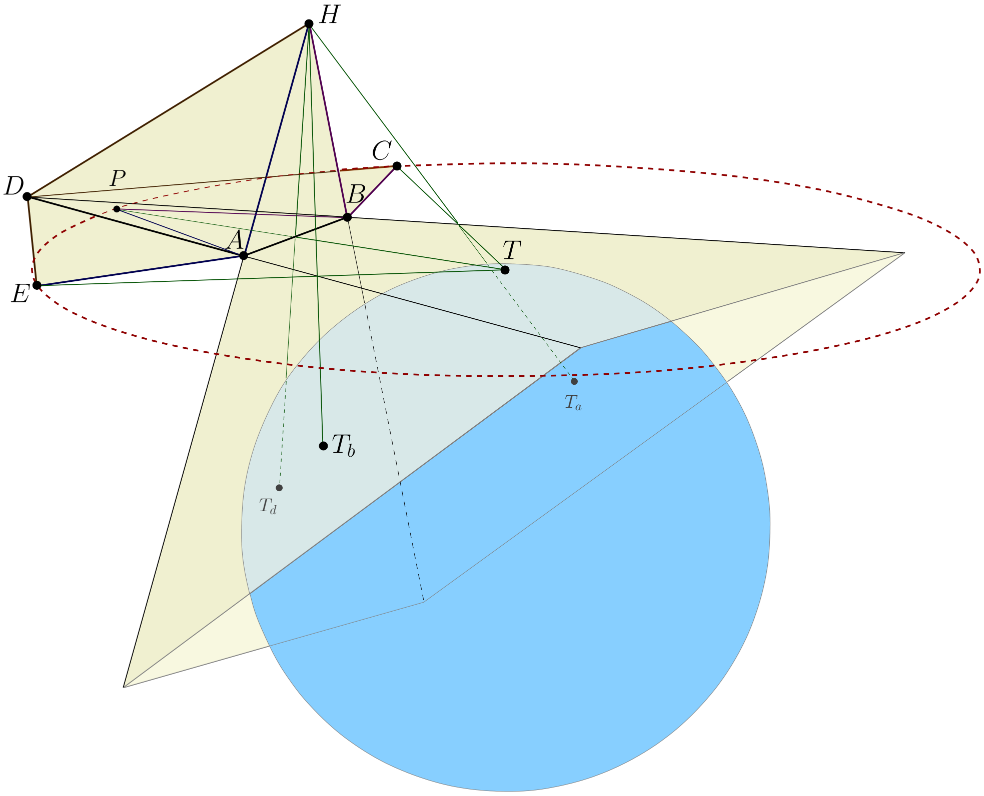
Comment 1. The condition $\angle EDC \neq 2 \cdot \angle ADB$ cannot be omitted. If D is the centre of circle CEP', then the condition on triangle areas is satisfied automatically, without having P on line CE.

Comment 2. The "only if" part of this problem is easier than the "if" part. For example, in the second part of Solution 2, the triangles EAQ and QBC are indirectly congruent, so the sum of their areas is 0, and $DCQE$ is a kite. Now one can easily see that $\angle(AQ, DE) = \angle(CD, CB)$ and $\angle(BQ, DC) = \angle(ED, EA)$, whence $\text{area}(BCD) = \text{area}(AQD) + \text{area}(EQA)$ and $\text{area}(ADE) = \text{area}(BDQ) + \text{area}(BQC)$, which yields the result.

Comment 3. The origin of the problem is the following observation. Let $ABDH$ be a tetrahedron and consider the sphere $\mathcal{S}$ that is tangent to the four face planes, internally to planes ADH and BDH and externally to ABD and ABH (or vice versa). It is known that the sphere $\mathcal{S}$ exists if and only if $\operatorname{area}(ADH) + \operatorname{area}(BDH) \neq \operatorname{area}(ABH) + \operatorname{area}(ABD)$; this relation comes from the usual formula for the volume of the tetrahedron.

Let T, T_a, T_b, T_d be the points of tangency between the sphere and the four planes, as shown in the picture. Rotate the triangle ABH inward, the triangles BDH and ADH outward, into the triangles ABP, BDC and ADE, respectively, in the plane ABD. Notice that the points T_d, T_a, T_b are rotated to T, so we have $HT_a = HT_b = HT_d = PT = CT = ET$. Therefore, the point T is the centre of the circle CEP. Hence, if the sphere exists then C, E, P cannot be collinear.

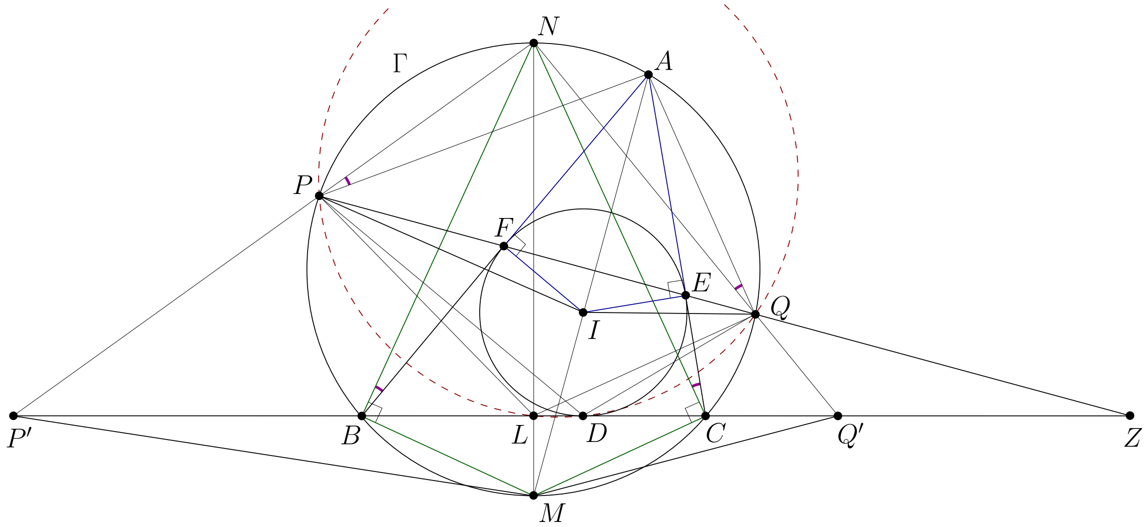
If the condition $\angle EDC \neq 2 \cdot \angle ADB$ is replaced by the constraint that the angles $\angle EDA$, $\angle ADB$ and $\angle BDC$ satisfy the triangle inequality, it enables reconstructing the argument with the tetrahedron and the tangent sphere.



G6. Let I be the incentre of acute-angled triangle ABC. Let the incircle meet BC, CA, and AB at D, E, and F, respectively. Let line EF intersect the circumcircle of the triangle at P and Q, such that F lies between E and P. Prove that $\angle DPA + \angle AQD = \angle QIP$.

(Slovakia)

Solution 1. Let N and M be the midpoints of the arcs $\widehat{BC}$ of the circumcircle, containing and opposite vertex A, respectively. By $\angle FAE = \angle BAC = \angle BNC$, the right-angled kites $AFIE$ and $NBMC$ are similar. Consider the spiral similarity φ (dilation in case of $AB = AC$) that moves $AFIE$ to $NBMC$. The directed angle in which φ changes directions is $\angle(AF, NB)$, same as $\angle(AP, NP)$ and $\angle(AQ, NQ)$; so lines AP and AQ are mapped to lines NP and NQ, respectively. Line EF is mapped to BC; we can see that the intersection points $P = EF \cap AP$ and $Q = EF \cap AQ$ are mapped to points $BC \cap NP$ and $BC \cap NQ$, respectively. Denote these points by P' and Q', respectively.



Let L be the midpoint of BC. We claim that points P, Q, D, L are concyclic (if $D = L$ then line BC is tangent to circle PQD). Let PQ and BC meet at Z. By applying Menelaus' theorem to triangle ABC and line EFZ, we have

$$\frac{BD}{DC} = \frac{BF}{FA} \cdot \frac{AE}{EC} = -\frac{BZ}{ZC},$$

so the pairs B, C and D, Z are harmonic. It is well-known that this implies $ZB \cdot ZC = ZD \cdot ZL$. (The inversion with pole Z that swaps B and C sends Z to infinity and D to the midpoint of BC, because the cross-ratio is preserved.) Hence, $ZD \cdot ZL = ZB \cdot ZC = ZP \cdot ZQ$ by the power of Z with respect to the circumcircle; this proves our claim.

By $\angle MPP' = \angle MQQ' = \angle MLP' = \angle MLQ' = 90^\circ$, the quadrilaterals $MLPP'$ and $MLQQ'$ are cyclic. Then the problem statement follows by

$$\begin{aligned}\angle DPA + \angle AQD &= 360^\circ - \angle PAQ - \angle QDP = 360^\circ - \angle PNQ - \angle QLP \\ &= \angle LPN + \angle NQL = \angle P'ML + \angle LMQ' = \angle P'MQ' = \angle PIQ.\end{aligned}$$

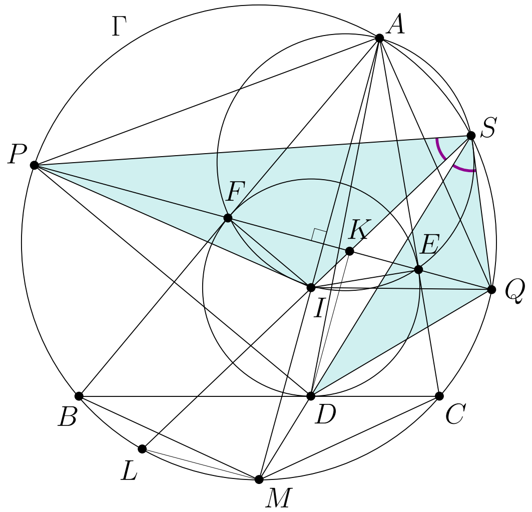
Solution 2. Define the point M and the same spiral similarity φ as in the previous solution. (The point N is not necessary.) It is well-known that the centre of the spiral similarity that maps F, E to B, C is the Miquel point of the lines FE, BC, BF and CE; that is, the second intersection of circles ABC and AEF. Denote that point by S.

By $\varphi(F) = B$ and $\varphi(E) = C$ the triangles SBF and SCE are similar, so we have

$$\frac{SB}{SC} = \frac{BF}{CE} = \frac{BD}{CD}.$$

By the converse of the angle bisector theorem, that indicates that line SD bisects $\angle BSC$ and hence passes through M.

Let K be the intersection point of lines EF and SI. Notice that φ sends points S, F, E, I to S, B, C, M, so $\varphi(K) = \varphi(FE \cap SI) = BC \cap SM = D$. By $\varphi(I) = M$, we have $KD \parallel IM$.



We claim that triangles SPI and SDQ are similar, and so are triangles SPD and SIQ. Let ray SI meet the circumcircle again at L. Note that the segment EF is perpendicular to the angle bisector AM. Then by $\angle AML = \angle ASL = \angle ASI = 90^\circ$, we have $ML \parallel PQ$. Hence, $\widehat{PL} = \widehat{MQ}$ and therefore $\angle PSL = \angle MSQ = \angle DSQ$. By $\angle QPS = \angle QMS$, the triangles SPK and SMQ are similar. Finally,

$$\frac{SP}{SI} = \frac{SP}{SK} \cdot \frac{SK}{SI} = \frac{SM}{SQ} \cdot \frac{SD}{SM} = \frac{SD}{SQ}$$

shows that triangles SPI and SDQ are similar. The second part of the claim can be proved analogously.

Now the problem statement can be proved by

$$\angle DPA + \angle AQD = \angle DPS + \angle SQD = \angle QIS + \angle SIP = \angle QIP.$$

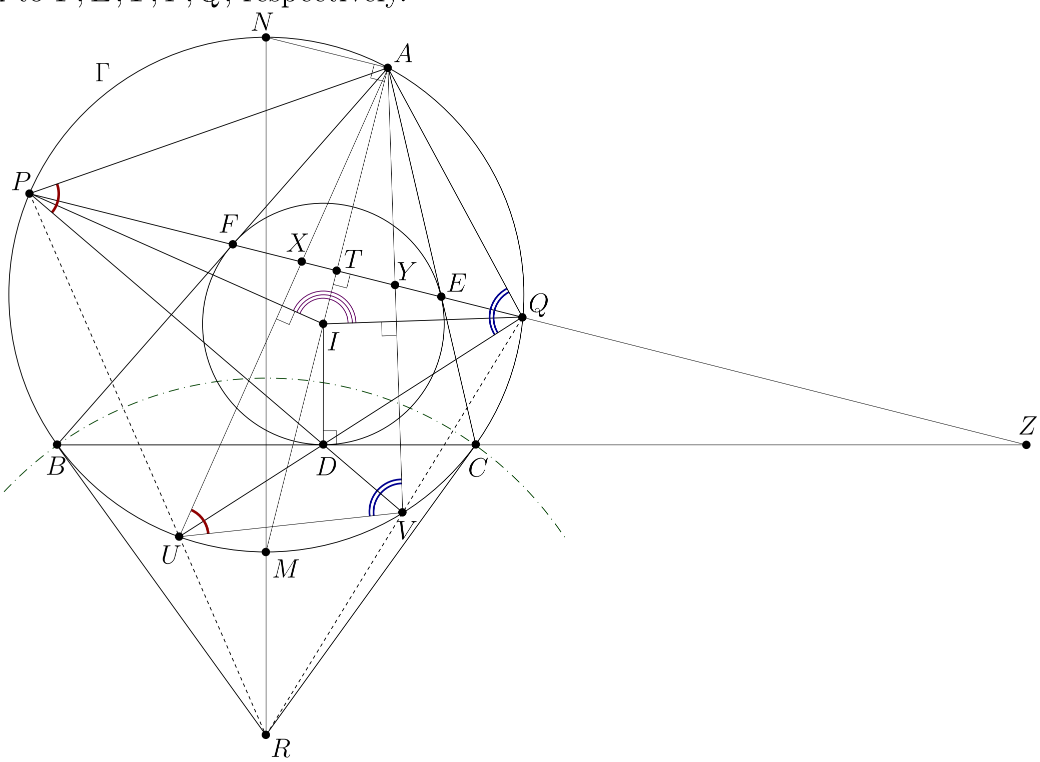
Solution 3. Denote the circumcircle of triangle ABC by Γ, and let rays PD and QD meet Γ again at V and U, respectively. We will show that $AU \perp IP$ and $AV \perp IQ$. Then the problem statement will follow as

$$\angle DPA + \angle AQD = \angle VUA + \angle AVU = 180^\circ - \angle UAV = \angle QIP.$$

Let M be the midpoint of arc $\widehat{BUVC}$ and let N be the midpoint of arc $\widehat{CAB}$; the lines AIM and AN being the internal and external bisectors of angle BAC, respectively, are perpendicular. Let the tangents drawn to Γ at B and C meet at R; let line PQ meet AU, AI, AV and BC at X, T, Y and Z, respectively.

As in Solution 1, we observe that the pairs B, C and D, Z are harmonic. Projecting these points from Q onto the circumcircle, we can see that B, C and U, P are also harmonic. Analogously, the pair V, Q is harmonic with B, C. Consider the inversion about the circle with centre R, passing through B and C. Points B and C are fixed points, so this inversion exchanges every point of Γ by its harmonic pair with respect to B, C. In particular, the inversion maps points B, C, N, U, V to points B, C, M, P, Q, respectively.

Combine the inversion with projecting Γ from A to line PQ; the points B, C, M, P, Q are projected to F, E, T, P, Q, respectively.



The combination of these two transformations is projective map from the lines AB, AC, AN, AU, AV to IF, IE, IT, IP, IQ, respectively. On the other hand, we have $AB \perp IF$, $AC \perp IE$ and $AN \perp AT$, so the corresponding lines in these two pencils are perpendicular. This proves $AU \perp IP$ and $AV \perp IQ$, and hence completes the solution.

G7. The incircle ω of acute-angled scalene triangle ABC has centre I and meets sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at $Q \neq P$. Prove that lines DI and PQ meet on the external bisector of angle BAC.

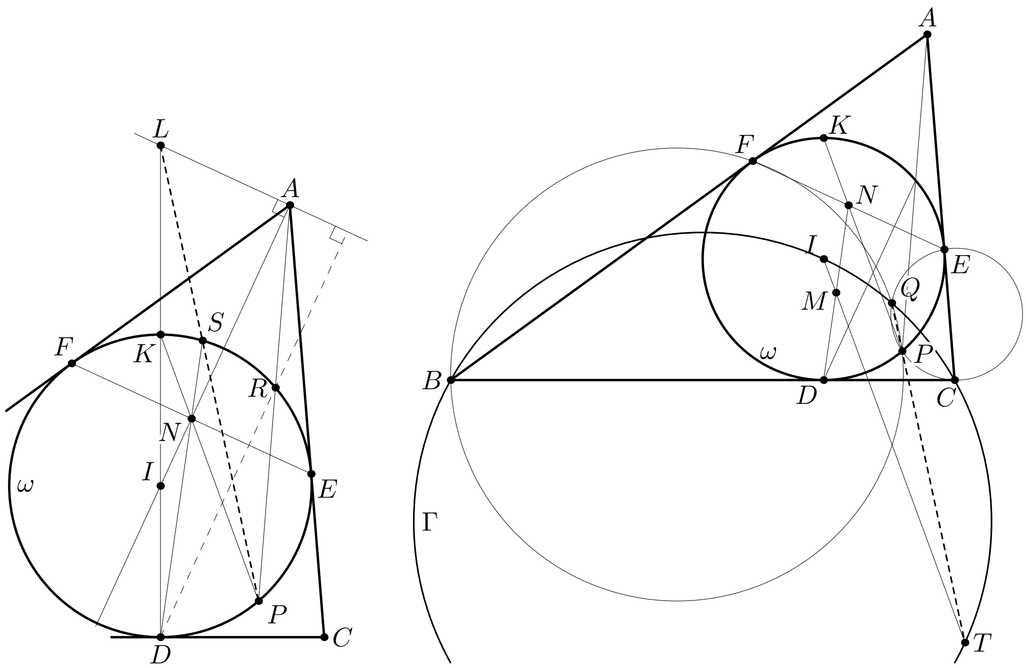
(India)

Common remarks. Throughout the solution, $\angle(a, b)$ denotes the directed angle between lines a and b, measured modulo π.

Solution 1.

Step 1. The external bisector of $\angle BAC$ is the line through A perpendicular to IA. Let DI meet this line at L and let DI meet ω at K. Let N be the midpoint of EF, which lies on IA and is the pole of line AL with respect to ω. Since $AN \cdot AI = AE^2 = AR \cdot AP$, the points R, N, I, and P are concyclic. As $IR = IP$, the line NI is the external bisector of $\angle PNR$, so PN meets ω again at the point symmetric to R with respect to AN – i.e. at K.

Let DN cross ω again at S. Opposite sides of any quadrilateral inscribed in the circle ω meet on the polar line of the intersection of the diagonals with respect to ω. Since L lies on the polar line AL of N with respect to ω, the line PS must pass through L. Thus it suffices to prove that the points S, Q, and P are collinear.

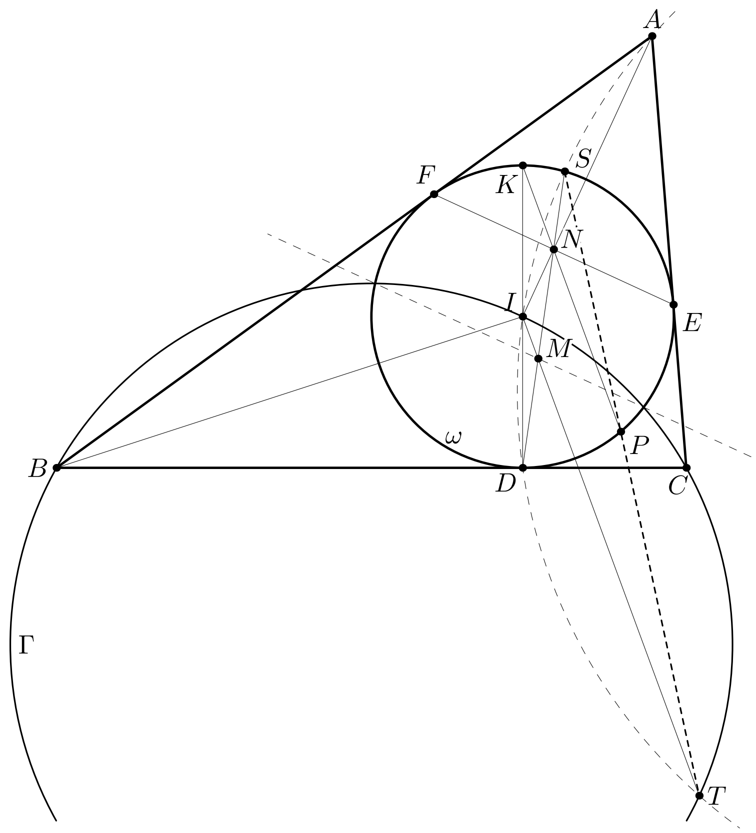


Step 2. Let Γ be the circumcircle of $\triangle BIC$. Notice that

$$\begin{aligned}\angle(BQ, QC) &= \angle(BQ, QP) + \angle(PQ, QC) = \angle(BF, FP) + \angle(PE, EC) \\ &= \angle(EF, EP) + \angle(FP, FE) = \angle(FP, EP) = \angle(DF, DE) = \angle(BI, IC),\end{aligned}$$

so Q lies on Γ. Let QP meet Γ again at T. It will now suffice to prove that S, P, and T are collinear. Notice that $\angle(BI, IT) = \angle(BQ, QT) = \angle(BF, FP) = \angle(FK, KP)$. Note $FD \perp FK$ and $FD \perp BI$ so $FK \parallel BI$ and hence IT is parallel to the line KNP. Since $DI = IK$, the line IT crosses DN at its midpoint M.

Step 3. Let F' and E' be the midpoints of DE and DF, respectively. Since $DE' \cdot E'F = DE'^2 = BE' \cdot E'I$, the point E' lies on the radical axis of ω and Γ; the same holds for F'. Therefore, this radical axis is $E'F'$, and it passes through M. Thus $IM \cdot MT = DM \cdot MS$, so S, I, D, and T are concyclic. This shows $\angle(DS, ST) = \angle(DI, IT) = \angle(DK, KP) = \angle(DS, SP)$, whence the points S, P, and T are collinear, as desired.



Comment. Here is a longer alternative proof in step 1 that P, S, and L are collinear, using a circular inversion instead of the fact that opposite sides of a quadrilateral inscribed in a circle ω meet on the polar line with respect to ω of the intersection of the diagonals. Let G be the foot of the altitude from N to the line $DIKL$. Observe that N, G, K, S are concyclic (opposite right angles) so

$$\angle DIP = 2\angle DKP = \angle GKN + \angle DSP = \angle GSN + \angle NSP = \angle GSP\,,$$

hence I, G, S, P are concyclic. We have $IG \cdot IL = IN \cdot IA = r^2$ since $\triangle IGN \sim \triangle IAL$. Inverting the circle $IGSP$ in circle ω, points P and S are fixed and G is taken to L so we find that P, S, and L are collinear.

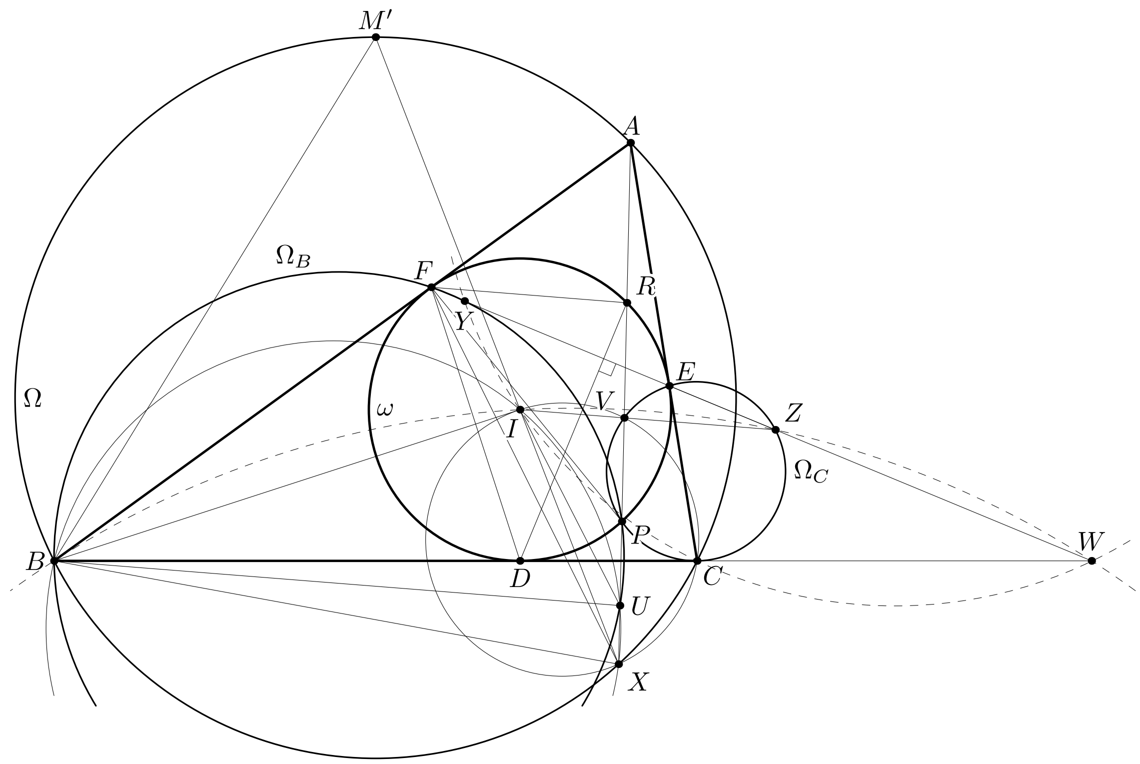
Solution 2. We start as in Solution 1. Namely, we introduce the same points K, L, N, and S, and show that the triples (P, N, K) and (P, S, L) are collinear. We conclude that K and R are symmetric in AI, and reduce the problem statement to showing that P, Q, and S are collinear.

Step 1. Let AR meet the circumcircle Ω of ABC again at X. The lines AR and AK are isogonal in the angle BAC; it is well known that in this case X is the tangency point of Ω with the A-mixtilinear circle. It is also well known that for this point X, the line XI crosses Ω again at the midpoint M' of arc BAC.

Step 2. Denote the circles BFP and CEP by Ω_B and Ω_C, respectively. Let Ω_B cross AR and EF again at U and Y, respectively. We have

$$\angle(UB, BF) = \angle(UP, PF) = \angle(RP, PF) = \angle(RF, FA),$$

so $UB \parallel RF$.



Next, we show that the points B, I, U, and X are concyclic. Since

$$\angle(UB, UX) = \angle(RF, RX) = \angle(AF, AR) + \angle(FR, FA) = \angle(M'B, M'X) + \angle(DR, DF),$$

it suffices to prove $\angle(IB, IX) = \angle(M'B, M'X) + \angle(DR, DF)$, or $\angle(IB, M'B) = \angle(DR, DF)$. But both angles equal $\angle(CI, CB)$, as desired. (This is where we used the fact that M' is the midpoint of arc BAC of Ω.)

It follows now from circles $BUIX$ and $BPUFY$ that

$$\angle(IU, UB) = \angle(IX, BX) = \angle(M'X, BX) = \frac{\pi - \angle A}{2}$$
$$= \angle(EF, AF) = \angle(YF, BF) = \angle(YU, BU),$$

so the points Y, U, and I are collinear.

Let EF meet BC at W. We have

$$\angle(IY, YW) = \angle(UY, FY) = \angle(UB, FB) = \angle(RF, AF) = \angle(CI, CW),$$

so the points W, Y, I, and C are concyclic.

Similarly, if V and Z are the second meeting points of Ω_C with AR and EF, we get that the 4-tuples (C, V, I, X) and (B, I, Z, W) are both concyclic.

Step 3. Let $Q' = CY \cap BZ$. We will show that $Q' = Q$.

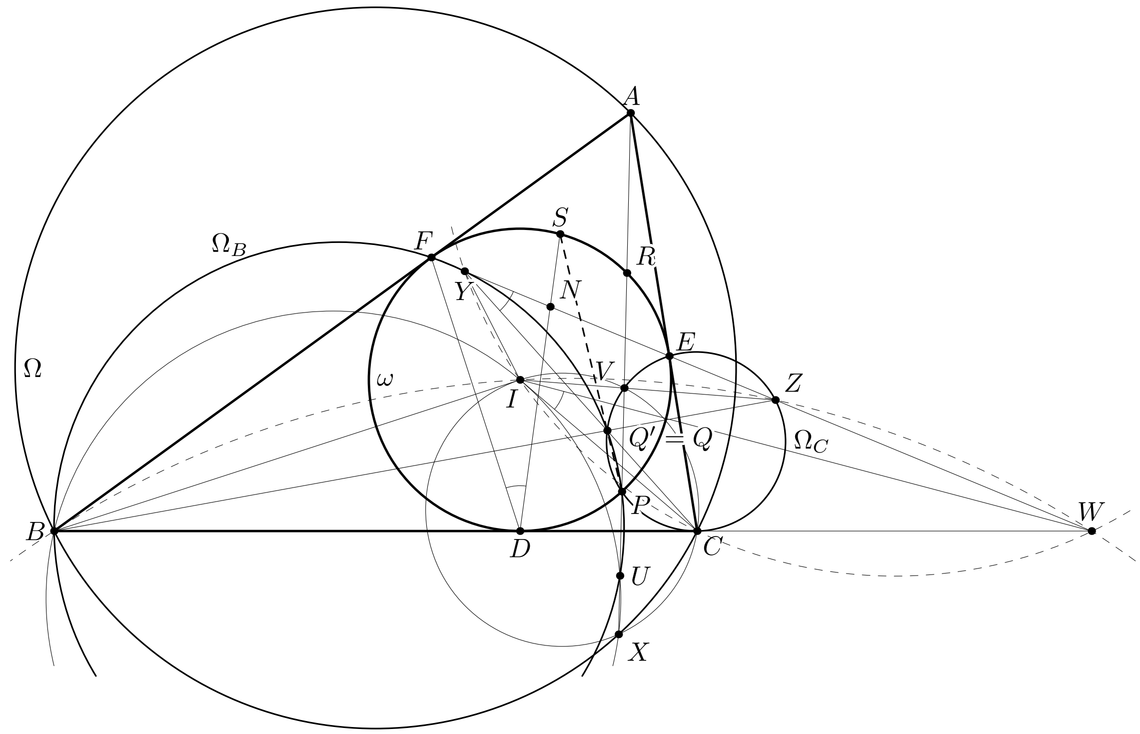
First of all, we have

$$\begin{aligned}\angle(Q'Y, Q'B) &= \angle(CY, ZB) = \angle(CY, ZY) + \angle(ZY, BZ) \\ &= \angle(CI, IW) + \angle(IW, IB) = \angle(CI, IB) = \frac{\pi - \angle A}{2} = \angle(FY, FB),\end{aligned}$$

so $Q' \in \Omega_B$. Similarly, $Q' \in \Omega_C$. Thus $Q' \in \Omega_B \cap \Omega_C = \{P, Q\}$ and it remains to prove that $Q' \neq P$. If we had $Q' = P$, we would have $\angle(PY, PZ) = \angle(Q'Y, Q'Z) = \angle(IC, IB)$. This would imply

$$\angle(PY, YF) + \angle(EZ, ZP) = \angle(PY, PZ) = \angle(IC, IB) = \angle(PE, PF),$$

so circles Ω_B and Ω_C would be tangent at P. That is excluded in the problem conditions, so $Q' = Q$.



Step 4. Now we are ready to show that P, Q, and S are collinear.

Notice that A and D are the poles of EW and DW with respect to ω, so W is the pole of AD. Hence, $WI \perp AD$. Since $CI \perp DE$, this yields $\angle(IC, WI) = \angle(DE, DA)$. On the other hand, DA is a symmedian in $\triangle DEF$, so $\angle(DE, DA) = \angle(DN, DF) = \angle(DS, DF)$. Therefore,

$$\begin{aligned}\angle(PS, PF) &= \angle(DS, DF) = \angle(DE, DA) = \angle(IC, IW) \\ &= \angle(YC, YW) = \angle(YQ, YF) = \angle(PQ, PF),\end{aligned}$$

which yields the desired collinearity.

G8. Let $\mathcal{L}$ be the set of all lines in the plane and let f be a function that assigns to each line $\ell \in \mathcal{L}$ a point $f(\ell)$ on ℓ. Suppose that for any point X, and for any three lines ℓ_1, ℓ_2, ℓ_3 passing through X, the points $f(\ell_1)$, $f(\ell_2)$, $f(\ell_3)$ and X lie on a circle.

Prove that there is a unique point P such that $f(\ell) = P$ for any line ℓ passing through P.

(Australia)

Common remarks. The condition on f is equivalent to the following: There is some function g that assigns to each point X a circle $g(X)$ passing through X such that for any line ℓ passing through X, the point $f(\ell)$ lies on $g(X)$. (The function g may not be uniquely defined for all points, if some points X have at most one value of $f(\ell)$ other than X; for such points, an arbitrary choice is made.)

If there were two points P and Q with the given property, $f(PQ)$ would have to be both P and Q, so there is at most one such point, and it will suffice to show that such a point exists.

Solution 1. We provide a complete characterisation of the functions satisfying the given condition.

Write $\angle(\ell_1, \ell_2)$ for the *directed* angle modulo 180° between the lines ℓ_1 and ℓ_2. Given a point P and an angle $\alpha \in (0, 180^\circ)$, for each line ℓ, let ℓ' be the line through P satisfying $\angle(\ell', \ell) = \alpha$, and let $h_{P,\alpha}(\ell)$ be the intersection point of ℓ and ℓ'. We will prove that there is some pair (P, α) such that f and $h_{P,\alpha}$ are the same function. Then P is the unique point in the problem statement.

Given an angle α and a point P, let a line ℓ be called (P, α)-*good* if $f(\ell) = h_{P,\alpha}(\ell)$. Let a point $X \neq P$ be called (P, α)-*good* if the circle $g(X)$ passes through P and some point $Y \neq P, X$ on $g(X)$ satisfies $\angle(PY, YX) = \alpha$. It follows from this definition that if X is (P, α)-good then every point $Y \neq P, X$ of $g(X)$ satisfies this angle condition, so $h_{P,\alpha}(XY) = Y$ for every $Y \in g(X)$. Equivalently, $f(\ell) \in \{X, h_{P,\alpha}(\ell)\}$ for each line ℓ passing through X. This shows the following lemma.

Lemma 1. If X is (P, α)-good and ℓ is a line passing through X then either $f(\ell) = X$ or ℓ is (P, α)-good.

Lemma 2. If X and Y are different (P, α)-good points, then line XY is (P, α)-good.

Proof. If XY is not (P, α)-good then by the previous Lemma, $f(XY) = X$ and similarly $f(XY) = Y$, but clearly this is impossible as $X \neq Y$. $\square$

Lemma 3. If ℓ_1 and ℓ_2 are different (P, α)-good lines which intersect at $X \neq P$, then either $f(\ell_1) = X$ or $f(\ell_2) = X$ or X is (P, α)-good.

Proof. If $f(\ell_1), f(\ell_2) \neq X$, then $g(X)$ is the circumcircle of X, $f(\ell_1)$ and $f(\ell_2)$. Since ℓ_1 and ℓ_2 are (P, α)-good lines, the angles

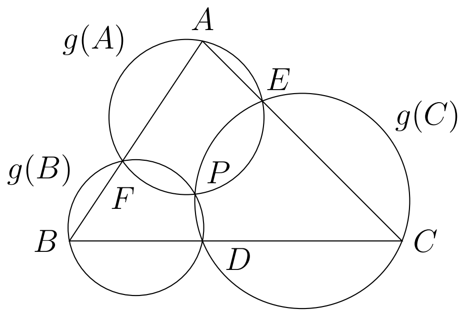
$$\angle(Pf(\ell_1), f(\ell_1)X) = \angle(Pf(\ell_2), f(\ell_2)X) = \alpha,$$

so P lies on $g(X)$. Hence, X is (P, α)-good. $\square$

Lemma 4. If ℓ_1, ℓ_2 and ℓ_3 are different (P, α)-good lines which intersect at $X \neq P$, then X is (P, α)-good.

Proof. This follows from the previous Lemma since at most one of the three lines ℓ_i can satisfy $f(\ell_i) = X$ as the three lines are all (P, α)-good. $\square$

Lemma 5. If ABC is a triangle such that A, B, C, $f(AB)$, $f(AC)$ and $f(BC)$ are all different points, then there is some point P and some angle α such that A, B and C are (P, α)-good points and AB, BC and CA are (P, α)-good lines.



Proof. Let D, E, F denote the points $f(BC)$, $f(AC)$, $f(AB)$, respectively. Then $g(A)$, $g(B)$ and $g(C)$ are the circumcircles of AEF, BDF and CDE, respectively. Let $P \neq F$ be the second intersection of circles $g(A)$ and $g(B)$ (or, if these circles are tangent at F, then $P = F$). By Miquel's theorem (or an easy angle chase), $g(C)$ also passes through P. Then by the cyclic quadrilaterals, the directed angles

$$\angle(PD, DC) = \angle(PF, FB) = \angle(PE, EA) = \alpha,$$

for some angle α. Hence, lines AB, BC and CA are all (P, α)-good, so by Lemma 3, A, B and C are (P, α)-good. (In the case where $P = D$, the line PD in the equation above denotes the line which is tangent to $g(B)$ at $P = D$. Similar definitions are used for PE and PF in the cases where $P = E$ or $P = F$.) □

Consider the set Ω of all points (x, y) with integer coordinates $1 \leqslant x, y \leqslant 1000$, and consider the set L_Ω of all horizontal, vertical and diagonal lines passing through at least one point in Ω. A simple counting argument shows that there are 5998 lines in L_Ω. For each line ℓ in L_Ω we colour the point $f(\ell)$ red. Then there are at most 5998 red points. Now we partition the points in Ω into 10000 ten by ten squares. Since there are at most 5998 red points, at least one of these squares Ω_{10} contains no red points. Let (m, n) be the bottom left point in Ω_{10}. Then the triangle with vertices (m, n), $(m + 1, n)$ and $(m, n + 1)$ satisfies the condition of Lemma 5, so these three vertices are all (P, α)-good for some point P and angle α, as are the lines joining them. From this point on, we will simply call a point or line *good* if it is (P, α)-good for this particular pair (P, α). Now by Lemma 1, the line $x = m + 1$ is good, as is the line $y = n + 1$. Then Lemma 3 implies that $(m+1, n+1)$ is good. By applying these two lemmas repeatedly, we can prove that the line $x + y = m + n + 2$ is good, then the points $(m, n + 2)$ and $(m + 2, n)$ then the lines $x = m+2$ and $y = n+2$, then the points $(m+2, n+1)$, $(m+1, n+2)$ and $(m+2, n+2)$ and so on until we have prove that all points in Ω_{10} are good.

Now we will use this to prove that every point $S \neq P$ is good. Since $g(S)$ is a circle, it passes through at most two points of Ω_{10} on any vertical line, so at most 20 points in total. Moreover, any line ℓ through S intersects at most 10 points in Ω_{10}. Hence, there are at least eight lines ℓ through S which contain a point Q in Ω_{10} which is not on $g(S)$. Since Q is not on $g(S)$, the point $f(\ell) \neq Q$. Hence, by Lemma 1, the line ℓ is good. Hence, at least eight good lines pass through S, so by Lemma 4, the point S is good. Hence, every point $S \neq P$ is good, so by Lemma 2, every line is good. In particular, every line ℓ passing through P is good, and therefore satisfies $f(\ell) = P$, as required.

Solution 2. Note that for any distinct points X, Y, the circles $g(X)$ and $g(Y)$ meet on XY at the point $f(XY) \in g(X) \cap g(Y) \cap (XY)$. We write $s(X, Y)$ for the second intersection point of circles $g(X)$ and $g(Y)$.

Lemma 1. Suppose that X, Y and Z are not collinear, and that $f(XY) \notin \{X, Y\}$ and similarly for YZ and ZX. Then $s(X, Y) = s(Y, Z) = s(Z, X)$.

Proof. The circles $g(X)$, $g(Y)$ and $g(Z)$ through the vertices of triangle XYZ meet pairwise on the corresponding edges (produced). By Miquel's theorem, the second points of intersection of any two of the circles coincide. (See the diagram for Lemma 5 of Solution 1.) □

Now pick any line ℓ and any six different points $Y_1, \ldots, Y_6$ on $\ell \setminus \{f(\ell)\}$. Pick a point X not on ℓ or any of the circles $g(Y_i)$. Reordering the indices if necessary, we may suppose that $Y_1, \ldots, Y_4$ do not lie on $g(X)$, so that $f(XY_i) \notin \{X, Y_i\}$ for $1 \leqslant i \leqslant 4$. By applying the above lemma to triangles XY_iY_j for $1 \leqslant i < j \leqslant 4$, we find that the points $s(Y_i, Y_j)$ and $s(X, Y_i)$ are all equal, to point O say. Note that either O does not lie on ℓ, or $O = f(\ell)$, since $O \in g(Y_i)$.

Now consider an arbitrary point X' not on ℓ or any of the circles $g(Y_i)$ for $1 \leqslant i \leqslant 4$. As above, we see that there are two indices $1 \leqslant i < j \leqslant 4$ such that Y_i and Y_j do not lie on $g(X')$. By applying the above lemma to triangle $X'Y_iY_j$ we see that $s(X', Y_i) = O$, and in particular $g(X')$ passes through O.

We will now show that $f(\ell') = O$ for all lines ℓ' through O. By the above note, we may assume that $\ell' \neq \ell$. Consider a variable point $X' \in \ell' \setminus \{O\}$ not on ℓ or any of the circles $g(Y_i)$ for $1 \leqslant i \leqslant 4$. We know that $f(\ell') \in g(X') \cap \ell' = \{X', O\}$. Since X' was suitably arbitrary, we have $f(\ell') = O$ as desired.

Solution 3. Notice that, for any two different points X and Y, the point $f(XY)$ lies on both $g(X)$ and $g(Y)$, so any two such circles meet in at least one point. We refer to two circles as *cutting* only in the case where they cross, and so meet at exactly two points, thus excluding the cases where they are tangent or are the same circle.

Lemma 1. Suppose there is a point P such that all circles $g(X)$ pass through P. Then P has the given property.

Proof. Consider some line ℓ passing through P, and suppose that $f(\ell) \neq P$. Consider some $X \in \ell$ with $X \neq P$ and $X \neq f(\ell)$. Then $g(X)$ passes through all of P, $f(\ell)$ and X, but those three points are collinear, a contradiction. □

Lemma 2. Suppose that, for all $\epsilon > 0$, there is a point P_ϵ with $g(P_\epsilon)$ of radius at most ϵ. Then there is a point P with the given property.

Proof. Consider a sequence $\epsilon_i = 2^{-i}$ and corresponding points P_{ϵ_i}. Because the two circles $g(P_{\epsilon_i})$ and $g(P_{\epsilon_j})$ meet, the distance between P_{ϵ_i} and P_{ϵ_j} is at most $2^{1-i} + 2^{1-j}$. As $\sum_i \epsilon_i$ converges, these points converge to some point P. For all $\epsilon > 0$, the point P has distance at most 2ϵ from P_ϵ, and all circles $g(X)$ pass through a point with distance at most 2ϵ from P_ϵ, so distance at most 4ϵ from P. A circle that passes distance at most 4ϵ from P for all $\epsilon > 0$ must pass through P, so by Lemma 1 the point P has the given property. □

Lemma 3. Suppose no two of the circles $g(X)$ cut. Then there is a point P with the given property.

Proof. Consider a circle $g(X)$ with centre Y. The circle $g(Y)$ must meet $g(X)$ without cutting it, so has half the radius of $g(X)$. Repeating this argument, there are circles with arbitrarily small radius and the result follows by Lemma 2. □

Lemma 4. Suppose there are six different points A, B_1, B_2, B_3, B_4, B_5 such that no three are collinear, no four are concyclic, and all the circles $g(B_i)$ cut pairwise at A. Then there is a point P with the given property.

Proof. Consider some line ℓ through A that does not pass through any of the B_i and is not tangent to any of the $g(B_i)$. Fix some direction along that line, and let X_ϵ be the point on ℓ that has distance ϵ from A in that direction. In what follows we consider only those ϵ for which X_ϵ does not lie on any $g(B_i)$ (this restriction excludes only finitely many possible values of ϵ).

Consider the circle $g(X_\epsilon)$. Because no four of the B_i are concyclic, at most three of them lie on this circle, so at least two of them do not. There must be some sequence of $\epsilon \to 0$ such that it is the same two of the B_i for all ϵ in that sequence, so now restrict attention to that sequence, and suppose without loss of generality that B_1 and B_2 do not lie on $g(X_\epsilon)$ for any ϵ in that sequence.

Then $f(X_\epsilon B_1)$ is not B_1, so must be the other point of intersection of $X_\epsilon B_1$ with $g(B_1)$, and the same applies with B_2. Now consider the three points X_ϵ, $f(X_\epsilon B_1)$ and $f(X_\epsilon B_2)$. As $\epsilon \to 0$, the angle at X_ϵ tends to $\angle B_1AB_2$ or $180^\circ - \angle B_1AB_2$, which is not 0 or 180° because no three of the points were collinear. All three distances between those points are bounded above by constant multiples of ϵ (in fact, if the triangle is scaled by a factor of $1/\epsilon$, it tends to a fixed triangle). Thus the circumradius of those three points, which is the radius of $g(X_\epsilon)$, is also bounded above by a constant multiple of ϵ, and so the result follows by Lemma 2. □

Lemma 5. Suppose there are two points A and B such that $g(A)$ and $g(B)$ cut. Then there is a point P with the given property.

Proof. Suppose that $g(A)$ and $g(B)$ cut at C and D. One of those points, without loss of generality C, must be $f(AB)$, and so lie on the line AB. We now consider two cases, according to whether D also lies on that line.

Case 1: D does not lie on that line.

In this case, consider a sequence of X_ϵ at distance ϵ from D, tending to D along some line that is not a tangent to either circle, but perturbed slightly (by at most ϵ^2) to ensure that no three of the points A, B and X_ϵ are collinear and no four are concyclic.

Consider the points $f(X_\epsilon A)$ and $f(X_\epsilon B)$, and the circles $g(X_\epsilon)$ on which they lie. The point $f(X_\epsilon A)$ might be either A or the other intersection of $X_\epsilon A$ with the circle $g(A)$, and the same applies for B. If, for some sequence of $\epsilon \to 0$, both those points are the other point of intersection, the same argument as in the proof of Lemma 4 applies to find arbitrarily small circles. Otherwise, we have either infinitely many of those circles passing through A, or infinitely many passing through B; without loss of generality, suppose infinitely many through A.

We now show we can find five points B_i satisfying the conditions of Lemma 4 (together with A). Let B_1 be any of the X_ϵ for which $g(X_\epsilon)$ passes through A. Then repeat the following four times, for $2 \leqslant i \leqslant 5$.

Consider some line $\ell = X_\epsilon A$ (different from those considered for previous i) that is not tangent to any of the $g(B_j)$ for $j < i$, and is such that $f(\ell) = A$, so $g(Y)$ passes through A for all Y on that line. If there are arbitrarily small circles $g(Y)$ we are done by Lemma 2, so the radii of such circles must be bounded below. But as $Y \to A$, along any line not tangent to $g(B_j)$, the radius of a circle through Y and tangent to $g(B_j)$ at A tends to 0. So there must be some Y such that $g(Y)$ cuts $g(B_j)$ at A rather than being tangent to it there, for all of the previous B_j, and we may also pick it such that no three of the B_i and A are collinear and no four are concyclic. Let B_i be this Y. Now the result follows by Lemma 4.

Case 2: D does lie on that line.

In this case, we follow a similar argument, but the sequence of X_ϵ needs to be slightly different. C and D both lie on the line AB, so one must be A and the other must be B. Consider a sequence of X_ϵ tending to B. Rather than tending to B along a straight line (with small perturbations), let the sequence be such that all the points are inside the two circles, with the angle between $X_\epsilon B$ and the tangent to $g(B)$ at B tending to 0.

Again consider the points $f(X_\epsilon A)$ and $f(X_\epsilon B)$. If, for some sequence of $\epsilon \to 0$, both those points are the other point of intersection with the respective circles, we see that the angle at X_ϵ tends to the angle between AB and the tangent to $g(B)$ at B, which is not 0 or 180°, while the distances tend to 0 (although possibly slower than any multiple of ϵ), so we have arbitrarily small circumradii and the result follows by Lemma 2. Otherwise, we have either infinitely many of the circles $g(X_\epsilon)$ passing through A, or infinitely many passing through B, and the same argument as in the previous case enables us to reduce to Lemma 4. □

Lemmas 3 and 5 together cover all cases, and so the required result is proved.

Comment. From the property that all circles $g(X)$ pass through the same point P, it is possible to deduce that the function f has the form given in Solution 1. For any line ℓ not passing through P we may define a corresponding angle $\alpha(\ell)$, which we must show is the same for all such lines. For any point $X \neq P$, with at least one line ℓ through X and not through P, such that $f(\ell) \neq X$, this angle must be equal for all such lines through X (by (directed) angles in the same segment of $g(X)$).

Now consider all horizontal and all vertical lines not through P. For any pair consisting of a horizontal line ℓ_1 and a vertical line ℓ_2, we have $\alpha(\ell_1) = \alpha(\ell_2)$ unless $f(\ell_1)$ or $f(\ell_2)$ is the point of intersection of those lines. Consider the bipartite graph whose vertices are those lines and where an edge joins a horizontal and a vertical line with the same value of α. Considering a subgraph induced by n horizontal and n vertical lines, it must have at least $n^2 - 2n$ edges, so some horizontal line has edges to at least $n - 2$ of the vertical lines. Thus, in the original graph, all but at most two of the vertical lines have the same value of α, and likewise all but at most two of the horizontal lines have the same value of α, and, restricting attention to suitable subsets of those lines, we see that this value must be the same for the vertical lines and for the horizontal lines.

But now we can extend this to *all* vertical and horizontal lines not through P (and thus to lines in other directions as well, since the only requirement for 'vertical' and 'horizontal' above is that they are any two nonparallel directions). Consider any horizontal line ℓ_1 not passing through P, and we wish to show that $\alpha(\ell_1)$ has the same value α it has for all but at most two lines not through P in any direction. Indeed, we can deduce this by considering the intersection with any but at most five of the vertical lines: the only ones to exclude are the one passing through P, the one passing through $f(\ell_1)$, at most two such that $\alpha(\ell) \neq \alpha$, and the one passing through $h_{P,\alpha}(\ell_1)$ (defined as in Solution 1). So all lines ℓ not passing through P have the same value of $\alpha(\ell)$.

Solution 4. For any point X, denote by $t(X)$ the line tangent to $g(X)$ at X; notice that $f(t(X)) = X$, so f is surjective.

Step 1: We find a point P for which there are at least two different lines p_1 and p_2 such that $f(p_i) = P$.

Choose any point X. If X does not have this property, take any $Y \in g(X) \setminus \{X\}$; then $f(XY) = Y$. If Y does not have the property, $t(Y) = XY$, and the circles $g(X)$ and $g(Y)$ meet again at some point Z. Then $f(XZ) = Z = f(YZ)$, so Z has the required property.

We will show that P is the desired point. From now on, we fix two different lines p_1 and p_2 with $f(p_1) = f(p_2) = P$. Assume for contradiction that $f(\ell) = Q \neq P$ for some line ℓ through P. We fix ℓ, and note that $Q \in g(P)$.

Step 2: We prove that $P \in g(Q)$.

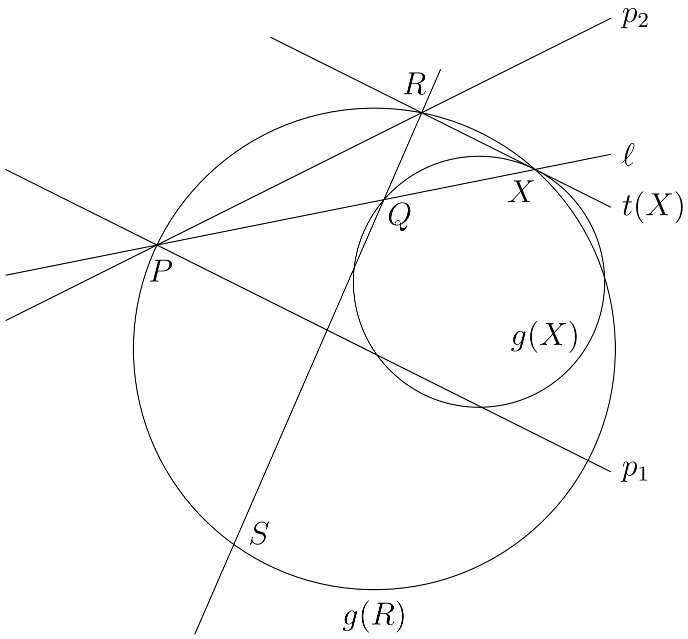
Take an arbitrary point $X \in \ell \setminus \{P, Q\}$. Two cases are possible for the position of $t(X)$ in relation to the p_i; we will show that each case (and subcase) occurs for only finitely many positions of X, yielding a contradiction.

Case 2.1: $t(X)$ is parallel to one of the p_i; say, to p_1.

Let $t(X)$ cross p_2 at R. Then $g(R)$ is the circle (PRX), as $f(RP) = P$ and $f(RX) = X$. Let RQ cross $g(R)$ again at S. Then $f(RQ) \in \{R, S\} \cap g(Q)$, so $g(Q)$ contains one of the points R and S.

If $R \in g(Q)$, then R is one of finitely many points in the intersection $g(Q) \cap p_2$, and each of them corresponds to a unique position of X, since RX is parallel to p_1.

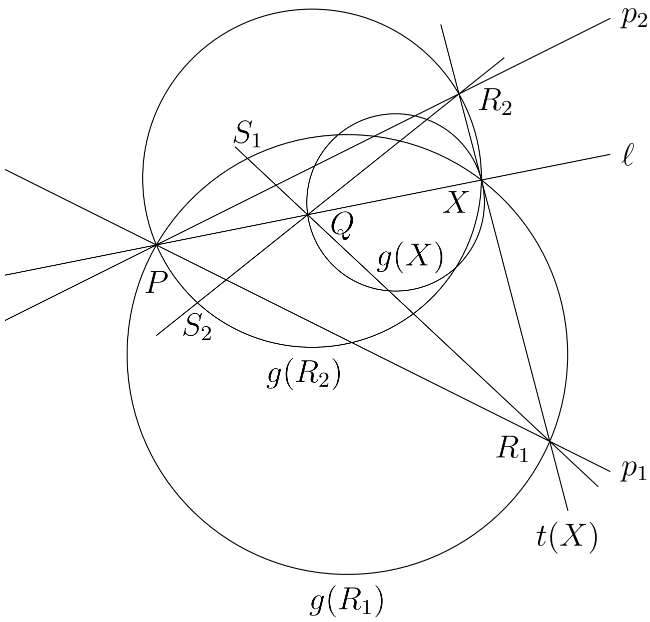
If $S \in g(Q)$, then $\angle(QS, SP) = \angle(RS, SP) = \angle(RX, XP) = \angle(p_1, \ell)$, so $\angle(QS, SP)$ is constant for all such points X, and all points S obtained in such a way lie on one circle γ passing through P and Q. Since $g(Q)$ does not contain P, it is different from γ, so there are only finitely many points S. Each of them uniquely determines R and thus X.



So, Case 2.1 can occur for only finitely many points X.

Case 2.2: $t(X)$ crosses p_1 and p_2 at R_1 and R_2, respectively.

Clearly, $R_1 \neq R_2$, as $t(X)$ is the tangent to $g(X)$ at X, and $g(X)$ meets ℓ only at X and Q. Notice that $g(R_i)$ is the circle (PXR_i). Let R_iQ meet $g(R_i)$ again at S_i; then $S_i \neq Q$, as $g(R_i)$ meets ℓ only at P and X. Then $f(R_iQ) \in \{R_i, S_i\}$, and we distinguish several subcases.



Subcase 2.2.1: $f(R_1Q) = S_1$, $f(R_2Q) = S_2$; so $S_1, S_2 \in g(Q)$.

In this case we have $0 = \angle(R_1X, XP) + \angle(XP, R_2X) = \angle(R_1S_1, S_1P) + \angle(S_2P, S_2R_2) = \angle(QS_1, S_1P) + \angle(S_2P, S_2Q)$, which shows $P \in g(Q)$.

Subcase 2.2.2: $f(R_1Q) = R_1$, $f(R_2Q) = R_2$; so $R_1, R_2 \in g(Q)$.

This can happen for at most four positions of X – namely, at the intersections of ℓ with a line of the form K_1K_2, where $K_i \in g(Q) \cap p_i$.

Subcase 2.2.3: $f(R_1Q) = S_1$, $f(R_2Q) = R_2$ (the case $f(R_1Q) = R_1$, $f(R_2Q) = S_2$ is similar).

In this case, there are at most two possible positions for R_2 – namely, the meeting points of $g(Q)$ with p_2. Consider one of them. Let X vary on ℓ. Then R_1 is the projection of X to p_1 via R_2, S_1 is the projection of R_1 to $g(Q)$ via Q. Finally, $\angle(QS_1, S_1X) = \angle(R_1S_1, S_1X) = \angle(R_1P, PX) = \angle(p_1, \ell) \neq 0$, so X is obtained by a fixed projective transform $g(Q) \to \ell$ from S_1. So, if there were three points X satisfying the conditions of this subcase, the composition of the three projective transforms would be the identity. But, if we apply it to $X = Q$, we successively get some point R_1', then R_2, and then some point different from Q, a contradiction.

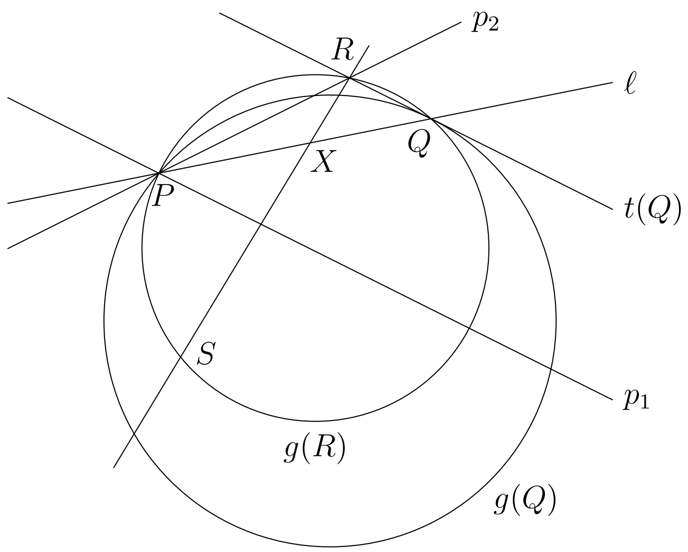
Thus Case 2.2 also occurs for only finitely many points X, as desired.

Step 3: We show that $f(PQ) = P$, as desired.

The argument is similar to that in Step 2, with the roles of Q and X swapped. Again, we show that there are only finitely many possible positions for a point $X \in \ell \setminus \{P, Q\}$, which is absurd.

Case 3.1: $t(Q)$ is parallel to one of the p_i; say, to p_1.

Let $t(Q)$ cross p_2 at R; then $g(R)$ is the circle (PRQ). Let RX cross $g(R)$ again at S. Then $f(RX) \in \{R, S\} \cap g(X)$, so $g(X)$ contains one of the points R and S.



Subcase 3.1.1: $S = f(RX) \in g(X)$.

We have $\angle(t(X), QX) = \angle(SX, SQ) = \angle(SR, SQ) = \angle(PR, PQ) = \angle(p_2, \ell)$. Hence $t(X) \parallel p_2$. Now we recall Case 2.1: we let $t(X)$ cross p_1 at R', so $g(R') = (PR'X)$, and let $R'Q$ meet $g(R')$ again at S'; notice that $S' \neq Q$. Excluding one position of X, we may assume that $R' \notin g(Q)$, so $R' \neq f(R'Q)$. Therefore, $S' = f(R'Q) \in g(Q)$. But then, as in Case 2.1, we get $\angle(t(Q), PQ) = \angle(QS', S'P) = \angle(R'X, XP) = \angle(p_2, \ell)$. This means that $t(Q)$ is parallel to p_2, which is impossible.

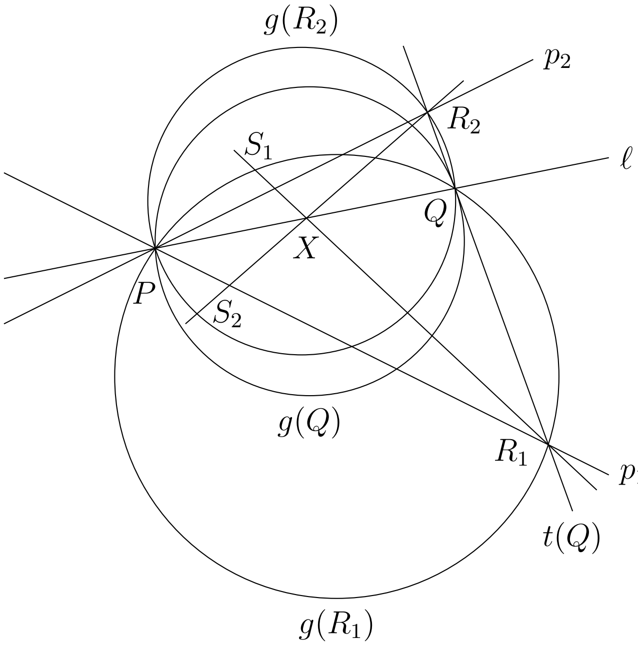
Subcase 3.1.2: $R = f(RX) \in g(X)$.

In this case, we have $\angle(t(X), \ell) = \angle(RX, RQ) = \angle(RX, p_1)$. Again, let $R' = t(X) \cap p_1$; this point exists for all but at most one position of X. Then $g(R') = (R'XP)$; let $R'Q$ meet $g(R')$ again at S'. Due to $\angle(R'X, XR) = \angle(QX, QR) = \angle(\ell, p_1)$, R' determines X in at most two ways, so for all but finitely many positions of X we have $R' \notin g(Q)$. Therefore, for those positions we have $S' = f(R'Q) \in g(Q)$. But then $\angle(RX, p_1) = \angle(R'X, XP) = \angle(R'S', S'P) = \angle(QS', S'P) = \angle(t(Q), QP)$ is fixed, so this case can hold only for one specific position of X as well.

Thus, in Case 3.1, there are only finitely many possible positions of X, yielding a contradiction.

Case 3.2: $t(Q)$ crosses p_1 and p_2 at R_1 and R_2, respectively.

By Step 2, $R_1 \neq R_2$. Notice that $g(R_i)$ is the circle (PQR_i). Let R_iX meet $g(R_i)$ at S_i; then $S_i \neq X$. Then $f(R_iX) \in \{R_i, S_i\}$, and we distinguish several subcases.



Subcase 3.2.1: $f(R_1X) = S_1$ and $f(R_2X) = S_2$, so $S_1, S_2 \in g(X)$.

As in Subcase 2.2.1, we have $0 = \angle(R_1Q, QP) + \angle(QP, R_2Q) = \angle(XS_1, S_1P) + \angle(S_2P, S_2X)$, which shows $P \in g(X)$. But $X, Q \in g(X)$ as well, so $g(X)$ meets ℓ at three distinct points, which is absurd.

Subcase 3.2.2: $f(R_1X) = R_1$, $f(R_2X) = R_2$, so $R_1, R_2 \in g(X)$.

Now three distinct collinear points R_1, R_2, and Q belong to $g(X)$, which is impossible.

Subcase 3.2.3: $f(R_1X) = S_1$, $f(R_2X) = R_2$ (the case $f(R_1X) = R_1$, $f(R_2X) = S_2$ is similar).

We have $\angle(XR_2, R_2Q) = \angle(XS_1, S_1Q) = \angle(R_1S_1, S_1Q) = \angle(R_1P, PQ) = \angle(p_1, \ell)$, so this case can occur for a unique position of X.

Thus, in Case 3.2, there is only a unique position of X, again yielding the required contradiction.

Number Theory

N1. Find all pairs (m, n) of positive integers satisfying the equation

$$(2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}) = m! \tag{1}$$

(El Salvador)

Answer: The only such pairs are $(1, 1)$ and $(3, 2)$.

Common remarks. In all solutions, for any prime p and positive integer N, we will denote by $v_p(N)$ the exponent of the largest power of p that divides N. The left-hand side of (1) will be denoted by L_n; that is, $L_n = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1})$.

Solution 1. We will get an upper bound on n from the speed at which $v_2(L_n)$ grows.

From

$$L_n = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}) = 2^{1+2+\cdots+(n-1)}(2^n - 1)(2^{n-1} - 1) \cdots (2^1 - 1)$$

we read

$$v_2(L_n) = 1 + 2 + \cdots + (n - 1) = \frac{n(n-1)}{2}.$$

On the other hand, $v_2(m!)$ is expressed by the Legendre formula as

$$v_2(m!) = \sum_{i=1}^{\infty} \left\lfloor \frac{m}{2^i} \right\rfloor.$$

As usual, by omitting the floor functions,

$$v_2(m!) < \sum_{i=1}^{\infty} \frac{m}{2^i} = m.$$

Thus, $L_n = m!$ implies the inequality

$$\frac{n(n-1)}{2} < m. \tag{2}$$

In order to obtain an opposite estimate, observe that

$$L_n = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}) < (2^n)^n = 2^{n^2}.$$

We claim that

$$2^{n^2} < \left(\frac{n(n-1)}{2}\right)! \quad \text{for } n \geqslant 6. \tag{3}$$

For $n = 6$ the estimate (3) is true because $2^{6^2} < 6.9 \cdot 10^{10}$ and $\left(\frac{n(n-1)}{2}\right)! = 15! > 1.3 \cdot 10^{12}$.

For $n \geqslant 7$ we prove (3) by the following inequalities:

$$\begin{aligned}\left(\frac{n(n-1)}{2}\right)! &= 15! \cdot 16 \cdot 17 \cdots \frac{n(n-1)}{2} > 2^{36} \cdot 16^{\frac{n(n-1)}{2} - 15} \\ &= 2^{2n(n-1)-24} = 2^{n^2} \cdot 2^{n(n-2)-24} > 2^{n^2}.\end{aligned}$$

Putting together (2) and (3), for $n \geqslant 6$ we get a contradiction, since

$$L_n < 2^{n^2} < \left(\frac{n(n-1)}{2}\right)! < m! = L_n.$$

Hence $n \geqslant 6$ is not possible.

Checking manually the cases $n \leqslant 5$ we find

$$L_1 = 1 = 1!, \qquad L_2 = 6 = 3!, \qquad 5! < L_3 = 168 < 6!,$$
$$7! < L_4 = 20\,160 < 8! \qquad \text{and} \qquad 10! < L_5 = 9\,999\,360 < 11!.$$

So, there are two solutions:

$$(m,n) \in \{(1,1),(3,2)\}.$$

Solution 2. Like in the previous solution, the cases $n = 1,2,3,4$ are checked manually. We will exclude $n \geqslant 5$ by considering the exponents of 3 and 31 in (1).

For odd primes p and distinct integers a, b, coprime to p, with $p \mid a - b$, the Lifting The Exponent lemma asserts that

$$v_p(a^k - b^k) = v_p(a-b) + v_p(k).$$

Notice that 3 divides $2^k - 1$ if only if k is even; moreover, by the Lifting The Exponent lemma we have

$$v_3(2^{2k}-1) = v_3(4^k - 1) = 1 + v_3(k) = v_3(3k).$$

Hence,

$$v_3(L_n) = \sum_{2k \leqslant n} v_3(4^k - 1) = \sum_{k \leqslant \lfloor \frac{n}{2} \rfloor} v_3(3k).$$

Notice that the last expression is precisely the exponent of 3 in the prime factorisation of $\left(3\lfloor\frac{n}{2}\rfloor\right)!$. Therefore

$$v_3(m!) = v_3(L_n) = v_3\left(\left(3\lfloor\tfrac{n}{2}\rfloor\right)!\right)$$

$$3\left\lfloor\frac{n}{2}\right\rfloor \leqslant m \leqslant 3\left\lfloor\frac{n}{2}\right\rfloor + 2. \tag{4}$$

Suppose that $n \geqslant 5$. Note that every fifth factor in L_n is divisible by $31 = 2^5 - 1$, and hence we have $v_{31}(L_n) \geqslant \lfloor\frac{n}{5}\rfloor$. Then

$$\frac{n}{10} \leqslant \left\lfloor\frac{n}{5}\right\rfloor \leqslant v_{31}(L_n) = v_{31}(m!) = \sum_{k=1}^{\infty}\left\lfloor\frac{m}{31^k}\right\rfloor < \sum_{k=1}^{\infty}\frac{m}{31^k} = \frac{m}{30}. \tag{5}$$

By combining (4) and (5),

$$3n < m \leqslant \frac{3n}{2} + 2$$

so $n < \frac{4}{3}$ which is inconsistent with the inequality $n \geqslant 5$.

Comment 1. There are many combinations of the ideas above; for example combining (2) and (4) also provides $n < 5$. Obviously, considering the exponents of any two primes in (1), or considering one prime and the magnitude orders lead to an upper bound on n and m.

Comment 2. This problem has a connection to group theory. Indeed, the left-hand side is the order of the group $GL_n(\mathbb{F}_2)$ of invertible n-by-n matrices with entries modulo 2, while the right-hand side is the order of the symmetric group S_m on m elements. The result thus shows that the only possible isomorphisms between these groups are $GL_1(\mathbb{F}_2) \cong S_1$ and $GL_2(\mathbb{F}_2) \cong S_3$, and there are in fact isomorphisms in both cases. In general, $GL_n(\mathbb{F}_2)$ is a simple group for $n \geqslant 3$, as it is isomorphic to $PSL_n(\mathbb{F}_2)$.

There is also a near-solution of interest: the left-hand side for $n = 4$ is *half* of the right-hand side when $m = 8$; this turns out to correspond to an isomorphism $GL_4(\mathbb{F}_2) \cong A_8$ with the alternating group on eight elements.

However, while this indicates that the problem is a useful one, knowing group theory is of no use in solving it!

N2. Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

(Nigeria)

Answer: The solutions are $(1, 2, 3)$ and its permutations.

Common remarks. Note that the equation is symmetric. In all solutions, we will assume without loss of generality that $a \geqslant b \geqslant c$, and prove that the only solution is $(a, b, c) = (3, 2, 1)$.

The first two solutions all start by proving that $c = 1$.

Solution 1. We will start by proving that $c = 1$. Note that

$$3a^3 \geqslant a^3 + b^3 + c^3 > a^3 .$$

So $3a^3 \geqslant (abc)^2 > a^3$ and hence $3a \geqslant b^2c^2 > a$. Now $b^3 + c^3 = a^2(b^2c^2 - a) \geqslant a^2$, and so

$$18b^3 \geqslant 9(b^3 + c^3) \geqslant 9a^2 \geqslant b^4c^4 \geqslant b^3c^5 ,$$

so $18 \geqslant c^5$ which yields $c = 1$.

Now, note that we must have $a > b$, as otherwise we would have $2b^3 + 1 = b^4$ which has no positive integer solutions. So

$$a^3 - b^3 \geqslant (b + 1)^3 - b^3 > 1$$

and

$$2a^3 > 1 + a^3 + b^3 > a^3 ,$$

which implies $2a^3 > a^2b^2 > a^3$ and so $2a > b^2 > a$. Therefore

$$4(1 + b^3) = 4a^2(b^2 - a) \geqslant 4a^2 > b^4 ,$$

so $4 > b^3(b - 4)$; that is, $b \leqslant 4$.

Now, for each possible value of b with $2 \leqslant b \leqslant 4$ we obtain a cubic equation for a with constant coefficients. These are as follows:

$$\begin{aligned} b = 2: &\qquad a^3 - 4a^2 + 9 = 0 \\ b = 3: &\qquad a^3 - 9a^2 + 28 = 0 \\ b = 4: &\qquad a^3 - 16a^2 + 65 = 0. \end{aligned}$$

The only case with an integer solution for a with $b \leqslant a$ is $b = 2$, leading to $(a, b, c) = (3, 2, 1)$.

Comment 1.1. Instead of writing down each cubic equation explicitly, we could have just observed that $a^2 \mid b^3 + 1$, and for each choice of b checked each square factor of $b^3 + 1$ for a^2.

We could also have observed that, with $c = 1$, the relation $18b^3 \geqslant b^4c^4$ becomes $b \leqslant 18$, and we can simply check all possibilities for b (instead of working to prove that $b \leqslant 4$). This check becomes easier after using the factorisation $b^3 + 1 = (b + 1)(b^2 - b + 1)$ and observing that no prime besides 3 can divide both of the factors.

Comment 1.2. Another approach to finish the problem after establishing that $c \leqslant 1$ is to set $k = b^2c^2 - a$, which is clearly an integer and must be positive as it is equal to $(b^3 + c^3)/a^2$. Then we divide into cases based on whether $k = 1$ or $k \geqslant 2$; in the first case, we have $b^3 + 1 = a^2 = (b^2 - 1)^2$ whose only positive root is $b = 2$, and in the second case we have $b^2 \leqslant 3a$, and so

$$b^4 \leqslant (3a)^2 \leqslant \frac{9}{2}(ka^2) = \frac{9}{2}(b^3 + 1),$$

which implies that $b \leqslant 4$.

Solution 2. Again, we will start by proving that $c = 1$. Suppose otherwise that $c \geqslant 2$. We have $a^3 + b^3 + c^3 \leqslant 3a^3$, so $b^2c^2 \leqslant 3a$. Since $c \geqslant 2$, this tells us that $b \leqslant \sqrt{3a/4}$. As the right-hand side of the original equation is a multiple of a^2, we have $a^2 \leqslant 2b^3 \leqslant 2(3a/4)^{3/2}$. In other words, $a \leqslant \frac{27}{16} < 2$, which contradicts the assertion that $a \geqslant c \geqslant 2$. So there are no solutions in this case, and so we must have $c = 1$.

Now, the original equation becomes $a^3 + b^3 + 1 = a^2b^2$. Observe that $a \geqslant 2$, since otherwise $a = b = 1$ as $a \geqslant b$.

The right-hand side is a multiple of a^2, so the left-hand side must be as well. Thus, $b^3 + 1 \geqslant a^2$. Since $a \geqslant b$, we also have

$$b^2 = a + \frac{b^3 + 1}{a^2} \leqslant 2a + \frac{1}{a^2}$$

and so $b^2 \leqslant 2a$ since b^2 is an integer. Thus $(2a)^{3/2} + 1 \geqslant b^3 + 1 \geqslant a^2$, from which we deduce $a \leqslant 8$.

Now, for each possible value of a with $2 \leqslant a \leqslant 8$ we obtain a cubic equation for b with constant coefficients. These are as follows:

$$\begin{aligned}
a = 2: &\qquad b^3 - 4b^2 + 9 = 0\\
a = 3: &\qquad b^3 - 9b^2 + 28 = 0\\
a = 4: &\qquad b^3 - 16b^2 + 65 = 0\\
a = 5: &\qquad b^3 - 25b^2 + 126 = 0\\
a = 6: &\qquad b^3 - 36b^2 + 217 = 0\\
a = 7: &\qquad b^3 - 49b^2 + 344 = 0\\
a = 8: &\qquad b^3 - 64b^2 + 513 = 0.
\end{aligned}$$

The only case with an integer solution for b with $a \geqslant b$ is $a = 3$, leading to $(a, b, c) = (3, 2, 1)$.

Comment 2.1. As in Solution 1, instead of writing down each cubic equation explicitly, we could have just observed that $b^2 \mid a^3 + 1$, and for each choice of a checked each square factor of $a^3 + 1$ for b^2.

Comment 2.2. This solution does not require initially proving that $c = 1$, in which case the bound would become $a \leqslant 108$. The resulting cases could, in principle, be checked by a particularly industrious student.

Solution 3. Set $k = (b^3 + c^3)/a^2 \leqslant 2a$, and rewrite the original equation as $a + k = (bc)^2$. Since b^3 and c^3 are positive integers, we have $(bc)^3 \geqslant b^3 + c^3 - 1 = ka^2 - 1$, so

$$a + k \geqslant (ka^2 - 1)^{2/3}.$$

As in Comment 1.2, k is a positive integer; for each value of $k \geqslant 1$, this gives us a polynomial inequality satisfied by a:

$$k^2a^4 - a^3 - 5ka^2 - 3k^2a - (k^3 - 1) \leqslant 0.$$

We now prove that $a \leqslant 3$. Indeed,

$$0 \geqslant \frac{k^2a^4 - a^3 - 5ka^2 - 3k^2a - (k^3 - 1)}{k^2} \geqslant a^4 - a^3 - 5a^2 - 3a - k \geqslant a^4 - a^3 - 5a^2 - 5a,$$

which fails when $a \geqslant 4$.

This leaves ten triples with $3 \geqslant a \geqslant b \geqslant c \geqslant 1$, which may be checked manually to give $(a, b, c) = (3, 2, 1)$.

Solution 4. Again, observe that $b^3 + c^3 = a^2(b^2c^2 - a)$, so $b \leqslant a \leqslant b^2c^2 - 1$.

We consider the function $f(x) = x^2(b^2c^2 - x)$. It can be seen that that on the interval $[0, b^2c^2 - 1]$ the function f is increasing if $x < \frac{2}{3}b^2c^2$ and decreasing if $x > \frac{2}{3}b^2c^2$. Consequently, it must be the case that

$$b^3 + c^3 = f(a) \geqslant \min\Big(f(b), f(b^2c^2 - 1)\Big).$$

First, suppose that $b^3 + c^3 \geqslant f(b^2c^2 - 1)$. This may be written $b^3 + c^3 \geqslant (b^2c^2 - 1)^2$, and so

$$2b^3 \geqslant b^3 + c^3 \geqslant (b^2c^2 - 1)^2 > b^4c^4 - 2b^2c^2 \geqslant b^4c^4 - 2b^3c^4.$$

Thus, $(b-2)c^4 < 2$, and the only solutions to this inequality have $(b, c) = (2, 2)$ or $b \leqslant 3$ and $c = 1$. It is easy to verify that the only case giving a solution for $a \geqslant b$ is $(a, b, c) = (3, 2, 1)$.

Otherwise, suppose that $b^3 + c^3 = f(a) \geqslant f(b)$. Then, we have

$$2b^3 \geqslant b^3 + c^3 = a^2(b^2c^2 - a) \geqslant b^2(b^2c^2 - b).$$

Consequently $bc^2 \leqslant 3$, with strict inequality in the case that $b \neq c$. Hence $c = 1$ and $b \leqslant 2$. Both of these cases have been considered already, so we are done.

Comment 4.1. Instead of considering which of $f(b)$ and $f(b^2c^2 - 1)$ is less than $f(a)$, we may also proceed by explicitly dividing into cases based on whether $a \geqslant \frac{2}{3}b^2c^2$ or $a < \frac{2}{3}b^2c^2$. The first case may now be dealt with as follows. We have $b^3c^3 + 1 \geqslant b^3 + c^3$ as b^3 and c^3 are positive integers, so we have

$$b^3c^3 + 1 \geqslant b^3 + c^3 \geqslant a^2 \geqslant \frac{4}{9}b^4c^4.$$

This implies $bc \leqslant 2$, and hence $c = 1$ and $b \leqslant 2$.

N3. We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \ldots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

(Czech Republic)

Answer: The set $\mathbb{Z}$ of all integers is the only such rootiful set.

Solution 1. The set $\mathbb{Z}$ of all integers is clearly rootiful. We shall prove that any rootiful set S containing all the numbers of the form $2^a - 2^b$ for $a, b \in \mathbb{Z}_{>0}$ must be all of $\mathbb{Z}$.

First, note that $0 = 2^1 - 2^1 \in S$ and $2 = 2^2 - 2^1 \in S$. Now, $-1 \in S$, since it is a root of $2x + 2$, and $1 \in S$, since it is a root of $2x^2 - x - 1$. Also, if $n \in S$ then $-n$ is a root of $x + n$, so it suffices to prove that all positive integers must be in S.

Now, we claim that any positive integer n has a multiple in S. Indeed, suppose that $n = 2^\alpha \cdot t$ for $\alpha \in \mathbb{Z}_{\geqslant 0}$ and t odd. Then $t \mid 2^{\phi(t)} - 1$, so $n \mid 2^{\alpha+\phi(t)+1} - 2^{\alpha+1}$. Moreover, $2^{\alpha+\phi(t)+1} - 2^{\alpha+1} \in S$, and so S contains a multiple of every positive integer n.

We will now prove by induction that all positive integers are in S. Suppose that $0, 1, \ldots, n-1 \in S$; furthermore, let N be a multiple of n in S. Consider the base-n expansion of N, say $N = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$. Since $0 \leqslant a_i < n$ for each a_i, we have that all the a_i are in S. Furthermore, $a_0 = 0$ since N is a multiple of n. Therefore, $a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n - N = 0$, so n is a root of a polynomial with coefficients in S. This tells us that $n \in S$, completing the induction.

Solution 2. As in the previous solution, we can prove that $0, 1$ and -1 must all be in any rootiful set S containing all numbers of the form $2^a - 2^b$ for $a, b \in \mathbb{Z}_{>0}$.

We show that, in fact, every integer k with $|k| > 2$ can be expressed as a root of a polynomial whose coefficients are of the form $2^a - 2^b$. Observe that it suffices to consider the case where k is positive, as if k is a root of $a_n x^n + \cdots + a_1 x + a_0 = 0$, then $-k$ is a root of $(-1)^n a_n x^n + \cdots - a_1 x + a_0 = 0$.

Note that

$$(2^{a_n} - 2^{b_n}) k^n + \cdots + (2^{a_0} - 2^{b_0}) = 0$$

is equivalent to

$$2^{a_n} k^n + \cdots + 2^{a_0} = 2^{b_n} k^n + \cdots + 2^{b_0}.$$

Hence our aim is to show that two numbers of the form $2^{a_n} k^n + \cdots + 2^{a_0}$ are equal, for a fixed value of n. We consider such polynomials where every term $2^{a_i} k^i$ is at most $2k^n$; in other words, where $2 \leqslant 2^{a_i} \leqslant 2k^{n-i}$, or, equivalently, $1 \leqslant a_i \leqslant 1 + (n-i)\log_2 k$. Therefore, there must be $1 + \lfloor (n-i)\log_2 k \rfloor$ possible choices for a_i satisfying these constraints.

The number of possible polynomials is then

$$\prod_{i=0}^{n} (1 + \lfloor (n-i)\log_2 k \rfloor) \geqslant \prod_{i=0}^{n-1} (n-i)\log_2 k = n!(\log_2 k)^n$$

where the inequality holds as $1 + \lfloor x \rfloor \geqslant x$.

As there are $(n+1)$ such terms in the polynomial, each at most $2k^n$, such a polynomial must have value at most $2k^n(n+1)$. However, for large n, we have $n!(\log_2 k)^n > 2k^n(n+1)$. Therefore there are more polynomials than possible values, so some two must be equal, as required.

N4. Let $\mathbb{Z}_{>0}$ be the set of positive integers. A positive integer constant C is given. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that, for all positive integers a and b satisfying $a + b > C$,

$$a + f(b) \mid a^2 + b\,f(a). \qquad (*)$$

(Croatia)

Answer: The functions satisfying $(*)$ are exactly the functions $f(a) = ka$ for some constant $k \in \mathbb{Z}_{>0}$ (irrespective of the value of C).

Common remarks. It is easy to verify that the functions $f(a) = ka$ satisfy $(*)$. Thus, in the proofs below, we will only focus on the converse implication: that condition $(*)$ implies that $f = ka$.

A common minor part of these solutions is the derivation of some relatively easy bounds on the function f. An upper bound is easily obtained by setting $a = 1$ in $(*)$, giving the inequality

$$f(b) \leqslant b \cdot f(1)$$

for all sufficiently large b. The corresponding lower bound is only marginally more difficult to obtain: substituting $b = 1$ in the original equation shows that

$$a + f(1) \mid (a^2 + f(a)) - (a - f(1)) \cdot (a + f(1)) = f(1)^2 + f(a)$$

for all sufficiently large a. It follows from this that one has the lower bound

$$f(a) \geqslant a + f(1) \cdot (1 - f(1)),$$

again for all sufficiently large a.

Each of the following proofs makes use of at least one of these bounds.

Solution 1. First, we show that $b \mid f(b)^2$ for all b. To do this, we choose a large positive integer n so that $nb - f(b) \geqslant C$. Setting $a = nb - f(b)$ in $(*)$ then shows that

$$nb \mid (nb - f(b))^2 + bf(nb - f(b))$$

so that $b \mid f(b)^2$ as claimed.

Now in particular we have that $p \mid f(p)$ for every prime p. If we write $f(p) = k(p) \cdot p$, then the bound $f(p) \leqslant f(1) \cdot p$ (valid for p sufficiently large) shows that some value k of $k(p)$ must be attained for infinitely many p. We will show that $f(a) = ka$ for all positive integers a. To do this, we substitute $b = p$ in $(*)$, where p is any sufficiently large prime for which $k(p) = k$, obtaining

$$a + kp \mid (a^2 + pf(a)) - a(a + kp) = pf(a) - pka.$$

For suitably large p we have $\gcd(a + kp, p) = 1$, and hence we have

$$a + kp \mid f(a) - ka.$$

But the only way this can hold for arbitrarily large p is if $f(a) - ka = 0$. This concludes the proof.

Comment. There are other ways to obtain the divisibility $p \mid f(p)$ for primes p, which is all that is needed in this proof. For instance, if $f(p)$ were not divisible by p then the arithmetic progression $p^2 + bf(p)$ would attain prime values for infinitely many b by Dirichlet's Theorem: hence, for these pairs p, b, we would have $p + f(b) = p^2 + bf(p)$. Substituting $a \mapsto b$ and $b \mapsto p$ in $(*)$ then shows that $(f(p)^2 - p^2)(p - 1)$ is divisible by $b + f(p)$ and hence vanishes, which is impossible since $p \nmid f(p)$ by assumption.

Solution 2. First, we substitute $b = 1$ in $(*)$ and rearrange to find that

$$\frac{f(a)+f(1)^2}{a+f(1)} = f(1) - a + \frac{a^2+f(a)}{a+f(1)}$$

is a positive integer for sufficiently large a. Since $f(a) \leqslant af(1)$, for all sufficiently large a, it follows that $\frac{f(a)+f(1)^2}{a+f(1)} \leqslant f(1)$ also and hence there is a positive integer k such that $\frac{f(a)+f(1)^2}{a+f(1)} = k$ for infinitely many values of a. In other words,

$$f(a) = ka + f(1) \cdot (k - f(1))$$

for infinitely many a.

Fixing an arbitrary choice of a in $(*)$, we have that

$$\frac{a^2 + bf(a)}{a + kb + f(1) \cdot (k - f(1))}$$

is an integer for infinitely many b (the same b as above, maybe with finitely many exceptions). On the other hand, for b taken sufficiently large, this quantity becomes arbitrarily close to $\frac{f(a)}{k}$; this is only possible if $\frac{f(a)}{k}$ is an integer and

$$\frac{a^2 + bf(a)}{a + kb + f(1) \cdot (k - f(1))} = \frac{f(a)}{k}$$

for infinitely many b. This rearranges to

$$\frac{f(a)}{k} \cdot \big(a + f(1) \cdot (k - f(1))\big) = a^2. \tag{$**$}$$

Hence a^2 is divisible by $a + f(1) \cdot (k - f(1))$, and hence so is $f(1)^2(k - f(1))^2$. The only way this can occur for all a is if $k = f(1)$, in which case $(**)$ provides that $f(a) = ka$ for all a, as desired.

Solution 3. Fix any two distinct positive integers a and b. From $(*)$ it follows that the two integers

$$(a^2 + cf(a)) \cdot (b + f(c)) \text{ and } (b^2 + cf(b)) \cdot (a + f(c))$$

are both multiples of $(a + f(c)) \cdot (b + f(c))$ for all sufficiently large c. Taking an appropriate linear combination to eliminate the $cf(c)$ term, we find after expanding out that the integer

$$\big[a^2 f(b) - b^2 f(a)\big] \cdot f(c) + \big[(b-a)f(a)f(b)\big] \cdot c + \big[ab(af(b) - bf(a))\big] \tag{$\dagger$}$$

is also a multiple of $(a + f(c)) \cdot (b + f(c))$.

But as c varies, $(\dagger)$ is bounded above by a positive multiple of c while $(a + f(c)) \cdot (b + f(c))$ is bounded below by a positive multiple of c^2. The only way that such a divisibility can hold is if in fact

$$\big[a^2 f(b) - b^2 f(a)\big] \cdot f(c) + \big[(b-a)f(a)f(b)\big] \cdot c + \big[ab(af(b) - bf(a))\big] = 0 \tag{$\dagger\dagger$}$$

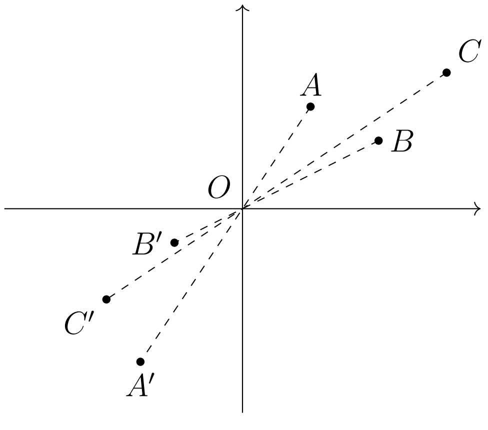
for sufficiently large c. Since the coefficient of c in this linear relation is nonzero, it follows that there are constants k, ℓ such that $f(c) = kc + \ell$ for all sufficiently large c; the constants k and ℓ are necessarily integers.

The value of ℓ satisfies

$$\big[a^2 f(b) - b^2 f(a)\big] \cdot \ell + \big[ab(af(b) - bf(a))\big] = 0 \tag{$\dagger\dagger\dagger$}$$

and hence $b \mid \ell a^2 f(b)$ for all a and b. Taking b sufficiently large so that $f(b) = kb + \ell$, we thus have that $b \mid \ell^2 a^2$ for all sufficiently large b; this implies that $\ell = 0$. From $(\dagger\dagger\dagger)$ it then follows that $\frac{f(a)}{a} = \frac{f(b)}{b}$ for all $a \neq b$, so that there is a constant k such that $f(a) = ka$ for all a (k is equal to the constant defined earlier).

Solution 4. Let Γ denote the set of all points $(a, f(a))$, so that Γ is an infinite subset of the upper-right quadrant of the plane. For a point $A = (a, f(a))$ in Γ, we define a point $A' = (-f(a), -f(a)^2/a)$ in the lower-left quadrant of the plane, and let Γ' denote the set of all such points A'.



Claim. For any point $A \in \Gamma$, the set Γ is contained in finitely many lines through the point A'.

Proof. Let $A = (a, f(a))$. The functional equation (with a and b interchanged) can be rewritten as $b + f(a) \mid af(b) - bf(a)$, so that all but finitely many points in Γ are contained in one of the lines with equation

$$ay - f(a)x = m(x + f(a))$$

for m an integer. Geometrically, these are the lines through $A' = (-f(a), -f(a)^2/a)$ with gradient $\frac{f(a)+m}{a}$. Since Γ is contained, with finitely many exceptions, in the region $0 \leqslant y \leqslant f(1) \cdot x$ and the point A' lies strictly in the lower-left quadrant of the plane, there are only finitely many values of m for which this line meets Γ. This concludes the proof of the claim. $\square$

Now consider any distinct points $A, B \in \Gamma$. It is clear that A' and B' are distinct. A line through A' and a line through B' only meet in more than one point if these two lines are equal to the line $A'B'$. It then follows from the above claim that the line $A'B'$ must contain all but finitely many points of Γ. If C is another point of Γ, then the line $A'C'$ also passes through all but finitely many points of Γ, which is only possible if $A'C' = A'B'$.

We have thus seen that there is a line ℓ passing through all points of Γ' and through all but finitely many points of Γ. We claim that this line passes through the origin O and passes through every point of Γ. To see this, note that by construction A, O, A' are collinear for every point $A \in \Gamma$. Since $\ell = AA'$ for all but finitely many points $A \in \Gamma$, it thus follows that $O \in \ell$. Thus any $A \in \Gamma$ lies on the line $\ell = A'O$.

Since Γ is contained in a line through O, it follows that there is a real constant k (the gradient of ℓ) such that $f(a) = ka$ for all a. The number k is, of course, a positive integer.

Comment. Without the $a + b > C$ condition, this problem is approachable by much more naive methods. For instance, using the given divisibility for $a, b \in \{1, 2, 3\}$ one can prove by a somewhat tedious case-check that $f(2) = 2f(1)$ and $f(3) = 3f(1)$; this then forms the basis of an induction establishing that $f(n) = nf(1)$ for all n.

N5. Let a be a positive integer. We say that a positive integer b is *a-good* if $\binom{an}{b}-1$ is divisible by $an+1$ for all positive integers n with $an \geqslant b$. Suppose b is a positive integer such that b is a-good, but $b+2$ is not a-good. Prove that $b+1$ is prime.

(Netherlands)

Solution 1. For p a prime and n a nonzero integer, we write $v_p(n)$ for the *p-adic valuation of n*: the largest integer t such that $p^t \mid n$.

We first show that b is a-good if and only if b is even, and $p \mid a$ for all primes $p \leqslant b$.

To start with, the condition that $an+1 \mid \binom{an}{b}-1$ can be rewritten as saying that

$$\frac{an(an-1)\cdots(an-b+1)}{b!} \equiv 1 \pmod{an+1}. \tag{1}$$

Suppose, on the one hand, there is a prime $p \leqslant b$ with $p \nmid a$. Take $t=v_p(b!)$. Then there exist positive integers c such that $ac \equiv 1 \pmod{p^{t+1}}$. If we take c big enough, and then take $n=(p-1)c$, then $an=a(p-1)c \equiv p-1 \pmod{p^{t+1}}$ and $an \geqslant b$. Since $p \leqslant b$, one of the terms of the numerator $an(an-1)\cdots(an-b+1)$ is $an-p+1$, which is divisible by p^{t+1}. Hence the p-adic valuation of the numerator is at least $t+1$, but that of the denominator is exactly t. This means that $p \mid \binom{an}{b}$, so $p \nmid \binom{an}{b}-1$. As $p \mid an+1$, we get that $an+1 \nmid \binom{an}{b}$, so b is not a-good.

On the other hand, if for all primes $p \leqslant b$ we have $p \mid a$, then every factor of $b!$ is coprime to $an+1$, and hence invertible modulo $an+1$: hence $b!$ is also invertible modulo $an+1$. Then equation (1) reduces to:

$$an(an-1)\cdots(an-b+1) \equiv b! \pmod{an+1}.$$

However, we can rewrite the left-hand side as follows:

$$an(an-1)\cdots(an-b+1) \equiv (-1)(-2)\cdots(-b) \equiv (-1)^b b! \pmod{an+1}.$$

Provided that $an>1$, if b is even we deduce $(-1)^b b! \equiv b!$ as needed. On the other hand, if b is odd, and we take $an+1>2(b!)$, then we will not have $(-1)^b b! \equiv b!$, so b is not a-good. This completes the claim.

To conclude from here, suppose that b is a-good, but $b+2$ is not. Then b is even, and $p \mid a$ for all primes $p \leqslant b$, but there is a prime $q \leqslant b+2$ for which $q \nmid a$: so $q=b+1$ or $q=b+2$. We cannot have $q=b+2$, as that is even too, so we have $q=b+1$: in other words, $b+1$ is prime.

Solution 2. We show only half of the claim of the previous solution: we show that if b is a-good, then $p \mid a$ for all primes $p \leqslant b$. We do this with Lucas' theorem.

Suppose that we have $p \leqslant b$ with $p \nmid a$. Then consider the expansion of b in base p; there will be some digit (not the final digit) which is nonzero, because $p \leqslant b$. Suppose it is the p^t digit for $t \geqslant 1$.

Now, as n varies over the integers, $an+1$ runs over all residue classes modulo p^{t+1}; in particular, there is a choice of n (with $an>b$) such that the p^0 digit of an is $p-1$ (so $p \mid an+1$) and the p^t digit of an is 0. Consequently, $p \mid an+1$ but $p \mid \binom{an}{b}$ (by Lucas' theorem) so $p \nmid \binom{an}{b}-1$. Thus b is not a-good.

Now we show directly that if b is a-good but $b+2$ fails to be so, then there must be a prime dividing $an+1$ for some n, which also divides $(b+1)(b+2)$. Indeed, the ratio between $\binom{an}{b+2}$ and $\binom{an}{b}$ is $(b+1)(b+2)/(an-b)(an-b-1)$. We know that there must be a choice of $an+1$ such that the former binomial coefficient is 1 modulo $an+1$ but the latter is not, which means that the given ratio must not be 1 mod $an+1$. If $b+1$ and $b+2$ are both coprime to $an+1$ then the ratio *is* 1, so that must not be the case. In particular, as any prime less than b divides a, it must be the case that either $b+1$ or $b+2$ is prime.

However, we can observe that b must be even by insisting that $an+1$ is prime (which is possible by Dirichlet's theorem) and hence $\binom{an}{b} \equiv (-1)^b = 1$. Thus $b+2$ cannot be prime, so $b+1$ must be prime.

N6. Let $H = \{ \lfloor i\sqrt{2} \rfloor : i \in \mathbb{Z}_{>0} \} = \{1, 2, 4, 5, 7, \ldots\}$, and let n be a positive integer. Prove that there exists a constant C such that, if $A \subset \{1, 2, \ldots, n\}$ satisfies $|A| \geqslant C\sqrt{n}$, then there exist $a, b \in A$ such that $a - b \in H$. (Here $\mathbb{Z}_{>0}$ is the set of positive integers, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

(Brazil)

Common remarks. In all solutions, we will assume that A is a set such that $\{a - b : a, b \in A\}$ is disjoint from H, and prove that $|A| < C\sqrt{n}$.

Solution 1. First, observe that if n is a positive integer, then $n \in H$ exactly when

$$\left\{ \frac{n}{\sqrt{2}} \right\} > 1 - \frac{1}{\sqrt{2}}. \tag{1}$$

To see why, observe that $n \in H$ if and only if $0 < i\sqrt{2} - n < 1$ for some $i \in \mathbb{Z}_{>0}$. In other words, $0 < i - n/\sqrt{2} < 1/\sqrt{2}$, which is equivalent to (1).

Now, write $A = \{a_1 < a_2 < \cdots < a_k\}$, where $k = |A|$. Observe that the set of differences is not altered by shifting A, so we may assume that $A \subseteq \{0, 1, \ldots, n-1\}$ with $a_1 = 0$.

From (1), we learn that $\{a_i/\sqrt{2}\} < 1 - 1/\sqrt{2}$ for each $i > 1$ since $a_i - a_1 \notin H$. Furthermore, we must have $\{a_i/\sqrt{2}\} < \{a_j/\sqrt{2}\}$ whenever $i < j$; otherwise, we would have

$$-\left(1 - \frac{1}{\sqrt{2}}\right) < \left\{ \frac{a_j}{\sqrt{2}} \right\} - \left\{ \frac{a_i}{\sqrt{2}} \right\} < 0.$$

Since $\{(a_j - a_i)/\sqrt{2}\} = \{a_j/\sqrt{2}\} - \{a_i/\sqrt{2}\} + 1$, this implies that $\{(a_j - a_i)/\sqrt{2}\} > 1/\sqrt{2} > 1 - 1/\sqrt{2}$, contradicting (1).

Now, we have a sequence $0 = a_1 < a_2 < \cdots < a_k < n$, with

$$0 = \left\{ \frac{a_1}{\sqrt{2}} \right\} < \left\{ \frac{a_2}{\sqrt{2}} \right\} < \cdots < \left\{ \frac{a_k}{\sqrt{2}} \right\} < 1 - \frac{1}{\sqrt{2}}.$$

We use the following fact: for any $d \in \mathbb{Z}$, we have

$$\left\{ \frac{d}{\sqrt{2}} \right\} > \frac{1}{2d\sqrt{2}}. \tag{2}$$

To see why this is the case, let $h = \lfloor d/\sqrt{2} \rfloor$, so $\{d/\sqrt{2}\} = d/\sqrt{2} - h$. Then

$$\left\{ \frac{d}{\sqrt{2}} \right\} \left(\frac{d}{\sqrt{2}} + h \right) = \frac{d^2 - 2h^2}{2} \geqslant \frac{1}{2},$$

since the numerator is a positive integer. Because $d/\sqrt{2} + h < 2d/\sqrt{2}$, inequality (2) follows.

Let $d_i = a_{i+1} - a_i$, for $1 \leqslant i < k$. Then $\{a_{i+1}/\sqrt{2}\} - \{a_i/\sqrt{2}\} = \{d_i/\sqrt{2}\}$, and we have

$$1 - \frac{1}{\sqrt{2}} > \sum_i \left\{ \frac{d_i}{\sqrt{2}} \right\} > \frac{1}{2\sqrt{2}} \sum_i \frac{1}{d_i} \geqslant \frac{(k-1)^2}{2\sqrt{2}} \frac{1}{\sum_i d_i} > \frac{(k-1)^2}{2\sqrt{2}} \cdot \frac{1}{n}. \tag{3}$$

Here, the first inequality holds because $\{a_k/\sqrt{2}\} < 1 - 1/\sqrt{2}$, the second follows from (2), the third follows from an easy application of the AM–HM inequality (or Cauchy–Schwarz), and the fourth follows from the fact that $\sum_i d_i = a_k < n$.

Rearranging this, we obtain

$$\sqrt{2\sqrt{2} - 2} \cdot \sqrt{n} > k - 1,$$

which provides the required bound on k.

Solution 2. Let $\alpha = 2 + \sqrt{2}$, so $(1/\alpha) + (1/\sqrt{2}) = 1$. Thus, $J = \{\lfloor i\alpha \rfloor : i \in \mathbb{Z}_{>0}\}$ is the complementary Beatty sequence to H (in other words, H and J are disjoint with $H \cup J = \mathbb{Z}_{>0}$). Write $A = \{a_1 < a_2 < \cdots < a_k\}$. Suppose that A has no differences in H, so all its differences are in J and we can set $a_i - a_1 = \lfloor \alpha b_i \rfloor$ for $b_i \in \mathbb{Z}_{>0}$.

For any $j > i$, we have $a_j - a_i = \lfloor \alpha b_j \rfloor - \lfloor \alpha b_i \rfloor$. Because $a_j - a_i \in J$, we also have $a_j - a_i = \lfloor \alpha t \rfloor$ for some positive integer t. Thus, $\lfloor \alpha t \rfloor = \lfloor \alpha b_j \rfloor - \lfloor \alpha b_i \rfloor$. The right hand side must equal either $\lfloor \alpha(b_j - b_i) \rfloor$ or $\lfloor \alpha(b_j - b_i) \rfloor - 1$, the latter of which is not a member of J as $\alpha > 2$. Therefore, $t = b_j - b_i$ and so we have $\lfloor \alpha b_j \rfloor - \lfloor \alpha b_i \rfloor = \lfloor \alpha(b_j - b_i) \rfloor$.

For $1 \leqslant i < k$ we now put $d_i = b_{i+1} - b_i$, and we have

$$\left\lfloor \alpha \sum_i d_i \right\rfloor = \lfloor \alpha b_k \rfloor = \sum_i \lfloor \alpha d_i \rfloor ;$$

that is, $\sum_i \{\alpha d_i\} < 1$. We also have

$$1 + \left\lfloor \alpha \sum_i d_i \right\rfloor = 1 + a_k - a_1 \leqslant a_k \leqslant n$$

so $\sum_i d_i \leqslant n/\alpha$.

With the above inequalities, an argument similar to (3) (which uses the fact that $\{\alpha d\} = \{d\sqrt{2}\} > 1/(2d\sqrt{2})$ for positive integers d) proves that $1 > \big((k-1)^2/(2\sqrt{2})\big)(\alpha/n)$, which again rearranges to give

$$\sqrt{2\sqrt{2} - 2} \cdot \sqrt{n} > k - 1.$$

Comment. The use of Beatty sequences in Solution 2 is essentially a way to bypass (1). Both Solutions 1 and 2 use the fact that $\sqrt{2} < 2$; the statement in the question would still be true if $\sqrt{2}$ did not have this property (for instance, if it were replaced with α), but any argument along the lines of Solutions 1 or 2 would be more complicated.

Solution 3. Again, define $J = \mathbb{Z}_{>0} \setminus H$, so all differences between elements of A are in J. We start by making the following observation. Suppose we have a set $B \subseteq \{1, 2, \ldots, n\}$ such that all of the differences between elements of B are in H. Then $|A| \cdot |B| \leqslant 2n$.

To see why, observe that any two sums of the form $a + b$ with $a \in A, b \in B$ are different; otherwise, we would have $a_1 + b_1 = a_2 + b_2$, and so $|a_1 - a_2| = |b_2 - b_1|$. However, then the left hand side is in J whereas the right hand side is in H. Thus, $\{a + b : a \in A, b \in B\}$ is a set of size $|A| \cdot |B|$ all of whose elements are no greater than $2n$, yielding the claimed inequality.

With this in mind, it suffices to construct a set B, all of whose differences are in H and whose size is at least $C'\sqrt{n}$ for some constant $C' > 0$.

To do so, we will use well-known facts about the negative Pell equation $X^2 - 2Y^2 = -1$; in particular, that there are infinitely many solutions and the values of X are given by the recurrence $X_1 = 1, X_2 = 7$ and $X_m = 6X_{m-1} - X_{m-2}$. Therefore, we may choose X to be a solution with $\sqrt{n}/6 < X \leqslant \sqrt{n}$.

Now, we claim that we may choose $B = \{X, 2X, \ldots, \lfloor (1/3)\sqrt{n} \rfloor X\}$. Indeed, we have

$$\left(\frac{X}{\sqrt{2}} - Y\right)\left(\frac{X}{\sqrt{2}} + Y\right) = \frac{-1}{2}$$

and so

$$0 > \left(\frac{X}{\sqrt{2}} - Y\right) \geqslant \frac{-3}{\sqrt{2n}},$$

from which it follows that $\{X/\sqrt{2}\} > 1 - (3/\sqrt{2n})$. Combined with (1), this shows that all differences between elements of B are in H.

Comment. Some of the ideas behind Solution 3 may be used to prove that the constant $C = \sqrt{2\sqrt{2}-2}$ (from Solutions 1 and 2) is optimal, in the sense that there are arbitrarily large values of n and sets $A_n \subseteq \{1, 2, \ldots, n\}$ of size roughly $C\sqrt{n}$, all of whose differences are contained in J.

To see why, choose X to come from a sufficiently large solution to the Pell equation $X^2 - 2Y^2 = 1$, so $\{X/\sqrt{2}\} \approx 1/(2X\sqrt{2})$. In particular, $\{X\}, \{2X\}, \ldots, \{\lfloor 2X\sqrt{2}(1-1/\sqrt{2})\rfloor X\}$ are all less than $1 - 1/\sqrt{2}$. Thus, by (1) any positive integer of the form iX for $1 \leqslant i \leqslant \lfloor 2X\sqrt{2}(1-1/\sqrt{2})\rfloor$ lies in J.

Set $n \approx 2X^2\sqrt{2}(1-1/\sqrt{2})$. We now have a set $A = \{iX : i \leqslant \lfloor 2X\sqrt{2}(1-1/\sqrt{2})\rfloor\}$ containing roughly $2X\sqrt{2}(1-1/\sqrt{2})$ elements less than or equal to n such that all of the differences lie in J, and we can see that $|A| \approx C\sqrt{n}$ with $C = \sqrt{2\sqrt{2}-2}$.

Solution 4. As in Solution 3, we will provide a construction of a large set $B \subseteq \{1, 2, \ldots, n\}$, all of whose differences are in H.

Choose Y to be a solution to the Pell-like equation $X^2 - 2Y^2 = \pm 1$; such solutions are given by the recurrence $Y_1 = 1, Y_2 = 2$ and $Y_m = 2Y_{m-1} + Y_{m-2}$, and so we can choose Y such that $n/(3\sqrt{2}) < Y \leqslant n/\sqrt{2}$. Furthermore, it is known that for such a Y and for $1 \leqslant x < Y$,

$$\{x\sqrt{2}\} + \{(Y-x)\sqrt{2}\} = \{Y/\sqrt{2}\} \tag{4}$$

if $X^2 - 2Y^2 = 1$, and

$$\{x\sqrt{2}\} + \{(Y-x)\sqrt{2}\} = 1 + \{Y/\sqrt{2}\} \tag{5}$$

if $X^2 - 2Y^2 = -1$. Indeed, this is a statement of the fact that X/Y is a best rational approximation to $\sqrt{2}$, from below in the first case and from above in the second.

Now, consider the sequence $\{\sqrt{2}\}, \{2\sqrt{2}\}, \ldots, \{(Y-1)\sqrt{2}\}$. The Erdős–Szekeres theorem tells us that this sequence has a monotone subsequence with at least $\sqrt{Y-2}+1 > \sqrt{Y}$ elements; if that subsequence is decreasing, we may reflect (using (4) or (5)) to ensure that it is increasing. Call the subsequence $\{y_1\sqrt{2}\}, \{y_2\sqrt{2}\}, \ldots, \{y_t\sqrt{2}\}$ for $t > \sqrt{Y}$.

Now, set $B = \{\lfloor y_i\sqrt{2}\rfloor : 1 \leqslant i \leqslant t\}$. We have $\lfloor y_j\sqrt{2}\rfloor - \lfloor y_i\sqrt{2}\rfloor = \lfloor (y_j - y_i)\sqrt{2}\rfloor$ for $i < j$ (because the corresponding inequality for the fractional parts holds by the ordering assumption on the $\{y_i\sqrt{2}\}$), which means that all differences between elements of B are indeed in H. Since $|B| > \sqrt{Y} > \sqrt{n}/\sqrt{3\sqrt{2}}$, this is the required set.

Comment. Any solution to this problem will need to use the fact that $\sqrt{2}$ cannot be approximated well by rationals, either directly or implicitly (for example, by using facts about solutions to Pell-like equations). If $\sqrt{2}$ were replaced by a value of θ with very good rational approximations (from below), then an argument along the lines of Solution 3 would give long arithmetic progressions in $\{\lfloor i\theta\rfloor : 0 \leqslant i < n\}$ (with initial term 0) for certain values of n.

N7. Prove that there is a constant $c > 0$ and infinitely many positive integers n with the following property: there are infinitely many positive integers that cannot be expressed as the sum of fewer than $cn\log(n)$ pairwise coprime n^{th} powers.

(Canada)

Solution 1. Suppose, for an integer n, that we can find another integer N satisfying the following property:

$$n \text{ is divisible by } \varphi(p^e) \text{ for every prime power } p^e \text{ exactly dividing } N. \qquad (\dagger)$$

This property ensures that all n^{th} powers are congruent to 0 or 1 modulo each such prime power p^e, and hence that any sum of m pairwise coprime n^{th} powers is congruent to m or $m-1$ modulo p^e, since at most one of the n^{th} powers is divisible by p. Thus, if k denotes the number of distinct prime factors of N, we find by the Chinese Remainder Theorem at most $2^k m$ residue classes modulo N which are sums of at most m pairwise coprime n^{th} powers. In particular, if $N > 2^k m$ then there are infinitely many positive integers not expressible as a sum of at most m pairwise coprime n^{th} powers.

It thus suffices to prove that there are arbitrarily large pairs (n, N) of integers satisfying ($\dagger$) such that

$$N > c \cdot 2^k n \log(n)$$

for some positive constant c.

We construct such pairs as follows. Fix a positive integer t and choose (distinct) prime numbers $p \mid 2^{2^{t-1}} + 1$ and $q \mid 2^{2^t} + 1$; we set $N = pq$. It is well-known that $2^t \mid p-1$ and $2^{t+1} \mid q-1$, hence

$$n = \frac{(p-1)(q-1)}{2^t}$$

is an integer and the pair (n, N) satisfies ($\dagger$).

Estimating the size of N and n is now straightforward. We have

$$\log_2(n) \leqslant 2^{t-1} + 2^t - t < 2^{t+1} < 2 \cdot \frac{N}{n},$$

which rearranges to

$$N > \frac{1}{8} \cdot 2^2 n \log_2(n)$$

and so we are done if we choose $c < \frac{1}{8\log(2)} \approx 0.18$.

Comment 1. The trick in the above solution was to find prime numbers p and q congruent to 1 modulo some $d = 2^t$ and which are not too large. An alternative way to do this is via Linnik's Theorem, which says that there are absolute constants b and $L > 1$ such that for any coprime integers a and d, there is a prime congruent to a modulo d and of size $\leqslant bd^L$. If we choose some d not divisible by 3 and choose two distinct primes $p, q \leqslant b \cdot (3d)^L$ congruent to 1 modulo d (and, say, distinct modulo 3), then we obtain a pair (n, N) satisfying ($\dagger$) with $N = pq$ and $n = \frac{(p-1)(q-1)}{d}$. A straightforward computation shows that

$$N > Cn^{1+\frac{1}{2L-1}}$$

for some constant C, which is in particular larger than any $c \cdot 2^2 n \log(n)$ for p large. Thus, the statement of the problem is true for any constant c. More strongly, the statement of the problem is still true when $cn\log(n)$ is replaced by $n^{1+\delta}$ for a sufficiently small $\delta > 0$.

Solution 2, obtaining better bounds. As in the preceding solution, we seek arbitrarily large pairs of integers n and N satisfying (†) such that $N > c2^k n\log(n)$.

This time, to construct such pairs, we fix an integer $x \geqslant 4$, set N to be the lowest common multiple of $1, 2, \ldots, 2x$, and set n to be twice the lowest common multiple of $1, 2, \ldots, x$. The pair (n, N) does indeed satisfy the condition, since if p^e is a prime power divisor of N then $\frac{\varphi(p^e)}{2} \leqslant x$ is a factor of $\frac{n}{2} = \operatorname{lcm}_{r\leqslant x}(r)$.

Now $2N/n$ is the product of all primes having a power lying in the interval $(x, 2x]$, and hence $2N/n > x^{\pi(2x)-\pi(x)}$. Thus for sufficiently large x we have

$$\log\left(\frac{2N}{2^{\pi(2x)}n}\right) > (\pi(2x)-\pi(x))\log(x) - \log(2)\pi(2x) \sim x,$$

using the Prime Number Theorem $\pi(t) \sim t/\log(t)$.

On the other hand, n is a product of at most $\pi(x)$ prime powers less than or equal to x, and so we have the upper bound

$$\log(n) \leqslant \pi(x)\log(x) \sim x,$$

again by the Prime Number Theorem. Combined with the above inequality, we find that for any $\epsilon > 0$, the inequality

$$\log\left(\frac{N}{2^{\pi(2x)}n}\right) > (1-\epsilon)\log(n)$$

holds for sufficiently large x. Rearranging this shows that

$$N > 2^{\pi(2x)}n^{2-\epsilon} > 2^{\pi(2x)}n\log(n)$$

for all sufficiently large x and we are done.

Comment 2. The stronger bound $N > 2^{\pi(2x)}n^{2-\epsilon}$ obtained in the above proof of course shows that infinitely many positive integers cannot be written as a sum of at most $n^{2-\epsilon}$ pairwise coprime n^{th} powers.

By refining the method in Solution 2, these bounds can be improved further to show that infinitely many positive integers cannot be written as a sum of at most n^α pairwise coprime n^{th} powers for any positive $\alpha > 0$. To do this, one fixes a positive integer d, sets N equal to the product of the primes at most dx which are congruent to 1 modulo d, and $n = d\operatorname{lcm}_{r\leqslant x}(r)$. It follows as in Solution 2 that (n, N) satisfies (†).

Now the Prime Number Theorem in arithmetic progressions provides the estimates $\log(N) \sim \frac{d}{\varphi(d)}x$, $\log(n) \sim x$ and $\pi(dx) \sim \frac{dx}{\log(x)}$ for any fixed d. Combining these provides a bound

$$N > 2^{\pi(dx)}n^{d/\varphi(d)-\epsilon}$$

for any positive ϵ, valid for x sufficiently large. Since the ratio $\frac{d}{\varphi(d)}$ can be made arbitrarily large by a judicious choice of d, we obtain the n^α bound claimed.

Comment 3. While big results from analytic number theory such as the Prime Number Theorem or Linnik's Theorem certainly can be used in this problem, they do not seem to substantially simplify matters: all known solutions involve first reducing to condition (†), and even then analytic results do not make it clear how to proceed. For this reason, we regard this problem as suitable for the IMO.

Rather than simplifying the problem, what nonelementary results from analytic number theory allow one to achieve is a strengthening of the main bound, typically replacing the $n\log(n)$ growth with a power $n^{1+\delta}$. However, we believe that such stronger bounds are unlikely to be found by students in the exam.

The strongest bound we know how to achieve using purely elementary methods is a bound of the form $N > 2^k n\log(n)^M$ for any positive integer M. This is achieved by a variant of the argument in Solution 1, choosing primes $p_0, \ldots, p_M$ with $p_i \mid 2^{2^{t+i-1}} + 1$ and setting $N = \prod_i p_i$ and $n = 2^{-tM}\prod_i(p_i - 1)$.

N8. Let a and b be two positive integers. Prove that the integer

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square. (Here $\lceil z \rceil$ denotes the least integer greater than or equal to z.)

(Russia)

Solution 1. Arguing indirectly, assume that

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil = (a+k)^2, \quad \text{or} \quad \left\lceil \frac{(2a)^2}{b} \right\rceil = (2a+k)k.$$

Clearly, $k \geqslant 1$. In other words, the equation

$$\left\lceil \frac{c^2}{b} \right\rceil = (c+k)k \tag{1}$$

has a positive integer solution (c, k), with an even value of c.

Choose a positive integer solution of (1) with minimal possible value of k, without regard to the parity of c. From

$$\frac{c^2}{b} > \left\lceil \frac{c^2}{b} \right\rceil - 1 = ck + k^2 - 1 \geqslant ck$$

and

$$\frac{(c-k)(c+k)}{b} < \frac{c^2}{b} \leqslant \left\lceil \frac{c^2}{b} \right\rceil = (c+k)k$$

it can be seen that $c > bk > c - k$, so

$$c = kb + r \quad \text{with some } 0 < r < k.$$

By substituting this in (1) we get

$$\left\lceil \frac{c^2}{b} \right\rceil = \left\lceil \frac{(bk+r)^2}{b} \right\rceil = k^2 b + 2kr + \left\lceil \frac{r^2}{b} \right\rceil$$

and

$$(c+k)k = (kb + r + k)k = k^2 b + 2kr + k(k-r),$$

so

$$\left\lceil \frac{r^2}{b} \right\rceil = k(k-r). \tag{2}$$

Notice that relation (2) provides another positive integer solution of (1), namely $c' = r$ and $k' = k - r$, with $c' > 0$ and $0 < k' < k$. That contradicts the minimality of k, and hence finishes the solution.

Solution 2. Suppose that

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil = c^2$$

with some positive integer $c > a$, so

$$\begin{aligned} c^2 - 1 < a^2 + \frac{4a^2}{b} &\leqslant c^2, \\ 0 \leqslant c^2 b - a^2(b+4) &< b. \end{aligned} \tag{3}$$

Let $d = c^2 b - a^2(b+4)$, $x = c + a$ and $y = c - a$; then we have $c = \dfrac{x+y}{2}$ and $a = \dfrac{x-y}{2}$, and (3) can be re-written as follows:

$$\begin{gathered} \left(\frac{x+y}{2}\right)^2 b - \left(\frac{x-y}{2}\right)^2 (b+4) = d, \\ x^2 - (b+2)xy + y^2 + d = 0; \qquad 0 \leqslant d < b. \end{gathered} \tag{4}$$

So, by the indirect assumption, the equation (4) has some positive integer solution (x, y).

Fix b and d, and take a pair (x, y) of positive integers, satisfying (4), such that $x + y$ is minimal. By the symmetry in (4) we may assume that $x \geqslant y \geqslant 1$.

Now we perform a usual "Vieta jump". Consider (4) as a quadratic equation in variable x, and let z be its second root. By the Vieta formulas,

$$x + z = (b+2)y, \quad \text{and} \quad zx = y^2 + d,$$

so

$$z = (b+2)y - x = \frac{y^2 + d}{x}.$$

The first formula shows that z is an integer, and by the second formula z is positive. Hence (z, y) is another positive integer solution of (4). From

$$\begin{aligned} (x-1)(z-1) &= xz - (x+z) + 1 = (y^2 + d) - (b+2)y + 1 \\ &< (y^2 + b) - (b+2)y + 1 = (y-1)^2 - b(y-1) \leqslant (y-1)^2 \leqslant (x-1)^2 \end{aligned}$$

we can see that $z < x$ and therefore $z + y < x + y$. But this contradicts the minimality of $x + y$ among the positive integer solutions of (4).

The activities of the Problem Selection Committee were supported by

Trinity College, Cambridge
University of Cambridge

lunes, 21. septiembre 2020

Problema 1. Considere el cuadrilátero convexo $ABCD$. El punto P está en el interior de $ABCD$. Asuma las siguientes igualdades de razones:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Demuestre que las siguientes tres rectas concurren en un punto: la bisectriz interna del ángulo $\angle ADP$, la bisectriz interna del ángulo $\angle PCB$ y la mediatriz del segmento AB.

Problema 2. Los números reales a, b, c, d son tales que $a \geq b \geq c \geq d > 0$ y $a + b + c + d = 1$. Demuestre que

$$(a + 2b + 3c + 4d)\, a^a\, b^b\, c^c\, d^d < 1.$$

Problema 3. Hay $4n$ piedritas de pesos $1, 2, 3, \ldots, 4n$. Cada piedrita se colorea de uno de n colores de manera que hay cuatro piedritas de cada color. Demuestre que podemos colocar las piedritas en dos montones de tal forma que las siguientes dos condiciones se satisfacen:

- Los pesos totales de ambos montones son iguales.
- Cada montón contiene dos piedritas de cada color.

Tiempo: 4 horas y 30 minutos.
Cada problema vale 7 puntos.

martes, 22. septiembre 2020

Problema 4. Sea $n > 1$ un entero. A lo largo de la pendiente de una montaña hay n^2 estaciones, todas a diferentes altitudes. Dos compañías de teleférico, A y B, operan k teleféricos cada una. Cada teleférico realiza el servicio desde una estación a otra de mayor altitud (sin paradas intermedias). Los teleféricos de la compañía A parten de k estaciones diferentes y acaban en k estaciones diferentes; igualmente, si un teleférico parte de una estación más alta que la de otro, también acaba en una estación más alta que la del otro. La compañía B satisface las mismas condiciones. Decimos que dos estaciones están *unidas* por una compañía si uno puede comenzar por la más baja y llegar a la más alta con uno o más teleféricos de esa compañía (no se permite otro tipo de movimientos entre estaciones).

Determine el menor entero positivo k para el cual se puede garantizar que hay dos estaciones unidas por ambas compañías.

Problema 5. Se tiene una baraja de $n > 1$ cartas, con un entero positivo escrito en cada carta. La baraja tiene la propiedad de que la media aritmética de los números escritos en cada par de cartas es también la media geométrica de los números escritos en alguna colección de una o más cartas.

¿Para qué valores de n se tiene que los números escritos en las cartas son todos iguales?

Problema 6. Pruebe que existe una constante positiva c para la que se satisface la siguiente afirmación:

Sea $n > 1$ un entero y sea $\mathcal{S}$ un conjunto de n puntos del plano tal que la distancia entre cualesquiera dos puntos diferentes de $\mathcal{S}$ es al menos 1. Entonces existe una recta ℓ separando $\mathcal{S}$ tal que la distancia de cualquier punto de $\mathcal{S}$ a ℓ es al menos $cn^{-1/3}$.

(Una recta ℓ *separa* un conjunto de puntos $\mathcal{S}$ si ℓ corta a alguno de los segmentos que une dos puntos de $\mathcal{S}$.)

Nota. Los resultados más débiles que se obtienen al sustituir $cn^{-1/3}$ por $cn^{-\alpha}$ se podrán valorar dependiendo del valor de la constante $\alpha > 1/3$.

Tiempo: 4 horas y 30 minutos.
Cada problema vale 7 puntos.

Monday, 21. September 2020

Problem 1. Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Problem 2. The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)\, a^a\, b^b\, c^c\, d^d < 1.$$

Problem 3. There are $4n$ pebbles of weights $1, 2, 3, \ldots, 4n$. Each pebble is coloured in one of n colours and there are four pebbles of each colour. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

- The total weights of both piles are the same.
- Each pile contains two pebbles of each colour.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Tuesday, 22. September 2020

Problem 4. There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are *linked* by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed).

Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Problem 5. A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards.

For which n does it follow that the numbers on the cards are all equal?

Problem 6. Prove that there exists a positive constant c such that the following statement is true:

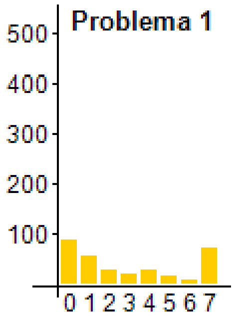
Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. It follows that there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $cn^{-1/3}$.

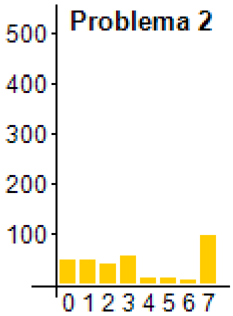
(A line ℓ *separates* a set of points $\mathcal{S}$ if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

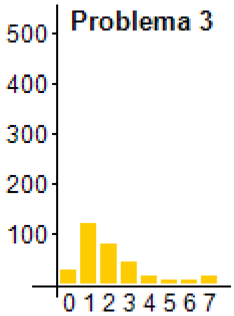
Note. Weaker results with $cn^{-1/3}$ replaced by $cn^{-\alpha}$ may be awarded points depending on the value of the constant $\alpha > 1/3$.

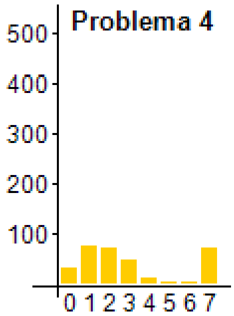
31ᴬ OIM 1990

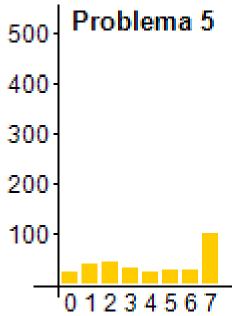
Estadística

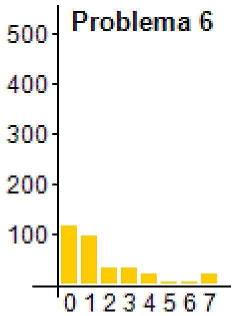








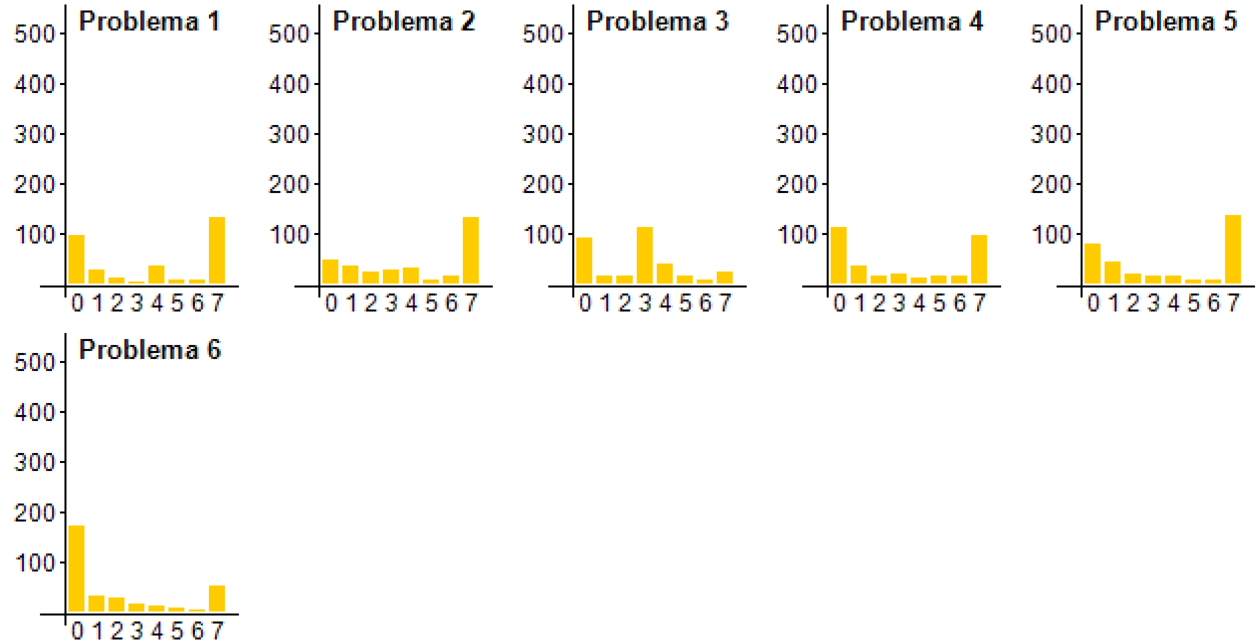




	P1	P2	P3	P4	P5	P6
Num(P# = 0)	88	47	25	31	21	115
Num(P# = 1)	54	46	118	76	39	93
Num(P# = 2)	28	37	79	70	43	29
Num(P# = 3)	19	54	42	46	31	31
Num(P# = 4)	26	11	16	10	22	18
Num(P# = 5)	14	9	6	3	26	2
Num(P# = 6)	8	8	6	3	26	3
Num(P# = 7)	71	96	16	69	100	17
Mean(P#)	2,877	3,542	2,091	2,955	4,195	1,503
Max(P#)	7	7	7	7	7	7
σ(P#)	2,745	2,678	1,682	2,421	2,489	1,865
Corr(P#, Sum)	0,604	0,741	0,652	0,702	0,676	0,683
Corr(P#, P1)		0,239	0,384	0,276	0,119	0,365
Corr(P#, P2)	0,239		0,308	0,407	0,564	0,386
Corr(P#, P3)	0,384	0,308		0,414	0,300	0,435
Corr(P#, P4)	0,276	0,407	0,414		0,365	0,388
Corr(P#, P5)	0,119	0,564	0,300	0,365		0,339
Corr(P#, P6)	0,365	0,386	0,435	0,388	0,339	

32ª OIM 1991

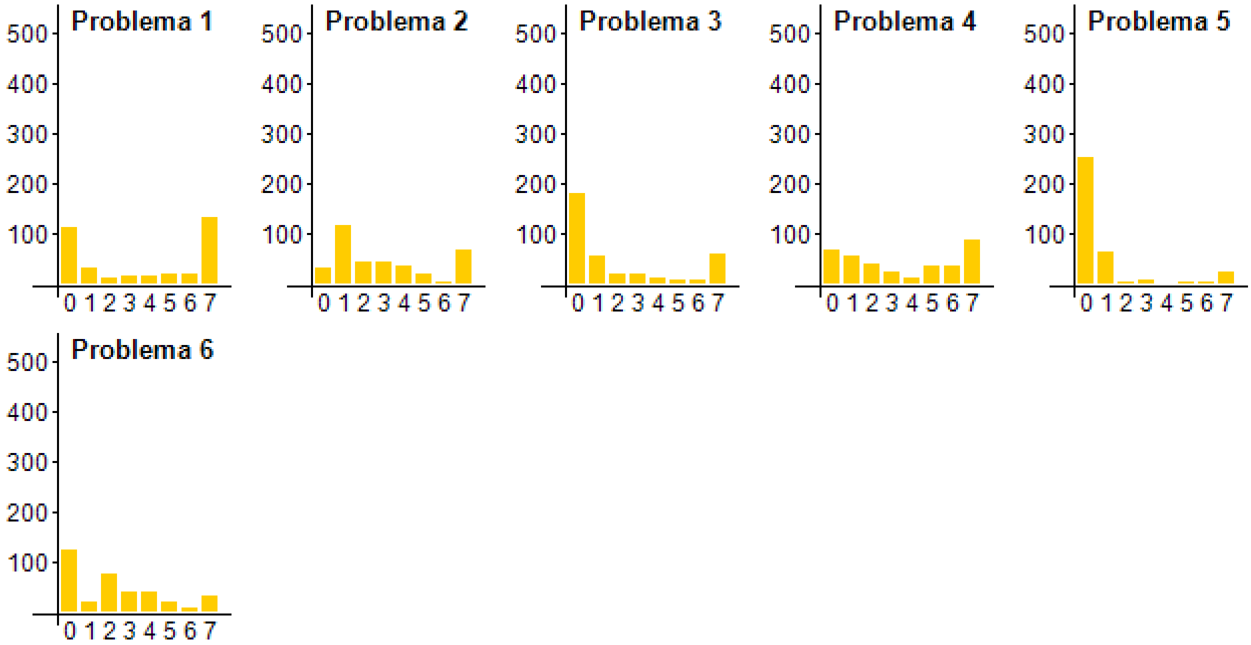
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	93	47	90	112	77	172
Num(P# = 1)	27	34	14	34	42	29
Num(P# = 2)	10	21	16	15	18	27
Num(P# = 3)	3	27	111	18	14	16
Num(P# = 4)	34	31	37	10	14	10
Num(P# = 5)	8	6	16	15	6	5
Num(P# = 6)	6	16	5	13	7	3
Num(P# = 7)	131	130	23	95	134	50
Mean(P#)	3,798	4,221	2,558	3,128	3,801	1,808
Max(P#)	7	7	7	7	7	7
σ(P#)	3,087	2,770	2,069	3,028	3,061	2,597
Corr(P#, Sum)	0,720	0,826	0,695	0,747	0,737	0,733
Corr(P#, P1)		0,532	0,344	0,385	0,497	0,364
Corr(P#, P2)	0,532		0,547	0,588	0,530	0,487
Corr(P#, P3)	0,344	0,547		0,477	0,376	0,521
Corr(P#, P4)	0,385	0,588	0,477		0,360	0,504
Corr(P#, P5)	0,497	0,530	0,376	0,360		0,454
Corr(P#, P6)	0,364	0,487	0,521	0,504	0,454	

33ª OIM 1992

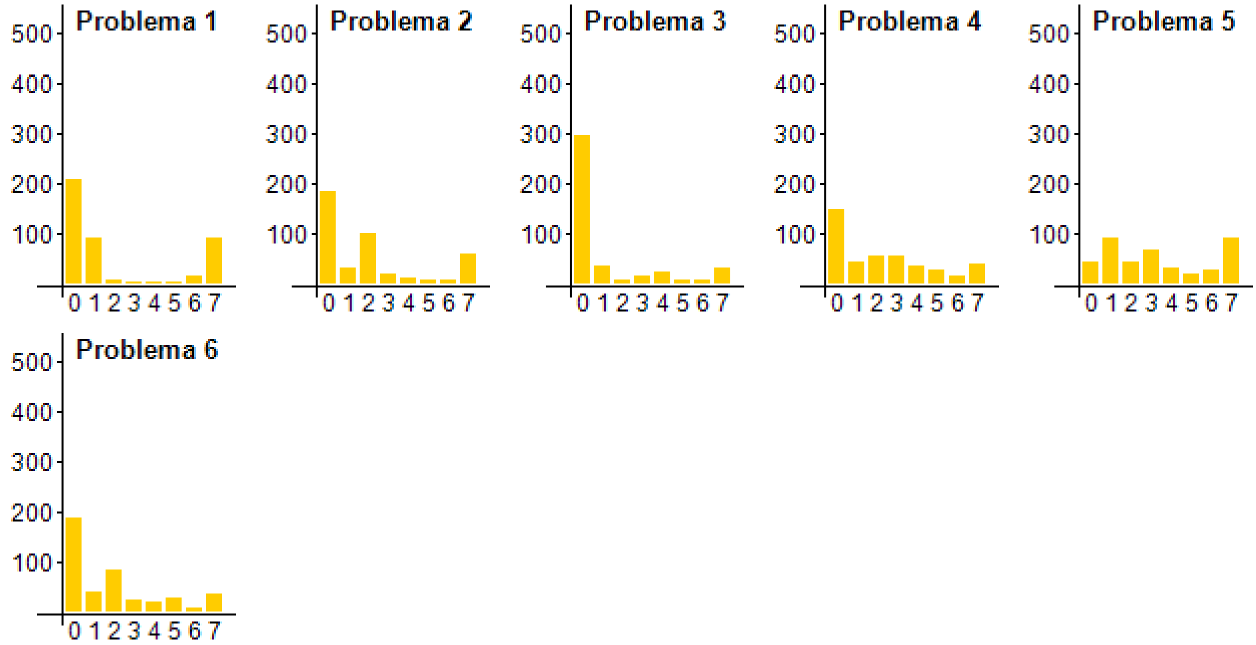
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	110	31	177	68	252	123
Num(P# = 1)	32	113	55	55	62	18
Num(P# = 2)	11	41	19	39	3	75
Num(P# = 3)	14	43	17	22	7	40
Num(P# = 4)	14	34	11	10	0	37
Num(P# = 5)	20	18	7	36	2	19
Num(P# = 6)	18	3	5	34	2	8
Num(P# = 7)	131	67	59	86	22	30
Mean(P#)	3,649	2,963	1,903	3,500	0,757	2,254
Max(P#)	7	7	7	7	7	7
σ(P#)	3,086	2,376	2,640	2,736	1,802	2,220
Corr(P#, Sum)	0,738	0,705	0,712	0,619	0,532	0,716
Corr(P#, P1)		0,410	0,377	0,352	0,244	0,440
Corr(P#, P2)	0,410		0,448	0,350	0,229	0,408
Corr(P#, P3)	0,377	0,448		0,230	0,350	0,468
Corr(P#, P4)	0,352	0,350	0,230		0,192	0,283
Corr(P#, P5)	0,244	0,229	0,350	0,192		0,364
Corr(P#, P6)	0,440	0,408	0,468	0,283	0,364	

34ª OIM 1993

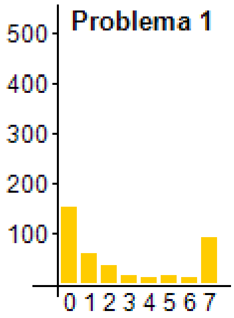
Estadística

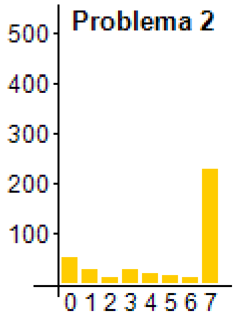


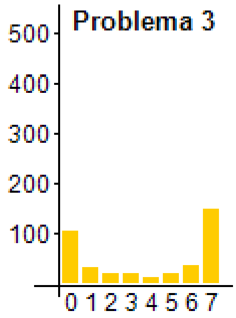
	P1	P2	P3	P4	P5	P6
Num(P# = 0)	208	183	293	145	43	186
Num(P# = 1)	91	32	33	43	92	38
Num(P# = 2)	6	98	6	55	44	81
Num(P# = 3)	1	19	15	55	65	22
Num(P# = 4)	1	12	21	35	32	20
Num(P# = 5)	1	6	7	26	19	26
Num(P# = 6)	13	6	8	15	27	6
Num(P# = 7)	92	57	30	39	91	34
Mean(P#)	2,027	1,932	1,131	2,303	3,383	1,816
Max(P#)	7	7	7	7	7	7
σ(P#)	2,888	2,413	2,163	2,323	2,483	2,228
Corr(P#, Sum)	0,720	0,637	0,605	0,643	0,751	0,677
Corr(P#, P1)		0,336	0,265	0,353	0,469	0,359
Corr(P#, P2)	0,336		0,203	0,399	0,348	0,283
Corr(P#, P3)	0,265	0,203		0,276	0,402	0,391
Corr(P#, P4)	0,353	0,399	0,276		0,316	0,277
Corr(P#, P5)	0,469	0,348	0,402	0,316		0,482
Corr(P#, P6)	0,359	0,283	0,391	0,277	0,482	

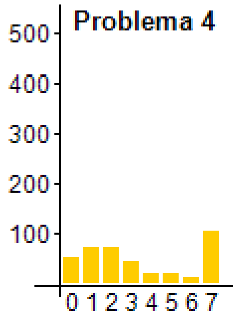
35ª OIM 1994

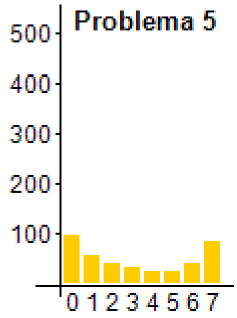
Estadística

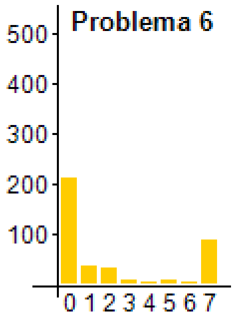








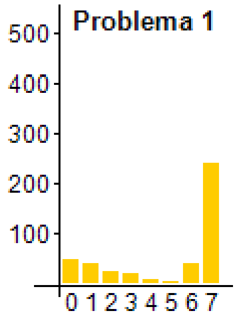


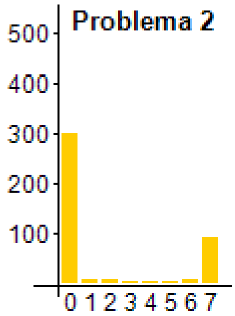


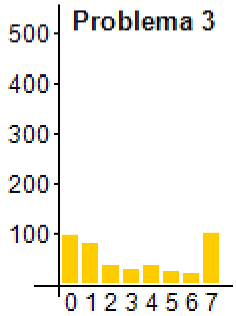
	P1	P2	P3	P4	P5	P6
Num(P# = 0)	151	52	104	52	95	211
Num(P# = 1)	60	25	30	69	54	34
Num(P# = 2)	33	12	18	70	40	30
Num(P# = 3)	15	26	19	43	29	7
Num(P# = 4)	10	17	11	19	22	4
Num(P# = 5)	13	14	19	20	23	8
Num(P# = 6)	11	12	36	11	38	3
Num(P# = 7)	92	227	148	101	84	88
Mean(P#)	2,561	5,003	3,932	3,343	3,221	2,091
Max(P#)	7	7	7	7	7	7
σ(P#)	2,879	2,720	3,023	2,590	2,741	2,891
Corr(P#, Sum)	0,709	0,601	0,726	0,731	0,717	0,646
Corr(P#, P1)		0,262	0,408	0,443	0,423	0,379
Corr(P#, P2)	0,262		0,247	0,389	0,365	0,255
Corr(P#, P3)	0,408	0,247		0,463	0,395	0,441
Corr(P#, P4)	0,443	0,389	0,463		0,509	0,265
Corr(P#, P5)	0,423	0,365	0,395	0,509		0,294
Corr(P#, P6)	0,379	0,255	0,441	0,265	0,294	

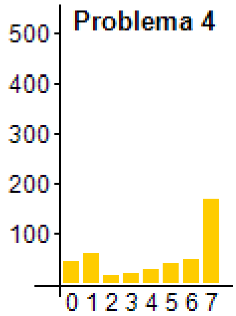
36ª OIM 1995

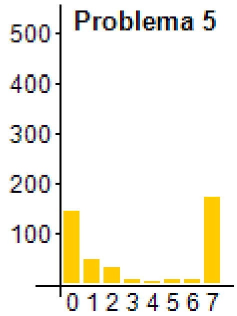
Estadística

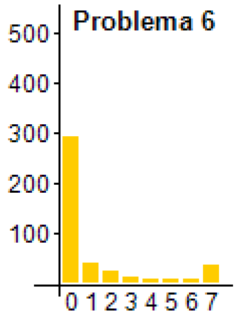








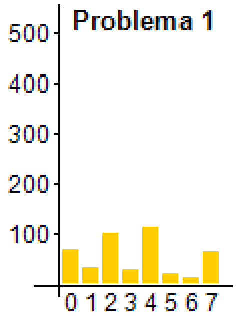


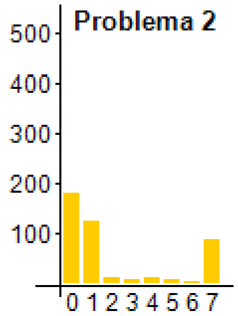


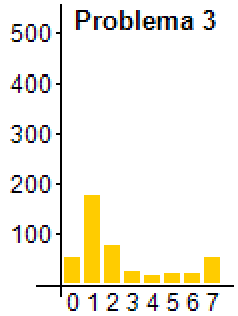
	P1	P2	P3	P4	P5	P6
Num(P# = 0)	47	299	96	43	141	291
Num(P# = 1)	38	7	80	57	48	37
Num(P# = 2)	21	5	33	16	30	22
Num(P# = 3)	18	1	28	18	8	12
Num(P# = 4)	7	1	34	25	1	5
Num(P# = 5)	4	4	24	37	7	5
Num(P# = 6)	38	5	17	48	5	6
Num(P# = 7)	239	90	100	168	172	34
Mean(P#)	5,056	1,709	3,126	4,592	3,410	1,058
Max(P#)	7	7	7	7	7	7
σ(P#)	2,718	2,931	2,740	2,644	3,211	2,124
Corr(P#, Sum)	0,689	0,679	0,787	0,759	0,735	0,607
Corr(P#, P1)		0,306	0,410	0,499	0,448	0,260
Corr(P#, P2)	0,306		0,456	0,374	0,352	0,374
Corr(P#, P3)	0,410	0,456		0,605	0,461	0,432
Corr(P#, P4)	0,499	0,374	0,605		0,433	0,341
Corr(P#, P5)	0,448	0,352	0,461	0,433		0,337
Corr(P#, P6)	0,260	0,374	0,432	0,341	0,337	

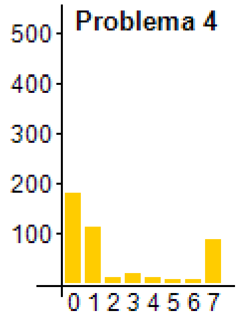
37ª OIM 1996

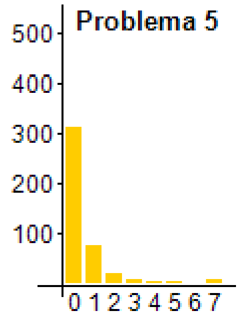
Estadística

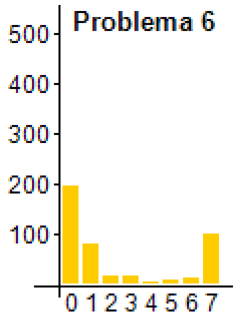








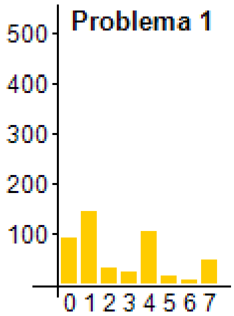


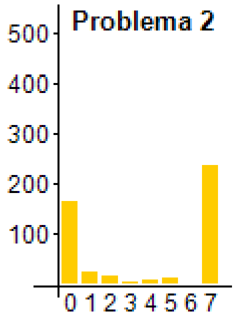


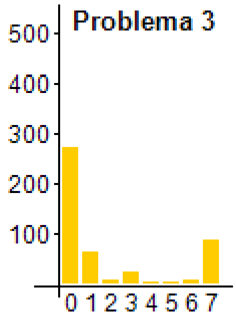
	P1	**P2**	**P3**	**P4**	**P5**	**P6**
Num(P# = 0)	67	178	49	177	311	196
Num(P# = 1)	30	124	176	109	74	79
Num(P# = 2)	98	10	73	11	18	16
Num(P# = 3)	26	5	22	17	7	15
Num(P# = 4)	111	11	14	11	4	3
Num(P# = 5)	17	6	19	7	4	6
Num(P# = 6)	12	2	19	6	0	10
Num(P# = 7)	63	88	52	86	6	99
Mean(P#)	3,175	2,031	2,399	2,120	0,493	2,243
Max(P#)	7	7	7	7	7	7
σ(P#)	2,227	2,735	2,256	2,739	1,153	2,895
Corr(P#, Sum)	0,639	0,666	0,789	0,728	0,418	0,703
Corr(P#, P1)		0,206	0,446	0,367	0,076	0,415
Corr(P#, P2)	0,206		0,409	0,430	0,329	0,235
Corr(P#, P3)	0,446	0,409		0,479	0,244	0,541
Corr(P#, P4)	0,367	0,430	0,479		0,246	0,293
Corr(P#, P5)	0,076	0,329	0,244	0,246		0,185
Corr(P#, P6)	0,415	0,235	0,541	0,293	0,185	

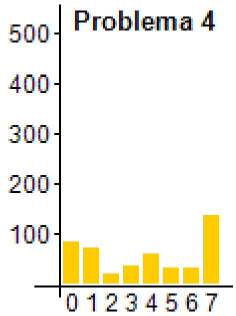
38ª OIM 1997

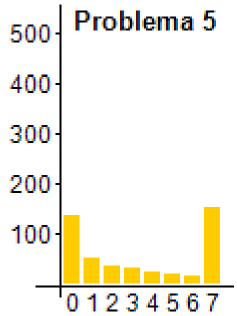
Estadística

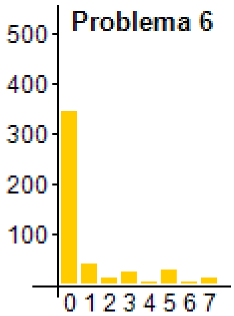








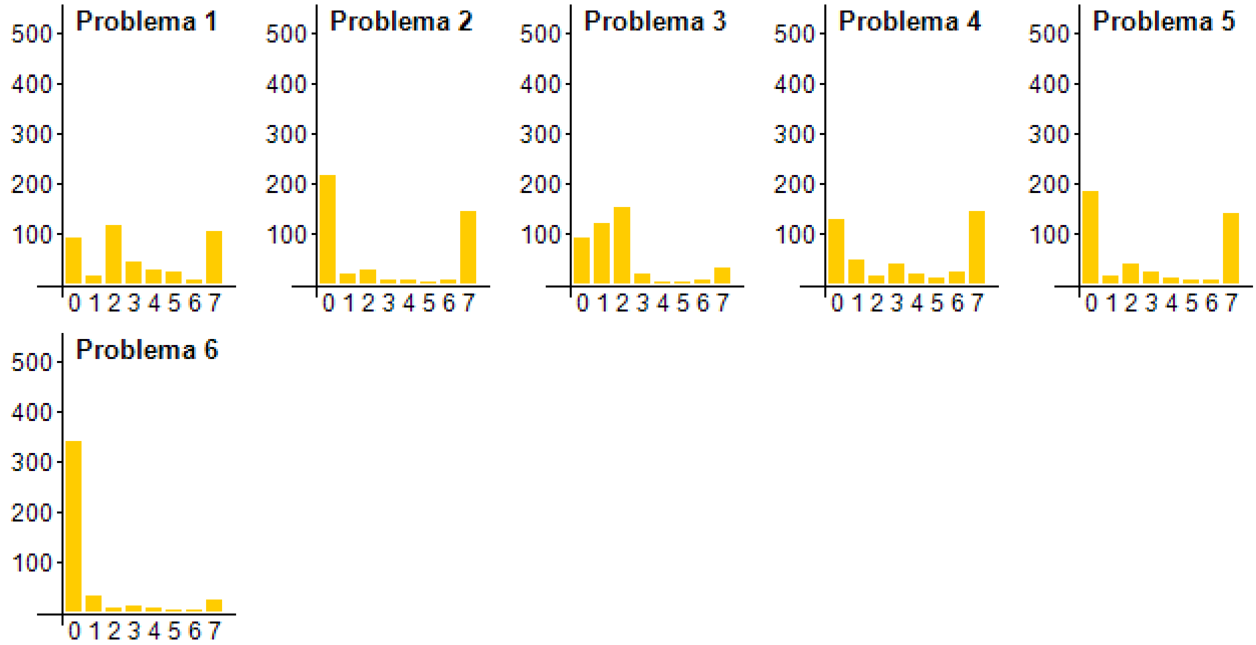




	P1	P2	P3	P4	P5	P6
Num(P# = 0)	91	164	269	81	136	341
Num(P# = 1)	143	24	64	72	51	40
Num(P# = 2)	30	16	5	20	36	12
Num(P# = 3)	23	4	22	34	31	22
Num(P# = 4)	102	7	4	58	21	4
Num(P# = 5)	15	9	2	29	19	27
Num(P# = 6)	8	0	6	31	14	4
Num(P# = 7)	48	236	88	135	152	10
Mean(P#)	2,476	3,898	1,778	3,743	3,354	0,815
Max(P#)	7	7	7	7	7	7
σ(P#)	2,234	3,309	2,746	2,717	2,971	1,715
Corr(P#, Sum)	0,722	0,687	0,725	0,734	0,748	0,646
Corr(P#, P1)		0,304	0,491	0,494	0,454	0,470
Corr(P#, P2)	0,304		0,306	0,404	0,425	0,295
Corr(P#, P3)	0,491	0,306		0,431	0,421	0,488
Corr(P#, P4)	0,494	0,404	0,431		0,423	0,361
Corr(P#, P5)	0,454	0,425	0,421	0,423		0,392
Corr(P#, P6)	0,470	0,295	0,488	0,361	0,392	

39^A OIM 1998

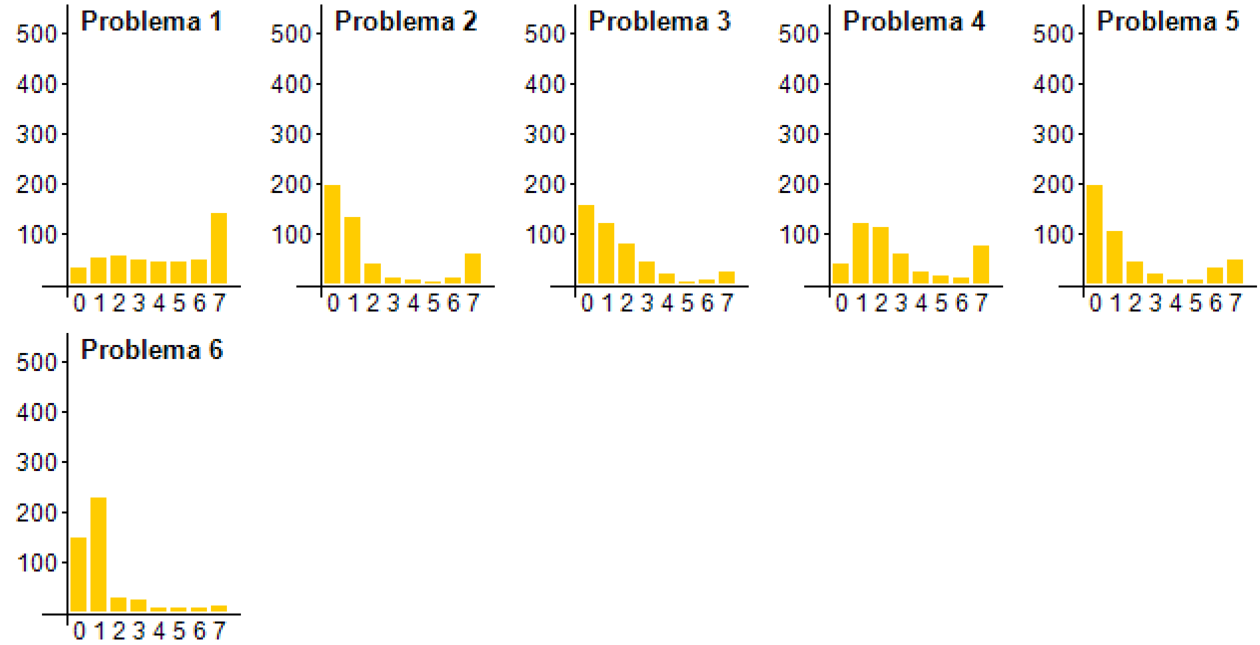
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	89	213	91	125	184	340
Num(P# = 1)	16	19	117	47	16	29
Num(P# = 2)	115	26	152	14	38	6
Num(P# = 3)	41	6	20	38	22	10
Num(P# = 4)	27	5	3	20	9	7
Num(P# = 5)	23	4	1	10	5	1
Num(P# = 6)	5	5	5	23	6	2
Num(P# = 7)	103	141	30	142	139	24
Mean(P#)	3,205	2,735	1,761	3,463	2,931	0,678
Max(P#)	7	7	7	7	7	7
σ(P#)	2,573	3,208	1,780	3,002	3,114	1,785
Corr(P#, Sum)	0,734	0,678	0,617	0,762	0,712	0,571
Corr(P#, P1)		0,299	0,375	0,510	0,488	0,334
Corr(P#, P2)	0,299		0,363	0,351	0,325	0,313
Corr(P#, P3)	0,375	0,363		0,422	0,267	0,328
Corr(P#, P4)	0,510	0,351	0,422		0,422	0,357
Corr(P#, P5)	0,488	0,325	0,267	0,422		0,257
Corr(P#, P6)	0,334	0,313	0,328	0,357	0,257	

40^{A} OIM 1999

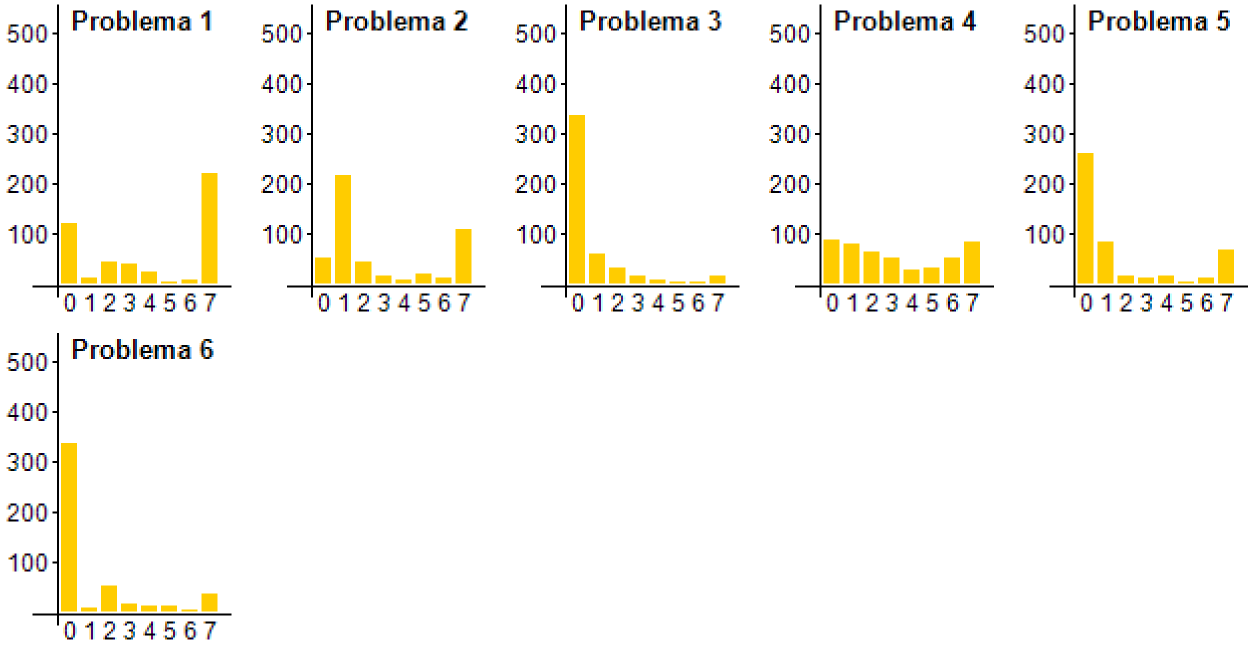
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	29	195	154	40	195	145
Num(P# = 1)	52	129	119	119	103	225
Num(P# = 2)	53	37	79	109	44	28
Num(P# = 3)	45	10	44	59	18	21
Num(P# = 4)	42	5	20	24	5	8
Num(P# = 5)	41	4	3	15	7	7
Num(P# = 6)	48	11	8	9	31	5
Num(P# = 7)	140	59	23	75	47	11
Mean(P#)	4,298	1,671	1,584	2,809	1,811	1,151
Max(P#)	7	7	7	7	7	7
σ(P#)	2,423	2,393	1,833	2,258	2,422	1,454
Corr(P#, Sum)	0,720	0,768	0,483	0,725	0,717	0,552
Corr(P#, P1)		0,433	0,277	0,417	0,376	0,279
Corr(P#, P2)	0,433		0,220	0,477	0,497	0,351
Corr(P#, P3)	0,277	0,220		0,183	0,177	0,211
Corr(P#, P4)	0,417	0,477	0,183		0,428	0,338
Corr(P#, P5)	0,376	0,497	0,177	0,428		0,268
Corr(P#, P6)	0,279	0,351	0,211	0,338	0,268	

41ᴬ OIM 2000

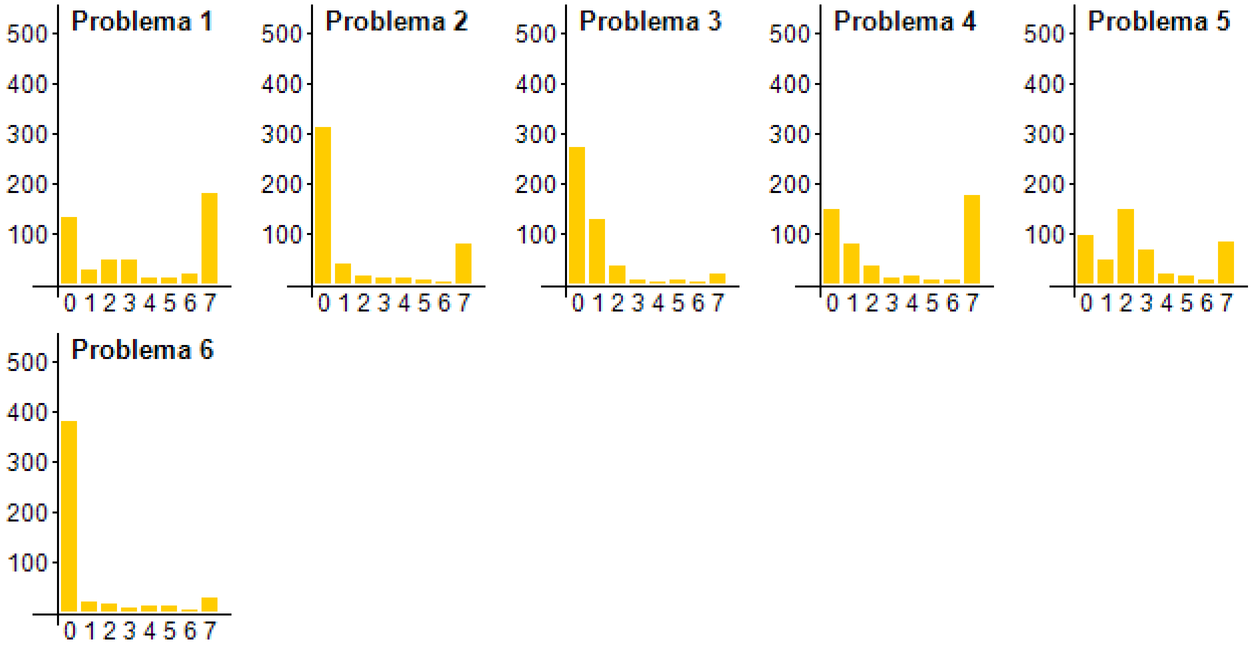
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	118	50	335	85	257	335
Num(P# = 1)	11	213	60	78	82	5
Num(P# = 2)	44	41	29	62	16	51
Num(P# = 3)	37	13	13	49	12	15
Num(P# = 4)	22	6	6	28	14	11
Num(P# = 5)	4	18	2	29	1	9
Num(P# = 6)	5	12	1	49	11	2
Num(P# = 7)	220	108	15	81	68	33
Mean(P#)	4,095	2,768	0,655	3,182	1,633	1,050
Max(P#)	7	7	7	7	7	7
σ(P#)	3,018	2,648	1,482	2,553	2,551	2,044
Corr(P#, Sum)	0,714	0,734	0,610	0,580	0,784	0,646
Corr(P#, P1)		0,405	0,260	0,228	0,430	0,411
Corr(P#, P2)	0,405		0,338	0,265	0,513	0,410
Corr(P#, P3)	0,260	0,338		0,404	0,410	0,363
Corr(P#, P4)	0,228	0,265	0,404		0,367	0,102
Corr(P#, P5)	0,430	0,513	0,410	0,367		0,454
Corr(P#, P6)	0,411	0,410	0,363	0,102	0,454	

42^{A} OIM 2001

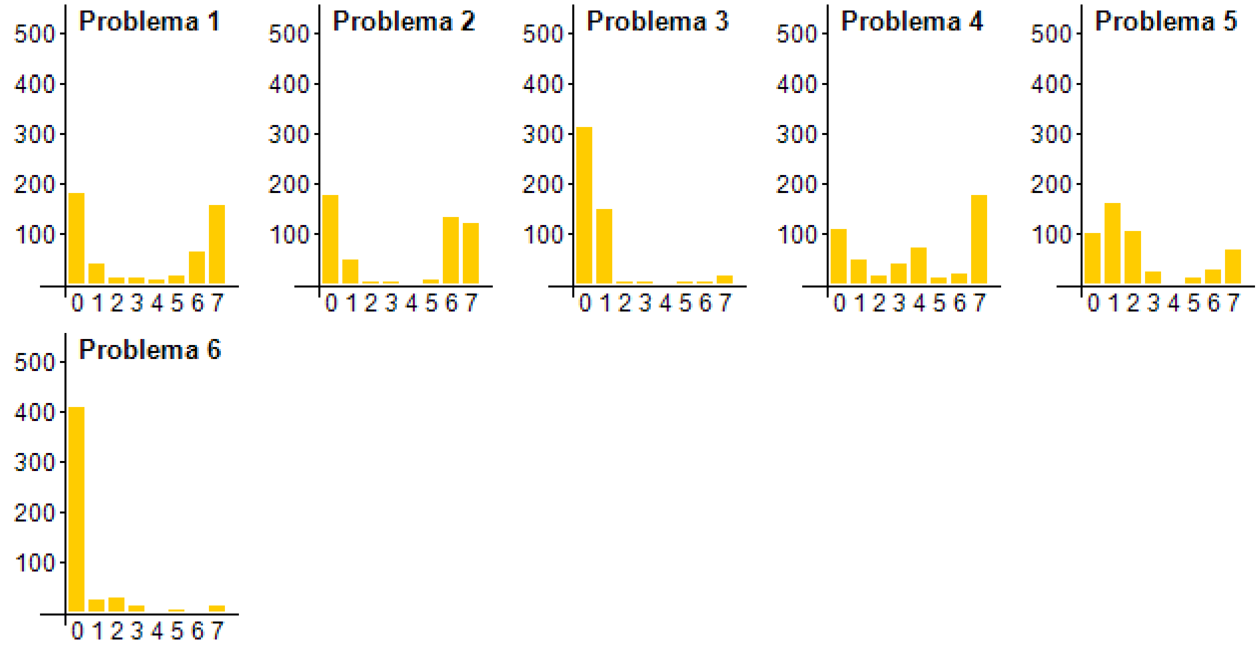
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	132	311	272	147	93	380
Num(P# = 1)	28	40	127	79	48	20
Num(P# = 2)	47	15	36	34	145	16
Num(P# = 3)	45	9	8	11	67	8
Num(P# = 4)	12	11	1	15	17	10
Num(P# = 5)	12	7	6	6	16	9
Num(P# = 6)	20	3	3	8	5	3
Num(P# = 7)	177	77	20	173	82	27
Mean(P#)	3,645	1,550	0,877	3,233	2,729	0,778
Max(P#)	7	7	7	7	7	7
σ(P#)	2,988	2,632	1,614	3,093	2,334	1,888
Corr(P#, Sum)	0,791	0,679	0,535	0,726	0,714	0,642
Corr(P#, P1)		0,404	0,316	0,500	0,524	0,364
Corr(P#, P2)	0,404		0,289	0,325	0,365	0,385
Corr(P#, P3)	0,316	0,289		0,274	0,263	0,339
Corr(P#, P4)	0,500	0,325	0,274		0,375	0,319
Corr(P#, P5)	0,524	0,365	0,263	0,375		0,423
Corr(P#, P6)	0,364	0,385	0,339	0,319	0,423	

43ª OIM 2002

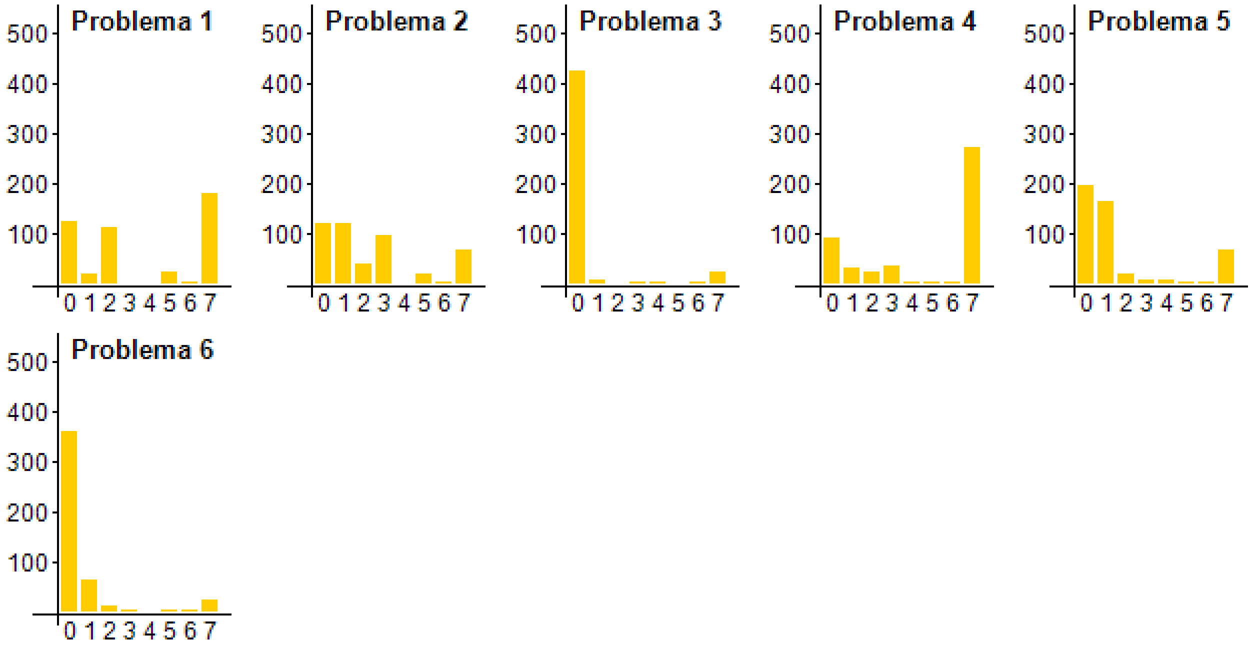
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	179	173	311	106	97	408
Num(P# = 1)	39	46	145	45	159	21
Num(P# = 2)	11	1	1	15	101	25
Num(P# = 3)	11	4	2	38	21	12
Num(P# = 4)	8	0	0	70	0	0
Num(P# = 5)	15	6	4	9	10	1
Num(P# = 6)	61	129	2	20	25	0
Num(P# = 7)	155	120	14	176	66	12
Mean(P#)	3,449	3,557	0,591	3,896	2,267	0,409
Max(P#)	7	7	7	7	7	7
σ(P#)	3,158	3,132	1,338	2,841	2,363	1,261
Corr(P#, Sum)	0,739	0,744	0,589	0,709	0,773	0,461
Corr(P#, P1)		0,390	0,301	0,426	0,437	0,220
Corr(P#, P2)	0,390		0,326	0,366	0,515	0,230
Corr(P#, P3)	0,301	0,326		0,325	0,482	0,354
Corr(P#, P4)	0,426	0,366	0,325		0,417	0,197
Corr(P#, P5)	0,437	0,515	0,482	0,417		0,354
Corr(P#, P6)	0,220	0,230	0,354	0,197	0,354	

44ª OIM 2003

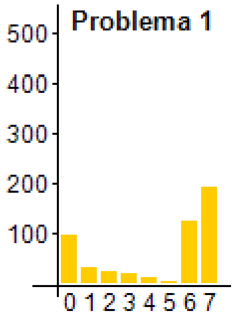
Estadística

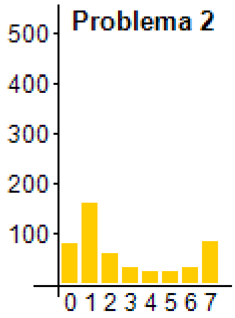


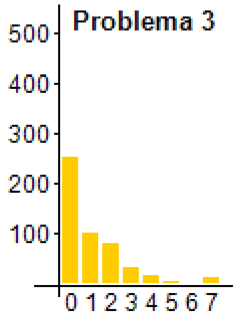
	P1	P2	P3	P4	P5	P6
Num(P# = 0)	123	117	424	90	193	357
Num(P# = 1)	18	119	5	31	161	64
Num(P# = 2)	111	40	0	23	20	9
Num(P# = 3)	0	93	3	35	6	1
Num(P# = 4)	0	0	1	1	5	0
Num(P# = 5)	24	20	0	3	1	1
Num(P# = 6)	1	1	1	2	4	1
Num(P# = 7)	180	67	23	272	67	24
Mean(P#)	3,558	2,304	0,405	4,632	1,613	0,578
Max(P#)	7	7	7	7	7	7
σ(P#)	3,007	2,343	1,577	3,003	2,403	1,616
Corr(P#, Sum)	0,675	0,740	0,528	0,672	0,717	0,609
Corr(P#, P1)		0,379	0,214	0,276	0,332	0,269
Corr(P#, P2)	0,379		0,330	0,388	0,457	0,393
Corr(P#, P3)	0,214	0,330		0,154	0,332	0,415
Corr(P#, P4)	0,276	0,388	0,154		0,359	0,255
Corr(P#, P5)	0,332	0,457	0,332	0,359		0,380
Corr(P#, P6)	0,269	0,393	0,415	0,255	0,380	

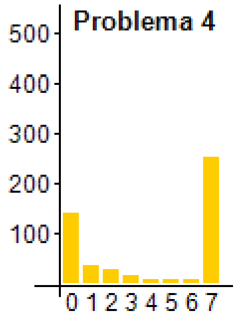
45ª OIM 2004

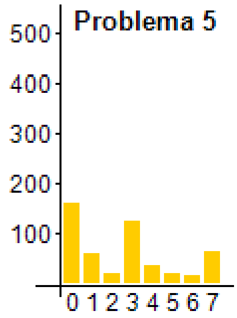
ESTADÍSTICA

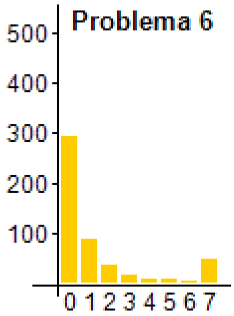








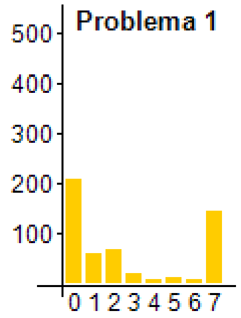


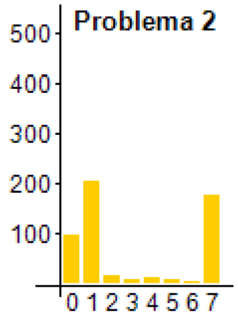


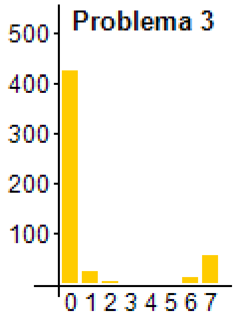
	P1	P2	P3	P4	P5	P6
Num(P# = 0)	94	79	249	140	157	289
Num(P# = 1)	29	158	100	33	57	85
Num(P# = 2)	21	57	80	27	18	33
Num(P# = 3)	19	32	30	16	124	14
Num(P# = 4)	9	23	15	8	35	7
Num(P# = 5)	2	23	1	6	18	6
Num(P# = 6)	121	31	0	6	13	4
Num(P# = 7)	191	83	11	250	64	48
Mean(P#)	4,603	2,761	1,012	4,080	2,512	1,257
Max(P#)	7	7	7	7	7	7
σ(P#)	2,831	2,523	1,432	3,188	2,397	2,191
Corr(P#, Sum)	0,708	0,754	0,589	0,787	0,723	0,700
Corr(P#, P1)		0,359	0,292	0,503	0,454	0,278
Corr(P#, P2)	0,359		0,371	0,509	0,449	0,532
Corr(P#, P3)	0,292	0,371		0,322	0,372	0,498
Corr(P#, P4)	0,503	0,509	0,322		0,422	0,418
Corr(P#, P5)	0,454	0,449	0,372	0,422		0,419
Corr(P#, P6)	0,278	0,532	0,498	0,418	0,419	

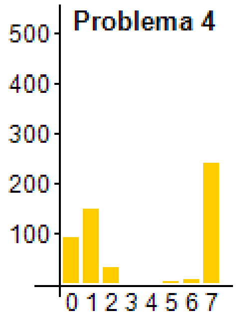
46ª OIM 2005

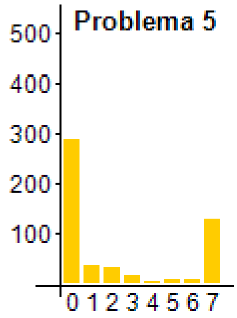
ESTADÍSTICA

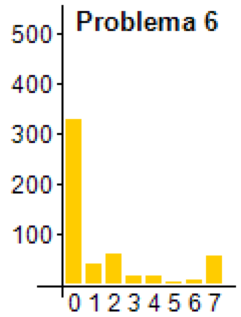








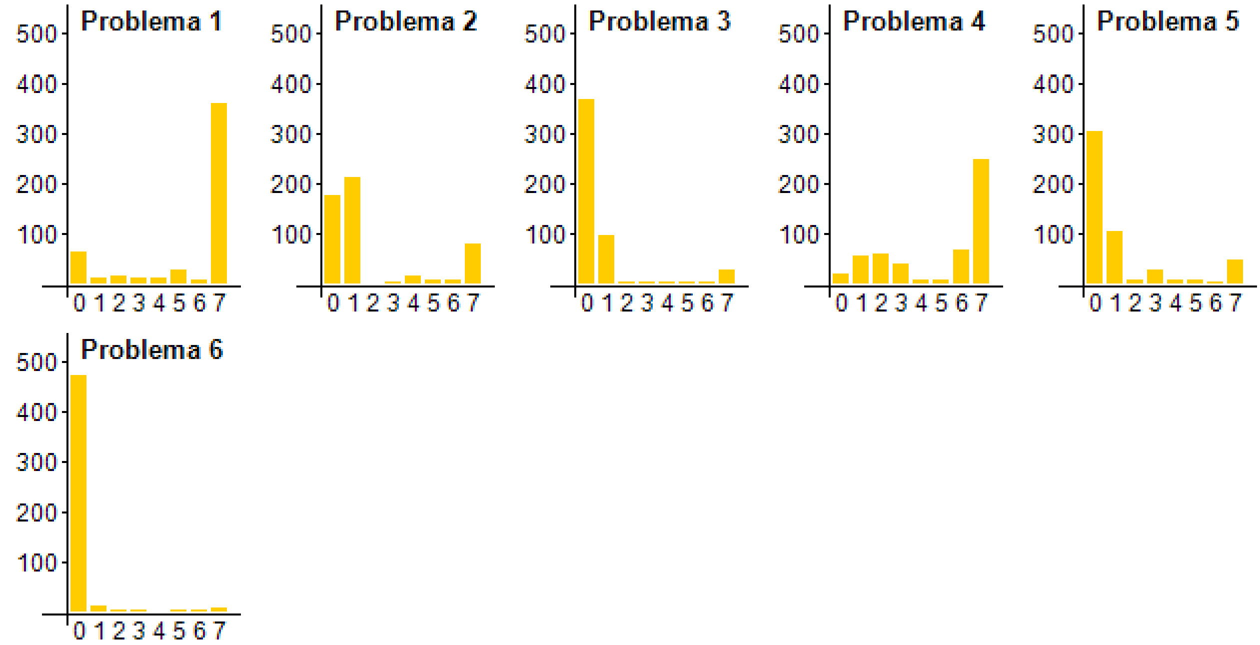




	P1	P2	P3	P4	P5	P6
Num(P# = 0)	207	96	423	90	288	325
Num(P# = 1)	59	202	23	145	36	39
Num(P# = 2)	65	13	3	30	30	57
Num(P# = 3)	20	7	0	0	16	13
Num(P# = 4)	5	11	0	0	4	15
Num(P# = 5)	11	7	0	3	6	2
Num(P# = 6)	5	2	9	7	8	6
Num(P# = 7)	141	175	55	238	125	56
Mean(P#)	2,614	3,051	0,912	3,758	2,170	1,345
Max(P#)	7	7	7	7	7	7
σ(P#)	2,941	2,975	2,263	3,124	2,966	2,297
Corr(P#, Sum)	0,716	0,767	0,634	0,720	0,724	0,701
Corr(P#, P1)		0,438	0,325	0,408	0,457	0,374
Corr(P#, P2)	0,438		0,339	0,472	0,428	0,564
Corr(P#, P3)	0,325	0,339		0,322	0,449	0,409
Corr(P#, P4)	0,408	0,472	0,322		0,377	0,409
Corr(P#, P5)	0,457	0,428	0,449	0,377		0,340
Corr(P#, P6)	0,374	0,564	0,409	0,409	0,340	

47ᴬ OIM 2006

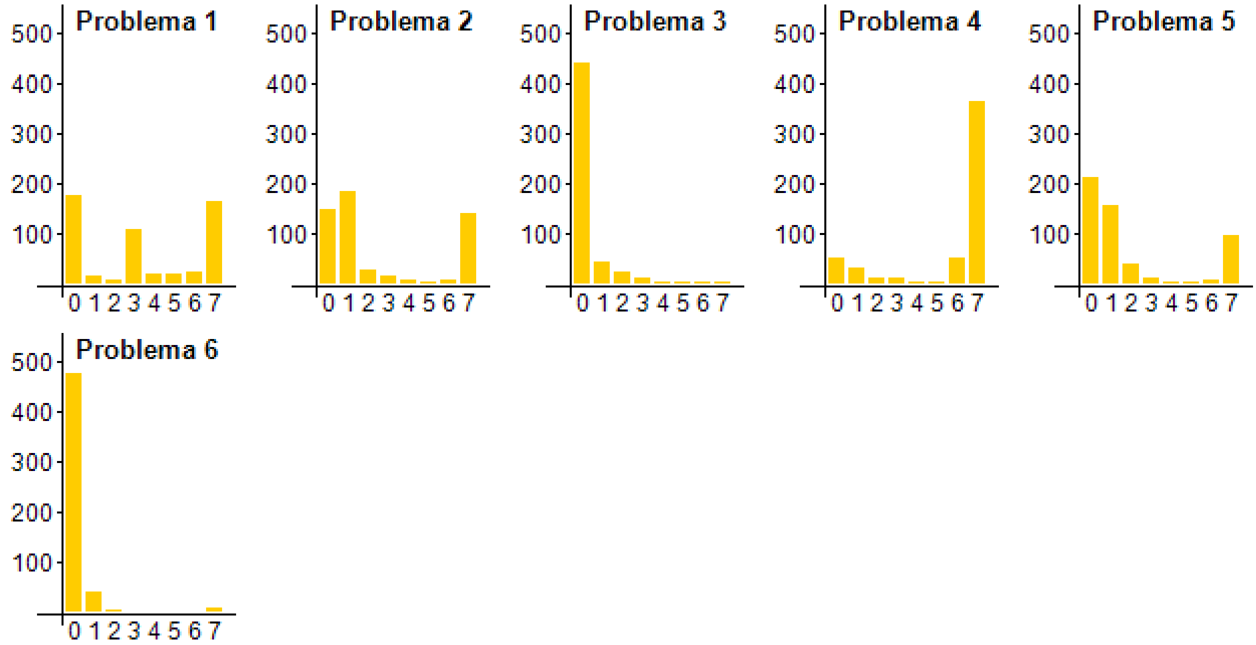
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	61	176	365	18	303	471
Num(P# = 1)	9	210	95	54	101	11
Num(P# = 2)	14	0	4	59	8	3
Num(P# = 3)	12	3	1	38	25	3
Num(P# = 4)	10	15	1	6	6	0
Num(P# = 5)	27	8	2	7	5	1
Num(P# = 6)	7	8	2	68	2	1
Num(P# = 7)	358	78	28	248	48	8
Mean(P#)	5,614	1,833	0,659	4,998	1,183	0,187
Max(P#)	7	7	7	7	7	7
σ(P#)	2,497	2,500	1,685	2,480	2,152	0,988
Corr(P#, Sum)	0,676	0,699	0,547	0,722	0,694	0,416
Corr(P#, P1)		0,300	0,191	0,491	0,259	0,105
Corr(P#, P2)	0,300		0,259	0,330	0,400	0,273
Corr(P#, P3)	0,191	0,259		0,243	0,343	0,262
Corr(P#, P4)	0,491	0,330	0,243		0,357	0,108
Corr(P#, P5)	0,259	0,400	0,343	0,357		0,330
Corr(P#, P6)	0,105	0,273	0,262	0,108	0,330	

48ª OIM 2007

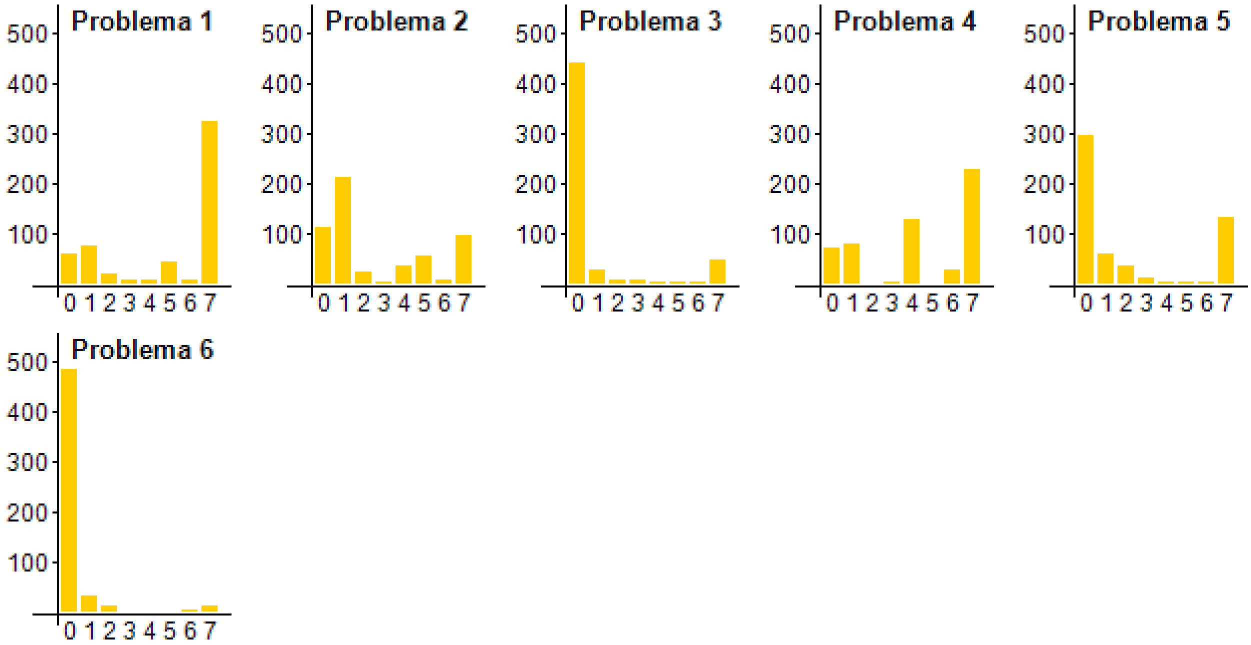
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	176	147	437	51	210	473
Num(P# = 1)	13	181	42	30	155	40
Num(P# = 2)	8	26	23	9	38	2
Num(P# = 3)	105	15	11	9	10	0
Num(P# = 4)	18	5	3	3	3	0
Num(P# = 5)	18	1	1	4	4	0
Num(P# = 6)	21	8	1	51	6	0
Num(P# = 7)	161	137	2	363	94	5
Mean(P#)	3,383	2,519	0,304	5,681	1,898	0,152
Max(P#)	7	7	7	7	7	7
σ(P#)	2,916	2,851	0,868	2,456	2,592	0,735
Corr(P#, Sum)	0,747	0,767	0,452	0,632	0,767	0,361
Corr(P#, P1)		0,385	0,275	0,362	0,428	0,201
Corr(P#, P2)	0,385		0,295	0,337	0,521	0,216
Corr(P#, P3)	0,275	0,295		0,126	0,366	0,133
Corr(P#, P4)	0,362	0,337	0,126		0,288	0,110
Corr(P#, P5)	0,428	0,521	0,366	0,288		0,297
Corr(P#, P6)	0,201	0,216	0,133	0,110	0,297	

49ª OIM 2008

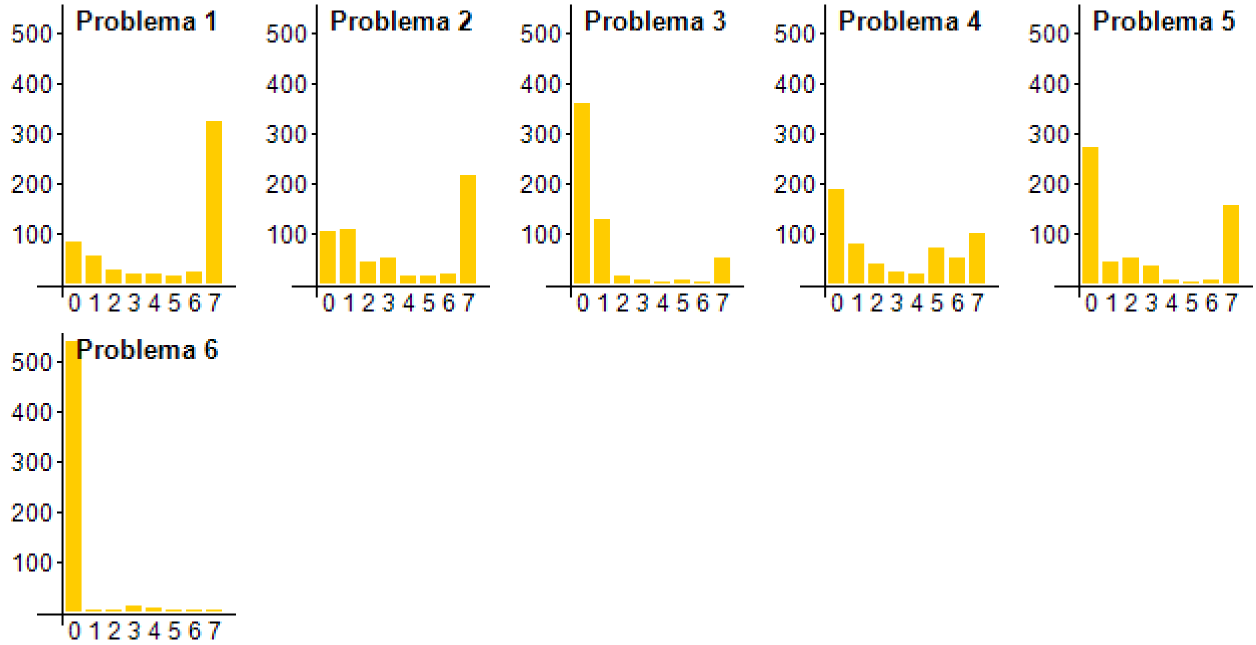
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	59	110	438	69	295	482
Num(P# = 1)	74	209	27	80	58	31
Num(P# = 2)	17	24	8	0	33	9
Num(P# = 3)	5	3	8	4	11	0
Num(P# = 4)	8	35	3	128	1	0
Num(P# = 5)	44	53	1	0	4	0
Num(P# = 6)	7	7	4	27	1	1
Num(P# = 7)	321	94	46	227	132	12
Mean(P#)	4,979	2,563	0,804	4,402	2,077	0,260
Max(P#)	7	7	7	7	7	7
σ(P#)	2,779	2,580	2,054	2,697	2,933	1,106
Corr(P#, Sum)	0,717	0,754	0,654	0,755	0,761	0,393
Corr(P#, P1)		0,408	0,266	0,526	0,403	0,164
Corr(P#, P2)	0,408		0,502	0,444	0,450	0,243
Corr(P#, P3)	0,266	0,502		0,337	0,413	0,291
Corr(P#, P4)	0,526	0,444	0,337		0,467	0,152
Corr(P#, P5)	0,403	0,450	0,413	0,467		0,249
Corr(P#, P6)	0,164	0,243	0,291	0,152	0,249	

50^A OIM 2009

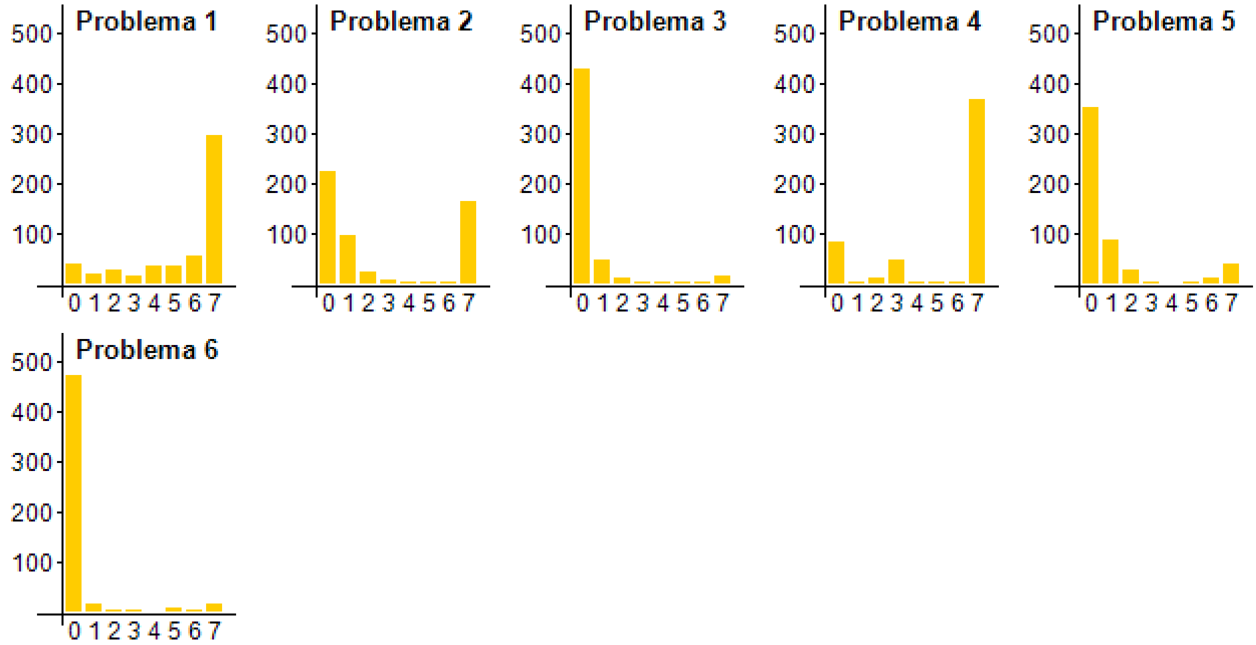
ESTADÍSTICA



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	83	101	357	188	270	540
Num(P# = 1)	56	106	127	79	42	2
Num(P# = 2)	28	43	16	37	50	1
Num(P# = 3)	20	51	5	23	33	10
Num(P# = 4)	17	16	2	17	6	6
Num(P# = 5)	16	15	5	69	4	2
Num(P# = 6)	21	19	2	52	7	1
Num(P# = 7)	324	214	51	100	153	3
Mean(P#)	4,804	3,710	1,019	2,915	2,474	0,168
Max(P#)	7	7	7	7	7	7
σ(P#)	2,858	2,903	2,051	2,791	2,982	0,851
Corr(P#, Sum)	0,728	0,802	0,690	0,772	0,842	0,361
Corr(P#, P1)		0,488	0,349	0,417	0,528	0,148
Corr(P#, P2)	0,488		0,445	0,577	0,571	0,161
Corr(P#, P3)	0,349	0,445		0,400	0,568	0,350
Corr(P#, P4)	0,417	0,577	0,400		0,557	0,233
Corr(P#, P5)	0,528	0,571	0,568	0,557		0,265
Corr(P#, P6)	0,148	0,161	0,350	0,233	0,265	

51ª OIM 2010

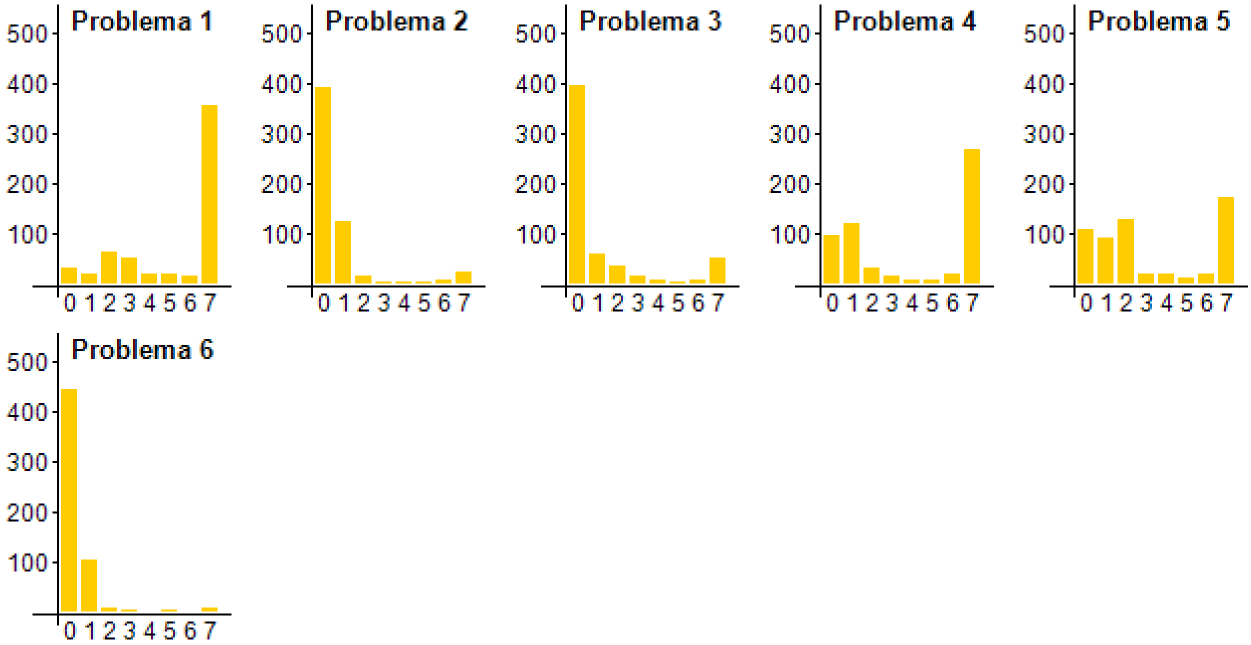
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	39	223	428	84	352	470
Num(P# = 1)	17	93	48	2	85	14
Num(P# = 2)	27	23	10	10	26	4
Num(P# = 3)	16	8	4	47	3	3
Num(P# = 4)	34	2	4	2	0	0
Num(P# = 5)	35	4	4	4	2	6
Num(P# = 6)	54	2	2	2	11	4
Num(P# = 7)	294	161	16	365	37	15
Mean(P#)	5,450	2,578	0,465	5,345	0,932	0,368
Max(P#)	7	7	7	7	7	7
σ(P#)	2,293	3,088	1,414	2,718	1,982	1,403
Corr(P#, Sum)	0,633	0,768	0,540	0,689	0,437	0,441
Corr(P#, P1)		0,337	0,194	0,373	0,135	0,098
Corr(P#, P2)	0,337		0,372	0,415	0,132	0,228
Corr(P#, P3)	0,194	0,372		0,195	0,206	0,246
Corr(P#, P4)	0,373	0,415	0,195		0,067	0,149
Corr(P#, P5)	0,135	0,132	0,206	0,067		0,211
Corr(P#, P6)	0,098	0,228	0,246	0,149	0,211	

52ª OIM 2011

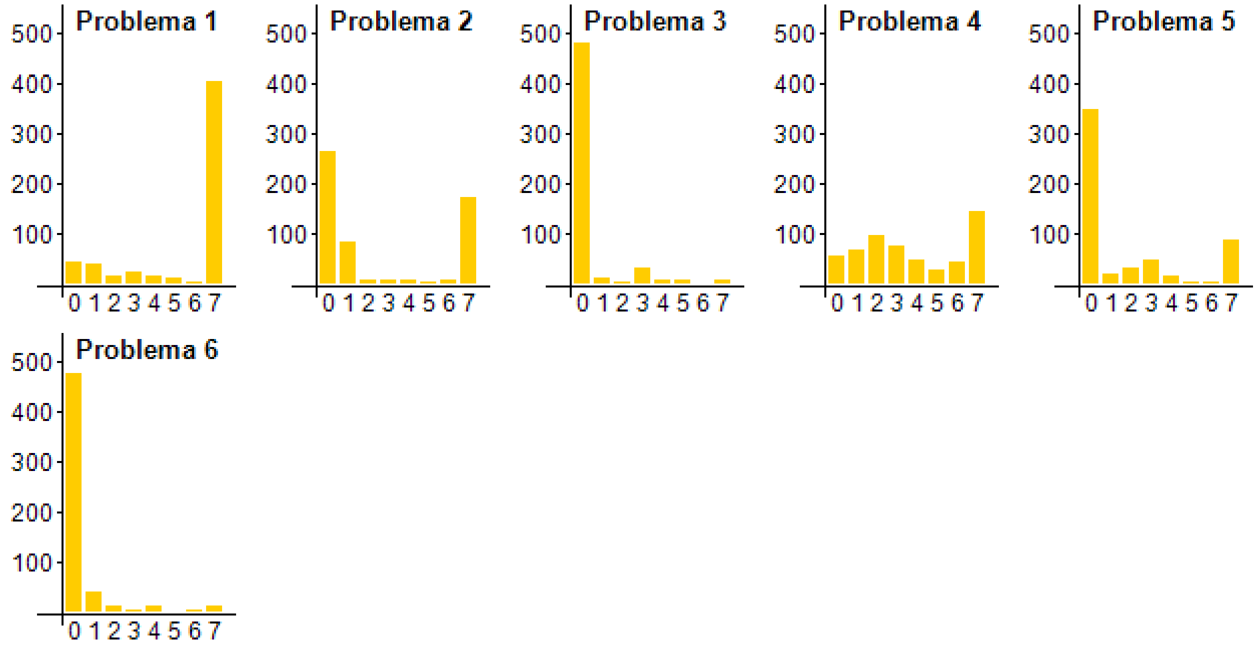
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	29	390	393	93	106	443
Num(P# = 1)	17	124	57	120	92	102
Num(P# = 2)	63	14	34	31	127	7
Num(P# = 3)	52	2	13	16	20	2
Num(P# = 4)	18	4	7	8	20	0
Num(P# = 5)	17	2	3	8	9	3
Num(P# = 6)	14	5	5	20	19	0
Num(P# = 7)	353	22	51	267	170	6
Mean(P#)	5,348	0,654	1,055	4,069	3,259	0,318
Max(P#)	7	7	7	7	7	7
σ(P#)	2,365	1,537	2,128	3,038	2,782	0,904
Corr(P#, Sum)	0,733	0,508	0,622	0,806	0,834	0,357
Corr(P#, P1)		0,236	0,267	0,551	0,514	0,159
Corr(P#, P2)	0,236		0,219	0,335	0,300	0,134
Corr(P#, P3)	0,267	0,219		0,281	0,468	0,333
Corr(P#, P4)	0,551	0,335	0,281		0,589	0,119
Corr(P#, P5)	0,514	0,300	0,468	0,589		0,225
Corr(P#, P6)	0,159	0,134	0,333	0,119	0,225	

53ª OIM 2012

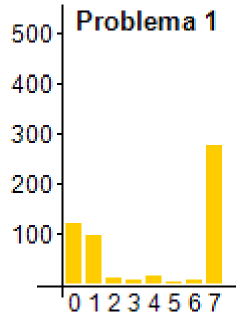
Estadística

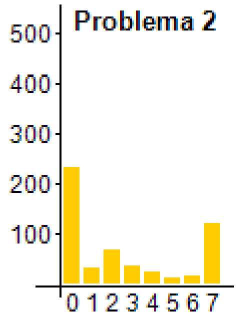


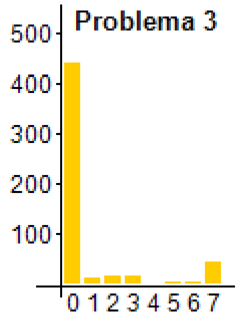
	P1	P2	P3	P4	P5	P6
Num(P# = 0)	41	263	480	53	348	473
Num(P# = 1)	37	83	11	65	17	39
Num(P# = 2)	15	8	4	95	29	12
Num(P# = 3)	24	5	31	74	45	3
Num(P# = 4)	16	8	7	47	15	9
Num(P# = 5)	11	2	6	26	4	0
Num(P# = 6)	2	7	0	44	3	1
Num(P# = 7)	401	171	8	143	86	10
Mean(P#)	5,625	2,550	0,413	3,766	1,664	0,336
Max(P#)	7	7	7	7	7	7
σ(P#)	2,440	3,145	1,259	2,473	2,596	1,149
Corr(P#, Sum)	0,652	0,748	0,535	0,721	0,727	0,483
Corr(P#, P1)		0,325	0,140	0,463	0,331	0,110
Corr(P#, P2)	0,325		0,329	0,369	0,408	0,256
Corr(P#, P3)	0,140	0,329		0,268	0,315	0,538
Corr(P#, P4)	0,463	0,369	0,268		0,395	0,217
Corr(P#, P5)	0,331	0,408	0,315	0,395		0,322
Corr(P#, P6)	0,110	0,256	0,538	0,217	0,322	

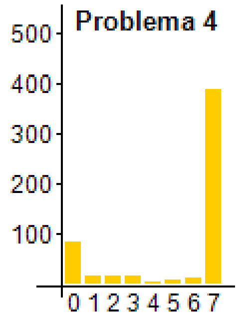
54ª OIM 2013

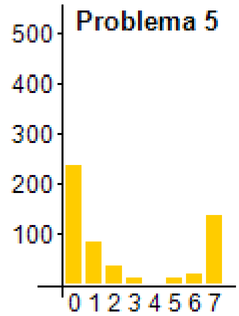
Estadística

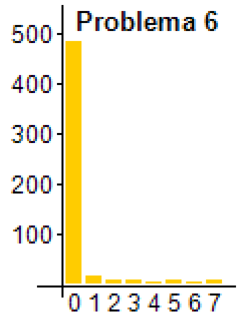








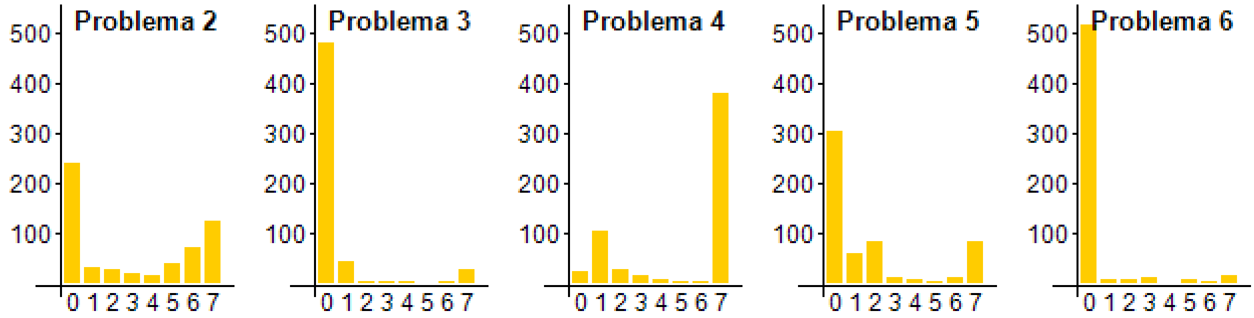




	P1	P2	P3	P4	P5	P6
Num(P# = 0)	118	229	438	82	235	481
Num(P# = 1)	96	32	10	16	84	15
Num(P# = 2)	9	65	15	14	33	6
Num(P# = 3)	6	33	16	14	11	6
Num(P# = 4)	14	22	0	2	0	2
Num(P# = 5)	3	12	3	5	10	6
Num(P# = 6)	5	16	4	9	19	4
Num(P# = 7)	276	118	41	385	135	7
Mean(P#)	4,108	2,526	0,786	5,442	2,452	0,296
Max(P#)	7	7	7	7	7	7
σ(P#)	3,170	2,836	2,004	2,733	2,985	1,165
Corr(P#, Sum)	0,792	0,728	0,521	0,648	0,820	0,447
Corr(P#, P1)		0,501	0,277	0,429	0,541	0,229
Corr(P#, P2)	0,501		0,158	0,302	0,532	0,328
Corr(P#, P3)	0,277	0,158		0,218	0,413	0,208
Corr(P#, P4)	0,429	0,302	0,218		0,388	0,141
Corr(P#, P5)	0,541	0,532	0,413	0,388		0,336
Corr(P#, P6)	0,229	0,328	0,208	0,141	0,336	

55ª OIM 2014

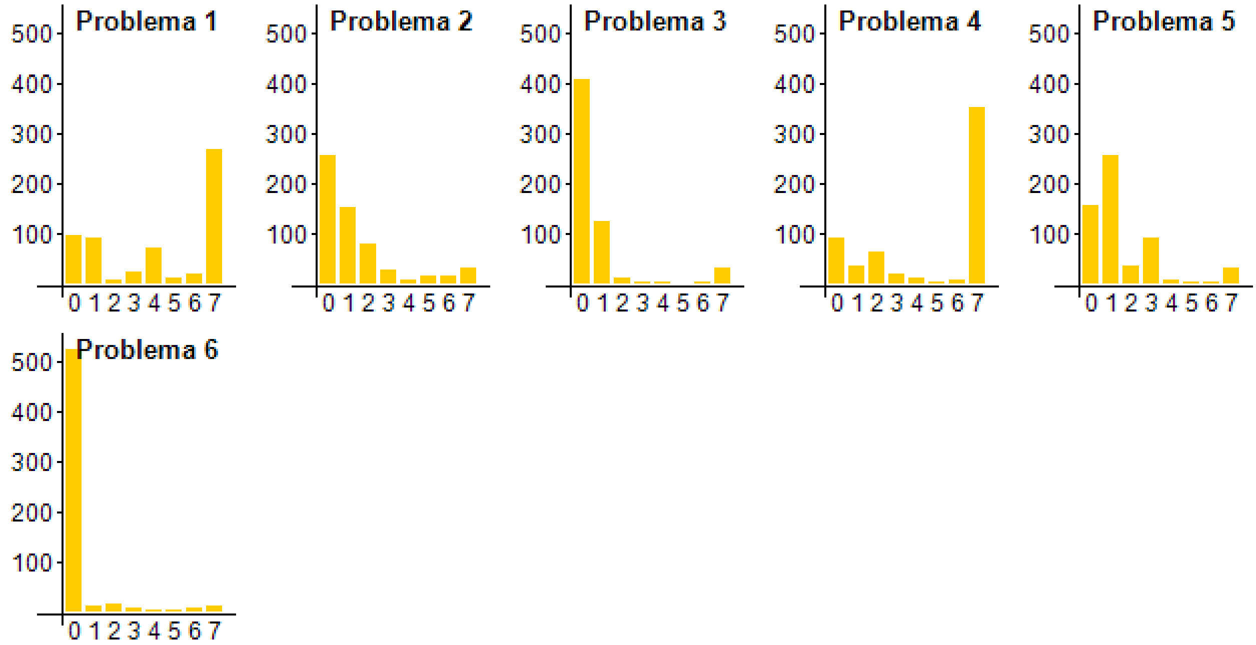
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	75	240	479	24	301	514
Num(P# = 1)	23	32	43	103	60	7
Num(P# = 2)	14	25	1	28	83	7
Num(P# = 3)	22	17	2	16	10	11
Num(P# = 4)	15	14	3	5	8	0
Num(P# = 5)	18	39	0	3	3	5
Num(P# = 6)	23	71	4	3	11	1
Num(P# = 7)	370	122	28	378	84	15
Mean(P#)	5,348	2,971	0,505	5,189	1,709	0,339
Max(P#)	7	7	7	7	7	7
σ(P#)	2,641	3,010	1,629	2,694	2,522	1,313
Corr(P#, Sum)	0,733	0,792	0,505	0,723	0,732	0,472
Corr(P#, P1)		0,466	0,194	0,563	0,368	0,153
Corr(P#, P2)	0,466		0,275	0,440	0,520	0,289
Corr(P#, P3)	0,194	0,275		0,206	0,325	0,371
Corr(P#, P4)	0,563	0,440	0,206		0,346	0,153
Corr(P#, P5)	0,368	0,520	0,325	0,346		0,359
Corr(P#, P6)	0,153	0,289	0,371	0,153	0,359	

56ª OIM 2015

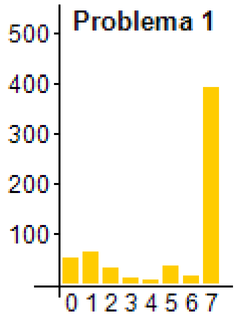
Estadística

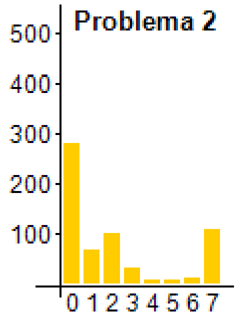


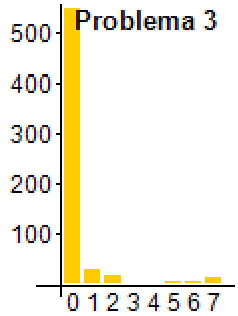
	P1	P2	P3	P4	P5	P6
Num(P# = 0)	93	256	408	91	153	521
Num(P# = 1)	89	151	122	36	255	11
Num(P# = 2)	5	77	12	61	34	15
Num(P# = 3)	21	27	1	18	90	6
Num(P# = 4)	72	8	3	11	8	3
Num(P# = 5)	12	13	0	1	4	3
Num(P# = 6)	20	14	1	8	3	7
Num(P# = 7)	265	31	30	351	30	11
Mean(P#)	4,307	1,359	0,653	4,794	1,513	0,355
Max(P#)	7	7	7	7	7	7
σ(P#)	2,870	1,914	1,605	2,905	1,709	1,294
Corr(P#, Sum)	0,752	0,706	0,572	0,743	0,666	0,436
Corr(P#, P1)		0,378	0,249	0,450	0,385	0,211
Corr(P#, P2)	0,378		0,363	0,377	0,476	0,278
Corr(P#, P3)	0,249	0,363		0,272	0,333	0,288
Corr(P#, P4)	0,450	0,377	0,272		0,357	0,148
Corr(P#, P5)	0,385	0,476	0,333	0,357		0,172
Corr(P#, P6)	0,211	0,278	0,288	0,148	0,172	

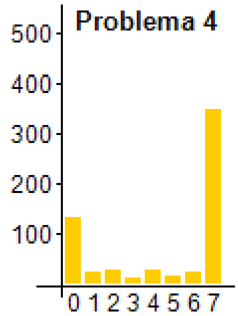
57ª OIM 2016

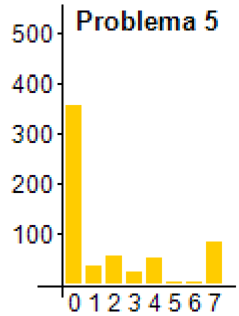
Estadística

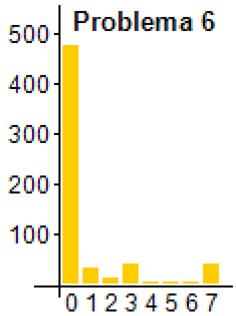








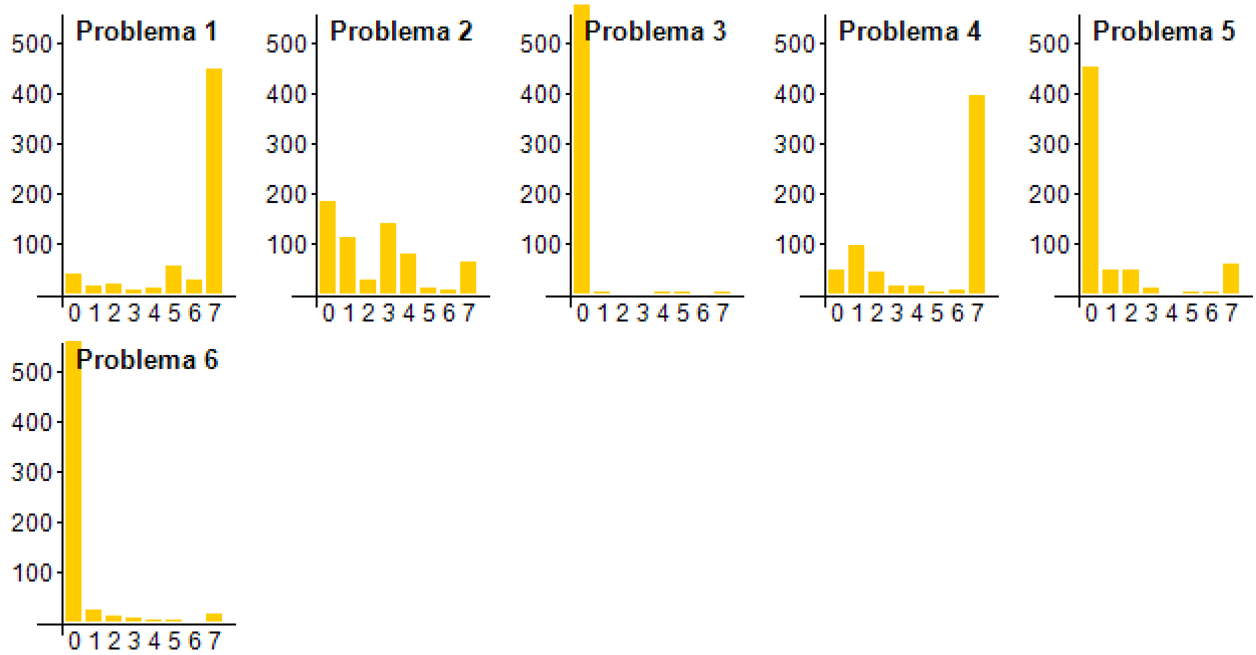




	P1	P2	P3	P4	P5	P6
Num(P# = 0)	52	277	548	132	353	474
Num(P# = 1)	63	65	25	22	36	31
Num(P# = 2)	32	99	14	26	55	9
Num(P# = 3)	9	30	0	10	21	39
Num(P# = 4)	6	7	0	26	50	4
Num(P# = 5)	35	8	2	15	2	4
Num(P# = 6)	14	9	3	24	4	4
Num(P# = 7)	391	107	10	347	81	37
Mean(P#)	5,272	2,033	0,251	4,744	1,678	0,806
Max(P#)	7	7	7	7	7	7
σ(P#)	2,632	2,617	1,071	2,974	2,484	1,889
Corr(P#, Sum)	0,711	0,762	0,428	0,786	0,746	0,538
Corr(P#, P1)		0,354	0,131	0,627	0,345	0,185
Corr(P#, P2)	0,354		0,281	0,464	0,548	0,354
Corr(P#, P3)	0,131	0,281		0,155	0,358	0,307
Corr(P#, P4)	0,627	0,464	0,155		0,427	0,222
Corr(P#, P5)	0,345	0,548	0,358	0,427		0,334
Corr(P#, P6)	0,185	0,354	0,307	0,222	0,334	

58ª OIM 2017

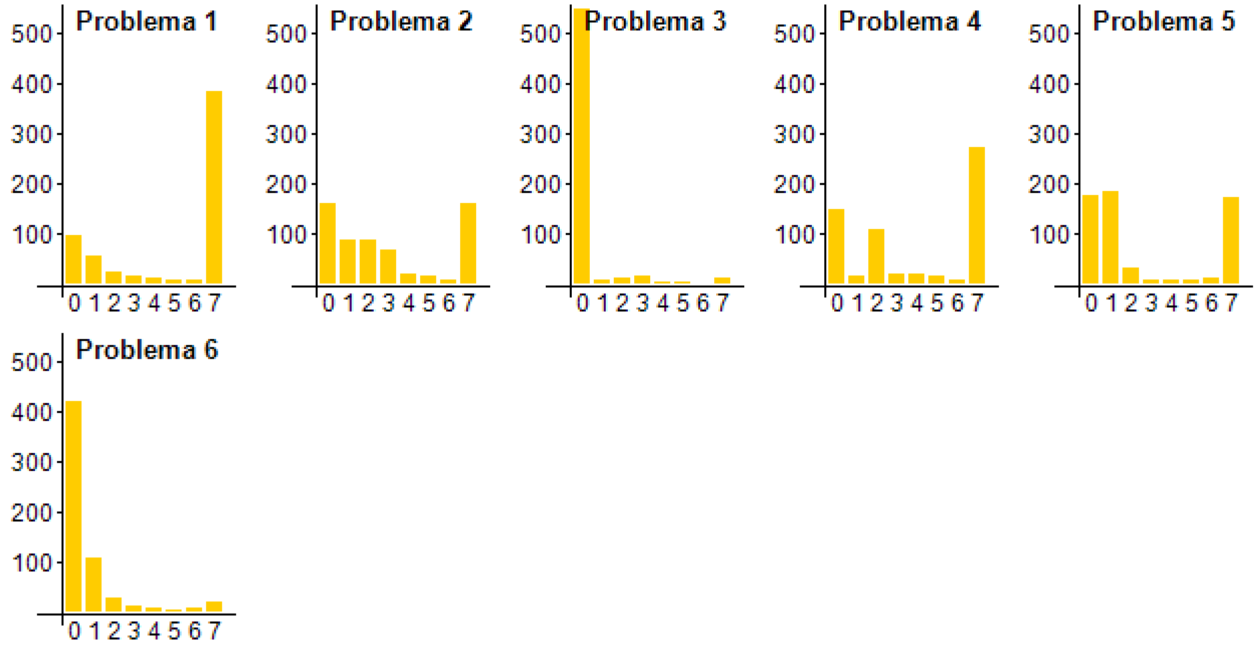
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	40	183	608	47	451	557
Num(P# = 1)	16	110	3	93	46	24
Num(P# = 2)	17	26	0	42	47	9
Num(P# = 3)	5	138	0	14	9	5
Num(P# = 4)	12	79	1	15	0	4
Num(P# = 5)	54	10	1	4	2	2
Num(P# = 6)	25	8	0	6	1	0
Num(P# = 7)	446	61	2	394	59	14
Mean(P#)	5,943	2,304	0,042	5,029	0,969	0,294
Max(P#)	7	7	7	7	7	7
σ(P#)	2,096	2,202	0,479	2,763	2,101	1,177
Corr(P#, Sum)	0,691	0,737	0,124	0,756	0,561	0,414
Corr(P#, P1)		0,358	0,045	0,462	0,208	0,118
Corr(P#, P2)	0,358		0,099	0,410	0,292	0,279
Corr(P#, P3)	0,045	0,099		0,005	0,008	0,027
Corr(P#, P4)	0,462	0,410	0,005		0,179	0,167
Corr(P#, P5)	0,208	0,292	0,008	0,179		0,159
Corr(P#, P6)	0,118	0,279	0,027	0,167	0,159	

59ª OIM 2018

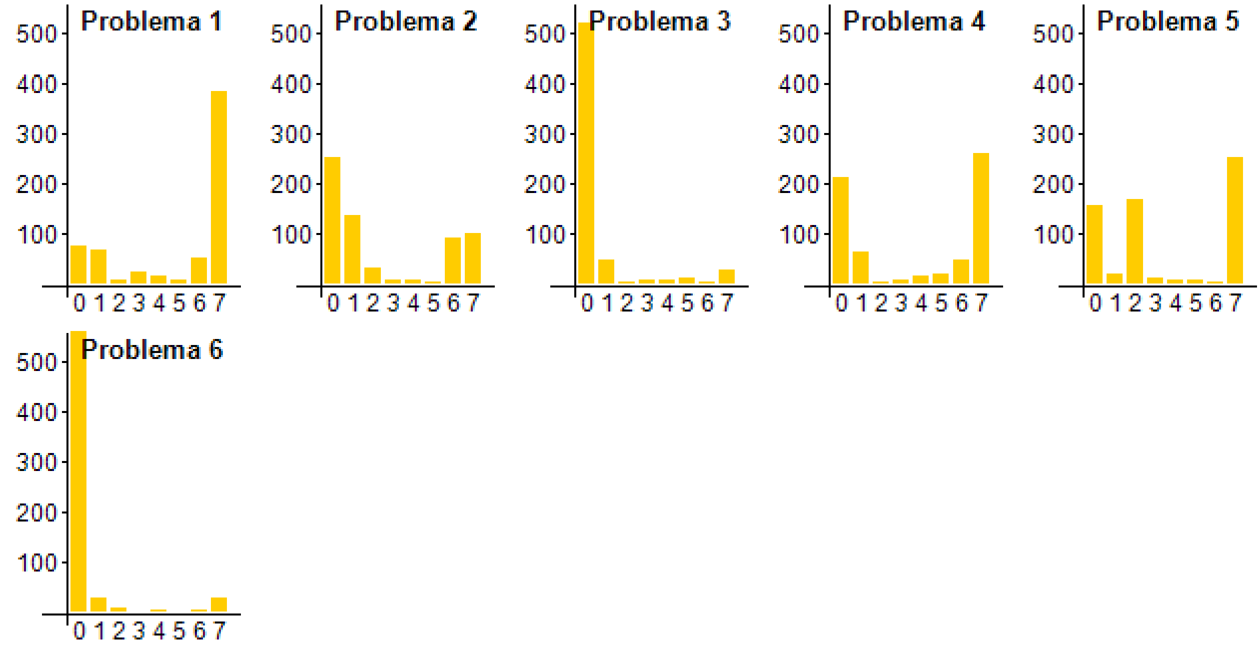
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	96	158	548	148	175	419
Num(P# = 1)	54	85	7	13	184	108
Num(P# = 2)	24	87	9	106	31	26
Num(P# = 3)	15	66	14	18	7	11
Num(P# = 4)	10	18	4	18	6	5
Num(P# = 5)	7	16	1	15	8	2
Num(P# = 6)	7	7	0	5	11	5
Num(P# = 7)	381	157	11	271	172	18
Mean(P#)	4,934	2,946	0,278	3,961	2,695	0,638
Max(P#)	7	7	7	7	7	7
σ(P#)	2,924	2,754	1,125	3,015	2,952	1,457
Corr(P#, Sum)	0,733	0,842	0,429	0,799	0,807	0,543
Corr(P#, P1)		0,565	0,174	0,464	0,436	0,249
Corr(P#, P2)	0,565		0,303	0,588	0,607	0,383
Corr(P#, P3)	0,174	0,303		0,219	0,307	0,334
Corr(P#, P4)	0,464	0,588	0,219		0,570	0,337
Corr(P#, P5)	0,436	0,607	0,307	0,570		0,369
Corr(P#, P6)	0,249	0,383	0,334	0,337	0,369	

60^{A} OIM 2019

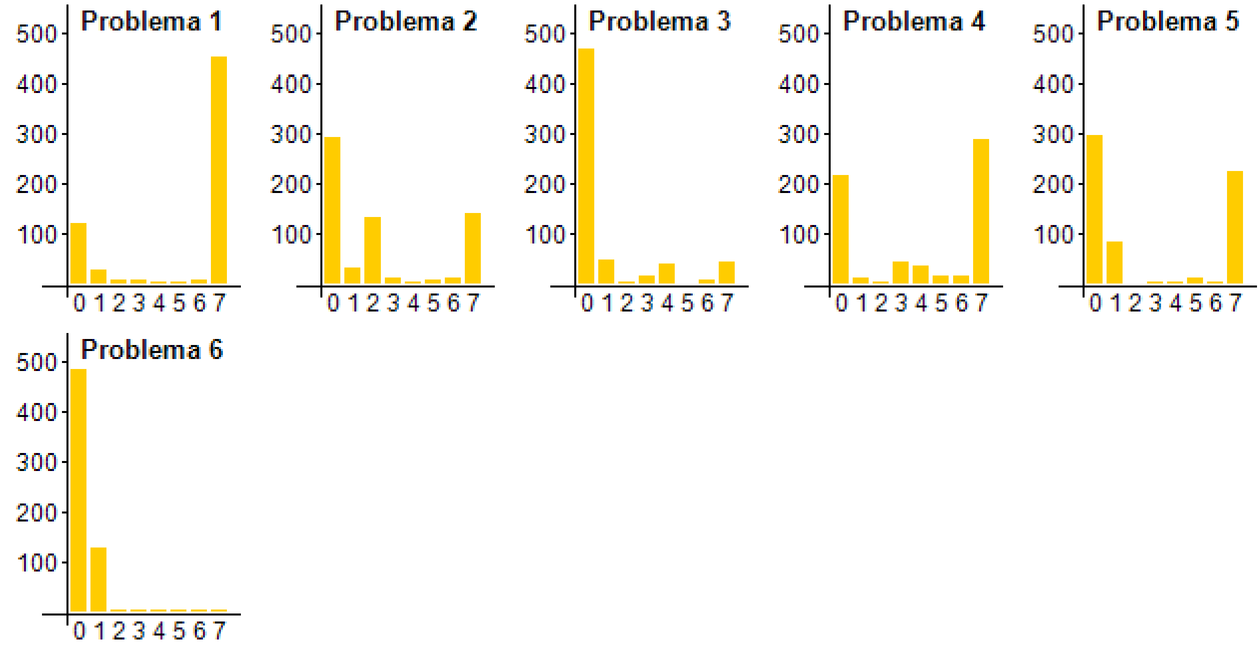
Estadística



	P1	P2	P3	P4	P5	P6
Num(P# = 0)	73	251	520	211	156	558
Num(P# = 1)	65	135	46	63	20	25
Num(P# = 2)	6	30	3	4	168	7
Num(P# = 3)	24	6	6	7	12	0
Num(P# = 4)	14	6	5	13	5	1
Num(P# = 5)	5	3	9	19	7	0
Num(P# = 6)	52	92	4	47	3	3
Num(P# = 7)	382	98	28	257	250	27
Mean(P#)	5,179	2,399	0,572	3,736	3,567	0,403
Max(P#)	7	7	7	7	7	7
σ(P#)	2,718	2,843	1,669	3,221	2,979	1,501
Corr(P#, Sum)	0,745	0,737	0,527	0,817	0,788	0,473
Corr(P#, P1)		0,415	0,215	0,603	0,509	0,168
Corr(P#, P2)	0,415		0,341	0,491	0,422	0,331
Corr(P#, P3)	0,215	0,341		0,252	0,364	0,343
Corr(P#, P4)	0,603	0,491	0,252		0,571	0,236
Corr(P#, P5)	0,509	0,422	0,364	0,571		0,278
Corr(P#, P6)	0,168	0,331	0,343	0,236	0,278	

61ª OIM 2020

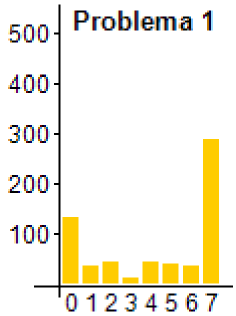
Estadística

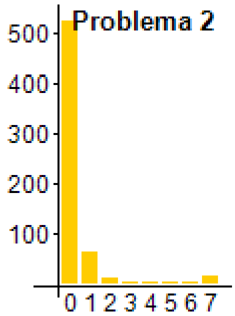


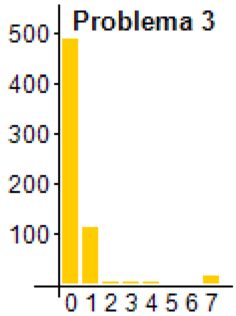
	P1	P2	P3	P4	P5	P6
Num(P# = 0)	117	291	465	213	294	481
Num(P# = 1)	26	29	47	11	83	126
Num(P# = 2)	5	129	3	3	0	1
Num(P# = 3)	5	9	14	42	2	1
Num(P# = 4)	2	4	40	35	1	1
Num(P# = 5)	3	7	0	14	9	1
Num(P# = 6)	7	9	5	13	4	1
Num(P# = 7)	451	138	42	285	223	4
Mean(P#)	5,313	2,248	0,940	3,938	2,797	0,282
Max(P#)	7	7	7	7	7	7
σ(P#)	2,894	2,802	2,022	3,170	3,272	0,767
Corr(P#, Sum)	0,676	0,702	0,535	0,779	0,805	0,494
Corr(P#, P1)		0,375	0,200	0,406	0,399	0,190
Corr(P#, P2)	0,375		0,280	0,374	0,466	0,255
Corr(P#, P3)	0,200	0,280		0,315	0,291	0,364
Corr(P#, P4)	0,406	0,374	0,315		0,572	0,342
Corr(P#, P5)	0,399	0,466	0,291	0,572		0,407
Corr(P#, P6)	0,190	0,255	0,364	0,342	0,407	

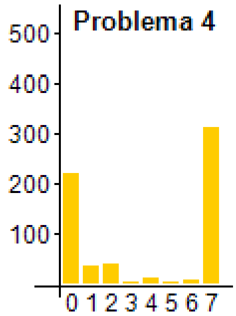
62^{A} OIM 2021

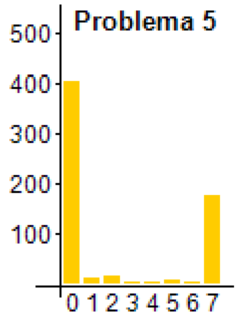
Estadística

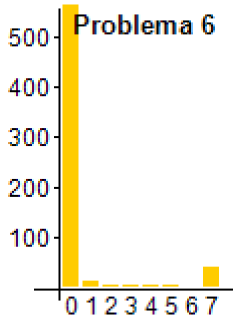








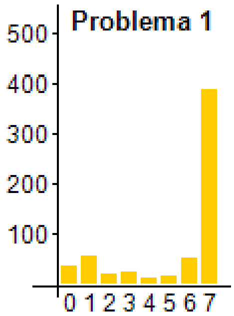


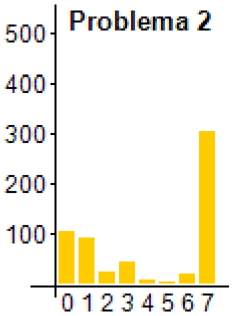


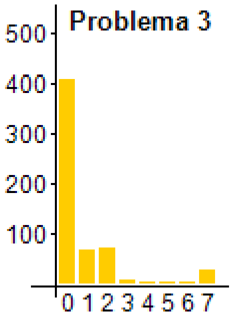
	P1	P2	P3	P4	P5	P6
Num(P# = 0)	131	522	488	218	404	562
Num(P# = 1)	36	61	110	33	12	12
Num(P# = 2)	41	12	4	39	13	2
Num(P# = 3)	10	2	1	2	4	3
Num(P# = 4)	41	3	1	12	2	1
Num(P# = 5)	38	1	0	1	5	2
Num(P# = 6)	36	2	0	5	4	0
Num(P# = 7)	286	16	15	309	175	37
Mean(P#)	4,394	0,375	0,372	3,817	2,152	0,481
Max(P#)	7	7	7	7	7	7
σ(P#)	2,913	1,251	1,137	3,296	3,142	1,697
Corr(P#, Sum)	0,763	0,483	0,480	0,764	0,766	0,557
Corr(P#, P1)		0,232	0,240	0,537	0,459	0,215
Corr(P#, P2)	0,232		0,202	0,221	0,325	0,330
Corr(P#, P3)	0,240	0,202		0,290	0,240	0,377
Corr(P#, P4)	0,537	0,221	0,290		0,382	0,229
Corr(P#, P5)	0,459	0,325	0,240	0,382		0,391
Corr(P#, P6)	0,215	0,330	0,377	0,229	0,391	

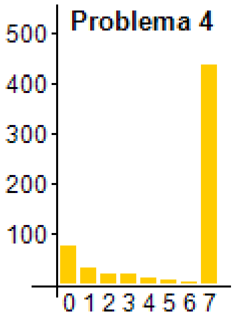
63ª OIM 2022

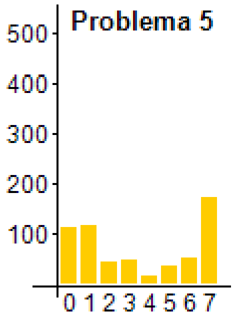
Estadística

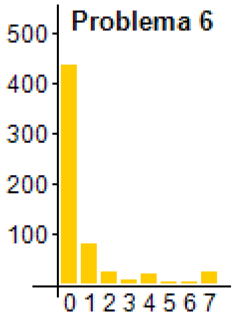








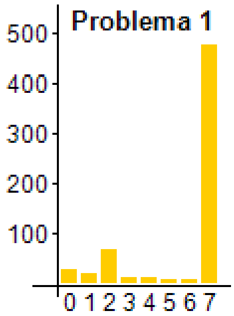


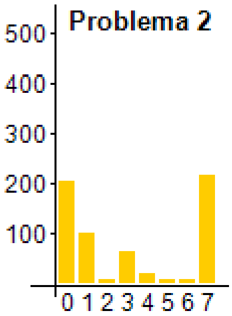


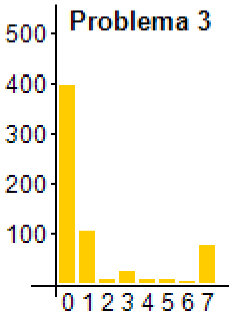
	P1	P2	P3	P4	P5	P6
Num(P# = 0)	34	104	407	74	112	434
Num(P# = 1)	56	89	68	31	115	80
Num(P# = 2)	19	23	69	18	44	22
Num(P# = 3)	21	41	6	17	48	7
Num(P# = 4)	10	7	3	10	15	19
Num(P# = 5)	13	3	4	5	35	3
Num(P# = 6)	51	19	4	1	49	2
Num(P# = 7)	385	303	28	433	171	22
Mean(P#)	5,540	4,306	0,808	5,467	3,520	0,683
Max(P#)	7	7	7	7	7	7
σ(P#)	2,413	3,014	1,688	2,667	2,820	1,588
Corr(P#, Sum)	0,724	0,852	0,618	0,679	0,848	0,544
Corr(P#, P1)		0,585	0,273	0,381	0,538	0,237
Corr(P#, P2)	0,585		0,412	0,483	0,682	0,344
Corr(P#, P3)	0,273	0,412		0,224	0,470	0,585
Corr(P#, P4)	0,381	0,483	0,224		0,488	0,178
Corr(P#, P5)	0,538	0,682	0,470	0,488		0,357
Corr(P#, P6)	0,237	0,344	0,585	0,178	0,357	

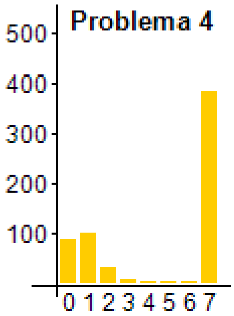
64ª OIM 2023

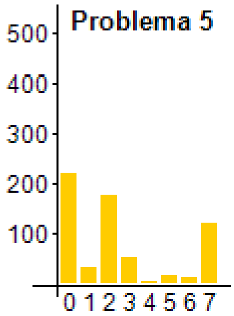
Estadística

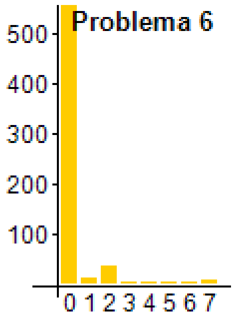












	P1	**P2**	**P3**	**P4**	**P5**	**P6**
Num(P# = 0)	26	202	396	86	219	555
Num(P# = 1)	19	100	102	100	29	11
Num(P# = 2)	67	6	7	32	174	36
Num(P# = 3)	9	62	23	8	52	4
Num(P# = 4)	9	20	8	4	4	1
Num(P# = 5)	6	7	6	1	13	1
Num(P# = 6)	8	6	3	3	9	4
Num(P# = 7)	474	215	73	384	118	6
Mean(P#)	5,845	3,162	1,256	4,717	2,417	0,275
Max(P#)	7	7	7	7	7	7
σ(P#)	2,227	3,050	2,320	3,000	2,568	1,004
Corr(P#, Sum)	0,677	0,780	0,708	0,801	0,746	0,457
Corr(P#, P1)		0,422	0,271	0,576	0,383	0,125
Corr(P#, P2)	0,422		0,482	0,520	0,412	0,311
Corr(P#, P3)	0,271	0,482		0,377	0,502	0,480
Corr(P#, P4)	0,576	0,520	0,377		0,512	0,193
Corr(P#, P5)	0,383	0,412	0,502	0,512		0,306
Corr(P#, P6)	0,125	0,311	0,480	0,193	0,306	

IMO 1997 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 1997 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. In the plane there is an infinite chessboard. For any pair of positive integers m and n, consider a right-angled triangle with vertices at lattice points and whose legs, of lengths m and n, lie along edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

(a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

2. It is known that $\angle BAC$ is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T.

Show that $AU = TB + TC$.

3. Let x_1, x_2, …, x_n be real numbers satisfying the conditions:

$$|x_1 + x_2 + \cdots + x_n| = 1$$
$$|x_i| \le \frac{n+1}{2} \qquad \text{for } i = 1, 2, \dots, n$$

Show that there exists a permutation y_1, y_2, …, y_n of x_1, x_2, …, x_n such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

4. An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \dots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \dots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

5. Find all pairs (a,b) of positive integers satisfying

$$a^{b^2} = b^a.$$

6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways: 4; $2+2$; $2+1+1$; $1+1+1+1$.

Prove that for any integer $n \ge 3$ we have $2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}$.

§1 Solutions to Day 1

§1.1 IMO 1997/1

Available online at https://aops.com/community/p356696.

Problem statement

In the plane there is an infinite chessboard. For any pair of positive integers m and n, consider a right-angled triangle with vertices at lattice points and whose legs, of lengths m and n, lie along edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

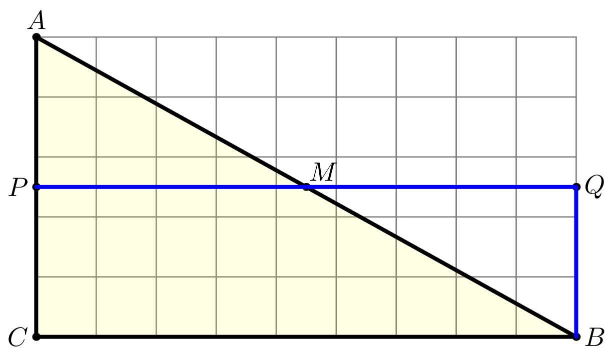
(a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

In general, we say the *discrepancy* of a region in the plane equals its black area minus its white area. We allow negative discrepancies, so discrepancy is additive and $f(m,n)$ equals the absolute value of the discrepancy of a right triangle with legs m and n.

For (a), the answers are 0 and 1/2 respectively. To see this, consider the figure shown below.



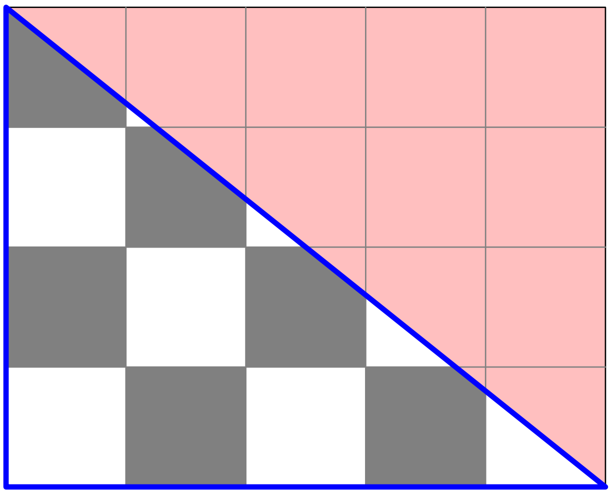
Notice that triangles APM and BQM are congruent, and when $m \equiv n \pmod 2$, their colorings actually coincide. Consequently, the discrepancy of the triangle is exactly equal to the discrepancy of $CPQB$, which is an $m \times n/2$ rectangle and hence equal to 0 or 1/2 according to parity.

For (b), note that a triangle with legs m and n, with m even and n odd, can be dissected into one right triangle with legs m and $n-1$ plus a thin triangle of area 1/2 which has height m and base 1. The former region has discrepancy 0 by (a), and the latter region obviously has discrepancy at most its area of $m/2$, hence $f(m,n) \le m/2$ as needed. (An alternative slower approach, which requires a few cases, is to prove that two adjacent columns have at most discrepancy 1/2.)

For (c), we prove:

Claim — For each $k \geq 1$, we have
$$f(2k, 2k+1) = \frac{2k-1}{6}.$$

Proof. An illustration for $k = 2$ is shown below, where we use $(0,0)$, $(0,2k)$, $(2k+1,0)$ as the three vertices.



WLOG, the upper-left square is black, as above. The $2k$ small white triangles just below the diagonal have area sum
$$\frac{1}{2} \cdot \frac{1}{2k+1} \cdot \frac{1}{2k}\left[1^2 + 2^2 + \dots + (2k)^2\right] = \frac{4k+1}{12}$$

The area of the $2k$ black polygons sums just below the diagonal to
$$\sum_{i=1}^{2k} \left(1 - \frac{1}{2} \cdot \frac{1}{2k+1} \cdot \frac{1}{2k} \cdot i^2\right) = 2k - \frac{4k+1}{12} = \frac{20k-1}{12}.$$

Finally, in the remaining $1 + 2 + \dots + 2k$ squares, there are k more white squares than black squares. So, it follows
$$f(2k, 2k+1) = \left| -k + \frac{20k-1}{12} - \frac{4k+1}{12} \right| = \frac{2k-1}{6}.$$

□

§1.2 IMO 1997/2

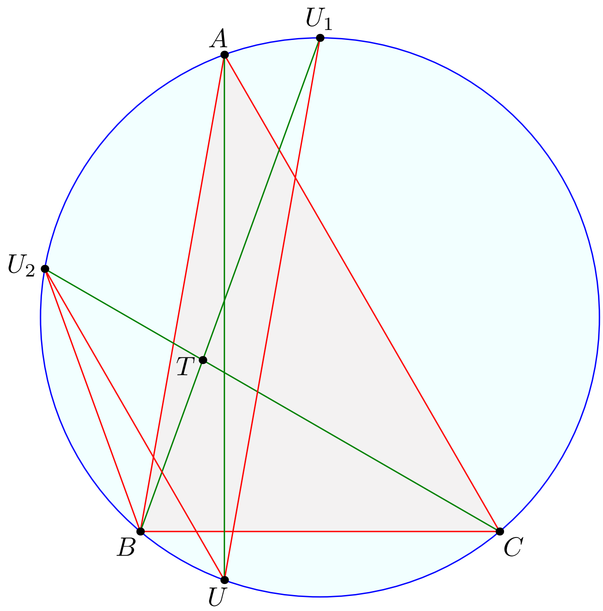
Available online at https://aops.com/community/p356701.

Problem statement

It is known that $\angle BAC$ is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T.

Show that $AU = TB + TC$.

Let $\overline{BTV}$ meet the circle again at U_1, so that AU_1UB is an isosceles trapezoid. Define U_2 similarly.



Now from the isosceles trapezoids we get

$$AU = BU_1 = BT + TU_1 = BT + TC$$

as desired.

§1.3 IMO 1997/3

Available online at https://aops.com/community/p356706.

Problem statement

Let $x_1, x_2, \dots, x_n$ be real numbers satisfying the conditions:

$$|x_1 + x_2 + \dots + x_n| = 1$$
$$|x_i| \le \frac{n+1}{2} \qquad \text{for } i = 1, 2, \dots, n$$

Show that there exists a permutation $y_1, y_2, \dots, y_n$ of $x_1, x_2, \dots, x_n$ such that

$$|y_1 + 2y_2 + \dots + ny_n| \le \frac{n+1}{2}.$$

WLOG $\sum x_i = 1$ (by negating x_i) and $x_1 \le x_2 \le \dots \le x_n$. Notice that

- The largest possible value of the sum in question is
$$A = x_1 + 2x_2 + 3x_3 + \dots + nx_n.$$
while the smallest value is
$$B = nx_1 + (n-1)x_2 + \dots + x_n.$$
- Meanwhile, the *average* value across all permutations is
$$1 \cdot \frac{n+1}{2} + 2 \cdot \frac{n+1}{2} + \dots + n \cdot \frac{n+1}{2} = \frac{n+1}{2}.$$

Now imagine we transform the sum A to the sum B, one step at a time, by swapping adjacent elements. Every time we do a swap of two neighboring $u < v$, the sum decreases by
$$(iu + (i+1)v) - (iv + (i+1)u) = v - u < n + 1.$$

We want to prove we land in the interval
$$I = \left[-\frac{n+1}{2}, \frac{n+1}{2} \right]$$
at some point during this transformation. But since $B \le \frac{n+1}{2} \le A$ (since $\frac{n+1}{2}$ was the average) and our step sizes were at most the length of the interval I, this is clear.

§2 Solutions to Day 2

§2.1 IMO 1997/4

Available online at https://aops.com/community/p611.

Problem statement

An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \dots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \dots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

For (a), define a *cross* to be the union of the ith row and ith column. Every cell of the matrix not on the diagonal is contained in exactly two crosses, while each cell on the diagonal is contained in one cross.

On the other hand, if a silver matrix existed for $n = 1997$, then each element of S is in all 1997 crosses, so it must appear at least once on the diagonal since 1997 is odd. However, $|S| = 3993$ while there are only 1997 diagonal cells. This is a contradiction.

For (b), we construct a silver matrix M_e for $n = 2^e$ for each $e \ge 1$. We write the first three explicitly for concreteness:

$$M_1 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 3 & 1 & 6 & 7 \\ 7 & 5 & 1 & 2 \\ 6 & 4 & 3 & 1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 2 & 4 & 5 & 8 & 9 & 11 & 12 \\ 3 & 1 & 6 & 7 & 10 & 15 & 13 & 14 \\ 7 & 5 & 1 & 2 & 14 & 12 & 8 & 9 \\ 6 & 4 & 3 & 1 & 13 & 11 & 10 & 15 \\ 15 & 9 & 11 & 12 & 1 & 2 & 4 & 5 \\ 10 & 8 & 13 & 14 & 3 & 1 & 6 & 7 \\ 14 & 12 & 15 & 9 & 7 & 5 & 1 & 2 \\ 13 & 11 & 10 & 8 & 6 & 4 & 3 & 1 \end{bmatrix}$$

The construction is described recursively as follows. Let

$$M_e' = \left[\begin{array}{c|c} M_{e-1} & M_{e-1} + (2^e - 1) \\ \hline M_{e-1} + (2^e - 1) & M_{e-1} \end{array}\right].$$

Then to get from M_e' to M_e, replace half of the 2^e's with $2^{e+1} - 1$: in the northeast quadrant, the even-indexed ones, and in the southwest quadrant, the odd-indexed ones.

§2.2 IMO 1997/5

Available online at https://aops.com/community/p3845.

Problem statement

Find all pairs (a, b) of positive integers satisfying

$$a^{b^2} = b^a.$$

The answer is $(1, 1)$, $(16, 2)$ and $(27, 3)$.

We assume $a, b > 1$ for convenience. Let T denote the set of non perfect powers other than 1.

Claim — Every integer greater than 1 is uniquely of the form t^n for some $t \in T$, $n \in \mathbb{N}$.

Proof. Clear. □

Let $a = s^m$, $b = t^n$.

$$s^{m \cdot (t^n)^2} = t^{n \cdot s^m}.$$

Hence $s = t$ and we have

$$m \cdot t^{2n} = n \cdot t^m \implies t^{2n-m} = \frac{n}{m}.$$

Let $n = t^e m$ and $2 \cdot t^e m - m = e$, or

$$e + m = 2t^e \cdot m.$$

We resolve this equation by casework

- If $e > 0$, then $2t^e \cdot m > 2e \cdot m > e + m$.
- If $e = 0$ we have $m = n$ and $m = 2m$, contradiction.
- If $e = -1$ we apparently have
$$\frac{2}{t} \cdot m = m - 1 \implies m = \frac{t}{t-2}$$
so $(t, m) = (3, 3)$ or $(t, m) = (4, 2)$.
- If $e = -2$ we apparently have
$$\frac{2}{t^2} \cdot m = m - 2 \implies m = \frac{2}{1 - 2/t^2} = \frac{2t^2}{t^2 - 2}.$$
This gives $(t, m) = (2, 2)$.
- If $e \le -3$ then let $k = -e \ge 3$, so the equation is
$$m - k = \frac{2m}{t^k} \iff m = \frac{k \cdot t^k}{t^k - 2} = k + \frac{2k}{t^k - 2}.$$
However, for $k \ge 3$ and $t \ge 2$, we always have $2k \le t^k - 2$, with equality only when $(t, k) = (2, 3)$; this means $m = 4$, which is not a new solution.

§2.3 IMO 1997/6

Available online at https://aops.com/community/p356713.

Problem statement

For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways: 4; $2+2$; $2+1+1$; $1+1+1+1$.

Prove that for any integer $n \ge 3$ we have $2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}$.

It's clear that f is non-decreasing. By sorting by the number of 1's we used, we have the equation

$$f(N) = f\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + f\left(\left\lfloor \frac{N}{2} \right\rfloor - 1\right) + f\left(\left\lfloor \frac{N}{2} \right\rfloor - 2\right) + \dots + f(1) + f(0). \qquad (\bigstar)$$

¶ Upper bound. We now prove the upper bound by induction. Indeed, the base case is trivial and for the inductive step we simply use $(\bigstar)$:

$$f(2^n) = f(2^{n-1}) + f(2^{n-1}-1) + \dots < 2^{n-1} f(2^{n-1}) < 2^{n-1} \cdot 2^{\frac{(n-1)^2}{2}} = 2^{\frac{n^2}{2} - \frac{1}{2}}.$$

¶ Lower bound. First, we contend that f is convex. We'll first prove this in the even case to save ourselves some annoyance:

Claim (f is basically convex) — If $2 \mid a+b$ then we have $f(2a) + f(2b) \ge 2f(a+b)$.

Proof. Since $f(2k+1) = f(2k)$, we will only prove the first equation. Assume WLOG $a \ge b$ and use $(\bigstar)$ on all three f expressions here; after subtracting repeated terms, the inequality then rewrites as

$$\sum_{(a+b)/2 \le x \le a} f(x) \ge \sum_{b \le x \le (a+b)/2} f(x).$$

This is true since there are an equal number of terms on each side and f is nondecreasing. $\square$

Claim — For each $1 \le k < 2^{n-1}$, we have

$$f(2^{n-1} - k) + f(k+1) \ge 2f(2^{n-2})$$

Proof. Use the fact that $f(2t+1) = f(2t)$ for all t and then apply convexity as above. $\square$

Now we can carry out the induction:

$$f(2^n) = f(2^{n-1}) + f(2^{n-1}-1) + \dots > 2^{n-1} f(2^{n-2}) + f(0) > 2^{n-1} 2^{\frac{(n-2)^2}{4}} = 2^{\frac{n^2}{4}}.$$

IMO 1998 Solution Notes

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2 June 2023

This is a compilation of solutions for the 1998 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

2. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges, their ratings coincide for at most k contestants. Prove that
$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

3. For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that
$$\frac{\tau(n^2)}{\tau(n)} = m.$$

4. Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

5. Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

6. Classify all functions $f \colon \mathbb{N} \to \mathbb{N}$ satisfying the identity
$$f(n^2 f(m)) = m f(n)^2.$$

§1 Solutions to Day 1

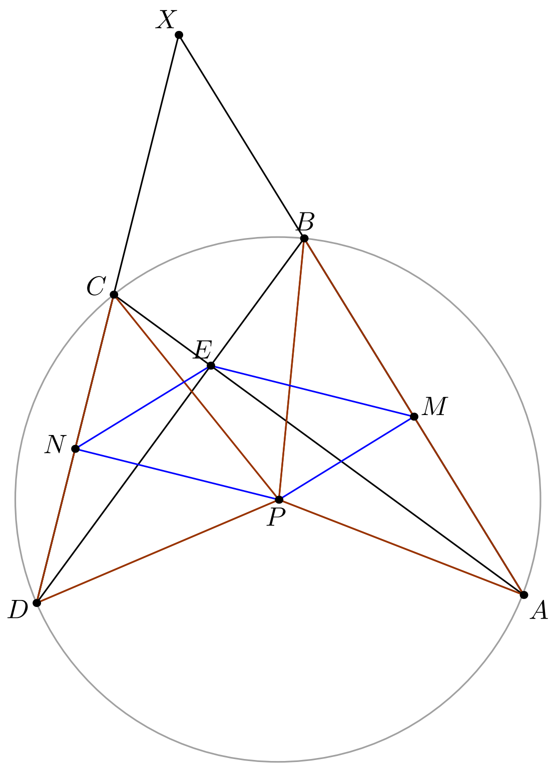
§1.1 IMO 1998/1

Available online at https://aops.com/community/p124387.

Problem statement

A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

If $ABCD$ is cyclic, then P is the circumcenter, and $\angle APB + \angle PCD = 180^\circ$. The hard part is the converse.



Let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$.

Claim — Unconditionally, we have $\measuredangle NEM = \measuredangle MPN$.

Proof. Note that $\overline{EN}$ is the median of right triangle $\triangle ECD$, and similarly for $\overline{EM}$. Hence $\measuredangle NED = \measuredangle EDN = \measuredangle BDC$, while $\measuredangle AEM = \measuredangle ACB$. Since $\measuredangle DEA = 90^\circ$, by looking at quadrilateral $XDEA$ where $X = \overline{CD} \cap \overline{AB}$, we derive that $\measuredangle NED + \measuredangle AEM + \measuredangle DXA = 90^\circ$, so

$$\measuredangle NEM = \measuredangle NED + \measuredangle AEM + 90^\circ = -\measuredangle DXA = -\measuredangle NXM = -\measuredangle NPM$$

as needed. $\square$

However, the area condition in the problem tells us

$$\frac{EN}{EM} = \frac{CN}{CM} = \frac{PM}{PN}.$$

Finally, we have $\angle MEN > 90^\circ$ from the configuration. These properties uniquely determine the point E: it is the reflection of P across line MN.

So $EMPN$ is a parallelogram, and thus $\overline{ME} \perp \overline{CD}$. This implies $\measuredangle BAE = \measuredangle CEM = \measuredangle EDC$ giving $ABCD$ cyclic.

§1.2 IMO 1998/2

Available online at https://aops.com/community/p124458.

Problem statement

In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

This is a "routine" problem with global ideas. We count pairs of coinciding ratings, i.e. the number N of tuples

$$(\{J_1, J_2\}, C)$$

of two distinct judges and a contestant for which the judges gave the same rating.

On the one hand, if we count by the judges, we have

$$N \leq \binom{b}{2} k$$

by he problem condition.

On the other hand, if $b = 2m+1$, then each contestant C contributes at least $\binom{m}{2} + \binom{m+1}{2} = m^2$ to N, and so

$$N \geq a \cdot \left(\frac{b-1}{2}\right)^2$$

Putting together the two estimates for N yields the conclusion.

§1.3 IMO 1998/3

Available online at https://aops.com/community/p124439.

Problem statement

For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that

$$\frac{\tau(n^2)}{\tau(n)} = m.$$

The answer is odd integers m only. If we write $n = p_1^{e_1} \dots p_k^{e_k}$ we get

$$\prod \frac{2e_i + 1}{e_i + 1} = m.$$

It's clear now that m must be odd, since every fraction has odd numerator.

We now endeavor to construct odd numbers. The proof is by induction, in which we are curating sets of fractions of the form $\frac{2e+1}{e+1}$ that multiply to a given target.

The base cases are easy to verify by hand. Generally, assume $p = 2^t k - 1$ is odd, where k is odd. Then we can write

$$\frac{2^{2t}k - 2^t(k+1) + 1}{2^{2t-1}k - 2^{t-1}(k+1) + 1} \cdot \frac{2^{2t-1}k - 2^{t-1}(k+1) + 1}{2^{2t-2}k - 2^{t-2}(k+1) + 1} \cdot \dots \cdot \frac{2^{t+1}k - 2(k+1) + 1}{2^t k - 2^0(k+1) + 1}.$$

Note that $2^{2t}k - 2^t(k+1) + 1 = (2^t k - 1)(2^t - 1)$, and $2^t k - k = k(2^t - 1)$, so the above fraction simplifies to

$$\frac{2^t k - 1}{k}$$

meaning we just need to multiply by k, which we can do using induction hypothesis.

§2 Solutions to Day 2

§2.1 IMO 1998/4

Available online at https://aops.com/community/p124428.

Problem statement

Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

The answer is $(7k^2, 7k)$ for all $k \geq 1$, as well as $(11, 1)$ and $(49, 1)$.

We are given $xy^2 + y + 7 \mid x^2y + x + y$. Multiplying the right-hand side by y gives

$$xy^2 + y + 7 \mid x^2y^2 + xy + y^2$$

Then subtracting x times the left-hand side gives

$$xy^2 + y + 7 \mid y^2 - 7x.$$

We consider cases based on the sign of $y^2 = 7x$.

- If $y^2 > 7x$, then $0 < y^2 - 7x < xy^2 + y + 7$, contradiction.
- If $y^2 = 7x$, let $y = 7k$, so $x = 7k^2$. Plugging this back in to the original equation reads
$$343k^4 + 7k + 7 \mid 343k^5 + 7k^2 + 7k$$
which is always valid, hence these are all solutions.
- If $y^2 < 7x$, then $|y^2 - 7x| \leq 7x$, so $y \in \{1, 2\}$.

  When $y = 1$ we get
$$x + 8 \mid x^2 + x + 1 \iff x + 8 \mid 64 - 8 + 1 = 57.$$
  This has solutions $x = 11$ and $x = 49$.

  When $y = 2$
$$\begin{aligned} 4x + 9 &\mid 2x^2 + x + 2 \\ \implies 4x + 9 &\mid 16x^2 + 8x + 16 \\ \implies 4x + 9 &\mid 81 - 18 + 16 = 79 \end{aligned}$$
  which never occurs.

§2.2 IMO 1998/5

Available online at https://aops.com/community/p121417.

Problem statement

Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

Observe that $\triangle MKL$ is acute with circumcenter I. We now present two proofs.

¶ First simple proof (grobber) The problem is equivalent to showing $BI^2 > BR \cdot BS$. But from

$$\triangle BRK \sim \triangle MKL \sim \triangle BLS$$

we conclude

$$BR = t \cdot \frac{MK}{ML}, \qquad BS = t \cdot \frac{ML}{MK}$$

where $t = BK = BL$ is the length of the tangent from B. Hence $BR \cdot BS = t^2$. Since $BI > t$ is clear, we are done.

¶ Second projective proof Let N be the midpoint of $\overline{KL}$, and let ray MN meet the incircle again at P.

Note that line $\overline{RBS}$ is the polar of N. By Brokard's theorem, lines MK and PL should thus meet the polar of N, so we conclude $R = \overline{MK} \cap \overline{PL}$. Analogously, $S = \overline{ML} \cap \overline{PK}$.

Again by Brokard's theorem, $\triangle NRS$ is self-polar, so N is the orthocenter of $\triangle RIS$. Since N lies between I and B we are done.

§2.3 IMO 1998/6

Available online at https://aops.com/community/p124426.

Problem statement

Classify all functions $f\colon \mathbb{N} \to \mathbb{N}$ satisfying the identity

$$f(n^2 f(m)) = m f(n)^2.$$

Let $\mathcal{P}$ be the set of primes, and let $g\colon \mathcal{P} \to \mathcal{P}$ be any involution on them. Extend g to a completely multiplicative function on $\mathbb{N}$. Then $f(n) = dg(n)$ is a solution for any $d \in \mathbb{N}$ which is fixed by g.

It's straightforward to check these all work, since $g\colon \mathbb{N} \to \mathbb{N}$ is an involution on them. So we prove these are the only functions.

Let $d = f(1)$.

Claim — We have $df(n) = f(dn)$ and $d \cdot f(ab) = f(a)f(b)$.

Proof. Let $P(m, n)$ denote the assertion in the problem statement. Off the bat,

- $P(1, 1)$ implies $f(d) = d^2$.
- $P(n, 1)$ implies $f(f(n)) = d^2 n$. In particular, f is injective.
- $P(1, n)$ implies $f(dn^2) = f(n)^2$.

Then

$$\begin{aligned} f(a)^2 f(b)^2 &= f(da^2) f(b)^2 && \text{by third bullet} \\ &= f(b^2 f(f(da^2))) && \text{by problem statement} \\ &= f(b^2 \cdot d^2 \cdot da^2) && \text{by second bullet} \\ &= f(dab)^2 && \text{by third bullet} \\ \implies f(a)f(b) &= f(dab). \end{aligned}$$

This implies the first claim by taking $(a, b) = (1, n)$. Then $df(a) = f(da)$, and so we actually have $f(a)f(b) = df(ab)$. □

Claim — All values of f are divisible by d.

Proof. We have

$$\begin{aligned} f(n^2) &= \frac{1}{d} f(n)^2 \\ f(n^3) &= \frac{f(n^2)f(n)}{d} = \frac{f(n)^3}{d^2} \\ f(n^4) &= \frac{f(n^3)f(n)}{d} = \frac{f(n)^4}{d^3} \end{aligned}$$

and so on, which implies the result. □

Then, define $g(n) = f(n)/d$. We conclude that g is completely multiplicative, with $g(1) = 1$. However, $f(f(n)) = d^2 n$ also implies $g(g(n)) = n$, i.e. g is an involution. Moreover, since $f(d) = d^2$, $g(d) = d$.

All that remains is to check that g must map primes to primes to finish the description in the problem. This is immediate; since g is multiplicative and $g(1) = 1$, if $g(g(p)) = p$ then $g(p)$ can have at most one prime factor, hence $g(p)$ is itself prime.

> **Remark.** The IMO problem actually asked for the least value of $f(1998)$. But for instruction purposes, it is probably better to just find all f. Since $1998 = 2 \cdot 3^3 \cdot 37$, this answer is $2^3 \cdot 3 \cdot 5 = 120$, anyways.

IMO 1999 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 1999 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. A set S of points from the space will be called completely symmetric if it has at least three elements and fulfills the condition that for every two distinct points A and B from S, the perpendicular bisector plane of the segment AB is a plane of symmetry for S. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

2. Find the least constant C such that for any integer $n > 1$ the inequality

$$\sum_{1 \le i < j \le n} x_i x_j (x_i^2 + x_j^2) \le C \left(\sum_{1 \le i \le n} x_i \right)^4$$

holds for all real numbers $x_1, \dots, x_n \ge 0$. Determine the cases of equality.

3. Let n be an even positive integer. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

4. Find all pairs of positive integers (x, p) such that p is a prime and x^{p-1} is a divisor of $(p-1)^x + 1$.

5. Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B Lines MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

6. Find all the functions $f \colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

§1 Solutions to Day 1

§1.1 IMO 1999/1

Available online at https://aops.com/community/p131833.

Problem statement

A set S of points from the space will be called completely symmetric if it has at least three elements and fulfills the condition that for every two distinct points A and B from S, the perpendicular bisector plane of the segment AB is a plane of symmetry for S. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

Let G be the centroid of S.

Claim — All points of S lie on a sphere Γ centered at G.

Proof. Each perpendicular bisector plane passes through G. So if $A, B \in S$ it follows $GA = GB$. $\square$

Claim — Consider any plane passing through three or more points of S. The points of S in the plane form a regular polygon.

Proof. The cross section is a circle because we are intersecting a plane with sphere Γ. Now if A, B, C are three adjacent points on this circle, by taking the perpendicular bisector we have $AB = BC$. $\square$

If the points of S all lie in a plane, we are done. Otherwise, the points of S determine a polyhedron Π inscribed in Γ. All of the faces of Π are evidently regular polygons, of the same side length s.

Claim — Every face of Π is an equilateral triangle.

Proof. Suppose on the contrary some face $A_1A_2 \dots A_n$ has $n > 3$. Let B be any vertex adjacent to A_1 in Π other than A_2 or A_n. Consider the plane determined by $\triangle A_1A_3B$. This is supposed to be a regular polygon, but arc A_1A_3 is longer than arc A_1B, and by construction there are no points inside these arcs. This is a contradiction. $\square$

Hence, Π has faces all congruent equilateral triangles. This implies it is a regular polyhedron — either a regular tetrahedron, regular octahedron, or regular icosahedron. We can check the regular icosahedron fails by taking two antipodal points as our counterexample. This finishes the problem.

§1.2 IMO 1999/2

Available online at https://aops.com/community/p131846.

Problem statement

Find the least constant C such that for any integer $n > 1$ the inequality

$$\sum_{1 \le i < j \le n} x_i x_j (x_i^2 + x_j^2) \le C \left(\sum_{1 \le i \le n} x_i \right)^4$$

holds for all real numbers $x_1, \dots, x_n \ge 0$. Determine the cases of equality.

Answer: $C = \frac{1}{8}$, with equality when two x_i are equal and the remaining x_i are equal to zero.

We present two proofs of the bound.

¶ First solution by smoothing Fix $\sum x_i = 1$. The sum on the left-hand side can be interpreted as $\sum_{i=1}^n x_i^3 \sum_{j \neq i} x_j = \sum_{i=1}^n x_i^3 (1 - x_i)$, so we may rewrite the inequality as: Then it becomes

$$\sum_i (x_i^3 - x_i^4) \le C.$$

Claim (Smoothing) — Let $f(x) = x^3 - x^4$. If $u + v \le \frac{3}{4}$, then $f(u) + f(v) \le f(0) + f(u+v)$.

Proof. Note that

$$\begin{aligned} &(u^3 - u^4) + (v^3 - v^4) \le (u+v)^3 - (u+v)^4 \\ \iff &uv(4u^2 + 4v^2 + 6uv) \le 3uv(u+v) \end{aligned}$$

If $u + v \le \frac{3}{4}$ this is obvious as $4u^2 + 4v^2 + 6uv \le 4(u+v)^2$. □

Observe that if three nonnegative reals have pairwise sums exceeding $\frac{3}{4}$ then they have sum at least $\frac{9}{8}$. Hence we can smooth until $n - 2$ of the terms are zero. Hence it follows

$$C = \max_{a+b=1} (a^3 + b^3 - a^4 - b^4)$$

which is routine computation giving $C = \frac{1}{8}$.

¶ Second solution by AM-GM (Nairit Sarkar) Write

$$\begin{aligned} \text{LHS} &\le \left(\sum_{1 \le k \le n} x_k^2 \right) \left(\sum_{1 \le i < j \le n} x_i x_j \right) = \frac{1}{2} \left(\sum_{1 \le k \le n} x_k^2 \right) \left(\sum_{1 \le i < j \le n} 2 x_i x_j \right) \\ &\le \frac{1}{2} \left(\frac{\sum_k x_k^2 + 2 \sum_{i<j} x_i x_j}{2} \right)^2 = \frac{1}{8} \left(\sum_{1 \le i < n} x_i \right)^4 \end{aligned}$$

as desired.

§1.3 IMO 1999/3

Available online at https://aops.com/community/p131873.

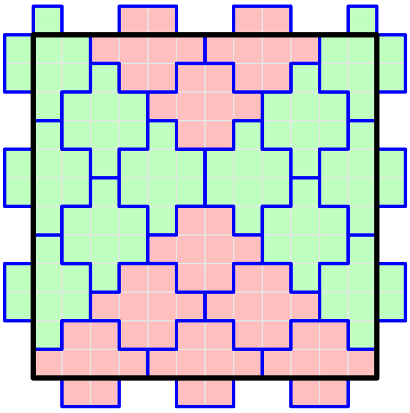
Problem statement

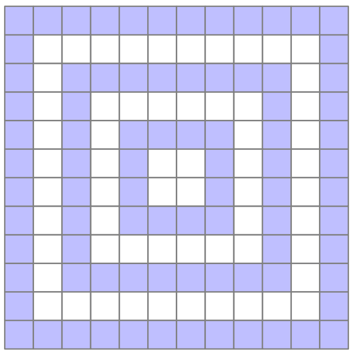
Let n be an even positive integer. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

For every marked cell, consider the marked cell adjacent to it; in this way we have a *domino* of two cells. For each domino, its *aura* consists of all the cells which are adjacent to a cell of the domino. There are up to eight squares in each aura, but some auras could be cut off by the boundary of the board, which means that there could be as few as five squares.

We will prove that $\frac{1}{2}n(n+2)$ is the minimum number of auras needed to cover the board (the auras need not be disjoint).

- A construction is shown on the left below, showing that $\frac{1}{2}n(n+2)$.
- Color the board as shown to the right into “rings”. Every aura takes covers exactly (!) four blue cells. Since there are $2n(n+2)$ blue cells, this implies the lower bound.





Note that this proves that a partition into disjoint auras actually always has exactly $\frac{1}{2}n(n+2)$ auras, thus also implying EGMO 2019/2.

§2 Solutions to Day 2

§2.1 IMO 1999/4

Available online at https://aops.com/community/p131811.

Problem statement

Find all pairs of positive integers (x, p) such that p is a prime and x^{p-1} is a divisor of $(p-1)^x + 1$.

If $p = 2$ then $x \in \{1, 2\}$, and if $p = 3$ then $x \in \{1, 3\}$, since this is IMO 1990/3. Also, $x = 1$ gives a solution for any prime p. We show that there are no other solutions.

Assume $x > 1$ and let q be smallest prime divisor of x. We have $q > 2$ since $(p-1)^x + 1$ is odd. Then

$$(p-1)^x \equiv -1 \pmod q \implies (p-1)^{2x} \equiv 1 \pmod q$$

so the order of $p - 1 \bmod q$ is even and divides $\gcd(q - 1, 2x) \le 2$. This means that

$$p - 1 \equiv -1 \pmod q \implies p = q.$$

In other words $p \mid x$ and we get $x^{p-1} \mid (p-1)^x + 1$. By exponent lifting lemma, we now have

$$0 < (p-1)\nu_p(x) \le 1 + \nu_p(x).$$

This forces $p = 3$, which we already addressed.

§2.2 IMO 1999/5

Available online at https://aops.com/community/p131838.

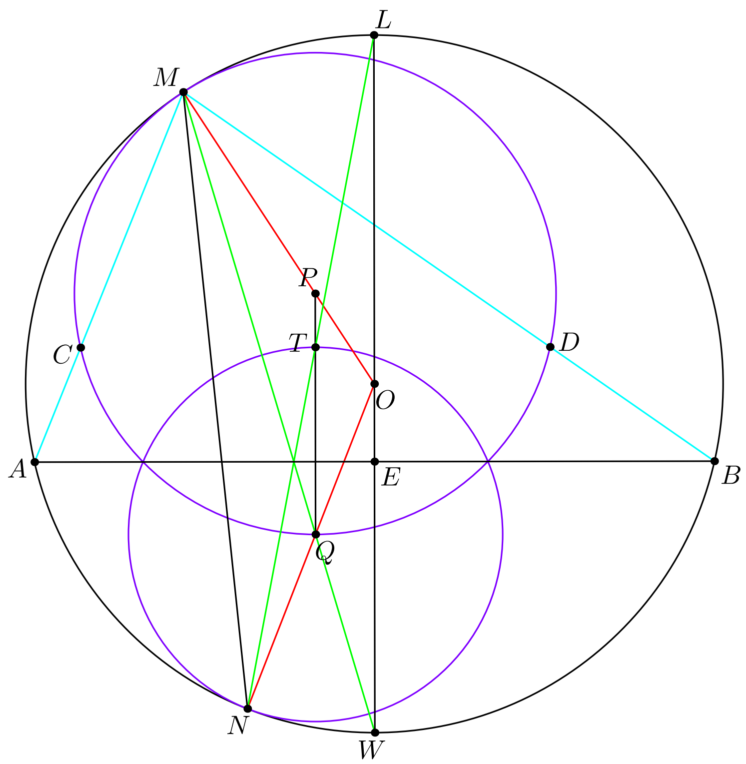
Problem statement

Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B Lines MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

Let P and Q be the centers of Ω_1 and Ω_2.

Let line MQ meet Ω_1 again at W, the homothetic image of Q under $\Omega_1 \to \Omega$.

Meanwhile, let T be the intersection of segment PQ with Ω_2, and let L be its homothetic image on Ω. Since $\overline{PTQ} \perp \overline{AB}$, it follows $\overline{LW}$ is a diameter of Ω. Let O be its center.



Claim — $MNTQ$ is cyclic.

Proof. By Reim: $\measuredangle TQM = \measuredangle LWM = \measuredangle LNM = \measuredangle TNM$. $\square$

Let E be the midpoint of $\overline{AB}$.

Claim — $OEMN$ is cyclic.

Proof. By radical axis, the lines MM, NN, AEB meet at a point R. Then $OEMN$ is on the circle with diameter $\overline{OR}$. $\square$

Claim — MTE are collinear.

Proof. $\measuredangle NMT = \measuredangle TQN = \measuredangle LON = \measuredangle NOE = \measuredangle NME$. $\square$

Now consider the homothety mapping $\triangle WAB$ to $\triangle QCD$. It should map E to a point on line ME which is also on the line through Q perpendicular to $\overline{AB}$; that is, to point T. Hence TCD are collinear, and it's immediate that T is the desired tangency point.

§2.3 IMO 1999/6

Available online at https://aops.com/community/p131856.

Problem statement

Find all the functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

The answer is $f(x) = -\frac{1}{2}x^2 + 1$ which obviously works.

For the other direction, first note that

$$P(f(y), y) \implies 2f(f(y)) + f(y)^2 - 1 = f(0).$$

We introduce the notation $c = \frac{f(0)-1}{2}$, and $S = \operatorname{img} f$. Then the above assertion says

$$f(s) = -\frac{1}{2}s^2 + (c+1).$$

Thus, the given functional equation can be rewritten as

$$Q(x,s) : f(x-s) = -\frac{1}{2}s^2 + sx + f(x) - c.$$

Claim (Main claim) — We can find a function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x-z) = zx + f(x) + g(z). \qquad (\spadesuit).$$

Proof. If $z \neq 0$, the idea is to fix a nonzero value $s_0 \in S$ (it exists) and then choose x_0 such that $-\frac{1}{2}s_0^2 + s_0x_0 - c = z$. Then, $Q(x_0, s)$ gives an pair (u, v) with $u - v = z$.

But now for any x, using $Q(x+v, u)$ and $Q(x, -v)$ gives

$$\begin{aligned} f(x-z) - f(x) = f(x-u+v) - f(x) &= f(x+v) - f(x) + u(x+v) - \frac{1}{2}u^2 + c \\ &= -vx - \frac{1}{2}s^2 - c + u(x+v) - \frac{1}{2}u^2 + c \\ &= -vx - \frac{1}{2}v^2 + u(x+v) - \frac{1}{2}u^2 = zx + g(z) \end{aligned}$$

where $g(z) = -\frac{1}{2}(u^2 + v^2)$ depends only on z. $\square$

Now, let

$$h(x) := \frac{1}{2}x^2 + f(x) - (2c+1),$$

so $h(0) = 0$.

Claim — The function h is additive.

Proof. We just need to rewrite $(\spadesuit)$. Letting $x = z$ in $(\spadesuit)$, we find that actually $g(x) = f(0) - x^2 - f(x)$. Using the definition of h now gives

$$h(x-z) = h(x) + h(z). \qquad \square$$

To finish, we need to remember that f, hence h, is known on the image

$$S = \{f(x) \mid x \in \mathbb{R}\} = \left\{h(x) - \frac{1}{2}x^2 + (2c+1) \mid x \in \mathbb{R}\right\}.$$

Thus, we derive

$$h\left(h(x) - \frac{1}{2}x^2 + (2c+1)\right) = -c \qquad \forall x \in \mathbb{R}. \qquad (\heartsuit)$$

We can take the following two instances of $\heartsuit$:

$$\begin{aligned} h\left(h(2x) - 2x^2 + (2c+1)\right) &= -c \\ h\left(2h(x) - x^2 + 2(2c+1)\right) &= -2c. \end{aligned}$$

Now subtracting these and using $2h(x) = h(2x)$ gives

$$c = h\left(-x^2 - (2c+1)\right).$$

Together with h additive, this implies readily h is constant. That means $c = 0$ and the problem is solved.

IMO 2000 Solution Notes

EVAN CHEN《陳誼廷》

29 June 2023

This is a compilation of solutions for the 2000 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

2. Let a, b, c be positive real numbers with $abc = 1$. Show that
$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

3. Let $n \ge 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\frac{BC}{AB} = \lambda$.

   Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

   How many ways are there to put the cards in the three boxes so that the trick works?

5. Does there exist a positive integer n such that n has exactly 2000 distinct prime divisors and n divides $2^n + 1$?

6. Let $\overline{AH_1}$, $\overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute triangle ABC. The incircle ω of triangle ABC touches the sides BC, CA and AB at T_1, T_2 and T_3, respectively. Consider the reflections of the lines H_1H_2, H_2H_3, and H_3H_1 with respect to the lines T_1T_2, T_2T_3, and T_3T_1. Prove that these images form a triangle whose vertices lie on ω.

§1 Solutions to Day 1

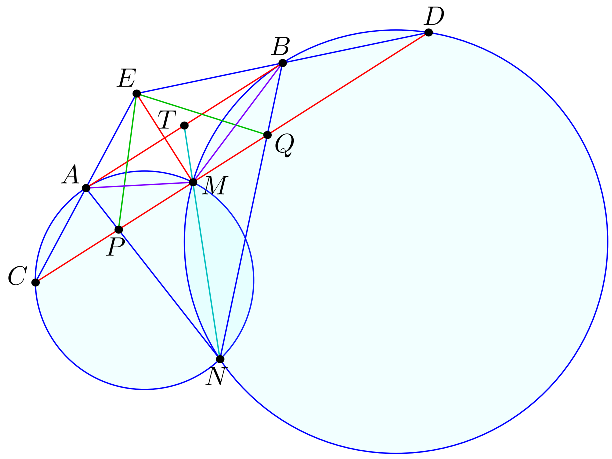
§1.1 IMO 2000/1

Available online at https://aops.com/community/p354110.

Problem statement

Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

First, we have $\measuredangle EAB = \measuredangle ACM = \measuredangle BAM$ and similarly $\measuredangle EBA = \measuredangle BDM = \measuredangle ABM$. Consequently, $\overline{AB}$ bisects $\angle EAM$ and $\angle EBM$, and hence $\triangle EAB \cong \triangle MAB$.



Now it is well-known that $\overline{MN}$ bisects $\overline{AB}$ and since $\overline{AB} \parallel \overline{PQ}$ we deduce that M is the midpoint of $\overline{PQ}$. As $\overline{AB}$ is the perpendicular bisector of $\overline{EM}$, it follows that $EP = EQ$ as well.

§1.2 IMO 2000/2

Available online at https://aops.com/community/p354109.

Problem statement

Let a, b, c be positive real numbers with $abc = 1$. Show that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

Let $a = x/y$, $b = y/z$, $c = z/x$ for $x, y, z > 0$. Then the inequality rewrites as

$$(-x + y + z)(x - y + z)(x + y - z) \le xyz$$

which when expanded is equivalent to Schur's inequality. Alternatively, if one wants to avoid appealing to Schur, then the following argument works:

- At most one term on the left-hand side is negative; if that occurs we are done from $xyz > 0 > (-x + y + z)(x - y + z)(x + y - z)$.
- If all terms in the left-hand side are nonnegative, let us denote $m = -x + y + z \ge 0$, $n = x - y + z \ge 0$, $p = x + y - z \ge 0$. Then it becomes

  $$mnp \le \frac{(m + n)(n + p)(p + m)}{8}$$

  which follows by AM-GM.

§1.3 IMO 2000/3

Available online at https://aops.com/community/p354112.

Problem statement

Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\frac{BC}{AB} = \lambda$.

Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

The answer is $\lambda \geq \frac{1}{n-1}$.

We change the problem by replacing the fleas with **bowling balls** B_1, B_2, ..., B_n in that order. Bowling balls aren't exactly great at jumping, so each move can now be described as follows:

- Select two indices $i < j$. Then ball B_i moves to B_{i+1}'s location, B_{i+1} moves to B_{i+2}'s location, and so on; until B_{j-1} moves to B_j's location,
- Finally, B_j moves some distance forward; the distance is at most $\lambda \cdot |B_j B_i|$ and B_j may not pass B_{j+1}.

Claim — If $\lambda < \frac{1}{n-1}$ the bowling balls have bounded movement.

Proof. Let $a_i \geq 0$ denote the initial distance between B_i and B_{i+1}, and let Δ_i denote the distance travelled by ball i. Of course we have $\Delta_1 \leq a_1 + \Delta_2$, $\Delta_2 \leq a_2 + \Delta_3$, ..., $\Delta_{n-1} \leq a_{n-1} + \Delta_n$ by the relative ordering of the bowling balls. Finally, distance covered by B_n is always λ times distance travelled by other bowling balls, so

$$\begin{aligned} \Delta_n &\leq \lambda \sum_{i=1}^{n-1} \Delta_i \leq \lambda \sum_{i=1}^{n-1} \left((a_i + a_{i+1} + \cdots + a_{n-1}) + \Delta_n \right) \\ &= (n-1)\lambda \cdot \Delta_n + \sum_{i=1}^{n-1} i a_i \end{aligned}$$

and since $(n-1)\lambda > 1$, this gives an upper bound. $\square$

Remark. Equivalently, you can phrase the proof without bowling balls as follows: if $x_1 < \cdots < x_n$ are the positions of the fleas, the quantity

$$L = x_n - \lambda(x_1 + \cdots + x_{n-1})$$

is a monovariant which never increases; i.e. L is bounded above. Since $L > (1 - (n-1)\lambda)x_n$, it follows $\lambda < \frac{1}{n-1}$ is enough to stop the fleas.

Claim — When $\lambda \ge \frac{1}{n-1}$, it suffices to always jump the leftmost flea over the rightmost flea.

Proof. If we let x_i denote the distance travelled by B_1 in the ith step, then $x_i = a_i$ for $1 \le i \le n-1$ and $x_i = \lambda(x_{i-1} + x_{i-2} + \dots + x_{i-(n-1)})$.

In particular, if $\lambda \ge \frac{1}{n-1}$ then each x_i is at least the average of the previous $n-1$ terms. So if the a_i are not all zero, then $\{x_n, \dots, x_{2n-2}\}$ are all positive and thereafter $x_i \ge \min\{x_n, \dots, x_{2n-2}\} > 0$ for every $i \ge 2n-1$. So the partial sums of x_i are unbounded, as desired. $\square$

Remark. Other inductive constructions are possible. Here is the idea of one of them, although the details are more complicated.

We claim in general that given $n-1$ fleas at 0 and one flea at 1, we can get all the fleas arbitrarily close to $\frac{1}{1-(n-1)\lambda}$ (or as far as we want if $\lambda > \frac{1}{n-1}$.). The proof is induction by $n \ge 2$; for $n = 2$ we get a geometric series. For $n \ge 3$, we leave one flea at zero and move the remainder close to $\frac{1}{1-(n-2)\lambda}$, then jump the last flea to $\frac{1+\lambda}{1-(n-2)\lambda}$.

Now we're in the same situation, except we shifted $\frac{1}{1-(n-2)\lambda}$ right and have then scaled everything by $r = \frac{\lambda}{1-(n-2)\lambda}$. If we repeat this process again and check the geometric series, we see the fleas converge to

$$\frac{1}{1-(n-2)\lambda}\left(1 + r + r^2 + r^3 + \dots\right) = \frac{1}{1-(n-2)\lambda} \cdot \frac{1}{1-r} = \frac{1}{1-(n-1)\lambda}.$$

§2 Solutions to Day 2

§2.1 IMO 2000/4

Available online at https://aops.com/community/p354114.

Problem statement

A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

How many ways are there to put the cards in the three boxes so that the trick works?

There are $2 \cdot 3! = 12$ ways, which amount to:

- Partitioning the cards modulo 3, or
- Placing 1 alone in a box, 100 alone in a second box, and all remaining cards in the third box.

These are easily checked to work so we prove they are the only ones.

¶ **First solution** We proceed by induction on $n \geq 3$ with the base case being immediate.

For the inductive step, consider a working partition of $\{1, 2, \dots, n\}$. Then either n is in its own box; or deleting n gives a working partition of $\{1, 2, \dots, n-1\}$. Similarly, either 1 is in its own box; or deleting 1 gives a working partition of $\{2, 3, \dots, n\}$, and we can reduce all numbers by 1 to get a working partition of $\{1, 2, \dots, n-1\}$.

Therefore, we only need to consider there cases.

- If 1 and n are both in their own box, this yields one type of solution we already found.
- If n is not in a box by itself, then by induction hypothesis the cards 1 through $n-1$ are either arranged mod 3, or as $\{1\} \cup \{2, 3, \dots, n-2\} \cup \{n-1\}$.
  - In the former mod 3 situation, since $n + (n-3) = (n-1) + (n-2)$, so n must be in the same box as $n-3$.
  - In the latter case and for $n > 4$, since $n + 1 = 2 + (n-1)$, n must be in the same box as 1. But now $n + 2 = (n-1) + 3$ for $n > 4$, contradiction.
- The case where 1 is in a box by itself is analogous.

This exhausts all cases, completing the proof.

¶ **Second solution** Let A, B, C be the sets of cards in the three boxes. Then $A + B$, $B + C$, $C + A$ should be disjoint, and contained in $\{3, 4, \dots, 199\}$. On the other hand, we have the following famous fact.

Lemma

Let X and Y be finite nonempty sets of real numbers. We have $|X+Y| \ge |X|+|Y|-1$, with equality if and only if X and Y are arithmetic progressions with the same common difference, or one of X and Y is a singleton set.

Putting these two together gives the estimates

$$197 \ge |A+B| + |B+C| + |C+A| \ge 2\left(|A|+|B|+|C|\right) - 3 = 197.$$

So all the inequalities must be sharp. Consequently we conclude that:

Claim — Either the sets A, B, C are disjoint arithmetic progressions with the same common difference $d = \min_{x \neq y \text{ in same set}} |x-y|$, or two of the sets are two singleton. Moreover, $\{3,4,\dots,199\} = (A+B) \sqcup (B+C) \sqcup (C+A)$.

From here it is not hard to deduce the layouts above are the only ones, but there are some details. First, we make the preliminary observation that $3 = 1+2$, $4 = 1+3$, $198 = 98+100$, $199 = 99+100$ and these numbers can't be decomposed in other ways; thus from the remark about the disjoint union:

Claim (Convenient corollary) — The pairs $(1,2)$, $(1,3)$, $(98,100)$, $(99,100)$ are all in different sets.

We now consider the four cases.

- If two of the boxes are singletons, it follows from the corollary that we should have $A = \{1\}$, $B = \{100\}$ and $C = \{2,\dots,99\}$, up to permutation.
- Otherwise A, B, C are disjoint arithmetic progressions with the same common difference d. As two of $\{1,2,3,4\}$ are in the same box (by pigeonhole), we must have $d \le 3$.
  - If $d = 3$, then no two elements of different residues modulo 3 can be in the same box, so we must be in the first construction claimed earlier.
  - If $d = 2$, then the convenient corollary tells us we may assume WLOG that $1 \in A$ and $2 \in B$, hence $3 \in C$ (since $3 \notin A$ by convenient corollary, and $3 \notin B$ because common difference 2). Thus we must have $A = \{1\}$, $B = \{2,4,\dots,100\}$ and $C = \{3,5,\dots 99\}$ which does not work since $1+4 = 2+3$. Therefore there are no solutions in this case.
  - If $d = 1$, then by convenient corollary the numbers 1 and 2 are in different sets, as are 99 and 100. So we must have $A = \{1\}$, $B = \{2,\dots,99\}$, $C = \{100\}$ which we have already seen is valid.

§2.2 IMO 2000/5

Available online at https://aops.com/community/p354115.

Problem statement

Does there exist a positive integer n such that n has exactly 2000 distinct prime divisors and n divides $2^n + 1$?

Answer: Yes.

We say that n is *Korean* if $n \mid 2^n + 1$. First, observe that $n = 9$ is Korean. Now, the problem is solved upon the following claim:

Claim — If $n > 3$ is Korean, there exists a prime p not dividing n such that np is Korean too.

Proof. I claim that one can take any primitive prime divisor p of $2^{2n} - 1$, which exists by Zsigmondy theorem. Obviously $p \neq 2$. Then:

- Since $p \nmid 2^{\varphi(n)} - 1$ it follows then that $p \nmid n$.
- Moreover, $p \mid 2^n + 1$ since $p \nmid 2^n - 1$.

Hence $np \mid 2^n + 1 \mid 2^{np} + 1$ by Chinese Theorem, since $\gcd(n, p) = 1$. $\square$

§2.3 IMO 2000/6

Available online at https://aops.com/community/p351094.

Problem statement

Let $\overline{AH_1}$, $\overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute triangle ABC. The incircle ω of triangle ABC touches the sides BC, CA and AB at T_1, T_2 and T_3, respectively. Consider the reflections of the lines H_1H_2, H_2H_3, and H_3H_1 with respect to the lines T_1T_2, T_2T_3, and T_3T_1. Prove that these images form a triangle whose vertices lie on ω.

We use complex numbers with ω the unit circle. Let $T_1 = a$, $T_2 = b$, $T_3 = c$. The main content of the problem is to show that the triangle in question has vertices ab/c, bc/a, ca/b (which is evident from a good diagram).

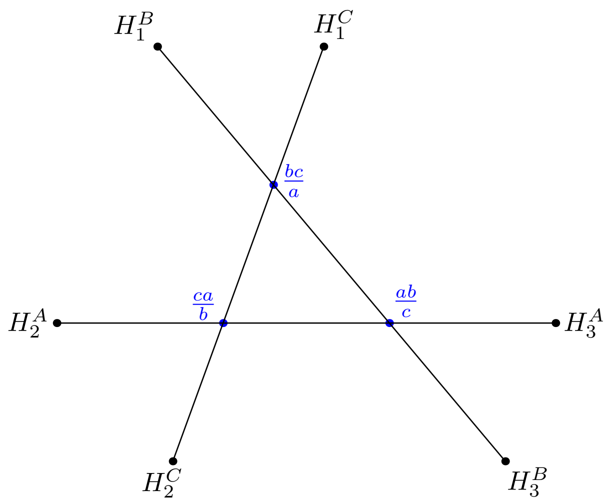
Since $A = \frac{2bc}{b+c}$, we have

$$H_1 = \frac{1}{2}\left(\frac{2bc}{b+c} + a + a - a^2 \cdot \frac{2}{b+c}\right) = \frac{ab+bc+ca-a^2}{b+c}.$$

The reflection of H_1 over $\overline{T_1T_2}$ is

$$\begin{aligned} H_1^C &= a + b - ab\overline{H_1} = a + b - b \cdot \frac{ac+ab+a^2-bc}{a(b+c)} \\ &= \frac{a(a+b)(b+c) - b(a^2+ab+ac-bc)}{a(b+c)} = \frac{c(a^2+b^2)}{a(b+c)}. \end{aligned}$$

Now, we claim that H_1^C lies on the chord joining $\frac{ca}{b}$ and $\frac{cb}{a}$; by symmetry so will H_2^C and this will imply the problem (it means that the desired triangle has vertices ab/c, bc/a, ca/b). A cartoon of this is shown below.



To see this, it suffices to compute

$$\begin{aligned} H_1^C + \left(\frac{ca}{b}\right)\left(\frac{cb}{a}\right)\overline{H_1^C} &= \frac{c(a^2+b^2)}{a(b+c)} + c^2 \frac{\frac{1}{c} \cdot \frac{a^2+b^2}{a^2b^2}}{\frac{1}{a}\left(\frac{b+c}{bc}\right)} \\ &= \frac{c(a^2+b^2)}{a(b+c)} + \frac{c(a^2+b^2)}{abc^{-1}(b+c)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{c(a^2+b^2)}{a(b+c)}\left(\frac{b+c}{b}\right) \\
&= \frac{c(a^2+b^2)}{ab} = \frac{ca}{b} + \frac{cb}{a}
\end{aligned}
$$

as desired.

IMO 2001 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2001 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

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Corrections and comments are welcome!

Contents

§0 Problems

1. Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \ge \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.

2. Let a, b, c be positive reals. Prove that
$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \ge 1.$$

3. Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that each contestant solved at most six problems, and for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Show that there is a problem that was solved by at least three girls and at least three boys.

4. Let n be an odd integer greater than 1 and let c_1, c_2, …, c_n be integers. For each permutation $a = (a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$, define $S(a) = \sum_{i=1}^n c_i a_i$. Prove that there exist two permutations $a \ne b$ of $\{1, 2, \dots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

5. Let ABC be a triangle. Let $\overline{AP}$ bisect $\angle BAC$ and let $\overline{BQ}$ bisect $\angle ABC$, with P on $\overline{BC}$ and Q on $\overline{AC}$. If $AB + BP = AQ + QB$ and $\angle BAC = 60^\circ$, what are the angles of the triangle?

6. Let $a > b > c > d > 0$ be integers satisfying
$$ac + bd = (b + d + a - c)(b + d - a + c).$$
Prove that $ab + cd$ is not prime.

§1 Solutions to Day 1

§1.1 IMO 2001/1

Available online at https://aops.com/community/p119192.

Problem statement

Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.

The conclusion rewrites as

$$\begin{aligned} & \angle COP < 90^\circ - \angle A = \angle OCP \\ \iff & PC < PO \\ \iff & PC^2 < PO^2 \\ \iff & PC^2 < R^2 - PB \cdot PC \\ \iff & PC \cdot BC < R^2 \\ \iff & ab \cos C < R^2 \\ \iff & \sin A \sin B \cos C < \frac{1}{4}. \end{aligned}$$

Now

$$\cos C \sin B = \frac{1}{2} (\sin(C+B) - \sin(C-B)) \leq \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

which finishes when combined with $\sin A < 1$.

Remark. If we allow ABC to be right then equality holds when $\angle A = 90^\circ$, $\angle C = 60^\circ$, $\angle B = 30^\circ$. This motivates the choice of estimates after reducing to a trig inequality.

§1.2 IMO 2001/2

Available online at https://aops.com/community/p119168.

Problem statement

Let a, b, c be positive reals. Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\ge 1.$$

By Holder, we have

$$\left(\sum_{\text{cyc}}\frac{a}{\sqrt{a^2+8bc}}\right)^2\left(\sum_{\text{cyc}}a(a^2+8bc)\right)\ge(a+b+c)^3.$$

So it suffices to show $(a+b+c)^3 \ge a^3+b^3+c^3+24abc$ which is clear by expanding.

§1.3 IMO 2001/3

Available online at https://aops.com/community/p119191.

Problem statement

Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that each contestant solved at most six problems, and for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Show that there is a problem that was solved by at least three girls and at least three boys.

We will show the contrapositive. That is, assume that

- For each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy.
- Assume every problem is either solved mostly by girls (at most two boys) or mostly by boys (at most two girls).

Then we will prove that then some contestant solved more than six problems.

Create a 21×21 grid with boys as columns and girls as rows, and in each cell write the name of a problem solved by the pair. Color the cell **green** if at most two girls solved that problem, and color it **blue** if at most two boys solved that problem. (G for girl, B for boy. It's possible both colors are used for some cell.)

WLOG, there are more green cells than blue, so (by pigeonhole) take a column (boy) with that property. That means the boy's column has at least 11 green squares. By hypothesis, those corresponds to at least 6 different problems solved. Now there are two cases:

- If there are any blue-only squares, then that square means a seventh distinct problems.
- If the entire column is green, then among the 21 green squares there are at least 11 distinct problems solved in that column.

Remark. The number 21 cannot be decreased. Here is an example of 20 boys and 20 girls who solve problems named A-J and 0-9, which motivates the solution to begin with.

0000000000AABBCCDDEE
0000000000AABBCCDDEE
1111111111AABBCCDDEE
1111111111AABBCCDDEE
2222222222AABBCCDDEE
2222222222AABBCCDDEE
3333333333AABBCCDDEE
3333333333AABBCCDDEE
4444444444AABBCCDDEE
4444444444AABBCCDDEE
FFGGHHIIJJ5555555555
FFGGHHIIJJ5555555555
FFGGHHIIJJ6666666666
FFGGHHIIJJ6666666666
FFGGHHIIJJ7777777777

FFGGHHIIJJ7777777777
FFGGHHIIJJ8888888888
FFGGHHIIJJ8888888888
FFGGHHIIJJ9999999999
FFGGHHIIJJ9999999999

Remark. This took me embarrassingly long, but part of the work of the problem seemed to be finding the right "data structure" to get a foothold. I think the grid is good because we want to fill each intersection, then we consider for each cell a problem to put.

I initially wanted to capture the full data by writing in each green cell the row index of the other girl who solved it, and similarly for the blue cells. (That led naturally to the colors, there were two types of cells.) This was actually helpful for finding the equality case above, but once I realized the equality case I also realized that I could discard the extra information and only remember the colors.

§2 Solutions to Day 2

§2.1 IMO 2001/4

Available online at https://aops.com/community/p119174.

Problem statement

Let n be an odd integer greater than 1 and let c_1, c_2, ..., c_n be integers. For each permutation $a = (a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist two permutations $a \neq b$ of $\{1, 2, \dots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

Assume for contradiction that all the $S(a)$ are distinct modulo $n!$. Then summing across all permutations gives

$$\begin{aligned} 1 + 2 + \cdots + n! &\equiv \sum_a S(a) \\ &= \sum_a \sum_{i=1}^{n} c_i a_i \\ &= \sum_{i=1}^{n} c_i \sum_a a_i \\ &= \sum_{i=1}^{n} c_i \cdot ((n-1)! \cdot (1 + \cdots + n)) \\ &= (n-1)! \cdot \frac{n(n+1)}{2} \sum_{i=1}^{n} c_i \\ &= n! \cdot \frac{n+1}{2} \sum_{i=1}^{n} c_i \\ &\equiv 0 \end{aligned}$$

since $\frac{1}{2}(n+1)$ is an integer. But on the other hand $1 + 2 + \cdots + n! = \frac{n!(n!+1)}{2}$ which is not divisible by $n!$ if $n > 1$, as the quotient is the non-integer $\frac{n!+1}{2}$. This is a contradiction.

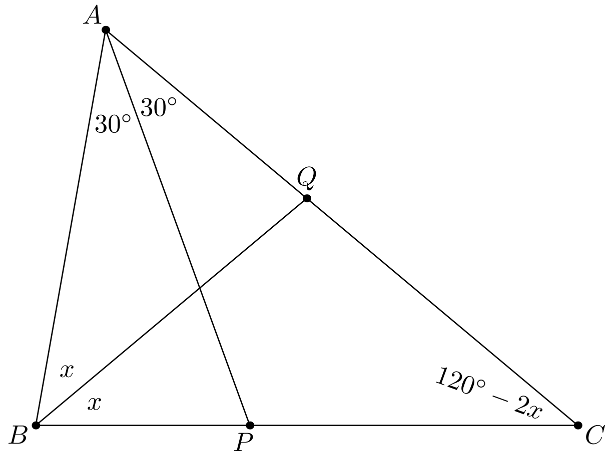
§2.2 IMO 2001/5

Available online at https://aops.com/community/p119207.

Problem statement

Let ABC be a triangle. Let $\overline{AP}$ bisect $\angle BAC$ and let $\overline{BQ}$ bisect $\angle ABC$, with P on $\overline{BC}$ and Q on $\overline{AC}$. If $AB + BP = AQ + QB$ and $\angle BAC = 60^\circ$, what are the angles of the triangle?

The answer is $\angle B = 80^\circ$ and $\angle C = 40^\circ$. Set $x = \angle ABQ = \angle QBC$, so that $\angle QCB = 120^\circ - 2x$. We observe $\angle AQB = 120^\circ - x$ and $\angle APB = 150^\circ - 2x$.



Now by the law of sines, we may compute

$$\begin{aligned} BP &= AB \cdot \frac{\sin 30^\circ}{\sin(150^\circ - 2x)} \\ AQ &= AB \cdot \frac{\sin x}{\sin(120^\circ - x)} \\ QB &= AB \cdot \frac{\sin 60^\circ}{\sin(120^\circ - x)}. \end{aligned}$$

So, the relation $AB + BP = AQ + QB$ is exactly

$$1 + \frac{\sin 30^\circ}{\sin(150^\circ - 2x)} = \frac{\sin x + \sin 60^\circ}{\sin(120^\circ - x)}.$$

This is now a trig problem, and we simply solve for x. There are many possible approaches and we just present one.

First of all, we can write

$$\sin x + \sin 60^\circ = 2 \sin\left(\frac{1}{2}(x + 60^\circ)\right) \cos\left(\frac{1}{2}(x - 60^\circ)\right).$$

On the other hand, $\sin(120^\circ - x) = \sin(x + 60^\circ)$ and

$$\sin(x + 60^\circ) = 2 \sin\left(\frac{1}{2}(x + 60^\circ)\right) \cos\left(\frac{1}{2}(x + 60^\circ)\right)$$

so

$$\frac{\sin x + \sin 60^\circ}{\sin(120^\circ - x)} = \frac{\cos\left(\frac{1}{2}x - 30^\circ\right)}{\cos\left(\frac{1}{2}x + 30^\circ\right)}.$$

Let $y = \frac{1}{2}x$ for brevity now. Then

$$\begin{aligned} \frac{\cos(y-30^\circ)}{\cos(y+30^\circ)} - 1 &= \frac{\cos(y-30^\circ) - \cos(y+30^\circ)}{\cos(y+30^\circ)} \\ &= \frac{2\sin(30^\circ)\sin y}{\cos(y+30^\circ)} \\ &= \frac{\sin y}{\cos(y+30^\circ)}. \end{aligned}$$

Hence the problem is just

$$\frac{\sin 30^\circ}{\sin(150^\circ - 4y)} = \frac{\sin y}{\cos(y+30^\circ)}.$$

Equivalently,

$$\begin{aligned} \cos(y+30^\circ) &= 2\sin y \sin(150^\circ - 4y) \\ &= \cos(5y - 150^\circ) - \cos(150^\circ - 3y) \\ &= -\cos(5y+30^\circ) + \cos(3y+30^\circ). \end{aligned}$$

Now we are home free, because $3y + 30^\circ$ is the average of $y + 30^\circ$ and $5y + 30^\circ$. That means we can write

$$\frac{\cos(y+30^\circ) + \cos(5y+30^\circ)}{2} = \cos(3y+30^\circ)\cos(2y).$$

Hence

$$\cos(3y+30^\circ)\left(2\cos(2y) - 1\right) = 0.$$

Recall that

$$y = \frac{1}{2}x = \frac{1}{4}\angle B < \frac{1}{4}\left(180^\circ - \angle A\right) = 30^\circ.$$

Hence it is not possible that $\cos(2y) = \frac{1}{2}$, since the smallest positive value of y that satisfies this is $y = 30^\circ$. So $\cos(3y+30^\circ) = 0$.

The only permissible value of y is then $y = 20^\circ$, giving $\angle B = 80^\circ$ and $\angle C = 40^\circ$.

§2.3 IMO 2001/6

Available online at https://aops.com/community/p119217.

Problem statement

Let $a > b > c > d > 0$ be integers satisfying

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

The problem condition is equivalent to

$$ac + bd = (b + d)^2 - (a - c)^2$$

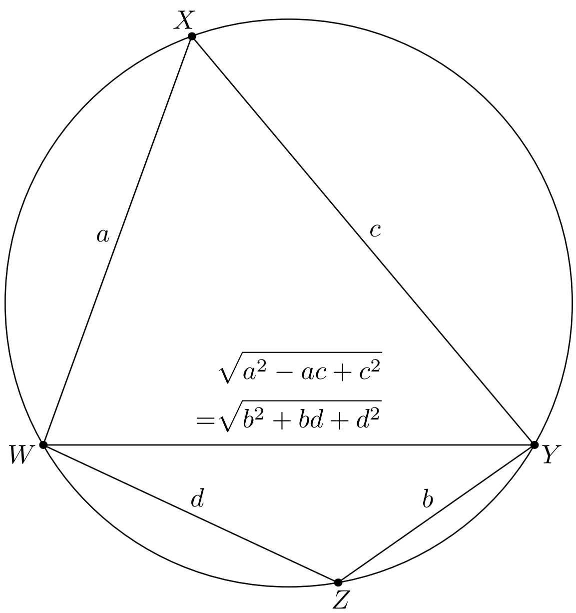
or

$$a^2 - ac + c^2 = b^2 + bd + d^2.$$

Let us construct a quadrilateral $WXYZ$ such that $WX = a$, $XY = c$, $YZ = b$, $ZW = d$, and

$$WY = \sqrt{a^2 - ac + c^2} = \sqrt{b^2 + bd + d^2}.$$

Then by the law of cosines, we obtain $\angle WXY = 60^\circ$ and $\angle WZY = 120^\circ$. Hence this quadrilateral is cyclic.



By the more precise version of Ptolemy's theorem, we find that

$$WY^2 = \frac{(ab + cd)(ad + bc)}{ac + bd}.$$

Now assume for contradiction that that $ab + cd$ is a prime p. Recall that we assumed $a > b > c > d$. It follows, for example by rearrangement inequality, that

$$p = ab + cd > ac + bd > ad + bc.$$

Let $y = ac + bd$ and $x = ad + bc$ now. The point is that

$$p \cdot \frac{x}{y}$$

can never be an integer if p is prime and $x < y < p$. But $WY^2 = a^2 - ac + c^2$ is clearly an integer, and this is a contradiction.

Hence $ab + cd$ cannot be prime.

IMO 2002 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

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Corrections and comments are welcome!

Contents

§0 Problems

1. Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers with $x + y < n$. Each point of T is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') of T with $x' \le x$ and $y' \le y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

2. Let BC be a diameter of circle ω with center O. Let A be a point of circle ω such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of arc AB not containing C. Line ℓ passes through O and is parallel to line AD. Line ℓ intersects line AC at J. The perpendicular bisector of segment OA intersects circle ω at E and F. Prove that J is the incenter of triangle CEF.

3. Find all pairs of positive integers $m, n \ge 3$ for which there exist infinitely many positive integers a such that
$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$
is itself an integer.

4. Let $n \ge 2$ be a positive integer with divisors $1 = d_1 < d_2 < \dots < d_k = n$. Prove that $d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$ is always less than n^2, and determine when it is a divisor of n^2.

5. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that
$$(f(x) + f(z))\,(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$
for all real numbers x, y, z, t.

6. Let $n \ge 3$ be a positive integer. Let C_1, C_2, …, C_n be unit circles in the plane, with centers O_1, O_2, …, O_n respectively. If no line meets more than two of the circles, prove that
$$\sum_{1 \le i < j \le n} \frac{1}{O_i O_j} \le \frac{(n-1)\pi}{4}.$$

§1 Solutions to Day 1

§1.1 IMO 2002/1

Available online at https://aops.com/community/p118710.

Problem statement

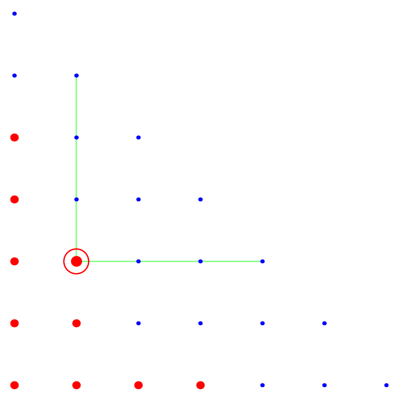
Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers with $x + y < n$. Each point of T is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') of T with $x' \le x$ and $y' \le y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

Let a_x denote the number of blue points with a given x-coordinate. Define b_y to be the number of blue points with a given y-coordinate.

We actually claim that

Claim — The multisets $\mathcal{A} := \{a_x \mid x\}$ and $\mathcal{B} := \{b_y \mid y\}$ are equal.

Proof. By induction on the number of red points. If there are no red points at all, then $\mathcal{A} = \mathcal{B} = \{1, \dots, n\}$.



The proof consists of two main steps. First, suppose we color a single point $P = (x, y)$ from blue to red (while preserving the condition). Before the coloring, we have $a_x = b_y = n - (x + y)$; afterwards $a_x = b_y = n - (x + y) - 1$ and no other numbers change, as desired.

We also must show that this operation (repeatedly adding a single point P) reaches all possible shapes of red points. This is well-known as the red points form a Young tableaux; for example, one way is to add all the points with $x = 0$ first one by one, then all the points with $x = 1$, and so on. So the induction implies the result. $\square$

Finally,

$$A = \prod_{x=0}^{n-1} a_x = \prod_{y=0}^{n-1} b_y = B.$$

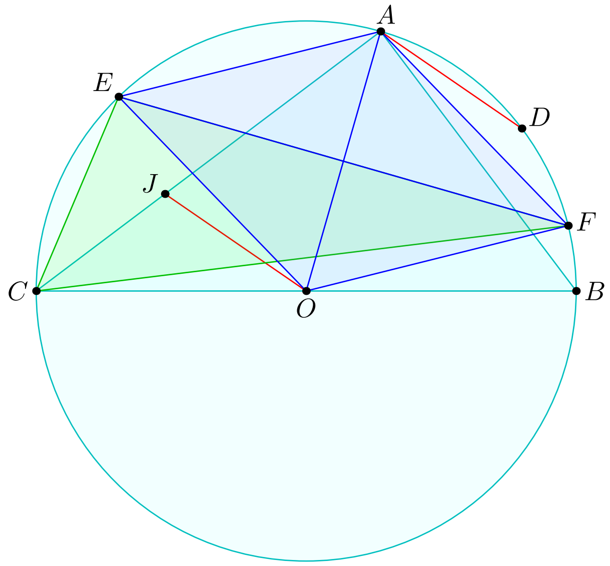
§1.2 IMO 2002/2

Available online at https://aops.com/community/p118672.

Problem statement

Let BC be a diameter of circle ω with center O. Let A be a point of circle ω such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of arc AB not containing C. Line ℓ passes through O and is parallel to line AD. Line ℓ intersects line AC at J. The perpendicular bisector of segment OA intersects circle ω at E and F. Prove that J is the incenter of triangle CEF.

By construction, $AEOF$ is a rhombus with 60°-120° angles. Consequently, we may set $s = AO = AE = AF = EO = EF$.



Claim — We have $AJ = s$ too.

Proof. It suffices to show $AJ = AO$ which is angle chasing. Let $\theta = \angle BOD = \angle DOA$, so $\angle BOA = 2\theta$. Thus $\angle CAO = \frac{1}{2}\angle BOA = \theta$. However $\angle AOJ = \angle OAD = 90^\circ - \frac{1}{2}\theta$, as desired. $\square$

Then, since $AE = AJ = AF$, we are done by the infamous Fact 5.

§1.3 IMO 2002/3, proposed by Laurentiu Panaitopol (ROM)

Available online at https://aops.com/community/p118695.

Problem statement

Find all pairs of positive integers $m, n \ge 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

The condition is equivalent to $a^n + a^2 - 1$ dividing $a^m + a - 1$ as polynomials. The big step is the following analytic one.

Claim — We must have $m \le 2n$.

Proof. Assume on contrary $m > 2n$ and let $0 < r < 1$ be the unique real number with $r^n + r^2 = 1$, hence $r^m + r = 1$. But now

$$\begin{aligned} 0 = r^m + r - 1 &< r(r^n)^2 + r - 1 = r\left((1-r^2)^2 + 1\right) - 1 \\ &= -(1-r)\left(r^4 + r^3 - r^2 - r + 1\right). \end{aligned}$$

As $1 - r > 0$ and $r^4 + r^3 - r^2 - r + 1 > 0$, this is a contradiction □

Now for the algebraic part. Obviously $m > n$.

$$\begin{aligned} & a^n + a^2 - 1 \mid a^m + a - 1 \\ \iff & a^n + a^2 - 1 \mid (a^m + a - 1)(a+1) = a^m(a+1) + (a^2 - 1) \\ \iff & a^n + a^2 - 1 \mid a^m(a+1) - a^n \\ \iff & a^n + a^2 - 1 \mid a^{m-n}(a+1) - 1. \end{aligned}$$

The right-hand side has degree $m - n + 1 \le n + 1$, and the leading coefficients are both $+1$. So the only possible situations are

$$\begin{aligned} a^{m-n}(a+1) - 1 &= (a+1)\left(a^n + a^2 - 1\right) \\ a^{m-n}(a+1) + 1 &= a^n + a^2 - 1. \end{aligned}$$

The former fails by just taking $a = -1$; the latter implies $(m, n) = (5, 3)$. As our work was reversible, this also implies $(m, n) = (5, 3)$ works, done.

§2 Solutions to Day 2

§2.1 IMO 2002/4

Available online at https://aops.com/community/p118687.

Problem statement

Let $n \ge 2$ be a positive integer with divisors $1 = d_1 < d_2 < \dots < d_k = n$. Prove that $d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$ is always less than n^2, and determine when it is a divisor of n^2.

We always have

$$\begin{aligned} d_k d_{k-1} + d_{k-1} d_{k-2} + \dots + d_2 d_1 &< n \cdot \frac{n}{2} + \frac{n}{2} \cdot \frac{n}{3} + \dots \\ &= \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots \right) n^2 = n^2. \end{aligned}$$

This proves the first part.

For the second, we claim that this only happens when n is prime (in which case we get $d_1 d_2 = n$). Assume n is not prime (equivalently $k \ge 2$) and let p be the smallest prime dividing n. Then

$$d_k d_{k-1} + d_{k-1} d_{k-2} + \dots + d_2 d_1 > d_k d_{k-1} = \frac{n^2}{p}$$

exceeds the largest proper divisor of n^2, but is less than n^2, so does not divide n^2.

§2.2 IMO 2002/5

Available online at https://aops.com/community/p118703.

Problem statement

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(f(x)+f(z))\,(f(y)+f(t)) = f(xy-zt)+f(xt+yz)$$

for all real numbers x, y, z, t.

The answer is $f(x) \equiv 0$, $f(x) \equiv 1/2$ and $f(x) \equiv x^2$ which are easily seen to work. Let's prove they are the only ones; we show two solutions.

¶ First solution (multiplicativity) Let $P(x,y,z,t)$ denote the given statement.

- By comparing $P(x,1,0,0)$ and $P(0,0,1,x)$ we get $\boxed{f \text{ even}}$.
- By $P(0,y,0,t)$ we get for nonconstant f that $f(0)=0$. If f is constant we get the solutions earlier, so in the sequel assume $\boxed{f(0)=0}$.
- By $P(x,y,0,0)$ we get $\boxed{f(xy)=f(x)f(y)}$. Note in particular that for any real number x we now have

$$f(x) = f(|x|) = f\left(\sqrt{|x|}\right)^2 \geq 0$$

  that is, $f \geq 0$.

From $P(x,y,y,x)$ we now have

$$f(x^2+y^2) = (f(x)+f(y))^2 = f(x^2)+2f(x)f(y)+f(y^2) \geq f(x^2)$$

so f is weakly increasing. Combined with f multiplicative and nonconstant, this implies $f(x)=|x|^r$ for some real number r.

Finally, $P(1,1,1,1)$ gives $f(2)=4f(1)$, so $f(x)\equiv x^2$.

¶ Second solution (ELMO) Let $P(x,y,z,t)$ denote the statement. Assume f is nonconstant, as before we derive that f is even, $f(0)=0$, and $f(x)\geq 0$ for all x.

Now comparing $P(x,y,z,t)$ and $P(z,y,x,t)$ we obtain

$$f(xy-zt)+f(xt+yz) = (f(x)+f(z))\,(f(y)+f(t)) = f(xy+zt)+f(xt-yz)$$

which in particular implies that

$$f(a-d)+f(b+c) = f(a+d)+f(b-c) \qquad \text{if } ad=bc \text{ and } a,b,c,d>0.$$

Thus the restriction of f to $(0,\infty)$ satisfies **ELMO 2011, problem 4** which implies that $f(x)=kx^2+\ell$ for constants k and ℓ. From here we recover the original.

(Minor note: technically ELMO 2011/4 is $f\colon (0,\infty)\to(0,\infty)$ but we only have $f\geq 0$, however the proof for the ELMO problem works as long as f is bounded below; we could also just apply the ELMO problem to $f+0.01$ instead.)

§2.3 IMO 2002/6

Available online at https://aops.com/community/p118677.

Problem statement

Let $n \geq 3$ be a positive integer. Let $C_1, C_2, \ldots, C_n$ be unit circles in the plane, with centers $O_1, O_2, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$

For brevity, let d_{ij} be the length of O_{ij} and let $\angle(ijk)$ be shorthand for $\angle O_iO_jO_k$ (or its measure in radians).

First, we eliminate the circles completely and reduce the problem to angles using the following fact (which is in part motivated by the mysterious presence of π on right-hand side, and also brings d_{ij}^{-1} into the picture).

Lemma

For any indices i, j, m we have the inequalities

$$\angle(imj) \geq \max\left(\frac{2}{d_{mi}}, \frac{2}{d_{mj}}\right) \quad \text{and} \quad \pi - \angle(imj) \geq \max\left(\frac{2}{d_{mi}}, \frac{2}{d_{mj}}\right).$$

Proof. We first prove the former line. Consider the altitude from O_i to O_mO_j. The altitude must have length at least 2, otherwise its perpendicular bisector passes intersects all of C_i , C_m, C_j. Thus

$$2 \leq d_{mi} \sin\angle(imj) \leq \angle(imj)$$

proving the first line. The second line follows by considering the external angle formed by lines O_mO_i and O_mO_j instead of the internal one. $\square$

Our idea now is for any index m we will make an estimate on $\sum_{\substack{1 \leq i \leq n \\ i \neq b}} \frac{1}{d_{bi}}$ for each index b. If the centers formed a convex polygon, this would be much simpler, but because we do not have this assumption some more care is needed.

Claim — Suppose O_a, O_b, O_c are consecutive vertices of the convex hull. Then

$$\frac{n-1}{n-2}\angle(abc) \geq \frac{2}{d_{1b}} + \frac{2}{d_{2b}} + \cdots + \frac{2}{d_{nb}}$$

where the term $\frac{2}{d_{bb}}$ does not appear (obviously).

Proof. WLOG let's suppose $(a, b, c) = (2, 1, n)$ and that rotating ray O_2O_1 hits O_3, O_4, $\ldots$, O_n in that order. We have

$$\frac{2}{d_{12}} \leq \angle(213)$$
$$\frac{2}{d_{13}} \leq \min\{\angle(213), \angle(314)\}$$

$$\begin{aligned}
\frac{2}{d_{14}} &\le \min\left\{\angle(314), \angle(415)\right\} \\
&\vdots \\
\frac{2}{d_{1(n-1)}} &\le \min\left\{\angle((n-2)1(n-1)), \angle((n-1)1n)\right\} \\
\frac{2}{d_{1n}} &\le \angle\left((n-1)1n\right).
\end{aligned}$$

Of the $n-1$ distinct angles appearing on the right-hand side, we let κ denote the smallest of them. We have $\kappa \le \frac{1}{n-2}\angle(21n)$ by pigeonhole principle. Then we pick the minimums on the right-hand side in the unique way such that summing gives

$$\begin{aligned}
\sum_{i=2}^{n} \frac{2}{d_{1i}} &\ge \left(\angle(213) + \angle(314) + \dots + \angle((n-1)1n)\right) + \kappa \\
&\ge \angle(21n) + \frac{1}{n-2}\angle(21n) = \frac{n-1}{n-2}\angle(21n)
\end{aligned}$$

as desired. $\square$

Next we show a bound that works for any center, even if it does not lie on the convex hull $\mathcal{H}$.

Claim — For any index b we have

$$\frac{n-1}{n-2}\pi \ge \frac{2}{d_{1b}} + \frac{2}{d_{2b}} + \dots + \frac{2}{d_{nb}}$$

where the term $\frac{2}{d_{bb}}$ does not appear (obviously).

Proof. This is the same argument as in the previous proof, with the modification that because O_b could lie inside the convex hull now, our rotation argument should use lines instead of rays (in order for the angle to be π rather than 2π). This is why the first lemma is stated with two cases; we need it now.

Again WLOG $b = 1$. Consider line O_1O_2 (rather than just the ray) and imagine rotating it counterclockwise through O_2; suppose that this line passes through O_3, O_4, ..., O_n in that order before returning to O_2 again. We let $\measuredangle(i1j) \in \{\angle(i1j), \pi - \angle(i1j)\} \in [0, \pi)$ be the counterclockwise rotations obtained in this way, so that

$$\measuredangle(21n) = \measuredangle(213) + \measuredangle(314) + + \dots + \measuredangle((n-1)1n).$$

(This is not "directed angles", but related.)

Then we get bounds

$$\begin{aligned}
\frac{2}{d_{12}} &\le \measuredangle(213) \\
\frac{2}{d_{13}} &\le \min\left\{\measuredangle(213), \measuredangle(314)\right\} \\
&\vdots \\
\frac{2}{d_{1(n-1)}} &\le \min\left\{\measuredangle((n-2)1(n-1)), \measuredangle((n-1)1n)\right\} \\
\frac{2}{d_{1n}} &\le \measuredangle\left\{(n-1)1n\right\}
\end{aligned}$$

as in the last proof, and so as before we get

$$\sum_{i=1}^{n} \frac{2}{d_{1i}} \le \frac{n-1}{n-2} \measuredangle(21n)$$

which is certainly less than $\frac{n-1}{n-2}\pi$. $\square$

Now suppose there were r vertices in the convex hull. If we sum the first claim across all b on the hull, and the second across all b not on the hull (inside it), we get

$$\begin{aligned} \sum_{1 \le i < j \le n} \frac{2}{d_{ij}} &= \frac{1}{2} \sum_{b} \sum_{i \ne b} \frac{2}{d_{bi}} \\ &\le \frac{1}{2} \cdot \frac{n-1}{n-2} \left((r-2)\pi + (n-r)\pi \right) \\ &= \frac{(n-1)\pi}{4} \end{aligned}$$

as needed (with $(r-2)\pi$ being the sum of angles in the hull).

IMO 2003 Solution Notes

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29 June 2023

This is a compilation of solutions for the 2003 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let A be a 101-element subset of $S = \{1, 2, \dots, 10^6\}$. Prove that there exist numbers $t_1, t_2, \dots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \dots, 100$$

are pairwise disjoint.

2. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

3. Each pair of opposite sides of convex hexagon has the property that the distance between their midpoints is $\frac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that the hexagon is equiangular.

4. Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of perpendiculars from D to lines $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively. Show that $PQ = QR$ if and only if the bisectors of angles ABC and ADC meet on segment $\overline{AC}$.

5. Let n be a positive integer and let $x_1 \le x_2 \le \dots \le x_n$ be real numbers. Prove that

$$\left(\sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j| \right)^2 \le \frac{2(n^2-1)}{3} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2$$

with equality if and only if $x_1, x_2, \dots, x_n$ form an arithmetic sequence.

6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

§1 Solutions to Day 1

§1.1 IMO 2003/1

Available online at https://aops.com/community/p261.

Problem statement

Let A be a 101-element subset of $S = \{1, 2, \dots, 10^6\}$. Prove that there exist numbers $t_1, t_2, \dots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \dots, 100$$

are pairwise disjoint.

A greedy algorithm works: suppose we have picked

$$T = \{t_1, \dots, t_n\}$$

as large as possible, meaning it's impossible to add any more elements to T. That means, for each $t \in \left\{1, \dots, 10^6\right\}$ either $t \in T$ already or there exists two distinct elements $a, b \in A$ and $t_i \in T$ such that

$$t = t_i + b - a \qquad (\star).$$

There are at most $|T| \cdot |A| \cdot (|A| - 1) = n \cdot 101 \cdot 100$ possible values for the right-hand side of $(\star)$. So we therefore must have

$$101 \cdot 100 \cdot n + n \ge 10^6$$

which implies $n > 99$, as desired.

Remark. It is possible to improve the bound significantly with a small optimization; rather than adding any t, we require that $t_1 < \dots < t_n$ and that at each step we add the *least* $t \in S$ which is permitted. In that case, one finds we only need to consider $b > a$ in $(\star)$, and so this will essentially save us a factor of $2 + o(1)$ as the main term $101 \cdot 100$ becomes $\binom{101}{2}$ instead. See, e.g., https://aops.com/community/p22959828.

§1.2 IMO 2003/2

Available online at https://aops.com/community/p262.

Problem statement

Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

The answer is $(a, b) = (2\ell, 1)$, $(a, b) = (\ell, 2\ell)$ and $(a, b) = (8\ell^4 - \ell, 2\ell)$, for any ℓ. Check these work.

In the sequel, assume $b > 1$, and integers a, b, k obey $k = \frac{a^2}{2ab^2 - b^3 + 1}$. Expanding, we have the polynomial

$$X^2 - 2kb^2 \cdot X + k(b^3 - 1) = 0$$

has two integer roots, one of which is $X = a$. This means solutions to the original problem come in pairs (even with k fixed):

$$(a, b) \longleftrightarrow \left(2kb^2 - a, b\right) = \left(\frac{k(b^3 - 1)}{a}, b\right).$$

(Here, the first representation ensures $2kb^2 - a \in \mathbb{Z}$, while the latter representation and the hypothesis $b > 1$ ensures that $\frac{k(b^3-1)}{a} > 0$.)

On the other hand, we claim that:

Claim — For any solution (a, b), either $2a = b$ or $a > b$.

Proof. Since the denominator is positive, $a \ge b/2$. Now,

$$a^2 \ge 2ab^2 - b^3 + 1 \iff a^2 \ge b^2(2a - b) + 1$$

and so if $2a - b > 0$ then $a^2 > b^2 \implies a > b$. □

Now assume we have pair (a_1, b) and (a_2, b) of solutions with $b \neq 2a_1, 2a_2$. Then assume $a_1 > a_2 > b$ and

$$\begin{aligned} a_1 + a_2 &= 2k \cdot b^2 \\ a_1 a_2 &= k(b^3 - 1) \end{aligned}$$

That's impossible, since then $a_1 > \frac{a_1 + a_2}{2} = kb^2$ and hence $a_1 a_2 > kb^2 \cdot b = kb^3$. Thus the only solutions are the ones we claimed at the beginning.

Remark. Important to notice that the problem is *positive divides*, not just divides. There is an implicit inequality built in to the problem statement and it is essentially impossible to solve without. I would be interested in a pair (a, b) for which $k < 0$, $k \in \mathbb{Z}$ yet $a, b > 0$.

§1.3 IMO 2003/3

Available online at https://aops.com/community/p263.

Problem statement

Each pair of opposite sides of convex hexagon has the property that the distance between their midpoints is $\frac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that the hexagon is equiangular.

Unsurprisingly, this is a geometric inequality. Denote the hexagon by $ABCDEF$. Then we have that

$$\left|\frac{\vec{D}+\vec{E}}{2}-\frac{\vec{A}+\vec{B}}{2}\right| = \sqrt{3}\cdot\frac{\left|\vec{B}-\vec{A}\right|+\left|\vec{E}-\vec{D}\right|}{2} \ge \sqrt{3}\cdot\left|\frac{(\vec{B}-\vec{A})-(\vec{E}-\vec{D})}{2}\right|$$

and cyclic variations. Suppose we define the right-hand sides as variables

$$\begin{aligned}\vec{x} &= (\vec{B}-\vec{A})-(\vec{E}-\vec{D})\\ \vec{y} &= (\vec{D}-\vec{C})-(\vec{A}-\vec{F})\\ \vec{z} &= (\vec{F}-\vec{E})-(\vec{C}-\vec{B}).\end{aligned}$$

Then we now have

$$\begin{aligned}|\vec{y}-\vec{z}| &\ge \sqrt{3}\,|\vec{x}|\\ |\vec{z}-\vec{x}| &\ge \sqrt{3}\,|\vec{y}|\\ |\vec{x}-\vec{y}| &\ge \sqrt{3}\,|\vec{z}|\,.\end{aligned}$$

We square all sides (using $|\vec{v}|^2 = \vec{v}\cdot\vec{v}$) and then sum to get

$$\sum_{\text{cyc}}(\vec{y}-\vec{z})\cdot(\vec{y}-\vec{z}) \ge 3\sum_{\text{cyc}}\vec{x}\cdot\vec{x}$$

which rearranges to

$$-\left|\vec{x}+\vec{y}+\vec{z}\right|^2 \ge 0.$$

This can only happen if $\vec{x}+\vec{y}+\vec{z}=0$, and moreover all the inequalities above were actually equalities. That means that our triangle inequalities above were actually sharp (and already we have $\overline{AB}\parallel\overline{DE}$ and so on).

Working with just x and y now we have

$$\begin{aligned}3(\vec{x}\cdot\vec{x}) &= (2\vec{y}-\vec{x})\cdot(2\vec{y}-\vec{x})\\ &= \vec{x}\cdot\vec{x}-4\vec{y}\cdot\vec{x}+4\vec{y}\cdot\vec{y}\\ \implies -\vec{x}\cdot\vec{x}+2(\vec{y}\cdot\vec{y}) &= 2\vec{x}\cdot\vec{y}\\ 2(\vec{x}\cdot\vec{x})-\vec{y}\cdot\vec{y} &= 2\vec{x}\cdot\vec{y}.\end{aligned}$$

which implies $\vec{x}\cdot\vec{x}=\vec{y}\cdot\vec{y}$, that is, $\vec{x}$ and $\vec{y}$ have the same magnitude. In this way we find $\vec{x}$, $\vec{y}$, $\vec{z}$ all have the same magnitude, and since $\vec{x}+\vec{y}+\vec{z}=0$ they are related by 120° rotations, as desired.

Remark. In fact one can show further that the equiangular hexagons which work are exactly those formed by taking an equilateral triangle and cutting off equally sized corners. This equality case helps motivate the solution.

Remark. One can note this "must" be an inequality because the space of such hexagons is 2-dimensional, even though *a priori* the space of hexagons satisfying three given conditions should have dimension $9 - 3 = 6$.

§2 Solutions to Day 2

§2.1 IMO 2003/4

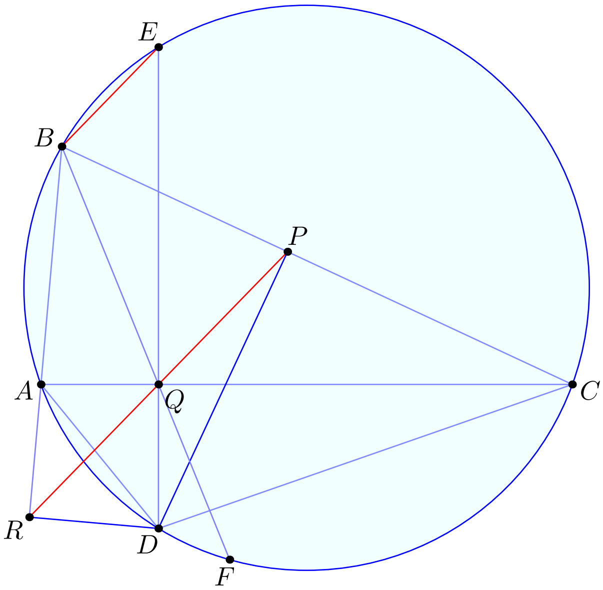
Available online at https://aops.com/community/p264.

Problem statement

Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of perpendiculars from D to lines $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively. Show that $PQ = QR$ if and only if the bisectors of angles ABC and ADC meet on segment $\overline{AC}$.

Let γ denote the circumcircle of $ABCD$. The condition on bisectors is equivalent to $(AC; BD)_\gamma = -1$. Meanwhile if ∞ denotes the point at infinity along Simson line $\overline{PQR}$ then $PQ = QR$ if and only if $(PR; Q\infty) = -1$.

Let rays BQ and DQ meet the circumcircle again at F and E.



Lemma (EGMO Proposition 4.1)
Then $\overline{BE} \parallel \overline{PQR}$.

Proof. Since $\measuredangle DPR = \measuredangle DAR = \measuredangle DAB = \measuredangle DEB$. $\square$

Now we have

$$(PR; Q\infty) \overset{B}{=} (CA; FE)_\gamma \overset{Q}{=} (AC; BD)_\gamma$$

as desired.

§2.2 IMO 2003/5

Available online at https://aops.com/community/p265.

Problem statement

Let n be a positive integer and let $x_1 \le x_2 \le \dots \le x_n$ be real numbers. Prove that

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2$$

with equality if and only if $x_1, x_2, \dots, x_n$ form an arithmetic sequence.

Let $d_1 = x_2 - x_1, \dots, d_{n-1} = x_n - x_{n-1}$. The inequality in question becomes:

$$\left(\sum_i i(n-i)d_i\right)^2 \le \frac{n^2-1}{3} \cdot \left(\sum_i i(n-i)d_i^2 + 2\sum_{i<j} i(n-j)d_id_j\right).$$

Clearing the square on the right-hand side we want to show

$$\sum_{i<j}\left(3ij(n-i)(n-j) - (n^2-1)i(n-j)\right) \cdot 2d_id_j \le \sum_i (n^2-1-3i(n-i)) \cdot i(n-i)d_i^2.$$

We use AM-GM directly on $2d_id_j \le d_i^2 + d_j^2$: this actually solves the problem. The annoying part is to check that the coefficients actually match:

Claim (Big bash) — For an index $1 \le k \le n-1$, we have

$$\begin{aligned} &\sum_{i<k}\left(3ik(n-i)(n-k) - (n^2-1)i(n-k)\right) \\ +&\sum_{j>k}\left(3kj(n-k)(n-j) - (n^2-1)k(n-j)\right) \\ =&(n^2-1-3k(n-k)) \cdot k(n-k). \end{aligned}$$

Proof. Rewrite as:

$$\begin{aligned} 3k(n-k)\left(-k(n-k) + \sum_i i(n-i)\right) &= (n^2-1)\left((n-k)\sum_{i<k} i + k\sum_{j>k}(n-j)\right) \\ &\quad + (n^2-1-3k(n-k)) \cdot k(n-k) \\ \iff 3k(n-k)\sum_i i(n-i) &= (n^2-1)\left((n-k)\sum_{i<k} i + k\sum_{j>k}(n-j)\right) \\ &\quad + (n^2-1)k(n-k) - 3k^2(n-k)^2 \\ \iff 3k(n-k)\left(\sum_i i(n-i)\right) &= (n^2-1)\left((n-k)\sum_{i\le k} i + k\sum_{i<n-k} i\right) \\ \iff 3k(n-k)\frac{(n-1)n(n+1)}{6} &= (n^2-1)\left((n-k)\frac{k(k+1)}{2}\right) \end{aligned}$$

$$+ (n^2-1)\left(k\frac{(n-k)(n-k-1)}{2}\right)$$

$$\iff 3k(n-k)\frac{(n-1)n(n+1)}{6} = (n^2-1)k(n-k)\cdot\frac{n}{2}$$

which is visibly true. $\square$

Equality occurs only if all d_i are equal because the coefficient of $d_i d_j$ is nonzero for any $i \le n/2$ and $j \ge n/2$.

§2.3 IMO 2003/6

Available online at https://aops.com/community/p266.

Problem statement

Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

By orders, we must have $q = pk + 1$ for this to be possible (since if $q \not\equiv 1 \pmod p$, then n^p can be any residue modulo q). Since $p \equiv n^p \pmod q \implies p^k \equiv 1 \pmod q$, it suffices to prevent the latter situation from happening.

So we need a prime $q \equiv 1 \pmod p$ such that $p^k \not\equiv 1 \pmod q$. To do this, we first recall the following lemma.

Lemma

Let $\Phi_p(X) = 1 + X + X^2 + \cdots + X^{p-1}$. For any integer a, if q is a prime divisor of $\Phi_p(a)$ other than p, then $a \pmod q$ has order p. (In particular, $q \equiv 1 \pmod p$.)

Proof. We have $a^p - 1 \equiv 0 \pmod q$, so either the order is 1 or p. If it is 1, then $a \equiv 1 \pmod q$, so $q \mid \Phi_p(1) = p$, hence $q = p$. $\square$

Now the idea is to extract a prime factor q from the cyclotomic polynomial

$$\Phi_p(p) = \frac{p^p - 1}{p - 1} \equiv 1 + p \pmod{p^2}$$

such that $q \not\equiv 1 \pmod{p^2}$; hence $k \not\equiv 0 \pmod p$, and as $p \pmod q$ has order p we have $p^k \not\equiv 1 \pmod q$.

IMO 2004 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

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Corrections and comments are welcome!

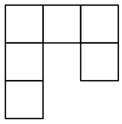
Contents

§0 Problems

1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

2. Find all polynomials P with real coefficients such that for all reals a, b, c such that $ab + bc + ca = 0$, we have
$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c).$$

3. Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Which $m \times n$ rectangles can be tiled by hooks?

4. Let $n \geq 3$ be an integer and $t_1, t_2, \dots, t_n$ positive real numbers such that
$$n^2 + 1 > (t_1 + t_2 + \dots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \dots + \frac{1}{t_n}\right).$$
Show that t_i, t_j, t_k are the sides of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

5. In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies
$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$
Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

6. We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n which have an alternating multiple.

§1 Solutions to Day 1

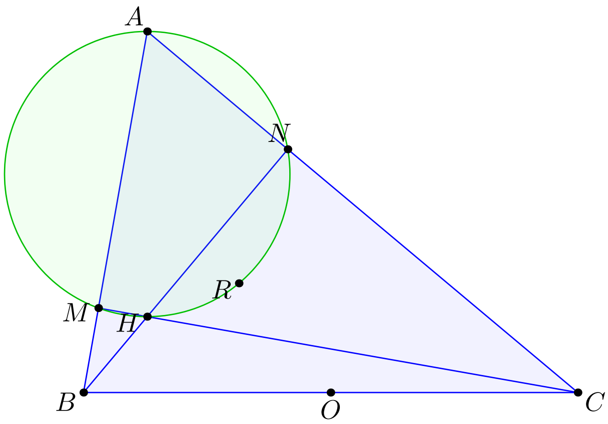
§1.1 IMO 2004/1

Available online at https://aops.com/community/p99445.

Problem statement

Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

By Miquel's theorem it's enough to show $AMRN$ is cyclic.



In fact, since the bisector of $\angle MON$ is just the perpendicular bisector of $\overline{MN}$, the point R is actually just the arc midpoint of $\widehat{MN}$ of (AMN) as desired.

§1.2 IMO 2004/2

Available online at https://aops.com/community/p99448.

Problem statement

Find all polynomials P with real coefficients such that for all reals a, b, c such that $ab + bc + ca = 0$, we have

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c).$$

The answer is

$$P(x) = \alpha x^4 + \beta x^2$$

which can be checked to work, for any real numbers α and β.

It is easy to obtain that P is even and $P(0) = 0$. The trick is now to choose $(a, b, c) = (6x, 3x, -2x)$ and then compare the leading coefficients to get

$$3^n + 5^n + 8^n = 2 \cdot 7^n$$

for $n = \deg f$ (which is even). As $n \geq 7 \implies (8/7)^n > 2$, this means that we must have $n \leq 6$. Moreover, taking modulo 7 we have $3^n + 5^n \equiv 6 \pmod 7$ which gives $n \equiv 2, 4 \pmod 6$.

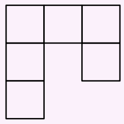
Thus $\deg P \leq 4$, which (combined with P even) resolves the problem.

§1.3 IMO 2004/3

Available online at https://aops.com/community/p99450.

Problem statement

Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Which $m \times n$ rectangles can be tiled by hooks?

The answer is that one requires:

- $\{1, 2, 5\} \notin \{m, n\}$,
- $3 \mid m$ or $3 \mid n$,
- $4 \mid m$ or $4 \mid n$.

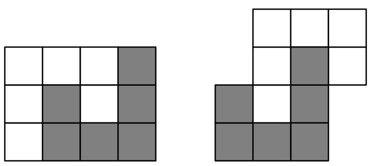
First, we check all of these work, in fact we claim:

Claim — Any rectangle satisfying these conditions can be tiled by 3×4 rectangles (and hence by hooks).

Proof. In fact it will be sufficient to tile with 3×4 rectangles. If $3 \mid m$ and $4 \mid n$, this is clear. Else suppose $12 \mid m$ but $3 \nmid n$, $4 \nmid n$. Then $n \geq 7$, so it can be written in the form $3a + 4b$ for nonengative integers a and b, which permits a tiling. $\square$

We now prove these conditions are necessary. It is not hard to see that $m, n \in \{1, 2, 5\}$ is necessary.

We thus turn our attention to divisibility conditions. Each hook has a *hole*, and if we associate each hook with the one that fills its hole, we get a bijective pairing of hooks. Thus the number of cells is divisible by 12, and the cells come in two types of **tiles** shown below (rotations and reflections permitted).



In particular, the total number of cells is divisible by 12. Thus the problem is reduce to proving that:

Claim — if a $6a \times 2b$ rectangle is tiled by tiles, then at least one of a and b is even.

Proof. Note that the tiles come in two forms:

- **First type**: These tiles have exactly four columns, each with exactly three cells (an odd number). Moreover, all rows have an even number of cells (either 2 or 4).
- **Second type**: vice-versa. These tiles have exactly four rows, each with exactly three cells (an odd number). Moreover, all rows have an odd number of cells.

We claim that any tiling uses an even number of each type, which is enough.

By symmetry we prove an even number of first-type tiles. Color red every fourth column of the rectangle. The number of cells colored is red. The tiles of the second type cover an even number of red cells, and the tiles of the first type cover an odd number of red cells. Hence the number of tiles of the first type must be even. $\square$

> **Remark.** This shows that a rectangle can be tiled by hooks iff it can be tiled by 3×4 rectangles. But there exists tilings which do not decompose into 3×4; see e.g. https://aops.com/community/c6h14023p99881.

§2 Solutions to Day 2

§2.1 IMO 2004/4

Available online at https://aops.com/community/p99756.

Problem statement

Let $n \geq 3$ be an integer and t_1, t_2, $\dots$, t_n positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \dots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \dots + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are the sides of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

Let $a = t_1$, $b = t_2$, $c = t_3$. Expand:

$$\begin{aligned}
n^2 + 1 &> (t_1 + t_2 + \dots + t_n)\left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right) \\
&= n + \sum_{1 \leq i < j \leq n} \left(\frac{t_i}{t_j} + \frac{t_j}{t_i}\right) \\
&= n + \sum_{1 \leq i < j \leq n} \left(\frac{t_i}{t_j} + \frac{t_j}{t_i}\right) \\
&\geq n + \sum_{1 \leq i < j \leq 3} \left(\frac{t_i}{t_j} + \frac{t_j}{t_i}\right) + \sum_{\substack{1 \leq i < j \leq n \\ j > 3}} 2 \\
&= n + 2\left(\binom{n}{2} - 3\right) + \left(\frac{a}{b} + \frac{b}{a}\right) + \frac{a+b}{c} + \frac{c}{b} + \frac{c}{a} \\
&\geq n + 2\left(\binom{n}{2} - 3\right) + 2 + \frac{a+b}{c} + c \cdot \frac{4}{a+b}
\end{aligned}$$

So, we conclude that

$$\frac{a+b}{c} + \frac{4c}{a+b} < 5$$

which rearranges to

$$(4c - (a+b))\,(c - (a+b)) < 0.$$

This is enough to imply $c \leq a + b$.

Remark. A variant of the same argument allows one to improve the left-hand side to $(n + \sqrt{10} - 3)^2$. One does so by writing

$$\text{RHS} \geq \left(\sqrt{(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} + (n-3)\right)^2$$

and estimating the square root as in the previous approach.

In addition, $(n + \sqrt{10} - 3)^2$ is best possible, as seen by taking $(a, b, c) = (2, 1, 1)$ and $t_4 = t_5 = \dots = \frac{2}{5}\sqrt{10}$.

§2.2 IMO 2004/5, proposed by Waldemar Pompe

Available online at https://aops.com/community/p99759.

Problem statement

In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Apply barycentric coordinates to $\triangle PBD$ with $P = (1,0,0)$, $B = (0,1,0)$ and $D = (0,0,1)$. Define $a = BD$, $b = DP$ and $c = PB$.

Since A and C are isogonal conjugates with respect to $\triangle PBD$, we set

$$A = (au : bv : cw) \qquad \text{and} \qquad C = \left(\frac{a}{u} : \frac{b}{v} : \frac{c}{w}\right).$$

For brevity define $M = au + bv + cw$ and $N = avw + bwu + cuv$.

We now compute each condition.

Claim — Quadrilateral $ABCD$ is cyclic if and only if $N^2 = u^2M^2$.

Proof. W know a circle through B and D is a locus of points with

$$\frac{a^2yz + b^2zx + c^2xy}{x(x+y+z)}$$

is equal to some constant. Therefore quadrilateral $ABCD$ is cyclic if and only if $\frac{abc \cdot N}{au \cdot M}$ is equal to $\frac{abc \cdot uvw \cdot M}{avw \cdot N}$ which rearranges to $N^2 = u^2M^2$. □

Claim — We have $PA = PC$ if and only if $N^2 = u^2M^2$.

Proof. We have the displacement vector $\overrightarrow{PA} = \frac{1}{M}(bv + cw, -bv, -cw)$. Therefore,

$$\begin{aligned} M^2 \cdot |PA|^2 &= -a^2(bv)(cw) + b^2(cw)(bv+cw) + c^2(bv)(bv+cw) \\ &= bc(-a^2vw + (bw+cv)(bv+cw)). \end{aligned}$$

In a similar way (by replacing u, v, w with their inverses) we have

$$\begin{aligned} &\left(\frac{N}{uvw}\right)^2 \cdot |PC|^2 = (vw)^{-2} \cdot bc(-a^2vw + (bv+cw)(bw+cv)) \\ \iff &N^2 \cdot |PC|^2 = u^2bc(-a^2vw + (bw+cv)(bv+cw)) \end{aligned}$$

These are equal if and only if $N^2 = u^2M^2$, as desired. □

§2.3 IMO 2004/6

Available online at https://aops.com/community/p99760.

Problem statement

We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n which have an alternating multiple.

If $20 \mid n$, then clearly n has no alternating multiple since the last two digits are both even. We will show the other values of n all work.

The construction is just rush-down do-it. The meat of the solution is the two following steps.

Claim (Tail construction) — For every even integer $w \geq 2$,

- there exists an even alternating multiple $g(w)$ of 2^{w+1} with exactly w digits, and
- there exists an even alternating multiple $h(w)$ of 5^w with exactly w digits.

(One might note this claim is implied by the problem, too.)

Proof. We prove the first point by induction on w. If $w = 2$, take $g(2) = 32$. In general, we can construct $g(w+2)$ from $g(w)$ by adding some element in

$$10^w \cdot \{10, 12, 14, 16, 18, 30, \dots, 98\}$$

to $g(w)$, corresponding to the digits we want to append to the start. This multiple is automatically divisible by 2^{w+1}, and also can take any of the four possible values modulo 2^{w+3}.

The second point is a similar induction, with base case $h(2) = 50$. The same set above consists of numbers divisible by 5^w, and covers all residues modulo 5^{w+2}. Careful readers might recognize the second part as essentially USAMO 2003/1. $\square$

Claim (Head construction) — If $\gcd(n, 10) = 1$, then for any b, there exists an even alternating number $f(b \bmod n)$ which is $b \pmod n$.

Proof. A standard argument shows that

$$10 \cdot \frac{100^m - 1}{99} = \underbrace{1010 \dots 10}_{m \text{ 10's}} \equiv 0 \pmod n$$

for any m divisible by $\varphi(99n)$. Take a very large such m, and then add on b distinct numbers of the form $10^{\varphi(n)r}$ for various even values of r; these all are $1 \pmod n$ and change some of the 1's to 3's. $\square$

Now, we can solve the problem. Consider three cases:

- If $n = 2^k m$ where $\gcd(m, 10) = 1$ and $k \ge 2$ is even, then the concatenated number
$$10^k f\left(-\frac{g(k)}{10^k} \bmod m\right) + g(k)$$
works fine.
- If $n = 5^k m$ where $\gcd(m, 10) = 1$ and $k \ge 2$ is even, then the concatenated number
$$10^k f\left(-\frac{h(k)}{10^k} \bmod m\right) + h(k)$$
works fine.
- If $n = 50m$ where $\gcd(m, 10) = 1$, then the concatenated number
$$100 f\left(-\frac{1}{2} \bmod m\right) + 50$$
works fine.

Since every non-multiple of 20 divides such a number, we are done.

IMO 2005 Solution Notes

EVAN CHEN《陳誼廷》

2 June 2023

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Corrections and comments are welcome!

Contents

§0 Problems

1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

2. Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \dots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence.

3. Let $x, y, z > 0$ satisfy $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

4. Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \quad n \geq 1.$$

5. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and $\overline{BC} \nparallel \overline{DA}$. Let two variable points E and F lie on the sides BC and DA, respectively, and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

6. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $\frac{2}{5}$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

§1 Solutions to Day 1

§1.1 IMO 2005/1

Available online at https://aops.com/community/p281571.

Problem statement

Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

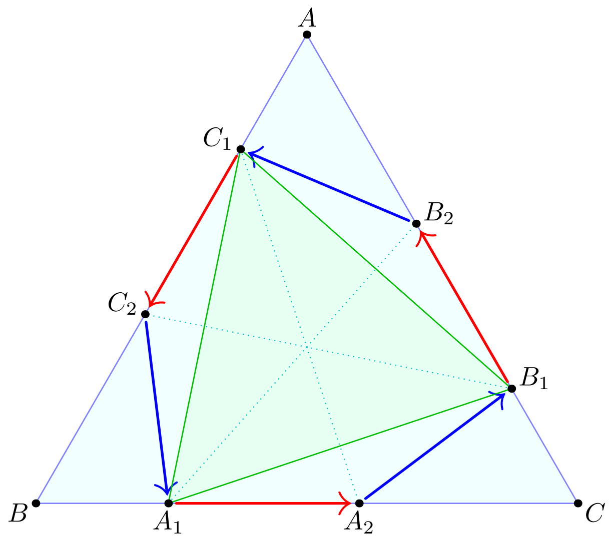
The six sides of the hexagon, when oriented, comprise six vectors with vanishing sum. However note that

$$\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 0.$$

Thus

$$\overrightarrow{A_2B_1} + \overrightarrow{B_2C_1} + \overrightarrow{C_2A_1} = 0$$

and since three unit vectors with vanishing sum must be rotations of each other by $120°$, it follows they must also form an equilateral triangle.



Consequently, triangles $A_1A_2B_1$, $B_1B_2C_1$, $C_1C_2A_1$ are congruent, as $\angle A_2 = \angle B_2 = \angle C_2$. So triangle $A_1B_1C_1$ is equilateral and the diagonals are concurrent at the center.

§1.2 IMO 2005/2

Available online at https://aops.com/community/p281572.

Problem statement

Let a_1, a_2, ... be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers a_1, a_2, ..., a_n leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence.

Obviously every integer appears at most once (otherwise take n much larger). So we will prove every integer appears at least once.

Claim — For any $i < j$ we have $|a_i - a_j| < j$.

Proof. Otherwise, let $n = |a_i - a_j| \neq 0$. Then $i, j \in [1, n]$ and $a_i \equiv a_j \pmod n$, contradiction. □

Claim — For any n, the set $\{a_1, \dots, a_n\}$ is of the form $\{k+1, \dots, k+n\}$ for some integer k.

Proof. By induction, with the base case $n = 1$ being vacuous. For the inductive step, suppose $\{a_1, \dots, a_n\} = \{k+1, \dots, k+n\}$ are determined. Then

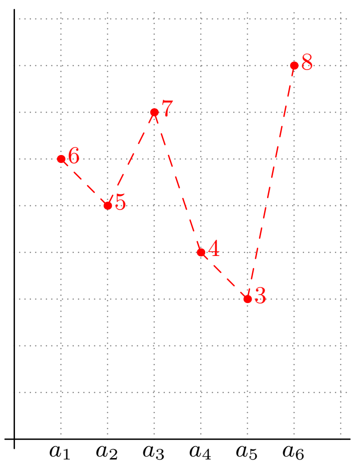
$$a_{n+1} \equiv k \pmod{n+1}.$$

Moreover by the earlier claim we have

$$|a_{n+1} - a_1| < n + 1.$$

From this we deduce $a_{n+1} \in \{k, k+n+1\}$ as desired. □

This gives us actually a complete description of all possible sequences satisfying the hypothesis: choose any value of a_1 to start. Then, for the nth term, the set $S = \{a_1, \dots, a_{n-1}\}$ is (in some order) a set of $n-1$ consecutive integers. We then let $a_n = \max S + 1$ or $a_n = \min S - 1$. A picture of six possible starting terms is shown below.



Finally, we observe that the condition that the sequence has infinitely many positive and negative terms (which we have not used until now) implies it is unbounded above and below. Thus it must contain every integer.

§1.3 IMO 2005/3

Available online at https://aops.com/community/p281573.

Problem statement

Let $x, y, z > 0$ satisfy $xyz \ge 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \ge 0.$$

Negating both sides and adding 3 eliminates the minus signs:

$$\sum_{\text{cyc}} \frac{1}{x^5 + y^2 + z^2} \le \frac{3}{x^2 + y^2 + z^2}.$$

Thus we only need to consider the case $xyz = 1$.

Direct expansion and Muirhead works now! As advertised, once we show it suffices to analyze if $xyz = 1$ the inequality becomes more economically written as

$$S = \sum_{\text{cyc}} x^2(x^2 - yz)(y^4 + x^3z + xz^3)(z^4 + x^3y + xy^3) \overset{?}{\ge} 0.$$

So, clearing all the denominators gives

$$\begin{aligned}
S &= \sum_{\text{cyc}} x^2(x^2 - yz)\left[y^4z^4 + x^3y^5 + xy^7 + x^3z^5 + x^6yz + x^4y^3z + xz^7 + x^4yz^3 + x^2y^3z^3\right] \\
&= \sum_{\text{cyc}} \left[x^4y^4z^4 + x^7y^5 + x^5y^7 + x^7z^5 + x^{10}yz + x^8y^3z + x^5z^7 + x^8yz^3 + x^6y^3z^3\right] \\
&- \sum_{\text{cyc}} \left[x^2y^5z^5 + x^5y^6z + x^3y^8z + x^5yz^6 + x^8y^2z^2 + x^6y^4z^2 + x^3yz^8 + x^6y^2z^4 + x^4y^4z^4\right] \\
&= \sum_{\text{cyc}} \left[x^7y^5 + x^5y^7 + x^7z^5 + x^{10}yz + x^5z^7 + x^6y^3z^3\right] \\
&- \sum_{\text{cyc}} \left[x^2y^5z^5 + x^5y^6z + x^5yz^6 + x^8y^2z^2 + x^6y^4z^2 + x^6y^2z^4\right]
\end{aligned}$$

In other words we need to show

$$\sum_{\text{sym}} \left(2x^7y^5 + \frac{1}{2}x^{10}yz + \frac{1}{2}x^6y^3z^3\right) \ge \sum_{\text{sym}} \left(\frac{1}{2}x^8y^2z^2 + \frac{1}{2}x^5y^5z^2 + x^6y^4z^2 + x^6y^5z\right).$$

which follows by summing

$$\begin{aligned}
\sum_{\text{sym}} \frac{x^{10}yz + x^6y^3z^3}{2} &\ge \sum_{\text{sym}} x^8y^2z^2 \\
\frac{1}{2}\sum_{\text{sym}} x^8y^2z^2 &\ge \frac{1}{2}\sum_{\text{sym}} x^6y^4z^2 \\
\frac{1}{2}\sum_{\text{sym}} x^7y^5 &\ge \frac{1}{2}\sum_{\text{sym}} x^5y^5z^2
\end{aligned}$$

$$\begin{aligned} \frac{1}{2}\sum_{\text{sym}} x^7y^5 &\ge \frac{1}{2}\sum_{\text{sym}} x^6y^4z^2 \\ \sum_{\text{sym}} x^7y^5 &\ge \sum_{\text{sym}} x^6y^5z. \end{aligned}$$

The first line here comes from AM-GM, the rest come from Muirhead.

Remark. More elegant approach is to use Cauchy in the form

$$\frac{1}{x^5+y^2+z^2} \le \frac{x^{-1}+y^2+z^2}{(x^2+y^2+z^2)^2}.$$

§2 Solutions to Day 2

§2.1 IMO 2005/4

Available online at https://aops.com/community/p282138.

Problem statement

Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \quad n \geq 1.$$

The answer is 1 only (which works).

It suffices to show there are no primes. For the primes $p = 2$ and $p = 3$, take $a_2 = 48$. For any prime $p \geq 5$ notice that

$$\begin{aligned} a_{p-2} &= 2^{p-2} + 3^{p-2} + 6^{p-2} - 1 \\ &\equiv \frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 \pmod{p} \\ &\equiv 0 \pmod{p} \end{aligned}$$

so no other larger prime works.

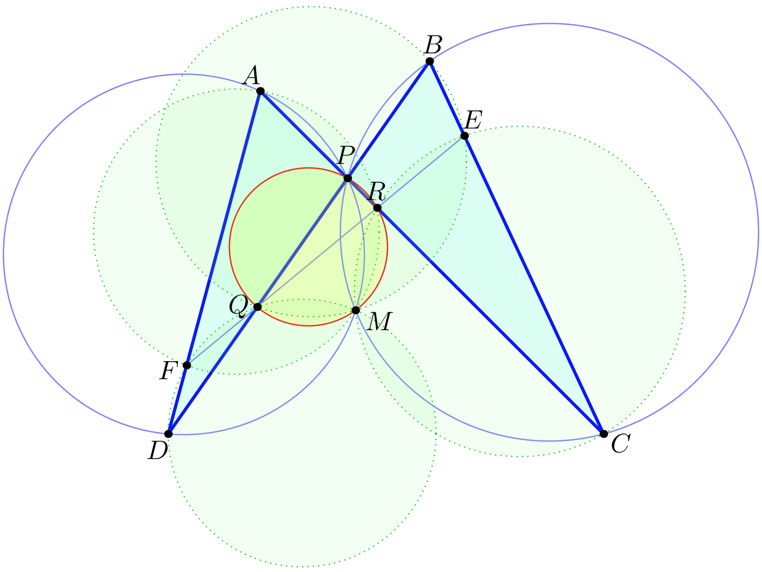
§2.2 IMO 2005/5

Available online at https://aops.com/community/p282140.

Problem statement

Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and $\overline{BC} \nparallel \overline{DA}$. Let two variable points E and F lie on the sides BC and DA, respectively, and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

Let M be the Miquel point of complete quadrilateral $ADBC$; in other words, let M be the second intersection point of the circumcircles of $\triangle APD$ and $\triangle BPC$. (A good diagram should betray this secret; all the points are given in the picture.) This makes lots of sense since we know E and F will be sent to each other under the spiral similarity too.



Thus M is the Miquel point of complete quadrilateral $FACE$. As $R = \overline{FE} \cap \overline{AC}$ we deduce $FARM$ is a cyclic quadrilateral (among many others, but we'll only need one).

Now look at complete quadrilateral $AFQP$. Since M lies on (DFQ) and (RAF), it follows that M is in fact the Miquel point of $AFQP$ as well. So M lies on (PQR).

Thus M is the fixed point that we wanted.

Remark. Naturally, the congruent length condition can be relaxed to $DF/DA = BE/BC$.

§2.3 IMO 2005/6, proposed by Radu Gologan, Dan Schwartz

Available online at https://aops.com/community/p282141.

Problem statement

In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $\frac{2}{5}$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

Assume not and at most one contestant solved five problems. By adding in solves, we can assume WLOG that one contestant solved problems one through five, and every other contestant solved four of the six problems.

We split the remaining contestants based on whether they solved P6. Let a_i denote the number of contestants who solved $\{1, 2, \dots, 5\} \setminus \{i\}$ (and missed P6). Let b_{ij} denote the number of contestants who solved $\{1, 2, \dots, 5, 6\} \setminus \{i, j\}$, for $1 \le i < j \le 5$ (thus in particular they solved P6). Thus

$$n = 1 + \sum_{1 \le i \le 5} a_i + \sum_{1 \le i < j \le 5} b_{ij}$$

denotes the total number of contestants.

Considering contestants who solved P1/P6 we have

$$t_1 := b_{23} + b_{24} + b_{25} + b_{34} + b_{35} + b_{45} \ge \frac{2}{5}n + \frac{1}{5}$$

and we similarly define t_2, t_3, t_4, t_5. (We have written $\frac{2}{5}n + \frac{1}{5}$ since we know the left-hand side is an integer strictly larger than $\frac{2}{5}n$.) Also, by considering contestants who solved P1/P2 we have

$$t_{12} = 1 + a_3 + a_4 + a_5 + b_{34} + b_{35} + b_{45} \ge \frac{2}{5}n + \frac{1}{5}$$

and we similarly define t_{ij} for $1 \le i < j \le 5$.

Claim — The number $\frac{2n+1}{5}$ is equal to some integer k, fourteen of the t's are equal to k, and the last one is equal to $k + 1$.

Proof. First, summing all fifteen equations gives

$$\begin{aligned} 6n + 4 = 10 + 6(n-1) &= 10 + \sum_{1 \le i \le 5} 6a_i + \sum_{1 \le i < j \le 5} 6b_{ij} \\ &= \sum_{1 \le i \le 5} t_i + \sum_{1 \le i < j \le 5} t_{ij}. \end{aligned}$$

Thus the sum of the 15 t's is $6n+4$. But since all the t's are integers at least $\frac{2n+1}{5} = \frac{6n+3}{15}$, the conclusion follows. $\square$

However, we will also manipulate the equations to get the following.

Claim — We have

$$t_{45} \equiv 1 + t_1 + t_2 + t_3 + t_{12} + t_{23} + t_{31} \pmod 3.$$

Proof. This follows directly by computing the coefficient of the a's and b's. We will nonetheless write out a derivation of this equation, to motivate it, but the proof stands without it.

Let $B = \sum_{1 \le i < j \le 5} b_{ij}$ be the sum of all b's. First, note that

$$\begin{aligned} t_1 + t_2 &= B + b_{34} + b_{45} + b_{35} - b_{12} \\ &= B + (t_{12} - 1 - a_3 - a_4 - a_5) - b_{12} \\ \implies b_{12} &= B - (t_1 + t_2) + t_{12} - 1 - (a_3 + a_4 + a_5). \end{aligned}$$

This means we have more or less solved for each b_{ij} in terms of only t and a variables. Now

$$\begin{aligned} t_{45} &= 1 + a_1 + a_2 + a_3 + b_{12} + b_{23} + b_{31} \\ &= 1 + a_1 + a_2 + a_3 \\ &\quad + [B - (t_1 + t_2) + t_{12} - 1 - (a_3 + a_4 + a_5)] \\ &\quad + [B - (t_2 + t_3) + t_{23} - 1 - (a_1 + a_4 + a_5)] \\ &\quad + [B - (t_3 + t_1) + t_{13} - 1 - (a_2 + a_4 + a_5)] \\ &\equiv 1 + t_1 + t_2 + t_3 + t_{12} + t_{23} + t_{31} \pmod 3 \end{aligned}$$

as desired. □

However, we now show the two claims are incompatible (and this is easy, many ways to do this). There are two cases.

- Say $t_5 = k + 1$ and the others are k. Then the equation for t_{45} gives that $k \equiv 6k + 1 \pmod 3$. But now the equation for t_{12} give $k \equiv 6k \pmod 3$.
- Say $t_{45} = k + 1$ and the others are k. Then the equation for t_{45} gives that $k + 1 \equiv 6k \pmod 3$. But now the equation for t_{12} give $k \equiv 6k + 1 \pmod 3$.

Remark. It is significantly easier to prove that there is at least one contestant who solved five problems. One can see it by dropping the $+10$ in the proof of the claim, and arrives at a contradiction. In this situation it is not even necessary to set up the many a and b variables; just note that the expected number of contestants solving any particular pair of problems is $\frac{\binom{4}{2} n}{\binom{6}{2}} = \frac{2}{5} n$.

The fact that $\frac{2n+1}{5}$ should be an integer also follows quickly, since if not one can improve the bound to $\frac{2n+2}{5}$ and quickly run into a contradiction. Again one can get here without setting up a and b.

The main difficulty seems to be the precision required in order to nail down the second 5-problem solve.

Remark. The second claim may look miraculous, but the proof shows that it is not too unnatural to consider $t_1 + t_2 - t_{12}$ to isolate b_{12} in terms of a's and t's. The main trick is: why mod 3?

The reason is that if one looks closely, for a fixed k we have a system of 15 equations in 15 variables. Unless the determinant D of that system happens to be zero, this means there will be a rational solution in a and b, whose denominators are bounded by D. However if

$p \mid D$ then we may conceivably run into mod p issues.

This motivates the choice $p = 3$, since it is easy to see the determinant is divisible by 3, since constant shifts of $\vec{a}$ and $\vec{b}$ are also solutions mod 3. (The choice $p = 2$ is a possible guess as well for this reason, but the problem seems to have better 3-symmetry.)

IMO 2006 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

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Corrections and comments are welcome!

Contents

§0 Problems

1. Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies
$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$
Show that $AP \geq AI$ and that equality holds if and only if $P = I$.

2. Let P be a regular 2006-gon. A diagonal is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*. Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

3. Determine the least real number M such that the inequality
$$\left|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)\right| \leq M\left(a^2+b^2+c^2\right)^2$$
holds for all real numbers a, b and c.

4. Determine all pairs (x, y) of integers such that
$$1 + 2^x + 2^{2x+1} = y^2.$$

5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial
$$Q(x) = P(P(\dots P(P(x))\dots))$$
where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

6. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

§1 Solutions to Day 1

§1.1 IMO 2006/1

Available online at https://aops.com/community/p571966.

Problem statement

Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$ and that equality holds if and only if $P = I$.

The condition rewrites as

$$\angle PBC + \angle PCB = (\angle B - \angle PBC) + (\angle C - \angle PCB) \implies \angle PBC + \angle PCB = \frac{\angle B + \angle C}{2}$$

which means that

$$\angle BPC = 180^\circ - \frac{\angle B + \angle C}{2} = 90^\circ + \frac{\angle A}{2} = \angle BIC.$$

Since P and I are both inside $\triangle ABC$ that implies P lies on the circumcircle of $\triangle BIC$.

It's well-known (by "Fact 5") that the circumcenter of $\triangle BIC$ is the arc midpoint M of $\widehat{BC}$. Therefore

$$AI + IM = AM \leq AP + PM \implies AI \leq AP$$

with equality holding iff A, P, M are collinear, or $P = I$.

§1.2 IMO 2006/2

Available online at https://aops.com/community/p571973.

Problem statement

Let P be a regular 2006-gon. A diagonal is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*. Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Call a triangle with the desired property *special*. We prove the maximum number of special triangles is 1003, achieved by paring up the sides of the polygon.

We present two solutions for the upper bound. Both of them rely first on two geometric notes:

- In a special triangle, the good sides are congruent (and not congruent to the third side).
- No two isosceles triangles share a good side.

Solution using bijections: Call a good diagonal **special** if it's part of a special triangle; special diagonals come in pairs. Consider the minor arc cut out by a special diagonal d, which has an odd number of sides. Since special diagonals come in pairs, one can associate to d a side of the polygon not covered by any special diagonals from d. Hence there are at most 2006 special diagonals, so at most 1003 special triangles.

Solution using graph theory: Consider the tree T formed by the 2004 triangles in the dissection, with obvious adjacency. Let F be the forest obtained by deleting any edge corresponding to a good diagonal. Then the resulting graph F has only degrees 1 and 3, with special triangles only occurring at degree 1 vertices.

If there are k good diagonals drawn, then this forest consists of $k+1$ trees. A tree with n_i vertices $(0 \le i \le k)$ consequently has $\frac{n_i+2}{2}$ leaves. However by the earlier remark at least k leaves don't give special triangles (one on each side of a special diagonal); so the number of leaves that do give good triangles is at most

$$-k + \sum_i \frac{n_i+2}{2} = -k + \frac{2004 + 2(k+1)}{2} = 1003.$$

§1.3 IMO 2006/3

Available online at https://aops.com/community/p571945.

Problem statement

Determine the least real number M such that the inequality

$$\left|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)\right| \le M\left(a^2+b^2+c^2\right)^2$$

holds for all real numbers a, b and c.

It's the same as

$$|(a-b)(b-c)(c-a)(a+b+c)| \le M\left(a^2+b^2+c^2\right)^2.$$

Let $x=a-b$, $y=b-c$, $z=c-a$, $s=a+b+c$. Then we want to have

$$|xyzs| \le \frac{M}{9}(x^2+y^2+z^2+s^2)^2.$$

Here $x+y+z=0$.

Now if x and y have the same sign, we can replace them with the average (this increases the LHS and decreases RHS). So we can have $x=y$, $z=-2x$. Now WLOG $x>0$ to get

$$2x^3\cdot s \le \frac{M}{9}\left(6x^2+s^2\right)^2.$$

After this routine calculation gives $M=\frac{9}{32}\sqrt{2}$ works and is optimal (by $6x^2+s^2=2x^2+2x^2+2x^2+s^2$ and AM-GM).

§2 Solutions to Day 2

§2.1 IMO 2006/4, proposed by Zuming Feng (USA)

Available online at https://aops.com/community/p572815.

Problem statement

Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Answers: $(0, \pm 2)$, $(4, \pm 23)$, which work.

Assume $x \geq 4$.

$$2^x \left(1 + 2^{x+1}\right) = 2^x + 2^{2x+1} = y^2 - 1 = (y-1)(y+1).$$

So either:

- $y = 2^{x-1}m + 1$ for some odd m, so

$$1 + 2^{x+1} = m\left(2^{x-2}m + 1\right) \implies 2^x = \frac{4(1-m)}{m^2 - 8}.$$

- $y = 2^{x-1}m - 1$ for some odd m, so

$$1 + 2^{x+1} = m\left(2^{x-2}m - 1\right) \implies 2^x = \frac{4(1+m)}{m^2 - 8}.$$

In particular we need $4|1 \pm m| \geq 2^4|m^2 - 8|$, which is enough to imply $m < 5$. From here easily recover $x = 4$, $m = 3$ as the last solution (in the second case).

§2.2 IMO 2006/5

Available online at https://aops.com/community/p572821.

Problem statement

Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial

$$Q(x) = P(P(\dots P(P(x)) \dots))$$

where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

First, we prove that:

Claim (Putnam 2000 et al) — If a number is periodic under P then in fact it's fixed by $P \circ P$.

Proof. Let x_1, x_2, …, x_n be a minimal orbit. Then

$$x_i - x_{i+1} \mid P(x_i) - P(x_{i+1}) = x_{i+1} - x_{i+2}$$

and so on cyclically.

If any of the quantities are zero we are done. Else, we must eventually have $x_i - x_{i+1} = -(x_{i+1} - x_{i+2})$, so $x_i = x_{i+2}$ and we get 2-periodicity. □

The tricky part is to study the 2-orbits. Suppose there exists a fixed pair $u \neq v$ with $P(u) = v$, $P(v) = u$. (If no such pair exists, we are already done.) Let (a, b) be any other pair with $P(a) = b$, $P(b) = a$, possibly even $a = b$, but $\{a, b\} \cap \{u, v\} = \varnothing$. Then we should have

$$u - a \mid P(u) - P(a) = v - b \mid P(v) - P(b) = u - a$$

and so $u - a$ and $v - b$ divide each other (and are nonzero). Similarly, $u - b$ and $v - a$ divide each other.

Hence $u - a = \pm(v - b)$ and $u - b = \pm(v - a)$. We consider all four cases:

- If $u - a = v - b$ and $u - b = v - a$ then $u - v = b - a = a - b$, contradiction.
- If $u - a = -(v - b)$ and $u - b = -(v - a)$ then $u + v = u - v = a + b$.
- If $u - a = -(v - b)$ and $u - b = v - a$, we get $a + b = u + v$ from the first one (discarding the second).
- If $u - a = v - b$ and $u - b = -(v - a)$, we get $a + b = u + v$ from the second one (discarding the first one).

Thus in all possible situations we have

$$a + b = c \coloneqq u + v$$

a fixed constant.

Therefore, any pair (a, b) with $P(a) = b$ and $P(b) = a$ actually satisfies $P(a) = c - a$. And since $\deg P > 1$, this means there are at most n roots to $a + P(a) = c$, as needed.

§2.3 IMO 2006/6

Available online at https://aops.com/community/p572824.

Problem statement

Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

We say a polygon in *almost convex* if all its angles are at most 180°.

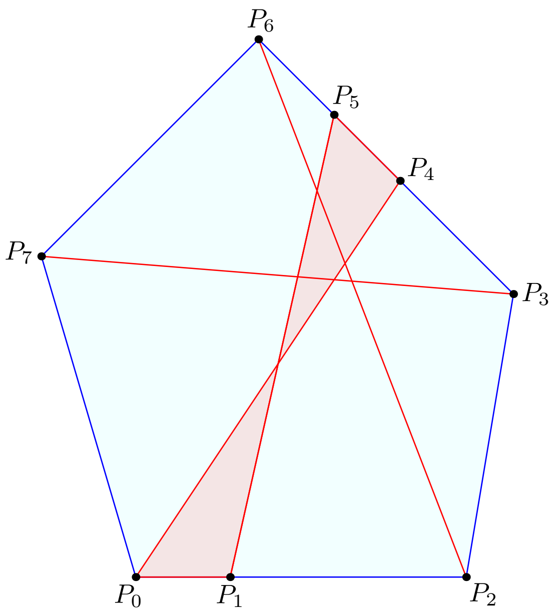
Note that given any convex or almost convex polygon, we can take any side b and add another vertex on it, and the sum of the labels doesn't change (since the label of a side is the length of the side times the distance of the farthest point).

Lemma

Let N be an even integer. Then any almost convex N-gon with area S should have an inscribed triangle with area at least $2S/N$.

The main work is the proof of the lemma.

Label the polygon $P_0P_1 \dots P_{N-1}$. Consider the $N/2$ major diagonals of the almost convex N-gon, $P_0P_{N/2}$, $P_1P_{N/2+1}$, et cetera. A *butterfly* refers to a self-intersecting quadrilateral $P_iP_{i+1}P_{i+1+N/2}P_{i+N/2}$. An example of a butterfly is shown below for $N = 8$.



Claim — Every point X in the polygon is contained in the wingspan of some butterfly.

Proof. Consider a windmill-like process which

- starts from some oriented red line $P_0P_{N/2}$, oriented to face $P_0P_{N/2}$

- rotates through $P_0P_{N/2} \cap P_1P_{N/2+1}$ to get line $P_1P_{N/2+1}$,
- rotates through $P_1P_{N/2+1} \cap P_2P_{N/2+2}$ to get line $P_2P_{N/2+2}$,
- ...et cetera, until returning to line $P_{N/2}P_0$, but in the reverse orientation.

At the end of the process, every point in the plane has switched sides with our moving line. The moment that X crosses the moving red line, we get it contained in a butterfly, as needed. □

Claim — If $ABDC = P_iP_{i+1}P_{i+1+N/2}P_{i+N/2}$ is a butterfly, one of the triangles ABC, BCD, CDA, DAB has area at least that of the butterfly.

Proof. Let the diagonals of the butterfly meet at O, and let $a = AO$, $b = BO$, $c = CO$, $d = DO$. If we assume WLOG $d = \min(a, b, c, d)$ then it follows $[ABC] = [AOB] + [BOC] \geq [AOB] + [COD]$, as needed. □

Now, since the $N/2$ butterflies cover an area of S, it follows that one of the butterflies has area at least $S/(N/2) = 2S/N$, and so that butterfly gives a triangle with area at least $2S/N$, completing the proof of the lemma.

Main proof: Let $a_1, \dots, a_n$ be the numbers assigned to the sides. Assume for contradiction $a_1 + \dots + a_n < 2S$. We pick even integers $m_1, m_2, \dots, m_n$ such that

$$\begin{aligned} \frac{a_1}{S} &< \frac{2m_1}{m_1 + \dots + m_n} \\ \frac{a_2}{S} &< \frac{2m_2}{m_1 + \dots + m_n} \\ &\vdots \\ \frac{a_n}{S} &< \frac{2m_n}{m_1 + \dots + m_n}. \end{aligned}$$

which is possible by rational approximation, since the right-hand sides sum to 2 and the left-hand sides sum to strictly less than 2.

Now we break every side of P into m_i equal parts to get an almost convex N-gon, where $N = m_1 + \dots + m_n$.

The main lemma then gives us a triangle Δ of the almost convex N-gon which has area at least $\frac{2S}{N}$. If Δ used the ith side then it then follows the label a_i on that side should be at least $m_i \cdot \frac{2S}{N}$, contradiction.

IMO 2007 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

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§0 Problems

1. Real numbers $a_1, a_2, \dots, a_n$ are fixed. For each $1 \le i \le n$ we let $d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : i \le j \le n\}$ and let $d = \max\{d_i : 1 \le i \le n\}$.

   (a) Prove that for any real numbers $x_1 \le \dots \le x_n$ we have
   $$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \frac{1}{2} d.$$

   (b) Moreover, show that there exists some choice of $x_1 \le \dots \le x_n$ which achieves equality.

2. Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

3. In a mathematical competition some competitors are (mutual) friends. Call a group of competitors a *clique* if each two of them are friends. Given that the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

4. In triangle ABC the bisector of $\angle BCA$ meets the circumcircle again at R, the perpendicular bisector of $\overline{BC}$ at P, and the perpendicular bisector of $\overline{AC}$ at Q. The midpoint of $\overline{BC}$ is K and the midpoint of $\overline{AC}$ is L. Prove that the triangles RPK and RQL have the same area.

5. Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

6. Let n be a positive integer. Consider
   $$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \dots, n\},\ x + y + z > 0\}$$
   as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

§1 Solutions to Day 1

§1.1 IMO 2007/1

Available online at https://aops.com/community/p893741.

Problem statement

Real numbers $a_1, a_2, \dots, a_n$ are fixed. For each $1 \le i \le n$ we let $d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : i \le j \le n\}$ and let $d = \max\{d_i : 1 \le i \le n\}$.

(a) Prove that for any real numbers $x_1 \le \dots \le x_n$ we have

$$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \frac{1}{2}d.$$

(b) Moreover, show that there exists some choice of $x_1 \le \dots \le x_n$ which achieves equality.

Note that we can dispense of d_i immediately by realizing that the definition of d just says

$$d = \max_{1 \le i \le j \le n} (a_i - a_j).$$

If $a_1 \le \dots \le a_n$ are already nondecreasing then $d = 0$ and there is nothing to prove (for the equality case, just let $x_i = a_i$), so we will no longer consider this case.

Otherwise, consider any indices $i < j$ with $a_i > a_j$. We first prove (a) by applying the following claim with $p = a_i$ and $q = a_j$:

Claim — For any $p \le q$, we have either $|p - a_i| \ge \frac{1}{2}(a_i - a_j)$ or $|q - a_j| \ge \frac{1}{2}(a_i - a_j)$.

Proof. Assume for contradiction both are false. Then $p > a_i - \frac{1}{2}(a_i - a_j) = a_j + \frac{1}{2}(a_i - a_j) > q$, contradiction. $\square$

As for (b), we let $i < j$ be any indices for which $a_i - a_j = d > 0$ achieves the maximal difference. We then define $x_\bullet$ in three steps:

- We set $x_k = \frac{a_i + a_j}{2}$ for $k = i, \dots, j$.
- We recursively set $x_k = \max(x_{k-1}, a_k)$ for $k = j+1, j+2, \dots$.
- We recursively set $x_k = \min(x_{k+1}, a_k)$ for $k = i-1, i-2, \dots$.

By definition, these $x_\bullet$ are weakly increasing. To prove this satisfies (b) we only need to check that

$$|x_k - a_k| \le \frac{a_i - a_j}{2} \qquad (\star)$$

for any index k (as equality holds for $k = i$ or $k = j$).

We note $(\star)$ holds for $i < k < j$ by construction. For $k > j$, note that $x_k \in \{a_j, a_{j+1}, \dots, a_k\}$ by construction, so $(\star)$ follows from our choice of i and j giving the largest possible difference; the case $k < i$ is similar.

§1.2 IMO 2007/2, proposed by Charles Leytem (LUX)

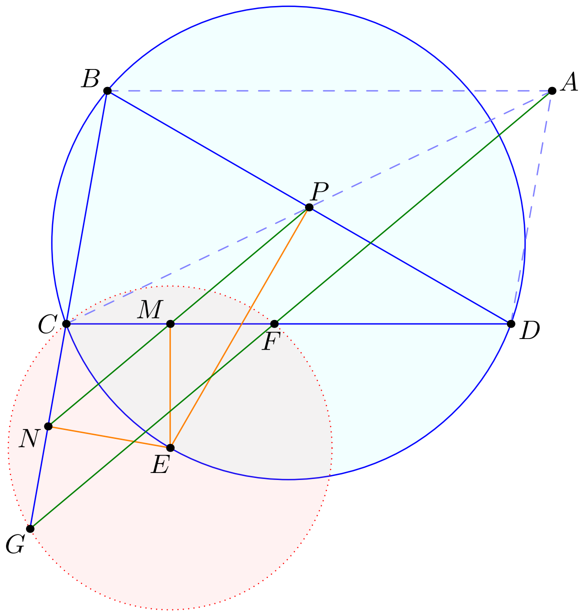
Available online at https://aops.com/community/p893744.

Problem statement

Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

Let M, N, P denote the midpoints of $\overline{CF}$, $\overline{CG}$, $\overline{AC}$ (noting P is also the midpoint of $\overline{BD}$).

By a homothety at C with ratio $\frac{1}{2}$, we find $\overline{MNP}$ is the image of line $\ell \equiv \overline{AGF}$.



However, since we also have $\overline{EM} \perp \overline{CF}$ and $\overline{EN} \perp \overline{CG}$ (from $EF = EG = EC$) we conclude $\overline{PMN}$ is the Simson line of E with respect to $\triangle BCD$, which implies $\overline{EP} \perp \overline{BD}$. In other words, $\overline{EP}$ is the perpendicular bisector of $\overline{BD}$, so E is the midpoint of arc $\widehat{BCD}$.

Finally,

$$\begin{aligned} \measuredangle(\overline{AB}, \ell) &= \measuredangle(\overline{CD}, \overline{MNP}) = \measuredangle CMN = \measuredangle CEN \\ &= 90^\circ - \measuredangle NCE = 90^\circ + \measuredangle ECB \end{aligned}$$

which means that ℓ is parallel to a bisector of $\angle BCD$, and hence to one of $\angle BAD$. (Moreover since F lies on the interior of $\overline{CD}$, it is actually the internal bisector)

§1.3 IMO 2007/3, proposed by Vasily Astakhov (RUS)

Available online at https://aops.com/community/p893746.

Problem statement

In a mathematical competition some competitors are (mutual) friends. Call a group of competitors a *clique* if each two of them are friends. Given that the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Take the obvious graph interpretation G. We paint red any vertices in one of the maximal cliques K, which we assume has $2r$ vertices, and paint the remaining vertices green. We let $\alpha(\bullet)$ denote the clique number.

Initially, let the two rooms $A = K$, $B = G - K$.

Claim — We can move at most r vertices of A into B to arrive at $\alpha(A) \le \alpha(B) \le \alpha(A) + 1$.

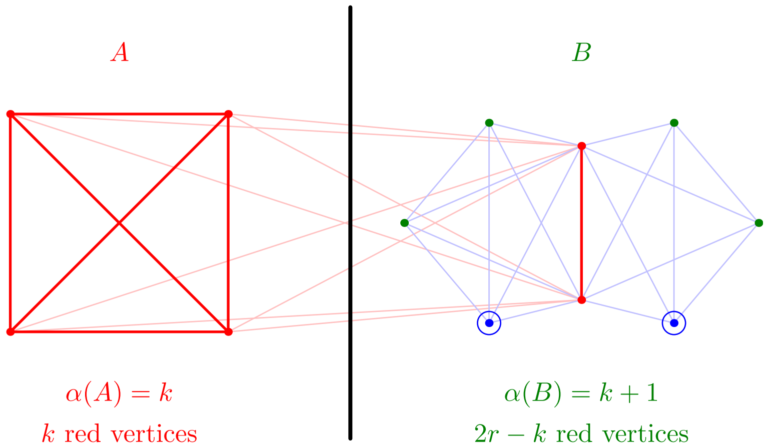
Proof. This is actually obvious by discrete continuity. We move one vertex at a time, noting $\alpha(A)$ decreases by one at each step, while $\alpha(B)$ increases by either zero or one at each step.

We stop once $\alpha(B) \ge \alpha(A)$, which happens before we have moved r vertices (since then we have $\alpha(B) \ge r = \alpha(A)$). The conclusion follows. □

So let's consider the situation

$$\alpha(A) = k \ge r \qquad \text{and} \qquad \alpha(B) = k + 1.$$

At this point A is a set of k red vertices, while B has the remaining $2r - k$ red vertices (and all the green ones). An example is shown below with $k = 4$ and $2r = 6$.



Now, if we can move any red vertex from B back to A without changing the clique number of B, we do so, and win.

Otherwise, it must be the case that *every* $(k+1)$-clique in B uses *every* red vertex in B. For each $(k+1)$-clique in B (in arbitrary order), we do the following procedure.

- If all $k+1$ vertices are still green, pick one and re-color it blue. This is possible since $k+1 > 2r-k$.

- Otherwise, do nothing.

Then we move all the blue vertices from B to A, one at a time, in the same order we re-colored them. This forcibly decreases the clique number of B to k, since the clique number is $k+1$ just before the last blue vertex is moved, and strictly less than $k+1$ (hence equal to k) immediately after that.

Claim — After this, $\alpha(A) = k$ still holds.

Proof. Assume not, and we have a $(k+1)$-clique which uses b blue vertices and $(k+1)-b$ red vertices in A. Together with the $2r-k$ red vertices already in B we then get a clique of size

$$b + ((k+1-b)) + (2r-k) = 2r+1$$

which is a contradiction. □

Remark. Dragomir Grozev posted the following motivation on his blog:

> I think, it's a natural idea to place all students in one room and begin moving them one by one into the other one. Then the max size of the cliques in the first and second room increase (resp. decrease) at most with one. So, there would be a moment both sizes are almost the same. At that moment we may adjust something.
>
> Trying the idea, I had some difficulties keeping track of the maximal cliques in the both rooms. It seemed easier all the students in one of the rooms to comprise a clique. It could be achieved by moving only the members of the maximal clique. Following this path the remaining obstacles can be overcome naturally.

§2 Solutions to Day 2

§2.1 IMO 2007/4, proposed by Marek Pechal (CZE)

Available online at https://aops.com/community/p894655.

Problem statement

In triangle ABC the bisector of $\angle BCA$ meets the circumcircle again at R, the perpendicular bisector of $\overline{BC}$ at P, and the perpendicular bisector of $\overline{AC}$ at Q. The midpoint of $\overline{BC}$ is K and the midpoint of $\overline{AC}$ is L. Prove that the triangles RPK and RQL have the same area.

We first begin by proving the following claim.

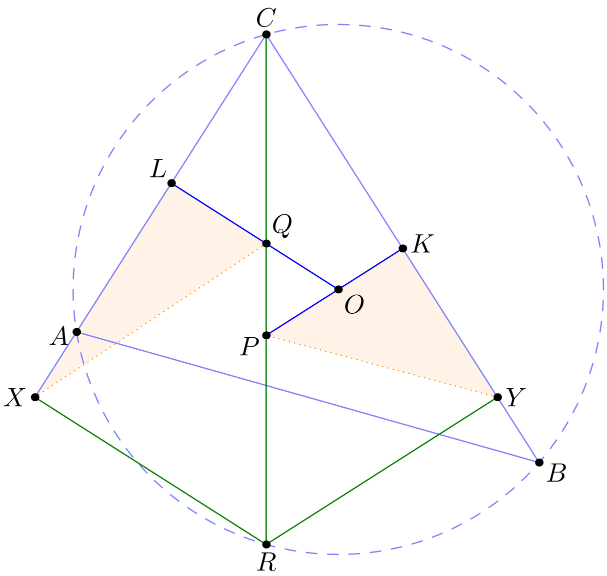
Claim — We have $CQ = PR$ (equivalently, $CP = QR$).

Proof. Let $O = \overline{LQ} \cap \overline{KP}$ be the circumcenter. Then

$$\measuredangle OPQ = \measuredangle KPC = 90^\circ - \measuredangle PCK = 90^\circ - \measuredangle LCQ = \measuredangle\measuredangle CQL = \measuredangle PQO.$$

Thus $OP = OQ$. Since $OC = OR$ as well, we get the conclusion. $\square$

Denote by X and Y the feet from R to $\overline{CA}$ and $\overline{CB}$, so $\triangle CXR \cong \triangle CYR$. Then, let $t = \frac{CQ}{CR} = 1 - \frac{CP}{CR}$.



Then it follows that

$$[RQL] = [XQL] = t(1-t) \cdot [XRC] = t(1-t) \cdot [YCR] = [YKP] = [RKP]$$

as needed.

Remark. Trigonometric approaches are very possible (and easier to find) as well: both areas work out to be $\frac{1}{8}ab \tan \frac{1}{2}C$.

§2.2 IMO 2007/5, proposed by Kevin Buzzard, Edward Crane (UNK)

Available online at https://aops.com/community/p894656.

Problem statement

Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

As usual,

$$4ab-1 \mid (4a^2-1)^2 \iff 4ab-1 \mid (4ab \cdot a - b)^2 \iff 4ab-1 \mid (a-b)^2.$$

Then we use a typical Vieta jumping argument. Define

$$k = \frac{(a-b)^2}{4ab-1}.$$

Note that $k=0 \iff a=b$. So we will prove that $k>0$ leads to a contradiction.

Indeed, suppose (a,b) is a minimal solution with $a>b$ (we have $a \neq b$ since $k \neq 0$). By Vieta jumping, $(b, \frac{b^2+k}{a})$ is also such a solution. But now

$$\begin{aligned} \frac{b^2+k}{a} \ge a &\implies k \ge a^2 - b^2 \\ &\implies \frac{(a-b)^2}{4ab-1} \ge a^2-b^2 \\ &\implies a-b \ge (4ab-1)(a+b) \end{aligned}$$

which is absurd for $a,b \in \mathbb{Z}_{>0}$. (In the last step we divided by $a-b>0$.)

§2.3 IMO 2007/6

Available online at https://aops.com/community/p894658.

Problem statement

Let n be a positive integer. Consider

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \dots, n\},\ x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

The answer is $3n$. Here are two examples of constructions with $3n$ planes:

- $x + y + z = i$ for $i = 1, \dots, 3n$.
- $x = i$, $y = i$, $z = i$ for $i = 1, \dots, n$.

Suppose for contradiction we have $N < 3n$ planes. Let them be $a_i x + b_i y + c_i z + 1 = 0$, for $i = 1, \dots, N$. Define the polynomials

$$A(x, y, z) = \prod_{i=1}^{n} (x - i) \prod_{i=1}^{n} (y - i) \prod_{i=1}^{n} (z - i)$$
$$B(x, y, z) = \prod_{i=1}^{N} (a_i x + b_i y + c_i z + 1).$$

Note that $A(0, 0, 0) = (-1)^n (n!)^3 \neq 0$ and $B(0, 0, 0) = 1 \neq 0$, but $A(x, y, z) = B(x, y, z) = 0$ for any $(x, y, z) \in S$. Also, the coefficient of $x^n y^n z^n$ in A is 1, while the coefficient of $x^n y^n z^n$ in B is 0.

Now, define

$$P(x, y, z) := A(x, y, z) - \lambda B(x, y, z).$$

where $\lambda = \frac{A(0,0,0)}{B(0,0,0)} = (-1)^n (n!)^3$. We now have that

- $P(x, y, z) = 0$ for any $x, y, z \in \{0, 1, \dots, n\}^3$.
- But the coefficient of $x^n y^n z^n$ is 1.

This is a contradiction to Alon's combinatorial nullstellensatz.

IMO 2008 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2008 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, C_2 are concyclic.

2. Let x, y, z be real numbers with $xyz = 1$, all different from 1. Prove that
$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \ge 1$$
and show that equality holds for infinitely many choices of rational numbers x, y, z.

3. Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime factor greater than $2n + \sqrt{2n}$.

4. Find all functions f from the positive reals to the positive reals such that
$$\frac{f(w)^2 + f(x)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$
for all positive real numbers w, x, y, z satisfying $wx = yz$.

5. Let n and k be positive integers with $k \ge n$ and $k - n$ an even number. There are $2n$ lamps labelled 1, 2, ..., $2n$ each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on). Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off. Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on. Determine $\frac{N}{M}$.

6. Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

§1 Solutions to Day 1

§1.1 IMO 2008/1

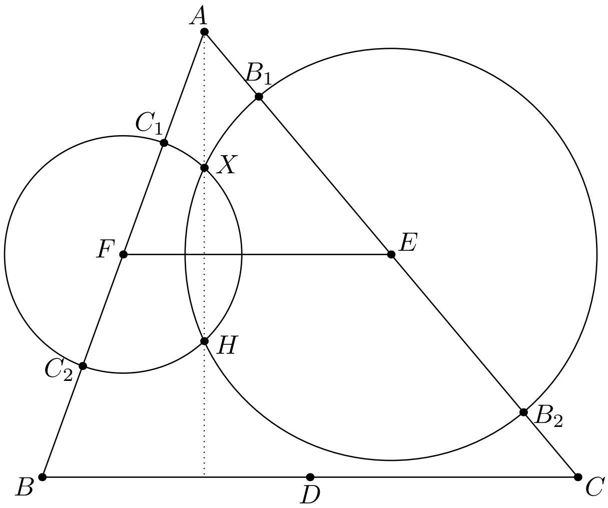
Available online at https://aops.com/community/p1190553.

Problem statement

Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, C_2 are concyclic.

Let D, E, F be the centers of Γ_A, Γ_B, Γ_C (in other words, the midpoints of the sides).

We first show that B_1, B_2, C_1, C_2 are concyclic. It suffices to prove that A lies on the radical axis of the circles Γ_B and Γ_C.



Let X be the second intersection of Γ_B and Γ_C. Clearly $\overline{XH}$ is perpendicular to the line joining the centers of the circles, namely $\overline{EF}$. But $\overline{EF} \parallel \overline{BC}$, so $\overline{XH} \perp \overline{BC}$. Since $\overline{AH} \perp \overline{BC}$ as well, we find that A, X, H are collinear, as needed.

Thus, B_1, B_2, C_1, C_2 are concyclic. Similarly, C_1, C_2, A_1, A_2 are concyclic, as are A_1, A_2, B_1, B_2. Now if any two of these three circles coincide, we are done; else the pairwise radical axii are not concurrent, contradiction. (Alternatively, one can argue directly that O is the center of all three circles, by taking the perpendicular bisectors.)

§1.2 IMO 2008/2

Available online at https://aops.com/community/p1190551.

Problem statement

Let x, y, z be real numbers with $xyz = 1$, all different from 1. Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \ge 1$$

and show that equality holds for infinitely many choices of rational numbers x, y, z.

Let $x = a/b$, $y = b/c$, $z = c/a$, so we want to show

$$\left(\frac{a}{a-b}\right)^2 + \left(\frac{b}{b-c}\right)^2 + \left(\frac{c}{c-a}\right)^2 \ge 1.$$

A very boring computation shows this is equivalent to

$$\frac{(a^2b + b^2c + c^2a - 3abc)^2}{(a-b)^2(b-c)^2(c-a)^2} \ge 0$$

which proves the inequality (and it is unsurprising we are in such a situation, given that there is an infinite curve of rationals).

For equality, it suffices to show there are infinitely many integer solutions to

$$a^2b + b^2c + c^2a = 3abc \iff \frac{a}{c} + \frac{b}{a} + \frac{c}{a} = 3$$

or equivalently that there are infinitely many rational solutions to

$$u + v + \frac{1}{uv} = 3.$$

For any $0 \ne u \in \mathbb{Q}$ the real solution for u is

$$v = \frac{-u + (u-1)\sqrt{1 - 4/u} + 3}{2}$$

and there are certainly infinitely many rational numbers u for which $1 - 4/u$ is a rational square (say, $u = \frac{-4}{q^2-1}$ for $q \ne \pm 1$ a rational number).

§1.3 IMO 2008/3

Available online at https://aops.com/community/p1190546.

Problem statement

Prove that there are infinitely many positive integers n such that n^2+1 has a prime factor greater than $2n+\sqrt{2n}$.

The idea is to pick the prime p first!

Select any large prime $p \ge 2013$, and let $h = \left\lceil \sqrt{p} \right\rceil$. We will try to find an n such that

$$n \le \frac{1}{2}(p-h) \quad \text{and} \quad p \mid n^2+1.$$

This implies $p \ge 2n + \sqrt{p}$ which is enough to ensure $p \ge 2n + \sqrt{2n}$.

Assume $p \equiv 1 \pmod 8$ henceforth. Then there exists some $\frac{1}{2}p < x < p$ such that $x^2 \equiv -1 \pmod p$, and we set

$$x = \frac{p+1}{2} + t.$$

Claim — We have $t \ge \frac{h-1}{2}$ and hence may take $n = p - x$.

Proof. Assume for contradiction this is false; then

$$\begin{aligned} 0 &\equiv 4(x^2+1) \pmod p \\ &= (p+1+2t)^2 + 4 \\ &\equiv (2t+1)^2 + 4 \pmod p \\ &< h^2 + 4 \end{aligned}$$

So we have that $(2t+1)^2+4$ is positive and divisible by p, yet at most $\left\lceil \sqrt{p} \right\rceil^2 + 4 < 2p$. So it must be the case that $(2t+1)^2 + 4 = p$, but this has no solutions modulo 8. $\square$

§2 Solutions to Day 2

§2.1 IMO 2008/4

Available online at https://aops.com/community/p1191683.

Problem statement

Find all functions f from the positive reals to the positive reals such that

$$\frac{f(w)^2 + f(x)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z satisfying $wx = yz$.

The answers are $f(x) \equiv x$ and $f(x) \equiv 1/x$. These work, so we show they are the only ones.

First, setting (t, t, t, t) gives $f(t^2) = f(t)^2$. In particular, $f(1) = 1$. Next, setting $(t, 1, \sqrt{t}, \sqrt{t})$ gives

$$\frac{f(t)^2 + 1}{2f(t)} = \frac{t^2 + 1}{2t}$$

which as a quadratic implies $f(t) \in \{t, 1/t\}$.

Now assume $f(a) = a$ and $f(b) = 1/b$. Setting $(\sqrt{a}, \sqrt{b}, 1, \sqrt{ab})$ gives

$$\frac{a + 1/b}{f(ab) + 1} = \frac{a + b}{ab + 1}.$$

One can check the two cases on $f(ab)$ each imply $a = 1$ and $b = 1$ respectively. Hence the only answers are those claimed.

§2.2 IMO 2008/5

Available online at https://aops.com/community/p1191679.

Problem statement

Let n and k be positive integers with $k \geq n$ and $k-n$ an even number. There are $2n$ lamps labelled 1, 2, ..., $2n$ each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on). Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off. Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on. Determine $\frac{N}{M}$.

The answer is 2^{k-n}.

Consider the following map Ψ from N-sequences to M-sequences:

- change every instance of $n+1$ to 1;
- change every instance of $n+2$ to 2;

  $\vdots$

- change every instance of $2n$ to n.

(For example, suppose $k=9$, $n=3$; then $144225253 \mapsto 111222223$.)

Clearly this is map is well-defined and surjective. So all that remains is:

Claim — Every M-sequence has exactly 2^{k-n} pre-images under Ψ.

Proof. Indeed, suppose that there are c_1 instances of lamp 1. Then we want to pick an odd subset of the 1's to change to $n+1$'s, so 2^{c_1-1} ways to do this. And so on. Hence the number of pre-images is

$$\prod_i 2^{c_i-1} = 2^{k-n}. \qquad \square$$

§2.3 IMO 2008/6

Available online at https://aops.com/community/p1191671.

Problem statement

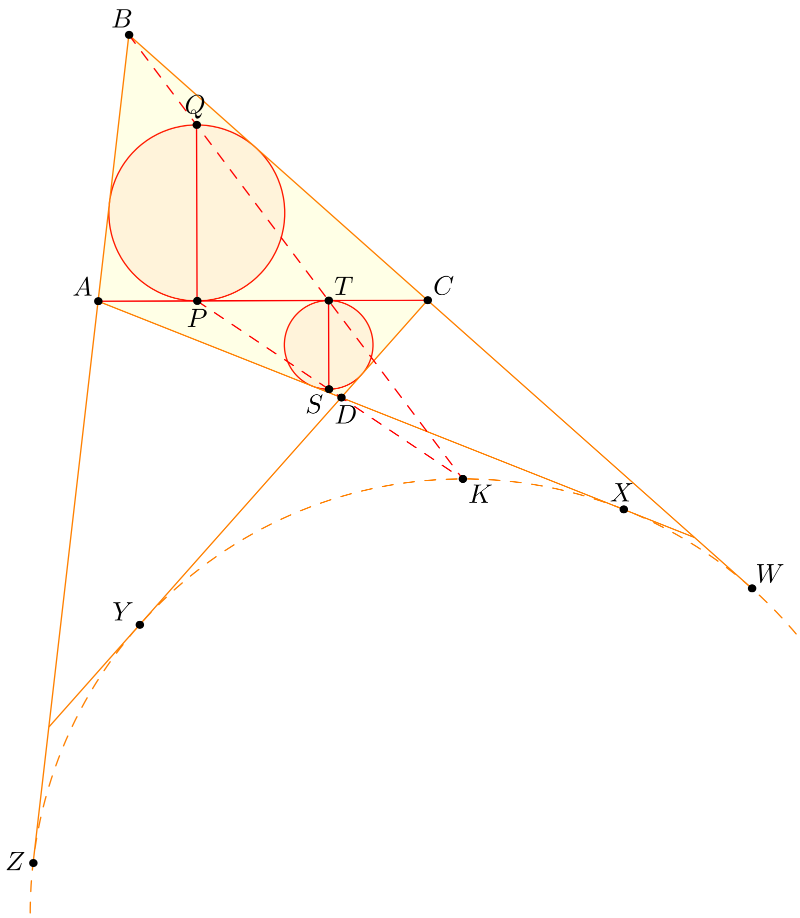
Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

By the external version of Pitot theorem, the existence of ω implies that

$$BA + AD = CB + CD.$$

Let $\overline{PQ}$ and $\overline{ST}$ be diameters of ω_1 and ω_2 with $P, T \in \overline{AC}$. Then the length relation on $ABCD$ implies that P and T are reflections about the midpoint of $\overline{AC}$.

Now orient AC horizontally and let K be the "uppermost" point of ω, as shown.



Consequently, a homothety at B maps Q, T, K to each other (since T is the uppermost of the excircle, Q of the incircle). Similarly, a homothety at D maps P, S, K to each other. As $\overline{PQ}$ and $\overline{ST}$ are parallel diameters it then follows K is the exsimilicenter of ω_1 and ω_2.

IMO 2009 Solution Notes

Evan Chen《陳誼廷》

29 June 2023

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Corrections and comments are welcome!

Contents

§0 Problems

1. Let $n, k \geq 2$ be positive integers and let a_1, a_2, a_3, $\dots$, a_k be distinct integers in the set $\{1, 2, \dots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \dots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

2. Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L M be the midpoints of $\overline{BP}$, $\overline{CQ}$, $\overline{PQ}$, respectively, and let Γ be the circumcircle of $\triangle KLM$. Suppose that $\overline{PQ}$ is tangent to Γ. Prove that $OP = OQ$.

3. Suppose that $s_1, s_2, s_3, \dots$ is a strictly increasing sequence of positive integers such that the sub-sequences $s_{s_1}, s_{s_2}, s_{s_3}, \dots$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$ are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \dots$ is itself an arithmetic progression.

4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

5. Find all functions $f \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for positive integers a and b, the numbers
$$a, \qquad f(b), \qquad f(b + f(a) - 1)$$
are the sides of a non-degenerate triangle.

6. Let a_1, a_2, $\dots$, a_n be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + \dots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths a_1, a_2, $\dots$, a_n in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

§1 Solutions to Day 1

§1.1 IMO 2009/1

Available online at https://aops.com/community/p1561571.

Problem statement

Let $n, k \geq 2$ be positive integers and let $a_1, a_2, a_3, \dots, a_k$ be distinct integers in the set $\{1, 2, \dots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \dots, k - 1$. Prove that n does not divide $a_k(a_1 - 1)$.

We proceed indirectly and assume that

$$a_i(a_{i+1} - 1) \equiv 0 \pmod n$$

for $i = 1, \dots, k$ (indices taken modulo k). We claim that this implies all the a_i are equal modulo n.

Let $q = p^e$ be any prime power dividing n. Then, $a_1(a_2 - 1) \equiv 0 \pmod q$, so p divides either a_1 or a_2.

- If $p \mid a_1$, then $p \nmid a_1 - 1$. Then

  $$a_k(a_1 - 1) \equiv 0 \pmod q \implies a_k \equiv 0 \pmod q.$$

  In particular, $p \mid a_k$. So repeating this argument, we get $a_{k-1} \equiv 0 \pmod q$, $a_{k-2} \equiv 0 \pmod q$, and so on.

- Similarly, if $p \mid a_2 - 1$ then $p \nmid a_2$, and from

  $$a_2(a_3 - 1) \equiv 0 \pmod q \implies a_3 \equiv 1 \pmod q.$$

  In particular, $p \mid a_3 - 1$. So repeating this argument, we get $a_4 \equiv 0 \pmod q$, $a_5 \equiv 0 \pmod q$, and so on.

Either way, we find $a_i \pmod q$ is constant (and either 0 or 1).

Since q was an arbitrary prime power dividing n, by Chinese remainder theorem we conclude that $a_i \pmod n$ is constant as well. But this contradicts the assumption of distinctness.

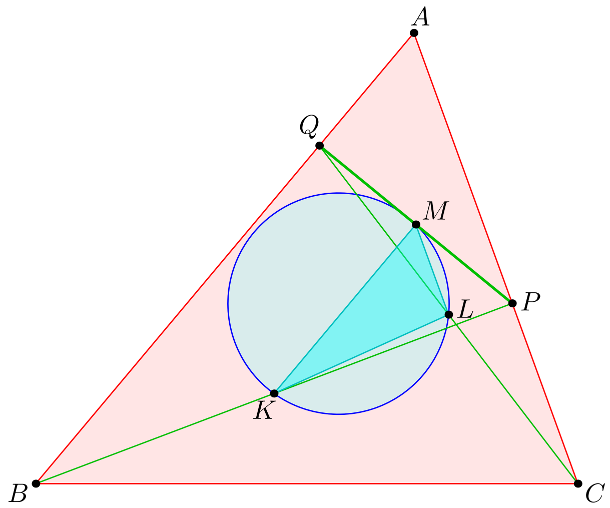
§1.2 IMO 2009/2

Available online at https://aops.com/community/p1561572.

Problem statement

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L M be the midpoints of $\overline{BP}$, $\overline{CQ}$, $\overline{PQ}$, respectively, and let Γ be the circumcircle of $\triangle KLM$. Suppose that $\overline{PQ}$ is tangent to Γ. Prove that $OP = OQ$.

By power of a point, we have $-AQ \cdot QB = OQ^2 - R^2$ and $-AP \cdot PC = OP^2 - R^2$. Therefore, it suffices to show $AQ \cdot QB = AP \cdot PC$.



As $\overline{ML} \parallel \overline{AC}$ and $\overline{MK} \parallel \overline{AB}$ we have that

$$\measuredangle APQ = \measuredangle LMP = \measuredangle LKM$$
$$\measuredangle PQA = \measuredangle KMQ = \measuredangle MLK$$

and consequently we have that $\triangle APQ \sim \triangle MKL$ (with opposite orientations). Therefore

$$\frac{AQ}{AP} = \frac{ML}{MK} = \frac{2ML}{2MK} = \frac{PC}{QB}$$

id est $AQ \cdot QB = AP \cdot PC$, which is what we wanted to prove.

§1.3 IMO 2009/3, proposed by Gabriel Carroll

Available online at https://aops.com/community/p1561573.

Problem statement

Suppose that $s_1, s_2, s_3, \dots$ is a strictly increasing sequence of positive integers such that the sub-sequences $s_{s_1}, s_{s_2}, s_{s_3}, \dots$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$ are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \dots$ is itself an arithmetic progression.

We present two solutions.

¶ First solution (Alex Zhai) Let $s(n) \coloneqq s_n$ and write

$$\begin{aligned} s(s(n)) &= Dn + A \\ s(s(n)+1) &= D'n + B. \end{aligned}$$

In light of the bounds $s(s(n)) \le s(s(n)+1) \le s(s(n+1))$ we right away recover $D = D'$ and $A \le B$.

Let $d_n = s(n+1) - s(n)$. Note that $\sup d_n < \infty$ since d_n is bounded above by A.

Then we let

$$m \coloneqq \min d_n, \qquad M \overset{\text{def}}{=} \max d_n.$$

Now suppose a achieves the maximum, meaning $s(a+1) - s(a) = M$. Then

$$\begin{aligned} \underbrace{d_{s(s(a))} + \dots + d_{s(s(a+1))-1}}_{D \text{ terms}} &= \boxed{s(s(s(a+1))) - s(s(s(a)))} \\ &= (D \cdot s(a+1) + A) - (D \cdot s(a) + A) = DM. \end{aligned}$$

Now M was maximal hence $M = d_{s(s(a))} = \dots = d_{s(s(a+1))-1}$. But $d_{s(s(a))} = B - A$ is a constant. Hence $M = B - A$. In the same way $m = B - A$ as desired.

¶ Second solution We retain the notation D, A, B above, as well as $m = \min_n s(n+1) - s(n) \ge B - A$. We do the involution trick first as:

$$D = \boxed{s(s(s(n)+1)) - s(s(s(n)))} = s(Dn+B) - s(Dn+A)$$

and hence we recover $D \ge m(B-A)$.

The edge case $D = B - A$ is easy since then $m = 1$ and $D = s(Dn+B) - s(Dn+A)$ forces s to be a constant shift. So henceforth assume $D > B - A$.

The idea is that right now the B terms are "too big", so we want to use the involution trick in a way that makes as many "A minus B" shape expressions as possible. This motivates considering $s(s(s(n+1))) - s(s(s(n)+1)+1) > 0$, since the first expression will have all A's and the second expression will have all B's. Calculation gives:

$$\begin{aligned} s(D(n+1)+A) - s(Dn+B+1) &= \boxed{s(s(s(n+1))) - s(s(s(n)+1)+1)} \\ &= (Ds(n+1)+A) - (D(s(n)+1)+B) \\ &= D\left(s(n+1) - s(n)\right) + A - B - D. \end{aligned}$$

Then by picking n achieving the minimum m,

$$\underbrace{m(D+A-B-1)}_{>0} \le s(s(s(n+1))) - s(s(s(n)+1)+1) \le Dm + A - B - D$$

which becomes

$$(D - m(B-A)) + ((B-A) - m) \le 0.$$

Since both of these quantities were supposed to be nonnegative, we conclude $m = B - A$ and $D = m^2$. Now the estimate $D = s(Dn+B) - s(Dn+A) \ge m(B-A)$ is actually sharp, so it follows that $s(n)$ is arithmetic.

§2 Solutions to Day 2

§2.1 IMO 2009/4

Available online at https://aops.com/community/p1562847.

Problem statement

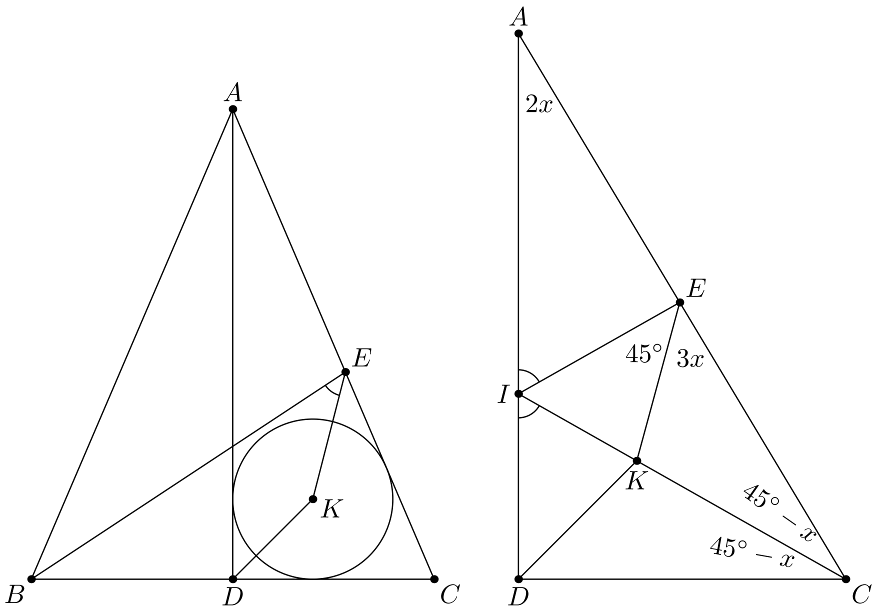
Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

Here is the solution presented in my book *EGMO*.

Let I be the incenter of ABC, and set $\angle DAC = 2x$ (so that $0^\circ < x < 45^\circ$). From $\angle AIE = \angle DIC$, it is easy to compute

$$\angle KIE = 90^\circ - 2x, \ \angle ECI = 45^\circ - x, \ \angle IEK = 45^\circ, \ \angle KEC = 3x.$$

Having chased all the angles we want, we need a relationship. We can find it by considering the side ratio $\frac{IK}{KC}$. Using the angle bisector theorem, we can express this in terms of triangle IDC; however we can also express it in terms of triangle IEC.



By the law of sines, we obtain

$$\frac{IK}{KC} = \frac{\sin 45^\circ \cdot \frac{EK}{\sin(90^\circ - 2x)}}{\sin(3x) \cdot \frac{EK}{\sin(45^\circ - x)}} = \frac{\sin 45^\circ \sin(45^\circ - x)}{\sin(3x) \sin(90^\circ - 2x)}.$$

Also, by the angle bisector theorem on $\triangle IDC$, we have

$$\frac{IK}{KC} = \frac{ID}{DC} = \frac{\sin(45^\circ - x)}{\sin(45^\circ + x)}.$$

Equating these and cancelling $\sin(45^\circ - x) \neq 0$ gives

$$\sin 45^\circ \sin(45^\circ + x) = \sin 3x \sin(90^\circ - 2x).$$

Applying the product-sum formula (again, we are just trying to break down things as much as possible), this just becomes

$$\cos(x) - \cos(90^\circ + x) = \cos(5x - 90^\circ) - \cos(90^\circ + x)$$

or $\cos x = \cos(5x - 90^\circ)$.

At this point we are basically done; the rest is making sure we do not miss any solutions and write up the completion nicely. One nice way to do this is by using product-sum in reverse as

$$0 = \cos(5x - 90^\circ) - \cos x = 2\sin(3x - 45^\circ)\sin(2x - 45^\circ).$$

This way we merely consider the two cases

$$\sin(3x - 45^\circ) = 0 \text{ and } \sin(2x - 45^\circ) = 0.$$

Notice that $\sin\theta = 0$ if and only θ is an integer multiple of 180°. Using the bound $0^\circ < x < 45^\circ$, it is easy to see that that the permissible values of x are $x = 15^\circ$ and $x = \frac{45}{2}^\circ$. As $\angle A = 4x$, this corresponds to $\angle A = 60^\circ$ and $\angle A = 90^\circ$, which can be seen to work.

§2.2 IMO 2009/5

Available online at https://aops.com/community/p1562848.

Problem statement

Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for positive integers a and b, the numbers

$$a, \qquad f(b), \qquad f(b+f(a)-1)$$

are the sides of a non-degenerate triangle.

The only function is the identity function (which works). We prove it is the only one.

Let $P(a,b)$ denote the given statement.

Claim — We have $f(1) = 1$, and $f(f(n)) = n$. (In particular f is a bijection.)

Proof. Note that

$$P(1,b) \implies f(b) = f(b+f(1)-1).$$

Otherwise, the function f is periodic modulo $N = f(1) - 1 \geq 1$. This is impossible since we can fix b and let a be arbitrarily large in some residue class modulo N.

Hence $f(1) = 1$, so taking $P(n,1)$ gives $f(f(n)) = n$. □

Claim — Let $\delta = f(2) - 1 > 0$. Then for every n,

$$f(n+1) = f(n) + \delta \quad \text{or} \quad f(n-1) = f(n) + \delta$$

Proof. Use

$$P(2, f(n)) \implies n - 2 < f(f(n)+\delta) < n+2.$$

Let $y = f(f(n)+\delta)$, hence $n-2 < y < n+2$ and $f(y) = f(n) + \delta$. But, remark that if $y = n$, we get $\delta = 0$, contradiction. So $y \in \{n+1, n-1\}$ and that is all. □

We now show f is an arithmetic progression with common difference $+\delta$. Indeed we already know $f(1) = 1$ and $f(2) = 1 + \delta$. Now suppose $f(1) = 1, \dots, f(n) = 1 + (n-1)\delta$. Then by induction for any $n \geq 2$, the second case can't hold, so we have $f(n+1) = f(n) + \delta$, as desired.

Combined with $f(f(n)) = n$, we recover that f is the identity.

§2.3 IMO 2009/6

Available online at https://aops.com/community/p1562840.

Problem statement

Let a_1, a_2, ..., a_n be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths a_1, a_2, ..., a_n in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

The proof is by induction on n. Assume $a_1 < \cdots < a_n$ and call each element of M a *mine*. Let $x = s - a_n$. We consider four cases, based on whether x has a mine and whether there is a mine past x.

- If x has no mine, and there is a mine past x, then at most $n-2$ mines in $[0, x]$ and so we use induction to reach x, then leap from x to s and win.
- If x has no mine but there is also no mine to the right of x, then let m be the maximal mine. By induction hypothesis on $M \setminus \{m\}$, there is a path to x using $\{a_1, \ldots, a_{n-1}\}$ which avoids mines except possibly m. If the path hits the mine m on the hop of length a_k, we then swap that hop with a_n, and finish.
- If x has a mine, but there are no mines to the right of x, we can repeat the previous case with $m = x$.
- Now suppose x has a mine, *and* there is a mine past x. There should exist an integer $1 \le i \le n-1$ such that $s - a_i$ and $y = s - a_i - a_n$ both have no mine. By induction hypothesis, we can then reach y in $n-2$ steps (as there are two mines to the right of y), and then $y \to s - a_i \to s$ finishes.

Remark. It seems much of the difficulty of the problem is realizing that induction will actually work. Attempts at induction are, indeed, a total minefield (ha!), and given the position P6 of the problem, it is expected that many contestants will abandon induction after some cursory attempts fail.

IMO 2010 Solution Notes

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2 June 2023

This is a compilation of solutions for the 2010 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,
$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor .$$

2. Let I be the incenter of a triangle ABC and let Γ be its circumcircle. Let line AI intersect Γ again at D. Let E be a point on arc $\widehat{BDC}$ and F a point on side BC such that
$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$
Finally, let G be the midpoint of $\overline{IF}$. Prove that $\overline{DG}$ and $\overline{EI}$ intersect on Γ.

3. Find all functions $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that
$$(g(m) + n)\,(g(n) + m)$$
is always a perfect square.

4. Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, M, respectively. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

5. Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following two types of operations are allowed:
   a) Choose a non-empty box B_j, $1 \le j \le 5$, remove one coin from B_j and add two coins to B_{j+1};
   b) Choose a non-empty box B_k, $1 \le k \le 4$, remove one coin from B_k and swap the contents (possibly empty) of the boxes B_{k+1} and B_{k+2}.

   Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

6. Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers, and s be a positive integer, such that
$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\} \text{ for all } n > s.$$
Prove there exist positive integers $\ell \le s$ and N, such that
$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \ge N.$$

§1 Solutions to Day 1

§1.1 IMO 2010/1

Available online at https://aops.com/community/p1935849.

Problem statement

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor .$$

The only solutions are $f(x) \equiv c$, where $c = 0$ or $1 \le c < 2$. It's easy to see these work.

Plug in $x = 0$ to get $f(0) = f(0) \lfloor f(y) \rfloor$, so either

$$1 \le f(y) < 2 \quad \forall y \qquad \text{or} \qquad f(0) = 0$$

In the first situation, plug in $y = 0$ to get $f(x) \lfloor f(0) \rfloor = f(0)$, thus f is constant. Thus assume henceforth $f(0) = 0$.

Now set $x = y = 1$ to get

$$f(1) = f(1) \lfloor f(1) \rfloor$$

so either $f(1) = 0$ or $1 \le f(1) < 2$. We split into cases:

- If $f(1) = 0$, pick $x = 1$ to get $f(y) \equiv 0$.
- If $1 \le f(1) < 2$, then $y = 1$ gives

  $$f(\lfloor x \rfloor) = f(x)$$

  from $y = 1$, in particular $f(x) = 0$ for $0 \le x < 1$. Choose $(x, y) = \left(2, \frac{1}{2}\right)$ to get $f(1) = f(2) \left\lfloor f\left(\frac{1}{2}\right) \right\rfloor = 0$.

§1.2 IMO 2010/2

Available online at https://aops.com/community/p1935927.

Problem statement

Let I be the incenter of a triangle ABC and let Γ be its circumcircle. Let line AI intersect Γ again at D. Let E be a point on arc $\widehat{BDC}$ and F a point on side BC such that

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of $\overline{IF}$. Prove that $\overline{DG}$ and $\overline{EI}$ intersect on Γ.

Let $\overline{EI}$ meet Γ again at K. Then it suffices to show that $\overline{KD}$ bisects $\overline{IF}$. Let $\overline{AF}$ meet Γ again at H, so $\overline{HE} \parallel \overline{BC}$. By Pascal theorem on

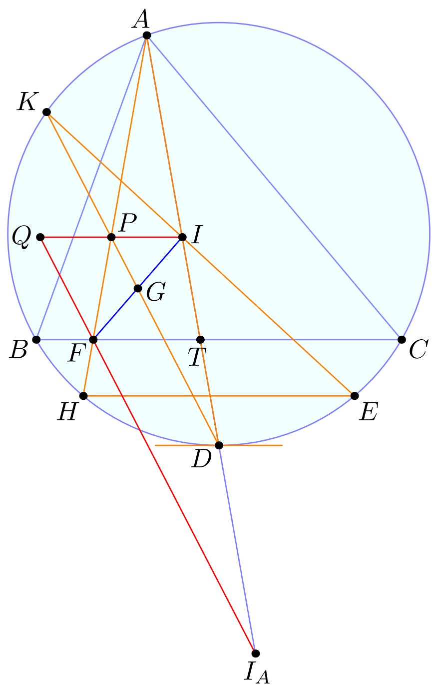
$$AHEKDD$$

we then obtain that $P = \overline{AH} \cap \overline{KD}$ lies on a line through I parallel to $\overline{BC}$.

Let I_A be the A-excenter, and set $Q = \overline{I_AF} \cap \overline{IP}$, and $T = \overline{AIDI_A} \cap \overline{BFC}$. Then

$$-1 = (AI;TI_A) \overset{F}{=} (IQ;\infty P)$$

where ∞ is the point at infinity along $\overline{IPQ}$. Thus P is the midpoint of $\overline{IQ}$. Since D is the midpoint of $\overline{II_A}$ by "Fact 5", it follows that $\overline{DP}$ bisects $\overline{IF}$.



§1.3 IMO 2010/3, proposed by Gabriel Carroll (USA)

Available online at https://aops.com/community/p1935854.

Problem statement

Find all functions $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$(g(m) + n)\,(g(n) + m)$$

is always a perfect square.

For $c \ge 0$, the function $g(n) = n + c$ works; we prove this is the only possibility.

First, the main point of the problem is that

Claim — We have $g(n) \equiv g(n') \pmod p \implies n \equiv n' \pmod p$.

Proof. Pick a large integer M such that

$$\nu_p(M + g(n)), \quad \nu_p(M + g(n')) \quad \text{are both odd.}$$

(It's not hard to see this is always possible.) Now, since each of

$$\begin{gathered}(M + g(n))\,(n + g(M)) \\ \left(M + g(n')\right)\left(n' + g(M)\right)\end{gathered}$$

is a square, we get $g(n) \equiv g(n') \equiv -M \pmod p$. $\square$

This claim implies that

- The numbers $g(n)$ and $g(n + 1)$ differ by ± 1 for any n, and
- The function g is injective.

It follows g is a linear function with slope ± 1, hence done.

§2 Solutions to Day 2

§2.1 IMO 2010/4

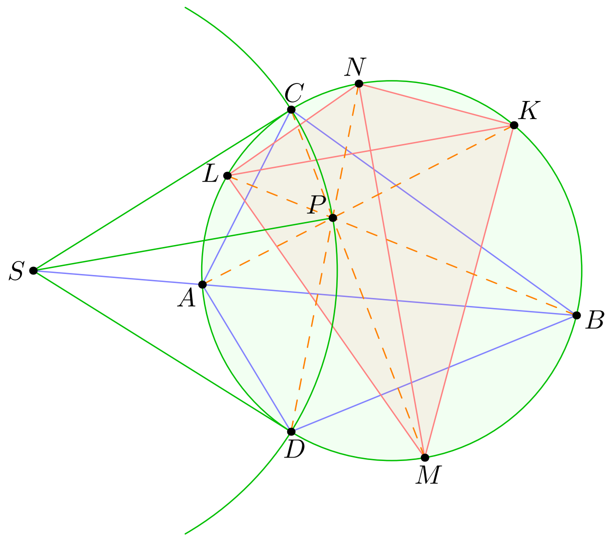
Available online at https://aops.com/community/p1936916.

Problem statement

Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, M, respectively. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

We present two solutions using harmonic bundles.

¶ First solution (Evan Chen) Let N be the antipode of M, and let NP meet Γ again at D. Focus only on $CDMN$ for now (ignoring the condition). Then C and D are feet of altitudes in $\triangle MNP$; it is well-known that the circumcircle of $\triangle CDP$ is orthogonal to Γ (passing through the orthocenter of $\triangle MPN$).



Now, we are given that point S is such that $\overline{SC}$ is tangent to Γ, and $SC = SP$. It follows that S is the circumcenter of $\triangle CDP$, and hence $\overline{SC}$ and $\overline{SD}$ are tangents to Γ.

Then $-1 = (AB;CD) \overset{P}{=} (KL;MN)$. Since $\overline{MN}$ is a diameter, this implies $MK = ML$.

> **Remark.** I think it's more natural to come up with this solution in reverse. Namely, suppose we define the points the other way: let $\overline{SD}$ be the other tangent, so $(AB;CD) = -1$. Then project through P to get $(KL;MN) = -1$, where N is the second intersection of $\overline{DP}$. However, if $ML = MK$ then $KMLN$ must be a kite. Thus one can recover the solution in reverse.

¶ Second solution (Sebastian Jeon) We have

$$SP^2 = SC^2 = SA \cdot SB \implies \measuredangle SPA = \measuredangle PBA = \measuredangle LBA = \measuredangle LKA = \measuredangle LKP$$

(the latter half is Reim's theorem). Therefore $\overline{SP}$ and $\overline{LK}$ are *parallel.*

Now, let $\overline{SP}$ meet Γ again at X and Y, and let Q be the antipode of P on (S). Then

$$SP^2 = SQ^2 = SX \cdot SY \implies (PQ; XY) = -1 \implies \angle QCP = 90^\circ$$

that $\overline{CP}$ bisects $\angle XCY$. Since $\overline{XY} \parallel \overline{KL}$, it follows $\overline{CP}$ bisects to $\angle LCK$ too.

§2.2 IMO 2010/5, proposed by Netherlands

Available online at https://aops.com/community/p1936917.

Problem statement

Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following two types of operations are allowed:

1. Choose a non-empty box B_j, $1 \le j \le 5$, remove one coin from B_j and add two coins to B_{j+1};
2. Choose a non-empty box B_k, $1 \le k \le 4$, remove one coin from B_k and swap the contents (possibly empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

First,

$$\begin{aligned}(1,1,1,1,1,1) &\to (0,3,1,0,3,1) \to (0,0,7,0,0,7)\\ &\to (0,0,6,2,0,7) \to (0,0,6,1,2,7) \to (0,0,6,1,0,11)\\ &\to (0,0,6,0,11,0) \to (0,0,5,11,0,0).\end{aligned}$$

and henceforth we ignore boxes B_1 and B_2, looking at just the last four boxes; so we write the current position as $(5, 11, 0, 0)$.

We prove a lemma:

Claim — Let $k \ge 0$ and $n > 0$. From $(k, n, 0, 0)$ we may reach $(k-1, 2^n, 0, 0)$.

Proof. Working with only the last three boxes for now,

$$\begin{aligned}(n,0,0) &\to (n-1,2,0) \to (n-1,0,4)\\ &\to (n-2,4,0) \to (n-2,0,8)\\ &\to (n-3,8,0) \to (n-3,0,16)\\ &\to \cdots \to (1,2^{n-1},0) \to (1,0,2^n) \to (0,2^n,0).\end{aligned}$$

Finally we have $(k, n, 0, 0) \to (k, 0, 2^n, 0) \to (k-1, 2^n, 0, 0)$. $\square$

Now from $(5, 11, 0, 0)$ we go as follows:

$$\begin{aligned}(5,11,0,0) \to (4,2^{11},0,0) \to \left(3,2^{2^{11}},0,0\right) \to \left(2,2^{2^{2^{11}}},0,0\right)\\ \to \left(1,2^{2^{2^{2^{11}}}},0,0\right) \to \left(0,2^{2^{2^{2^{2^{11}}}}},0,0\right).\end{aligned}$$

Let $A = 2^{2^{2^{2^{2^{11}}}}} > 2010^{2010^{2010}} = B$. Then by using move 2 repeatedly on the fourth box (i.e., throwing away several coins by swapping the empty B_5 and B_6), we go from $(0, A, 0, 0)$ to $(0, B/4, 0, 0)$. From there we reach $(0, 0, 0, B)$.

§2.3 IMO 2010/6

Available online at https://aops.com/community/p1936918.

Problem statement

Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers, and s be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\} \text{ for all } n > s.$$

Prove there exist positive integers $\ell \le s$ and N, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \ge N.$$

Let

$$w_1 = \frac{a_1}{1}, \quad w_2 = \frac{a_2}{2}, \quad \dots, \quad w_s = \frac{a_s}{s}.$$

(The choice of the letter w is for "weight".) We claim the right choice of ℓ is the one maximizing w_ℓ.

Our plan is to view each a_n as a linear combination of the weights $w_1, \dots, w_s$ and track their coefficients.

To this end, let's define an *n-type* to be a vector $T = \langle t_1, \dots, t_s \rangle$ of nonnegative integers such that

- $n = t_1 + \dots + t_s$; and
- t_i is divisible by i for every i.

We then define its *valuation* as $v(T) = \sum_{i=1}^{s} w_i t_i$.

Now we define a n-type to be *valid* according to the following recursive rule. For $1 \le n \le s$ the only valid n-types are

$$\begin{aligned} T_1 &= \langle 1, 0, 0, \dots, 0 \rangle \\ T_2 &= \langle 0, 2, 0, \dots, 0 \rangle \\ T_3 &= \langle 0, 0, 3, \dots, 0 \rangle \\ &\vdots \\ T_s &= \langle 0, 0, 0, \dots, s \rangle \end{aligned}$$

for $n = 1, \dots, s$, respectively. Then for any $n > s$, an n-type is valid if it can be written as the sum of a valid k-type and a valid $(n-k)$-type, componentwise. These represent the linear combinations possible in the recursion; in other words the recursion in the problem is phrased as

$$a_n = \max_{T \text{ is a valid } n\text{-type}} v(T).$$

In fact, we have the following description of valid n-types:

Claim — Assume $n > s$. Then an n-type $\langle t_1, \dots, t_s \rangle$ is valid if and only if either

- there exist indices $i < j$ with $i + j > s$, $t_i \ge i$ and $t_j \ge j$; or
- there exists an index $i > s/2$ with $t_i \ge 2i$.

Proof. Immediate by forwards induction on $n > s$ that all n-types have this property.

The reverse direction is by downwards induction on n. Indeed if $\sum_i \frac{t_i}{i} > 2$, then we may subtract off on of $\{T_1, \dots, T_s\}$ while preserving the condition; and the case $\sum_i \frac{t_i}{i} = 2$ is essentially by definition. $\square$

> **Remark.** The claim is a bit confusingly stated in its two cases; really the latter case should be thought of as the situation $i = j$ but requiring that t_i/i is counted with multiplicity.

Now, for each $n > s$ we pick a valid n-type T_n with $a_n = v(T_n)$; if there are ties, we pick one for which the ℓth entry is as large as possible.

> **Claim** — For any $n > s$ and index $i \neq \ell$, the ith entry of T_n is at most $2s + \ell i$.

Proof. If not, we can go back $i\ell$ steps to get a valid $(n - i\ell)$-type T achieved by decreasing the ith entry of T_n by $i\ell$. But then we can add ℓ to the ℓth entry i times to get another n-type T' which obviously has valuation at least as large, but with larger ℓth entry. $\square$

Now since all other entries in T_n are bounded, eventually the sequence $(T_n)_{n>s}$ just consists of repeatedly adding 1 to the ℓth entry, as required.

> **Remark.** One big step is to consider $w_k = a_k/k$. You can get this using wishful thinking or by examining small cases. (In addition this normalization makes it easier to see why the largest w plays an important role, since then in the definition of type, the n-types all have a sum of n. Unfortunately, it makes the characterization of valid n-types somewhat clumsier too.)

IMO 2011 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2011 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \le i < j \le 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

2. Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

   Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

3. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies
$$f(x + y) \le yf(x) + f(f(x))$$
   for all real numbers x and y. Prove that $f(x) = 0$ for all $x \le 0$.

4. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \dots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

5. Let $f \colon \mathbb{Z} \to \mathbb{Z}_{>0}$ be a function such that $f(m - n) \mid f(m) - f(n)$ for $m, n \in \mathbb{Z}$. Prove that if $m, n \in \mathbb{Z}$ satisfy $f(m) \le f(n)$ then $f(m) \mid f(n)$.

6. Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b, ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, and ℓ_c is tangent to the circle Γ.

§1 Solutions to Day 1

§1.1 IMO 2011/1, proposed by Fernando Campos (MEX)

Available online at https://aops.com/community/p2363530.

Problem statement

Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \le i < j \le 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

There are two curves of solutions, namely $\{x, 5x, 7x, 11x\}$ and $\{x, 11x, 19x, 29x\}$, for any positive integer x, achieving $n_A = 4$ (easy to check). We'll show that $n_A \le 4$ and equality holds only in one of the curves.

Let $A = \{a < b < c < d\}$.

Claim — We have $n_A \le 4$ with equality iff

$$a + b \mid c + d, \quad a + c \mid b + d, \quad a + d = b + c.$$

Proof. Note $a + b \mid s_A \iff a + b \mid c + d$ etc. Now $c + d \nmid a + b$ and $b + d \nmid a + c$ for size reasons, so we already have $n_A \le 4$; moreover $a + d \mid b + c$ and $b + c \mid a + d$ if and only if $a + d = b + c$. □

We now show the equality curve is the one above.

$$a + c \mid b + d \iff a + c \mid -a + 2b + c \iff a + c \mid 2(b - a).$$

Since $a + c > |b - a|$, so we must have $a + c = 2(b - a)$. So we now have

$$\begin{aligned} c &= 2b - 3a \\ d &= b + c - a = 3b + c - 4a. \end{aligned}$$

The last condition is

$$a + b \mid c + d = 5b - 7a \iff a + b \mid 12a.$$

Now, let $x = \gcd(a, b)$. The expressions for c and d above imply that $x \mid c, d$ so we may scale down so that $x = 1$. Then $\gcd(a + b, a) = \gcd(a, b) = 1$ and so $a + b \mid 12$.

We have $c > b$, so $3a < b$. The only pairs (a, b) with $3a < 2b$, $\gcd(a, b) = 1$ and $a + b \mid 12$ are $(a, b) \in \{(1, 5), (1, 11)\}$ which give the solutions earlier.

§1.2 IMO 2011/2, proposed by Geoff Smith (UNK)

Available online at https://aops.com/community/p2363537.

Problem statement

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Orient ℓ in some direction, and color the plane such that its left half is red and right half is blue. The critical observation is that:

Claim — The number of points on the red side of ℓ does not change, nor does the number of points on the blue side (except at a moment when ℓ contains two points).

Thus, if $|\mathcal{S}| = n+1$, it suffices to pick the initial configuration so that there are $\lfloor n/2 \rfloor$ red and $\lceil n/2 \rceil$ blue points. Then when the line ℓ does a full 180° rotation, the red and blue sides "switch", so the windmill has passed through every point.

(See official shortlist for verbose write-up; this is deliberately short to make a point.)

§1.3 IMO 2011/3, proposed by Igor Voronovich, BLR

Available online at https://aops.com/community/p2363539.

Problem statement

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \le yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \le 0$.

We begin by rewriting the given as

$$f(z) \le (z-x)f(x) + f(f(x)) \quad \forall x, z \in \mathbb{R} \qquad (\heartsuit)$$

(which is better anyways since control over inputs to f is more valuable). We start by eliminating the double f: let $z = f(w)$ to get

$$f(f(w)) \le (f(w) - x)f(x) + f(f(x))$$

and then use the *symmetry trick* to write

$$f(f(x)) \le (f(x) - w)f(w) + f(f(w))$$

so that when we sum we get

$$wf(w) + xf(x) \le 2f(x)f(w).$$

Next we use *cancellation trick*: set $w = 2f(x)$ in the above to get

$$xf(x) \le 0 \quad \forall x \in \mathbb{R}. \qquad (\spadesuit)$$

Claim — For every $p \in \mathbb{R}$, we have $f(p) \le 0$.

Proof. Assume $f(p) > 0$ for some $p \in \mathbb{R}$. Then for any negative number z,

$$0 \overset{(\spadesuit)}{\le} f(z) \overset{(\heartsuit)}{\le} (z-p)f(p) + f(f(p)).$$

which is false if we let $z \to -\infty$. □

Together with $(\spadesuit)$ we derive $f(x) = 0$ for $x < 0$. Finally, letting x and z be any negative numbers in $(\heartsuit)$, we get $f(0) \ge 0$, so $f(0) = 0$ too.

Remark. As another corollary of the claim, $f(f(x)) = 0$ for all x.

Remark. A nontrivial example of a working f is to take

$$f(x) = \begin{cases} -\exp(\exp(\exp(x))) & x > 0 \\ 0 & x \le 0. \end{cases}$$

or some other negative function growing rapidly in absolute value for $x > 0$.

§2 Solutions to Day 2

§2.1 IMO 2011/4, proposed by Morteza Saghafian (IRN)

Available online at https://aops.com/community/p2365036.

Problem statement

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \dots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

The answer is $a_n = (2n-1)!!$. We refer to what we're counting as a *valid n-sequence*: an order of which weights to place, and whether to place them on the left or right pan.

We use induction, with $n = 1$ being obvious. Now consider the weight $2^0 = 1$.

- If we delete it from any valid n-sequence, we get a valid $(n-1)$-sequence with all weights doubled.
- Given a valid $(n-1)$-sequence with all weights doubled, we may insert $2^0 = 1$ it into $2n-1$ ways. Indeed, we may insert it anywhere, and designate it either left or right, except we may not designate right if we choose to insert $2^0 = 1$ at the very beginning.

Consequently, we have that

$$a_n = (2n-1) \cdot a_{n-1}.$$

Since $a_1 = 1$, the conclusion follows.

§2.2 IMO 2011/5, proposed by Mahyar Sefidgaran (IRN)

Available online at https://aops.com/community/p2365041.

Problem statement

Let $f\colon \mathbb{Z} \to \mathbb{Z}_{>0}$ be a function such that $f(m-n) \mid f(m) - f(n)$ for $m, n \in \mathbb{Z}$. Prove that if $m, n \in \mathbb{Z}$ satisfy $f(m) \le f(n)$ then $f(m) \mid f(n)$.

Let $P(m,n)$ denote the given assertion. First, we claim f is even. This is straight calculation:

- $P(x,0) \implies f(x) \mid f(x) - f(0) \implies f(x) \mid M := f(0)$.
- $P(0,x) \implies f(-x) \mid M - f(x) \implies f(-x) \mid f(x)$. Analogously, $f(x) \mid f(-x)$. So $f(x) = f(-x)$ and f is even.

Claim — Let x, y, z be integers with $x+y+z=0$. Then among $f(x)$, $f(y)$, $f(z)$, two of them are equal and divide the third.

Proof. Let $a = f(\pm x)$, $b = f(\pm y)$, $c = f(\pm z)$ be positive integers. Note that

$$\begin{aligned} a &\mid b - c \\ b &\mid c - a \\ c &\mid a - b \end{aligned}$$

from $P(y,-z)$ and similarly. WLOG $c = \max(a,b,c)$; then $c > |a-b|$ so $a = b$. Thus $a = b \mid c$ from the first two. $\square$

This implies the problem, by taking x and y in the previous claim to be the integers m and n.

Remark. At https://aops.com/community/c6h418981p2381909, Davi Medeiros gives the following characterization of functions f satisfying the hypothesis.

Pick $f(0)$, k positive integers, a chain $d_1 \mid d_2 \mid \cdots \mid d_k$ of divisors of $f(0)$, and positive integers $a_1, a_2, \dots, a_{k-1}$, greater than 1 (if $k=1$, a_i doesn't exist, for every i). We'll define f as follows:

- $f(n) = d_1$, for every integer n that is not divisible by a_1;
- $f(a_1 n) = d_2$, for every integer n that is not divisible by a_2;
- $f(a_1 a_2 n) = d_3$, for every integer n that is not divisible by a_3;
- $f(a_1 a_2 a_3 n) = d_4$, for every integer n that is not divisible by a_4;
- $\dots$
- $f(a_1 a_2 \dots a_{k-1} n) = d_k$, for every integer n;

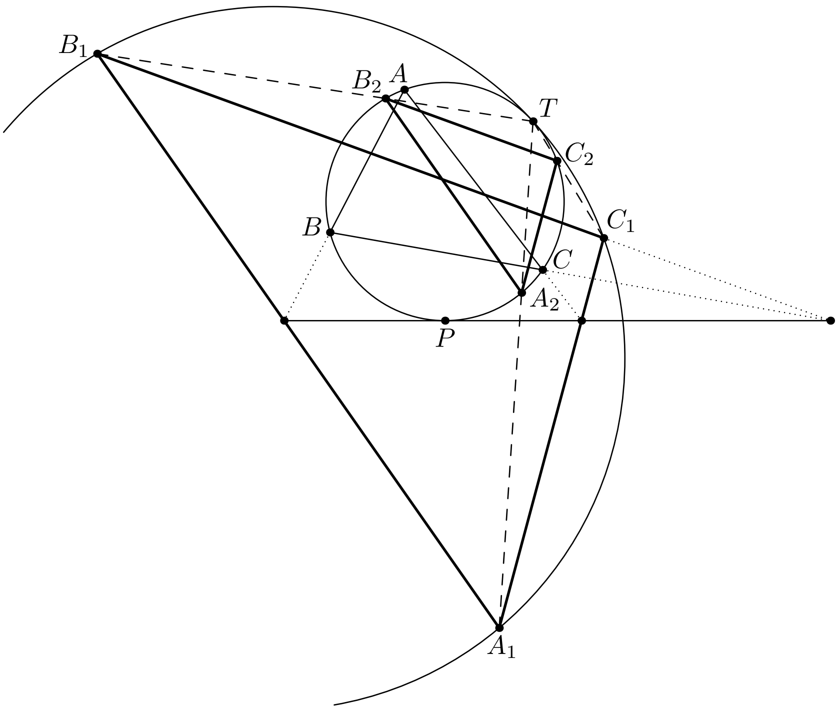
§2.3 IMO 2011/6, proposed by Japan

Available online at https://aops.com/community/p2365045.

Problem statement

Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b, ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, and ℓ_c is tangent to the circle Γ.

This is a hard problem with many beautiful solutions. The following solution is not very beautiful but not too hard to find during an olympiad, as the only major insight it requires is the construction of A_2, B_2, and C_2.



We apply complex numbers with ω the unit circle and $p = 1$. Let $A_1 = \ell_B \cap \ell_C$, and let $a_2 = a^2$ (in other words, A_2 is the reflection of P across the diameter of ω through A). Define the points B_1, C_1, B_2, C_2 similarly.

We claim that $\overline{A_1A_2}$, $\overline{B_1B_2}$, $\overline{C_1C_2}$ concur at a point on Γ.

We begin by finding A_1. If we reflect the points $1+i$ and $1-i$ over $\overline{AB}$, then we get two points Z_1, Z_2 with

$$\begin{aligned} z_1 &= a + b - ab(1-i) = a + b - ab + abi \\ z_2 &= a + b - ab(1+i) = a + b - ab - abi. \end{aligned}$$

Therefore,

$$\begin{aligned} z_1 - z_2 &= 2abi \\ \overline{z_1}z_2 - \overline{z_2}z_1 &= -2i\left(a + b + \frac{1}{a} + \frac{1}{b} - 2\right). \end{aligned}$$

Now ℓ_C is the line $\overline{Z_1Z_2}$, so with the analogous equation ℓ_B we obtain:

$$\begin{aligned} a_1 &= \frac{-2i\left(a+b+\frac{1}{a}+\frac{1}{b}-2\right)(2aci)+2i\left(a+c+\frac{1}{a}+\frac{1}{c}-2\right)(2abi)}{\left(-\frac{2}{ab}i\right)(2aci)-\left(-\frac{2}{ac}i\right)(2abi)} \\ &= \frac{[c-b]\,a^2+\left[\frac{c}{b}-\frac{b}{c}-2c+2b\right]a+(c-b)}{\frac{c}{b}-\frac{b}{c}} \\ &= a+\frac{(c-b)\left[a^2-2a+1\right]}{(c-b)(c+b)/bc} \\ &= a+\frac{bc}{b+c}(a-1)^2. \end{aligned}$$

Then the second intersection of $\overline{A_1A_2}$ with ω is given by

$$\begin{aligned} \frac{a_1-a_2}{1-a_2\overline{a_1}} &= \frac{a+\frac{bc}{b+c}(a-1)^2-a^2}{1-a-a^2\cdot\frac{(1-1/a)^2}{b+c}} \\ &= \frac{a+\frac{bc}{b+c}(1-a)}{1-\frac{1}{b+c}(1-a)} \\ &= \frac{ab+bc+ca-abc}{a+b+c-1}. \end{aligned}$$

Thus, the claim is proved.

Finally, it suffices to show $\overline{A_1B_1} \parallel \overline{A_2B_2}$. One can also do this with complex numbers; it amounts to showing a^2-b^2, $a-b$, i (corresponding to $\overline{A_2B_2}$, $\overline{A_1B_1}$, $\overline{PP}$) have their arguments an arithmetic progression, equivalently

$$\frac{(a-b)^2}{i(a^2-b^2)} \in \mathbb{R} \iff \frac{(a-b)^2}{i(a^2-b^2)} = \frac{\left(\frac{1}{a}-\frac{1}{b}\right)^2}{\frac{1}{i}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}$$

which is obvious.

Remark. One can use directed angle chasing for this last part too. Let $\overline{BC}$ meet ℓ at K and $\overline{B_2C_2}$ meet ℓ at L. Evidently

$$\begin{aligned} -\measuredangle B_2LP &= \measuredangle LPB_2+\measuredangle PB_2L \\ &= 2\measuredangle KPB+\measuredangle PB_2C_2 \\ &= 2\measuredangle KPB+2\measuredangle PBC \\ &= -2\measuredangle PKB \\ &= \measuredangle PKB_1 \end{aligned}$$

as required.

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Evan Chen《陳誼廷》

2 June 2023

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Corrections and comments are welcome!

Contents

§0 Problems

1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

2. Let a_2, a_3, $\dots$, a_n be positive reals with product 1, where $n \ge 3$. Show that
$$(1+a_2)^2(1+a_3)^3 \dots (1+a_n)^n > n^n.$$

3. The liar's guessing game is a game played between two players A and B. The rules of the game depend on two fixed positive integers k and n which are known to both players.

   At the start of the game A chooses integers x and N with $1 \le x \le N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

   After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

   (a) If $n \ge 2^k$, then B can guarantee a win.

   (b) For all sufficiently large k, there exists an integer $n \ge (1.99)^k$ such that B cannot guarantee a win.

4. Find all functions $f \colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a+b+c=0$, the following equality holds:
$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

5. Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let $M = \overline{AL} \cap \overline{BK}$. Prove that $MK = ML$.

6. Find all positive integers n for which there exist non-negative integers $a_1, a_2, \dots, a_n$ such that
$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \dots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \dots + \frac{n}{3^{a_n}} = 1.$$

§1 Solutions to Day 1

§1.1 IMO 2012/1

Available online at https://aops.com/community/p2736397.

Problem statement

Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

We employ barycentric coordinates with reference $\triangle ABC$. As usual $a = BC$, $b = CA$, $c = AB$, $s = \frac{1}{2}(a+b+c)$.

It's obvious that $K = (-(s-c) : s : 0)$, $M = (0 : s-b : s-c)$. Also, $J = (-a : b : c)$. We then obtain

$$G = \left(-a : b : \frac{-as + (s-c)b}{s-b}\right).$$

It follows that

$$T = \left(0 : b : \frac{-as + (s-c)}{s-b}\right) = (0 : b(s-b) : b(s-c) - as).$$

Normalizing, we see that $T = \left(0, -\frac{b}{a}, 1 + \frac{b}{a}\right)$, from which we quickly obtain $MT = s$. Similarly, $MS = s$, so we're done.

§1.2 IMO 2012/2

Available online at https://aops.com/community/p2736375.

Problem statement

Let $a_2, a_3, \dots, a_n$ be positive reals with product 1, where $n \ge 3$. Show that

$$(1+a_2)^2(1+a_3)^3\dots(1+a_n)^n > n^n.$$

Try the dumbest thing possible: by AM-GM,

$$\begin{aligned}(1+a_2)^2 &\ge 2^2 a_2 \\ (1+a_3)^3 = \left(\frac{1}{2}+\frac{1}{2}+a_3\right)^3 &\ge \frac{3^3}{2^2}a_3 \\ (1+a_4)^4 = \left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+a_4\right)^4 &\ge \frac{4^4}{3^3}a_4 \\ &\vdots\end{aligned}$$

and so on. Multiplying these all gives the result. The inequality is strict since it's not possible that $a_2 = 1$, $a_3 = \frac{1}{2}$, et cetera.

§1.3 IMO 2012/3

Available online at https://aops.com/community/p2736406.

Problem statement

The liar's guessing game is a game played between two players A and B. The rules of the game depend on two fixed positive integers k and n which are known to both players.

At the start of the game A chooses integers x and N with $1 \le x \le N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

(a) If $n \ge 2^k$, then B can guarantee a win.

(b) For all sufficiently large k, there exists an integer $n \ge (1.99)^k$ such that B cannot guarantee a win.

Call the players Alice and Bob.

Part (a): We prove the following.

Claim — If $N \ge 2^k + 1$, then in $2k+1$ questions, Bob can rule out some number in $\{1, \dots, 2^k+1\}$ form being equal to x.

Proof. First, Bob asks the question $S_0 = \{2^k+1\}$ until Alice answers "yes" or until Bob has asked $k+1$ questions. If Alice answers "no" to all of these then Bob rules out 2^k+1. So let's assume Alice just said "yes".

Now let $T = \{1, \dots, 2^k\}$. Then, he asks k-follow up questions $S_1, \dots, S_k$ defined as follows:

- $S_1 = \{1, 3, 5, 7, \dots, 2^k-1\}$ consists of all numbers in T whose least significant digit in binary is 1.
- $S_2 = \{2, 3, 6, 7, \dots, 2^k-2, 2^k-1\}$ consists of all numbers in T whose second least significant digit in binary is 1.
- More generally S_i consists of all numbers in T whose ith least significant digit in binary is 1.

WLOG Alice answers these all as "yes" (the other cases are similar). Among the last $k+1$ answers at least one must be truthful, and the number 2^k (having zeros in all relevant digits) does not appear in any of $S_0, \dots, S_k$ and is ruled out. $\square$

Thus in this way Bob can repeatedly find non-possibilities for x (and then relabel the remaining candidates $1, \dots, N-1$) until he arrives at a set of at most 2^k numbers.

Part (b): It suffices to consider $n = \lceil 1.99^k \rceil$ and $N = n+1$ for large k. At the tth step, Bob asks some question S_t; we phrase each of Alice's answers in the form "$x \notin B_t$", where B_t is either S_t or its complement. (You may think of these as "bad sets"; the idea is to show we can avoid having any number appear in $k+1$ consecutive bad sets, preventing Bob from ruling out any numbers.)

Main idea: for every number $1 \le x \le N$, at time step t we define its *weight* to be

$$w(x) = 1.998^e$$

where e is the largest number such that $x \in B_{t-1} \cap B_{t-2} \cap \dots \cap B_{t-e}$.

Claim — Alice can ensure the total weight never exceeds 1.998^{k+1} for large k.

Proof. Let W_t denote the sum of weights after the tth question. We have $W_0 = N < 1000n$. We will prove inductively that $W_t < 1000n$ always.

At time t, Bob specifies a question S_t. We have Alice choose B_t as whichever of S_t or $\overline{S_t}$ has lesser total weight, hence at most $W_t/2$. The weights of for B_t increase by a factor of 1.998, while the weights for $\overline{B_t}$ all reset to 1. So the new total weight after time t is

$$W_{t+1} \le 1.998 \cdot \frac{W_t}{2} + \#\overline{B_t} \le 0.999W_t + n.$$

Thus if $W_t < 1000n$ then $W_{t+1} < 1000n$.

To finish, note that $1000n < 1000\left(1.99^k + 1\right) < 1.998^{k+1}$ for k large. $\square$

In particular, no individual number can have weight 1.998^{k+1}. Thus for every time step t we have

$$B_t \cap B_{t+1} \cap \dots \cap B_{t+k} = \varnothing.$$

Then once Bob stops, if he declares a set of n positive integers, and x is an integer Bob did not choose, then Alice's question history is consistent with x being Alice's number, as among any $k+1$ consecutive answers she claimed that $x \in \overline{B_t}$ for some t in that range.

Remark (Motivation)**.** In our B_t setup, let's think backwards. The problem is equivalent to avoiding $e = k+1$ at any time step t, for any number x. That means

- have at most two elements with $e = k$ at time $t-1$,
- thus have at most four elements with $e = k-1$ at time $t-2$,
- thus have at most eight elements with $e = k-2$ at time $t-3$,
- and so on.

We already exploited this in solving part (a). In any case it's now natural to try letting $w(x) = 2^e$, so that all the cases above sum to "equally bad" situations: since $8 \cdot 2^{k-2} = 4 \cdot 2^{k-1} = 2 \cdot 2^k$, say.

However, we then get $W_{t+1} \le \frac{1}{2}(2W_t) + n$, which can increase without bound due to contributions from numbers resetting to zero. The way to fix this is to change the weight to $w(x) = 1.998^e$, taking advantage of the little extra space we have due to having $n \ge 1.99^k$ rather than $n \ge 2^k$.

§2 Solutions to Day 2

§2.1 IMO 2012/4

Available online at https://aops.com/community/p2737336.

Problem statement

Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a+b+c=0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

Answer: for arbitrary $k \in \mathbb{Z}$, we have

(i) $f(x) = kx^2$,

(ii) $f(x) = 0$ for even x, and $f(x) = k$ for odd x, and

(iii) $f(x) = 0$ for $x \equiv 0 \pmod 4$, $f(x) = k$ for odd x, and $f(x) = 4k$ for $x \equiv 2 \pmod 4$.

These can be painfully seen to work. (It's more natural to think of these as $f(x) = x^2$, $f(x) = x^2 \pmod 4$, $f(x) = x^2 \pmod 8$, and multiples thereof.)

Set $a = b = c = 0$ to get $f(0) = 0$. Then set $c = 0$ to get $f(a) = f(-a)$, so f is even. Now

$$f(a)^2 + f(b)^2 + f(a+b)^2 = 2f(a+b)\left(f(a) + f(b)\right) + 2f(a)f(b)$$

or

$$\left(f(a+b) - (f(a) + f(b))\right)^2 = 4f(a)f(b).$$

Hence $f(a)f(b)$ is a perfect square for all $a, b \in \mathbb{Z}$. So there exists a λ such that $f(n) = \lambda g(n)^2$, where $g(n) \geq 0$. From here we recover

$$\boxed{g(a+b) = \pm g(a) \pm g(b)}.$$

Also $g(0) = 0$.

Let $k = g(1) \neq 0$. We now split into cases on $g(2)$:

- $g(2) = 0$. Put $b = 2$ in original to get $g(a+2) = \pm g(a) = +g(a)$.
- $g(2) = 2k$. Cases on $g(4)$:
  - $g(4) = 0$, then we get $(g(n))_{n\geq 0} = (0, 1, 2, 1, 0, 1, 2, 1, \dots)$. This works.
  - $g(4) = 4k$. This only happens when $g(1) = k$, $g(2) = 2k$, $g(3) = 3k$, $g(4) = 4k$. Then
    - $g(5) = \pm 3k \pm 2k = \pm 4k \pm k$.
    - $g(6) = \pm 4k \pm 2k = \pm 5k \pm k$.
    - $\dots$

    and so by induction $g(n) = nk$.

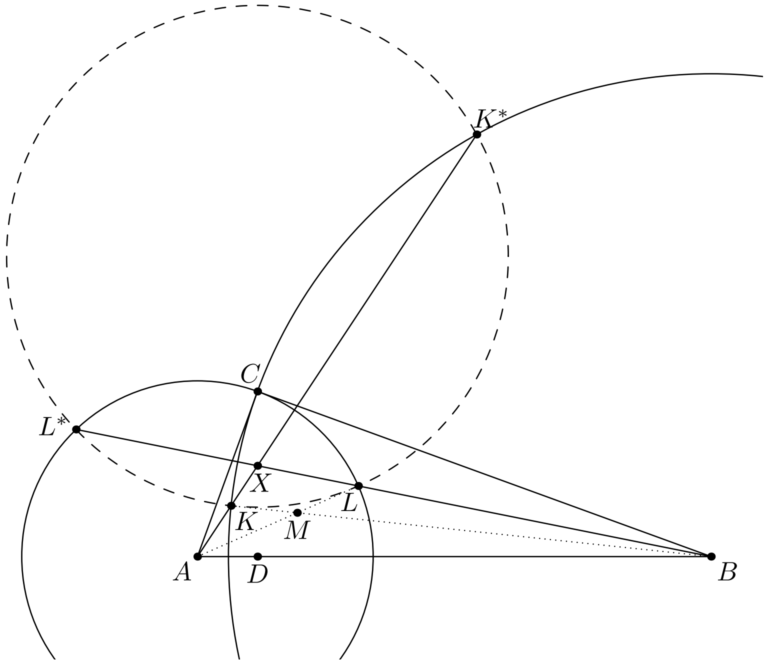
§2.2 IMO 2012/5

Available online at https://aops.com/community/p2737425.

Problem statement

Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let $M = \overline{AL} \cap \overline{BK}$. Prove that $MK = ML$.

Let ω_A and ω_B be the circles through C centered at A and B; extend rays AK and BL to hit ω_B and ω_A again at K^*, L^*. By radical center X, we have KLK^*L^* is cyclic, say with circumcircle ω.



By orthogonality of (A) and (B) we find that $\overline{AL}$, $\overline{AL^*}$, $\overline{BK}$, $\overline{BK^*}$ are tangents to ω (in particular, KLK^*L^* is harmonic). In particular $\overline{MK}$ and $\overline{ML}$ are tangents to ω, so $MK = ML$.

§2.3 IMO 2012/6, proposed by Dusan Djukic (SRB)

Available online at https://aops.com/community/p2737435.

Problem statement

Find all positive integers n for which there exist non-negative integers $a_1, a_2, \dots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

The answer is $n \equiv 1, 2 \pmod 4$. To see these are necessary, note that taking the latter equation modulo 2 gives

$$1 = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} \equiv 1 + 2 + .. + n \pmod 2.$$

Now we prove these are sufficient. The following nice construction was posted on AOPS by the user `cfheolpiixn`.

Claim — If $n = 2k - 1$ works then so does $n = 2k$.

Proof. Replace

$$\frac{k}{3^r} = \frac{k}{3^{r+1}} + \frac{2k}{3^{r+1}}. \qquad (*)$$

□

Claim — If $n = 4k + 2$ works then so does $n = 4k + 13$.

Proof. First use the identity

$$\frac{k+2}{3^r} = \frac{k+2}{3^{r+2}} + \frac{4k+3}{3^{r+3}} + \frac{4k+5}{3^{r+3}} + \frac{4k+7}{3^{r+3}} + \frac{4k+9}{3^{r+3}} + \frac{4k+11}{3^{r+3}} + \frac{4k+13}{3^{r+3}}$$

to fill in the odd numbers. The even numbers can then be instantiated with $(*)$ too. □

Thus it suffices to construct base cases for $n = 1$, $n = 5$, $n = 9$. They are

$$\begin{aligned} 1 &= \frac{1}{3^0} \\ &= \frac{1}{3^2} + \frac{2}{3^2} + \frac{3}{3^2} + \frac{4}{3^3} + \frac{5}{3^3} \\ &= \frac{1}{3^2} + \frac{2}{3^3} + \frac{3}{3^3} + \frac{4}{3^3} + \frac{5}{3^3} + \frac{6}{3^4} + \frac{7}{3^4} + \frac{8}{3^4} + \frac{9}{3^4}. \end{aligned}$$

IMO 2013 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2013 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let k and n be positive integers. Prove that there exist positive integers $m_1, \dots, m_k$ such that
$$1+\frac{2^k-1}{n} = \left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\dots\left(1+\frac{1}{m_k}\right).$$

2. A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:
   (i) No line passes through any point of the configuration.
   (ii) No region contains points of both colors.

   Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

3. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcenter of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

4. Let ABC be an acute triangle with orthocenter H, and let W be a point on the side $\overline{BC}$, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. Suppose ω_1 is the circumcircle of triangle BWN and X is a point such that $\overline{WX}$ is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that $\overline{WY}$ is a diameter of ω_2. Show that the points X, Y, and H are collinear.

5. Suppose a function $f : \mathbb{Q}_{>0} \to \mathbb{R}$ satisfies:
   (i) If $x, y \in \mathbb{Q}_{>0}$, then $f(x)f(y) \ge f(xy)$.
   (ii) If $x, y \in \mathbb{Q}_{>0}$, then $f(x+y) \ge f(x) + f(y)$.
   (iii) There exists a rational number $a > 1$ with $f(a) = a$.

   Prove that f is the identity function.

6. Let $n \ge 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \dots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c. Let M be the number of beautiful labelings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \le n$ and $\gcd(x, y) = 1$. Prove that $M = N + 1$.

§1 Solutions to Day 1

§1.1 IMO 2013/1, proposed by Japan

Available online at https://aops.com/community/p5720240.

Problem statement

Let k and n be positive integers. Prove that there exist positive integers m_1, ..., m_k such that

$$1+\frac{2^k-1}{n}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\dots\left(1+\frac{1}{m_k}\right).$$

By induction on $k \geq 1$. When $k=1$ there is nothing to prove.

For the inductive step, if n is even, write

$$\frac{n+(2^k-1)}{n}=\left(1+\frac{1}{n+(2^k-2)}\right)\cdot\frac{\frac{n}{2}+(2^{k-1}-1)}{\frac{n}{2}}$$

and use inductive hypothesis on the second term. On the other hand if n is odd then write

$$\frac{n+(2^k-1)}{n}=\left(1+\frac{1}{n}\right)\cdot\frac{\frac{n+1}{2}+(2^{k-1}-1)}{\frac{n+1}{2}}$$

and use inductive hypothesis on the second term.

§1.2 IMO 2013/2, proposed by Ivan Guo (AUS)

Available online at https://aops.com/community/p5720110.

Problem statement

A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:

(i) No line passes through any point of the configuration.

(ii) No region contains points of both colors.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

The answer is $k \geq 2013$.

To see that $k = 2013$ is necessary, consider a regular 4026-gon and alternatively color the points red and blue, then place the last blue point anywhere in general position (it doesn't matter). Each side of the 4026 is a red-blue line segment which needs to be cut by one of the k lines, and each line can cut at most two of the segments.

Now, we prove that $k = 2013$ lines is sufficient. Consider the convex hull of all the points.

- If the convex hull has any red points, cut that red point off from everyone else by a single line. Then, for each of the remaining 2012 red points, break them into 1006 pairs arbitrarily, and for each pair $\{A, B\}$ draw two lines parallel to AB and close to them.
- If the convex hull has two consecutive blue points, cut those two blue points off from everyone else by a single line. Then repeat the above construction for the remaining 2012 blue points.

The end.

§1.3 IMO 2013/3, proposed by Alexander A. Polyansky (RUS)

Available online at https://aops.com/community/p5720184.

Problem statement

Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcenter of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

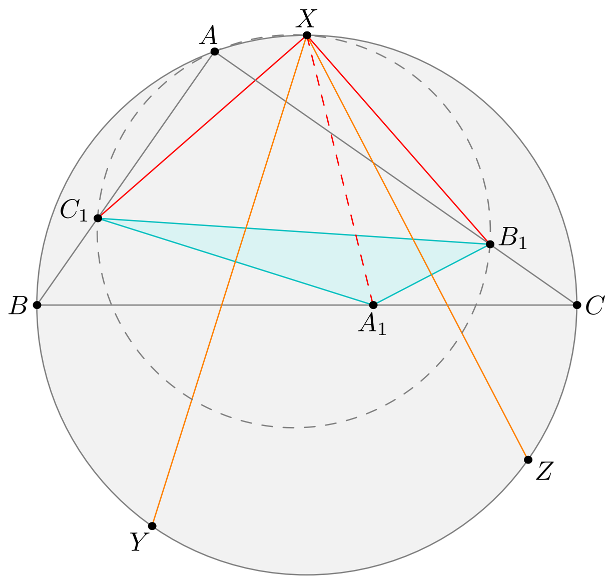
We ignore for now the given condition and prove the following important lemma.

Lemma

Let (AB_1C_1) meet (ABC) again at X. From $BC_1 = B_1C$ follows $XC_1 = XB_1$, and X is the midpoint of major arc $\widehat{BC}$.

Proof. This follows from the fact that we have a spiral similarity $\triangle XBC_1 \sim \triangle XCB_1$ which must actually be a spiral congruence since $BC_1 = B_1C$. $\square$

We define the arc midpoints Y and Z similarly, which lie on the perpendicular bisectors of $\overline{A_1C_1}$, $\overline{A_1B_1}$.



We now turn to the problem condition which asserts the circumcenter W of $\triangle A_1B_1C_1$ lies on (ABC).

Claim — We may assume WLOG that $W = X$.

Proof. This is just configuration analysis, since we already knew that the arc midpoints both lie on (ABC) and the relevant perpendicular bisectors.

Point W lies on (ABC) and hence outside $\triangle ABC$, hence outside $\triangle A_1B_1C_1$. Thus we may assume WLOG that $\angle B_1A_1C_1 > 90^\circ$. Then A and X lie on the same side of line $\overline{B_1C_1}$, and since W is supposed to lie both on (ABC) and the perpendicular bisector of $\overline{B_1C_1}$ it follows $W = X$. $\square$

Consequently, $\overline{XY}$ and $\overline{XZ}$ are exactly the perpendicular bisectors of $\overline{A_1C_1}$, $\overline{A_1B_1}$. The rest is angle chase, the fastest one is

$$\begin{aligned}\angle A &= \angle C_1XB_1 = \angle C_1XA_1 + \angle A_1XB_1 = 2\angle YXA_1 + 2\angle A_1XZ \\ &= 2\angle YXZ = 180^\circ - \angle A\end{aligned}$$

which solves the problem.

Remark. Angle chasing is also possible even without the points Y and Z, though it takes much longer. Introduce the Bevan point V and use the fact that VA_1B_1C is cyclic (with diameter $\overline{VC}$) and similarly VA_1C_1B is cyclic; a calculation then gives $\angle CVB = 180^\circ - \frac{1}{2}\angle A$. Thus V lies on the circle with diameter $\overline{I_bI_c}$.

§2 Solutions to Day 2

§2.1 IMO 2013/4, proposed by Warut Suksompong, Potcharapol Suteparuk (THA)

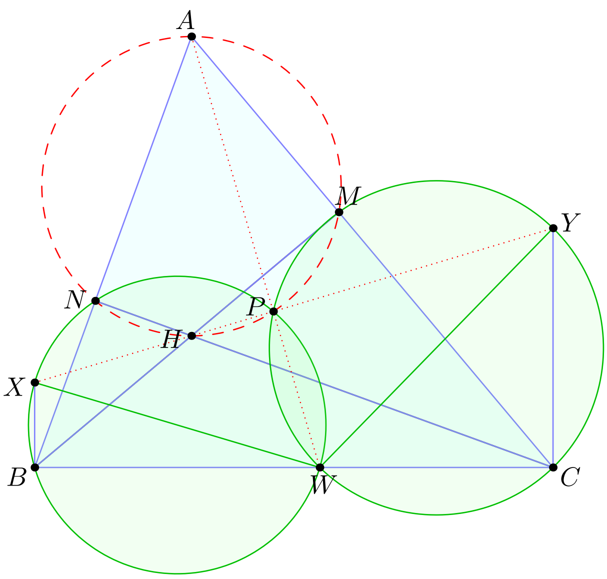
Available online at https://aops.com/community/p5720174.

Problem statement

Let ABC be an acute triangle with orthocenter H, and let W be a point on the side $\overline{BC}$, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. Suppose ω_1 is the circumcircle of triangle BWN and X is a point such that $\overline{WX}$ is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that $\overline{WY}$ is a diameter of ω_2. Show that the points X, Y, and H are collinear.

We present two solutions, an elementary one and then an advanced one by moving points.

¶ First solution, classical Let P be the second intersection of ω_1 and ω_2; this is the Miquel point, so P also lies on the circumcircle of AMN, which is the circle with diameter $\overline{AH}$.



We now contend:

Claim — Points P, H, X collinear. (Similarly, points P, H, Y are collinear.)

Proof using power of a point. By radical axis on $BNMC$, ω_1, ω_2, it follows that A, P, W are collinear. We know that $\angle APH = 90°$, and also $\angle XPW = 90°$ by construction. Thus P, H, X are collinear. $\square$

Proof using angle chasing. This is essentially Reim's theorem:

$$\measuredangle NPH = \measuredangle NAH = \measuredangle BAH = \measuredangle ABX = \measuredangle NBX = \measuredangle NPX$$

as desired. Alternatively, one may prove A, P, W are collinear by $\measuredangle NPA = \measuredangle NMA = \measuredangle NMC = \measuredangle NBC = \measuredangle NBW = \measuredangle NPW$. $\square$

¶ Second solution, by moving points Fix $\triangle ABC$ and vary W. Let ∞ be the point at infinity perpendicular to $\overline{BC}$ for brevity.

By spiral similarity, the point X moves linearly on $\overline{B\infty}$ as W varies linearly on $\overline{BC}$. Similarly, so does Y. So in other words, the map

$$X \mapsto W \mapsto Y$$

is linear. However, the map

$$X \mapsto Y' := \overline{XH} \cap \overline{C\infty}$$

is linear too.

To show that these maps are the same, it suffices to check it thus at two points.

- When $W = B$, the circle (BNW) degenerates to the circle through B tangent to $\overline{BC}$, and $X = \overline{CN} \cap \overline{B\infty}$. We have $Y = Y' = C$.
- When $W = C$, the result is analogous.
- Although we don't need to do so, it's also easy to check the result if W is the foot from A since then $XHWB$ and $YHWC$ are rectangles.

§2.2 IMO 2013/5, proposed by Bulgaria

Available online at https://aops.com/community/p5720286.

Problem statement

Suppose a function $f : \mathbb{Q}_{>0} \to \mathbb{R}$ satisfies:

(i) If $x, y \in \mathbb{Q}_{>0}$, then $f(x)f(y) \ge f(xy)$.

(ii) If $x, y \in \mathbb{Q}_{>0}$, then $f(x+y) \ge f(x) + f(y)$.

(iii) There exists a rational number $a > 1$ with $f(a) = a$.

Prove that f is the identity function.

First, we dispense of negative situations by proving:

Claim — For any integer $n > 0$, we have $f(n) \ge n$.

Proof. Note by induction on (ii) we have $f(nx) \ge nf(x)$. Taking $(x, y) = (a, 1)$ in (i) gives $f(1) \ge 1$, and hence $f(n) \ge n$. □

Claim — The f takes only positive values, and hence by (ii) is strictly increasing.

Proof, suggested by Gopal Goel. Let $p, q > 0$ be integers. Then $f(q)f(p/q) \ge f(p)$, and since both $\min(f(p), f(q)) > 0$ it follows $f(p/q) > 0$. □

Claim — For any $x > 1$ we have $f(x) \ge x$.

Proof. Note that

$$f(x)^N \ge f(x^N) \ge f\left(\lfloor x^N \rfloor\right) \ge \lfloor x^N \rfloor > x^N - 1$$

for any integer N. Since N can be arbitrarily large, we conclude $f(x) \ge x$ for $x > 1$. □

On the other hand, f has arbitrarily large fixed points (namely powers of a) so from (ii) we're essentially done. First, for $x > 1$ pick a large m and note

$$a^m = f(a^m) \ge f(a^m - x) + f(x) \ge (a^m - x) + x = a^m.$$

Finally, for $x \le 1$ use

$$nf(x) = f(n)f(x) \ge f(nx) \ge nf(x)$$

for large n.

Remark. Note that $a > 1$ is essential; if $b \ge 1$ then $f(x) = bx^2$ works with unique fixed point $1/b \le 1$.

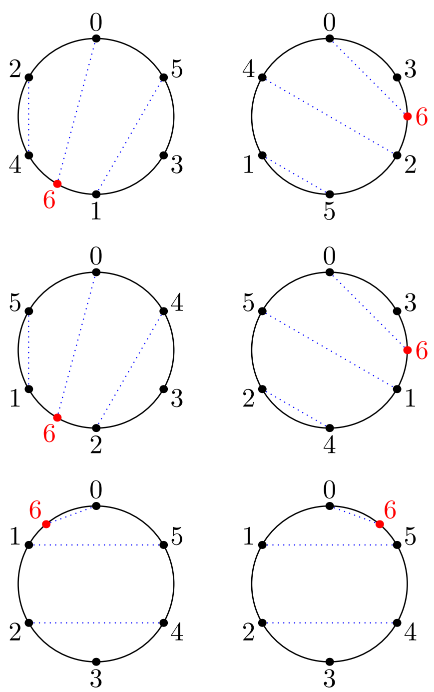
§2.3 IMO 2013/6, proposed by Russia

Available online at https://aops.com/community/p5720264.

Problem statement

Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \dots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c. Let M be the number of beautiful labelings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that $M = N + 1$.

First, here are half of the beautiful labellings up to reflection for $n = 6$, just for concreteness.



Abbreviate "beautiful labelling of points around a circle" to ring. Moreover, throughout the solution we will allow degenerate chords that join a point to itself; this has no effect on the problem statement.

The idea is to proceed by induction in the following way. A ring of $[0, n]$ is called *linear* if it is an arithmetic progression modulo $n + 1$. For example, the first two rings in the diagram and the last one are linear for $n = 6$, while the other three are not.

Of course we can move from any ring on $[0, n]$ to a ring on $[0, n-1]$ by deleting n. We are going to prove that:

- Each linear ring on $[0, n-1]$ yields exactly two rings of $[0, n]$, and
- Each nonlinear ring on $[0, n-1]$ yields exactly one rings of $[0, n]$.

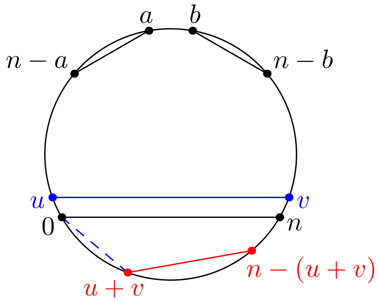
In light of the fact there are obviously $\varphi(n)$ linear rings on $[0, n]$, the conclusion will follow by induction.

We say a set of chords (possibly degenerate) is *pseudo-parallel* if for any three of them, one of them separates the two. (Pictorially, one can perturb the endpoints along the circle in order to make them parallel in Euclidean sense.) The main structure lemma is going to be:

Lemma

In any ring, the chords of sum k (even including degenerate ones) are pseudo-parallel.

Proof. By induction on n. By shifting, we may assume that one of the chords is $\{0, k\}$ and discard all numbers exceeding k; that is, assume $n = k$. Suppose the other two chords are $\{a, n-a\}$ and $\{b, n-b\}$.



We consider the chord $\{u, v\}$ directly above $\{0, n\}$, drawn in blue. There are now three cases.

- If $u + v = n$, then delete 0 and n and decrease everything by 1. Then the chords $\{a-1, n-a-1\}$, $\{b-1, n-b-1\}$, $\{u-1, v-1\}$ contradict the induction hypothesis.
- If $u + v < n$, then search for the chord $\{u + v, n - (u + v)\}$. It lies on the other side of $\{0, n\}$ in light of chord $\{0, u + v\}$. Now again delete 0 and n and decrease everything by 1. Then the chords $\{a-1, n-a-1\}$, $\{b-1, n-b-1\}$, $\{u-1, v-1\}$ contradict the induction hypothesis.
- If $u + v > n$, apply the map $t \mapsto n - t$ to the entire ring. This gives the previous case as now $(n - u) + (n - v) < n$. $\square$

Next, we give another characterization of linear rings.

Lemma

A ring on $[0, n-1]$ is linear if and only if the point 0 does not lie between two chords of sum n.

Proof. It's obviously true for linear rings. Conversely, assume the property holds for some ring. Note that the chords with sum $n-1$ are pseudo-parallel and encompass every point, so they are *actually* parallel. Similarly, the chords of sum n are *actually* parallel and encompass every point other than 0. So the map

$$t \mapsto n - t \mapsto (n-1) - (n-t) = t - 1 \pmod{n}$$

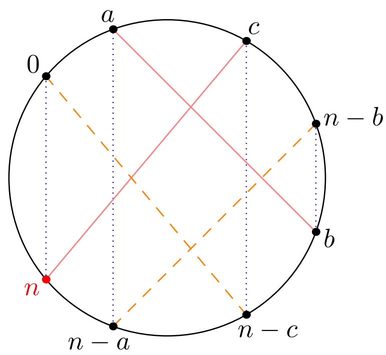
is rotation as desired. $\square$

Lemma

Every nonlinear ring on $[0, n-1]$ induces exactly one ring on $[0, n]$.

Proof. Because the chords of sum n are pseudo-parallel, there is at most one possibility for the location n.

Conversely, we claim that this works. The chords of sum n (and less than n) are OK by construction, so assume for contradiction that there exists $a, b, c \in \{1, \dots, n-1\}$ such that $a + b = n + c$. Then, we can "reflect" them using the (pseudo-parallel) chords of length n to find that $(n-a) + (n-b) = 0 + (n-c)$, and the chords joining 0 to $n-c$ and $n-a$ to $n-b$ intersect, by definition.



This is a contradiction that the original numbers on $[0, n-1]$ form a ring. □

Lemma

Every linear ring on $[0, n-1]$ induces exactly two rings on $[0, n]$.

Proof. Because the chords of sum n are pseudo-parallel, the point n must lie either directly to the left or right of 0. For the same reason as in the previous proof, both of them work. □

IMO 2014 Solution Notes

Evan Chen《陳誼廷》

29 June 2023

This is a compilation of solutions for the 2014 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \ge 1$ such that
$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \le a_{n+1}.$$

2. Let $n \ge 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$. Point H is the foot of the perpendicular from A to $\overline{BD}$. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and
$$\angle CHS - \angle CSB = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ.$$
Prove that line BD is tangent to the circumcircle of triangle TSH.

4. Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be points on $\overline{AP}$ and $\overline{AQ}$, respectively, such that P is the midpoint of $\overline{AM}$ and Q is the midpoint of $\overline{AN}$. Prove that $\overline{BM}$ and $\overline{CN}$ meet on the circumcircle of $\triangle ABC$.

5. For every positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

§1 Solutions to Day 1

§1.1 IMO 2014/1, proposed by Gerhard Woeginger (AUT)

Available online at https://aops.com/community/p3542095.

Problem statement

Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

Fedor Petrov presents the following nice solution. Let us define the sequence

$$b_n = (a_n - a_{n-1}) + \cdots + (a_n - a_1).$$

Since $(a_i)_i$ is increasing, this sequence is unbounded, and moreover $b_1 = 0$. The problem requires an n such that

$$b_n < a_0 \leq b_{n+1}$$

which obviously exists and is unique.

§1.2 IMO 2014/2, proposed by Croatia

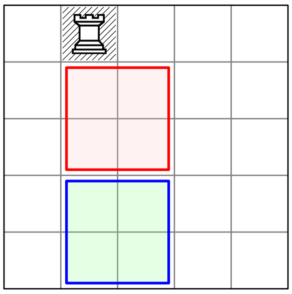
Available online at https://aops.com/community/p3542094.

Problem statement

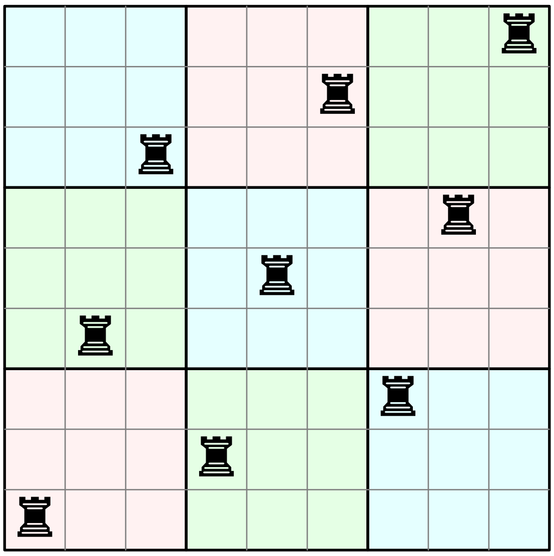
Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

The answer is $k = \lfloor \sqrt{n-1} \rfloor$, sir.

First, assume $n > k^2$ for some k. We will prove we can find an empty $k \times k$ square. Indeed, let R be a rook in the uppermost column, and draw k squares of size $k \times k$ directly below it, aligned. There are at most $k-1$ rooks among these squares, as desired.



Now for the construction for $n = k^2$. We draw the example for $k = 3$ (with the generalization being obvious);



To show that this works, consider for each rook drawing an $k \times k$ square of X's whose bottom-right hand corner is the rook (these may go off the board). These indicate positions where one cannot place the upper-left hand corner of any square. It's easy to see that these cover the entire board, except parts of the last $k-1$ columns, which don't matter anyways.

It remains to check that $n \le k^2$ also all work (omitting this step is a common mistake). For this, we can delete rows and column to get an $n \times n$ board, and then fill in any gaps where we accidentally deleted a rook.

§1.3 IMO 2014/3, proposed by Iran

Available online at https://aops.com/community/p3542092.

Problem statement

Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to $\overline{BD}$. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

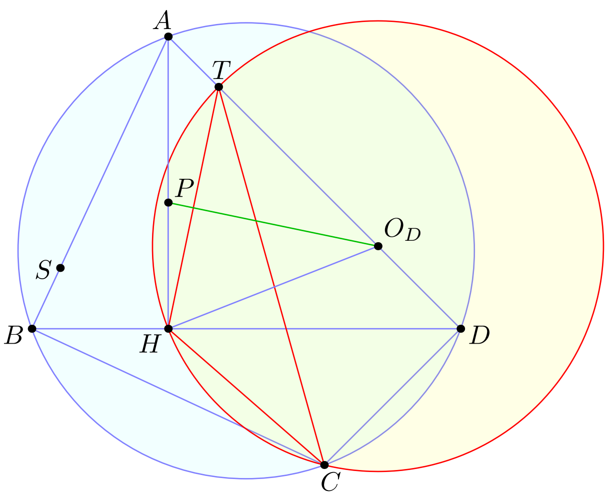
¶ **First solution (mine)** First we rewrite the angle condition in a suitable way.

Claim — We have $\angle ATH = \angle TCH + 90°$. Thus the circumcenter of $\triangle CTH$ lies on $\overline{AD}$. Similarly the circumcenter of $\triangle CSH$ lies on $\overline{AB}$.

Proof.

$$\begin{aligned} \measuredangle ATH &= \measuredangle DTH \\ &= \measuredangle DTC + \measuredangle CTH \\ &= \measuredangle DTC - \measuredangle THC - \measuredangle HCT \\ &= 90° - \measuredangle HCT = 90° + \measuredangle TCH. \end{aligned}$$

which implies conclusion. □



Let the perpendicular bisector of TH meet AH at P now. It suffices to show that $\frac{AP}{PH}$ is symmetric in $b = AD$ and $d = AB$, because then P will be the circumcenter of $\triangle TSH$. To do this, set $AH = \frac{bd}{2R}$ and $AC = 2R$.

Let O denote the circumcenter of $\triangle CHT$. Use the Law of Cosines on $\triangle ACO$ and $\triangle AHO$, using variables $x = AO$ and $r = HO$. We get that

$$r^2 = x^2 + AH^2 - 2x \cdot AH \cdot \frac{d}{2R} = x^2 + (2R)^2 - 2bx.$$

By the angle bisector theorem, $\frac{AP}{PH} = \frac{AO}{HO}$.

The rest is computation: notice that

$$r^2 - x^2 = h^2 - 2xh \cdot \frac{d}{2R} = (2R)^2 - 2bx$$

where $h = AH = \frac{bd}{2R}$, whence

$$x = \frac{(2R)^2 - h^2}{2b - 2h \cdot \frac{d}{2R}}.$$

Moreover,

$$\frac{1}{2}\left(\frac{r^2}{x^2} - 1\right) = \frac{1}{x}\left(\frac{2}{x}R^2 - b\right).$$

Now, if we plug in the x in the right-hand side of the above, we obtain

$$\frac{2b - 2h \cdot \frac{d}{2R}}{4R^2 - h^2}\left(\frac{2b - 2h \cdot \frac{d}{2R}}{4R^2 - h^2} \cdot 2R^2 - b\right) = \frac{2h}{(4R^2 - h^2)^2}\left(b - h \cdot \frac{d}{2R}\right)\left(-2hdR + bh^2\right).$$

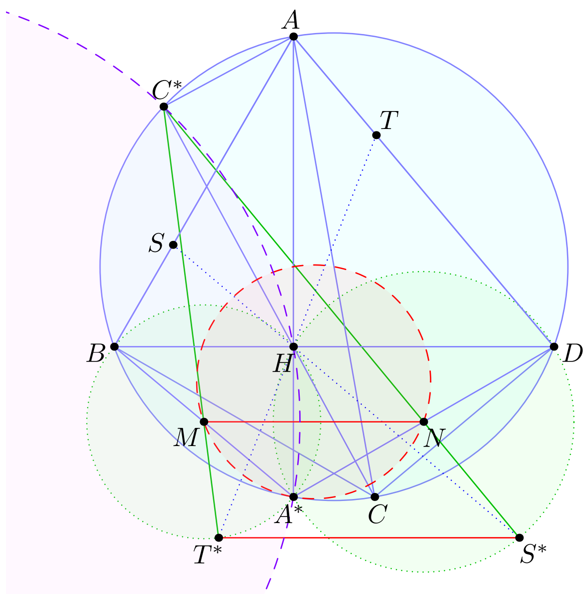
Pulling out a factor of $-2Rh$ from the rightmost term, we get something that is symmetric in b and d, as required.

¶ Second solution (Victor Reis) Here is the fabled solution using inversion at H. First, we rephrase the angle conditions in the following ways:

- $\overline{AD} \perp (THC)$, which is equivalent to the claim from the first solution.
- $\overline{AB} \perp (SHC)$, by symmetry.
- $\overline{AC} \perp (ABCD)$, by definition.

Now for concreteness we will use a negative inversion at H which swaps B and D and overlay it on the original diagram. As usual we denote inverses with stars.

Let us describe the inverted problem. We let M and N denote the midpoints of $\overline{A^*B^*}$ and $\overline{A^*D^*}$, which are the centers of (HA^*B^*) and (HA^*D^*). From $\overline{T^*C^*} \perp (HA^*D^*)$, we know have C^*, M, T^* collinear. Similarly, C^*, N, S^* are collinear. We have that (A^*HC^*) is orthogonal to $(ABCD)$ which remains fixed. We wish to show $\overline{T^*S^*}$ and $\overline{MN}$ are parallel.



Lot ω denote the circumcircle of $\triangle A^*HC^*$, which is orthogonal to the original circle $(ABCD)$. It would suffices to show (A^*HC^*) is an H-Apollonius circle with respect to $\overline{MN}$, from which we would get $C^*M/HM = C^*N/HN$.

However, ω through H and A, hence it center lies on line MN. Moreover ω is orthogonal to (A^*MN) (since (A^*MN) and (A^*BD) are homothetic). This is enough (for example, if we let O denote the center of ω, we now have $\mathrm{r}(\omega)^2 = OH^2 = OM \cdot ON$). (Note in this proof that the fact that C^* lies on $(ABCD)$ is not relevant.)

§2 Solutions to Day 2

§2.1 IMO 2014/4, proposed by Giorgi Arabidze (GEO)

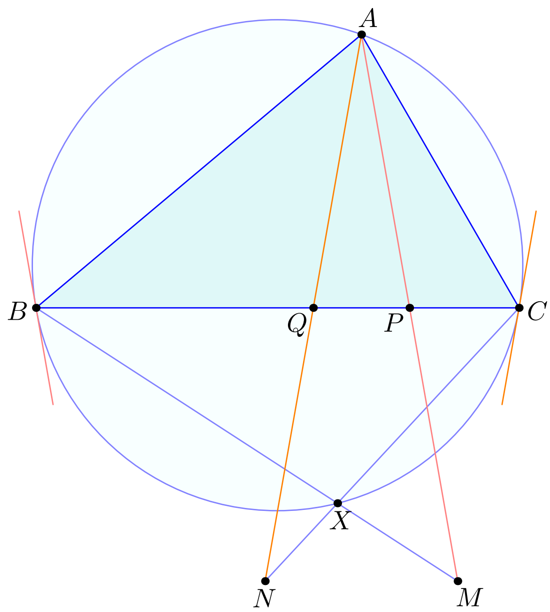
Available online at https://aops.com/community/p3543136.

Problem statement

Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be points on $\overline{AP}$ and $\overline{AQ}$, respectively, such that P is the midpoint of $\overline{AM}$ and Q is the midpoint of $\overline{AN}$. Prove that $\overline{BM}$ and $\overline{CN}$ meet on the circumcircle of $\triangle ABC$.

We give three solutions.

¶ First solution by harmonic bundles Let $\overline{BM}$ intersect the circumcircle again at X.



The angle conditions imply that the tangent to (ABC) at B is parallel to $\overline{AP}$. Let ∞ be the point at infinity along line AP. Then

$$-1 = (AM; P\infty) \stackrel{B}{=} (AX; BC).$$

Similarly, if $\overline{CN}$ meets the circumcircle at Y then $(AY; BC) = -1$ as well. Hence $X = Y$, which implies the problem.

¶ Second solution by similar triangles Once one observes $\triangle CAQ \sim \triangle CBA$, one can construct D the reflection of B across A, so that $\triangle CAN \sim \triangle CBD$. Similarly, letting E be the reflection of C across A, we get $\triangle BAP \sim \triangle BCA \implies \triangle BAM \sim \triangle BCE$. Now to show $\angle ABM + \angle ACN = 180^\circ$ it suffices to show $\angle EBC + \angle BCD = 180^\circ$, which follows since $BCDE$ is a parallelogram.

¶ Third solution by barycentric coordinates Since $PB = c^2/a$ we have

$$P = (0 : a^2 - c^2 : c^2)$$

so the reflection $\vec{M} = 2\vec{P} - \vec{A}$ has coordinates

$$M = (-a^2 : 2(a^2 - c^2) : 2c^2).$$

Similarly $N = (-a^2 : 2b^2 : 2(b^2 - a^2))$. Thus

$$\overline{BM} \cap \overline{CN} = (-a^2 : 2b^2 : 2c^2)$$

which clearly lies on the circumcircle, and is in fact the point identified in the first solution.

§2.2 IMO 2014/5, proposed by Luxembourg

Available online at https://aops.com/community/p3543144.

Problem statement

For every positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

We'll prove the result for at most $k - \frac{k}{2k+1}$ with k groups. First, perform the following optimizations.

- If any coin of size $\frac{1}{2m}$ appears twice, then replace it with a single coin of size $\frac{1}{m}$.
- If any coin of size $\frac{1}{2m+1}$ appears $2m+1$ times, group it into a single group and induct downwards.

Apply this operation repeatedly until it cannot be done anymore.

Now construct boxes B_0, B_1, ..., B_{k-1}. In box B_0 put any coins of size $\frac{1}{2}$ (clearly there is at most one). In the other boxes B_m, put coins of size $\frac{1}{2m+1}$ and $\frac{1}{2m+2}$ (at most $2m$ of the former and at most one of the latter). Note that the total weight in the box is less than 1. Finally, place the remaining "light" coins of size at most $\frac{1}{2k+1}$ in a pile.

Then just toss coins from the pile into the boxes arbitrarily, other than the proviso that no box should have its weight exceed 1. We claim this uses up all coins in the pile. Assume not, and that some coin remains in the pile when all the boxes are saturated. Then all the boxes must have at least $1 - \frac{1}{2k+1}$, meaning the total amount in the boxes is strictly greater than

$$k\left(1 - \frac{1}{2k+1}\right) > k - \tfrac{1}{2}$$

which is a contradiction.

Remark. This gets a stronger bound $k - \frac{k}{2k+1}$ than the requested $k - \frac{1}{2}$.

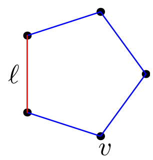
§2.3 IMO 2014/6, proposed by Austria

Available online at https://aops.com/community/p3543151.

Problem statement

A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

Suppose we have colored k of the lines blue, and that it is not possible to color any additional lines. That means any of the $n-k$ non-blue lines is the side of some finite region with an otherwise entirely blue perimeter. For each such line ℓ, select one such region, and take the next counterclockwise vertex; this is the intersection of two blue lines v. We'll say ℓ is the *eyelid* of v.

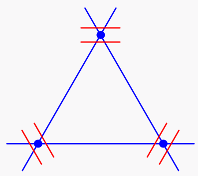


You can prove without too much difficulty that every intersection of two blue lines has at most two eyelids. Since there are $\binom{k}{2}$ such intersections, we see that

$$n - k \le 2\binom{k}{2} = k^2 - k$$

so $n \le k^2$, as required.

Remark. In fact, $k = \sqrt{n}$ is "sharp for greedy algorithms", as illustrated below for $k = 3$:



IMO 2015 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

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Corrections and comments are welcome!

Contents

§0 Problems

1. We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points A, B and C in $\mathcal{S}$, there are no points P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \ge 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \ge 3$ for which there exists a balanced centre-free set consisting of n points.

2. Find all positive integers a, b, c such that each of $ab - c$, $bc - a$, $ca - b$ is a power of 2 (possibly including $2^0 = 1$).

3. Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of $\overline{BC}$. Let Q be the point on Γ such that $\angle HQA = 90^\circ$ and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

4. Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let $K = (BDF) \cap \overline{AB} \neq B$ and $L = (CGE) \cap \overline{AC} \neq C$ and assume these points do not lie on line FG. Define $X = \overline{FK} \cap \overline{GL}$. Prove that X lies on the line AO.

5. Solve the functional equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for $f\colon \mathbb{R} \to \mathbb{R}$.

6. The sequence $a_1, a_2, \dots$ of integers satisfies the conditions:

(i) $1 \le a_j \le 2015$ for all $j \ge 1$,

(ii) $k + a_k \neq \ell + a_\ell$ for all $1 \le k < \ell$.

Prove that there exist two positive integers b and N for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \le 1007^2$$

for all integers m and n such that $n > m \ge N$.

§1 Solutions to Day 1

§1.1 IMO 2015/1, proposed by Netherlands

Available online at https://aops.com/community/p5079689.

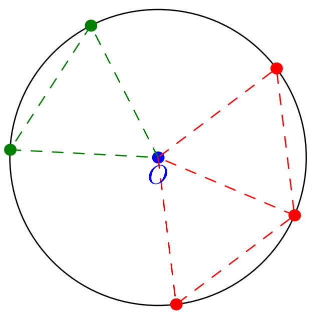
Problem statement

We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points A, B and C in $\mathcal{S}$, there are no points P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

For part (a), take a circle centered at a point O, and add $n-1$ additional points by adding pairs of points separated by an arc of $60°$ or similar triples. An example for $n = 6$ is shown below.



For part (b), the answer is odd n, achieved by taking a regular n-gon. To show even n fail, note that some point is on the perpendicular bisector of

$$\left\lceil \frac{1}{n}\binom{n}{2} \right\rceil = \frac{n}{2}$$

pairs of points, which is enough. (This is a standard double-counting argument.)

As an aside, there is a funny joke about this problem. There are two types of people in the world: those who solve (b) quickly and then take forever to solve (a), and those who solve (a) quickly and then can't solve (b) at all. (Empirically true when the Taiwan IMO 2014 team was working on it.)

§1.2 IMO 2015/2, proposed by Serbia

Available online at https://aops.com/community/p5079630.

Problem statement

Find all positive integers a, b, c such that each of $ab-c$, $bc-a$, $ca-b$ is a power of 2 (possibly including $2^0=1$).

Here is the solution of **Telv Cohl**, which is the shortest solution I am aware of. We will prove the only solutions are $(2,2,2)$, $(2,2,3)$, $(2,6,11)$ and $(3,5,7)$ and permutations.

WLOG assume $a \ge b \ge c > 1$, so $ab-c \ge ca-b \ge bc-a$. We consider the following cases:

- If a is even, then

$$\begin{aligned} ca-b &= \gcd(ab-c, ca-b) \le \gcd(ab-c, a(ca-b)+ab-c) \\ &= \gcd\left(ab-c, c(a^2-1)\right). \end{aligned}$$

  As a^2-1 is odd, we conclude $ca-b \le c$. This implies $a=b=c=2$.

- If a, b, c are all odd, then $a>b>c>1$ follows. Then as before

$$ca-b \le \gcd(ab-c, c(a^2-1)) \le 2^{\nu_2(a^2-1)} \le 2a+2 \le 3a-b$$

  so $c=3$ and $a=b+2$. As $3a-b=ca-b \ge 2(bc-a)=6b-2a$ we then conclude $a=7$ and $b=5$.

- If a is odd and b, c are even, then $bc-a=1$ and hence $bc^2-b-c=ca-b$. Then from the miraculous identity

$$c^3-b-c = (1-c^2)(ab-c) + a(\underbrace{bc^2-b-c}_{=ca-b}) + (ca-b)$$

  so we conclude $\gcd(ab-c, ca-b) = \gcd(ab-c, c^3-b-c)$, in other words

$$bc^2-b-c = ca-b = \gcd(ab-c, ca-b) = \gcd(ab-c, c^3-b-c).$$

  We thus consider two more cases:

  - If $c^3-b-c \ne 0$ then the above implies $|c^3-b-c| \ge bc^2-b-c$. As $b \ge c > 1$, we must actually have $b=c$, thus $a=c^2-1$. Finally $ab-c=c(c^2-2)$ is a power of 2, hence $b=c=2$, so $a=3$.
  - In the second case, assume $c^3-b-c=0$, hence c^3-c. From $bc-a=1$ we obtain $a=c^4-c^2-1$, hence

$$ca-b = c^5-2c^3 = c^3(c^2-2)$$

    is a power of 2, hence again $c=2$. Thus $a=11$ and $b=6$.

This finishes all cases, so the proof is done.

§1.3 IMO 2015/3, proposed by Ukraine

Available online at https://aops.com/community/p5079655.

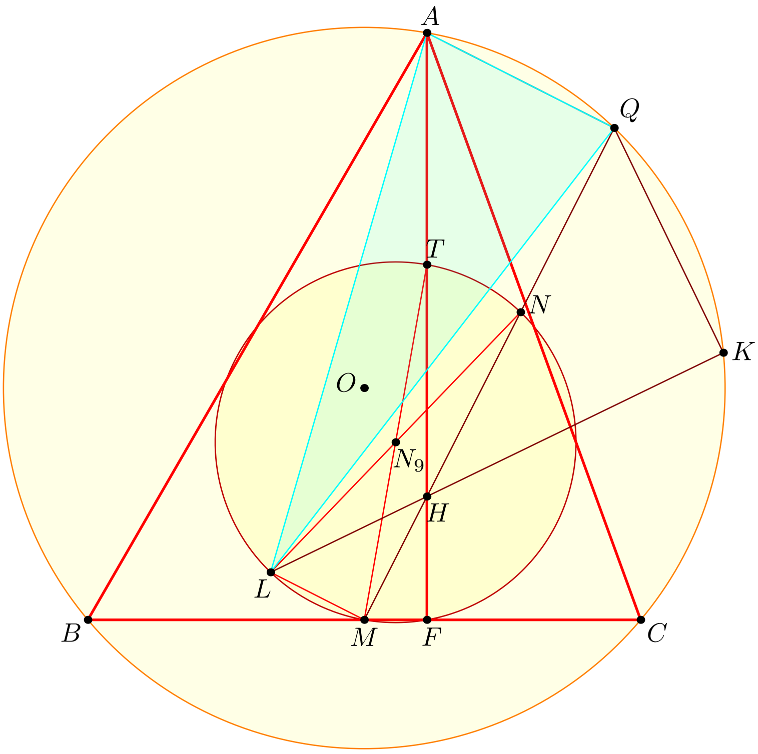
Problem statement

Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of $\overline{BC}$. Let Q be the point on Γ such that $\angle HQA = 90^\circ$ and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Let L be on the nine-point circle with $\angle HML = 90^\circ$. The negative inversion at H swapping Γ and nine-point circle maps

$$A \longleftrightarrow F, \quad Q \longleftrightarrow M, \quad K \longleftrightarrow L.$$

In the inverted statement, we want line ML to be tangent to (AQL).



Claim — $\overline{LM} \parallel \overline{AQ}$.

Proof. Both are perpendicular to $\overline{MHQ}$. □

Claim — $LA = LQ$.

Proof. Let N and T be the midpoints of $\overline{HQ}$ and $\overline{AH}$, and O the circumcenter. As $\overline{MT}$ is a diameter, we know $LTNM$ is a rectangle, so $\overline{LT}$ passes through O. Since $\overline{LOT} \perp \overline{AQ}$ and $OA = OQ$, the proof is complete. $\square$

Together these two claims solve the problem.

§2 Solutions to Day 2

§2.1 IMO 2015/4, proposed by Silouanos Brazitikos (HEL)

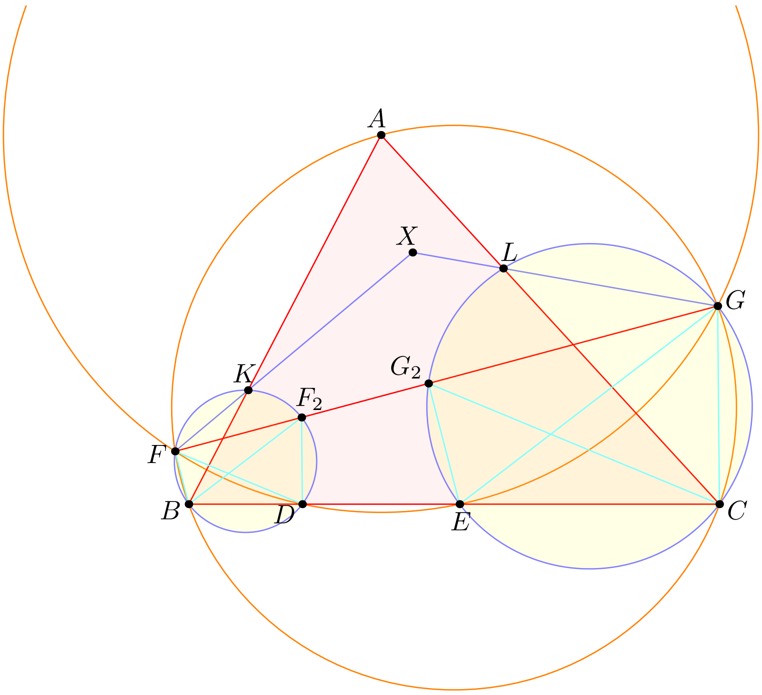
Available online at https://aops.com/community/p5083464.

Problem statement

Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let $K = (BDF) \cap \overline{AB} \neq B$ and $L = (CGE) \cap \overline{AC} \neq C$ and assume these points do not lie on line FG. Define $X = \overline{FK} \cap \overline{GL}$. Prove that X lies on the line AO.

Since $\overline{AO} \perp \overline{FG}$ for obvious reasons, we will only need to show that $XF = XG$, or that $\measuredangle KFG = \measuredangle LGF$.

Let line FG meet (BDF) and (CGE) again at F_2 and G_2.



Claim — Quadrilaterals $FBDF_2$ and G_2ECG are similar, actually homothetic through $\overline{FG} \cap \overline{BC}$.

Proof. This is essentially a repeated application of being "anti-parallel" through $\angle(FG, BC)$. Note the four angle relations

$$\begin{aligned}
\measuredangle(FD, FG) = \measuredangle(BC, GE) = \measuredangle(G_2C, FG) &\implies \overline{FD} \parallel \overline{G_2C} \\
\measuredangle(F_2B, FG) = \measuredangle(BC, FD) = \measuredangle(GE, FG) &\implies \overline{F_2B} \parallel \overline{GE} \\
\measuredangle(FB, FG) = \measuredangle(BC, GC) = \measuredangle(G_2E, FG) &\implies \overline{FB} \parallel \overline{G_2E} \\
\measuredangle(F_2D, FG) = \measuredangle(BC, FB) = \measuredangle(GC, FG) &\implies \overline{F_2D} \parallel \overline{GC}.
\end{aligned}$$

This gives the desired homotheties. □

To finish the angle chase,

$$\begin{aligned}\measuredangle GFK = \measuredangle F_2BK &= \measuredangle F_2BF - \measuredangle ABF = \measuredangle F_2DF - \measuredangle ABF \\ &= \measuredangle F_2DF - \measuredangle GCA = \measuredangle GCG_2 - \measuredangle GCA \\ &= \measuredangle LCG_2 = \measuredangle LGF\end{aligned}$$

as needed. (Here $\measuredangle ABF = \measuredangle GCA$ since $AF = AG$.)

§2.2 IMO 2015/5, proposed by Dorlir Ahmeti (ALB)

Available online at https://aops.com/community/p5083463.

Problem statement

Solve the functional equation

$$f(x+f(x+y))+f(xy)=x+f(x+y)+yf(x)$$

for $f\colon \mathbb{R}\to\mathbb{R}$.

The answers are $f(x)\equiv x$ and $f(x)\equiv 2-x$. Obviously, both of them work.

Let $P(x,y)$ be the given assertion. We also will let $S=\{t \mid f(t)=t\}$ be the set of fixed points of f.

- From $P(0,0)$ we get $f(f(0))=0$.
- From $P(0,f(0))$ we get $2f(0)=f(0)^2$ and hence $f(0)\in\{0,2\}$.
- From $P(x,1)$ we find that $x+f(x+1)\in S$ for all x.

We now solve the case $f(0)=2$.

Claim — If $f(0)=2$ then $f(x)\equiv 2-x$.

Proof. Let $t\in S$ be any fixed point. Then $P(0,t)$ gives $2=2t$ or $t=1$; so $S=\{1\}$. But we also saw $x+f(x+1)\in S$, which implies $f(x)\equiv 2-x$. □

Henceforth, assume $f(0)=0$.

Claim — If $f(0)=0$ then f is odd.

Proof. Note that $P(1,-1) \implies f(1)+f(-1)=1-f(1)$ and $P(-1,1) \implies f(-1)+f(-1)=-1+f(1)$, together giving $f(1)=1$ and $f(-1)=-1$. To prove f odd we now obtain more fixed points:

- From $P(x,0)$ we find that $x+f(x)\in S$ for all $x\in\mathbb{R}$.
- From $P(x-1,1)$ we find that $x-1+f(x)\in S$ for all $x\in\mathbb{R}$.
- From $P(1,f(x)+x-1)$ we find $x+1+f(x)\in S$ for all $x\in\mathbb{R}$.

Finally $P(x,-1)$ gives f odd. □

To finish from f odd, notice that

$$\begin{aligned} P(x,-x) &\implies f(x)+f(-x^2)=x-xf(x) \\ P(-x,x) &\implies f(-x)+f(-x^2)=-x+xf(-x) \end{aligned}$$

which upon subtracting gives $f(x)\equiv x$.

§2.3 IMO 2015/6, proposed by Australia

Available online at https://aops.com/community/p5083494.

Problem statement

The sequence $a_1, a_2, \dots$ of integers satisfies the conditions:

(i) $1 \le a_j \le 2015$ for all $j \ge 1$,

(ii) $k + a_k \neq \ell + a_\ell$ for all $1 \le k < \ell$.

Prove that there exist two positive integers b and N for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \le 1007^2$$

for all integers m and n such that $n > m \ge N$.

We give two equivalent solutions with different presentations, one with "arrows" and the other by "juggling".

¶ First solution (arrows) Consider the map

$$f \colon k \mapsto k + a_k.$$

This map is injective, so if we draw all arrows of the form $k \mapsto f(k)$ we get a partition of $\mathbb{N}$ into one or more ascending chains (which skip by at most 2015).

There are at most 2015 such chains, since among any 2015 consecutive points in $\mathbb{N}$ every chain must have an element.

We claim we may take b to be the number of such chains, and N to be the largest of the start-points of all the chains.

Consider an interval $I = [m+1, n]$. We have that

$$\sum_{m<j\le n} a_j = \sum_{\text{chain } c} \left[\min\left\{x > n, x \in c\right\} - \min\left\{x > m, x \in c\right\} \right].$$

Thus the upper bound is proved by the calculation

$$\begin{aligned} \sum_{m<j\le n} (a_j - b) &= \sum_{\text{chain } c} \left[\left(\min\left\{x > n, x \in c\right\} - n\right) - \left(\min\left\{x > m, x \in c\right\} - m\right) \right] \\ &= \sum_{\text{chain } c} \left[\left(\min\left\{x > n, x \in c\right\} - n\right) \right] - \sum_{\text{chain } c} \left[\min\left\{x > m, x \in c\right\} - m \right] \\ &\le \left(1 + 2015 + 2014 + \dots + (2015 - (b-2))\right) - (1 + 2 + \dots + b) = (b-1)(2015-b) \end{aligned}$$

from above (noting that $n+1$ has to belong to some chain). The lower bound is similar.

¶ Second solution (juggling) This solution is essentially the same, but phrased as a juggling problem. Here is a solution in this interpretation: we will consider several balls thrown in the air, which may be at heights $0, 1, 2, \dots, 2014$. The process is as follows:

- Initially, at time $t = 0$, there are no balls in the air.

- Then at each integer time t thereafter, if there is a ball at height 0, it is caught; otherwise a ball is added to the juggler's hand. This ball (either caught or added) is then thrown to a height of a_t.
- Immediately afterwards, all balls have their height decreased by one.

The condition $a_k + k \neq \ell + a_\ell$ thus ensures that no two balls are ever at the same height. In particular, there will never be more than 2016 balls, since there are only 2015 possible heights.

We claim we may set.

$$\begin{aligned} b &= \text{number of balls in entire process} \\ N &= \text{last moment in time at which a ball is added.} \end{aligned}$$

Indeed, the key fact is that if we let S_t denote the sum of the height of all the balls just after time $t + \frac{1}{2}$, then

$$S_{t+1} - S_t = a_{t+1} - b$$

After all, at each time step t, the caught ball is thrown to height a_t, and then all balls have their height decreased by 1, from which the conclusion follows. Hence the quantity in the problem is exactly equal to

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| = \left| S_m - S_n \right|.$$

For a fixed b, we easily have the inequalities $0 + 1 + \dots + (b-1) \le S_t \le 2014 + 2013 + \dots + (2015 - b)$. Hence $|S_m - S_n| \le (b-1)(2015-b) \le 1007^2$ as desired.

IMO 2016 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2016 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. In convex pentagon $ABCDE$ with $\angle B > 90^\circ$, let F be a point on $\overline{AC}$ such that $\angle FBC = 90^\circ$. It is given that $FA = FB$, $DA = DC$, $EA = ED$, and rays $\overline{AC}$ and $\overline{AD}$ trisect $\angle BAE$. Let M be the midpoint of $\overline{CF}$. Let X be the point such that $AMXE$ is a parallelogram. Show that $\overline{FX}$, $\overline{EM}$, $\overline{BD}$ are concurrent.

2. Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:
   - In each row and column, one third of the entries are I, one third are M and one third are O; and
   - in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

   Note that an $n \times n$ table has $4n - 2$ diagonals.

3. Let $P = A_1A_2 \cdots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \dots, A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the smallest possible positive integer value of b such that there exists a non-negative integer a for which the set

   $$\{P(a+1), P(a+2), \dots, P(a+b)\}$$

   is fragrant?

5. The equation

   $$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

   is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

6. There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n - 1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

   (a) Prove that Geoff can always fulfill his wish if n is odd.

   (b) Prove that Geoff can never fulfill his wish if n is even.

§1 Solutions to Day 1

§1.1 IMO 2016/1, proposed by Art Waeterschoot (BEL)

Available online at https://aops.com/community/p6637656.

Problem statement

In convex pentagon $ABCDE$ with $\angle B > 90^\circ$, let F be a point on $\overline{AC}$ such that $\angle FBC = 90^\circ$. It is given that $FA = FB$, $DA = DC$, $EA = ED$, and rays $\overline{AC}$ and $\overline{AD}$ trisect $\angle BAE$. Let M be the midpoint of $\overline{CF}$. Let X be the point such that $AMXE$ is a parallelogram. Show that $\overline{FX}$, $\overline{EM}$, $\overline{BD}$ are concurrent.

Here is a "long" solution which I think shows where the "power" in the configuration comes from (it should be possible to come up with shorter solutions by cutting more directly to the desired conclusion). Throughout the proof, we let

$$\theta = \angle FAB = \angle FBA = \angle DAC = \angle DCA = \angle EAD = \angle EDA.$$

We begin by focusing just on $ABCD$ with point F, ignoring for now the points E and X (and to some extent even point M). It turns out this is a very familiar configuration.

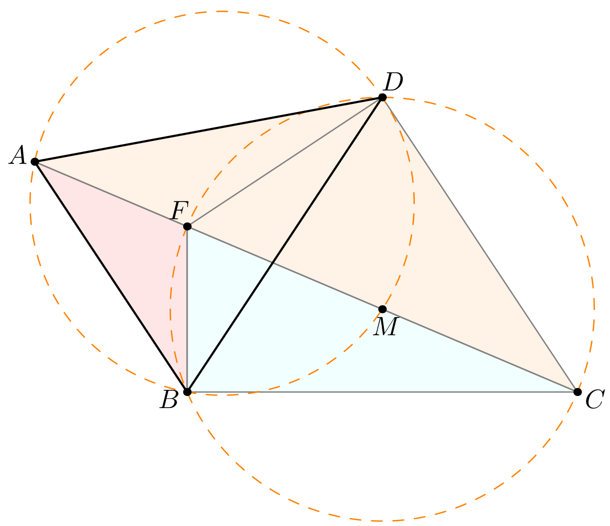
Lemma (Central lemma)

The points F and C are the incenter and A-excenter of $\triangle DAB$. Moreover, $\triangle DAB$ is isosceles with $DA = DB$.

Proof. The proof uses three observations:

- We already know that $\overline{FAC}$ is the angle bisector of $\angle ABD$.
- We were given $\angle FBC = 90^\circ$.
- Next, note that $\triangle AFB \sim \triangle ADC$ (they are similar isosceles triangles). From this it follows that $AF \cdot AC = AB \cdot AD$.

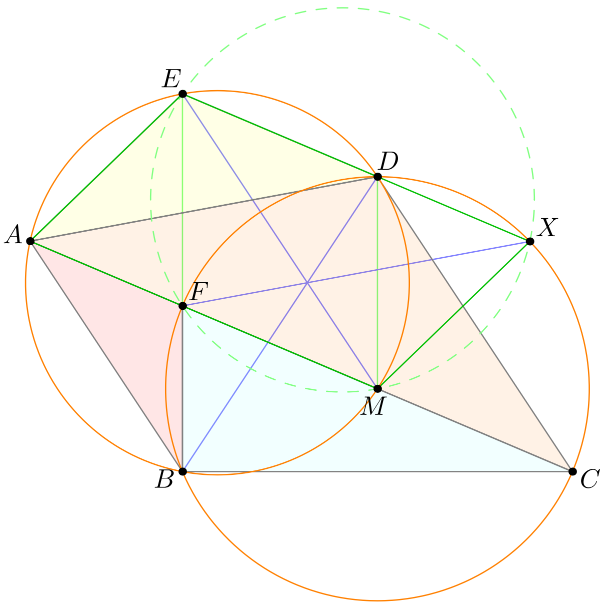
These three facts, together with F lying inside $\triangle ABD$, are enough to imply the result. $\square$



Corollary
The point M is the midpoint of arc $\widehat{BD}$ of (DAB), and the center of cyclic quadrilateral $FDCB$.

Proof. Fact 5. ☐

Using these observations as the anchor for everything that follows, we now prove several claims about X and E in succession.



Claim — Point E is the midpoint of arc $\widehat{AD}$ in $(ABMD)$, and hence lies on ray BF.

Proof. This follows from $\angle EDA = \theta = \angle EBA$. ☐

Claim — Points X is the second intersection of ray $\overline{ED}$ with $(BFDC)$.

Proof. First, $\overline{ED} \parallel \overline{AC}$ already since $\angle AED = 180^\circ - 2\theta$ and $\angle CAE = 2\theta$.
Now since $DB = DA$, we get $MB = MD = ED = EA$. Thus, $MX = AE = MB$, so X also lies on the circle $(BFDC)$ centered at M. ☐

Claim — The quadrilateral $EXMF$ is an isosceles trapezoid.

Proof. We already know $\overline{EX} \parallel \overline{FM}$. Since $\angle EFA = 180^\circ - \angle AFB = 2\theta = \angle FAE$, we have $EF = EA$ as well (and $F \neq A$). As $EXMA$ was a parallelogram, it follows $EXMF$ is an isosceles trapezoid. ☐

The problem then follows by radical axis theorem on the three circles $(AEDMB)$, $(BFDXC)$ and $(EXMF)$.

§1.2 IMO 2016/2, proposed by Trevor Tau (AUS)

Available online at https://aops.com/community/p6637677.

Problem statement

Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

- In each row and column, one third of the entries are I, one third are M and one third are O; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Note that an $n \times n$ table has $4n-2$ diagonals.

The answer is n divisible by 9.

First we construct $n=9$ and by extension every multiple of 9.

I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M
I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M
I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M

We now prove $9 \mid n$ is necessary.

Let $n = 3k$, which divides the given grid into k^2 sub-boxes (of size 3×3 each). We say a multiset of squares S is *clean* if the letters distribute equally among them; note that unions of clean multisets are clean.

Consider the following clean sets (given to us by problem statement):

- All columns indexed $2 \pmod 3$,
- All rows indexed $2 \pmod 3$, and
- All $4k-2$ diagonals mentioned in the problem.

Take their union. This covers the center of each box four times, and every other cell exactly once. We conclude the set of k^2 center squares are clean, hence $3 \mid k^2$ and so $9 \mid n$, as desired.

Shown below is the sums over all diagonals only, and of the entire union.

1		1	1		1	1		1
	2			2			2	
1		1	1		1	1		1
1		1	1		1	1		1
	2			2			2	
1		1	1		1	1		1
1		1	1		1	1		1
	2			2			2	
1		1	1		1	1		1

1	1	1	1	1	1	1	1	1
1	4	1	1	4	1	1	4	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
1	4	1	1	4	1	1	4	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
1	4	1	1	4	1	1	4	1
1	1	1	1	1	1	1	1	1

§1.3 IMO 2016/3, proposed by Russia

Available online at https://aops.com/community/p6637660.

Problem statement

Let $P = A_1A_2\cdots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \dots, A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

Solution by Jeck Lim: We will prove the result just for $n = p^e$ where p is an odd prime and $e \geq 1$. The case $k = 3$ is resolved by Heron's formula directly: we have $S = \frac{1}{4}\sqrt{2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4}$, so if $p^e \mid \gcd(a^2, b^2, c^2)$ then $p^{2e} \mid S^2$.

Now we show we can pick a diagonal and induct down on k by using inversion.

Let the polygon be $A_1A_2 \dots A_{k+1}$ and suppose for contradiction that all sides are divisible by p^e but no diagonals are. Let $O = A_{k+1}$ for notational convenience. By applying inversion around O with radius 1, we get the "generalized Ptolemy theorem"

$$\frac{A_1A_2}{OA_1 \cdot OA_2} + \frac{A_2A_3}{OA_2 \cdot OA_3} + \dots + \frac{A_{k-1}A_k}{OA_{k-1} \cdot OA_k} = \frac{A_1A_k}{OA_1 \cdot OA_k}$$

or, making use of square roots,

$$\sqrt{\frac{A_1A_2^2}{OA_1^2 \cdot OA_2^2}} + \sqrt{\frac{A_2A_3^2}{OA_2^2 \cdot OA_3^2}} + \dots + \sqrt{\frac{A_{k-1}A_k^2}{OA_{k-1}^2 \cdot OA_k^2}} = \sqrt{\frac{A_1A_k^2}{OA_1^2 \cdot OA_k^2}}$$

Suppose ν_p of all diagonals is strictly less than e. Then the relation becomes

$$\sqrt{q_1} + \dots + \sqrt{q_{k-1}} = \sqrt{q}$$

where q_i are positive rational numbers. Since there are no nontrivial relations between square roots (see this link) there is a positive rational number b such that $r_i = \sqrt{q_i/b}$ and $r = \sqrt{q/b}$ are all rational numbers. Then

$$\sum r_i = r.$$

However, the condition implies that $\nu_p(q_i^2) > \nu_p(q^2)$ for all i (check this for $i = 1$, $i = k-1$ and $2 \leq i \leq k-2$), and hence $\nu_p(r_i) > \nu_p(r)$. This is absurd.

Remark. I think you basically have to use some Ptolemy-like geometric property, and also all correct solutions I know of $n = p^e$ depend on finding a diagonal and inducting down. (Actually, the case $k = 4$ is pretty motivating; Ptolemy implies one can cut in two.)

§2 Solutions to Day 2

§2.1 IMO 2016/4, proposed by Luxembourg

Available online at https://aops.com/community/p6642559.

Problem statement

A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the smallest possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \dots, P(a+b)\}$$

is fragrant?

The answer is $b = 6$.

First, we prove $b \ge 6$ must hold. It is not hard to prove the following divisibilities by Euclid:

$$\begin{aligned} \gcd(P(n), P(n+1)) &\mid 1 \\ \gcd(P(n), P(n+2)) &\mid 7 \\ \gcd(P(n), P(n+3)) &\mid 3 \\ \gcd(P(n), P(n+4)) &\mid 19. \end{aligned}$$

Now assume for contradiction $b \le 5$. Then any GCD's among $P(a+1), \dots, P(a+b)$ must be among $\{3, 7, 19\}$. Consider a multi-graph on $\{a+1, \dots, a+b\}$ where we join two elements with nontrivial GCD and label the edge with the corresponding prime. Then we readily see there is at most one edge each of $\{3, 7, 19\}$: id est at most one edge of gap 2, 3, 4 (and no edges of gap 1). (By the gap of an edge $e = \{u, v\}$ we mean $|u - v|$.) But one can see that it's now impossible for every vertex to have nonzero degree, contradiction.

To construct $b = 6$ we use the Chinese remainder theorem: select a with

$$\begin{aligned} a+1 &\equiv 7 \pmod{19} \\ a+5 &\equiv 11 \pmod{19} \\ a+2 &\equiv 2 \pmod 7 \\ a+4 &\equiv 4 \pmod 7 \\ a+3 &\equiv 1 \pmod 3 \\ a+6 &\equiv 1 \pmod 3 \end{aligned}$$

which does the trick.

§2.2 IMO 2016/5, proposed by Russia

Available online at https://aops.com/community/p6642565.

Problem statement

The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

The answer is 2016. Obviously this is necessary in order to delete duplicated factors. We now prove it suffices to deleted $2 \pmod 4$ and $3 \pmod 4$ guys from the left-hand side, and $0 \pmod 4$, $1 \pmod 4$ from the right-hand side.

Consider the 1008 inequalities

$$\begin{aligned} (x-1)(x-4) &< (x-2)(x-3) \\ (x-5)(x-8) &< (x-6)(x-7) \\ (x-9)(x-12) &< (x-10)(x-11) \\ &\vdots \\ (x-2013)(x-2016) &< (x-2014)(x-2015). \end{aligned}$$

Notice that in all these inequalities, at most one of them has non-positive numbers in it, and we never have both zero. If there is exactly one negative term among the $1008 \cdot 2 = 2016$ sides, it is on the left and we can multiply all together. Thus the only case that remains is if $x \in (4m-2, 4m-1)$ for some m, say the mth inequality. In that case, the two sides of that inequality differ by a factor of at least 9.

Claim — We have
$$\prod_{k\ge 0} \frac{(4k+2)(4k+3)}{(4k+1)(4k+4)} < e.$$

Proof of claim using logarithms. To see this, note that it's equivalent to prove

$$\sum_{k\ge 0} \log\left(1 + \frac{2}{(4k+1)(4k+4)}\right) < 1.$$

To this end, we use the deep fact that $\log(1+t) \le t$, and thus it follows from $\sum_{k\ge 0} \frac{1}{(4k+1)(4k+4)} < \frac{1}{2}$, which one can obtain for example by noticing it's less than $\frac{1}{4}\frac{\pi^2}{6}$. □

Elementary proof of claim, given by Espen Slettnes. For each $N \ge 0$, the partial product is bounded by

$$\prod_{k=0}^{N} \frac{(4k+2)(4k+3)}{(4k+1)(4k+4)} = \frac{2}{1}\cdot\left(\frac{3}{4}\cdot\frac{6}{5}\right)\cdot\left(\frac{7}{8}\cdot\frac{10}{9}\right)\cdot\dots\cdot\frac{4N+3}{4N+4}$$

$$< 2 \cdot 1 \cdot 1 \cdot \dots \cdot \frac{4N+3}{4N+4} < 2 < e. \qquad \square$$

This solves the problem, because then the factors being multiplied on by the positive inequalities before the mth one are both less than e, and $e^2 < 9$. In symbols, for $4m-2 < x < 4m-1$ we should have

$$\frac{(x-(4m-6))(x-(4m-5))}{(x-(4m-7))(x-(4m-4))} \times \dots \times \frac{(x-2)(x-3)}{(x-1)(x-4)} < e$$

and

$$\frac{(x-(4m+2))(x-(4m+3))}{(x-(4m+1))(x-(4m+4))} \times \dots \times \frac{(x-2014)(x-2015)}{(x-2013)(x-2016)} < e$$

because the $(k+1)$st term of each left-hand side is at most $\frac{(4k+2)(4k+3)}{(4k+1)(4k+4)}$, for $k \ge 0$. As $e^2 < 9$, we're okay.

§2.3 IMO 2016/6, proposed by Josef Tkadlec (CZE)

Available online at https://aops.com/community/p6642576.

Problem statement

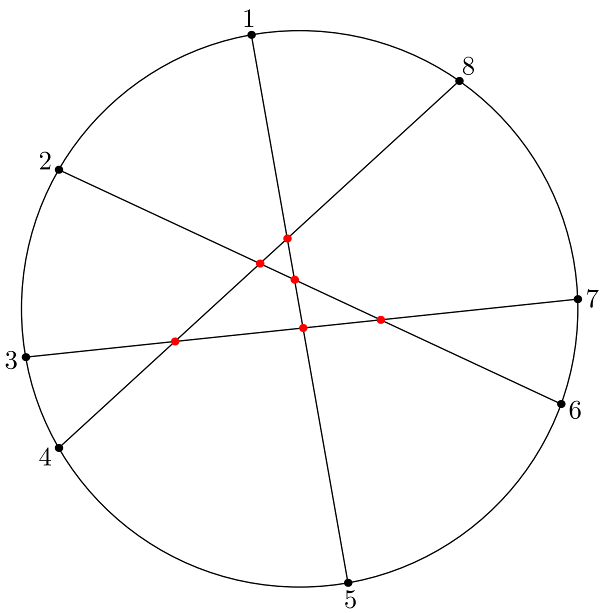
There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfill his wish if n is odd.

(b) Prove that Geoff can never fulfill his wish if n is even.

The following solution was communicated to me by Yang Liu.

Imagine taking a larger circle ω encasing all $\binom{n}{2}$ intersection points. Denote by P_1, P_2, $\dots$, P_{2n} the order of the points on ω in clockwise order; we imagine placing the frogs on P_i instead. Observe that, in order for every pair of segments to meet, each line segment must be of the form P_iP_{i+n}.



Then:

(a) Place the frogs on P_1, P_3, $\dots$, P_{2n-1}. A simple parity arguments shows this works.

(b) Observe that we cannot place frogs on consecutive P_i, so the n frogs must be placed on alternating points. But since we also are supposed to not place frogs on diametrically opposite points, for even n we immediately get a contradiction.

Remark. Yang says: this is easy to guess if you just do a few small cases and notice that the pairs of "violating points" just forms a large cycle around.

IMO 2017 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

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These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \dots$, by

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise} \end{cases}$$

for each $n \ge 0$. Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

2. Solve over $\mathbb{R}$ the functional equation

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

3. A hunter and an invisible rabbit play a game in the plane. The rabbit and hunter start at points $A_0 = B_0$. In the nth round of the game ($n \ge 1$), three things occur in order:

(i) The rabbit moves invisibly from A_{n-1} to a point A_n such that $A_{n-1}A_n = 1$.

(ii) The hunter has a tracking device (e.g. dog) which reports an approximate location P_n of the rabbit, such that $P_nA_n \le 1$.

(iii) The hunter moves visibly from B_{n-1} to a point B_n such that $B_{n-1}B_n = 1$.

Let $N = 10^9$. Can the hunter guarantee that $A_NB_N < 100$?

4. Let R and S be different points on a circle Ω such that $\overline{RS}$ is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of $\overline{RT}$. Point J is chosen on minor arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ closer to R. Line AJ meets Ω again at K. Prove that line KT is tangent to Γ.

5. Fix $N \ge 1$. A collection of $N(N+1)$ soccer players of distinct heights stand in a row. Sir Alex Song wishes to remove $N(N-1)$ players from this row to obtain a new row of $2N$ players in which the following N conditions hold: no one stands between the two tallest players, no one stands between the third and fourth tallest players, ..., no one stands between the two shortest players. Prove that this is possible.

6. An *irreducible lattice point* is an ordered pair of integers (x, y) satisfying $\gcd(x, y) = 1$. Prove that if S is a finite set of irreducible lattice points then there exists a nonconstant *homogeneous* polynomial $f(x, y)$ with integer coefficients such that $f(x, y) = 1$ for each $(x, y) \in S$.

§1 Solutions to Day 1

§1.1 IMO 2017/1, proposed by Stephan Wagner (SAF)

Available online at https://aops.com/community/p8633268.

Problem statement

For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots,$ by

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise} \end{cases}$$

for each $n \ge 0$. Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

The answer is $a_0 \equiv 0 \pmod 3$ only.

¶ First solution We first compute the minimal term of any sequence, periodic or not.

Lemma

Let c be the smallest term in a_n. Then either $c \equiv 2 \pmod 3$ or $c = 3$.

Proof. Clearly $c \neq 1, 4$. Assume $c \not\equiv 2 \pmod 3$. As c is not itself a square, the next perfect square after c in the sequence is one of $(\lfloor\sqrt{c}\rfloor + 1)^2$, $(\lfloor\sqrt{c}\rfloor + 2)^2$, or $(\lfloor\sqrt{c}\rfloor + 3)^2$. So by minimality we require

$$c \le \lfloor\sqrt{c}\rfloor + 3 \le \sqrt{c} + 3$$

which requires $c \le 5$. Since $c \neq 1, 2, 4, 5$ we conclude $c = 3$. □

Now we split the problem into two cases:

- If $a_0 \equiv 0 \pmod 3$, then all terms of the sequence are $0 \pmod 3$. The smallest term of the sequence is thus 3 by the lemma and we have

$$3 \to 6 \to 9 \to 3$$

  so $A = 3$ works fine.

- If $a_0 \not\equiv 0 \pmod 3$, then no term of the sequence is $0 \pmod 3$, and so in particular 3 does not appear in the sequence. So the smallest term of the sequence is $2 \pmod 3$ by lemma. But since no squares are $2 \pmod 3$, the sequence a_k grows without bound forever after, so no such A can exist.

Hence the answer is $a_0 \equiv 0 \pmod 3$ only.

¶ **Second solution** We clean up the argument by proving the following lemma.

Lemma

If a_n is constant modulo 3 and not $2 \pmod 3$, then a_n must eventually cycle in the form $(m, m+3, m+6, \dots, m^2)$, with no squares inside the cycle except m^2.

Proof. Observe that a_n must eventually hit a square, say $a_k = c^2$; the next term is $a_{k+1} = c$. Then it is forever impossible to exceed c^2 again, by what is essentially discrete intermediate value theorem. Indeed, suppose $a_\ell > c^2$ and take $\ell > k$ minimal (in particular $a_\ell \neq \sqrt{a_{\ell-1}}$). Thus $a_{\ell-1} \in \{c^2-2, c^2-1, c^2\}$ and thus for modulo 3 reasons we have $a_{\ell-1} = c^2$. But that should imply $a_\ell = c < c^2$, contradiction.

We therefore conclude $\sup\{a_n, a_{n+1}, \dots\}$ is a decreasing integer sequence in n. It must eventually stabilize, say at m^2. Now we can't hit a square between m and m^2, and so we are done. $\square$

Now, we contend that all $a_0 \equiv 0 \pmod 3$ work. Indeed, for such a_0 we have $a_n \equiv 0 \pmod 3$ for all n, so the lemma implies that the problem statement is valid.

Next, we observe that if $a_i \equiv 2 \pmod 3$, then the sequence grows without bound afterwards since no squares are $2 \pmod 3$. In particular, if $a_0 \equiv 2 \pmod 3$ the answer is no.

Finally, we claim that if $a_0 \equiv 1 \pmod 3$, then eventually some term is $2 \pmod 3$. Assume for contradiction this is not so; then $a_n \equiv 1 \pmod 3$ must hold forever, and the lemma applies to give us a cycle of the form $(m, m+3, \dots, m^2)$ where $m \equiv 1 \pmod 3$. In particular $m \geq 4$ and

$$m \leq (m-2)^2 < m^2$$

but $(m-2)^2 \equiv 1 \pmod 3$ which is a contradiction.

§1.2 IMO 2017/2, proposed by Dorlir Ahmeti (ALB)

Available online at https://aops.com/community/p8633190.

Problem statement

Solve over $\mathbb{R}$ the functional equation

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

The only solutions are $f(x) = 0$, $f(x) = x - 1$ and $f(x) = 1 - x$, which clearly work.

Note that

- If f is a solution, so is $-f$.
- Moreover, if $f(0) = 0$ then setting $y = 0$ gives $f \equiv 0$. So henceforth we assume $f(0) > 0$.

Claim — We have $f(z) = 0 \iff z = 1$. Also, $f(0) = 1$ and $f(1) = 0$.

Proof. For the forwards direction, if $f(z) = 0$ and $z \neq 1$ one may put $(x, y) = \left(z, z(z-1)^{-1}\right)$ (so that $x + y = xy$) we deduce $f(0) = 0$ which is a contradiction.

For the reverse, $f(f(0)^2) = 0$ by setting $x = y = 0$, and use the previous part. We also conclude $f(1) = 0$, $f(0) = 1$. □

Claim — If f is injective, we are done.

Proof. Setting $y = 0$ in the original equation gives $f(f(x)) = 1 - f(x)$. We apply this three times on the expression $f^3(x)$:

$$f(1 - f(x)) = f(f(f(x))) = 1 - f(f(x)) = f(x).$$

Hence $1 - f(x) = x$ or $f(x) = 1 - x$. □

Remark. The result $f(f(x)) + f(x) = 1$ also implies that surjectivity would solve the problem.

Claim — f is injective.

Proof. Setting $y = 1$ in the original equation gives $f(x+1) = f(x) - 1$, and by induction

$$f(x+n) = f(x) - n. \tag{1}$$

Assume now $f(a) = f(b)$. By using (1) we may shift a and b to be large enough that we may find x and y obeying $x + y = a + 1$, $xy = b$. Setting these gives

$$\begin{aligned} f(f(x)f(y)) &= f(xy) - f(x+y) = f(b) - f(a+1) \\ &= f(b) + 1 - f(a) = 1 \end{aligned}$$

from which we conclude

$$f\left(f(x)f(y)+1\right)=0.$$

Hence by the first claim we have $f(x)f(y)+1=1$, so $f(x)f(y)=0$. Applying the first claim again gives $1 \in \{x,y\}$. But that implies $a=b$. $\square$

Remark. Jessica Wan points out that for any $a \neq b$, at least one of $a^2 > 4(b-1)$ and $b^2 > 4(a-1)$ is true. So shifting via (1) is actually unnecessary for this proof.

Remark. One can solve the problem over $\mathbb{Q}$ using only (1) and the easy parts. Indeed, that already implies $f(n)=1-n$ for all n. Now we induct to show $f(p/q)=1-p/q$ for all $0<p<q$ (on q). By choosing $x=1+p/q$, $y=1+q/p$, we cause $xy=x+y$, and hence $0=f\left(f(1+p/q)f(1+q/p)\right)$ or $1=f(1+p/q)f(1+q/p)$.

By induction we compute $f(1+q/p)$ and this gives $f(p/q+1)=f(p/q)-1$.

§1.3 IMO 2017/3, proposed by Gerhard Woeginger (AUT)

Available online at https://aops.com/community/p8633324.

Problem statement

A hunter and an invisible rabbit play a game in the plane. The rabbit and hunter start at points $A_0 = B_0$. In the nth round of the game ($n \ge 1$), three things occur in order:

(i) The rabbit moves invisibly from A_{n-1} to a point A_n such that $A_{n-1}A_n = 1$.

(ii) The hunter has a tracking device (e.g. dog) which reports an approximate location P_n of the rabbit, such that $P_nA_n \le 1$.

(iii) The hunter moves visibly from B_{n-1} to a point B_n such that $B_{n-1}B_n = 1$.

Let $N = 10^9$. Can the hunter guarantee that $A_NB_N < 100$?

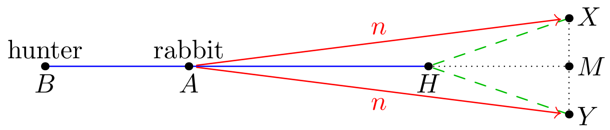
No, the hunter cannot. We will show how to increase the distance in the following way:

Claim — Suppose the rabbit is at a distance $d \ge 1$ from the hunter at some point in time. Then it can increase its distance to at least $\sqrt{d^2 + \frac{1}{2}}$ in $4d$ steps regardless of what the hunter already knows about the rabbit.

Proof. Consider a positive integer $n > d$, to be chosen later. Let the hunter start at B and the rabbit at A, as shown. Let ℓ denote line AB.

Now, we may assume the rabbit reveals its location A, so that all previous information becomes irrelevant.

The rabbit chooses two points X and Y symmetric about ℓ such that $XY = 2$ and $AX = AY = n$, as shown. The rabbit can then hop to either X or Y, pinging the point P_n on the ℓ each time. This takes n hops.



Now among all points H the hunter can go to, $\min \max\{HX, HY\}$ is clearly minimized with $H \in \ell$ by symmetry. So the hunter moves to a point H such that $BH = n$ as well. In that case the new distance is $HX = HY$.

We now compute

$$\begin{aligned} HX^2 = 1 + HM^2 &= 1 + \left(\sqrt{AX^2 - 1} - AH\right)^2 \\ &= 1 + \left(\sqrt{n^2-1} - (n-d)\right)^2 \\ &\ge 1 + \left(\left(n - \frac{1}{n}\right) - (n-d)\right)^2 \\ &= 1 + (d - 1/n)^2 \end{aligned}$$

which exceeds $d^2 + \frac{1}{2}$ whenever $n \ge 4d$. $\square$

In particular we can always take $n = 400$ even very crudely; applying the lemma $2 \cdot 100^2$ times, this gives a bound of $400 \cdot 2 \cdot 100^2 < 10^9$, as desired.

> **Remark.** The step of revealing the location of the rabbit seems critical because as far as I am aware it is basically impossible to keep track of ping locations in the problem.

> **Remark.** Reasons to believe the answer is "no": the 10^9 constant, and also that "follow the last ping" is losing for the hunter.

> **Remark.** I think there are roughly two ways you can approach the problem once you recognize the answer.
>
> (i) Try and control the location of the pings
>
> (ii) Abandon the notion of controlling possible locations, and try to increase the distance by a little bit, say from d to $\sqrt{d^2 + \varepsilon}$. This involves revealing the location of the rabbit before each iteration of several jumps.
>
> I think it's clear that the difficulty of my approach is realizing that (ii) is possible; once you do, the two-point approach is more or less the only one possible.
>
> My opinion is that (ii) is not that magical; as I said it was the first idea I had. But I am biased, because when I test-solved the problem at the IMO it was called "C5" and not "IMO3"; this effectively told me it was unlikely that the official solution was along the lines of (i), because otherwise it would have been placed much later in the shortlist.

§2 Solutions to Day 2

§2.1 IMO 2017/4, proposed by Charles Leytem (LUX)

Available online at https://aops.com/community/p8639236.

Problem statement

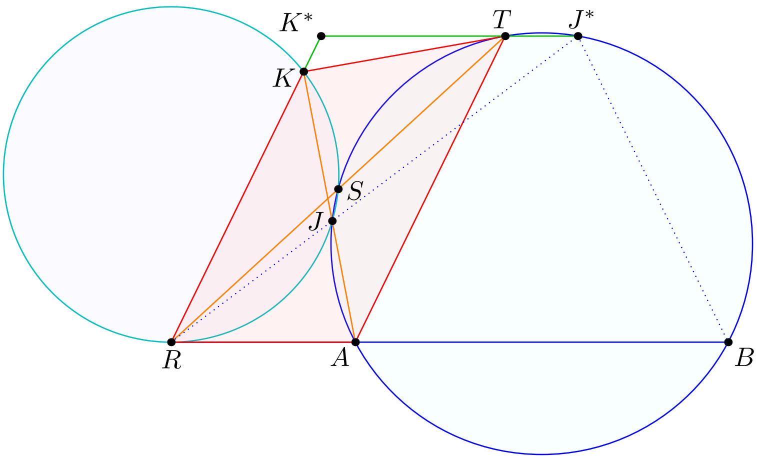
Let R and S be different points on a circle Ω such that $\overline{RS}$ is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of $\overline{RT}$. Point J is chosen on minor arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ closer to R. Line AJ meets Ω again at K. Prove that line KT is tangent to Γ.

¶ First solution (elementary) First, note

$$\measuredangle RKA = \measuredangle RKJ = \measuredangle RSJ = \measuredangle TSJ = \measuredangle TAJ = \measuredangle TAK$$

so $\overline{RK} \parallel \overline{AT}$. Now,

- $\overline{RA}$ is tangent at R iff $\triangle KRS \sim \triangle RTA$ (oppositely), because both equate to $-\measuredangle RKS = \measuredangle SKR = \measuredangle SRA = \measuredangle TRA$.
- Similarly, $\overline{TK}$ is tangent at T iff $\triangle KTS \sim \triangle ART$.
- The two similarities are equivalent because $RS = ST$ the SAS gives $KR \cdot TA = RS \cdot RT = TS \cdot TR$.



> **Remark.** The problem is actually symmetric with respect to two circles; $\overline{RA}$ is tangent at R if and only if $\overline{TK}$ at T.

¶ Second solution (inversion) Consider an inversion at R fixing the circumcircle Γ of $TSJA$. Then:

- T and S swap,

- A and B swap, where B is the second intersection of ℓ with Γ.
- Circle Ω inverts to the line through T parallel to $\overline{RAB}$, call it ℓ.
- J^* is the second intersection of ℓ with Γ.
- K^* is the intersection of ℓ with the circumcircle of RBJ^*; this implies RK^*J^*B is an isosceles trapezoid. In particular, one reads $\overline{RK^*} \parallel \overline{AT}$ from this, hence RK^*TA is a parallelogram.

Thus we wish to show the circumcircle of RSK^* is tangent to Γ. But that follows from the final parallelogram observed: S is the center of the parallelogram since it is the midpoint of the diagonal.

> **Remark.** This also implies $RKTB$ is cyclic, from $\overline{K^*SA}$ collinear. Moreover, quadrilateral KK^*TS is cyclic (by power of a point); this leads to the second official solution to the problem.

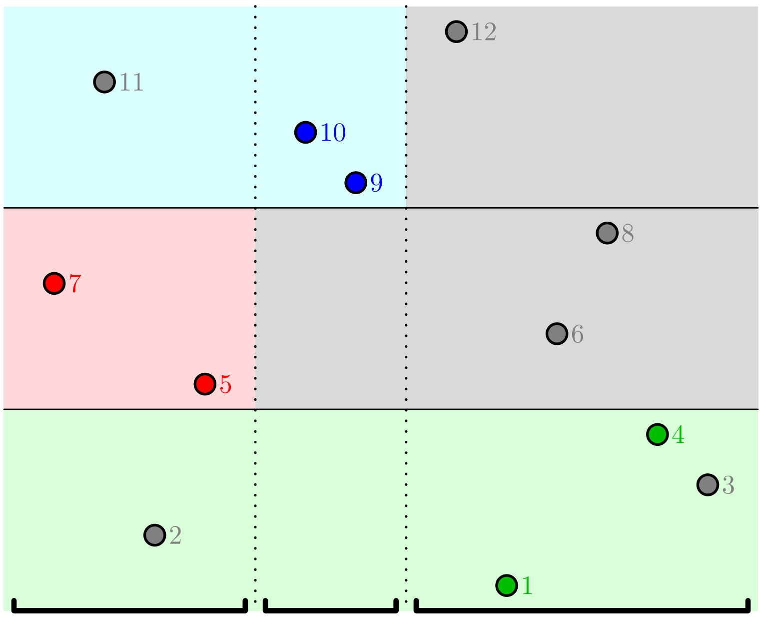
§2.2 IMO 2017/5, proposed by Grigory Chelnokov (RUS)

Available online at https://aops.com/community/p8639240.

Problem statement

Fix $N \geq 1$. A collection of $N(N+1)$ soccer players of distinct heights stand in a row. Sir Alex Song wishes to remove $N(N-1)$ players from this row to obtain a new row of $2N$ players in which the following N conditions hold: no one stands between the two tallest players, no one stands between the third and fourth tallest players, $\ldots$, no one stands between the two shortest players. Prove that this is possible.

Some opening remarks: **location and height are symmetric to each other**, if one thinks about this problem as permutation pattern avoidance. So while officially there are multiple solutions, they are basically isomorphic to one another, and I am not aware of any solution otherwise.



Take a partition of N groups in order by height: $G_1 = \{1, \ldots, N+1\}$, $G_2 = \{N+2, \ldots, 2N+2\}$, and so on. We will pick two people from each group G_k.

Scan from the left until we find two people in the same group G_k. Delete all people scanned and also everyone in G_k. All the groups still have at least N people left, so we can induct down with the non-deleted people; the chosen pair is to the far left anyways.

> **Remark.** The important bit is to *scan by position* but *group by height*, and moreover not change the groups as we scan. Dually, one can have a solution which scans by height but groups by position.

§2.3 IMO 2017/6, proposed by John Berman (USA)

Available online at https://aops.com/community/p8639242.

Problem statement

An *irreducible lattice point* is an ordered pair of integers (x, y) satisfying $\gcd(x, y) = 1$. Prove that if S is a finite set of irreducible lattice points then there exists a nonconstant *homogeneous* polynomial $f(x, y)$ with integer coefficients such that $f(x, y) = 1$ for each $(x, y) \in S$.

We present two solutions.

¶ First solution (Dan Carmon, Israel) We prove the result by induction on $|S|$, with the base case being Bezout's Lemma ($n = 1$). For the inductive step, suppose we want to add a given pair (a_{m+1}, b_{m+1}) to $\{(a_1, \dots, a_m), (b_1, \dots, b_m)\}$. By a suitable linear transformation assume $(a_{m+1}, b_{m+1}) = (1, 0)$. (The transformation is not necessary to proceed but cleans up the presentation that follows.)

Let $g(x, y)$ be a polynomial which works on the latter set. We claim we can choose the new polynomial f of the form

$$f(x, y) = g(x, y)^M - Cx^{\deg g \cdot M - m} \prod_{i=1}^{m} (b_i x - a_i y).$$

where C and M are integer parameters we may adjust.

Since $f(a_i, b_i) = 1$ by construction we just need

$$1 = f(1, 0) = g(1, 0)^M - C \prod b_i.$$

If $\prod b_i = 0$ we are done, since $b_i = 0 \implies a_i = \pm 1$ in that case and so $g(1, 0) = \pm 1$, thus take $M = 2$. So it suffices to prove:

Claim — We have $\gcd(g(1, 0), b_i) = 1$ when $b_i \neq 0$.

Proof. Fix i. If $b_i = 0$ then $a_i = \pm 1$ and $g(\pm 1, 0) = \pm 1$. Otherwise know

$$1 = g(a_i, b_i) \equiv g(a_i, 0) \pmod{b_i}$$

and since the polynomial is homogeneous with $\gcd(a_i, b_i) = 1$ it follows $g(1, 0) \not\equiv 0 \pmod{b_i}$ as well. □

Then take M a large even multiple of $\varphi(\prod b_i)$ and we're done.

¶ Second solution (Lagrange) The main claim is that:

Claim — For every positive integer N, there is a homogeneous polynomial $P(x, y)$ such that $P(x, y) \equiv 1 \pmod{N}$ whenever $\gcd(x, y) = 1$.

(This claim is actually implied by the problem.)

Proof. For $N = p^e$ a prime take $(x^{p-1} + y^{p-1})^{\varphi(N)}$ when p is odd, and $(x^2 + xy + y^2)^{\varphi(N)}$ for $p = 2$.

Now, if N is a product of primes, we can collate coefficient by coefficient using the Chinese remainder theorem. $\square$

Let $S = \{(a_i, b_i) \mid i = 1, \dots, m\}$. We have the natural homogeneous "Lagrange polynomials" $L_k(x, y) = \prod_{i \neq k}(b_i x - a_i y)$. Now let $N = \prod_k L_k(x_k, y_k)$ and take P as above. Then we can take a large power of P, and for each i subtract an appropriate multiple of $L_i(x, y)$.

IMO 2018 Solution Notes

EVAN CHEN《陳誼廷》

2 June 2023

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Corrections and comments are welcome!

Contents

§0 Problems

1. Let Γ be the circumcircle of acute triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of $\overline{BD}$ and $\overline{CE}$ intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel.

2. Find all integers $n \ge 3$ for which there exist real numbers $a_1, a_2, \dots, a_n$ satisfying

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \dots, n$, where indices are taken modulo n.

3. An *anti-Pascal triangle* is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + \cdots + 2018$?

4. A *site* is any point (x, y) in the plane for which $x, y \in \{1, \dots, 20\}$. Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones on unoccupied sites, with Amy going first; Amy has the additional restriction that no two of her stones may be at a distance equal to $\sqrt{5}$. They stop once either player cannot move. Find the greatest K such that Amy can ensure that she places at least K stones.

5. Let a_1, a_2, ... be an infinite sequence of positive integers, and N a positive integer. Suppose that for all integers $n \ge N$, the expression

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that (a_n) is eventually constant.

6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180°$.

§1 Solutions to Day 1

§1.1 IMO 2018/1, proposed by Silouanos Brazitikos, Vangelis Psyxas, Michael Sarantis (HEL)

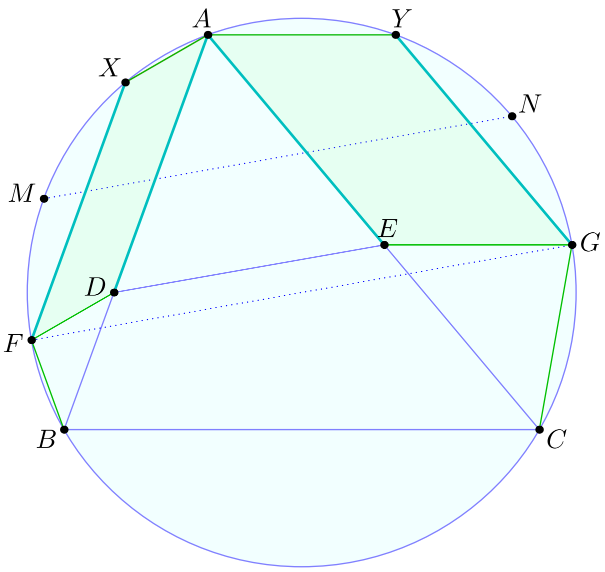
Available online at https://aops.com/community/p10626500.

Problem statement

Let Γ be the circumcircle of acute triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of $\overline{BD}$ and $\overline{CE}$ intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel.

We present a synthetic solution from the IMO shortlist as well as a complex numbers approach. We also outline a trig solution (the one I found at IMO), and a fourth solution from Derek Liu.

¶ Synthetic solution (from Shortlist) Construct parallelograms $AXFD$ and $AEGY$, noting that X and Y lie on Γ. As $\overline{XF} \parallel \overline{AB}$ we can let M denote the midpoint of minor arcs $\widehat{XF}$ and $\widehat{AB}$ (which coincide). Define N similarly.



Observe that $XF = AD = AE = YG$, so arcs $\widehat{XF}$ and $\widehat{YG}$ have equal measure; hence arcs $\widehat{MF}$ and $\widehat{NG}$ have equal measure; therefore $\overline{MN} \parallel \overline{FG}$.

Since $\overline{MN}$ and $\overline{DE}$ are both perpendicular to the $\angle A$ bisector, so we're done.

¶ Complex numbers solution Let b, c, f, g, a be as usual. Note that

$$d - a = \left(2 \cdot \frac{f + a + b - ab\overline{f}}{2} - b\right) - a = f - \frac{ab}{f}$$
$$e - a = g - \frac{ac}{g}$$

We are given $AD = AE$ from which one deduces

$$\left(\frac{e-a}{d-a}\right)^2 = \frac{c}{b} \implies \frac{(g^2-ac)^2}{(f^2-ab)^2} = \frac{g^2c}{f^2b}$$
$$\implies bc(bg^2 - cf^2)a^2 = g^2f^4c - f^2g^4b = f^2g^2(f^2c - g^2b)$$
$$\implies bc \cdot a^2 = (fg)^2 \implies \left(-\frac{fg}{a}\right)^2 = bc.$$

Since $\frac{-fg}{a}$ is the point X on the circle with $\overline{AX} \perp \overline{FG}$, we conclude $\overline{FG}$ is either parallel or perpendicular to the $\angle A$-bisector; it must the latter since the $\angle A$-bisector separates the two minor arcs.

¶ Trig solution (outline) Let ℓ denote the $\angle A$ bisector. Fix D and F. We define the phantom point G' such that $\overline{FG'} \perp \ell$ and E' on side $\overline{AC}$ such that $GE' = GC$.

Claim (Converse of the IMO problem) — We have $AD = AE'$, so that $E = E'$.

Proof. Since $\overline{FG'} \perp \ell$, one can deduce $\angle FBD = \frac{1}{2}C + x$ and $\angle GCA = \frac{1}{2}B + x$ for some x. (One fast way to see this is to note that $\overline{FG} \parallel \overline{MN}$ where M and N are in the first solution.) Then $\angle FAB = \frac{1}{2}C - x$ and $\angle GAC = \frac{1}{2}B - x$.

Let R be the circumradius. Now, by the law of sines,

$$BF = 2R \sin\left(\frac{1}{2}C - x\right).$$

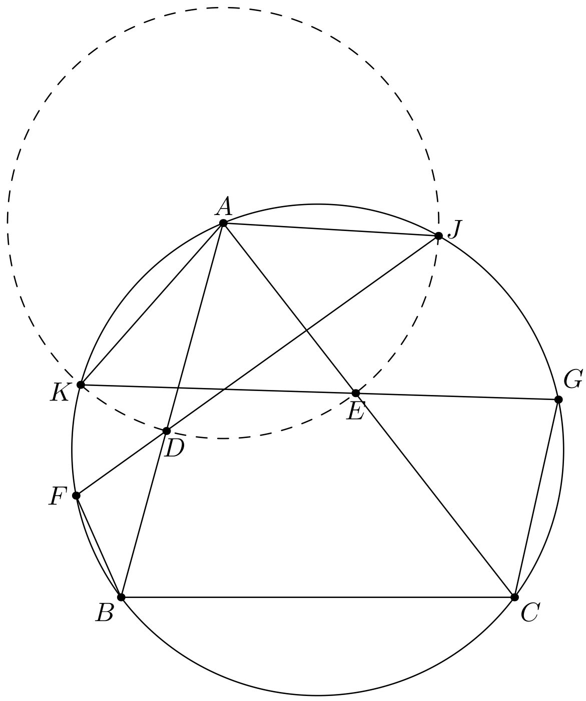
From there we get

$$\begin{aligned} BD &= 2 \cdot BF \cos\left(\frac{1}{2}C + x\right) = 4R \cos\left(\frac{1}{2}C + x\right) \sin\left(\frac{1}{2}C - x\right) \\ DA &= AB - BD = 2R \sin C - 4R \cos\left(\frac{1}{2}C + x\right) \sin\left(\frac{1}{2}C - x\right) \\ &= 2R\left[\sin C - 2\cos\left(\frac{1}{2}C + x\right) \sin\left(\frac{1}{2}C - x\right)\right] \\ &= 2R\left[\sin C - (\sin C - \sin 2x)\right] = 2R \sin 2x. \end{aligned}$$

A similar calculation gives $AE' = 2R \sin 2x$ as needed. □

Thus, $\overline{FG'} \parallel \overline{DE}$, so $G = G'$ as well. This concludes the proof.

¶ Synthetic solution from Derek Liu Let lines FD and GE intersect Γ again at J and K, respectively.



Notice that $\triangle BFD \sim \triangle JAD$; as $FB = FD$, it follows that $AJ = AD$. Likewise, $\triangle CGE \sim \triangle KAE$ and $GC = GE$, so $AK = AE$. Hence,

$$AK = AE = AD = AJ,$$

so $DEJK$ is cyclic with center A.

It follows that

$$\measuredangle KED = \measuredangle KJD = \measuredangle KJF = \measuredangle KGF,$$

so we're done.

> **Remark.** Note that K and J must be distinct for this solution to work. Since G and K lie on opposite sides of AC, K is on major arc ABC. As $AK = AD = AE \le \min(AB, AC)$, K lies on minor arc AB. Similarly, J lies on minor arc AC, so $K \neq J$.

§1.2 IMO 2018/2, proposed by Patrik Bak (SVK)

Available online at https://aops.com/community/p10626524.

Problem statement

Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \dots, a_n$ satisfying

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \dots, n$, where indices are taken modulo n.

The answer is $3 \mid n$, achieved by $(-1, -1, 2, -1, -1, 2, \dots)$. We present two solutions.

¶ First solution by inequalities We compute $a_i a_{i+1} a_{i+2}$ in two ways:

$$\begin{aligned} a_i a_{i+1} a_{i+2} &= [a_{i+2} - 1] a_{i+2} = a_{i+2}^2 - a_{i+2} \\ &= a_i [a_{i+3} - 1] = a_i a_{i+3} - a_i. \end{aligned}$$

Cyclically summing $a_{i+2}^2 - a_{i+2} = a_i a_{i+3} - a_i$ then gives

$$\sum_i a_{i+2}^2 = \sum_i a_i a_{i+3} \iff \sum_{\text{cyc}} (a_i - a_{i+3})^2 = 0.$$

This means for inequality reasons the sequence is 3-periodic. Since the sequence is clearly not 1-periodic, as $x^2 + 1 = x$ has no real solutions. Thus $3 \mid n$.

¶ Second solution by sign counting Extend a_n to be a periodic sequence. The idea is to look at the signs, and show the sequence of the signs must be $--+$ repeated. This takes several steps:

- The pattern $---$ is impossible. Obvious, since the third term should be > 1.
- The pattern $++$ is impossible. Then the sequence becomes strictly increasing, hence may not be periodic.
- Zeros are impossible. If $a_1 = 0$, then $a_2 = 0$, $a_3 > 0$, $a_4 > 0$, which gives the impossible $++$.
- The pattern $--+-+$ is impossible. Compute the terms:

$$\begin{aligned} a_1 &= -x < 0 \\ a_2 &= -y < 0 \\ a_3 &= 1 + xy > 1 \\ a_4 &= 1 - y(1 + xy) < 0 \\ a_5 &= 1 + (1 + xy)(1 - y(1 + xy)) < 1. \end{aligned}$$

  But now

$$a_6 - a_5 = (1 + a_5 a_4) - (1 + a_3 a_4) = a_4(a_5 - a_3) > 0$$

  since $a_5 > 1 > a_3$. This means we have the impossible $++$ pattern.

- The infinite alternating pattern $- + - + - + - + \dots$ is impossible. Note that

$$a_1 a_2 + 1 = a_3 < 0 < a_4 = 1 + a_2 a_3 \implies a_1 < a_3$$

since $a_2 > 0$; extending this we get $a_1 < a_3 < a_5 < \dots$ which contradicts the periodicity.

We finally collate the logic of sign patterns. Since the pattern is not alternating, there must be $--$ somewhere. Afterwards must be $+$, and then after that must be two minus signs (since one minus sign is impossible by impossibility of $--+-+$ and $---$ is also forbidden); thus we get the periodic $--+$ as desired.

§1.3 IMO 2018/3, proposed by Morteza Saghafian (IRN)

Available online at https://aops.com/community/p10626557.

Problem statement

An *anti-Pascal triangle* is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

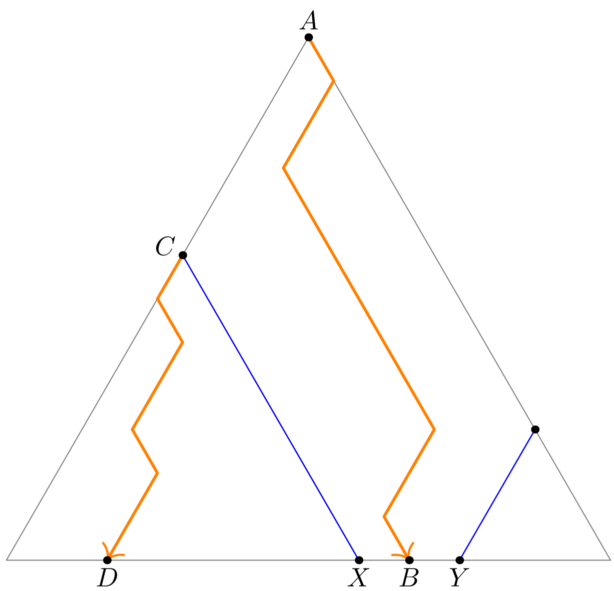
$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + \cdots + 2018$?

The answer is no, there is no anti-Pascal triangle with the required properties.

Let $n = 2018$ and $N = 1 + 2 + \cdots + n$. For every number d not in the bottom row, draw an arrow from d to the larger of the two numbers below it (i.e. if $d = a - b$, draw $d \to a$). This creates an *oriented forest* (which looks like lightning strikes).

Consider the directed path starting from the top vertex A. Starting from the first number, it increments by at least $1 + 2 + \cdots + n$, since the increments at each step in the path are distinct; therefore equality must hold and thus the path from the top ends at $N = 1 + 2 + \cdots + n$ with all the numbers $\{1, 2, \ldots, n\}$ being close by. Let B be that position.



Consider the two left/right neighbors X and Y of the endpoint B. Assume that B is to the right of the midpoint of the bottom side, and complete the equilateral triangle as shown to an apex C. Consider the lightning strike from C hitting the bottom at D. It travels at least $\lfloor n/2-1 \rfloor$ steps, by construction. But the increases must be at least $n+1$, $n+2$, ...since $1, 2, \dots, n$ are close to the $A \to B$ lightning path. Then the number at D is at least

$$(n+1)+(n+2)+\cdots+(n+(\lfloor n/2-1 \rfloor)) > 1+2+\cdots+n$$

for $n \geq 2018$, contradiction.

§2 Solutions to Day 2

§2.1 IMO 2018/4, proposed by Armenia

Available online at https://aops.com/community/p10632348.

Problem statement

A *site* is any point (x, y) in the plane for which $x, y \in \{1, \dots, 20\}$. Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones on unoccupied sites, with Amy going first; Amy has the additional restriction that no two of her stones may be at a distance equal to $\sqrt{5}$. They stop once either player cannot move. Find the greatest K such that Amy can ensure that she places at least K stones.

The answer is $K = 100$.

First, we show Amy can always place at least 100 stones. Indeed, treat the problem as a grid with checkerboard coloring. Then Amy can choose to always play on one of the 200 black squares. In this way, she can guarantee half the black squares, i.e. she can get $\frac{1}{2} \cdot 200 = 100$ stones.

Second, we show Ben can prevent Amy from placing more than 100 stones. Divide into several 4×4 squares and then further partition each 4×4 squares as shown in the grid below.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

The squares with each label form 4-cycles by knight jumps. For each such cycle, whenever Amy plays in the cycle, Ben plays in the opposite point of the cycle, preventing Amy from playing any more stones in that original cycle. Hence Amy can play at most in $1/4$ of the stones, as desired.

§2.2 IMO 2018/5, proposed by Mongolia

Available online at https://aops.com/community/p10632353.

Problem statement

Let $a_1, a_2, \dots$ be an infinite sequence of positive integers, and N a positive integer. Suppose that for all integers $n \ge N$, the expression

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that (a_n) is eventually constant.

The condition implies that the difference

$$S(n) = \frac{a_{n+1} - a_n}{a_1} + \frac{a_n}{a_{n+1}}$$

is an integer for all $n > N$. We proceed by p-adic valuation only henceforth; fix a prime p. Then analyzing the ν_p, we immediately get that for $n > N$:

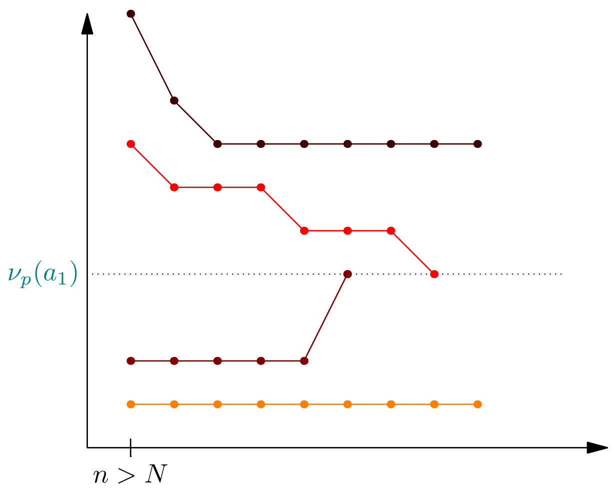
- If $\nu_p(a_n) < \nu_p(a_{n+1})$, then $\nu_p(a_{n+1}) = \nu_p(a_1)$.
- If $\nu_p(a_n) = \nu_p(a_{n+1})$, no conclusion.
- If $\nu_p(a_n) > \nu_p(a_{n+1})$, then $\nu_p(a_{n+1}) \ge \nu_p(a_1)$.

In other words:

Claim — Let p be a prime. Consider the sequence $\nu_p(a_{N+1}), \nu_p(a_{N+2}), \dots$. Then either:

- We have $\nu_p(a_{N+1}) \ge \nu_p(a_{N+2}) \ge \dots$ and so on, i.e. the sequence is weakly decreasing immediately; or
- For some index $K > N$ we have $\nu_p(a_K) < \nu_p(a_{K+1}) = \nu_p(a_{K+2}) = \dots = \nu_p(a_1)$, i.e. the sequence "jumps" to $\nu_p(a_1)$ at some point and then stays there forever after. Note this requires $\nu_p(a_1) > 0$.

A cartoon of the situation is drawn below.



As only finitely many primes p divide a_1, after some time $\nu_p(a_n)$ is fixed for all such $p \mid a_1$. Afterwards, the sequence satisfies $a_{n+1} \mid a_n$ for each n, and thus must be eventually constant.

Remark. This solution is almost completely p-adic, in the sense that I think a similar result holds if one replaces $a_n \in \mathbb{Z}$ by $a_n \in \mathbb{Z}_p$ for any particular prime p. In other words, the primes almost do not talk to each other.

There is one caveat: if x_n is an integer sequence such that $\nu_p(x_n)$ is eventually constant for each prime then x_n may not be constant. For example, take x_n to be the nth prime! That's why in the first claim (applied to co-finitely many of the primes), we need the stronger non-decreasing condition, rather than just eventually constant.

Remark. An alternative approach is to show that, when the fractions a_n/a_1 is written in simplest form for $n = N+1, N+2, \dots$, the numerator and denominator are both weakly decreasing. Hence it must eventually be constant; in which case it equals $\frac{1}{1}$.

§2.3 IMO 2018/6, proposed by Poland

Available online at https://aops.com/community/p10632360.

Problem statement

A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180°$.

We present two solutions by inversion. The first is the official one. The second is a solution via inversion, completed by USA5 Michael Ren.

¶ Official solution by inversion In what follows a convex quadrilateral is called *quasi-harmonic* if $AB \cdot CD = BC \cdot DA$.

Claim — A quasi-harmonic quadrilateral is determined up to similarity by its angles.

(This could be expected by degrees of freedom; a quadrilateral has four degrees of freedom up to similarity; the pseudo-harmonic condition is one while the angles provide three conditions.)

Proof. Do some inequalities. □

Performing an inversion at X, one obtains a second quasi-harmonic quadrilateral $A^*B^*C^*D^*$ which has the same angles as the original one, $\angle D^* = \angle A$, $\angle A^* = \angle B$, and so on. Thus by the claim we obtain similarity

$$D^*A^*B^*C^* \sim ABCD.$$

If one then maps $D^*A^*B^*C^*$, onto $ABCD$, the image of X^* becomes a point isogonally conjugate to X. In other words, X has an isogonal conjugate in $ABCD$.

It is well-known that this is equivalent to $\angle BXA + \angle DXC = 180°$, for example by inscribing an ellipse with foci X and X^*.

¶ Second solution: "rhombus inversion", by Michael Ren Since

$$\frac{AB}{AD} = \frac{CB}{CD}$$

and

$$\frac{BA}{BC} = \frac{DA}{DC}$$

it follows that B and D lie on an Apollonian circle ω_{AC} through A and C, while A and C lie on an Apollonian circle ω_{BD} through B and D. We let these two circles intersect at a point P inside $ABCD$.

The main idea is then to perform an inversion about P with radius 1. We obtain:

Lemma

The image of $ABCD$ is a rhombus.

Proof. By the inversion distance formula, we have

$$\frac{1}{A'B'} = \frac{PA}{AB} \cdot PB = \frac{PC}{BC} \cdot PB = \frac{1}{B'C'}$$

and so $A'B' = B'C'$. In a similar way, we derive $B'C' = C'D' = D'A'$, so the image is a rhombus as claimed. ☐

Let us now translate the angle conditions. We were given that $\measuredangle XAB = \measuredangle XCD$, but

$$\begin{aligned} \measuredangle XAB &= \measuredangle XAP + \measuredangle PAB = \measuredangle PX'A' + \measuredangle A'B'P \\ \measuredangle XCD &= \measuredangle XCP + \measuredangle PCD = \measuredangle PX'C' + \measuredangle C'D'P \end{aligned}$$

so subtracting these gives

$$\begin{aligned} \measuredangle A'X'C' &= \measuredangle A'B'P + \measuredangle PD'C' = \measuredangle(A'B', B'P) + \measuredangle(PD', C'D') \\ &= \measuredangle(A'B', B'P) + \measuredangle(PD', A'B') = \measuredangle D'PB'. \end{aligned} \tag{1}$$

since $\overline{A'B'} \parallel \overline{C'D'}$. Similarly, we obtain

$$\measuredangle B'X'D' = \measuredangle A'PC'. \tag{2}$$

We now translate the desired condition. Since

$$\begin{aligned} \measuredangle AXB &= \measuredangle AXP + \measuredangle PXB = \measuredangle PA'X' + \measuredangle X'B'P \\ \measuredangle CXD &= \measuredangle CXP + \measuredangle PXD = \measuredangle PC'X' + \measuredangle X'DP' \end{aligned}$$

we compute

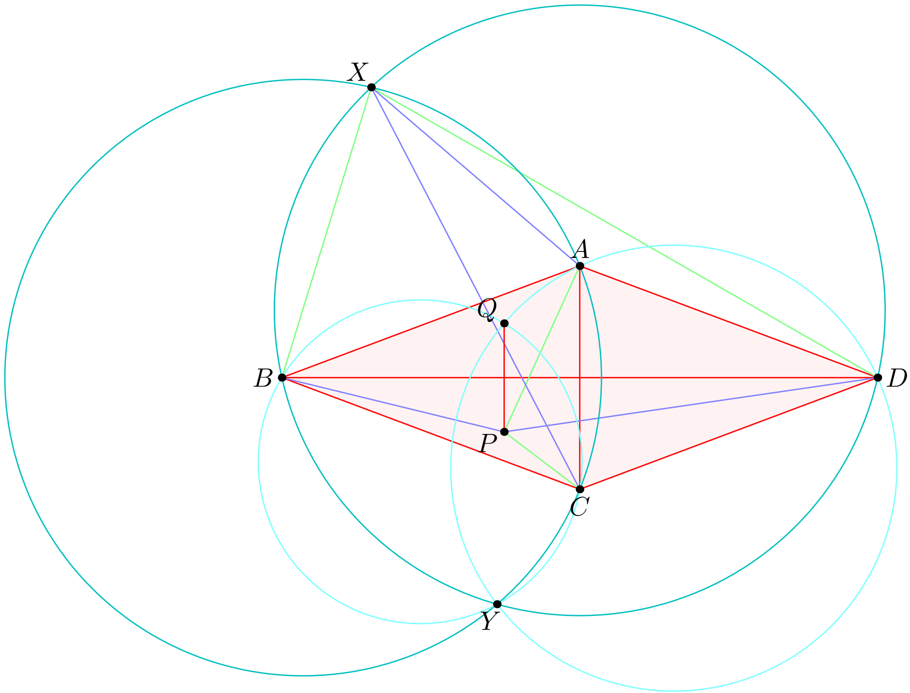
$$\begin{aligned} \measuredangle AXB + \measuredangle CXD &= (\measuredangle PA'X' + \measuredangle X'B'P) + (\measuredangle PC'X' + \measuredangle X'D'P) \\ &= -\left[\left(\measuredangle A'X'P + \measuredangle X'PA'\right) + \left(\measuredangle PX'B' + \measuredangle B'PX'\right)\right] \\ &\quad - \left[\left(\measuredangle C'X'P + \measuredangle X'PC'\right) + \left(\measuredangle PX'D' + \measuredangle D'PX'\right)\right] \\ &= \left[\measuredangle PX'A' + \measuredangle BX'P + \measuredangle PX'C' + \measuredangle D'X'P\right] \\ &\quad + \left[\measuredangle A'PX' + \measuredangle X'PB' + \measuredangle C'PX' + \measuredangle X'PD'\right] \\ &= \measuredangle A'PB' + \measuredangle C'PD' + \measuredangle B'X'C + \measuredangle D'X'A \end{aligned}$$

and we wish to show this is equal to zero, i.e. the desired becomes

$$\measuredangle A'PB' + \measuredangle C'PD' + \measuredangle B'X'C + \measuredangle D'X'A = 0. \tag{3}$$

In other words, the problem is to show (1) and (2) implies (3).

Henceforth drop apostrophes. Here is the inverted diagram (with apostrophes dropped).



Let Q denote the reflection of P and let Y denote the second intersection of (BQC) and (AQD). Then

$$\begin{aligned} -\measuredangle AXC &= -\measuredangle DPB = \measuredangle BQD = \measuredangle BQY + \measuredangle YQD = \measuredangle BCY + \measuredangle YAD \\ &= \measuredangle(BC, CY) + \measuredangle(YA, AD) = \measuredangle YCA = -\measuredangle AYC. \end{aligned}$$

Hence $XACY$ is concyclic; similarly $XBDY$ is concyclic.

Claim — $X \neq Y$.

Proof. To see this: Work pre-inversion assuming $AB < AC$. Then Q was the center of ω_{BD}. If T was the second intersection of BA with (QBC), then $QB = QD = QT = \sqrt{QA \cdot QC}$, by shooting lemma. Since $\angle BAD < 180^\circ$, it follows $(QBCY)$ encloses $ABCD$ (pre-inversion). (This part is where the hypothesis that $ABCD$ is convex with X inside is used.) □

Finally, we do an angle chase to finish:

$$\begin{aligned} \measuredangle DXA &= \measuredangle DXY + \measuredangle YXA = \measuredangle DBY + \measuredangle YCA \\ &= \measuredangle(DB, YB) + \measuredangle(CY, CA) = \measuredangle CYB + 90^\circ \\ &= \measuredangle CQB + 90^\circ = -\measuredangle APB + 90^\circ. \end{aligned} \tag{4}$$

Similarly,

$$\measuredangle BXC = \measuredangle DPC + 90^\circ. \tag{5}$$

Summing (4) and (5) gives (3).

Remark. A difficult part of the problem in many solutions is that the conclusion is false in the directed sense, if the point X is allowed to lie outside the quadrilateral. We are saved in the first solution because the equivalence of the isogonal conjugation requires X inside the quadrilateral. On the other hand, in the second solution, the issue appears in the presence of the second point Y.

IMO 2019 Solution Notes

EVAN CHEN《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2019 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Solve over $\mathbb{Z}$ the functional equation $f(2a) + 2f(b) = f(f(a+b))$.

2. In triangle ABC point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that $\overline{PQ} \parallel \overline{AB}$. Point P_1 is chosen on ray PB_1 beyond B_1 such that $\angle PP_1C = \angle BAC$. Point Q_1 is chosen on ray QA_1 beyond A_1 such that $\angle CQ_1Q = \angle CBA$. Prove that points P_1, Q_1, P, Q are cyclic.

3. A social network has 2019 users, some pairs of which are friends (friendship is symmetric). If A, B, C are three users such that AB are friends and AC are friends but BC is not, then the administrator may perform the following operation: change the friendships such that BC are friends, but AB and AC are no longer friends.

   Initially, 1009 users have 1010 friends and 1010 users have 1009 friends. Prove that the administrator can make a sequence of operations such that all users have at most 1 friend.

4. Solve over positive integers the equation
$$k! = \prod_{i=0}^{n-1}(2^n - 2^i) = (2^n - 1)(2^n - 2)(2^n - 4)\dots(2^n - 2^{n-1}).$$

5. Let n be a positive integer. Harry has n coins lined up on his desk, which can show either heads or tails. He does the following operation: if there are k coins which show heads and $k > 0$, then he flips the kth coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \to HHT \to HTT \to TTT$, which takes three steps.)

   Prove the process will always terminate, and determine the average number of steps this takes over all 2^n configurations.

6. Let ABC be a triangle with incenter I and incircle ω. Let D, E, F denote the tangency points of ω with $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. The line through D perpendicular to $\overline{EF}$ meets ω again at R (other than D), and line AR meets ω again at P (other than R). Suppose the circumcircles of $\triangle PCE$ and $\triangle PBF$ meet again at Q (other than P). Prove that lines DI and PQ meet on the external $\angle A$-bisector.

§1 Solutions to Day 1

§1.1 IMO 2019/1, proposed by Liam Baker (SAF)

Available online at https://aops.com/community/p12744859.

Problem statement

Solve over $\mathbb{Z}$ the functional equation $f(2a) + 2f(b) = f(f(a + b))$.

Notice that $f(x) \equiv 0$ or $f(x) \equiv 2x + k$ work and are clearly the only linear solutions. We now prove all solutions are linear.

Let $P(a, b)$ be the assertion.

Claim — For each $x \in \mathbb{Z}$ we have $f(2x) = 2f(x) - f(0)$.

Proof. Compare $P(0, x)$ and $P(x, 0)$. □

Now, $P(a, b)$ and $P(0, a + b)$ give

$$\begin{aligned} f(f(a+b)) &= f(2a) + 2f(b) = f(0) + 2f(a+b) \\ \implies [2f(a) - f(0)] + 2f(b) &= f(0) + 2f(a+b) \\ \implies (f(a) - f(0)) + (f(b) - f(0)) &= (f(a+b) - f(0)). \end{aligned}$$

Thus the map $x \mapsto f(x) - f(0)$ is additive, therefore linear.

Remark. The same proof works on the functional equation

$$f(2a) + 2f(b) = g(a + b)$$

where g is an arbitrary function (it implies that f is linear).

§1.2 IMO 2019/2, proposed by Anton Trygub (UKR)

Available online at https://aops.com/community/p12744870.

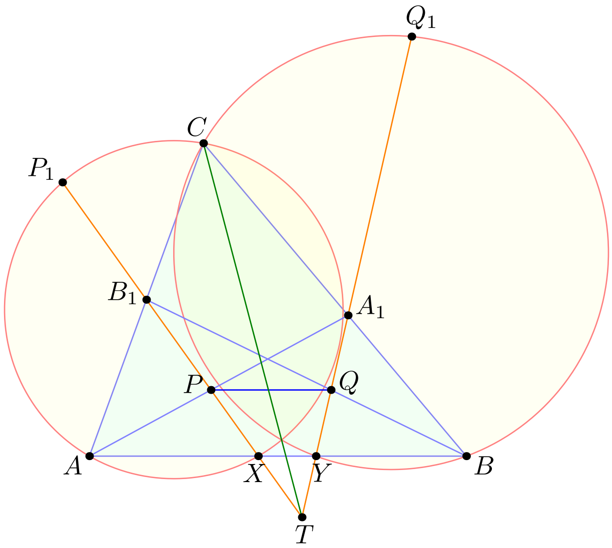
Problem statement

In triangle ABC point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that $\overline{PQ} \parallel \overline{AB}$. Point P_1 is chosen on ray PB_1 beyond B_1 such that $\angle PP_1C = \angle BAC$. Point Q_1 is chosen on ray QA_1 beyond A_1 such that $\angle CQ_1Q = \angle CBA$. Prove that points P_1, Q_1, P, Q are cyclic.

We present two solutions.

¶ First solution by bary (Evan Chen) Let PB_1 and QA_1 meet line AB at X and Y. Since $\overline{XY} \parallel \overline{PQ}$ it is equivalent to show P_1XYQ_1 is cyclic (Reim's theorem) Note that P_1CXA and Q_1CYB are cyclic.

Letting $T = \overline{PX} \cap \overline{QY}$ (possibly at infinity), it suffices to show that the radical axis of $\triangle CXA$ and $\triangle CYB$ passes through T, because that would imply P_1XYQ_1 is cyclic (by power of a point when T is Euclidean, and because it is an isosceles trapezoid if T is at infinity).



To this end we use barycentric coordinates on $\triangle ABC$. We begin by writing

$$P = (u+t : s : r), \quad Q = (t : u+s : r)$$

from which it follows that $A_1 = (0 : s : r)$ and $B_1 = (t : 0 : r)$.

Next, compute $X = \left(\det \begin{bmatrix} u+t & r \\ t & r \end{bmatrix} : \det \begin{bmatrix} s & r \\ 0 & r \end{bmatrix} : 0\right) = (u : s : 0)$. Similarly, $Y = (t : u : 0)$. So we have computed all points.

Claim — Line B_1X has equation $-rs \cdot x + ru \cdot y + st \cdot z = 0$, while line C_1Y has equation $ru \cdot x - rt \cdot y + st \cdot z = 0$.

Proof. Line B_1X is $0 = \det(B_1, X, -) = \det \begin{bmatrix} t & 0 & r \\ u & s & 0 \\ x & y & z \end{bmatrix}$. Line C_1Y is analogous. $\square$

Claim — The radical axis $(u+t)y - (u+s)x = 0$.

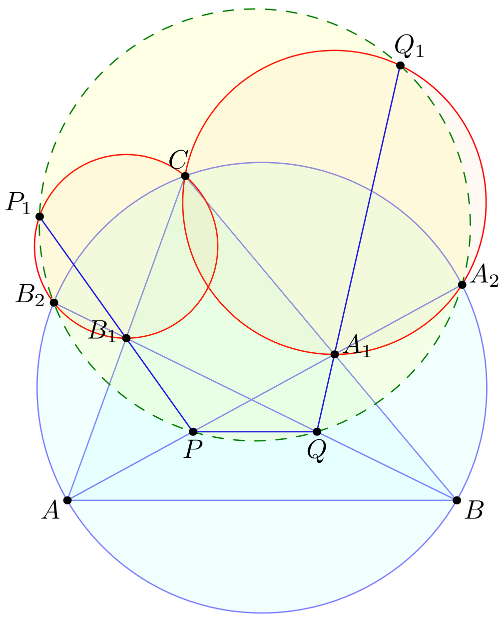
Proof. Circle (AXC) is given by $-a^2yz - b^2zx - c^2xy + (x+y+z)\cdot \frac{c^2\cdot u}{u+s}y = 0$. Similarly, circle (BYC) has equation $-a^2yz - b^2zx - c^2xy + (x+y+z)\cdot \frac{c^2\cdot u}{u+t}x = 0$. Subtracting gives the radical axis. $\square$

Finally, to see these three lines are concurrent, we now compute

$$\begin{aligned} \det \begin{bmatrix} -rs & ru & st \\ ru & -rt & st \\ -(u+s) & u+t & 0 \end{bmatrix} &= rst\left[[u(u+t)-t(u+s)] + [s(u+t)-u(u+s)]\right] \\ &= rst\left[(u^2-st)+(st-u^2)\right] = 0. \end{aligned}$$

This completes the proof.

¶ Second official solution by tricky angle chasing Let lines AA_1 and BB_1 meet at the circumcircle of $\triangle ABC$ again at points A_2 and B_2. By Reim's theorem, PQA_2B_2 are cyclic.



Claim — The points P, Q, A_2, Q_1 are cyclic. Similarly the points P, Q, B_2, P_1 are cyclic.

Proof. Note that $CA_1A_2Q_1$ is cyclic since $\measuredangle CQ_1A_1 = \measuredangle CQ_1Q = \measuredangle CBA = \measuredangle CA_2A = \measuredangle CA_2A_1$. Then $\measuredangle QQ_1A_2 = \measuredangle A_1Q_1A_2 = \measuredangle A_1CA_2 = \measuredangle BCA_2 = \measuredangle BAA_2 = \measuredangle QPA_2$. $\square$

This claim obviously solves the problem.

§1.3 IMO 2019/3, proposed by Adrian Beker (HRV)

Available online at https://aops.com/community/p12744851.

Problem statement

A social network has 2019 users, some pairs of which are friends (friendship is symmetric). If A, B, C are three users such that AB are friends and AC are friends but BC is not, then the administrator may perform the following operation: change the friendships such that BC are friends, but AB and AC are no longer friends.

Initially, 1009 users have 1010 friends and 1010 users have 1009 friends. Prove that the administrator can make a sequence of operations such that all users have at most 1 friend.

We take the obvious graph formulation and call the move a *toggle*.

Claim — Let G be a connected graph. Then one can toggle G without disconnecting the graph, unless G is a clique, a cycle, or a tree.

Proof. Assume G is connected and not a tree, so it has a cycle. Take the smallest cycle C; by hypothesis $C \neq G$.

If C is not a triangle (equivalently, G is triangle-free), then let $b \notin C$ be a vertex adjacent to C, say at a. Take a vertex c of the cycle adjacent to a (hence not to b). Then we can toggle abc.

Now assume there exists a triangle; let K be the maximal clique. By hypothesis, $K \neq G$. We take an edge $e = ab$ dangling off the clique, with $a \in K$ and $b \notin K$. Note some vertex c of K is not adjacent to b; now toggle abc. □

Back to the original problem; let G_{imo} be the given graph. The point is that we can apply toggles (by the claim) repeatedly, without disconnecting the graph, until we get a tree. This is because

- G_{imo} is connected, since any two vertices which are not adjacent have a common neighbor by pigeonhole ($1009 + 1009 + 2 > 2019$).
- G_{imo} cannot become a cycle, because it initially has an odd-degree vertex, and toggles preserve parity of degree!
- G_{imo} is obviously not a clique initially (and hence not afterwards).

So, we can eventually get G_{imo} to be a tree.

Once G_{imo} is a tree the problem follows by repeatedly applying toggles arbitrarily until no more are possible; the graph (although now disconnected) remains acyclic (in particular having no triangles) and therefore can only terminate in the desired situation.

Remark. The above proof in fact shows the following better result:

> The task is possible if and only if G_{imo} is a connected graph which is not a clique and has any vertex of odd degree.

The "only if" follows from the observation that toggles preserve parity of degree.

Thus the given condition about the degrees of vertices being 1009 and 1010 is largely a red herring; it's a somewhat strange way of masking the correct and more natural both-sufficient-and-necessary condition.

§2 Solutions to Day 2

§2.1 IMO 2019/4, proposed by Gabriel Chicas Reyes (SLV)

Available online at https://aops.com/community/p12752761.

Problem statement

Solve over positive integers the equation

$$k! = \prod_{i=0}^{n-1}(2^n - 2^i) = (2^n-1)(2^n-2)(2^n-4)\dots(2^n-2^{n-1}).$$

The answer is $(n,k) = (1,1)$ and $(n,k) = (2,3)$ which work.

Let $A = \prod_i (2^n - 2^k)$, and assume $A = k!$ for some $k \ge 3$. Recall by exponent lifting that

$$\nu_3(2^t - 1) = \begin{cases} 0 & t \text{ odd} \\ 1 + \nu_3(t) & t \text{ even.} \end{cases}$$

Consequently, we can compute

$$\begin{aligned} k > \nu_2(k!) = \nu_2(A) &= 1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2} \\ \left\lfloor \frac{k}{3} \right\rfloor \le \nu_3(k!) = \nu_3(A) &= \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor + \dots < \frac{3}{4}n. \end{aligned}$$

where the very first inequality can be justified say by Legendre's formula $\nu_2(k!) = k - s_2(k)$.

In this way, we get

$$\frac{9}{4}n + 3 > k > \frac{n(n-1)}{2}$$

which means $n \le 6$; a manual check then shows the solutions we claimed earlier are the only ones.

Remark. An amusing corollary of the problem pointed out in the Shortlist is that the symmetric group S_k cannot be isomorphic to the group $\mathrm{GL}_n(\mathbb{F}_2)$ unless $(n,k) = (1,1)$ or $(n,k) = (2,3)$, which indeed produce isomorphisms.

§2.2 IMO 2019/5, proposed by David Altizio (USA)

Available online at https://aops.com/community/p12752847.

Problem statement

Let n be a positive integer. Harry has n coins lined up on his desk, which can show either heads or tails. He does the following operation: if there are k coins which show heads and $k > 0$, then he flips the kth coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \to HHT \to HTT \to TTT$, which takes three steps.)

Prove the process will always terminate, and determine the average number of steps this takes over all 2^n configurations.

The answer is

$$E_n = \frac{1}{2}(1 + \cdots + n) = \frac{1}{4}n(n+1)$$

which is finite.

We'll represent the operation by a directed graph G_n on vertices $\{0,1\}^n$ (each string points to its successor) with 1 corresponding to heads and 0 corresponding to tails. For $b \in \{0,1\}$ we let $\overline{b} = 1 - b$, and denote binary strings as a sequence of n symbols.

The main claim is that G_n can be described explicitly in terms of G_{n-1}:

- We take two copies X and Y of G_{n-1}.
- In X, we take each string of length $n-1$ and just append a 0 to it. In symbols, we replace $s_1 \dots s_{n-1} \mapsto s_1 \dots s_{n-1}0$.
- In Y, we toggle every bit, then reverse the order, and then append a 1 to it. In symbols, we replace $s_1 \dots s_{n-1} \mapsto \overline{s}_{n-1}\overline{s}_{n-2} \dots \overline{s}_1 1$.
- Finally, we add one new edge from Y to X by $11\dots1 \to 11\dots110$.

An illustration of G_4 is given below.

$$\begin{tikzcd}
0001 \arrow[d] & 0101 \arrow[l] & 0111 \arrow[l] & 0011 \arrow[l] \\
1001 \arrow[d] & 1011 \arrow[l] & & \\
1101 \arrow[d] & & & \\
1111 \arrow[d, Rightarrow] & & & \\
1110 \arrow[d] & 1010 \arrow[l] & 0010 \arrow[l] & 0110 \arrow[l] \\
1100 \arrow[d] & 0100 \arrow[l] & & \\
1000 \arrow[d] & & & \\
0000 & & &
\end{tikzcd}$$

To prove this claim, we need only show the arrows of this directed graph remain valid. The graph X is correct as a subgraph of G_n, since the extra 0 makes no difference. As for Y, note that if $s = s_1 \dots s_{n-1}$ had k ones, then the modified string has $(n-1-k)+1 = n-k$

ones, ergo $\overline{s}_{n-1} \dots \overline{s}_1 1 \mapsto \overline{s}_{n-1} \dots \overline{s}_{k+1} s_k \overline{s}_{k-1} \dots \overline{s}_1 1$ which is what we wanted. Finally, the one edge from Y to X is obviously correct.

To finish, let E_n denote the desired expected value. Since $1 \dots 1$ takes n steps to finish we have

$$E_n = \frac{1}{2}\left[E_{n-1} + (E_{n-1} + n)\right]$$

based on cases on whether the chosen string is in X or Y or not. By induction, we have $E_n = \frac{1}{2}(1 + \dots + n) = \frac{1}{4}n(n+1)$, as desired.

Remark. Actually, the following is true: if the indices of the 1's are $1 \le i_1 < \dots < i_\ell \le n$, then the number of operations required is

$$2(i_1 + \dots + i_\ell) - \ell^2.$$

This problem also has an interpretation as a Turing machine: the head starts at a position on the tape (the binary string). If it sees a 1, it changes the cell to a 0 and moves left; if it sees a 0, it changes the cell to a 1 and moves right.

§2.3 IMO 2019/6, proposed by Anant Mudgal (IND)

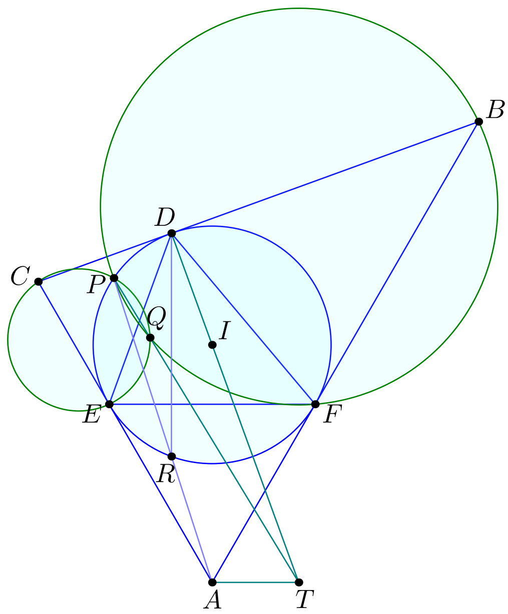
Available online at https://aops.com/community/p12752769.

Problem statement

Let ABC be a triangle with incenter I and incircle ω. Let D, E, F denote the tangency points of ω with $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. The line through D perpendicular to $\overline{EF}$ meets ω again at R (other than D), and line AR meets ω again at P (other than R). Suppose the circumcircles of $\triangle PCE$ and $\triangle PBF$ meet again at Q (other than P). Prove that lines DI and PQ meet on the external $\angle A$-bisector.

We present three solutions.

¶ First solution by complex numbers (Evan Chen, with Yang Liu) We use complex numbers with $D = x$, $E = y$, $F = z$.



Then $A = \frac{2yz}{y+z}$, $R = \frac{-yz}{x}$ and so

$$P = \frac{A - R}{1 - R\overline{A}} = \frac{\frac{2yz}{y+z} + \frac{yz}{x}}{1 + \frac{yz}{x} \cdot \frac{2}{y+z}} = \frac{yz(2x + y + z)}{2yz + x(y + z)}.$$

We now compute

$$O_B = \det \begin{bmatrix} P & P\overline{P} & 1 \\ F & F\overline{F} & 1 \\ B & B\overline{B} & 1 \end{bmatrix} \div \det \begin{bmatrix} P & \overline{P} & 1 \\ F & \overline{F} & 1 \\ B & \overline{B} & 1 \end{bmatrix} = \det \begin{bmatrix} P & 1 & 1 \\ z & 1 & 1 \\ \frac{2xz}{x+z} & \frac{4xz}{(x+z)^2} & 1 \end{bmatrix} \div \det \begin{bmatrix} P & 1/P & 1 \\ z & 1/z & 1 \\ \frac{2xz}{x+z} & \frac{2}{x+z} & 1 \end{bmatrix}$$

$$= \frac{1}{x+z} \det \begin{bmatrix} P & 0 & 1 \\ z & 0 & 1 \\ 2xz(x+z) & -(x-z)^2 & (x+z)^2 \end{bmatrix} \div \det \begin{bmatrix} P & 1/P & 1 \\ z & 1/z & 1 \\ 2xz & 2 & x+z \end{bmatrix}$$

$$= \frac{(x-z)^2}{x+z} \cdot \frac{P - z}{(x+z)(P/z - z/P) + 2z - 2x + \frac{2xz}{P} - 2P}$$

$$
\begin{aligned}
&= \frac{(x-z)^2}{x+z} \cdot \frac{P-z}{(\frac{x}{z}-1)P - 2(x-z) + (xz-z^2)\frac{1}{P}} \\
&= \frac{x-z}{x+z} \cdot \frac{P-z}{P/z + z/P - 2} = \frac{x-z}{x+z} \cdot \frac{P-z}{\frac{(P-z)^2}{Pz}} = \frac{x-z}{x+z} \cdot \frac{1}{\frac{1}{z} - \frac{1}{P}} \\
&= \frac{x-z}{x+z} \cdot \frac{y(2x+y+z)}{y(2x+y+z) - (2yz+xy+xz)} = \frac{x-z}{x+z} \cdot \frac{yz(2x+y+z)}{xy+y^2-yz-xz} \\
&= \frac{x-z}{x+z} \cdot \frac{yz(2x+y+z)}{(y-z)(x+y)}.
\end{aligned}
$$

Similarly

$$O_C = \frac{x-y}{x+y} \cdot \frac{yz(2x+y+z)}{(z-y)(x+z)}.$$

Therefore, subtraction gives

$$O_B - O_C = \frac{yz(2x+y+z)}{(x+y)(x+z)(y-z)}\left[(x-z)+(x-y)\right] = \frac{yz(2x+y+z)(2x-y-z)}{(x+y)(x+z)(z-y)}.$$

It remains to compute T. Since $T \in \overline{ID}$ we have $t/x \in \mathbb{R}$ so $\overline{t} = t/x^2$. Also,

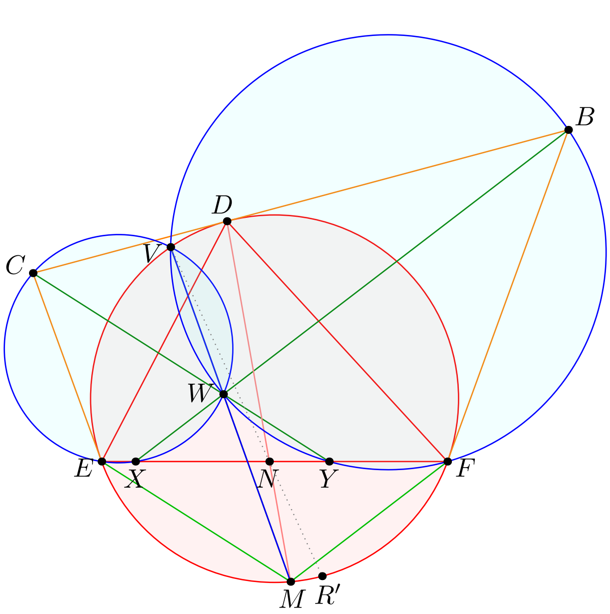
$$
\begin{aligned}
\frac{t - \frac{2yz}{y+z}}{y+z} \in i\mathbb{R} \implies 0 &= \frac{t - \frac{2yz}{y+z}}{y+z} + \frac{\frac{t}{x^2} - \frac{2}{y+z}}{\frac{1}{y}+\frac{1}{z}} \\
&= \frac{1+\frac{yz}{x^2}}{y+z}t - \frac{2yz}{(y+z)^2} - \frac{2yz}{(y+z)^2} \\
\implies t &= \frac{x^2}{x^2+yz} \cdot \frac{4yz}{y+z}
\end{aligned}
$$

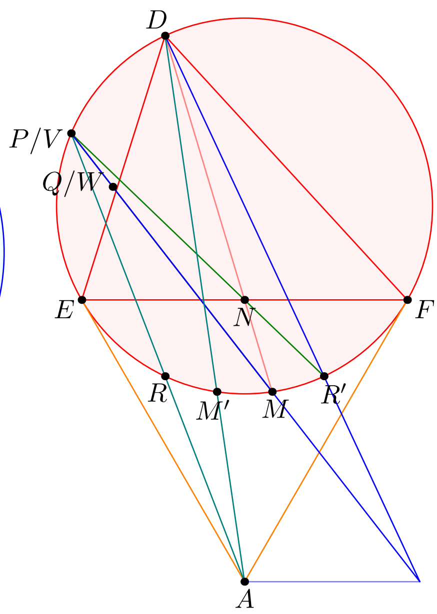
Thus

$$
\begin{aligned}
P - T &= \frac{yz(2x+y+z)}{2yz+x(y+z)} - \frac{4x^2yz}{(x^2+yz)(y+z)} \\
&= yz \cdot \frac{(2x+y+z)(x^2+yz)(y+z) - 4x^2(2yz+xy+xz)}{(y+z)(x^2+yz)(2yz+xy+xz)} \\
&= -yz \cdot \frac{(2x-y-z)(x^2y+x^2z+4xyz+y^2z+yz^2)}{(y+z)(x^2+yz)(2yz+xy+xz)}.
\end{aligned}
$$

This gives $\overline{PT} \perp \overline{O_BO_C}$ as needed.

¶ Second solution by tethered moving points, with optimization (Evan Chen) Fix $\triangle DEF$ and ω, with $B = \overline{DD} \cap \overline{FF}$ and $C = \overline{DD} \cap \overline{EE}$. We consider a variable point M on ω and let X, Y be on $\overline{EF}$ with $\overline{CY} \cap \parallel \overline{ME}$, $\overline{BX} \cap \parallel \overline{MF}$. We define $W = \overline{CY} \cap \overline{BX}$. Also, let line MW meet ω again at V.





> **Claim** (Angle chasing) — Pentagons $CVWXE$ and $BVWYF$ are cyclic.

Proof. By $\measuredangle EVW = \measuredangle EVM = \measuredangle EFM = \measuredangle CEM = \measuredangle ECW$ and $\measuredangle EXW = \measuredangle EFM = \measuredangle CEM = \measuredangle ECW$. $\square$

Let $N = \overline{DM} \cap \overline{EF}$ and R' be the D-antipode on ω.

> **Claim** (Black magic) — The points V, N, R' are collinear.

Proof. We use tethered moving points with $\triangle DEF$ fixed.

Obviously the map $\omega \mapsto \overline{EF} \mapsto \omega$ by $M \mapsto N \mapsto \overline{R'N} \cap \omega$ is projective. Also, the map $\omega \mapsto \overline{EF} \mapsto \omega$ by $M \mapsto X \mapsto V$ is also projective (the first by projection to the line at infinity at back; the second say by inversion at E).

So it suffices to check for three points. When $M = E$ we get $N = E$ so $\overline{R'N} \cap \omega = E$, while $W = E$ and thus $V = E$. The case $M = F$ is similar. Finally, if $M = R'$, then W is the center of ω and so $V = \overline{R'N} \cap \overline{EF} = D$. $\square$

We now address the original problem by specializing M: choose it so that N is the midpoint of $\overline{EF}$. Let $M' = \overline{DA} \cap (DEF)$.

> **Claim** — After this specialization, $V = P$ and $W = Q$.

Proof. Thus $\overline{RR'}$ and $\overline{MM'}$ are parallel to $\overline{EF}$. From $(EF;PR) = -1 = (EF;N\infty) \overset{R'}{=} (EF;NV)$, we derive that $P = V$ and $Q = R$, proving (i). $\square$

Finally, the concurrence requested follows by Pascal theorem on $M'MDR'PR$.

¶ Third solution by power of a point linearity (Luke Robitaille) Let us define

$$f(\bullet) = \operatorname{Pow}(\bullet, (CPE)) - \operatorname{Pow}(\bullet, (BPF))$$

which is a linear function from the plane to $\mathbb{R}$.

Define $W = \overline{BA} \cap \overline{PE}$, $V = \overline{AC} \cap \overline{PF}$. Also, let $W_1 = \overline{ER} \cap \overline{AB}$, $V_1 = \overline{FR} \cap \overline{AC}$. Note that

$$-1 = (PR; EF) \stackrel{E}{=} (WA; W_1F)$$

and similarly $(VA; V_1E) = -1$.

Claim — We have

$$\begin{aligned} f(F) &= \frac{|EF| \cdot (s-c) \sin C/2}{\sin B/2} \\ f(E) &= -\frac{|EF| \cdot (s-b) \sin B/2}{\sin C/2}. \end{aligned}$$

Proof. We have

$$f(W) = WF^2 - WB \cdot WF = WF \cdot BF$$

where lengths are directed. Next,

$$\begin{aligned} f(F) &= \frac{AF \cdot f(W) + FW \cdot f(A)}{AW} \\ &= \frac{AF \cdot WF \cdot BF + FW \cdot (AE \cdot AC - AF \cdot AB)}{AW} \\ &= \frac{WF(AF \cdot BF + AF \cdot AB) + FW \cdot AE \cdot AC}{AW} \\ &= \frac{WF \cdot AF^2 - WF \cdot AE \cdot AC}{AW} = \frac{WF}{AW} \cdot (AE^2 - AE \cdot AC) \\ &= \frac{WF}{AW} \cdot AE \cdot CE = -\frac{W_1F}{AW_1} \cdot AE \cdot CE. \end{aligned}$$

Since $\triangle DEF$ is acute, the point R lies inside $\triangle AEF$. Thus W_1 lies inside segment $\overline{AF}$ and the ratio $\frac{W_1F}{AW_1}$ is positive. We now determine its value: by the ratio lemma

$$\begin{aligned} \frac{|W_1F|}{|AW_1|} &= \frac{|EF| \sin \angle W_1EF}{|AE| \sin \angle AEW_1} \\ &= \frac{|EF| \sin \angle REF}{|AE| \sin \angle AER} \\ &= \frac{|EF| \sin \angle RDF}{|AE| \sin \angle EDR} \\ &= \frac{|EF| \sin C/2}{|AE| \sin B/2}. \end{aligned}$$

Also, we have $AE \cdot CE < 0$ since E lies inside $\overline{AC}$. Hence

$$f(F) = -\frac{|EF| \sin C/2}{|AE| \sin B/2} \cdot \cdot AE \cdot CE = |EF| \cdot \frac{|CE| \sin B/2}{\sin C/2} = |EF| \cdot \frac{(s-c) \sin B/2}{\sin C/2}.$$

The calculation for $f(E)$ is similar, (noting the sign flips since f is anti-symmetric in terms of B and C). $\square$

Let $Z \in \overline{DI}$ with $\angle ZAI = 90^\circ$ be the point requested in the problem now. Our goal is to show $f(Z) = 0$. We assume WLOG that $AB < AC$, so $\frac{ZA}{EF} > 0$. Then

$$|ZA| = |AI| \cdot \tan \angle AIZ$$

$$
\begin{aligned}
&= |AI| \cdot \tan \angle(\overline{AI}, \overline{DI}) \\
&= \frac{s-a}{\cos A/2} \cdot \tan(\overline{BC}, \overline{EF}) \\
&= \frac{s-a}{\cos A/2} \tan(B/2 - C/2).
\end{aligned}
$$

To this end we compute

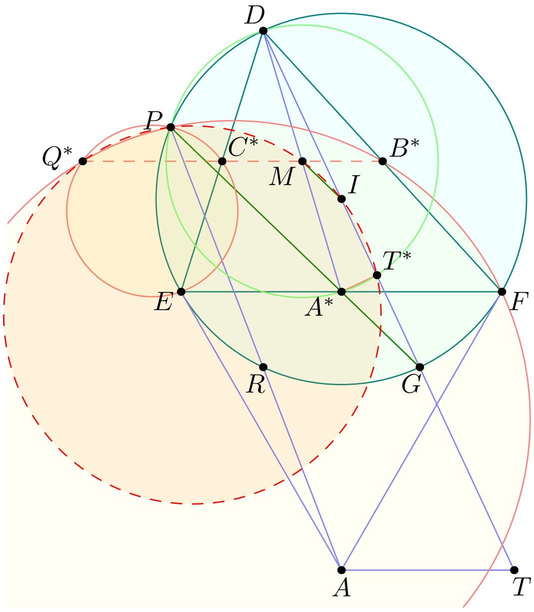
$$
\begin{aligned}
f(Z) &= f(A) + [f(Z) - f(A)] = f(A) + \frac{ZA}{EF}\left[f(E) - f(F)\right] \\
&= f(A) - \frac{ZA}{EF}\left[\frac{|EF| \cdot (s-b)\sin B/2}{\sin C/2} + \frac{|EF| \cdot (s-c)\sin C/2}{\sin B/2}\right] \\
&= f(A) - |ZA|\left[\frac{(s-b)\sin B/2}{\sin C/2} + \frac{(s-c)\sin C/2}{\sin B/2}\right] \\
&= [b(s-a) - c(s-a)] - |ZA|\left[\frac{(s-b)\sin B/2}{\sin C/2} + \frac{(s-c)\sin C/2}{\sin B/2}\right] \\
&= (b-c)(s-a) - \frac{s-a}{\cos A/2}\tan(B/2 - C/2)\left[\frac{(s-b)\sin B/2}{\sin C/2} + \frac{(s-c)\sin C/2}{\sin B/2}\right].
\end{aligned}
$$

Dividing out,

$$
\begin{aligned}
\frac{f(Z)}{s-a} &= (b-c) - \frac{1}{\cos A/2}\tan(B/2 - C/2)\left[\frac{r\cos B/2}{\sin C/2} + \frac{r\cos C/2}{\sin B/2}\right] \\
&= (b-c) - \frac{r\tan(B/2 - C/2)}{\cos A/2} \cdot \frac{\cos B/2 \sin B/2 + \cos C/2 \sin C/2}{\sin C/2 \sin B/2} \\
&= (b-c) - \frac{r\tan(B/2 - C/2)}{\cos A/2} \cdot \frac{\sin B + \sin C}{2\sin C/2 \sin B/2} \\
&= (b-c) - \frac{r\tan(B/2 - C/2)}{\cos A/2} \cdot \frac{\sin(B/2 + C/2)\cos(B/2 - C/2)}{\sin C/2 \sin B/2} \\
&= (b-c) - r\frac{\sin(B/2 - C/2)}{\sin B/2 \sin C/2} \\
&= (b-c) - r(\cot C/2 - \cot B/2) = (b-c) - ((s-c) - (s-b)) = 0.
\end{aligned}
$$

¶ Fourth solution by incircle inversion (USA IMO live stream, led by Andrew Gu)
Let T be the intersection of line DI and the external $\angle A$-bisector. Also, let G be the antipode of D on ω.

We perform inversion around ω, using $\bullet^*$ for the inverse. Then $\triangle A^*B^*C^*$ is the medial triangle of $\triangle DEF$, and T^* is the foot from A^* on to $\overline{DI}$. If we denote Q^* as the second intersection of (PC^*E) and (PB^*F), then the goal it show that Q^* lies on (PIT^*).



Claim — Points Q^*, B^*, C^* are collinear.

Proof. $\measuredangle PQ^*C^* = \measuredangle PEC^* = \measuredangle PED = \measuredangle PFD = \measuredangle PFB^* = \measuredangle PQ^*B^*$. $\square$

Claim (cf Brazil 2011/5) — Points P, A^*, G are collinear.

Proof. Project harmonic quadrilateral $PERF$ through G, noting $\overline{GR} \parallel \overline{EF}$. $\square$

Denote by M the center of parallelogram $DC^*A^*B^*$. Note that it is the center of the circle with diameter $\overline{DA^*}$, which passes through P and T^*. Also, $\overline{MI} \parallel \overline{PA^*G}$.

Claim — Points P, M, I, T^* are cyclic.

Proof. $\measuredangle IT^*P = \measuredangle DT^*P = \measuredangle DA^*P = \measuredangle MA^*P = \measuredangle A^*PM = \measuredangle IMP$. $\square$

Claim — Points P, M, I, Q^* are cyclic.

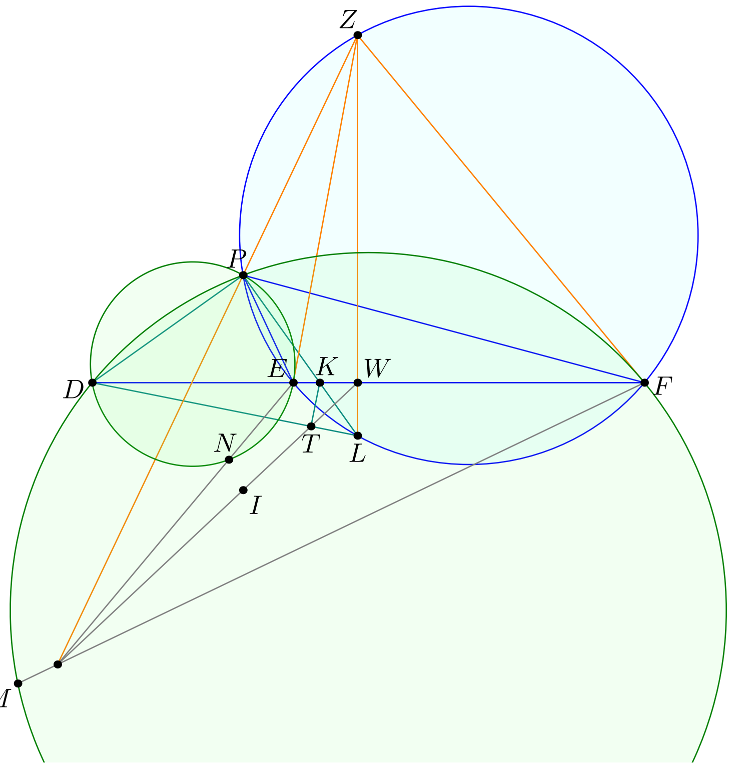
Proof. $\measuredangle MQ^*P = \measuredangle C^*Q^*P = \measuredangle C^*EP = \measuredangle DEP = \measuredangle DGP = \measuredangle GPI = \measuredangle MIP$. $\square$

¶ Fifth solution by double inversion (Brandon Wang, Luke Robitaille, Michael Ren, Evan Chen) We outline one final approach. After inverting about ω as in the previous approach, we then apply another inversion around P. Dropping the apostrophes/stars/etc now one can check that the problem we arrive at becomes the following.

Proposition (Doubly inverted problem)

In $\triangle PEF$, the P-symmedian meets $\overline{EF}$ and (PEF) at K, L. Let $D \in \overline{EF}$ with $\angle DPK = 90^\circ$, and let T be the foot from K to $\overline{DL}$. Denote by I the reflection of P about $\overline{EF}$. Finally, let $PDNE$ and $PDMF$ be cyclic harmonic quadrilaterals. Then lines EN, MF, TI, are concurrent.

The proof proceeds in three steps. Suppose the line through L perpendicular to $\overline{EF}$ meets $\overline{EF}$ at W and (PEF) at Z.



1. Since $\measuredangle ZEP = \measuredangle WLP = \measuredangle WDP$, it follows $\overline{ZE}$ is tangent to $(PDNE)$. Similarly, $\overline{ZF}$ is tangent to $(PDMF)$.
2. $\triangle WTP$ is the orthic triangle of $\triangle DKL$, so $\overline{WD}$ bisects $\angle PWT$ and $\overline{WTI}$ collinear.
3. $-1 = E(PN;DZ) = F(PM;DZ) = W(PI;DZ)$, so $\overline{EN}$, $\overline{FM}$, $\overline{WI}$ meet on $\overline{PZ}$.

IMO 2020 Solution Notes

EVAN CHEN《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2020 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

2. The real numbers a, b, c, d are such that $a \ge b \ge c \ge d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1.$$

3. There are $4n$ pebbles of weights $1, 2, 3, \dots, 4n$. Each pebble is coloured in one of n colours and there are four pebbles of each colour. Show that we can arrange the pebbles into two piles the total weights of both piles are the same, and each pile contains two pebbles of each colour.

4. There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

5. A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that the numbers on the cards are all equal?

6. Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. Prove there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $\Omega(n^{-1/3})$.

   (A line ℓ separates a set of points S if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

§1 Solutions to Day 1

§1.1 IMO 2020/1, proposed by Dominik Burek (POL)

Available online at https://aops.com/community/p17821635.

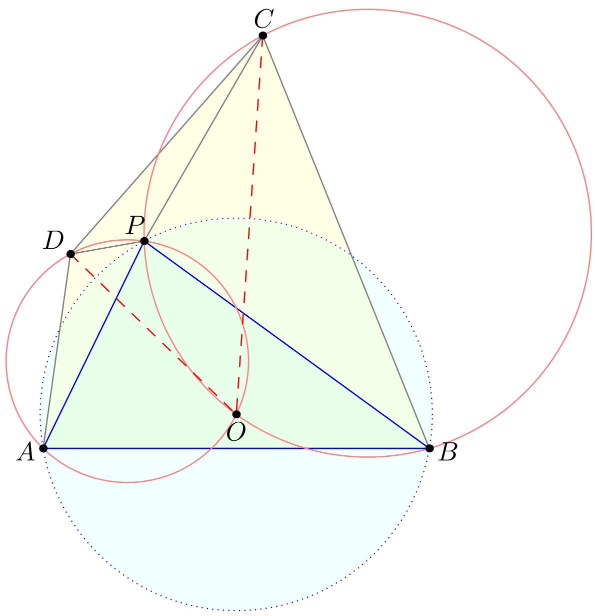
Problem statement

Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Let O denote the circumcenter of $\triangle PAB$. We claim it is the desired concurrency point.



Indeed, O obviously lies on the perpendicular bisector of AB. Now

$$\begin{aligned} \measuredangle BCP &= \measuredangle CBP + \measuredangle BPC \\ &= 2\measuredangle BAP = \measuredangle BOP \end{aligned}$$

it follows $BOPC$ are cyclic. And since $OP = OB$, it follows that O is on the bisector of $\angle PCB$, as needed.

Remark. The angle equality is only used insomuch $\angle BAP$ is the average of $\angle CBP$ and $\angle BPC$, i.e. only $\frac{1+3}{2} = 2$ matters.

§1.2 IMO 2020/2, proposed by BEL

Available online at https://aops.com/community/p17821569.

Problem statement

The real numbers a, b, c, d are such that $a \ge b \ge c \ge d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1.$$

By weighted AM-GM we have

$$a^a b^b c^c d^d \le \sum_{\text{cyc}} \frac{a}{a+b+c+d} \cdot a = a^2 + b^2 + c^2 + d^2.$$

Consequently, it is enough to prove that

$$(a^2 + b^2 + c^2 + d^2)(a + 2b + 3c + 4d) \le 1 = (a + b + c + d)^3.$$

Expand both sides to get

$$\begin{array}{cccc} +a^3 & +b^2a & +c^2a & +d^2a \\ +2a^2b & +2b^3 & +2b^2c & +2d^2b \\ +3a^2c & +3b^2c & +3c^3 & +3d^2c \\ +4a^2d & +4b^2d & +4c^2d & +4d^3 \end{array} < \begin{array}{cccc} +a^3 & +3b^2a & +3c^2a & +3d^2a \\ +3a^2b & +b^3 & +3b^2c & +3d^2b \\ +3a^2c & +3b^2c & +c^3 & +3d^2c \\ +3a^2d & +3b^2d & +3c^2d & +d^3 \\ +6abc & +6bcd & +6cda & +6dab \end{array}$$

In other words, we need to prove that

$$\begin{array}{cccc} & +b^3 & & \\ & & +2c^3 & \\ +a^2d & +b^2d & +c^2d & +3d^3 \end{array} < \begin{array}{cccc} & +2b^2a & +2c^2a & +2d^2a \\ +a^2b & & +b^2c & +d^2b \\ & & & \\ +6abc & +6bcd & +6cda & +6dab \end{array}$$

This follows since

$$\begin{aligned} 2b^2a &\ge b^3 + c^2d \\ 2c^2a &\ge 2c^3 \\ 2d^2a &\ge 2d^3 \\ a^2b &\ge a^2d \\ b^2c &\ge b^2d \\ d^2b &\ge d^3 \end{aligned}$$

and $6(abc + bcd + cda + dab) > 0$.

Remark. Fedor Petrov provides the following motivational comments for why the existence of this solution is not surprising:

> Better to think about mathematics. You have to bound from above a product $(a+2b+3c+4d)(a^2+b^2+c^2+d^2)$, the coefficients $1, 2, 3, 4$ are increasing and so play on your side, so plausibly $(a+b+c+d)^3$ should majorize this term-wise, you check it and this appears to be true.

§1.3 IMO 2020/3, proposed by Milan Haiman (HUN), Carl Schildkraut (USA)

Available online at https://aops.com/community/p17821656.

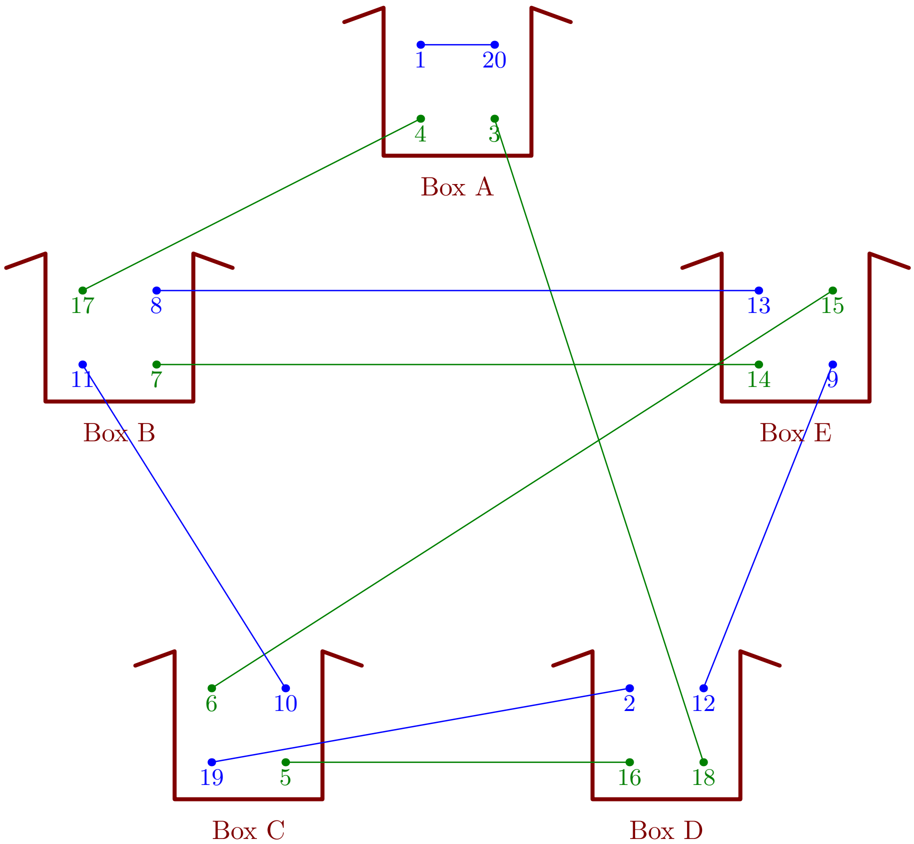
Problem statement

There are $4n$ pebbles of weights $1, 2, 3, \dots, 4n$. Each pebble is coloured in one of n colours and there are four pebbles of each colour. Show that we can arrange the pebbles into two piles the total weights of both piles are the same, and each pile contains two pebbles of each colour.

The first key idea is the deep fact that

$$1 + 4n = 2 + (4n - 1) = 3 + (4n - 2) = \dots.$$

So, place all four pebbles of the same colour in a box (hence n boxes). For each $k = 1, 2, \dots, 2n$ we tape a piece of string between pebble k and $4n + 1 - k$. To solve the problem, it suffices to paint each string either blue or green such that each box has two blue strings and two green strings (where a string between two pebbles in the same box counts double).



We can therefore rephrase the problem as follows, if we view boxes as vertices and strings as edges:

Claim — Given a 4-regular multigraph on n vertices (where self-loops are allowed and have degree 2), one can color the edges blue and green such that each vertex has two blue and two green edges.

Proof. Each connected component of the graph can be decomposed into an Eulerian circuit, since 4 is even. A connected component with k vertices has $2k$ edges in its Eulerian circuit, so we may color the edges in this circuit alternating green and blue. This may be checked to work. $\square$

§2 Solutions to Day 2

§2.1 IMO 2020/4, proposed by Tejaswi Navilarekallu (IND)

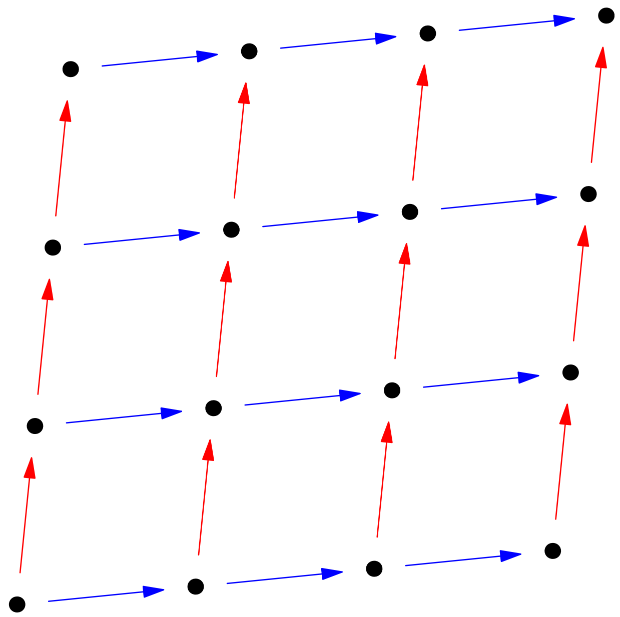
Available online at https://aops.com/community/p17821585.

Problem statement

There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Answer: $k = n^2 - n + 1$.

When $k = n^2 - n$, the construction for $n = 4$ is shown below which generalizes readily. (We draw A in red and B in blue.)



To see this is sharp, view A and B as graphs whose connected components are paths (possibly with 0 edges; the direction of these edges is irrelevant). Now, if $k = n^2 - n + 1$ it follows that A and B each have exactly $n - 1$ connected components.

But in particular some component of A has at least $n + 1$ vertices. This component has two vertices in the same component of B, as desired.

Remark. The main foothold for this problem is the hypothesis that the number of stations should be n^2 rather than, say, n. This gives a big hint towards finding the construction which in turn shows how the bound can be computed.

On the other hand, the hypothesis that "a cable car which starts higher also finishes

> higher" appears to be superfluous.

§2.2 IMO 2020/5, proposed by Oleg Košik (EST)

Available online at https://aops.com/community/p17821528.

Problem statement

A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that the numbers on the cards are all equal?

The assertion is true for all n.

Setup (boilerplate). Suppose that $a_1, \dots, a_n$ satisfy the required properties but are not all equal. Let $d = \gcd(a_1, \dots, a_n) > 1$ then replace $a_1, \dots, a_n$ by $\frac{a_1}{d}, \dots, \frac{a_n}{d}$. Hence without loss of generality we may assume

$$\gcd(a_1, a_2, \dots, a_n) = 1.$$

WLOG we also assume

$$a_1 \ge a_2 \ge \dots \ge a_n.$$

Main proof. As $a_1 \ge 2$, let p be a prime divisor of a_1. Let k be smallest index such that $p \nmid a_k$ (which must exist). In particular, note that $a_1 \ne a_k$.

Consider the mean $x = \frac{a_1+a_k}{2}$; by assumption, it equals some geometric mean, hence

$$\sqrt[m]{a_{i_1} \dots a_{i_m}} = \frac{a_1 + a_k}{2} > a_k.$$

Since the arithmetic mean is an integer not divisible by p, all the indices $i_1, i_2, \dots, i_m$ must be at least k. But then the GM is at most a_k, contradiction.

Remark. A similar approach could be attempted by using the smallest numbers rather than the largest ones, but one must then handle the edge case $a_n = 1$ separately since no prime divides 1.

Remark. Since $\frac{27+9}{2} = 18 = \sqrt[3]{27 \cdot 27 \cdot 8}$, it is not true that in general the AM of two largest different cards is not the GM of other numbers in the sequence (say the cards are $27, 27, 9, 8, \dots$).

§2.3 IMO 2020/6, proposed by Ting-Feng Lin, Hung-Hsun Hans Yu (TWN)

Available online at https://aops.com/community/p17821732.

Problem statement

Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. Prove there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $\Omega(n^{-1/3})$.

(A line ℓ separates a set of points S if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

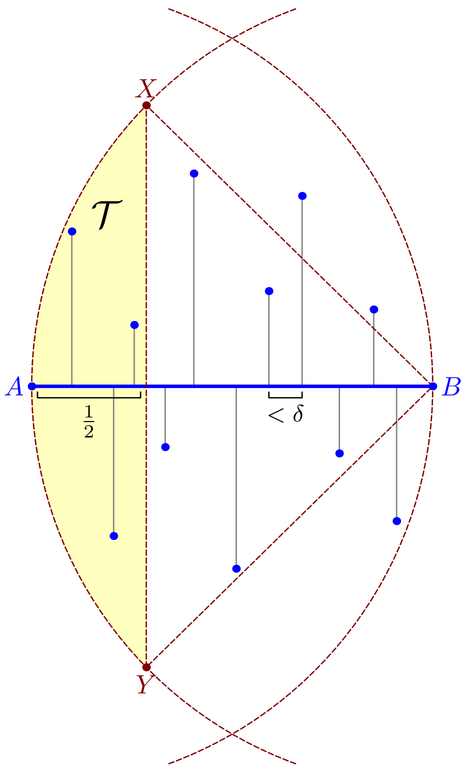
We present the official solution given by the Problem Selection Committee.

Let's suppose that among all projections of points in $\mathcal{S}$ onto some line m, the maximum possible distance between two consecutive projections is δ. We will prove that $\delta \geq \Omega(n^{-1/3})$, solving the problem.

We make the following the definitions:

- Define A and B as the two points farthest apart in $\mathcal{S}$. This means that all points lie in the intersections of the circles centered at A and B with radius $R = AB \geq 1$.
- We pick chord $\overline{XY}$ of $\odot(B)$ such that $\overline{XY} \perp \overline{AB}$ and the distance from A to $\overline{XY}$ is exactly $\frac{1}{2}$.
- We denote by $\mathcal{T}$ the smaller region bound by $\odot(B)$ and chord $\overline{XY}$.

The figure is shown below with $\mathcal{T}$ drawn in yellow, and points of $\mathcal{S}$ drawn in blue.



Claim (Length of AB + Pythagorean theorem) — We have $XY < 2\sqrt{n\delta}$.

Proof. First, note that we have $R = AB < (n-1)\cdot\delta$, since the n projections of points onto AB are spaced at most δ apart. The Pythagorean theorem gives

$$XY = 2\sqrt{R^2 - \left(R - \frac{1}{2}\right)^2} = 2\sqrt{R - \frac{1}{4}} < 2\sqrt{n\delta}. \qquad \square$$

Claim ($|\mathcal{T}|$ lower bound + narrowness) — We have $XY > \frac{\sqrt{3}}{2}\left(\frac{1}{2}\delta^{-1} - 1\right)$.

Proof. Because $\mathcal{T}$ is so narrow (has width $\frac{1}{2}$ only), the projections of points in $\mathcal{T}$ onto line XY are spaced at least $\frac{\sqrt{3}}{2}$ apart (more than just δ). This means

$$XY > \frac{\sqrt{3}}{2}\left(|\mathcal{T}| - 1\right).$$

But projections of points in $\mathcal{T}$ onto the segment of length $\frac{1}{2}$ are spaced at most δ apart, so apparently

$$|\mathcal{T}| > \frac{1}{2}\cdot\delta^{-1}.$$

This implies the result. $\square$

Combining these two this implies $\delta \geq \Omega(n^{-1/3})$ as needed.

Remark. The constant $1/3$ in the problem is actually optimal and cannot be improved; the constructions give an example showing $\Theta(n^{-1/3}\log n)$.

IMO 2021 Solution Notes

EVAN CHEN《陳誼廷》

2 June 2023

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Corrections and comments are welcome!

Contents

§0 Problems

1. Let $n \ge 100$ be an integer. Ivan writes the numbers $n, n+1, \dots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

2. Show that the inequality
$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \le \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$
holds for all real numbers x_1, x_2, $\dots$, x_n.

3. Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

4. Let Γ be a circle with center I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that
$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

5. Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the kth move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

   Prove that there exists a value of k such that, on the kth move, Jumpy swaps some walnuts a and b such that $a < k < b$.

6. Let $m \ge 2$ be an integer, A a finite set of integers (not necessarily positive) and B_1, B_2, $\dots$, B_m subsets of A. Suppose that, for every $k = 1, 2, \dots, m$, the sum of the elements of B_k is m^k. Prove that A contains at least $\frac{m}{2}$ elements.

§1 Solutions to Day 1

§1.1 IMO 2021/1, proposed by Australia

Available online at https://aops.com/community/p22698392.

Problem statement

Let $n \geq 100$ be an integer. Ivan writes the numbers $n, n+1, \dots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

We will find three cards $a < b < c$ such that

$$\begin{aligned} b+c &= (2k+1)^2 \\ c+a &= (2k)^2 \\ a+b &= (2k-1)^2 \end{aligned}$$

for some integer k. Solving for a, b, c gives

$$\begin{aligned} a &= \frac{(2k)^2+(2k-1)^2-(2k+1)^2}{2} = 2k^2-4k \\ b &= \frac{(2k+1)^2+(2k-1)^2-(2k)^2}{2} = 2k^2+1 \\ c &= \frac{(2k+1)^2+(2k)^2-(2k-1)^2}{2} = 2k^2+4k \end{aligned}$$

We need to show that when $n \geq 100$, one can find a suitable k.

Let

$$\begin{aligned} I_k &:= \{n \in \mathbb{Z} \mid n \leq a < b < c \leq 2n\} \\ &= \{n \in \mathbb{Z} \mid k^2+2k \leq n \leq 2k^2-4k\} \end{aligned}$$

be the interval such that when $n \in I_k$, the problem dies for that choice of k. It would be sufficient to show these intervals I_k cover all the integers ≥ 100. Starting from $I_9 = \{99 \leq n \leq 126\}$, we have

$$k \geq 9 \implies 2k^2-4k \geq (k+1)^2+2(k+1)$$

which means the right endpoint of I_k exceeds the left endpoint of I_{k+1}. Hence for $n \geq 99$ in fact the problem is true.

Remark. The problem turns out to be false for $n = 98$, surprisingly. The counterexample is for one pile to be

$$\{98, 100, 102, \dots, 126\} \cup \{129, 131, 135, \dots, 161\} \cup \{162, 164, \dots, 196\}.$$

§1.2 IMO 2021/2, proposed by Calvin Deng

Available online at https://aops.com/community/p22697952.

Problem statement

Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \le \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds for all real numbers x_1, x_2, $\dots$, x_n.

The proof is by induction on $n \ge 1$ with the base cases $n = 1$ and $n = 2$ being easy to verify by hand.

In the general situation, consider replacing the tuple $(x_i)_i$ with $(x_i + t)_i$ for some parameter $t \in \mathbb{R}$. The inequality becomes

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \le \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j + 2t|}.$$

The left-hand side is independent of t.

Claim — The right-hand side, viewed as a function $F(t)$ of t, is minimized when $2t = -(x_i + x_j)$ for some i and j.

Proof. Since $F(t)$ is the sum of piecewise concave functions, it is hence itself piecewise concave. Moreover F increases without bound if $|t| \to \infty$.

On each of the finitely many intervals on which $F(t)$ is concave, the function is minimized at its endpoints. Hence the minimum value must occur at one of the endpoints. □

If $t = -x_i$ for some i, this is the same as shifting all the variables so that $x_i = 0$. In that case, we may apply induction on $n - 1$ variables, deleting the variable x_i.

If $t = -\frac{x_i + x_j}{2}$, then notice

$$x_i + t = -(x_j + t)$$

so it's the same as shifting all the variables such that $x_i = -x_j$. In that case, we may apply induction on $n - 2$ variables, after deleting x_i and x_j.

§1.3 IMO 2021/3, proposed by Mykhalio Shtandenko (UKR)

Available online at https://aops.com/community/p22698068.

Problem statement

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

This problem and solution were contributed by Abdullahil Kafi.

Claim — Quadrilateral $BCEF$ is cyclic.

Proof. Let D' be the isogonal conjugate of the point D. The angle condition implies quadrilateral $CEDD'$ and $BFDD'$ are cyclic. By power of point we have

$$AE \cdot AC = AD \cdot AD' = AF \cdot AB$$

So $BCEF$ is cyclic. □

Claim — Line ZD is tangent to the circles (BCD) and (DEF) where $Z = EF \cap BC$.

Proof. Let $\angle CAD = \angle BAD = \alpha$, $\angle BCD = \beta$, $\angle DBC = \gamma$, $\angle ACD = \phi$, $\angle ABD = \epsilon$. From $\triangle ABC$ we have $2\alpha + \beta + \gamma + \phi + \epsilon = 180^\circ$. Let ℓ be a line tangent to (BCD) and K be a point on it in the same side of AD as C and $L = AD \cap BC$. From our labeling we have,

$$\angle AFE = \beta + \phi \qquad \angle BFD = \alpha + \gamma \qquad \angle DFE = \alpha + \phi \qquad \angle CDL = \alpha + \phi$$

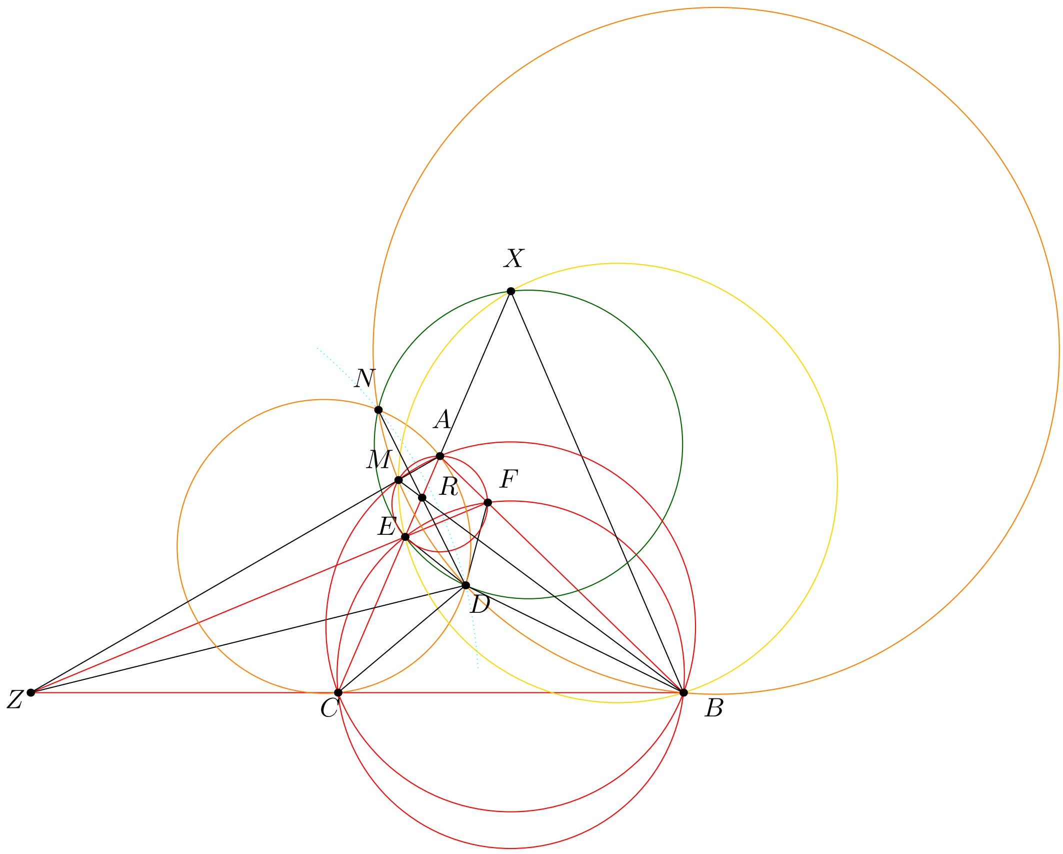
Now $\angle CDJ = 180^\circ - \gamma - \beta - (\alpha + \phi) = \alpha + \epsilon$. So $\angle DFE = \angle EDK = \alpha + \epsilon$, which means ℓ is also tangent to (DEF). Now by the radical center theorem we have ℓ passes through Z. □

Let M be the Miquel point of the cyclic quadrilateral $BCEF$. From the Miquel configuration we have A, M, Z are collinear and $(AFEM)$, $(ZCEM)$ are cyclic.

Claim — Points B, X, M, E are cyclic.

Proof. Notice that $\angle EMB = 180^\circ - \angle AMB - \angle EMZ = 180^\circ - 2\angle ACB = \angle EXB$. □

Let N be the other intersection of circles (ACD) and (DEX) and let R be the intersection of AC and BM.



Claim — Points B, D, M, N are cyclic.

Proof. By power of point we have

$$\operatorname{Pow}(R,(ACD)) = RC \cdot RA = RM \cdot RB = RE \cdot RX = \operatorname{Pow}(R,(DEX)).$$

Hence R lies on the radical axis of (ACD) and (DEX), so N, R, D are collinear. Also

$$RN \cdot RD = RA \cdot RC = RM \cdot RB$$

So $BDMN$ is cyclic. □

Notice that (ACD), $(BDMN)$, (DEX) are coaxial so their centers are collinear. Now we just need to prove the centers of (ACD), $(BDMN)$ and Z are collinear. To prove this, take a circle ω with radius ZD centered at Z. Notice that by power of point

$$ZC \cdot ZB = ZD^2 = ZE \cdot ZF = ZM \cdot ZA$$

which means inversion circle ω swaps (ACD) and $(BDMN)$. So the centers of (ACD) and $(BDMN)$ must have to be collinear with the center of inversion circle, as desired.

§2 Solutions to Day 2

§2.1 IMO 2021/4, proposed by Dominik Burek (POL) and Tomasz Ciesla (POL)

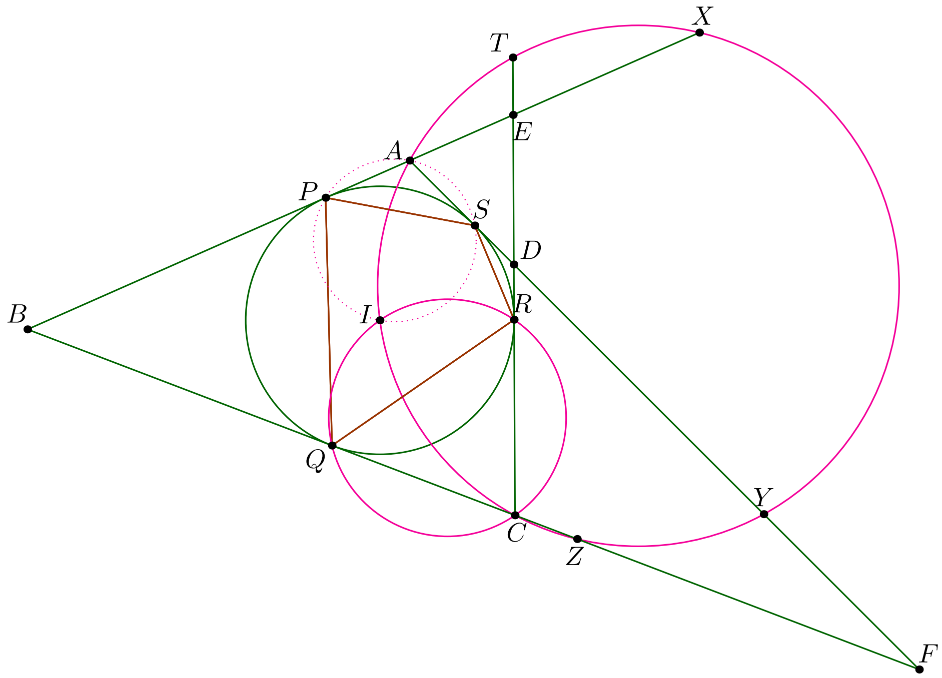
Available online at https://aops.com/community/p22698001.

Problem statement

Let Γ be a circle with center I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Let $PQRS$ be the contact points of Γ an $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$.



Claim — We have $\triangle IQZ \cong \triangle IRT$. Similarly, $\triangle IPX \cong \triangle ISY$.

Proof. By considering $(CQIR)$ and $(CITZ)$, there is a spiral similarity similarity mapping $\triangle IQZ$ to $\triangle IRT$. Since $IQ = IR$, it is in fact a congruence. □

This congruence essentially solves the problem. First, it implies:

Claim — $TX = YZ$.

Proof. Because we saw $IX = IY$ and $IT = IZ$. □

Then, we can compute

$$\begin{aligned} AD + DT + XA &= AD + (RT - RD) + (XP - AP) \\ &= (AD - RD - AP) + RT + XP = RT + XP \end{aligned}$$

and

$$\begin{aligned} CD + DY + ZC &= CD + (SY - SD) + (ZQ - QC) \\ &= (CD - SD - QC) + SY + ZQ = SY + ZQ \end{aligned}$$

but $ZQ = RT$ and $XP = SY$, as needed.

§2.2 IMO 2021/5, proposed by Spain

Available online at https://aops.com/community/p22697921.

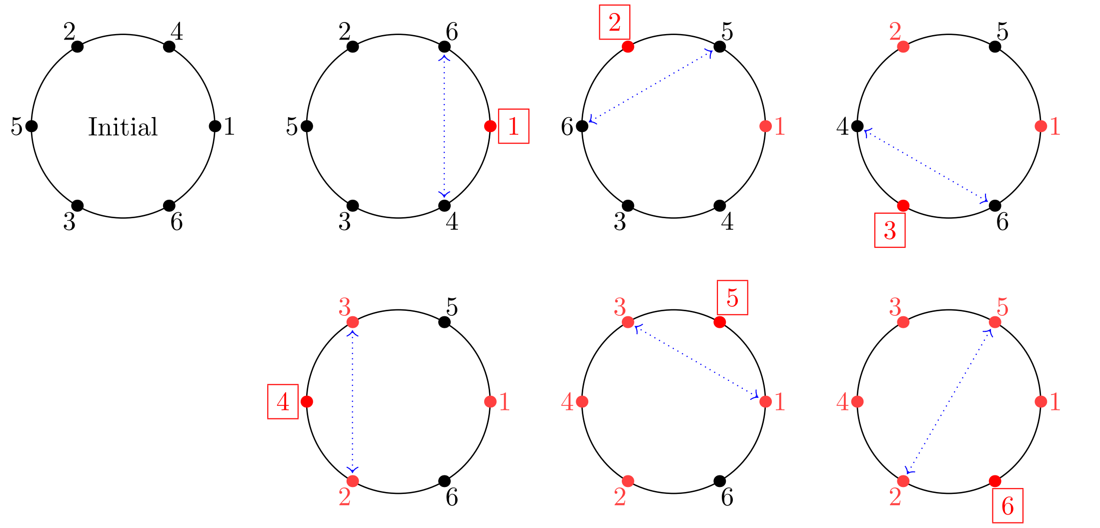
Problem statement

Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the kth move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the kth move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Assume for contradiction no such k exists.

This process takes exactly 2021 steps. Right after the kth move, we consider a situation where we color walnut k red as well, so at the kth step there are k ones. For brevity, a non-red walnut is called black. An example is illustrated below with 2021 replaced by 6.



Claim — At each step, the walnut that becomes red is between two non-red or two red walnuts.

Proof. By definition. □

On the other hand, if there are 2021 walnuts, one obtains a parity obstruction to this simplified process:

Claim — After the first step, there is always a consecutive block of black walnuts positive even length.

Proof. After the first step, there is a block of 2020 black walnuts.

Thereafter, note that a length 2 block of black walnuts can never be changed. Meanwhile for even lengths at least 4, if one places a red walnut inside it, the even length block splits into an odd length block and an even length block. □

Remark. The statement is true with 2021 replaced by any odd number, and false for any even number.

The motivation comes from the following rephrasing of the problem:

> Start with all 0's and at each step change a 0 between two matching numbers from a 0 to a 1.

Although the coloring (or 0/1) argument may appear to lose information at first, I think it should be *equivalent* to the original process; the "extra" information comes down to the choice of which walnut to color red at each step.

§2.3 IMO 2021/6, proposed by Austria

Available online at https://aops.com/community/p22698082.

Problem statement

Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive) and B_1, B_2, ..., B_m subsets of A. Suppose that, for every $k = 1, 2, \ldots, m$, the sum of the elements of B_k is m^k. Prove that A contains at least $\frac{m}{2}$ elements.

If $0 \leq X < m^{m+1}$ is a multiple of m, then write it in base m as

$$X = \sum_{i=1}^{m} c_i m^i \qquad c_i \in \{0, 1, 2, \ldots, m-1\}$$

Then swapping the summation to over A through the B_i's gives

$$X = \sum_{i=1}^{n} \left(\sum_{b \in B_i} b \right) c_i = \sum_{a \in A} f_a(X) a \quad \text{where} \quad f_a(X) := \sum_{i : a \in B_i} c_i.$$

Evidently, $0 \leq f_a(X) \leq n(m-1)$ for any a and X. So, setting $|A| = n$, the right-hand side of the display takes on at most $(n(m-1)+1)^n$ distinct values. This means

$$m^m \leq (n(m-1))^n$$

which implies $n \geq m/2$.

Remark (Motivation comments from USJL)**.** In linear algebra terms, we have some n-dimensional 0/1 vectors $\vec{v_1}$, ..., $\vec{v_m}$ and an n-dimensional vector $\vec{a}$ such that $\vec{v_i} \cdot \vec{a} = m^i$ for $i = 1, \ldots, m$. The intuition is that if n is too small, then there should be lots of linear dependences between $\vec{v_i}$.

In fact, *Siegel's lemma* is a result that says, if there are many more vectors than the dimension of the ambient space, there exist linear dependences whose coefficients are not-too-big integers. On the other hand, any linear dependence between m, m^2, ..., m^m is going to have coefficients that are pretty big; at least one of them needs to exceed m.

Applying Siegel's lemma turns out to solve the problem (and is roughly equivalent to the solution above).

Remark. In https://aops.com/community/p23185192, `dgrozev` shows the stronger bound $n \geq \left(\frac{2}{3} + \frac{c}{\log m}\right) m$ elements, for some absolute constant $c > 0$.

IMO 2022 Solution Notes

Evan Chen《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2022 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \le 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to \cdots$.

   Find all pairs (n, k) with $1 \le k \le 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

2. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying
$$xf(y) + yf(x) \le 2.$$

3. Let k be a positive integer and let S be a finite set of odd prime numbers. Prove that there is at most one way (up to rotation and reflection) to place the elements of S around the circle such that the product of any two neighbors is of the form $x^2 + x + k$ for some positive integer x.

4. Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD, TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

5. Find all triples (a, b, p) of positive integers with p prime and
$$a^p = b! + p.$$

6. Let n be a positive integer. A *Nordic square* is an $n \times n$ board containing all the integers from 1 to n^2 so that each cell contains exactly one number. An *uphill path* is a sequence of one or more cells such that:

   a) the first cell in the sequence is a *valley*, meaning the number written is less than all its orthogonal neighbors,

   b) each subsequent cell in the sequence is orthogonally adjacent to the previous cell, and

   c) the numbers written in the cells in the sequence are in increasing order.

   Find, as a function of n, the smallest possible total number of uphill paths in a Nordic square.

§1 Solutions to Day 1

§1.1 IMO 2022/1, proposed by Baptiste Serraille (FRA)

Available online at https://aops.com/community/p25635135.

Problem statement

The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \le 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to \cdots$.

Find all pairs (n, k) with $1 \le k \le 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Answer: $n \le k \le \lceil \frac{3}{2} n \rceil$.

Call a maximal chain a *block*. Then the line can be described as a sequence of blocks: it's one of:

$$\underbrace{A\dots A}_{e_1}\underbrace{B\dots B}_{e_2}\underbrace{A\dots A}_{e_3}\dots\underbrace{A\dots A}_{e_m} \text{ for odd } m$$
$$\underbrace{A\dots A}_{e_1}\underbrace{B\dots B}_{e_2}\underbrace{A\dots A}_{e_3}\dots\underbrace{B\dots B}_{e_m} \text{ for even } m$$

or the same thing with the roles of A and B flipped.

The main claim is the following:

Claim — The number m of blocks will never increase after an operation. Moreover, it stays the same if and only if

- $k \le e_1$; or
- m is even and $e_m \ge 2n + 1 - k$.

Proof. This is obvious, just run the operation and see! □

The problem asks for which values of k we always reach $m = 2$ eventually; we already know that it's non-increasing. We consider a few cases:

- If $k < n$, then any configuration with $e_1 = n - 1$ will never change.
- If $k > \lceil 3n/2 \rceil$, then take $m = 4$ and $e_1 = e_2 = \lfloor n/2 \rfloor$ and $e_3 = e_4 = \lceil n/2 \rceil$. This configuration retains $m = 4$ always: the blocks simply rotate.
- Conversely, suppose $k \ge n$ has the property that $m > 2$ stays fixed. If after the first three operations m hasn't changed, then we must have $m \ge 4$ even, and $e_m, e_{m-1}, e_{m-2} \ge 2n + 1 - k$. Now,

$$n \ge e_m + e_{m-2} \ge 2(2n + 1 - k) \implies k \ge \frac{3}{2} n + 1$$

so this completes the proof.

§1.2 IMO 2022/2, proposed by Merlijn Staps (NLD)

Available online at https://aops.com/community/p25635138.

Problem statement

Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \le 2.$$

The answer is $f(x) \equiv 1/x$ which obviously works (here $y = x$).

For the converse, assume we have f such that each $x \in \mathbb{R}^+$ has a *friend* y with $xf(y) + yf(x) \le 2$. By symmetry y is also the friend of x.

Claim — In fact every number is its own friend.

Proof. Assume for contradiction $a \ne b$ are friends. Then we know that $af(a) + af(a) > 2 \implies f(a) > \frac{1}{a}$. Analogously, $f(b) > \frac{1}{b}$. However, we then get

$$2 \ge af(b) + bf(a) > \frac{a}{b} + \frac{b}{a} \overset{\text{AMGM}}{\ge} 2$$

which is impossible. $\square$

The problem condition now simplifies to saying

$$f(x) \le \frac{1}{x} \text{ for all } x, \qquad xf(y) + yf(x) > 2 \text{ for all } x \ne y.$$

In particular, for any $x > 0$ and $\varepsilon > 0$ we have

$$\begin{aligned} 2 &< xf(x+\varepsilon) + (x+\varepsilon)f(x) \le \frac{x}{x+\varepsilon} + (x+\varepsilon)f(x) \\ \implies f(x) &> \frac{x+2\varepsilon}{(x+\varepsilon)^2} = \frac{1}{x + \frac{\varepsilon^2}{x+2\varepsilon}}. \end{aligned}$$

Since this holds for all $\varepsilon > 0$ this forces $f(x) \ge \frac{1}{x}$ as well. We're done.

§1.3 IMO 2022/3, proposed by Ankan Bhattacharya (USA)

Available online at https://aops.com/community/p25635143.

Problem statement

Let k be a positive integer and let S be a finite set of odd prime numbers. Prove that there is at most one way (up to rotation and reflection) to place the elements of S around the circle such that the product of any two neighbors is of the form $x^2 + x + k$ for some positive integer x.

We replace "positive integer x" with "nonnegative integer x", and say numbers of the form $x^2 + x + k$ are *good*. We could also replace "nonnegative integer x" with "integer x" owing to the obvious map $x \mapsto 1 - x$.

Claim — If p is an odd prime, there are at most two odd primes q and r less than p for which $pq = x^2 + x + k$ and $pr = y^2 + y + k$ are good.

Moreover, if the above occurs and $x, y \geq 0$, then $x + y + 1 = p$ and $xy \equiv k \pmod{p}$.

Proof. The equation $T^2 + T + k \equiv 0 \pmod{p}$ has at most two solutions modulo p, i.e. at most two solutions in the interval $[0, p-1]$. Because $0 \leq x, y < p$ from $p > \max(q, r)$ and $k > 0$, the first half follows.

For the second half, Vieta also gives $x + y \equiv -1 \pmod{p}$ and $xy \equiv k \pmod{p}$, and we know $0 < x + y < 2p$. □

Claim — If two such primes do exist as above, then qr is also good (!).

Proof. Let $pq = x^2 + x + k$ and $pr = y^2 + y + k$ for $x, y \geq 0$ as before. Fix $\alpha \in \mathbb{C}$ such that $\alpha^2 + \alpha + k = 0$; then for any $n \in \mathbb{Z}$, we have

$$n^2 + n + k = \operatorname{Norm}(n - \alpha).$$

Hence

$$pq \cdot pr = \operatorname{Norm}\Big((x - \alpha)(y - \alpha)\Big) = \operatorname{Norm}\Big((xy - k) - (x + y + 1)\alpha\Big)$$

But $\operatorname{Norm}(p) = p^2$, so combining with the second half of the previous claim gives

$$qr = \operatorname{Norm}(\frac{1}{p}(xy - k) - \alpha)$$

as needed. □

These two claims imply the claim directly by induction on $|S|$, since one can now delete the largest element of S.

Remark. To show that the condition is not vacuous, the author gave a ring of 385 primes for $k = 41$; see https://aops.com/community/p26068963.

§2 Solutions to Day 2

§2.1 IMO 2022/4, proposed by Patrik Bak (SVK)

Available online at https://aops.com/community/p25635154.

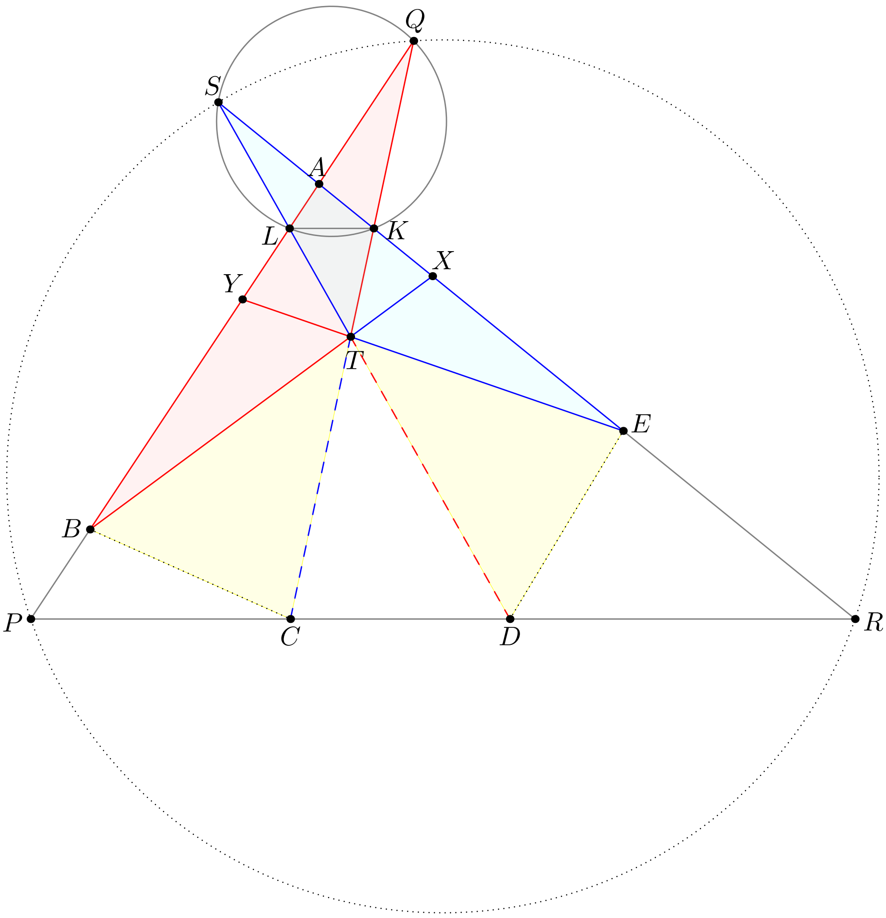
Problem statement

Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD, TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

The conditions imply

$$\triangle BTC \cong \triangle DTE, \qquad \text{and} \qquad \triangle BTY \overset{-}{\sim} \triangle ETX.$$

Define $K = \overline{CT} \cap \overline{AE}$, $L = \overline{DT} \cap \overline{AB}$, $X = \overline{BT} \cap \overline{AE}$, $Y = \overline{ET} \cap \overline{BY}$.



Claim (Main claim) — We have

$$\triangle BTQ \overset{-}{\sim} \triangle ETS, \qquad \text{and} \qquad BY : YL : LQ = EX : XK : KS.$$

In other words, $TBYLQ \overset{-}{\sim} TEXKS$.

Proof. We know $\triangle BTY \overset{-}{\sim} \triangle ETX$. Also, $\measuredangle BTL = \measuredangle BTD = \measuredangle CTE = \measuredangle KTE$ and $\measuredangle BTQ = \measuredangle BTC = \measuredangle DTE = \measuredangle STE$. $\square$

It follows from the claim that:

- $TL/TQ = TK/TS$, ergo $TL \cdot TS = TK \cdot TQ$, so $KLSQ$ is cyclic; and
- $TC/TK = TE/TK = TB/TL = TD/TL$, so $\overline{KL} \parallel \overline{PCDR}$.

With these two bullets, we're done by Reim theorem.

§2.2 IMO 2022/5, proposed by Tijs Buggenhout (BEL)

Available online at https://aops.com/community/p25635158.

Problem statement

Find all triples (a, b, p) of positive integers with p prime and

$$a^p = b! + p.$$

The answer is $(2, 2, 2)$ and $(3, 4, 3)$ only, which work.

In what follows we assume $a \ge 2$.

Claim — We have $b \le 2p - 2$, and hence $a < p^2$.

Proof. For the first half, assume first for contradiction that $b \ge 2p$. Then $b! + p \equiv p \pmod{p^2}$, so $\nu_p(b! + p) = 1$, but $\nu_p(a^p) = 1$ never occurs.

We can also rule out $b = 2p - 1$ since that would give

$$(2p-1)! + p = p\left[(p-1)!(p+1)(p+2)\dots(2p-1) + 1\right]$$

By Wilson theorem the inner bracket is $(-1)^2 + 1 \equiv 2 \pmod p$ exactly, contradiction for $p > 2$. And when $p = 2$, $3! + 2 = 8$ is not a perfect square.

The second half follows as $a^p \le (2p-2)! + p < p^{2p}$. (Here we used the crude estimate $(2p-2)! = \prod_{k=1}^{p-1} k \cdot (2p - 1 - k) < (p(p-1))^{p-1}$). $\square$

Claim — We have $a \ge p$, and hence $b \ge p$.

Proof. For the first half, assume for contradiction that $p > a$. Then

$$b! + p = a^p \ge a^{p-1} + p \ge a^a + p > a! + p \implies b > a.$$

Then taking modulo a now gives $0 \equiv 0 + p \pmod a$, which is obviously impossible.

The second half follows from $b! = a^p - p \ge p! - p > (p-1)!$. $\square$

Claim — We have $a = p$ exactly.

Proof. We know $p \ge b$ hence $p \mid b! + p$, so let $a = pk$ for $k < p$. Then $k \mid b!$ yet $k \nmid a^p - p$, contradiction. $\square$

Let's get the small p out of the way:

- For $p = 2$, checking $2 \le b \le 3$ gives $(a, b) = (2, 2)$ only.
- For $p = 3$, checking $3 \le b \le 5$ gives $(a, b) = (3, 4)$ only.

Once $p \ge 5$, if $b! = p^p - p = p(p^{p-1} - 1)$ then applying Zsigmondy gets a prime factor $q \equiv 1 \pmod{p-1}$ which divides $p^{p-1} - 1$. Yet $q \le b \le 2p - 2$ and $q \ne p$, contradiction.

§2.3 IMO 2022/6, proposed by Nikola Petrović (SRB)

Available online at https://aops.com/community/p25635163.

Problem statement

Let n be a positive integer. A *Nordic square* is an $n \times n$ board containing all the integers from 1 to n^2 so that each cell contains exactly one number. An *uphill path* is a sequence of one or more cells such that:

1. the first cell in the sequence is a *valley*, meaning the number written is less than all its orthogonal neighbors,
2. each subsequent cell in the sequence is orthogonally adjacent to the previous cell, and
3. the numbers written in the cells in the sequence are in increasing order.

Find, as a function of n, the smallest possible total number of uphill paths in a Nordic square.

Answer: $2n^2 - 2n + 1$.

¶ Bound The lower bound is the "obvious" one:

- For any pair of adjacent cells, say $a > b$, one can extend it to a downhill path (the reverse of an uphill path) by walking downwards until one reaches a valley. This gives $2n(n-1) = 2n^2 - 2n$ uphill paths of length ≥ 2.
- There is always at least one uphill path of length 1, namely the single cell $\{1\}$ (or indeed any valley).

¶ Construction For the construction, the ideas it build a tree T on the grid such that no two cells not in T are adjacent.

An example of such a grid is shown below for $n = 15$ with T in yellow and cells not in T in black; it generalizes to any $3 \mid n$, and then to any n by deleting the last $n \bmod 3$ rows and either/both of the leftmost/rightmost column.

1	2	3	32		36	62	63	64	93		97	123	124	125
	4		33	34	35		65		94	95	96		126	
6	5	7		37		67	66	68		98		128	127	129
	8		39	38	40		69		100	99	101		130	
10	9	11		41		71	70	72		102		132	131	133
	12		43	42	44		73		104	103	105		134	
14	13	15		45		75	74	76		106		136	135	137
	16		47	46	48		77		108	107	109		138	
18	17	19		49		79	78	80		110		140	139	141
	20		51	50	52		81		112	111	113		142	
22	21	23		53		83	82	84		114		144	143	145
	24		55	54	56		85		116	115	117		146	
26	25	27		57		87	86	88		118		148	147	149
	28		59	58	60		89		120	119	121		150	
30	29	31		61		91	90	92		122		152	151	153

Place 1 anywhere in T and then place all the small numbers at most $|T|$ adjacent to previously placed numbers (example above). Then place the remaining numbers outside T arbitrarily.

By construction, as 1 is the only valley, any uphill path must start from 1. And by construction, it may only reach a given pair of terminal cells in one way, i.e. the downhill paths we mentioned are the only one. End proof.

sábado 8 de julio de 2023

Problema 1. Determina todos los enteros compuestos $n > 1$ que satisfacen la siguiente propiedad: si $d_1, d_2, \ldots, d_k$ son todos los divisores positivos de n con $1 = d_1 < d_2 < \cdots < d_k = n$, entonces d_i divide a $d_{i+1} + d_{i+2}$ para cada $1 \leqslant i \leqslant k-2$.

Problema 2. Sea ABC un triángulo acutángulo con $AB < AC$. Sea Ω el circuncírculo de ABC. Sea S el punto medio del arco CB de Ω que contiene a A. La perpendicular por A a BC corta al segmento BS en D y a Ω de nuevo en $E \neq A$. La paralela a BC por D corta a la recta BE en L. Sea ω el circuncírculo del triángulo BDL. Las circunferencias ω y Ω se cortan de nuevo en $P \neq B$. Demuestra que la recta tangente a ω en P corta a la recta BS en un punto de la bisectriz interior del ángulo $\angle BAC$.

Problema 3. Para cada entero $k \geqslant 2$, determina todas las sucesiones infinitas de enteros positivos $a_1, a_2, \ldots$ para las cuales existe un polinomio P de la forma $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, con $c_0, c_1, \ldots, c_{k-1}$ enteros no negativos, tal que

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

para todo entero $n \geqslant 1$.

Tiempo: 4 horas y 30 minutos.
Cada problema vale 7 puntos.

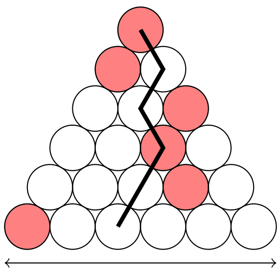
domingo 9 de julio de 2023

Problema 4. Sean $x_1, x_2, \ldots, x_{2023}$ números reales positivos, todos distintos entre sí, tales que

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

es entero para todo $n = 1, 2, \ldots, 2023$. Demuestra que $a_{2023} \geqslant 3034$.

Problema 5. Sea n un entero positivo. Un *triángulo japonés* consiste en $1 + 2 + \cdots + n$ círculos iguales acomodados en forma de triángulo equilátero de modo que para cada $i = 1, 2, \ldots, n$, la fila número i contiene i círculos, de los cuales exactamente uno de ellos se pinta de rojo. Un *camino ninja* en un triángulo japonés es una sucesión de n círculos que comienza con el círculo de la fila superior y termina en la fila inferior, pasando sucesivamente de un círculo a uno de los dos círculos inmediatamente debajo de él. En el siguiente dibujo se muestra un ejemplo de un triángulo japonés con $n = 6$, junto con un camino ninja en ese triángulo que contiene dos círculos rojos.



En términos de n, determina el mayor k tal que cada triángulo japonés tiene un camino ninja que contiene al menos k círculos rojos.

Problema 6. Sea ABC un triángulo equilátero. Sean A_1, B_1, C_1 puntos interiores de ABC tales que $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, y

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480°.$$

Las rectas BC_1 y CB_1 se cortan en A_2, las rectas CA_1 y AC_1 se cortan en B_2, y las rectas AB_1 y BA_1 se cortan en C_2.
Demuestra que si el triángulo $A_1B_1C_1$ es escaleno, entonces los tres circuncírculos de los triángulos AA_1A_2, BB_1B_2 y CC_1C_2 pasan todos por dos puntos comunes.

(Nota: un triángulo escaleno es un triángulo cuyos tres lados tienen longitudes distintas.)

Tiempo: 4 horas y 30 minutos.
Cada problema vale 7 puntos.

Saturday, 8. July 2023

Problem 1. Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leqslant i \leqslant k-2$.

Problem 2. Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$.
Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

Problem 3. For each integer $k \geqslant 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, where $c_0, c_1, \ldots, c_{k-1}$ are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

for every integer $n \geqslant 1$.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

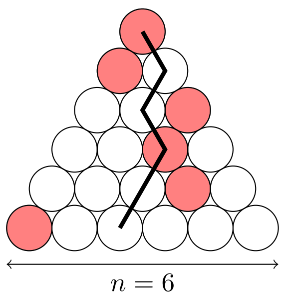
Sunday, 9. July 2023

Problem 4. Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geqslant 3034$.

Problem 5. Let n be a positive integer. A *Japanese triangle* consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$, the i^{th} row contains exactly i circles, exactly one of which is coloured red. A *ninja path* in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles.



In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

Problem 6. Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.$$

Let BC_1 and CB_1 meet at A_2, let CA_1 and AC_1 meet at B_2, and let AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, then the three circumcircles of triangles AA_1A_2, BB_1B_2 and CC_1C_2 all pass through two common points.

(Note: a scalene triangle is one where no two sides have equal length.)

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

IMO 2020

61st International
Mathematical
Olympiad
Saint Petersburg
Russia

SHORTLISTED PROBLEMS

(with solutions)

61st International Mathematical Olympiad

Saint-Petersburg — Russia, 18–28 September 2020

Shortlisted Problems

(with solutions)

61st International Mathematical Olympiad

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Note of Confidentiality

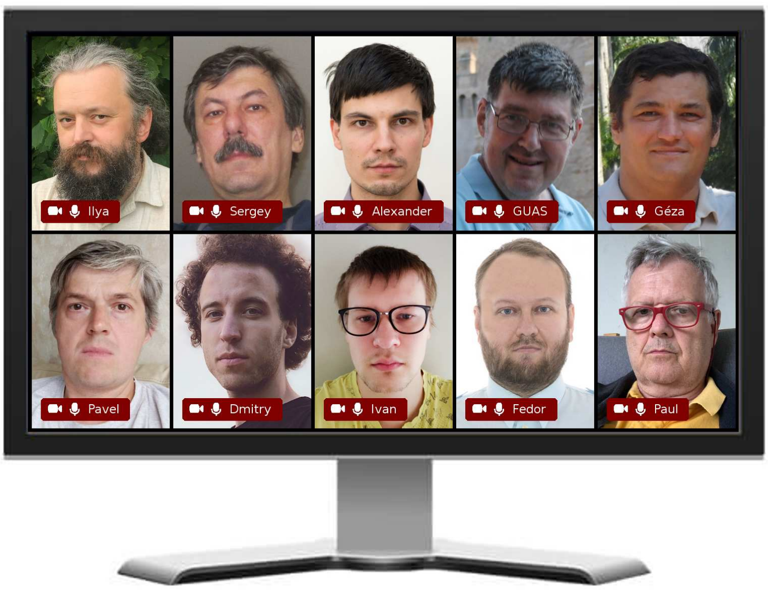
> **The Shortlist has to be kept strictly confidential until the conclusion of the following International Mathematical Olympiad.**
>
> **IMO General Regulations §6.6**

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2020 thank the following 39 countries for contributing 149 problem proposals:

Armenia, Australia, Austria, Belgium, Brazil, Canada, Croatia, Cuba, Cyprus, Czech Republic, Denmark, Estonia, France, Georgia, Germany, Hong Kong, Hungary, India, Iran, Ireland, Israel, Kosovo, Latvia, Luxembourg, Mongolia, Netherlands, North Macedonia, Philippines, Poland, Slovakia, Slovenia, South Africa, Taiwan, Thailand, Trinidad and Tobago, Ukraine, United Kingdom, USA, Venezuela

Problem Selection Committee



Ilya I. Bogdanov, Sergey Berlov, Alexander Gaifullin, Alexander S. Golovanov, Géza Kós, Pavel Kozhevnikov, Dmitry Krachun, Ivan Mitrofanov, Fedor Petrov, Paul Vaderlind

Problems

Algebra

A1. *Version 1.* Let n be a positive integer, and set $N = 2^n$. Determine the smallest real number a_n such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant a_n(x-1)^2 + x.$$

Version 2. For every positive integer N, determine the smallest real number b_N such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant b_N(x-1)^2 + x.$$

(Ireland)

A2. Let $\mathcal{A}$ denote the set of all polynomials in three variables x, y, z with integer coefficients. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ formed by all polynomials which can be expressed as

$$(x+y+z)P(x,y,z) + (xy+yz+zx)Q(x,y,z) + xyzR(x,y,z)$$

with $P, Q, R \in \mathcal{A}$. Find the smallest non-negative integer n such that $x^i y^j z^k \in \mathcal{B}$ for all non-negative integers i, j, k satisfying $i + j + k \geqslant n$.

(Venezuela)

A3. Suppose that a, b, c, d are positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

(Israel)

A4. Let a, b, c, d be four real numbers such that $a \geqslant b \geqslant c \geqslant d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)\, a^a\, b^b\, c^c\, d^d < 1.$$

(Belgium)

A5. A magician intends to perform the following trick. She announces a positive integer n, along with $2n$ real numbers $x_1 < \ldots < x_{2n}$, to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree n with real coefficients, computes the $2n$ values $P(x_1), \ldots, P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience.

Can the magician find a strategy to perform such a trick?

(Luxembourg)

A6. Determine all functions $f: \mathbb{Z} \to \mathbb{Z}$ such that

$$f^{a^2+b^2}(a+b) = af(a) + bf(b) \qquad \text{for every } a, b \in \mathbb{Z}.$$

Here, f^n denotes the n^{th} iteration of f, i.e., $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for all $n \geqslant 0$.

(Slovakia)

A7. Let n and k be positive integers. Prove that for $a_1, \ldots, a_n \in [1, 2^k]$ one has

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{a_1^2 + \ldots + a_i^2}} \leqslant 4\sqrt{kn}.$$

(Iran)

A8. Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all positive real numbers x and y,

$$f\big(x + f(xy)\big) + y = f(x)f(y) + 1.$$

(Ukraine)

Combinatorics

C1. Let n be a positive integer. Find the number of permutations $a_1, a_2, \ldots, a_n$ of the sequence $1, 2, \ldots, n$ satisfying

$$a_1 \leqslant 2a_2 \leqslant 3a_3 \leqslant \ldots \leqslant na_n.$$

(United Kingdom)

C2. In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, \ldots, Q_{24}$ whose corners are vertices of the 100-gon, so that

- the quadrilaterals $Q_1, \ldots, Q_{24}$ are pairwise disjoint, and
- every quadrilateral Q_i has three corners of one color and one corner of the other color.

(Austria)

C3. Let n be an integer with $n \geqslant 2$. On a slope of a mountain, n^2 checkpoints are marked, numbered from 1 to n^2 from the bottom to the top. Each of two cable car companies, A and B, operates k cable cars numbered from 1 to k; each cable car provides a transfer from some checkpoint to a higher one. For each company, and for any i and j with $1 \leqslant i < j \leqslant k$, the starting point of car j is higher than the starting point of car i; similarly, the finishing point of car j is higher than the finishing point of car i. Say that two checkpoints are *linked* by some company if one can start from the lower checkpoint and reach the higher one by using one or more cars of that company (no movement on foot is allowed).

Determine the smallest k for which one can guarantee that there are two checkpoints that are linked by each of the two companies.

(India)

C4. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 1$. Given an integer $n \geqslant 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \ldots, n$ there exist some $x, y \in S$ such that $x - y = F_k$.

(Croatia)

C5. Let p be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \ldots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote by $r(n)$ the fraction of integers in $\{1, 2, \ldots, n\}$ that are red.

Prove that there exists a positive integer $a \in \{1, 2, \ldots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \ldots, N$.

(Netherlands)

C6. $4n$ coins of weights $1, 2, 3, \ldots, 4n$ are given. Each coin is colored in one of n colors and there are four coins of each color. Show that all these coins can be partitioned into two sets with the same total weight, such that each set contains two coins of each color.

(Hungary)

C7. Consider any rectangular table having finitely many rows and columns, with a real number $a(r,c)$ in the cell in row r and column c. A pair (R,C), where R is a set of rows and C a set of columns, is called a *saddle pair* if the following two conditions are satisfied:

(i) For each row r', there is $r \in R$ such that $a(r,c) \geqslant a(r',c)$ for all $c \in C$;

(ii) For each column c', there is $c \in C$ such that $a(r,c) \leqslant a(r,c')$ for all $r \in R$.

A saddle pair (R,C) is called a *minimal pair* if for each saddle pair (R',C') with $R' \subseteq R$ and $C' \subseteq C$, we have $R' = R$ and $C' = C$.

Prove that any two minimal pairs contain the same number of rows.

(Thailand)

C8. Players A and B play a game on a blackboard that initially contains 2020 copies of the number 1. In every round, player A erases two numbers x and y from the blackboard, and then player B writes one of the numbers $x+y$ and $|x-y|$ on the blackboard. The game terminates as soon as, at the end of some round, one of the following holds:

(1) one of the numbers on the blackboard is larger than the sum of all other numbers;

(2) there are only zeros on the blackboard.

Player B must then give as many cookies to player A as there are numbers on the blackboard. Player A wants to get as many cookies as possible, whereas player B wants to give as few as possible. Determine the number of cookies that A receives if both players play optimally.

(Austria)

Geometry

G1. Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside side AB such that $AD < DB$. Let P and Q be two points inside sides BC and CA, respectively, such that $\angle DPB = \angle DQA = 90^\circ$. Let the perpendicular bisector of PQ meet line segment CQ at E, and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C.

Suppose that P, E, F are collinear. Prove that $\angle ACB = 90^\circ$.

(Luxembourg)

G2. Let $ABCD$ be a convex quadrilateral. Suppose that P is a point in the interior of $ABCD$ such that

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

The internal bisectors of angles ADP and PCB meet at a point Q inside the triangle ABP. Prove that $AQ = BQ$.

(Poland)

G3. Let $ABCD$ be a convex quadrilateral with $\angle ABC > 90^\circ$, $\angle CDA > 90^\circ$, and $\angle DAB = \angle BCD$. Denote by E and F the reflections of A in lines BC and CD, respectively. Suppose that the segments AE and AF meet the line BD at K and L, respectively. Prove that the circumcircles of triangles BEK and DFL are tangent to each other.

(Slovakia)

G4. In the plane, there are $n \geqslant 6$ pairwise disjoint disks $D_1, D_2, \ldots, D_n$ with radii $R_1 \geqslant R_2 \geqslant \ldots \geqslant R_n$. For every $i = 1, 2, \ldots, n$, a point P_i is chosen in disk D_i. Let O be an arbitrary point in the plane. Prove that

$$OP_1 + OP_2 + \ldots + OP_n \geqslant R_6 + R_7 + \ldots + R_n.$$

(A disk is assumed to contain its boundary.)

(Iran)

G5. Let $ABCD$ be a cyclic quadrilateral with no two sides parallel. Let K, L, M, and N be points lying on sides AB, BC, CD, and DA, respectively, such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let ω_1, ω_2, ω_3, and ω_4 be the incircles of triangles ANK, BKL, CLM, and DMN, respectively. Prove that the internal common tangents to ω_1 and ω_3 and the internal common tangents to ω_2 and ω_4 are concurrent.

(Poland)

G6. Let I and I_A be the incenter and the A-excenter of an acute-angled triangle ABC with $AB < AC$. Let the incircle meet BC at D. The line AD meets BI_A and CI_A at E and F, respectively. Prove that the circumcircles of triangles AID and I_AEF are tangent to each other.

(Slovakia)

G7. Let P be a point on the circumcircle of an acute-angled triangle ABC. Let D, E, and F be the reflections of P in the midlines of triangle ABC parallel to BC, CA, and AB, respectively. Denote by ω_A, ω_B, and ω_C the circumcircles of triangles ADP, BEP, and CFP, respectively. Denote by ω the circumcircle of the triangle formed by the perpendicular bisectors of segments AD, BE and CF.

Show that ω_A, ω_B, ω_C, and ω have a common point.

(Denmark)

G8. Let Γ and I be the circumcircle and the incenter of an acute-angled triangle ABC. Two circles ω_B and ω_C passing through B and C, respectively, are tangent at I. Let ω_B meet the shorter arc AB of Γ and segment AB again at P and M, respectively. Similarly, let ω_C meet the shorter arc AC of Γ and segment AC again at Q and N, respectively. The rays PM and QN meet at X, and the tangents to ω_B and ω_C at B and C, respectively, meet at Y.

Prove that the points A, X, and Y are collinear.

(Netherlands)

G9. Prove that there exists a positive constant c such that the following statement is true:

Assume that n is an integer with $n \geqslant 2$, and let $\mathcal{S}$ be a set of n points in the plane such that the distance between any two distinct points in $\mathcal{S}$ is at least 1. Then there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $cn^{-1/3}$.

(A line ℓ *separates* a point set $\mathcal{S}$ if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

(Taiwan)

Number Theory

N1. Given a positive integer k, show that there exists a prime p such that one can choose distinct integers $a_1, a_2, \ldots, a_{k+3} \in \{1, 2, \ldots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \ldots, k$.

(South Africa)

N2. For each prime p, there is a kingdom of p-Landia consisting of p islands numbered $1, 2, \ldots, p$. Two distinct islands numbered n and m are connected by a bridge if and only if p divides $(n^2 - m + 1)(m^2 - n + 1)$. The bridges may pass over each other, but cannot cross. Prove that for infinitely many p there are two islands in p-Landia not connected by a chain of bridges.

(Denmark)

N3. Let n be an integer with $n \geqslant 2$. Does there exist a sequence $(a_1, \ldots, a_n)$ of positive integers with not all terms being equal such that the arithmetic mean of every two terms is equal to the geometric mean of some (one or more) terms in this sequence?

(Estonia)

N4. For any odd prime p and any integer n, let $d_p(n) \in \{0, 1, \ldots, p-1\}$ denote the remainder when n is divided by p. We say that $(a_0, a_1, a_2, \ldots)$ is a *p-sequence*, if a_0 is a positive integer coprime to p, and $a_{n+1} = a_n + d_p(a_n)$ for $n \geqslant 0$.

(a) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_n > b_n$ for infinitely many n, and $b_n > a_n$ for infinitely many n?

(b) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_0 < b_0$, but $a_n > b_n$ for all $n \geqslant 1$?

(United Kingdom)

N5. Determine all functions f defined on the set of all positive integers and taking non-negative integer values, satisfying the three conditions:

(i) $f(n) \neq 0$ for at least one n;

(ii) $f(xy) = f(x) + f(y)$ for every positive integers x and y;

(iii) there are infinitely many positive integers n such that $f(k) = f(n-k)$ for all $k < n$.

(Croatia)

N6. For a positive integer n, let $d(n)$ be the number of positive divisors of n, and let $\varphi(n)$ be the number of positive integers not exceeding n which are coprime to n. Does there exist a constant C such that

$$\frac{\varphi(d(n))}{d(\varphi(n))} \leqslant C$$

for all $n \geqslant 1$?

(Cyprus)

N7. Let $\mathcal{S}$ be a set consisting of $n \geqslant 3$ positive integers, none of which is a sum of two other distinct members of $\mathcal{S}$. Prove that the elements of $\mathcal{S}$ may be ordered as $a_1, a_2, \ldots, a_n$ so that a_i does not divide $a_{i-1} + a_{i+1}$ for all $i = 2, 3, \ldots, n-1$.

(Ukraine)

Solutions

Algebra

A1. *Version 1.* Let n be a positive integer, and set $N = 2^n$. Determine the smallest real number a_n such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant a_n(x-1)^2 + x.$$

Version 2. For every positive integer N, determine the smallest real number b_N such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant b_N(x-1)^2 + x.$$

(Ireland)

Answer for both versions : $a_n = b_N = N/2$.

Solution 1 (for Version 1). First of all, assume that $a_n < N/2$ satisfies the condition. Take $x = 1 + t$ for $t > 0$, we should have

$$\frac{(1+t)^{2N}+1}{2} \leqslant (1 + t + a_n t^2)^N.$$

Expanding the brackets we get

$$(1 + t + a_n t^2)^N - \frac{(1+t)^{2N}+1}{2} = \left(Na_n - \frac{N^2}{2}\right)t^2 + c_3t^3 + \ldots + c_{2N}t^{2N} \tag{1}$$

with some coefficients $c_3, \ldots, c_{2N}$. Since $a_n < N/2$, the right hand side of (1) is negative for sufficiently small t. A contradiction.

It remains to prove the following inequality

$$\sqrt[N]{\frac{1+x^{2N}}{2}} \leqslant x + \frac{N}{2}(x-1)^2, \tag*{$\mathcal{I}(N,x)$}$$

where $N = 2^n$.

Use induction in n. The base case $n = 0$ is trivial: $N = 1$ and both sides of $\mathcal{I}(N,x)$ are equal to $(1+x^2)/2$. For completing the induction we prove $\mathcal{I}(2N,x)$ assuming that $\mathcal{I}(N,y)$ is established for all real y. We have

$$\begin{aligned}\left(x + N(x-1)^2\right)^2 &= x^2 + N^2(x-1)^4 + N(x-1)^2\frac{(x+1)^2-(x-1)^2}{2} \\ &= x^2 + \frac{N}{2}(x^2-1)^2 + \left(N^2 - \frac{N}{2}\right)(x-1)^4 \geqslant x^2 + \frac{N}{2}(x^2-1)^2 \geqslant \sqrt[N]{\frac{1+x^{4N}}{2}},\end{aligned}$$

where the last inequality is $\mathcal{I}(N,x^2)$. Since

$$x + N(x-1)^2 \geqslant x + \frac{(x-1)^2}{2} = \frac{x^2+1}{2} \geqslant 0,$$

taking square root we get $\mathcal{I}(2N,x)$. The inductive step is complete.

Solution 2.1 (for Version 2). Like in Solution 1 of Version 1, we conclude that $b_N \geqslant N/2$. It remains to prove the inequality $\mathcal{I}(N, x)$ for an arbitrary positive integer N.

First of all, $\mathcal{I}(N, 0)$ is obvious. Further, if $x > 0$, then the left hand sides of $\mathcal{I}(N, -x)$ and $\mathcal{I}(N, x)$ coincide, while the right hand side of $\mathcal{I}(N, -x)$ is larger than that of $\mathcal{I}(N, -x)$ (their difference equals $2(N-1)x \geqslant 0$). Therefore, $\mathcal{I}(N, -x)$ follows from $\mathcal{I}(N, x)$. So, hereafter we suppose that $x > 0$.

Divide $\mathcal{I}(N, x)$ by x and let $t = (x-1)^2/x = x - 2 + 1/x$; then $\mathcal{I}(n, x)$ reads as

$$f_N := \frac{x^N + x^{-N}}{2} \leqslant \left(1 + \frac{N}{2}t\right)^N. \tag{2}$$

The key identity is the expansion of f_N as a polynomial in t:

Lemma.

$$f_N = N \sum_{k=0}^{N} \frac{1}{N+k} \binom{N+k}{2k} t^k. \tag{3}$$

Proof. Apply induction on N. We will make use of the straightforward recurrence relation

$$f_{N+1} + f_{N-1} = (x + 1/x) f_N = (2+t) f_N. \tag{4}$$

The base cases $N = 1, 2$ are straightforward:

$$f_1 = 1 + \frac{t}{2}, \quad f_2 = \frac{1}{2}t^2 + 2t + 1.$$

For the induction step from $N-1$ and N to $N+1$, we compute the coefficient of t^k in f_{N+1} using the formula $f_{N+1} = (2+t)f_N - f_{N-1}$. For $k = 0$ that coefficient equals 1, for $k > 0$ it equals

$$\begin{aligned}
&2\frac{N}{N+k}\binom{N+k}{2k} + \frac{N}{N+k-1}\binom{N+k-1}{2k-2} - \frac{N-1}{N+k-1}\binom{N+k-1}{2k} \\
&= \frac{(N+k-1)!}{(2k)!(N-k)!}\left(2N + \frac{2k(2k-1)N}{(N+k-1)(N-k+1)} - \frac{(N-1)(N-k)}{N+k-1}\right) \\
&= \frac{(N+k-1)!}{(2k)!(N-k+1)!}\left(2N(N-k+1) + 3kN + k - N^2 - N\right) = \frac{\binom{N+k+1}{2k}}{(N+k+1)}(N+1),
\end{aligned}$$

that completes the induction. $\square$

Turning back to the problem, in order to prove (2) we write

$$\left(1 + \frac{N}{2}t\right)^N - f_N = \left(1 + \frac{N}{2}t\right)^N - N\sum_{k=0}^{N} \frac{1}{N+k}\binom{N+k}{2k} t^k = \sum_{k=0}^{N} \alpha_k t^k,$$

where

$$\begin{aligned}
\alpha_k &= \left(\frac{N}{2}\right)^k \binom{N}{k} - \frac{N}{N+k}\binom{N+k}{2k} \\
&= \left(\frac{N}{2}\right)^k \binom{N}{k}\left(1 - 2^k \frac{(1+1/N)(1+2/N)\cdot \ldots \cdot (1+(k-1)/N)}{(k+1)\cdot \ldots \cdot (2k)}\right) \\
&\geqslant \left(\frac{N}{2}\right)^k \binom{N}{k}\left(1 - 2^k \frac{2\cdot 3\cdot \ldots \cdot k}{(k+1)\cdot \ldots \cdot (2k)}\right) = \left(\frac{N}{2}\right)^k \binom{N}{k}\left(1 - \prod_{j=1}^{k} \frac{2j}{k+j}\right) \geqslant 0,
\end{aligned}$$

and (2) follows.

Solution 2.2 (for Version 2). Here we present another proof of the inequality (2) for $x > 0$, or, equivalently, for $t = (x-1)^2/x \geqslant 0$. Instead of finding the coefficients of the polynomial $f_N = f_N(t)$ we may find its roots, which is in a sense more straightforward. Note that the recurrence (4) and the initial conditions $f_0 = 1$, $f_1 = 1 + t/2$ imply that f_N is a polynomial in t of degree N. It also follows by induction that $f_N(0) = 1$, $f_N'(0) = N^2/2$: the recurrence relations read as $f_{N+1}(0) + f_{N-1}(0) = 2f_N(0)$ and $f'_{N+1}(0) + f'_{N-1}(0) = 2f'_N(0) + f_N(0)$, respectively.

Next, if $x_k = \exp(\frac{i\pi(2k-1)}{2N})$ for $k \in \{1, 2, \ldots, N\}$, then

$$-t_k := 2 - x_k - \frac{1}{x_k} = 2 - 2\cos\frac{\pi(2k-1)}{2N} = 4\sin^2\frac{\pi(2k-1)}{4N} > 0$$

and

$$f_N(t_k) = \frac{x_k^N + x_k^{-N}}{2} = \frac{\exp\left(\frac{i\pi(2k-1)}{2}\right) + \exp\left(-\frac{i\pi(2k-1)}{2}\right)}{2} = 0.$$

So the roots of f_N are $t_1, \ldots, t_N$ and by the AM–GM inequality we have

$$f_N(t) = \left(1 - \frac{t}{t_1}\right)\left(1 - \frac{t}{t_2}\right)\ldots\left(1 - \frac{t}{t_N}\right) \leqslant \left(1 - \frac{t}{N}\left(\frac{1}{t_1} + \ldots + \frac{1}{t_n}\right)\right)^N = \left(1 + \frac{tf_N'(0)}{N}\right)^N = \left(1 + \frac{N}{2}t\right)^N.$$

Comment. The polynomial $f_N(t)$ equals to $\frac{1}{2}T_N(t+2)$, where T_n is the n^{th} Chebyshev polynomial of the first kind: $T_n(2\cos s) = 2\cos ns$, $T_n(x + 1/x) = x^n + 1/x^n$.

Solution 2.3 (for Version 2). Here we solve the problem when $N \geqslant 1$ is an arbitrary real number. For a real number a let

$$f(x) = \left(\frac{x^{2N}+1}{2}\right)^{\frac{1}{N}} - a(x-1)^2 - x.$$

Then $f(1) = 0$,

$$f'(x) = \left(\frac{x^{2N}+1}{2}\right)^{\frac{1}{N}-1} x^{2N-1} - 2a(x-1) - 1 \quad \text{and} \quad f'(1) = 0;$$

$$f''(x) = (1-N)\left(\frac{x^{2N}+1}{2}\right)^{\frac{1}{N}-2} x^{4N-2} + (2N-1)\left(\frac{x^{2N}+1}{2}\right)^{\frac{1}{N}-1} x^{2N-2} - 2a \quad \text{and} \quad f''(1) = N - 2a.$$

So if $a < \frac{N}{2}$, the function f has a strict local minimum at point 1, and the inequality $f(x) \leqslant 0 = f(1)$ does not hold. This proves $b_N \geqslant N/2$.

For $a = \frac{N}{2}$ we have $f''(1) = 0$ and

$$f'''(x) = \frac{1}{2}(1-N)(1-2N)\left(\tfrac{x^{2N}+1}{2}\right)^{\frac{1}{N}-3} x^{2N-3}(1 - x^{2N}) \quad \begin{cases} > 0 & \text{if } 0 < x < 1 \text{ and} \\ < 0 & \text{if } x > 1. \end{cases}$$

Hence, $f''(x) < 0$ for $x \neq 1$; $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$, finally $f(x) < 0$ for $x \neq 1$.

Comment. Version 2 is much more difficult, of rather A5 or A6 difficulty. The induction in Version 1 is rather straightforward, while all three above solutions of Version 2 require some creativity.

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A2. Let $\mathcal{A}$ denote the set of all polynomials in three variables x, y, z with integer coefficients. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ formed by all polynomials which can be expressed as

$$(x+y+z)P(x,y,z)+(xy+yz+zx)Q(x,y,z)+xyzR(x,y,z)$$

with $P, Q, R \in \mathcal{A}$. Find the smallest non-negative integer n such that $x^i y^j z^k \in \mathcal{B}$ for all non-negative integers i, j, k satisfying $i+j+k \geqslant n$.

(Venezuela)

Answer: $n = 4$.

Solution. We start by showing that $n \leqslant 4$, i.e., any monomial $f = x^i y^j z^k$ with $i+j+k \geqslant 4$ belongs to $\mathcal{B}$. Assume that $i \geqslant j \geqslant k$, the other cases are analogous.

Let $x+y+z=p$, $xy+yz+zx=q$ and $xyz=r$. Then

$$0=(x-x)(x-y)(x-z)=x^3-px^2+qx-r,$$

therefore $x^3 \in \mathcal{B}$. Next, $x^2y^2 = xyq-(x+y)r \in \mathcal{B}$.

If $k \geqslant 1$, then r divides f, thus $f \in \mathcal{B}$. If $k=0$ and $j \geqslant 2$, then x^2y^2 divides f, thus we have $f \in \mathcal{B}$ again. Finally, if $k=0, j \leqslant 1$, then x^3 divides f and $f \in \mathcal{B}$ in this case also.

In order to prove that $n \geqslant 4$, we show that the monomial x^2y does not belong to $\mathcal{B}$. Assume the contrary:

$$x^2y = pP+qQ+rR \tag{1}$$

for some polynomials P, Q, R. If polynomial P contains the monomial x^2 (with nonzero coefficient), then $pP+qQ+rR$ contains the monomial x^3 with the same nonzero coefficient. So P does not contain x^2, y^2, z^2 and we may write

$$x^2y=(x+y+z)(axy+byz+czx)+(xy+yz+zx)(dx+ey+fz)+gxyz,$$

where a, b, c; d, e, f; g are the coefficients of xy, yz, zx; x, y, z; xyz in the polynomials P; Q; R, respectively (the remaining coefficients do not affect the monomials of degree 3 in $pP+qQ+rR$). By considering the coefficients of xy^2 we get $e=-a$, analogously $e=-b$, $f=-b$, $f=-c$, $d=-c$, thus $a=b=c$ and $f=e=d=-a$, but then the coefficient of x^2y in the right hand side equals $a+d=0 \neq 1$.

Comment 1. The general question is the following. Call a polynomial $f(x_1, \ldots, x_n)$ with integer coefficients *nice*, if $f(0,0,\ldots,0)=0$ and $f(x_{\pi_1},\ldots,x_{\pi_n}) = f(x_1,\ldots,x_n)$ for any permutation π of $1,\ldots,n$ (in other words, f is symmetric and its constant term is zero.) Denote by $\mathcal{I}$ the set of polynomials of the form

$$p_1q_1+p_2q_2+\ldots+p_mq_m, \tag{2}$$

where m is an integer, $q_1,\ldots,q_m$ are polynomials with integer coefficients, and $p_1,\ldots,p_m$ are nice polynomials. Find the least N for which any monomial of degree at least N belongs to $\mathcal{I}$.

The **answer** is $n(n-1)/2+1$. The lower bound follows from the following claim: the polynomial

$$F(x_1,\ldots,x_n)=x_2x_3^2x_4^3\cdot\ldots\cdot x_n^{n-1}$$

does not belong to $\mathcal{I}$.

Assume that $F=\sum p_iq_i$, according to (2). By taking only the monomials of degree $n(n-1)/2$, we can additionally assume that every p_i and every q_i is homogeneous, $\deg p_i > 0$, and $\deg p_i + \deg q_i = \deg F = n(n-1)/2$ for all i.

Consider the alternating sum

$$\sum_{\pi}\operatorname{sign}(\pi)F(x_{\pi_1},\ldots,x_{\pi_n})=\sum_{i=1}^{m}p_i\sum_{\pi}\operatorname{sign}(\pi)q_i(x_{\pi_1},\ldots,x_{\pi_n}):=S, \tag{3}$$

where the summation is done over all permutations π of $1, \ldots n$, and $\operatorname{sign}(\pi)$ denotes the sign of the permutation π. Since $\deg q_i = n(n-1)/2 - \deg p_i < n(n-1)/2$, in any monomial Q of q_i, there are at least two variables, say x_α and x_β, with equal exponents. Therefore $\sum_\pi \operatorname{sign}(\pi) Q(x_{\pi_1}, \ldots, x_{\pi_n}) = 0$, because each pair of terms that corresponds to permutations which differ by the transposition of α and β, cancels out. This holds for any $i = 1, \ldots, m$ and any monomial of q_i, so $S = 0$. But the left hand side of (3) is a non-zero polynomial. This is a contradiction.

Let us now prove, using induction on n, that any monomial $h = x_1^{c_1} \ldots x_n^{c_n}$ of degree $n(n-1)/2+1$ belongs to $\mathcal{I}$, and additionally all p_i, q_i in the representation (2) can be chosen homogeneous with sum of degrees equal to $n(n-1)/2+1$. (Obviously, any monomial of degree at least $n(n-1)/2+1$ is divisible by a monomial of degree exactly $n(n-1)/2+1$, thus this suffices.) The proposition is true for $n = 1$, so assume that $n > 1$ and that the proposition is proved for smaller values of n.

We proceed by an internal induction on $S := |\{i : c_i = 0\}|$. In the base case $S = 0$ the monomial h is divisible by the nice polynomial $x_1 \cdot \ldots \cdot x_n$, therefore $h \in \mathcal{I}$. Now assume that $S > 0$ and that the claim holds for smaller values of S. Let $T = n - S$. We may assume that $c_{T+1} = \ldots = c_n = 0$ and $h = x_1 \cdot \ldots \cdot x_T g(x_1, \ldots, x_{n-1})$, where $\deg g = n(n-1)/2 - T + 1 \geqslant (n-1)(n-2)/2 + 1$. Using the outer induction hypothesis we represent g as $p_1 q_1 + \ldots + p_m q_m$, where $p_i(x_1, \ldots, x_{n-1})$ are nice polynomials in $n-1$ variables. There exist nice homogeneous polynomials $P_i(x_1, \ldots, x_n)$ such that $P_i(x_1, \ldots, x_{n-1}, 0) = p_i(x_1, \ldots, x_{n-1})$. In other words, $\Delta_i := p_i(x_1, \ldots, x_{n-1}) - P_i(x_1, \ldots, x_{n-1}, x_n)$ is divisible by x_n, let $\Delta_i = x_n g_i$. We get

$$h = x_1 \cdot \ldots \cdot x_T \sum p_i q_i = x_1 \cdot \ldots \cdot x_T \sum (P_i + x_n g_i) q_i = (x_1 \cdot \ldots \cdot x_T x_n) \sum g_i q_i + \sum P_i q_i \in \mathcal{I}.$$

The first term belongs to $\mathcal{I}$ by the inner induction hypothesis. This completes both inductions.

Comment 2. The solutions above work smoothly for the versions of the original problem and its extensions to the case of n variables, where all polynomials are assumed to have real coefficients. In the version with integer coefficients, the argument showing that $x^2 y \notin \mathcal{B}$ can be simplified: it is not hard to show that in every polynomial $f \in \mathcal{B}$, the sum of the coefficients of $x^2 y$, $x^2 z$, $y^2 x$, $y^2 z$, $z^2 x$ and $z^2 y$ is even. A similar fact holds for any number of variables and also implies that $N \geqslant n(n-1)/2+1$ in terms of the previous comment.

A3. Suppose that a, b, c, d are positive real numbers satisfying $(a+c)(b+d) = ac+bd$. Find the smallest possible value of

$$S = \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

(Israel)

Answer: The smallest possible value is 8.

Solution 1. To show that $S \geqslant 8$, apply the AM–GM inequality twice as follows:

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\left(\frac{b}{c}+\frac{d}{a}\right) \geqslant 2\sqrt{\frac{ac}{bd}}+2\sqrt{\frac{bd}{ac}} = \frac{2(ac+bd)}{\sqrt{abcd}} = \frac{2(a+c)(b+d)}{\sqrt{abcd}} \geqslant 2\cdot\frac{2\sqrt{ac}\cdot 2\sqrt{bd}}{\sqrt{abcd}} = 8\,.$$

The above inequalities turn into equalities when $a = c$ and $b = d$. Then the condition $(a+c)(b+d) = ac+bd$ can be rewritten as $4ab = a^2+b^2$. So it is satisfied when $a/b = 2\pm\sqrt{3}$. Hence, S attains value 8, e.g., when $a = c = 1$ and $b = d = 2+\sqrt{3}$.

Solution 2. By homogeneity we may suppose that $abcd = 1$. Let $ab = C$, $bc = A$ and $ca = B$. Then a, b, c can be reconstructed from A, B and C as $a = \sqrt{BC/A}$, $b = \sqrt{AC/B}$ and $c = \sqrt{AB/C}$. Moreover, the condition $(a+c)(b+d) = ac+bd$ can be written in terms of A, B, C as

$$A+\frac{1}{A}+C+\frac{1}{C} = bc+ad+ab+cd = (a+c)(b+d) = ac+bd = B+\frac{1}{B}.$$

We then need to minimize the expression

$$\begin{aligned} S &:= \frac{ad+bc}{bd}+\frac{ab+cd}{ac} = \left(A+\frac{1}{A}\right)B+\left(C+\frac{1}{C}\right)\frac{1}{B} \\ &= \left(A+\frac{1}{A}\right)\left(B-\frac{1}{B}\right)+\left(A+\frac{1}{A}+C+\frac{1}{C}\right)\frac{1}{B} \\ &= \left(A+\frac{1}{A}\right)\left(B-\frac{1}{B}\right)+\left(B+\frac{1}{B}\right)\frac{1}{B}. \end{aligned}$$

Without loss of generality assume that $B \geqslant 1$ (otherwise, we may replace B by $1/B$ and swap A and C, this changes neither the relation nor the function to be maximized). Therefore, we can write

$$S \geqslant 2\left(B-\frac{1}{B}\right)+\left(B+\frac{1}{B}\right)\frac{1}{B} = 2B+\left(1-\frac{1}{B}\right)^2 =: f(B).$$

Clearly, f increases on $[1,\infty)$. Since

$$B+\frac{1}{B} = A+\frac{1}{A}+C+\frac{1}{C} \geqslant 4,$$

we have $B \geqslant B'$, where $B' = 2+\sqrt{3}$ is the unique root greater than 1 of the equation $B'+1/B' = 4$. Hence,

$$S \geqslant f(B) \geqslant f(B') = 2\left(B'-\frac{1}{B'}\right)+\left(B'+\frac{1}{B'}\right)\frac{1}{B'} = 2B'-\frac{2}{B'}+\frac{4}{B'} = 8.$$

It remains to note that when $A = C = 1$ and $B = B'$ we have the equality $S = 8$.

Solution 3. We present another proof of the inequality $S \geqslant 8$. We start with the estimate

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\left(\frac{b}{c}+\frac{d}{a}\right) \geqslant 2\sqrt{\frac{ac}{bd}}+2\sqrt{\frac{bd}{ac}}.$$

Let $y=\sqrt{ac}$ and $z=\sqrt{bd}$, and assume, without loss of generality, that $ac \geqslant bd$. By the AM–GM inequality, we have

$$y^2+z^2=ac+bd=(a+c)(b+d) \geqslant 2\sqrt{ac}\cdot 2\sqrt{bd}=4yz.$$

Substituting $x=y/z$, we get $4x \leqslant x^2+1$. For $x \geqslant 1$, this holds if and only if $x \geqslant 2+\sqrt{3}$.

Now we have

$$2\sqrt{\frac{ac}{bd}}+2\sqrt{\frac{bd}{ac}}=2\left(x+\frac{1}{x}\right).$$

Clearly, this is minimized by setting $x(\geqslant 1)$ as close to 1 as possible, i.e., by taking $x=2+\sqrt{3}$. Then $2(x+1/x)=2((2+\sqrt{3})+(2-\sqrt{3}))=8$, as required.

A4. Let a, b, c, d be four real numbers such that $a \geqslant b \geqslant c \geqslant d > 0$ and $a+b+c+d=1$. Prove that

$$(a+2b+3c+4d)\, a^a\, b^b\, c^c\, d^d < 1.$$

(Belgium)

Solution 1. The weighted AM–GM inequality with weights a, b, c, d gives

$$a^a b^b c^c d^d \leqslant a \cdot a + b \cdot b + c \cdot c + d \cdot d = a^2 + b^2 + c^2 + d^2,$$

so it suffices to prove that $(a+2b+3c+4d)(a^2+b^2+c^2+d^2) < 1 = (a+b+c+d)^3$. This can be done in various ways, for example:

$$\begin{aligned}(a+b+c+d)^3 &> a^2(a+3b+3c+3d) + b^2(3a+b+3c+3d)\\ &\qquad + c^2(3a+3b+c+3d) + d^2(3a+3b+3c+d)\\ &\geqslant (a^2+b^2+c^2+d^2)\cdot(a+2b+3c+4d).\end{aligned}$$

Solution 2. From $b \geqslant d$ we get

$$a+2b+3c+4d \leqslant a+3b+3c+3d = 3-2a.$$

If $a < \frac{1}{2}$, then the statement can be proved by

$$(a+2b+3c+4d)\, a^a b^b c^c d^d \leqslant (3-2a)a^a a^b a^c a^d = (3-2a)a = 1-(1-a)(1-2a) < 1.$$

From now on we assume $\frac{1}{2} \leqslant a < 1$.

By $b, c, d < 1-a$ we have

$$b^b c^c d^d < (1-a)^b \cdot (1-a)^c \cdot (1-a)^d = (1-a)^{1-a}.$$

Therefore,

$$(a+2b+3c+4d)a^a b^b c^c d^d < (3-2a)\, a^a\, (1-a)^{1-a}.$$

For $0 < x < 1$, consider the functions

$$f(x) = (3-2x)x^x(1-x)^{1-x} \quad \text{and} \quad g(x) = \log f(x) = \log(3-2x) + x\log x + (1-x)\log(1-x);$$

hereafter, log denotes the natural logarithm. It is easy to verify that

$$g''(x) = -\frac{4}{(3-2x)^2} + \frac{1}{x} + \frac{1}{1-x} = \frac{1+8(1-x)^2}{x(1-x)(3-2x)^2} > 0,$$

so g is strictly convex on $(0,1)$.

By $g\left(\frac{1}{2}\right) = \log 2 + 2\cdot\frac{1}{2}\log\frac{1}{2} = 0$ and $\lim\limits_{x\to 1-} g(x) = 0$, we have $g(x) \leqslant 0$ (and hence $f(x) \leqslant 1$) for all $x \in \left[\frac{1}{2}, 1\right)$, and therefore

$$(a+2b+3c+4d)a^a b^b c^c d^d < f(a) \leqslant 1.$$

Comment. For a large number of variables $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$ with $\sum_i a_i = 1$, the inequality

$$\left(\sum_i i a_i\right) \prod_i a_i^{a_i} \leqslant 1$$

does not necessarily hold. Indeed, let $a_2 = a_3 = \ldots = a_n = \varepsilon$ and $a_1 = 1-(n-1)\varepsilon$, where n and $\varepsilon \in (0, 1/n)$ will be chosen later. Then

$$\left(\sum_i i a_i\right) \prod_i a_i^{a_i} = \left(1 + \frac{n(n-1)}{2}\varepsilon\right) \varepsilon^{(n-1)\varepsilon}(1-(n-1)\varepsilon)^{1-(n-1)\varepsilon}. \tag{1}$$

If $\varepsilon = C/n^2$ with an arbitrary fixed $C > 0$ and $n \to \infty$, then the factors $\varepsilon^{(n-1)\varepsilon} = \exp((n-1)\varepsilon\log\varepsilon)$ and $(1-(n-1)\varepsilon)^{1-(n-1)\varepsilon}$ tend to 1, so the limit of (1) in this set-up equals $1 + C/2$. This is not simply greater than 1, but it can be arbitrarily large.

A5. A magician intends to perform the following trick. She announces a positive integer n, along with $2n$ real numbers $x_1 < \ldots < x_{2n}$, to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree n with real coefficients, computes the $2n$ values $P(x_1), \ldots, P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience.

Can the magician find a strategy to perform such a trick?

(Luxembourg)

Answer: No, she cannot.

Solution. Let $x_1 < x_2 < \ldots < x_{2n}$ be real numbers chosen by the magician. We will construct two distinct polynomials $P(x)$ and $Q(x)$, each of degree n, such that the member of audience will write down the same sequence for both polynomials. This will mean that the magician cannot distinguish P from Q.

Claim. There exists a polynomial $P(x)$ of degree n such that $P(x_{2i-1}) + P(x_{2i}) = 0$ for $i = 1, 2, \ldots, n$.

Proof. We want to find a polynomial $a_n x^n + \ldots + a_1 x + a_0$ satisfying the following system of equations:

$$\begin{cases} (x_1^n + x_2^n)a_n + (x_1^{n-1} + x_2^{n-1})a_{n-1} + \ldots + 2a_0 = 0 \\ (x_3^n + x_4^n)a_n + (x_3^{n-1} + x_4^{n-1})a_{n-1} + \ldots + 2a_0 = 0 \\ \ldots \\ (x_{2n-1}^n + x_{2n}^n)a_n + (x_{2n-1}^{n-1} + x_{2n}^{n-1})a_{n-1} + \ldots + 2a_0 = 0 \end{cases}$$

We use the well known fact that a homogeneous system of n linear equations in $n+1$ variables has a nonzero solution. (This fact can be proved using induction on n, via elimination of variables.) Applying this fact to the above system, we find a nonzero polynomial $P(x)$ of degree not exceeding n such that its coefficients $a_0, \ldots, a_n$ satisfy this system. Therefore $P(x_{2i-1}) + P(x_{2i}) = 0$ for all $i = 1, 2, \ldots, n$. Notice that P has a root on each segment $[x_{2i-1}, x_{2i}]$ by the Intermediate Value theorem, so n roots in total. Since P is nonzero, we get $\deg P = n$. $\square$

Now consider a polynomial $P(x)$ provided by the Claim, and take $Q(x) = -P(x)$. The properties of $P(x)$ yield that $P(x_{2i-1}) = Q(x_{2i})$ and $Q(x_{2i-1}) = P(x_{2i})$ for all $i = 1, 2, \ldots, n$. It is also clear that $P \neq -P = Q$ and $\deg Q = \deg P = n$.

Comment. It can be shown that for any positive integer n the magician can choose $2n+1$ distinct real numbers so as to perform such a trick. Moreover, she can perform such a trick with almost all (in a proper sense) $(2n+1)$-tuples of numbers.

A6. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f^{a^2+b^2}(a+b) = af(a) + bf(b) \qquad \text{for every } a, b \in \mathbb{Z}.$$

Here, f^n denotes the n^{th} iteration of f, i.e., $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for all $n \geqslant 0$.

(Slovakia)

Answer: Either $f(x) = 0$ for all $x \in \mathbb{Z}$, or $f(x) = x + 1$ for all $x \in \mathbb{Z}$.

Solution. Refer to the main equation as $E(a, b)$.

$E(0, b)$ reads as $f^{b^2}(b) = bf(b)$. For $b = -1$ this gives $f(-1) = 0$.

Now $E(a, -1)$ reads as

$$f^{a^2+1}(a-1) = af(a) = f^{a^2}(a). \tag{1}$$

For $x \in \mathbb{Z}$ define the *orbit of x* by $\mathcal{O}(x) = \{x, f(x), f(f(x)), \dots\} \subseteq \mathbb{Z}$. We see that the orbits $\mathcal{O}(a-1)$ and $\mathcal{O}(a)$ differ by finitely many terms. Hence, any two orbits differ by finitely many terms. In particular, this implies that either all orbits are finite or all orbits are infinite.

Case 1: All orbits are finite.

Then $\mathcal{O}(0)$ is finite. Using $E(a, -a)$ we get

$$a\big(f(a) - f(-a)\big) = af(a) - af(-a) = f^{2a^2}(0) \in \mathcal{O}(0).$$

For $|a| > \max\limits_{z \in \mathcal{O}(0)} |z|$, this yields $f(a) = f(-a)$ and $f^{2a^2}(0) = 0$. Therefore, the sequence $\big(f^k(0)\colon k = 0, 1, \dots\big)$ is purely periodic with a minimal period T which divides $2a^2$. Analogously, T divides $2(a+1)^2$, therefore, $T \mid \gcd\big(2a^2, 2(a+1)^2\big) = 2$, i.e., $f(f(0)) = 0$ and $a\big(f(a) - f(-a)\big) = f^{2a^2}(0) = 0$ for all a. Thus,

$$f(a) = f(-a) \qquad \text{for all } a \neq 0; \tag{$\clubsuit$}$$

$$\text{in particular,} \quad f(1) = f(-1) = 0. \tag{$\spadesuit$}$$

Next, for each $n \in \mathbb{Z}$, by $E(n, 1-n)$ we get

$$nf(n) + (1-n)f(1-n) = f^{n^2+(1-n)^2}(1) = f^{2n^2-2n}(0) = 0. \tag{$\heartsuit$}$$

Assume that there exists some $m \neq 0$ such that $f(m) \neq 0$. Choose such an m for which $|m|$ is minimal possible. Then $|m| > 1$ due to ($\spadesuit$); $f(|m|) \neq 0$ due to ($\clubsuit$); and $f(1 - |m|) \neq 0$ due to ($\heartsuit$) for $n = |m|$. This contradicts to the minimality assumption.

So, $f(n) = 0$ for $n \neq 0$. Finally, $f(0) = f^3(0) = f^4(2) = 2f(2) = 0$. Clearly, the function $f(x) \equiv 0$ satisfies the problem condition, which provides the first of the two answers.

Case 2: All orbits are infinite.

Since the orbits $\mathcal{O}(a)$ and $\mathcal{O}(a-1)$ differ by finitely many terms for all $a \in \mathbb{Z}$, each two orbits $\mathcal{O}(a)$ and $\mathcal{O}(b)$ have infinitely many common terms for arbitrary $a, b \in \mathbb{Z}$.

For a minute, fix any $a, b \in \mathbb{Z}$. We claim that all pairs (n, m) of nonnegative integers such that $f^n(a) = f^m(b)$ have the same difference $n - m$. Arguing indirectly, we have $f^n(a) = f^m(b)$ and $f^p(a) = f^q(b)$ with, say, $n - m > p - q$, then $f^{p+m+k}(b) = f^{p+n+k}(a) = f^{q+n+k}(b)$, for all nonnegative integers k. This means that $f^{\ell+(n-m)-(p-q)}(b) = f^{\ell}(b)$ for all sufficiently large ℓ, i.e., that the sequence $\big(f^n(b)\big)$ is eventually periodic, so $\mathcal{O}(b)$ is finite, which is impossible.

Now, for every $a, b \in \mathbb{Z}$, denote the common difference $n - m$ defined above by $X(a, b)$. We have $X(a-1, a) = 1$ by (1). Trivially, $X(a, b) + X(b, c) = X(a, c)$, as if $f^n(a) = f^m(b)$ and $f^p(b) = f^q(c)$, then $f^{p+n}(a) = f^{p+m}(b) = f^{q+m}(c)$. These two properties imply that $X(a, b) = b - a$ for all $a, b \in \mathbb{Z}$.

But (1) yields $f^{a^2+1}(f(a-1)) = f^{a^2}(f(a))$, so

$$1 = X\big(f(a-1), f(a)\big) = f(a) - f(a-1) \quad \text{for all } a \in \mathbb{Z}.$$

Recalling that $f(-1) = 0$, we conclude by (two-sided) induction on x that $f(x) = x + 1$ for all $x \in \mathbb{Z}$.

Finally, the obtained function also satisfies the assumption. Indeed, $f^n(x) = x + n$ for all $n \geqslant 0$, so

$$f^{a^2+b^2}(a+b) = a + b + a^2 + b^2 = af(a) + bf(b).$$

Comment. There are many possible variations of the solution above, but it seems that finiteness of orbits seems to be a crucial distinction in all solutions. However, the case distinction could be made in different ways; in particular, there exist some versions of Case 1 which work whenever there is at least *one* finite orbit.

We believe that Case 2 is conceptually harder than Case 1.

A7. Let n and k be positive integers. Prove that for $a_1, \ldots, a_n \in [1, 2^k]$ one has

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{a_1^2 + \ldots + a_i^2}} \leqslant 4\sqrt{kn}.$$

(Iran)

Solution 1. Partition the set of indices $\{1, 2, \ldots, n\}$ into disjoint subsets $M_1, M_2, \ldots, M_k$ so that $a_\ell \in [2^{j-1}, 2^j]$ for $\ell \in M_j$. Then, if $|M_j| =: p_j$, we have

$$\sum_{\ell \in M_j} \frac{a_\ell}{\sqrt{a_1^2 + \ldots + a_\ell^2}} \leqslant \sum_{i=1}^{p_j} \frac{2^j}{2^{j-1}\sqrt{i}} = 2\sum_{i=1}^{p_j} \frac{1}{\sqrt{i}},$$

where we used that $a_\ell \leqslant 2^j$ and in the denominator every index from M_j contributes at least $(2^{j-1})^2$. Now, using $\sqrt{i} - \sqrt{i-1} = \frac{1}{\sqrt{i}+\sqrt{i-1}} \geqslant \frac{1}{2\sqrt{i}}$, we deduce that

$$\sum_{\ell \in M_j} \frac{a_\ell}{\sqrt{a_1^2 + \ldots + a_\ell^2}} \leqslant 2\sum_{i=1}^{p_j} \frac{1}{\sqrt{i}} \leqslant 2\sum_{i=1}^{p_j} 2(\sqrt{i} - \sqrt{i-1}) = 4\sqrt{p_j}.$$

Therefore, summing over $j = 1, \ldots, k$ and using the QM–AM inequality, we obtain

$$\sum_{\ell=1}^{n} \frac{a_\ell}{\sqrt{a_1^2 + \ldots + a_\ell^2}} \leqslant 4\sum_{j=1}^{k} \sqrt{|M_j|} \leqslant 4\sqrt{k\sum_{j=1}^{k} |M_j|} = 4\sqrt{kn}.$$

Comment. Consider the function $f(a_1, \ldots, a_n) = \sum_{i=1}^{n} \frac{a_i}{\sqrt{a_1^2+\ldots+a_i^2}}$. One can see that rearranging the variables in increasing order can only increase the value of $f(a_1, \ldots, a_n)$. Indeed, if $a_j > a_{j+1}$ for some index j then we have

$$f(a_1, \ldots, a_{j-1}, a_{j+1}, a_j, a_{j+2}, \ldots, a_n) - f(a_1, \ldots, a_n) = \frac{a}{S} + \frac{b}{\sqrt{S^2 - a^2}} - \frac{b}{S} - \frac{a}{\sqrt{S^2 - b^2}}$$

where $a = a_j, b = a_{j+1}$, and $S = \sqrt{a_1^2 + \ldots + a_{j+1}^2}$. The positivity of the last expression above follows from

$$\frac{b}{\sqrt{S^2 - a^2}} - \frac{b}{S} = \frac{a^2 b}{S\sqrt{S^2 - a^2} \cdot (S + \sqrt{S^2 - a^2})} > \frac{ab^2}{S\sqrt{S^2 - b^2} \cdot (S + \sqrt{S^2 - b^2})} = \frac{a}{\sqrt{S^2 - b^2}} - \frac{a}{S}.$$

Comment. If $k < n$, the example $a_m := 2^{k(m-1)/n}$ shows that the problem statement is sharp up to a multiplicative constant. For $k \geqslant n$ the trivial upper bound n becomes sharp up to a multiplicative constant.

Solution 2. Apply induction on n. The base $n \leqslant 16$ is clear: our sum does not exceed $n \leqslant 4\sqrt{nk}$. For the inductive step from $1, \ldots, n-1$ to $n \geqslant 17$ consider two similar cases.

Case 1: $n = 2t$.

Let $x_\ell = \frac{a_\ell}{\sqrt{a_1^2+\ldots+a_\ell^2}}$. We have

$$\exp(-x_{t+1}^2 - \ldots - x_{2t}^2) \geqslant \left(1 - x_{t+1}^2\right) \ldots \left(1 - x_{2t}^2\right) = \frac{a_1^2 + \ldots + a_t^2}{a_1^2 + \ldots + a_{2t}^2} \geqslant \frac{1}{1 + 4^k},$$

where we used that the product is telescopic and then an estimate $a_{t+i} \leqslant 2^k a_i$ for $i = 1, \ldots, t$. Therefore, $x_{t+1}^2 + \ldots + x_{2t}^2 \leqslant \log(4^k + 1) \leqslant 2k$, where log denotes the natural logarithm. This implies $x_{t+1} + \ldots + x_{2t} \leqslant \sqrt{2kt}$. Hence, using the inductive hypothesis for $n = t$ we get

$$\sum_{\ell=1}^{2t} x_\ell \leqslant 4\sqrt{kt} + \sqrt{2kt} \leqslant 4\sqrt{2kt}.$$

Case 2: $n = 2t + 1$.

Analogously, we get $x_{t+2}^2 + \ldots + x_{2t+1}^2 \leqslant \log(4^k + 1) \leqslant 2k$ and

$$\sum_{\ell=1}^{2t+1} x_\ell \leqslant 4\sqrt{k(t+1)} + \sqrt{2kt} \leqslant 4\sqrt{k(2t+1)}.$$

The last inequality is true for all $t \geqslant 8$ since

$$4\sqrt{2t+1} - \sqrt{2t} \geqslant 3\sqrt{2t} = \sqrt{18t} \geqslant \sqrt{16t+16} = 4\sqrt{t+1}.$$

A8. Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all positive real numbers x and y,

$$f\big(x + f(xy)\big) + y = f(x)f(y) + 1. \tag{$*$}$$

(Ukraine)

Answer: $f(x) = x + 1$.

Solution 1. A straightforward check shows that $f(x) = x + 1$ satisfies $(*)$. We divide the proof of the converse statement into a sequence of steps.

Step 1: f is injective.

Put $x = 1$ in $(*)$ and rearrange the terms to get

$$y = f(1)f(y) + 1 - f\big(1 + f(y)\big).$$

Therefore, if $f(y_1) = f(y_2)$, then $y_1 = y_2$.

Step 2: f is (strictly) monotone increasing.

For any fixed $y \in \mathbb{R}^+$, the function

$$g(x) := f\big(x + f(xy)\big) = f(x)f(y) + 1 - y$$

is injective by Step 1. Therefore, $x_1 + f(x_1y) \neq x_2 + f(x_2y)$ for all $y, x_1, x_2 \in \mathbb{R}^+$ with $x_1 \neq x_2$. Plugging in $z_i = x_iy$, we arrive at

$$\frac{z_1 - z_2}{y} \neq f(z_2) - f(z_1), \quad \text{or} \quad \frac{1}{y} \neq \frac{f(z_2) - f(z_1)}{z_1 - z_2}$$

for all $y, z_1, z_2 \in \mathbb{R}^+$ with $z_1 \neq z_2$. This means that the right-hand side of the rightmost relation is always non-positive, i.e., f is monotone non-decreasing. Since f is injective, it is strictly monotone.

Step 3: There exist constants a and b such that $f(y) = ay + b$ for all $y \in \mathbb{R}^+$.

Since f is monotone and bounded from below by 0, for each $x_0 \geqslant 0$, there exists a right limit $\lim_{x\searrow x_0} f(x) \geqslant 0$. Put $p = \lim_{x\searrow 0} f(x)$ and $q = \lim_{x\searrow p} f(x)$.

Fix an arbitrary y and take the limit of $(*)$ as $x \searrow 0$. We have $f(xy) \searrow p$ and hence $f\big(x + f(xy)\big) \searrow q$; therefore, we obtain

$$q + y = pf(y) + 1, \quad \text{or} \quad f(y) = \frac{q + y - 1}{p}.$$

(Notice that $p \neq 0$, otherwise $q + y = 1$ for all y, which is absurd.) The claim is proved.

Step 4: $f(x) = x + 1$ for all $x \in \mathbb{R}^+$.

Based on the previous step, write $f(x) = ax + b$. Putting this relation into $(*)$ we get

$$a\,(x + axy + b) + b + y = (ax + b)(ay + b) + 1,$$

which can be rewritten as

$$(a - ab)x + (1 - ab)y + ab + b - b^2 - 1 = 0 \qquad \text{for all } x, y \in \mathbb{R}^+.$$

This identity may hold only if all the coefficients are 0, i.e.,

$$a - ab = 1 - ab = ab + b - b^2 - 1 = 0.$$

Hence, $a = b = 1$.

Solution 2. We provide another proof that $f(x)=x+1$ is the only function satisfying $(*)$.

Put $a=f(1)$. Define the function $\phi\colon \mathbb{R}^+\to\mathbb{R}$ by

$$\phi(x)=f(x)-x-1.$$

Then equation $(*)$ reads as

$$\phi(x+f(xy))=f(x)f(y)-f(xy)-x-y. \tag{1}$$

Since the right-hand side of (1) is symmetric under swapping x and y, we obtain

$$\phi\big(x+f(xy)\big)=\phi\big(y+f(xy)\big).$$

In particular, substituting $(x,y)=(t,1/t)$ we get

$$\phi(a+t)=\phi\left(a+\frac{1}{t}\right), \qquad t\in\mathbb{R}^+. \tag{2}$$

Notice that the function f is bounded from below by a positive constant. Indeed, for each $y\in\mathbb{R}^+$, the relation $(*)$ yields $f(x)f(y)>y-1$, hence

$$f(x)>\frac{y-1}{f(y)} \qquad \text{for all } x\in\mathbb{R}^+.$$

If $y>1$, this provides a desired positive lower bound for $f(x)$.

Now, let $M=\inf\limits_{x\in\mathbb{R}^+} f(x)>0$. Then, for all $y\in\mathbb{R}^+$,

$$M\geqslant\frac{y-1}{f(y)}, \quad \text{or} \quad f(y)\geqslant\frac{y-1}{M}. \tag{3}$$

Lemma 1. The function $f(x)$ (and hence $\phi(x)$) is bounded on any segment $[p,q]$, where $0<p<q<+\infty$.

Proof. f is bounded from below by M. It remains to show that f is bounded from above on $[p,q]$. Substituting $y=1$ into $(*)$, we get

$$f\big(x+f(x)\big)=af(x). \tag{4}$$

Take $z\in[p,q]$ and put $s=f(z)$. By (4), we have

$$f(z+s)=as \quad\text{and}\quad f(z+s+as)=f\big(z+s+f(z+s)\big)=a^2s.$$

Plugging in $(x,y)=\big(z,1+\frac{s}{z}\big)$ to $(*)$ and using (3), we obtain

$$f(z+as)=f\big(z+f(z+s)\big)=sf\left(1+\frac{s}{z}\right)-\frac{s}{z}\geqslant\frac{s^2}{Mz}-\frac{s}{z}.$$

Now, substituting $(x,y)=\big(z+as,\frac{z}{z+as}\big)$ to $(*)$ and applying the above estimate and the estimate $f(y)\geqslant M$, we obtain

$$\begin{aligned}a^2s=f(z+s+as)=f\big(z+as+f(z)\big)&=f(z+as)f\left(\frac{z}{z+as}\right)+1-\frac{z}{z+as}\\ &\geqslant Mf(z+as)\geqslant\frac{s^2}{z}-\frac{Ms}{z}\geqslant\frac{s^2}{q}-\frac{Ms}{p}.\end{aligned}$$

This yields $s\leqslant q\big(\frac{M}{p}+a^2\big)=:L$, and f is bounded from above by L on $[p,q]$. $\square$

Applying Lemma 1 to the segment $[a, a+1]$, we see that ϕ is bounded on it. By (2) we get that ϕ is also bounded on $[a+1, +\infty)$, and hence on $[a, +\infty)$. Put $C = \max\{a, 3\}$.

Lemma 2. For all $x \geqslant C$, we have $\phi(x) = 0$ (and hence $f(x) = x+1$).

Proof. Substituting $y = x$ to (1), we obtain

$$\phi\big(x + f(x^2)\big) = f(x)^2 - f(x^2) - 2x,$$

hence,

$$\phi\big(x + f(x^2)\big) + \phi(x^2) = f(x)^2 - (x+1)^2 = \phi(x)\big(f(x) + x + 1\big). \tag{5}$$

Since $f(x) + x + 1 \geqslant C + 1 \geqslant 4$, we obtain that

$$|\phi(x)| \leqslant \frac{1}{4}\left(\big|\phi\big(x + f(x^2)\big)\big| + \big|\phi(x^2)\big|\right). \tag{6}$$

Since $C \geqslant a$, there exists a finite supremum $S = \sup\limits_{x \geqslant C} |\phi(x)|$. For each $x \in [C, +\infty)$, both $x + f(x^2)$ and x^2 are greater than x; hence they also lie in $[C, +\infty)$. Therefore, taking the supremum of the left-hand side of (6) over $x \in [C, +\infty)$, we obtain $S \leqslant S/2$ and hence $S = 0$. Thus, $\phi(x) = 0$ for all $x \geqslant C$. □

It remains to show that $f(y) = y + 1$ when $0 < y < C$. For each y, choose $x > \max\big\{C, \frac{C}{y}\big\}$. Then all three numbers x, xy, and $x + f(xy)$ are greater than C, so $(*)$ reads as

$$(x + xy + 1) + 1 + y = (x+1)f(y) + 1, \quad \text{hence} \quad f(y) = y + 1.$$

Comment 1. It may be useful to rewrite $(*)$ in the form

$$\phi\big(x + f(xy)\big) + \phi(xy) = \phi(x)\phi(y) + x\phi(y) + y\phi(x) + \phi(x) + \phi(y).$$

This general identity easily implies both (1) and (5).

Comment 2. There are other ways to prove that $f(x) \geqslant x + 1$. Once one has proved this, they can use this stronger estimate instead of (3) in the proof of Lemma 1. Nevertheless, this does not make this proof simpler. So proving that $f(x) \geqslant x + 1$ does not seem to be a serious progress towards the solution of the problem. In what follows, we outline one possible proof of this inequality.

First of all, we improve inequality (3) by noticing that, in fact, $f(x)f(y) \geqslant y - 1 + M$, and hence

$$f(y) \geqslant \frac{y-1}{M} + 1. \tag{7}$$

Now we divide the argument into two steps.

Step 1: We show that $M \leqslant 1$.

Suppose that $M > 1$; recall the notation $a = f(1)$. Substituting $y = 1/x$ in $(*)$, we get

$$f(x+a) = f(x)f\left(\frac{1}{x}\right) + 1 - \frac{1}{x} \geqslant Mf(x),$$

provided that $x \geqslant 1$. By a straightforward induction on $\lceil (x-1)/a \rceil$, this yields

$$f(x) \geqslant M^{(x-1)/a}. \tag{8}$$

Now choose an arbitrary $x_0 \in \mathbb{R}^+$ and define a sequence $x_0, x_1, \ldots$ by $x_{n+1} = x_n + f(x_n) \geqslant x_n + M$ for all $n \geqslant 0$; notice that the sequence is unbounded. On the other hand, by (4) we get

$$ax_{n+1} > af(x_n) = f(x_{n+1}) \geqslant M^{(x_{n+1}-1)/a},$$

which cannot hold when x_{n+1} is large enough.

Step 2: We prove that $f(y) \geqslant y + 1$ for all $y \in \mathbb{R}^+$.

Arguing indirectly, choose $y \in \mathbb{R}^+$ such that $f(y) < y+1$, and choose μ with $f(y) < \mu < y+1$. Define a sequence $x_0, x_1, \ldots$ by choosing a large $x_0 \geqslant 1$ and setting $x_{n+1} = x_n + f(x_n y) \geqslant x_n + M$ for all $n \geqslant 0$ (this sequence is also unbounded). If x_0 is large enough, then (7) implies that $\big(\mu - f(y)\big) f(x_n) \geqslant 1 - y$ for all n. Therefore,

$$f(x_{n+1}) = f(y)f(x_n) + 1 - y \leqslant \mu f(x_n).$$

On the other hand, since $M \leqslant 1$, inequality (7) implies that $f(z) \geqslant z$, provided that $z \geqslant 1$. Hence, if x_0 is large enough, we have $x_{n+1} \geqslant x_n(1 + y)$ for all n. Therefore,

$$x_0(1 + y)^n \leqslant x_n \leqslant f(x_n) \leqslant \mu^n f(x_0),$$

which cannot hold when n is large enough.

Combinatorics

C1. Let n be a positive integer. Find the number of permutations $a_1, a_2, \dots, a_n$ of the sequence $1, 2, \dots, n$ satisfying

$$a_1 \leqslant 2a_2 \leqslant 3a_3 \leqslant \dots \leqslant na_n. \qquad (*)$$

(United Kingdom)

Answer: The number of such permutations is F_{n+1}, where F_k is the k^{th} Fibonacci number: $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$.

Solution 1. Denote by P_n the number of permutations that satisfy $(*)$. It is easy to see that $P_1 = 1$ and $P_2 = 2$.

Lemma 1. Let $n \geqslant 3$. If a permutation $a_1, \dots, a_n$ satisfies $(*)$ then either $a_n = n$, or $a_{n-1} = n$ and $a_n = n-1$.

Proof. Let k be the index for which $a_k = n$. If $k = n$ then we are done.

If $k = n-1$ then, by $(*)$, we have $n(n-1) = (n-1)a_{n-1} \leqslant na_n$, so $a_n \geqslant n-1$. Since $a_n \neq a_{n-1} = n$, the only choice for a_n is $a_n = n-1$.

Now suppose that $k \leqslant n-2$. For every $k < i < n$ we have $kn = ka_k \leqslant ia_i < na_i$, so $a_i \geqslant k+1$. Moreover, $na_n \geqslant (n-1)a_{n-1} \geqslant (n-1)(k+1) = nk + (n-1-k) > nk$, so $a_n \geqslant k+1$. Now the $n-k+1$ numbers $a_k, a_{k+1}, \dots, a_n$ are all greater than k; but there are only $n-k$ such values; this is not possible. □

If $a_n = n$ then $a_1, a_2, \dots, a_{n-1}$ must be a permutation of the numbers $1, \dots, n-1$ satisfying $a_1 \leqslant 2a_2 \leqslant \dots \leqslant (n-1)a_{n-1}$; there are P_{n-1} such permutations. The last inequality in $(*)$, $(n-1)a_{n-1} \leqslant na_n = n^2$, holds true automatically.

If $(a_{n-1}, a_n) = (n, n-1)$, then $a_1, \dots, a_{n-2}$ must be a permutation of $1, \dots, n-2$ satisfying $a_1 \leqslant \dots \leqslant (n-2)a_{n-2}$; there are P_{n-2} such permutations. The last two inequalities in $(*)$ hold true automatically by $(n-2)a_{n-2} \leqslant (n-2)^2 < n(n-1) = (n-1)a_{n-1} = na_n$.

Hence, the sequence $(P_1, P_2, \dots)$ satisfies the recurrence relation $P_n = P_{n-1} + P_{n-2}$ for $n \geqslant 3$. The first two elements are $P_1 = F_2$ and $P_2 = F_3$, so by a trivial induction we have $P_n = F_{n+1}$.

Solution 2. We claim that all sought permutations are of the following kind. Split $\{1, 2, \dots, n\}$ into singletons and pairs of adjacent numbers. In each pair, swap the two numbers and keep the singletons unchanged.

Such permutations correspond to tilings of a $1 \times n$ chessboard using dominoes and unit squares; it is well-known that the number of such tilings is the Fibonacci number F_{n+1}.

The claim follows by induction from

Lemma 2. Assume that $a_1, \dots, a_n$ is a permutation satisfying $(*)$, and k is an integer such that $1 \leqslant k \leqslant n$ and $\{a_1, a_2, \dots, a_{k-1}\} = \{1, 2, \dots, k-1\}$. (If $k = 1$, the condition is empty.) Then either $a_k = k$, or $a_k = k+1$ and $a_{k+1} = k$.

Proof. Choose t with $a_t = k$. Since $k \notin \{a_1, \dots, a_{k-1}\}$, we have either $t = k$ or $t > k$. If $t = k$ then we are done, so assume $t > k$.

Notice that one of the numbers among the $t-k$ numbers $a_k, a_{k+1}, \dots, a_{t-1}$ is at least t, because there are only $t-k-1$ values between k and t. Let i be an index with $k \leqslant i < t$ and $a_i \geqslant t$; then $kt = ta_t \geqslant ia_i \geqslant it \geqslant kt$, so that all the inequalities turn into equalities, hence $i = k$ and $a_k = t$. If $t = k+1$, we are done.

Suppose that $t > k+1$. Then the chain of inequalities $kt = ka_k \leqslant \dots \leqslant ta_t = kt$ should also turn into a chain of equalities. From this point we can find contradictions in several ways; for example by pointing to $a_{t-1} = \frac{kt}{t-1} = k + \frac{k}{t-1}$ which cannot be an integer, or considering the product of the numbers $(k+1)a_{k+1}, \ldots, (t-1)a_{t-1}$; the numbers $a_{k+1}, \ldots, a_{t-1}$ are distinct and greater than k, so

$$(kt)^{t-k-1} = (k+1)a_{k+1} \cdot (k+2)a_{k+2} \cdot \ldots \cdot (t-1)a_{t-1} \geqslant \big((k+1)(k+2) \cdot \ldots \cdot (t-1)\big)^2.$$

Notice that $(k+i)(t-i) = kt + i(t-k-i) > kt$ for $1 \leqslant i < t-k$. This leads to the contradiction

$$(kt)^{t-k-1} \geqslant \big((k+1)(k+2) \cdot \ldots \cdot (t-1)\big)^2 = \prod_{i=1}^{t-k-1} (k+i)(t-i) > (kt)^{t-k-1}.$$

Therefore, the case $t > k+1$ is not possible. □

C2. In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, \ldots, Q_{24}$ whose corners are vertices of the 100-gon, so that

- the quadrilaterals $Q_1, \ldots, Q_{24}$ are pairwise disjoint, and
- every quadrilateral Q_i has three corners of one color and one corner of the other color.

(Austria)

Solution. Call a quadrilateral *skew-colored*, if it has three corners of one color and one corner of the other color. We will prove the following

Claim. If the vertices of a convex $(4k+1)$-gon P are colored black and white such that each color is used at least k times, then there exist k pairwise disjoint skew-colored quadrilaterals whose vertices are vertices of P. (One vertex of P remains unused.)

The problem statement follows by removing 3 arbitrary vertices of the 100-gon and applying the Claim to the remaining 97 vertices with $k = 24$.

Proof of the Claim. We prove by induction. For $k = 1$ we have a pentagon with at least one black and at least one white vertex. If the number of black vertices is even then remove a black vertex; otherwise remove a white vertex. In the remaining quadrilateral, there are an odd number of black and an odd number of white vertices, so the quadrilateral is skew-colored.

For the induction step, assume $k \geqslant 2$. Let b and w be the numbers of black and white vertices, respectively; then $b, w \geqslant k$ and $b + w = 4k + 1$. Without loss of generality we may assume $w \geqslant b$, so $k \leqslant b \leqslant 2k$ and $2k + 1 \leqslant w \leqslant 3k + 1$.

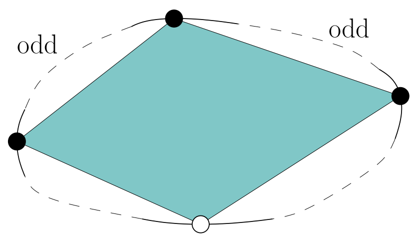
We want to find four consecutive vertices such that three of them are white, the fourth one is black. Denote the vertices by $V_1, V_2, \ldots, V_{4k+1}$ in counterclockwise order, such that V_{4k+1} is black, and consider the following k groups of vertices:

$$(V_1, V_2, V_3, V_4), (V_5, V_6, V_7, V_8), \ldots, (V_{4k-3}, V_{4k-2}, V_{4k-1}, V_{4k})$$

In these groups there are w white and $b-1$ black vertices. Since $w > b-1$, there is a group, $(V_i, V_{i+1}, V_{i+2}, V_{i+3})$ that contains more white than black vertices. If three are white and one is black in that group, we are done. Otherwise, if $V_i, V_{i+1}, V_{i+2}, V_{i+3}$ are all white then let V_j be the first black vertex among $V_{i+4}, \ldots, V_{4k+1}$ (recall that V_{4k+1} is black); then V_{j-3}, V_{j-2} and V_{j-1} are white and V_j is black.

Now we have four consecutive vertices $V_i, V_{i+1}, V_{i+2}, V_{i+3}$ that form a skew-colored quadrilateral. The remaining vertices form a convex $(4k-3)$-gon; $w-3$ of them are white and $b-1$ are black. Since $b - 1 \geqslant k - 1$ and $w - 3 \geqslant (2k+1) - 3 > k - 1$, we can apply the Claim with $k - 1$. $\square$

Comment. It is not true that the vertices of the 100-gon can be split into 25 skew-colored quadrilaterals. A possible counter-example is when the vertices $V_1, V_3, V_5, \ldots, V_{81}$ are black and the other vertices, $V_2, V_4, \ldots, V_{80}$ and $V_{82}, V_{83}, \ldots, V_{100}$ are white. For having 25 skew-colored quadrilaterals, there should be 8 containing three black vertices. But such a quadrilateral splits the other 96 vertices into four sets in such a way that at least two sets contain odd numbers of vertices and therefore they cannot be grouped into disjoint quadrilaterals.



C3. Let n be an integer with $n \geqslant 2$. On a slope of a mountain, n^2 checkpoints are marked, numbered from 1 to n^2 from the bottom to the top. Each of two cable car companies, A and B, operates k cable cars numbered from 1 to k; each cable car provides a transfer from some checkpoint to a higher one. For each company, and for any i and j with $1 \leqslant i < j \leqslant k$, the starting point of car j is higher than the starting point of car i; similarly, the finishing point of car j is higher than the finishing point of car i. Say that two checkpoints are *linked* by some company if one can start from the lower checkpoint and reach the higher one by using one or more cars of that company (no movement on foot is allowed).

Determine the smallest k for which one can guarantee that there are two checkpoints that are linked by each of the two companies.

(India)

Answer: $k = n^2 - n + 1$.

Solution. We start with showing that for any $k \leqslant n^2 - n$ there may be no pair of checkpoints linked by both companies. Clearly, it suffices to provide such an example for $k = n^2 - n$.

Let company A connect the pairs of checkpoints of the form $(i, i+1)$, where $n \nmid i$. Then all pairs of checkpoints (i, j) linked by A satisfy $\lceil i/n \rceil = \lceil j/n \rceil$.

Let company B connect the pairs of the form $(i, i+n)$, where $1 \leqslant i \leqslant n^2 - n$. Then pairs of checkpoints (i, j) linked by B satisfy $i \equiv j \pmod{n}$. Clearly, no pair (i, j) satisfies both conditions, so there is no pair linked by both companies.

Now we show that for $k = n^2 - n + 1$ there always exist two required checkpoints. Define an *A-chain* as a sequence of checkpoints $a_1 < a_2 < \ldots < a_t$ such that company A connects a_i with a_{i+1} for all $1 \leqslant i \leqslant t-1$, but there is no A-car transferring from some checkpoint to a_1 and no A-car transferring from a_t to any other checkpoint. Define *B-chains* similarly. Moving forth and back, one easily sees that any checkpoint is included in a unique A-chain (possibly consisting of that single checkpoint), as well as in a unique B-chain. Now, put each checkpoint into a correspondence to the pair of the A-chain and the B-chain it belongs to.

All finishing points of A-cars are distinct, so there are $n^2 - k = n - 1$ checkpoints that are not such finishing points. Each of them is a starting point of a unique A-chain, so the number of A-chains is $n-1$. Similarly, the number of B-chains also equals $n-1$. Hence, there are $(n-1)^2$ pairs consisting of an A- and a B-chain. Therefore, two of the n^2 checkpoints correspond to the same pair, so that they belong to the same A-chain, as well as to the same B-chain. This means that they are linked by both companies, as required.

Comment 1. The condition that the i^{th} car starts and finishes lower than the j^{th} one is used only in the "moving forth and back" argument and in the counting of starting points of the chains. In both cases, the following weaker assumption suffices: No two cars of the same company start at the same checkpoint, and no two such cars finish at the same checkpoint.

Thus, the problem conditions could be weakened in this way,, with no affect on the solution.

Comment 2. If the number of checkpoints were N, then the answer would be $N - \lceil \sqrt{N} \rceil + 1$. The solution above works verbatim for this generalization.

C4. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 1$. Given an integer $n \geqslant 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \ldots, n$ there exist some $x, y \in S$ such that $x - y = F_k$.

(Croatia)

Answer: $\lceil n/2 \rceil + 1$.

Solution. First we show that if a set $S \subset \mathbb{Z}$ satisfies the conditions then $|S| \geqslant \frac{n}{2} + 1$.

Let $d = \lceil n/2 \rceil$, so $n \leqslant 2d \leqslant n + 1$. In order to prove that $|S| \geqslant d + 1$, construct a graph as follows. Let the vertices of the graph be the elements of S. For each $1 \leqslant k \leqslant d$, choose two elements $x, y \in S$ such that $x - y = F_{2k-1}$, and add the pair (x, y) to the graph as edge. (Note that by the problem's constraints, there must be a pair (x, y) with $x - y = F_{2k-1}$ for every $3 \leqslant 2k - 1 \leqslant 2d - 1 \leqslant n$; moreover, due to $F_1 = F_2$ we have a pair with $x - y = F_1$ as well.) We will say that the *length* of the edge (x, y) is $|x - y|$.

We claim that the graph contains no cycle. For the sake of contradiction, suppose that the graph contains a cycle $(x_1, \ldots, x_\ell)$, and let the longest edge in the cycle be (x_1, x_ℓ) with length F_{2m+1}. The other edges $(x_1, x_2), \ldots, (x_{\ell-1}, x_\ell)$ in the cycle are shorter than F_{2m+1} and distinct, their lengths form a subset of $\{F_1, F_3, \ldots, F_{2m-1}\}$. But this is not possible because

$$\begin{aligned} F_{2m+1} = |x_\ell - x_1| &\leqslant \sum_{i=1}^{\ell-1} |x_{i+1} - x_i| \leqslant F_1 + F_3 + F_5 + \ldots + F_{2m-1} \\ &= F_2 + (F_4 - F_2) + (F_6 - F_4) + \ldots + (F_{2m} - F_{2m-2}) = F_{2m} < F_{2m+1}. \end{aligned}$$

Hence, the graph has d edges and cannot contain a cycle, therefore it must contain at least $d + 1$ vertices, so $|S| \geqslant d + 1$.

Now we show a suitable set with $d + 1$ elements. Let

$$S = \{F_0, F_2, F_4, F_5, \ldots, F_{2d}\}.$$

For $1 \leqslant k \leqslant d$ we have $F_0, F_{2k-2}, F_{2k} \in S$ with differences $F_{2k} - F_{2k-2} = F_{2k-1}$ and $F_{2k} - F_0 = F_{2k}$, so each of $F_1, F_2, \ldots, F_{2d}$ occurs as difference between two elements in S. So this set containing $d + 1$ numbers is suitable.

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C5. Let p be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \ldots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote by $r(n)$ the fraction of integers in $\{1, 2, \ldots, n\}$ that are red.

Prove that there exists a positive integer $a \in \{1, 2, \ldots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \ldots, N$.

(Netherlands)

Solution. Denote by $R(n)$ the number of red numbers in $\{1, 2, \ldots, n\}$, i.e., $R(n) = nr(n)$. Similarly, denote by $B(n)$ and $b(n) = B(n)/n$ the number and proportion of blue numbers in $\{1, 2, \ldots, n\}$, respectively. Notice that $B(n) + R(n) = n$ and $b(n) + r(n) = 1$. Therefore, the statement of the problem does not change after swapping the colors.

Arguing indirectly, for every $a \in \{1, 2, \ldots, p-1\}$ choose some positive integer n_a such that $r(n_a) = a/p$ and, hence, $R(n_a) = an_a/p$. Clearly, $p \mid n_a$, so that $n_a = pm_a$ for some positive integer m_a, and $R(n_a) = am_a$. Without loss of generality, we assume that $m_1 < m_{p-1}$, as otherwise one may swap the colors. Notice that

$$m_a \leqslant \frac{N}{p} < \frac{p^2-1}{4} \qquad \text{for all } a = 1, 2, \ldots, p-1. \tag{1}$$

The solution is based on a repeated application of the following simple observation.

Claim. Assume that $m_a < m_b$ for some $a, b \in \{1, 2, \ldots, p-1\}$. Then

$$m_b \geqslant \frac{a}{b} m_a \qquad \text{and} \qquad m_b \geqslant \frac{p-a}{p-b} m_a.$$

Proof. The first inequality follows from $bm_b = R(n_b) \geqslant R(n_a) = am_a$. The second inequality is obtained by swapping colors . $\square$

Let $q = (p-1)/2$. We distinguish two cases.

Case 1: All q numbers $m_1, m_2, \ldots, m_q$ are smaller than m_{p-1}.

Let m_a be the maximal number among $m_1, m_2, \ldots, m_q$; then $m_a \geqslant q \geqslant a$. Applying the Claim, we get

$$m_{p-1} \geqslant \frac{p-a}{p-(p-1)} m_a \geqslant (p-q)q = \frac{p^2-1}{4},$$

which contradicts (1).

Case 2: There exists $k \leqslant q$ such that $m_k > m_{p-1}$.

Choose k to be the smallest index satisfying $m_k > m_{p-1}$; by our assumptions, we have $1 < k \leqslant q < p-1$.

Let m_a be the maximal number among $m_1, m_2, \ldots, m_{k-1}$; then $a \leqslant k-1 \leqslant m_a < m_{p-1}$. Applying the Claim, we get

$$\begin{aligned} m_k \geqslant \frac{p-1}{k} m_{p-1} &\geqslant \frac{p-1}{k} \cdot \frac{p-a}{p-(p-1)} m_a \\ &\geqslant \frac{p-1}{k} \cdot (p-k+1)(k-1) \geqslant \frac{k-1}{k} \cdot (p-1)(p-q) \geqslant \frac{1}{2} \cdot \frac{p^2-1}{2} \end{aligned}$$

which contradicts (1) again.

Comment 1. The argument in Case 2, after a slight modification of estimates at the end, applies as soon as there exists $k < \frac{3(p+1)}{4}$ with $a_k < a_{p-1}$. However, this argument does not seem to work if there is no such k.

Comment 2. If p is small enough, then one can color $\{1, 2, \ldots, N+1\}$ so that there exist numbers m_1, m_2, $\ldots$, m_{p-1} satisfying $r(pm_a) = a/p$. For $p = 3, 5, 7$, one can find colorings providing the following sequences:

$$(m_1, m_2) = (1, 2), \qquad (m_1, m_2, m_3, m_4) = (1, 2, 3, 6), \quad \text{and} \quad (m_1, \ldots, m_6) = (1, 2, 3, 4, 6, 12),$$

respectively.

Thus, for small values of p, the number N in the problem statement cannot be increased. However, a careful analysis of the estimates shows that this number can be slightly increased for $p \geqslant 11$.

C6. $4n$ coins of weights $1, 2, 3, \ldots, 4n$ are given. Each coin is colored in one of n colors and there are four coins of each color. Show that all these coins can be partitioned into two sets with the same total weight, such that each set contains two coins of each color.

(Hungary)

Solution 1. Let us pair the coins with weights summing up to $4n+1$, resulting in the set S of $2n$ pairs: $\{1, 4n\}$, $\{2, 4n-1\}$, $\ldots$, $\{2n, 2n+1\}$. It suffices to partition S into two sets, each consisting of n pairs, such that each set contains two coins of each color.

Introduce a multi-graph G (i.e., a graph with loops and multiple edges allowed) on n vertices, so that each vertex corresponds to a color. For each pair of coins from S, we add an edge between the vertices corresponding to the colors of those coins. Note that each vertex has degree 4. Also, a desired partition of the coins corresponds to a coloring of the edges of G in two colors, say red and blue, so that each vertex has degree 2 with respect to each color (i.e., each vertex has equal red and blue degrees).

To complete the solution, it suffices to provide such a coloring for each component G' of G. Since all degrees of the vertices are even, in G' there exists an Euler circuit C (i.e., a circuit passing through each edge of G' exactly once). Note that the number of edges in C is even (it equals twice the number of vertices in G'). Hence all the edges can be colored red and blue so that any two edges adjacent in C have different colors (one may move along C and color the edges one by one alternating red and blue colors). Thus in G' each vertex has equal red and blue degrees, as desired.

Comment 1. To complete Solution 1, any partition of the edges of G into circuits of even lengths could be used. In the solution above it was done by the reference to the well-known Euler Circuit Lemma: Let G be a connected graph with all its vertices of even degrees. Then there exists a circuit passing through each edge of G exactly once.

Solution 2. As in Solution 1, we will show that it is possible to partition $2n$ pairs $\{1, 4n\}$, $\{2, 4n-1\}$, $\ldots$, $\{2n, 2n+1\}$ into two sets, each consisting of n pairs, such that each set contains two coins of each color.

Introduce a multi-graph (i.e., a graph with multiple edges allowed) Γ whose vertices correspond to coins; thus we have $4n$ vertices of n colors so that there are four vertices of each color. Connect pairs of vertices $\{1, 4n\}$, $\{2, 4n-1\}$, $\ldots$, $\{2n, 2n+1\}$ by $2n$ *black* edges.

Further, for each monochromatic quadruple of vertices i, j, k, ℓ we add a pair of *grey* edges forming a matching, e.g., (i, j) and (k, ℓ). In each of n colors of coins we can choose one of three possible matchings; this results in 3^n ways of constructing grey edges. Let us call each of 3^n possible graphs Γ a *cyclic* graph. Note that in a cyclic graph Γ each vertex has both black and grey degrees equal to 1. Hence Γ is a union of disjoint cycles, and in each cycle black and grey edges alternate (in particular, all cycles have even lengths).

It suffices to find a cyclic graph with all its cycle lengths divisible by 4. Indeed, in this case, for each cycle we start from some vertex, move along the cycle and recolor the black edges either to red or to blue, alternating red and blue colors. Now blue and red edges define the required partition, since for each monochromatic quadruple of vertices the grey edges provide a bijection between the endpoints of red and blue edges.

Among all possible cyclic graphs, let us choose graph Γ_0 having the minimal number of components (i.e., cycles). The following claim completes the solution.

Claim. In Γ_0, all cycle lengths are divisible by 4.

Proof. Assuming the contrary, choose a cycle C_1 with an odd number of grey edges. For some color c the cycle C_1 contains exactly one grey edge joining two vertices i, j of color c, while the other edge joining two vertices k, ℓ of color c lies in another cycle C_2. Now delete edges (i, j) and (k, ℓ) and add edges (i, k) and (j, ℓ). By this switch we again obtain a cyclic graph Γ_0' and decrease the number of cycles by 1. This contradicts the choice of Γ_0. $\square$

Comment 2. Use of an auxiliary graph and reduction to a new problem in terms of this graph is one of the crucial steps in both solutions presented. In fact, graph G from Solution 1 could be obtained from any graph Γ from Solution 2 by merging the vertices of the same color.

C7. Consider any rectangular table having finitely many rows and columns, with a real number $a(r, c)$ in the cell in row r and column c. A pair (R, C), where R is a set of rows and C a set of columns, is called a *saddle pair* if the following two conditions are satisfied:

(i) For each row r', there is $r \in R$ such that $a(r, c) \geqslant a(r', c)$ for all $c \in C$;

(ii) For each column c', there is $c \in C$ such that $a(r, c) \leqslant a(r, c')$ for all $r \in R$.

A saddle pair (R, C) is called a *minimal pair* if for each saddle pair (R', C') with $R' \subseteq R$ and $C' \subseteq C$, we have $R' = R$ and $C' = C$.

Prove that any two minimal pairs contain the same number of rows.

(Thailand)

Solution 1. We say that a pair (R', C') of nonempty sets is a *subpair* of a pair (R, C) if $R' \subseteq R$ and $C' \subseteq C$. The subpair is *proper* if at least one of the inclusions is strict.

Let (R_1, C_1) and (R_2, C_2) be two saddle pairs with $|R_1| > |R_2|$. We will find a saddle subpair (R', C') of (R_1, C_1) with $|R'| \leqslant |R_2|$; clearly, this implies the desired statement.

Step 1: We construct maps $\rho\colon R_1 \to R_1$ and $\sigma\colon C_1 \to C_1$ such that $|\rho(R_1)| \leqslant |R_2|$, and $a\big(\rho(r_1), c_1\big) \geqslant a\big(r_1, \sigma(c_1)\big)$ for all $r_1 \in R_1$ and $c_1 \in C_1$.

Since (R_1, C_1) is a saddle pair, for each $r_2 \in R_2$ there is $r_1 \in R_1$ such that $a(r_1, c_1) \geqslant a(r_2, c_1)$ for all $c_1 \in C_1$; denote one such an r_1 by $\rho_1(r_2)$. Similarly, we define four functions

$$\begin{array}{llllll}
\rho_1\colon R_2 \to R_1 & \text{such that} & a\big(\rho_1(r_2), c_1\big) \geqslant a(r_2, c_1) & \text{for all} & r_2 \in R_2, & c_1 \in C_1; \\
\rho_2\colon R_1 \to R_2 & \text{such that} & a\big(\rho_2(r_1), c_2\big) \geqslant a(r_1, c_2) & \text{for all} & r_1 \in R_1, & c_2 \in C_2; \\
\sigma_1\colon C_2 \to C_1 & \text{such that} & a\big(r_1, \sigma_1(c_2)\big) \leqslant a(r_1, c_2) & \text{for all} & r_1 \in R_1, & c_2 \in C_2; \\
\sigma_2\colon C_1 \to C_2 & \text{such that} & a\big(r_2, \sigma_2(c_1)\big) \leqslant a(r_2, c_1) & \text{for all} & r_2 \in R_2, & c_1 \in C_1.
\end{array} \tag{1}$$

Set now $\rho = \rho_1 \circ \rho_2\colon R_1 \to R_1$ and $\sigma = \sigma_1 \circ \sigma_2\colon C_1 \to C_1$. We have

$$|\rho(R_1)| = |\rho_1(\rho_2(R_1))| \leqslant |\rho_1(R_2)| \leqslant |R_2|.$$

Moreover, for all $r_1 \in R_1$ and $c_1 \in C_1$, we get

$$\begin{aligned}
a\big(\rho(r_1), c_1\big) = a\big(\rho_1(\rho_2(r_1)), c_1\big) \geqslant a\big(\rho_2(r_1), c_1\big) \geqslant a\big(\rho_2(r_1), \sigma_2(c_1)\big) \\
\geqslant a\big(r_1, \sigma_2(c_1)\big) \geqslant a\big(r_1, \sigma_1(\sigma_2(c_1))\big) = a\big(r_1, \sigma(c_1)\big),
\end{aligned} \tag{2}$$

as desired.

Step 2: Given maps ρ and σ, we construct a proper saddle subpair (R', C') of (R_1, C_1).

The properties of ρ and σ yield that

$$a\big(\rho^i(r_1), c_1\big) \geqslant a\big(\rho^{i-1}(r_1), \sigma(c_1)\big) \geqslant \ldots \geqslant a\big(r_1, \sigma^i(c_1)\big),$$

for each positive integer i and all $r_1 \in R_1$, $c_1 \in C_1$.

Consider the images $R^i = \rho^i(R_1)$ and $C^i = \sigma^i(C_1)$. Clearly, $R_1 = R^0 \supseteq R^1 \supseteq R^2 \supseteq \ldots$ and $C_1 = C^0 \supseteq C^1 \supseteq C^2 \supseteq \ldots$. Since both chains consist of finite sets, there is an index n such that $R^n = R^{n+1} = \ldots$ and $C^n = C^{n+1} = \ldots$. Then $\rho^n(R^n) = R^{2n} = R^n$, so ρ^n restricted to R^n is a bijection. Similarly, σ^n restricted to C^n is a bijection from C^n to itself. Therefore, there exists a positive integer k such that ρ^{nk} acts identically on R^n, and σ^{nk} acts identically on C^n.

We claim now that (R^n, C^n) is a saddle subpair of (R_1, C_1), with $|R^n| \leqslant |R^1| = |\rho(R_1)| \leqslant |R_2|$, which is what we needed. To check that this is a saddle pair, take any row r'; since (R_1, C_1) is a saddle pair, there exists $r_1 \in R_1$ such that $a(r_1, c_1) \geqslant a(r', c_1)$ for all $c_1 \in C_1$. Set now $r_* = \rho^{nk}(r_1) \in R^n$. Then, for each $c \in C^n$ we have $c = \sigma^{nk}(c)$ and hence

$$a(r_*, c) = a\big(\rho^{nk}(r_1), c\big) \geqslant a\big(r_1, \sigma^{nk}(c)\big) = a(r_1, c) \geqslant a(r', c),$$

which establishes condition (i). Condition (ii) is checked similarly.

Solution 2. Denote by $\mathcal{R}$ and $\mathcal{C}$ the set of all rows and the set of all columns of the table, respectively. Let $\mathcal{T}$ denote the given table; for a set R of rows and a set C of columns, let $\mathcal{T}[R, C]$ denote the subtable obtained by intersecting rows from R and columns from C.

We say that row r_1 *exceeds* row r_2 *in range of columns* C (where $C \subseteq \mathcal{C}$) and write $r_1 \succeq_C r_2$ or $r_2 \preceq_C r_1$, if $a(r_1, c) \geqslant a(r_2, c)$ for all $c \in C$. We say that a row r_1 is *equal* to a row r_2 *in range of columns* C and write $r_1 \equiv_C r_2$, if $a(r_1, c) = a(r_2, c)$ for all $c \in C$. We introduce similar notions, and use the same notation, for columns. Then conditions (i) and (ii) in the definition of a saddle pair can be written as (i) for each $r' \in \mathcal{R}$ there exists $r \in R$ such that $r \succeq_C r'$; and (ii) for each $c' \in \mathcal{C}$ there exists $c \in C$ such that $c \preceq_R c'$.

Lemma. Suppose that (R, C) is a minimal pair. Remove from the table several rows outside of R and/or several columns outside of C. Then (R, C) remains a minimal pair in the new table.

Proof. Obviously, (R, C) remains a saddle pair. Suppose (R', C') is a proper subpair of (R, C). Since (R, C) is a saddle pair, for each row r^* of the initial table, there is a row $r \in R$ such that $r \succeq_C r^*$. If (R', C') became saddle after deleting rows not in R and/or columns not in C, there would be a row $r' \in R'$ satisfying $r' \succeq_{C'} r$. Therefore, we would obtain that $r' \succeq_{C'} r^*$, which is exactly condition (i) for the pair (R', C') in the initial table; condition (ii) is checked similarly. Thus, (R', C') was saddle in the initial table, which contradicts the hypothesis that (R, C) was minimal. Hence, (R, C) remains minimal after deleting rows and/or columns. □

By the Lemma, it suffices to prove the statement of the problem in the case $\mathcal{R} = R_1 \cup R_2$ and $\mathcal{C} = C_1 \cup C_2$. Further, suppose that there exist rows that belong both to R_1 and R_2. Duplicate every such row, and refer one copy of it to the set R_1, and the other copy to the set R_2. Then (R_1, C_1) and (R_2, C_2) will remain minimal pairs in the new table, with the same numbers of rows and columns, but the sets R_1 and R_2 will become disjoint. Similarly duplicating columns in $C_1 \cap C_2$, we make C_1 and C_2 disjoint. Thus it is sufficient to prove the required statement in the case $R_1 \cap R_2 = \varnothing$ and $C_1 \cap C_2 = \varnothing$.

The rest of the solution is devoted to the proof of the following claim including the statement of the problem.

Claim. Suppose that (R_1, C_1) and (R_2, C_2) are minimal pairs in table $\mathcal{T}$ such that $R_2 = \mathcal{R} \setminus R_1$ and $C_2 = \mathcal{C} \setminus C_1$. Then $|R_1| = |R_2|$, $|C_1| = |C_2|$; moreover, there are four bijections

$$\begin{aligned}
&\rho_1 \colon R_2 \to R_1 \quad \text{such that} \quad \rho_1(r_2) \equiv_{C_1} r_2 \quad \text{for all} \quad r_2 \in R_2;\\
&\rho_2 \colon R_1 \to R_2 \quad \text{such that} \quad \rho_2(r_1) \equiv_{C_2} r_1 \quad \text{for all} \quad r_1 \in R_1;\\
&\sigma_1 \colon C_2 \to C_1 \quad \text{such that} \quad \sigma_1(c_2) \equiv_{R_1} c_2 \quad \text{for all} \quad c_2 \in C_2;\\
&\sigma_2 \colon C_1 \to C_2 \quad \text{such that} \quad \sigma_2(c_1) \equiv_{R_2} c_1 \quad \text{for all} \quad c_1 \in C_1.
\end{aligned} \tag{3}$$

We prove the Claim by induction on $|\mathcal{R}| + |\mathcal{C}|$. In the base case we have $|R_1| = |R_2| = |C_1| = |C_2| = 1$; let $R_i = \{r_i\}$ and $C_i = \{c_i\}$. Since (R_1, C_1) and (R_2, C_2) are saddle pairs, we have $a(r_1, c_1) \geqslant a(r_2, c_1) \geqslant a(r_2, c_2) \geqslant a(r_1, c_2) \geqslant a(r_1, c_1)$, hence, the table consists of four equal numbers, and the statement follows.

To prove the inductive step, introduce the maps ρ_1, ρ_2, σ_1, and σ_2 as in Solution 1, see (1). Suppose first that all four maps are surjective. Then, in fact, we have $|R_1| = |R_2|$, $|C_1| = |C_2|$, and all maps are bijective. Moreover, for all $r_2 \in R_2$ and $c_2 \in C_2$ we have

$$\begin{aligned}
a(r_2, c_2) \leqslant a\big(r_2, \sigma_2^{-1}(c_2)\big) \leqslant a\big(\rho_1(r_2), \sigma_2^{-1}(c_2)\big) \leqslant a\big(\rho_1(r_2), \sigma_1^{-1} \circ \sigma_2^{-1}(c_2)\big)&\\
\leqslant a\big(\rho_2 \circ \rho_1(r_2), \sigma_1^{-1} \circ \sigma_2^{-1}(c_2)\big).&
\end{aligned} \tag{4}$$

Summing up, we get

$$\sum_{\substack{r_2 \in R_2 \\ c_2 \in C_2}} a(r_2, c_2) \leqslant \sum_{\substack{r_2 \in R_2 \\ c_2 \in C_2}} a\big(\rho_2 \circ \rho_1(r_2), \sigma_1^{-1} \circ \sigma_2^{-1}(c_2)\big).$$

Since $\rho_1 \circ \rho_2$ and $\sigma_1^{-1} \circ \sigma_2^{-1}$ are permutations of R_2 and C_2, respectively, this inequality is in fact equality. Therefore, all inequalities in (4) turn into equalities, which establishes the inductive step in this case.

It remains to show that all four maps are surjective. For the sake of contradiction, we assume that ρ_1 is not surjective. Now let $R_1' = \rho_1(R_2)$ and $C_1' = \sigma_1(C_2)$, and set $R^* = R_1 \setminus R_1'$ and $C^* = C_1 \setminus C_1'$. By our assumption, $R^* \neq \varnothing$.

Let $\mathcal{Q}$ be the table obtained from $\mathcal{T}$ by removing the rows in R^* and the columns in C^*; in other words, $\mathcal{Q} = \mathcal{T}[R_1' \cup R_2, C_1' \cup C_2]$. By the definition of ρ_1, for each $r_2 \in R_2$ we have $\rho_1(r_2) \succeq_{C_1} r_2$, so a fortiori $\rho_1(r_2) \succeq_{C_1'} r_2$; moreover, $\rho_1(r_2) \in R_1'$. Similarly, $C_1' \ni \sigma_1(c_2) \preceq_{R_1'} c_2$ for each $c_2 \in C_2$. This means that (R_1', C_1') is a saddle pair in $\mathcal{Q}$. Recall that (R_2, C_2) remains a minimal pair in $\mathcal{Q}$, due to the Lemma.

Therefore, $\mathcal{Q}$ admits a minimal pair $(\overline{R}_1, \overline{C}_1)$ such that $\overline{R}_1 \subseteq R_1'$ and $\overline{C}_1 \subseteq C_1'$. For a minute, confine ourselves to the subtable $\overline{\mathcal{Q}} = \mathcal{Q}[\overline{R}_1 \cup R_2, \overline{C}_1 \cup C_2]$. By the Lemma, the pairs $(\overline{R}_1, \overline{C}_1)$ and (R_2, C_2) are also minimal in $\overline{\mathcal{Q}}$. By the inductive hypothesis, we have $|R_2| = |\overline{R}_1| \leqslant |R_1'| = |\rho_1(R_2)| \leqslant |R_2|$, so all these inequalities are in fact equalities. This implies that $\overline{R}_2 = R_2'$ and that ρ_1 is a bijection $R_2 \to R_1'$. Similarly, $\overline{C}_1 = C_1'$, and σ_1 is a bijection $C_2 \to C_1'$. In particular, (R_1', C_1') is a minimal pair in $\mathcal{Q}$.

Now, by inductive hypothesis again, we have $|R_1'| = |R_2|$, $|C_1'| = |C_2|$, and there exist four bijections

$$\begin{array}{llll}
\rho_1' \colon R_2 \to R_1' & \text{such that} & \rho_1'(r_2) \equiv_{C_1'} r_2 & \text{for all} \quad r_2 \in R_2; \\
\rho_2' \colon R_1' \to R_2 & \text{such that} & \rho_2'(r_1) \equiv_{C_2} r_1 & \text{for all} \quad r_1 \in R_1'; \\
\sigma_1' \colon C_2 \to C_1' & \text{such that} & \sigma_1'(c_2) \equiv_{R_1'} c_2 & \text{for all} \quad c_2 \in C_2; \\
\sigma_2' \colon C_1' \to C_2 & \text{such that} & \sigma_2'(c_1) \equiv_{R_2} c_1 & \text{for all} \quad c_1 \in C_1'.
\end{array}$$

Notice here that σ_1 and σ_1' are two bijections $C_2 \to C_1'$ satisfying $\sigma_1'(c_2) \equiv_{R_1'} c_2 \succeq_{R_1} \sigma_1(c_2)$ for all $c_2 \in C_2$. Now, if $\sigma_1'(c_2) \neq \sigma_1(c_2)$ for some $c_2 \in C_2$, then we could remove column $\sigma_1'(c_2)$ from C_1' obtaining another saddle pair $\big(R_1', C_1' \setminus \{\sigma_1'(c_2)\}\big)$ in $\mathcal{Q}$. This is impossible for a minimal pair (R_1', C_1'); hence the maps σ_1 and σ_1' coincide.

Now we are prepared to show that (R_1', C_1') is a saddle pair in $\mathcal{T}$, which yields a desired contradiction (since (R_1, C_1) is not minimal). By symmetry, it suffices to find, for each $r' \in \mathcal{R}$, a row $r_1 \in R_1'$ such that $r_1 \succeq_{C_1'} r'$. If $r' \in R_2$, then we may put $r_1 = \rho_1(r')$; so, in the sequel we assume $r' \in R_1$.

There exists $r_2 \in R_2$ such that $r' \preceq_{C_2} r_2$; set $r_1 = (\rho_2')^{-1}(r_2) \in R_1'$ and recall that $r_1 \equiv_{C_2} r_2 \succeq_{C_2} r'$. Therefore, implementing the bijection $\sigma_1 = \sigma_1'$, for each $c_1 \in C_1'$ we get

$$a(r', c_1) \leqslant a\big(r', \sigma_1^{-1}(c_1)\big) \leqslant a\big(r_1, \sigma_1^{-1}(c_1)\big) = a\big(r_1, \sigma_1' \circ \sigma_1^{-1}(c_1)\big) = a\big(r_1, c_1\big),$$

which shows $r' \preceq_{C_1'} r_1$, as desired. The inductive step is completed.

Comment 1. For two minimal pairs (R_1, C_1) and (R_2, C_2), Solution 2 not only proves the required equalities $|R_1| = |R_2|$ and $|C_1| = |C_2|$, but also shows the existence of bijections (3). In simple words, this means that the four subtables $\mathcal{T}[R_1, C_1]$, $\mathcal{T}[R_1, C_2]$, $\mathcal{T}[R_2, C_1]$, and $\mathcal{T}[R_2, C_2]$ differ only by permuting rows/columns. Notice that the existence of such bijections immediately implies that (R_1, C_2) and (R_2, C_1) are also minimal pairs.

This stronger claim may also be derived directly from the arguments in Solution 1, even without the assumptions $R_1 \cap R_2 = \varnothing$ and $C_1 \cap C_2 = \varnothing$. Indeed, if $|R_1| = |R_2|$ and $|C_1| = |C_2|$, then similar arguments show that $R^n = R_1$, $C^n = C_1$, and for any $r \in R^n$ and $c \in C^n$ we have

$$a(r, c) = a\big(\rho^{nk}(r), c\big) \geqslant a\big(\rho^{nk-1}(r), \sigma(c)\big) \geqslant \ldots \geqslant a\big(r, \sigma^{nk}(c)\big) = a(r, c).$$

This yields that all above inequalities turn into equalities. Moreover, this yields that all inequalities in (2) turn into equalities. Hence ρ_1, ρ_2, σ_1, and σ_2 satisfy (3).

It is perhaps worth mentioning that one cannot necessarily find the maps in (3) so as to satisfy $\rho_1 = \rho_2^{-1}$ and $\sigma_1 = \sigma_2^{-1}$, as shown by the table below.

1	0	0	1
0	1	1	0
1	0	1	0
0	1	0	1

Comment 2. One may use the following, a bit more entertaining formulation of the same problem.

On a specialized market, a finite number of products are being sold, and there are finitely many retailers each selling all the products by some prices. Say that retailer r_1 *dominates* retailer r_2 with respect to a set of products P if r_1's price of each $p \in P$ does not exceed r_2's price of p. Similarly, product p_1 *exceeds* product p_2 with respect to a set of retailers R, if r's price of p_1 is not less than r's price of p_2, for each $r \in R$.

Say that a set R of retailers and a set P of products form a *saddle pair* if for each retailer r' there is $r \in R$ dominating r' with respect to P, and for each product p' there is $p \in P$ exceeding p' with respect to R. A saddle pair (R, P) is called a *minimal pair* if for each saddle pair (R', P') with $R' \subseteq R$ and $P' \subseteq P$, we have $R' = R$ and $P' = P$.

Prove that any two minimal pairs contain the same number of retailers.

C8. Players A and B play a game on a blackboard that initially contains 2020 copies of the number 1. In every round, player A erases two numbers x and y from the blackboard, and then player B writes one of the numbers $x+y$ and $|x-y|$ on the blackboard. The game terminates as soon as, at the end of some round, one of the following holds:

(1) one of the numbers on the blackboard is larger than the sum of all other numbers;

(2) there are only zeros on the blackboard.

Player B must then give as many cookies to player A as there are numbers on the blackboard. Player A wants to get as many cookies as possible, whereas player B wants to give as few as possible. Determine the number of cookies that A receives if both players play optimally.

(Austria)

Answer: 7.

Solution. For a positive integer n, we denote by $S_2(n)$ the sum of digits in its binary representation. We prove that, in fact, if a board initially contains an even number $n>1$ of ones, then A can guarantee to obtain $S_2(n)$, but not more, cookies. The binary representation of 2020 is $2020=\overline{11111100100}_2$, so $S_2(2020)=7$, and the answer follows.

A strategy for A. *At any round, while possible, A chooses two equal nonzero numbers on the board.* Clearly, while A can make such choice, the game does not terminate. On the other hand, A can follow this strategy unless the game has already terminated. Indeed, if A always chooses two equal numbers, then each number appearing on the board is either 0 or a power of 2 with non-negative integer exponent, this can be easily proved using induction on the number of rounds. At the moment when A is unable to follow the strategy all nonzero numbers on the board are distinct powers of 2. If the board contains at least one such power, then the largest of those powers is greater than the sum of the others. Otherwise there are only zeros on the blackboard, in both cases the game terminates.

For every number on the board, define its *range* to be the number of ones it is obtained from. We can prove by induction on the number of rounds that for any nonzero number k written by B its range is k, and for any zero written by B its range is a power of 2. Thus at the end of each round all the ranges are powers of two, and their sum is n. Since $S_2(a+b)\leqslant S_2(a)+S_2(b)$ for any positive integers a and b, the number n cannot be represented as a sum of less than $S_2(n)$ powers of 2. Thus at the end of each round the board contains at least $S_2(n)$ numbers, while A follows the above strategy. So A can guarantee at least $S_2(n)$ cookies for himself.

Comment. There are different proofs of the fact that the presented strategy guarantees at least $S_2(n)$ cookies for A. For instance, one may denote by Σ the sum of numbers on the board, and by z the number of zeros. Then the board contains at least $S_2(\Sigma)+z$ numbers; on the other hand, during the game, the number $S_2(\Sigma)+z$ does not decrease, and its initial value is $S_2(n)$. The claim follows.

A strategy for B. Denote $s=S_2(n)$.

Let $x_1,\dots,x_k$ be the numbers on the board at some moment of the game after B's turn or at the beginning of the game. Say that a collection of k signs $\varepsilon_1,\dots,\varepsilon_k\in\{+1,-1\}$ is *balanced* if

$$\sum_{i=1}^{k}\varepsilon_i x_i=0.$$

We say that a situation on the board is *good* if 2^{s+1} does not divide the number of balanced collections. An appropriate strategy for B can be explained as follows: *Perform a move so that the situation remains good, while it is possible.* We intend to show that in this case B will not lose more than $S_2(n)$ cookies. For this purpose, we prove several lemmas.

For a positive integer k, denote by $\nu_2(k)$ the exponent of the largest power of 2 that divides k. Recall that, by Legendre's formula, $\nu_2(n!)=n-S_2(n)$ for every positive integer n.

Lemma 1. The initial situation is good.

Proof. In the initial configuration, the number of balanced collections is equal to $\binom{n}{n/2}$. We have

$$\nu_2\left(\binom{n}{n/2}\right) = \nu_2(n!) - 2\nu_2\big((n/2)!\big) = \big(n - S_2(n)\big) - 2\left(\frac{n}{2} - S_2(n/2)\right) = S_2(n) = s.$$

Hence 2^{s+1} does not divide the number of balanced collections, as desired. □

Lemma 2. B may play so that after each round the situation remains good.

Proof. Assume that the situation $(x_1, \ldots, x_k)$ before a round is good, and that A erases two numbers, x_p and x_q.

Let N be the number of all balanced collections, N_+ be the number of those having $\varepsilon_p = \varepsilon_q$, and N_- be the number of other balanced collections. Then $N = N_+ + N_-$. Now, if B replaces x_p and x_q by $x_p + x_q$, then the number of balanced collections will become N_+. If B replaces x_p and x_q by $|x_p - x_q|$, then this number will become N_-. Since 2^{s+1} does not divide N, it does not divide one of the summands N_+ and N_-, hence B can reach a good situation after the round. □

Lemma 3. Assume that the game terminates at a good situation. Then the board contains at most s numbers.

Proof. Suppose, one of the numbers is greater than the sum of the other numbers. Then the number of balanced collections is 0 and hence divisible by 2^{s+1}. Therefore, the situation is not good.

Then we have only zeros on the blackboard at the moment when the game terminates. If there are k of them, then the number of balanced collections is 2^k. Since the situation is good, we have $k \leqslant s$. □

By Lemmas 1 and 2, B may act in such way that they keep the situation good. By Lemma 3, when the game terminates, the board contains at most s numbers. This is what we aimed to prove.

Comment 1. If the initial situation had some odd number $n > 1$ of ones on the blackboard, player A would still get $S_2(n)$ cookies, provided that both players act optimally. The proof of this fact is similar to the solution above, after one makes some changes in the definitions. Such changes are listed below.

Say that a collection of k signs $\varepsilon_1, \ldots, \varepsilon_k \in \{+1, -1\}$ is *positive* if

$$\sum_{i=1}^{k} \varepsilon_i x_i > 0.$$

For every index $i = 1, 2, \ldots, k$, we denote by N_i the number of positive collections such that $\varepsilon_i = 1$. Finally, say that a situation on the board is *good* if 2^{s-1} does not divide at least one of the numbers N_i. Now, a strategy for B again consists in preserving the situation good, after each round.

Comment 2. There is an easier strategy for B, allowing, in the game starting with an even number n of ones, to lose no more than $d = \lfloor \log_2(n+2) \rfloor - 1$ cookies. If the binary representation of n contains at most two zeros, then $d = S_2(n)$, and hence the strategy is optimal in that case. We outline this approach below.

First of all, we can assume that A never erases zeros from the blackboard. Indeed, A may skip such moves harmlessly, ignoring the zeros in the further process; this way, A's win will just increase.

We say that a situation on the blackboard is *pretty* if the numbers on the board can be partitioned into two groups with equal sums. Clearly, if the situation before some round is pretty, then B may play so as to preserve this property after the round. The strategy for B is as follows:

- B always chooses a move that leads to a pretty situation.

• If both possible moves of B lead to pretty situations, then B writes the sum of the two numbers erased by A.

Since the situation always remains pretty, the game terminates when all numbers on the board are zeros.

Suppose that, at the end of the game, there are $m \geqslant d+1 = \lfloor \log_2(n+2) \rfloor$ zeros on the board; then $2^m - 1 > n/2$.

Now we analyze the whole process of the play. Let us number the zeros on the board in order of appearance. During the play, each zero had appeared after subtracting two equal numbers. Let $n_1, \ldots, n_m$ be positive integers such that the first zero appeared after subtracting n_1 from n_1, the second zero appeared after subtracting n_2 from n_2, and so on. Since the sum of the numbers on the blackboard never increases, we have $2n_1 + \ldots + 2n_m \leqslant n$, hence

$$n_1 + \ldots + n_m \leqslant n/2 < 2^m - 1.$$

There are 2^m subsets of the set $\{1, 2, \ldots, m\}$. For $I \subseteq \{1, 2, \ldots, m\}$, denote by $f(I)$ the sum $\sum_{i \in I} n_i$. There are less than 2^m possible values for $f(I)$, so there are two distinct subsets I and J with $f(I) = f(J)$. Replacing I and J with $I \setminus J$ and $J \setminus I$, we assume that I and J are disjoint.

Let i_0 be the smallest number in $I \cup J$; without loss of generality, $i_0 \in I$. Consider the round when A had erased two numbers equal to n_{i_0}, and B had put the ${i_0}^{\text{th}}$ zero instead, and the situation *before* that round.

For each nonzero number z which is on the blackboard at this moment, we can keep track of it during the further play until it enters one of the numbers n_i, $i \geqslant i_0$, which then turn into zeros. For every $i = i_0, i_0+1, \ldots, m$, we denote by X_i the collection of all numbers on the blackboard that finally enter the first copy of n_i, and by Y_i the collection of those finally entering the second copy of n_i. Thus, each of X_{i_0} and Y_{i_0} consists of a single number. Since A never erases zeros, the $2(m - i_0 + 1)$ defined sets are pairwise disjoint.

Clearly, for either of the collections X_i and Y_i, a signed sum of its elements equals n_i, for a proper choice of the signs. Therefore, for every $i = i_0, i_0 + 1, \ldots, m$ one can endow numbers in $X_i \cup Y_i$ with signs so that their sum becomes any of the numbers $-2n_i$, 0, or $2n_i$. Perform such endowment so as to get $2n_i$ from each collection $X_i \cup Y_i$ with $i \in I$, $-2n_j$ from each collection $X_j \cup Y_j$ with $j \in J$, and 0 from each remaining collection. The obtained signed sum of all numbers on the blackboard equals

$$\sum_{i \in I} 2n_i - \sum_{i \in J} 2n_i = 0,$$

and the numbers in X_{i_0} and Y_{i_0} have the same (positive) sign.

This means that, at this round, B could add up the two numbers n_{i_0} to get a pretty situation. According to the strategy, B should have performed that, instead of subtracting the numbers. This contradiction shows that $m \leqslant d$, as desired.

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Geometry

G1. Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside side AB such that $AD < DB$. Let P and Q be two points inside sides BC and CA, respectively, such that $\angle DPB = \angle DQA = 90°$. Let the perpendicular bisector of PQ meet line segment CQ at E, and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C.

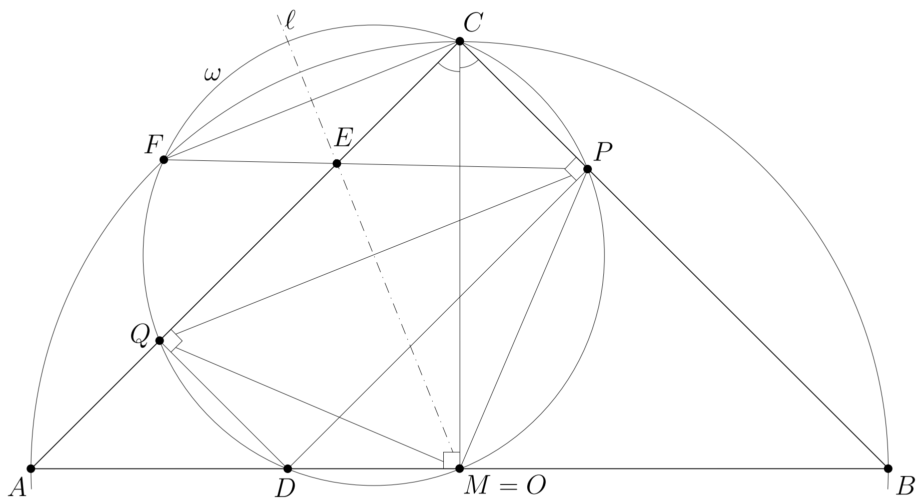
Suppose that P, E, F are collinear. Prove that $\angle ACB = 90°$.

(Luxembourg)

Solution 1. Let ℓ be the perpendicular bisector of PQ, and denote by ω the circle $CFPQ$. By $DP \perp BC$ and $DQ \perp AC$, the circle ω passes through D; moreover, CD is a diameter of ω.

The lines QE and PE are symmetric about ℓ, and ℓ is a symmetry axis of ω as well; it follows that the chords CQ and FP are symmetric about ℓ, hence C and F are symmetric about ℓ. Therefore, the perpendicular bisector of CF coincides with ℓ. Thus ℓ passes through the circumcenter O of ABC.

Let M be the midpoint of AB. Since $CM \perp DM$, M also lies on ω. By $\angle ACM = \angle BCM$, the chords MP and MQ of ω are equal. Then, from $MP = MQ$ it follows that ℓ passes through M.



Finally, both O and M lie on lines ℓ and CM, therefore $O = M$, and $\angle ACB = 90°$ follows.

Solution 2. Like in the first solution, we conclude that points C, P, Q, D, F and the midpoint M of AB lie on one circle ω with diameter CD, and M lies on ℓ, the perpendicular bisector of PQ.

Let BF and CM meet at G and let $\alpha = \angle ABF$. Then, since E lies on ℓ, and the quadrilaterals $FCBA$ and $FCPQ$ are cyclic, we have

$$\angle CQP = \angle FPQ = \angle FCQ = \angle FCA = \angle FBA = \alpha.$$

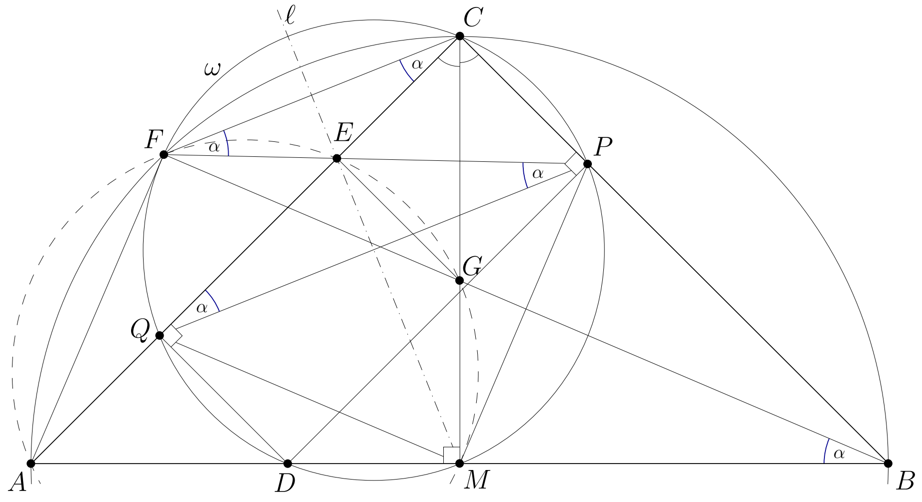
Since points P, E, F are collinear, we have

$$\angle FEM = \angle FEQ + \angle QEM = 2\alpha + (90° - \alpha) = 90° + \alpha.$$

But $\angle FGM = 90° + \alpha$, so $FEGM$ is cyclic. Hence

$$\angle EGC = \angle EFM = \angle PFM = \angle PCM.$$

Thus $GE \parallel BC$. It follows that $\angle FAC = \angle CBF = \angle EGF$, so $FEGA$ is cyclic, too. Hence $\angle ACB = \angle AFB = \angle AFG = 180° - \angle AMG = 90°$, that completes the proof.



Comment 1. The converse statement is true: if $\angle ACB = 90^\circ$ then points P, E and F are collinear. This direction is easier to prove.

Comment 2. The statement of the problem remains true if the projection P of D onto BC lies outside line segment BC. The restriction that P lies inside line segment BC is given to reduce case-sensitivity.

G2. Let $ABCD$ be a convex quadrilateral. Suppose that P is a point in the interior of $ABCD$ such that

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

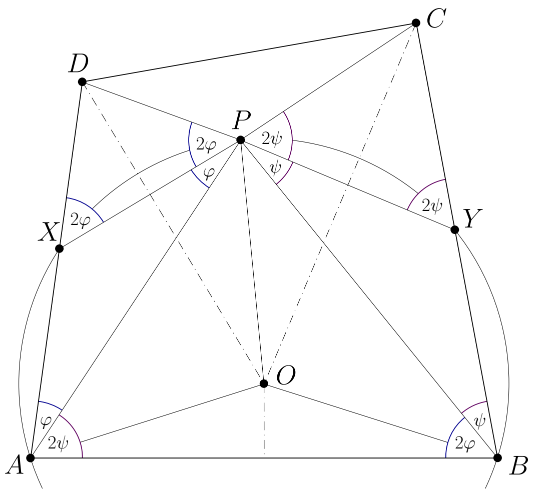
The internal bisectors of angles ADP and PCB meet at a point Q inside the triangle ABP. Prove that $AQ = BQ$.

(Poland)

Solution 1. Let $\varphi = \angle PAD$ and $\psi = \angle CBP$; then we have $\angle PBA = 2\varphi$, $\angle DPA = 3\varphi$, $\angle BAP = 2\psi$ and $\angle BPC = 3\psi$. Let X be the point on segment AD with $\angle XPA = \varphi$. Then

$$\angle PXD = \angle PAX + \angle XPA = 2\varphi = \angle DPA - \angle XPA = \angle DPX.$$

It follows that triangle DPX is isosceles with $DX = DP$ and therefore the internal angle bisector of $\angle ADP$ coincides with the perpendicular bisector of XP. Similarly, if Y is a point on BC such that $\angle BPY = \psi$, then the internal angle bisector of $\angle PCB$ coincides with the perpendicular bisector of PY. Hence, we have to prove that the perpendicular bisectors of XP, PY, and AB are concurrent.



Notice that

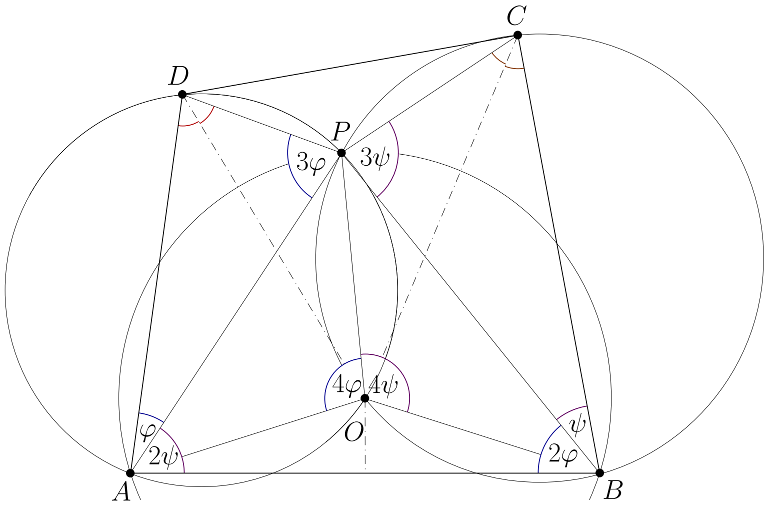
$$\angle AXP = 180^\circ - \angle PXD = 180^\circ - 2\varphi = 180^\circ - \angle PBA.$$

Hence the quadrilateral $AXPB$ is cyclic; in other words, X lies on the circumcircle of triangle APB. Similarly, Y lies on the circumcircle of triangle APB. It follows that the perpendicular bisectors of XP, PY, and AB all pass through the center of circle $(ABYPX)$. This finishes the proof.

Comment. Introduction of points X and Y seems to be the key step in the solution above. Note that the same point X could be introduced in different ways, e.g., as the point on the ray CP beyond P such that $\angle PBX = \varphi$, or as a point where the circle (APB) meets again AB. Different definitions of X could lead to different versions of the further solution.

Solution 2. We define the angles $\varphi = \angle PAD$, $\psi = \angle CBP$ and use $\angle PBA = 2\varphi$, $\angle DPA = 3\varphi$, $\angle BAP = 2\psi$ and $\angle BPC = 3\psi$ again. Let O be the circumcenter of $\triangle APB$.

Notice that $\angle ADP = 180^\circ - \angle PAD - \angle DPA = 180^\circ - 4\varphi$, which, in particular, means that $4\varphi < 180^\circ$. Further, $\angle POA = 2\angle PBA = 4\varphi = 180^\circ - \angle ADP$, therefore the quadrilateral $ADPO$ is cyclic. By $AO = OP$, it follows that $\angle ADO = \angle ODP$. Thus DO is the internal bisector of $\angle ADP$. Similarly, CO is the internal bisector of $\angle PCB$.



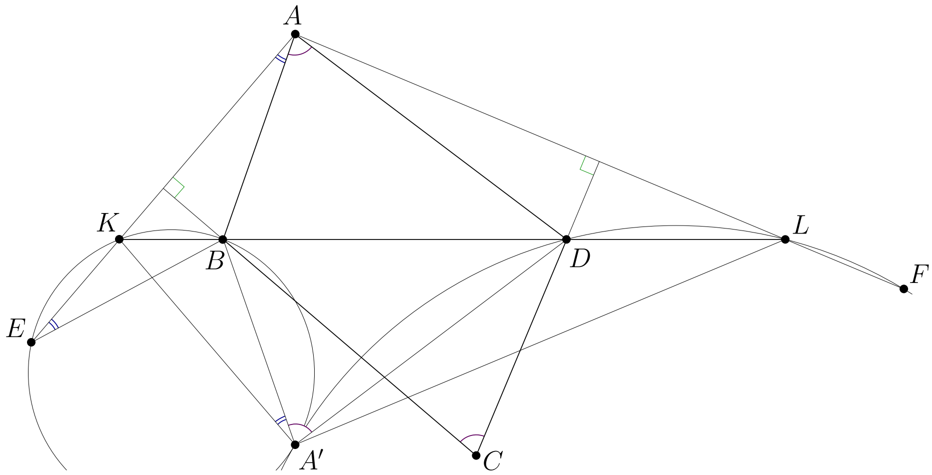
Finally, O lies on the perpendicular bisector of AB as it is the circumcenter of $\triangle APB$. Therefore the three given lines in the problem statement concur at point O.

G3. Let $ABCD$ be a convex quadrilateral with $\angle ABC > 90^\circ$, $\angle CDA > 90^\circ$, and $\angle DAB = \angle BCD$. Denote by E and F the reflections of A in lines BC and CD, respectively. Suppose that the segments AE and AF meet the line BD at K and L, respectively. Prove that the circumcircles of triangles BEK and DFL are tangent to each other.

(Slovakia)

Solution 1. Denote by A' the reflection of A in BD. We will show that that the quadrilaterals $A'BKE$ and $A'DLF$ are cyclic, and their circumcircles are tangent to each other at point A'.

From the symmetry about line BC we have $\angle BEK = \angle BAK$, while from the symmetry in BD we have $\angle BAK = \angle BA'K$. Hence $\angle BEK = \angle BA'K$, which implies that the quadrilateral $A'BKE$ is cyclic. Similarly, the quadrilateral $A'DLF$ is also cyclic.



For showing that circles $A'BKE$ and $A'DLF$ are tangent it suffices to prove that

$$\angle A'KB + \angle A'LD = \angle BA'D.$$

Indeed, by $AK \perp BC$, $AL \perp CD$, and again the symmetry in BD we have

$$\angle A'KB + \angle A'LD = 180^\circ - \angle KA'L = 180^\circ - \angle KAL = \angle BCD = \angle BAD = \angle BA'D,$$

as required.

Comment 1. The key to the solution above is introducing the point A'; then the angle calculations can be done in many different ways.

Solution 2. Note that $\angle KAL = 180^\circ - \angle BCD$, since AK and AL are perpendicular to BC and CD, respectively. Reflect both circles (BEK) and (DFL) in BD. Since $\angle KEB = \angle KAB$ and $\angle DFL = \angle DAL$, the images are the circles (KAB) and (LAD), respectively; so they meet at A. We need to prove that those two reflections are tangent at A.

For this purpose, we observe that

$$\angle AKB + \angle ALD = 180^\circ - \angle KAL = \angle BCD = \angle BAD.$$

Thus, there exists a ray AP inside angle $\angle BAD$ such that $\angle BAP = \angle AKB$ and $\angle DAP = \angle DLA$. Hence the line AP is a common tangent to the circles (KAB) and (LAD), as desired.

Comment 2. The statement of the problem remains true for a more general configuration, e.g., if line BD intersect the extension of segment AE instead of the segment itself, etc. The corresponding restrictions in the statement are given to reduce case sensitivity.

G4. In the plane, there are $n \geqslant 6$ pairwise disjoint disks $D_1, D_2, \ldots, D_n$ with radii $R_1 \geqslant R_2 \geqslant \ldots \geqslant R_n$. For every $i = 1, 2, \ldots, n$, a point P_i is chosen in disk D_i. Let O be an arbitrary point in the plane. Prove that

$$OP_1 + OP_2 + \ldots + OP_n \geqslant R_6 + R_7 + \ldots + R_n.$$

(A disk is assumed to contain its boundary.)

(Iran)

Solution. We will make use of the following lemma.

Lemma. Let $D_1, \ldots, D_6$ be disjoint disks in the plane with radii $R_1, \ldots, R_6$. Let P_i be a point in D_i, and let O be an arbitrary point. Then there exist indices i and j such that $OP_i \geqslant R_j$.

Proof. Let O_i be the center of D_i. Consider six rays $OO_1, \ldots, OO_6$ (if $O = O_i$, then the ray OO_i may be assumed to have an arbitrary direction). These rays partition the plane into six angles (one of which may be non-convex) whose measures sum up to $360°$; hence one of the angles, say $\angle O_iOO_j$, has measure at most $60°$. Then O_iO_j cannot be the unique largest side in (possibly degenerate) triangle OO_iO_j, so, without loss of generality, $OO_i \geqslant O_iO_j \geqslant R_i + R_j$. Therefore, $OP_i \geqslant OO_i - R_i \geqslant (R_i + R_j) - R_i = R_j$, as desired. □

Now we prove the required inequality by induction on $n \geqslant 5$. The base case $n = 5$ is trivial. For the inductive step, apply the Lemma to the six largest disks, in order to find indices i and j such that $1 \leqslant i, j \leqslant 6$ and $OP_i \geqslant R_j \geqslant R_6$. Removing D_i from the configuration and applying the inductive hypothesis, we get

$$\sum_{k \neq i} OP_k \geqslant \sum_{\ell \geqslant 7} R_\ell.$$

Adding up this inequality with $OP_i \geqslant R_6$ we establish the inductive step.

Comment 1. It is irrelevant to the problem whether the disks contain their boundaries or not. This condition is included for clarity reasons only. The problem statement remains true, and the solution works verbatim, if the disks are assumed to have disjoint interiors.

Comment 2. There are several variations of the above solution. In particular, while performing the inductive step, one may remove the disk with the largest value of OP_i and apply the inductive hypothesis to the remaining disks (the Lemma should still be applied to the six largest disks).

Comment 3. While proving the Lemma, one may reduce it to a particular case when the disks are congruent, as follows: Choose the smallest radius r of the disks in the Lemma statement, and then replace, for each i, the i^{th} disk with its homothetic copy, using the homothety centered at P_i with ratio r/R_i.

This argument shows that the Lemma is tightly connected to a circle packing problem, see, e.g., https://en.wikipedia.org/wiki/Circle_packing_in_a_circle. The known results on that problem provide versions of the Lemma for different numbers of disks, which lead to different inequalities of the same kind. E.g., for 4 disks the best possible estimate in the Lemma is $OP_i \geqslant (\sqrt{2} - 1)R_j$, while for 13 disks it has the form $OP_i \geqslant \sqrt{5}\, R_j$. Arguing as in the above solution, one obtains the inequalities

$$\sum_{i=1}^{n} OP_i \geqslant (\sqrt{2} - 1) \sum_{j=4}^{n} R_j \quad \text{and} \quad \sum_{i=1}^{n} OP_i \geqslant \sqrt{5} \sum_{j=13}^{n} R_j.$$

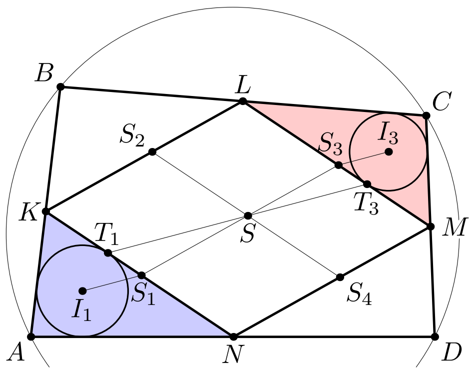
However, there are some harder arguments which allow to improve these inequalities, meaning that the R_j with large indices may be taken with much greater factors.

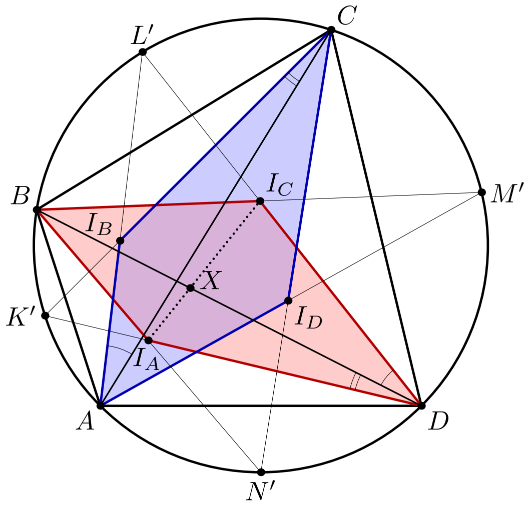
G5. Let $ABCD$ be a cyclic quadrilateral with no two sides parallel. Let K, L, M, and N be points lying on sides AB, BC, CD, and DA, respectively, such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let ω_1, ω_2, ω_3, and ω_4 be the incircles of triangles ANK, BKL, CLM, and DMN, respectively. Prove that the internal common tangents to ω_1 and ω_3 and the internal common tangents to ω_2 and ω_4 are concurrent.

(Poland)

Solution 1. Let I_i be the center of ω_i, and let r_i be its radius for $i = 1, 2, 3, 4$. Denote by T_1 and T_3 the points of tangency of ω_1 and ω_3 with NK and LM, respectively. Suppose that the internal common tangents to ω_1 and ω_3 meet at point S, which is the center of homothety h with negative ratio (namely, with ratio $-\frac{r_3}{r_1}$) mapping ω_1 to ω_3. This homothety takes T_1 to T_3 (since the tangents to ω_1 and ω_3 at T_1 to T_3 are parallel), hence S is a point on the segment T_1T_3 with $T_1S : ST_3 = r_1 : r_3$.

Construct segments $S_1S_3 \parallel KL$ and $S_2S_4 \parallel LM$ through S with $S_1 \in NK$, $S_2 \in KL$, $S_3 \in LM$, and $S_4 \in MN$. Note that h takes S_1 to S_3, hence $I_1S_1 \parallel I_3S_3$, and $S_1S : SS_3 = r_1 : r_3$. We will prove that $S_2S : SS_4 = r_2 : r_4$ or, equivalently, $KS_1 : S_1N = r_2 : r_4$. This will yield the problem statement; indeed, applying similar arguments to the intersection point S' of the internal common tangents to ω_2 and ω_4, we see that S' satisfies similar relations, and there is a unique point inside $KLMN$ satisfying them. Therefore, $S' = S$.





Further, denote by I_A, I_B, I_C, I_D and r_A, r_B, r_C, r_D the incenters and inradii of triangles DAB, ABC, BCD, and CDA, respectively. One can shift triangle CLM by $\overrightarrow{LK}$ to glue it with triangle AKN into a quadrilateral $AKC'N$ similar to $ABCD$. In particular, this shows that $r_1 : r_3 = r_A : r_C$; similarly, $r_2 : r_4 = r_B : r_D$. Moreover, the same shift takes S_3 to S_1, and it also takes I_3 to the incenter I_3' of triangle $KC'N$. Since $I_1S_1 \parallel I_3S_3$, the points I_1, S_1, I_3' are collinear. Thus, in order to complete the solution, it suffices to apply the following Lemma to quadrilateral $AKC'N$.

Lemma 1. Let $ABCD$ be a cyclic quadrilateral, and define I_A, I_C, r_B, and r_D as above. Let I_AI_C meet BD at X; then $BX : XD = r_B : r_D$.

Proof. Consider an inversion centered at X; the images under that inversion will be denoted by primes, e.g., A' is the image of A.

By properties of inversion, we have

$$\angle I_C'I_A'D' = \angle XI_A'D' = \angle XDI_A = \angle BDA/2 = \angle BCA/2 = \angle ACI_B.$$

We obtain $\angle I_A'I_C'D' = \angle CAI_B$ likewise; therefore, $\triangle I_C'I_A'D' \sim \triangle ACI_B$. In the same manner, we get $\triangle I_C'I_A'B' \sim \triangle ACI_D$, hence the quadrilaterals $I_C'B'I_A'D'$ and AI_DCI_B are also similar. But the diagonals AC and I_BI_D of quadrilateral AI_DCI_B meet at a point Y such that I_BY :

$YI_D = r_B : r_D$. By similarity, we get $D'X : B'X = r_B : r_D$ and hence $BX : XD = D'X : B'X = r_B : r_D$. □

Comment 1. The solution above shows that the problem statement holds also for any *parallelogram* $KLMN$ whose sides are parallel to the diagonals of $ABCD$, as no property specific for a rhombus has been used. This solution works equally well when two sides of quadrilateral $ABCD$ are parallel.

Comment 2. The problem may be reduced to Lemma 1 by using different tools, e.g., by using mass point geometry, linear motion of K, L, M, and N, etc.

Lemma 1 itself also can be proved in different ways. We present below one alternative proof.

Proof. In the circumcircle of $ABCD$, let K', L'. M', and N' be the midpoints of arcs AB, BC, CD, and DA containing no other vertices of $ABCD$, respectively. Thus, $K' = CI_B \cap DI_A$, etc. In the computations below, we denote by $[P]$ the area of a polygon P. We use similarities $\triangle I_ABK' \sim \triangle I_ADN'$, $\triangle I_BK'L' \sim \triangle I_BAC$, etc., as well as congruences $\triangle I_BK'L' = \triangle BK'L'$ and $\triangle I_DM'N' = \triangle DM'N'$ (e.g., the first congruence holds because $K'L'$ is a common internal bisector of angles $BK'I_B$ and $BL'I_B$).

We have

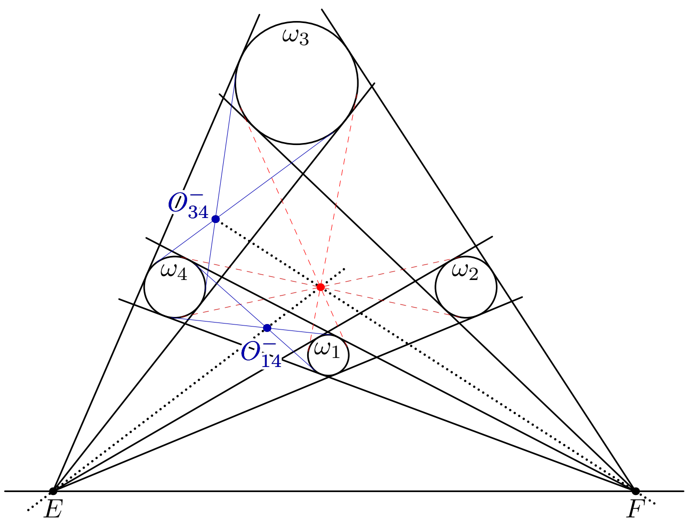
$$\begin{aligned}\frac{BX}{DX} = \frac{[I_ABI_C]}{[I_ADI_C]} = \frac{BI_A \cdot BI_C \cdot \sin I_ABI_C}{DI_A \cdot DI_C \cdot \sin I_ADI_C} = \frac{BI_A}{DI_A} \cdot \frac{BI_C}{DI_C} \cdot \frac{\sin N'BM'}{\sin K'DL'} \\ = \frac{BK'}{DN'} \cdot \frac{BL'}{DM'} \cdot \frac{\sin N'DM'}{\sin K'BL'} = \frac{BK' \cdot BL' \cdot \sin K'BL'}{DN' \cdot DM' \cdot \sin N'DM'} \cdot \frac{\sin^2 N'DM'}{\sin^2 K'BL'} \\ = \frac{[K'BL']}{[M'DN']} \cdot \frac{N'M'^2}{K'L'^2} = \frac{[K'I_BL'] \cdot \frac{A'C'^2}{K'L'^2}}{[M'I_DN'] \cdot \frac{A'C'^2}{N'M'^2}} = \frac{[AI_BC]}{[AI_DC]} = \frac{r_B}{r_D},\end{aligned}$$

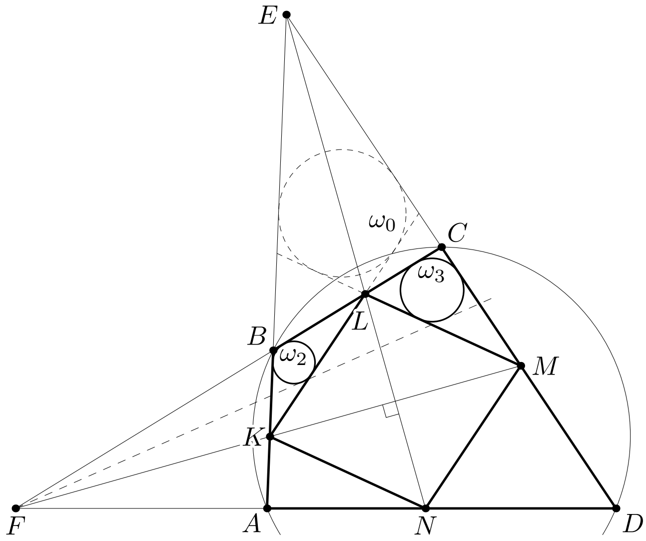
as required. □

Solution 2. This solution is based on the following general Lemma.

Lemma 2. Let E and F be distinct points, and let ω_i, $i = 1, 2, 3, 4$, be circles lying in the same halfplane with respect to EF. For distinct indices $i, j \in \{1, 2, 3, 4\}$, denote by O^+_{ij} (respectively, O^-_{ij}) the center of homothety with positive (respectively, negative) ratio taking ω_i to ω_j. Suppose that $E = O^+_{12} = O^+_{34}$ and $F = O^+_{23} = O^+_{41}$. Then $O^-_{13} = O^-_{24}$.

Proof. Applying Monge's theorem to triples of circles $\omega_1, \omega_2, \omega_4$ and $\omega_1, \omega_3, \omega_4$, we get that both points O^-_{24} and O^-_{13} lie on line EO^-_{14}. Notice that this line is distinct from EF. Similarly we obtain that both points O^-_{24} and O^-_{13} lie on FO^-_{34}. Since the lines EO^-_{14} and FO^-_{34} are distinct, both points coincide with the meeting point of those lines. □





Turning back to the problem, let AB intersect CD at E and let BC intersect DA at F. Assume, without loss of generality, that B lies on segments AE and CF. We will show that the points E and F, and the circles ω_i satisfy the conditions of Lemma 2, so the problem statement follows. In the sequel, we use the notation of $O_{ij}^{\pm}$ from the statement of Lemma 2, applied to circles ω_1, ω_2, ω_3, and ω_4.

Using the relations $\triangle ECA \sim \triangle EBD$, $KN \parallel BD$, and $MN \parallel AC$. we get

$$\frac{AN}{ND} = \frac{AN}{AD} \cdot \frac{AD}{ND} = \frac{KN}{BD} \cdot \frac{AC}{NM} = \frac{AC}{BD} = \frac{AE}{ED}.$$

Therefore, by the angle bisector theorem, point N lies on the internal angle bisector of $\angle AED$. We prove similarly that L also lies on that bisector, and that the points K and M lie on the internal angle bisector of $\angle AFB$.

Since $KLMN$ is a rhombus, points K and M are symmetric in line ELN. Hence, the convex quadrilateral determined by the lines EK, EM, KL, and ML is a kite, and therefore it has an incircle ω_0. Applying Monge's theorem to ω_0, ω_2, and ω_3, we get that O_{23}^{+} lies on KM. On the other hand, O_{23}^{+} lies on BC, as BC is an external common tangent to ω_2 and ω_3. It follows that $F = O_{23}^{+}$. Similarly, $E = O_{12}^{+} = O_{34}^{+}$, and $F = O_{41}^{+}$.

Comment 3. The reduction to Lemma 2 and the proof of Lemma 2 can be performed with the use of different tools, e.g., by means of Menelaus theorem, by projecting harmonic quadruples, by applying Monge's theorem in some other ways, etc.

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G6. Let I and I_A be the incenter and the A-excenter of an acute-angled triangle ABC with $AB < AC$. Let the incircle meet BC at D. The line AD meets BI_A and CI_A at E and F, respectively. Prove that the circumcircles of triangles AID and I_AEF are tangent to each other.

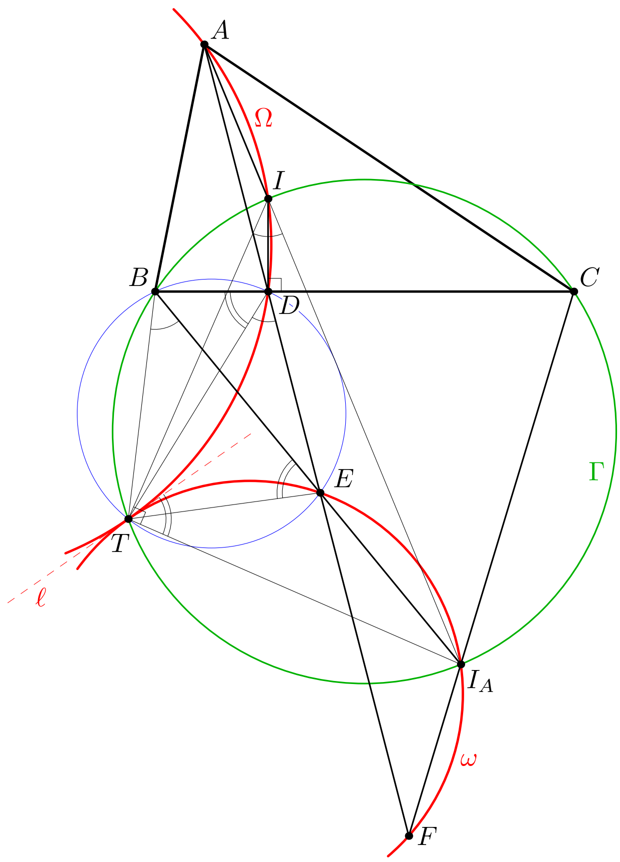
(Slovakia)

Solution 1. Let $\measuredangle(p, q)$ denote the directed angle between lines p and q.

The points B, C, I, and I_A lie on the circle Γ with diameter II_A. Let ω and Ω denote the circles (I_AEF) and (AID), respectively. Let T be the second intersection point of ω and Γ. Then T is the Miquel point of the complete quadrilateral formed by the lines BC, BI_A, CI_A, and DEF, so T also lies on circle (BDE) (as well as on circle (CDF)). We claim that T is a desired tangency point of ω and Ω.

In order to show that T lies on Ω, use cyclic quadrilaterals $BDET$ and BII_AT to write

$$\measuredangle(DT, DA) = \measuredangle(DT, DE) = \measuredangle(BT, BE) = \measuredangle(BT, BI_A) = \measuredangle(IT, II_A) = \measuredangle(IT, IA).$$



To show that ω and Ω are tangent at T, let ℓ be the tangent to ω at T, so that $\measuredangle(TI_A, \ell) = \measuredangle(EI_A, ET)$. Using circles $(BDET)$ and $(BICI_A)$, we get

$$\measuredangle(EI_A, ET) = \measuredangle(EB, ET) = \measuredangle(DB, DT).$$

Therefore,

$$\measuredangle(TI, \ell) = 90^\circ + \measuredangle(TI_A, \ell) = 90^\circ + \measuredangle(DB, DT) = \measuredangle(DI, DT),$$

which shows that ℓ is tangent to Ω at T.

Solution 2. We use the notation of circles Γ, ω, and Ω as in the previous solution.

Let L be the point opposite to I in circle Ω. Then $\angle IAL = \angle IDL = 90^\circ$, which means that L is the foot of the external bisector of $\angle A$ in triangle ABC. Let LI cross Γ again at M.

Let T be the foot of the perpendicular from I onto I_AL. Then T is the second intersection point of Γ and Γ. We will show that T is the desired tangency point.

First, we show that T lies on circle ω. Notice that

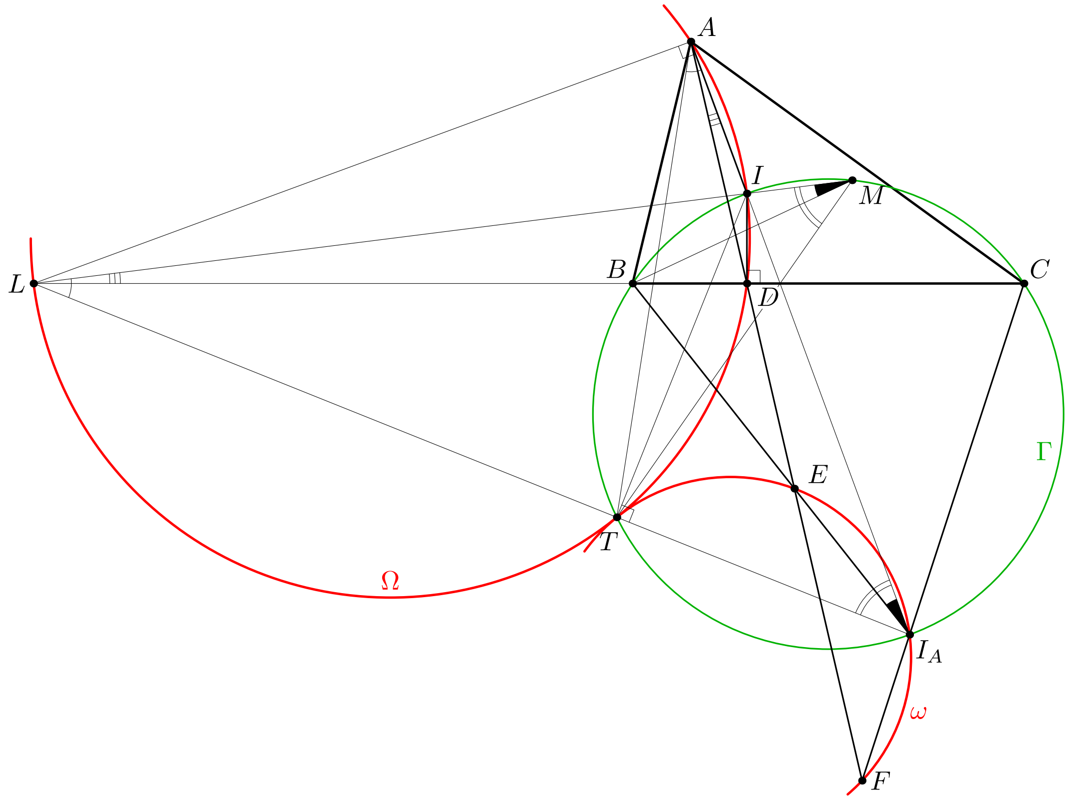
$$\measuredangle(LT, LM) = \measuredangle(AT, AI) \quad \text{and} \quad \measuredangle(MT, ML) = \measuredangle(MT, MI) = \measuredangle(I_AT, I_AI),$$

which shows that triangles TML and TI_AA are similar and equioriented. So there exists a rotational homothety τ mapping TML to TI_AA.

Since $\measuredangle(ML, LD) = \measuredangle(AI, AD)$, we get $\tau(BC) = AD$. Next, since

$$\measuredangle(MB, ML) = \measuredangle(MB, MI) = \measuredangle(I_AB, I_AI) = \measuredangle(I_AE, I_AA),$$

we get $\tau(B) = E$. Similarly, $\tau(C) = F$. Since the points M, B, C, and T are concyclic, so are their τ-images, which means that T lies on $\omega = \tau(\Gamma)$.



Finally, since $\tau(L) = A$ and $\tau(B) = E$, triangles ATL and ETB are similar so that

$$\measuredangle(AT, AL) = \measuredangle(ET, EB) = \measuredangle(EI_A, ET).$$

This means that the tangents to Ω and ω at T make the same angle with the line I_ATL, so the circles are indeed tangent at T.

Comment. In both solutions above, a crucial step is a guess that the desired tangency point lies on Γ. There are several ways to recognize this helpful property.

E.g. one may perform some angle chasing to see that the tangents to Ω at L and to ω at I_A are parallel (and the circles lie on different sides of the tangents). This yields that, under the assumption that the circles are tangent externally, the tangency point must lie on I_AL. Since IL is a diameter in Ω, this, in turn, implies that T is the projection of I onto I_AL.

Another way to see the same fact is to perform a homothety centered at A and mapping I to I_A (and D to some point D'). The image Ω' of Ω is tangent to ω at I_A, because $\angle BI_AA + \angle CI_AD' = 180^\circ$. This yields that the tangents to Ω at I and to ω at I_A are parallel.

There are other ways to describe the tangency point. The next solution presents one of them.

Solution 3. We also use the notation of circles ω, and Ω from the previous solutions.

Perform an inversion centered at D. The images of the points will be denoted by primes, e.g., A' is the image of A.

For convenience, we use the notation $\angle BID = \beta$, $\angle CID = \gamma$, and $\alpha = 180^\circ - \beta - \gamma = 90^\circ - \angle BAI$. We start with computing angles appearing after inversion. We get

$$\angle DB'I' = \beta, \quad \angle DC'I' = \gamma, \quad \text{and hence} \quad \angle B'I'C' = \alpha;$$
$$\angle E'I'_AF' = \angle E'I'_AD - \angle F'I'_AD = \angle I_AED - \angle I_AFD = \angle EI_AF = 180^\circ - \alpha.$$

Next, we have

$$\angle A'E'B' = \angle DE'B' = \angle DBE = \beta = 90^\circ - \frac{\angle DBA}{2} = 90^\circ - \frac{\angle E'A'B'}{2},$$

which yields that triangle $A'B'E'$ is isosceles with $A'B' = A'E'$. Similarly, $A'F' = A'C'$.

Finally, we get

$$\begin{aligned}\angle A'B'I' = \angle I'B'D - \angle A'B'D = \beta - \angle BAD = \beta - (90^\circ - \alpha) + \angle IAD \\ = \angle ICD + \angle IAD = \angle C'I'D + \angle A'I'D = \angle C'I'A';\end{aligned}$$

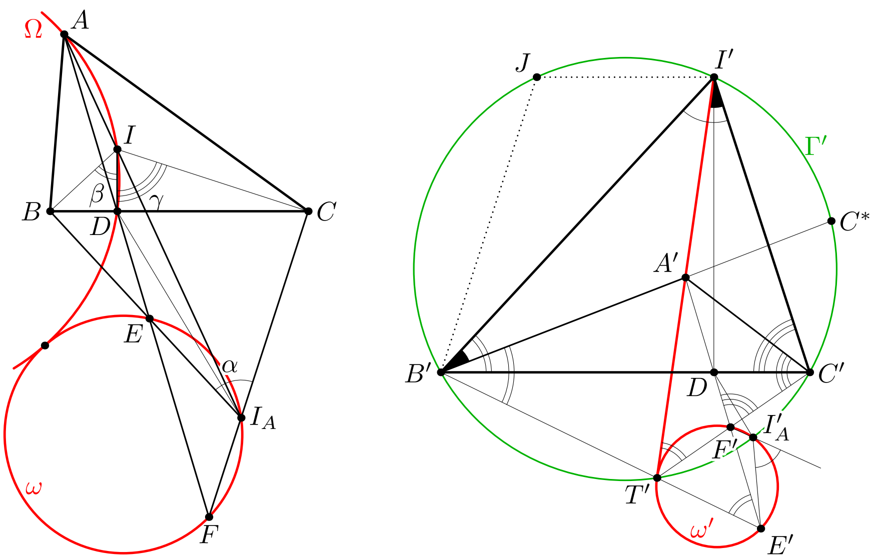
similarly, $\angle A'C'I' = \angle A'I'B'$, so that triangles $A'B'I'$ and $A'I'C'$ are similar. Therefore, $A'I'^2 = A'B' \cdot A'C'$.

Recall that we need to prove the tangency of line $A'I' = \Omega'$ with circle $(E'F'I'_A) = \omega'$. A desired tangency point T' must satisfy $A'T'^2 = A'E' \cdot A'F'$; the relations obtained above yield

$$A'E' \cdot A'F' = A'B' \cdot A'C' = A'I'^2,$$

so that T' should be symmetric to I' with respect to A'.

Thus, let us *define* a point T' as the reflection of I' in A', and show that T' lies on circle Ω'; the equalities above will then imply that $A'T'$ is tangent to Ω', as desired.



The property that triangles $B'A'I'$ and $I'A'C'$ are similar means that quadrilateral $B'I'C'T'$ is harmonic. Indeed, let C^* be the reflection of C' in the perpendicular bisector of $I'T'$; then C^* lies on $B'A'$ by $\angle B'A'I' = \angle A'I'C' = \angle T'I'C^*$, and then C^* lies on circle $(I'B'T')$ since $A'B' \cdot A'C^* = A'B' \cdot A'C' = A'I'^2 = A'I' \cdot A'T'$. Therefore, C' also lies on that circle (and the circle is $(B'I'C') = \Gamma'$). Moreover, $B'C^*$ is a median in triangle $B'I'T'$, so $B'C'$ is its symmedian, which establishes harmonicity.

Now we have $\angle A'B'T' = \angle I'B'C' = \beta = \angle A'B'E'$; which shows that E' lies on $B'T'$. Similarly, F' lies on $C'T'$. Hence, $\angle E'T'F' = \angle B'I'C' = 180^\circ - \angle E'I'_AF'$, which establishes $T' \in \omega'$.

Comment 2. The solution above could be finished without use of harmonicity. E.g., one may notice that both triangles $A'T'F'$ and $A'E'T'$ are similar to triangle $B'I'J$, where J is the point symmetric to I' in the perpendicular bisector of $B'C'$; indeed, we have $\angle T'A'E' = \gamma - \beta = \angle I'B'J'$ and $\frac{B'I'}{B'J'} = \frac{B'I'}{C'I'} = \frac{B'A'}{A'I'} = \frac{A'E'}{A'T'}$. This also allows to compute $\angle E'T'F' = \angle E'T'A' - \angle F'T'A' = \angle I'JB' - \angle JI'B' = \alpha$.

Comment 3. Here we list several properties of the configuration in the problem, which can be derived from the solutions above.

The quadrilateral $IBTC$ (as well as $I'B'T'C'$) is harmonic. Hence, line IT contains the meeting point of tangents to Γ at B and C, i.e., the midpoint N of arc BAC in the circumcircle of $\triangle ABC$.

G7. Let P be a point on the circumcircle of an acute-angled triangle ABC. Let D, E, and F be the reflections of P in the midlines of triangle ABC parallel to BC, CA, and AB, respectively. Denote by ω_A, ω_B, and ω_C the circumcircles of triangles ADP, BEP, and CFP, respectively. Denote by ω the circumcircle of the triangle formed by the perpendicular bisectors of segments AD, BE and CF.

Show that ω_A, ω_B, ω_C, and ω have a common point.

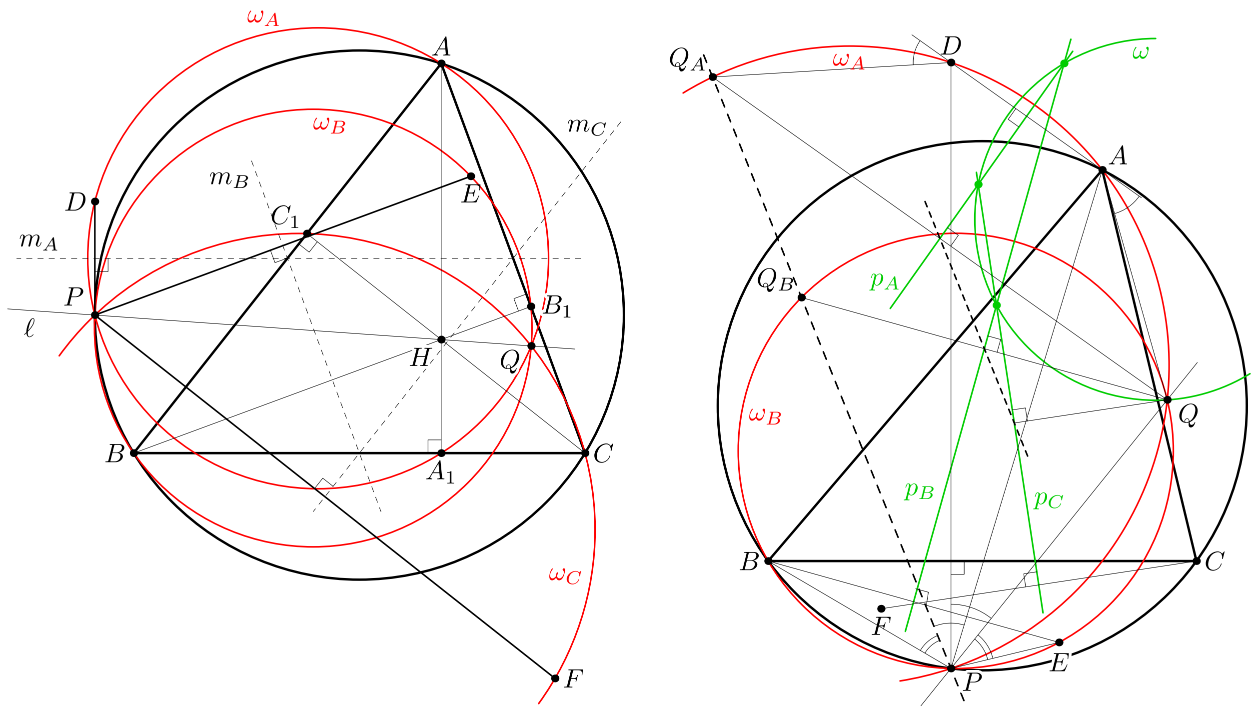
(Denmark)

Solution. Let AA_1, BB_1, and CC_1 be the altitudes in triangle ABC, and let m_A, m_B, and m_C be the midlines parallel to BC, CA, and AB, respectively. We always denote by $\measuredangle(p, q)$ the directed angle from a line p to a line q, taken modulo 180°.

Step 1: Circles ω_A, ω_B, and ω_C share a common point Q different from P.

Notice that m_A is the perpendicular bisector of PD, so ω_A is symmetric with respect to m_A. Since A and A_1 are also symmetric to each other in m_A, the point A_1 lies on ω_A. Similarly, B_1 and C_1 lie on ω_B and ω_C, respectively.

Let H be the orthocenter of $\triangle ABC$. Quadrilaterals ABA_1B_1 and BCB_1C_1 are cyclic, so $AH \cdot HA_1 = BH \cdot HB_1 = CH \cdot HC_1$. This means that H lies on pairwise radical axes of ω_A, ω_B, and ω_C. Point P also lies on those radical axes; hence the three circles have a common radical axis $\ell = PH$, and the second meeting point Q of ℓ with ω_A is the second common point of the three circles. Notice here that H lies inside all three circles, hence $Q \neq P$.



Step 2: Point Q lies on ω.

Let p_A, p_B, and p_C denote the perpendicular bisectors of AD, BE, and CF, respectively; denote by Δ the triangle formed by those perpendicular bisectors. By Simson's theorem, in order to show that Q lies on the circumcircle ω of Δ, it suffices to prove that the projections of Q onto the sidelines p_A, p_B, and p_C are collinear. Alternatively, but equivalently, it suffices to prove that the reflections Q_A, Q_B, and Q_C of Q in those lines, respectively, are collinear. In fact, we will show that four points P, Q_A, Q_B, and Q_C are collinear.

Since p_A is the common perpendicular bisector of AD and QQ_A, the point Q_A lies on ω_A, and, moreover, $\measuredangle(DA, DQ_A) = \measuredangle(AQ, AD)$. Therefore,

$$\measuredangle(PA, PQ_A) = \measuredangle(DA, DQ_A) = \measuredangle(AQ, AD) = \measuredangle(PQ, PD) = \measuredangle(PQ, BC) + 90^\circ.$$

Similarly, we get $\measuredangle(PB, PQ_B) = \measuredangle(PQ, CA) + 90^\circ$. Therefore,

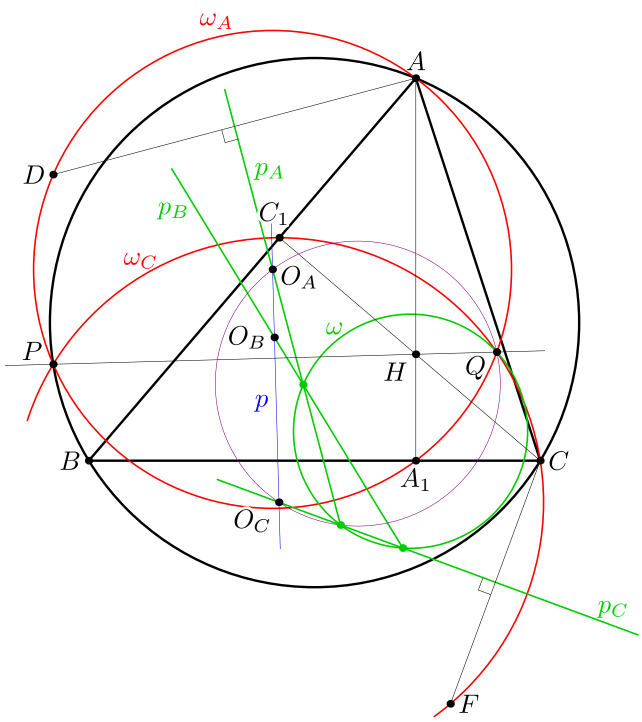
$$\begin{aligned}\measuredangle(PQ_A, PQ_B) &= \measuredangle(PQ_A, PA) + \measuredangle(PA, PB) + \measuredangle(PB, PQ_B)\\ &= \measuredangle(BC, PQ) + 90^\circ + \measuredangle(CA, CB) + \measuredangle(PQ, CA) + 90^\circ = 0,\end{aligned}$$

which shows that P, Q_A, and Q_B are collinear. Similarly, Q_C also lies on PQ_A.

Comment 1. There are several variations of Step 2. In particular, let O_A, O_B, and O_C denote the centers of ω_A, ω_B, and ω_C, respectively; they lie on p_A, p_B, and p_C, respectively. Moreover, all those centers lie on the perpendicular bisector p of PQ. Now one can show that $\measuredangle(QO_A, p_A) = \measuredangle(QO_B, p_B) = \measuredangle(QO_C, p_C)$, and then finish by applying generalized Simson's theorem, Alternatively, but equivalently, those relations show that Q is the Miquel point of the lines p_A, p_B, p_C, and p.

To establish $\measuredangle(QO_A, p_A) = \measuredangle(QO_C, p_C)$, notice that it is equivalent to $\measuredangle(QO_A, QO_C) = \measuredangle(p_A, p_C)$ which may be obtained, e.g., as follows:

$$\begin{aligned}\measuredangle(QO_A, QO_C) &= \measuredangle(QO_A, p) + \measuredangle(p, QO_C) = \measuredangle(AQ, AP) + \measuredangle(CP, CQ)\\ &= \measuredangle(AQ, CQ) + \measuredangle(CP, AP) = \measuredangle(AQ, PQ) + \measuredangle(PQ, CQ) + \measuredangle(CB, AB)\\ &= \measuredangle(AD, AA_1) + \measuredangle(CC_1, CF) + \measuredangle(AA_1, CC_1) = \measuredangle(AD, CF) = \measuredangle(p_A, p_C).\end{aligned}$$



Comment 2. The inversion at H with (negative) power $-AH \cdot HA_1$ maps P to Q, and the circumcircle of $\triangle ABC$ to its Euler circle. Therefore, Q lies on that Euler circle.

G8. Let Γ and I be the circumcircle and the incenter of an acute-angled triangle ABC. Two circles ω_B and ω_C passing through B and C, respectively, are tangent at I. Let ω_B meet the shorter arc AB of Γ and segment AB again at P and M, respectively. Similarly, let ω_C meet the shorter arc AC of Γ and segment AC again at Q and N, respectively. The rays PM and QN meet at X, and the tangents to ω_B and ω_C at B and C, respectively, meet at Y.

Prove that the points A, X, and Y are collinear.

(Netherlands)

Solution 1. Let AI, BI, and CI meet Γ again at D, E, and F, respectively. Let ℓ be the common tangent to ω_B and ω_C at I. We always denote by $\measuredangle(p, q)$ the directed angle from a line p to a line q, taken modulo $180°$.

Step 1: We show that Y lies on Γ.

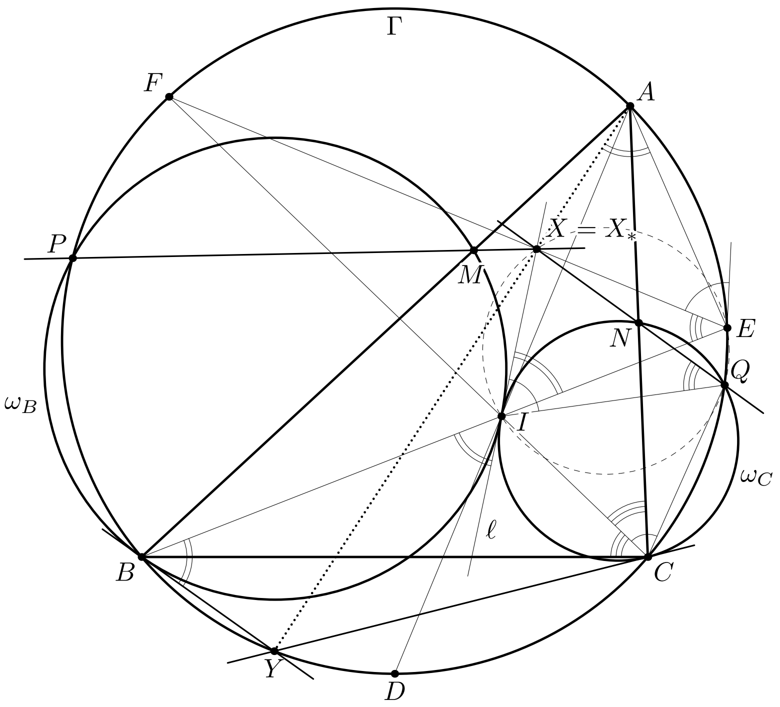
Recall that any chord of a circle makes complementary directed angles with the tangents to the circle at its endpoints. Hence,

$$\measuredangle(BY, BI) + \measuredangle(CI, CY) = \measuredangle(IB, \ell) + \measuredangle(\ell, IC) = \measuredangle(IB, IC).$$

Therefore,

$$\begin{aligned}\measuredangle(BY, BA) + \measuredangle(CA, CY) &= \measuredangle(BI, BA) + \measuredangle(BY, BI) + \measuredangle(CI, CY) + \measuredangle(CA, CI)\\ &= \measuredangle(BC, BI) + \measuredangle(IB, IC) + \measuredangle(CI, CB) = 0,\end{aligned}$$

which yields $Y \in \Gamma$.



Step 2: We show that $X = \ell \cap EF$.

Let $X_* = \ell \cap EF$. To prove our claim, it suffices to show that X_* lies on both PM and QN; this will yield $X_* = X$. Due to symmetry, it suffices to show $X_* \in QN$.

Notice that

$$\measuredangle(IX_*, IQ) = \measuredangle(CI, CQ) = \measuredangle(CF, CQ) = \measuredangle(EF, EQ) = \measuredangle(EX_*, EQ);$$

therefore, the points X_*, I, Q, and E are concyclic (if $Q = E$, then the direction of EQ is supposed to be the direction of a tangent to Γ at Q; in this case, the equality means that the circle (X_*IQ) is tangent to Γ at Q). Then we have

$$\measuredangle(QX_*, QI) = \measuredangle(EX_*, EI) = \measuredangle(EF, EB) = \measuredangle(CA, CF) = \measuredangle(CN, CI) = \measuredangle(QN, QI),$$

which shows that $X_* \in QN$.

Step 3: We finally show that A, X, and Y are collinear.

Recall that I is the orthocenter of triangle DEF, and A is symmetric to I with respect to EF. Therefore,

$$\measuredangle(AX, AE) = \measuredangle(IE, IX) = \measuredangle(BI, \ell) = \measuredangle(BY, BI) = \measuredangle(BY, BE) = \measuredangle(AY, AE),$$

which yields the desired collinearity.

Comment 1. Step 2 in the above solution seems to be crucial. After it has been performed (even without Step 1), there are different ways of finishing the solution.

E.g., one may involve the notion of *isogonal conjugacy*. Let X_1 and Y_1 be isogonal conjugates of X and Y, respectively, with respect to triangle ABC. Since $XA = XI$, triangle AIX is isosceles, and hence the lines AX and XI form equal angles with the internal bisector AI of $\angle BAC$. This means that $AX_1 \parallel XI$, or $AX_1 \parallel \ell$.

On the other hand, the lines BY and ℓ form equal angles with BI, so that $BY_1 \parallel \ell$. Similarly, $CY_1 \parallel \ell$. This means that Y_1 is an ideal point, and $AY_1 \parallel \ell$ as well. Therefore, points A, X_1, and Y_1 are collinear, and hence A, X, and Y are such.

Solution 2. Perform an inversion centered at I; the images of the points are denoted by primes, e.g., A' is the image of A.

On the inverted figure, I and Γ' are the orthocenter and the circumcircle of triangle $A'B'C'$, respectively. The points P' and Q' lie on Γ' such that $B'P' \parallel C'Q'$ (since $B'P' = \omega'_B$ and $C'Q' = \omega'_C$). The points M' and N' are the second intersections of lines $B'P'$ and $C'Q'$ with the circumcircles γ_B and γ_C of triangles $A'IB'$ and $A'IC'$, respectively. Notice here that γ_C is obtained from γ_B by the translation at $\overrightarrow{B'C'}$; the same translation maps line $B'P'$ to $C'Q'$, and hence M' to N'. In other words, $B'M'N'C'$ is a parallelogram, and $P'Q'$ partitions it into two isosceles trapezoids.

Point X' is the second intersection point of circles $(IP'M')$ and $(IQ'N')$ that is — the reflection of I in their line of centers. But the centers lie on the common perpendicular bisector p of $P'M'$ and $Q'N'$, so p is that line of centers. Hence, $IX' \parallel B'P'$, as both lines are perpendicular to p.

Finally, the point Y satisfies $\measuredangle(BY, BI) = \measuredangle(PB, PI)$ and $\measuredangle(CY, CI) = \measuredangle(QC, QI)$, which yields $\measuredangle(Y'B', Y'I) = \measuredangle(B'P', B'I)$ and $\measuredangle(Y'C', Y'I) = \measuredangle(C'Q', C'I)$. Therefore,

$$\measuredangle(Y'B', Y'C') = \measuredangle(B'P', B'I) + \measuredangle(C'I, C'Q') = \measuredangle(C'I, B'I) = \measuredangle(A'B', A'C'),$$

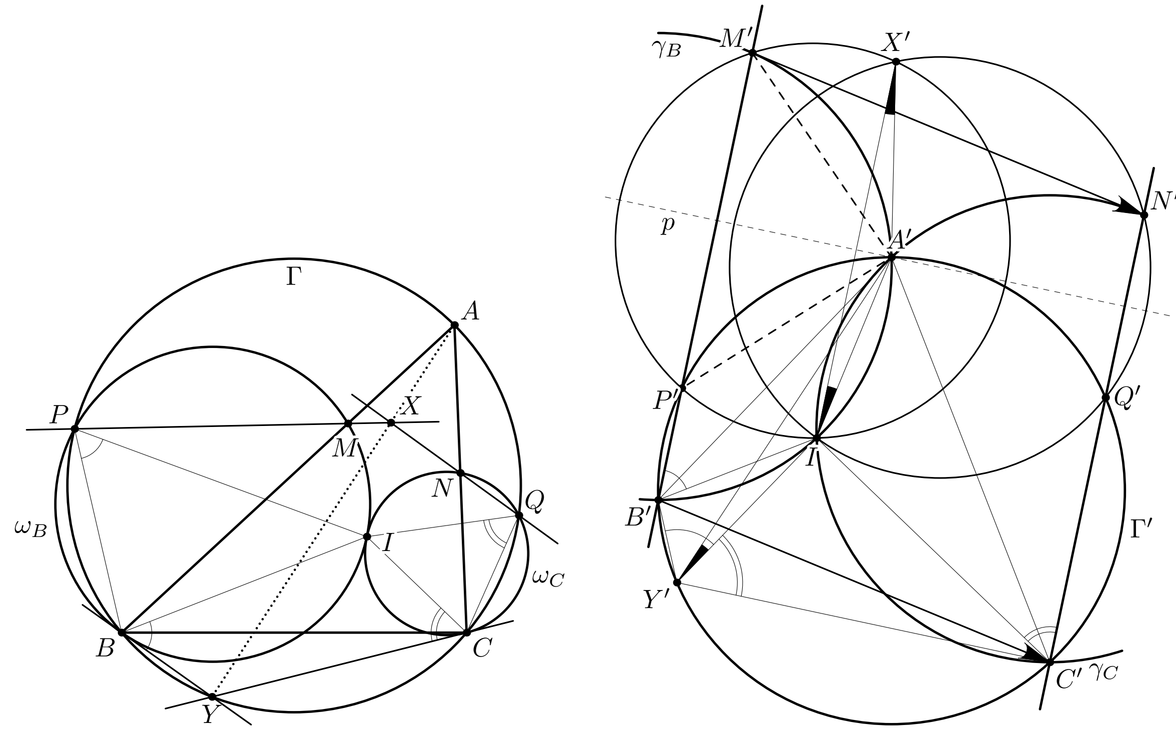
which shows that $Y' \in \Gamma'$.

In congruent circles Γ' and γ_B, the chords $A'P'$ and $A'M'$ subtend the same angle $\angle A'B'P'$; therefore, $A'P' = A'M'$, and hence $A' \in p$. This yields $A'X' = A'I$, and hence $\measuredangle(IA', IX') = \measuredangle(X'I, X'A')$.

Finally, we have

$$\begin{aligned}\measuredangle(Y'I, Y'A') &= \measuredangle(Y'I, Y'B') + \measuredangle(Y'B', Y'A')\\ &= \measuredangle(B'I, B'P') + \measuredangle(IA', IB') = \measuredangle(IA', B'P') = \measuredangle(IA', IX') = \measuredangle(X'I, X'A'),\end{aligned}$$

which yields that the points A', X', Y', and I are concyclic. This means exactly that A, X, and Y are collinear.



Comment 2. An inversion at I may also help in establishing Step 2 in Solution 1. Indeed, relation $A'X' = A'I$ yields $XA = XI$, so that $X \in EF$. On the other hand, $IX' \parallel B'P'$ yields $IX \parallel \ell$, i.e., $X \in \ell$.

G9. Prove that there exists a positive constant c such that the following statement is true:

Assume that n is an integer with $n \geqslant 2$, and let $\mathcal{S}$ be a set of n points in the plane such that the distance between any two distinct points in $\mathcal{S}$ is at least 1. Then there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $cn^{-1/3}$.

(A line ℓ *separates* a point set $\mathcal{S}$ if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

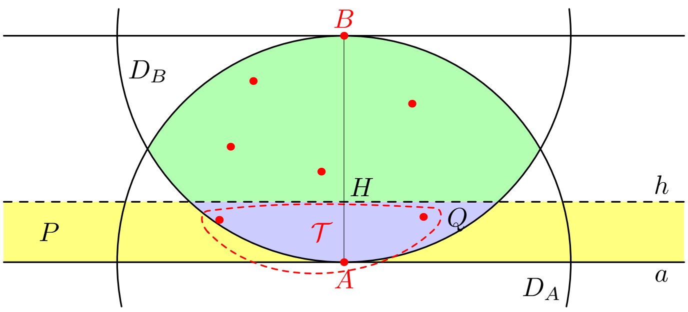
(Taiwan)

Solution. We prove that the desired statement is true with $c = \frac{1}{8}$. Set $\delta = \frac{1}{8}n^{-1/3}$. For any line ℓ and any point X, let X_ℓ denote the projection of X to ℓ; a similar notation applies to sets of points.

Suppose that, for some line ℓ, the set $\mathcal{S}_\ell$ contains two adjacent points X and Y with $XY = 2d$. Then the line perpendicular to ℓ and passing through the midpoint of segment XY separates $\mathcal{S}$, and all points in $\mathcal{S}$ are at least d apart from ℓ. Thus, if $d \geqslant \delta$, then a desired line has been found. For the sake of contradiction, we assume that no such points exist, in any projection.

Choose two points A and B in $\mathcal{S}$ with the maximal distance $M = AB$ (i.e., AB is a *diameter* of $\mathcal{S}$); by the problem condition, $M \geqslant 1$. Denote by ℓ the line AB. The set $\mathcal{S}$ is contained in the intersection of two disks D_A and D_B of radius M centered at A and B, respectively. Hence, the projection $\mathcal{S}_\ell$ is contained in the segment AB. Moreover, the points in $\mathcal{S}_\ell$ divide that segment into at most $n-1$ parts, each of length less than 2δ. Therefore,

$$M < n \cdot 2\delta. \tag{1}$$



Choose a point H on segment AB with $AH = \frac{1}{2}$. Let P be a strip between the lines a and h perpendicular to AB and passing through A and H, respectively; we assume that P contains its boundary, which consists of lines a and h. Set $\mathcal{T} = P \cap \mathcal{S}$ and let $t = |\mathcal{T}|$. By our assumption, segment AH contains at least $\left\lceil \frac{1}{2} : (2\delta) \right\rceil$ points of S_ℓ, which yields

$$t \geqslant \frac{1}{4\delta}. \tag{2}$$

Notice that $\mathcal{T}$ is contained in $Q = P \cap D_B$. The set Q is a circular segment, and its projection Q_a is a line segment of length

$$2\sqrt{M^2 - \left(M - \frac{1}{2}\right)^2} < 2\sqrt{M}.$$

On the other hand, for any two points $X, Y \in \mathcal{T}$, we have $XY \geqslant 1$ and $X_\ell Y_\ell \leqslant \frac{1}{2}$, so $X_a Y_a = \sqrt{XY^2 - X_\ell Y_\ell^2} \geqslant \frac{\sqrt{3}}{2}$. To summarize, t points constituting $\mathcal{T}_a$ lie on the segment of length less than $2\sqrt{M}$, and are at least $\frac{\sqrt{3}}{2}$ apart from each other. This yields $2\sqrt{M} > (t-1)\frac{\sqrt{3}}{2}$, or

$$t < 1 + \frac{4\sqrt{M}}{\sqrt{3}} < 4\sqrt{M}, \tag{3}$$

as $M \geqslant 1$.

Combining the estimates (1), (2), and (3), we finally obtain

$$\frac{1}{4\delta} \leqslant t < 4\sqrt{M} < 4\sqrt{2n\delta}, \quad \text{or} \quad 512n\delta^3 > 1,$$

which does not hold for the chosen value of δ.

Comment 1. As the proposer mentions, the exponent $-1/3$ in the problem statement is optimal. In fact, for any $n \geqslant 2$, there is a configuration $\mathcal{S}$ of n points in the plane such that any two points in $\mathcal{S}$ are at least 1 apart, but every line ℓ separating $\mathcal{S}$ is at most $c'n^{-1/3}\log n$ apart from some point in $\mathcal{S}$; here c' is some absolute constant.

The original proposal suggested to prove the estimate of the form $cn^{-1/2}$. That version admits much easier solutions. E.g., setting $\delta = \frac{1}{16}n^{-1/2}$ and applying (1), we see that $\mathcal{S}$ is contained in a disk D of radius $\frac{1}{8}n^{1/2}$. On the other hand, for each point X of $\mathcal{S}$, let D_X be the disk of radius $\frac{1}{2}$ centered at X; all these disks have disjoint interiors and lie within the disk concentric to D, of radius $\frac{1}{16}n^{1/2} + \frac{1}{2} < \frac{1}{2}n^{1/2}$. Comparing the areas, we get

$$n \cdot \frac{\pi}{4} \leqslant \pi\left(\frac{n^{1/2}}{16} + \frac{1}{2}\right)^2 < \frac{\pi n}{4},$$

which is a contradiction.

The Problem Selection Committee decided to choose a harder version for the Shortlist.

Comment 2. In this comment, we discuss some versions of the solution above, which avoid concentrating on the diameter of $\mathcal{S}$. We start with introducing some terminology suitable for those versions.

Put $\delta = cn^{-1/3}$ for a certain sufficiently small positive constant c. For the sake of contradiction, suppose that, for some set $\mathcal{S}$ satisfying the conditions in the problem statement, there is no separating line which is at least δ apart from each point of $\mathcal{S}$.

Let C be the convex hull of $\mathcal{S}$. A line is separating if and only if it meets C (we assume that a line passing through a point of $\mathcal{S}$ is always separating). Consider a strip between two parallel separating lines a and a' which are, say, $\frac{1}{4}$ apart from each other. Define a *slice* determined by the strip as the intersection of $\mathcal{S}$ with the strip. The *length* of the slice is the diameter of the projection of the slice to a.

In this terminology, the arguments used in the proofs of (2) and (3) show that for any slice $\mathcal{T}$ of length L, we have

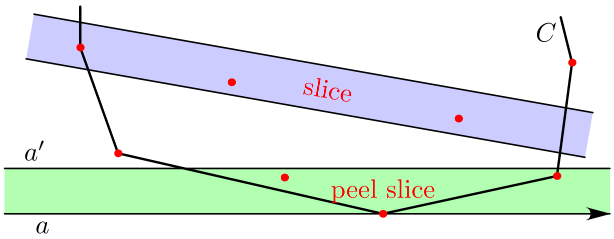
$$\frac{1}{8\delta} \leqslant |\mathcal{T}| \leqslant 1 + \frac{4}{\sqrt{15}}L. \tag{4}$$

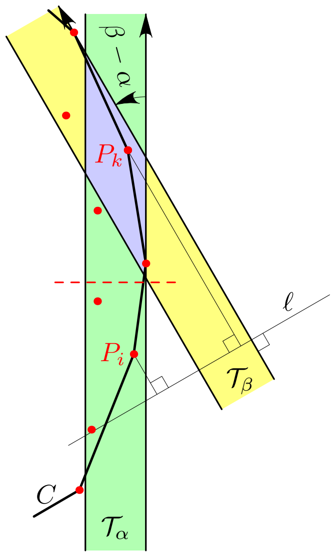
The key idea of the solution is to apply these estimates to a *peel slice*, where line a does not cross the interior of C. In the above solution, this idea was applied to one carefully chosen peel slice. Here, we outline some different approach involving many of them. We always assume that n is sufficiently large.

Consider a peel slice determined by lines a and a', where a contains no interior points of C. We orient a so that C lies to the left of a. Line a is called a *supporting line* of the slice, and the obtained direction is the *direction* of the slice; notice that the direction determines uniquely the supporting line and hence the slice. Fix some direction $\mathbf{v}_0$, and for each $\alpha \in [0, 2\pi)$ denote by $\mathcal{T}_\alpha$ the peel slice whose direction is $\mathbf{v}_0$ rotated by α counterclockwise.

When speaking about the slice, we always assume that the figure is rotated so that its direction is vertical from the bottom to the top; then the points in $\mathcal{T}$ get a natural order from the bottom to the top. In particular, we may speak about the *top half* $\mathsf{T}(\mathcal{T})$ consisting of $\lfloor |\mathcal{T}|/2 \rfloor$ topmost points in $\mathcal{T}$, and similarly about its *bottom half* $\mathsf{B}(\mathcal{T})$. By (4), each half contains at least 10 points when n is large.

Claim. Consider two angles $\alpha, \beta \in [0, \pi/2]$ with $\beta - \alpha \geqslant 40\delta =: \phi$. Then all common points of $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ lie in $\mathsf{T}(\mathcal{T}_\alpha) \cap \mathsf{B}(\mathcal{T}_\beta)$.





Proof. By symmetry, it suffices to show that all those points lie in $\mathsf{T}(\mathcal{T}_\alpha)$. Let a be the supporting line of $\mathcal{T}_\alpha$, and let ℓ be a line perpendicular to the direction of $\mathcal{T}_\beta$. Let $P_1, \dots, P_k$ list all points in $\mathcal{T}_\alpha$, numbered from the bottom to the top; by (4), we have $k \geqslant \frac{1}{8}\delta^{-1}$.

Introduce the Cartesian coordinates so that the (oriented) line a is the y-axis. Let P_i be any point in $\mathsf{B}(\mathcal{T}_\alpha)$. The difference of ordinates of P_k and P_i is at least $\frac{\sqrt{15}}{4}(k-i) > \frac{1}{3}k$, while their abscissas differ by at most $\frac{1}{4}$. This easily yields that the projections of those points to ℓ are at least

$$\frac{k}{3}\sin\phi - \frac{1}{4} \geqslant \frac{1}{24\delta} \cdot 20\delta - \frac{1}{4} > \frac{1}{4}$$

apart from each other, and P_k is closer to the supporting line of $\mathcal{T}_\beta$ than P_i, so that P_i does not belong to $\mathcal{T}_\beta$. $\square$

Now, put $\alpha_i = 40\delta i$, for $i = 0, 1, \dots, \left\lfloor \frac{1}{40}\delta^{-1} \cdot \frac{\pi}{2} \right\rfloor$, and consider the slices $\mathcal{T}_{\alpha_i}$. The Claim yields that each point in $\mathcal{S}$ is contained in at most two such slices. Hence, the union $\mathcal{U}$ of those slices contains at least

$$\frac{1}{2} \cdot \frac{1}{8\delta} \cdot \frac{1}{40\delta} \cdot \frac{\pi}{2} = \frac{\lambda}{\delta^2}$$

points (for some constant λ), and each point in $\mathcal{U}$ is at most $\frac{1}{4}$ apart from the boundary of C.

It is not hard now to reach a contradiction with (1). E.g., for each point $X \in \mathcal{U}$, consider a closest point $f(X)$ on the boundary of C. Obviously, $f(X)f(Y) \geqslant XY - \frac{1}{2} \geqslant \frac{1}{2}XY$ for all $X, Y \in \mathcal{U}$. This yields that the perimeter of C is at least $\mu\delta^{-2}$, for some constant μ, and hence the diameter of $\mathcal{S}$ is of the same order.

Alternatively, one may show that the projection of $\mathcal{U}$ to the line at the angle of $\pi/4$ with $\mathbf{v}_0$ has diameter at least $\mu\delta^{-2}$ for some constant μ.

Number Theory

N1. Given a positive integer k, show that there exists a prime p such that one can choose distinct integers $a_1, a_2, \ldots, a_{k+3} \in \{1, 2, \ldots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \ldots, k$.

(South Africa)

Solution. First we choose distinct positive rational numbers $r_1, \ldots, r_{k+3}$ such that

$$r_i r_{i+1} r_{i+2} r_{i+3} = i \quad \text{for } 1 \leqslant i \leqslant k.$$

Let $r_1 = x$, $r_2 = y$, $r_3 = z$ be some distinct primes greater than k; the remaining terms satisfy $r_4 = \frac{1}{r_1 r_2 r_3}$ and $r_{i+4} = \frac{i+1}{i} r_i$. It follows that if r_i are represented as irreducible fractions, the numerators are divisible by x for $i \equiv 1 \pmod 4$, by y for $i \equiv 2 \pmod 4$, by z for $i \equiv 3 \pmod 4$ and by none for $i \equiv 0 \pmod 4$. Notice that $r_i < r_{i+4}$; thus the sequences $r_1 < r_5 < r_9 < \ldots$, $r_2 < r_6 < r_{10} < \ldots$, $r_3 < r_7 < r_{11} < \ldots$, $r_4 < r_8 < r_{12} < \ldots$ are increasing and have no common terms, that is, all r_i are distinct.

If each r_i is represented by an irreducible fraction $\frac{u_i}{v_i}$, choose a prime p which divides neither v_i, $1 \leqslant i \leqslant k+1$, nor $v_i v_j (r_i - r_j) = v_j u_i - v_i u_j$ for $i < j$, and define a_i by the congruence $a_i v_i \equiv u_i \pmod p$. Since $r_i r_{i+1} r_{i+2} r_{i+3} = i$, we have

$$\begin{aligned} i v_i v_{i+1} v_{i+2} v_{i+3} &= r_i v_i r_{i+1} v_{i+1} r_{i+2} v_{i+2} r_{i+3} v_{i+3} \\ &= u_i u_{i+1} u_{i+2} u_{i+3} \equiv a_i v_i a_{i+1} v_{i+1} a_{i+2} v_{i+2} a_{i+3} v_{i+3} \pmod p \end{aligned}$$

and therefore $a_i a_{i+1} a_{i+2} a_{i+3} \equiv i \pmod p$ for $1 \leqslant i \leqslant k$.

If $a_i \equiv a_j \pmod p$, then $u_i v_j \equiv a_i v_i v_j \equiv u_j v_i \pmod p$, a contradiction.

Comment. One can explicitly express residues $b_i \equiv a_1 a_2 \cdot \ldots \cdot a_i \pmod p$ in terms of b_1, b_2, b_3 and $b_0 = 1$:

$$b_{i+3} = i(i-4)(i-8) \cdot \ldots \cdot (i-4k+4) b_r,$$

where $i + 3 = 4k + r$, $0 \leqslant r < 4$. Then the numbers a_i are found from the congruences $b_{i-1} a_i \equiv b_i \pmod p$, and choosing p so that a_i are not congruent modulo p is done in a way very similar to the above solution.

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N2. For each prime p, there is a kingdom of p-Landia consisting of p islands numbered $1, 2, \ldots, p$. Two distinct islands numbered n and m are connected by a bridge if and only if p divides $(n^2 - m + 1)(m^2 - n + 1)$. The bridges may pass over each other, but cannot cross. Prove that for infinitely many p there are two islands in p-Landia not connected by a chain of bridges.

(Denmark)

Solution 1. We prove that for each prime $p > 3$ dividing a number of the form $x^2 - x + 1$ with integer x there are two unconnected islands in p-Landia.

For brevity's sake, when a bridge connects the islands numbered m and n, we shall speak simply that it connects m and n.

A bridge connects m and n if $n \equiv m^2 + 1 \pmod p$ or $m \equiv n^2 + 1 \pmod p$. If $m^2 + 1 \equiv n \pmod p$, we draw an arrow starting at m on the bridge connecting m and n. Clearly only one arrow starts at m if $m^2 + 1 \not\equiv m \pmod p$, and no arrows otherwise. The total number of bridges does not exceed the total number of arrows.

Suppose $x^2 - x + 1 \equiv 0 \pmod p$. We may assume that $1 \leqslant x \leqslant p$; then there is no arrow starting at x. Since $(1-x)^2 - (1-x) + 1 = x^2 - x + 1$, $(p+1-x)^2 + 1 \equiv (p+1-x) \pmod p$, and there is also no arrow starting at $p + 1 - x$. If $x = p + 1 - x$, that is, $x = \frac{p+1}{2}$, then $4(x^2 - x + 1) = p^2 + 3$ and therefore $x^2 - x + 1$ is not divisible by p. Thus the islands x and $p + 1 - x$ are different, and no arrows start at either of them. It follows that the total number of bridges in p-Landia does not exceed $p - 2$.

Let $1, 2, \ldots, p$ be the vertices of a graph G_p, where an edge connects m and n if and only if there is a bridge between m and n. The number of vertices of G_p is p and the number of edges is less than $p - 1$. This means that the graph is not connected, which means that there are two islands not connected by a chain of bridges.

It remains to prove that there are infinitely many primes p dividing $x^2 - x + 1$ for some integer x. Let $p_1, p_2, \ldots, p_k$ be any finite set of such primes. The number $(p_1 p_2 \cdot \ldots \cdot p_k)^2 - p_1 p_2 \cdot \ldots \cdot p_k + 1$ is greater than 1 and not divisible by any p_i; therefore it has another prime divisor with the required property.

Solution 2. One can show, by using only arithmetical methods, that for infinitely many p, the kingdom of p-Ladia contains two islands connected to no other island, except for each other.

Let arrows between islands have the same meaning as in the previous solution. Suppose that positive $a < p$ satisfies the congruence $x^2 - x + 1 \equiv 0 \pmod p$. We have seen in the first solution that $b = p + 1 - a$ satisfies it too, and $b \neq a$ when $p > 3$. It follows that $ab \equiv a(1-a) \equiv 1 \pmod p$. If an arrow goes from t to a, then t must satisfy the congruence $t^2 + 1 \equiv a \equiv a^2 + 1 \pmod p$; the only such $t \neq a$ is $p - a$. Similarly, the only arrow going to b goes from $p - b$. If one of the numbers $p - a$ and $p - b$, say, $p - a$, is not at the end of any arrow, the pair a, $p - a$ is not connected with the rest of the islands. This is true if at least one of the congruences $x^2 + 1 \equiv -a$, $x^2 + 1 \equiv -b$ has no solutions, that is, either $-a - 1$ or $-b - 1$ is a quadratic non-residue modulo p.

Note that $x^2 - x + 1 \equiv x^2 - (a+b)x + ab \equiv (x-a)(x-b) \pmod p$. Substituting $x = -1$ we get $(-1-a)(-1-b) \equiv 3 \pmod p$. If 3 is a quadratic non-residue modulo p, so is one of the numbers $-1 - a$ and $-1 - b$.

Thus it is enough to find infinitely many primes $p > 3$ dividing $x^2 - x + 1$ for some integer x and such that 3 is a quadratic non-residue modulo p.

If $x^2 - x + 1 \equiv 0 \pmod p$ then $(2x-1)^2 \equiv -3 \pmod p$, that is, -3 is a quadratic residue modulo p, so 3 is a quadratic non-residue if and only if -1 is also a non-residue, in other words, $p \equiv -1 \pmod 4$.

Similarly to the first solution, let $p_1, \ldots, p_k$ be primes congruent to -1 modulo 4 and dividing numbers of the form $x^2 - x + 1$. The number $(2p_1 \cdot \ldots \cdot p_k)^2 - 2p_1 \cdot \ldots \cdot p_k + 1$ is not divisible by any p_i and is congruent to -1 modulo 4, therefore, it has some prime divisor $p \equiv -1 \pmod 4$ which has the required properties.

N3. Let n be an integer with $n \geqslant 2$. Does there exist a sequence $(a_1, \ldots, a_n)$ of positive integers with not all terms being equal such that the arithmetic mean of every two terms is equal to the geometric mean of some (one or more) terms in this sequence?

(Estonia)

Answer: No such sequence exists.

Solution 1. Suppose that $a_1, \ldots, a_n$ satisfy the required properties. Let $d = \gcd(a_1 \ldots, a_n)$. If $d > 1$ then replace the numbers $a_1, \ldots, a_n$ by $\frac{a_1}{d}, \ldots, \frac{a_n}{d}$; all arithmetic and all geometric means will be divided by d, so we obtain another sequence satisfying the condition. Hence, without loss of generality, we can assume that $\gcd(a_1 \ldots, a_n) = 1$.

We show two numbers, a_m and a_k such that their arithmetic mean, $\frac{a_m+a_k}{2}$ is different from the geometric mean of any (nonempty) subsequence of $a_1 \ldots, a_n$. That proves that there cannot exist such a sequence.

Choose the index $m \in \{1, \ldots, n\}$ such that $a_m = \max(a_1, \ldots, a_n)$. Note that $a_m \geqslant 2$, because $a_1, \ldots, a_n$ are not all equal. Let p be a prime divisor of a_m.

Let $k \in \{1, \ldots, n\}$ be an index such that $a_k = \max\{a_i : p \nmid a_i\}$. Due to $\gcd(a_1 \ldots, a_n) = 1$, not all a_i are divisible by p, so such a k exists. Note that $a_m > a_k$ because $a_m \geqslant a_k$, $p \mid a_m$ and $p \nmid a_k$.

Let $b = \frac{a_m+a_k}{2}$; we will show that b cannot be the geometric mean of any subsequence of $a_1, \ldots, a_n$.

Consider the geometric mean, $g = \sqrt[t]{a_{i_1} \cdot \ldots \cdot a_{i_t}}$ of an arbitrary subsequence of $a_1, \ldots, a_n$.

If none of $a_{i_1}, \ldots, a_{i_t}$ is divisible by p, then they are not greater than a_k, so

$$g = \sqrt[t]{a_{i_1} \cdot \ldots \cdot a_{i_t}} \leqslant a_k < \frac{a_m + a_k}{2} = b,$$

and therefore $g \neq b$.

Otherwise, if at least one of $a_{i_1}, \ldots, a_{i_t}$ is divisible by p, then $2g = 2\sqrt[t]{a_{i_1} \cdot \ldots \cdot a_{i_t}}$ is either not an integer or is divisible by p, while $2b = a_m + a_k$ is an integer not divisible by p, so $g \neq b$ again.

Solution 2. Like in the previous solution, we assume that the numbers $a_1, \ldots, a_n$ have no common divisor greater than 1. The arithmetic mean of any two numbers in the sequence is half of an integer; on the other hand, it is a (some integer order) root of an integer. This means each pair's mean is an integer, so all terms in the sequence must be of the same parity; hence they all are odd. Let $d = \min\left\{\gcd(a_i, a_j) \colon a_i \neq a_j\right\}$. By reordering the sequence we can assume that $\gcd(a_1, a_2) = d$, the sum $a_1 + a_2$ is maximal among such pairs, and $a_1 > a_2$.

We will show that $\frac{a_1+a_2}{2}$ cannot be the geometric mean of any subsequence of $a_1 \ldots, a_n$.

Let $a_1 = xd$ and $a_2 = yd$ where x, y are coprime, and suppose that there exist some $b_1, \ldots, b_t \in \{a_1, \ldots, a_n\}$ whose geometric mean is $\frac{a_1+a_2}{2}$. Let $d_i = \gcd(a_1, b_i)$ for $i = 1, 2, \ldots, t$ and let $D = d_1 d_2 \cdot \ldots \cdot d_t$. Then

$$D = d_1 d_2 \cdot \ldots \cdot d_t \mid b_1 b_2 \cdot \ldots \cdot b_t = \left(\frac{a_1 + a_2}{2}\right)^t = \left(\frac{x + y}{2}\right)^t d^t.$$

We claim that $D \mid d^t$. Consider an arbitrary prime divisor p of D. Let $\nu_p(x)$ denote the exponent of p in the prime factorization of x. If $p \mid \frac{x+y}{2}$, then $p \nmid x, y$, so p is coprime with x; hence, $\nu_p(d_i) \leqslant \nu_p(a_1) = \nu_p(xd) = \nu_p(d)$ for every $1 \leqslant i \leqslant t$, therefore $\nu_p(D) = \sum_i \nu_p(d_i) \leqslant t\nu_p(d) = \nu_p(d^t)$. Otherwise, if p is coprime to $\frac{x+y}{2}$, we have $\nu_p(D) \leqslant \nu_p(d^t)$ trivially. The claim has been proved.

Notice that $d_i = \gcd(b_i, a_1) \geqslant d$ for $1 \leqslant i \leqslant t$: if $b_i \neq a_1$ then this follows from the definition of d; otherwise we have $b_i = a_1$, so $d_i = a_1 \geqslant d$. Hence, $D = d_1 \cdot \ldots \cdot d_t \geqslant d^t$, and the claim forces $d_1 = \ldots = d_t = d$.

Finally, by $\frac{a_1+a_2}{2} > a_2$ there must be some b_k which is greater than a_2. From $a_1 > a_2 \geqslant d = \gcd(a_1, b_k)$ it follows that $a_1 \neq b_k$. Now the have a pair a_1, b_k such that $\gcd(a_1, b_k) = d$ but $a_1 + b_k > a_1 + a_2$; that contradicts the choice of a_1 and a_2.

Comment. The original problem proposal contained a second question asking if there exists a non-constant sequence $(a_1, \ldots, a_n)$ of positive integers such that the geometric mean of every two terms is equal the arithmetic mean of some terms.

For $n \geqslant 3$ such a sequence is $(4, 1, 1, \ldots, 1)$. The case $n = 2$ can be done by the trivial estimates

$$\min(a_1, a_2) < \sqrt{a_1 a_2} < \frac{a_1 + a_2}{2} < \max(a_1, a_2).$$

The Problem Selection Committee found this variant less interesting and suggests using only the first question.

N4. For any odd prime p and any integer n, let $d_p(n) \in \{0, 1, \ldots, p-1\}$ denote the remainder when n is divided by p. We say that $(a_0, a_1, a_2, \ldots)$ is a *p-sequence*, if a_0 is a positive integer coprime to p, and $a_{n+1} = a_n + d_p(a_n)$ for $n \geqslant 0$.

(a) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_n > b_n$ for infinitely many n, and $b_n > a_n$ for infinitely many n?

(b) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_0 < b_0$, but $a_n > b_n$ for all $n \geqslant 1$?

(United Kingdom)

Answer: Yes, for both parts.

Solution. Fix some odd prime p, and let T be the smallest positive integer such that $p \mid 2^T - 1$; in other words, T is the multiplicative order of 2 modulo p.

Consider any p-sequence $(x_n) = (x_0, x_1, x_2, \ldots)$. Obviously, $x_{n+1} \equiv 2x_n \pmod{p}$ and therefore $x_n \equiv 2^n x_0 \pmod{p}$. This yields $x_{n+T} \equiv x_n \pmod{p}$ and therefore $d(x_{n+T}) = d(x_n)$ for all $n \geqslant 0$. It follows that the sum $d(x_n) + d(x_{n+1}) + \ldots + d(x_{n+T-1})$ does not depend on n and is thus a function of x_0 (and p) only; we shall denote this sum by $S_p(x_0)$, and extend the function $S_p(\cdot)$ to all (not necessarily positive) integers. Therefore, we have $x_{n+kT} = x_n + kS_p(x_0)$ for all positive integers n and k. Clearly, $S_p(x_0) = S_p(2^t x_0)$ for every integer $t \geqslant 0$.

In both parts, we use the notation

$$S_p^+ = S_p(1) = \sum_{i=0}^{T-1} d_p(2^i) \quad \text{and} \quad S_p^- = S_p(-1) = \sum_{i=0}^{T-1} d_p(p - 2^i).$$

(a) Let $q > 3$ be a prime and p a prime divisor of $2^q + 1$ that is greater than 3. We will show that p is suitable for part (a). Notice that $9 \nmid 2^q + 1$, so that such a p exists. Moreover, for any two odd primes $q < r$ we have $\gcd(2^q + 1, 2^r + 1) = 2^{\gcd(q,r)} + 1 = 3$, thus there exist infinitely many such primes p.

For the chosen p, we have $T = 2q$. Since $2^q \equiv -1 \pmod{p}$, we have $S_p^+ = S_p^-$. Now consider the p-sequences (a_n) and (b_n) with $a_0 = p + 1$ and $b_0 = p - 1$; we claim that these sequences satisfy the required conditions. We have $a_0 > b_0$ and $a_1 = p + 2 < b_1 = 2p - 2$. It follows then that

$$a_{k \cdot 2q} = a_0 + kS_p^+ > b_0 + kS_p^+ = b_{k \cdot 2q} \quad \text{and} \quad a_{k \cdot 2q+1} = a_1 + kS_p^+ < b_1 + kS_p^+ = b_{k \cdot 2q+1}$$

for all $k = 0, 1, \ldots$, as desired.

(b) Let q be an odd prime and p a prime divisor of $2^q - 1$; thus we have $T = q$. We will show that p is suitable for part (b). Notice that the numbers of the form $2^q - 1$ are pairwise coprime (since $\gcd(2^q - 1, 2^r - 1) = 2^{\gcd(q,r)} - 1 = 1$ for any two distinct primes q and r), thus there exist infinitely many such primes p. Notice that $d_p(x) + d_p(p - x) = p$ for all x with $p \nmid x$, so that the sum $S_p^+ + S_p^- = pq$ is odd, which yields $S_p^+ = S_p(1) \neq S_p(-1) = S_p^-$.

Assume that (x_n) and (y_n) are two p-sequences with $S_p(x_0) > S_p(y_0)$ but $x_0 < y_0$. The first condition yields that

$$x_{Mq+r} - y_{Mq+r} = (x_r - y_r) + M\big(S_p(x_0) - S_p(y_0)\big) \geqslant (x_r - y_r) + M$$

for all nonnegative integers M and every $r = 0, 1, \ldots, q-1$. Thus, we have $x_n > y_n$ for every $n \geqslant q + q \cdot \max\{y_r - x_r : r = 0, 1, \ldots, q-1\}$. Now, since $x_0 < y_0$, there exists the largest n_0 with $x_{n_0} < y_{n_0}$. In this case the p-sequences $a_n = x_{n-n_0}$ and $b_n = y_{n-n_0}$ possess the desired property (notice here that $x_n \neq y_n$ for all $n \geqslant 0$, as otherwise we would have $S_p(x_0) = S_p(x_n) = S_p(y_n) = S_p(y_0)$).

It remains to find p-sequences (x_n) and (y_n) satisfying the two conditions. Recall that $S_p^+ \neq S_p^-$. Now, if $S_p^+ > S_p^-$, then we can put $x_0 = 1$ and $y_0 = p - 1$. Otherwise, if $S_p^+ < S_p^-$, then we put $x_0 = p - 1$ and $y_0 = p + 1$.

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N5. Determine all functions f defined on the set of all positive integers and taking non-negative integer values, satisfying the three conditions:

(*i*) $f(n) \neq 0$ for at least one n;

(*ii*) $f(xy) = f(x) + f(y)$ for every positive integers x and y;

(*iii*) there are infinitely many positive integers n such that $f(k) = f(n-k)$ for all $k < n$.

(Croatia)

Answer: The sought functions are those of the form $f(n) = c \cdot \nu_p(n)$, where p is some prime, c is a nonnegative integer, and $\nu_p(n)$ denotes the exponent of p in the prime decomposition of n.

Solution 1. If a number n is a product of primes, $n = p_1 p_2 \cdot \ldots \cdot p_k$, then

$$f(n) = f(p_1) + \ldots + f(p_k),$$

in particular, $f(1) = 0$ (since $f(1) = f(1) + f(1)$).

It is also clear that $f(n) = 0$ implies $f(p) = 0$ for all primes p dividing n.

Let us call positive integer n *good* if $f(k) = f(n-k)$ for $0 < k < n$. If n is good then each its divisor d is also good; indeed, if $n = dm$ then

$$f(k) = f(mk) - f(m) = f(n-mk) - f(m) = f(m(d-k)) - f(m) = f(d-k)$$

for $0 < k < d$. Thus, good numbers are products of good primes.

It follows immediately from (i) that there exists a prime p such that $f(p) \neq 0$; let p be the smallest such prime. Then $f(r) = 0$ for all $r < p$ (since all prime divisors of $r < p$ are less than p). Now every good number $n > p$ must be divisible by p. Indeed, if $n = pk + r$ is a good number, $k > 0$, $0 < r < p$, then $f(p) \leqslant f(pk) = f(n - pk) = f(r) = 0$, a contradiction. Since any divisor of a good number is also good, this means that if a divisor r of a good number is not divisible by p, it is less than p. Thus all good numbers have the form $r \cdot p^k$ with $r < p$. The condition (iii) implies that k can be arbitrarily large, consequently all powers of p are good.

If $q \neq p$ is a prime, $p^{q-1} - 1$ is divisible by q and p^{q-1} is good. Then $f(q) \leqslant f(p^{q-1} - 1) = f(1) = 0$, that is, $f(q) = 0$.

Now we see that $f(n) = \nu_p(n) \cdot c$, where $c = f(p)$. The conditions (i) and (ii) for all such functions with $c \neq 0$ are obvious; the condition (iii) holds for all $n = p^m$, since $\nu_p(p^m - k) = \nu_p(k)$ when $0 < k < p^m$.

Solution 2. We use the notion of a good number from the previous solution. As above, we also denote by $\nu_p(n)$ the exponent of a prime p in the prime decomposition of n.

Say that a positive integer k is *big* if $f(k) > 0$. Let $\mathcal{B}$ be the set of big primes, and let $p_1 < p_2 < \ldots$ list the elements of $\mathcal{B}$ (this set might be either finite or infinite). By the problem conditions, we have

$$f(n) = \sum_i \nu_{p_i}(n) f(p_i); \tag{1}$$

thus, the big numbers are those divisible by at least one big prime.

For a positive integer k, define its *essence* $e(k)$ to be the largest product e of (not necessarily different) big primes such that $e \mid k$. In other words,

$$e(n) = \prod_{p_i \in \mathcal{B}} p_i^{\nu_{p_i}(n)}.$$

This yields that $k/e(k)$ is not big, so $f(k) = f(e(k)) + f(k/e(k)) = f(e(k))$.

Lemma. Assume that n is a good number. Then $e(k) = e(n-k)$ for all $k < n$.

Proof. Arguing indirectly, choose a minimal k for which the claim of the lemma is violated. Clearly, k is big, as otherwise $f(k) = f(n-k) = 0$ and hence $e(k) = e(n-k) = 1$.

There are $t = k/e(k)$ multiples of $e(k)$ in each of the segments $[1, k]$ and $[n-k, n-1]$. On the other hand, there are $t-1$ such multiples on $[1, k-1]$ — and, by minimality of k, on $[n-k+1, n-1]$ as well. This yields that $n-k$ is a multiple of $e(k)$. Therefore,

$$f(e(k)) = f(k) = f(n-k) = f(e(k)) + f\left(\frac{n-k}{e(k)}\right),$$

so the last summand vanishes, hence $\frac{n-k}{e(k)}$ has no big prime divisors, that is, $e(n-k) = e(k)$. This contradicts our choice. $\square$

Back to the problem, assume that $|\mathcal{B}| \geqslant 2$. Take any good number $n > p_1 p_2$, and let p_1^α be the largest power of p_1 smaller than n, so that $n \leqslant p_1^{\alpha+1} < p_1^\alpha p_2$. By the lemma, $e(n - p_1^\alpha) = e(p_1^\alpha) = p_1^\alpha$, which yields $p_1^\alpha \mid n$. Similarly, $p_2 \mid n$, so that $n \geqslant p_1^\alpha p_2$. This contradiction shows that $|\mathcal{B}| \leqslant 1$, which by (1) yields that f is listed in the answer.

Solution 3. We have $f(\prod p_i^{\alpha_i}) = \sum \alpha_i f(p_i)$. Note that

$$f(n-1) + f(n-2) + \ldots + \ldots f(n-k) \geqslant f(1) + \ldots + f(k)$$

for all $k = 1, 2, \ldots, n-1$, since the difference LHS$-$RHS is just $f(\binom{n-1}{k})$. Assume that $f(p) > 0$. If $f(k) = f(n-k)$ for all k, it implies that $\binom{n-1}{k}$ is not divisible by p for all $k = 1, 2, \ldots, n-2$. It is well known that it implies $n = a \cdot p^s$, $a < p$. If there are two primes p, q such that $f(p) > 0, f(q) > 0$, there exist only finitely many n which are equal both to $a \cdot p^s$, $a < p$, and $b \cdot q^t, b < q$. So there exists at most one such p, and therefore $f(n) = C \cdot \nu_p(n)$ for some constant C.

Solution 4. We call a function $f : \mathbb{N} \to \mathbb{N}_0$ satisfying (ii) *additive*. We call a pair (f, n), where f is an additive function and $n \in \mathbb{N}$, *good*, if for all $k < n$ it holds $f(k) = f(n-k)$. For an additive function f and a prime number p the number $\frac{f(p)}{\ln p}$ is denoted by $g(f, p)$.

Let (f, n) be a good pair such that $f(p) > 0$ for at least two primes less than n. Let p_0 be the prime with maximal $g(f, p)$ among all primes $p < n$. Let a_0 be the maximal exponent such that $p_0^{a_0} < n$. Then $f(k) < f(p_0^{a_0})$ for all $k < p_0^{a_0}$. Indeed, if $k = p_1^{a_1} \ldots p_m^{a_m} < p_0^{a_0}$, then

$$\begin{aligned} f(k) &= a_1 f(p_1) + \ldots + a_m f(p_m) = g(f, p_1) a_1 \ln p_1 + \ldots + g(f, p_m) a_m \ln a_m \\ &< g(f, p_0) a_0 \ln p_0 = f(p_0^{a_0}). \end{aligned}$$

Let $n = b p_0^{a_0} + r$, where $0 < r < p_0^{a_0}$. Then $f(r) = f(b p_0^{a_0}) \geqslant f(p_0^{a_0})$. This contradiction shows that $p_0^{a_0} | n$. Then $n = p_0^{\nu_{p_0}(n)} n'$, where $n' \leqslant p_0$.

The functions $f_1(m) := f(p_0) \nu_{p_0}(m)$ and $f_2 := f - f_1$ are additive (obviously $f(m) \geqslant f(p_0^{\nu_{p_0}(m)}) = f_1(m)$, since $p_0^{\nu_{p_0}(m)}$ divides m). For $k < n$, $\nu_p(k) = \nu_p(n-k)$. Hence the pair (f_2, n) is also good. Note that $f_2(p_0) = 0$.

Choose among all primes $p < n$ the prime q_0 with maximal $g(f_2, p)$. As above we can prove that $n = q_0^{\nu_{q_0}(n)} n''$ with $n'' < q_0$. Since $p_0 \neq q_0$, we get a contradiction. Thus $f(n) = f(p) \cdot \nu_p(n)$.

N6. For a positive integer n, let $d(n)$ be the number of positive divisors of n, and let $\varphi(n)$ be the number of positive integers not exceeding n which are coprime to n. Does there exist a constant C such that

$$\frac{\varphi(d(n))}{d(\varphi(n))} \leqslant C$$

for all $n \geqslant 1$?

(Cyprus)

Answer: No, such constant does not exist.

Solution 1. Fix $N > 1$, let $p_1, \ldots, p_k$ be all primes between 1 and N and $p_{k+1}, \ldots, p_{k+s}$ be all primes between $N+1$ and $2N$. Since for $j \leqslant k+s$ all prime divisors of $p_j - 1$ do not exceed N, we have

$$\prod_{j=1}^{k+s}(p_j - 1) = \prod_{i=1}^{k} p_i^{c_i},$$

with some fixed exponents $c_1, \ldots, c_k$. Choose a huge prime number q and consider a number

$$n = (p_1 \cdot \ldots \cdot p_k)^{q-1} \cdot (p_{k+1} \cdot \ldots \cdot p_{k+s}).$$

Then

$$\varphi(d(n)) = \varphi(q^k \cdot 2^s) = q^{k-1}(q-1)2^{s-1}$$

and

$$d(\varphi(n)) = d\left((p_1 \cdot \ldots \cdot p_k)^{q-2} \prod_{i=1}^{k+s}(p_i - 1)\right) = d\left(\prod_{i=1}^{k} p_i^{q-2+c_i}\right) = \prod_{i=1}^{k}(q-1+c_i),$$

so

$$\frac{\varphi(d(n))}{d(\varphi(n))} = \frac{q^{k-1}(q-1)2^{s-1}}{\prod_{i=1}^{k}(q-1+c_i)} = 2^{s-1} \cdot \frac{q-1}{q} \cdot \prod_{i=1}^{k} \frac{q}{q-1+c_i},$$

which can be made arbitrarily close to 2^{s-1} by choosing q large enough. It remains to show that s can be arbitrarily large, i.e. that there can be arbitrarily many primes between N and $2N$.

This follows, for instance, from the well-known fact that $\sum \frac{1}{p} = \infty$, where the sum is taken over the set $\mathbb{P}$ of prime numbers. Indeed, if, for some constant C, there were always at most C primes between 2^ℓ and $2^{\ell+1}$, we would have

$$\sum_{p \in \mathbb{P}} \frac{1}{p} = \sum_{\ell=0}^{\infty} \sum_{\substack{p \in \mathbb{P} \\ p \in [2^\ell, 2^{\ell+1})}} \frac{1}{p} \leqslant \sum_{\ell=0}^{\infty} \frac{C}{2^\ell} < \infty,$$

which is a contradiction.

Comment 1. Here we sketch several alternative elementary self-contained ways to perform the last step of the solution above. In particular, they avoid using divergence of $\sum \frac{1}{p}$.

Suppose that for some constant C and for every $k = 1, 2, \ldots$ there exist at most C prime numbers between 2^k and 2^{k+1}. Consider the prime factorization of the factorial $(2^n)! = \prod p^{\alpha_p}$. We have $\alpha_p = \lfloor 2^n/p \rfloor + \lfloor 2^n/p^2 \rfloor + \ldots$. Thus, for $p \in [2^k, 2^{k+1})$, we get $\alpha_p \leqslant 2^n/2^k + 2^n/2^{k+1} + \ldots = 2^{n-k+1}$, therefore $p^{\alpha_p} \leqslant 2^{(k+1)2^{n-k+1}}$. Combining this with the bound $(2m)! \geqslant m(m+1) \cdot \ldots \cdot (2m-1) \geqslant m^m$ for $m = 2^{n-1}$ we get

$$2^{(n-1) \cdot 2^{n-1}} \leqslant (2^n)! \leqslant \prod_{k=1}^{n-1} 2^{C(k+1)2^{n-k+1}},$$

or

$$\sum_{k=1}^{n-1} C(k+1)2^{1-k} \geqslant \frac{n-1}{2}$$

that fails for large n since $C(k+1)2^{1-k} < 1/3$ for all but finitely many k.

In fact, a much stronger inequality can be obtained in an elementary way: Note that the formula for $\nu_p(n!)$ implies that if p^α is the largest power of p dividing $\binom{n}{n/2}$, then $p^\alpha \leqslant n$. By looking at prime factorization of $\binom{n}{n/2}$ we instantaneously infer that

$$\pi(n) \geqslant \log_n \binom{n}{n/2} \geqslant \frac{\log(2^n/n)}{\log n} \geqslant \frac{n}{2\log n}.$$

This, in particular, implies that for infinitely many n there are at least $\frac{n}{3\log n}$ primes between n and $2n$.

Solution 2. In this solution we will use the Prime Number Theorem which states that

$$\pi(m) = \frac{m}{\log m} \cdot (1 + o(1)),$$

as m tends to infinity. Here and below $\pi(m)$ denotes the number of primes not exceeding m, and log the natural logarithm.

Let $m > 5$ be a large positive integer and let $n := p_1 p_2 \cdot \ldots \cdot p_{\pi(m)}$ be the product of all primes not exceeding m. Then $\varphi(d(n)) = \varphi\left(2^{\pi(m)}\right) = 2^{\pi(m)-1}$. Consider the number

$$\varphi(n) = \prod_{k=1}^{\pi(m)} (p_k - 1) = \prod_{s=1}^{\pi(m/2)} q_s^{\alpha_s},$$

where $q_1, \ldots, q_{\pi(m/2)}$ are primes not exceeding $m/2$. Note that every term $p_k - 1$ contributes at most one prime $q_s > \sqrt{m}$ into the product $\prod_s q_s^{\alpha_s}$, so we have

$$\sum_{s:\, q_s > \sqrt{m}} \alpha_s \leqslant \pi(m) \quad \Longrightarrow \quad \sum_{s:\, q_s > \sqrt{m}} (1 + \alpha_s) \leqslant \pi(m) + \pi(m/2).$$

Hence, applying the AM–GM inequality and the inequality $(A/x)^x \leqslant e^{A/e}$, we obtain

$$\prod_{s:\, q_s > \sqrt{m}} (\alpha_s + 1) \leqslant \left(\frac{\pi(m) + \pi(m/2)}{\ell}\right)^{\ell} \leqslant \exp\left(\frac{\pi(m) + \pi(m/2)}{e}\right),$$

where ℓ is the number of primes in the interval $(\sqrt{m}, m]$.

We then use a trivial bound $\alpha_i \leqslant \log_2(\varphi(n)) \leqslant \log_2 n < \log_2(m^m) < m^2$ for each i with $q_i < \sqrt{m}$ to obtain

$$\prod_{s=1}^{\pi(\sqrt{m})} (\alpha_s + 1) \leqslant \left(m^2\right)^{\sqrt{m}} = m^{2\sqrt{m}}.$$

Putting this together we obtain

$$d(\varphi(n)) = \prod_{s=1}^{\pi(m/2)} (\alpha_s + 1) \leqslant \exp\left(2\sqrt{m} \cdot \log m + \frac{\pi(m) + \pi(m/2)}{e}\right).$$

The prime number theorem then implies that

$$\limsup_{m\to\infty} \frac{\log(d(\varphi(n)))}{m/\log m} \leqslant \limsup_{m\to\infty} \frac{2\sqrt{m} \cdot \log m}{m/\log m} + \limsup_{m\to\infty} \frac{\pi(m) + \pi(m/2)}{e \cdot m/\log m} = \frac{3}{2e}.$$

Whereas, again by prime number theorem, we have

$$\liminf_{m\to\infty} \frac{\log(\varphi(d(n)))}{m/\log m} = \liminf_{m\to\infty} \frac{\log\left(2^{\pi(m)-1}\right)}{m/\log m} = \log 2.$$

Since $\frac{3}{2e} < \frac{3}{5} < \log 2$, this implies that $\varphi(d(n))/d(\varphi(n))$ can be arbitrarily large.

Comment 2. The original formulation of the problem was asking whether $d(\varphi(n)) \geqslant \varphi(d(n))$ for all but finitely many values of n. The Problem Selection Committee decided that the presented version is better suited for the Shortlist.

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N7. Let $\mathcal{S}$ be a set consisting of $n \geqslant 3$ positive integers, none of which is a sum of two other distinct members of $\mathcal{S}$. Prove that the elements of $\mathcal{S}$ may be ordered as $a_1, a_2, \ldots, a_n$ so that a_i does not divide $a_{i-1} + a_{i+1}$ for all $i = 2, 3, \ldots, n-1$.

(Ukraine)

Common remarks. In all solutions, we call a set $\mathcal{S}$ of positive integers *good* if no its element is a sum of two other distinct members of $\mathcal{S}$. We will use the following simple observation.

Observation A. If a, b, and c are three distinct elements of a good set $\mathcal{S}$ with $b > a, c$, then $b \nmid a + c$. Otherwise, since $b \neq a + c$, we would have $b \leqslant (a+c)/2 < \max\{a, c\}$.

Solution 1. We prove the following stronger statement.

Claim. Let $\mathcal{S}$ be a good set consisting of $n \geqslant 2$ positive integers. Then the elements of $\mathcal{S}$ may be ordered as $a_1, a_2, \ldots, a_n$ so that $a_i \nmid a_{i-1} + a_{i+1}$ and $a_i \nmid a_{i-1} - a_{i+1}$, for all $i = 2, 3, \ldots, n-1$.

Proof. Say that the ordering $a_1, \ldots, a_n$ of $\mathcal{S}$ is *nice* if it satisfies the required property.

We proceed by induction on n. The base case $n = 2$ is trivial, as there are no restrictions on the ordering.

To perform the step of induction, suppose that $n \geqslant 3$. Let $a = \max \mathcal{S}$, and set $\mathcal{T} = \mathcal{S} \setminus \{a\}$. Use the inductive hypothesis to find a nice ordering $b_1, \ldots, b_{n-1}$ of $\mathcal{T}$. We will show that a may be inserted into this sequence so as to reach a nice ordering of $\mathcal{S}$. In other words, we will show that there exists a $j \in \{1, 2, \ldots, n\}$ such that the ordering

$$N_j = (b_1, \ldots, b_{j-1}, a, b_j, b_{j+1}, \ldots, b_{n-1})$$

is nice.

Assume that, for some j, the ordering N_j is not nice, so that some element x in it divides either the sum or the difference of two adjacent ones. This did not happen in the ordering of $\mathcal{T}$, hence $x \in \{b_{j-1}, a, b_j\}$ (if, say, b_{j-1} does not exist, then $x \in \{a, b_j\}$; a similar agreement is applied hereafter). But the case $x = a$ is impossible: a cannot divide $b_{j-1} - b_j$, since $0 < |b_{j-1} - b_j| < a$, while $a \nmid b_{j-1} + b_j$ by Observation A. Therefore $x \in \{b_{j-1}, b_j\}$. In this case, assign the number x to the index j.

Suppose now that none of the N_j is nice. Since there are n possible indices j, and only $n-1$ elements in $\mathcal{T}$, one of those elements (say, b_k) is assigned to two different indices, which then should equal k and $k+1$. This means that b_k divides the numbers $b_{k-1} + \varepsilon_1 a$ and $a + \varepsilon_2 b_{k+1}$, for some signs $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. But then

$$b_{k-1} \equiv -\varepsilon_1 a \equiv \varepsilon_1 \varepsilon_2 b_{k+1} \pmod{b_k},$$

and therefore $b_k \mid b_{k-1} - \varepsilon_1 \varepsilon_2 b_{k+1}$, which means that the ordering of $\mathcal{T}$ was not nice. This contradiction proves the step of induction. $\square$

Solution 2. We again prove a stronger statement.

Claim. Let $\mathcal{S}$ be an arbitrary set of $n \geqslant 3$ positive integers. Then its elements can be ordered as $a_1, \ldots, a_n$ so that, if $a_i \mid a_{i-1} + a_{i+1}$, then $a_i = \max \mathcal{S}$.

The claim easily implies what we need to prove, due to Observation A.

To prove the Claim, introduce the function f which assigns to any two elements $a, b \in \mathcal{S}$ with $a < b$ the unique integer $f(a, b) \in \{1, 2, \ldots, a\}$ such that $a \mid b + f(a, b)$. Hence, if $b \mid a + c$ for some $a, b, c \in \mathcal{S}$ with $a < b < c$, then $a = f(b, c)$. Therefore, the Claim is a consequence of the following combinatorial lemma.

Lemma. Let $\mathcal{S}$ be a set of $n \geqslant 3$ positive integers, and let f be a function which assigns to any $a, b \in \mathcal{S}$ with $a < b$ some integer from the range $\{1, \ldots, a\}$. Then the elements of $\mathcal{S}$ may be ordered as $a_1, a_2, \ldots, a_n$ so as to satisfy the following two conditions simultaneously:

(*i*) **Unimodality:** There exists a $j \in \{1, 2, \ldots, n\}$ such that $a_1 < a_2 < \ldots < a_j > a_{j+1} > \ldots > a_n$; and

(*ii*) f**-avoidance:** If $a < b$ are two elements of $\mathcal{S}$, which are adjacent in the ordering, then $f(a, b)$ is not adjacent to a.

Proof. We call an ordering of $\mathcal{S}$ satisfying (*i*) and (*ii*) f*-nice*. We agree that $f(x, y) = x$ for $x \geqslant y$; this agreement puts no extra restriction.

We proceed by induction; for the base case $n = 3$, it suffices to put the maximal element in $\mathcal{S}$ onto the middle position.

To perform the step of induction, let $p < q$ be the two minimal elements of $\mathcal{S}$, and set $\mathcal{T} = \mathcal{S} \setminus \{p\}$. Define a function g by assigning to any elements $a < b$ of $\mathcal{T}$ the value

$$g(a, b) = \begin{cases} q, & \text{if } f(a, b) = p; \\ f(a, b), & \text{otherwise.} \end{cases} \tag{1}$$

Notice that $g(a, b) \leqslant a$ for all $a, b \in \mathcal{T}$.

Use the inductive hypothesis to get a g-nice ordering $b_1, b_2, \ldots, b_{n-1}$ of $\mathcal{T}$. By unimodality, either b_1 or b_{n-1} equals q; these cases differ only by reverting the order, so we assume $b_1 = q$.

Notice that, according to (1), the number $f(b_2, b_3)$ differs from both p and q. On the other hand, the number $f(b_{n-1}, b_{n-2})$ differs from at least one of them — say, from r; set $s = p + q - r$, so that $\{r, s\} = \{p, q\}$. Now, order $\mathcal{S}$ as

$$s, b_2, b_3, \ldots, b_{n-1}, r.$$

By the induction hypothesis and the above choice, this ordering is nice. □

Comment. In the original proposal, the numbers in the set were assumed to be odd (which implies that none is a sum of two others); moreover, the proposal requested to arrange in a row all numbers but one.

On the other hand, Solution 2 shows that the condition of $\mathcal{S}$ being good may be relaxed to the condition that the maximal element of $\mathcal{S}$ is not a sum of two other elements in $\mathcal{S}$. On the other hand, the set $\{1, 2, 3\}$ shows that the condition cannot be merely omitted.

The Problem Selection Committee considered several versions of the problem and chose the best version in their opinion for the Shortlist.

IMO 2020

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Mathematical
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62nd International Mathematical Olympiad Saint-Petersburg Russia

Shortlisted Problems

(with solutions)

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62nd International Mathematical Olympiad

Saint-Petersburg — Russia, 16th–24th July 2021

Note of Confidentiality

The Shortlist has to be kept strictly confidential until the conclusion of the following International Mathematical Olympiad.

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Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2021 thank the following 51 countries for contributing 175 problem proposals:

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Problem Selection Committee

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Problems

Algebra

A1. Let n be an integer, and let A be a subset of $\{0, 1, 2, 3, \ldots, 5^n\}$ consisting of $4n+2$ numbers. Prove that there exist $a, b, c \in A$ such that $a < b < c$ and $c + 2a > 3b$.

A2. For every integer $n \geqslant 1$ consider the $n \times n$ table with entry $\left\lfloor \frac{ij}{n+1} \right\rfloor$ at the intersection of row i and column j, for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Determine all integers $n \geqslant 1$ for which the sum of the n^2 entries in the table is equal to $\frac{1}{4}n^2(n-1)$.

A3. Given a positive integer n, find the smallest value of $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor$ over all permutations $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$.

A4. Show that for all real numbers $x_1, \ldots, x_n$ the following inequality holds:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}.$$

A5. Let $n \geqslant 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\sum_{k=1}^{n}\frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

A6. Let A be a finite set of (not necessarily positive) integers, and let $m \geqslant 2$ be an integer. Assume that there exist non-empty subsets $B_1, B_2, B_3, \ldots, B_m$ of A whose elements add up to the sums $m^1, m^2, m^3, \ldots, m^m$, respectively. Prove that A contains at least $m/2$ elements.

A7. Let $n \geqslant 1$ be an integer, and let $x_0, x_1, \ldots, x_{n+1}$ be $n+2$ non-negative real numbers that satisfy $x_i x_{i+1} - x_{i-1}^2 \geqslant 1$ for all $i = 1, 2, \ldots, n$. Show that

$$x_0 + x_1 + \cdots + x_n + x_{n+1} > \left(\frac{2n}{3}\right)^{3/2}.$$

A8. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$\big(f(a) - f(b)\big)\big(f(b) - f(c)\big)\big(f(c) - f(a)\big) = f(ab^2 + bc^2 + ca^2) - f(a^2b + b^2c + c^2a)$$

for all real numbers a, b, c.

Combinatorics

C1. Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a, b, c, d \in S$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three pairwise distinct $x, y, z \in S$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

C2. Let $n \geqslant 3$ be an integer. An integer $m \geqslant n + 1$ is called *n-colourful* if, given infinitely many marbles in each of n colours $C_1, C_2, \ldots, C_n$, it is possible to place m of them around a circle so that in any group of $n + 1$ consecutive marbles there is at least one marble of colour C_i for each $i = 1, \ldots, n$.

Prove that there are only finitely many positive integers which are not n-colourful. Find the largest among them.

C3. A thimblerigger has 2021 thimbles numbered from 1 through 2021. The thimbles are arranged in a circle in arbitrary order. The thimblerigger performs a sequence of 2021 moves; in the k^{th} move, he swaps the positions of the two thimbles adjacent to thimble k.

Prove that there exists a value of k such that, in the k^{th} move, the thimblerigger swaps some thimbles a and b such that $a < k < b$.

C4. The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a *path from X to Y* is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called *diverse* if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

C5. Let n and k be two integers with $n > k \geqslant 1$. There are $2n + 1$ students standing in a circle. Each student S has $2k$ *neighbours*— namely, the k students closest to S on the right, and the k students closest to S on the left.

Suppose that $n + 1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbours.

C6. A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share a side). The hunter wins if after some finite time either

- the rabbit cannot move; or
- the hunter can determine the cell in which the rabbit started.

Decide whether there exists a winning strategy for the hunter.

C7. Consider a checkered $3m \times 3m$ square, where m is an integer greater than 1. A frog sits on the lower left corner cell S and wants to get to the upper right corner cell F. The frog can hop from any cell to either the next cell to the right or the next cell upwards.

Some cells can be sticky, and the frog gets trapped once it hops on such a cell. A set X of cells is called *blocking* if the frog cannot reach F from S when all the cells of X are sticky. A blocking set is *minimal* if it does not contain a smaller blocking set.

(a) Prove that there exists a minimal blocking set containing at least $3m^2 - 3m$ cells.

(b) Prove that every minimal blocking set contains at most $3m^2$ cells.

Note. An example of a minimal blocking set for $m = 2$ is shown below. Cells of the set X are marked by letters x.

					F
x	x				
		x			
			x		
				x	
S		x			

C8. Determine the largest N for which there exists a table T of integers with N rows and 100 columns that has the following properties:

(i) Every row contains the numbers 1, 2, ..., 100 in some order.

(ii) For any two distinct rows r and s, there is a column c such that $|T(r,c) - T(s,c)| \geqslant 2$.

Here $T(r,c)$ means the number at the intersection of the row r and the column c.

Geometry

G1. Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B. The circumcircle of the triangle ACD meets the segment PD again at Q, and the circumcircle of the triangle APQ meets the segment PC again at R. Prove that the lines CD, AQ, and BR are concurrent.

G2. Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Prove that the perimeters of the (possibly self-intersecting) quadrilaterals $ADTX$ and $CDYZ$ are equal.

G3.

Version 1. Let n be a fixed positive integer, and let S be the set of points (x, y) on the Cartesian plane such that both coordinates x and y are nonnegative integers smaller than $2n$ (thus $|\mathsf{S}| = 4n^2$). Assume that $\mathcal{F}$ is a set consisting of n^2 quadrilaterals such that all their vertices lie in S, and each point in S is a vertex of exactly one of the quadrilaterals in $\mathcal{F}$.

Determine the largest possible sum of areas of all n^2 quadrilaterals in $\mathcal{F}$.

Version 2. Let n be a fixed positive integer, and let S be the set of points (x, y) on the Cartesian plane such that both coordinates x and y are nonnegative integers smaller than $2n$ (thus $|\mathsf{S}| = 4n^2$). Assume that $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in S, and each point in S is a vertex of exactly one of the polygons in $\mathcal{F}$.

Determine the largest possible sum of areas of all polygons in $\mathcal{F}$.

G4. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that $TE \parallel CD$ and $TF \parallel AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

G5. Let $ABCD$ be a cyclic quadrilateral whose sides have pairwise different lengths. Let O be the circumcentre of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1, respectively. Let O_B be the centre of the circle which passes through B and is tangent to AC at D_1. Similarly, let O_D be the centre of the circle which passes through D and is tangent to AC at B_1.

Assume that $BD_1 \parallel DB_1$. Prove that O lies on the line O_BO_D.

G6. Determine all integers $n \geqslant 3$ satisfying the following property: every convex n-gon whose sides all have length 1 contains an equilateral triangle of side length 1.

(Every polygon is assumed to contain its boundary.)

G7. A point D is chosen inside an acute-angled triangle ABC with $AB > AC$ so that $\angle BAD = \angle DAC$. A point E is constructed on the segment AC so that $\angle ADE = \angle DCB$. Similarly, a point F is constructed on the segment AB so that $\angle ADF = \angle DBC$. A point X is chosen on the line AC so that $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and DXE. Prove that the lines BC, EF, and O_1O_2 are concurrent.

G8. Let ω be the circumcircle of a triangle ABC, and let Ω_A be its excircle which is tangent to the segment BC. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y, respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $AR \perp BC$.

Number Theory

N1. Determine all integers $n \geqslant 1$ for which there exists a pair of positive integers (a, b) such that no cube of a prime divides $a^2 + b + 3$ and

$$\frac{ab + 3b + 8}{a^2 + b + 3} = n.$$

N2. Let $n \geqslant 100$ be an integer. The numbers $n, n+1, \ldots, 2n$ are written on $n+1$ cards, one number per card. The cards are shuffled and divided into two piles. Prove that one of the piles contains two cards such that the sum of their numbers is a perfect square.

N3. Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for every $i = 1, 2, \ldots, k$, the number $d_1 + \cdots + d_i$ is a perfect square.

N4. Alice is given a rational number $r > 1$ and a line with two points $B \neq R$, where point R contains a red bead and point B contains a blue bead. Alice plays a solitaire game by performing a sequence of moves. In every move, she chooses a (not necessarily positive) integer k, and a bead to move. If that bead is placed at point X, and the other bead is placed at Y, then Alice moves the chosen bead to point X' with $\overrightarrow{YX'} = r^k\,\overrightarrow{YX}$.

Alice's goal is to move the red bead to the point B. Find all rational numbers $r > 1$ such that Alice can reach her goal in at most 2021 moves.

N5. Prove that there are only finitely many quadruples (a, b, c, n) of positive integers such that

$$n! = a^{n-1} + b^{n-1} + c^{n-1}.$$

N6. Determine all integers $n \geqslant 2$ with the following property: every n pairwise distinct integers whose sum is not divisible by n can be arranged in some order $a_1, a_2, \ldots, a_n$ so that n divides $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$.

N7. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers such that a_{n+2m} divides $a_n + a_{n+m}$ for all positive integers n and m. Prove that this sequence is eventually periodic, i.e. there exist positive integers N and d such that $a_n = a_{n+d}$ for all $n > N$.

N8. For a polynomial $P(x)$ with integer coefficients let $P^1(x) = P(x)$ and $P^{k+1}(x) = P(P^k(x))$ for $k \geqslant 1$. Find all positive integers n for which there exists a polynomial $P(x)$ with integer coefficients such that for every integer $m \geqslant 1$, the numbers $P^m(1), \ldots, P^m(n)$ leave exactly $\lceil n/2^m \rceil$ distinct remainders when divided by n.

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Solutions

Algebra

A1. Let n be an integer, and let A be a subset of $\{0, 1, 2, 3, \ldots, 5^n\}$ consisting of $4n+2$ numbers. Prove that there exist $a, b, c \in A$ such that $a < b < c$ and $c + 2a > 3b$.

Solution 1. (By contradiction) Suppose that there exist $4n+2$ non-negative integers $x_0 < x_1 < \cdots < x_{4n+1}$ that violate the problem statement. Then in particular $x_{4n+1} + 2x_i \leqslant 3x_{i+1}$ for all $i = 0, \ldots, 4n-1$, which gives

$$x_{4n+1} - x_i \geqslant \frac{3}{2}(x_{4n+1} - x_{i+1}).$$

By a trivial induction we then get

$$x_{4n+1} - x_i \geqslant \left(\frac{3}{2}\right)^{4n-i}(x_{4n+1} - x_{4n}),$$

which for $i = 0$ yields the contradiction

$$x_{4n+1} - x_0 \geqslant \left(\frac{3}{2}\right)^{4n}(x_{4n+1} - x_{4n}) = \left(\frac{81}{16}\right)^{n}(x_{4n+1} - x_{4n}) > 5^n \cdot 1.$$

Solution 2. Denote the maximum element of A by c. For $k = 0, \ldots, 4n-1$ let

$$A_k = \left\{x \in A : (1 - (2/3)^k)c \leqslant x < (1 - (2/3)^{k+1})c\right\}.$$

Note that

$$(1 - (2/3)^{4n})c = c - (16/81)^n c > c - (1/5)^n c \geqslant c - 1,$$

which shows that the sets $A_0, A_1, \ldots, A_{4n-1}$ form a partition of $A \setminus \{c\}$. Since $A \setminus \{c\}$ has $4n+1$ elements, by the pigeonhole principle some set A_k does contain at least two elements of $A \setminus \{c\}$. Denote these two elements a and b and assume $a < b$, so that $a < b < c$. Then

$$c + 2a \geqslant c + 2(1 - (2/3)^k)c = (3 - 2(2/3)^k)c = 3(1 - (2/3)^{k+1})c > 3b,$$

as desired.

A2. For every integer $n \geqslant 1$ consider the $n \times n$ table with entry $\left\lfloor \frac{ij}{n+1} \right\rfloor$ at the intersection of row i and column j, for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Determine all integers $n \geqslant 1$ for which the sum of the n^2 entries in the table is equal to $\frac{1}{4}n^2(n-1)$.

Answer: All integers n for which $n+1$ is a prime.

Solution 1. First, observe that every pair x, y of real numbers for which the sum $x + y$ is integer satisfies

$$\lfloor x \rfloor + \lfloor y \rfloor \geqslant x + y - 1. \tag{1}$$

The inequality is strict if x and y are integers, and it holds with equality otherwise.

We estimate the sum S as follows.

$$\begin{aligned} 2S = \sum_{1 \leqslant i,j \leqslant n} \left(\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{ij}{n+1} \right\rfloor \right) &= \sum_{1 \leqslant i,j \leqslant n} \left(\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{(n+1-i)j}{n+1} \right\rfloor \right) \\ &\geqslant \sum_{1 \leqslant i,j \leqslant n} (j-1) = \frac{(n-1)n^2}{2}. \end{aligned}$$

The inequality in the last line follows from (1) by setting $x = ij/(n+1)$ and $y = (n+1-i)j/(n+1)$, so that $x + y = j$ is integral.

Now $S = \frac{1}{4}n^2(n-1)$ if and only if the inequality in the last line holds with equality, which means that none of the values $ij/(n+1)$ with $1 \leqslant i, j \leqslant n$ may be integral.

Hence, if $n+1$ is composite with factorisation $n+1 = ab$ for $2 \leqslant a, b \leqslant n$, one gets a strict inequality for $i = a$ and $j = b$. If $n+1$ is a prime, then $ij/(n+1)$ is never integral and $S = \frac{1}{4}n^2(n-1)$.

Solution 2. To simplify the calculation with indices, extend the table by adding a phantom column of index 0 with zero entries (which will not change the sum of the table). Fix a row i with $1 \leqslant i \leqslant n$, and let $d := \gcd(i, n+1)$ and $k := (n+1)/d$. For columns $j = 0, \ldots, n$, define the remainder $r_j := ij \bmod (n+1)$. We first prove the following

Claim. For every integer g with $1 \leqslant g \leqslant d$, the remainders r_j with indices j in the range

$$(g-1)k \leqslant j \leqslant gk - 1 \tag{2}$$

form a permutation of the k numbers $0 \cdot d,\ 1 \cdot d,\ 2 \cdot d,\ \ldots,\ (k-1) \cdot d$.

Proof. If $r_{j'} = r_j$ holds for two indices j' and j in (2), then $i(j'-j) \equiv 0 \bmod (n+1)$, so that $j'-j$ is a multiple of k; since $|j'-j| \leqslant k-1$, this implies $j' = j$. Hence, the k remainders are pairwise distinct. Moreover, each remainder $r_j = ij \bmod (n+1)$ is a multiple of $d = \gcd(i, n+1)$. This proves the claim.

We then have

$$\sum_{j=0}^{n} r_j = \sum_{g=1}^{d} \sum_{\ell=0}^{(n+1)/d-1} \ell d = d^2 \cdot \frac{1}{2} \left(\frac{n+1}{d} - 1 \right) \frac{n+1}{d} = \frac{(n+1-d)(n+1)}{2}. \tag{3}$$

By using (3), compute the sum S_i of row i as follows:

$$\begin{aligned} S_i &= \sum_{j=0}^{n} \left\lfloor \frac{ij}{n+1} \right\rfloor = \sum_{j=0}^{n} \frac{ij - r_j}{n+1} = \frac{i}{n+1} \sum_{j=0}^{n} j - \frac{1}{n+1} \sum_{j=0}^{n} r_j \\ &= \frac{i}{n+1} \cdot \frac{n(n+1)}{2} - \frac{1}{n+1} \cdot \frac{(n+1-d)(n+1)}{2} = \frac{(in - n - 1 + d)}{2}. \end{aligned} \tag{4}$$

Equation (4) yields the following lower bound on the row sum S_i, which holds with equality if and only if $d = \gcd(i, n+1) = 1$:

$$S_i \geqslant \frac{(in - n - 1 + 1)}{2} = \frac{n(i-1)}{2}. \tag{5}$$

By summing up the bounds (5) for the rows $i = 1, \ldots, n$, we get the following lower bound for the sum of all entries in the table

$$\sum_{i=1}^{n} S_i \geqslant \sum_{i=1}^{n} \frac{n}{2}(i-1) = \frac{n^2(n-1)}{4}. \tag{6}$$

In (6) we have equality if and only if equality holds in (5) for each $i = 1, \ldots, n$, which happens if and only if $\gcd(i, n+1) = 1$ for each $i = 1, \ldots, n$, which is equivalent to the fact that $n+1$ is a prime. Thus the sum of the table entries is $\frac{1}{4}n^2(n-1)$ if and only if $n+1$ is a prime.

Comment. To simplify the answer, in the problem statement one can make a change of variables by introducing $m := n+1$ and writing everything in terms of m. The drawback is that the expression for the sum will then be $\frac{1}{4}(m-1)^2(m-2)$ which seems more artificial.

A3. Given a positive integer n, find the smallest value of $\left\lfloor\frac{a_1}{1}\right\rfloor+\left\lfloor\frac{a_2}{2}\right\rfloor+\cdots+\left\lfloor\frac{a_n}{n}\right\rfloor$ over all permutations $(a_1,a_2,\ldots,a_n)$ of $(1,2,\ldots,n)$.

Answer: The minimum of such sums is $\lfloor\log_2 n\rfloor+1$; so if $2^k\leqslant n<2^{k+1}$, the minimum is $k+1$.

Solution 1. Suppose that $2^k\leqslant n<2^{k+1}$ with some nonnegative integer k. First we show a permutation $(a_1,a_2,\ldots,a_n)$ such that $\left\lfloor\frac{a_1}{1}\right\rfloor+\left\lfloor\frac{a_2}{2}\right\rfloor+\cdots+\left\lfloor\frac{a_n}{n}\right\rfloor=k+1$; then we will prove that $\left\lfloor\frac{a_1}{1}\right\rfloor+\left\lfloor\frac{a_2}{2}\right\rfloor+\cdots+\left\lfloor\frac{a_n}{n}\right\rfloor\geqslant k+1$ for every permutation. Hence, the minimal possible value will be $k+1$.

I. Consider the permutation

$$(a_1)=(1),\quad (a_2,a_3)=(3,2),\quad (a_4,a_5,a_6,a_7)=(7,4,5,6),\quad \ldots$$
$$(a_{2^{k-1}},\ldots,a_{2^k-1})=(2^k-1,2^{k-1},2^{k-1}+1,\ldots,2^k-2),$$
$$(a_{2^k},\ldots,a_n)=(n,2^k,2^k+1,\ldots,n-1).$$

This permutation consists of $k+1$ cycles. In every cycle $(a_p,\ldots,a_q)=(q,p,p+1,\ldots,q-1)$ we have $q<2p$, so

$$\sum_{i=p}^{q}\left\lfloor\frac{a_i}{i}\right\rfloor=\left\lfloor\frac{q}{p}\right\rfloor+\sum_{i=p+1}^{q}\left\lfloor\frac{i-1}{i}\right\rfloor=1;$$

The total sum over all cycles is precisely $k+1$.

II. In order to establish the lower bound, we prove a more general statement.

Claim. If $b_1,\ldots,b_{2^k}$ are distinct positive integers then

$$\sum_{i=1}^{2^k}\left\lfloor\frac{b_i}{i}\right\rfloor\geqslant k+1.$$

From the Claim it follows immediately that $\displaystyle\sum_{i=1}^{n}\left\lfloor\frac{a_i}{i}\right\rfloor\geqslant\sum_{i=1}^{2^k}\left\lfloor\frac{a_i}{i}\right\rfloor\geqslant k+1$.

Proof of the Claim. Apply induction on k. For $k=1$ the claim is trivial, $\left\lfloor\frac{b_1}{1}\right\rfloor\geqslant 1$. Suppose the Claim holds true for some positive integer k, and consider $k+1$.

If there exists an index j such that $2^k<j\leqslant 2^{k+1}$ and $b_j\geqslant j$ then

$$\sum_{i=1}^{2^{k+1}}\left\lfloor\frac{b_i}{i}\right\rfloor\geqslant\sum_{i=1}^{2^k}\left\lfloor\frac{b_i}{i}\right\rfloor+\left\lfloor\frac{b_j}{j}\right\rfloor\geqslant(k+1)+1$$

by the induction hypothesis, so the Claim is satisfied.

Otherwise we have $b_j<j\leqslant 2^{k+1}$ for every $2^k<j\leqslant 2^{k+1}$. Among the 2^{k+1} distinct numbers $b_1,\ldots,b_{2^{k+1}}$ there is some b_m which is at least 2^{k+1}; that number must be among $b_1\ldots,b_{2^k}$. Hence, $1\leqslant m\leqslant 2^k$ and $b_m\geqslant 2^{k+1}$.

We will apply the induction hypothesis to the numbers

$$c_1=b_1,\ldots,c_{m-1}=b_{m-1},\quad c_m=b_{2^k+1},\quad c_{m+1}=b_{m+1},\ldots,c_{2^k}=b_{2^k},$$

so take the first 2^k numbers but replace b_m with b_{2^k+1}. Notice that

$$\left\lfloor\frac{b_m}{m}\right\rfloor\geqslant\left\lfloor\frac{2^{k+1}}{m}\right\rfloor=\left\lfloor\frac{2^k+2^k}{m}\right\rfloor\geqslant\left\lfloor\frac{b_{2^k+1}+m}{m}\right\rfloor=\left\lfloor\frac{c_m}{m}\right\rfloor+1.$$

For the other indices i with $1 \leqslant i \leqslant 2^k$, $i \neq m$ we have $\left\lfloor \frac{b_i}{i} \right\rfloor = \left\lfloor \frac{c_i}{i} \right\rfloor$, so

$$\sum_{i=1}^{2^{k+1}} \left\lfloor \frac{b_i}{i} \right\rfloor = \sum_{i=1}^{2^k} \left\lfloor \frac{b_i}{i} \right\rfloor \geqslant \sum_{i=1}^{2^k} \left\lfloor \frac{c_i}{i} \right\rfloor + 1 \geqslant (k+1) + 1.$$

That proves the Claim and hence completes the solution. $\square$

Solution 2. We present a different proof for the lower bound.

Assume again $2^k \leqslant n < 2^{k+1}$, and let $P = \{2^0, 2^1, \ldots, 2^k\}$ be the set of powers of 2 among $1, 2, \ldots, n$. Call an integer $i \in \{1, 2, \ldots, n\}$ and the interval $[i, a_i]$ *good* if $a_i \geqslant i$.

Lemma 1. The good intervals cover the integers $1, 2, \ldots, n$.
Proof. Consider an arbitrary $x \in \{1, 2 \ldots, n\}$; we want to find a good interval $[i, a_i]$ that covers x; i.e., $i \leqslant x \leqslant a_i$. Take the cycle of the permutation that contains x, that is $(x, a_x, a_{a_x}, \ldots)$. In this cycle, let i be the first element with $a_i \geqslant x$; then $i \leqslant x \leqslant a_i$. $\square$

Lemma 2. If a good interval $[i, a_i]$ covers p distinct powers of 2 then $\left\lfloor \frac{a_i}{i} \right\rfloor \geqslant p$; more formally, $\left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \left|[i, a_i] \cap P\right|$.
Proof. The ratio of the smallest and largest powers of 2 in the interval is at least 2^{p-1}. By Bernoulli's inequality, $\frac{a_i}{i} \geqslant 2^{p-1} \geqslant p$; that proves the lemma. $\square$

Now, by Lemma 1, the good intervals cover P. By applying Lemma 2 as well, we obtain that

$$\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor = \sum_{i \text{ is good}}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \sum_{i \text{ is good}}^{n} \left|[i, a_i] \cap P\right| \geqslant |P| = k+1.$$

Solution 3. We show yet another proof for the lower bound, based on the following inequality.

Lemma 3.

$$\left\lfloor \frac{a}{b} \right\rfloor \geqslant \log_2 \frac{a+1}{b}$$

for every pair a, b of positive integers.
Proof. Let $t = \left\lfloor \frac{a}{b} \right\rfloor$, so $t \leqslant \frac{a}{b}$ and $\frac{a+1}{b} \leqslant t+1$. By applying the inequality $2^t \geqslant t+1$, we obtain

$$\left\lfloor \frac{a}{b} \right\rfloor = t \geqslant \log_2(t+1) \geqslant \log_2 \frac{a+1}{b}.$$ $\square$

By applying the lemma to each term, we get

$$\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \sum_{i=1}^{n} \log_2 \frac{a_i+1}{i} = \sum_{i=1}^{n} \log_2(a_i+1) - \sum_{i=1}^{n} \log_2 i.$$

Notice that the numbers $a_1+1, a_2+1, \ldots, a_n+1$ form a permutation of $2, 3, \ldots, n+1$. Hence, in the last two sums all terms cancel out, except for $\log_2(n+1)$ in the first sum and $\log_2 1 = 0$ in the second sum. Therefore,

$$\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \log_2(n+1) > k.$$

As the left-hand side is an integer, it must be at least $k+1$.

A4. Show that for all real numbers $x_1, \ldots, x_n$ the following inequality holds:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}.$$

Solution 1. If we add t to all the variables then the left-hand side remains constant and the right-hand side becomes

$$H(t) := \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j + 2t|}.$$

Let T be large enough such that both $H(-T)$ and $H(T)$ are larger than the value L of the left-hand side of the inequality we want to prove. Not necessarily distinct points $p_{i,j} := -(x_i+x_j)/2$ together with T and $-T$ split the real line into segments and two rays such that on each of these segments and rays the function $H(t)$ is concave since $f(t) := \sqrt{|\ell + 2t|}$ is concave on both intervals $(-\infty, -\ell/2]$ and $[-\ell/2, +\infty)$. Let $[a, b]$ be the segment containing zero. Then concavity implies $H(0) \geqslant \min\{H(a), H(b)\}$ and, since $H(\pm T) > L$, it suffices to prove the inequalities $H(-(x_i + x_j)/2) \geqslant L$, that is to prove the original inequality in the case when all numbers are shifted in such a way that two variables x_i and x_j add up to zero. In the following we denote the shifted variables still by x_i.

If $i = j$, i.e. $x_i = 0$ for some index i, then we can remove x_i which will decrease both sides by $2\sum_k \sqrt{|x_k|}$. Similarly, if $x_i + x_j = 0$ for distinct i and j we can remove both x_i and x_j which decreases both sides by

$$2\sqrt{2|x_i|} + 2 \cdot \sum_{k\neq i,j}\left(\sqrt{|x_k + x_i|} + \sqrt{|x_k + x_j|}\right).$$

In either case we reduced our inequality to the case of smaller n. It remains to note that for $n = 0$ and $n = 1$ the inequality is trivial.

Solution 2. For real p consider the integral

$$I(p) = \int_0^{\infty} \frac{1 - \cos(px)}{x\sqrt{x}}\, dx,$$

which clearly converges to a strictly positive number. By changing the variable $y = |p|x$ one notices that $I(p) = \sqrt{|p|}I(1)$. Hence, by using the trigonometric formula $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$ we obtain

$$\sqrt{|a+b|}-\sqrt{|a-b|} = \frac{1}{I(1)}\int_0^{\infty}\frac{\cos((a-b)x) - \cos((a+b)x)}{x\sqrt{x}}\, dx = \frac{1}{I(1)}\int_0^{\infty}\frac{2\sin(ax)\sin(bx)}{x\sqrt{x}}\, dx,$$

from which our inequality immediately follows:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|} - \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} = \frac{2}{I(1)}\int_0^{\infty}\frac{\left(\sum_{i=1}^{n}\sin(x_i x)\right)^2}{x\sqrt{x}}\, dx \geqslant 0.$$

Comment 1. A more general inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|^r \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|x_i + x_j|^r$$

holds for any $r \in [0, 2]$. The first solution can be repeated verbatim for any $r \in [0, 1]$ but not for $r > 1$. In the second solution, by putting x^{r+1} in the denominator in place of $x\sqrt{x}$ we can prove the inequality for any $r \in (0, 2)$ and the cases $r = 0, 2$ are easy to check by hand.

Comment 2. In fact, the integral from Solution 2 can be computed explicitly, we have $I(1) = \sqrt{2\pi}$.

A5. Let $n \geqslant 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

Solution 1. For all $k \leqslant n$, let

$$s_k = a_1 + a_2 + \cdots + a_k \qquad \text{and} \qquad b_k = \frac{a_k s_{k-1}^2}{1-a_k},$$

with the convention that $s_0 = 0$. Note that b_k is exactly a summand in the sum we need to estimate. We shall prove the inequality

$$b_k < \frac{s_k^3 - s_{k-1}^3}{3}. \tag{1}$$

Indeed, it suffices to check that

$$\begin{aligned}
(1) \iff & 0 < (1-a_k)\left((s_{k-1}+a_k)^3 - s_{k-1}^3\right) - 3a_k s_{k-1}^2 \\
\iff & 0 < (1-a_k)\left(3s_{k-1}^2 + 3s_{k-1}a_k + a_k^2\right) - 3s_{k-1}^2 \\
\iff & 0 < -3a_k s_{k-1}^2 + 3(1-a_k)s_{k-1}a_k + (1-a_k)a_k^2 \\
\iff & 0 < 3(1-a_k-s_{k-1})s_{k-1}a_k + (1-a_k)a_k^2
\end{aligned}$$

which holds since $a_k + s_{k-1} = s_k \leqslant 1$ and $a_k \in (0,1)$.

Thus, adding inequalities (1) for $k = 1, \ldots, n$, we conclude that

$$b_1 + b_2 + \cdots + b_n < \frac{s_n^3 - s_0^3}{3} = \frac{1}{3},$$

as desired.

Comment 1. There are many ways of proving (1) which can be written as

$$\frac{as^2}{1-a} - \frac{(a+s)^3 - s^3}{3} < 0, \tag{2}$$

for non-negative a and s satisfying $a + s \leqslant 1$ and $a > 0$.

E.g., note that for any fixed a the expression in (2) is quadratic in s with the leading coefficient $a/(1-a) - a > 0$. Hence, it is convex as a function in s, so it suffices to check the inequality at $s = 0$ and $s = 1 - a$. The former case is trivial and in the latter case the inequality can be rewritten as

$$as - \frac{3as(a+s) + a^3}{3} < 0,$$

which is trivial since $a + s = 1$.

Solution 2. First, let us define

$$S(a_1, \ldots, a_n) := \sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2.$$

For some index i, denote $a_1 + \cdots + a_{i-1}$ by s. If we replace a_i with two numbers $a_i/2$ and $a_i/2$, i.e. replace the tuple $(a_1, \ldots, a_n)$ with $(a_1, \ldots, a_{i-1}, a_i/2, a_i/2, a_{i+1}, \ldots, a_n)$, the sum will increase by

$$\begin{aligned} S(a_1, \ldots, a_i/2, a_i/2, \ldots, a_n) - S(a_1, \ldots, a_n) &= \frac{a_i/2}{1-a_i/2}\left(s^2 + (s + a_i/2)^2\right) - \frac{a_i}{1-a_i}s^2 \\ &= a_i \frac{(1-a_i)(2s^2 + sa_i + a_i^2/4) - (2-a_i)s^2}{(2-a_i)(1-a_i)} \\ &= a_i \frac{(1-a_i-s)sa_i + (1-a_i)a_i^2/4}{(2-a_i)(1-a_i)}, \end{aligned}$$

which is strictly positive. So every such replacement strictly increases the sum. By repeating this process and making maximal number in the tuple tend to zero, we keep increasing the sum which will converge to

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

This completes the proof.

Solution 3. We sketch a probabilistic version of the first solution. Let x_1, x_2, x_3 be drawn uniformly and independently at random from the segment $[0, 1]$. Let $I_1 \cup I_2 \cup \cdots \cup I_n$ be a partition of $[0, 1]$ into segments of length $a_1, a_2, \ldots, a_n$ in this order. Let $J_k := I_1 \cup \cdots \cup I_{k-1}$ for $k \geqslant 2$ and $J_1 := \varnothing$. Then

$$\begin{aligned} \frac{1}{3} &= \sum_{k=1}^{n} \mathbb{P}\{x_1 \geqslant x_2, x_3;\ x_1 \in I_k\} \\ &= \sum_{k=1}^{n} \Big(\mathbb{P}\{x_1 \in I_k;\ x_2, x_3 \in J_k\} + 2 \cdot \mathbb{P}\{x_1 \geqslant x_2;\ x_1, x_2 \in I_k;\ x_3 \in J_k\} \\ &\qquad\qquad + \mathbb{P}\{x_1 \geqslant x_2, x_3;\ x_1, x_2, x_3 \in I_k\} \Big) \\ &= \sum_{k=1}^{n} \left(a_k(a_1 + \cdots + a_{k-1})^2 + 2 \cdot \frac{a_k^2}{2} \cdot (a_1 + \cdots + a_{k-1}) + \frac{a_k^3}{3} \right) \\ &> \sum_{k=1}^{n} \left(a_k(a_1 + \cdots + a_{k-1})^2 + a_k^2(a_1 + \cdots + a_{k-1}) \cdot \frac{a_1 + \cdots + a_{k-1}}{1 - a_k} \right), \end{aligned}$$

where for the last inequality we used that $1 - a_k \geqslant a_1 + \cdots + a_{k-1}$. This completes the proof since

$$a_k + \frac{a_k^2}{1-a_k} = \frac{a_k}{1-a_k}.$$

A6. Let A be a finite set of (not necessarily positive) integers, and let $m \geqslant 2$ be an integer. Assume that there exist non-empty subsets $B_1, B_2, B_3, \ldots, B_m$ of A whose elements add up to the sums $m^1, m^2, m^3, \ldots, m^m$, respectively. Prove that A contains at least $m/2$ elements.

Solution. Let $A = \{a_1, \ldots, a_k\}$. Assume that, on the contrary, $k = |A| < m/2$. Let

$$s_i := \sum_{j: a_j \in B_i} a_j$$

be the sum of elements of B_i. We are given that $s_i = m^i$ for $i = 1, \ldots, m$.

Now consider all m^m expressions of the form

$$f(c_1, \ldots, c_m) := c_1 s_1 + c_2 s_2 + \ldots + c_m s_m, \ c_i \in \{0, 1, \ldots, m-1\} \text{ for all } i = 1, 2, \ldots, m.$$

Note that every number $f(c_1, \ldots, c_m)$ has the form

$$\alpha_1 a_1 + \ldots + \alpha_k a_k, \ \alpha_i \in \{0, 1, \ldots, m(m-1)\}.$$

Hence, there are at most $(m(m-1)+1)^k < m^{2k} < m^m$ distinct values of our expressions; therefore, at least two of them coincide.

Since $s_i = m^i$, this contradicts the uniqueness of representation of positive integers in the base-m system.

Comment 1. For other rapidly increasing sequences of sums of B_i's the similar argument also provides lower estimates on $k = |A|$. For example, if the sums of B_i are equal to $1!$, $2!$, $3!$, …, $m!$, then for any fixed $\varepsilon > 0$ and large enough m we get $k \geqslant (1/2 - \varepsilon)m$. The proof uses the fact that the combinations $\sum c_i i!$ with $c_i \in \{0, 1, \ldots, i\}$ are all distinct.

Comment 2. The problem statement holds also if A is a set of real numbers (not necessarily integers), the above proofs work in the real case.

A7. Let $n \geqslant 1$ be an integer, and let $x_0, x_1, \ldots, x_{n+1}$ be $n+2$ non-negative real numbers that satisfy $x_i x_{i+1} - x_{i-1}^2 \geqslant 1$ for all $i = 1, 2, \ldots, n$. Show that

$$x_0 + x_1 + \cdots + x_n + x_{n+1} > \left(\frac{2n}{3}\right)^{3/2}.$$

Solution 1.

Lemma 1.1. If a, b, c are non-negative numbers such that $ab - c^2 \geqslant 1$, then

$$(a + 2b)^2 \geqslant (b + 2c)^2 + 6.$$

Proof. $(a+2b)^2 - (b+2c)^2 = (a-b)^2 + 2(b-c)^2 + 6(ab - c^2) \geqslant 6$. □

Lemma 1.2. $\sqrt{1} + \cdots + \sqrt{n} > \frac{2}{3} n^{3/2}$.

Proof. Bernoulli's inequality $(1+t)^{3/2} > 1 + \frac{3}{2}t$ for $0 > t \geqslant -1$ (or, alternatively, a straightforward check) gives

$$(k-1)^{3/2} = k^{3/2}\left(1 - \frac{1}{k}\right)^{3/2} > k^{3/2}\left(1 - \frac{3}{2k}\right) = k^{3/2} - \frac{3}{2}\sqrt{k}. \qquad (*)$$

Summing up $(*)$ over $k = 1, 2, \ldots, n$ yields

$$0 > n^{3/2} - \frac{3}{2}\left(\sqrt{1} + \cdots + \sqrt{n}\right).$$ □

Now put $y_i := 2x_i + x_{i+1}$ for $i = 0, 1, \ldots, n$. We get $y_0 \geqslant 0$ and $y_i^2 \geqslant y_{i-1}^2 + 6$ for $i = 1, 2, \ldots, n$ by Lemma 1.1. Thus, an easy induction on i gives $y_i \geqslant \sqrt{6i}$. Using this estimate and Lemma 1.2 we get

$$3(x_0 + \ldots + x_{n+1}) \geqslant y_1 + \ldots + y_n \geqslant \sqrt{6}\left(\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n}\right) > \sqrt{6} \cdot \frac{2}{3} n^{3/2} = 3\left(\frac{2n}{3}\right)^{3/2}.$$

Solution 2. Say that an index $i \in \{0, 1, \ldots, n+1\}$ is *good*, if $x_i \geqslant \sqrt{\frac{2}{3} i}$, otherwise call the index i *bad*.

Lemma 2.1. There are no two consecutive bad indices.

Proof. Assume the contrary and consider two bad indices $j, j+1$ with minimal possible j. Since 0 is good, we get $j > 0$, thus by minimality $j - 1$ is a good index and we have

$$\frac{2}{3}\sqrt{j(j+1)} > x_j x_{j+1} \geqslant x_{j-1}^2 + 1 \geqslant \frac{2}{3}(j-1) + 1 = \frac{2}{3} \cdot \frac{j + (j+1)}{2}$$

that contradicts the AM–GM inequality for numbers j and $j+1$. □

Lemma 2.2. If an index $j \leqslant n-1$ is good, then

$$x_{j+1} + x_{j+2} \geqslant \sqrt{\frac{2}{3}}\left(\sqrt{j+1} + \sqrt{j+2}\right).$$

Proof. We have

$$x_{j+1} + x_{j+2} \geqslant 2\sqrt{x_{j+1} x_{j+2}} \geqslant 2\sqrt{x_j^2 + 1} \geqslant 2\sqrt{\frac{2}{3} j + 1} \geqslant \sqrt{\frac{2}{3} j + \frac{2}{3}} + \sqrt{\frac{2}{3} j + \frac{4}{3}},$$

the last inequality follows from concavity of the square root function, or, alternatively, from the AM–QM inequality for the numbers $\sqrt{\frac{2}{3} j + \frac{2}{3}}$ and $\sqrt{\frac{2}{3} j + \frac{4}{3}}$. □

Let $S_i = x_1 + \ldots + x_i$ and $T_i = \sqrt{\frac{2}{3}}(\sqrt{1} + \ldots + \sqrt{i})$.

Lemma 2.3. If an index i is good, then $S_i \geqslant T_i$.

Proof. Induction on i. The base case $i = 0$ is clear. Assume that the claim holds for good indices less than i and prove it for a good index $i > 0$.

If $i-1$ is good, then by the inductive hypothesis we get $S_i = S_{i-1} + x_i \geqslant T_{i-1} + \sqrt{\frac{2}{3}i} = T_i$.

If $i-1$ is bad, then $i > 1$, and $i-2$ is good by Lemma 2.1. Then using Lemma 2.2 and the inductive hypothesis we get

$$S_i = S_{i-2} + x_{i-1} + x_i \geqslant T_{i-2} + \sqrt{\frac{2}{3}}\left(\sqrt{i-1} + \sqrt{i}\right) = T_i. \qquad \square$$

Since either n or $n+1$ is good by Lemma 2.1, Lemma 2.3 yields in both cases $S_{n+1} \geqslant T_n$, and it remains to apply Lemma 1.2 from Solution 1.

Comment 1. Another way to get $(*)$ is the integral bound

$$k^{3/2} - (k-1)^{3/2} = \int_{k-1}^{k} \frac{3}{2}\sqrt{x}\,dx < \frac{3}{2}\sqrt{k}.$$

Comment 2. If $x_i = \sqrt{2/3} \cdot (\sqrt{i} + 1)$, the conditions of the problem hold. Indeed, the inequality to check is

$$(\sqrt{i}+1)(\sqrt{i+1}+1) - (\sqrt{i-1}+1)^2 \geqslant 3/2,$$

that rewrites as

$$\sqrt{i} + \sqrt{i+1} - 2\sqrt{i-1} \geqslant (i + 1/2) - \sqrt{i(i+1)} = \frac{1/4}{i + 1/2 + \sqrt{i(i+1)}},$$

which follows from

$$\sqrt{i} - \sqrt{i-1} = \frac{1}{\sqrt{i} + \sqrt{i-1}} > \frac{1}{2i}.$$

For these numbers we have $x_0 + \ldots + x_{n+1} = \left(\frac{2n}{3}\right)^{3/2} + O(n)$, thus the multiplicative constant $(2/3)^{3/2}$ in the problem statement is sharp.

A8. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$\big(f(a)-f(b)\big)\big(f(b)-f(c)\big)\big(f(c)-f(a)\big)=f(ab^2+bc^2+ca^2)-f(a^2b+b^2c+c^2a)$$

for all real numbers a,b,c.

Answer: $f(x)=\alpha x+\beta$ or $f(x)=\alpha x^3+\beta$ where $\alpha\in\{-1,0,1\}$ and $\beta\in\mathbb{R}$.

Solution. It is straightforward to check that above functions satisfy the equation. Now let $f(x)$ satisfy the equation, which we denote $E(a,b,c)$. Then clearly $f(x)+C$ also does; therefore, we may suppose without loss of generality that $f(0)=0$.We start with proving

Lemma. Either $f(x)\equiv 0$ or f is injective.

Proof. Denote by $\Theta\subseteq\mathbb{R}^2$ the set of points (a,b) for which $f(a)=f(b)$. Let $\Theta^*=\{(x,y)\in\Theta: x\neq y\}$. The idea is that if $(a,b)\in\Theta$, then by $E(a,b,x)$ we get

$$H_{a,b}(x):=(ab^2+bx^2+xa^2,a^2b+b^2x+x^2a)\in\Theta$$

for all real x. Reproducing this argument starting with $(a,b)\in\Theta^*$, we get more and more points in Θ. There are many ways to fill in the details, we give below only one of them.

Assume that $(a,b)\in\Theta^*$. Note that

$$g_-(x):=(ab^2+bx^2+xa^2)-(a^2b+b^2x+x^2a)=(a-b)(b-x)(x-a)$$

and

$$g_+(x):=(ab^2+bx^2+xa^2)+(a^2b+b^2x+x^2a)=(x^2+ab)(a+b)+x(a^2+b^2).$$

Hence, there exists x for which both $g_-(x)\neq 0$ and $g_+(x)\neq 0$. This gives a point $(\alpha,\beta)=H_{a,b}(x)\in\Theta^*$ for which $\alpha\neq-\beta$. Now compare $E(\alpha,1,0)$ and $E(\beta,1,0)$. The left-hand side expressions coincide, on right-hand side we get $f(\alpha)-f(\alpha^2)=f(\beta)-f(\beta^2)$, respectively. Hence, $f(\alpha^2)=f(\beta^2)$ and we get a point $(\alpha_1,\beta_1):=(\alpha^2,\beta^2)\in\Theta^*$ with both coordinates α_1,β_1 non-negative. Continuing squaring the coordinates, we get a point $(\gamma,\delta)\in\Theta^*$ for which $\delta>5\gamma\geqslant 0$. Our nearest goal is to get a point $(0,r)\in\Theta^*$. If $\gamma=0$, this is already done. If $\gamma>0$, denote by x a real root of the quadratic equation $\delta\gamma^2+\gamma x^2+x\delta^2=0$, which exists since the discriminant $\delta^4-4\delta\gamma^3$ is positive. Also $x<0$ since this equation cannot have non-negative root. For the point $H_{\delta,\gamma}(x)=:(0,r)\in\Theta$ the first coordinate is 0. The difference of coordinates equals $-r=(\delta-\gamma)(\gamma-x)(x-\delta)<0$, so $r\neq 0$ as desired.

Now, let $(0,r)\in\Theta^*$. We get $H_{0,r}(x)=(rx^2,r^2x)\in\Theta$. Thus $f(rx^2)=f(r^2x)$ for all $x\in\mathbb{R}$. Replacing x to $-x$ we get $f(rx^2)=f(r^2x)=f(-r^2x)$, so f is even: $(a,-a)\in\Theta$ for all a. Then $H_{a,-a}(x)=(a^3-ax^2+xa^2,-a^3+a^2x+x^2a)\in\Theta$ for all real a,x. Putting $x=\frac{1+\sqrt5}{2}a$ we obtain $(0,(1+\sqrt5)a^3)\in\Theta$ which means that $f(y)=f(0)=0$ for every real y. $\square$

Hereafter we assume that f is injective and $f(0)=0$. By $E(a,b,0)$ we get

$$f(a)f(b)(f(a)-f(b))=f(a^2b)-f(ab^2).\qquad(\heartsuit)$$

Let $\kappa:=f(1)$ and note that $\kappa=f(1)\neq f(0)=0$ by injectivity. Putting $b=1$ in $(\heartsuit)$ we get

$$\kappa f(a)(f(a)-\kappa)=f(a^2)-f(a).\qquad(\clubsuit)$$

Subtracting the same equality for $-a$ we get

$$\kappa(f(a)-f(-a))(f(a)+f(-a)-\kappa)=f(-a)-f(a).$$

Now, if $a \neq 0$, by injectivity we get $f(a) - f(-a) \neq 0$ and thus

$$f(a) + f(-a) = \kappa - \kappa^{-1} =: \lambda. \tag{$\spadesuit$}$$

It follows that

$$f(a) - f(b) = f(-b) - f(-a)$$

for all non-zero a, b. Replace non-zero numbers a, b in ($\heartsuit$) with $-a, -b$, respectively, and add the two equalities. Due to ($\spadesuit$) we get

$$(f(a) - f(b))(f(a)f(b) - f(-a)f(-b)) = 0,$$

thus $f(a)f(b) = f(-a)f(-b) = (\lambda - f(a))(\lambda - f(b))$ for all non-zero $a \neq b$. If $\lambda \neq 0$, this implies $f(a) + f(b) = \lambda$ that contradicts injectivity when we vary b with fixed a. Therefore, $\lambda = 0$ and $\kappa = \pm 1$. Thus f is odd. Replacing f with $-f$ if necessary (this preserves the original equation) we may suppose that $f(1) = 1$.

Now, ($\clubsuit$) yields $f(a^2) = f^2(a)$. Summing relations ($\heartsuit$) for pairs (a, b) and $(a, -b)$, we get $-2f(a)f^2(b) = -2f(ab^2)$, i.e. $f(a)f(b^2) = f(ab^2)$. Putting $b = \sqrt{x}$ for each non-negative x we get $f(ax) = f(a)f(x)$ for all real a and non-negative x. Since f is odd, this multiplicativity relation is true for all a, x. Also, from $f(a^2) = f^2(a)$ we see that $f(x) \geqslant 0$ for $x \geqslant 0$. Next, $f(x) > 0$ for $x > 0$ by injectivity.

Assume that $f(x)$ for $x > 0$ does not have the form $f(x) = x^\tau$ for a constant τ. The known property of multiplicative functions yields that the graph of f is dense on $(0, \infty)^2$. In particular, we may find positive $b < 1/10$ for which $f(b) > 1$. Also, such b can be found if $f(x) = x^\tau$ for some $\tau < 0$. Then for all x we have $x^2 + xb^2 + b \geqslant 0$ and so $E(1, b, x)$ implies that

$$f(b^2 + bx^2 + x) = f(x^2 + xb^2 + b) + (f(b) - 1)(f(x) - f(b))(f(x) - 1) \geqslant 0 - ((f(b) - 1)^3/4$$

is bounded from below (the quadratic trinomial bound $(t - f(1))(t - f(b)) \geqslant -(f(b) - 1)^2/4$ for $t = f(x)$ is used). Hence, f is bounded from below on $(b^2 - \frac{1}{4b}, +\infty)$, and since f is odd it is bounded from above on $(0, \frac{1}{4b} - b^2)$. This is absurd if $f(x) = x^\tau$ for $\tau < 0$, and contradicts to the above dense graph condition otherwise.

Therefore, $f(x) = x^\tau$ for $x > 0$ and some constant $\tau > 0$. Dividing $E(a, b, c)$ by $(a - b)(b - c)(c - a) = (ab^2 + bc^2 + ca^2) - (a^2b + b^2c + c^2a)$ and taking a limit when a, b, c all go to 1 (the divided ratios tend to the corresponding derivatives, say, $\frac{a^\tau - b^\tau}{a - b} \to (x^\tau)'_{x=1} = \tau$), we get $\tau^3 = \tau \cdot 3^{\tau - 1}$, $\tau^2 = 3^{\tau - 1}$, $F(\tau) := 3^{\tau/2 - 1/2} - \tau = 0$. Since function F is strictly convex, it has at most two roots, and we get $\tau \in \{1, 3\}$.

Combinatorics

C1. Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a, b, c, d \in S$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three pairwise distinct $x, y, z \in S$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

Solution. There exists $\alpha \in S$ so that $\{\gcd(\alpha, s) \mid s \in S,\ s \neq \alpha\}$ contains at least two elements. Since α has only finitely many divisors, there is a $d \mid \alpha$ such that the set $B = \{\beta \in S \mid \gcd(\alpha, \beta) = d\}$ is infinite. Pick $\gamma \in S$ so that $\gcd(\alpha, \gamma) \neq d$. Pick $\beta_1, \beta_2 \in B$ so that $\gcd(\beta_1, \gamma) = \gcd(\beta_2, \gamma) =: d'$. If $d = d'$, then $\gcd(\alpha, \beta_1) = \gcd(\gamma, \beta_1) \neq \gcd(\alpha, \gamma)$. If $d \neq d'$, then either $\gcd(\alpha, \beta_1) = \gcd(\alpha, \beta_2) = d$ and $\gcd(\beta_1, \beta_2) \neq d$ or $\gcd(\gamma, \beta_1) = \gcd(\gamma, \beta_2) = d'$ and $\gcd(\beta_1, \beta_2) \neq d'$.

Comment. The situation can be modelled as a complete graph on the infinite vertex set S, where every edge $\{s, t\}$ is colored by $c(s, t) := \gcd(s, t)$. For every vertex the incident edges carry only finitely many different colors, and by the problem statement at least two different colors show up on the edge set. The goal is to show that there exists a bi-colored triangle (a triangle, whose edges carry exactly two different colors).

For the proof, consider a vertex v whose incident edges carry at least two different colors. Let $X \subset S$ be an infinite subset so that $c(v, x) \equiv c_1$ for all $x \in X$. Let $y \in S$ be a vertex so that $c(v, y) \neq c_1$. Let $x_1, x_2 \in X$ be two vertices with $c(y, x_1) = c(y, x_2) = c_2$. If $c_1 = c_2$, then the triangle v, y, x_1 is bi-colored. If $c_1 \neq c_2$, then one of v, x_1, x_2 and y, x_1, x_2 is bi-colored.

C2. Let $n \geqslant 3$ be an integer. An integer $m \geqslant n+1$ is called *n-colourful* if, given infinitely many marbles in each of n colours $C_1, C_2, \ldots, C_n$, it is possible to place m of them around a circle so that in any group of $n+1$ consecutive marbles there is at least one marble of colour C_i for each $i = 1, \ldots, n$.

Prove that there are only finitely many positive integers which are not n-colourful. Find the largest among them.

Answer: $m_{max} = n^2 - n - 1$.

Solution. First suppose that there are $n(n-1)-1$ marbles. Then for one of the colours, say blue, there are at most $n-2$ marbles, which partition the non-blue marbles into at most $n-2$ groups with at least $(n-1)^2 > n(n-2)$ marbles in total. Thus one of these groups contains at least $n+1$ marbles and this group does not contain any blue marble.

Now suppose that the total number of marbles is at least $n(n-1)$. Then we may write this total number as $nk+j$ with some $k \geqslant n-1$ and with $0 \leqslant j \leqslant n-1$. We place around a circle $k-j$ copies of the colour sequence $[1, 2, 3, \ldots, n]$ followed by j copies of the colour sequence $[1, 1, 2, 3, \ldots, n]$.

C3. A thimblerigger has 2021 thimbles numbered from 1 through 2021. The thimbles are arranged in a circle in arbitrary order. The thimblerigger performs a sequence of 2021 moves; in the k^{th} move, he swaps the positions of the two thimbles adjacent to thimble k.

Prove that there exists a value of k such that, in the k^{th} move, the thimblerigger swaps some thimbles a and b such that $a < k < b$.

Solution. Assume the contrary. Say that the k^{th} thimble is the *central thimble* of the k^{th} move, and its position on that move is the *central position* of the move.

Step 1: Black and white colouring.

Before the moves start, let us paint all thimbles in white. Then, after each move, we repaint its central thimble in black. This way, at the end of the process all thimbles have become black.

By our assumption, in every move k, the two swapped thimbles have the same colour (as their numbers are either both larger or both smaller than k). At every moment, assign the colours of the thimbles to their current positions; then the only position which changes its colour in a move is its central position. In particular, each position is central for exactly one move (when it is being repainted to black).

Step 2: Red and green colouring.

Now we introduce a colouring of the *positions*. If in the k^{th} move, the numbers of the two swapped thimbles are both less than k, then we paint the central position of the move in red; otherwise we paint that position in green. This way, each position has been painted in red or green exactly once. We claim that among any two adjacent positions, one becomes green and the other one becomes red; this will provide the desired contradiction since 2021 is odd.

Consider two adjacent positions A and B, which are central in the a^{th} and in the b^{th} moves, respectively, with $a < b$. Then in the a^{th} move the thimble at position B is white, and therefore has a number greater than a. After the a^{th} move, position A is green and the thimble at position A is black. By the arguments from Step 1, position A contains only black thimbles after the a^{th} step. Therefore, on the b^{th} move, position A contains a black thimble whose number is therefore less than b, while thimble b is at position B. So position B becomes red, and hence A and B have different colours.

Comment 1. Essentially, Step 1 provides the proof of the following two assertions (under the indirect assumption):

(1) Each position P becomes central in exactly one move (denote that move's number by k); and

(2) Before the k^{th} move, position P always contains a thimble whose number is larger than the number of the current move, while after the k^{th} move the position always contains a thimble whose number is smaller than the number of the current move.

Both (1) and (2) can be proved without introduction of colours, yet the colours help to visualise the argument.

After these two assertions have been proved, Step 2 can be performed in various ways, e.g., as follows.

At any moment in the process, the black positions are split into several groups consisting of one or more contiguous black positions each; different groups are separated by white positions. Now one can prove by induction on k that, after the k^{th} move, all groups have odd sizes. Indeed, in every move, the new black position either forms a separate group, or merges two groups (say, of lengths a and b) into a single group of length $a + b + 1$.

However, after the 2020^{th} move the black positions should form one group of length 2020. This is a contradiction.

This argument has several variations; e.g., one can check in a similar way that, after the process starts, at least one among the groups of *white* positions has an even size.

Comment 2. The solution above works equally well for any *odd* number of thimbles greater than 1, instead of 2021. On the other hand, a similar statement with an *even* number $n = 2k \geqslant 4$ of thimbles is wrong. To show that, the thimblerigger can enumerate positions from 1 through n clockwise, and then put thimbles $1, 2, \ldots, k$ at the odd positions, and thimbles $k+1, k+2, \ldots, 2k$ at the even positions.

C4. The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a *path from X to Y* is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called *diverse* if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Solution 1. We write $X \to Y$ (or $Y \leftarrow X$) if the road between X and Y goes from X to Y. Notice that, if there is any route moving from X to Y (possibly passing through some cities more than once), then there is a path from X to Y consisting of some roads in the route. Indeed, any cycle in the route may be removed harmlessly; after some removals one obtains a path.

Say that a path is *short* if it consists of one or two roads.

Partition all cities different from A and B into four groups, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{A}$, and $\mathcal{B}$ according to the following rules: for each city C,

$$C \in \mathcal{I} \iff A \to C \leftarrow B; \qquad C \in \mathcal{O} \iff A \leftarrow C \to B;$$
$$C \in \mathcal{A} \iff A \to C \to B; \qquad C \in \mathcal{B} \iff A \leftarrow C \leftarrow B.$$

Lemma. Let $\mathcal{P}$ be a diverse collection consisting of p paths from A to B. Then there exists a diverse collection consisting of at least p paths from A to B and containing all short paths from A to B.

Proof. In order to obtain the desired collection, modify $\mathcal{P}$ as follows.

If there is a direct road $A \to B$ and the path consisting of this single road is not in $\mathcal{P}$, merely add it to $\mathcal{P}$.

Now consider any city $C \in \mathcal{A}$ such that the path $A \to C \to B$ is not in $\mathcal{P}$. If $\mathcal{P}$ contains at most one path containing a road $A \to C$ or $C \to B$, remove that path (if it exists), and add the path $A \to C \to B$ to $\mathcal{P}$ instead. Otherwise, $\mathcal{P}$ contains two paths of the forms $A \to C \dashrightarrow B$ and $A \dashrightarrow C \to B$, where $C \dashrightarrow B$ and $A \dashrightarrow C$ are some paths. In this case, we recombine the edges to form two new paths $A \to C \to B$ and $A \dashrightarrow C \dashrightarrow B$ (removing cycles from the latter if needed). Now we replace the old two paths in $\mathcal{P}$ with the two new ones.

After any operation described above, the number of paths in the collection does not decrease, and the collection remains diverse. Applying such operation to each $C \in \mathcal{A}$, we obtain the desired collection. □

Back to the problem, assume, without loss of generality, that there is a road $A \to B$, and let a and b denote the numbers of roads going out from A and B, respectively. Choose a diverse collection $\mathcal{P}$ consisting of N_{AB} paths from A to B. We will transform it into a diverse collection $\mathcal{Q}$ consisting of at least $N_{AB} + (b - a)$ paths from B to A. This construction yields

$$N_{BA} \geqslant N_{AB} + (b - a); \qquad \text{similarly, we get} \quad N_{AB} \geqslant N_{BA} + (a - b),$$

whence $N_{BA} - N_{AB} = b - a$. This yields the desired equivalence.

Apply the lemma to get a diverse collection $\mathcal{P}'$ of at least N_{AB} paths containing all $|\mathcal{A}| + 1$ short paths from A to B. Notice that the paths in $\mathcal{P}'$ contain no edge of a short path from B to A. Each non-short path in $\mathcal{P}'$ has the form $A \to C \dashrightarrow D \to B$, where $C \dashrightarrow D$ is a path from some city $C \in \mathcal{I}$ to some city $D \in \mathcal{O}$. For each such path, put into $\mathcal{Q}$ the path $B \to C \dashrightarrow D \to A$; also put into $\mathcal{Q}$ all short paths from B to A. Clearly, the collection $\mathcal{Q}$ is diverse.

Now, all roads going out from A end in the cities from $\mathcal{I} \cup \mathcal{A} \cup \{B\}$, while all roads going out from B end in the cities from $\mathcal{I} \cup \mathcal{B}$. Therefore,

$$a = |\mathcal{I}| + |\mathcal{A}| + 1, \quad b = |\mathcal{I}| + |\mathcal{B}|, \qquad \text{and hence} \quad a - b = |\mathcal{A}| - |\mathcal{B}| + 1.$$

On the other hand, since there are $|\mathcal{A}| + 1$ short paths from A to B (including $A \to B$) and $|\mathcal{B}|$ short paths from B to A, we infer

$$|\mathcal{Q}| = |\mathcal{P}'| - (|\mathcal{A}| + 1) + |\mathcal{B}| \geqslant N_{AB} + (b - a),$$

as desired.

Solution 2. We recall some graph-theoretical notions. Let G be a finite graph, and let V be the set of its vertices; fix two distinct vertices $s, t \in V$. An *(s,t)-cut* is a partition of V into two parts $V = S \sqcup T$ such that $s \in S$ and $t \in T$. The *cut-edges* in the cut (S,T) are the edges going from S to T, and the *size* $e(S,T)$ of the cut is the number of cut-edges.

We will make use of the following theorem (which is a partial case of the Ford–Fulkerson "min-cut max-flow" theorem).

Theorem (Menger). Let G be a directed graph, and let s and t be its distinct vertices. Then the maximal number of edge-disjoint paths from s to t is equal to the minimal size of an (s,t)-cut.

Back to the problem. Consider a directed graph G whose vertices are the cities, and edges correspond to the roads. Then N_{AB} is the maximal number of edge-disjoint paths from A to B in this graph; the number N_{BA} is interpreted similarly.

As in the previous solution, denote by a and b the out-degrees of vertices A and B, respectively. To solve the problem, we show that for any (A,B)-cut (S_A, T_A) in our graph there exists a (B,A)-cut (S_B, T_B) satisfying

$$e(S_B, T_B) = e(S_A, T_A) + (b - a).$$

This yields

$$N_{BA} \leqslant N_{AB} + (b - a); \qquad \text{similarly, we get} \quad N_{AB} \leqslant N_{BA} + (a - b),$$

whence again $N_{BA} - N_{AB} = b - a$.

The construction is simple: we put $S_B = S_A \cup \{B\} \setminus \{A\}$ and hence $T_B = T_A \cup \{A\} \setminus \{B\}$. To show that it works, let A and B denote the sets of cut-edges in (S_A, T_A) and (S_B, T_B), respectively. Let a_s and $a_t = a - a_s$ denote the numbers of edges going from A to S_A and T_A, respectively. Similarly, denote by b_s and $b_t = b - b_s$ the numbers of edges going from B to S_B and T_B, respectively.

Notice that any edge incident to none of A and B either belongs to both A and B, or belongs to none of them. Denote the number of such edges belonging to A by c. The edges in A which are not yet accounted for split into two categories: those going out from A to T_A (including $A \to B$ if it exists), and those going from $S_A \setminus \{A\}$ to B — in other words, going from S_B to B. The numbers of edges in the two categories are a_t and $|S_B| - 1 - b_s$, respectively. Therefore,

$$|\mathsf{A}| = c + a_t + (|S_B| - b_s - 1). \qquad \text{Similarly, we get} \quad |\mathsf{B}| = c + b_t + (|S_A| - a_s - 1),$$

and hence

$$|\mathsf{B}| - |\mathsf{A}| = (b_t + b_s) - (a_t + a_s) = b - a,$$

since $|S_A| = |S_B|$. This finishes the solution.

C5. Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ *neighbours*— namely, the k students closest to S on the right, and the k students closest to S on the left.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbours.

Solution. We replace the girls by 1's, and the boys by 0's, getting the numbers $a_1, a_2, \ldots, a_{2n+1}$ arranged in a circle. We extend this sequence periodically by letting $a_{2n+1+k} = a_k$ for all $k \in \mathbb{Z}$. We get an infinite periodic sequence

$$\ldots, a_1, a_2, \ldots, a_{2n+1}, a_1, a_2, \ldots, a_{2n+1}, \ldots.$$

Consider the numbers $b_i = a_i + a_{i-k-1} - 1 \in \{-1, 0, 1\}$ for all $i \in \mathbb{Z}$. We know that

$$b_{m+1} + b_{m+2} + \cdots + b_{m+2n+1} = 1 \qquad (m \in \mathbb{Z}); \tag{1}$$

in particular, this yields that there exists some i_0 with $b_{i_0} = 1$. Now we want to find an index i such that

$$b_i = 1 \quad \text{and} \quad b_{i+1} + b_{i+2} + \cdots + b_{i+k} \geqslant 0. \tag{2}$$

This will imply that $a_i = 1$ and

$$(a_{i-k} + a_{i-k+1} + \cdots + a_{i-1}) + (a_{i+1} + a_{i+2} + \cdots + a_{i+k}) \geqslant k,$$

as desired.

Suppose, to the contrary, that for every index i with $b_i = 1$ the sum $b_{i+1} + b_{i+2} + \cdots + b_{i+k}$ is negative. We start from some index i_0 with $b_{i_0} = 1$ and construct a sequence i_0, i_1, i_2, ..., where i_j $(j > 0)$ is the smallest possible index such that $i_j > i_{j-1} + k$ and $b_{i_j} = 1$. We can choose two numbers among i_0, i_1, ..., i_{2n+1} which are congruent modulo $2n+1$ (without loss of generality, we may assume that these numbers are i_0 and i_T).

On the one hand, for every j with $0 \leqslant j \leqslant T-1$ we have

$$S_j := b_{i_j} + b_{i_j+1} + b_{i_j+2} + \cdots + b_{i_{j+1}-1} \leqslant b_{i_j} + b_{i_j+1} + b_{i_j+2} + \cdots + b_{i_j+k} \leqslant 0$$

since $b_{i_j+k+1}, \ldots, b_{i_{j+1}-1} \leqslant 0$. On the other hand, since $(i_T - i_0) \mid (2n+1)$, from (1) we deduce

$$S_0 + \cdots + S_{T-1} = \sum_{i=i_0}^{i_T-1} b_i = \frac{i_T - i_0}{2n+1} > 0.$$

This contradiction finishes the solution.

Comment 1. After the problem is reduced to finding an index i satisfying (2), one can finish the solution by applying the (existence part of) following statement.

Lemma (Raney). If $\langle x_1, x_2, \ldots, x_m \rangle$ is any sequence of integers whose sum is $+1$, exactly one of the cyclic shifts $\langle x_1, x_2, \ldots, x_m \rangle, \langle x_2, \ldots, x_m, x_1 \rangle, \ldots, \langle x_m, x_1, \ldots, x_{m-1} \rangle$ has all of its partial sums positive.

A (possibly wider known) version of this lemma, which also can be used in order to solve the problem, is the following

Claim (Gas stations problem). Assume that there are several fuel stations located on a circular route which together contain just enough gas to make one trip around. Then one can make it all the way around, starting at the right station with an empty tank.

Both Raney's theorem and the Gas stations problem admit many different (parallel) proofs. Their ideas can be disguised in direct solutions of the problem at hand (as it, in fact, happens in the above solution); such solutions may avoid the introduction of the b_i. Below, in Comment 2 we present a variant of such solution, while in Comment 3 we present an alternative proof of Raney's theorem.

Comment 2. Here is a version of the solution which avoids the use of the b_i.

Suppose the contrary. Introduce the numbers a_i as above. Starting from any index s_0 with $a_{s_0} = 1$, we construct a sequence $s_0, s_1, s_2, \dots$ by letting s_i to be the smallest index larger than $s_{i-1}+k$ such that $a_{s_i} = 1$, for $i = 1, 2, \dots$. Choose two indices among $s_1, \dots, s_{2n+1}$ which are congruent modulo $2n + 1$; we assume those two are s_0 and s_T, with $s_T - s_0 = t(2n + 1)$. Notice here that $s_{T+1} - s_T = s_1 - s_0$.

For every $i = 0, 1, 2, \dots, T$, put

$$L_i = s_{i+1} - s_i \quad \text{and} \quad S_i = a_{s_i} + a_{s_i+1} + \dots + a_{s_{i+1}-1}.$$

Now, by the indirect assumption, for every $i = 1, 2, \dots, T$, we have

$$a_{s_i-k} + a_{s_i-k+1} + \dots + a_{s_i+k} \leqslant a_{s_i} + (k - 1) = k.$$

Recall that $a_j = 0$ for all j with $s_i + k < j < a_{s_{i+1}}$. Therefore,

$$S_{i-1} + S_i = \sum_{j=s_{i-1}}^{s_i+k} a_j = \sum_{j=s_{i-1}}^{s_i-k-1} a_j + \sum_{j=s_i-k}^{s_i+k} a_j \leqslant (s_i - s_{i-1} - k) + k = L_{i-1}.$$

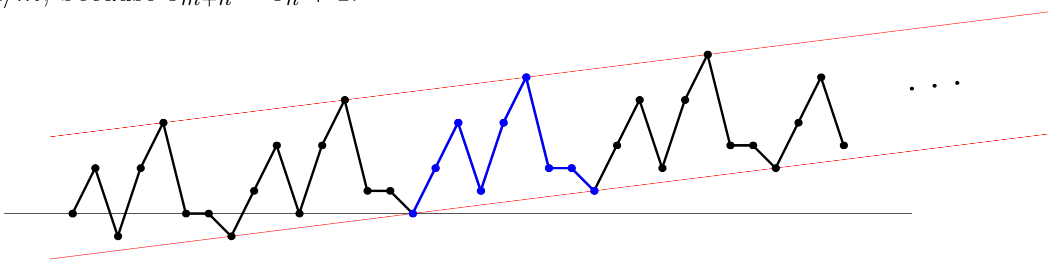
Summing up these equalities over $i = 1, 2, \dots, T$ we get

$$2t(n + 1) = \sum_{i=1}^{T}(S_{i-1} + S_i) \leqslant \sum_{i=1}^{T} L_{i-1} = (2n + 1)t,$$

which is a contradiction.

Comment 3. Here we present a proof of Raney's lemma different from the one used above.

If we plot the partial sums $s_n = x_1 + \dots + x_n$ as a function of n, the graph of s_n has an average slope of $1/m$, because $s_{m+n} = s_n + 1$.



The entire graph can be contained between two lines of slope $1/m$. In general these bounding lines touch the graph just once in each cycle of m points, since lines of slope $1/m$ hit points with integer coordinates only once per m units. The unique (in one cycle) lower point of intersection is the only place in the cycle from which all partial sums will be positive.

Comment 4. The following example shows that for different values of k the required girl may have different positions: 0110 0110 1.

C6. A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share a side). The hunter wins if after some finite time either

- the rabbit cannot move; or
- the hunter can determine the cell in which the rabbit started.

Decide whether there exists a winning strategy for the hunter.

Answer: Yes, there exists a colouring that yields a winning strategy for the hunter.

Solution. A central idea is that several colourings $C_1, C_2, \ldots, C_k$ can be merged together into a single product colouring $C_1 \times C_2 \times \cdots \times C_k$ as follows: the colours in the product colouring are ordered tuples $(c_1, \ldots, c_n)$ of colours, where c_i is a colour used in C_i, so that each cell gets a tuple consisting of its colours in the individual colourings C_i. This way, any information which can be determined from one of the individual colourings can also be determined from the product colouring.

Now let the hunter merge the following colourings:

- The first two colourings C_1 and C_2 allow the tracking of the horizontal and vertical movements of the rabbit.

  The colouring C_1 colours the cells according to the residue of their x-coordinates modulo 3, which allows to determine whether the rabbit moves left, moves right, or moves vertically. Similarly, the colouring C_2 uses the residues of the y-coordinates modulo 3, which allows to determine whether the rabbit moves up, moves down, or moves horizontally.
- Under the condition that the rabbit's x-coordinate is unbounded, colouring C_3 allows to determine the exact value of the x-coordinate:

  In C_3, the columns are coloured white and black so that the gaps between neighboring black columns are pairwise distinct. As the rabbit's x-coordinate is unbounded, it will eventually visit two black cells in distinct columns. With the help of colouring C_1 the hunter can catch that moment, and determine the difference of x-coordinates of those two black cells, hence deducing the precise column.

  Symmetrically, under the condition that the rabbit's y-coordinate is unbounded, there is a colouring C_4 that allows the hunter to determine the exact value of the y-coordinate.
- Finally, under the condition that the sum $x + y$ of the rabbit's coordinates is unbounded, colouring C_5 allows to determine the exact value of this sum: The diagonal lines $x + y =$ const are coloured black and white, so that the gaps between neighboring black diagonals are pairwise distinct.

Unless the rabbit gets stuck, at least two of the three values x, y and $x+y$ must be unbounded as the rabbit keeps moving. Hence the hunter can eventually determine two of these three values; thus he does know all three. Finally the hunter works backwards with help of the colourings C_1 and C_2 and computes the starting cell of the rabbit.

Comment. There are some variations of the solution above: e.g., the colourings C_3, C_4 and C_5 can be replaced with different ones. However, such alternatives are more technically involved, and we do not present them here.

C7. Consider a checkered $3m \times 3m$ square, where m is an integer greater than 1. A frog sits on the lower left corner cell S and wants to get to the upper right corner cell F. The frog can hop from any cell to either the next cell to the right or the next cell upwards.

Some cells can be sticky, and the frog gets trapped once it hops on such a cell. A set X of cells is called *blocking* if the frog cannot reach F from S when all the cells of X are sticky. A blocking set is *minimal* if it does not contain a smaller blocking set.

(a) Prove that there exists a minimal blocking set containing at least $3m^2 - 3m$ cells.

(b) Prove that every minimal blocking set contains at most $3m^2$ cells.

Note. An example of a minimal blocking set for $m = 2$ is shown below. Cells of the set X are marked by letters x.

					F
x	x				
		x			
			x		
				x	
S		x			

Solution for part (a). In the following example the square is divided into m stripes of size $3 \times 3m$. It is easy to see that X is a minimal blocking set. The first and the last stripe each contains $3m - 1$ cells from the set X; every other stripe contains $3m - 2$ cells, see Figure 1. The total number of cells in the set X is $3m^2 - 2m + 2$.

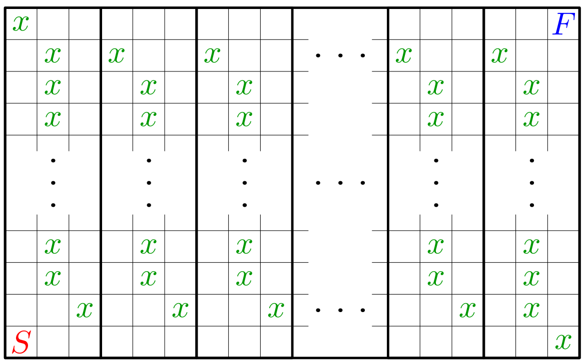


Figure 1

Solution 1 for part (b). For a given blocking set X, say that a non-sticky cell is *red* if the frog can reach it from S via some hops without entering set X. We call a non-sticky cell *blue* if the frog can reach F from that cell via hops without entering set X. One can regard the blue cells as those reachable from F by *anti-hops*, i.e. moves downwards and to the left. We also colour all cells in X *green*. It follows from the definition of the blocking set that no cell will be coloured twice. In Figure 2 we show a sample of a blocking set and the corresponding colouring.

Now assume that X is a minimal blocking set. We denote by R (resp., B and G) be the total number of red (resp., blue and green) cells.

We claim that $G \leqslant R + 1$ and $G \leqslant B + 1$. Indeed, there are at most $2R$ possible frog hops from red cells. Every green or red cell (except for S) is accessible by such hops. Hence $2R \geqslant G + (R - 1)$, or equivalently $G \leqslant R + 1$. In order to prove the inequality $G \leqslant B + 1$, we turn over the board and apply the similar arguments.

Therefore we get $9m^2 \geqslant B + R + G \geqslant 3G - 2$, so $G \leqslant 3m^2$.

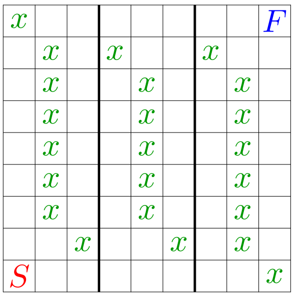


Figure 2 (a)

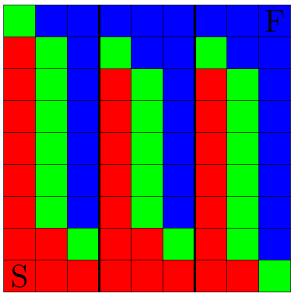
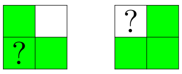


Figure 2 (b)

Solution 2 for part (b). We shall use the same colouring as in the above solution. Again, assume that X is a minimal blocking set.

Note that any 2×2 square cannot contain more than 2 green cells. Indeed, on Figure 3(a) the cell marked with "?" does not block any path, while on Figure 3(b) the cell marked with "?" should be coloured red and blue simultaneously. So we can split all green cells into chains consisting of three types of links shown on Figure 4 (diagonal link in the other direction is not allowed, corresponding green cells must belong to different chains). For example, there are 3 chains in Figure 2(b).



(a) (b)

Figure 3

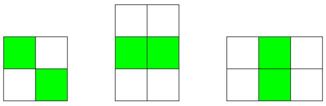


Figure 4

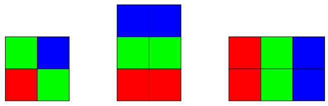
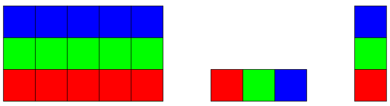


Figure 5

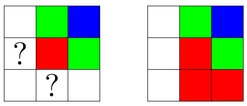
We will inscribe green chains in disjoint axis-aligned rectangles so that the number of green cells in each rectangle will not exceed 1/3 of the area of the rectangle. This will give us the bound $G \leqslant 3m^2$. Sometimes the rectangle will be the minimal bounding rectangle of the chain, sometimes minimal bounding rectangles will be expanded in one or two directions in order to have sufficiently large area.

Note that for any two consecutive cells in the chain the colouring of some neighbouring cells is uniquely defined (see Figure 5). In particular, this observation gives a corresponding rectangle for the chains of height (or width) 1 (see Figure 6(a)). A separate green cell can be inscribed in 1×3 or 3×1 rectangle with one red and one blue cell, see Figure 6(b)–(c), otherwise we get one of impossible configurations shown in Figure 3.



(a) (b) (c)

Figure 6



(a) (b)

Figure 7

Any diagonal chain of length 2 is always inscribed in a 2×3 or 3×2 rectangle without another green cells. Indeed, one of the squares marked with "?" in Figure 7(a) must be red. If it is the bottom question mark, then the remaining cell in the corresponding 2×3 rectangle must have the same colour, see Figure 7(b).

A longer chain of height (or width) 2 always has a horizontal (resp., vertical) link and can be inscribed into a $3 \times a$ rectangle. In this case we expand the minimal bounding rectangle across the long side which touches the mentioned link. On Figure 8(a) the corresponding expansion of the minimal bounding rectangle is coloured in light blue. The upper right corner cell must be also blue. Indeed it cannot be red or green. If it is not coloured in blue, see Figure 8(b), then all anti-hop paths from F to "?" are blocked with green cells. And these green cells are surrounded by blue ones, what is impossible. In this case the green chain contains a cells, which is exactly 1/3 of the area of the rectangle.

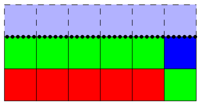


Figure 8 (a)

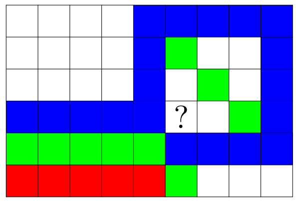


Figure 8 (b)

In the remaining case the minimal bounding rectangle of the chain is of size $a \times b$ where $a, b \geqslant 3$. Denote by ℓ the *length* of the chain (i.e. the number of cells in the chain).

If the chain has at least two diagonal links (see Figure 9), then $\ell \leqslant a+b-3 \leqslant ab/3$.

If the chain has only one diagonal link then $\ell = a+b-2$. In this case the chain has horizontal and vertical end-links, and we expand the minimal bounding rectangle in two directions to get an $(a+1)\times(b+1)$ rectangle. On Figure 10 a corresponding expansion of the minimal bounding rectangle is coloured in light red. Again the length of the chain does not exceed 1/3 of the rectangle's area: $\ell \leqslant a+b-2 \leqslant (a+1)(b+1)/3$.

On the next step we will use the following statement: all cells in constructed rectangles are coloured red, green or blue (the cells upwards and to the right of green cells are blue; the cells downwards and to the left of green cells are red). The proof repeats the same arguments as before (see Figure 8(b).)

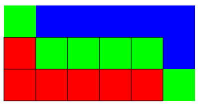


Figure 9

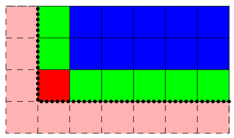
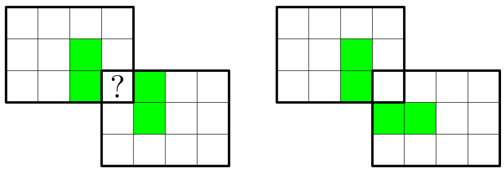


Figure 10



(a) (b)

Figure 11

Note that all constructed rectangles are disjoint. Indeed, assume that two rectangles have a common cell. Using the above statement, one can see that the only such cell can be a common corner cell, as shown in Figure 11. Moreover, in this case both rectangles should be expanded, otherwise they would share a green corner cell.

If they were expanded along the same axis (see Figure 11(a)), then again the common corner cannot be coloured correctly. If they were expanded along different axes (see Figure 11(b)) then the two chains have a common point and must be connected in one chain. (These arguments work for 2×3 and 1×3 rectangles in a similar manner.)

Comment 1. We do not *a priori* know whether all points are either red, or blue, or green. One might colour the remaining cells in black. The arguments from Solution 2 allow to prove that black cells do not exist. (One can start with a black cell which is nearest to S. Its left and downward neighbours must be coloured green or blue. In all cases one gets a configuration similar to Figure 8(b).)

Comment 2. The maximal possible size of a minimal blocking set in $3m\times 3m$ rectangle seems to be $3m^2-2m+2$.

One can prove a more precise upper bound on the cardinality of the minimal blocking set: $G \leqslant 3m^2-m+2$. Denote by D_R the number of red branching cells (i.e. such cells which have 2 red subsequent neighbours). And let D_B be the number of similar blue cells. Then a double counting argument allows to prove that $G \leqslant R-D_R+1$ and $G \leqslant B-D_B+1$. Thus, we can bound G in terms of D_B and D_R as

$$9m^2 \geqslant R+B+G \geqslant 3G+D_R+D_B-2.$$

Now one can estimate the number of branching cells in order to obtain that $G \leqslant 3m^2-m+2$.

Comment 3. An example with $3m^2 - 2m + 2$ green cells may look differently; see, e.g., Figure 12.

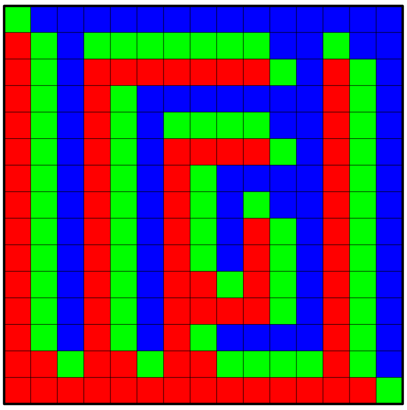


Figure 12

C8. Determine the largest N for which there exists a table T of integers with N rows and 100 columns that has the following properties:

(i) Every row contains the numbers 1, 2, ..., 100 in some order.

(ii) For any two distinct rows r and s, there is a column c such that $|T(r,c) - T(s,c)| \geqslant 2$.

Here $T(r,c)$ means the number at the intersection of the row r and the column c.

Answer: The largest such integer is $N = 100!/2^{50}$.

Solution 1.

Non-existence of a larger table. Let us consider some fixed row in the table, and let us replace (for $k = 1, 2, \ldots, 50$) each of two numbers $2k-1$ and $2k$ respectively by the symbol x_k. The resulting *pattern* is an arrangement of 50 symbols $x_1, x_2, \ldots, x_{50}$, where every symbol occurs exactly twice. Note that there are $N = 100!/2^{50}$ distinct patterns $P_1, \ldots, P_N$.

If two rows $r \neq s$ in the table have the same pattern P_i, then $|T(r,c) - T(s,c)| \leqslant 1$ holds for all columns c. As this violates property (ii) in the problem statement, different rows have different patterns. Hence there are at most $N = 100!/2^{50}$ rows.

Existence of a table with N rows. We construct the table by translating every pattern P_i into a corresponding row with the numbers $1, 2, \ldots, 100$. We present a procedure that inductively replaces the symbols by numbers. The translation goes through steps $k = 1, 2, \ldots, 50$ in increasing order and at step k replaces the two occurrences of symbol x_k by $2k-1$ and $2k$.

- The left occurrence of x_1 is replaced by 1, and its right occurrence is replaced by 2.
- For $k \geqslant 2$, we already have the number $2k-2$ somewhere in the row, and now we are looking for the places for $2k-1$ and $2k$. We make the three numbers $2k-2, 2k-1, 2k$ show up (ordered from left to right) either in the order $2k-2, 2k-1, 2k$, or as $2k, 2k-2, 2k-1$, or as $2k-1, 2k, 2k-2$. This is possible, since the number $2k-2$ has been placed in the preceding step, and shows up before / between / after the two occurrences of the symbol x_k.

We claim that the N rows that result from the N patterns yield a table with the desired property (ii). Indeed, consider the r-th and the s-th row ($r \neq s$), which by construction result from patterns P_r and P_s. Call a symbol x_i *aligned*, if it occurs in the same two columns in P_r and in P_s. Let k be the largest index, for which symbol x_k is not aligned. Note that $k \geqslant 2$. Consider the column c' with $T(r,c') = 2k$ and the column c'' with $T(s,c'') = 2k$. Then $T(r,c'') \leqslant 2k$ and $T(s,c') \leqslant 2k$, as all symbols x_i with $i \geqslant k+1$ are aligned.

- If $T(r,c'') \leqslant 2k-2$, then $|T(r,c'') - T(s,c'')| \geqslant 2$ as desired.
- If $T(s,c') \leqslant 2k-2$, then $|T(r,c') - T(s,c')| \geqslant 2$ as desired.
- If $T(r,c'') = 2k-1$ and $T(s,c') = 2k-1$, then the symbol x_k is aligned; contradiction.

In the only remaining case we have $c' = c''$, so that $T(r,c') = T(s,c') = 2k$ holds. Now let us consider the columns d' and d'' with $T(r,d') = 2k-1$ and $T(s,d'') = 2k-1$. Then $d \neq d''$ (as the symbol x_k is not aligned), and $T(r,d'') \leqslant 2k-2$ and $T(s,d') \leqslant 2k-2$ (as all symbols x_i with $i \geqslant k+1$ are aligned).

- If $T(r,d'') \leqslant 2k-3$, then $|T(r,d'') - T(s,d'')| \geqslant 2$ as desired.
- If $T(s,c') \leqslant 2k-3$, then $|T(r,d') - T(s,d')| \geqslant 2$ as desired.

In the only remaining case we have $T(r,d'') = 2k-2$ and $T(s,d') = 2k-2$. Now the row r has the numbers $2k-2, 2k-1, 2k$ in the three columns d', d'', c'. As one of these triples violates the ordering property of $2k-2, 2k-1, 2k$, we have the final contradiction. $\square$

Comment 1. We can identify rows of the table T with permutations of $\mathcal{M} := \{1, \ldots, 100\}$; also for every set $S \subset \mathcal{M}$ each row induces a subpermutation of S obtained by ignoring all entries not from S.

The example from Solution 1 consists of all permutations for which all subpermutations of the 50 sets $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}, \ldots, \{98, 99, 100\}$ are even.

Solution 2. We provide a bit different proof why the example from Solution 1 (see also Comment 1) works.

Lemma. Let π_1 and π_2 be two permutations of the set $\{1, 2, \ldots, n\}$ such that $|\pi_1(i) - \pi_2(i)| \leqslant 1$ for every i. Then there exists a set of disjoint pairs $(i, i+1)$ such that π_2 is obtained from π_1 by swapping elements in each pair from the set.

Proof. We may assume that $\pi_1(i) = i$ for every i and proceed by induction on n. The case $n = 1$ is trivial. If $\pi_2(n) = n$, we simply apply the induction hypothesis. If $\pi_2(n) = n - 1$, then $\pi_2(i) = n$ for some $i < n$. It is clear that $i = n - 1$, and we can also use the induction hypothesis. □

Now let π_1 and π_2 be two rows (which we identify with permutations of $\{1, 2, \ldots, 100\}$) of the table constructed in Solution 1. Assume that $|\pi_1(i) - \pi_2(i)| \leqslant 1$ for any i. From the Lemma it follows that there exists a set $S \subset \{1, \ldots, 99\}$ such that any two numbers from S differ by at least 2 and π_2 is obtained from π_1 by applying the permutations $(j, j+1)$, $j \in S$. Let $r = \min(S)$. If $r = 2k - 1$ is odd, then π_1 and π_2 induce two subpermutations of $\{2k-2, 2k-1, 2k\}$ (or of $\{1, 2\}$ for $k = 1$) of opposite parities. Thus $r = 2k$ is even. Since π_1 and π_2 induce subpermutations of the same (even) parity of $\{2k, 2k+1, 2k+2\}$, we must have $2k + 2 \in S$. Next, $2k + 4 \in S$ and so on, we get $98 \in S$, but then the parities of the subpermutations of $\{98, 99, 100\}$ in π_1, π_2 are opposite. A contradiction. □

Comment 2. In Solution 2 we only used that for each set from $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}, \ldots, \{98, 99, 100\}$ any two rows of T induce a subpermutation of the same parity, not necessarily even.

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Geometry

G1. Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B. The circumcircle of the triangle ACD meets the segment PD again at Q, and the circumcircle of the triangle APQ meets the segment PC again at R. Prove that the lines CD, AQ, and BR are concurrent.

Common remarks. The introductory steps presented here are used in all solutions below.

Since $AC = BC = AD$, we have $\angle ABC = \angle BAC = \angle ACD = \angle ADC$. Since the quadrilaterals $APRQ$ and $AQCD$ are cyclic, we obtain

$$\angle CRA = 180^\circ - \angle ARP = 180^\circ - \angle AQP = \angle DQA = \angle DCA = \angle CBA,$$

so the points A, B, C, and R lie on some circle γ.

Solution 1. Introduce the point $X = AQ \cap CD$; we need to prove that B, R and X are collinear.

By means of the circle $(APRQ)$ we have

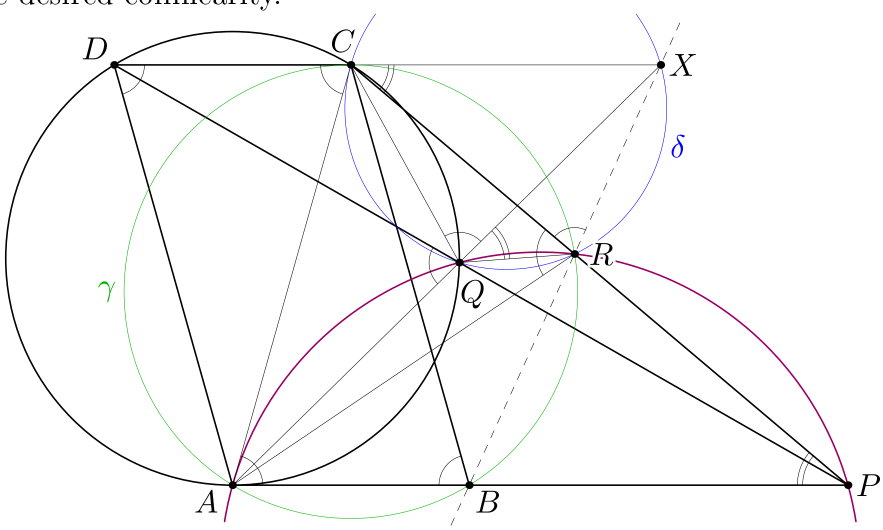
$$\angle RQX = 180^\circ - \angle AQR = \angle RPA = \angle RCX$$

(the last equality holds in view of $AB \parallel CD$), which means that the points C, Q, R, and X also lie on some circle δ.

Using the circles δ and γ we finally obtain

$$\angle XRC = \angle XQC = 180^\circ - \angle CQA = \angle ADC = \angle BAC = 180^\circ - \angle CRB,$$

that proves the desired collinearity.



Solution 2. Let α denote the circle $(APRQ)$. Since

$$\angle CAP = \angle ACD = \angle AQD = 180^\circ - \angle AQP,$$

the line AC is tangent to α.

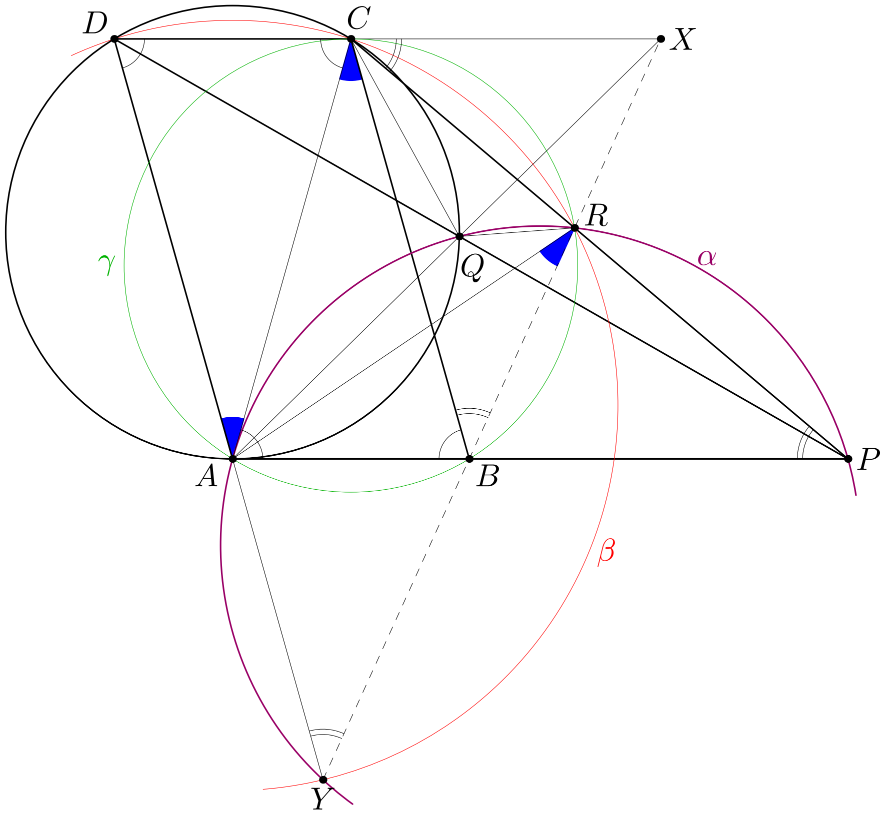
Now, let AD meet α again at a point Y (which necessarily lies on the extension of DA beyond A). Using the circle γ, along with the fact that AC is tangent to α, we have

$$\angle ARY = \angle CAD = \angle ACB = \angle ARB,$$

so the points Y, B, and R are collinear.

Applying Pascal's theorem to the hexagon $AAYRPQ$ (where AA is regarded as the tangent to α at A), we see that the points $AA \cap RP = C$, $AY \cap PQ = D$, and $YR \cap QA$ are collinear. Hence the lines CD, AQ, and BR are concurrent.

Comment 1. Solution 2 consists of two parts: (1) showing that BR and DA meet on α; and (2) showing that this yields the desired concurrency. Solution 3 also splits into those parts, but the proofs are different.



Solution 3. As in Solution 1, we introduce the point $X = AQ \cap CD$ and aim at proving that the points B, R, and X are collinear. As in Solution 2, we denote $\alpha = (APQR)$; but now we define Y to be the second meeting point of RB with α.

Using the circle α and noticing that CD is tangent to γ, we obtain

$$\angle RYA = \angle RPA = \angle RCX = \angle RBC. \tag{1}$$

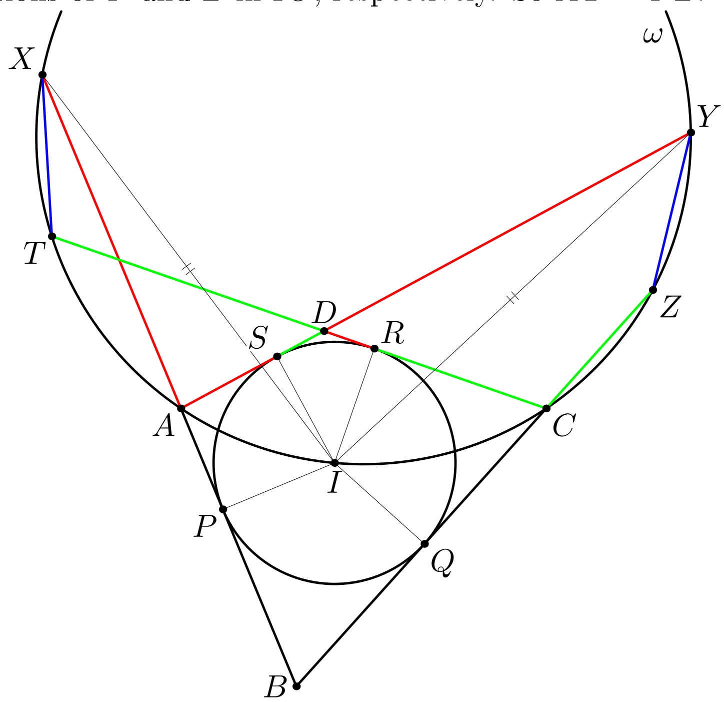
So $AY \parallel BC$, and hence Y lies on DA.

Now the chain of equalities (1) shows also that $\angle RYD = \angle RCX$, which implies that the points C, D, Y, and R lie on some circle β. Hence, the lines CD, AQ, and YBR are the pairwise radical axes of the circles $(AQCD)$, α, and β, so those lines are concurrent.

Comment 2. The original problem submission contained an additional assumption that $BP = AB$. The Problem Selection Committee removed this assumption as superfluous.

G2. Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Prove that the perimeters of the (possibly self-intersecting) quadrilaterals $ADTX$ and $CDYZ$ are equal.

Solution. The point I is the intersection of the external bisector of the angle TCZ with the circumcircle ω of the triangle TCZ, so I is the midpoint of the arc TCZ and $IT = IZ$. Similarly, I is the midpoint of the arc YAX and $IX = IY$. Let O be the centre of ω. Then X and T are the reflections of Y and Z in IO, respectively. So $XT = YZ$.



Let the incircle of $ABCD$ touch AB, BC, CD, and DA at points P, Q, R, and S, respectively.

The right triangles IXP and IYS are congruent, since $IP = IS$ and $IX = IY$. Similarly, the right triangles IRT and IQZ are congruent. Therefore, $XP = YS$ and $RT = QZ$.

Denote the perimeters of $ADTX$ and $CDYZ$ by P_{ADTX} and P_{CDYZ} respectively. Since $AS = AP$, $CQ = RC$, and $SD = DR$, we obtain

$$\begin{aligned} P_{ADTX} &= XT + XA + AS + SD + DT = XT + XP + RT \\ &= YZ + YS + QZ = YZ + YD + DR + RC + CZ = P_{CDYZ}, \end{aligned}$$

as required.

Comment 1. After proving that X and T are the reflections of Y and Z in IO, respectively, one can finish the solution as follows. Since $XT = YZ$, the problem statement is equivalent to

$$XA + AD + DT = YD + DC + CZ. \tag{1}$$

Since $ABCD$ is circumscribed, $AB - AD = BC - CD$. Adding this to (1), we come to an equivalent equality $XA + AB + DT = YD + BC + CZ$, or

$$XB + DT = YD + BZ. \tag{2}$$

Let $\lambda = \frac{XZ}{AC} = \frac{TY}{AC}$. Since $XACZ$ is cyclic, the triangles ZBX and ABC are similar, hence

$$\frac{XB}{BC} = \frac{BZ}{AB} = \frac{XZ}{AC} = \lambda.$$

It follows that $XB = \lambda BC$ and $BZ = \lambda AB$. Likewise, the triangles TDY and ADC are similar, hence

$$\frac{DT}{AD} = \frac{DY}{CD} = \frac{TY}{AC} = \lambda.$$

Therefore, (2) rewrites as $\lambda BC + \lambda AD = \lambda CD + \lambda AB$.

This is equivalent to $BC + AD = CD + AB$ which is true as $ABCD$ is circumscribed.

Comment 2. Here is a more difficult modification of the original problem, found by the PSC.

Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Let $U = BC \cap AD$ and $V = BA \cap CD$. Let I_U be the incentre of UYZ and let J_V be the V-excentre of VXT. Then $I_U J_V \perp BD$.

G3.

Version 1. Let n be a fixed positive integer, and let S be the set of points (x,y) on the Cartesian plane such that both coordinates x and y are nonnegative integers smaller than $2n$ (thus $|\mathsf{S}| = 4n^2$). Assume that $\mathcal{F}$ is a set consisting of n^2 quadrilaterals such that all their vertices lie in S, and each point in S is a vertex of exactly one of the quadrilaterals in $\mathcal{F}$.

Determine the largest possible sum of areas of all n^2 quadrilaterals in $\mathcal{F}$.

Version 2. Let n be a fixed positive integer, and let S be the set of points (x,y) on the Cartesian plane such that both coordinates x and y are nonnegative integers smaller than $2n$ (thus $|\mathsf{S}| = 4n^2$). Assume that $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in S, and each point in S is a vertex of exactly one of the polygons in $\mathcal{F}$.

Determine the largest possible sum of areas of all polygons in $\mathcal{F}$.

Answer for both Versions: The largest possible sum of areas is $\Sigma(n) := \frac{1}{3}n^2(2n+1)(2n-1)$.

Common remarks. Throughout all solutions, the area of a polygon P will be denoted by $[P]$.

We say that a polygon is *legal* if all its vertices belong to S. Let $O = \left(n-\frac{1}{2}, n-\frac{1}{2}\right)$ be the centre of S. We say that a legal square is *central* if its centre is situated at O. Finally, say that a set $\mathcal{F}$ of polygons is *acceptable* if it satisfies the problem requirements, i.e. if all polygons in $\mathcal{F}$ are legal, and each point in S is a vertex of exactly one polygon in $\mathcal{F}$. For an acceptable set $\mathcal{F}$, we denote by $\Sigma(\mathcal{F})$ the sum of areas of polygons in $\mathcal{F}$.

Solution 1, for both Versions. Each point in S is a vertex of a unique central square. Thus the set $\mathcal{G}$ of central squares is acceptable. We will show that

$$\Sigma(\mathcal{F}) \leqslant \Sigma(\mathcal{G}) = \Sigma(n), \tag{1}$$

thus establishing the answer.

We will use the following key lemma.

Lemma 1. Let $P = A_1A_2\ldots A_m$ be a polygon, and let O be an arbitrary point in the plane. Then

$$[P] \leqslant \frac{1}{2}\sum_{i=1}^{m} OA_i^2; \tag{2}$$

moreover, if P is a square centred at O, then the inequality (2) turns into an equality.

Proof. Put $A_{n+1} = A_1$. For each $i = 1, 2, \ldots, m$, we have

$$[OA_iA_{i+1}] \leqslant \frac{OA_i \cdot OA_{i+1}}{2} \leqslant \frac{OA_i^2 + OA_{i+1}^2}{4}.$$

Therefore,

$$[P] \leqslant \sum_{i=1}^{m}[OA_iA_{i+1}] \leqslant \frac{1}{4}\sum_{i=1}^{m}(OA_i^2 + OA_{i+1}^2) = \frac{1}{2}\sum_{i=1}^{m} OA_i^2,$$

which proves (2). Finally, all the above inequalities turn into equalities when P is a square centred at O. □

Back to the problem, consider an arbitrary acceptable set $\mathcal{F}$. Applying Lemma 1 to each element in $\mathcal{F}$ and to each element in $\mathcal{G}$ (achieving equality in the latter case), we obtain

$$\Sigma(\mathcal{F}) \leqslant \frac{1}{2}\sum_{A\in\mathsf{S}} OA^2 = \Sigma(\mathcal{G}),$$

which establishes the left inequality in (1).

It remains to compute $\Sigma(\mathcal{G})$. We have

$$\begin{aligned}\Sigma(\mathcal{G}) &= \frac{1}{2}\sum_{A\in\mathsf{S}} OA^2 = \frac{1}{2}\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1}\left(\left(n-\frac{1}{2}-i\right)^2+\left(n-\frac{1}{2}-j\right)^2\right)\\ &= \frac{1}{8}\cdot 4\cdot 2n\sum_{i=0}^{n-1}(2n-2i-1)^2 = n\sum_{j=0}^{n-1}(2j+1)^2 = n\left(\sum_{j=1}^{2n} j^2-\sum_{j=1}^{n}(2j)^2\right)\\ &= n\left(\frac{2n(2n+1)(4n+1)}{6}-4\cdot\frac{n(n+1)(2n+1)}{6}\right) = \frac{n^2(2n+1)(2n-1)}{3} = \Sigma(n).\end{aligned}$$

Comment. There are several variations of the above solution, also working for both versions of the problem. E.g., one may implement only the inequality $[OA_iA_{i+1}] \leqslant \frac{1}{2}OA_i \cdot OA_{i+1}$ to obtain

$$\Sigma(\mathcal{F}) \leqslant \frac{1}{2}\sum_{i=1}^{4n^2} OK_i \cdot OL_i,$$

where both (K_i) and (L_i) are permutations of all points in S. The right hand side can then be bounded from above by means of the rearrangement inequality; the bound is also achieved on the collection $\mathcal{G}$.

However, Version 2 seems to be more difficult than Version 1. First of all, the optimal model for this version is much less easy to guess, until one finds an idea for proving the upper bound. Moreover, Version 1 allows different solutions which do not seem to be generalized easily — such as Solution 2 below.

Solution 2, for Version 1. Let $\mathcal{F}$ be an accessible set of quadrilaterals. For every quadrilateral $ABCD$ in $\mathcal{F}$ write

$$[ABCD] = \frac{AC\cdot BD}{2}\sin\phi \leqslant \frac{AC^2+BD^2}{4}, \tag{3}$$

where ϕ is the angle between AC and BD. Applying this estimate to all members in $\mathcal{F}$ we obtain

$$\Sigma(\mathcal{F}) \leqslant \frac{1}{4}\sum_{i=1}^{2n^2} A_iB_i^2,$$

where $A_1, A_2, \ldots, A_{2n^2}, B_1, B_2, \ldots, B_{2n^2}$ is some permutation of S. For brevity, denote

$$f\big((A_i),(B_i)\big) := \sum_{i=1}^{2n^2} A_iB_i^2.$$

The rest of the solution is based on the following lemma.

Lemma 2. The maximal value of $f\big((A_i),(B_i)\big)$ over all permutations of S equals $\frac{4}{3}n^2(4n^2-1)$ and is achieved when A_i is symmetric to B_i with respect to O, for every $i = 1, 2, \ldots, 2n^2$.

Proof. Let $A_i = (p_i, q_i)$ and $B_i = (r_i, s_i)$, for $i = 1, 2, \ldots, 2n^2$. We have

$$f\big((A_i),(B_i)\big) = \sum_{i=1}^{2n^2}(p_i-r_i)^2+\sum_{i=1}^{2n^2}(q_i-s_i)^2;$$

it suffices to bound the first sum, the second is bounded similarly. This can be done, e.g., by means of the QM–AM inequality as follows:

$$\begin{aligned}\sum_{i=1}^{2n^2}(p_i-r_i)^2 &= \sum_{i=1}^{2n^2}\bigl(2p_i^2+2r_i^2-(p_i+r_i)^2\bigr) = 4n\sum_{j=0}^{2n-1} j^2 - \sum_{i=1}^{2n^2}(p_i+r_i)^2 \\ &\leqslant 4n\sum_{j=0}^{2n-1} j^2 - \frac{1}{2n^2}\left(\sum_{i=1}^{2n^2}(p_i+r_i)\right)^2 = 4n\sum_{j=0}^{2n-1} j^2 - \frac{1}{2n^2}\left(2n\cdot\sum_{j=0}^{2n-1} j\right)^2 \\ &= 4n\cdot\frac{2n(2n-1)(4n-1)}{6} - 2n^2(2n-1)^2 = \frac{2n^2(2n-1)(2n+1)}{3}.\end{aligned}$$

All the estimates are sharp if $p_i + r_i = 2n - 1$ for all i. Thus,

$$f\bigl((A_i),(B_i)\bigr) \leqslant \frac{4n^2(4n^2-1)}{3},$$

and the estimate is sharp when $p_i + r_i = q_i + s_i = 2n - 1$ for all i, i.e. when A_i and B_i are symmetric with respect to O. $\square$

Lemma 2 yields

$$\Sigma(\mathcal{F}) \leqslant \frac{1}{4}\cdot\frac{4n^2(4n^2-1)}{3} = \frac{n^2(2n-1)(2n+1)}{3}.$$

Finally, all estimates are achieved simultaneously on the set $\mathcal{G}$ of central squares.

Comment 2. Lemma 2 also allows different proofs. E.g., one may optimize the sum $\sum_i p_i r_i$ step by step: if $p_i < p_j$ and $r_i < r_j$, then a swap $r_i \leftrightarrow r_j$ increases the sum. By applying a proper chain of such replacements (possibly swapping elements in some pairs (p_i, r_i)), one eventually comes to a permutation where $p_i + r_i = 2n - 1$ for all i.

Comment 3. Version 2 can also be considered for a square grid with odd number n of points on each side. If we allow a polygon consisting of one point, then Solution 1 is applied verbatim, providing an answer $\frac{1}{12}n^2(n^2-1)$. If such polygons are not allowed, then one needs to subtract $\frac{1}{2}$ from the answer.

G4. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that $TE \parallel CD$ and $TF \parallel AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

Solution 1. Let the segments TE and TF cross AC at P and Q, respectively. Since $PE \parallel CD$ and ED is tangent to the circumcircle of $ABCD$, we have

$$\angle EPA = \angle DCA = \angle EDA,$$

and so the points A, P, D, and E lie on some circle α. Similarly, the points C, Q, D, and F lie on some circle γ.

We now want to prove that the line DT is tangent to both α and γ at D. Indeed, since $\angle FCD + \angle EAD = 180°$, the circles α and γ are tangent to each other at D. To prove that T lies on their common tangent line at D (i.e., on their radical axis), it suffices to check that $TP \cdot TE = TQ \cdot TF$, or that the quadrilateral $PEFQ$ is cyclic. This fact follows from

$$\angle QFE = \angle ADE = \angle APE.$$

Since $TD = TK$, we have $\angle TKD = \angle TDK$. Next, as TD and DE are tangent to α and Ω, respectively, we obtain

$$\angle TKD = \angle TDK = \angle EAD = \angle BDE,$$

which implies $TK \parallel BD$.

Next we prove that the five points T, P, Q, D, and K lie on some circle τ. Indeed, since TD is tangent to the circle α we have

$$\angle EPD = \angle TDF = \angle TKD,$$

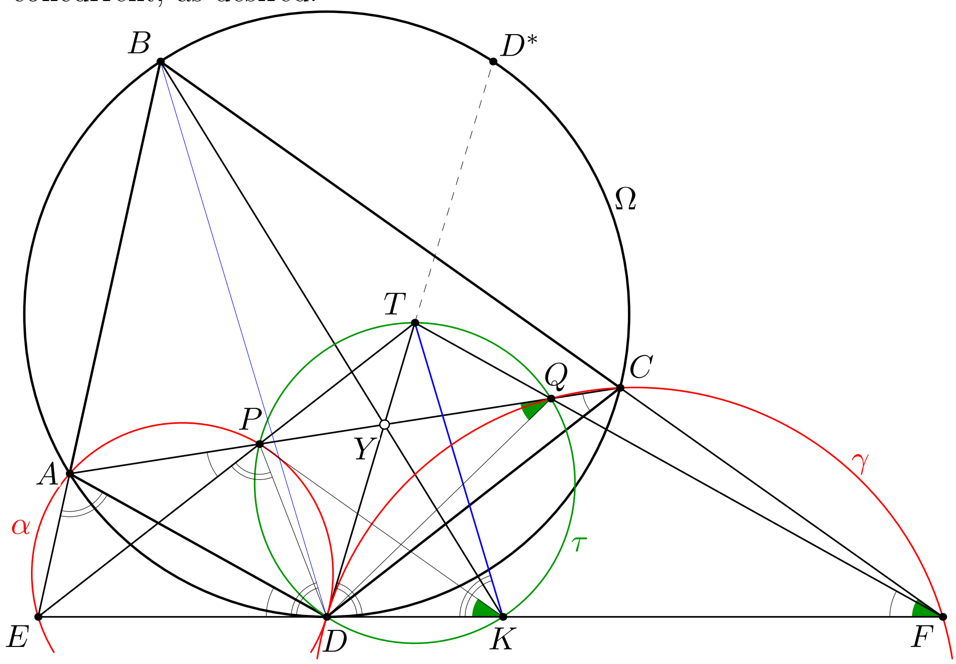
which means that the point P lies on the circle (TDK). Similarly, we have $Q \in (TDK)$.

Finally, we prove that $PK \parallel BC$. Indeed, using the circles τ and γ we conclude that

$$\angle PKD = \angle PQD = \angle DFC,$$

which means that $PK \parallel BC$.

Triangles TPK and DCB have pairwise parallel sides, which implies the fact that TD, PC and KB are concurrent, as desired.



Comment 1. There are several variations of the above solution.

E.g., after finding circles α and γ, one can notice that there exists a homothety h mapping the triangle TPQ to the triangle DCA; the centre of that homothety is $Y = AC \cap TD$. Since

$$\angle DPE = \angle DAE = \angle DCB = \angle DQT,$$

the quadrilateral $TPDQ$ is inscribed in some circle τ. We have $h(\tau) = \Omega$, so the point $D^* = h(D)$ lies on Ω.

Finally, by

$$\angle DCD^* = \angle TPD = \angle BAD,$$

the points B and D^* are symmetric with respect to the diameter of Ω passing through D. This yields $DB = DD^*$ and $BD^* \parallel EF$, so $h(K) = B$, and BK passes through Y.

Solution 2. Consider the spiral similarity ϕ centred at D which maps B to F. Recall that for any two points X and Y, the triangles $DX\phi(X)$ and $DY\phi(Y)$ are similar.

Define $T' = \phi(E)$. Then

$$\angle CDF = \angle FBD = \angle \phi(B)BD = \angle \phi(E)ED = \angle T'ED,$$

so $CD \parallel T'E$. Using the fact that DE is tangent to (ABD) and then applying ϕ we infer

$$\angle ADE = \angle ABD = \angle T'FD,$$

so $AD \parallel T'F$; hence T' coincides with T. Therefore,

$$\angle BDE = \angle FDT = \angle DKT,$$

whence $TK \parallel BD$.

Let $BK \cap TD = X$, $AC \cap TD = Y$, and $AC \cap TF = Q$. Notice that $TK \parallel BD$ implies

$$\frac{TX}{XD} = \frac{TK}{BD} = \frac{TD}{BD}.$$

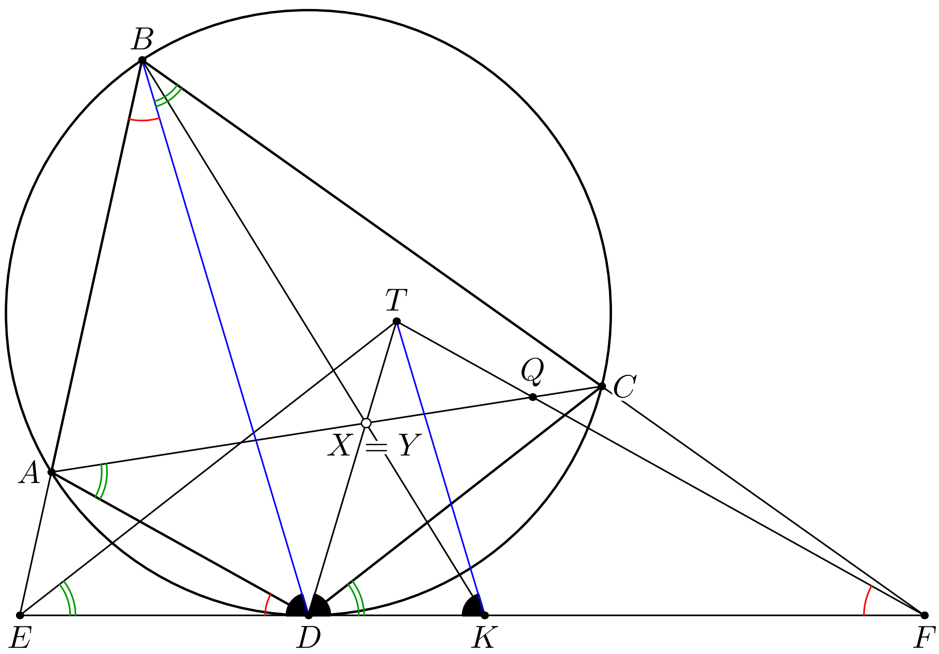
So we wish to prove that $\frac{TY}{YD}$ is equal to the same ratio.

We first show that $\phi(A) = Q$. Indeed,

$$\angle DA\phi(A) = \angle DBF = \angle DAC,$$

and so $\phi(A) \in AC$. Together with $\phi(A) \in \phi(EB) = TF$ this yields $\phi(A) = Q$. It follows that

$$\frac{TQ}{AE} = \frac{TD}{ED}.$$



Now, since $TF \parallel AD$ and $\triangle EAD \sim \triangle EDB$, we have

$$\frac{TY}{YD} = \frac{TQ}{AD} = \frac{TQ}{AE} \cdot \frac{AE}{AD} = \frac{TD}{ED} \cdot \frac{ED}{BD} = \frac{TD}{BD},$$

which completes the proof.

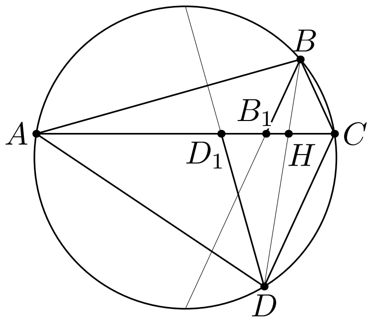
Comment 2. The point D is the Miquel point for any 4 of the 5 lines BA, BC, TE, TF and AC. Essentially, this is proved in both solutions by different methods.

G5. Let $ABCD$ be a cyclic quadrilateral whose sides have pairwise different lengths. Let O be the circumcentre of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1, respectively. Let O_B be the centre of the circle which passes through B and is tangent to AC at D_1. Similarly, let O_D be the centre of the circle which passes through D and is tangent to AC at B_1.

Assume that $BD_1 \parallel DB_1$. Prove that O lies on the line O_BO_D.

Common remarks. We introduce some objects and establish some preliminary facts common for all solutions below.

Let Ω denote the circle $(ABCD)$, and let γ_B and γ_D denote the two circles from the problem statement (their centres are O_B and O_D, respectively). Clearly, all three centres O, O_B, and O_D are distinct.



Assume, without loss of generality, that $AB > BC$. Suppose that $AD > DC$, and let $H = AC \cap BD$. Then the rays BB_1 and DD_1 lie on one side of BD, as they contain the midpoints of the arcs ADC and ABC, respectively. However, if $BD_1 \parallel DB_1$, then B_1 and D_1 should be separated by H. This contradiction shows that $AD < CD$.

Let γ_B and γ_D meet Ω again at T_B and T_D, respectively. The common chord BT_B of Ω and γ_B is perpendicular to their line of centres O_BO; likewise, $DT_D \perp O_DO$. Therefore, $O \in O_BO_D \iff O_BO \parallel O_DO \iff BT_B \parallel DT_D$, and the problem reduces to showing that

$$BT_B \parallel DT_D. \tag{1}$$

Comment 1. It seems that the discussion of the positions of points is necessary for both Solutions below. However, this part automatically follows from the angle chasing in Comment 2.

Solution 1. Let the diagonals AC and BD cross at H. Consider the homothety h centred at H and mapping B to D. Since $BD_1 \parallel DB_1$, we have $h(D_1) = B_1$.

Let the tangents to Ω at B and D meet AC at L_B and L_D, respectively. We have

$$\angle L_BBB_1 = \angle L_BBC + \angle CBB_1 = \angle BAL_B + \angle B_1BA = \angle BB_1L_B,$$

which means that the triangle L_BBB_1 is isosceles, $L_BB = L_BB_1$. The powers of L_B with respect to Ω and γ_D are L_BB^2 and $L_BB_1^2$, respectively; so they are equal, whence L_B lies on the radical axis T_DD of those two circles. Similarly, L_D lies on the radical axis T_BB of Ω and γ_B.

By the sine rule in the triangle BHL_B, we obtain

$$\frac{HL_B}{\sin\angle HBL_B} = \frac{BL_B}{\sin\angle BHL_B} = \frac{B_1L_B}{\sin\angle BHL_B}; \tag{2}$$

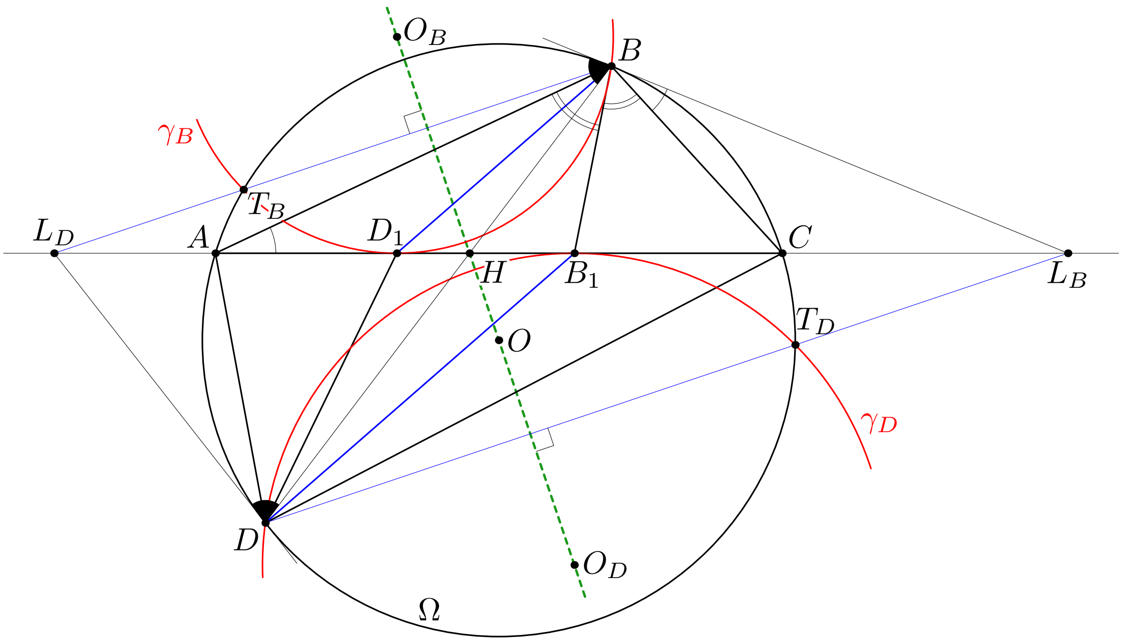
similarly,

$$\frac{HL_D}{\sin\angle HDL_D} = \frac{DL_D}{\sin\angle DHL_D} = \frac{D_1L_D}{\sin\angle DHL_D}. \tag{3}$$

Clearly, $\angle BHL_B = \angle DHL_D$. In the circle Ω, tangent lines BL_B and DL_D form equal angles with the chord BD, so $\sin\angle HBL_B = \sin\angle HDL_D$ (this equality does not depend on the picture). Thus, dividing (2) by (3) we get

$$\frac{HL_B}{HL_D} = \frac{B_1L_B}{D_1L_D}, \qquad \text{and hence} \qquad \frac{HL_B}{HL_D} = \frac{HL_B - B_1L_B}{HL_D - D_1L_D} = \frac{HB_1}{HD_1}.$$

Since $h(D_1) = B_1$, the obtained relation yields $h(L_D) = L_B$, so h maps the line L_DB to L_BD, and these lines are parallel, as desired.



Comment 2. In the solution above, the key relation $h(L_D) = L_B$ was obtained via a short computation in sines. Here we present an alternative, pure synthetical way of establishing that.

Let the external bisectors of $\angle ABC$ and $\angle ADC$ cross AC at B_2 and D_2, respectively; assume that $\widehat{AB} > \widehat{CB}$. In the right-angled triangle BB_1B_2, the point L_B is a point on the hypothenuse such that $L_BB_1 = L_BB$, so L_B is the midpoint of B_1B_2.

Since DD_1 is the internal angle bisector of $\angle ADC$, we have

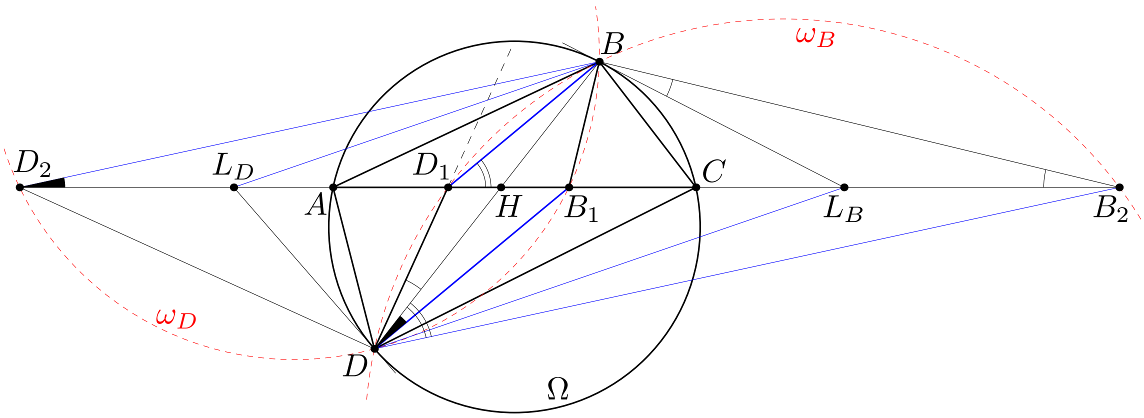
$$\angle BDD_1 = \frac{\angle BDA - \angle CDB}{2} = \frac{\angle BCA - \angle CAB}{2} = \angle BB_2D_1,$$

so the points B, B_2, D, and D_1 lie on some circle ω_B. Similarly, L_D is the midpoint of D_1D_2, and the points D, D_2, B, and B_1 lie on some circle ω_D.

Now we have

$$\angle B_2DB_1 = \angle B_2DB - \angle B_1DB = \angle B_2D_1B - \angle B_1D_2B = \angle D_2BD_1.$$

Therefore, the corresponding sides of the triangles DB_1B_2 and BD_1D_2 are parallel, and the triangles are homothetical (in H). So their corresponding medians DL_B and BL_D are also parallel.



Yet alternatively, after obtaining the circles ω_B and ω_D, one may notice that H lies on their radical axis BD, whence $HB_1 \cdot HD_2 = HD_1 \cdot HB_2$, or

$$\frac{HB_1}{HD_1} = \frac{HB_2}{HD_1}.$$

Since $h(D_1) = B_1$, this yields $h(D_2) = B_2$ and hence $h(L_D) = L_B$.

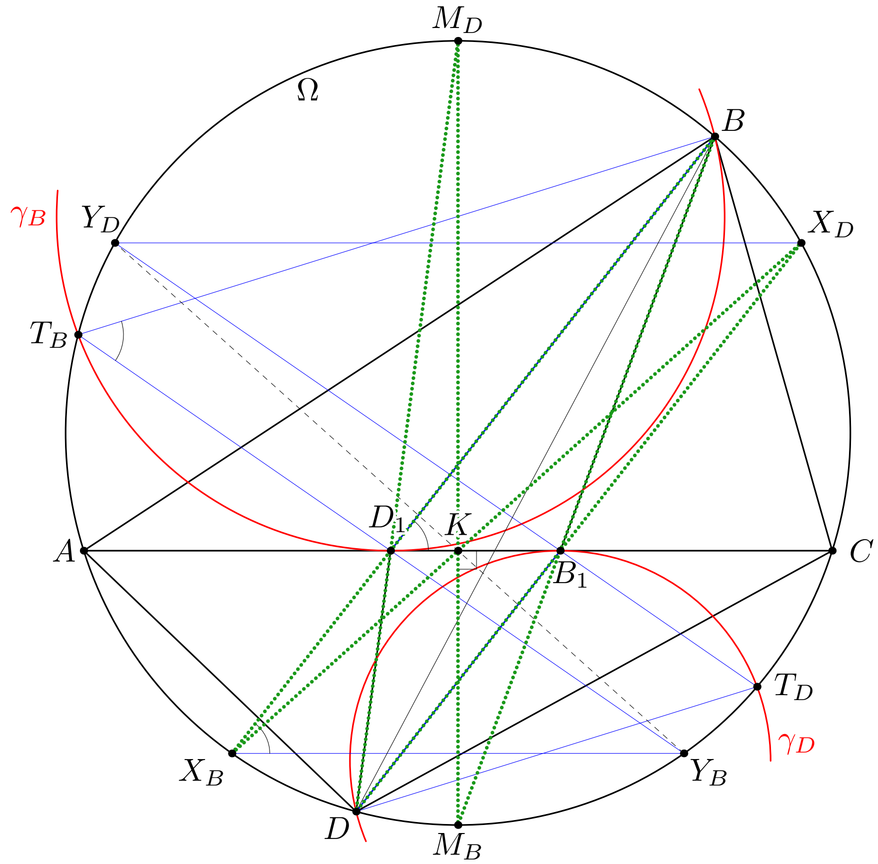
Comment 3. Since h preserves the line AC and maps $B \mapsto D$ and $D_1 \mapsto B_1$, we have $h(\gamma_B) = \gamma_D$. Therefore, $h(O_B) = O_D$; in particular, H also lies on O_BO_D.

Solution 2. Let BD_1 and T_BD_1 meet Ω again at X_B and Y_B, respectively. Then

$$\angle BD_1C = \angle BT_BD_1 = \angle BT_BY_B = \angle BX_BY_B,$$

which shows that $X_BY_B \parallel AC$. Similarly, let DB_1 and T_DB_1 meet Ω again at X_D and Y_D, respectively; then $X_DY_D \parallel AC$.

Let M_D and M_B be the midpoints of the arcs ABC and ADC, respectively; then the points D_1 and B_1 lie on DM_D and BM_B, respectively. Let K be the midpoint of AC (which lies on M_BM_D). Applying Pascal's theorem to $M_DDX_DX_BBM_B$, we obtain that the points $D_1 = M_DD \cap X_BB$, $B_1 = DX_D \cap BM_B$, and $X_DX_B \cap M_BM_D$ are collinear, which means that X_BX_D passes through K. Due to symmetry, the diagonals of an isosceles trapezoid $X_BY_BX_DY_D$ cross at K.



Let b and d denote the distances from the lines X_BY_B and X_DY_D, respectively, to AC. Then we get

$$\frac{X_BY_B}{X_DY_D} = \frac{b}{d} = \frac{D_1X_B}{B_1X_D},$$

where the second equation holds in view of $D_1X_B \parallel B_1X_D$. Therefore, the triangles $D_1X_BY_B$ and $B_1X_DY_D$ are similar. The triangles D_1T_BB and B_1T_DD are similar to them and hence to each other. Since $BD_1 \parallel DB_1$, these triangles are also homothetical. This yields $BT_B \parallel DT_D$, as desired.

Comment 4. The original problem proposal asked to prove that the relations $BD_1 \parallel DB_1$ and $O \in O_1O_2$ are equivalent. After obtaining $BD_1 \parallel DB_1 \Rightarrow O \in O_1O_2$, the converse proof is either repeated backwards *mutatis mutandis*, or can be obtained by the usual procedure of varying some points in the construction.

The Problem Selection Committee chose the current version, because it is less technical, yet keeps most of the ideas.

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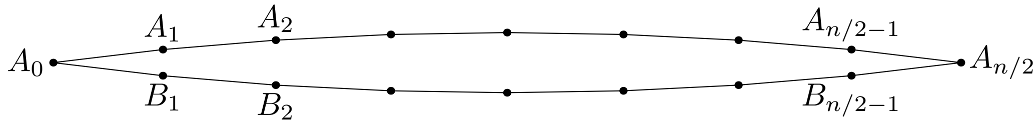
G6. Determine all integers $n \geqslant 3$ satisfying the following property: every convex n-gon whose sides all have length 1 contains an equilateral triangle of side length 1.

(Every polygon is assumed to contain its boundary.)

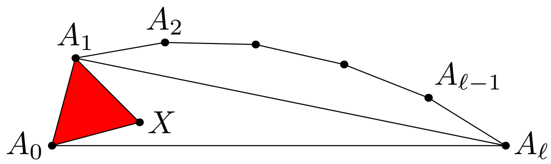
Answer: All odd $n \geqslant 3$.

Solution. First we show that for every even $n \geqslant 4$ there exists a polygon violating the required statement. Consider a regular k-gon $A_0A_1,\dots A_{k-1}$ with side length 1. Let $B_1, B_2, \dots, B_{n/2-1}$ be the points symmetric to $A_1, A_2, \dots, A_{n/2-1}$ with respect to the line $A_0A_{n/2}$. Then $P = A_0A_1A_2\dots A_{n/2-1}A_{n/2}B_{n/2-1}B_{n/2-2}\dots B_2B_1$ is a convex n-gon whose sides all have length 1. If k is big enough, P is contained in a strip of width $1/2$, which clearly does not contain any equilateral triangle of side length 1.



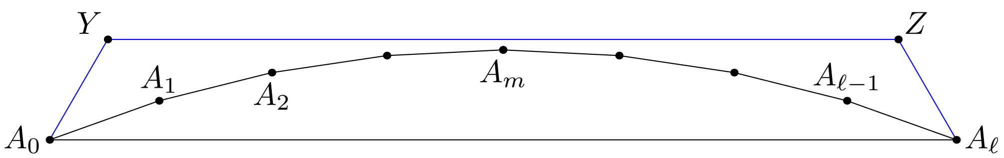
Assume now that $n = 2k+1$. As the case $k = 1$ is trivially true, we assume $k \geqslant 2$ henceforth. Consider a convex $(2k+1)$-gon P whose sides all have length 1. Let d be its longest diagonal. The endpoints of d split the perimeter of P into two polylines, one of which has length at least $k+1$. Hence we can label the vertices of P so that $P = A_0A_1\dots A_{2k}$ and $d = A_0A_\ell$ with $\ell \geqslant k+1$. We will show that, in fact, the polygon $A_0A_1\dots A_\ell$ contains an equilateral triangle of side length 1.

Suppose that $\angle A_\ell A_0A_1 \geqslant 60^\circ$. Since d is the longest diagonal, we have $A_1A_\ell \leqslant A_0A_\ell$, so $\angle A_0A_1A_\ell \geqslant \angle A_\ell A_0A_1 \geqslant 60^\circ$. It follows that there exists a point X inside the triangle $A_0A_1A_\ell$ such that the triangle A_0A_1X is equilateral, and this triangle is contained in P. Similar arguments apply if $\angle A_{\ell-1}A_\ell A_0 \geqslant 60^\circ$.



From now on, assume $\angle A_\ell A_0A_1 < 60^\circ$ and $A_{\ell-1}A_\ell A_0 < 60^\circ$.

Consider an isosceles trapezoid A_0YZA_ℓ such that $A_0A_\ell \parallel YZ$, $A_0Y = ZA_\ell = 1$, and $\angle A_\ell A_0Y = \angle ZA_\ell A_0 = 60^\circ$. Suppose that $A_0A_1\dots A_\ell$ is contained in A_0YZA_ℓ. Note that the perimeter of $A_0A_1\dots A_\ell$ equals $\ell + A_0A_\ell$ and the perimeter of A_0YZA_ℓ equals $2A_0A_\ell + 1$.



Recall a well-known fact stating that if a convex polygon P_1 is contained in a convex polygon P_2, then the perimeter of P_1 is at most the perimeter of P_2. Hence we obtain

$$\ell + A_0A_\ell \leqslant 2A_0A_\ell + 1, \quad \text{i.e.} \quad \ell - 1 \leqslant A_0A_\ell.$$

On the other hand, the triangle inequality yields

$$A_0A_\ell < A_\ell A_{\ell+1} + A_{\ell+1}A_{\ell+2} + \dots + A_{2k}A_0 = 2k+1-\ell \leqslant \ell - 1,$$

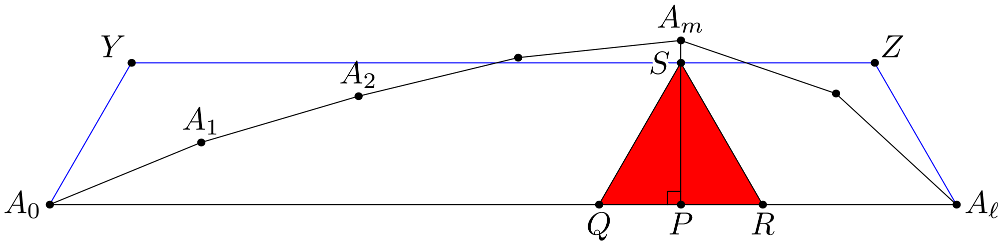
which gives a contradiction.

Therefore, there exists a vertex A_m of $A_0A_1\dots A_\ell$ which lies outside A_0YZA_ℓ. Since

$$\angle A_\ell A_0A_1 < 60^\circ = \angle A_\ell A_0Y \quad \text{and} \quad A_{\ell-1}A_\ell A_0 < 60^\circ = \angle ZA_\ell A_0, \tag{1}$$

the distance between A_m and A_0A_ℓ is at least $\sqrt{3}/2$.

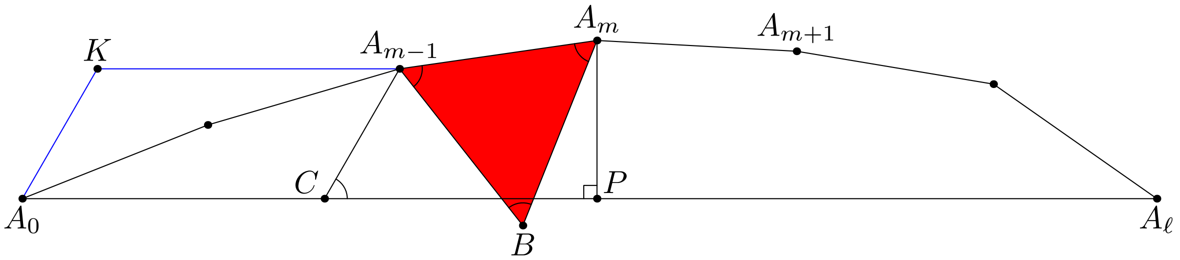
Let P be the projection of A_m to A_0A_ℓ. Then $PA_m \geqslant \sqrt{3}/2$, and by (1) we have $A_0P > 1/2$ and $PA_\ell > 1/2$. Choose points $Q \in A_0P$, $R \in PA_\ell$, and $S \in PA_m$ such that $PQ = PR = 1/2$ and $PS = \sqrt{3}/2$. Then QRS is an equilateral triangle of side length 1 contained in $A_0A_1 \ldots A_\ell$.



Comment. In fact, for every odd n a stronger statement holds, which is formulated in terms defined in the solution above: there exists an equilateral triangle $A_iA_{i+1}B$ contained in $A_0A_1 \ldots A_\ell$ for some $0 \leqslant i < \ell$. We sketch an indirect proof below.

As above, we get $\angle A_\ell A_0 A_1 < 60°$ and $A_{\ell-1}A_\ell A_0 < 60°$. Choose an index $m \in [1, \ell-1]$ maximising the distance between A_m and A_0A_ℓ. Arguments from the above solution yield $1 < m < \ell - 1$. Then $\angle A_0A_{m-1}A_m > 120°$ and $\angle A_{m-1}A_mA_\ell > \angle A_0A_mA_\ell \geqslant 60°$. We construct an equilateral triangle $A_{m-1}A_mB$ as in the figure below. If B lies in $A_0A_{m-1}A_mA_\ell$, then we are done. Otherwise B and A_m lie on different sides of A_0A_ℓ. As before, let P be the projection of A_m to A_0A_ℓ. We will show that

$$A_0A_1 + A_1A_2 + \ldots + A_{m-1}A_m < A_0P + 1/2. \tag{2}$$



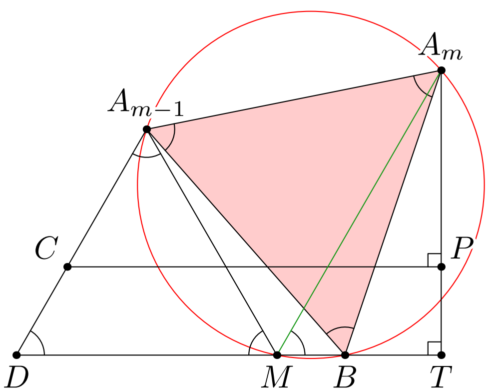
There exists a point C on the segment A_0P such that $\angle A_{m-1}CP = 60°$. Construct a parallelogram $A_0CA_{m-1}K$. Then the polyline $A_0A_1 \ldots A_{m-1}$ is contained in the triangle $A_{m-1}KA_0$, so

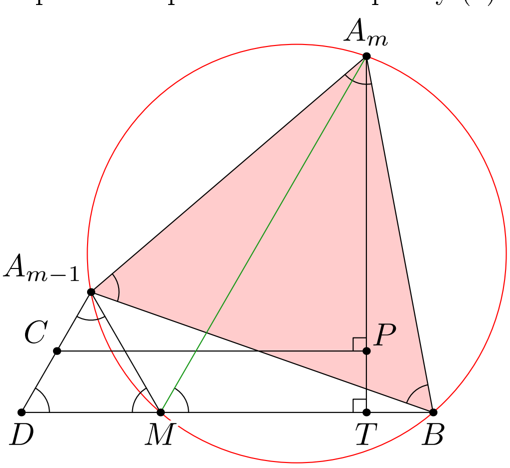
$$A_0A_1 + A_1A_2 + \ldots + A_{m-2}A_{m-1} + A_{m-1}A_m \leqslant A_0K + KA_{m-1} + A_{m-1}A_m = A_0C + CA_{m-1} + 1.$$

To prove (2), it suffices to show that $CA_{m-1} < CP - 1/2$. Let the line through B parallel to CP intersect the rays $A_{m-1}C$ and A_mP at D and T, respectively. It is easy to see that the desired inequality will follow from $DA_{m-1} \leqslant DT - 1/2$.

Two possible arrangements of points are shown in the figures below.

Observe that $\angle DA_{m-1}B \geqslant 60°$, so there is a point M on the segment DB such that the triangle DMA_{m-1} is equilateral. Then $\angle A_{m-1}MD = 60° = \angle A_{m-1}A_mB$, so $A_{m-1}MBA_m$ is a cyclic quadrilateral. Therefore, $\angle A_mMB = 60°$. Thus, T lies on the ray MB and we have to show that $MT \geqslant 1/2$. Indeed, $MT = A_mM/2$ and $A_mM \geqslant A_mB = 1$. This completes the proof of the inequality (2).





Similarly, either there exists an equilateral triangle $A_mA_{m+1}B'$ contained in $A_0A_1\ldots A_\ell$, or

$$A_mA_{m+1} + A_{m+1}A_{m+2} + \ldots + A_{\ell-1}A_\ell < A_\ell P + 1/2. \tag{3}$$

Adding (2) and (3) yields $A_0A_1 + A_1A_2 + \ldots + A_{\ell-1}A_\ell < A_0A_\ell + 1$, which gives a contradiction.

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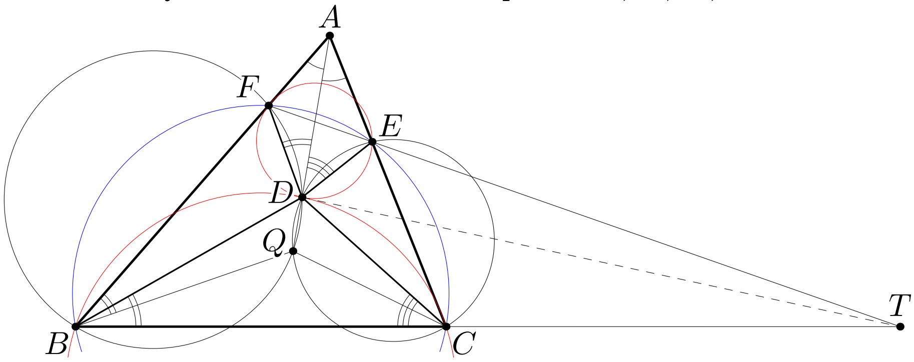
G7. A point D is chosen inside an acute-angled triangle ABC with $AB > AC$ so that $\angle BAD = \angle DAC$. A point E is constructed on the segment AC so that $\angle ADE = \angle DCB$. Similarly, a point F is constructed on the segment AB so that $\angle ADF = \angle DBC$. A point X is chosen on the line AC so that $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and DXE. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Common remarks. Let Q be the isogonal conjugate of D with respect to the triangle ABC. Since $\angle BAD = \angle DAC$, the point Q lies on AD. Then $\angle QBA = \angle DBC = \angle FDA$, so the points Q, D, F, and B are concyclic. Analogously, the points Q, D, E, and C are concyclic. Thus $AF \cdot AB = AD \cdot AQ = AE \cdot AC$ and so the points B, F, E, and C are also concyclic.



Let T be the intersection of BC and FE.

Claim. $TD^2 = TB \cdot TC = TF \cdot TE$.

Proof. We will prove that the circles (DEF) and (BDC) are tangent to each other. Indeed, using the above arguments, we get

$$\begin{aligned} &\angle BDF = \angle AFD - \angle ABD = (180^\circ - \angle FAD - \angle FDA) - (\angle ABC - \angle DBC) \\ &= 180^\circ - \angle FAD - \angle ABC = 180^\circ - \angle DAE - \angle FEA = \angle FED + \angle ADE = \angle FED + \angle DCB, \end{aligned}$$

which implies the desired tangency.

Since the points B, C, E, and F are concyclic, the powers of the point T with respect to the circles (BDC) and (EDF) are equal. So their radical axis, which coincides with the common tangent at D, passes through T, and hence $TD^2 = TE \cdot TF = TB \cdot TC$. □

Solution 1. Let TA intersect the circle (ABC) again at M. Due to the circles $(BCEF)$ and $(AMCB)$, and using the above Claim, we get $TM \cdot TA = TF \cdot TE = TB \cdot TC = TD^2$; in particular, the points A, M, E, and F are concyclic.

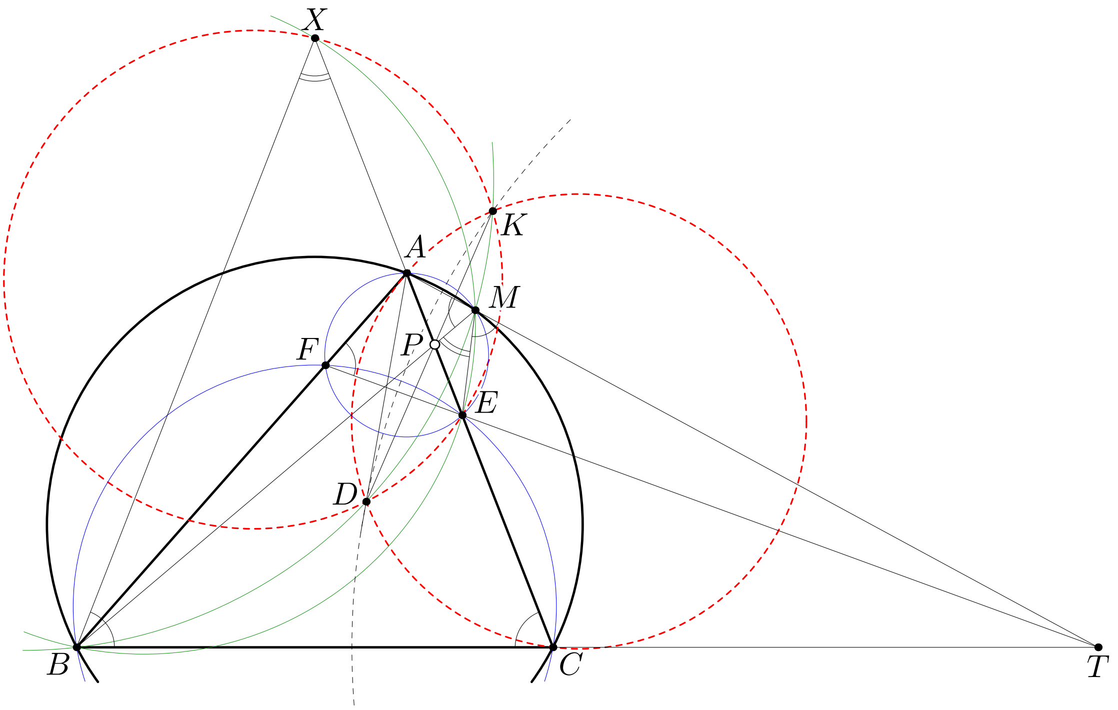
Under the inversion with centre T and radius TD, the point M maps to A, and B maps to C, which implies that the circle (MBD) maps to the circle (ADC). Their common point D lies on the circle of the inversion, so the second intersection point K also lies on that circle, which means $TK = TD$. It follows that the point T and the centres of the circles (KDE) and (ADC) lie on the perpendicular bisector of KD.

Since the center of (ADC) is O_1, it suffices to show now that the points D, K, E, and X are concyclic (the center of the corresponding circle will be O_2).

The lines BM, DK, and AC are the pairwise radical axes of the circles $(ABCM)$, $(ACDK)$ and $(BMDK)$, so they are concurrent at some point P. Also, M lies on the circle (AEF), thus

$$\begin{aligned} \measuredangle(EX, XB) &= \measuredangle(CX, XB) = \measuredangle(XC, BC) + \measuredangle(BC, BX) = 2\measuredangle(AC, CB) \\ &= \measuredangle(AC, CB) + \measuredangle(EF, FA) = \measuredangle(AM, BM) + \measuredangle(EM, MA) = \measuredangle(EM, BM), \end{aligned}$$

so the points M, E, X, and B are concyclic. Therefore, $PE \cdot PX = PM \cdot PB = PK \cdot PD$, so the points E, K, D, and X are concyclic, as desired.



Comment 1. We present here a different solution which uses similar ideas.

Perform the inversion ι with centre T and radius TD. It swaps B with C and E with F; the point D maps to itself. Let $X' = \iota(X)$. Observe that the points E, F, X, and X' are concyclic, as well as the points B, C, X, and X'. Then

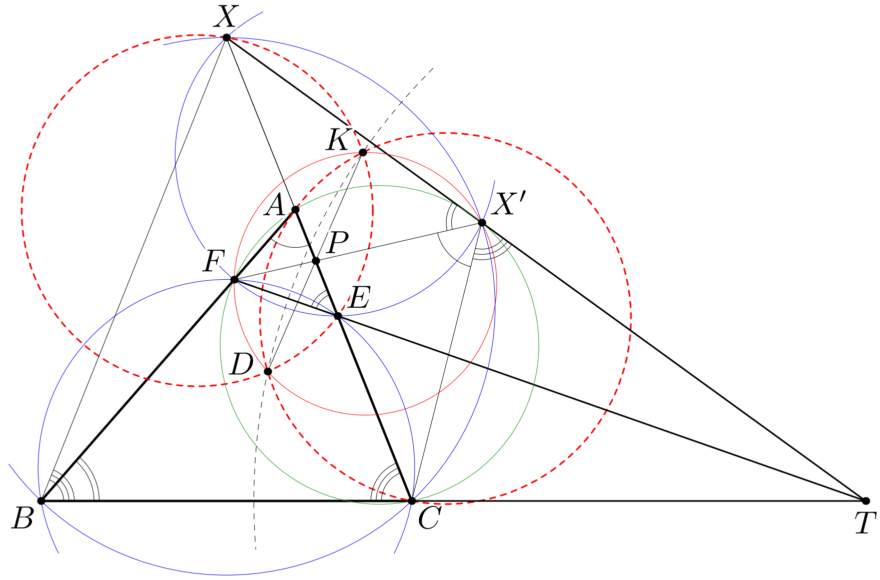
$$\begin{aligned}\measuredangle(CX', X'F) &= \measuredangle(CX', X'X) + \measuredangle(X'X, X'F) = \measuredangle(CB, BX) + \measuredangle(EX, EF)\\ &= \measuredangle(XC, CB) + \measuredangle(EC, EF) = \measuredangle(CA, CB) + \measuredangle(BC, BF) = \measuredangle(CA, AF),\end{aligned}$$

therefore the points C, X', A, and F are concyclic.

Let $X'F$ intersect AC at P, and let K be the second common point of DP and the circle (ACD). Then

$$PK \cdot PD = PA \cdot PC = PX' \cdot PF = PE \cdot PX;$$

hence, the points K, X, D, and E lie on some circle ω_1, while the points K, X', D, and F lie on some circle ω_2. (These circles are distinct since $\angle EXF + \angle EDF < \angle EAF + \angle DCB + \angle DBC < 180°$). The inversion ι swaps ω_1 with ω_2 and fixes their common point D, so it fixes their second common point K. Thus $TD = TK$ and the perpendicular bisector of DK passes through T, as well as through the centres of the circles $(CDKA)$ and $(DEKX)$.



Solution 2. We use only the first part of the Common remarks, namely, the facts that the tuples (C, D, Q, E) and (B, C, E, F) are both concyclic. We also introduce the point $T = BC \cap EF$. Let the circle (CDE) meet BC again at E_1. Since $\angle E_1CQ = \angle DCE$, the arcs DE and QE_1 of the circle (CDQ) are equal, so $DQ \parallel EE_1$.

Since $BFEC$ is cyclic, the line AD forms equal angles with BC and EF, hence so does EE_1. Therefore, the triangle EE_1T is isosceles, $TE = TE_1$, and T lies on the common perpendicular bisector of EE_1 and DQ.

Let U and V be the centres of circles (ADE) and $(CDQE)$, respectively. Then UO_1 is the perpendicular bisector of AD. Moreover, the points U, V, and O_2 belong to the perpendicular bisector of DE. Since $UO_1 \parallel VT$, in order to show that O_1O_2 passes through T, it suffices to show that

$$\frac{O_2U}{O_2V} = \frac{O_1U}{TV}. \tag{1}$$

Denote angles A, B, and C of the triangle ABC by α, β, and γ, respectively. Projecting onto AC we obtain

$$\frac{O_2U}{O_2V} = \frac{(XE - AE)/2}{(XE + EC)/2} = \frac{AX}{CX} = \frac{AX}{BX} = \frac{\sin(\gamma - \beta)}{\sin \alpha} \tag{2}$$

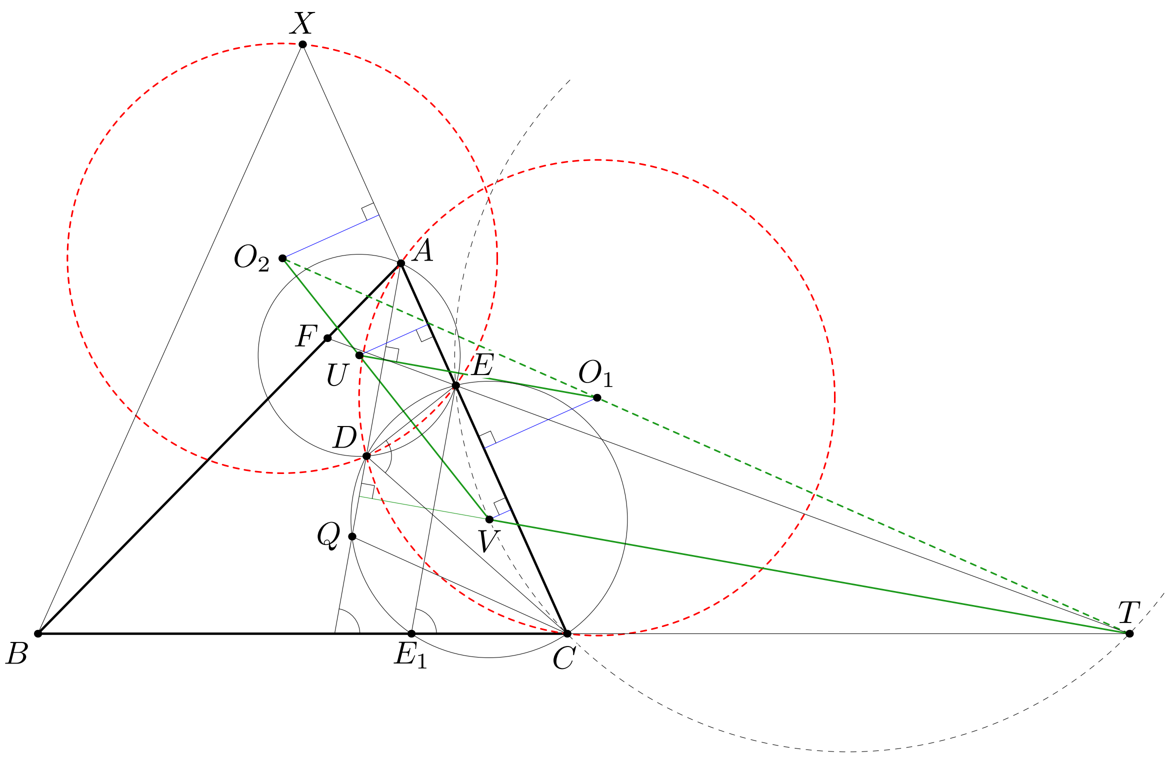
The projection of O_1U onto AC is $(AC - AE)/2 = CE/2$; the angle between O_1U and AC is $90^\circ - \alpha/2$, so

$$\frac{O_1U}{EC} = \frac{1}{2\sin(\alpha/2)} \tag{3}$$

Next, we claim that E, V, C, and T are concyclic. Indeed, the point V lies on the perpendicular bisector of CE, as well as on the internal angle bisector of $\angle CTF$. Therefore, V coincides with the midpoint of the arc CE of the circle (TCE).

Now we have $\angle EVC = 2\angle EE_1C = 180^\circ - (\gamma - \beta)$ and $\angle VET = \angle VE_1T = 90^\circ - \angle E_1EC = 90^\circ - \alpha/2$. Therefore,

$$\frac{EC}{TV} = \frac{\sin \angle ETC}{\sin \angle VET} = \frac{\sin(\gamma - \beta)}{\cos(\alpha/2)}. \tag{4}$$



Recalling (2) and multiplying (3) and (4) we establish (1):

$$\frac{O_2U}{O_2V} = \frac{\sin(\gamma-\beta)}{\sin\alpha} = \frac{1}{2\sin(\alpha/2)} \cdot \frac{\sin(\gamma-\beta)}{\cos(\alpha/2)} = \frac{O_1U}{EC} \cdot \frac{EC}{TV} = \frac{O_1U}{TV}.$$

Solution 3. Notice that $\angle AQE = \angle QCB$ and $\angle AQF = \angle QBC$; so, if we replace the point D with Q in the problem set up, the points E, F, and T remain the same. So, by the Claim, we have $TQ^2 = TB \cdot TC = TD^2$.

Thus, there exists a circle Γ centred at T and passing through D and Q. We denote the second meeting point of the circles Γ and (ADC) by K. Let the line AC meet the circle (DEK) again at Y; we intend to prove that $Y = X$. As in Solution 1, this will yield that the point T, as well as the centres O_1 and O_2, all lie on the perpendicular bisector of DK.

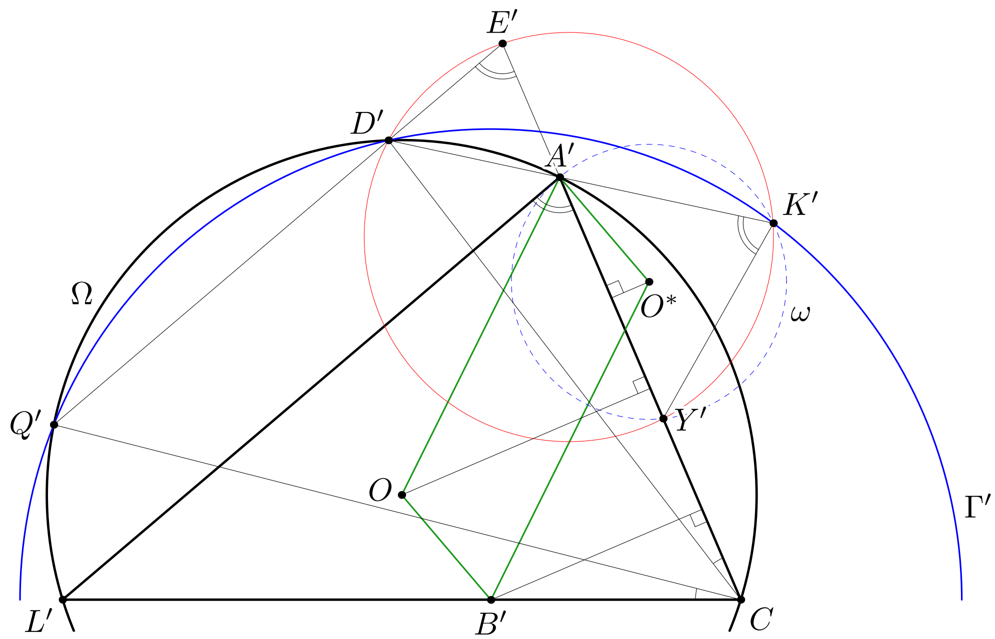
Let $L = AD \cap BC$. We perform an inversion centred at C; the images of the points will be denoted by primes, e.g., A' is the image of A. We obtain the following configuration, constructed in a triangle $A'CL'$.

The points D' and Q' are chosen on the circumcircle Ω of $A'L'C$ such that $\measuredangle(L'C, D'C) = \measuredangle(Q'C, A'C)$, which means that $A'L' \parallel D'Q'$. The lines $D'Q'$ and $A'C$ meet at E'.

A circle Γ' centred on CL' passes through D' and Q'. Notice here that B' lies on the segment CL', and that $\angle A'B'C = \angle BAC = 2\angle LAC = 2\angle A'L'C$, so that $B'L' = B'A'$, and B' lies on the perpendicular bisector of $A'L'$ (which coincides with that of $D'Q'$). All this means that B' is the centre of Γ'.

Finally, K' is the second meeting point of $A'D'$ and Γ', and Y' is the second meeting point of the circle $(D'K'E')$ and the line $A'E'$, We have $\measuredangle(Y'K', K'A') = \measuredangle(Y'E', E'D') = \measuredangle(Y'A', A'L')$, so $A'L'$ is tangent to the circumcircle ω of the triangle $Y'A'K'$.

Let O and O^* be the centres of Ω and ω, respectively. Then $O^*A' \perp A'L' \perp B'O$. The projections of vectors $\overrightarrow{O^*A'}$ and $\overrightarrow{B'O}$ onto $K'D'$ are equal to $\overrightarrow{K'A'}/2 = \overrightarrow{K'D'}/2 - \overrightarrow{A'D'}/2$. So $\overrightarrow{O^*A'} = \overrightarrow{B'O}$, or equivalently $\overrightarrow{A'O} = \overrightarrow{O^*B'}$. Projecting this equality onto $A'C$, we see that the projection of $\overrightarrow{O^*B'}$ equals $\overrightarrow{A'C}/2$. Since O^* is projected to the midpoint of $A'Y'$, this yields that B' is projected to the midpoint of CY', i.e., $B'Y' = B'C$ and $\angle B'Y'C = \angle B'CY'$. In the original figure, this rewrites as $\angle CBY = \angle BCY$, so Y lies on the perpendicular bisector of BC, as desired.



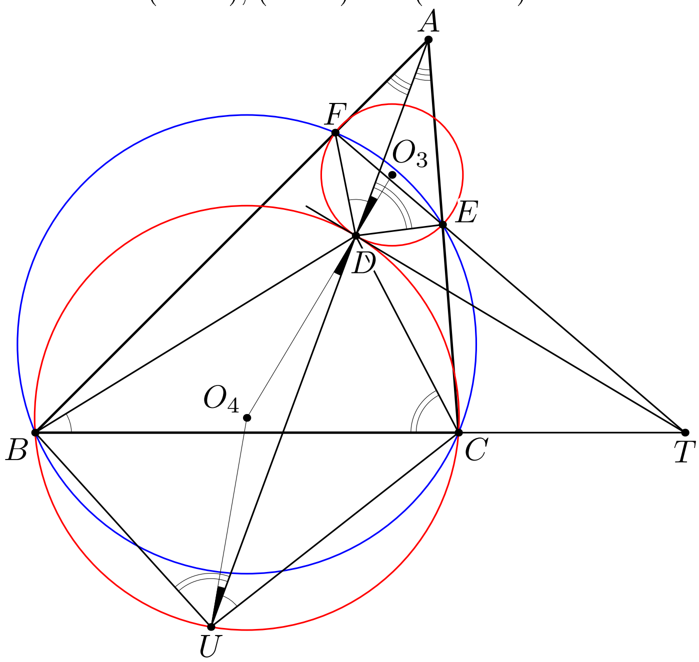
Comment 2. The point K appears to be the same in Solutions 1 and 3 (and Comment 1 as well). One can also show that K lies on the circle passing through A, X, and the midpoint of the arc BAC.

Comment 3. There are different proofs of the facts from the Common remarks, namely, the cyclicity of B, C, E, and F, and the Claim. We present one such alternative proof here.

We perform the composition ϕ of a homothety with centre A and the reflection in AD, which maps E to B. Let $U = \phi(D)$. Then $\measuredangle(BC, CD) = \measuredangle(AD, DE) = \measuredangle(BU, UD)$, so the points B, U, C, and D are concyclic. Therefore, $\measuredangle(CU, UD) = \measuredangle(CB, BD) = \measuredangle(AD, DF)$, so $\phi(F) = C$. Then the coefficient of the homothety is $AC/AF = AB/AE$, and thus points C, E, F, and B are concyclic.

Denote the centres of the circles (EDF) and $(BUCD)$ by O_3 and O_4, respectively. Then $\phi(O_3) = O_4$, hence $\measuredangle(O_3D, DA) = -\measuredangle(O_4U, UA) = \measuredangle(O_4D, DA)$, whence the circle (BDC) is tangent to the circle (EDF).

Now, the radical axes of circles (DEF), (BDC) and $(BCEF)$ intersect at T, and the claim follows.



This suffices for Solution 1 to work. However, Solutions 2 and 3 need properties of point Q, established in Common remarks before Solution 1.

Comment 4. In the original problem proposal, the point X was hidden. Instead, a circle γ was constructed such that D and E lie on γ, and its center is collinear with O_1 and T. The problem requested to prove that, in a fixed triangle ABC, independently from the choice of D on the bisector of $\angle BAC$, all circles γ pass through a fixed point.

G8. Let ω be the circumcircle of a triangle ABC, and let Ω_A be its excircle which is tangent to the segment BC. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y, respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $AR \perp BC$.

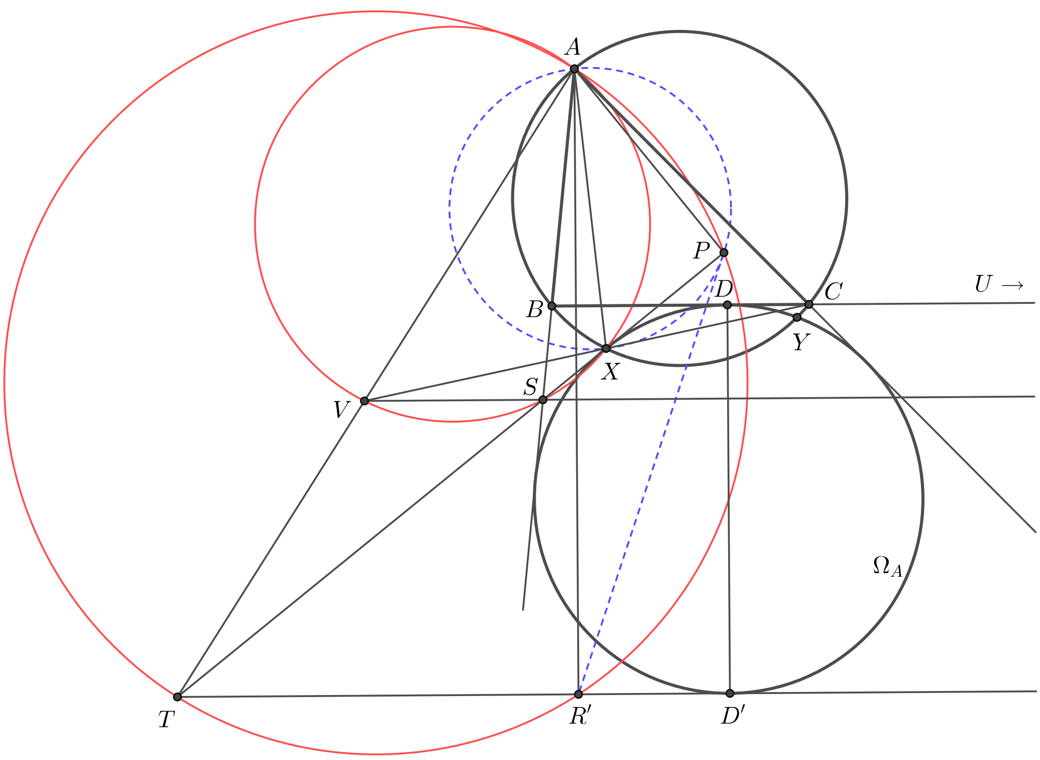
Solution 1. Let D be the point of tangency of BC and Ω_A. Let D' be the point such that DD' is a diameter of Ω_A. Let R' be (the unique) point such that $AR' \perp BC$ and $R'D' \parallel BC$. We shall prove that R' coincides with R.

Let PX intersect AB and $D'R'$ at S and T, respectively. Let U be the ideal common point of the parallel lines BC and $D'R'$. Note that the (degenerate) hexagon $ASXTUC$ is circumscribed around Ω_A, hence by the Brianchon theorem AT, SU, and XC concur at a point which we denote by V. Then $VS \parallel BC$. It follows that $\sphericalangle(SV, VX) = \sphericalangle(BC, CX) = \sphericalangle(BA, AX)$, hence $AXSV$ is cyclic. Therefore, $\sphericalangle(PX, XA) = \sphericalangle(SV, VA) = \sphericalangle(R'T, TA)$. Since $\angle APT = \angle AR'T = 90^\circ$, the quadrilateral $APR'T$ is cyclic. Hence,

$$\sphericalangle(XA, AP) = 90^\circ - \sphericalangle(PX, XA) = 90^\circ - \sphericalangle(R'T, TA) = \sphericalangle(TA, AR') = \sphericalangle(TP, PR').$$

It follows that PR' is tangent to the circle (APX).

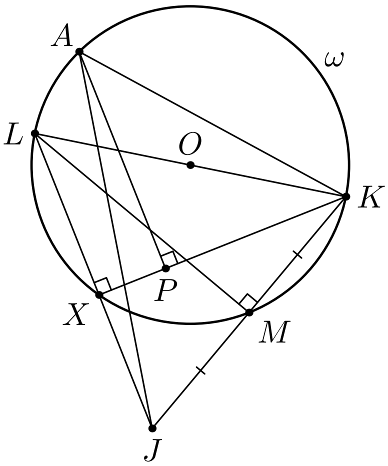
Analogous argument shows that QR' is tangent to the circle (AQY). Therefore, $R = R'$ and $AR \perp BC$.

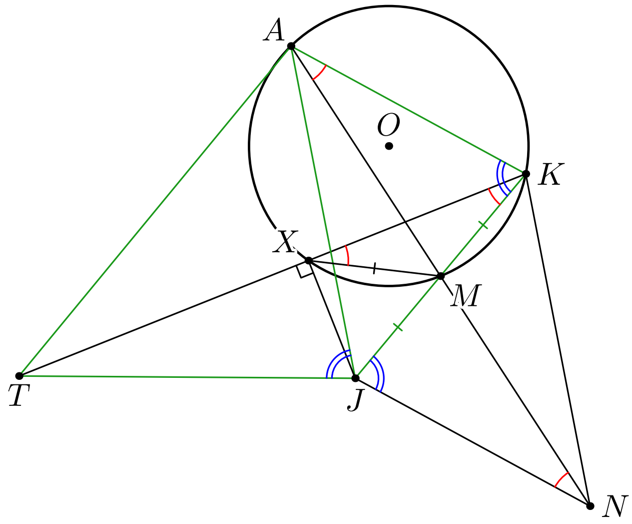


Comment 1. After showing $\sphericalangle(PX, XA) = \sphericalangle(R'T, TA)$ one can finish the solution as follows. There exists a spiral similarity mapping the triangle ATR' to the triangle AXP. So the triangles ATX and $AR'P$ are similar and equioriented. Thus, $\sphericalangle(TX, XA) = \sphericalangle(R'P, PA)$, which implies that PR' is tangent to the circle (APX).

Solution 2. Let J and r be the center and the radius of Ω_A. Denote the diameter of ω by d and its center by O. By Euler's formula, $OJ^2 = (d/2)^2 + dr$, so the power of J with respect to ω equals dr.

Let JX intersect ω again at L. Then $JL = d$. Let LK be a diameter of ω and let M be the midpoint of JK. Since $JL = LK$, we have $\angle LMK = 90°$, so M lies on ω. Let R' be the point such that $R'P$ is tangent to the circle (APX) and $AR' \perp BC$. Note that the line AR' is symmetric to the line AO with respect to AJ.



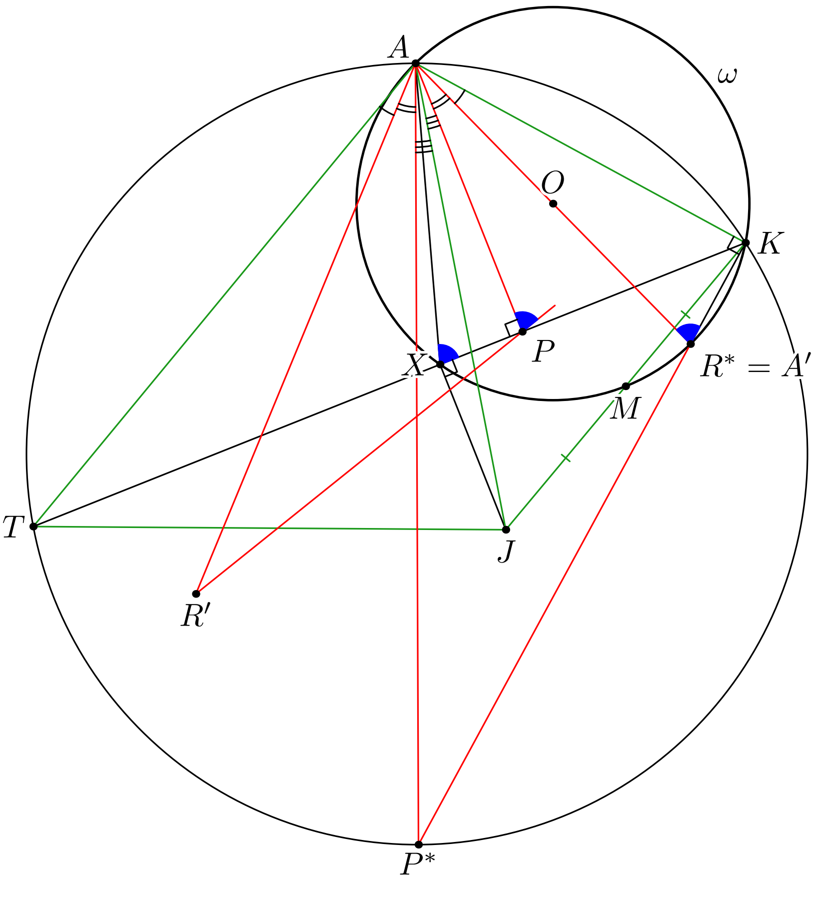


Lemma. Let M be the midpoint of the side JK in a triangle AJK. Let X be a point on the circle (AMK) such that $\angle JXK = 90°$. Then there exists a point T on the line KX such that the triangles AKJ and AJT are similar and equioriented.

Proof. Note that $MX = MK$. We construct a parallelogram $AJNK$. Let T be a point on KX such that $\measuredangle(NJ, JA) = \measuredangle(KJ, JT)$. Then

$$\measuredangle(JN, NA) = \measuredangle(KA, AM) = \measuredangle(KX, XM) = \measuredangle(MK, KX) = \measuredangle(JK, KT).$$

So there exists a spiral similarity with center J mapping the triangle AJN to the triangle TJK. Therefore, the triangles NJK and AJT are similar and equioriented. It follows that the triangles AKJ and AJT are similar and equioriented. $\square$



Back to the problem, we construct a point T as in the lemma. We perform the composition ϕ of inversion with centre A and radius AJ and reflection in AJ. It is known that every triangle AEF is similar and equioriented to $A\phi(F)\phi(E)$.

So $\phi(K) = T$ and $\phi(T) = K$. Let $P^* = \phi(P)$ and $R^* = \phi(R')$. Observe that $\phi(TK)$ is a circle with diameter AP^*. Let AA' be a diameter of ω. Then $P^*K \perp AK \perp A'K$, so A' lies on P^*K. The triangles $AR'P$ and AP^*R^* are similar and equioriented, hence

$$\measuredangle(AA', A'P^*) = \measuredangle(AA', A'K) = \measuredangle(AX, XP) = \measuredangle(AX, XP) = \measuredangle(AP, PR') = \measuredangle(AR^*, R^*P^*),$$

so A, A', R^*, and P^* are concyclic. Since A' and R^* lie on AO, we obtain $R^* = A'$. So $R' = \phi(A')$, and $\phi(A')P$ is tangent to the circle (APX).

An identical argument shows that $\phi(A')Q$ is tangent to the circle (AQY). Therefore, $R = \phi(A')$ and $AR \perp BC$.

Comment 2. One of the main ideas of Solution 2 is to get rid of the excircle, along with points B and C. After doing so we obtain the following fact, which is, essentially, proved in Solution 2.

Let ω be the circumcircle of a triangle AK_1K_2. Let J be a point such that the midpoints of JK_1 and JK_2 lie on ω. Points X and Y are chosen on ω so that $\angle JXK_1 = \angle JYK_2 = 90°$. Let P and Q be the projections of A onto XK_1 and YK_2, respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Then the reflection of the line AR in AJ passes through the centre O of ω.

Number Theory

N1. Determine all integers $n \geqslant 1$ for which there exists a pair of positive integers (a, b) such that no cube of a prime divides $a^2 + b + 3$ and

$$\frac{ab + 3b + 8}{a^2 + b + 3} = n.$$

Answer: The only integer with that property is $n = 2$.

Solution. As $b \equiv -a^2 - 3 \pmod{a^2 + b + 3}$, the numerator of the given fraction satisfies

$$ab + 3b + 8 \equiv a(-a^2 - 3) + 3(-a^2 - 3) + 8 \equiv -(a + 1)^3 \pmod{a^2 + b + 3}.$$

As $a^2 + b + 3$ is not divisible by p^3 for any prime p, if $a^2 + b + 3$ divides $(a + 1)^3$ then it does also divide $(a + 1)^2$. Since

$$0 < (a + 1)^2 < 2(a^2 + b + 3),$$

we conclude $(a + 1)^2 = a^2 + b + 3$. This yields $b = 2(a - 1)$ and $n = 2$. The choice $(a, b) = (2, 2)$ with $a^2 + b + 3 = 9$ shows that $n = 2$ indeed is a solution.

N2. Let $n \geqslant 100$ be an integer. The numbers $n, n+1, \ldots, 2n$ are written on $n+1$ cards, one number per card. The cards are shuffled and divided into two piles. Prove that one of the piles contains two cards such that the sum of their numbers is a perfect square.

Solution. To solve the problem it suffices to find three squares and three cards with numbers a, b, c on them such that pairwise sums $a+b, b+c, a+c$ are equal to the chosen squares. By choosing the three consecutive squares $(2k-1)^2$, $(2k)^2$, $(2k+1)^2$ we arrive at the triple

$$(a, b, c) = \left(2k^2 - 4k, \quad 2k^2 + 1, \quad 2k^2 + 4k\right).$$

We need a value for k such that

$$n \leqslant 2k^2 - 4k, \quad \text{and} \quad 2k^2 + 4k \leqslant 2n.$$

A concrete k is suitable for all n with

$$n \in \left[k^2 + 2k, 2k^2 - 4k + 1\right] =: I_k.$$

For $k \geqslant 9$ the intervals I_k and I_{k+1} overlap because

$$(k+1)^2 + 2(k+1) \leqslant 2k^2 - 4k + 1.$$

Hence $I_9 \cup I_{10} \cup \ldots = [99, \infty)$, which proves the statement for $n \geqslant 99$.

Comment 1. There exist approaches which only work for sufficiently large n.

One possible approach is to consider three cards with numbers $70k^2, 99k^2, 126k^2$ on them. Then their pairwise sums are perfect squares and so it suffices to find k such that $70k^2 \geqslant n$ and $126k^2 \leqslant 2n$ which exists for sufficiently large n.

Another approach is to prove, arguing by contradiction, that a and $a-2$ are in the same pile provided that n is large enough and a is sufficiently close to n. For that purpose, note that every pair of neighbouring numbers in the sequence $a, x^2 - a, a + (2x+1), x^2 + 2x + 3 - a, a - 2$ adds up to a perfect square for any x; so by choosing $x = \lfloor \sqrt{2a} \rfloor + 1$ and assuming that n is large enough we conclude that a and $a-2$ are in the same pile for any $a \in [n+2, 3n/2]$. This gives a contradiction since it is easy to find two numbers from $[n+2, 3n/2]$ of the same parity which sum to a square.

It then remains to separately cover the cases of small n which appears to be quite technical.

Comment 2. An alternative formulation for this problem could ask for a proof of the statement for all $n > 10^6$. An advantage of this formulation is that some solutions, e.g. those mentioned in Comment 1 need not contain a technical part which deals with the cases of small n. However, the original formulation seems to be better because the bound it gives for n is almost sharp, see the next comment for details.

Comment 3. The statement of the problem is false for $n = 98$. As a counterexample, the first pile may contain the even numbers from 98 to 126, the odd numbers from 129 to 161, and the even numbers from 162 to 196.

N3. Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for every $i = 1, 2, \ldots, k$, the number $d_1 + \cdots + d_i$ is a perfect square.

Answer: $n = 1$ and $n = 3$.

Solution. For $i = 1, 2, \ldots, k$ let $d_1 + \ldots + d_i = s_i^2$, and define $s_0 = 0$ as well. Obviously $0 = s_0 < s_1 < s_2 < \ldots < s_k$, so

$$s_i \geqslant i \quad \text{and} \quad d_i = s_i^2 - s_{i-1}^2 = (s_i + s_{i-1})(s_i - s_{i-1}) \geqslant s_i + s_{i-1} \geqslant 2i - 1. \tag{1}$$

The number 1 is one of the divisors $d_1, \ldots, d_k$ but, due to $d_i \geqslant 2i - 1$, the only possibility is $d_1 = 1$.

Now consider d_2 and $s_2 \geqslant 2$. By definition, $d_2 = s_2^2 - 1 = (s_2 - 1)(s_2 + 1)$, so the numbers $s_2 - 1$ and $s_2 + 1$ are divisors of n. In particular, there is some index j such that $d_j = s_2 + 1$.

Notice that

$$s_2 + s_1 = s_2 + 1 = d_j \geqslant s_j + s_{j-1}; \tag{2}$$

since the sequence $s_0 < s_1 < \ldots < s_k$ increases, the index j cannot be greater than 2. Hence, the divisors $s_2 - 1$ and $s_2 + 1$ are listed among d_1 and d_2. That means $s_2 - 1 = d_1 = 1$ and $s_2 + 1 = d_2$; therefore $s_2 = 2$ and $d_2 = 3$.

We can repeat the above process in general.

Claim. $d_i = 2i - 1$ and $s_i = i$ for $i = 1, 2, \ldots, k$.

Proof. Apply induction on i. The Claim has been proved for $i = 1, 2$. Suppose that we have already proved $d = 1$, $d_2 = 3$, $\ldots$, $d_i = 2i - 1$, and consider the next divisor d_{i+1}:

$$d_{i+1} = s_{i+1}^2 - s_i^2 = s_{i+1}^2 - i^2 = (s_{i+1} - i)(s_{i+1} + i).$$

The number $s_{i+1} + i$ is a divisor of n, so there is some index j such that $d_j = s_{i+1} + i$.

Similarly to (2), by (1) we have

$$s_{i+1} + s_i = s_{i+1} + i = d_j \geqslant s_j + s_{j-1}; \tag{3}$$

since the sequence $s_0 < s_1 < \ldots < s_k$ increases, (3) forces $j \leqslant i + 1$. On the other hand, $d_j = s_{i+1} + i > 2i > d_i > d_{i-1} > \ldots > d_1$, so $j \leqslant i$ is not possible. The only possibility is $j = i + 1$.

Hence,

$$s_{i+1} + i = d_{i+1} = s_{i+1}^2 - s_i^2 = s_{i+1}^2 - i^2;$$
$$s_{i+1}^2 - s_{i+1} = i(i + 1).$$

By solving this equation we get $s_{i+1} = i + 1$ and $d_{i+1} = 2i + 1$, that finishes the proof. □

Now we know that the positive divisors of the number n are $1, 3, 5, \ldots, n-2, n$. The greatest divisor is $d_k = 2k - 1 = n$ itself, so n must be odd. The second greatest divisor is $d_{k-1} = n - 2$; then $n - 2$ divides $n = (n - 2) + 2$, so $n - 2$ divides 2. Therefore, n must be 1 or 3.

The numbers $n = 1$ and $n = 3$ obviously satisfy the requirements: for $n = 1$ we have $k = 1$ and $d_1 = 1^2$; for $n = 3$ we have $k = 2$, $d_1 = 1^2$ and $d_1 + d_2 = 1 + 3 = 2^2$.

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N4. Alice is given a rational number $r > 1$ and a line with two points $B \neq R$, where point R contains a red bead and point B contains a blue bead. Alice plays a solitaire game by performing a sequence of moves. In every move, she chooses a (not necessarily positive) integer k, and a bead to move. If that bead is placed at point X, and the other bead is placed at Y, then Alice moves the chosen bead to point X' with $\overrightarrow{YX'} = r^k\,\overrightarrow{YX}$.

Alice's goal is to move the red bead to the point B. Find all rational numbers $r > 1$ such that Alice can reach her goal in at most 2021 moves.

Answer: All $r = (b+1)/b$ with $b = 1, \ldots, 1010$.

Solution. Denote the red and blue beads by $\mathcal{R}$ and $\mathcal{B}$, respectively. Introduce coordinates on the line and identify the points with their coordinates so that $R = 0$ and $B = 1$. Then, during the game, the coordinate of $\mathcal{R}$ is always smaller than the coordinate of $\mathcal{B}$. Moreover, the distance between the beads always has the form r^ℓ with $\ell \in \mathbb{Z}$, since it only multiplies by numbers of this form. Denote the value of the distance after the m^{th} move by $d_m = r^{\alpha_m}$, $m = 0, 1, 2, \ldots$ (after the 0^{th} move we have just the initial position, so $\alpha_0 = 0$).

If some bead is moved in two consecutive moves, then Alice could instead perform a single move (and change the distance from d_i directly to d_{i+2}) which has the same effect as these two moves. So, if Alice can achieve her goal, then she may as well achieve it in fewer (or the same) number of moves by alternating the moves of $\mathcal{B}$ and $\mathcal{R}$. In the sequel, we assume that Alice alternates the moves, and that $\mathcal{R}$ is shifted altogether t times.

If $\mathcal{R}$ is shifted in the m^{th} move, then its coordinate increases by $d_m - d_{m+1}$. Therefore, the total increment of $\mathcal{R}$'s coordinate, which should be 1, equals

$$\text{either} \qquad (d_0 - d_1) + (d_2 - d_3) + \cdots + (d_{2t-2} - d_{2t-1}) = 1 + \sum_{i=1}^{t-1} r^{\alpha_{2i}} - \sum_{i=1}^{t} r^{\alpha_{2i-1}},$$

$$\text{or} \qquad (d_1 - d_2) + (d_3 - d_4) + \cdots + (d_{2t-1} - d_{2t}) = \sum_{i=1}^{t} r^{\alpha_{2i-1}} - \sum_{i=1}^{t} r^{\alpha_{2i}},$$

depending on whether $\mathcal{R}$ or $\mathcal{B}$ is shifted in the first move. Moreover, in the former case we should have $t \leqslant 1011$, while in the latter one we need $t \leqslant 1010$. So both cases reduce to an equation

$$\sum_{i=1}^{n} r^{\beta_i} = \sum_{i=1}^{n-1} r^{\gamma_i}, \qquad \beta_i, \gamma_i \in \mathbb{Z}, \tag{1}$$

for some $n \leqslant 1011$. Thus, if Alice can reach her goal, then this equation has a solution for $n = 1011$ (we can add equal terms to both sums in order to increase n).

Conversely, if (1) has a solution for $n = 1011$, then Alice can compose a corresponding sequence of distances $d_0, d_1, d_2, \ldots, d_{2021}$ and then realise it by a sequence of moves. So the problem reduces to the solvability of (1) for $n = 1011$.

Assume that, for some rational r, there is a solution of (1). Write r in lowest terms as $r = a/b$. Substitute this into (1), multiply by the common denominator, and collect all terms on the left hand side to get

$$\sum_{i=1}^{2n-1} (-1)^i a^{\mu_i} b^{N-\mu_i} = 0, \qquad \mu_i \in \{0, 1, \ldots, N\}, \tag{2}$$

for some $N \geqslant 0$. We assume that there exist indices j_- and j_+ such that $\mu_{j_-} = 0$ and $\mu_{j_+} = N$.

Reducing (2) modulo $a-b$ (so that $a \equiv b$), we get

$$0=\sum_{i=1}^{2n-1}(-1)^i a^{\mu_i} b^{N-\mu_i} \equiv \sum_{i=1}^{2n-1}(-1)^i b^{\mu_i} b^{N-\mu_i}=-b^N \pmod{a-b}.$$

Since $\gcd(a-b, b)=1$, this is possible only if $a-b=1$.

Reducing (2) modulo $a+b$ (so that $a \equiv -b$), we get

$$0=\sum_{i=1}^{2n-1}(-1)^i a^{\mu_i} b^{N-\mu_i} \equiv \sum_{i=1}^{2n-1}(-1)^i(-1)^{\mu_i} b^{\mu_i} b^{N-\mu_i}=S b^N \pmod{a+b}$$

for some odd (thus nonzero) S with $|S| \leqslant 2n-1$. Since $\gcd(a+b, b)=1$, this is possible only if $a+b \mid S$. So $a+b \leqslant 2n-1$, and hence $b=a-1 \leqslant n-1=1010$.

Thus we have shown that any sought r has the form indicated in the answer. It remains to show that for any $b=1,2, \ldots, 1010$ and $a=b+1$, Alice can reach the goal. For this purpose, in (1) we put $n=a$, $\beta_1=\beta_2=\cdots=\beta_a=0$, and $\gamma_1=\gamma_2=\cdots=\gamma_b=1$.

Comment 1. Instead of reducing modulo $a+b$, one can reduce modulo a and modulo b. The first reduction shows that the number of terms in (2) with $\mu_i=0$ is divisible by a, while the second shows that the number of terms with $\mu_i=N$ is divisible by b.

Notice that, in fact, $N>0$, as otherwise (2) contains an alternating sum of an odd number of equal terms, which is nonzero. Therefore, all terms listed above have different indices, and there are at least $a+b$ of them.

Comment 2. Another way to investigate the solutions of equation (1) is to consider the Laurent polynomial

$$L(x)=\sum_{i=1}^{n} x^{\beta_i}-\sum_{i=1}^{n-1} x^{\gamma_i}.$$

We can pick a sufficiently large integer d so that $P(x)=x^d L(x)$ is a polynomial in $\mathbb{Z}[x]$. Then

$$P(1)=1, \tag{3}$$

and

$$1 \leqslant|P(-1)| \leqslant 2021. \tag{4}$$

If $r=p/q$ with integers $p>q \geqslant 1$ is a rational number with the properties listed in the problem statement, then $P(p/q)=L(p/q)=0$. As $P(x)$ has integer coefficients,

$$(p-q x) \mid P(x). \tag{5}$$

Plugging $x=1$ into (5) gives $(p-q) \mid P(1)=1$, which implies $p=q+1$. Moreover, plugging $x=-1$ into (5) gives $(p+q) \mid P(-1)$, which, along with (4), implies $p+q \leqslant 2021$ and $q \leqslant 1010$. Hence $x=(q+1)/q$ for some integer q with $1 \leqslant q \leqslant 1010$.

N5. Prove that there are only finitely many quadruples (a, b, c, n) of positive integers such that

$$n! = a^{n-1} + b^{n-1} + c^{n-1}.$$

Solution. For fixed n there are clearly finitely many solutions; we will show that there is no solution with $n > 100$. So, assume $n > 100$. By the AM–GM inequality,

$$\begin{aligned} n! &= 2n(n-1)(n-2)(n-3)\cdot(3\cdot 4\cdots(n-4)) \\ &\leqslant 2(n-1)^4\left(\frac{3+\cdots+(n-4)}{n-6}\right)^{n-6} = 2(n-1)^4\left(\frac{n-1}{2}\right)^{n-6} < \left(\frac{n-1}{2}\right)^{n-1}, \end{aligned}$$

thus $a, b, c < (n-1)/2$.

For every prime p and integer $m \neq 0$, let $\nu_p(m)$ denote the p-adic valuation of m; that is, the greatest non-negative integer k for which p^k divides m. Legendre's formula states that

$$\nu_p(n!) = \sum_{s=1}^{\infty}\left\lfloor \frac{n}{p^s}\right\rfloor,$$

and a well-know corollary of this formula is that

$$\nu_p(n!) < \sum_{s=1}^{\infty}\frac{n}{p^s} = \frac{n}{p-1}. \qquad (\heartsuit)$$

If n is odd then $a^{n-1}, b^{n-1}, c^{n-1}$ are squares, and by considering them modulo 4 we conclude that a, b and c must be even. Hence, $2^{n-1} \mid n!$ but that is impossible for odd n because $\nu_2(n!) = \nu_2((n-1)!) < n-1$ by $(\heartsuit)$.

From now on we assume that n is even. If all three numbers $a+b, b+c, c+a$ are powers of 2 then a, b, c have the same parity. If they all are odd, then $n! = a^{n-1} + b^{n-1} + c^{n-1}$ is also odd which is absurd. If all a, b, c are divisible by 4, this contradicts $\nu_2(n!) \leqslant n-1$. If, say, a is not divisible by 4, then $2a = (a+b)+(a+c)-(b+c)$ is not divisible by 8, and since all $a+b$, $b+c$, $c+a$ are powers of 2, we get that one of these sums equals 4, so two of the numbers of a, b, c are equal to 2. Say, $a = b = 2$, then $c = 2^r - 2$ and, since $c \mid n!$, we must have $c \mid a^{n-1} + b^{n-1} = 2^n$ implying $r = 2$, and so $c = 2$, which is impossible because $n! \equiv 0 \not\equiv 3 \cdot 2^{n-1} \pmod 5$.

So now we assume that the sum of two numbers among a, b, c, say $a+b$, is not a power of 2, so it is divisible by some odd prime p. Then $p \leqslant a+b < n$ and so $c^{n-1} = n! - (a^{n-1} + b^{n-1})$ is divisible by p. If p divides a and b, we get $p^{n-1} \mid n!$, contradicting $(\heartsuit)$. Next, using $(\heartsuit)$ and the Lifting the Exponent Lemma we get

$$\nu_p(1)+\nu_p(2)+\cdots+\nu_p(n) = \nu_p(n!) = \nu_p(n!-c^{n-1}) = \nu_p\left(a^{n-1}+b^{n-1}\right) = \nu_p(a+b)+\nu_p(n-1). \quad (\diamondsuit)$$

In view of $(\diamondsuit)$, no number of $1, 2, \ldots, n$ can be divisible by p, except $a+b$ and $n-1 > a+b$. On the other hand, $p|c$ implies that $p < n/2$ and so there must be at least two such numbers. Hence, there are two multiples of p among $1, 2, \ldots, n$, namely $a+b = p$ and $n-1 = 2p$. But this is another contradiction because $n-1$ is odd. This final contradiction shows that there is no solution of the equation for $n > 100$.

Comment 1. The original version of the problem asked to find all solutions to the equation. The solution to that version is not much different but is more technical.

Comment 2. To find all solutions we can replace the bound $a, b, c < (n-1)/2$ for all n with a weaker bound $a, b, c \leqslant n/2$ only for even n, which is a trivial application of AM–GM to the tuple $(2, 3, \ldots, n)$. Then we may use the same argument for odd n (it works for $n \geqslant 5$ and does not require any bound on a, b, c), and for even n the same solution works for $n \geqslant 6$ unless we have $a + b = n - 1$ and $2\nu_p(n-1) = \nu_p(n!)$. This is only possible for $p = 3$ and $n = 10$ in which case we can consider the original equation modulo 7 to deduce that $7 \mid abc$ which contradicts the fact that $7^9 > 10!$. Looking at $n \leqslant 4$ we find four solutions, namely,

$$(a, b, c, n) = (1, 1, 2, 3), (1, 2, 1, 3), (2, 1, 1, 3), (2, 2, 2, 4).$$

Comment 3. For sufficiently large n, the inequality $a, b, c < (n-1)/2$ also follows from Stirling's formula.

N6. Determine all integers $n \geqslant 2$ with the following property: every n pairwise distinct integers whose sum is not divisible by n can be arranged in some order $a_1, a_2, \ldots, a_n$ so that n divides $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$.

Answer: All odd integers and all powers of 2.

Solution. If $n = 2^k a$, where $a \geqslant 3$ is odd and k is a positive integer, we can consider a set containing the number $2^k + 1$ and $n - 1$ numbers congruent to 1 modulo n. The sum of these numbers is congruent to 2^k modulo n and therefore is not divisible by n; for any permutation $(a_1, a_2, \ldots, a_n)$ of these numbers

$$1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n \equiv 1 + \cdots + n \equiv 2^{k-1}a(2^k a + 1) \not\equiv 0 \pmod{2^k}$$

and *a fortiori* $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$ is not divisible by n.

From now on, we suppose that n is either odd or a power of 2. Let S be the given set of integers, and s be the sum of elements of S.

Lemma 1. If there is a permutation (a_i) of S such that (n, s) divides $\sum_{i=1}^{n} i a_i$, then there is a permutation (b_i) of S such that n divides $\sum_{i=1}^{n} i b_i$.

Proof. Let $r = \sum_{i=1}^{n} i a_i$. Consider the permutation (b_i) defined by $b_i = a_{i+x}$, where $a_{j+n} = a_j$. For this permutation, we have

$$\sum_{i=1}^{n} i b_i = \sum_{i=1}^{n} i a_{i+x} \equiv \sum_{i=1}^{n} (i - x) a_i \equiv r - sx \pmod{n}.$$

Since (n, s) divides r, the congruence $r - sx \equiv 0 \pmod{n}$ admits a solution.

Lemma 2. Every set T of km integers, $m > 1$, can be partitioned into m sets of k integers so that in every set either the sum of elements is not divisible by k or all the elements leave the same remainder upon division by k.

Proof. The base case, $m = 2$. If T contains k elements leaving the same remainder upon division by k, we form one subset A of these elements; the remaining elements form a subset B. If k does not divide the sum of all elements of B, we are done. Otherwise it is enough to exchange any element of A with any element of B not congruent to it modulo k, thus making sums of both A and B not divisible by k. This cannot be done only when all the elements of T are congruent modulo k; in this case any partition will do.

If no k elements of T have the same residue modulo k, there are three elements $a, b, c \in T$ leaving pairwise distinct remainders upon division by k. Let t be the sum of elements of T. It suffices to find $A \subset T$ such that $|A| = k$ and $\sum_{x \in A} x \not\equiv 0, t \pmod{k}$: then neither the sum of elements of A nor the sum of elements of $B = T \setminus A$ is divisible by k. Consider $U' \subset T \setminus \{a, b, c\}$ with $|U'| = k - 1$. The sums of elements of three sets $U' \cup \{a\}$, $U' \cup \{b\}$, $U' \cup \{c\}$ leave three different remainders upon division by k, and at least one of them is not congruent either to 0 or to t.

Now let $m > 2$. If T contains k elements leaving the same remainder upon division by k, we form one subset A of these elements and apply the inductive hypothesis to the remaining $k(m-1)$ elements. Otherwise, we choose any $U \subset T$, $|U| = k - 1$. Since all the remaining elements cannot be congruent modulo k, there is $a \in T \setminus U$ such that $a \not\equiv -\sum_{x \in U} x \pmod{k}$. Now we can take $A = U \cup \{a\}$ and apply the inductive hypothesis to $T \setminus A$.

Now we are ready to prove the statement of the problem for all odd n and $n = 2^k$. The proof is by induction.

If n is prime, the statement follows immediately from Lemma 1, since in this case $(n, s) = 1$. Turning to the general case, we can find prime p and an integer t such that $p^t \mid n$ and $p^t \nmid s$. By Lemma 2, we can partition S into p sets of $\frac{n}{p} = k$ elements so that in every set either the sum of numbers is not divisible by k or all numbers have the same residue modulo k.

For sets in the first category, by the inductive hypothesis there is a permutation (a_i) such that $k \mid \sum_{i=1}^{k} ia_i$.

If n (and therefore k) is odd, then for each permutation (b_i) of a set in the second category we have

$$\sum_{i=1}^{k} ib_i \equiv b_1 \frac{k(k+1)}{2} \equiv 0 \pmod{k}.$$

By combining such permutation for all sets of the partition, we get a permutation (c_i) of S such that $k \mid \sum_{i=1}^{n} ic_i$. Since this sum is divisible by k, and k is divisible by (n, s), we are done by Lemma 1.

If $n = 2^s$, we have $p = 2$ and $k = 2^{s-1}$. Then for each of the subsets there is a permutation $(a_1, \dots, a_k)$ such that $\sum_{i=1}^{k} ia_i$ is divisible by $2^{s-2} = \frac{k}{2}$: if the subset belongs to the first category, the expression is divisible even by k, and if it belongs to the second one,

$$\sum_{i=1}^{k} ia_i \equiv a_1 \frac{k(k+1)}{2} \equiv 0 \left(\bmod\ \frac{k}{2}\right).$$

Now the numbers of each permutation should be multiplied by all the odd or all the even numbers not exceeding n in increasing order so that the resulting sums are divisible by k:

$$\sum_{i=1}^{k}(2i-1)a_i \equiv \sum_{i=1}^{k} 2ia_i \equiv 2\sum_{i=1}^{k} ia_i \equiv 0 \pmod{k}.$$

Combining these two sums, we again get a permutation (c_i) of S such that $k \mid \sum_{i=1}^{n} ic_i$, and finish the case by applying Lemma 1.

Comment. We cannot dispense with the condition that n does not divide the sum of all elements. Indeed, for each $n > 1$ and the set consisting of 1, -1, and $n-2$ elements divisible by n the required permutation does not exist.

N7. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers such that a_{n+2m} divides $a_n + a_{n+m}$ for all positive integers n and m. Prove that this sequence is eventually periodic, i.e. there exist positive integers N and d such that $a_n = a_{n+d}$ for all $n > N$.

Solution. We will make repeated use of the following simple observation:

Lemma 1. If a positive integer d divides a_n and a_{n-m} for some m and $n > 2m$, it also divides a_{n-2m}. If d divides a_n and a_{n-2m}, it also divides a_{n-m}.

Proof. Both parts are obvious since a_n divides $a_{n-2m} + a_{n-m}$. $\square$

Claim. The sequence (a_n) is bounded.

Proof. Suppose the contrary. Then there exist infinitely many indices n such that a_n is greater than each of the previous terms $a_1, a_2, \ldots, a_{n-1}$. Let $a_n = k$ be such a term, $n > 10$. For each $s < \frac{n}{2}$ the number $a_n = k$ divides $a_{n-s} + a_{n-2s} < 2k$, therefore

$$a_{n-s} + a_{n-2s} = k.$$

In particular,

$$a_n = a_{n-1} + a_{n-2} = a_{n-2} + a_{n-4} = a_{n-4} + a_{n-8},$$

that is, $a_{n-1} = a_{n-4}$ and $a_{n-2} = a_{n-8}$. It follows from Lemma 1 that a_{n-1} divides a_{n-1-3s} for $3s < n-1$ and a_{n-2} divides a_{n-2-6s} for $6s < n-2$. Since at least one of the numbers a_{n-1} and a_{n-2} is at least $a_n/2$, so is some a_i with $i \leqslant 6$. However, a_n can be arbitrarily large, a contradiction. $\square$

Since (a_n) is bounded, there exist only finitely many i for which a_i appears in the sequence finitely many times. In other words, there exists N such that if $a_i = t$ and $i > N$, then $a_j = t$ for infinitely many j.

Clearly the sequence $(a_{n+N})_{n>0}$ satisfies the divisibility condition, and it is enough to prove that this sequence is eventually periodic. Thus truncating the sequence if necessary, we can assume that each number appears infinitely many times in the sequence. Let k be the maximum number appearing in the sequence.

Lemma 2. If a positive integer d divides a_n for some n, then the numbers i such that d divides a_i form an arithmetical progression with an odd difference.

Proof. Let $i_1 < i_2 < i_3 < \ldots$ be all the indices i such that d divides a_i. If $i_s + i_{s+1}$ is even, it follows from Lemma 1 that d also divides $a_{\frac{i_s+i_{s+1}}{2}}$, impossible since $i_s < \frac{i_s+i_{s+1}}{2} < i_{s+1}$. Thus i_s and i_{s+1} are always of different parity, and therefore $i_s + i_{s+2}$ is even. Applying Lemma 1 again, we see that d divides $a_{\frac{i_s+i_{s+2}}{2}}$, hence $\frac{i_s+i_{s+2}}{2} = i_{s+1}$, $\square$

We are ready now to solve the problem.

The number of positive divisors of all terms of the progression is finite. Let d_s be the difference of the progression corresponding to s, that is, s divides a_n if and only if it divides a_{n+td_s} for any positive integer t. Let D be the product of all d_s. Then each s dividing a term of the progression divides a_n if and only if it divides a_{n+D}. This means that the sets of divisors of a_n and a_{n+D} coincide, and $a_{n+D} = a_n$. Thus D is a period of the sequence.

Comment. In the above solution we did not try to find the exact structure of the periodic part of (a_n). A little addition to the argument above shows that the period of the sequence has one of the following three forms:

(i) t (in this case the sequence is eventually constant);

(ii) t, $2t$, $3t$ or $2t$, t, $3t$ (so the period is 3);

(iii) t, t, $\ldots$, $2t$ (the period can be any odd number).

In these three cases t can be any positive integer. It is easy to see that all three cases satisfy the original condition.

We again denote by k be the maximum number appearing in the sequence. All the indices i such that $a_i = k$ form an arithmetical progression. If the difference of this progression is 1, the sequence (a_n) is constant, and we get the case (i). Assume that the difference T is at least 3.

Take an index n such that $a_n = k$ and let $a = a_{n-2}$, $b = a_{n-1}$. We have $a, b < k$ and therefore $k = a_n = a_{n-1} + a_{n-2} = a + b$. If $a = b = \frac{k}{2}$, then all the terms $a_1, a_2, \ldots, a_n$ are divisible by $k/2$, that is, are equal to k or $k/2$. Since the indices i such that $a_i = k$ form an arithmetical progression with odd diference, we get the case (iii).

Suppose now that $a \neq b$.

Claim. For $\frac{n}{2} < m < n$ we have $a_m = a$ if $m \equiv n-2 \pmod 3$ and $a_m = b$ if $m \equiv n-1 \pmod 3$.

Proof. The number $k = a_n$ divides $a_{n-2} + a_{n-1} = a + b$ and $a_{n-4} + a_{n-2} = a_{n-4} + a$ and is therefore equal to these sums (since $a, b < k$ and $a_i \leqslant k$ for all i). Therefore $a_{n-1} = a_{n-4} = b$, that is, $a_{n-4} < k$, $a_{n-4} + a_{n-8} = k$ and $a_{n-8} = a_{n-2} = a$. One of the numbers a and b is greater than $k/2$.

If $b = a_{n-1} = a_{n-4} > \frac{k}{2}$, it follows from Lemma 1 that a_{n-1} divides a_{n-1-3s} when $3s < n-1$, and therefore $a_{n-1-3s} = b$ when $3s < n-1$. When $6s < n-4$, k also divides $a_{n-4-6s} + a_{n-2-3s} = b + a_{n-2-3s}$, thus, $a_{n-2-3s} = k - b = a$.

If $a = a_{n-2} = a_{n-8} > \frac{k}{2}$, all the terms a_{n-2-6s} with $6s < n-2$ are divisible by a, that is, the indices i for which a divides a_i form a progression with difference dividing 6. Since this difference is odd and greater than 1, it must be 3, that is, $a_{n-2-3s} = a$ when $3s < n-2$. Similarly to the previous case, we have $a_{n-1-3s} = a_n - a_{n-2-6s} = k - a = b$ when $6s < n-2$. □

Let a_n and a_{n+T} be two consecutive terms of the sequence equal to k. If n is large enough, $\frac{n+T}{2} < n-2$, and applying the claim to $n+T$ instead of n we see that the three consecutive terms $a_{n-2} = a$, $a_{n-1} = b$, $a_n = k$ must be equal to a_{n+T-2}, a_{n+T-1} and a_{n+T} respectively. Thus, for some i we have $a_{i+3s} = a$ and $a_{i+1+3s} = b$ for all s. Truncating the sequence again if necessary, we may assume that $a_{3s+1} = a$ and $a_{3s+2} = b$ for all s. We know also that $a_n = k$ if and only if n is divisible by T (incidentally, this proves that T is divisible by 3).

If $a_{3s} = c$ for some integer s, each of the numbers a, b, c divides the sum of the other two. It is easy to see that these numbers are proportional to one of the triplets (1, 1, 1), (1, 1, 2) and (1, 2, 3) in some order. It follows that the greater of the two numbers a and b is the smaller multiplied by 2, 3 or 3/2. The last two cases are impossible because then c cannot be the maximum element in the triplet (a, b, c), while $c = k = a + b$ for infinitely many s. Thus the only possible case is 2, the numbers a and b are $k/3$ and $2k/3$ in some order, and the only possible values of c are k and $k/3$. Suppose that $a_{3s} = k/3$ for some $s > 1$. We can choose s so that $a_{3s+3} = k$. Therefore T, which we already know to be odd and divisible by 3, is greater than 3, that is, at least 9. Then $a_{3s-3} \neq k$, and the only other possibility is $a_{3s-3} = k/3$. However, $a_{3s+3} = k$ must divide $a_{3s} + a_{3s-3} = 2k/3$, which is impossible. We have proved then that $a_{3s} = k$ for all $s > 1$, which is the case (ii).

N8. For a polynomial $P(x)$ with integer coefficients let $P^1(x) = P(x)$ and $P^{k+1}(x) = P(P^k(x))$ for $k \geqslant 1$. Find all positive integers n for which there exists a polynomial $P(x)$ with integer coefficients such that for every integer $m \geqslant 1$, the numbers $P^m(1), \ldots, P^m(n)$ leave exactly $\lceil n/2^m \rceil$ distinct remainders when divided by n.

Answer: All powers of 2 and all primes.

Solution. Denote the set of residues modulo ℓ by $\mathbb{Z}_\ell$. Observe that P can be regarded as a function $\mathbb{Z}_\ell \to \mathbb{Z}_\ell$ for any positive integer ℓ. Denote the cardinality of the set $P^m(\mathbb{Z}_\ell)$ by $f_{m,\ell}$. Note that $f_{m,n} = \lceil n/2^m \rceil$ for all $m \geqslant 1$ if and only if $f_{m+1,n} = \lceil f_{m,n}/2 \rceil$ for all $m \geqslant 0$.

Part 1. The required polynomial exists when n is a power of 2 or a prime.

If n is a power of 2, set $P(x) = 2x$.

If $n = p$ is an odd prime, every function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ coincides with some polynomial with integer coefficients. So we can pick the function that sends $x \in \{0, 1, \ldots, p-1\}$ to $\lfloor x/2 \rfloor$.

Part 2. The required polynomial does not exist when n is not a prime power.

Let $n = ab$ where $a, b > 1$ and $\gcd(a,b) = 1$. Note that, since $\gcd(a,b) = 1$,

$$f_{m,ab} = f_{m,a} f_{m,b}$$

by the Chinese remainder theorem. Also, note that, if $f_{m,\ell} = f_{m+1,\ell}$, then P permutes the image of P^m on $\mathbb{Z}_\ell$, and therefore $f_{s,\ell} = f_{m,\ell}$ for all $s > m$. So, as $f_{m,ab} = 1$ for sufficiently large m, we have for each m

$$f_{m,a} > f_{m+1,a} \quad \text{or} \quad f_{m,a} = 1, \qquad f_{m,b} > f_{m+1,b} \quad \text{or} \quad f_{m,b} = 1.$$

Choose the smallest m such that $f_{m+1,a} = 1$ or $f_{m+1,b} = 1$. Without loss of generality assume that $f_{m+1,a} = 1$. Then $f_{m+1,ab} = f_{m+1,b} < f_{m,b} \leqslant f_{m,ab}/2 \leqslant f_{m+1,ab}$, a contradiction.

Part 3. The required polynomial does not exist when n is an odd prime power that is not a prime.

Let $n = p^k$, where $p \geqslant 3$ is prime and $k \geqslant 2$. For $r \in \mathbb{Z}_p$ let S_r denote the subset of $\mathbb{Z}_{p^k}$ consisting of numbers congruent to r modulo p. We denote the cardinality of a set S by $|S|$.

Claim. For any residue r modulo p, either $|P(S_r)| = p^{k-1}$ or $|P(S_r)| \leqslant p^{k-2}$.

Proof. Recall that $P(r+h) = P(r) + hP'(r) + h^2 Q(r,h)$, where Q is an integer polynomial.

If $p \mid P'(r)$, then $P(r+ps) \equiv P(r) \pmod{p^2}$, hence all elements of $P(S_r)$ are congruent modulo p^2. So in this case $|P(S_r)| \leqslant p^{k-2}$.

Now we show that $p \nmid P'(r)$ implies $|P(S_r)| = p^{k-1}$ for all k.

Suppose the contrary: $|P(S_r)| < p^{k-1}$ for some $k > 1$. Let us choose the smallest k for which this is so. To each residue in $P(S_r)$ we assign its residue modulo p^{k-1}; denote the resulting set by $\overline{P}(S,r)$. We have $|\overline{P}(S,r)| = p^{k-2}$ by virtue of minimality of k. Then $|P(S_r)| < p^{k-1} = p \cdot |\overline{P}(S,r)|$, that is, there is $u = P(x) \in P(S_r)$ ($x \equiv r \pmod{p}$) and $t \not\equiv 0 \pmod{p}$ such that $u + p^{k-1}t \notin P(S_r)$.

Note that $P(x + p^{k-1}s) \equiv u + p^{k-1}sP'(x) \pmod{p^k}$. Since $P(x + p^{k-1}s) \not\equiv u + p^{k-1}t \pmod{p^k}$, the congruence $p^{k-1}sP'(x) \equiv p^{k-1}t \pmod{p^k}$ has no solutions. So the congruence $sP'(x) \equiv t \pmod{p}$ has no solutions, which contradicts $p \nmid P'(r)$. □

Since the image of P^m consists of one element for sufficiently large m, we can take the smallest m such that $|P^{m-1}(S_r)| = p^{k-1}$ for some $r \in \mathbb{Z}_p$, but $|P^m(S_q)| \leqslant p^{k-2}$ for all $q \in \mathbb{Z}_p$.

From now on, we fix m and r.

Since the image of $P^{m-1}(\mathbb{Z}_{p^k})\backslash P^{m-1}(S_r)$ under P contains $P^m(\mathbb{Z}_{p^k})\backslash P^m(S_r)$, we have

$$a := |P^m(\mathbb{Z}_{p^k})\backslash P^m(S_r)| \leqslant |P^{m-1}(\mathbb{Z}_{p^k})\backslash P^{m-1}(S_r)|,$$

thus

$$a + p^{k-1} \leqslant f_{m-1,p^k} \leqslant 2f_{m,p^k} \leqslant 2p^{k-2} + 2a,$$

so

$$(p-2)p^{k-2} \leqslant a.$$

Since $f_{i,p} = 1$ for sufficiently large i, there is exactly one $t \in \mathbb{Z}_p$ such that $P(t) \equiv t \pmod p$. Moreover, as i increases, the cardinality of the set $\{s \in \mathbb{Z}_p \mid P^i(s) \equiv t \pmod p\}$ increases (strictly), until it reaches the value p. So either

$$|\{s \in \mathbb{Z}_p \mid P^{m-1}(s) \equiv t \pmod p\}| = p \quad \text{or} \quad |\{s \in \mathbb{Z}_p \mid P^{m-1}(s) \equiv t \pmod p\}| \geqslant m.$$

Therefore, either $f_{m-1,p} = 1$ or there exists a subset $X \subset \mathbb{Z}_p$ of cardinality at least m such that $P^{m-1}(x) \equiv t \pmod p$ for all $x \in X$.

In the first case $|P^{m-1}(\mathbb{Z}_{p^k})| \leqslant p^{k-1} = |P^{m-1}(S_r)|$, so $a = 0$, a contradiction.

In the second case let Y be the set of all elements of $\mathbb{Z}_{p^k}$ congruent to some element of X modulo p. Let $Z = \mathbb{Z}_{p^k}\backslash Y$. Then $P^{m-1}(Y) \subset S_t$, $P(S_t) \subsetneq S_t$, and $Z = \bigcup_{i \in \mathbb{Z}_p \backslash X} S_i$, so

$$|P^m(Y)| \leqslant |P(S_t)| \leqslant p^{k-2} \quad \text{and} \quad |P^m(Z)| \leqslant |\mathbb{Z}_p\backslash X| \cdot p^{k-2} \leqslant (p-m)p^{k-2}.$$

Hence,

$$(p-2)p^{k-2} \leqslant a < |P^m(\mathbb{Z}_{p^k})| \leqslant |P^m(Y)| + |P^m(Z)| \leqslant (p-m+1)p^{k-2}$$

and $m < 3$. Then $|P^2(S_q)| \leqslant p^{k-2}$ for all $q \in \mathbb{Z}_p$, so

$$p^k/4 \leqslant |P^2(\mathbb{Z}_{p^k})| \leqslant p^{k-1},$$

which is impossible for $p \geqslant 5$. It remains to consider the case $p = 3$.

As before, let t be the only residue modulo 3 such that $P(t) \equiv t \pmod 3$.

If $3 \nmid P'(t)$, then $P(S_t) = S_t$ by the proof of the Claim above, which is impossible.

So $3 \mid P'(t)$. By substituting $h = 3^i s$ into the formula $P(t+h) = P(t) + hP'(t) + h^2Q(t,h)$, we obtain $P(t+3^i s) \equiv P(t) \pmod{3^{i+1}}$. Using induction on i we see that all elements of $P^i(S_t)$ are congruent modulo 3^{i+1}. Thus, $|P^{k-1}(S_t)| = 1$.

Note that $f_{1,3} \leqslant 2$ and $f_{2,3} \leqslant 1$, so $P^2(\mathbb{Z}_{3^k}) \subset S_t$. Therefore, $|P^{k+1}(\mathbb{Z}_{3^k})| \leqslant |P^{k-1}(S_t)| = 1$. It follows that $3^k \leqslant 2^{k+1}$, which is impossible for $k \geqslant 2$.

Comment. Here is an alternative version of the problem.

A function $f\colon \mathbb{Z} \to \mathbb{Z}$ is chosen so that $a - b \mid f(a) - f(b)$ for all $a, b \in \mathbb{Z}$ with $a \neq b$. Let $S_0 = \mathbb{Z}$, and for each positive integer m, let S_m denote the image of f on the set S_{m-1}. It is given that, for each nonnegative integer m, there are exactly $\lceil n/2^m \rceil$ distinct residues modulo n in the set S_m. Find all possible values of n.

Answer: All powers of primes.

Solution. Observe that f can be regarded as a function $\mathbb{Z}_\ell \to \mathbb{Z}_\ell$ for any positive integer ℓ. We use notations f^m and $f_{m,\ell}$ as in the above solution.

Part 1. There exists a function $f\colon \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^k}$ satisfying the desired properties.

For $x \in \mathbb{Z}_{p^k}$, let $\operatorname{rev}(x)$ denote the reversal of the base-p digits of x (we write every $x \in \mathbb{Z}_{p^k}$ with exactly k digits, adding zeroes at the beginning if necessary). Choose

$$f(x) = \operatorname{rev}\left(\left\lfloor \frac{\operatorname{rev}(x)}{2} \right\rfloor\right)$$

where, for dividing by 2, $\operatorname{rev}(x)$ is interpreted as an integer in the range $[0, p^k)$. It is easy to see that $f_{m+1,k} = \lceil f_{m,k}/2 \rceil$.

We claim that if $a, b \in \mathbb{Z}_{p^k}$ so that $p^m | a - b$, then $p^m | f(a) - f(b)$. Let $x = \operatorname{rev}(a)$, $y = \operatorname{rev}(b)$. The first m digits of x and y are the same, i.e $\lfloor x/p^{m-k} \rfloor = \lfloor y/p^{m-k} \rfloor$. For every positive integers c, d and z we have $\lfloor \lfloor z/c \rfloor /d \rfloor = \lfloor z/(cd) \rfloor = \lfloor \lfloor z/d \rfloor /c \rfloor$, so

$$\left\lfloor \lfloor x/2 \rfloor / p^{m-k} \right\rfloor = \left\lfloor \lfloor x/p^{m-k} \rfloor /2 \right\rfloor = \left\lfloor \lfloor y/p^{m-k} \rfloor /2 \right\rfloor = \left\lfloor \lfloor y/2 \rfloor / p^{m-k} \right\rfloor.$$

Thus, the first m digits of $\lfloor x/2 \rfloor$ and $\lfloor y/2 \rfloor$ are the same. So the last m digits of $f(a)$ and $f(b)$ are the same, i.e. $p^m | f(a) - f(b)$.

Part 2. Lifting the function $f\colon \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^k}$ to a function on all of $\mathbb{Z}$.

We show that, for any function $f\colon \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^k}$ for which $\gcd(p^k, a - b) \mid f(a) - f(b)$, there is a corresponding function $g\colon \mathbb{Z} \to \mathbb{Z}$ for which $a - b \mid g(a) - g(b)$ for all distinct integers a, b and $g(x) \equiv f(x) \pmod{p^k}$ for all $x \in \mathbb{Z}$, whence the proof will be completed. We will construct the values of such a function inductively; assume that we have constructed it for some interval $[a, b)$ and wish to define $g(b)$. (We will define $g(a-1)$ similarly.)

For every prime $q \leqslant |a - b|$, we choose the maximal α_q for which there exists $c_q \in [a, b)$, such that $b - c_q \,\vdots\, q^{\alpha_q}$, and choose one such c_q.

We apply Chinese remainder theorem to find $g(b)$ satisfying the following conditions:

$$g(b) \equiv g(c_q) \pmod{q^{\alpha_q}} \quad \text{for} \quad q \neq p, \quad \text{and}$$
$$g(b) \equiv g(c_p) \pmod{q^{\alpha_p}} \quad \text{if} \quad \alpha_p \geqslant k, \qquad g(b) \equiv f(b) \pmod{p^k} \quad \text{if} \quad \alpha_p < k.$$

It is not hard to verify that $b - c \mid g(b) - g(c)$ for every $c \in [a, b)$ and $g(b) \equiv f(b) \pmod{p^k}$.

Part 3. The required function does not exist if n has at least two different prime divisors.

The proof is identical to the polynomial version.

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