COMPENDIUM ELMO

Gerard Romo Garrido

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Toomates Coolección

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Versión de este documento: **13/12/2023**

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The name of the ELMO changes each year. Here are some example names:

Ego Loss May Occur
Everybody Lives at Most Once
Every Little Mistake => 0
English Language Master's Open
Exceedingly Luck-Based Math Olympiad
Entirely Legitimate (Junior) Math Olympiad
Eric Larsen Math Olympiad
Ex-experimental Math Olympiad
Easy Little Math Olympiad
Extremely Last-Minute Olympiad
e^log Math Olympiad
End Letter Missing
Exceedingly Loquacious Math Olympiad

Fuente:

https://web.evanchen.cc/problems.html

Exceedingly Luck-based Math Olympiad
Day 1

1. Determine all (not necessarily finite) sets S of points in the plane such that given any four distinct points in S, there is a circle passing through all four or a line passing through some three.

2. Let r and s be positive integers. Define $a_0 = 0$, $a_1 = 1$, and $a_n = ra_{n-1} + sa_{n-2}$ for $n \geq 2$. Let $f_n = a_1 a_2 \cdots a_n$. Prove that $\dfrac{f_n}{f_k f_{n-k}}$ is an integer for all integers n and k such that $0 < k < n$.

3. Let $n > 1$ be a positive integer. A 2-dimensional grid, infinite in all directions, is given. Each 1 by 1 square in a given n by n square has a counter on it. A *move* consists of taking n adjacent counters in a row or column and sliding them each by one space along that row or column. A *returning sequence* is a finite sequence of moves such that all counters again fill the original n by n square at the end of the sequence.

   (a) Assume that all counters are distinguishable except two, which are indistinguishable from each other. Prove that any distinguishable arrangement of counters in the n by n square can be reached by a returning sequence.

   (b) Assume all counters are distinguishable. Prove that there is no returning sequence that switches two counters and returns the rest to their original positions.

Exceedingly Luck-based Math Olympiad
Day 2

4. Determine all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $nf(f(n)) = f(n)^2$ for all positive integers n.

5. 2010 MOPpers are assigned numbers 1 through 2010. Each one is given a red slip and a blue slip of paper. Two positive integers, A and B, each less than or equal to 2010 are chosen. On the red slip of paper, each MOPper writes the remainder when the product of A and his or her number is divided by 2011. On the blue slip of paper, he or she writes the remainder when the product of B and his or her number is divided by 2011. The MOPpers may then perform either of the following two operations:

   - Each MOPper gives his or her red slip to the MOPper whose number is written on his or her blue slip.
   - Each MOPper gives his or her blue slip to the MOPper whose number is written on his or her red slip.

   Show that it is always possible to perform some number of these operations such that each MOPper is holding a red slip with his or her number written on it.

6. Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.

Exceedingly Luck-based Math Olympiad Solutions

1. Determine all (not necessarily finite) sets S of points in the plane such that given any four distinct points in S, there is a circle passing through all four or a line passing through some three.

Solution The answer is any subset of a fixed circle, any subset of a fixed line, any subset of a fixed line with one additional point not on the line, or four points on a circle, with a fifth point as the intersection of its diagonals or the intersection of a pair of its sides (outside the circle). It is clear that these sets all satisfy the needed condition.

First, assume that some four points on S lie on a circle, say A, B, C, and D, in that order. We claim that the rest of S lies on the circle, or S consists of exactly one more point, either the intersection of the diagonals of the quadrilateral formed by A, B, C, D, or the intersection of two sides of the quadrilateral outside the circle. Assume there exists a point E in S, not on the circle. Then, E, A, B, C are not concyclic, and A, B, C are not collinear, so E lies on one of segments AB, BC, CA. Without loss of generality, say E lies on AB. Now, consider E, B, C, D; by similar logic to before, E lies on BC, CD, or DB, but since E, A, B are collinear, and A, B, C are not collinear, we need E, B, D to be collinear, that is, $E = AB \cap CD$.

However, note that at most one of these intersection points can be in S, because if not, it is easy to check that we will get a triangle with a point in the interior in S, in which we have four points that cannot satisfy the given condition. Additionally, we can have at most four points on the circle, because if we have five, say A, B, C, D, E, and a sixth point P in S lies off the circle (we know that at most one such point exists, from before), then it must be the intersection of two lines formed by A, B, C, D; without loss of generality, say $P = AB \cap CD$. Also, it must be the intersection of two lines formed by A, B, C, E. But $P \in AB$, so $P \in CE$, which is impossible, since this means C, D, E are collinear.

We are now left with the case when no four points are concyclic, which means that any four points in S have some three collinear. Starting with four points A, B, C, D, some three are collinear, say A, B, C. But for any other point $E \in S$, some three of A, B, C, E are collinear, meaning all four must be collinear. Thus, all or all but one of our points must lie on the same line.

This exhausts all cases, and when there are fewer than four points in S, the statement is vacuously true. It follows that the only possible sets S are those described above.

2. Let r and s be positive integers. Define $a_0 = 0$, $a_1 = 1$, and $a_n = ra_{n-1} + sa_{n-2}$ for $n \geq 2$. Let $f_n = a_1 a_2 \cdots a_n$. Prove that $\dfrac{f_n}{f_k f_{n-k}}$ is an integer for all integers n and k such that $0 < k < n$.

Solution Lemma: For nonnegative integers x, y, $a_{x+y} = a_x a_{y+1} + sa_{x-1}a_y$. We will prove this by induction on y. We have two base cases, $y = 0$ and $y = 1$. When $y = 0$ we simply need to prove that $a_x = a_x$, which is trivial. When $y = 1$, we need to prove that $a_{x+1} = a_x a_1 + sa_{x-1}$. But $a_1 = r$, so this is true directly from the recurrence relation. Now suppose we know that $a_{x+y} = a_x a_{y+1} + sa_{x-1}a_y$ for $y = k$ and $y = k + 1$. Then we have $a_{x+k+2} = ra_{x+k} + sa_{x+k+1} = ra_x a_{k+1} + sa_x a_{k+2} + rsa_{x-1}a_k + s^2 a_{x-1}a_{k+1} = a_x a_{k+3} + a_{x-1}a_{k+2}$, which is exactly what we want to show for $y = k + 2$. This completes our induction.

Now for the main proof, let $f_0 = 1$. Then we will prove the claim by induction on n. The base cases, $n = 0$ or $k = 0$, are trivial. Suppose we know that $\frac{f_n}{f_k f_{n-k}}$ is an integer for all smaller n. Then we have $\frac{f_n}{f_k f_{n-k}} = \frac{f_{n-1}a_{n-k+k}}{f_k f_{n-k}} = \frac{f_{n-1}(a_{n-k}a_{k+1}+sa_{n-k-1}a_k)}{f_k f_{n-k}} = \frac{f_{n-1}a_{n-k}a_{k+1}}{f_k f_{n-k}} + \frac{f_{n-1}sa_{n-k-1}a_k}{f_k f_{n-k}} = \frac{f_{n-1}}{f_k f_{n-k-1}} \cdot a_{k+1} + \frac{f_{n-1}}{f_{k-1} f_{n-k}} \cdot sa_{n-k-1}$, which is clearly an integer by the inductive hypothesis. This completes the induction and the proof.

3. Let $n > 1$ be a positive integer. A 2-dimensional grid, infinite in all directions, is given. Each 1 by 1 square in a given n by n square has a counter on it. A *move* consists of taking n adjacent counters in a row or column and sliding them each by one space along that row or column. A *returning sequence* is a finite sequence of moves such that all counters again fill the original n by n square at the end of the sequence.

   (a) Assume that all counters are distinguishable except two, which are indistinguishable from each other. Prove that any distinguishable arrangement of counters in the n by n square can be reached by a returning sequence.

(b) Assume all counters are distinguishable. Prove that there is no returning sequence that switches two counters and returns the rest to their original positions.

Solution (a) First, we will find a way to 3-cycle some counters, and then use these cycles to construct any board.

Lemma 1. It is possible to cycle any three adjacent counters in an L-formation, while leaving all other counters unchanged.

Proof. Suppose we have counters c_1, c_2, and c_3 in such a formation. Suppose without loss of generality that c_1 is directly above c_2 and that c_3 is directly to the right of c_2. Make the following four moves:

i. Slide the column containing c_1 and c_2 down.
ii. Slide the row now containing c_1 and c_3 right.
iii. Slide the column now containing c_2 and c_3 up.
iv. Slide the row now containing c_1 and c_2 right.

This cycles the three counters. Note that performing this cycle twice is simply cycling the in the other direction.

Now we can use this cycle to get any grid we want. To show this, we think of this as starting from a given grid, from where we aim to get back to the original position. To show that this can be done, we do induction on n.

Base Case. $n = 2$. First, we do a cycle, if necessary, to get the correct counter into the top-left position. Then, we do another cycle, consisiting of the other three squares, to get the correct counter into the top-right position. Then we are done, because the remaining two counters are indistinguishable and thus will be correctly placed.

Inductive Step. Assume that such an algorithm is possible for an $(n-1)\times(n-1)$ board. In our $n\times n$ board, we can use these cycles to get the correct counters into the topmost row, one-by-one. We then finish the remaining positions in the leftmost column. We are now left with an $(n-1)\times(n-1)$ board, so we apply the inductive hypothesis to finish.

(b) First, I claim that any returning sequence must use an even number of moves. To see this, consider all of the positions that contain a counter, and let S be the sum of all the x-coordinates and y-coordinates of these positions. Any move will add either 1 or -1 to n of the x-coordinates or y-coordinates, thus changing S by n. If we look at S mod $2n$, this is equivalent to always adding n to S. In a returning sequence, S must be the same as it was originally, so there must be an even number of moves to make S agree with its original value mod $2n$.

Now, instead of thinking of these counters as being on an infinite grid, we only look at their coordinates mod n. Any valid move will simply cycle the coordinates (either x or y) mod n. Then, at any point, for any position (x, y), there will be exactly one counter that has those coordinates mod n, so each move is simply an n-cycle of these mod n coordinates. Since any returning sequence consists of an even number of moves, the coordinates will ultimately go through an even number of n cycles, and the composition of these cycles will be an even permuation. However, the transposition of any two counters is an odd permuation, so a returning sequence that switches only two counters is impossible.

4. Determine all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $nf(f(n)) = f(n)^2$ for all positive integers n.

Solution The answer is $f(n) = n$ for all $n = 1, 2, \ldots, N$ for some positive integer N, and $f(n) = an$ for fixed positive integer a for $n > N$. It is not difficult to check that all of these f work, since if $n \leq N$, $nf(f(n)) = n^2 = f(n)^2$, and if $n > N$, $nf(f(n)) = a^2n^2 = f(n)^2$.

First, say $f(n) = an$ for some positive integer n, such that $an \in \mathbb{N}$. Then, $nf(f(n)) = nf(an) = f(n)^2 = a^2n^2$, so $f(an) = a(an)$. It follows easily by induction that for all non-negative integers k, $f(a^kn) = a^{k+1}n$. In particular, $an, a^2n, \ldots$ are all integers, which implies that a itself is an integer, since if a prime p divides the denominator of a, when a is raised to a large enough power, the power of p can no longer divide n, making a^kn non-integral for large enough k.

Now, assume that $f(n_1) = an_1$ and $f(n_2) = bn_2$ for some distinct positive integers $a, b > 1$. Without loss of generality, say $a < b$. Choose a positive integer k such that $a^kn_1 > n_2$. Then, we have

$f(a^k n_1) = a^{k+1}n_1$, and $f(n_2) = bn_2$, so that $a^{k+1}n_1 > bn_2$, as f is strictly increasing. Applying f repeatedly to both sides, we find that $a^{k+e}n_1 > b^e n_2$ for all integers $e > 0$, but this is impossible for large enough e, as $b > a$. Thus, we must have $a = b$.

Thus, for some positive integer a, for all n, either $f(n) = n$ or $f(n) = an$. Let n be an integer such that $f(n) = an$, and $a > 1$. Then, assume we have some $m > n$ such that $f(m) = m$. For the unique k such that $a^k n \le m < a^{k+1}n$, note that $f(a^k n) = a^{k+1}n$. But since $m \ge a^k n$, as f is increasing, we need $f(m) = m \ge a^{k+1}n$, a contradiction. It follows that either $f(n) = an$ for all n, or there exists a positive integer N such that $f(n) = n$ for all $n \le N$ and $f(n) = an$ for $n > N$, as claimed.

5. 2010 MOPpers are assigned numbers 1 through 2010. Each one is given a red slip and a blue slip of paper. Two positive integers, A and B, each less than or equal to 2010 are chosen. On the red slip of paper, each MOPper writes the remainder when the product of A and his or her number is divided by 2011. On the blue slip of paper, he or she writes the remainder when the product of B and his or her number is divided by 2011. The MOPpers may then perform either of the following two operations:

   - Each MOPper gives his or her red slip to the MOPper whose number is written on his or her blue slip.
   - Each MOPper gives his or her blue slip to the MOPper whose number is written on his or her red slip.

   Show that it is always possible to perform some number of these operations such that each MOPper is holding a red slip with his or her number written on it.

Solution Note that 2011 is prime, so each slip of paper of a given color has a different number on it. All arithmetic from now on will be done modulo 2011 unless otherwise stated. Now suppose that person i has red slip Ai and blue slip Bi. Then person $B^{-1}i$ has blue slip i, so after performing the first operation, person i will have red slip $AB^{-1}i$ and still have blue slip Bi. Similarly, if the second operation were performed instead, then person i would have red slip Ai and blue slip $A^{-1}Bi$. This holds for every index i, so we can represent the operations simply as $(A, B) \to (AB^{-1}, B)$ and $(A, B) \to (A, A^{-1}B)$.

Now consider a primitive root g modulo 2011 and write $A = g^a$ and $B = g^b$ for some natural numbers a, b. Then, now considering arithmetic in natural numbers, we can write the operations as $(a, b) \to (a - b, b)$ and $(a, b) \to (a, b-a)$. These two operations allow us to apply the Euclidean algorithm to reduce one of these two values to 0. If a becomes 0, every MOPper has his or her red slip, and so we are done. If b becomes 0, then we notice that the second to last pair must have been (a, a), in which case we can simply go to $(0, a)$ instead. However, if we started at $(a, 0)$ then we cannot do this, so we apply the second operation repeatedly. We notice that as the multiples of a are cyclic modulo 2010 and these values are exponents of a primitive root, eventually we will reach a pair equivalent to (a, a), at which point we can perform the first operation to arrive at $(0, a)$, as desired.

6. Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.

Solution Note that the homothety around T taking the small circle to ω. This homothety takes D to M as the tangents are parallel, so T, D, M are collinear. Then note that $\angle MBD = \frac{1}{2} \overset{\frown}{MC} = \frac{1}{2} \overset{\frown}{MB} = \angle MTB$, so $\triangle MBD \sim \triangle MTB$, so $MD \cdot MT = MB^2$. Let ME intersect ω at R. Then it suffices to show that R, S, I_A are collinear. Note that $MB = MI_A = MI = MC$. Additionally, notice that E and R are the reflections across the perpendicular bisector of BC of D and T, respectively. Therefore, $MD = ME$ and $MT = MR$, so $MI_A^2 = ME \cdot MR$, so $\triangle MEI_A \sim \triangle MI_AR$ and so $\angle MI_AE = \angle MRI_A$. Additionally, as $I_AE \perp BC$, we have $I_AE \parallel ID$, so $\angle MI_AE = \angle MID$. Finally, $MI^2 = MD \cdot MT$, so $\angle MID = \angle MTI = \angle MRS$ because $MTRS$ is cyclic. Therefore, $\angle MRI_A = \angle MRS$, so R, S, I_A are collinear as desired.

ELMO Shortlist

A1 (Carl Lian + Brian Hamrick) Determine all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $nf(f(n)) = f(n)^2$ for all positive integers n.

A2 (Calvin Deng) Let a, b, c be positive reals. Prove that

$$\frac{(a-b)(a-c)}{2a^2+(b+c)^2} + \frac{(b-c)(b-a)}{2b^2+(c+a)^2} + \frac{(c-a)(c-b)}{2c^2+(a+b)^2} \geq 0.$$

A3 (George Xing) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = \max(f(x), y) + \min(f(y), x)$.

A4 (Evan O'Dorney) Let $-2 < x_1 < 2$ be a real number and define $x_2, x_3, \ldots$ by $x_{n+1} = x_n^2 - 2$ for $n \geq 1$. Assume that no x_n is 0 and define a number A, $0 \leq A \leq 1$ in the following way: The n^{th} digit after the decimal point in the binary representation of A is a 0 if $x_1 x_2 \cdots x_n$ is positive and 1 otherwise. Prove that $A = \frac{1}{\pi}\cos^{-1}\left(\frac{x_1}{2}\right)$.

A5 (Brian Hamrick) Given a prime p, let $d(a, b)$ be the number of integers c such that $1 \leq c < p$, and the remainders when ac and bc are divided by p are both at most $\frac{p}{3}$. Determine the maximum value of

$$\sqrt{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} d(a,b)(x_a+1)(x_b+1)} - \sqrt{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} d(a,b)x_a x_b}$$

over all $(p-1)$-tuples $(x_1, x_2, \ldots, x_{p-1})$ of real numbers.

A6 (In-Sung Na) For all positive real numbers a, b, c, prove that

$$\sqrt{\frac{a^4+2b^2c^2}{a^2+2bc}} + \sqrt{\frac{b^4+2c^2a^2}{b^2+2ca}} + \sqrt{\frac{c^4+2a^2b^2}{c^2+2ab}} \geq a+b+c.$$

A7 (Evan O'Dorney) Find the smallest real number M with the following property: Given nine nonnegative real numbers with sum 1, it is possible to arrange them in the cells of a 3×3 square so that the product of each row or column is at most M.

C1 (Brian Hamrick) For a permutation π of $\{1, 2, 3, \ldots, n\}$, let $\operatorname{Inv}(\pi)$ be the number of pairs (i, j) with $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$.

(a) Given n, what is $\sum \operatorname{Inv}(\pi)$ where the sum ranges over all permutations π of $\{1, 2, 3, \ldots, n\}$?

(b) Given n, what is $\sum (\operatorname{Inv}(\pi))^2$ where the sum ranges over all permutations π of $\{1, 2, 3, \ldots, n\}$?

C2 (Alex Zhu) For a positive integer n, let $s(n)$ be the number of ways that n can be written as the sum of strictly increasing perfect 2010^{th} powers. For instance, $s(2) = 0$ and $s(1^{2010} + 2^{2010}) = 1$. Show that for every real number x, there exists an integer N such that for all $n > N$,

$$\frac{\max_{1 \le i \le n} s(i)}{n} > x.$$

C3 (Brian Hamrick) 2010 MOPpers are assigned numbers 1 through 2010. Each one is given a red slip and a blue slip of paper. Two positive integers, A and B, each less than or equal to 2010 are chosen. On the red slip of paper, each MOPper writes the remainder when the product of A and his or her number is divided by 2011. On the blue slip of paper, he or she writes the remainder when the product of B and his or her number is divided by 2011. The MOPpers may then perform either of the following two operations:

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Show that it is always possible to perform some number of these operations such that each MOPper is holding a red slip with his or her number written on it.

C4 (Brian Hamrick) The numbers $1, 2, \ldots, n$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers x and y, erases them, and writes the number $2x + 2y$ on the board. This continues until only one number remains. Prove that this number is at least $\frac{4}{9}n^3$.

C5 (Mitchell Lee and Benjamin Gunby) Let $n > 1$ be a positive integer. A 2-dimensional grid, infinite in all directions, is given. Each 1 by 1 square in a given n by n square has a counter on it. A *move* consists of taking n adjacent counters in a row or column and sliding them each by one space along that row or column. A *returning sequence* is a finite sequence of moves such that all counters again fill the original n by n square at the end of the sequence.

(a) Assume that all counters are distinguishable except two, which are indistinguishable from each other. Prove that any distinguishable arrangement of counters in the n by n square can be reached by a returning sequence.

(b) Assume all counters are distinguishable. Prove that there is no returning sequence that switches two counters and returns the rest to their original positions.

C6 (Brian Hamrick) Hamster is playing a game on an $m \times n$ chessboard. He places a rook anywhere on the board and then moves it around with the restriction that every vertical move must be followed by a horizontal move and every horizontal move must be followed by a vertical move. For what values of m, n is it possible for the rook to visit every square of the chessboard exactly once? A square is only considered visited if the rook was initially placed there or if it ended one of its moves on it.

C7 (Brian Hamrick) The game of circulate is played with a deck of kn cards each with a number in $1, 2, \ldots, n$ such that there are k cards with each number. First, n piles numbered $1, 2, \ldots, n$ of k cards each are dealt out face down. The player then flips over a card from pile 1, places that card face up at the bottom of the pile, then next flips over a card from the pile whose number matches the number on the card just flipped. The player repeats this until he reaches a pile in which every card has already been flipped and wins if at that point every card has been flipped. Hamster has grown tired of losing every time, so he decides to cheat. He looks at the piles beforehand and rearranges the k cards in each pile as he pleases. When can Hamster perform this procedure such that he will win the game?

C8 (David Yang) A tree T is given. Starting with the complete graph on n vertices, subgraphs isomorphic to T are erased at random until no such subgraph remains. For what trees does there exist a positive constant c such that the expected number of edges remaining is at least cn^2 for all positive integers n?

G1 (Carl Lian) Let ABC be a triangle. Let A_1, A_2 be points on AB and AC respectively such that $A_1A_2 \parallel BC$ and the circumcircle of $\triangle AA_1A_2$ is tangent to BC at A_3. Define B_3, C_3 similarly. Prove that AA_3, BB_3, and CC_3 are concurrent.

G2 (Brian Hamrick) Given a triangle ABC, a point P is chosen on side BC. Points M and N lie on sides AB and AC, respectively, such that $MP \parallel AC$ and $NP \parallel AB$. Point P is reflected across MN to point Q. Show that triangle QMB is similar to triangle CNQ.

G3 (Evan O'Dorney) A circle ω not passing through any vertex of $\triangle ABC$ intersects each of the segments AB, BC, CA in 2 distinct points. Prove that the incenter of $\triangle ABC$ lies inside ω.

G4 (Amol Aggarwal) Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.

G5 (Carl Lian) Determine all (not necessarily finite) sets S of points in the plane such that given any four distinct points in S, there is a circle passing through all four or a line passing through some three.

G6 (Carl Lian) Let ABC be a triangle with circumcircle Ω. X and Y are points on Ω such that XY meets AB and AC at D and E, respectively. Show that the midpoints of XY, BE, CD, and DE are concyclic.

N1 (Wenyu Cao) For a positive integer n, let $\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not squarefree} \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ primes} \end{cases}$ and $\sigma(n)$ be the sum of the divisors of n. Prove that for all n we have

$$\left| \sum_{d|n} \frac{\mu(d)\sigma(d)}{d} \right| \geq \frac{1}{n}$$

and determine when equality holds.

N2 (Tim Chu) Given a prime p, show that

$$\left(1 + p\sum_{k=1}^{p-1} k^{-1}\right)^2 \equiv 1 + p^2 \sum_{k=1}^{p-1} k^{-2} \pmod{p^4}.$$

N3 (Travis Hance) Prove that there are infinitely many quadruples of integers (a, b, c, d) such that

$$\begin{aligned} a^2 + b^2 + 3 &= 4ab \\ c^2 + d^2 + 3 &= 4cd \\ 4c^3 - 3c &= a \end{aligned}$$

N4 (Evan O'Dorney) Let r and s be positive integers. Define $a_0 = 0$, $a_1 = 1$, and $a_n = ra_{n-1} + sa_{n-2}$ for $n \geq 2$. Let $f_n = a_1 a_2 \cdots a_n$. Prove that $\dfrac{f_n}{f_k f_{n-k}}$ is an integer for all integers n and k such that $0 < k < n$.

N5 (Brian Hamrick) Find the set S of primes such that $p \in S$ if and only if there exists an integer x such that $x^{2010} + x^{2009} + \cdots + 1 \equiv p^{2010} \pmod{p^{2011}}$.

English Language Master's Open

Day I   8:00 AM – 12:30 PM

June 18, 2011

Write your number and team abbreviation, but not your name, on top of all pages turned in.

1. Let $ABCD$ be a convex quadralateral. Let E, F, G, H be points on segments AB, BC, CD, DA, respectively, and let P be intersection of EG and FH. Given that quadrilaterals $HAEP, EBFP, FCGP, GDHP$ all have inscribed circles, prove that $ABCD$ also has an inscribed circle.

2. Wanda the Worm likes to eat Pascal's triangle. One day, she starts at the top of the triangle and eats $\binom{0}{0} = 1$. Each move, she travels to an adjacent positive integer and eats it, but she can never return to a spot that she has previously eaten. If Wanda can never eat numbers a, b, c such that $a + b = c$, proof that it is possible for her to eat 100,000 numbers in the first 2011 rows given that she is not restricted to traveling only in the first 2011 rows.

   (Here, the $n+1^{\text{st}}$ row of Pascal's triangle consists of entries of the form $\binom{n}{k}$ for integers $0 \le k \le n$. Thus, the entry $\binom{n}{k}$ is considered adjacent to the entries $\binom{n-1}{k-1}$, $\binom{n-1}{k}$, $\binom{n}{k-1}$, $\binom{n}{k+1}$, $\binom{n+1}{k}$, $\binom{n+1}{k+1}$.)

3. Determine whether there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers such that the following holds:

   - For all $n \ge 0$, $a_n \ne 0$.
   - There exist real numbers x and y such that $a_{n+2} = xa_{n+1} + ya_n$ for all $n \ge 0$.
   - For all positive real numbers r, there exists positive integers i and j such that $|a_i| < r < |a_j|$.

Note: Our Engrish level beginner. Please excuse us any typos and us help fix mistake.

English Language Master's Open

Dai II 8:00 AM – 12:30 PM

June 19, 2011

Write your number and team abbreviation, but not your name, on top of all pages turned in.

4. Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the positive reals, such that whenever $a > b > c > d > 0$ are reel numbers with $ad = bc$,

$$f(a + d) + f(b - c) = f(a - d) + f(b + c).$$

5. Let $p > 13$ be a prime of the the form $2q+1$, where q is prime. Find the number of ordered pairs of integers (m, n) such that $0 \leq m < n < p - 1$ and

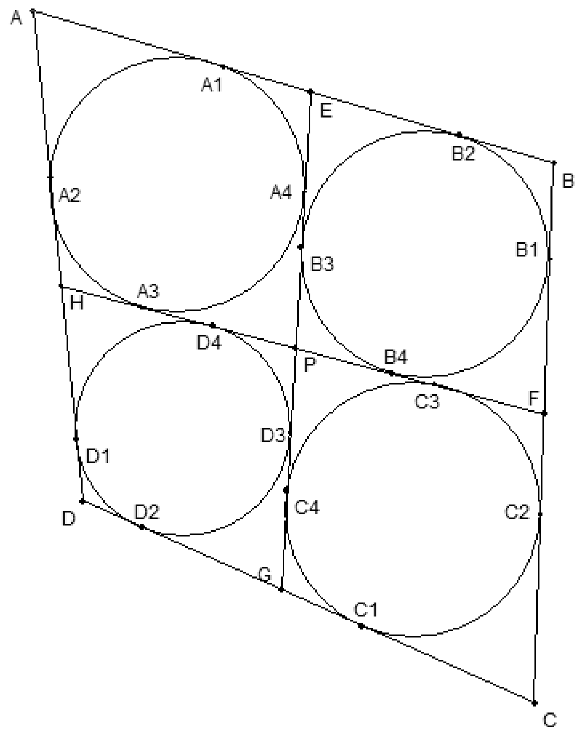
$$3^m + (-12)^m \equiv 3^n + (-12)^n \pmod{p}.$$

6. Consider the infinite grid of lattice points in $\mathbb{Z}^3$. Little D and Big Z play a game, where Little D first loses a shoe on an unmunched point in the grid. Then, Big Z munches a shoe-free plane perpendicular to one of the coordinate axes. They continue to alternature turns in this fashion, with Little D's goal to loose a shoe on each of n consecutive lattice points on a line parrallel to one of the coordinate axes. Determine all n for which Little D can accomplish his goal.

Note: Our Engrish level beginner. Please excuse us any typos and us help fix mistake.

37th English Language Master's Open

1. Let $ABCD$ be a convex quadralateral. Let E, F, G, H be points on segments AB, BC, CD, DA, respectively, and let P be intersection of EG and FH. Given that quadrilaterals $HAEP, EBFP, FCGP, GDHP$ all have inscribed circles, prove that $ABCD$ also has an inscribed circle.



Solution: Let us label the points of tangency of the four given incircles as shown in the diagram.

Then, to prove that $ABCD$ has an inscribed circle, it suffices to show that $AB + CD = AD + BC$. Since common tangents from a point to a circle share the same length, we get

$$AB + CD = AD + BC$$

$$\Leftrightarrow (AA_1+A_1B_2+B_2B)+(CC_1+C_1D_2+D_2D) = (AA_2+A_2D_1+D_1D)+(BB_1+B_1C_2+C_2C)$$

$$\Leftrightarrow \; A_1B_2 + C_1D_2 = A_2D_1 + B_1C_2.$$

We first want to show that $A_2D_1 = A_4D_3$. If $AD||EG$, then this is true because A_2, D_1, D_3, A_4 form the corners of a rectangle. Otherwise, consider the intersection of EG and AD. Note that, of the incircles of $AEPH$ and $HPGD$, one is an incircle and the other an excircle of the triangle with the intersection point as a vertex.

Consequently, A_4 is the reflection of D_3 over the midpoint of HP and we have

$$A_2D_1 = A_2H + HD_1 = HA_4 + HD_3 = PD_3 + A_4P = A_3P + PD_4 = A_3D_4.$$

Similarly, $B_1C_2 = B_3C_4$, $A_1B_2 = A_3B_4$, and $D_2C_1 = D_4C_3$.

Combining, we get

$$\begin{aligned} A_2D_1 + B_1C_2 =& A_4D_3 + B_3C_4 \\ =& PA_4 + PD_3 + PB + 3 + PC_4 \\ =& PA_3 + PD_4 + PB_4 + PC_3 \\ =& A_3B_4 + D_4C_3 \\ =& A_1B_2 + C_1D_2, \end{aligned}$$

so we are done.

This problem was proposed by Evan O'Dorney.

2. Wanda the Worm likes to eat Pascal's triangle. One day, she starts at the top of the triangle and eats $\binom{0}{0} = 1$. Each move, she travels to an adjacent positive integer and eats it, but she can never return to a spot that she has previously eaten. If Wanda can never eat numbers a, b, c such that $a + b = c$, proof that it is possible for her to eat $100,000$ numbers in the first 2011 rows given that she is not restricted to traveling only in the first 2011 rows.

   (Here, the $n+1^{\text{st}}$ row of Pascal's triangle consists of entries of the form $\binom{n}{k}$ for integers $0 \leq k \leq n$. Thus, the entry $\binom{n}{k}$ is considered adjacent to the entries $\binom{n-1}{k-1}$, $\binom{n-1}{k}$, $\binom{n}{k-1}$, $\binom{n}{k+1}$, $\binom{n+1}{k}$, $\binom{n+1}{k+1}$.)

   Solution: We will prove by induction on n that it is possible for Wanda to eat 3^n numbers in the first 2^n rows of Pascal's triangle. Our inductive hypothesis includes the following conditions on the first 2^n rows of Pascal's triangle when all the entries are taken modulo 2:

   - Row 2^n contains only odd numbers.
   - The 2^n rows contain a total of 3^n odd numbers.

- The triangle of rows has 120 degree rotational symmetry.
- There is a path for Wanda to munch that starts at any corner of these rows, contains all the odd numbers, and ends at any other corner.

Our base case is $n = 1$; it is not difficult to check that all of these conditions hold. Wanda's path in these two rows is $\binom{0}{0} \to \binom{0}{1} \to \binom{1}{1}$.

Now, assume that these hold for the first 2^m rows of Pascal's triangle. We will show that they also hold for the first 2^{m+1} rows. Note that a single 1 surrounded by $2^m - 1$ 0's to either side generated the first 2^m rows since each element is equal to the sum of the two numbers directly above it. However, by our inductive hypothesis, all of the entries in the 2^m row were 1's. Hence, the first and last entires of the 2^{m+1} row are also both 1, and the remainder of the entires are 0. Consequently, we note that these 1's and 0's generate two other copies of the first 2^m rows of Pascal's triangle, along with an inverted triangle of all 0's in the middle.

Now it suffices to check that our conditions hold:

- As row 2^{m+1} simply contains two side-by-side copies of the 2^mth row modulo 2, it also consists all of 1's.
- The first 2^{m+1} rows contain three copies of the first 2^m rows along with a triangle of 0's, so they contain $3(3^m) = 3^{m+1}$ odd numbers.
- As each of the three 2^m row triangles had rotational symmetry, so does the larger one.
- By our inductive hypothesis, Wanda can travel from $\binom{0}{0}$ to $\binom{2^m-1}{0}$ and eat all the odd numbers in those rows. She can then travel to $\binom{2^m}{0}$, eat all the numbers in the lower-left triangle and end at $\binom{2^{m+1}-1}{2^m-1}$, travel to $\binom{2^{m+1}-1}{2^m}$, eat all the odd numbers in the lower-right triangle, and finally end at $\binom{2^{m+1}-2}{2^{m+1}-1}$. Due to rotational symmetry, she can also start and end at any corner.

We have now proved our induction.

Note that if Wanda only eats odd numbers, then she will never eat three numbers a, b, c such that $a + b = c$. We have $2^{10} < 2011 < 2048 = 2^{11}$.

It suffices to check that there are sufficient odd numbers in the first 2011 rows. We have showed that there are 3^{11} odd numbers in the first 2048 rows. Also, row n has n elements and thus contains at most n odd numbers. Hence, there are at least

$$3^{11} - 2048 - 2047 - \ldots - 2012 = 3^{11} - \frac{1}{2}(2048 + 2012)(2048 - 2011) > 1000000$$

odd numbers in the first 2011 rows.

This problem was proposed by Linus Hamilton.

3. Determine whether there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers such that the following holds:

- For all $n \geq 0$, $a_n \neq 0$.
- There exist real numbers x and y such that $a_{n+2} = x a_{n+1} + y a_n$ for all $n \geq 0$.
- For all positive real numbers r, there exists positive integers i and j such that $|a_i| < r < |a_j|$.

Solution: The answer is *yes*.

Let $x_n = \underbrace{2^{2^{\cdots^{2}}}}_{2n\ 2\text{'s}}$. Then, let $\theta = \frac{\pi}{2}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots\right)$, and let $r = 2$ and $a_n = r^n \cos(n\theta)$.

We will prove that this sequences satisfies the three given conditions.

First, note that

$$a_{n+2} = 2r\cos(\theta)a_{n+1} - r^2 a_n$$

for all n by the addition formula for cosine, so the recursion condition is satisfied by setting $x = 2r\cos(\theta)$ and $y = -r^2$.

Second, we note that if there exists any integer n such that $a_n = 0$, then we would have $n\theta = \pi(k + \frac{1}{2})$ for some $k \in \{0, 1, 2, \ldots\}$, implying that $\frac{\theta}{\pi}$ is rational. However, we have

$$\frac{\theta}{\pi} = \frac{1}{2}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots\right),$$

which has a non-periodic binary expansion and is therefore irrational. Hence, we know hat the second condition is satisfied.

Third, consider the subsequence

$$\begin{aligned} b_n &= a_{x_n} \\ &= r^{x_n}\cos(x_n\theta) \\ &= r^{x_n} cos\left(\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{x_n}{x_k}\right). \end{aligned}$$

Where $a = \sum_{k=1}^{n} \frac{x_n}{x_k}$ is an odd integer, we note that

$$\begin{aligned}
|b_n| &< |r|^{x_n} \left| \cos\left(\frac{\pi}{2} \sum_{k=1}^{\infty} \frac{x_n}{x_k}\right) \right| \\
&= |r|^{x_n} \left| \cos\left(\frac{\pi}{2} \left(a + \sum_{k=n+1}^{\infty} \frac{x_n}{x_k}\right)\right) \right| \\
&= |r|^{x_n} \left| \sin\left(\frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{x_n}{x_k}\right) \right| \\
&\leq 2^{x_n} \frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{x_n}{x_k} \\
&\leq 2^{x_n} \frac{\pi}{2} \sum_{k=n+!}^{\infty} \frac{x_n}{x_{n+1} \cdot 2^{k-n-1}} \\
&= 2^{x_n} \cdot \pi \cdot \frac{x_n}{x_{n+1}},
\end{aligned}$$

which becomes arbitrarily small as n approaches infinity.

Consequently, $\{a_n\}$ has a subsequence with arbitrarily small magnitude. By Kronecker's Theorem, there is also a sequence $n_1, n_2, \ldots$ with $\{\frac{n_1\theta}{2\pi}\} \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ for $i = 1, 2, \ldots$. Then, the sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ tends to infinity. Thus, $\{a_n\}$ has both a subsequence with magnitude tending to 0 and a subsequence with magnitude tending to infinity, so the third property also holds.

This problem was proposed by Alex Zhu.

4. Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the positive reals, such that whenever $a > b > c > d > 0$ are reel numbers with $ad = bc$,

$$f(a+d) + f(b-c) = f(a-d) + f(b+c).$$

Solution: Since $f(a+d) - f(a-d)$ only depends on ad, we can have a function g mapping positive reals to reals such that whenever $a > d$,

$$g(ad) = f(a+d) - f(a-d).$$

Also,

$$\begin{aligned}
g(kad + k(k+1)d^2) &= g((a+(k+1)d)(kd)) \\
&= f(a+(2k+1)d) - f(a+d) \\
&\text{and} \\
g((k+1)ad + k(k+1)d^2) &= g((a+kd)((k+1)d)) \\
&= f(a+(2k+1)d) - f(a-d) \\
&= g(ad) + g(kad + k(k+1)d^2)
\end{aligned}$$

for any constant $k > 0$.

Let $a = 2d$, and let $x = d^2$. Then we have the following:

$$g((k^2+3k+2)x) = g(2x) + g((k^2+3k)x)$$

However, $(k^2+3k)x$ is surjective over the positive reals as $k > 0$, so if we let $y = (k^2+3k)x$, we obtain

$$g(x+y) = g(x) + g(y)$$

for all positive real numbers x and y. Consequently, for any positive real number x, we can always find a unique $\lambda > 0$ such that $\lambda(\lambda+1) = x$. Thus,

$$g(x) = g(\lambda(\lambda+1)) = f(2\lambda+1) - f(1) \geq -f(1)$$

Because g is bounded below and satisfies Cauchy's Functional Equation, there exists a real number a such that $g(x) = ax$ for all $x > 0$. That gives, for $u > 1$,

$$f(u) = f(1) + g((u^2-1)/4) = \frac{a}{4}u^2 + \frac{-a+4f(1)}{4}$$

and for $u < 1$

$$f(u) = f(1) - g((1-u^2)/4) = \frac{a}{4}u^2 + \frac{-a+4f(1)}{4}$$

Thus there exist constants c, d such that for $u \neq 1$, $f(u) = cu^2 + d$. Finally,

$$f(1) = f(4-3) = f(4+3) + f(6-2) - f(6+2) = 49c + d + 16c + d - 64c - d = c + d$$

Thus equations of the form $f(u) = cu^2 + d$ for all $u > 0$ are the only possible solutions. It is not hard to see that this is a solution to the functional equation if and only if c and d are nonnegative real numbers which are not both zero.

This problem was proposed by Calvin Deng.

5. Let $p > 13$ be a prime of the the form $2q+1$, where q is prime. Find the number of ordered pairs of integers (m, n) such that $0 \le m < n < p-1$ and

$$3^m + (-12)^m \equiv 3^n + (-12)^n \pmod{p}.$$

Solution:

Lemma 1: -4 is a primitive root modulo p.

Proof of Lemma 1: Note that $\mathrm{ord}_p(-4) | p-1 = 2q$, so $\mathrm{ord}_p(-4)$ is one of $1, 2, q, 2q$. Because $-4 \not\equiv 1 \pmod{p}$ we have $\mathrm{ord}_p(-4) \ne q$). As $16 = 4^n \not\equiv 1 \pmod{p}$, we have $\mathrm{ord}_p(-4) \ne 2$. Also, we have

$$\left(\frac{-4}{p}\right) = \left(\frac{-1}{p}\right) \cdot \left(\frac{4}{p}\right) = -1 \cdot 1 \cdot -1,$$

since $\left(\frac{-1}{p}\right) = 1$ follows from $p > 13 \Rightarrow \frac{p-1}{2} = q$ being odd.

Thus, following from the fact that -4 is not a quadratic residue modulo p, we have that

$$2 \not| \frac{p-1}{\mathrm{ord}_p(-4)} = \frac{2q}{\mathrm{ord}_p(-4)}$$

$$\Rightarrow 2 \cdot \mathrm{ord}_p(-4) \not| 2q$$

$$\Rightarrow \mathrm{ord}_p(-4) \not| q$$

Consequently, $\mathrm{ord}_p(-4) = 2q$, as desired.

Lemma 2: The order of 3 modulo p is exactly q.

Proof of Lemma 2: Note that $\mathrm{ord}_p(3) | p-1 = 2q$, so $\mathrm{ord}_p(3)$ is one of $1, 2, q, 2q$. Because $3 \not\equiv 1 \pmod{p}$, we have $\mathrm{ord}_p(3) \ne 1$. As $3^2 = 9 \not\equiv 1 \pmod{p}$, we have $\mathrm{ord}_p(3) \ne 2$. Then, we have

$$\begin{aligned} p &= 2q+1 \\ &\equiv 2 \cdot 1 + 1 \text{ or } 2 \cdot 2 + 1 \pmod{3} \\ &\equiv 0 \text{ or } 2 \pmod{3} \end{aligned}$$

Because $p > 13$, we know that $q \ne 3$ and $p \ne 3$, giving

$$\left(\frac{p}{3}\right) = \left(\frac{2}{3}\right) = 1.$$

Also, by quadratic reciprocity, we have

$$\begin{aligned}
\left(\frac{3}{p}\right)\cdot\left(\frac{p}{3}\right) &= (-1)^{\frac{(3-1)(p-1)}{4}} \\
\left(\frac{3}{p}\right)(-1) &= (-1)^q \\
&= -1 \\
\Rightarrow \left(\frac{3}{p}\right) &= 1.
\end{aligned}$$

We now know that 3 is a quadratic residue modulo p, so $\text{ord}_p(3) \neq 2q$, giving us $\text{ord}_p(3) = q$, as desired.

Lemma 3: -12 is a primitive root modulo p.

Proof of Lemma 3: Note that $\text{ord}_p(-12) | p - 1 = 2q$, so $\text{ord}_p(-12)$ is one of $1, 2, q, 2q$. Because $-12 \not\equiv 1 \pmod{p}$, we have $\text{ord}_p(-12) \neq 1$. As $(-12)^2 = 144 \not\equiv 1 \pmod{p}$, we have $\text{ord}_p(-12) \neq 2$. Then, after substituting for the values found in Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}
\left(\frac{-12}{p}\right) &= \left(\frac{-1}{p}\right)\cdot\left(\frac{4}{p}\right)\cdot\left(\frac{3}{p}\right) \\
&= (-1)\cdot(1)\cdot(1) \\
&= -1.
\end{aligned}$$

Thus, -12 is not a quadratic residue modulo p, giving us

$$\begin{aligned}
2 \nmid \frac{p-1}{\text{ord}_p(-12)} &= \frac{2q}{\text{ord}_p(-12)} \\
\Rightarrow \text{ord}_p(-12) &\nmid q
\end{aligned}$$

It follows that $\text{ord}_p(-12) = 2q$, as desired.

Main Proof: We now simplify the given equation:

$$\begin{aligned}
3^m + (-12)^m &\equiv 3^n + (-12)^n && \pmod{p} \\
&\equiv 3^{n-m}\cdot 3^m + 3^{n-m}\cdot 3^m\cdot(-4)^n && \pmod{p} \\
1 + (-4)^m &\equiv 3^{n-m} + 3^{n-m}\cdot(-1)^n && \pmod{p} \\
1 - 3^{n-m} &\equiv 3^{n-m}\cdot(-4)^n - (-4)^m && \pmod{p} \\
&\equiv (-4)^m\cdot((-12)^{n-m} - 1) && \pmod{p}.
\end{aligned}$$

We ignore for the moment the condition that $m < n$ and count all pairs $m, n \in \mathbb{Z}_{p-1} = \mathbb{Z}_{2q}$. So, if $n \not\equiv m \pmod{2q}$, then $(-12)^{n-m} - 1 \not\equiv 0 \pmod{p}$, giving us

$$(-4)^m \equiv (1 - 3^{n-m})((-12)^{n-m} - 1)^{-1} \pmod{p}.$$

Because -4 is primitive modulo p, we have that any non-zero residue of $(-4)^m \pmod{p}$ uniquely determines the residue of $m \pmod{2q}$. So each non-zero residue of $n-m \pmod{2q}$ uniquely determines $m \pmod{2q}$, so long as

$$3^{n-m} - 1 \not\equiv 0 \pmod{p} \Leftrightarrow q \not| n - m.$$

Consequently, for each $(n-m) \in \{1, 2, \ldots, q-1, q+1, q+2, \ldots, 2q-1\} \pmod{2q}$, we unitely determine the ordered pair $(m, n) \in \mathbb{Z}_{2q}^2$. However, on taking remainders on divison by $2q$ of m, n, we must have $m < n$. Thus, for each $x \not\equiv q, 0$, the solutions for $n - m \equiv x \pmod{2q}$ and $n - m \equiv -1 \pmod{2q}$ give exactly 1 solution (m, n) with $m < n$.

Thus, we have a total of $\frac{2q-2}{2} = q - 1$ solutions.

This problem was proposed by Alex Zhu.

6. Consider the infinite grid of lattice points in $\mathbb{Z}^3$. Little D and Big Z play a game, where Little D first loses a shoe on an unmunched point in the grid. Then, Big Z munches a shoe-free plane perpendicular to one of the coordinate axes. They continue to alternature turns in this fashion, with Little D's goal to loose a shoe on each of n consecutive lattice points on a line parrallel to one of the coordinate axes. Determine all n for which Little D can accomplish his goal.

Solution: We claim that Little D can accomplish this for all n.

We will start by separating out the three coordinate axes: thus, if Little D loses a shoe at the point (i, j, k) for integers i, j, and k, he plays on i on the x-axis, j on the y-axis, and k on the z-axis in the same move. Meanwhile, when Big Z munches a plane, he plays on only one point on one of the coordinate axes. Hence, since Big Z can only munch a shoe-free plane, he cannot munch point l on a particular axis if Little D has already placed a shoe there.

We will call a string of points marked (by shoes) on one of these coordinate axes *unbounded* if Big Z has not munched any point on that axis within $2n + 1$ of at least one endpoint of the string.

Lemma: For any integers m and l, Little D can create l unbounded strings of m consecutive points on a single coordinate axis.

Proof of lemma: We will prove this by induction on m.

Our base case is $m = 1$. Then, we note that if Little D makes $\lceil 1.5l \rceil$ triplets of moves over the three axes, making sure that he distributes any marked points in the same axis at least $5n$ apart, then Big Z can bound at most $\lceil 1.5l \rceil$ strings because he can only bound at most one string on each move. However, this leaves $3x$ unbounded strings of length 1; by the pigeonhole principle, at least l of these must be in the same cooordinate axis.

Now, suppose that this is true for some m. We will show that it is also true for $m + 1$. Without loss of generality, we note from our induction hypothesis that Little D can construct $\lceil \frac{x(m+1)}{2} \rceil$ unbounded strings of length m on the x-axis. Consequently, he can create $m + 3$ strings of length $m + 1$ in $m + 1$ moves: in each move, he lengthens one unbounded string of the x-axis, while on each of the y and z-axes he builds up a new string of $m+1$ marked points. However, Big Z can, in these $m+1$ moves, bound at most $m + 1$ of these strings. Hence, Little D can construct at least 2 strings of length $m + 1$ for every $m + 1$ strings of length m used up. It follows that he can achieve x unbounded strings of length $m + 1$. We have now proved our desired induction.

Main Proof: Now, without loss of generality, by our lemma Little D can mark n consecutive points on the x-axis. Then, he has established n consecutive yz-planes that Big Z can never much. Suppose that one of the points he has played on is (i, j, k) for some $i, j, k \in \mathbb{Z}$. Then, Big Z can never munch any part of the line $y = j, z = k$ in those n consecutive xy planes. Hence Little D can lose shoes in the remainder of those n points over his next few moves, at which point he has achieved his goal.

This problem was proposed by David Yang.

Every Little Mistake $\Longrightarrow$ 0

Lincoln, Nebraska

Day I 8 a.m. - 12:30 p.m.

June 16, 2012

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in an automatic 0 for that problem.

1. In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

2. Find all ordered pairs of positive integers (m, n) for which there exists a set $C = \{c_1, \ldots, c_k\}$ $(k \geq 1)$ of colors and an assignment of colors to each of the mn unit squares of a $m \times n$ grid such that for every color $c_i \in C$ and unit square S of color c_i, exactly two direct (non-diagonal) neighbors of S have color c_i.

3. Let f, g be polynomials with complex coefficients such that $\gcd(\deg f, \deg g) = 1$. Suppose that there exist polynomials $P(x, y)$ and $Q(x, y)$ with complex coefficients such that $f(x) + g(y) = P(x, y)Q(x, y)$. Show that one of P and Q must be constant.

Every Little Mistake $\implies$ 0

Lincoln, Nebraska

Day II 8 a.m. - 12:30 p.m.

June 17, 2012

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in an automatic 0 for that problem.

4. Let a_0, b_0 be positive integers, and define $a_{i+1} = a_i + \lfloor \sqrt{b_i} \rfloor$ and $b_{i+1} = b_i + \lfloor \sqrt{a_i} \rfloor$ for all $i \geq 0$. Show that there exists a positive integer n such that $a_n = b_n$.

5. Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

6. A diabolical combination lock has n dials (each with c possible states), where $n, c > 1$. The dials are initially set to states $d_1, d_2, \ldots, d_n$, where $0 \leq d_i \leq c-1$ for each $1 \leq i \leq n$. Unfortunately, the actual states of the dials (the d_i's) are concealed, and the initial settings of the dials are also unknown. On a given turn, one may advance each dial by an integer amount c_i ($0 \leq c_i \leq c-1$), so that every dial is now in a state $d_i' \equiv d_i + c_i \pmod{c}$ with $0 \leq d_i' \leq c-1$. After each turn, the lock opens if and only if all of the dials are set to the zero state; otherwise, the lock selects a random integer k and cyclically shifts the d_i's by k (so that for every i, d_i is replaced by d_{i-k}, where indices are taken modulo n).

   Show that the lock can always be opened, regardless of the choices of the initial configuration and the choices of k (which may vary from turn to turn), if and only if n and c are powers of the same prime.

Every Little Mistake $\implies$ 0 Shortlist

MOP 2012

June 12, 2012

Note: The problem czars' recommendations are bolded.

1 Geometry

G1. (Ray Li) In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

G2. (Ray Li) In triangle ABC, P is a point on altitude AD. Q, R are the feet of the perpendiculars from P to AB, AC, and QP, RP meet BC at S and T respectively. the circumcircles of BQS and CRT meet QR at X, Y.

a) Prove SX, TY, AD are concurrent at a point Z.

b) Prove Z is on QR iff $Z = H$, where H is the orthocenter of ABC.

G3. (Alex Zhu) ABC is a triangle with incenter I. The foot of the perpendicular from I to BC is D, and the foot of the perpendicular from I to AD is P. Prove that $\angle BPD = \angle DPC$.

G4. (Ray Li) Circles Ω and ω are internally tangent at point C. Chord AB of Ω is tangent to ω at E, where E is the midpoint of AB. Another circle, ω_1 is tangent to Ω, ω, and AB at D, Z, and F respectively. Rays CD and AB meet at P. If M is the midpoint of major arc AB, show that $\tan \angle ZEP = \frac{PE}{CM}$.

G5. (Calvin Deng) Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

G6. (Ray Li) In $\triangle ABC$, H is the orthocenter, and AD, BE are arbitrary cevians. Let ω_1, ω_2 denote the circles with diameters AD and BE, respectively. HD, HE meet ω_1, ω_2 again at F, G. DE meets ω_1, ω_2 again at P_1, P_2 respectively. FG meets ω_1, ω_2 again Q_1, Q_2 respectively. P_1H, Q_1H meet ω_1 at R_1, S_1 respectively. P_2H, Q_2H meet ω_2 at R_2, S_2 respectively. Let $P_1Q_1 \cap P_2Q_2 = X$, and $R_1S_1 \cap R_2S_2 = Y$. Prove that X, Y, H are collinear.

G7. (Alex Zhu) Let $\triangle ABC$ be an acute triangle with circumcenter O such that $AB < AC$, let Q be the intersection of the external bisector of $\angle A$ with BC, and let P be a point in the interior of $\triangle ABC$ such that $\triangle BPA$ is similar to $\triangle APC$. Show that $\angle QPA + \angle OQB = 90°$.

2 Algebra

A1. (Ray Li, Max Schindler) Let $x_1, x_2, x_3, y_1, y_2, y_3$ be nonzero real numbers satisfying $x_1 + x_2 + x_3 = 0, y_1 + y_2 + y_3 = 0$. Prove that

$$\frac{x_1x_2 + y_1y_2}{\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}} + \frac{x_2x_3 + y_2y_3}{\sqrt{(x_2^2 + y_2^2)(x_3^2 + y_3^2)}} + \frac{x_3x_1 + y_3y_1}{\sqrt{(x_3^2 + y_3^2)(x_1^2 + y_1^2)}} \geq -\frac{3}{2}$$

A2. (Owen Goff) Let a, b, c be three positive real numbers such that $a \leq b \leq c$ and $a + b + c = 1$. Prove that

$$\frac{a+c}{\sqrt{a^2+c^2}} + \frac{b+c}{\sqrt{b^2+c^2}} + \frac{a+b}{\sqrt{a^2+b^2}} \leq \frac{3\sqrt{6}(b+c)^2}{\sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}.$$

A3. (David Yang) Let a_0, b_0 be positive integers, and define $a_{i+1} = a_i + \lfloor \sqrt{b_i} \rfloor$ and $b_{i+1} = b_i + \lfloor \sqrt{a_i} \rfloor$ for all $i \geq 0$. Show that there exists a positive integer n such that $a_n = b_n$.

A4. (David Yang) Prove that if m, n are relatively prime positive integers, $x^m - y^n$ is irreducible in the complex numbers. (A polynomial $P(x, y)$ is irreducible if there do not exist nonconstant polynomials $f(x, y)$ and $g(x, y)$ such that $P(x, y) = f(x, y)g(x, y)$ for all x, y.)

A5. (Calvin Deng) Let $a, b, c \geq 0$. Show that

$$(a^2 + 2bc)^{2012} + (b^2 + 2ca)^{2012} + (c^2 + 2ab)^{2012} \leq (a^2 + b^2 + c^2)^{2012} + 2(ab + bc + ca)^{2012}.$$

A6. (Victor Wang) Let f, g be polynomials with complex coefficients such that $\gcd(\deg f, \deg g) = 1$. Suppose that there exist polynomials $P(x, y)$ and $Q(x, y)$ with complex coefficients such that $f(x) + g(y) = P(x, y)Q(x, y)$. Show that one of P and Q must be constant.

Note: A4 is a special case of A6, but is significantly easier.

A7. (Alex Zhu) Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x)f(y)f(x + y) = f(xy)(f(x) + f(y))$ for all $x, y \in \mathbb{Q}$.

A8. (David Yang) Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a cyclic octagon. Let B_i by the intersection of A_iA_{i+1} and $A_{i+3}A_{i+4}$. (Take $A_9 = A_1$, $A_{10} = A_2$, etc.) Prove that $B_1, B_2, \ldots, B_8$ lie on a conic.

3 Number Theory

N1. (David Yang, Alex Zhu) Find all positive integers n such that $4^n + 6^n + 9^n$ is a square.

N2. (Anderson Wang) For positive rational x, if x is written in the form $\frac{p}{q}$ with p, q positive relatively prime integers, define $f(x) = p + q$. For example, $f(1) = 2$. Prove that if $f(x) = f(\frac{mx}{n})$ for rational x and positive integers m, n, then $f(x)$ divides $|m - n|$.

Possible part (b): Let n be a positive integer. If all x which satisfy $f(x) = f(2^n x)$ also satisfy $f(x) = 2^n - 1$, find all possible values of n.

N3. (Alex Zhu) Let $s(k)$ be the number of ways to express k as the sum of distinct 2012^{th} powers. Show that for every real number c there exists an integer n such that $s(n) > cn$.

N4. (Lewis Chen) Do there exist positive integers $b, n > 1$ such that when n is expressed in base b, there are more than n distinct permutations of its digits? For example, when $b = 4$ and $n = 18$, $18 = 102_4$, but 102 only has 6 digit arrangements. (Leading zeros are allowed in the permutations.)

N5. (Ravi Jagadeesan) Let $n > 2$ be a positive integer and let p be a prime. Suppose that the nonzero integers are colored in n colors. Let $a_1, a_2, \ldots, a_n$ be integers such that for all $1 \le i \le n$, $p^i \nmid a_i$ and $p^{i-1} \mid a_i$. In terms of n, p, and $\{a_i\}_{i=1}^n$, determine if there must exist integers $x_1, x_2, \ldots, x_n$ of the same color such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

N6. (Calvin Deng) Prove that if a and b are positive integers and $ab > 1$, then

$$\left\lfloor \frac{(a-b)^2 - 1}{ab} \right\rfloor = \left\lfloor \frac{(a-b)^2 - 1}{ab - 1} \right\rfloor$$

Here $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

N7. (Bobby Shen) A diabolical combination lock has n dials (each with c possible states), where $n, c > 1$. The dials are initially set to states $d_1, d_2, \ldots, d_n$, where $0 \le d_i \le c-1$ for each $1 \le i \le n$. Unfortunately, the actual states of the dials (the d_i's) are concealed, and the initial settings of the dials are also unknown. On a given turn, one may advance each dial by an integer amount c_i ($0 \le c_i \le c - 1$), so that every dial is now in a state $d_i' \equiv d_i + c_i \pmod{c}$ with $0 \le d_i' \le c - 1$. After each turn, the lock opens if and only if all of the dials are set to the zero state; otherwise, the lock selects a random integer k and cyclically shifts the d_i's by k (so that for every i, d_i is replaced by d_{i-k}, where indices are taken modulo n).

Show that the lock can always be opened, regardless of the choices of the initial configuration and the choices of k (which may vary from turn to turn), if and only if n and c are powers of the same prime.

N8. (Victor Wang) Fix two positive integers $a, k \ge 2$, and let $f \in \mathbb{Z}[x]$ be a polynomial. Suppose that for all sufficiently large positive integers n, there exists a rational number x satisfying $f(x) = f(a^n)^k$. Prove that there exists a polynomial $g \in \mathbb{Q}[x]$ such that $f(g(x)) = f(x)^k$ for all real x.

N9. (David Yang) Are there positive integers m, n such that there exist 2012 positive integers x such that both $m - x^2$ and $n - x^2$ are perfect squares?

4 Combinatorics

C1. (David Yang) Let $n \geq 2$ be a positive integer. Given a sequence s_i of n distinct real numbers, define the "class" of the sequence to be the sequence $a_1, a_2, \ldots, a_{n-1}$, where a_i is 1 if $s_{i+1} > s_i$ and -1 otherwise.

Find the smallest integer m such that there exists a sequence w_i such that for every possible class of a sequence of length n, there is a subsequence of w_i that has that class.

C2. (David Yang) Let A be the set of positive integers with at most 10 digits and with all digits 0 or 1. Let B be the set of positive integers with at most 10 digits and with all digits 0,1,2, or 3. Define the difference set $X - Y$ of two sets of reals X, Y to be the set of elements z of the form $x - y$, where $x \in X$ and $y \in Y$. Prove that for any finite set of positive integers C, $|C-A| \leq |C-B| \leq 1024|C-A|$.

C3. (David Yang) Find all ordered pairs of positive integers (m, n) for which there exists a set $C = \{c_1, \ldots, c_k\}$ $(k \geq 1)$ of colors and an assignment of colors to each of the mn unit squares of a $m \times n$ grid such that for every color $c_i \in C$ and unit square S of color c_i, exactly two direct (non-diagonal) neighbors of S have color c_i.

C4. (Calvin Deng) A tournament on $2k$ vertices contains no 7-cycles. Show that its vertices can be partitioned into two sets, each with size k, such that the edges between vertices of the same set do not determine any 3-cycles.

C5. (Linus Hamilton) Form the infinite graph A by taking the set of primes p congruent to $1 \pmod 4$, and connecting p and q if they are quadratic residues modulo each other. Do the same for a graph B with the primes $1 \pmod 8$. Show A and B are isomorphic to each other.

C6. (Linus Hamilton) Consider a directed graph G with n vertices, where 1-cycles and 2-cycles are permitted. For any set S of vertices, let $N^+(S)$ denote the out-neighborhood of S (i.e. set of successors of S), and define $(N^+)^k(S) = N^+((N^+)^{k-1}(S))$ for $k \geq 2$.

For fixed n, let $f(n)$ denote the maximum possible number of distinct sets of vertices in $\{(N^+)^k(X)\}_{k=1}^{\infty}$. Show that there exists $n > 2012$ such that $f(n) < 1.0001^n$.

C7. (David Yang) We have a graph with n vertices and at least $n^2/10$ edges. Each edge is colored in one of c colors such that no two incident edges have the same color. Assume that no cycles of size 10 have the same set of colors. Prove that there is a constant k such that c is at least $kn^{\frac{8}{5}}$ for any n.

C8. (Victor Wang) Consider the equilateral triangular lattice in the complex plane defined by the Eisenstein integers; let the ordered pair (x, y) denote the complex number $x + y\omega$ for $\omega = e^{2\pi i/3}$. We define an ω-chessboard polygon to be a (non self-intersecting) polygon whose sides are situated along lines of the form $x = a$ or $y = b$, where a and b are integers. These lines divide the interior into unit triangles, which are shaded alternately black and white so that adjacent triangles have different colors. To tile an ω-chessboard polygon by lozenges is to exactly cover the polygon by non-overlapping rhombuses consisting of two bordering triangles. Finally, a *tasteful tiling* is one such that for every unit hexagon tiled by three lozenges, each lozenge has a black triangle on its left (defined by clockwise orientation) and a white triangle on its right (so the lozenges are BW, BW, BW in clockwise order).

a) Prove that if an ω-chessboard polygon can be tiled by lozenges, then it can be done so tastefully.

b) Prove that such a tasteful tiling is unique.

15$^{\text{th}}$ Everyone Lives at Most Once

Lincoln, Nebraska

Day I 8:00 AM - 12:30 PM

June 15, 2013

1. Let $a_1, a_2, \dots, a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \le i < j < k \le 9$ for which $a_i + a_j + a_k \ge 3m$. What is the minimum possible value of A?

2. Let a, b, c be positive reals satisfying $a + b + c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove that $a^a b^b c^c \ge 1$.

3. Let $m_1, m_2, \dots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \dots, A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, \dots, m_i - 1\}$ for $i = 1, 2, \dots, 2013$. Prove that there is a positive integer N such that

$$N \le (2\,|A_1| + 1)\,(2\,|A_2| + 1) \cdots (2\,|A_{2013}| + 1)$$

and for each $i = 1, 2, \dots, 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

15^{th} Everyone Lives at Most Once

Lincoln, Nebraska

Day II 8:00 AM - 12:30 PM

June 16, 2013

4. Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

5. For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \geq 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

6. Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $n \geq 0$, there are at most $0.001n^2$ pairs of integers (x, y) for which $f(x + y) \neq f(x) + f(y)$ and $\max\{|x|, |y|\} \leq n$. Is it possible that for some integer $n \geq 0$, there are more than n integers a such that $f(a) \neq a \cdot f(1)$ and $|a| \leq n$?

15$^{\text{th}}$ Everyone Lives at Most Once

ELMO 2013
Lincoln, Nebraska

OFFICIAL SOLUTIONS

1. Let $a_1, a_2, \dots, a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \le i < j < k \le 9$ for which $a_i + a_j + a_k \ge 3m$. What is the minimum possible value of A?

   Proposed by Ray Li.

   Answer. $A \ge 28$.

   Solution 1. Call a 3-set *good* iff it has average at least m, and let S be the family of good sets.

   The equality case $A = 28$ can be achieved when $a_1 = \cdots = a_8 = 0$ and $a_9 = 1$. Here $m = \frac{1}{9}$, and the good sets are precisely those containing a_9. This gives a total of $\binom{8}{2} = 28$.

   To prove the lower bound, suppose we have exactly N good 3-sets, and let $p = \frac{N}{\binom{9}{3}}$ denote the probability that a randomly chosen 3-set is good. Now, consider a random permutation π of $\{1, 2, \dots, 9\}$. Then the corresponding partition $\bigcup_{i=0}^{2} \{\pi(3i+1), \pi(3i+2), \pi(3i+3)\}$ has at least 1 good 3-set, so by the linearity of expectation,

$$\begin{aligned}
1 &\le \mathbb{E}\left[\sum_{i=0}^{2} [\{\pi(3i+1), \pi(3i+2), \pi(3i+3)\} \in S]\right] \\
&= \sum_{i=0}^{2} \left[\mathbb{E}[\{\pi(3i+1), \pi(3i+2), \pi(3i+3)\} \in S]\right] \\
&= \sum_{i=0}^{2} 1 \cdot p = 3p.
\end{aligned}$$

   Hence $N = p\binom{9}{3} \ge \frac{1}{3}\binom{9}{3} = 28$, establishing the lower bound. ■

   This problem and solution were proposed by Ray Li.

   Remark. One can use double-counting rather than expectation to prove $N \ge 28$. In any case, this method generalizes effortlessly to larger numbers.

   Solution 2. Proceed as above to get an upper bound of 28.

   On the other hand, we will show that we can partition the $\binom{9}{3} = 84$ 3-sets into 28 groups of 3, such that in any group, the elements $a_1, a_2, \cdots, a_9$ all appear. This will imply the conclusion, since if $A < 28$, then there are at least 57 sets with average at most m, but by pigeonhole three of them must be in such a group, which is clearly impossible.

   Consider a 3-set and the following array:

$$\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{matrix}$$

   Consider a set $|S| = 3$. We obtain the other two 3-sets in the group as follows:

- If S contains one element in each column, then shift the elements down cyclically mod 3.
- If S contains one element in each row, then shift the elements right cyclically mod 3. Note that the result coincides with the previous case if both conditions are satisfied.
- Otherwise, the elements of S are "constrained" in a 2×2 box, possibly shifted diagonally. In this case, we get an L-tromino. Then shift diagonally in the direction the L-tromino points in.

One can verify that this algorithm creates such a partition, so we conclude that $A \geq 28$. ■

This second solution was suggested by Lewis Chen.

2. Let a, b, c be positive reals satisfying $a + b + c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove that $a^a b^b c^c \geq 1$.

Proposed by Evan Chen.

Solution 1. By weighted AM-GM we have that

$$\begin{aligned} 1 &= \sum_{\text{cyc}} \left(\frac{\sqrt[7]{a}}{a+b+c} \right) \\ &= \sum_{\text{cyc}} \left(\frac{a}{a+b+c} \cdot \frac{1}{\sqrt[7]{a^6}} \right) \\ &\geq \left(\frac{1}{a^a b^b c^c} \right)^{\frac{6/7}{a+b+c}}. \end{aligned}$$

Rearranging yields $a^a b^b c^c \geq 1$. ■

This problem and solution were proposed by Evan Chen.

Remark. The problem generalizes easily to n variables, and exponents other than $\frac{1}{7}$. Specifically, if positive reals $x_1 + \cdots + x_n = x_1^r + \cdots + x_n^r$ for some real number $r \neq 1$, then $\prod_{i \geq 1} x_i^{x_i} \geq 1$ if and only if $r < 1$. When $r \leq 0$, a Jensen solution is possible using only the inequality $a + b + c \geq 3$.

Solution 2. First we claim that $a, b, c < 5$. Assume the contrary, that $a \geq 5$. Let $f(x) = x - \sqrt[7]{x}$. Since $f'(x) > 0$ for $x \geq 5$, we know that $f(a) \geq 5 - \sqrt[7]{5} > 3$. But this means that WLOG $b - \sqrt[7]{b} < -1.5$, which is clearly false since $b - \sqrt[7]{b} \geq 0$ for $b \geq 1$, and $b - \sqrt[7]{b} \geq -\sqrt[7]{b} \geq -1$ for $0 < b < 1$. So indeed $a, b, c < 5$.

Now rewrite the inequality as

$$\sum a \ln a \geq 0 \Leftrightarrow \sum \left(\frac{a^{\frac{1}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right) (a^{\frac{6}{7}} \ln a) \geq 0.$$

Now note that if $g(x) = x^{\frac{6}{7}} \ln x$, then $g''(x) = \frac{35 - 6 \ln x}{49 x^{\frac{8}{7}}} > 0$ for $x \in (0, 5)$. Therefore g is convex and we can use Jensen's Inequality to get

$$\sum \left(\frac{a^{\frac{1}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right) (a^{\frac{6}{7}} \ln a) \geq \left(\sum \frac{a^{\frac{8}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right)^{\frac{6}{7}} \ln \left(\sum \frac{a^{\frac{8}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right).$$

Since $\sum a = \sum a^{\frac{1}{7}}$, it suffices to show that $\sum a^{\frac{8}{7}} \geq \sum a$. But by weighted AM-GM we have

$$6a^{\frac{8}{7}} + a^{\frac{1}{7}} \geq 7a \implies a^{\frac{8}{7}} - a \geq \frac{1}{6}(a - \sqrt[7]{a}).$$

Adding up the analogous inequalities for b, c gives the desired result. ■

This second solution was suggested by David Stoner.

Solution 3. Here we unify the two solutions above.

It's well-known that weighted AM-GM follows from (and in fact, is equivalent to) the convexity of e^x (or equivalently, the concavity of $\ln x$), as $\sum w_i e^{x_i} \geq e^{\sum w_i x_i}$ for reals x_i and nonnegative weights w_i summing to 1. However, it also follows from the convexity of $y \ln y$ (or equivalently, the concavity of ye^y) for $y > 0$. Indeed, letting $y_i = e^{x_i} > 0$, and taking logs, weighted AM-GM becomes

$$\sum w_i y_i \cdot \frac{1}{y_i} \log \frac{1}{y_i} \geq (\sum w_i y_i) \frac{\sum w_i y_i \cdot \frac{1}{y_i}}{\sum w_i y_i} \log \frac{\sum w_i y_i \cdot \frac{1}{y_i}}{\sum w_i y_i},$$

which is clear.

To find Evan's solution, we can use the concavity of $\ln x$ to get $\sum a \ln a^{-s} \leq (\sum a) \ln \sum \frac{a \cdot a^{-s}}{\sum a} = 0$. (Here we take $s = 6/7 > 0$.)

For a cleaner version of David's solution, we can use the convexity of $x \ln x$ to get

$$\sum a \ln a^s = \sum a^{1-s} \cdot a^s \ln a^s \geq (\sum a^{1-s}) \frac{\sum a^{1-s} \cdot a^s}{\sum a^{1-s}} \ln \frac{\sum a^{1-s} \cdot a^s}{\sum a^{1-s}} = 0$$

(where we again take $s = 6/7 > 0$).

Both are pretty intuitive (but certainly not obvious) solutions once one realizes direct Jensen goes in the wrong direction. In particular, $s = 1$ doesn't work since we have $a + b + c \leq 3$ from the power mean inequality. ■

This third solution was suggested by Victor Wang.

Solution 4. From $e^t \geq 1 + t$ for $t = \ln x^{-\frac{6}{7}}$, we find $\frac{6}{7} \ln x \geq 1 - x^{-\frac{6}{7}}$. Thus

$$\frac{6}{7} \sum a \ln a \geq \sum a - a^{\frac{1}{7}} = 0,$$

as desired. ■

This fourth solution was suggested by chronodecay.

Remark. Polya once dreamed a similar proof of n-variable AM-GM: $x \geq 1 + \ln x$ for positive x, so $\sum x_i \geq n + \ln \prod x_i$. This establishes AM-GM when $\prod x_i = 1$; the rest follows by homogenizing.

3. Let $m_1, m_2, \ldots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \ldots, A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, \ldots, m_i - 1\}$ for $i = 1, 2, \ldots, 2013$. Prove that there is a positive integer N such that

$$N \leq (2|A_1| + 1)(2|A_2| + 1) \cdots (2|A_{2013}| + 1)$$

and for each $i = 1, 2, \ldots, 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

Proposed by Victor Wang.

Remark. As Solution 3 shows, the bound can in fact be tightened to $\prod_{i=1}^{2013}(|A_i| + 1)$.

Solution 1. We will show that the smallest integer N such that $N \notin A_i \pmod{m_i}$ is less than the bound provided.

The idea is to use pigeonhole and the "Lagrange interpolation"-esque representation of CRT systems. Define integers t_i satisfying $t_i \equiv 1 \pmod{m_i}$ and $t_i \equiv 0 \pmod{m_j}$ for $j \neq i$. If we find nonempty sets B_i of distinct residues mod m_i with $B_i - B_i \pmod{m_i}$ and $A_i \pmod{m_i}$ disjoint, then by pigeonhole, a positive integer solution with $N \le \frac{m_1 m_2 \cdots m_{2013}}{|B_1|\cdot|B_2|\cdots|B_{2013}|}$ must exist (more precisely, since

$$b_1 t_1 + \cdots + b_{2013} t_{2013} \pmod{m_1 m_2 \cdots m_{2013}}$$

is injective over $B_1 \times B_2 \times \cdots \times B_{2013}$, some two consecutively ordered solutions must differ by at most $\frac{m_1 m_2 \cdots m_{2013}}{|B_1|\cdot|B_2|\cdots|B_{2013}|}$).

On the other hand, since $0 \notin A_i$ for every i, we know such nonempty B_i must exist (e.g. take $B_i = \{0\}$). Now suppose $|B_i|$ is maximal; then every $x \pmod{m_i}$ lies in at least one of B_i, $B_i + A_i$, $B_i - A_i$ (note that $x - x = 0$ is not an issue when considering $(B_i \cup \{x\}) - (B_i \cup \{x\})$), or else $B_i \cup \{x\}$ would be a larger working set. Hence $m_i \le |B_i| + |B_i + A_i| + |B_i - A_i| \le |B_i|(1 + 2|A_i|)$, so we get an upper bound of $\prod_{i=1}^{2013} \frac{m_i}{|B_i|} \le \prod_{i=1}^{2013}(2|A_i| + 1)$, as desired. ■

Remark. We can often find $|B_i|$ significantly larger than $\frac{m_i}{2|A_i|+1}$ (the bounds $|B_i + A_i|, |B_i - A_i| \le |B_i| \cdot |A_i|$ seem really weak, and $B_i + A_i, B_i - A_i$ might not be that disjoint either). For instance, if $A_i \equiv -A_i \pmod{m_i}$, then we can get (the ceiling of) $\frac{m_i}{|A_i|+1}$.

Remark. By translation and repeated application of the problem, one can prove the following slightly more general statement: "Let $m_1, m_2, \ldots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \ldots, A_{2013}$ be 2013 (possibly empty) sets with A_i a proper subset of $\{1, 2, \ldots, m_i\}$ for $i = 1, 2, \ldots, 2013$. Then for every integer n, there exists an integer x in the range $(n, n + \prod_{i=1}^{2013}(2|A_i| + 1)]$ such that $x \notin A_i \pmod{m_i}$ for $i = 1, 2, \ldots, 2013$. (We say A is a *proper subset* of B if A is a subset of B but $A \ne B$.)"

Remark. Let f be a non-constant integer-valued polynomial with $\gcd(\ldots, f(-1), f(0), f(1), \ldots) = 1$. Then by the previous remark, we can easily prove that there exist infinitely many positive integers n such that the smallest prime divisor of $f(n)$ is at least $c \log n$, where $c > 0$ is any constant. (We take m_i the ith prime and $A_i \equiv \{n : m_i \mid f(n)\} \pmod{m_i}$—if $f = \frac{a}{b}x^d + \cdots$, then $|A_i| \le d$ for all sufficiently large i.)

Solution 2. We will mimic the proof of 2010 RMM Problem 1.

Suppose $1, 2, \ldots, N$ (for some $N \ge 1$) can be covered by the sets $A_i \pmod{m_i}$.

Observe that for fixed m and $1 \le a \le m$, exactly $1 + \lfloor \frac{N-a}{m} \rfloor$ of $1, 2, \ldots, N$ are $a \pmod{m}$. In particular, we have lower and upper bounds of $\frac{N-m}{m}$ and $\frac{N+m}{m}$, respectively, so PIE yields

$$N \le \sum_i |A_i| \frac{N + m_i}{m_i} - \sum_{i<j} |A_i| \cdot |A_j| \frac{N - m_i m_j}{m_i m_j} \pm \cdots .$$

It follows that

$$N \prod_i \left(1 - \frac{|A_i|}{m_i}\right) \le \prod_i (1 + |A_i|),$$

so $N \le \prod_i \frac{m_i}{m_i - |A_i|}(1 + |A_i|)$.

Note that $\frac{m_i}{m_i - |A_i|} \le \frac{2|A_i|+1}{|A_i|+1}$ iff $m_i \ge 2|A_i| + 1$, so we're done unless $m_i \le 2|A_i|$ for some i.

In this case, there exists (by induction) $1 \le N \le \prod_{j \ne i}(2|A_j| + 1)$ such that $N \notin m_i^{-1} A_j \pmod{m_j}$ for all $j \ne i$. Thus $m_i N \notin A_j \pmod{m_j}$ and we trivially have $m_i N \equiv 0 \notin A_i \pmod{m_i}$, so $m_i N \le \prod_k (2|A_k| + 1)$, as desired. ■

This problem and the above solutions were proposed by Victor Wang.

Solution 3. We can in fact get a bound of $\prod(|A_k| + 1)$ directly.

Let $t = 2013$. Suppose $1, 2, \ldots, N$ are covered by the $A_k \pmod{m_k}$; then

$$z_n = \prod_{1 \le k \le t, a \in A_k} \left(1 - e^{\frac{2\pi i}{m_k}(n-a)}\right)$$

is a linear recurrence in $e^{2\pi i \sum_{k=1}^{t} \frac{j_k}{m_k}}$ (where each j_k ranges from 0 to $|A_k|$). But $z_0 \neq 0 = z_1 = \cdots = z_N$, so N must be strictly less than the degree $\prod(|A_k|+1)$ of the linear recurrence. Thus $1, 2, \ldots, \prod(|A_k|+1)$ cannot all be covered, as desired. ■

This third solution was suggested by Zhi-Wei Sun.

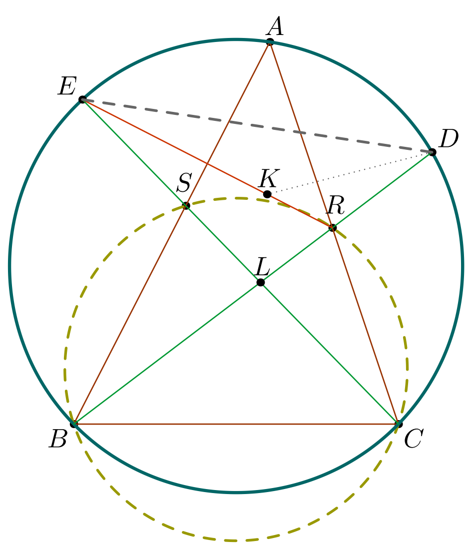
Remark. Solution 3 doesn't require the m_k to be coprime. Note that if $|A_1| = \cdots = |A_t| = b-1$, then a base b construction shows the bound of $\prod(b-1+1) = b^t$ is "tight" (if we remove the restriction that the m_k must be coprime).

However, Solutions 2 and 3 "ignore" the additive structure of CRT solution sets encapsulated in Solution 1's Lagrange interpolation representation.

4. Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

Proposed by Evan Chen.

Solution 1.



First, we claim that $BE = BR = BC$. Indeed, construct a circle with radius $BE = BR$ centered at B, and notice that $\angle ECR = \frac{1}{2}\angle EBR$, implying that it lies on the circle.

Now, CA bisects $\angle ECD$ and DB bisects $\angle EDC$, so R is the incenter of $\triangle CDE$. Then, K is the incenter of $\triangle LED$, so $\angle ELK = \frac{1}{2}\angle ELD = \frac{1}{2}\left(\frac{\widehat{ED}+\widehat{BC}}{2}\right) = \frac{1}{2}\frac{\widehat{BED}}{2} = \frac{1}{2}\angle BCD$. ■

This problem and solution were proposed by Evan Chen.

Solution 2. Note $\angle EBA = \angle ECA = \angle SCR = \angle SBR = \angle ABR$, so AB bisects $\angle EBR$. Then by symmetry $\angle BEA = \angle BRA$, so $\angle BCR = \angle BCA = 180 - \angle BEA = 180 - \angle BRA = \angle BRC$, so $BE = BR = BC$. Proceed as above. ■

This second solution was suggested by Michael Kural.

5. For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \geq 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

Proposed by Andre Arslan.

Answer. All P of the form $P(x) = cx^k$, where c is a nonzero integer and k is a nonnegative integer.

Solution. Suppose $P(x) = x^k Q(x)$ with $Q(0) \neq 0$ and Q is nonconstant; then there exist infinitely many primes p dividing some $Q(n)$; fix one of them not dividing $Q(0)$, and take a sequence of pairwise coprime integers $m_1, n_1, m_2, n_2, \ldots$ with $p \mid Q(m_i), Q(n_i)$ (we can do this with CRT).

Let $f(x, y, z)$ be the number written at (x, y, z). Note that $P(m)$ divides every $mn \times mn \times m$ grid and $P(n)$ divides every $mn \times mn \times n$ grid, so by Bezout's identity, $(P(m), P(n))$ divides every $mn \times mn \times (m, n)$ grid. It follows that p divides every $m_i n_i \times m_i n_i \times 1$ grid. Similarly, we find that p divides every $m_i n_i m_j n_j \times 1 \times 1$ grid whenever $i \neq j$, and finally, every $1 \times 1 \times 1$ grid. Since p was arbitrarily chosen from an infinite set, f must be identically zero, contradiction.

For the other direction, take a solution g to the one-dimensional case using repeated CRT (the key relation $\gcd(P(m), P(n)) = P(\gcd(m, n))$ prevents "conflicts"): start with a positive multiple of $P(1) \neq 0$ at zero, and then construct $g(1), g(-1), g(2), g(-2)$, etc. in that order using CRT. Now for the three-dimensional version, we can just let $f(x, y, z) = g(x)$. ■

This problem and solution were proposed by Andre Arslan.

Remark. The crux of the problem lies in the 1D case. (We use the same type of reasoning to "project" from d dimension to $d-1$ dimensions.) Note that the condition $P(n) \mid g(i) + \cdots + g(i+n-1)$ (for the 1D case) is "almost" the same as $P(n) \mid g(i) - g(i+n)$, so we immediately find $\gcd(P(m), P(n)) \mid g(i) - g(i + \gcd(m, n))$ by Bezout's identity. In particular, when m, n are coprime, we will intuitively be able to get $\gcd(P(m), P(n))$ as large as we want unless P is of the form cx^k (we formalize this by writing $P = x^k Q$ with $Q(0) \neq 0$).

Conversely, if $P = cx^k$, then $\gcd(P(m), P(n)) = P(\gcd(m, n))$ renders our derived restriction $\gcd(P(m), P(n)) \mid g(i) - g(i + \gcd(m, n))$ superfluous. So it "feels easy" to find *nonconstant* g with $P(n) \mid g(i) - g(i+n)$ for all i, n, just by greedily constructing $g(0), g(1), g(-1), \ldots$ in that order using CRT. Fortunately, $g(i) + \cdots + g(i+m-1) - g(i) - \cdots - g(i+n-1) = g(i+n) + \cdots + g(i+n+(m-n)-1)$ for $m > n$, so the inductive approach still works for the stronger condition $P(n) \mid g(i) + \cdots + g(i+n-1)$.

Remark. Note that polynomial constructions cannot work for $P = cx^{d+1}$ in d dimensions. Suppose otherwise, and take a minimal degree $f(x_1, \ldots, x_d)$; then f isn't constant, so $f'(x_1, \ldots, x_d) = f(x_1 + 1, \ldots, x_d + 1) - f(x_1, \ldots, x_d)$ is a working polynomial of strictly smaller degree.

6. Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $n \geq 0$, there are at most $0.001n^2$ pairs of integers (x, y) for which $f(x+y) \neq f(x) + f(y)$ and $\max\{|x|, |y|\} \leq n$. Is it possible that for some integer $n \geq 0$, there are more than n integers a such that $f(a) \neq a \cdot f(1)$ and $|a| \leq n$?

Proposed by David Yang.

Answer. No.

Solution. Call an integer conformist if $f(n) = n \cdot f(1)$. Call a pair (x, y) good if $f(x+y) = f(x) + f(y)$ and bad otherwise. Let $h(n)$ denote the number of conformist integers with absolute value at most n.

Let $\epsilon = 0.001$, S be the set of conformist integers, $T = \mathbb{Z} \setminus S$ be the set of non-conformist integers, and $X_n = [-n, n] \cap X$ for sets X and positive integers n (so $|S_n| = h(n)$); clearly $|T_n| = 2n + 1 - h(n)$.

First we can easily get $h(n) = 2n + 1$ ($-n$ to n are all conformist) for $n \le 10$.

Lemma 1. *Suppose a, b are positive integers such that $h(a) > a$ and $b \le 2h(a) - 2a - 1$. Then $h(b) \ge 2b(1 - \sqrt{\epsilon}) - 1$.*

Proof. For any integer t, we have

$$\begin{aligned}
|S_a \cap (t - S_a)| &= |S_a| + |t - S_a| - |S_a \cup (t - S_a)| \\
&\ge 2h(a) - (\max(S_a \cup (t - S_a)) - \min(S_a \cup (t - S_a)) + 1) \\
&\ge 2h(a) - (\max(a, t + a) - \min(-a, t - a) + 1) \\
&= 2h(a) - (|t| + 2a + 1) \\
&\ge b - |t|.
\end{aligned}$$

But (x, y) is bad whenever $x, y \in S$ yet $x + y \in T$, so summing over all $t \in T_b$ (assuming $|T_b| \ge 2$) yields

$$\begin{aligned}
\epsilon b^2 \ge g(b) &\ge \sum_{t \in T_b} |S_a \cap (t - S_a)| \\
&\ge \sum_{t \in T_b} (b - |t|) \ge \sum_{k=0}^{\lfloor |T_b|/2 \rfloor - 1} k + \sum_{k=0}^{\lceil |T_b|/2 \rceil - 1} k \ge 2\frac{1}{2}(|T_b|/2)(|T_b|/2 - 1),
\end{aligned}$$

where we use $\lfloor r/2 \rfloor + \lceil r/2 \rceil = r$ (for $r \in \mathbb{Z}$) and the convexity of $\frac{1}{2}x(x-1)$. We conclude that $|T_b| \le 2 + 2b\sqrt{\epsilon}$ (which obviously remains true without the assumption $|T_b| \ge 2$) and $h(b) = 2b + 1 - |T_b| \ge 2b(1 - \sqrt{\epsilon}) - 1$. □

Now we prove by induction on n that $h(n) \ge 2n(1 - \sqrt{\epsilon}) - 1$ for all $n \ge 10$, where the base case is clear. If we assume the result for $n - 1$ ($n > 10$), then in view of the lemma, it suffices to show that $2h(n-1) - 2(n-1) - 1 \ge n$, or equivalently, $2h(n-1) \ge 3n - 1$. But

$$2h(n-1) \ge 4(n-1)(1 - \sqrt{\epsilon}) - 2 \ge 3n - 1,$$

so we're done. (The second inequality is equivalent to $n(1 - 4\sqrt{\epsilon}) \ge 5 - 4\sqrt{\epsilon}$; $n \ge 11$ reduces this to $6 \ge 40\sqrt{\epsilon} = 40\sqrt{0.001} = 4\sqrt{0.1}$, which is obvious.) ■

This problem and solution were proposed by David Yang.

Everyone Lives at Most Once

June 2013
Lincoln, Nebraska

Problem Shortlist

Created and Managed by Evan Chen

ELMO regulation:
The shortlist problems should be kept strictly confidential until after the exam.

The Everyone Lives at Most Once committee gratefully acknowledges the receipt of 41 problem proposals from the following 13 contributors:

Andre Arslan	1 problem
Matthew Babbitt	3 problems
Evan Chen	7 problems
Eric Chen	1 problem
Calvin Deng	2 problems
Owen Goff	1 problem
Michael Kural	2 problems
Ray Li	6 problems
Allen Liu	2 problems
Bobby Shen	1 problem
David Stoner	8 problems
Victor Wang	4 problems
David Yang	3 problems

This list was last updated on June 21, 2014.

Part I

Problems

Algebra

A1*

A1*

Find all triples (f, g, h) of injective functions from $\mathbb{R}$ to $\mathbb{R}$ satisfying

$$\begin{aligned} f(x+f(y)) &= g(x)+h(y) \\ g(x+g(y)) &= h(x)+f(y) \\ h(x+h(y)) &= f(x)+g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(x) \neq F(y)$ whenever $x \neq y$.)

Evan Chen

A2

A2

Prove that for all positive reals a, b, c,

$$\frac{1}{a+\frac{1}{b}+1}+\frac{1}{b+\frac{1}{c}+1}+\frac{1}{c+\frac{1}{a}+1} \geq \frac{3}{\sqrt[3]{abc}+\frac{1}{\sqrt[3]{abc}}+1}.$$

David Stoner

A3

A3

Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $f(x)+f(y) = f(x+y)$ and $f(x^{2013}) = f(x)^{2013}$.

Calvin Deng

A4

A4

Positive reals a, b, and c obey $\frac{a^2+b^2+c^2}{ab+bc+ca} = \frac{ab+bc+ca+1}{2}$. Prove that

$$\sqrt{a^2+b^2+c^2} \leq 1+\frac{|a-b|+|b-c|+|c-a|}{2}.$$

Evan Chen

A5*

A5*

Let a, b, c be positive reals satisfying $a+b+c = \sqrt[7]{a}+\sqrt[7]{b}+\sqrt[7]{c}$. Prove that $a^a b^b c^c \geq 1$.

Evan Chen

A6

A6

Let a, b, c be positive reals such that $a + b + c = 3$. Prove that

$$18 \sum_{\text{cyc}} \frac{1}{(3-c)(4-c)} + 2(ab + bc + ca) \geq 15.$$

David Stoner

A7*

A7*

Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $n \geq 0$, there are at most $0.001n^2$ pairs of integers (x, y) for which $f(x + y) \neq f(x) + f(y)$ and $\max\{|x|, |y|\} \leq n$. Is it possible that for some integer $n \geq 0$, there are more than n integers a such that $f(a) \neq a \cdot f(1)$ and $|a| \leq n$?

David Yang

A8*

A8*

Let a, b, c be positive reals with $a^{2013} + b^{2013} + c^{2013} + abc = 4$. Prove that

$$\left(\sum a(a^2 + bc)\right) \left(\sum \left(\frac{a}{b} + \frac{b}{a}\right)\right) \geq \left(\sum \sqrt{(a+1)(a^3 + bc)}\right) \left(\sum \sqrt{a(a+1)(a+bc)}\right).$$

David Stoner

A9

A9

Let a, b, c be positive reals, and let $\sqrt[2013]{\frac{3}{a^{2013}+b^{2013}+c^{2013}}} = P$. Prove that

$$\prod_{\text{cyc}} \left(\frac{(2P + \frac{1}{2a+b})(2P + \frac{1}{a+2b})}{(2P + \frac{1}{a+b+c})^2} \right) \geq \prod_{\text{cyc}} \left(\frac{(P + \frac{1}{4a+b+c})(P + \frac{1}{3b+3c})}{(P + \frac{1}{3a+2b+c})(P + \frac{1}{3a+b+2c})} \right).$$

David Stoner

Combinatorics

C1

C1

Let $n \geq 2$ be a positive integer. The numbers $1, 2, \ldots, n^2$ are consecutively placed into squares of an $n \times n$, so the first row contains $1, 2, \ldots, n$ from left to right, the second row contains $n+1, n+2, \ldots, 2n$ from left to right, and so on. The *magic square value* of a grid is defined to be the number of rows, columns, and main diagonals whose elements have an average value of $\frac{n^2+1}{2}$. Show that the magic-square value of the grid stays constant under the following two operations: (1) a permutation of the rows; and (2) a permutation of the columns. (The operations can be used multiple times, and in any order.)

Ray Li

C2

C2

Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x+y)^4$ on the blackboard. Show that after $n-1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.

Calvin Deng

C3*

C3*

Let $a_1, a_2, \ldots, a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \leq i < j < k \leq 9$ for which $a_i + a_j + a_k \geq 3m$. What is the minimum possible value of A?

Ray Li

C4

C4

Let n be a positive integer. The numbers $\{1, 2, \ldots, n^2\}$ are placed in an $n \times n$ grid, each exactly once. The grid is said to be *Muirhead-able* if the sum of the entries in each column is the same, but for every $1 \leq i, k \leq n-1$, the sum of the first k entries in column i is at least the sum of the first k entries in column $i+1$. For which n can one construct a Muirhead-able array?

Evan Chen

C5

C5

There is a 2012×2012 grid with rows numbered $1, 2, \ldots 2012$ and columns numbered $1, 2, \ldots, 2012$, and we place some rectangular napkins on it such that the sides of the napkins all lie on grid lines. Each napkin has a positive integer thickness. (in micrometers!)

(a) Show that there exist 2012^2 unique integers $a_{i,j}$ where $i, j \in [1, 2012]$ such that for all $x, y \in [1, 2012]$, the sum

$$\sum_{i=1}^{x} \sum_{j=1}^{y} a_{i,j}$$

is equal to the sum of the thicknesses of all the napkins that cover the grid square in row x and column y.

(b) Show that if we use at most $500,000$ napkins, at least half of the $a_{i,j}$ will be 0.

Ray Li

C6

C6

A 4×4 grid has its 16 cells colored arbitrarily in three colors. A *swap* is an exchange between the colors of two cells. Prove or disprove that it always takes at most three swaps to produce a line of symmetry, regardless of the grid's initial coloring.

Matthew Babbitt

C7*

C7*

A $2^{2013} + 1$ by $2^{2013} + 1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer $n > 1$, there do not exist pairwise distinct black squares $s_1, s_2, \ldots, s_n$ such that s_i, s_{i+1} share an edge for $i = 1, 2, \ldots, n$ (here $s_{n+1} = s_1$). What is the maximum possible number of filled black squares?

David Yang

C8

C8

There are 20 people at a party. Each person holds some number of coins. Every minute, each person who has at least 19 coins simultaneously gives one coin to every other person at the party. (So, it is possible that A gives B a coin and B gives A a coin at the same time.) Suppose that this process continues indefinitely. That is, for any positive integer n, there exists a person who will give away coins during the nth minute. What is the smallest number of coins that could be at the party?

Ray Li

C9*

C9*

f_0 is the function from $\mathbb{Z}^2$ to $\{0,1\}$ such that $f_0(0,0)=1$ and $f_0(x,y)=0$ otherwise. For each $i>1$, let $f_i(x,y)$ be the remainder when

$$f_{i-1}(x,y)+\sum_{j=-1}^{1}\sum_{k=-1}^{1} f_{i-1}(x+j,y+k)$$

is divided by 2.

For each $i \geq 0$, let $a_i=\sum_{(x,y)\in\mathbb{Z}^2} f_i(x,y)$. Find a closed form for a_n (in terms of n).

Bobby Shen

C10*

C10*

Let $N \geq 2$ be a fixed positive integer. There are $2N$ people, numbered $1,2,\dots,2N$, participating in a tennis tournament. For any two positive integers i,j with $1 \leq i < j \leq 2N$, player i has a higher skill level than player j. Prior to the first round, the players are paired arbitrarily and each pair is assigned a unique court among N courts, numbered $1,2,\dots,N$.

During a round, each player plays against the other person assigned to his court (so that exactly one match takes place per court), and the player with higher skill wins the match (in other words, there are no upsets). Afterwards, for $i=2,3,\dots,N$, the winner of court i moves to court $i-1$ and the loser of court i stays on court i; however, the winner of court 1 stays on court 1 and the loser of court 1 moves to court N.

Find all positive integers M such that, regardless of the initial pairing, the players $2,3,\dots,N+1$ all change courts immediately after the Mth round.

Ray Li

Geometry

G1

G1

Let ABC be a triangle with incenter I. Let U, V and W be the intersections of the angle bisectors of angles A, B, and C with the incircle, so that V lies between B and I, and similarly with U and W. Let X, Y, and Z be the points of tangency of the incircle of triangle ABC with BC, AC, and AB, respectively. Let triangle UVW be the *David Yang triangle* of ABC and let XYZ be the *Scott Wu triangle* of ABC. Prove that the David Yang and Scott Wu triangles of a triangle are congruent if and only if ABC is equilateral.

Owen Goff

G2

G2

Let ABC be a scalene triangle with circumcircle Γ, and let D,E,F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z respectively. Prove that the perpendiculars from A, B, C to AX, BY, CZ respectively are concurrent.

Michael Kural

G3

G3

In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Allen Liu

G4*

G4*

Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

Evan Chen

G5

G5

Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter AB of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2, passing through O and A, touch ω_2 at F and G. Prove that $FGOB$ is cyclic.

Eric Chen

G6

G6

Let $ABCDEF$ be a non-degenerate cyclic hexagon with no two opposite sides parallel, and define $X = AB \cap DE$, $Y = BC \cap EF$, and $Z = CD \cap FA$. Prove that

$$\frac{XY}{XZ} = \frac{BE}{AD}\frac{\sin|\angle B - \angle E|}{\sin|\angle A - \angle D|}.$$

Victor Wang

G7*

G7*

Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to AC at C, and let O_2 be the center of the circle through E tangent to AB at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.

Michael Kural

G8

G8

Let ABC be a triangle, and let D, A, B, E be points on line AB, in that order, such that $AC = AD$ and $BE = BC$. Let ω_1, ω_2 be the circumcircles of $\triangle ABC$ and $\triangle CDE$, respectively, which meet at a point $F \neq C$. If the tangent to ω_2 at F cuts ω_1 again at G, and the foot of the altitude from G to FC is H, prove that $\angle AGH = \angle BGH$.

David Stoner

G9

G9

Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω whose diagonals meet at F. Lines AB and CD meet at E. Segment EF intersects ω at X. Lines BX and CD meet at M, and lines CX and AB meet at N. Prove that MN and BC concur with the tangent to ω at X.

Allen Liu

G10*

G10*

Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

David Stoner

G11

G11

Let $\triangle ABC$ be a nondegenerate isosceles triangle with $AB = AC$, and let D, E, F be the midpoints of BC, CA, AB respectively. BE intersects the circumcircle of $\triangle ABC$ again at G, and H is the midpoint of minor arc BC. $CF \cap DG = I, BI \cap AC = J$. Prove that $\angle BJH = \angle ADG$ if and only if $\angle BID = \angle GBC$.

David Stoner

G12*

G12*

Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

David Stoner

G13

G13

In $\triangle ABC$, $AB < AC$. D and P are the feet of the internal and external angle bisectors of $\angle BAC$, respectively. M is the midpoint of segment BC, and ω is the circumcircle of $\triangle APD$. Suppose Q is on the minor arc AD of ω such that MQ is tangent to ω. QB meets ω again at R, and the line through R perpendicular to BC meets PQ at S. Prove SD is tangent to the circumcircle of $\triangle QDM$.

Ray Li

G14

G14

Let O be a point (in the plane) and T be an infinite set of points such that $|P_1P_2| \leq 2012$ for every two distinct points $P_1, P_2 \in T$. Let $S(T)$ be the set of points Q in the plane satisfying $|QP| \leq 2013$ for at least one point $P \in T$.

Now let L be the set of lines containing exactly one point of $S(T)$. Call a line ℓ_0 passing through O *bad* if there does not exist a line $\ell \in L$ parallel to (or coinciding with) ℓ_0.

(a) Prove that L is nonempty.

(b) Prove that one can assign a line $\ell(i)$ to each positive integer i so that for every bad line ℓ_0 passing through O, there exists a positive integer n with $\ell(n) = \ell_0$.

David Yang

Number Theory

N1

N1

Find all ordered triples of non-negative integers (a, b, c) such that $a^2 + 2b + c$, $b^2 + 2c + a$, and $c^2 + 2a + b$ are all perfect squares.

Note: This problem was withdrawn from the ELMO Shortlist and used on ksun48's mock AIME.

Matthew Babbitt

N2*

N2*

For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \geq 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

Andre Arslan

N3

N3

Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct numbers of the form a^b, where $a \in \{3, 4, 5, 6\}$ and b is a positive integer.

Matthew Babbitt

N4

N4

Find all triples (a, b, c) of positive integers such that if n is not divisible by any integer less than 2013, then $n + c$ divides $a^n + b^n + n$.

Evan Chen

N5*

N5*

Let $m_1, m_2, \ldots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \ldots, A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, \ldots, m_i - 1\}$ for $i = 1, 2, \ldots, 2013$. Prove that there is a positive integer N such that

$$N \leq (2|A_1| + 1)(2|A_2| + 1) \cdots (2|A_{2013}| + 1)$$

and for each $i = 1, 2, \ldots, 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

Victor Wang

N6*

N6*

Find all positive integers m for which there exists a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n, and $f^{2013}(m) \neq m$. Here $f^k(n)$ denotes $\underbrace{f(f(\cdots f}_{k\ f\text{'s}}(n)\cdots))$.

Evan Chen

N7*

N7*

Let p be a prime satisfying $p^2 \mid 2^{p-1} - 1$, n be a positive integer, and $f(x) = \frac{(x-1)^{p^n} - (x^{p^n} - 1)}{p(x-1)}$. Find the largest positive integer N such that there exist polynomials $g, h \in \mathbb{Z}[x]$ and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Victor Wang

N8

N8

We define the *Fibonacci sequence* $\{F_n\}_{n \geq 0}$ by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$; we define the *Stirling number of the second kind* $S(n, k)$ as the number of ways to partition a set of $n \geq 1$ distinguishable elements into $k \geq 1$ indistinguishable nonempty subsets.

For every positive integer n, let $t_n = \sum_{k=1}^{n} S(n, k) F_k$. Let $p \geq 7$ be a prime. Prove that

$$t_{n+p^{2p}-1} \equiv t_n \pmod p$$

for all $n \geq 1$.

Victor Wang

Part II

Solutions

A1*

Find all triples (f, g, h) of injective functions from $\mathbb{R}$ to $\mathbb{R}$ satisfying

$$\begin{aligned} f(x+f(y)) &= g(x)+h(y) \\ g(x+g(y)) &= h(x)+f(y) \\ h(x+h(y)) &= f(x)+g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(x) \neq F(y)$ whenever $x \neq y$.)

Evan Chen

Answer. For all real numbers x, $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real number.

Solution 1. Let a, b, c denote the values $f(0)$, $g(0)$ and $h(0)$. Notice that by putting $y = 0$, we can get that $f(x + a) = g(x) + c$, etc. In particular, we can write

$$h(y) = f(y-c) + b$$

and

$$g(x) = h(x-b) + a = f(x-b-c) + a + b$$

So the first equation can be rewritten as

$$f(x+f(y)) = f(x-b-c) + f(y-c) + a + 2b.$$

At this point, we may set $x = y - c - f(y)$ and cancel the resulting equal terms to obtain

$$f(y - f(y) - (b+2c)) = -(a+2b).$$

Since f is injective, this implies that $y - f(y) - (b + 2c)$ is constant, so that $y - f(y)$ is constant. Thus, f is linear, and $f(y) = y + a$. Similarly, $g(x) = x + b$ and $h(x) = x + c$.

Finally, we just need to notice that upon placing $x = y = 0$ in all the equations, we get $2a = b+c$, $2b = c+a$ and $2c = a + b$, whence $a = b = c$.

So, the family of solutions is $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real. One can easily verify these solutions are valid. ■

This problem and solution were proposed by Evan Chen.

Remark. This is not a very hard problem. The basic idea is to view $f(0)$, $g(0)$ and $h(0)$ as constants, and write the first equation entirely in terms of $f(x)$, much like we would attempt to eliminate variables in a standard system of equations. At this point we still had two degrees of freedom, x and y, so it seems likely that the result would be easy to solve. Indeed, we simply select x in such a way that two of the terms cancel, and the rest is working out details.

Solution 2. First note that plugging $x = f(a), y = b; x = f(b), y = a$ into the first gives $g(f(a)) + h(b) = g(f(b)) + h(a) \implies g(f(a)) - h(a) = g(f(b)) - h(b)$. So $g(f(x)) = h(x) + a_1$ for a constant a_1. Similarly, $h(g(x)) = f(x) + a_2, f(h(x)) = g(x) + a_3$.

Now, we will show that $h(h(x)) - f(x)$ and $h(h(x)) - g(x)$ are both constant. For the second, just plug in $x = 0$ to the third equation. For the first, let $x = a_3, y = k$ in the original to get $g(f(h(k))) = h(a_3) + f(k)$. But $g(f(h(k))) = h(h(k)) + a_1$, so $h(h(k)) - f(k) = h(a_3) - a_1$ is constant as desired.

Now $f(x) - g(x)$ is constant, and by symmetry $g(x) - h(x)$ is also constant. Now let $g(x) = f(x) + p, h(x) = f(x) + q$. Then we get:

$$\begin{aligned} f(x+f(y)) &= f(x)+f(y)+p+q \\ f(x+f(y)+p) &= f(x)+f(y)+q-p \\ f(x+f(y)+q) &= f(x)+f(y)+p-q \end{aligned}$$

Now plugging in (x, y) and (y, x) into the first one gives $f(x+f(y)) = f(y+f(x)) \implies f(x)-x = f(y)-y$ from injectivity, $f(x) = x + c$. Plugging this in gives $2p = q, 2q = p, p + q = 0$ so $p = q = 0$ and $f(x) = x + c, g(x) = x + c, h(x) = x + c$ for a constant c are the only solutions. ■

This second solution was suggested by David Stoner.

A2

Prove that for all positive reals a, b, c,

$$\frac{1}{a+\frac{1}{b}+1}+\frac{1}{b+\frac{1}{c}+1}+\frac{1}{c+\frac{1}{a}+1} \geq \frac{3}{\sqrt[3]{abc}+\frac{1}{\sqrt[3]{abc}}+1}.$$

David Stoner

Solution. Let $a = N\frac{x}{y}$, $b = N\frac{y}{z}$ and $c = N\frac{z}{x}$. Then

$$\begin{aligned}
\sum_{\text{cyc}} \frac{1}{a+\frac{1}{b}+1} &= \sum_{\text{cyc}} \frac{y}{Nx+\frac{1}{N}z+y} \\
&= \sum_{\text{cyc}} \frac{y^2}{Nxy+\frac{1}{N}yz+y^2} \\
&\geq \frac{(x+y+z)^2}{(xy+yz+zx)\left(N+\frac{1}{N}\right)+x^2+y^2+z^2} \\
&= \frac{(x+y+z)^2}{(xy+yz+zx)\left(N+\frac{1}{N}-2\right)+(x+y+z)^2} \\
&= \frac{3}{3+\frac{3(xy+yz+zx)}{(x+y+z)^2}\left(N+\frac{1}{N}-2\right)} \\
&\geq \frac{3}{3+N+\frac{1}{N}-2} \\
&= \frac{3}{N+\frac{1}{N}+1} \\
&= \frac{3}{\sqrt[3]{abc}+\frac{1}{\sqrt[3]{abc}}+1}.
\end{aligned}$$

■

This problem and solution were proposed by David Stoner.

A3

Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $f(x) + f(y) = f(x + y)$ and $f(x^{2013}) = f(x)^{2013}$.

Calvin Deng

Answer. $f(x) = x$, $f(x) = -x$, and $f(x) \equiv 0$.

Solution. WLOG $f(1) \geq 0$ (since 2013 is odd); then $f(1) = f(1)^{2013} \implies f(1) \in \{0, 1\}$.

Hence for any reals x, y,

$$\begin{aligned}\sum_{k=0}^{2013} \binom{2013}{k} n^{2013-k} f(x)^k f(y)^{2013-k} &= [f(x) + nf(y)]^{2013} \\ &= f(x + ny)^{2013} \\ &= f((x + ny)^{2013}) \\ &= \sum_{k=0}^{2013} \binom{2013}{k} n^{2013-k} f(x^k y^{2013-k})\end{aligned}$$

for all positive integers n, so viewing this as a polynomial identity in n we get $f(x)^k f(y)^{2013-k} = f(x^k y^{2013-k})$ for $k = 0, 1, \ldots, 2013$.

If $f(1) = 1$, then $k = 2$ gives $f(x^2) = f(x)^2 \geq 0$ which is enough to get $f(x) = x$ for all x. Otherwise, if $f(1) = 0$, then $k = 1$ gives $f(x) = 0$ for all x. ■

This problem and solution were proposed by Calvin Deng.

A4

Positive reals a, b, and c obey $\frac{a^2+b^2+c^2}{ab+bc+ca} = \frac{ab+bc+ca+1}{2}$. Prove that

$$\sqrt{a^2+b^2+c^2} \le 1 + \frac{|a-b|+|b-c|+|c-a|}{2}.$$

Evan Chen

Solution 1. The given condition rearranges as $2(a^2+b^2+c^2)-(ab+bc+ca) = (ab+bc+ca)^2$. Homogenizing, this becomes:

$$|a-b|+|b-c|+|c-a| + \frac{2(ab+bc+ca)}{\sqrt{2(a^2+b^2+c^2)-(ab+bc+ca)}} \ge 2\sqrt{a^2+b^2+c^2}.$$

An application of Holder's inequality gives:

$$\begin{aligned} \text{LHS}^2 &\ge \frac{\left((a-b)^2+(b-c)^2+(c-a)^2+2(ab+bc+ca)\right)^3}{\left(\sum_{\text{cyc}}(a-b)^4 + 2(ab+bc+ca)\left(2(a^2+b^2+c^2)-(ab+bc+ca)\right)\right)^1} \\ &= \frac{(2a^2+2b^2+2c^2)^3}{2a^4+2b^4+2c^4+4a^2b^2+4a^2c^2+4c^2a^2} \\ &= \frac{8(a^2+b^2+c^2)^3}{2(a^2+b^2+c^2)^2} \\ &= 4(a^2+b^2+c^2) \end{aligned}$$

Upon taking square roots of both sides we are done. ■

This problem and solution were proposed by Evan Chen.

Solution 2. Let $x = ab+bc+ca$, so $1 \le \frac{a^2+b^2+c^2}{x} = \frac{x+1}{2}$ implies $x \ge 1$. If $\alpha = a-b$, $\beta = b-c$, $\gamma = c-a$, WLOG with $\alpha, \beta \ge 0$ (or equivalently $a \ge b \ge c$), then because $\alpha+\beta+\gamma = 0$, we have

$$2(\alpha^2+\alpha\beta+\beta^2) = \alpha^2+\beta^2+\gamma^2 = 2x\frac{x+1}{2} - 2x = x(x-1),$$

and we want to minimize $|\alpha|+|\beta|+|\gamma| = 2(\alpha+\beta)$. But $(\alpha+\beta)^2 \ge \alpha^2+\alpha\beta+\beta^2$ implies $\alpha+\beta \ge \sqrt{\frac{x(x-1)}{2}}$, with equality attained for some choice of (a,b,c) precisely when $\alpha\beta = 0$ and $(\alpha+\beta)\beta \le x$ (since $c \ge 0$). In particular, $\beta = 0$ works for any fixed $x \ge 1$, so the problem is equivalent to $\sqrt{\frac{x(x+1)}{2}} \le 1 + \sqrt{\frac{x(x-1)}{2}}$ for $x \ge 1$, which is easy after squaring both sides. ■

This second solution was suggested by Victor Wang.

A5*

Let a, b, c be positive reals satisfying $a+b+c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove that $a^a b^b c^c \ge 1$.

Evan Chen

Solution 1. By weighted AM-GM we have that

$$\begin{aligned} 1 &= \sum_{\text{cyc}} \left(\frac{\sqrt[7]{a}}{a+b+c} \right) \\ &= \sum_{\text{cyc}} \left(\frac{a}{a+b+c} \cdot \frac{1}{\sqrt[7]{a^6}} \right) \\ &\ge \left(\frac{1}{a^a b^b c^c} \right)^{\frac{6/7}{a+b+c}}. \end{aligned}$$

Rearranging yields $a^a b^b c^c \ge 1$. ■

This problem and solution were proposed by Evan Chen.

Remark. The problem generalizes easily to n variables, and exponents other than $\frac{1}{7}$. Specifically, if positive reals $x_1 + \cdots + x_n = x_1^r + \cdots + x_n^r$ for some real number $r \ne 1$, then $\prod_{i \ge 1} x_i^{x_i} \ge 1$ if and only if $r < 1$. When $r \le 0$, a Jensen solution is possible using only the inequality $a+b+c \ge 3$.

Solution 2. First we claim that $a, b, c < 5$. Assume the contrary, that $a \ge 5$. Let $f(x) = x - \sqrt[7]{x}$. Since $f'(x) > 0$ for $x \ge 5$, we know that $f(a) \ge 5 - \sqrt[7]{5} > 3$. But this means that WLOG $b - \sqrt[7]{b} < -1.5$, which is clearly false since $b - \sqrt[7]{b} \ge 0$ for $b \ge 1$, and $b - \sqrt[7]{b} \ge -\sqrt[7]{b} \ge -1$ for $0 < b < 1$. So indeed $a, b, c < 5$.

Now rewrite the inequality as

$$\sum a \ln a \ge 0 \Leftrightarrow \sum \left(\frac{a^{\frac{1}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right) (a^{\frac{6}{7}} \ln a) \ge 0.$$

Now note that if $g(x) = x^{\frac{6}{7}} \ln x$, then $g''(x) = \frac{35 - 6\ln x}{49 x^{\frac{8}{7}}} > 0$ for $x \in (0, 5)$. Therefore g is convex and we can use Jensen's Inequality to get

$$\sum \left(\frac{a^{\frac{1}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right) (a^{\frac{6}{7}} \ln a) \ge \left(\sum \frac{a^{\frac{8}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right)^{\frac{6}{7}} \ln \left(\sum \frac{a^{\frac{8}{7}}}{a^{\frac{1}{7}} + b^{\frac{1}{7}} + c^{\frac{1}{7}}} \right).$$

Since $\sum a = \sum a^{\frac{1}{7}}$, it suffices to show that $\sum a^{\frac{8}{7}} \ge \sum a$. But by weighted AM-GM we have

$$6a^{\frac{8}{7}} + a^{\frac{1}{7}} \ge 7a \implies a^{\frac{8}{7}} - a \ge \frac{1}{6}(a - \sqrt[7]{a}).$$

Adding up the analogous inequalities for b, c gives the desired result. ■

This second solution was suggested by David Stoner.

Solution 3. Here we unify the two solutions above.

It's well-known that weighted AM-GM follows from (and in fact, is equivalent to) the convexity of e^x (or equivalently, the concavity of $\ln x$), as $\sum w_i e^{x_i} \ge e^{\sum w_i x_i}$ for reals x_i and nonnegative weights w_i summing to 1. However, it also follows from the convexity of $y \ln y$ (or equivalently, the concavity of ye^y) for $y > 0$. Indeed, letting $y_i = e^{x_i} > 0$, and taking logs, weighted AM-GM becomes

$$\sum w_i y_i \cdot \frac{1}{y_i} \log \frac{1}{y_i} \ge \left(\sum w_i y_i\right) \frac{\sum w_i y_i \cdot \frac{1}{y_i}}{\sum w_i y_i} \log \frac{\sum w_i y_i \cdot \frac{1}{y_i}}{\sum w_i y_i},$$

which is clear.

To find Evan's solution, we can use the concavity of $\ln x$ to get $\sum a \ln a^{-s} \le (\sum a) \ln \sum \frac{a \cdot a^{-s}}{\sum a} = 0$. (Here we take $s = 6/7 > 0$.)

For a cleaner version of David's solution, we can use the convexity of $x \ln x$ to get

$$\sum a \ln a^s = \sum a^{1-s} \cdot a^s \ln a^s \ge (\sum a^{1-s}) \frac{\sum a^{1-s} \cdot a^s}{\sum a^{1-s}} \ln \frac{\sum a^{1-s} \cdot a^s}{\sum a^{1-s}} = 0$$

(where we again take $s = 6/7 > 0$).

Both are pretty intuitive (but certainly not obvious) solutions once one realizes direct Jensen goes in the wrong direction. In particular, $s = 1$ doesn't work since we have $a+b+c \le 3$ from the power mean inequality. ■

This third solution was suggested by Victor Wang.

Solution 4. From $e^t \ge 1 + t$ for $t = \ln x^{-\frac{6}{7}}$, we find $\frac{6}{7} \ln x \ge 1 - x^{-\frac{6}{7}}$. Thus

$$\frac{6}{7} \sum a \ln a \ge \sum a - a^{\frac{1}{7}} = 0,$$

as desired. ■

This fourth solution was suggested by chronodecay.

Remark. Polya once dreamed a similar proof of n-variable AM-GM: $x \ge 1 + \ln x$ for positive x, so $\sum x_i \ge n + \ln \prod x_i$. This establishes AM-GM when $\prod x_i = 1$; the rest follows by homogenizing.

A6

Let a, b, c be positive reals such that $a+b+c=3$. Prove that

$$18\sum_{\text{cyc}}\frac{1}{(3-c)(4-c)}+2(ab+bc+ca)\geq 15.$$

David Stoner

Solution. Since $0 \leq a, b, c \leq 3$ we have

$$\frac{1}{(3-c)(4-c)} \geq \frac{2c^2+c+3}{36} \iff c(c-1)^2(2c-9) \leq 0.$$

Then

$$2(ab+bc+ca)+18\sum_{\text{cyc}}\left(\frac{2c^2+c+3}{36}\right)=(a+b+c)^2+\frac{a+b+c+9}{2}=15.$$

■

This problem was proposed by David Stoner. This solution was given by Evan Chen.

A7*

Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $n \ge 0$, there are at most $0.001n^2$ pairs of integers (x, y) for which $f(x+y) \ne f(x) + f(y)$ and $\max\{|x|, |y|\} \le n$. Is it possible that for some integer $n \ge 0$, there are more than n integers a such that $f(a) \ne a \cdot f(1)$ and $|a| \le n$?

David Yang

Answer. No.

Solution. Call an integer conformist if $f(n) = n \cdot f(1)$. Call a pair (x, y) good if $f(x+y) = f(x) + f(y)$ and bad otherwise. Let $h(n)$ denote the number of conformist integers with absolute value at most n.

Let $\epsilon = 0.001$, S be the set of conformist integers, $T = \mathbb{Z} \setminus S$ be the set of non-conformist integers, and $X_n = [-n, n] \cap X$ for sets X and positive integers n (so $|S_n| = h(n)$); clearly $|T_n| = 2n + 1 - h(n)$.

First we can easily get $h(n) = 2n + 1$ ($-n$ to n are all conformist) for $n \le 10$.

Lemma 1. *Suppose a, b are positive integers such that $h(a) > a$ and $b \le 2h(a) - 2a - 1$. Then $h(b) \ge 2b(1 - \sqrt{\epsilon}) - 1$.*

Proof. For any integer t, we have

$$\begin{aligned}
|S_a \cap (t - S_a)| &= |S_a| + |t - S_a| - |S_a \cup (t - S_a)| \\
&\ge 2h(a) - (\max(S_a \cup (t - S_a)) - \min(S_a \cup (t - S_a)) + 1) \\
&\ge 2h(a) - (\max(a, t+a) - \min(-a, t-a) + 1) \\
&= 2h(a) - (|t| + 2a + 1) \\
&\ge b - |t|.
\end{aligned}$$

But (x, y) is bad whenever $x, y \in S$ yet $x + y \in T$, so summing over all $t \in T_b$ (assuming $|T_b| \ge 2$) yields

$$\begin{aligned}
\epsilon b^2 \ge g(b) &\ge \sum_{t \in T_b} |S_a \cap (t - S_a)| \\
&\ge \sum_{t \in T_b} (b - |t|) \ge \sum_{k=0}^{\lfloor |T_b|/2 \rfloor - 1} k + \sum_{k=0}^{\lceil |T_b|/2 \rceil - 1} k \ge 2\frac{1}{2}(|T_b|/2)(|T_b|/2 - 1),
\end{aligned}$$

where we use $\lfloor r/2 \rfloor + \lceil r/2 \rceil = r$ (for $r \in \mathbb{Z}$) and the convexity of $\frac{1}{2}x(x-1)$. We conclude that $|T_b| \le 2 + 2b\sqrt{\epsilon}$ (which obviously remains true without the assumption $|T_b| \ge 2$) and $h(b) = 2b + 1 - |T_b| \ge 2b(1 - \sqrt{\epsilon}) - 1$. □

Now we prove by induction on n that $h(n) \ge 2n(1 - \sqrt{\epsilon}) - 1$ for all $n \ge 10$, where the base case is clear. If we assume the result for $n - 1$ ($n > 10$), then in view of the lemma, it suffices to show that $2h(n-1) - 2(n-1) - 1 \ge n$, or equivalently, $2h(n-1) \ge 3n - 1$. But

$$2h(n-1) \ge 4(n-1)(1 - \sqrt{\epsilon}) - 2 \ge 3n - 1,$$

so we're done. (The second inequality is equivalent to $n(1 - 4\sqrt{\epsilon}) \ge 5 - 4\sqrt{\epsilon}$; $n \ge 11$ reduces this to $6 \ge 40\sqrt{\epsilon} = 40\sqrt{0.001} = 4\sqrt{0.1}$, which is obvious.) ■

This problem and solution were proposed by David Yang.

A8*

Let a, b, c be positive reals with $a^{2013} + b^{2013} + c^{2013} + abc = 4$. Prove that

$$\left(\sum a(a^2 + bc)\right)\left(\sum \left(\frac{a}{b} + \frac{b}{a}\right)\right) \geq \left(\sum \sqrt{(a+1)(a^3+bc)}\right)\left(\sum \sqrt{a(a+1)(a+bc)}\right).$$

David Stoner

Solution.

Lemma 1. *Let x, y, z be positive reals, not all strictly on the same side of 1. Then $\sum \frac{x}{y} + \frac{y}{x} \geq \sum x + \frac{1}{x}$.*

Proof. WLOG $(x-1)(y-1) \leq 0$; then

$$(x + y + z - 1)(x^{-1} + y^{-1} + z^{-1} - 1) \geq (xy + z)(x^{-1}y^{-1} + z) \geq 4$$

by Cauchy.

Alternatively, if $x, y \geq 1 \geq z$, one may smooth z up to 1 (e.g. by differentiating with respect to z and observing that $x^{-1} + y^{-1} - 1 \leq x + y - 1$) to reduce the inequality to $\frac{x}{y} + \frac{y}{x} \geq 2$. $\square$

Let $s_i = a^i + b^i + c^i$ and $p = abc$. The key is to Cauchy out s_3's from the RHS and use the lemma (in the form $s_1 s_{-1} - 3 \geq s_1 + s_{-1}$) on the LHS to reduce the problem to

$$(s_1 + s_{-1})^2 (s_3 + 3p)^2 \geq (3 + s_1)(3 + s_{-1})(s_3 + p s_{-1})(s_3 + p s_1).$$

By AM-GM on the RHS, it suffices to prove

$$\frac{\frac{s_1+s_{-1}}{2} + \frac{s_1+s_{-1}}{2}}{\frac{s_1+s_{-1}}{2} + 3} \geq \frac{s_3 + p\frac{s_1+s_{-1}}{2}}{s_3 + 3p},$$

or equivalently, since $\frac{s_1+s_{-1}}{2} \geq 3$, that $\frac{s_3}{p} \geq \frac{s_1+s_{-1}}{2}$. By the lemma, this boils down to $2\sum_{\text{cyc}} a^3 \geq \sum_{textcyc} a(b^2 + c^2)$, which is obvious. ■

This problem and solution were proposed by David Stoner.

Remark. The condition $a^{2013} + b^{2013} + c^{2013} + abc = 4$ can be replaced with anything that guarantees a, b, c are not all on the same side of 1. One can also propose the following more direct application of the lemma instead: "Let a, b, c be positive reals with $a^{2013} + b^{2013} + c^{2013} + abc = 4$. Prove that

$$\sum \left(\left(\frac{a}{b}\right)^{2012} + \left(\frac{b}{a}\right)^{2012}\right) \geq \sum \left(a^{2011} + \frac{1}{a^{2011}}\right).$$

" This is perhaps more motivated, but also significantly easier. Note that if one replaces the exponents in the inequality with something like 2013 and 2012, then one may use the PQR method to reduce the problem to the case when two of a, b, c are equal. Alternatively, if one changes the condition to $a^{2013}b + b^{2013}c + c^{2013}a + abc = 4$, then it's perfectly fine for the first exponent to be at least 2013 and the second to be at most 2013; however, this makes the lemma much more transparent.

A9

Let a, b, c be positive reals, and let $\sqrt[2013]{\frac{3}{a^{2013}+b^{2013}+c^{2013}}} = P$. Prove that

$$\prod_{\text{cyc}}\left(\frac{(2P+\frac{1}{2a+b})(2P+\frac{1}{a+2b})}{(2P+\frac{1}{a+b+c})^2}\right) \geq \prod_{\text{cyc}}\left(\frac{(P+\frac{1}{4a+b+c})(P+\frac{1}{3b+3c})}{(P+\frac{1}{3a+2b+c})(P+\frac{1}{3a+b+2c})}\right).$$

David Stoner

Solution. WLOG $P = 1$; we prove that any positive a, b, c (even those without $\sum a^{2013} = 3$) satisfy the inequality. The key is that $f(x) = \log(1 + x^{-1}) = \log(1 + x) - \log(x)$ is convex, since $f''(x) = -(1 + x)^{-2} + x^{-2} > 0$ for all x.

By Jensen's inequality, we have

$$\begin{aligned}
\frac{1}{2}f(2(2a+b)) + \frac{1}{2}f(2(2a+c)) &\geq f(4a+b+c) \\
\frac{1}{2}f(2(2b+c)) + \frac{1}{2}f(2(2c+b)) &\geq f(3b+3c) \\
-2f(2(a+b+c)) &\geq -f(3a+2b+c) - f(3c+2b+a).
\end{aligned}$$

Exponentiating and multiplying everything once (cyclically), we get the desired inequality. ■

This problem and solution were proposed by David Stoner.

C1

Let $n \geq 2$ be a positive integer. The numbers $1, 2, \ldots, n^2$ are consecutively placed into squares of an $n \times n$, so the first row contains $1, 2, \ldots, n$ from left to right, the second row contains $n+1, n+2, \ldots, 2n$ from left to right, and so on. The *magic square value* of a grid is defined to be the number of rows, columns, and main diagonals whose elements have an average value of $\frac{n^2+1}{2}$. Show that the magic-square value of the grid stays constant under the following two operations: (1) a permutation of the rows; and (2) a permutation of the columns. (The operations can be used multiple times, and in any order.)

Ray Li

Solution 1. The set of row sums and column sums is clearly preserved under operations (1) and (2), so we just have to consider the main diagonals. In configuration A, let a_{ij} denote the number in the ith row and jth column; then whenever $i \neq j$ and $k \neq l$, we have $a_{ij} + a_{kl} = a_{il} + a_{kj}$. But this property is invariant as well, so the main diagonal sums remain constant under the operations, and we're done. ■

This problem and solution were proposed by Ray Li.

Solution 2. We present a proof without words for the case $n = 4$, which easily generalizes to other values of n.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 \\ 8 & 8 & 8 & 8 \\ 12 & 12 & 12 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

■

This second solution was suggested by Evan Chen.

C2

Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x+y)^4$ on the blackboard. Show that after $n-1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.

Calvin Deng

Solution. We proceed by strong induction on n. For $n=1$ this is obvious; now assuming the result up to $n-1$ for some $n>1$, consider the two numbers on the blackboard after $n-2$ minutes. They must have been created "independently," where the first took $a-1$ minutes and the second took $b-1$ minutes for two positive integers a, b $(a+b=n)$. But 2^x is convex, so

$$2^{\frac{4a^2-4}{3}} + 2^{\frac{4b^2-4}{3}} \ge 2 \cdot 2^{\frac{2(a^2+b^2)-4}{3}} \ge 2 \cdot 2^{\frac{(a+b)^2-4}{3}} = 2^{\frac{(a+b)^2-1}{3}} = 2^{\frac{n^2-1}{3}},$$

completing the induction. ■

This problem and solution were proposed by Calvin Deng.

C3*

Let $a_1, a_2, \ldots, a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \le i < j < k \le 9$ for which $a_i + a_j + a_k \ge 3m$. What is the minimum possible value of A?

Ray Li

Answer. $A \ge 28$.

Solution 1. Call a 3-set *good* iff it has average at least m, and let S be the family of good sets.

The equality case $A = 28$ can be achieved when $a_1 = \cdots = a_8 = 0$ and $a_9 = 1$. Here $m = \frac{1}{9}$, and the good sets are precisely those containing a_9. This gives a total of $\binom{8}{2} = 28$.

To prove the lower bound, suppose we have exactly N good 3-sets, and let $p = \frac{N}{\binom{9}{3}}$ denote the probability that a randomly chosen 3-set is good. Now, consider a random permutation π of $\{1, 2, \ldots, 9\}$. Then the corresponding partition $\bigcup_{i=0}^{2} \{\pi(3i+1), \pi(3i+2), \pi(3i+3)\}$ has at least 1 good 3-set, so by the linearity of expectation,

$$\begin{aligned} 1 &\le \mathbb{E}\left[\sum_{i=0}^{2} [\{\pi(3i+1), \pi(3i+2), \pi(3i+3)\} \in S]\right] \\ &= \sum_{i=0}^{2} [\mathbb{E}[\{\pi(3i+1), \pi(3i+2), \pi(3i+3)\} \in S]] \\ &= \sum_{i=0}^{2} 1 \cdot p = 3p. \end{aligned}$$

Hence $N = p\binom{9}{3} \ge \frac{1}{3}\binom{9}{3} = 28$, establishing the lower bound. ■

This problem and solution were proposed by Ray Li.

Remark. One can use double-counting rather than expectation to prove $N \ge 28$. In any case, this method generalizes effortlessly to larger numbers.

Solution 2. Proceed as above to get an upper bound of 28.

On the other hand, we will show that we can partition the $\binom{9}{3} = 84$ 3-sets into 28 groups of 3, such that in any group, the elements $a_1, a_2, \cdots, a_9$ all appear. This will imply the conclusion, since if $A < 28$, then there are at least 57 sets with average at most m, but by pigeonhole three of them must be in such a group, which is clearly impossible.

Consider a 3-set and the following array:

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{array}$$

Consider a set $|S| = 3$. We obtain the other two 3-sets in the group as follows:

- If S contains one element in each column, then shift the elements down cyclically mod 3.
- If S contains one element in each row, then shift the elements right cyclically mod 3. Note that the result coincides with the previous case if both conditions are satisfied.
- Otherwise, the elements of S are "constrained" in a 2×2 box, possibly shifted diagonally. In this case, we get an L-tromino. Then shift diagonally in the direction the L-tromino points in.

One can verify that this algorithm creates such a partition, so we conclude that $A \ge 28$. ■

This second solution was suggested by Lewis Chen.

C4

Let n be a positive integer. The numbers $\{1, 2, \dots, n^2\}$ are placed in an $n \times n$ grid, each exactly once. The grid is said to be *Muirhead-able* if the sum of the entries in each column is the same, but for every $1 \le i, k \le n-1$, the sum of the first k entries in column i is at least the sum of the first k entries in column $i+1$. For which n can one construct a Muirhead-able array?

Evan Chen

Answer. All $n \neq 3$.

Solution. It's easy to prove $n = 3$ doesn't work since the top row must be 9,8,7 (each column sums to 15) and the first column is either 9,5,1 or 9,4,2.

A construction for even n is not hard to realize:

$$\begin{array}{cccc} n^2 & n^2-1 & \cdots & n^2-n+1 \\ n^2-n & n^2-n-1 & \cdots & n^2-2n+1 \\ \vdots & \vdots & \ddots & \vdots \\ n^2-(\frac{n}{2}-1)n & n^2-(\frac{n}{2}-1)n & \cdots & n^2-(\frac{n}{2})n+1 \\ n^2-(\frac{n}{2}+1)n+1 & n^2-(\frac{n}{2}+1)n+2 & \cdots & n^2-(\frac{n}{2})n \\ \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & \cdots & 2n \\ 1 & 2 & \cdots & n \end{array}$$

And we can just alter the even construction a bit for $n \ge 5$ odd; I'll just write it out for $n = 7$ since it generalizes easily: we modify

$$7\begin{pmatrix} 6&6&6&6&6&6&6\\ 5&5&5&5&5&5&5\\ 4&4&4&4&4&4&4\\ 3&3&3&3&3&3&3\\ 2&2&2&2&2&2&2\\ 1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 7&6&5&4&3&2&1\\ 7&6&5&4&3&2&1\\ 7&6&5&4&3&2&1\\ 4&4&4&4&4&4&4\\ 1&2&3&4&5&6&7\\ 1&2&3&4&5&6&7\\ 1&2&3&4&5&6&7 \end{pmatrix}$$

to get

$$7\begin{pmatrix} 6&6&6&6&6&6&6\\ 5&5&5&5&5&5&5\\ 4&4&4&4&4&4&4\\ 3&3&3&3&3&3&3\\ 2&2&2&2&2&2&2\\ 1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 7&6&5&4&3&2&1\\ 7&6&5&4&3&2&1\\ 5&6&7&1&2&3&4\\ 6&4&2&7&5&3&1\\ 1&2&3&4&5&6&7\\ 1&2&3&4&5&6&7\\ 1&2&3&4&5&6&7 \end{pmatrix}.$$

If we verify the majorization condition for the original one (without regard to distinctness) then we only have to check it in the new one for $k = 3 = \frac{n-1}{2}$ and $i = 1, 2, 4, 5, 6$ (in particular, we can skip $i = 3 = \frac{n-1}{2}$). ■

This problem and solution were proposed by Evan Chen.

C5

There is a 2012×2012 grid with rows numbered $1, 2, \ldots 2012$ and columns numbered $1, 2, \ldots, 2012$, and we place some rectangular napkins on it such that the sides of the napkins all lie on grid lines. Each napkin has a positive integer thickness. (in micrometers!)

(a) Show that there exist 2012^2 unique integers $a_{i,j}$ where $i, j \in [1, 2012]$ such that for all $x, y \in [1, 2012]$, the sum

$$\sum_{i=1}^{x}\sum_{j=1}^{y} a_{i,j}$$

is equal to the sum of the thicknesses of all the napkins that cover the grid square in row x and column y.

(b) Show that if we use at most $500,000$ napkins, at least half of the $a_{i,j}$ will be 0.

Ray Li

Solution 1. (a) Let $t_{i,j}$ be the total thickness at square (i, j) (row i, column j). For convenience, set $t_{i,j} = 0$ outside the boundary (i.e. if one of i, j is less than 1 or greater than 2012). By induction on $i + j \geq 2$ (over $i, j \in [2012]$), it's easy to see that the $a_{i,j}$ are uniquely defined as $t_{i,j} + t_{i-1,j-1} - t_{i-1,j} - t_{i,j-1}$ (and that this solution also works).

(b) One can easily check that $a_{i,j} = 0$ if no napkin corners lie at intersection of the ith vertical grid line (from the top) and the jth horizontal grid line (from the left). Indeed, if we color squares $(i-1, j-1)$ and (i, j) red, $(i-1, j)$ and $(i, j-1)$ blue, then if there are no such napkin corners, every napkin must hit an equal number of red and blue squares and thus contribute zero to the sum $t_{i,j} + t_{i-1,j-1} - t_{i-1,j} - t_{i,j-1}$. On the other hand, there are at most $4 \cdot 500000$ corners, and $2012^2 > 4000000 = 2(4 \cdot 500000)$ pairs $(i, j) \in [2012]^2$, so we're done. ■

Solution 2. Throughout this proof, rows go from bottom to top, and columns go from left to right.

Suppose we add a napkin with thickness x.

This affects the a-value only at the four corner points of the napkin. Corners are defined to be the bolded points in the following diagram. If the napkin shares an edge with the top boundary or the right boundary, some corners may not be considered for a-value valuation, which is even better for part (b). [Alternatively, for purists out there, define a-values for $i, j = 2013$.]

$$\begin{array}{|ccccc|}
\hline
0 & 0 & 0 & 0 & 0 \\
\mathbf{-1} & 0 & 0 & \mathbf{1} & 0 \\
\boxed{0} & \boxed{0} & \boxed{0} & 0 & 0 \\
\boxed{\mathbf{1}} & \boxed{0} & \boxed{0} & \mathbf{-1} & 0 \\
\hline
\end{array}$$

Boxes represent squares covered by napkins.

Specifically, the a-values of the bottom-left and top-right corners increment by x, and the bottom-right and top-left corners decrement by x. (Easy verification with diagram. This should be somewhat intuitive as well: think PIE.)

Notably, the process of adding a napkin is additive and reversible. Hence no matter how many napkins are placed on the table, we can just add a-values together.

So a-values exist, and can be consistently labeled. Furthermore, each napkin modifies at most 4 a-values, so with 500,000 napkins at most 2 million a-values are modified, which is less than half of 2012^2. ■

This problem and its solutions were proposed by Ray Li.

C6

A 4×4 grid has its 16 cells colored arbitrarily in three colors. A *swap* is an exchange between the colors of two cells. Prove or disprove that it always takes at most three swaps to produce a line of symmetry, regardless of the grid's initial coloring.

Matthew Babbitt

Answer. No.

Solution. We provide the following counterexample, in the colors red, white, and green:

$$\begin{matrix} W & W & G & W \\ R & W & W & R \\ R & R & R & G \\ R & W & W & G \end{matrix}$$

Suppose for contradiction that we can get a line of symmetry in 3 or less swaps. Clearly the symmetry must be over a diagonal.

If it is upper left to lower right, then there are 6 pairs of squares that reflect to each other over this diagonal and 4 squares on the diagonal. None of the 6 pairs are matched, so at least one square in each must be part of a swap. Also, there must be an even number of red squares on the diagonal, so one of the diagonal squares must be swapped, for a total of $7 > 3 \cdot 2$. This requires more than 3 swaps. The other diagonal works similarly. ■

This problem was proposed by Matthew Babbitt. This solution was given by Bobby Shen.

Remark. To construct counterexamples, we first put an odd number of one color (so symmetry must be over a diagonal), make no existing matches over the diagonal, and require that one or more of the diagonal squares be part of a swap.

C7*

A $2^{2013}+1$ by $2^{2013}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer $n>1$, there do not exist pairwise distinct black squares $s_1, s_2, \ldots, s_n$ such that s_i, s_{i+1} share an edge for $i=1,2,\ldots,n$ (here $s_{n+1}=s_1$). What is the maximum possible number of filled black squares?

David Yang

Answer. If $n=2^m+1$ is the dimension of the grid, the answer is $\frac{2}{3}n(n+1)-1$. In this particular instance, $m=2013$ and $n=2^{2013}+1$.

Solution. Let $n=2^m+1$. Double-counting square edges yields $3v+1 \le 4v-e \le 2n(n+1)$, so because $n \not\equiv 1 \pmod 3$, $v \le 2n(n+1)/3-1$. Observe that if $3 \nmid n-1$, equality is achieved iff (a) the graph formed by black squares is a connected forest (i.e. a tree) and (b) all but two square edges belong to at least one black square.

We prove by induction on $m \ge 1$ that equality can in fact be achieved. For $m=1$, take an "H-shape" (so if we set the center at $(0,0)$ in the coordinate plane, everything but $(0,\pm 1)$ is black); call this G_1. To go from G_m to G_{m+1}, fill in $(2x,2y)$ in G_{m+1} iff (x,y) is filled in G_m, and fill in (x,y) with x,y not both even iff $x+y$ is odd (so iff one of x,y is odd and the other is even). Each "newly-created" white square has both coordinates odd, and thus borders 4 (newly-created) black squares. In particular, there are no new white squares on the border (we only have the original two from G_1). Furthermore, no two white squares share an edge in G_{m+1}, since no square with odd coordinate sum is white. Thus G_{m+1} satisfies (b). To check that (a) holds, first we show that $(2x_1,2y_1)$ and $(2x_2,2y_2)$ are connected in G_{m+1} iff (x_1,y_1) and (x_2,y_2) are black squares (and thus connected) in G_m (the new black squares are essentially just "bridges"). Indeed, every path in G_{m+1} alternates between coordinates with odd and even sum, or equivalently, new and old black squares. But two black squares (x_1,y_1) and (x_2,y_2) are adjacent in G_m iff (x_1+x_2, y_1+y_2) is black and adjacent to $(2x_1,2y_1)$ and $(2x_2,2y_2)$ in G_{m+1}, whence the claim readily follows. The rest is clear: the set of old black squares must remain connected in G_{m+1}, and all new black squares (including those on the boundary) border at least one (old) black square (or else G_m would not satisfy (b)), so G_{m+1} is fully connected. On the other hand, G_{m+1} cannot have any cycles, or else we would get a cycle in G_m by removing the new black squares from a cycle in G_{m+1} (as every other square in a cycle would have to have odd coordinate sum). ■

This problem and solution were proposed by David Yang.

C8

There are 20 people at a party. Each person holds some number of coins. Every minute, each person who has at least 19 coins simultaneously gives one coin to every other person at the party. (So, it is possible that A gives B a coin and B gives A a coin at the same time.) Suppose that this process continues indefinitely. That is, for any positive integer n, there exists a person who will give away coins during the nth minute. What is the smallest number of coins that could be at the party?

Ray Li

Solution 1. Call a person giving his 19 coins away a *charity*. For any finite, fixed number of coins there are finitely many states, which implies that the states must cycle infinitely. Hence by doing individual charities one by one, there is a way to make it cycle infinitely (just take the charities that would normally happen at the same time and do them one by one all together before moving on). So this means we can reverse the charities and have it go on infinitely the other way, so call an inverse charity a *theft*. But after $k \le 20$ thefts, the number of coins among the people who have stolen at least once is at least $19 + 18 + \cdots + (20 - k)$ since the kth thief steals at most $k - 1$ coins from people who were already thieves but gains 19. So then we're done since for $k = 20$ this is 190. Of course, one construction is just when person j has $j - 1$ coins. ■

This first solution was suggested by Mark Sellke.

Solution 2. Like above, do the charities in arbitrary order among the ones that are "together." Assume there are at most 189 coins. Then the sum of squares of coins each guy has decreases each time, since if one guy loses 19 coins then the sums of squares decreases by at least 361, while giving 1 coin to everyone else increases it by $19+2$(number of coins they had before), and the number of coins they had before is less than 171 since the giving guy had 19 already, and so the sum of squares decreases since $361 > 19 + 2 \cdot 170$. ■

This second solution was suggested by Mark Sellke.

Remark. Compare with this problem in 102 Combinatorial Problems (paraphrased, St. Petersburg 1988): "119 residents live in a place with 120 apartments. Every day, in each apartment with at least 15 people, all the people move out into pairwise distinct apartments. Must this process terminate?"

This problem was proposed by Ray Li.

C9*

f_0 is the function from $\mathbb{Z}^2$ to $\{0,1\}$ such that $f_0(0,0)=1$ and $f_0(x,y)=0$ otherwise. For each $i>1$, let $f_i(x,y)$ be the remainder when

$$f_{i-1}(x,y)+\sum_{j=-1}^{1}\sum_{k=-1}^{1} f_{i-1}(x+j,y+k)$$

is divided by 2.

For each $i\ge 0$, let $a_i=\sum_{(x,y)\in\mathbb{Z}^2} f_i(x,y)$. Find a closed form for a_n (in terms of n).

Bobby Shen

Solution. a_i is simply the number of odd coefficients of $A_i(x,y)=A(x,y)^i$, where $A(x,y)=(x^2+x+1)(y^2+y+1)-xy$. Throughout this proof, we work in $\mathbb{F}_2$ and repeatedly make use of the Frobenius endomorphism in the form $A_{2^k m}(x,y)=A_m(x,y)^{2^k}=A_m(x^{2^k},y^{2^k})$ (*). We advise the reader to try the following simpler problem before proceeding: "Find (a recursion for) the number of odd coefficients of $(x^2+x+1)^{2^n-1}$."

First suppose n is not of the form 2^m-1, and has $i\ge 0$ ones before its first zero from the right. By direct exponent analysis (after using (*)), we obtain $a_n=a_{\frac{n-(2^i-1)}{2}}a_{2^i-1}$. Applying this fact repeatedly, we find that $a_n=a_{2^{\ell_1}-1}\cdots a_{2^{\ell_r}-1}$, where $\ell_1,\ell_2,\dots,\ell_r$ are the lengths of the r consecutive strings of ones in the binary representation of n. (When $n=2^m-1$, this is trivially true. When $n=0$, we take $r=0$ and a_0 to be the empty product 1, by convention.)

We now restrict our attention to the case $n=2^m-1$. The key is to look at the exponents of x and y modulo 2—in particular, $A_{2n}(x,y)=A_n(x^2,y^2)$ has only "$(0,0) \pmod 2$" terms for $i\ge 1$. This will allow us to find a recursion.

For convenience, let $U[B(x,y)]$ be the number of odd coefficients of $B(x,y)$, so $U[A_{2^n-1}(x,y)]=a_{2^n-1}$. Observe that

$$\begin{aligned}
A(x,y)&=(x^2+x+1)(y^2+y+1)-xy=(x^2+1)(y^2+1)+(x^2+1)y+x(y^2+1)\\
(x+1)A(x,y)&=(y^2+1)+(x^2+1)y+x^3(y^2+1)+(x^3+x)y\\
(x+1)(y+1)A(x,y)&=(x^2y^2+1)+(x^2y+y^3)+(x^3+xy^2)+(x^3y^3+xy)\\
(x+y)A(x,y)&=(x^2+y^2)+(x^2+1)(y^3+y)+(x^3+x)(y^2+1)+(x^3y+xy^3).
\end{aligned}$$

Hence for $n\ge 1$, we have (using (*) again)

$$\begin{aligned}
U[A_{2^n-1}(x,y)]&=U[A(x,y)A_{2^{n-1}-1}(x^2,y^2)]\\
&=U[(x+1)(y+1)A_{2^{n-1}-1}(x,y)]+U[(y+1)A_{2^{n-1}-1}(x,y)]+U[(x+1)A_{2^{n-1}-1}(x,y)]\\
&=U[(x+1)(y+1)A_{2^{n-1}-1}(x,y)]+2U[(x+1)A_{2^{n-1}-1}(x,y)].
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
U[(x+1)A_{2^n-1}]&=2U[(y+1)A_{2^{n-1}-1}]+2U[(x+1)A_{2^{n-1}-1}]=4U[(x+1)A_{2^{n-1}-1}]\\
U[(x+1)(y+1)A_{2^n-1}]&=2U[(xy+1)A_{2^{n-1}-1}]+2U[(x+y)A_{2^{n-1}-1}]=4U[(x+y)A_{2^{n-1}-1}]\\
U[(x+y)A_{2^n-1}]&=2U[(x+1)(y+1)A_{2^{n-1}-1}]+2U[(x+y)A_{2^{n-1}-1}].
\end{aligned}$$

Here we use the symmetry between x and y, and the identity $(xy+1)=y(x+y^{-1})$.) It immediately follows that

$$\begin{aligned}
U[(x+1)(y+1)A_{2^{n+1}-1}]&=4U[(x+y)A_{2^n-1}]\\
&=8U[(x+1)(y+1)A_{2^{n-1}-1}]+8\frac{U[(x+1)(y+1)A_{2^n-1}]}{4}
\end{aligned}$$

for all $n \geq 1$, and because $x - 4 \mid (x+2)(x-4) = x^2 - 2x - 8$,

$$U[A_{2^{n+2}-1}(x,y)] = 2U[A_{2^{n+1}-1}(x,y)] + 8U[A_{2^n-1}(x,y)]$$

as well. But $U[A_{2^0-1}] = 1$, $U[A_{2^1-1}] = 8$, and

$$U[A_{2^2-1}] = 4U[x+y] + 8U[x+1] = 24,$$

so the recurrence also holds for $n = 0$. Solving, we obtain $a_{2^n-1} = \frac{5 \cdot 4^n - 2(-2)^n}{3}$, so we're done. ■

This problem and solution were proposed by Bobby Shen.

Remark. The number of odd coefficients of $(x^2+x+1)^n$ is the Jacobsthal sequence (OEIS A001045) (up to translation). The sequence $\{a_n\}$ in the problem also has a (rather empty) OEIS entry. It may be interesting to investigate the generalization

$$\sum_{j=-1}^{1} \sum_{k=-1}^{1} c_{j,k} f_{i-1}(x+j, y+k)$$

for 9-tuples $(c_{j,k}) \in \{0,1\}^9$. Note that when all $c_{j,k}$ are equal to 1, we get $(x^2+x+1)^n(y^2+y+1)^n$, and thus the square of the Jacobsthal sequence.

Even more generally, one may ask the following: "Let f be an integer-coefficient polynomial in $n \geq 1$ variables, and p be a prime. For $i \geq 0$, let a_i denote the number of nonzero coefficients of f^{p^i-1} (in $\mathbb{F}_p$).

Under what conditions must there always exist an infinite arithmetic progression AP of positive integers for which $\{a_i : i \in AP\}$ satisfies a linear recurrence?"

C10*

Let $N \geq 2$ be a fixed positive integer. There are $2N$ people, numbered $1, 2, \ldots, 2N$, participating in a tennis tournament. For any two positive integers i, j with $1 \leq i < j \leq 2N$, player i has a higher skill level than player j. Prior to the first round, the players are paired arbitrarily and each pair is assigned a unique court among N courts, numbered $1, 2, \ldots, N$.

During a round, each player plays against the other person assigned to his court (so that exactly one match takes place per court), and the player with higher skill wins the match (in other words, there are no upsets). Afterwards, for $i = 2, 3, \ldots, N$, the winner of court i moves to court $i - 1$ and the loser of court i stays on court i; however, the winner of court 1 stays on court 1 and the loser of court 1 moves to court N.

Find all positive integers M such that, regardless of the initial pairing, the players $2, 3, \ldots, N + 1$ all change courts immediately after the Mth round.

Ray Li

Answer. $M \geq N + 1$.

Solution. It is enough to prove the claim for $M = N + 1$. (Why?)

After the kth move ($k \geq 0$), let $a_i^{(k)} \in [0, 2]$ be the number of rookies (players $N + 2, \ldots, 2N$) in court i so that $a_1^{(k)} + \cdots + a_N^{(k)} = N - 1$.

The operation from the perspective of the rookies can be described as follows: $a_i^{(k)} = 2$ for some $i \in \{2, \ldots, N\}$ means we "transfer" a rookie from court i to court $i - 1$ on the $(k + 1)$th move, and $a_1^{(k)} \geq 1$ means we "transfer" a rookie from court 1 to court N on the $(k+1)$th move. Note that if $a_i^{(k)} \geq 1$ for some $k \geq 0$ and $i \in \{2, \ldots, N\}$, we must have $a_i^{(k+r)} \geq 1$ for all $r \geq 0$. (*)

But we also know that all "excesses" can be traced back to "transfers". More precisely, if $a_i^{(k)} = 2$ for some $i \in \{2, \ldots, N - 1\}$ and $k \geq 1$, we must have $a_{i+1}^{(k-1)} = 2$; if $a_N^{(k)} = 2$, we must have $a_1^{(k-1)} \geq 1$; and if $a_1^{(k)} \geq 1$, we must either have (i) $a_2^{(k-1)} = 2$ or (ii) $a_1^{(k-1)} = 2$ and if $k \geq 2$, $a_2^{(k-2)} = 2$.

If $a_i^{(N)} = 2$ for some $i \in \{2, \ldots, N\}$ or $a_1^{(N)} \geq 1$, then by the previous paragraph and (*) we see that $a_i^{(N)} \geq 1$ for $i = 2, \ldots, N$, contradicting the fact that $a_1^{(N)} + \cdots + a_N^{(N)} = N - 1$. (Here possibility (ii) from the previous paragraph forces us to consider the Nth move rather than the $(N - 1)$th move.)

Hence $a_1^{(N)} = 0$, $a_2^{(N)} = \cdots = a_N^{(N)} = 1$, and of course player 1 stabilizes after at most $N - 1$ moves (he always wins), so we get a bound of $\geq 1 + \max(N - 1, N) = N + 1$.

We cannot replace the condition $M \geq N + 1$ with $M \geq N'$ for any $N' < N$. Indeed, any configuration with $(a_1^{(0)}, \ldots, a_N^{(0)}) = (2, 0, 0, 1, 1, 1, 1, \ldots, 1)$ shows that $N + 1$ is the "best bound possible." ■

This problem was proposed by Ray Li. This solution was given by Victor Wang.

Remark. The key idea (which can be easily found by working backwards) is to focus on the rookies. Asking for the minimum number of rounds required for stablization rather than giving the answer directly (here $N + 1$) may make the problem slightly more difficult, but once one conceives the idea of isolating rookies, even this version is not much harder.

G1

Let ABC be a triangle with incenter I. Let U, V and W be the intersections of the angle bisectors of angles A, B, and C with the incircle, so that V lies between B and I, and similarly with U and W. Let X, Y, and Z be the points of tangency of the incircle of triangle ABC with BC, AC, and AB, respectively. Let triangle UVW be the *David Yang triangle* of ABC and let XYZ be the *Scott Wu triangle* of ABC. Prove that the David Yang and Scott Wu triangles of a triangle are congruent if and only if ABC is equilateral.

Owen Goff

Solution. The angles of the triangles are $(\frac{A+B}{2}, \frac{B+C}{2}, \frac{C+A}{2})$ and $(\frac{\frac{A+B}{2}+\frac{B+C}{2}}{2}, \frac{\frac{B+C}{2}+\frac{C+A}{2}}{2}, \frac{\frac{C+A}{2}+\frac{A+B}{2}}{2})$ by quick angle chasing. Since the sets $(x, y, z), (\frac{x+y}{2}, \frac{y+z}{2}, \frac{z+x}{2})$ are equal iff $x = y = z$, we are done. ■

This problem and solution were proposed by Owen Goff.

G2

Let ABC be a scalene triangle with circumcircle Γ, and let D,E,F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z respectively. Prove that the perpendiculars from A, B, C to AX, BY, CZ respectively are concurrent.

Michael Kural

Solution 1. We claim that this point is the reflection of I the incenter over O the circumcenter. Since $\angle AEI = \angle AFI = \frac{\pi}{2}$, $AFIE$ is cyclic with diameter AI, so $\angle AXI = 90$. Also, if M is the midpoint of AX, then $OM \perp AX$, so clearly the reflection of I over O lies on each of the perpendiculars. ■

Solution 2. Extend BY and CZ, CZ and AZ, and AX and BY to meet at P, Q, R respectively. Note that P is the radical center of the circumcircles of BDF and CDE and Γ, so P lies on the radical axis DI of the two circumcircles (I lies on both circles as we showed before). Then the perpendiculars from P, Q, R to BC, AC, AB concur at I, so by Carnot's theorem

$$PB^2 - PC^2 + QC^2 - QA^2 + RA^2 - RB^2 = 0 \implies AQ^2 - AR^2 + BR^2 - BP^2 + CP^2 - CQ^2 = 0.$$

Again by Carnot's theorem the perpendiculars from A, B, C to QR, PR, PQ concur, which was what we wanted. (In other words, triangles ABC and PQR are *orthologic*.) ■

This problem and its solutions were proposed by Michael Kural.

G3

In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Allen Liu

Solution 1. Invert about A. We get triangle ABC with a variable point D on its circumcircle. CD meets AB at E, BD meets AC at F. The pole of EF is the intersection of AD and BC, so it lies on BC, and the fixed pole of BC lies on EF, proving the claim. Also, since pole of BC is the intersection of the tangents from B and C, the point lies on the symmedian, which is the median under inversion. ■

This first solution was suggested by Michael Kural.

Solution 2. Use barycentric coordinates with $A = (1, 0, 0)$, etc. Let $D = (0 : m : n)$ with $m + n = 1$. Then the circle ABD has equation $-a^2yz - b^2zx - c^2xy + (x + y + z)\left(a^2m \cdot z\right)$. To intersect it with side AC, put $y = 0$ to get $(x + z)(a^2mz) = b^2zx \implies \frac{b^2}{a^2m} \cdot x = x + z \implies \left(\frac{b^2}{a^2m} - 1\right) x = z$, so

$$F = (a^2m : 0 : b^2 - a^2m)$$

Similarly,

$$G = (a^2n : c^2 - a^2n : 0).$$

Then, the circle (AFG) has equation

$$-a^2yz - b^2zx - c^2xy + a^2(x + y + z)(my + nz) = 0.$$

Upon picking $y = z = 1$, we easily see there exists a t such that $(t : 1 : 1)$ is on the circle, implying the conclusion. ■

This second solution was suggested by Evan Chen.

Solution 3. Let M be the midpoint of BC. By power of a point, $c \cdot BE + b \cdot CF = a \cdot BD + a \cdot CD = a^2$ is constant. Fix a point D_0; and let $P_0 = AM \cap (AE_0F_0)$. For any other point D, we have $\frac{E_0E}{F_0F} = \frac{b}{c} = \frac{\sin \angle BAM}{\sin \angle CAM} = \frac{P_0E_0}{P_0F_0}$ from the extended law of sines, so triangles P_0E_0E and P_0F_0F are directly similar, whence AEP_0F is cyclic, as desired. ■

This third solution was suggested by Victor Wang.

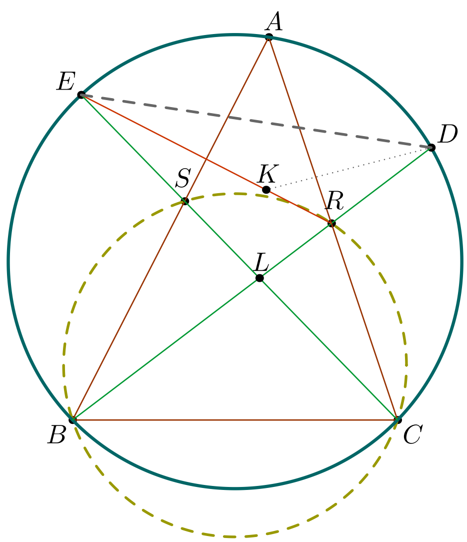
This problem was proposed by Allen Liu.

G4*

Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

Evan Chen

Solution 1.



First, we claim that $BE = BR = BC$. Indeed, construct a circle with radius $BE = BR$ centered at B, and notice that $\angle ECR = \frac{1}{2}\angle EBR$, implying that it lies on the circle.

Now, CA bisects $\angle ECD$ and DB bisects $\angle EDC$, so R is the incenter of $\triangle CDE$. Then, K is the incenter of $\triangle LED$, so $\angle ELK = \frac{1}{2}\angle ELD = \frac{1}{2}\left(\frac{\widehat{ED}+\widehat{BC}}{2}\right) = \frac{1}{2}\frac{\widehat{BED}}{2} = \frac{1}{2}\angle BCD$. ■

This problem and solution were proposed by Evan Chen.

Solution 2. Note $\angle EBA = \angle ECA = \angle SCR = \angle SBR = \angle ABR$, so AB bisects $\angle EBR$. Then by symmetry $\angle BEA = \angle BRA$, so $\angle BCR = \angle BCA = 180 - \angle BEA = 180 - \angle BRA = \angle BRC$, so $BE = BR = BC$. Proceed as above. ■

This second solution was suggested by Michael Kural.

G5

Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter AB of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2, passing through O and A, touch ω_2 at F and G. Prove that $FGOB$ is cyclic.

Eric Chen

Solution. Invert about ω_1. Then the problem becomes: "ω_1 and ω_2 are orthogonal circles. Show that if A is on ω_1 and outside of ω_2, and its tangents to ω_2 touch ω_2 at F, G, then its antipode B lies on FG."

Now let P be the center of ω_2, and let AP intersect FG at E. Then ω_1 is constant under an inversion with respect to ω_2, so E, the inverse of A, is on ω_1. Then $\angle AEB = \frac{\pi}{2}$, but $AE \perp FG$ so B is on FG and we are done. ■

This problem was proposed by Eric Chen. This solution was given by Michael Kural.

G6

Let $ABCDEF$ be a non-degenerate cyclic hexagon with no two opposite sides parallel, and define $X = AB \cap DE$, $Y = BC \cap EF$, and $Z = CD \cap FA$. Prove that

$$\frac{XY}{XZ} = \frac{BE}{AD} \frac{\sin|\angle B - \angle E|}{\sin|\angle A - \angle D|}.$$

Victor Wang

Solution. Use complex numbers with a, b, c, d, e, f on the unit circle, so $x = \frac{ab(d+e)-de(a+b)}{ab-de}$ and so on. It will be simpler to work with the conjugates of x, y, z, i.e. $\overline{x} = \frac{a+b-d-e}{ab-de}$, etc. Observing that

$$\begin{aligned}
\overline{x} - \overline{y} &= \frac{a+b-d-e}{ab-de} - \frac{b+c-e-f}{bc-ef} \\
&= \frac{(a-d)(cb-fe) - (c-f)(ab-de) + (b-e)(bc-ef-ab+de)}{(ab-de)(bc-ef)} \\
&= \frac{(b-e)(fa-cd+(bc-ef-ab+de))}{(ab-de)(bc-ef)},
\end{aligned}$$

we find (by "cyclically shifting" the variables by one so that $x - y \to z - x$) that

$$\frac{\overline{x-y}}{\overline{x-z}} = \frac{b-e}{a-d}\frac{af-cd}{bc-ef} = \frac{b-e}{a-d}\frac{a/c-d/f}{b/f-e/c},$$

from which the desired claim readily follows. ■

This problem and solution were proposed by Victor Wang.

G7*

Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to AC at C, and let O_2 be the center of the circle through E tangent to AB at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.

Michael Kural

Solution 1. Let M, N be the midpoints of AC, AB, respectively. Also, let BD, CE intersect (O_1) for a second time at X_1, Y_1, and let CE, BD intersect (O_2) for a second time at X_2, Y_2.

Now, by power of a point we have

$$MX_1 \cdot MD = MC^2 = MC \cdot MA = MD \cdot MB,$$

so $MX_1 = MB$, and X_1 is the reflection of B over M. Similarly, X_2 is the reflection of C over N.

(Alternatively, let X_1' be the reflection of B over M, and let D' be the intersection of the circles through X_1' tangent to AC at A, C respectively. Then by radical axes $X_1'D'$ bisects AC and $\angle ADC = 180 - \angle AX_1'C = 180 - \angle ABC$. This implies $D' = D$ and $X_1' = X_1$.)

Now let ZX_1X_2 be the antimedial triangle of ABC, and observe that $\angle X_2Y_1X_1 = \angle CDB = A = \angle CEB = \angle X_2Y_2X_1$. But $A = \angle X_2ZX_1$, so $X_1Y_1 \parallel EB$, $X_2Y_2 \parallel DC$, and $X_1X_2Y_2ZY_1$ is cyclic. Hence the lines through the centers of $(O_1), (ZX_1X_2)$, and $(ABC), (O_2)$ are parallel. In other words, $O_1H \parallel OO_2$ $O_1O \parallel HO_2$ (where O, H are the circumcenter and orthocenter of ABC), so O_1HO_2O is a parallelogram. Thus the midpoint of O_1O_2 is the midpoint N of OH. ■

This problem and solution were proposed by Michael Kural.

Remark. In fact, a -2 dilation about G sends B, D, C, E, O, A to X_1, Y_2, X_2, Y_1, H, Z.

Solution 2. Let (ABC) be the unit circle in the complex plane. Using the spiral similarities $D : CO_1 \to AO$ and $E : BO_2 \to AO$ (since AC is tangent to (O_1) and AB is tangent to (O_2)), it's easy to compute $o_1 = \frac{c(a+c-2b)}{c-b}$ and $o_2 = \frac{b(a+b-2c)}{b-c}$ (after solving for d, e via $\frac{bd(a+c)-ac(b+d)}{bd-ac} = m = \frac{a+c}{2}$), which gives us $o_1 + o_2 = a + b + c = 2n$. ■

This second solution was suggested by Victor Wang.

G8

Let ABC be a triangle, and let D, A, B, E be points on line AB, in that order, such that $AC = AD$ and $BE = BC$. Let ω_1, ω_2 be the circumcircles of $\triangle ABC$ and $\triangle CDE$, respectively, which meet at a point $F \neq C$. If the tangent to ω_2 at F cuts ω_1 again at G, and the foot of the altitude from G to FC is H, prove that $\angle AGH = \angle BGH$.

David Stoner

Solution 1. Let the centers of ω_1 and ω_2 be O_1 and O_2. Extend CA and CB to hit ω_2 again at K and L, respectively. Extend CO_2 to hit ω_2 again at R. Let M be the midpoint of arc $\widehat{AB}$, N the midpoint of arc $\widehat{FC}$ on ω_2, and T the intersection of FC and GM.

It's easy to see that $CK = CL = DE$, so O_2 is the C-excenter of triangle ABC. Hence C, M, and O_2 are collinear. Now $\angle CO_2O_1 = \angle CO_2N = 2\angle CRN = \angle CRF = \angle CFG = \angle CMG$, so MT is parallel to O_1O_2, and thus perpendicular to CF. But M is the midpoint of arc $\widehat{AB}$, so $\angle AGM = \angle MGB$, and we're done. ■

Solution 2. The observation that AO_2 is the perpendicular bisector of DC is not crucial; the key fact is just that $\angle GFC = \angle FEC$, since GF is tangent to ω_2. Indeed, this yields

$$\angle AGH = \angle AGF - \angle HGF = \angle ACF - 90^\circ + \angle GFC = \angle ACF - 90^\circ + \angle FEC.$$

But $\angle ACF = 180^\circ - \angle DCA - \angle FED$, $\alpha = \angle DCA$, and $\beta = \angle CEB = \angle FED - \angle FEC$, so $\angle AGH = 90^\circ - \alpha - \beta = \gamma$, where α, β, γ are half-angles. By symmetry, $\angle BGH = \gamma$ as well, so we're done. ■

This problem and its solutions were proposed by David Stoner.

G9

Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω whose diagonals meet at F. Lines AB and CD meet at E. Segment EF intersects ω at X. Lines BX and CD meet at M, and lines CX and AB meet at N. Prove that MN and BC concur with the tangent to ω at X.

Allen Liu

Solution. Let EF meet BC at P, and let K be the harmonic conjugate of P with respect to BC. View EP as a cevian of $\triangle EBC$. Since the cevians AC, BD and EP concur, it follows that AD passes through K. Similarly, MN passes through K. However, by Brokard's theorem, EF is the pole of K with respect to ω, so KX is tangent to ω. Therefore, the three lines in question concur at K. ■

This problem and solution were proposed by Allen Liu.

G10*

Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

David Stoner

Solution 1. Let J be the second intersection of ω and AC, and X be the intersection of BF and AC. It's well-known that $DJFC$ is harmonic; perspectivity wrt B implies $AJXC$ is also harmonic. Then $\frac{AJ}{JX} = \frac{AC}{CX} \implies (AJ)(CX) = (AC)(JX)$. This can be rearranged to get

$$(AJ)(CX) = (AJ + JX + XC)(JX) \implies 2(AJ)(CX) = (JX + AJ)(JX + XC) = (AX)(CJ),$$

so

$$\left(\frac{AX}{XC}\right)\left(\frac{CJ}{JA}\right) = 2.$$

But $\frac{CJ}{JA} = \frac{AD}{DB}$, so by Ceva's we have $BH = 2HC$, as desired. ■

Solution 2. Let J be the second intersection of ω and AC. It's well-known that $DJFC$ is harmonic; thus we have $(DJ)(FC) = (JF)(DC)$. By Ptolemy's, this means

$$(DF)(JC) = (DJ)(FC) + (JF)(DC) = 2(JD)(CF) \implies \left(\frac{JC}{JD}\right)\left(\frac{FD}{FC}\right) = 2.$$

Yet $JC = DB$ by symmetry, so this becomes

$$2 = \left(\frac{DB}{JD}\right)\left(\frac{FD}{FC}\right) = \left(\frac{\sin DJB}{\sin JBD}\right)\left(\frac{\sin FCD}{\sin FDC}\right) = \left(\frac{\sin DCB}{\sin ACD}\right)\left(\frac{\sin FBA}{\sin CBF}\right).$$

Thus by (trig) Ceva's we have $\frac{\sin BAH}{\sin CAH} = 2$, and since $AB = AC$ it follows that $BH = 2HC$, as desired. ■

This problem and its solutions were proposed by David Stoner.

G11

Let $\triangle ABC$ be a nondegenerate isosceles triangle with $AB = AC$, and let D, E, F be the midpoints of BC, CA, AB respectively. BE intersects the circumcircle of $\triangle ABC$ again at G, and H is the midpoint of minor arc BC. $CF \cap DG = I, BI \cap AC = J$. Prove that $\angle BJH = \angle ADG$ if and only if $\angle BID = \angle GBC$.

David Stoner

Solution. By barycentric coordinates on $\triangle ABC$ it is easy to obtain $G = (a^2+c^2 : -b^2 : a^2+c^2)$. Then, one can compute $I = (a^2+c^2 : a^2+c^2 : b^2+2(a^2+c^2))$, from which it follows that $J = (a^2+c^2 : 0 : b^2+2(a^2+c^2))$.

Now we use complex numbers. Set $D = 0$, $C = 1$, $B = -1$, $A = ri$ for $r \in \mathbb{R}^+$, $K = \frac{r}{3}$, and $H = -\frac{i}{r}$. Now, upon using the vector definition for barycentric coordinates, we obtain $I = \frac{(r^2+5)(ri)+(r^2+5)(-1)+(3r^2+11)(1)}{5r^2+21}$, or

$$I = \frac{2r^2+6}{5r^2+21} + \frac{r(r^2+5)}{5r^2+21}i.$$

Similarly, we can get

$$J = \frac{3r^2+11}{4r^2+16} + \frac{r(r^2+5)}{4r^2+16}i.$$

Claim. $\angle BID = \angle GBC \iff r^6 + 9r^4 - 17r^2 - 153 = 0.$

Proof. Let $V(a+bi) = \frac{b}{a}$ for $a, b \in \mathbb{R}$, and note $V(nz) = V(z)$ for all $n \in \mathbb{R}$. Then,

$$\angle BID = \angle GBC \iff V\left(\frac{D-I}{B-I}\right) = V\left(\frac{G-B}{C-B}\right)$$

Obviously the right-hand side is $\frac{r}{3}$. Meanwhile,

$$\begin{aligned}
\frac{-I}{1-I} &= \frac{I}{I+1} \\
&= \frac{\frac{2r^2+6}{5r^2+21} + \frac{r(r^2+5)}{5r^2+21}i}{\frac{7r^2+27}{5r^2+21} + \frac{r(r^2+5)}{5r^2+21}i} \\
&= \frac{1}{\text{real}}\left[\left((2r^2+6) + r(r^2+5)i\right)\left((7r^2+26) - r(r^2+5)i\right)\right] \\
&= \frac{1}{\text{real}}\left[\left(r^6 + 24r^4 + 121r^2 + 162\right) + (5r^2+21)(r)(r^2+5)i\right]
\end{aligned}$$

Hence, $V\left(\frac{I}{I+1}\right) = \frac{(5r^2+21)(r)(r^2+5)}{r^6+24r^4+121r^2+161}$. This is equal to $r/3$ if and only if

$$r^6 + 24r^4 + 121r^2 + 162 - 3(5r^2+21)(r^2+5) = 0.$$

Expanding gives the conclusion. □ □

Claim. $\angle BJH = \angle ADG \iff 2r^8 + 8r^6 - 28r^r - 136r^2 - 102 = 0.$

Proof. We proceed in the same spirit. It's evident that $V\left(\frac{K-D}{G-D}\right) = V(I)^{-1} = \frac{2r^2+6}{r(r^2+5)}$. On the other hand, we can compute

$$\begin{aligned}
\frac{-\frac{1}{r}\cdot i - J}{-1-J} &= \frac{rJ+i}{r(1+J)} \\
&= \frac{1}{r}\cdot\frac{\frac{r(3r^2+11)}{4r^2+16}+\frac{r^2(r^2+5)+(4r^2+16)}{4r^2+16}i}{\frac{7r^2+27}{4r^2+16}+\frac{r(r^2+5)}{4r^2+16}i} \\
&= \frac{1}{\text{real}}\left[\left(r(3r^2+11)+(r^4+9r^2+16)\right)i\right]\left[(7r^2+27)-r(r^2+5)i\right] \\
&= \frac{1}{\text{real}}\left[r(r^6+35r^4+219r^2+377)+i(4r^6+64r^4+300r^2+432)\right]
\end{aligned}$$

Hence, $V\left(\frac{H-J}{B-J}\right) = \frac{4r^6+64r^4+300r^2+432}{r(r^6+35r^4+219r^2+377)}$. So, the equality occurs when

$$(r^2+5)(4r^6+64r^4+300r^2+432)-(2r^2+6)(r^6+35r^4+219r^2+377)=0.$$

Expand again. □ □

Now all that's left to do is factor these polynomials! The former one is $(r^4-17)(r^2+9)$, and the latter is $2(r^2+1)(r^2+3)(r^4-17)$. Restricted to positive r we see that both are zero if and only if $r=\sqrt[4]{17}$. Therefore the conditions are equivalent, occuring if and only if $AD=\sqrt[4]{17}$. ■

This problem was proposed by David Stoner. This solution was given by Evan Chen.

G12*

Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

David Stoner

Solution. Let α be the half-angles of $\triangle ABC$, r inradius, and u, v, w tangent lengths to the incircle. Let $T = MP \cap NQ$ so that I is the incenter of $\triangle MNT$. Then $\angle IPT = \angle IXY = \alpha = \angle IYX = \angle IQT$ gives $\triangle TIP \sim \triangle TIQ$, so P, I, Q are collinear iff $\angle TIP = 90^\circ$ iff $\angle MTN = 180^\circ - 2\alpha$ iff $\angle MIN = 180^\circ - \alpha$ iff $MI^2 = MX \cdot MN$.

First suppose I is the center of γ. Since A, I are symmetric about XY, $\angle MAN = \angle MIN$. But P, I, Q are collinear iff $\angle MIN = 180^\circ - \alpha$, so because arcs AN and BM sum to 90°, P, I, Q are collinear iff arcs BM, MA have the same measure. Let $M' = CI \cap \omega$; then $\angle BM'I = \angle BM'C = 90^\circ - \angle BXI$, so $M'XIBZ$ is cyclic and $\angle M'XB = \angle M'IB = 180^\circ - \angle BIC = 45^\circ = \angle AXY$, as desired. (There are many other ways to finish as well.)

Conversely, if P, I, Q are collinear, then by power of a point, $m(m+2t) = MI^2 - r^2 = MX \cdot MN - r^2 = m(m+2t+n) - r^2$, so $mn = r^2$. But we also have $m(n+2t) = uv$ and $n(m+2t) = uw$, so

$$r^2 = mn = \frac{uv - r^2}{2t}\frac{uw - r^2}{2t} = \frac{\frac{uv(u+v)}{u+v+w}}{2r\cos\alpha}\frac{\frac{uw(u+w)}{u+v+w}}{2r\cos\alpha} = \frac{r^2}{4\cos^2\alpha}\frac{(u+v)(u+w)}{vw}.$$

Simplifying using $\cos^2\alpha = \frac{u^2}{u^2+r^2} = \frac{u(u+v+w)}{(u+v)(u+w)}$, we get

$$0 = (u+v)^2(u+w)^2 - 4uvw(u+v+w) = (u(u+v+w) - vw)^2,$$

which clearly implies $(u+v)^2 + (u+w)^2 = (v+w)^2$, as desired. ■

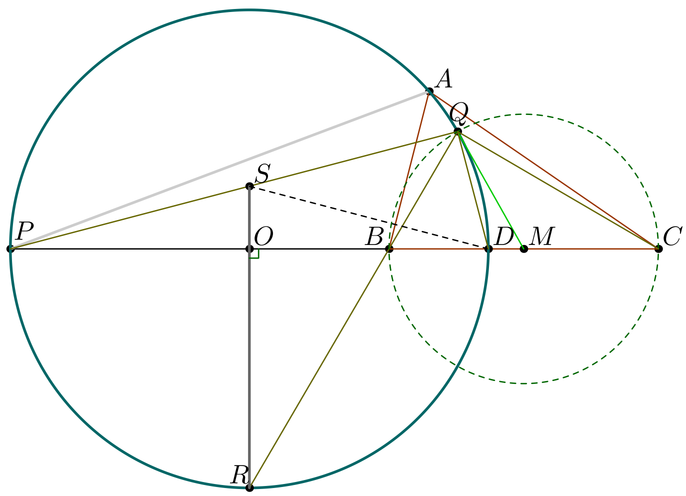
This problem was proposed by David Stoner. This solution was given by Victor Wang.

G13

In $\triangle ABC$, $AB < AC$. D and P are the feet of the internal and external angle bisectors of $\angle BAC$, respectively. M is the midpoint of segment BC, and ω is the circumcircle of $\triangle APD$. Suppose Q is on the minor arc AD of ω such that MQ is tangent to ω. QB meets ω again at R, and the line through R perpendicular to BC meets PQ at S. Prove SD is tangent to the circumcircle of $\triangle QDM$.

Ray Li

Solution.



We begin with a lemma.

Lemma 1. *Let $(A, B; C, D)$ be a harmonic bundle. Then the circles with diameter AB and CD are orthogonal.*

Proof. Let ω be the circle with diameter AB. Then D lies on the pole of C with respect to ω. Hence the inversion at ω sends C to D and vice-versa; so it fixes the circle with diameter CD, implying that the two circles are orthogonal. □ □

It's well known that $(P, D; B, C)$ is harmonic. Let O be the midpoint of PD. If we let Q' be the intersection of the circles with diameter PD and BC, then $\angle OQ'M = \frac{\pi}{2}$, implying that $Q' = Q$. It follows that Q lies on the circle with diameter BC; this is the key observation.

In that case, since $(P, D; B, C)$ is harmonic and $\angle PQD = \frac{\pi}{2}$, we see that QD is an angle bisector (this could also be realized via Apollonian circles). But $\angle BQC = \frac{\pi}{2}$ as well! So we find that $\angle PQB = \angle BQD = \angle DQC = \frac{\pi}{4}$. Then, R is the midpoint of arc PD, so $SP = SD$, insomuch as $SO \perp PD$.

Hence, we can just angle chase as $\angle DQM = \angle SPD = \angle SDP$, implying the conclusion. ■

This problem and solution were proposed by Ray Li.

G14

Let O be a point (in the plane) and T be an infinite set of points such that $|P_1P_2| \le 2012$ for every two distinct points $P_1, P_2 \in T$. Let $S(T)$ be the set of points Q in the plane satisfying $|QP| \le 2013$ for at least one point $P \in T$.

Now let L be the set of lines containing exactly one point of $S(T)$. Call a line ℓ_0 passing through O *bad* if there does not exist a line $\ell \in L$ parallel to (or coinciding with) ℓ_0.

(a) Prove that L is nonempty.

(b) Prove that one can assign a line $\ell(i)$ to each positive integer i so that for every bad line ℓ_0 passing through O, there exists a positive integer n with $\ell(n) = \ell_0$.

David Yang

Solution 1. (a) Instead of unique lines we work with *good directions* (e.g. northernmost points for the direction "north"). Since S is closed and bounded there is a diameter, say AB. Then B is the unique farthest point in the direction of the vector $\overrightarrow{AB}$ (if there was another point C that was the same or farther in that direction then AC would be longer than AB). ■

Solution 2. (b) We can work instead with the convex hull of S, since this does not change if directions are good. Note that bad directions correspond to lines segments that are boundaries of portions of the convex hull (i.e. "sides" of the convex hull). For each direction, consider the corresponding side. Now, consider the area 1 unit in front of the side. For distinct directions, these areas don't intersect, so there must be a countable number of them (more precisely, there are a finite number of squares with area in the interval $(\frac{1}{n+1}, \frac{1}{n}]$ for every positive integer n, and thus we can enumerate the bad directions.) ■

This problem and the above solutions were proposed by David Yang.

Solution 3. (b) Alternatively, take an interior point and look at the angle swept out by each side (positive numbers with finite sum). ■

This third solution was suggested by Mark Sellke.

Remark. We only need S to be a compact (closed and bounded) set in $\mathbb{R}^n$ for (a), and a compact set in $\mathbb{R}^2$ for (b). The current elementary formulation, however, preserves the essence of the problem. Note that the same proof works for (a), while a hyper-cylinder serves as a counterexample for (b) in $\mathbb{R}^n$ (more specifically, the set of points satisfying, say, $x_1^2 + x_2^2 \le 1$ and $0 \le x_3, \ldots, x_n \le 1$). Indeed, for each angle $\theta \in [0, 2\pi)$, the hyper-plane with equation $\sin\theta x_1 - \cos\theta x_2 = 0$ is tangent to the cylinder at the set of points of the form $(\cos\theta, \sin\theta, x_3, \ldots, x_n)$, yet $[0, 2\pi)$ (which bijects to the real numbers) is uncountable. More precisely, the set of points farthest $\langle\cos\theta, \sin\theta, 0, \ldots, 0\rangle$ direction is simply the set of points that maximize $\langle\cos\theta, \sin\theta, 0, \ldots, 0\rangle \cdot \langle x_1, x_2, 0, \ldots, 0\rangle$ (which is at most 1, by the Cauchy-Schwarz inequality), which is just the set of points of the form $(\cos\theta, \sin\theta, x_3, \ldots, x_n)$.

N1

Find all ordered triples of non-negative integers (a, b, c) such that $a^2 + 2b + c$, $b^2 + 2c + a$, and $c^2 + 2a + b$ are all perfect squares.

Note: This problem was withdrawn from the ELMO Shortlist and used on ksun48's mock AIME.

Matthew Babbitt

Answer. We have the trivial solutions $(a, b, c) = (0, 0, 0)$ and $(a, b, c) = (1, 1, 1)$, as well as the solution $(a, b, c) = (127, 106, 43)$ and its cyclic permutations.

Solution. The case $a = b = c = 0$ works. Without loss of generality, $a = \max\{a, b, c\}$. If b and c are both zero, it's obvious that we have no solution. So, via the inequality

$$a^2 < a^2 + 2b + c < (a + 2)^2$$

we find that $a^2 + 2b + c = (a + 1)^2 \implies 2a + 1 = 2b + c$. So,

$$a = b + \frac{c - 1}{2}.$$

Let $c = 2k + 1$ with $k \geq 0$; plugging into the given, we find that

$$b^2 + b + 2 + 5k \quad \text{and} \quad 4k^2 + 6k + 3b + 1$$

are both perfect squares. Multiplying both these quantities by 4, and setting $x = 2b + 1$ and $y = 4k + 3$, we find that

$$x^2 + 5y - 8 \quad \text{and} \quad y^2 + 6x - 11$$

are both even squares.

We may assume $x, y \geq 3$. We now have two cases, both of which aren't too bad:

- If $x \geq y$, then $x^2 < x^2 + 5y - 8 < (x + 3)^2$. Since the square is even, $x^2 + 5y - 8 = (x + 1)^2$. Then, $x = \frac{5y-9}{2}$ and we find that $y^2 + 15y - 38$ is an even square. Since $y^2 < y^2 + 15y - 38 < (y + 8)^2$, there are finitely many cases to check. The solutions are $(x, y) = (3, 3)$ and $(x, y) = (213, 87)$.
- Similarly, if $x \leq y$, then $y^2 < y^2 + 6x - 11 < (y + 3)^2$, so $y^2 + 6x - 11 = (y + 1)^2$. Then, $y = 3x - 6$ and we find that $x^2 + 15x - 38$ (!) is a perfect square. Amusingly, this is the exact same thing (whether this is just a coincidence due to me selecting the equality case to be $x = y$, I'm not sure). Here, the solutions are $(x, y) = (3, 3)$ and $(x, y) = (87, 255)$.

Converting back, we see the solutions are $(0, 0, 0)$, $(1, 1, 1)$ and $(127, 106, 43)$, and permutations. ■

This problem and solution were proposed by Matthew Babbitt.

N2*

For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \geq 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

Andre Arslan

Answer. All P of the form $P(x) = cx^k$, where c is a nonzero integer and k is a nonnegative integer.

Solution. Suppose $P(x) = x^k Q(x)$ with $Q(0) \neq 0$ and Q is nonconstant; then there exist infinitely many primes p dividing some $Q(n)$; fix one of them not dividing $Q(0)$, and take a sequence of pairwise coprime integers $m_1, n_1, m_2, n_2, \ldots$ with $p \mid Q(m_i), Q(n_i)$ (we can do this with CRT).

Let $f(x, y, z)$ be the number written at (x, y, z). Note that $P(m)$ divides every $mn \times mn \times m$ grid and $P(n)$ divides every $mn \times mn \times n$ grid, so by Bezout's identity, $(P(m), P(n))$ divides every $mn \times mn \times (m, n)$ grid. It follows that p divides every $m_i n_i \times m_i n_i \times 1$ grid. Similarly, we find that p divides every $m_i n_i m_j n_j \times 1 \times 1$ grid whenever $i \neq j$, and finally, every $1 \times 1 \times 1$ grid. Since p was arbitrarily chosen from an infinite set, f must be identically zero, contradiction.

For the other direction, take a solution g to the one-dimensional case using repeated CRT (the key relation $\gcd(P(m), P(n)) = P(\gcd(m, n))$ prevents "conflicts"): start with a positive multiple of $P(1) \neq 0$ at zero, and then construct $g(1), g(-1), g(2), g(-2)$, etc. in that order using CRT. Now for the three-dimensional version, we can just let $f(x, y, z) = g(x)$. ■

This problem and solution were proposed by Andre Arslan.

Remark. The crux of the problem lies in the 1D case. (We use the same type of reasoning to "project" from d dimension to $d-1$ dimensions.) Note that the condition $P(n) \mid g(i) + \cdots + g(i+n-1)$ (for the 1D case) is "almost" the same as $P(n) \mid g(i) - g(i+n)$, so we immediately find $\gcd(P(m), P(n)) \mid g(i) - g(i + \gcd(m, n))$ by Bezout's identity. In particular, when m, n are coprime, we will intuitively be able to get $\gcd(P(m), P(n))$ as large as we want unless P is of the form cx^k (we formalize this by writing $P = x^k Q$ with $Q(0) \neq 0$).

Conversely, if $P = cx^k$, then $\gcd(P(m), P(n)) = P(\gcd(m, n))$ renders our derived restriction $\gcd(P(m), P(n)) \mid g(i) - g(i + \gcd(m, n))$ superfluous. So it "feels easy" to find *nonconstant* g with $P(n) \mid g(i) - g(i+n)$ for all i, n, just by greedily constructing $g(0), g(1), g(-1), \ldots$ in that order using CRT. Fortunately, $g(i) + \cdots + g(i+m-1) - g(i) - \cdots - g(i+n-1) = g(i+n) + \cdots + g(i+n+(m-n)-1)$ for $m > n$, so the inductive approach still works for the stronger condition $P(n) \mid g(i) + \cdots + g(i+n-1)$.

Remark. Note that polynomial constructions cannot work for $P = cx^{d+1}$ in d dimensions. Suppose otherwise, and take a minimal degree $f(x_1, \ldots, x_d)$; then f isn't constant, so $f'(x_1, \ldots, x_d) = f(x_1 + 1, \ldots, x_d + 1) - f(x_1, \ldots, x_d)$ is a working polynomial of strictly smaller degree.

N3

Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct numbers of the form a^b, where $a \in \{3,4,5,6\}$ and b is a positive integer.

Matthew Babbitt

Solution. First, we prove a lemma.

Lemma 1. *Let $a_0 > a_1 > a_2 > \cdots > a_n$ be positive integers such that $a_0 - a_n < a_1 + a_2 + \cdots + a_n$. Then for some $1 \le i \le n$, we have*

$$0 \le a_0 - (a_1 + a_2 + \cdots + a_i) < a_i.$$

Proof. Proceed by contradiction; suppose the inequalities are all false. Use induction to show that $a_0 - (a_1 + \cdots + a_i) \ge a_i$ for each i. This becomes a contradiction at $i = n$. □ □

Let N be the integer we want to express in this form. We will prove the result by strong induction on N. The base cases will be $3 \le N \le 10 = 6 + 3 + 1$.

Let $x_1 > x_2 > x_3 > x_4$ be the largest powers of $3,4,5,6$ less than $N - 3$, in some order. If one of the inequalities of the form

$$3 \le N - (x_1 + \cdots + x_k) < x_k + 3; \quad 1 \le k \le 4$$

is true, then we are done, since we can subtract of $x_1, \dots, x_k$ from N to get an N' with $3 \le N' < N$ and then apply the inductive hypothesis; the construction for N' cannot use any of $\{x_1, \dots, x_k\}$ since $N' - x_k < 3$.

To see that this is indeed the case, first observe that $N - 3 > x_1$ by construction and compute

$$x_1 + x_2 + x_3 + x_4 + x_4 \ge (N-3) \cdot \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{6} \right) > N - 3.$$

So the hypothesis of the lemma applies with $a_0 = N - 3$ and $a_i = x_i$ for $1 \le i \le 4$.

Thus, we are done by induction. ■

This problem and solution were proposed by Matthew Babbitt.

Remark. While the approach of subtracting off large numbers and inducting is extremely natural, it is not immediately obvious that one should consider $3 \le N - (x_1 + \cdots + x_k) < x_k + 3$ rather than the stronger bound $3 \le N - (x_1 + \cdots + x_k) < x_k$. In particular, the solution method above does not work if one attempts to get the latter.

N4

Find all triples (a, b, c) of positive integers such that if n is not divisible by any integer less than 2013, then $n + c$ divides $a^n + b^n + n$.

Evan Chen

Answer. $(a, b, c) = (1, 1, 2)$.

Solution. Let p be an arbitrary prime such that $p \geq 2011 \cdot \max\{abc, 2013\}$. By the Chinese Remainder Theorem it is possible to select an integer n satisfying the following properties:

$$\begin{aligned} n &\equiv -c \pmod{p} \\ n &\equiv -1 \pmod{p-1} \\ n &\equiv -1 \pmod{q} \end{aligned}$$

for all primes $q \leq 2011$ not dividing $p - 1$. This will guarantee that n is not divisible by any integer less than 2013. Upon selecting this n, we find that

$$p \mid n + c \mid a^n + b^n + n$$

which implies that

$$a^n + b^n \equiv c \pmod{p}$$

But $n \equiv -1 \pmod{p-1}$; hence $a^n \equiv a^{-1} \pmod{p}$ by Euler's Little Theorem. Hence we may write

$$p \mid ab(a^{-1} + b^{-1} - c) = a + b - abc.$$

But since p is large, this is only possible if $a + b - abc$ is zero. The only triples of positive integers with that property are $(a, b, c) = (2, 2, 1)$ and $(a, b, c) = (1, 1, 2)$. One can check that of these, only $(a, b, c) = (1, 1, 2)$ is a valid solution. ■

This problem and solution were proposed by Evan Chen.

N5*

Let $m_1, m_2, \ldots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \ldots, A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, \ldots, m_i - 1\}$ for $i = 1, 2, \ldots, 2013$. Prove that there is a positive integer N such that

$$N \le (2|A_1| + 1)(2|A_2| + 1) \cdots (2|A_{2013}| + 1)$$

and for each $i = 1, 2, \ldots, 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

Victor Wang

Remark. As Solution 3 shows, the bound can in fact be tightened to $\prod_{i=1}^{2013}(|A_i| + 1)$.

Solution 1. We will show that the smallest integer N such that $N \notin A_i \pmod{m_i}$ is less than the bound provided.

The idea is to use pigeonhole and the "Lagrange interpolation"-esque representation of CRT systems. Define integers t_i satisfying $t_i \equiv 1 \pmod{m_i}$ and $t_i \equiv 0 \pmod{m_j}$ for $j \ne i$. If we find nonempty sets B_i of distinct residues mod m_i with $B_i - B_i \pmod{m_i}$ and $A_i \pmod{m_i}$ disjoint, then by pigeonhole, a positive integer solution with $N \le \frac{m_1 m_2 \cdots m_{2013}}{|B_1| \cdot |B_2| \cdots |B_{2013}|}$ must exist (more precisely, since

$$b_1 t_1 + \cdots + b_{2013} t_{2013} \pmod{m_1 m_2 \cdots m_{2013}}$$

is injective over $B_1 \times B_2 \times \cdots \times B_{2013}$, some two consecutively ordered solutions must differ by at most $\frac{m_1 m_2 \cdots m_{2013}}{|B_1| \cdot |B_2| \cdots |B_{2013}|}$).

On the other hand, since $0 \notin A_i$ for every i, we know such nonempty B_i must exist (e.g. take $B_i = \{0\}$). Now suppose $|B_i|$ is maximal; then every $x \pmod{m_i}$ lies in at least one of B_i, $B_i + A_i$, $B_i - A_i$ (note that $x - x = 0$ is not an issue when considering $(B_i \cup \{x\}) - (B_i \cup \{x\})$), or else $B_i \cup \{x\}$ would be a larger working set. Hence $m_i \le |B_i| + |B_i + A_i| + |B_i - A_i| \le |B_i|(1 + 2|A_i|)$, so we get an upper bound of $\prod_{i=1}^{2013} \frac{m_i}{|B_i|} \le \prod_{i=1}^{2013}(2|A_i| + 1)$, as desired. ■

Remark. We can often find $|B_i|$ significantly larger than $\frac{m_i}{2|A_i|+1}$ (the bounds $|B_i + A_i|, |B_i - A_i| \le |B_i| \cdot |A_i|$ seem really weak, and $B_i + A_i, B_i - A_i$ might not be that disjoint either). For instance, if $A_i \equiv -A_i \pmod{m_i}$, then we can get (the ceiling of) $\frac{m_i}{|A_i|+1}$.

Remark. By translation and repeated application of the problem, one can prove the following slightly more general statement: "Let $m_1, m_2, \ldots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \ldots, A_{2013}$ be 2013 (possibly empty) sets with A_i a proper subset of $\{1, 2, \ldots, m_i\}$ for $i = 1, 2, \ldots, 2013$. Then for every integer n, there exists an integer x in the range $(n, n + \prod_{i=1}^{2013}(2|A_i| + 1)]$ such that $x \notin A_i \pmod{m_i}$ for $i = 1, 2, \ldots, 2013$. (We say A is a *proper subset* of B if A is a subset of B but $A \ne B$.)"

Remark. Let f be a non-constant integer-valued polynomial with $\gcd(\ldots, f(-1), f(0), f(1), \ldots) = 1$. Then by the previous remark, we can easily prove that there exist infinitely many positive integers n such that the smallest prime divisor of $f(n)$ is at least $c \log n$, where $c > 0$ is any constant. (We take m_i the ith prime and $A_i \equiv \{n : m_i \mid f(n)\} \pmod{m_i}$—if $f = \frac{a}{b}x^d + \cdots$, then $|A_i| \le d$ for all sufficiently large i.)

Solution 2. We will mimic the proof of 2010 RMM Problem 1.

Suppose $1, 2, \ldots, N$ (for some $N \ge 1$) can be covered by the sets $A_i \pmod{m_i}$.

Observe that for fixed m and $1 \le a \le m$, exactly $1 + \lfloor \frac{N-a}{m} \rfloor$ of $1, 2, \ldots, N$ are $a \pmod{m}$. In particular, we have lower and upper bounds of $\frac{N-m}{m}$ and $\frac{N+m}{m}$, respectively, so PIE yields

$$N \le \sum_i |A_i| \frac{N + m_i}{m_i} - \sum_{i<j} |A_i| \cdot |A_j| \frac{N - m_i m_j}{m_i m_j} \pm \cdots.$$

It follows that

$$N \prod_i \left(1 - \frac{|A_i|}{m_i}\right) \le \prod_i (1 + |A_i|),$$

so $N \le \prod_i \frac{m_i}{m_i - |A_i|}(1 + |A_i|)$.

Note that $\frac{m_i}{m_i-|A_i|} \le \frac{2|A_i|+1}{|A_i|+1}$ iff $m_i \ge 2|A_i| + 1$, so we're done unless $m_i \le 2|A_i|$ for some i.

In this case, there exists (by induction) $1 \le N \le \prod_{j \ne i}(2|A_j| + 1)$ such that $N \notin m_i^{-1} A_j \pmod{m_j}$ for all $j \ne i$. Thus $m_i N \notin A_j \pmod{m_j}$ and we trivially have $m_i N \equiv 0 \notin A_i \pmod{m_i}$, so $m_i N \le \prod_k (2|A_k| + 1)$, as desired. ■

This problem and the above solutions were proposed by Victor Wang.

Solution 3. We can in fact get a bound of $\prod(|A_k| + 1)$ directly.

Let $t = 2013$. Suppose $1, 2, \dots, N$ are covered by the $A_k \pmod{m_k}$; then

$$z_n = \prod_{1 \le k \le t, a \in A_k} \left(1 - e^{\frac{2\pi i}{m_k}(n-a)}\right)$$

is a linear recurrence in $e^{2\pi i \sum_{k=1}^{t} \frac{j_k}{m_k}}$ (where each j_k ranges from 0 to $|A_k|$). But $z_0 \ne 0 = z_1 = \cdots = z_N$, so N must be strictly less than the degree $\prod(|A_k| + 1)$ of the linear recurrence. Thus $1, 2, \dots, \prod(|A_k| + 1)$ cannot all be covered, as desired. ■

This third solution was suggested by Zhi-Wei Sun.

Remark. Solution 3 doesn't require the m_k to be coprime. Note that if $|A_1| = \cdots = |A_t| = b - 1$, then a base b construction shows the bound of $\prod(b - 1 + 1) = b^t$ is "tight" (if we remove the restriction that the m_k must be coprime).

However, Solutions 2 and 3 "ignore" the additive structure of CRT solution sets encapsulated in Solution 1's Lagrange interpolation representation.

N6*

Find all positive integers m for which there exists a function $f:\mathbb{Z}^+\to\mathbb{Z}^+$ such that

$$f^{f^{f(n)}(n)}(n)=n$$

for every positive integer n, and $f^{2013}(m)\neq m$. Here $f^k(n)$ denotes $\underbrace{f(f(\cdots f}_{k\ f\text{'s}}(n)\cdots))$.

Evan Chen

Answer. All m not dividing 2013; that is, $\mathbb{Z}^+\setminus\{1,3,11,33,61,183,671,2013\}$.

Solution. First, it is easy to see that f is both surjective and injective, so f is a permutation of the positive integers. We claim that the functions f which satisfy the property are precisely those functions which satisfy $f^n(n)=n$ for every n.

For each integer n, let $\operatorname{ord}(n)$ denote the smallest integer k such that $f^k(n)$. These orders exist since $f^{f^{f(n)}(n)}(n)=n$, so $\operatorname{ord}(n)\le f^{f(n)}(n)$; in fact we actually have

$$\operatorname{ord}(n)\mid f^{f(n)}(n) \tag{8.1}$$

as a consequence of the division algorithm.

Since f is a permutation, it is immediate that $\operatorname{ord}(n)=\operatorname{ord}(f(n))$ for every n; this implies easily that $\operatorname{ord}(n)=\operatorname{ord}\left(f^k(n)\right)$ for every integer k. In particular, $\operatorname{ord}(n)=\operatorname{ord}\left(f^{f(n)-1}(n)\right)$. But then, applying (8.1) to $f^{f(n)-1}(n)$ gives

$$\begin{aligned}\operatorname{ord}(n)=\operatorname{ord}\left(f^{f(n)-1}(n)\right) &\mid f^{f\left(f^{f(n)-1}(n)\right)}\left(f^{f(n)-1}(n)\right)\\ &=f^{f^{f(n)}(n)+f(n)-1}(n)\\ &=f^{f(n)-1}\left(f^{f^{f(n)}(n)}(n)\right)\\ &=f^{f(n)-1}(n)\end{aligned}$$

Inductively, then, we are able to show that $\operatorname{ord}(n)\mid f^{f(n)-k}(n)$ for every integer k; in particular, $\operatorname{ord}(n)\mid n$, so $f^n(n)=n$. To see that this is actually sufficient, simply note that $\operatorname{ord}(n)=\operatorname{ord}(f(n))=\cdots$, which implies that $\operatorname{ord}(n)\mid f^k(n)$ for every k.

In particular, if $m\mid 2013$, then $\operatorname{ord}(m)\mid m\mid 2013$ and $f^{2013}(m)=m$. The construction for the other values of m is left as an easy exercise. ■

This problem and solution were proposed by Evan Chen.

Remark. There are many ways to express the same ideas.

For instance, the following approach ("unraveling indices") also works: It's not hard to show that f is a bijection with finite cycles (when viewed as a permutation). If $C=(n_0,n_1,\dots,n_{\ell-1})$ is one such cycle with $f(n_i)=n_{i+1}$ for all i (extending indices mod ℓ), then $f^{f^{f(n)}(n)}(n)=n$ holds on C iff $\ell\mid f^{f(n_i)}(n_i)=n_{i+n_{i+1}}$ for all i. But $\ell\mid n_j\implies \ell\mid n_{j-1+n_j}=n_{j-1}$ for fixed j, so the latter condition holds iff $\ell\mid n_i$ for all i. Thus $f^{2013}(n)=n$ is forced unlesss $n\nmid 2013$.

N7*

Let p be a prime satisfying $p^2 \mid 2^{p-1}-1$, n be a positive integer, and $f(x) = \frac{(x-1)^{p^n}-(x^{p^n}-1)}{p(x-1)}$. Find the largest positive integer N such that there exist polynomials $g, h \in \mathbb{Z}[x]$ and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Victor Wang

Answer. The largest possible N is $2p^{n-1}$.

Solution 1. Let $F(x) = \frac{x}{1} + \cdots + \frac{x^{p-1}}{p-1}$.

By standard methods we can show that $(x-1)^{p^n} - (x^{p^{n-1}}-1)^p$ has all coefficients divisible by p^2. But $p^2 \mid 2^{p-1}-1$ means p is odd, so working in $\mathbb{F}_p$, we have

$$(x-1)f(x) = \sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k}(-1)^{k-1}x^{p^{n-1}k} = \sum_{k=1}^{p-1}\binom{p-1}{k-1}(-1)^{k-1}\frac{x^{p^{n-1}k}}{k}$$
$$= \sum_{k=1}^{p-1}\frac{x^{p^{n-1}k}}{k^{p^{n-1}}} = F(x)^{p^{n-1}},$$

where we use Fermat's little theorem, $\binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod p$ for $k = 1, 2, \ldots, p-1$, and the well-known fact that $P(x^p) - P(x)^p$ has all coefficients divisible by p for any polynomial P with integer coefficients.

However, it is easy to verify that $p^2 \mid 2^{p-1}-1$ if and only if $p \mid F(-1)$, i.e. -1 is a root of F in $\mathbb{F}_p$. Furthermore, $F'(x) = \frac{x^{p-1}-1}{x-1} = (x+1)(x+2)\cdots(x+p-2)$ in $\mathbb{F}_p$, so -1 is a root of F with multiplicity 2; hence $N \geq 2p^{n-1}$. On the other hand, since F' has no double roots, F has no integer roots with multiplicity greater than 2. In particular, $N \leq 2p^{n-1}$ (note that the multiplicity of 1 is in fact $p^{n-1}-1$, since $F(1)=0$ by Wolstenholme's theorem but 1 is not a root of F'). ■

This problem and solution were proposed by Victor Wang.

Remark. The rth derivative of a polynomial P evaluated at 1 is simply the coefficient $[(x-1)^r]P$ (i.e. the coefficient of $(x-1)^r$ when P is written as a polynomial in $x-1$) divided by $r!$.

Solution 2. This is asking to find the greatest multiplicity of an integer root of f modulo p; I claim the answer is $2p^{n-1}$.

First, we shift x by 1 and take the negative (since this doesn't change the greatest multiplicity) for convenience, redefining f as $f(x) = \frac{(x+1)^{p^n}-x^{p^n}-1}{px}$.

Now, we expand this. We can show, by writing out and cancelling, that p^1 fully divides $\binom{p^n}{k}$ only when p^{n-1} divides k; thus, we can ignore all terms except the ones with degree divisible by p^{n-1} (since they still go away when taking it mod p), leaving $f(x) = \frac{1}{px}(\binom{p^n}{p^{n-1}}x^{p^n-p^{n-1}} + \cdots + \binom{p^n}{p^n-p^{n-1}}x^{p^{n-1}})$.

We can also show, by writing out/cancelling, that $\frac{1}{p}\binom{p^n}{kp^{n-1}} = \frac{1}{p}\binom{p}{k}$ modulo p. Simplifying using this, the expression above becomes $f(x) = \frac{1}{px}(\binom{p}{1}x^{p^n-p^{n-1}} + \cdots + \binom{p}{p-1}x^{p^{n-1}}) = \frac{1}{px}((x^{p^{n-1}}+1)^p - (x^{p^n}+1))$.

Now, we ignore the $1/x$ for the moment (all it does is reduce the multiplicity of the root at $x=0$ by 1) and just look at the rest, $P(x) = \frac{1}{p}((x^{p^{n-1}}+1)^p - (x^{p^n}+1))$.

Substituting $y = x^{p^{n-1}}$, this becomes $\frac{1}{p}((y+1)^p - (y^p+1))$; since $\frac{1}{p}\binom{p}{k} = \frac{1}{k}\binom{p-1}{k-1}$, this is equal to $P(x) = \frac{1}{1}\binom{p-1}{0}y^{p-1} + \cdots + \frac{1}{p-1}\binom{p-1}{p-2}y$. (We work mod p now; the ps can be cancelled before modding out.)

We now show that $P(x)$ has no integer roots of multiplicity greater than 2, by considering the root multiplicities of y times its reversal, or $Q(x) = \frac{1}{p-1}\binom{p-1}{p-2}y^{p-1} + \cdots + \frac{1}{1}\binom{p-1}{0}y$.

Note that some polynomial P has a root of multiplicity m at x iff P and its first $m-1$ derivatives all have zeroes at x. (We're using the formal derivatives here - we can prove this algebraically over $\mathbb{Z}$ mod p, if

$m < p$.) The derivative of Q is $\binom{p-1}{p-2}y^{p-2} + \cdots + \binom{p-1}{0}$, or $(y+1)^{p-1} - y^{p-1}$, which has as a root every residue except 0 and -1 by Fermat's little theorem; the second derivative is a constant multiple of $(y+1)^{p-2} - y^{p-2}$, which has no integer roots by Fermat's little theorem and unique inverses. Therefore, no integer root of Q has multiplicity greater than 2; we know that the factorization of a polynomial's reverse is just the reverse of its factorization, and integers have inverses mod p, so $P(x)$ doesn't have integer roots of multiplicity greater than 2 either.

Factoring $P(x)$ completely in y (over some extension of $\mathbb{F}_p$), we know that two distinct factors can't share a root; thus, at most 2 factors have any given integer root, and since their degrees (in x) are each p^{n-1}, this means no integer root has multiplicity greater than $2p^{n-1}$.

However, we see that $y = 1$ is a double root of P. This is because plugging in gives $P(1) = \frac{1}{p}((1+1)^p - (1^p + 1)) = \frac{1}{p}(2^p - 2)$; by the condition, p^2 divides $2^p - 2$, so this is zero mod p. Since 1 is its own inverse, it's a root of Q as well, and it's a root of Q's derivative so it's a double root (so $(y-1)^2$ is part of Q's factorization). Reversing, $(y-1)^2$ is part of P's factorization as well.

Applying a well-known fact, $y - 1 = x^{p^{n-1}} - 1 = (x-1)^{p^{n-1}}$ modulo p, so 1 is a root of P with multiplicity $2p^{n-1}$.

Since adding back in the factor of $1/x$ doesn't change this multiplicity, our answer is therefore $2p^{n-1}$. ■

This second solution was suggested by Alex Smith.

N8

We define the *Fibonacci sequence* $\{F_n\}_{n\ge 0}$ by $F_0 = 0$, $F_1 = 1$, and for $n \ge 2$, $F_n = F_{n-1} + F_{n-2}$; we define the *Stirling number of the second kind* $S(n,k)$ as the number of ways to partition a set of $n \ge 1$ distinguishable elements into $k \ge 1$ indistinguishable nonempty subsets.

For every positive integer n, let $t_n = \sum_{k=1}^{n} S(n,k)F_k$. Let $p \ge 7$ be a prime. Prove that

$$t_{n+p^{2p}-1} \equiv t_n \pmod p$$

for all $n \ge 1$.

Victor Wang

Solution. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. By convention we extend the definition to all $n, k \ge 0$ so that $S(0,0) = 1$ and for $m > 0$, $S(m,0) = S(0,m) = 0$. It will also be convenient to define the *falling factorial* $(x)_n = x(x-1)\cdots(x-n+1)$, where we take $(x)_0 = 1$. Then we can extend our sequence to t_0 by defining $t_n = \sum_{k=0}^{n} S(n,k)F_k$ instead (the $k = 0$ term vanishes for positive n).

A simple combinatorial interpretation yields the polynomial identity $\sum_{k=0}^{n} S(n,k)(x)_k = x^n$ (it is enough to establish the result just for positive integer x). Inspired by the methods of umbral calculus (we try to "exchange" $(x)_k, x^n$ with F_k, t_n), we consider the linear map $T : \mathbb{Z}[x] \to \mathbb{Z}$ satisfying $T((x)_k) = F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}$. Because the $(x)_k$ (for $k \ge 0$) form a basis of $\mathbb{Z}[x]$ (the standard one is $\{x^k\}_{k\ge 0}$), this uniquely determines such a map, and $t_n = T(x^n)$. Hence if $\ell = p^{2p} - 1$, we need to show that $p \mid T(x^n(1 - x^\ell))$ for all $n \ge 0$, or equivalently, that $p \mid T((x^\ell - 1)f(x))$ for all $f \in \mathbb{Z}[x]$.

Throughout this solution we will work in $\mathbb{F}_p$ and use the fact that $P(x^p) - P(x)^p$ has all coefficients divisible by p for any $P \in \mathbb{Z}[x]$. It is well-known (e.g. by Binet's formula) that $p \mid F_{n+p^2-1} - F_n$ for all $n \ge 0$ since $p \ne 2, 5$. But by a simple induction on $n \ge 0$ we find that $T((x)_n f(x)) = F_{n-1}T(f(x+n)) + F_n T(xf(x+n-1))$ for all $f \in \mathbb{Z}[x]$, so taking $n = p(p^2-1)$ yields $T((x^p - x)^{p^2-1} f(x)) = F_{-1}T(f(x)) + F_0 T(xf(x-1)) = T(f(x))$, where we use the fact that $x(x-1)\cdots(x-p+1) = x^p - x$, $F_{-1} = F_1 - F_0 = 1$, and $F_0 = 0$.

Since $T([(x^p - x)^{p^2-1} - 1]f(x)) = 0$, it suffices to show that $(x^p - x)^{p^2-1} - 1 \mid x^{p^{2p}-1} - 1$ (still in $\mathbb{F}_p$, of course). It will be convenient to work modulo $(x^p - x)^{p^2-1} - 1$. First note that

$$\begin{aligned}(x^p - x)^{p^2-1} - 1 &\mid (x^p - x)^{p^2} - (x^p - x) = x^{p^3} - x^{p^2} - x^p + x \\ &\mid (x^{p^3} - x^{p^2} - x^p + x)^p + (x^{p^3} - x^{p^2} - x^p + x) = x^{p^4} - 2x^{p^2} + x,\end{aligned}$$

so it's enough to prove that $x^{p^4} - 2x^{p^2} + x \mid x^{p^{2p}} - x$ (since $\gcd(x, (x^p - x)^{p^2-1} - 1) = 1$). But $(x^{p^4} - 2x^{p^2} + x)^{p^2} - (x^{p^4} - 2x^{p^2} + x) = x^{p^6} - 3x^{p^4} + 3x^{p^2} - x$; by a simple induction, we have $x^{p^4} - 2x^{p^2} + x \mid \sum_{k=0}^{m} (-1)^k \binom{m}{k} x^{p^{2m-2k}}$ for $m \ge 2$; for $m = p$ we obtain $x^{p^4} - 2x^{p^2} + x \mid x^{p^{2p}} - x$, as desired. ∎

This problem and solution were proposed by Victor Wang.

Remark. This is based off of the classical Bell number congruence $B_{n+\frac{p^p-1}{p-1}} \equiv B_n \pmod p$, where $B_n = \sum_{k=0}^{n} S(n,k)$ is the number of ways to partition a set of n distinguishable elements into indistinguishable nonempty sets (we take $S(0,0) = 1$ and for $m > 0$, $S(m,0) = S(0,m) = 0$, to deal with zero indices). We can replace $\{F_n\}_{n\ge 0}$ with any recurrence $\{a_n\}$ satisfying $a_n = a_{n-1} + a_{n-2}$, but Fibonacci numbers will still appear in the main part of the solution. There is a similar solution working in $\mathbb{F}_{p^2}$ (using Binet's formula more directly); we encourage the reader to find it. There is also an instructive solution using the generating function $\sum_{n\ge 0} a^k S(n,k)x^n = \frac{(ax)^k}{(1-x)(1-2x)\cdots(1-kx)}$ (which holds for all $k \ge 0$, and has a simple combinatorial interpretation) for $a = \alpha, \beta$ and working in $\mathbb{F}_{p^2}$ again; we also encourage the reader to explore this line of attack and realize its connections to umbral calculus.

Sunday, June 15, 2014
8:00 AM - 12:30 PM

Problem 1. Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x+f(y)) &= g(x)+h(y) \\ g(x+g(y)) &= h(x)+f(y) \\ h(x+h(y)) &= f(x)+g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Problem 2. Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3,4,5,6\}$ and n is a positive integer. Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Problem 3. We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

(a) Fix an integer n. Prove that if $n \geq 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

(b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Time limit: 4 hours and 30 minutes.
Each problem is worth 7 points.

Ego Loss May Occur

16th ELMO

Lincoln, Nebraska

Year: **2014**

Day: **2**

Saturday, June 21, 2014
8:00 AM - 12:30 PM

Problem 4. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be real numbers strictly between 0 and 1. For any subset S of $\{1, 2, \ldots, n\}$, define

$$f(S) = \prod_{i \in S} a_i \cdot \prod_{j \notin S} (1 - a_j).$$

Suppose that $\sum_{|S| \text{ odd}} f(S) = \frac{1}{2}$. Prove that $a_k = \frac{1}{2}$ for some k. (Here the sum ranges over all subsets of $\{1, 2, \ldots, n\}$ with an odd number of elements.)

Problem 5. Let ABC be a triangle with circumcenter O and orthocenter H. Let ω_1 and ω_2 denote the circumcircles of triangles BOC and BHC, respectively. Suppose the circle with diameter $\overline{AO}$ intersects ω_1 again at M, and line AM intersects ω_1 again at X. Similarly, suppose the circle with diameter $\overline{AH}$ intersects ω_2 again at N, and line AN intersects ω_2 again at Y. Prove that lines MN and XY are parallel.

Problem 6. A $2^{2014} + 1$ by $2^{2014} + 1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares $s_1, s_2, \ldots, s_n$ such that s_i and s_{i+1} share an edge for $i = 1, 2, \ldots, n$ (here $s_{n+1} = s_1$).

What is the maximum possible number of filled black squares?

Time limit: 4 hours and 30 minutes.
Each problem is worth 7 points.

16^{th} Ego Loss May Occur

ELMO 2014
Lincoln, Nebraska

OFFICIAL SOLUTIONS

1. Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$f(x + f(y)) = g(x) + h(y)$$
$$g(x + g(y)) = h(x) + f(y)$$
$$h(x + h(y)) = f(x) + g(y)$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Proposed by Evan Chen.

Answer. For all real numbers x, $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real number.

Solution 1. Let a, b, c denote the values $f(0)$, $g(0)$ and $h(0)$. Notice that by putting $y = 0$, we can get that $f(x + a) = g(x) + c$, etc. In particular, we can write

$$h(y) = f(y - c) + b$$

and

$$g(x) = h(x - b) + a = f(x - b - c) + a + b$$

So the first equation can be rewritten as

$$f(x + f(y)) = f(x - b - c) + f(y - c) + a + 2b.$$

At this point, we may set $x = y - c - f(y)$ and cancel the resulting equal terms to obtain

$$f(y - f(y) - (b + 2c)) = -(a + 2b).$$

Since f is injective, this implies that $y - f(y) - (b + 2c)$ is constant, so that $y - f(y)$ is constant. Thus, f is linear, and $f(y) = y + a$. Similarly, $g(x) = x + b$ and $h(x) = x + c$.

Finally, we just need to notice that upon placing $x = y = 0$ in all the equations, we get $2a = b + c$, $2b = c + a$ and $2c = a + b$, whence $a = b = c$.

So, the family of solutions is $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real. One can easily verify these solutions are valid. $\blacksquare$

This problem and solution were proposed by Evan Chen.

Remark. Although it may look intimidating, this is not a very hard problem. The basic idea is to view $f(0)$, $g(0)$ and $h(0)$ as constants, and write the first equation entirely in terms of $f(x)$, much like we would attempt to eliminate variables in a standard system of equations. At this point we still had two degrees of freedom, x and y, so it seems likely that the result would be easy to solve. Indeed, we simply select x in such a way that two of the terms cancel, and the rest is working out details.

Solution 2. First note that plugging $x = f(a), y = b; x = f(b), y = a$ into the first gives $g(f(a)) + h(b) = g(f(b)) + h(a) \implies g(f(a)) - h(a) = g(f(b)) - h(b)$. So $g(f(x)) = h(x) + a_1$ for a constant a_1. Similarly, $h(g(x)) = f(x) + a_2, f(h(x)) = g(x) + a_3$.

Now, we will show that $h(h(x)) - f(x)$ and $h(h(x)) - g(x)$ are both constant. For the second, just plug in $x = 0$ to the third equation. For the first, let $x = a_3, y = k$ in the original to get $g(f(h(k))) = h(a_3) + f(k)$. But $g(f(h(k))) = h(h(k)) + a_1$, so $h(h(k)) - f(k) = h(a_3) - a_1$ is constant as desired.

Now $f(x) - g(x)$ is constant, and by symmetry $g(x) - h(x)$ is also constant. Now let $g(x) = f(x) + p, h(x) = f(x) + q$. Then we get:

$$f(x + f(y)) = f(x) + f(y) + p + q$$
$$f(x + f(y) + p) = f(x) + f(y) + q - p$$
$$f(x + f(y) + q) = f(x) + f(y) + p - q$$

Now plugging in (x, y) and (y, x) into the first one gives $f(x + f(y)) = f(y + f(x)) \implies f(x)-x = f(y)-y$ from injectivity, $f(x) = x+c$. Plugging this in gives $2p = q, 2q = p, p+q = 0$ so $p = q = 0$ and $f(x) = x+c, g(x) = x+c, h(x) = x+c$ for a constant c are the only solutions. ■

This second solution was suggested by David Stoner.

Solution 3. By putting $(x, y) = (0, a)$ we derive that $f(f(a)) = g(0) + h(a)$ for each a, and the analogous counterparts for g and h. Thus we can derive from $(x, y) = (t, g(t))$ that

$$\begin{aligned} h(f(t) + h(g(t))) &= f(f(t)) + g(g(t)) \\ &= g(0) + h(t) + h(0) + f(t) \\ &= f(f(0)) + g(t + g(t)) \\ &= h(f(0) + h(t + g(t))) \end{aligned}$$

holds for all t. Thus by injectivity of h we derive that

$$f(x) + h(g(x)) = f(0) + h(x + g(x)) \quad (*)$$

holds for every x.

Now observe that placing $(x, y) = (g(a), a)$ gives

$$g(2g(a)) = g(g(a) + g(a)) = h(g(a)) + f(a)$$

while placing $(x, y) = (g(a) + a, 0)$ gives

$$g(g(a) + a + g(0)) = h(a + g(a)) + f(0).$$

Equating this via $(*)$ and applying injectivity of g again, we find that

$$2g(a) = g(a) + a + g(0)$$

for each a, whence $g(x) = x + b$ for some real number b. We can now proceed as in the earlier solutions. ■

This third solution was suggested by Mehtaab Sawhney.

Solution 4. In the first given, let $x = a + g(0)$ and $y = b$ to obtain

$$f(a + g(0) + f(b)) = g(a + g(0)) + h(b) = h(a) + h(b) + f(0).$$

Swapping the roles of a and b, we discover that

$$f(b + g(0) + f(a)) = f(a + g(0) + f(b)).$$

But f is injective; this implies $f(x) - x$ is constant, and we can the proceed as in the previous solutions. ■

This fourth solution was suggested by alibez.

2. Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3, 4, 5, 6\}$ and n is a positive integer. Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Proposed by Matthew Babbitt.

Solution. First, we prove a lemma.

Lemma 1. *Let $a_0 > a_1 > a_2 > \cdots > a_n$ be positive integers such that $a_0 - a_n < a_1 + a_2 + \cdots + a_n$. Then for some $1 \le i \le n$, we have*

$$0 \le a_0 - (a_1 + a_2 + \cdots + a_i) < a_i.$$

Proof. Proceed by contradiction; suppose the inequalities are all false. Use induction to show that $a_0 - (a_1 + \cdots + a_i) \ge a_i$ for each i. This becomes a contradiction at $i = n$. □ □

Let N be the integer we want to express in this form. We will prove the result by strong induction on N. The base cases will be $3 \le N \le 10 = 6 + 3 + 1$.

Let $x_1 > x_2 > x_3 > x_4$ be the largest powers of $3, 4, 5, 6$ less than $N - 3$, in some order. If one of the inequalities of the form

$$3 \le N - (x_1 + \cdots + x_k) < x_k + 3; \quad 1 \le k \le 4$$

is true, then we are done, since we can subtract of $x_1, \ldots, x_k$ from N to get an N' with $3 \le N' < N$ and then apply the inductive hypothesis; the construction for N' cannot use any of $\{x_1, \ldots, x_k\}$ since $N' - x_k < 3$.

To see that this is indeed the case, first observe that $N - 3 > x_1$ by construction and compute

$$x_1 + x_2 + x_3 + x_4 + x_4 \ge (N - 3) \cdot \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{6}\right) > N - 3.$$

So the hypothesis of the lemma applies with $a_0 = N - 3$ and $a_i = x_i$ for $1 \le i \le 4$.

Thus, we are done by induction. ■

This problem and solution were proposed by Matthew Babbitt.

Remark. While the approach of subtracting off large numbers and inducting is extremely natural, it is not immediately obvious that one should consider $3 \le N - (x_1 + \cdots + x_k) < x_k + 3$ rather than the stronger bound $3 \le N - (x_1 + \cdots + x_k) < x_k$. In particular, the solution method above does not work if one attempts to get the latter.

3. We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

(a) Fix an integer n. Prove that if $n \ge 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

(b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Proposed by Sammy Luo.

Answer. For part (b), the maximal size is 5.

Solution. For part (a), take a regular $(n+1)$-gon and number the vertices A_i $(i = 0, 1, 2, \dots, n)$ Now invert the polygon with center A_0 with arbitrary power. This gives a very set of size n. (This can be easy checked with angle chase, PoP, etc.) By scaling and translation, this shows the existence of a very set as in part (a).

It remains to prove uniqueness. Suppose points $A = P_1, P_2, \dots, P_n = B$ and $A = X_1, X_2, \dots, X_n = B$ are two very sets on $\overline{AB}$ in that order. Assume without loss of generality that $X_1X_2 > P_1P_2$. Then $X_2X_1^2 = X_2X_3 \cdot (X_1X_n - X_1X_2) \implies X_2X_3 > P_2P_3$. Proceeding inductively, we find $X_kX_{k+1} > P_kP_{k+1}$ for $k = 1, 2, \dots, n-1$. Thus, $X_1X_n > P_1P_n$, which is a contradiction.

For (b), let $P(A)$ (let's call this power, A is a point in space) be a function returning the radius of inversion with center A. Note that the power of endpoints of 1D very sets are equal, and these powers are the highest out of all points in the very set. Let the convex hull of our very set be H. Let the vertices be $A_1, A_2, ..., A_m$. (We have $m \geq 3$ since the points are not collinear.) Since A_1, A_2 are endpoints of a 1D very set, they have equal power. Going around the hull, all vertices have equal power.

Lemma 2. *Other than the vertices, no other points lie on the edges of H, and H is equilateral.*

Proof. Say X is on A_1A_2. Then X, A_3 are on opposite ends of a 1D very set, so they have equal power. Then $P(X) = P(A_1) = P(A_2)$ contradicting the fact the endpoints have the unique highest power. Therefore, since all sides only have 2 points on them, and all vertices have equal power, all sides are equal. □

Lemma 3. *H is a regular polygon.*

Proof. Let's look at the segment A_1A_3. Say that on it we have a very set of size $k-1$. By uniqueness and the construction in (a), and the fact that $P(A_1) = P(A_2) = P(A_3)$, we get that A_1, A_2, A_3 are 3 vertices of a regular k-gon. Now the very set on segment A_1A_3 under inversion at A_2 would map to a regular k-gon. So all vertices of this regular k-gon would be in our set. Assuming that not all angles are equal taking the largest angle who is adjacent to a smaller angle, we contradict convexity. So all angles are equal. Combining this with Lemma 1, H is a regular polygon. □

Lemma 4. *H cannot have more than 4 vertices.*

Proof. Firstly, note that no points can be strictly any of the triangles $A_iA_{i+1}A_{i+2}$. (*) Or else, inverting with center A_{i+1} we get a point outside H. First, let's do if m (number of vertices) is odd. Let $m = 2k+1$. $(k \geq 2)$ Look at the inversive image of A_{2k+1} under inversion with center A_2. Say it maps to X. Note that $P(X) < P(A_i)$ for any i. Now look at the line $A_{k+2}X$. Since A_{k+2} is an endpoint, but $P(X) < P(A_{k+2})$, the other endpoint of this 1D very set must be on ray $A_{k+2}X$ past X, contradicting (*), since no other vertices of H are on this ray. Similarly for m even and ≥ 6 we can also find 2 points like these who contain no other vertices in H on the line through them. □

Lemma 5. *We only have 2 distinct very sets in 2D (up to scaling), an equilateral triangle (when $n = 3$) and a square with its center (when $n = 5$).*

Proof. First if H has 3 points, then by (*) in Lemma 3, no other points can lie inside H. So we get an equilateral triangle. If H has 4 points, then by (*) in Lemma 3, the only other point that we can add into our set is the center of the square. This also must be added, and this gives a very set of size 5. □

Hence, the maximal size is 5. ■

This problem was proposed by Sammy Luo. This solution was given by Yang Liu.

4. Let n be a positive integer and let a_1, a_2, $\dots$, a_n be real numbers strictly between 0 and 1. For any subset S of $\{1, 2, \dots, n\}$, define

$$f(S) = \prod_{i \in S} a_i \cdot \prod_{j \notin S} (1 - a_j).$$

Suppose that $\sum_{|S| \text{ odd}} f(S) = \frac{1}{2}$. Prove that $a_k = \frac{1}{2}$ for some k. (Here the sum ranges over all subsets of $\{1, 2, \dots, n\}$ with an odd number of elements.)

Proposed by Kevin Sun.

Solution. Let $X = \sum_{|S| \text{ odd}} f(S)$. Consider n unfair coins which shows heads with probabilities a_1, a_2, $\dots$, a_n. Observe that X computes the probability that an odd number of heads is obtained. Thus, it is clear that if $a_k = \frac{1}{2}$ for some k, then $X = \frac{1}{2}$.

Consequently $X - \frac{1}{2}$ is divisible by the polynomial $\prod_{i=1}^{n} \left(a_i - \frac{1}{2}\right)$. Since both are degree n, they must be equal up to scaling. Thus the conclusion follows. ■

This problem and solution were proposed by Kevin Sun.

5. Let ABC be a triangle with circumcenter O and orthocenter H. Let ω_1 and ω_2 denote the circumcircles of triangles BOC and BHC, respectively. Suppose the circle with diameter $\overline{AO}$ intersects ω_1 again at M, and line AM intersects ω_1 again at X. Similarly, suppose the circle with diameter $\overline{AH}$ intersects ω_2 again at N, and line AN intersects ω_2 again at Y. Prove that lines MN and XY are parallel.

Proposed by Sammy Luo.

Remark. Originally, the problem was phrased with respect to arbitrary isogonal conjugates in place of O and H. The modified version admits additional properties. In this version, X is the intersection of the tangents at B and C, while Y is the reflection of A across the midpoint of $\overline{BC}$.

Solution 1.

Since $\angle PMX = \angle QNY = \frac{\pi}{2}$, we derive

$$\angle PBX = \angle QBY = \angle PCX = \angle QCY = \frac{\pi}{2}.$$

Thus

$$\angle ABY = \frac{\pi}{2} + \angle ABQ = \angle PBC + \frac{\pi}{2} = \pi - \angle CBX,$$

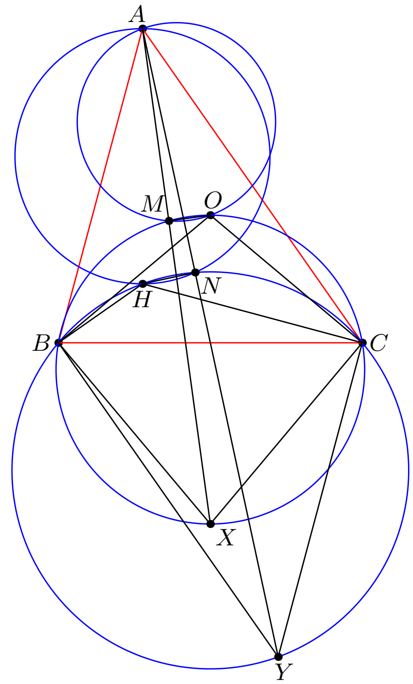
so X and Y are isogonal with respect to $\angle B$. However, similar angle chasing gives that they are isogonal with respect to $\angle C$. Thus they are isogonal conjugates with respect to ABC. (In particular, $\angle BAY = \angle XAC$.)

Also, $\angle ABY = \pi - \angle CBX = \pi - \angle CMX = \angle AMC$; hence $\triangle ABY \sim \triangle AMC$. Similarly, $\triangle ABN \sim \triangle AXC$. Thus $\frac{AN}{AB} = \frac{AC}{AX}$, and $\frac{AB}{AY} = \frac{AM}{AC}$. Multiplying, we get that $\frac{AN}{AY} = \frac{AM}{AX}$ which implies the conclusion. ■

This first solution was suggested by Kevin Sun.

Remark. The points M and N are also isogonal conjugates.

Solution 2. We apply barycentric coordinates with respect to triangle ABC (and as usual we apply Conway's Notation). Remark that the circle with diameter $\overline{AO}$ is the circumcircle of $A = (1, 0, 0)$ and the midpoints $M_B = (1 : 0 : 1)$ and $M_C = (1 : 1 : 0)$. Similarly, the circle with diameter $\overline{AH}$ is the circumcircle of $A = (1, 0, 0)$ and the feet of the altitudes $K_B = (S_C : 0 : S_A)$ and $K_C = (S_B : S_A : 0)$. It is then straightforward to derive the following



equations (using the standard formulas $2(S_{AB}+S_{BC}+S_{CA}) = a^2S_A + b^2S_B + c^2S_C = 16K^2$, where K is the area of ABC.)

$$
\begin{aligned}
(AM_BM_C) : a^2yz + b^zx + c^2xy &= (x+y+z)\left(\frac{1}{2}c^2y + \frac{1}{2}b^2z\right) \\
(AK_BK_C) : a^2yz + b^zx + c^2xy &= (x+y+z)\left(S_Bc^2y + S_Cz\right) \\
(BOC) : a^2yz + b^zx + c^2xy &= (x+y+z)\left(\frac{b^2c^2}{2S_A}x\right) \\
(BHC) : a^2yz + b^zx + c^2xy &= (x+y+z)\left(2S_ax\right)
\end{aligned}
$$

It is now straightforward to check $M = (2S_A : b^2 : c^2)$ and $N = (a^2 : 2S_A : 2S_A)$ are the coordinates of M and N (by checking that they lie on the respective required circles). Therefore $\overline{AM}$ is a symmedian, whence it is clear that the intersection of the two tangents $X = (-a^2 : b^2 : c^2)$ is the correct form for X (one can also verify directly that this lies on (BOC)). Analogously we find $Y = (-1 : 1 : 1)$ follows from $\overline{AN}$ being a median (and again this can also be verified using coordinates only).

It remains to prove that $\overline{MN}$ and $\overline{XY}$ are parallel. By normalizing and comparing the x-coordinates, we find that

$$\frac{AM}{AX} = \frac{1 - \frac{2S_A}{2S_A+b^2+c^2}}{1 - \frac{-a^2}{-a^2+b^2+c^2}} = \frac{-a^2+b^2+c^2}{-a^2+2b^2+2c^2}$$

and

$$\frac{AN}{NX} = \frac{1 - \frac{a^2}{a^2+4S_A}}{1-(-1)} = \frac{2S_A}{a^2+4S_A} = \frac{-a^2+b^2+c^2}{-a^2+2b^2+2c^2}$$

and we are done. ■

This second solution was suggested by Sam Korsky.

Remark. This solution is clearly back-constructed. If the points (and hence coordinates of) X and Y are predicted from a well-drawn diagram, then one can use single linear computations to obtain the points M and N (as opposed to quadratics). Simply parametrize M as $(t : b^2 : c^2)$ and then consider the radical axis of (AOM) and (BOC), obtained by merely subtracting the two circle's equations.

Solution 3. First, remark that $\overline{OX}$ is a diameter of (BOC), meaning X is the intersection of the tangents to (ABC) at B and C. In particular $\overline{AX}$ is a symmedian. Next, notice that $\overline{HY}$ is a diameter of (BHC), meaning Y is the reflection of A over the midpoint of $\overline{BC}$. In particular $\overline{AX}$ is a median.

Now we claim that (AMB) and (AMC) are tangent to AC and AB, respectively. This follows from angle chasing via

$$\angle ABM = \angle B - \angle MBC = \angle B - \angle MXC = \cdots = \angle MAC.$$

Similarly, we claim that (ANB) and (ANC) are both tangent to BC. This just follows from

$$\angle BAN = \angle NYC = \angle NBC.$$

Now invert at A with radius $\sqrt{AB \cdot AC}$ and then reflect around the angle bisector of A. This map sends B to C. Using the tangencies above, we see that M is mapped to Y and N is mapped to X, so $AM \cdot AX = AN \cdot AY = AB \cdot AC$ and the conclusion follows. $\blacksquare$

This third solution was suggested by Michael Ren.

This problem was proposed by Sammy Luo.

6. A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares $s_1, s_2, \dots, s_n$ such that s_i and s_{i+1} share an edge for $i = 1, 2, \dots, n$ (here $s_{n+1} = s_1$).

   What is the maximum possible number of filled black squares?

   Proposed by David Yang.

Answer. If $n = 2^m + 1$ is the dimension of the grid, the answer is $\frac{2}{3}n(n+1) - 1$. In this particular instance, $m = 2014$ and $n = 2^{2014} + 1$.

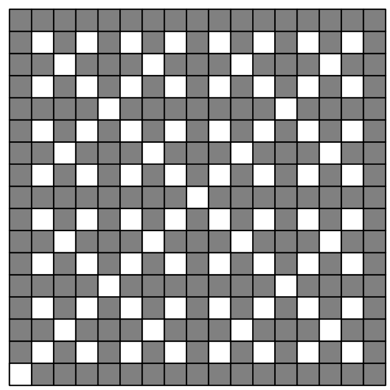
Solution 1. Let $n = 2^m + 1$. Double-counting square edges yields $3v + 1 \le 4v - e \le 2n(n+1)$, so because $n \not\equiv 1 \pmod 3$, $v \le 2n(n+1)/3 - 1$. Observe that if $3 \nmid n - 1$, equality is achieved iff (a) the graph formed by black squares is a connected forest (i.e. a tree) and (b) all but two square edges belong to at least one black square.

We prove by induction on $m \ge 1$ that equality can in fact be achieved. For $m = 1$, take an "H-shape" (so if we set the center at $(0, 0)$ in the coordinate plane, everything but $(0, \pm 1)$ is black); call this G_1. To go from G_m to G_{m+1}, fill in $(2x, 2y)$ in G_{m+1} iff (x, y) is filled in G_m, and fill in (x, y) with x, y not both even iff $x + y$ is odd (so iff one of x, y is odd and the other is even). Each "newly-created" white square has both coordinates odd, and thus borders 4 (newly-created) black squares. In particular, there are no new white squares on the border (we only have the original two from G_1). Furthermore, no two white squares share an edge in G_{m+1}, since no square with odd coordinate sum is white. Thus G_{m+1} satisfies (b). To check that (a) holds, first we show that $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ are connected in G_{m+1} iff (x_1, y_1) and (x_2, y_2) are black squares (and thus connected) in G_m (the new black squares are essentially just "bridges"). Indeed, every path in G_{m+1} alternates between coordinates with odd and even sum, or equivalently, new and old black squares. But two black squares (x_1, y_1)

and (x_2, y_2) are adjacent in G_m iff $(x_1 + x_2, y_1 + y_2)$ is black and adjacent to $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ in G_{m+1}, whence the claim readily follows. The rest is clear: the set of old black squares must remain connected in G_{m+1}, and all new black squares (including those on the boundary) border at least one (old) black square (or else G_m would not satisfy (b)), so G_{m+1} is fully connected. On the other hand, G_{m+1} cannot have any cycles, or else we would get a cycle in G_m by removing the new black squares from a cycle in G_{m+1} (as every other square in a cycle would have to have odd coordinate sum). ■

This problem and solution were proposed by David Yang.

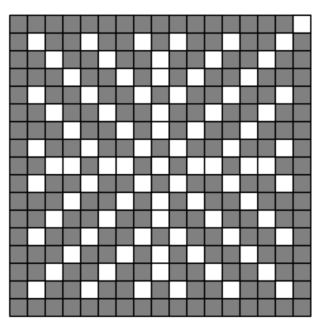
Solution 2. As above, we can show that there are at most $\frac{2}{3}n(n+1) - 1$ black squares. We provide a different construction now for $n = 2^k + 1$.



Consider the grid as a coordinate plane (x, y) where $0 \le x, y \le 2^m$. Color white the any square (x, y) for which there exists a positive integer k with $x \equiv y \equiv 2^{k-1}$ (mod 2)^k. Then, color white the square $(0, 0)$. Color the remaining squares black. Some calculations show that this is a valid construction which achieves $\frac{2}{3}n(n+1) - 1$. ■

This second solution was suggested by Kevin Sun.

Solution 3. We can achieve the bound of $\frac{2}{3}n(n+1) - 1$ as above. We will now give a construction which works for all $n = 6k + 5$. Let $M = 3k + 2$.



Consider the board as points (x, y) where $-M \le x, y \le M$. Paint white the following types of squares:

- The origin $(0, 0)$ and the corner (M, M).
- Squares of the form $(\pm a, 0)$ and $(0, \pm a)$, where $a \not\equiv 1 \pmod 3$ and $0 < a < M$.
- Any square $(\pm x, \pm y)$ such that $y - x \equiv 0 \pmod 3$ and $0 < x, y < M$.

Paint black the remaining squares. This yields the desired construction. ■

This third solution was suggested by Ashwin Sah.

Ego Loss May Occur

June 2014
Lincoln, Nebraska

Problem Shortlist

Created and Managed by Evan Chen

ELMO regulation:
The problems must be kept strictly confidential until disclosed publicly by the ELMO Committee.

The ELMO 2014 committee gratefully acknowledges the receipt of 43 problems from the following 16 authors:

Ryan Alweiss	3 problems
Matthew Babbitt	1 problem
Evan Chen	3 problems
AJ Dennis	1 problem
Shashwat Kishore	1 problem
Michael Kural	1 problem
Allen Liu	2 problems
Yang Liu	7 problems
Sammy Luo	12 problems
Robin Park	4 problems
Bobby Shen	1 problem
David Stoner	3 problems
Kevin Sun	1 problem
Victor Wang	1 problem
David Yang	1 problem
Jesse Zhang	1 problem

This list was last updated on June 30, 2014.

Part I

Problems

Algebra

A1

A1

In a non-obtuse triangle ABC, prove that

$$\frac{\sin A \sin B}{\sin C} + \frac{\sin B \sin C}{\sin A} + \frac{\sin C \sin A}{\sin B} \geq \frac{5}{2}.$$

Ryan Alweiss

A2

A2

Given positive reals a, b, c, p, q satisfying $abc = 1$ and $p \geq q$, prove that

$$p\left(a^2 + b^2 + c^2\right) + q\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq (p+q)(a+b+c).$$

AJ Dennis

A3

A3

Let a, b, c, d, e, f be positive real numbers. Given that $def + de + ef + fd = 4$, show that

$$((a+b)de + (b+c)ef + (c+a)fd)^2 \geq \ 12(abde + bcef + cafd).$$

Allen Liu

A4

A4

Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x + f(y)) &= g(x) + h(y) \\ g(x + g(y)) &= h(x) + f(y) \\ h(x + h(y)) &= f(x) + g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Evan Chen

A5

A5

Let $\mathbb{R}^*$ denote the set of nonzero reals. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ satisfying

$$f(x^2 + y) + 1 = f(x^2 + 1) + \frac{f(xy)}{f(x)}$$

for all $x, y \in \mathbb{R}^*$ with $x^2 + y \neq 0$.

Ryan Alweiss

A6

A6

Let a, b, c be positive reals such that $a + b + c = ab + bc + ca$. Prove that

$$(a + b)^{ab-bc}(b + c)^{bc-ca}(c + a)^{ca-ab} \geq a^{ca}b^{ab}c^{bc}.$$

Sammy Luo

A7

A7

Find all positive integers n with $n \geq 2$ such that the polynomial

$$P(a_1, a_2, \ldots, a_n) = a_1^n + a_2^n + \ldots + a_n^n - na_1a_2 \ldots a_n$$

in the n variables a_1, a_2, $\ldots$, a_n is irreducible over the real numbers, i.e. it cannot be factored as the product of two nonconstant polynomials with real coefficients.

Yang Liu

A8

A8

Let a, b, c be positive reals with $a^{2014} + b^{2014} + c^{2014} + abc = 4$. Prove that

$$\frac{a^{2013} + b^{2013} - c}{c^{2013}} + \frac{b^{2013} + c^{2013} - a}{a^{2013}} + \frac{c^{2013} + a^{2013} - b}{b^{2013}} \geq a^{2012} + b^{2012} + c^{2012}.$$

David Stoner

A9

A9

Let a, b, c be positive reals. Prove that

$$\sqrt{\frac{a^2(bc + a^2)}{b^2 + c^2}} + \sqrt{\frac{b^2(ca + b^2)}{c^2 + a^2}} + \sqrt{\frac{c^2(ab + c^2)}{a^2 + b^2}} \geq a + b + c.$$

Robin Park

Combinatorics

C1

C1

You have some cyan, magenta, and yellow beads on a non-reorientable circle, and you can perform only the following operations:

1. Move a cyan bead right (clockwise) past a yellow bead, and turn the yellow bead magenta.
2. Move a magenta bead left of a cyan bead, and insert a yellow bead left of where the magenta bead ends up.
3. Do either of the above, switching the roles of the words "magenta" and "left" with those of "yellow" and "right", respectively.
4. Pick any two disjoint consecutive pairs of beads, each either yellow-magenta or magenta-yellow, appearing somewhere in the circle, and swap the orders of each pair.
5. Remove four consecutive beads of one color.

Starting with the circle: "yellow, yellow, magenta, magenta, cyan, cyan, cyan", determine whether or not you can reach a) "yellow, magenta, yellow, magenta, cyan, cyan, cyan", b) "cyan, yellow, cyan, magenta, cyan", c) "magenta, magenta, cyan, cyan, cyan", d) "yellow, cyan, cyan, cyan".

Sammy Luo

C2

C2

A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares $s_1, s_2, \ldots, s_n$ such that s_i and s_{i+1} share an edge for $i = 1, 2, \ldots, n$ (here $s_{n+1} = s_1$).

What is the maximum possible number of filled black squares?

David Yang

C3

C3

We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

(a) Fix an integer n. Prove that if $n \geq 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

(b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Sammy Luo

C4

C4

Let r and b be positive integers. The game of *Monis*, a variant of Tetris, consists of a single column of red and blue blocks. If two blocks of the same color ever touch each other, they both vanish immediately. A red block falls onto the top of the column exactly once every r years, while a blue block falls exactly once every b years,

(a) Suppose that r and b are odd, and moreover the cycles are offset in such a way that no two blocks ever fall at exactly the same time. Consider a period of rb years in which the column is initially empty. Determine, in terms of r and b, the number of blocks in the column at the end.

(b) Now suppose r and b are relatively prime and $r+b$ is odd. At time $t=0$, the column is initially empty. Suppose a red block falls at times $t=r, 2r, \dots, (b-1)r$ years, while a blue block falls at times $t=b, 2b, \dots, (r-1)b$ years. Prove that at time $t=rb$, the number of blocks in the column is $|1+2(r-1)(b+r)-8S|$, where

$$S = \left\lfloor \frac{2r}{r+b} \right\rfloor + \left\lfloor \frac{4r}{r+b} \right\rfloor + \ldots + \left\lfloor \frac{(r+b-1)r}{r+b} \right\rfloor.$$

Sammy Luo

C5

C5

Let n be a positive integer. For any k, denote by a_k the number of permutations of $\{1, 2, \dots, n\}$ with exactly k disjoint cycles. (For example, if $n=3$ then $a_2=3$ since (1)(23), (2)(31), (3)(12) are the only such permutations.) Evaluate

$$a_n n^n + a_{n-1} n^{n-1} + \cdots + a_1 n.$$

Sammy Luo

C6

C6

Let f_0 be the function from $\mathbb{Z}^2$ to $\{0, 1\}$ such that $f_0(0,0)=1$ and $f_0(x,y)=0$ otherwise. For each positive integer m, let $f_m(x,y)$ be the remainder when

$$f_{m-1}(x,y) + \sum_{j=-1}^{1} \sum_{k=-1}^{1} f_{m-1}(x+j, y+k)$$

is divided by 2. Finally, for each nonnegative integer n, let a_n denote the number of pairs (x,y) such that $f_n(x,y)=1$. Find a closed form for a_n.

Bobby Shen

Geometry

G1

G1

Let ABC be a triangle with symmedian point K. Select a point A_1 on line BC such that the lines AB, AC, A_1K and BC are the sides of a cyclic quadrilateral. Define B_1 and C_1 similarly. Prove that A_1, B_1, and C_1 are collinear.

Sammy Luo

G2

G2

$ABCD$ is a cyclic quadrilateral inscribed in the circle ω. Let $AB \cap CD = E$, $AD \cap BC = F$. Let ω_1, ω_2 be the circumcircles of AEF, CEF, respectively. Let $\omega \cap \omega_1 = G$, $\omega \cap \omega_2 = H$. Show that AC, BD, GH are concurrent.

Yang Liu

G3

G3

Let $A_1A_2A_3 \cdots A_{2013}$ be a cyclic 2013-gon. Prove that for every point P not the circumcenter of the 2013-gon, there exists a point $Q \neq P$ such that $\frac{A_iP}{A_iQ}$ is constant for $i \in \{1, 2, 3, \cdots, 2013\}$.

Robin Park

G4

G4

Let $ABCD$ be a quadrilateral inscribed in circle ω. Define $E = AA \cap CD$, $F = AA \cap BC$, $G = BE \cap \omega$, $H = BE \cap AD$, $I = DF \cap \omega$, and $J = DF \cap AB$. Prove that GI, HJ, and the B-symmedian are concurrent.

Robin Park

G5

G5

Let P be a point in the interior of an acute triangle ABC, and let Q be its isogonal conjugate. Denote by ω_P and ω_Q the circumcircles of triangles BPC and BQC, respectively. Suppose the circle with diameter $\overline{AP}$ intersects ω_P again at M, and line AM intersects ω_P again at X. Similarly, suppose the circle with diameter $\overline{AQ}$ intersects ω_Q again at N, and line AN intersects ω_Q again at Y.

Prove that lines MN and XY are parallel. (Here, the points P and Q are *isogonal conjugates* with respect to $\triangle ABC$ if the internal angle bisectors of $\angle BAC$, $\angle CBA$, and $\angle ACB$ also bisect the angles $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$, respectively. For example, the orthocenter is the isogonal conjugate of the circumcenter.)

Sammy Luo

G6

G6

Let $ABCD$ be a cyclic quadrilateral with center O. Suppose the circumcircles of triangles AOB and COD meet again at G, while the circumcircles of triangles AOD and BOC meet again at H. Let ω_1 denote the circle passing through G as well as the feet of the perpendiculars from G to AB and CD. Define ω_2 analogously as the circle passing through H and the feet of the perpendiculars from H to BC and DA. Show that the midpoint of GH lies on the radical axis of ω_1 and ω_2.

Yang Liu

G7

G7

Let ABC be a triangle inscribed in circle ω with center O; let ω_A be its A-mixtilinear incircle, ω_B be its B-mixtilinear incircle, ω_C be its C-mixtilinear incircle, and X be the radical center of $\omega_A, \omega_B, \omega_C$. Let A', B', C' be the points at which ω_A, ω_B, ω_C are tangent to ω. Prove that AA', BB', CC' and OX are concurrent.

Robin Park

G8

G8

In triangle ABC with incenter I and circumcenter O, let A', B', C' be the points of tangency of its circumcircle with its A, B, C-mixtilinear circles, respectively. Let ω_A be the circle through A' that is tangent to AI at I, and define ω_B, ω_C similarly. Prove that $\omega_A, \omega_B, \omega_C$ have a common point X other than I, and that $\angle AXO = \angle OXA'$.

Sammy Luo

G9

G9

Let P be a point inside a triangle ABC such that $\angle PAC = \angle PCB$. Let the projections of P onto BC, CA, and AB be X, Y, Z respectively. Let O be the circumcenter of $\triangle XYZ$, H be the foot of the altitude from B to AC, N be the midpoint of AC, and T be the point such that $TYPO$ is a parallelogram. Show that $\triangle THN$ is similar to $\triangle PBC$.

Sammy Luo

G10

G10

We are given triangles ABC and DEF such that $D \in BC, E \in CA, F \in AB$, $AD \perp EF, BE \perp FD, CF \perp DE$. Let the circumcenter of DEF be O, and let the circumcircle of DEF intersect BC, CA, AB again at R, S, T respectively. Prove that the perpendiculars to BC, CA, AB through D, E, F respectively intersect at a point X, and the lines AR, BS, CT intersect at a point Y, such that O, X, Y are collinear.

Sammy Luo

G11

G11

Let ABC be a triangle with circumcenter O. Let P be a point inside ABC, so let the points D, E, F be on BC, AC, AB respectively so that the Miquel point of DEF with respect to ABC is P. Let the reflections of D, E, F over the midpoints of the sides that they lie on be R, S, T. Let the Miquel point of RST with respect to the triangle ABC be Q. Show that $OP = OQ$.

Yang Liu

G12

G12

Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

David Stoner

G13

G13

Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

David Stoner

Number Theory

N1

N1

Does there exist a strictly increasing infinite sequence of perfect squares $a_1, a_2, a_3, \ldots$ such that for all $k \in \mathbb{Z}^+$ we have that $13^k | a_k + 1$?

Jesse Zhang

N2

N2

Define the Fibanocci sequence recursively by $F_1 = 1$, $F_2 = 1$ and $F_{i+2} = F_i + F_{i+1}$ for all i. Prove that for all integers $b, c > 1$, there exists an integer n such that the sum of the digits of F_n when written in base b is greater than c.

Ryan Alweiss

N3

N3

Let t and n be fixed integers each at least 2. Find the largest positive integer m for which there exists a polynomial P, of degree n and with rational coefficients, such that the following property holds: exactly one of

$$\frac{P(k)}{t^k} \text{ and } \frac{P(k)}{t^{k+1}}$$

is an integer for each $k = 0, 1, \ldots, m$.

Michael Kural

N4

N4

Let $\mathbb{N}$ denote the set of positive integers, and for a function f, let $f^k(n)$ denote the function f applied k times. Call a function $f : \mathbb{N} \to \mathbb{N}$ *saturated* if

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n. Find all positive integers m for which the following holds: every saturated function f satisfies $f^{2014}(m) = m$.

Evan Chen

N5

N5

Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3, 4, 5, 6\}$ and n is a positive integer. Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Matthew Babbitt

N6

N6

Show that the numerator of

$$\frac{2^{p-1}}{p+1} - \left(\sum_{k=0}^{p-1} \frac{\binom{p-1}{k}}{(1-kp)^2}\right)$$

is a multiple of p^3 for any odd prime p.

Yang Liu

N7

N7

Find all triples (a, b, c) of positive integers such that if n is not divisible by any prime less than 2014, then $n + c$ divides $a^n + b^n + n$.

Evan Chen

N8

N8

Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that:

(i) The greatest common divisor of the sequence $f(1), f(2), \ldots$ is 1.

(ii) For all sufficiently large integers n, we have $f(n) \neq 1$ and

$$f(a)^n \mid f(a+b)^{a^{n-1}} - f(b)^{a^{n-1}}$$

for all positive integers a and b.

Yang Liu

N9

N9

Let d be a positive integer and let ε be any positive real. Prove that for all sufficiently large primes p with $\gcd(p-1, d) \neq 1$, there exists an positive integer less than p^r which is not a dth power modulo p, where r is defined by

$$\log r = \varepsilon - \frac{1}{\gcd(d, p-1)}.$$

Shashwat Kishore

N10

N10

Find all positive integer bases $b \geq 9$ so that the number

$$\frac{\overbrace{11\cdots1}^{n-1\ 1's}0\overbrace{77\cdots7}^{n-1\ 7's}8\overbrace{11\cdots1_b}^{n\ 1's}}{3}$$

is a perfect cube in base 10 for all sufficiently large positive integers n.

Yang Liu

N11

N11

Let p be a prime satisfying $p^2 \mid 2^{p-1} - 1$, and let n be a positive integer. Define

$$f(x) = \frac{(x-1)^{p^n} - (x^{p^n} - 1)}{p(x-1)}.$$

Find the largest positive integer N such that there exist polynomials $g(x)$, $h(x)$ with integer coefficients and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Victor Wang

Part II

Solutions

A2

Given positive reals a, b, c, p, q satisfying $abc = 1$ and $p \geq q$, prove that

$$p\left(a^2+b^2+c^2\right)+q\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq (p+q)(a+b+c).$$

AJ Dennis

Solution 1. First, note it suffices to prove that sum $a^2 + a^{-1}$ is at least twice sum a; in other words, the case $p = q$. Just multiply both sides by q and add $p - q$ times the inequality sum a^2 is at least sum a, which is due to Cauchy and $a + b + c \geq 3$.

So we must show that $a^2+b^2+c^2+1/a+1/b+1/c \geq 2(a+b+c)$. However, we have that $1/a+1/b+1/c \geq 3$ by AM-GM. So it suffices to have $a^2 + b^2 + c^2 + 1 + 1 + 1 \geq 2a + 2b + 2c$, but $a^2 + 1 \geq 2a$ and similar so this is obvious. ■

Solution 2. Note $\sum a^2 \geq \sum bc = \sum a^{-1}$ by AM-GM (or Cauchy-Schwarz), so $LHS \geq \frac{p+q}{2}\left(\sum a^2 + \sum bc\right)$. But

$$\sum a^2 + \sum bc = \sum(a^2 + \frac{1}{2}(ab+ac)) \geq 2\sum a^{3/2}b^{1/4}c^{1/4} = 2\sum a^{5/4}$$

Now we can finish by weighted AM-GM or (weighted) CS/Holder to get $\sum a^{5/4} \geq \sum a$, implying the result. ■

This problem and its solutions were proposed by AJ Dennis.

A3

Let a, b, c, d, e, f be positive real numbers. Given that $def + de + ef + fd = 4$, show that

$$((a+b)de + (b+c)ef + (c+a)fd)^2 \geq\ 12(abde + bcef + cafd).$$

Allen Liu

Solution 1. First, some beginning stuff. Note that the condition implies that $d = \frac{2m}{n+p}, e = \frac{2n}{m+p}, f = \frac{2p}{m+n}$ $\quad(*)$.

Also, the inequality $(a+b+c)^2 \geq (2\cos(X)+2)\cdot ab + (2\cos(Y)+2)\cdot ac + (2\cos(Z)+2)\cdot bc$, where X, Y, Z are angles of a triangle. (Note hard, just use quadratic discriminants).

Now rewrite the LHS as $(a(de+df) + b(de+ef) + c(df+ef))^2$ and then substitute $A = a(de+df), B = b(de+ef), C = c(df+ef)$. Then, the inequality becomes $(A+B+C)^2 \geq 12\sum_{cyc}\frac{BC}{(d+e)(d+f)}$. So now it suffices to find a triangle such that

$$\frac{12}{(d+e)(d+f)} \leq 2\cos(X) + 2$$

and its cyclic counterparts hold. But note that if the triangle has side lengths $y+z, x+z, x+y$, then $2\cos(X) + 2 = 4\frac{x(x+y+z)}{(x+y)(x+z)}$.

So we need

$$\frac{3}{(d+e)(d+f)} \leq \frac{x(x+y+z)}{(x+y)(x+z)}$$

So substitute in $(*)$ to get the equivalent statement

$$\frac{3(m+n)(m+p)(n+p)^2}{(m^2+mp+n^2+np)(m^2+mn+p^2+np)} \leq 4\frac{x(x+y+z)}{(x+y)(x+z)}$$

So choose $x = np(n+p), y = mp(m+p), z = mn(m+n)$. It is not hard to show that the above inequality reduces to

$$4\left(mn(m+n) + mp(m+p) + np(n+p)\right) \geq 3(m+n)(m+p)(n+p)$$

, which is immediate by expansion. ■

This problem and solution were proposed by Allen Liu.

Solution 2. Note that $de + ef + fe \geq 3$, so we have:

$$(e+f)^2(d+f)(e+d) \geq (3+d^2)(e+f)^2$$

$$\implies [(e+f)(d+f) - 3][(e+d)(e+f) - 3] \geq [3 - d(e+f)]^2$$

Therefore,

$$4\left[\frac{1}{(e+f)(d+f)} - \frac{3}{(e+f)^2(d+f)^2}\right]\left[\frac{1}{(e+d)(e+f)} - \frac{3}{(e+d)^2(e+f)^2}\right]$$
$$\geq \left[\frac{1}{(d+f)(f+e)} + \frac{1}{(d+e)(e+f)} - \frac{1}{(d+e)(d+f)} - \frac{6}{(d+e)(d+f)(e+f)^2}\right]^2$$

Therefore the quadratic expression:

$$y^2\left[\frac{1}{(e+f)(d+f)} - \frac{3}{(e+f)^2(d+f)^2}\right]$$
$$+yz\left[\frac{1}{(d+f)(f+e)} + \frac{1}{(d+e)(e+f)} - \frac{1}{(d+e)(d+f)} - \frac{6}{(d+e)(d+f)(e+f)^2}\right]$$
$$+z^2\left[\frac{1}{(e+d)(e+f)} - \frac{3}{(e+d)^2(e+f)^2}\right]$$

is always nonnegative. (The y^2 and constant coefficients are positive). So:

$$(y+z)\left[\frac{y}{(d+f)(e+f)}+\frac{z}{(d+e)(e+f)}\right] \geq \frac{yz}{e+f}+3\left(\frac{y}{(d+e)}+\frac{z}{(d+f)}\right)^2$$
$$\implies 4\left[(y+z)^2-\frac{12yz}{(d+e)(d+f)}\right] \geq \left[2(y+z)-\frac{12y}{(d+f)(e+f)}-\frac{12z}{(d+e)(e+f)}\right]^2.$$

So the quadratic expression:

$$x^2+x\left[2y+2z-\frac{12y}{(d+f)(e+f)}-\frac{12z}{(f+e)(e+f)}\right]+y^2+c^2+2yz-\frac{12yz}{(d+e)(d+f)}$$

is always nonnegative. (The x^2 and constant coefficients are positive). So:

$$(x+y+z)^2 \geq \sum_{\text{cyc}} \frac{x}{(d+e)(d+f)}$$

which is precisely what we want to show. (Let $x = a(de+df)$, et cetera.) ■

This second solution was suggested by David Stoner.

A4

Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x+f(y)) &= g(x)+h(y) \\ g(x+g(y)) &= h(x)+f(y) \\ h(x+h(y)) &= f(x)+g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Evan Chen

Answer. For all real numbers x, $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real number.

Solution 1. Let a, b, c denote the values $f(0)$, $g(0)$ and $h(0)$. Notice that by putting $y = 0$, we can get that $f(x + a) = g(x) + c$, etc. In particular, we can write

$$h(y) = f(y - c) + b$$

and

$$g(x) = h(x - b) + a = f(x - b - c) + a + b$$

So the first equation can be rewritten as

$$f(x + f(y)) = f(x - b - c) + f(y - c) + a + 2b.$$

At this point, we may set $x = y - c - f(y)$ and cancel the resulting equal terms to obtain

$$f(y - f(y) - (b + 2c)) = -(a + 2b).$$

Since f is injective, this implies that $y - f(y) - (b + 2c)$ is constant, so that $y - f(y)$ is constant. Thus, f is linear, and $f(y) = y + a$. Similarly, $g(x) = x + b$ and $h(x) = x + c$.

Finally, we just need to notice that upon placing $x = y = 0$ in all the equations, we get $2a = b + c$, $2b = c + a$ and $2c = a + b$, whence $a = b = c$.

So, the family of solutions is $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real. One can easily verify these solutions are valid. ■

This problem and solution were proposed by Evan Chen.

Remark. Although it may look intimidating, this is not a very hard problem. The basic idea is to view $f(0)$, $g(0)$ and $h(0)$ as constants, and write the first equation entirely in terms of $f(x)$, much like we would attempt to eliminate variables in a standard system of equations. At this point we still had two degrees of freedom, x and y, so it seems likely that the result would be easy to solve. Indeed, we simply select x in such a way that two of the terms cancel, and the rest is working out details.

Solution 2. First note that plugging $x = f(a), y = b; x = f(b), y = a$ into the first gives $g(f(a)) + h(b) = g(f(b)) + h(a) \implies g(f(a)) - h(a) = g(f(b)) - h(b)$. So $g(f(x)) = h(x) + a_1$ for a constant a_1. Similarly, $h(g(x)) = f(x) + a_2, f(h(x)) = g(x) + a_3$.

Now, we will show that $h(h(x)) - f(x)$ and $h(h(x)) - g(x)$ are both constant. For the second, just plug in $x = 0$ to the third equation. For the first, let $x = a_3, y = k$ in the original to get $g(f(h(k))) = h(a_3) + f(k)$. But $g(f(h(k))) = h(h(k)) + a_1$, so $h(h(k)) - f(k) = h(a_3) - a_1$ is constant as desired.

Now $f(x) - g(x)$ is constant, and by symmetry $g(x) - h(x)$ is also constant. Now let $g(x) = f(x) + p, h(x) = f(x) + q$. Then we get:

$$\begin{aligned} f(x+f(y)) &= f(x)+f(y)+p+q \\ f(x+f(y)+p) &= f(x)+f(y)+q-p \\ f(x+f(y)+q) &= f(x)+f(y)+p-q \end{aligned}$$

Now plugging in (x,y) and (y,x) into the first one gives $f(x+f(y)) = f(y+f(x)) \implies f(x)-x = f(y)-y$ from injectivity, $f(x) = x+c$. Plugging this in gives $2p = q, 2q = p, p+q = 0$ so $p = q = 0$ and $f(x) = x+c, g(x) = x+c, h(x) = x+c$ for a constant c are the only solutions. ■

This second solution was suggested by David Stoner.

Solution 3. By putting $(x,y) = (0,a)$ we derive that $f(f(a)) = g(0) + h(a)$ for each a, and the analogous counterparts for g and h. Thus we can derive from $(x,y) = (t, g(t))$ that

$$\begin{aligned} h(f(t) + h(g(t))) &= f(f(t)) + g(g(t)) \\ &= g(0) + h(t) + h(0) + f(t) \\ &= f(f(0)) + g(t + g(t)) \\ &= h(f(0) + h(t + g(t))) \end{aligned}$$

holds for all t. Thus by injectivity of h we derive that

$$f(x) + h(g(x)) = f(0) + h(x + g(x)) \quad (*)$$

holds for every x.

Now observe that placing $(x,y) = (g(a), a)$ gives

$$g(2g(a)) = g(g(a) + g(a)) = h(g(a)) + f(a)$$

while placing $(x,y) = (g(a) + a, 0)$ gives

$$g(g(a) + a + g(0)) = h(a + g(a)) + f(0).$$

Equating this via $(*)$ and applying injectivity of g again, we find that

$$2g(a) = g(a) + a + g(0)$$

for each a, whence $g(x) = x + b$ for some real number b. We can now proceed as in the earlier solutions. ■

This third solution was suggested by Mehtaab Sawhney.

Solution 4. In the first given, let $x = a + g(0)$ and $y = b$ to obtain

$$f(a + g(0) + f(b)) = g(a + g(0)) + h(b) = h(a) + h(b) + f(0).$$

Swapping the roles of a and b, we discover that

$$f(b + g(0) + f(a)) = f(a + g(0) + f(b)).$$

But f is injective; this implies $f(x) - x$ is constant, and we can the proceed as in the previous solutions. ■

This fourth solution was suggested by alibez.

A6

Let a, b, c be positive reals such that $a+b+c = ab+bc+ca$. Prove that

$$(a+b)^{ab-bc}(b+c)^{bc-ca}(c+a)^{ca-ab} \geq a^{ca}b^{ab}c^{bc}.$$

Sammy Luo

Solution 1. Note $f(x) = x\log x$ is convex. The key step: weighted Popoviciu gives

$$bf(a) + cf(b) + af(c) + (a+b+c)f\left(\frac{bc+ca+ab}{a+b+c}\right) \geq \sum_{\text{cyc}}(b+c)f\left(\frac{ab+bc}{b+c}\right).$$

Exponentiating gives

$$a^{ab} \cdot b^{bc} \cdot c^{ca} \cdot \left(\frac{bc+ca+ab}{a+b+c}\right)^{bc+ca+ab} \geq \prod_{\text{cyc}}\left(\frac{b(c+a)}{b+c}\right)^{bc+ab}$$

$$= \prod_{\text{cyc}} a^{ab+ca}(b+c)^{ab+ca-bc-ab}$$

Cancelling some terms and using $\frac{bc+ca+ab}{a+b+c} = 1$ gives

$$1 \geq \prod_{\text{cyc}} a^{ca}(a+b)^{bc-ab}$$

which rearranges to the result. ■

This problem and solution were proposed by Sammy Luo.

Solution 2. Let $a+b+c = ab+bc+ca = S$. We have

$$\prod_{\text{cyc}}\left(\frac{b(a+c)}{a+b}\right)^{ab} \leq \frac{1}{S}\sum_{\text{cyc}}\frac{ab^2(a+c)}{a+b} \leq 1$$

Where the last is true because:

$$(ab+bc+ca)^2 - (a+b+c)\left[\sum_{\text{cyc}}\frac{ab^2(a+c)}{a+b}\right] = \frac{abc(\sum_{\text{cyc}} a^3b - \sum a^2bc)}{(a+b)(b+c)(c+a)} \geq 0$$

as desired. ■

This second solution was suggested by David Stoner.

A7

Find all positive integers n with $n \geq 2$ such that the polynomial

$$P(a_1, a_2, \ldots, a_n) = a_1^n + a_2^n + \ldots + a_n^n - n a_1 a_2 \ldots a_n$$

in the n variables a_1, a_2, $\ldots$, a_n is irreducible over the real numbers, i.e. it cannot be factored as the product of two nonconstant polynomials with real coefficients.

Yang Liu

Answer. The permissible values are $n = 2$ and $n = 3$.

Solution. For $n = 2$ and $n = 3$ we respectively have the factorizations $(a_1 - a_2)^2$ and

$$\frac{1}{2}(a_1 + a_2 + a_3)(a_1^2 + a_2^2 + a_3^2 - a_1a_2 - a_2a_3 - a_3a_1).$$

For $n \geq 4$, we view P at as a polynomial in a_1 and note that the constant term is $a_2^n + a_3^n + \ldots + a_n^n$. So this polynomial must be reducible. We can set $a_5, a_6, \ldots, a_n = 0$, so now we need for $a_2^n + a_3^n + a_4^n$ to be irreducible over $\mathbb{C}$. Let $a = a_2$, $b = a_3$, $c = a_4$. Now we look at it as a polynomial in a, and it factors as

$$\prod_{i=1}^{n} \left(a + \omega_i \cdot \sqrt[n]{b^n + c^n} \right)$$

where the ω_i are the necessary roots of unity. Now we look how we can split this into two polynomials and look at their respective constant terms. So the constant terms would be $\omega(b^n + c^n)^{\frac{k}{n}}$ for some $0 < k < n$, and some root of unity ω. So the previous expression must be a polynomial, say $Q(x)$. But $(b^n + c^n)^k = Q(x)^n$. On the right-hand side, each root has multiplicity n, but since $b^n + c^n$ has no double roots, all roots on the left-hand side have multiplicity $k < n$, contradiction. ■

This problem and solution were proposed by Yang Liu.

A8

Let a, b, c be positive reals with $a^{2014} + b^{2014} + c^{2014} + abc = 4$. Prove that

$$\frac{a^{2013} + b^{2013} - c}{c^{2013}} + \frac{b^{2013} + c^{2013} - a}{a^{2013}} + \frac{c^{2013} + a^{2013} - b}{b^{2013}} \ge a^{2012} + b^{2012} + c^{2012}.$$

David Stoner

Solution. The problem follows readily from the following lemma.

Lemma 1. *Let x, y, z be positive reals, not all strictly on the same side of 1. Then $\sum \frac{x}{y} + \frac{y}{x} \ge \sum x + \frac{1}{x}$.*

Proof. WLOG $(x-1)(y-1) \le 0$; then

$$(x + y + z - 1)(x^{-1} + y^{-1} + z^{-1} - 1) \ge (xy + z)(x^{-1}y^{-1} + z) \ge 4$$

by Cauchy. Alternatively, if $x, y \ge 1 \ge z$, one may smooth z up to 1 (e.g. by differentiating with respect to z and observing that $x^{-1} + y^{-1} - 1 \le x + y - 1$) to reduce the inequality to $\frac{x}{y} + \frac{y}{x} \ge 2$. ☐

Now simply note that $\sum a^{2013} + a^{-2013} \ge \sum a^{2012} + a^{-2012}$. ■

This problem and solution were proposed by David Stoner.

Remark. An earlier (and harder) version of the problem asked to prove that

$$\left(\sum_{\text{cyc}} a(a^2 + bc)\right)\left(\sum_{\text{cyc}}\left(\frac{a}{b} + \frac{b}{a}\right)\right) \ge \left(\sum_{\text{cyc}} \sqrt{(a+1)(a^3 + bc)}\right)\left(\sum_{\text{cyc}} \sqrt{a(a+1)(a + bc)}\right).$$

However, it was vetoed by the benevolent dictator.

Here is the solution to the harder version. Let $s_i = a^i + b^i + c^i$ and $p = abc$. The key is to Cauchy out s_3's from the RHS and use the lemma (in the form $s_1 s_{-1} - 3 \ge s_1 + s_{-1}$) on the LHS to reduce the problem to

$$(s_1 + s_{-1})^2 (s_3 + 3p)^2 \ge (3 + s_1)(3 + s_{-1})(s_3 + ps_{-1})(s_3 + ps_1).$$

By AM-GM on the RHS, it suffices to prove

$$\frac{\frac{s_1 + s_{-1}}{2} + \frac{s_1 + s_{-1}}{2}}{\frac{s_1 + s_{-1}}{2} + 3} \ge \frac{s_3 + p\frac{s_1 + s_{-1}}{2}}{s_3 + 3p},$$

or equivalently, since $\frac{s_1 + s_{-1}}{2} \ge 3$, that $\frac{s_3}{p} \ge \frac{s_1 + s_{-1}}{2}$. By the lemma, this boils down to $2\sum_{\text{cyc}} a^3 \ge \sum_{\text{cyc}} a(b^2 + c^2)$, which is obvious.

A9

Let a, b, c be positive reals. Prove that

$$\sqrt{\frac{a^2(bc+a^2)}{b^2+c^2}}+\sqrt{\frac{b^2(ca+b^2)}{c^2+a^2}}+\sqrt{\frac{c^2(ab+c^2)}{a^2+b^2}}\geq a+b+c.$$

Robin Park

Remark. Equality occurs not only at $a=b=c$ but also when $a=b$ and $c=0$.

Solution. By Holder,

$$\left(\sum_{\text{cyc}}\sqrt{\frac{a^2(a^2+bc)}{b^2+c^2}}\right)^2\left(\sum_{\text{cyc}}a(a^2+bc)^2(b^2+c^2)\right)\geq\left(\sum_{\text{cyc}}a(a^2+bc)\right)^3.$$

So we need to prove that

$$\left(\sum_{\text{cyc}}a(a^2+bc)\right)^3\geq(a+b+c)^2\left(\sum_{\text{cyc}}a(a^2+bc)^2(b^2+c^2)\right).$$

Expanding this gives the following triangle in Chinese Dumbass Notation.

1
0 0
-1 9 -1
1 -3 -3 1
-1 -8 21 -8 -1
-1 8 -11 -11 8 -1
1 -8 -11 21 -11 -8 1
-1 -3 21 -11 -11 21 -3 1
0 9 -3 -8 8 -8 -3 9 0
1 0 -1 1 -1 -1 1 -1 0 1

This is the sum of the following seven inequalities:

$$0\leq\sum_{\text{cyc}}a^5(a^2-b^2)(a^2-c^2)$$

$$0\leq\sum_{\text{cyc}}b^3c^3(b+c)(b-c)^2$$

$$0\leq\sum_{\text{cyc}}3abc\cdot a^4(a-b)(a-c)$$

$$0\leq\sum_{\text{cyc}}2abc\cdot a^2(a^2-b^2)(a^2-c^2)$$

$$0\leq\sum_{\text{cyc}}2abc\cdot(b^4+c^4+2bc(b^2+c^2))(b-c)^2$$

$$0\leq\sum_{\text{cyc}}17(abc)^2\cdot a(a-b)(a-c)$$

$$0\leq\sum_{\text{cyc}}6(abc)^2\cdot a(b-c)^2.$$

Hence we're done. ■

This problem was proposed by Robin Park. This solution was given by Evan Chen.

C1

You have some cyan, magenta, and yellow beads on a non-reorientable circle, and you can perform only the following operations:

1. Move a cyan bead right (clockwise) past a yellow bead, and turn the yellow bead magenta.
2. Move a magenta bead left of a cyan bead, and insert a yellow bead left of where the magenta bead ends up.
3. Do either of the above, switching the roles of the words "magenta" and "left" with those of "yellow" and "right", respectively.
4. Pick any two disjoint consecutive pairs of beads, each either yellow-magenta or magenta-yellow, appearing somewhere in the circle, and swap the orders of each pair.
5. Remove four consecutive beads of one color.

Starting with the circle: "yellow, yellow, magenta, magenta, cyan, cyan, cyan", determine whether or not you can reach a) "yellow, magenta, yellow, magenta, cyan, cyan, cyan", b) "cyan, yellow, cyan, magenta, cyan", c) "magenta, magenta, cyan, cyan, cyan", d) "yellow, cyan, cyan, cyan".

Sammy Luo

Solution. So represent the beads in a string; write j for ma[u]j[/u]enta, i for [u]i[/u]ellow, C for cyan. Also, write k as a shorthand for ij, and 1 for (no beads). So $Ci = jC, Cj = kC, Ck = iC$. Also, $iiii = jjjj = 1$, $ij...ij = ji...ji$

We are reminded of quaternion multiplication. So what's C? We could ignore this question by moving all the Cs together; instead, we interpret the string as a series of operations (applied from left to right) to perform on a quaternion. Note that if a yellow bead corresponds to left multiplying by i and a magenta bead by j, i.e. an i in the string transforms $x = a + bi + cj + dk$ to $ix = -b + ai - dj + ck$, where $a, b, c, d \in \mathbb{R}$, then the operation $C(x) = a + ci + dj + bk$ that cyclicly permutes the i, j, k components satisfies

$$i(C(x)) = -c + ai - bj + dk = C(-c + di + aj - bk) = C(j(x)).$$

So $Ci = jC$ in the beads; similarly, $Cj = kC, Ck = iC$ as wanted.

So we let this be the cyan operation. Then, starting with the general quaternion $x = a + bi + cj + dk$, the initial state of the bead string, $iijjCCC$, gives $C(C(C(j(j(i(i(x))))))) = x$, since $C^3 = 1$. Since all the beads are invertible, starting the string at any other place in the circle will still produce the identity; all the allowed bead operations preserve the fact that the bead string composes to an identity (since removing 4 cyan beads will never be possible). Now we can check that the other strings do not compose to the identity.

- The first one is $ijijCCC$ which is multiplication by -1.
- The second is $CiCjC = jCkCC = jiCCC$, which is left multiplication by k.
- The third is $jjCCC$, again multiplication by -1.
- The fourth is $iCCC$, left multiplication by i.

So all are impossible. ■

This problem and solution were proposed by Sammy Luo.

C2

A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares $s_1, s_2, \ldots, s_n$ such that s_i and s_{i+1} share an edge for $i = 1, 2, \ldots, n$ (here $s_{n+1} = s_1$).

What is the maximum possible number of filled black squares?

David Yang

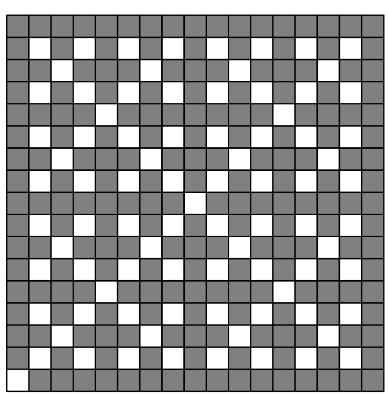
Answer. If $n = 2^m + 1$ is the dimension of the grid, the answer is $\frac{2}{3}n(n+1) - 1$. In this particular instance, $m = 2014$ and $n = 2^{2014} + 1$.

Solution 1. Let $n = 2^m + 1$. Double-counting square edges yields $3v + 1 \le 4v - e \le 2n(n+1)$, so because $n \not\equiv 1 \pmod 3$, $v \le 2n(n+1)/3 - 1$. Observe that if $3 \nmid n - 1$, equality is achieved iff (a) the graph formed by black squares is a connected forest (i.e. a tree) and (b) all but two square edges belong to at least one black square.

We prove by induction on $m \ge 1$ that equality can in fact be achieved. For $m = 1$, take an "H-shape" (so if we set the center at $(0,0)$ in the coordinate plane, everything but $(0, \pm 1)$ is black); call this G_1. To go from G_m to G_{m+1}, fill in $(2x, 2y)$ in G_{m+1} iff (x, y) is filled in G_m, and fill in (x, y) with x, y not both even iff $x + y$ is odd (so iff one of x, y is odd and the other is even). Each "newly-created" white square has both coordinates odd, and thus borders 4 (newly-created) black squares. In particular, there are no new white squares on the border (we only have the original two from G_1). Furthermore, no two white squares share an edge in G_{m+1}, since no square with odd coordinate sum is white. Thus G_{m+1} satisfies (b). To check that (a) holds, first we show that $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ are connected in G_{m+1} iff (x_1, y_1) and (x_2, y_2) are black squares (and thus connected) in G_m (the new black squares are essentially just "bridges"). Indeed, every path in G_{m+1} alternates between coordinates with odd and even sum, or equivalently, new and old black squares. But two black squares (x_1, y_1) and (x_2, y_2) are adjacent in G_m iff $(x_1 + x_2, y_1 + y_2)$ is black and adjacent to $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ in G_{m+1}, whence the claim readily follows. The rest is clear: the set of old black squares must remain connected in G_{m+1}, and all new black squares (including those on the boundary) border at least one (old) black square (or else G_m would not satisfy (b)), so G_{m+1} is fully connected. On the other hand, G_{m+1} cannot have any cycles, or else we would get a cycle in G_m by removing the new black squares from a cycle in G_{m+1} (as every other square in a cycle would have to have odd coordinate sum). ■

This problem and solution were proposed by David Yang.

Solution 2. As above, we can show that there are at most $\frac{2}{3}n(n+1) - 1$ black squares. We provide a different construction now for $n = 2^k + 1$.

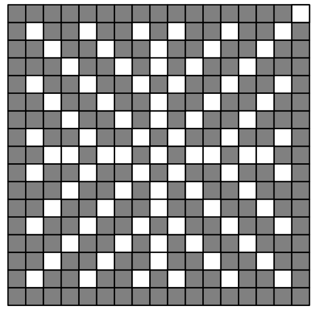


Consider the grid as a coordinate plane (x, y) where $0 \le x, y \le 2^m$. Color white the any square (x, y) for which there exists a positive integer k with $x \equiv y \equiv 2^{k-1} \pmod 2)^k$. Then, color white the square $(0, 0)$.

Color the remaining squares black. Some calculations show that this is a valid construction which achieves $\frac{2}{3}n(n+1)-1$. ■

This second solution was suggested by Kevin Sun.

Solution 3. We can achieve the bound of $\frac{2}{3}n(n+1)-1$ as above. We will now give a construction which works for all $n = 6k+5$. Let $M = 3k+2$.



Consider the board as points (x, y) where $-M \le x, y \le M$. Paint white the following types of squares:

- The origin $(0,0)$ and the corner (M, M).
- Squares of the form $(\pm a, 0)$ and $(0, \pm a)$, where $a \not\equiv 1 \pmod 3$ and $0 < a < M$.
- Any square $(\pm x, \pm y)$ such that $y - x \equiv 0 \pmod 3$ and $0 < x, y < M$.

Paint black the remaining squares. This yields the desired construction. ■

This third solution was suggested by Ashwin Sah.

C3

We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

(a) Fix an integer n. Prove that if $n \geq 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

(b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Sammy Luo

Answer. For part (b), the maximal size is 5.

Solution. For part (a), take a regular $(n+1)$-gon and number the vertices A_i $(i = 0, 1, 2, \ldots, n)$ Now invert the polygon with center A_0 with arbitrary power. This gives a very set of size n. (This can be easy checked with angle chase, PoP, etc.) By scaling and translation, this shows the existence of a very set as in part (a).

It remains to prove uniqueness. Suppose points $A = P_1, P_2, \ldots, P_n = B$ and $A = X_1, X_2, \ldots, X_n = B$ are two very sets on $\overline{AB}$ in that order. Assume without loss of generality that $X_1X_2 > P_1P_2$. Then $X_2X_1^2 = X_2X_3 \cdot (X_1X_n - X_1X_2) \implies X_2X_3 > P_2P_3$. Proceeding inductively, we find $X_kX_{k+1} > P_kP_{k+1}$ for $k = 1, 2, \ldots, n-1$. Thus, $X_1X_n > P_1P_n$, which is a contradiction.

For (b), let $P(A)$ (let's call this power, A is a point in space) be a function returning the radius of inversion with center A. Note that the power of endpoints of 1D very sets are equal, and these powers are the highest out of all points in the very set. Let the convex hull of our very set be H. Let the vertices be $A_1, A_2, ..., A_m$. (We have $m \geq 3$ since the points are not collinear.) Since A_1, A_2 are endpoints of a 1D very set, they have equal power. Going around the hull, all vertices have equal power.

Lemma 1. *Other than the vertices, no other points lie on the edges of H, and H is equilateral.*

Proof. Say X is on A_1A_2. Then X, A_3 are on opposite ends of a 1D very set, so they have equal power. Then $P(X) = P(A_1) = P(A_2)$ contradicting the fact the endpoints have the unique highest power. Therefore, since all sides only have 2 points on them, and all vertices have equal power, all sides are equal. □

Lemma 2. *H is a regular polygon.*

Proof. Let's look at the segment A_1A_3. Say that on it we have a very set of size $k-1$. By uniqueness and the construction in (a), and the fact that $P(A_1) = P(A_2) = P(A_3)$, we get that A_1, A_2, A_3 are 3 vertices of a regular k-gon. Now the very set on segment A_1A_3 under inversion at A_2 would map to a regular k-gon. So all vertices of this regular k-gon would be in our set. Assuming that not all angles are equal taking the largest angle who is adjacent to a smaller angle, we contradict convexity. So all angles are equal. Combining this with Lemma 1, H is a regular polygon. □

Lemma 3. *H cannot have more than 4 vertices.*

Proof. Firstly, note that no points can be strictly any of the triangles $A_iA_{i+1}A_{i+2}$. (*) Or else, inverting with center A_{i+1} we get a point outside H. First, let's do if m (number of vertices) is odd. Let $m = 2k+1$. $(k \geq 2)$ Look at the inversive image of A_{2k+1} under inversion with center A_2. Say it maps to X. Note that $P(X) < P(A_i)$ for any i. Now look at the line $A_{k+2}X$. Since A_{k+2} is an endpoint, but $P(X) < P(A_{k+2})$, the other endpoint of this 1D very set must be on ray $A_{k+2}X$ past X, contradicting (*), since no other vertices of H are on this ray. Similarly for m even and ≥ 6 we can also find 2 points like these who contain no other vertices in H on the line through them. □

Lemma 4. *We only have 2 distinct very sets in 2D (up to scaling), an equilateral triangle (when $n = 3$) and a square with its center (when $n = 5$).*

Proof. First if H has 3 points, then by (*) in Lemma 3, no other points can lie inside H. So we get an equilateral triangle. If H has 4 points, then by (*) in Lemma 3, the only other point that we can add into our set is the center of the square. This also must be added, and this gives a very set of size 5. □

Hence, the maximal size is 5. ■

This problem was proposed by Sammy Luo. This solution was given by Yang Liu.

C4

Let r and b be positive integers. The game of *Monis*, a variant of Tetris, consists of a single column of red and blue blocks. If two blocks of the same color ever touch each other, they both vanish immediately. A red block falls onto the top of the column exactly once every r years, while a blue block falls exactly once every b years,

(a) Suppose that r and b are odd, and moreover the cycles are offset in such a way that no two blocks ever fall at exactly the same time. Consider a period of rb years in which the column is initially empty. Determine, in terms of r and b, the number of blocks in the column at the end.

(b) Now suppose r and b are relatively prime and $r+b$ is odd. At time $t=0$, the column is initially empty. Suppose a red block falls at times $t = r, 2r, \ldots, (b-1)r$ years, while a blue block falls at times $t = b, 2b, \ldots, (r-1)b$ years. Prove that at time $t = rb$, the number of blocks in the column is $|1 + 2(r-1)(b+r) - 8S|$, where

$$S = \left\lfloor \frac{2r}{r+b} \right\rfloor + \left\lfloor \frac{4r}{r+b} \right\rfloor + \ldots + \left\lfloor \frac{(r+b-1)r}{r+b} \right\rfloor.$$

Sammy Luo

Remark. The second part of this problem was suggested by Allen Liu.

Answer. The answer is $2\gcd(r, b)$.

Solution 1. Consider strings of letters x, y, cancelling xx, Here yy. x, y correspond to red, blue blocks, respectively. I'll denote a way for the blocks to fall by (r, b, C), so r is the years between cycle of red blocks, b is cycle between blue blocks, and C is the cycle offset, more specifically how many years after the first red block falls does the first blue block fall. $C < 0$ is possible, that just means that the first blue block falls earlier than the first red block. To do this, we induct on $r + b$. Assume, $gcd(r, b) = 1$.

Now, let $r > b$ and $r = bk + q$, $0 \le q < b$. We have 2 similar cases to consider:

Case 1: q is odd. First we'll do if $C > 0$, and then by the problem statement, $C < b$. We'll actually show that this falling situation is the same as $(q, b, C) = (b, q, -C)$, and then we'll finish this case by induction. In this case, it's easy to see that the falling will result in a sequence like

$$x(y \ldots y)x(y \ldots y) \ldots x(y \ldots y).$$

Note that the $(y \ldots y)$ each have length either k or $k+1$, with exactly q of those strings having length $k+1$ and the other $b-q$ having length k. Note that k is even. Now for each of the $(y \ldots y)$ strings, reduce them to a single letter depending on parity. Now we are left with q y's and still b x's. We show the resultant string is equal to (q, b, C).

This is actually pretty clear using simple remainder arguments. Say that the first x block fell at time 0. Just note that the length of (some y) was $k+1$ iff the first y in the string of (some y) fell at time t and $0 < t \pmod r < q$ (then $t + kb < kb + q = r$, so another x would still have not appeared, but will appear next). So seeing all this, my claim becomes equivalent to the following assertion: Let l be the smallest positive integer such that $0 < (C + l \cdot b) \pmod r < q$. Let $t = (C + l \cdot b) \pmod r$ Let $j = \frac{(C + l \cdot b) - t}{r}$. Then j is also the smallest positive integer such that $(j+1) \cdot q > C$. The proof of this is pretty silly. Then $jr + t = C + l \cdot b$. Taking $\pmod b$ gives $C \equiv jq + t$, and since $0 < C < b$, $C \le jq + t < q(j+1)$. The converse follows from the fact that for anytime the 2 sides match $\pmod b$, we can solve for l. Why is it equivalent? Well, the first time $k+1$ y's appear consecutively in the initial sequence is when $0 < t \pmod r < q$, and the first time (since $k+1$ is odd) a y would appear in the reduced sequence is when $q(j+1) > C$. And these match! For the rest, just rotate the sequence and keep going. Now induction gives that it reduces to the string xy or yx.

Ok, now $C < 0$. So then our sequence would be $yyxyyyyxyyyxyy$ or something like that. What we do is the following: We rotate it by putting stuff on the back end, and then use the case $C > 0$, and associativity of cancellation:

$$\begin{aligned} yyxyyyyxyyyxyy &= (yyx)(xyy)yyxyyyyxyyyxyy \\ &= (yyx)(xyyyyxyyyyxyyy)(xyy) \\ &= (yyx)(xy)(xyy) \\ &= yx. \end{aligned}$$

(Computations show that it always ends up this way). So $C < 0$ is finished.

Case 2: q is even. Similar remainder arguments as above show that if $C > 0$, As above, it's equivalent to saying the minimal j with $(b-q)(j+1) > b-C$ is also the minimal j with $C+l\cdot b = j\cdot r+t$ and $q < t < b$. Taking $\pmod b$, we get $b-C \equiv j(b-q)-t$. But $0 < -t \pmod b < b-q$. So $b-C \le j(b-q)+(b-q) = (j+1)(b-q)$, as desired. ■

This first solution was suggested by Yang Liu.

Solution 2. As in Yang's solution have (r, b, C) represent the state. WLOG $r > b$ so we can set $0 < C < b$. Only $\lfloor C \rfloor$ actually matters so there are b possibilities. Before deletion, the sequence consists of b blue blocks in a cycle with some number of red blocks between each adjacent pair. We can see that taking any possible sequence and shifting the numbers of red blocks between each pair right one pair gives an equivalent sequence, but since $(r, b) = 1$ all of these are distinct, so they're the only possibilities.

So now every (r, b, C) is equivalent to (r, b, ϵ) where $0 < \epsilon < 1$, except shifted. Basically this yields xyS, where S is what would have resulted from all the nonsimultaneous blocks if we allowed $C = 0$. But by symmetry S is symmetric about its center $\frac{rb}{2}$, so everything cancels out in pairs from the center outwards, until we're left with xy.

Basically this leaves the issue of what the offset, in changing the point at which the cyclic sequence's wrap-over is broken, does. Let the unshifted string be $xySA$, where A is the part that is cut off and shifted to the left. Since SA must be a palindrome by the symmetry argument above, S is of the form $(A^{-1})(S')$, where A^{-1} is A in reverse and S' is a palindrome. Then the shifted string cancels to $AxyA^{-1}$. We claim this cancels with only two elements remaining. Indeed we can keep reducing the size of the A; since A's last element is the same as A^{-1}'s first, one of them has to cancel with one of x, y, leaving $A'yxA'^{-1}$, where $|A'| = |A| - 1$, and this continues until only xy or yx remains. ■

This second solution was suggested by Allen Liu.

This problem was proposed by Sammy Luo.

C6

Let f_0 be the function from $\mathbb{Z}^2$ to $\{0,1\}$ such that $f_0(0,0)=1$ and $f_0(x,y)=0$ otherwise. For each positive integer m, let $f_m(x,y)$ be the remainder when

$$f_{m-1}(x,y)+\sum_{j=-1}^{1}\sum_{k=-1}^{1}f_{m-1}(x+j,y+k)$$

is divided by 2. Finally, for each nonnegative integer n, let a_n denote the number of pairs (x,y) such that $f_n(x,y)=1$. Find a closed form for a_n.

Bobby Shen

Solution. Note that a_i is simply the number of odd coefficients of $A_i(x,y)=A(x,y)^i$, where $A(x,y)=(x^2+x+1)(y^2+y+1)-xy$. Throughout this proof, we work in $\mathbb{F}_2$ and repeatedly make use of the Frobenius endomorphism in the form $A_{2^k m}(x,y)=A_m(x,y)^{2^k}=A_m(x^{2^k},y^{2^k})$ (*). We advise the reader to try the following simpler problem before proceeding: "Find (a recursion for) the number of odd coefficients of $(x^2+x+1)^{2^n-1}$."

First suppose n is not of the form 2^m-1, and has $i\geq 0$ ones before its first zero from the right. By direct exponent analysis (after using (*)), we obtain $a_n=a_{\frac{n-(2^i-1)}{2}}a_{2^i-1}$. Applying this fact repeatedly, we find that $a_n=a_{2^{\ell_1}-1}\cdots a_{2^{\ell_r}-1}$, where $\ell_1,\ell_2,\dots,\ell_r$ are the lengths of the r consecutive strings of ones in the binary representation of n. (When $n=2^m-1$, this is trivially true. When $n=0$, we take $r=0$ and a_0 to be the empty product 1, by convention.)

We now restrict our attention to the case $n=2^m-1$. The key is to look at the exponents of x and y modulo 2 – in particular, $A_{2n}(x,y)=A_n(x^2,y^2)$ has only "$(0,0) \pmod 2$" terms for $i\geq 1$. This will allow us to find a recursion.

For convenience, let $U[B(x,y)]$ be the number of odd coefficients of $B(x,y)$, so $U[A_{2^n-1}(x,y)]=a_{2^n-1}$. Observe that

$$\begin{aligned}
A(x,y)&=(x^2+x+1)(y^2+y+1)-xy=(x^2+1)(y^2+1)+(x^2+1)y+x(y^2+1)\\
(x+1)A(x,y)&=(y^2+1)+(x^2+1)y+x^3(y^2+1)+(x^3+x)y\\
(x+1)(y+1)A(x,y)&=(x^2y^2+1)+(x^2y+y^3)+(x^3+xy^2)+(x^3y^3+xy)\\
(x+y)A(x,y)&=(x^2+y^2)+(x^2+1)(y^3+y)+(x^3+x)(y^2+1)+(x^3y+xy^3).
\end{aligned}$$

Hence for $n\geq 1$, we have (using (*) again)

$$\begin{aligned}
U[A_{2^n-1}(x,y)]&=U[A(x,y)A_{2^{n-1}-1}(x^2,y^2)]\\
&=U[(x+1)(y+1)A_{2^{n-1}-1}(x,y)]+U[(y+1)A_{2^{n-1}-1}(x,y)]+U[(x+1)A_{2^{n-1}-1}(x,y)]\\
&=U[(x+1)(y+1)A_{2^{n-1}-1}(x,y)]+2U[(x+1)A_{2^{n-1}-1}(x,y)].
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
U[(x+1)A_{2^n-1}]&=2U[(y+1)A_{2^{n-1}-1}]+2U[(x+1)A_{2^{n-1}-1}]=4U[(x+1)A_{2^{n-1}-1}]\\
U[(x+1)(y+1)A_{2^n-1}]&=2U[(xy+1)A_{2^{n-1}-1}]+2U[(x+y)A_{2^{n-1}-1}]=4U[(x+y)A_{2^{n-1}-1}]\\
U[(x+y)A_{2^n-1}]&=2U[(x+1)(y+1)A_{2^{n-1}-1}]+2U[(x+y)A_{2^{n-1}-1}].
\end{aligned}$$

Here we use the symmetry between x and y, and the identity $(xy+1)=y(x+y^{-1})$.) It immediately follows that

$$\begin{aligned}
U[(x+1)(y+1)A_{2^{n+1}-1}]&=4U[(x+y)A_{2^n-1}]\\
&=8U[(x+1)(y+1)A_{2^{n-1}-1}]+8\frac{U[(x+1)(y+1)A_{2^n-1}]}{4}
\end{aligned}$$

for all $n \geq 1$, and because $x-4 \mid (x+2)(x-4) = x^2-2x-8$,

$$U[A_{2^{n+2}-1}(x,y)] = 2U[A_{2^{n+1}-1}(x,y)] + 8U[A_{2^n-1}(x,y)]$$

as well. But $U[A_{2^0-1}] = 1$, $U[A_{2^1-1}] = 8$, and

$$U[A_{2^2-1}] = 4U[x+y] + 8U[x+1] = 24,$$

so the recurrence also holds for $n=0$. Solving, we obtain $a_{2^n-1} = \frac{5 \cdot 4^n - 2(-2)^n}{3}$, so we're done. ■

This problem and solution were proposed by Bobby Shen.

Remark. The number of odd coefficients of $(x^2+x+1)^n$ is the Jacobsthal sequence (OEIS A001045) (up to translation). The sequence $\{a_n\}$ in the problem also has a (rather empty) OEIS entry. It may be interesting to investigate the generalization

$$\sum_{j=-1}^{1} \sum_{k=-1}^{1} c_{j,k} f_{i-1}(x+j, y+k)$$

for 9-tuples $(c_{j,k}) \in \{0,1\}^9$. Note that when all $c_{j,k}$ are equal to 1, we get $(x^2+x+1)^n(y^2+y+1)^n$, and thus the square of the Jacobsthal sequence.

Even more generally, one may ask the following: "Let f be an integer-coefficient polynomial in $n \geq 1$ variables, and p be a prime. For $i \geq 0$, let a_i denote the number of nonzero coefficients of f^{p^i-1} (in $\mathbb{F}_p$).

Under what conditions must there always exist an infinite arithmetic progression AP of positive integers for which $\{a_i : i \in AP\}$ satisfies a linear recurrence?"

G1

Let ABC be a triangle with symmedian point K. Select a point A_1 on line BC such that the lines AB, AC, A_1K and BC are the sides of a cyclic quadrilateral. Define B_1 and C_1 similarly. Prove that A_1, B_1, and C_1 are collinear.

Sammy Luo

Solution 1. Let KA_1 intersect AC, AB at A_b, A_c respectively, and analogously define the points B_c, B_a, C_a, C_b. We claim that $A_bA_cB_cB_aC_aC_b$ is cyclic with center K. It's well known that $KA_b = KA_c$, etc. due to the antiparallelisms. Now note $\angle B_cA_cK = \angle AA_cA_b = \angle BCA = \angle B_aB_cB = \angle KB_cA_c$ so we also have $KA_c = KB_c$, etc. So all six segments from K are equal. Now Pascal on $A_bA_cB_cB_aC_aC_b$ gives A_1, B_1, C_1 collinear as wanted. ■

This problem and solution were proposed by Sammy Luo.

Solution 2. Let DEF be the triangle formed by the tangents to the circumcircle of ABC at A, B, and C. Let A', B', C' be $EF \cap BC$, $DF \cap AC$, and $DE \cap AB$, respectively. Since EF is a tangent, it is antiparallel to BC through A, so $A_1K \parallel EF$. Then $A_1B = A_1K \cdot \frac{A'B}{A'E}$, and $A_1C = A_1K \cdot \frac{A'C}{A'F}$ by similar triangles, so

$$\begin{aligned} \frac{A_1B}{A_1C}\frac{B_1C}{B_1A}\frac{C_1A}{C_1B} &= \frac{A'B \cdot A'F}{A'C \cdot A'E} \cdot \frac{B'C \cdot B'D}{B'A \cdot B'F} \cdot \frac{C'A \cdot C'E}{C'B \cdot C'D} \\ &= \frac{BA'}{A'C}\frac{CB'}{B'A}\frac{AC'}{C'B} \cdot \frac{FA'}{A'E}\frac{EC'}{C'D}\frac{DB'}{B'F} \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

by Menelaus. (DEF is collinear, since it is the symmedian line) Thus by the converse of Menelaus, A_1, B_1, and C_1 are collinear. ■

This second solution was suggested by Kevin Sun.

G2

$ABCD$ is a cyclic quadrilateral inscribed in the circle ω. Let $AB \cap CD = E$, $AD \cap BC = F$. Let ω_1, ω_2 be the circumcircles of AEF, CEF, respectively. Let $\omega \cap \omega_1 = G$, $\omega \cap \omega_2 = H$. Show that AC, BD, GH are concurrent.

Yang Liu

Solution. Let $AC \cap BD = Q$, $AC \cap GH = Q'$ (assuming $Q \neq Q'$), and let the radical center of ω, ω_1, and ω_2 be P, so P is the intersection of EF, AG, and HC. By Brokard's on $ABCD$, FQE is self-polar, so P (on EF) is on the polar of Q. Similarly, by Brokard's on $AGCH$, Q' is on the polar of P. Thus QQ' is the polar of P, so AC is the polar of P, which is clearly absurd. ■

This problem and solution were proposed by Yang Liu.

G3

Let $A_1A_2A_3\cdots A_{2013}$ be a cyclic 2013-gon. Prove that for every point P not the circumcenter of the 2013-gon, there exists a point $Q \neq P$ such that $\frac{A_iP}{A_iQ}$ is constant for $i \in \{1, 2, 3, \cdots, 2013\}$.

Robin Park

Solution. Let ω be the circumcircle of $A_1A_2A_3\cdots A_{2013}$. We just need Q such that ω is the Apollonius circle of P, Q for some ratio r. Let the center of ω be O, and let PO intersect ω at X, Y. Pick point Q on line XY such that $\frac{XP}{XQ} = \frac{YP}{YQ}$, i.e. $XPYQ$ is harmonic. Now, ω is a circle with center on PQ that has two points X, Y with the same ratio of distances to P, Q, so ω is an Apollonius circle of P, Q; the ratio of distances from any point on ω to P, Q is constant, implying the problem. ■

This problem was proposed by Robin Park. This solution was given by Sammy Luo.

G4

Let $ABCD$ be a quadrilateral inscribed in circle ω. Define $E = AA \cap CD$, $F = AA \cap BC$, $G = BE \cap \omega$, $H = BE \cap AD$, $I = DF \cap \omega$, and $J = DF \cap AB$. Prove that GI, HJ, and the B-symmedian are concurrent.

Robin Park

Solution. The main point of this problem is to show that $AICG$ is harmonic. Indeed, because of similar triangles and the Law of Sines, $AI = \frac{AD \cdot FI}{AF}$ and $CI = 2R\sin(\angle FBI) = 2R \cdot \frac{FI}{FB} \cdot \sin(\angle BID) = \frac{FI \cdot BD}{BF}$. So

$$\frac{AI}{CI} = \frac{AD}{BD} \cdot \frac{BF}{AB} = \frac{AD \cdot AB}{BD \cdot AC} = \frac{AG}{CG}$$

since it's symmetric in B, D.

Therefore, $AICG$ is harmonic. Let $AA \cap CC = K$. Note that I, G, K are collinear. By Pascal's Theorem on $AABGID$, we get that K, H, J are collinear. By the Symmedian Lemma, the B-symmedian passes through K, so HJ, IG, and the B-symmedian all pass through K ■

This problem was proposed by Robin Park. This solution was given by Yang Liu.

G5

Let P be a point in the interior of an acute triangle ABC, and let Q be its isogonal conjugate. Denote by ω_P and ω_Q the circumcircles of triangles BPC and BQC, respectively. Suppose the circle with diameter $\overline{AP}$ intersects ω_P again at M, and line AM intersects ω_P again at X. Similarly, suppose the circle with diameter $\overline{AQ}$ intersects ω_Q again at N, and line AN intersects ω_Q again at Y.

Prove that lines MN and XY are parallel. (Here, the points P and Q are *isogonal conjugates* with respect to $\triangle ABC$ if the internal angle bisectors of $\angle BAC$, $\angle CBA$, and $\angle ACB$ also bisect the angles $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$, respectively. For example, the orthocenter is the isogonal conjugate of the circumcenter.)

Sammy Luo

Solution. We are given that P and Q are isogonal conjugates.

Since $\angle PMX = \angle QNY = \frac{\pi}{2}$, we derive

$$\angle PBX = \angle QBY = \angle PCX = \angle QCY = \frac{\pi}{2}.$$

Thus

$$\angle ABY = \frac{\pi}{2} + \angle ABQ = \angle PBC + \frac{\pi}{2} = \pi - \angle CBX,$$

so X and Y are isogonal with respect to $\angle B$. However, similar angle chasing gives that they are isogonal with respect to $\angle C$. Thus they are isogonal conjugates with respect to ABC. (In particular, $\angle BAY = \angle XAC$.)

Also, $\angle ABY = \pi - \angle CBX = \pi - \angle CMX = \angle AMC$; hence $\triangle ABY \sim \triangle AMC$. Similarly, $\triangle ABN \sim \triangle AXC$. Thus $\frac{AN}{AB} = \frac{AC}{AX}$, and $\frac{AB}{AY} = \frac{AM}{AC}$. Multiplying, we get that $\frac{AN}{AY} = \frac{AM}{AX}$ which implies the conclusion. ■

This problem was proposed by Sammy Luo. This solution was given by Kevin Sun.

Remark. The points M and N are also isogonal conjugates.

G6

Let $ABCD$ be a cyclic quadrilateral with center O. Suppose the circumcircles of triangles AOB and COD meet again at G, while the circumcircles of triangles AOD and BOC meet again at H. Let ω_1 denote the circle passing through G as well as the feet of the perpendiculars from G to AB and CD. Define ω_2 analogously as the circle passing through H and the feet of the perpendiculars from H to BC and DA. Show that the midpoint of GH lies on the radical axis of ω_1 and ω_2.

Yang Liu

Solution 1. Let $F = AB\cap CD$, $E = AD\cap BC$. Let P be the intersection of the diagonals of the quadrilateral ($AC \cap BD$) Then simple angle chasing gives that $APGD$ is cyclic. (Just show that $\angle APD = \angle AGD = \angle AGO + \angle DGO$, both which are easy to find).

Similarly, $BPGC$ is cyclic. Now we show that $\angle PGO = \angle PGA + \angle OGA = \angle PDA + \angle OBA = \pi/2$.

Now by Radical Axis on $BPGC, APGD, ABCD$, we get that E, P, G are collinear. By Radical Axis on $ABGO, CDGO, ABCD$, we get that F, O, G are collinear. Therefore, $\angle EGF = \pi - \angle PGO = \pi/2$. Similarly, $\angle EHF = \pi/2$. So $EFGH$ is cyclic. Similarly, O, H, E are collinear.

Now, the finish is easy. Let M be the midpoint of GH. And let line MGH hit ω_1 at G', and ω_2 at H'. Note that $\angle EH'H = \pi/2 = \angle EGF$, and $\angle EHH' = \angle EFG$. So $\triangle EH'H \sim \triangle EGF \implies HH' = \frac{EH\cdot GF}{EF} = GG'$ by symmetry. So $MH \cdot MH' = MH \cdot (MH + HH') = MG \cdot (MG + GG') = MG \cdot MG'$, so M has the same power wrt both circles, so it's on the radical axis. ■

This problem and solution were proposed by Yang Liu.

Solution 2. Let $P = AB \cap CD, Q = AD \cap BC, R = AC \cap BD$. It's easy to show by angle chasing that the Miquel point M of a cyclic $ABCD$ with center O lies on (AOC). So G, H are the Miquel points of $ACBD, ABDC$ respectively. It's also well-known (by Brokard and a spiral similarity, see here) that G, H are then the feet of the altitudes from O to QR, RP respectively (and O is the orthocenter of PQR).

Note that ω_1, ω_2 are the circles with diameters GP, HQ respectively (due to the right angles). Now, $PQGH$ is cyclic due to the right angles, so the radical center of $(PQGH), \omega_1, \omega_2$ is $GP \cap HQ = O$. Let F be the midpoint of PQ, M the midpoint of GH, and O_1, O_2 the centers of ω_1, ω_2 respectively (so, the midpoints of PG, QH respectively). Now it suffices to show that $OM \perp O_1O_2$. But notice that O_1, O_2 are the feet of perpendiculars from F to PG, QH respectively, and so the line through O that is perpendicular to O_1O_2 is isogonal to OF w.r.t. angle POQ. But since $GHPQ$ is cyclic, GH, PQ are antiparallel wrt this angle, so since OM bisects segment GH, OM is the O-symmedian in $\triangle POQ$, and so is isogonal to OF, and thus perpendicular to O_1O_2 as wanted. So M is on the radical axis as wanted. ■

This second solution was suggested by Sammy Luo.

G7

Let ABC be a triangle inscribed in circle ω with center O; let ω_A be its A-mixtilinear incircle, ω_B be its B-mixtilinear incircle, ω_C be its C-mixtilinear incircle, and X be the radical center of $\omega_A, \omega_B, \omega_C$. Let A', B', C' be the points at which ω_A, ω_B, ω_C are tangent to ω. Prove that AA', BB', CC' and OX are concurrent.

Robin Park

Solution. Let the incenter be I, and the tangency points of the incircle to the 3 sides be T_A, T_B, T_C. Also, let ω_A be tangent to the sides AB, AC at A_B, A_C, respectively (and similar for the other circles and sides). Let the midpoints of the arcs be M_A, M_B, M_C, and the midpoints of T_A, I be N_A, etc.

It's pretty well-known that I is the midpoint of A_B, A_C, and similar. Now we show that the radical axis of ω_B, ω_C contains N_A and M_A. First we show that N_A is on the radical axis. Let (X, ω) denote the power of a point X w.r.t. some circle ω. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function such that $f(P) = (P, \omega_B) - (P, \omega_C)$. Then $f(I) = -IB_C^2 + IC_B^2$ and $f(T_A) = T_AB_C^2 - T_AC_B^2$, so it follows by Pythagorean Theorem that

$$f(I) + f(T_A) = (IC_B^2 - T_AC_B^2) - (IB_C^2 - T_AB_C^2) = IT_A^2 - IT_A^2 = 0.$$

Since f is linear in P, we have that $f(N_A) = \frac{f(I)+f(T_A)}{2} = 0$. Hence N_A lies on the radical axis of ω_B and ω_C.

Now we show that M_A lies on the radical axis. Let l_B be the length of the tangent from M_A to the circle ω_B. By Casey's Theorem on the circles B, M_A, C, ω_B, we get that

$$BM_A \cdot CB_C + CM_A \cdot BB_C = l_B \cdot BC \implies l_B = BM_A = CM_A$$

. Similarly, $l_C = BM_A = CM_A$ (tangent from M_A to ω_C), so M_A lies on their radical axis. Now by simple angle chasing, $M_AM_B \parallel N_AN_B$, so the triangles $M_AM_BM_C$ and $N_AN_BN_C$ are homothetic, so M_AN_A, M_BN_B, M_CN_C are concurrent on IO (the lines through their centers). ■

This problem was proposed by Robin Park. This solution was given by Yang Liu and Robin Park.

G8

In triangle ABC with incenter I and circumcenter O, let A', B', C' be the points of tangency of its circumcircle with its A, B, C-mixtilinear circles, respectively. Let ω_A be the circle through A' that is tangent to AI at I, and define ω_B, ω_C similarly. Prove that $\omega_A, \omega_B, \omega_C$ have a common point X other than I, and that $\angle AXO = \angle OXA'$.

Sammy Luo

Solution. For the sake of simplicity, let D, E, and F be the points of tangency of the circumcircle to the mixtilinear incircles.

Invert with respect to the incircle; $\triangle ABC$ is mapped to $\triangle A'B'C'$. Since the circumcircles of $\triangle A'B'I$, $\triangle B'C'I$, and $\triangle C'A'I$ concur at I, by a well-known lemma I is the orthocenter of $A'B'C'$. Let D', etc. be the images of D, etc., under this inversion. We claim that D' is the reflection of I over the midpoint of $B'C'$. This is clear because A', B', C', and D' are concyclic and ID is a symmedian of $\triangle IBC$, implying that ID' is a median of $\triangle IB'C'$. Therefore D' is also the antipode of A' with respect to the circumcircle of $\triangle A'B'C'$. Similarly, E' and F' are the antipodes of B' and C', respectively.

ω_A is mapped to a line parallel to $A'I$ passing through D', and ω_B, ω_C are mapped similarly. Clearly ω'_A, ω'_B, and ω'_C concur at the orthocenter of $\triangle D'E'F'$, since $B'C' \parallel E'F'$, $C'A' \parallel F'D'$, and $A'B' \parallel D'E'$. Let this point be X'. Note that $\angle X'A'I = \angle X'D'I$.

We claim that I, X', and O are collinear. If P is the circumcenter of $\triangle A'B'C'$, then note that P is the midpoint of IX' because there exists a homothety centered at O with ratio -1 sending $\triangle A'B'C'$ to $\triangle D'E'F'$ (X' is the de Longchamps point of $\triangle A'B'C'$). Hence O, I, and P are collinear and so it follows that I, X', and O are collinear.

Inverting back to our original diagram, we see that $\angle X'A'I = \angle X'D'I$ implies that $\angle AXO = \angle AXI = \angle IXD = \angle OXD$, as desired. ■

This problem was proposed by Sammy Luo. This solution was given by Robin Park.

G9

Let P be a point inside a triangle ABC such that $\angle PAC = \angle PCB$. Let the projections of P onto BC, CA, and AB be X, Y, Z respectively. Let O be the circumcenter of $\triangle XYZ$, H be the foot of the altitude from B to AC, N be the midpoint of AC, and T be the point such that $TYPO$ is a parallelogram. Show that $\triangle THN$ is similar to $\triangle PBC$.

Sammy Luo

Solution 1. Let Q be the isogonal conjugate of P with respect to ABC. It's well-known that O is the midpoint of PQ. Also, the given angle condition gives $\angle BAQ = \angle PAC = \angle PCB = \angle BCP$, so $\triangle BPC \sim \triangle BQA$. Now let B', P' be the reflections of B, P over AC, respectively, and let T' be the midpoint of QP'. We have $\triangle B'P'C \sim \triangle BPC \sim \triangle BQA$; furthermore, $B'P'C$ and BQA are oriented the same way, so their average (the triangle formed by the midpoints of the segments formed by corresponding points in the triangles), $HT'N$, is directly similar to both of them (for a proof, do some spiral similarity stuff). So it suffices to show $T' = T$. But OYT' is the medial triangle of $P'QP$, so $OT' \parallel PY$ and $YT' \parallel OP$, and so $T' = T$ and we're done. ■

This problem and solution were proposed by Sammy Luo.

Solution 2. Let Q be the reflection of P over O. It's quite well-known and easy to show that Q is the isogonal conjugate of P. Since $\angle PAC = \angle PCB$, $\angle BAQ = \angle PAC = \angle PCB = \angle BCP$. Thus $\triangle BPC \sim \triangle BQA$

Let $S = AP \cap CQ$. Since $\angle CAP = \angle ABQ$, $\triangle CAS$ is isosceles, so $SN \perp AC$. Let P' and Y' are the reflection of P and Y over NS. Since $YP \perp AC \perp NS$, $YPP'Y'$ is a rectangle. Let T' is the reflection of Y over T. Then P, P', Q, and O are the translations of Y, Y', T', and T under vector YP. Thus $Y'T' \parallel P'Q$, so $NT \parallel P'Q$ (since $Y'T'$ is the dilation by 2 from C of NT).

Thus $NT \parallel CQ$, so $\angle HNT = \angle HCQ = \angle PCB$.

Let B' be the reflection of B over PC, and let D be the foot of the perpendicular from B to PC. Then $\triangle B'PC \cong \triangle BPC \sim \triangle BQA$. If we average these triangles, we get that $\triangle BQA \sim \triangle DON$, since D, O, and N, are the midpoints of AC, PQ, and BB' respectively.

Since $NT \parallel CQ$, $\angle HNT = \angle HCQ = \angle PCB = \angle DNO$, so $\angle TNO = \angle HND$.

Now, we know that $\angle CHB = \angle CDB = \frac{\pi}{2}$, so $CHDB$ is cyclic, so $\angle NHD = \angle CHD = \pi - \angle CBD = \pi - (\frac{\pi}{2} - \angle DCB) = \frac{\pi}{2} + \angle PCB = \frac{\pi}{2} + \angle ACQ = \frac{\pi}{2} + \angle ANT = \angle NTO$. Thus $\triangle NHD \sim \triangle NTO$, so $\triangle THN \sim \triangle ODN \sim \triangle QBA \sim \triangle PBC$. ■

This second solution was suggested by Kevin Sun.

G10

We are given triangles ABC and DEF such that $D \in BC, E \in CA, F \in AB$, $AD \perp EF, BE \perp FD, CF \perp DE$. Let the circumcenter of DEF be O, and let the circumcircle of DEF intersect BC, CA, AB again at R, S, T respectively. Prove that the perpendiculars to BC, CA, AB through D, E, F respectively intersect at a point X, and the lines AR, BS, CT intersect at a point Y, such that O, X, Y are collinear.

Sammy Luo

Solution 1. Start with a triangle DEF, circumcircle ω and orthocenter H. Let $DH \cap EF = D_1$, $EH \cap DF = E_1$, $FH \cap DE = F_1$. We already showed that from this a unique triangle ABC. We first show that $HR \perp BC$ and similar stuff. To do this, phantom R', S', T' on $\odot DEF$ so that $HR' \perp R'F$ and similar for S', T'. Let $A' = DR' \cap ES'$, and similar for B', C'. By Radical Axis Theorem on $\odot FT'HF_1, \odot HS'ED_1, \omega$ we get that A', D_1, H are collinear, so $A'D \perp EF$. Since ABC is unique, $R' = R, S' = S, T' = T$. So $HR \perp BC$.

Now we show that $FS \cap ET = K, O, H$ are collinear. For this part we use complex numbers. Let ω be the unit circle. Then $h = d + e + f$. First we find s. s satisfies

$$\frac{s-e}{\overline{s-e}} = -\frac{s-h}{\overline{s-h}}$$

Using $\bar{x} = \frac{1}{x}$ for x on the unit circle, we simplify this to $\frac{s-(d+e+f)}{\frac{1}{s}-\left(\frac{de+df+ef}{def}\right)} = se$, and now we solve for s to find $s = \frac{df(d+f+2e)}{de+ef+2df}$. Now let $K' = OH \cap FS$. Since K' is on OH, we can write it's complex number as $k' = p(d+e+f)$ for a real number p. Now we compute $f - s = f - \frac{df(d+f+2e)}{de+ef+2df} = f\left(1 - \frac{d(d+f+2e)}{de+ef+2df}\right) = f\left(\frac{(f-d)(d+e)}{de+ef+2df}\right)$. Now its pretty easy to compute that $\frac{f-s}{\overline{f-s}} = -\frac{df^2(d+f+2e)}{de+ef+2df}$. So $\frac{k'-f}{\overline{k'-f}} = \frac{f-s}{\overline{f-s}} = -\frac{df^2(d+f+2e)}{de+ef+2df}$. Rearranging, we get

$$k' + \bar{k'} \cdot \frac{df^2(d+f+2e)}{de+ef+2df} = f + \frac{df(d+f+2e)}{de+ef+2df} \implies$$

$$p\left((d+e+f) + \frac{f(d+f+2e)(de+ef+df)}{e(de+ef+2df)}\right) = f\left(\frac{d^2+ef+3de+3df}{de+ef+2df}\right)$$

Now, if we be smart with some manipulation (just use distributive property a lot), we can simplify the above to (after multiplying both sides by $de + ef + 2df$),

$$p\left(\frac{def(d+e+f) + (e+f)(d+e+f)(de+ef+df) + ef(de+ef+df)}{ef}\right) = (d^2+ef+3de+3df)$$

. Now it's easy to see that p will be symmetric in e, f so ET also passes through K'.

Finally, to finish, use Pappus's Theorem on BTF, CSE. Let $BS \cap CT = Y, CF \cap BE = H, FS \cap ET = K$ are collinear. But note that O, H, K are collinear, and that X is the reflection of H over O (since $HR \perp BC$ and similar stuff). So O, X, Y are collinear, as desired. ∎

This problem and solution were proposed by Sammy Luo.

Solution 2. This is the same as above, except we will provide a synthetic proof that K, O, and H are collinear. Invert about H. H maps to the incenter of $D'E'F'$. S' is the intersection of the exterior angle bisector of E' with $(D'E'F')$, and T' is defined similarly for F'. Thus S', T' are midpoints of arcs DEF and DFE. We want to prove that H, $K' = (HF'S') \cap (HE'T')$, and the center of $D'E'F'$ are collinear. Let U be the center of this circle and $W = F'S' \cap E'T'$. Since $F'S'E'T'$ is cyclic, W lies on HK', so it suffices to show U, W, H are collinear. Let E_0, F_0 be the other arc midpoints of $D'E', D'F'$. Then Pascal on $DED_0E_0S'T'$ gives U, W, H collinear, so we are done. ∎

This second solution was suggested by Michael Kural.

G11

Let ABC be a triangle with circumcenter O. Let P be a point inside ABC, so let the points D, E, F be on BC, AC, AB respectively so that the Miquel point of DEF with respect to ABC is P. Let the reflections of D, E, F over the midpoints of the sides that they lie on be R, S, T. Let the Miquel point of RST with respect to the triangle ABC be Q. Show that $OP = OQ$.

Yang Liu

Solution 1. Let the midpoints of the sides be M_A, M_B, M_C, respectively.

Lemma 1. *Let D, E, F be points on BC, AC, AB respectively. Then there exists a point P such that such that $\angle PFB = \angle PDC = \angle PEA = \alpha$ if and only if*

$$BF^2 + CD^2 + AE^2 = BD^2 + CE^2 + AF^2 + 4K\cot\alpha$$

where K is the area of $\triangle ABC$.

Proof. We apply the Law of Cosines to the triangles $PFB, PFA, PEA, PEC, PDC, PBD$ to get the three equations

$$\begin{aligned} PF^2 + BF^2 - 2\cdot PF\cdot BF\cos\alpha &= PD^2 + BD^2 + 2\cdot PD\cdot BD\cos\alpha \\ PE^2 + AE^2 - 2\cdot PE\cdot AE\cos\alpha &= PF^2 + AF^2 + 2\cdot PF\cdot AD\cos\alpha \\ PD^2 + CD^2 - 2\cdot PD\cdot CD\cos\alpha &= PE^2 + CE^2 + 2\cdot PE\cdot CE\cos\alpha \end{aligned}$$

Summing this and rearranging terms gives

$$\begin{aligned} BF^2 + AE^2 + CD^2 &= BD^2 + CE^2 + AF^2 \\ &\qquad + 2\cos\alpha\,(PF\cdot BF + PD\cdot BD + PE\cdot AE + PF\cdot AD + PD\cdot CD + PE\cdot CE) \\ &= BD^2 + CE^2 + AF^2 + 2\cos\alpha\cdot\frac{2K}{\sin\alpha} \\ &= BD^2 + CE^2 + AF^2 + 4K\cot\alpha \end{aligned}$$

For the "if" part, just use that if we fix P, D, E, the there is only one point F on AB such that $\angle PFB = \angle PDC = \angle PEA = \alpha$. Also, the equation above only has one solution on the side AB as we move F around. So those 2 points must be the same. □

Lemma 2. *The reflections of PD, PE, PF over M_AO, M_BO, M_CO concur at Q.*

Proof. Since $\angle PFB = \angle PDC = \angle PEA$ (all cyclic quadrilaterals), we can just apply Lemma 1, and do some easy calculations to see that the reflections concur. So let the common intersection point be Q'. Then because opposite angles sum to π, $Q'SCR, Q'TAS, Q'TBR$ all are cyclic, so $Q' = Q$. □

To finish, let $QS \cap PE = Y, QT \cap PF = Z$. By easy angle chasing, $PQYZ$ is cyclic (the points are in some order). Note that $YM_B \cap ZM_C = O$. But also, since YM_B, ZM_C bisect the angles $\angle EYS, \angle FZT$ respectively, the meet at one of the arc midpoints of PQ on the circumcircle of $PQYZ$. So O is the arc midpoint of PQ on the circle $PQYZ$, so $OP = OQ$ as claimed. ■

This problem and solution were proposed by Yang Liu.

Solution 2. Let M_A, M_B, M_C be the midpoints of BC, AC, AB.

I guess we should use directed angles. Let $X = PD\cap QR$, $Y = PE\cap QS$, $Z = PF\cap QT$. Let $\alpha = \angle PDB = \angle PFA = \angle PEC$, and $\beta = \angle CRQ = \angle ASQ = \angle BTQ$. $\angle PXQ = -\angle BDP - \angle QRC = \alpha + \beta$. Similarly, $\angle PYQ = \angle PZQ = \alpha + \beta$. Thus P, Q, X, Y, and Z are concyclic.

Let $G = (AEF) \cap (AST)$, $H = (BFD) \cap (BTR)$, $I = (CDE) \cap (CRS)$. $\angle PGQ = \angle AGQ - \angle AGP = \angle ATQ - \angle AFP = \alpha + \beta$. Similarly, $\angle PHQ = \angle PIQ = \alpha + \beta$, so G, H, and I are on the circle, so P,Q,G,H,I,X,Y,Z are concyclic.

Now, I claim that AG, BH, and CI concur. Consider $\frac{\sin BAG}{\sin GAC} = \frac{\sin FAG}{\sin GAE} = \frac{\sin FEG}{\sin GFE} = \frac{FG}{GE}$. Since $\triangle FGT \sim \triangle EGS$ (due to cyclic quads), $\frac{FG}{GE} = \frac{FT}{ES}$. Thus $\frac{\sin BAG}{\sin GAC}\frac{\sin ACI}{\sin ICB}\frac{\sin CBH}{\sin HBA} = \frac{FT}{ES}\frac{ES}{DR}\frac{DR}{FT} = 1$, so by Ceva's theorem, AG, BH, and CI concur.

Also, since $\triangle FGE \sim \triangle TGS$, spiral similarity gives that $\triangle FGE \sim \triangle TGS \sim M_CGM_B$. Then AM_CGOM_B is cyclic.

Now, let $J = AG \cap BH \cap CI$. Since $\angle AM_CO = \frac{\pi}{2}$, $\angle AGO = \frac{\pi}{2}$, so $\angle JGO = \frac{\pi}{2}$. Similarly, $\angle JHO = \angle JIO = \frac{\pi}{2}$, so J, O, G, H, and I are cyclic with diameter JO. However, from earlier we have that the circumcircle of GHI contains points P, Q, X, Y, Z. Thus $GHIJOPQXYZ$ is a cyclic decagon with diameter OJ.

Then $\angle PDB = \angle PFA = \angle PGA = \angle PGJ = \angle PXJ$, so $BC \parallel JX$. Since OJ is a diameter, $OX \perp XJ$, and since M_A is a midpoint, $OM_A \perp BC$. However, $BC \parallel JX$, so M_A is on OX. However, $DM_A = RM_A$, so $\triangle DM_AX \cong \triangle RM_AX$, so $\angle DXM_A = \angle M_AXR$, so $\angle PXO = \angle OXQ$, so $\angle OPQ = -\angle OQP$, which means that $OP = OQ$. ■

This second solution was suggested by Kevin Sun.

Solution 3. Let AQ meet $APEF$ at L, BQ meet $BPDF$ at K, CQ meet $CPDE$ at G. Let the midpoint of K, Q be M, and the midpoints of the sides by M_A, M_B, M_C. Note that $KDF \sim QRT$ since

$$\angle KDF = \angle KBF = \angle QBT = \angle QRT$$

and similarly $\angle KFD = \angle QTR$, so averaging these two triangles yields another similar triangle MM_AM_C. Then $\angle M_CMM_A = \angle DKF = \pi - \angle DBF$, so BM_CMM_A is cyclic. But clearly this quadrilateral has diameter BO, so $OM \perp BM$. Thus $OQ = OK (= OL = OG)$ by similar arguments. We claim $PKQG$ is cyclic. Indeed,

$$\angle KPG + \angle KQG = 2\pi - \angle KPD - \angle GPD + \angle KQG = \angle BQC + \angle QCB + \angle CBQ = \pi$$

So this quadrilateral is cyclic. Then P lies on cyclic $QKLG$ with center O, so we are done. ■

This third solution was suggested by Michael Kural.

Solution 4. Let A', B', C' be the antipodes of A, B, C, respectively, in $(AEF), (BFD), (CDE)$ respectively; let A'', B'', C'' be the antipodes of A, B, C, respectively, in $(AST), (BTR), (CRS)$, respectively. Now, B', C' are both on the perpendicular to BC through D, and so forth. So note that B', B'' are reflections about O, since the feet from B', B'' to BC, BA are both symmetric about the corresponding midpoints.

Also, note (using directed angles): $\angle PB'B = \angle PFB = \angle PFA = \angle PEA = \angle PA'A = \angle PEC = \angle PDC = \angle PC'C$ and $\angle BPB' = \angle APA' = \angle CPC' = 90°$ so $BB'P, CC'P, AA'P$ are all directly similar; thus P is the center of a spiral similarity (with angle $90°$) from $A'B'C'$ to ABC, which we will call S_P. Similarly, Q is the center of a spiral similarity (with angle $90°$) from ABC to $A''B''C''$, which we call S_Q.

Now consider the composition S_QS_P (S_P is applied first). This maps $A'B'C'$ to $A''B''C''$. But these two triangles are reflections of each other about O, so O is at the same position relative to both (in fact, it's their center of rotation!); thus S_QS_P maps O to itself. In particular, since $A'B'C', A''B''C''$ are congruent, S_P, S_Q must have scale factors that are multiplicative inverses; say the scale factor of S_P is r.

So let O' be the image of O under S_P. So $OPO' = 90°$ and $O'QO = 90°$; $\frac{O'P}{OP} = r = \frac{O'Q}{OQ}$. This is enough to show OPO', OQO' congruent, so $OP = OQ$ as desired. ■

This fourth solution was suggested by Sammy Luo.

Remark. This is quite similar in flavor to IMO Shortlist 2012, Problem G6, and a comment given by user **proglote** in that thread can be used to solve this problem.

Remark. In fact, a further generalization of this problem of this problem is possible. Let P a point, and XYZ be its pedal triangle. A_1, B_1, and C_1 are points on BC, AC, and AB, and A_2, B_2, and C_2 are their reflections over X, Y, and Z, If the Miquel point of A_1, B_1, C_1 is P_1 and the Miquel point of A_2, B_2, C_2 is P_2, then $PP_1 = PP_2$.

G12

Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

David Stoner

Solution 1. Let J be the second intersection of ω and AC, and X be the intersection of BF and AC. It's well-known that $DJFC$ is harmonic; perspectivity wrt B implies $AJXC$ is also harmonic. Then $\frac{AJ}{JX} = \frac{AC}{CX} \implies (AJ)(CX) = (AC)(JX)$. This can be rearranged to get

$$(AJ)(CX) = (AJ + JX + XC)(JX) \implies 2(AJ)(CX) = (JX + AJ)(JX + XC) = (AX)(CJ),$$

so

$$\left(\frac{AX}{XC}\right)\left(\frac{CJ}{JA}\right) = 2.$$

But $\frac{CJ}{JA} = \frac{AD}{DB}$, so by Ceva's we have $BH = 2HC$, as desired. ■

Solution 2. Let J be the second intersection of ω and AC. It's well-known that $DJFC$ is harmonic; thus we have $(DJ)(FC) = (JF)(DC)$. By Ptolemy's, this means

$$(DF)(JC) = (DJ)(FC) + (JF)(DC) = 2(JD)(CF) \implies \left(\frac{JC}{JD}\right)\left(\frac{FD}{FC}\right) = 2.$$

Yet $JC = DB$ by symmetry, so this becomes

$$2 = \left(\frac{DB}{JD}\right)\left(\frac{FD}{FC}\right) = \left(\frac{\sin DJB}{\sin JBD}\right)\left(\frac{\sin FCD}{\sin FDC}\right) = \left(\frac{\sin DCB}{\sin ACD}\right)\left(\frac{\sin FBA}{\sin CBF}\right).$$

Thus by (trig) Ceva's we have $\frac{\sin BAH}{\sin CAH} = 2$, and since $AB = AC$ it follows that $BH = 2HC$, as desired. ■

This problem and its solutions were proposed by David Stoner.

G13

Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

David Stoner

Solution. Let α be the half-angles of $\triangle ABC$, r inradius, and u, v, w tangent lengths to the incircle. Let $T = MP \cap NQ$ so that I is the incenter of $\triangle MNT$. Then $\angle IPT = \angle IXY = \alpha = \angle IYX = \angle IQT$ gives $\triangle TIP \sim \triangle TIQ$, so P, I, Q are collinear iff $\angle TIP = 90^\circ$ iff $\angle MTN = 180^\circ - 2\alpha$ iff $\angle MIN = 180^\circ - \alpha$ iff $MI^2 = MX \cdot MN$. First suppose I is the center of γ. Since A, I are symmetric about XY, $\angle MAN = \angle MIN$. But P, I, Q are collinear iff $\angle MIN = 180^\circ - \alpha$, so because arcs AN and BM sum to 90°, P, I, Q are collinear iff arcs BM, MA have the same measure. Let $M' = CI \cap \omega$; then $\angle BM'I = \angle BM'C = 90^\circ - \angle BXI$, so $M'XIBZ$ is cyclic and $\angle M'XB = \angle M'IB = 180^\circ - \angle BIC = 45^\circ = \angle AXY$, as desired. (There are many other ways to finish as well.) Conversely, if P, I, Q are collinear, then by power of a point, $m(m+2t) = MI^2 - r^2 = MX \cdot MN - r^2 = m(m+2t+n) - r^2$, so $mn = r^2$. But we also have $m(n+2t) = uv$ and $n(m+2t) = uw$, so

$$r^2 = mn = \frac{uv - r^2}{2t}\frac{uw - r^2}{2t} = \frac{\frac{uv(u+v)}{u+v+w}}{2r\cos\alpha}\frac{\frac{uw(u+w)}{u+v+w}}{2r\cos\alpha} = \frac{r^2}{4\cos^2\alpha}\frac{(u+v)(u+w)}{vw}.$$

Simplifying using $\cos^2\alpha = \frac{u^2}{u^2+r^2} = \frac{u(u+v+w)}{(u+v)(u+w)}$, we get

$$0 = (u+v)^2(u+w)^2 - 4uvw(u+v+w) = (u(u+v+w) - vw)^2,$$

which clearly implies $(u+v)^2 + (u+w)^2 = (v+w)^2$, as desired. ■

This problem was proposed by David Stoner. This solution was given by Victor Wang.

N1

Does there exist a strictly increasing infinite sequence of perfect squares $a_1, a_2, a_3, ...$ such that for all $k \in \mathbb{Z}^+$ we have that $13^k | a_k + 1$?

Jesse Zhang

Solution. We have that 5 is a solution to $x^2 + 1 = 0 \mod 13$. Now assume that we have a solution x_k to $f(x) = x^2 + 1 = 0 \mod 13^k$. Note that $f'(x) = 2x \neq 0 \mod 13$ clearly, so by Hensel there is a solution x_{k+1} to $f(x) = x^2 + 1 = 0 \mod 13^{k+1}$. Then just add 13^{k+1} to x_{k+1} to make it strictly larger than x_k, and we're done. ■

This problem was proposed by Jesse Zhang. This solution was given by Michael Kural.

N2

Define the Fibanocci sequence recursively by $F_1 = 1$, $F_2 = 1$ and $F_{i+2} = F_i + F_{i+1}$ for all i. Prove that for all integers $b, c > 1$, there exists an integer n such that the sum of the digits of F_n when written in base b is greater than c.

Ryan Alweiss

Solution. It's well known that if N is a positive integer multiple of $b^k - 1$, then the base b digital sum of N is at least $k(b-1)$. Now just apply the lemma with k sufficiently large and pick n with $b^k - 1 \mid F_n$. ■

This problem and solution were proposed by Ryan Alweiss.

N3

Let t and n be fixed integers each at least 2. Find the largest positive integer m for which there exists a polynomial P, of degree n and with rational coefficients, such that the following property holds: exactly one of

$$\frac{P(k)}{t^k} \text{ and } \frac{P(k)}{t^{k+1}}$$

is an integer for each $k = 0, 1, ..., m$.

Michael Kural

Answer. The maximal value of m is n.

Solution 1. Note that if $t^{k+1} \parallel P(k+1)$ and $t^k \parallel P(k)$, then $t^k \parallel P(k+1) - P(k)$. A simple induction on $\deg P$ then establishes an upper bound of n. To achieve this, simply put $P(k) = t^k$ for each $0 \le k \le n$. ■

This problem and solution were proposed by Michael Kural.

Solution 2. By Lagrange Interpolation, we can find a polynomial satisfying $P(k) = t^k$ for $0 \le k \le n$ with rational coefficients. By Newtonian Interpolation, $P(n+1) = \sum_{i=0}^{n} \binom{n}{i} P(i)(-1)^{n-i}$. Taking $\pmod t$, $P(n+1) = (-1)^n \cdot P(0) \neq 0 \pmod t$. ■

This second solution was suggested by Yang Liu.

N4

Let $\mathbb{N}$ denote the set of positive integers, and for a function f, let $f^k(n)$ denote the function f applied k times. Call a function $f : \mathbb{N} \to \mathbb{N}$ *saturated* if

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n. Find all positive integers m for which the following holds: every saturated function f satisfies $f^{2014}(m) = m$.

Evan Chen

Answer. All m dividing 2014; that is, $\{1, 2, 19, 38, 53, 106, 1007, 2014\}$.

Solution. First, it is easy to see that f is both surjective and injective, so f is a permutation of the positive integers. We claim that the functions f which satisfy the property are precisely those functions which satisfy $f^n(n) = n$ for every n.

For each integer n, let $\operatorname{ord}(n)$ denote the smallest integer k such that $f^k(n)$. These orders exist since $f^{f^{f(n)}(n)}(n) = n$, so $\operatorname{ord}(n) \le f^{f(n)}(n)$; in fact we actually have

$$\operatorname{ord}(n) \mid f^{f(n)}(n) \tag{8.1}$$

as a consequence of the division algorithm.

Since f is a permutation, it is immediate that $\operatorname{ord}(n) = \operatorname{ord}(f(n))$ for every n; this implies easily that $\operatorname{ord}(n) = \operatorname{ord}\left(f^k(n)\right)$ for every integer k. In particular, $\operatorname{ord}(n) = \operatorname{ord}\left(f^{f(n)-1}(n)\right)$. But then, applying (8.1) to $f^{f(n)-1}(n)$ gives

$$\begin{aligned}
\operatorname{ord}(n) = \operatorname{ord}\left(f^{f(n)-1}(n)\right) &\mid f^{f\left(f^{f(n)-1}(n)\right)}\left(f^{f(n)-1}(n)\right) \\
&= f^{f^{f(n)}(n)+f(n)-1}(n) \\
&= f^{f(n)-1}\left(f^{f^{f(n)}(n)}(n)\right) \\
&= f^{f(n)-1}(n)
\end{aligned}$$

Inductively, then, we are able to show that $\operatorname{ord}(n) \mid f^{f(n)-k}(n)$ for every integer k; in particular, $\operatorname{ord}(n) \mid f^0(n) = n$, which implies that $f^n(n) = n$. To see that this is actually sufficient, simply note that $\operatorname{ord}(n) = \operatorname{ord}(f(n)) = \cdots$, which implies that $\operatorname{ord}(n) \mid f^k(n)$ for every k.

In particular, if $m \mid 2014$, then $\operatorname{ord}(m) \mid m \mid 2014$ and $f^{2014}(m) = m$. The construction for the other values of m (showing that they are not forced) is left as an easy exercise. ■

This problem and solution were proposed by Evan Chen.

Remark. There are many ways to express the same ideas. For instance, the following approach ("unraveling indices") also works: It's not hard to show that f is a bijection with finite cycles (when viewed as a permutation). If $C = (n_0, n_1, \ldots, n_{\ell-1})$ is one such cycle with $f(n_i) = n_{i+1}$ for all i (extending indices mod ℓ), then $f^{f^{f(n)}(n)}(n) = n$ holds on C iff $\ell \mid f^{f(n_i)}(n_i) = n_{i+n_{i+1}}$ for all i. But $\ell \mid n_j \implies \ell \mid n_{j-1+n_j} = n_{j-1}$ for fixed j, so the latter condition holds iff $\ell \mid n_i$ for all i. Thus $f^{2014}(n) = n$ is forced unless and only unless $n \nmid 2014$.

N5

Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3, 4, 5, 6\}$ and n is a positive integer. Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Matthew Babbitt

Solution. First, we prove a lemma.

Lemma 1. *Let $a_0 > a_1 > a_2 > \cdots > a_n$ be positive integers such that $a_0 - a_n < a_1 + a_2 + \cdots + a_n$. Then for some $1 \le i \le n$, we have*

$$0 \le a_0 - (a_1 + a_2 + \cdots + a_i) < a_i.$$

Proof. Proceed by contradiction; suppose the inequalities are all false. Use induction to show that $a_0 - (a_1 + \cdots + a_i) \ge a_i$ for each i. This becomes a contradiction at $i = n$. □ □

Let N be the integer we want to express in this form. We will prove the result by strong induction on N. The base cases will be $3 \le N \le 10 = 6 + 3 + 1$.

Let $x_1 > x_2 > x_3 > x_4$ be the largest powers of $3, 4, 5, 6$ less than $N - 3$, in some order. If one of the inequalities of the form

$$3 \le N - (x_1 + \cdots + x_k) < x_k + 3; \quad 1 \le k \le 4$$

is true, then we are done, since we can subtract of $x_1, \dots, x_k$ from N to get an N' with $3 \le N' < N$ and then apply the inductive hypothesis; the construction for N' cannot use any of $\{x_1, \dots, x_k\}$ since $N' - x_k < 3$.

To see that this is indeed the case, first observe that $N - 3 > x_1$ by construction and compute

$$x_1 + x_2 + x_3 + x_4 + x_4 \ge (N - 3) \cdot \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{6} \right) > N - 3.$$

So the hypothesis of the lemma applies with $a_0 = N - 3$ and $a_i = x_i$ for $1 \le i \le 4$.

Thus, we are done by induction. ■

This problem and solution were proposed by Matthew Babbitt.

Remark. While the approach of subtracting off large numbers and inducting is extremely natural, it is not immediately obvious that one should consider $3 \le N - (x_1 + \cdots + x_k) < x_k + 3$ rather than the stronger bound $3 \le N - (x_1 + \cdots + x_k) < x_k$. In particular, the solution method above does not work if one attempts to get the latter.

N6

Show that the numerator of

$$\frac{2^{p-1}}{p+1} - \left(\sum_{k=0}^{p-1} \frac{\binom{p-1}{k}}{(1-kp)^2} \right)$$

is a multiple of p^3 for any odd prime p.

Yang Liu

Solution. Remark $(1-kp)^2(1+2pk+3p^2k^2) \equiv 3k^4p^4-4k^3p^3+1 \equiv 1 \pmod{p^3}$, so $\frac{1}{(1-kp)^2} \equiv (1+2pk+3p^2k^2) \pmod{p^3}$. Thus

$$\begin{aligned}
\left(\sum_{k=0}^{p-1} \frac{\binom{p-1}{k}}{(1-kp)^2} \right) &\equiv \sum_{k=0}^{p-1} \binom{p-1}{k}(1+2pk+3p^2k^2) \pmod{p^3} \\
&= \sum_{k=0}^{p-1} \binom{p-1}{k} + \sum_{k=0}^{p-1} 2pk\binom{p-1}{k} + \sum_{k=0}^{p-1} 3p^2k^2\binom{p-1}{k} \\
&= 2^{p-1} + \sum_{k=0}^{p-1} pk\binom{p-1}{k} + \sum_{k=0}^{p-1} p(p-1-k)\binom{p-1}{k} + \sum_{k=0}^{p-1} 3p^2k^2\binom{p-1}{k} \\
&= 2^{p-1} + \sum_{k=0}^{p-1} p(p-1)\binom{p-1}{k} + \sum_{k=0}^{p-1} 3p^2k^2\binom{p-1}{k} \\
&= (p^2-p+1)2^{p-1} + \sum_{k=0}^{p-1} 3p^2k^2\binom{p-1}{k} \\
&\equiv (p^2-p+1)2^{p-1} + \sum_{k=0}^{p-1} 3p^2k^2(-1)^k \pmod{p^3} \\
&\equiv (p^2-p+1)2^{p-1} + 3p^3\frac{p-1}{2} \pmod{p^3} \\
&\equiv \frac{2^{p-1}}{p+1} \pmod{p^3}
\end{aligned}$$

■

This problem and solution were proposed by Yang Liu.

N7

Find all triples (a, b, c) of positive integers such that if n is not divisible by any prime less than 2014, then $n + c$ divides $a^n + b^n + n$.

Evan Chen

Answer. $(a, b, c) = (1, 1, 2)$.

Solution. Let p be an arbitrary prime such that $p \geq 2011 \cdot \max\{abc, 2013\}$. By the Chinese Remainder Theorem it is possible to select an integer n satisfying the following properties:

$$\begin{aligned} n &\equiv -c \pmod{p} \\ n &\equiv -1 \pmod{p-1} \\ n &\equiv -1 \pmod{q} \end{aligned}$$

for all primes $q \leq 2011$ not dividing $p - 1$. This will guarantee that n is not divisible by any integer less than 2013. Upon selecting this n, we find that

$$p \mid n + c \mid a^n + b^n + n$$

which implies that

$$a^n + b^n \equiv c \pmod{p}$$

But $n \equiv -1 \pmod{p-1}$; hence $a^n \equiv a^{-1} \pmod{p}$ by Euler's Little Theorem. Hence we may write

$$p \mid ab(a^{-1} + b^{-1} - c) = a + b - abc.$$

But since p is large, this is only possible if $a + b - abc$ is zero. The only triples of positive integers with that property are $(a, b, c) = (2, 2, 1)$ and $(a, b, c) = (1, 1, 2)$. One can check that of these, only $(a, b, c) = (1, 1, 2)$ is a valid solution. ■

This problem and solution were proposed by Evan Chen.

N8

Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that:

(i) The greatest common divisor of the sequence $f(1), f(2), \dots$ is 1.

(ii) For all sufficiently large integers n, we have $f(n) \neq 1$ and

$$f(a)^n \mid f(a+b)^{a^{n-1}} - f(b)^{a^{n-1}}$$

for all positive integers a and b.

Yang Liu

Answer. The only such function is the constant function $f(b) = b$.

Solution. Let (ii) hold for $n \geq C$. First we claim $f(a) \mid a$ for all a. Let p be any prime dividing $f(a)$. Choose b so that $p \nmid f(a+b), f(b)$ (possible via (i)). So

$$p \mid f(a+b)^{a^{C-1}} - f(b)^{a^{C-1}}.$$

Now let

$$\upsilon_p\left(f(a+b)^{a^{C-1}} - f(b)^{a^{C-1}}\right) = k.$$

By the divisibility for all $n > C$,

$$n\upsilon_p(f(a)) \leq \upsilon_p\left(f(a+b)^{a^{n-1}} - f(b)^{a^{n-1}}\right) = k + (n-C)\upsilon_p(a)$$

by Lifting the Exponent. Now it's clear that $\upsilon_p(f(a)) \leq \upsilon_p(a)$, so $f(a) \mid a$.

Note that for sufficiently large primes p since $f(p) \mid p$, and then $f(p) \neq 1$, $f(p) = p$. Now plug in $a = p$, and by Fermat's Little Theorem, $p \mid f(b+p) - f(b)$ for all b and sufficiently large p. In fact, this then gives that

$$p \mid f(b+kp) - f(b)$$

for any integer k. Now choose $p > b$. If $f(b+p) \neq b+p$, then

$$f(b+p) \leq \frac{b+p}{2} < p.$$

But $p \mid f(b+p) - f(b)$ for all large enough p. Therefore $f(b+p) = f(b)$ for all sufficiently large primes p. By our condition, $f(b) \neq 1$ now, so take a prime $q \mid f(b)$. Then $q \mid b$ and therefore, $q \mid f(b+p) - f(p) = f(b) - f(p) \implies q \mid p$ for any sufficiently large p. So $q = 1$, contradiction. So $f(b+p) = b+p$. Since $0 < f(b+p) - f(b) = b+p-f(b) < b+p < 2p$ and $p \mid f(b+p) - f(b)$, $f(b) = b$ for all b. You can check that this solution works with LTE. ■

This problem and solution were proposed by Yang Liu.

N11

Let p be a prime satisfying $p^2 \mid 2^{p-1}-1$, and let n be a positive integer. Define

$$f(x) = \frac{(x-1)^{p^n} - (x^{p^n}-1)}{p(x-1)}.$$

Find the largest positive integer N such that there exist polynomials $g(x)$, $h(x)$ with integer coefficients and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Victor Wang

Answer. The largest possible N is $2p^{n-1}$.

Solution 1. Let $F(x) = \frac{x}{1} + \cdots + \frac{x^{p-1}}{p-1}$.

By standard methods we can show that $(x-1)^{p^n} - (x^{p^{n-1}}-1)^p$ has all coefficients divisible by p^2. But $p^2 \mid 2^{p-1}-1$ means p is odd, so working in $\mathbb{F}_p$, we have

$$(x-1)f(x) = \sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k}(-1)^{k-1}x^{p^{n-1}k} = \sum_{k=1}^{p-1}\binom{p-1}{k-1}(-1)^{k-1}\frac{x^{p^{n-1}k}}{k}$$
$$= \sum_{k=1}^{p-1}\frac{x^{p^{n-1}k}}{k^{p^{n-1}}} = F(x)^{p^{n-1}},$$

where we use Fermat's little theorem, $\binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod p$ for $k = 1, 2, \dots, p-1$, and the well-known fact that $P(x^p) - P(x)^p$ has all coefficients divisible by p for any polynomial P with integer coefficients.

However, it is easy to verify that $p^2 \mid 2^{p-1}-1$ if and only if $p \mid F(-1)$, i.e. -1 is a root of F in $\mathbb{F}_p$. Furthermore, $F'(x) = \frac{x^{p-1}-1}{x-1} = (x+1)(x+2)\cdots(x+p-2)$ in $\mathbb{F}_p$, so -1 is a root of F with multiplicity 2; hence $N \ge 2p^{n-1}$. On the other hand, since F' has no double roots, F has no integer roots with multiplicity greater than 2. In particular, $N \le 2p^{n-1}$ (note that the multiplicity of 1 is in fact $p^{n-1}-1$, since $F(1) = 0$ by Wolstenholme's theorem but 1 is not a root of F'). ■

This problem and solution were proposed by Victor Wang.

Remark. The rth derivative of a polynomial P evaluated at 1 is simply the coefficient $[(x-1)^r]P$ (i.e. the coefficient of $(x-1)^r$ when P is written as a polynomial in $x-1$) divided by $r!$.

Solution 2. This is asking to find the greatest multiplicity of an integer root of f modulo p; I claim the answer is $2p^{n-1}$.

First, we shift x by 1 and take the negative (since this doesn't change the greatest multiplicity) for convenience, redefining f as $f(x) = \frac{(x+1)^{p^n} - x^{p^n} - 1}{px}$.

Now, we expand this. We can show, by writing out and cancelling, that p^1 fully divides $\binom{p^n}{k}$ only when p^{n-1} divides k; thus, we can ignore all terms except the ones with degree divisible by p^{n-1} (since they still go away when taking it mod p), leaving $f(x) = \frac{1}{px}(\binom{p^n}{p^{n-1}}x^{p^n-p^{n-1}} + \cdots + \binom{p^n}{p^n-p^{n-1}}x^{p^{n-1}})$.

We can also show, by writing out/cancelling, that $\frac{1}{p}\binom{p^n}{kp^{n-1}} = \frac{1}{p}\binom{p}{k}$ modulo p. Simplifying using this, the expression above becomes $f(x) = \frac{1}{px}(\binom{p}{1}x^{p^n-p^{n-1}} + \cdots + \binom{p}{p-1}x^{p^{n-1}}) = \frac{1}{px}((x^{p^{n-1}}+1)^p - (x^{p^n}+1))$.

Now, we ignore the $1/x$ for the moment (all it does is reduce the multiplicity of the root at $x = 0$ by 1) and just look at the rest, $P(x) = \frac{1}{p}((x^{p^{n-1}}+1)^p - (x^{p^n}+1))$.

Substituting $y = x^{p^{n-1}}$, this becomes $\frac{1}{p}((y+1)^p - (y^p+1))$; since $\frac{1}{p}\binom{p}{k} = \frac{1}{k}\binom{p-1}{k-1}$, this is equal to $P(x) = \frac{1}{1}\binom{p-1}{0}y^{p-1} + \cdots + \frac{1}{p-1}\binom{p-1}{p-2}y$. (We work mod p now; the ps can be cancelled before modding out.)

We now show that $P(x)$ has no integer roots of multiplicity greater than 2, by considering the root multiplicities of y times its reversal, or $Q(x) = \frac{1}{p-1}\binom{p-1}{p-2}y^{p-1} + \cdots + \frac{1}{1}\binom{p-1}{0}y$.

Note that some polynomial P has a root of multiplicity m at x iff P and its first $m-1$ derivatives all have zeroes at x. (We're using the formal derivatives here - we can prove this algebraically over $\mathbb{Z}$ mod p, if $m < p$.) The derivative of Q is $\binom{p-1}{p-2}y^{p-2} + \cdots + \binom{p-1}{0}$, or $(y+1)^{p-1} - y^{p-1}$, which has as a root every residue except 0 and -1 by Fermat's little theorem; the second derivative is a constant multiple of $(y+1)^{p-2} - y^{p-2}$, which has no integer roots by Fermat's little theorem and unique inverses. Therefore, no integer root of Q has multiplicity greater than 2; we know that the factorization of a polynomial's reverse is just the reverse of its factorization, and integers have inverses mod p, so $P(x)$ doesn't have integer roots of multiplicity greater than 2 either.

Factoring $P(x)$ completely in y (over some extension of $\mathbb{F}_p$), we know that two distinct factors can't share a root; thus, at most 2 factors have any given integer root, and since their degrees (in x) are each p^{n-1}, this means no integer root has multiplicity greater than $2p^{n-1}$.

However, we see that $y = 1$ is a double root of P. This is because plugging in gives $P(1) = \frac{1}{p}((1+1)^p - (1^p + 1)) = \frac{1}{p}(2^p - 2)$; by the condition, p^2 divides $2^p - 2$, so this is zero mod p. Since 1 is its own inverse, it's a root of Q as well, and it's a root of Q's derivative so it's a double root (so $(y-1)^2$ is part of Q's factorization). Reversing, $(y-1)^2$ is part of P's factorization as well.

Applying a well-known fact, $y - 1 = x^{p^{n-1}} - 1 = (x-1)^{p^{n-1}}$ modulo p, so 1 is a root of P with multiplicity $2p^{n-1}$.

Since adding back in the factor of $1/x$ doesn't change this multiplicity, our answer is therefore $2p^{n-1}$. ■

This second solution was suggested by Alex Smith.

Ex-Lincoln Math Olympiad

Year: **2015**

Day: **1**

17th ELMO
PITTSBURGH, PA

June 20, 2015
1:00 PM – 6:00 PM

Problem 1. Define the sequence $a_1 = 2$ and $a_n = 2^{a_{n-1}} + 2$ for all integers $n \geq 2$. Prove that a_{n-1} divides a_n for all integers $n \geq 2$.

Problem 2. Let m, n, and x be positive integers. Prove that

$$\sum_{i=1}^{n} \min\left(\left\lfloor \frac{x}{i} \right\rfloor, m\right) = \sum_{i=1}^{m} \min\left(\left\lfloor \frac{x}{i} \right\rfloor, n\right).$$

Problem 3. Let ω be a circle and C a point outside it; distinct points A and B are selected on ω so that $\overline{CA}$ and $\overline{CB}$ are tangent to ω. Let X be the reflection of A across the point B, and denote by γ the circumcircle of triangle BXC. Suppose γ and ω meet at $D \neq B$ and line CD intersects ω at $E \neq D$. Prove that line EX is tangent to the circle γ.

Problem 4. Let $a > 1$ be a positive integer. Prove that for some nonnegative integer n, the number $2^{2^n} + a$ is not prime.

Problem 5. Let $m, n, k > 1$ be positive integers. For a set S of positive integers, define $S(i, j)$ for $i < j$ to be the number of elements in S strictly between i and j. We say two sets (X, Y) are a *fat* pair if

$$X(i, j) \equiv Y(i, j) \pmod{n}$$

for every $i, j \in X \cap Y$. (In particular, if $|X \cap Y| < 2$ then (X, Y) is fat.)

If there are m distinct sets of k positive integers such that no two form a fat pair, show that $m < n^{k-1}$.

Time limit: 5 hours.
Each problem is worth 7 points.

17^{th} Ex-Lincoln Math Olympiad

ELMO 2015

Pittsburgh, PA

OFFICIAL SOLUTIONS

1. Define the sequence $a_1 = 2$ and $a_n = 2^{a_{n-1}} + 2$ for all integers $n \geq 2$. Prove that a_{n-1} divides a_n for all integers $n \geq 2$.

   Proposed by Sam Korsky.

   Solution. We prove by induction that both $a_{n-1} \mid a_n$ and $a_{n-1} - 1 \mid a_n - 1$ are true for all positive integers $n \geq 2$. We have $1 \mid 5$ and $2 \mid 6$ so the base case works.

   For the inductive step $k \to k+1$, note that

   $$a_{k-1} \mid a_k \implies 2^{a_{k-1}} + 1 \mid 2^{a_k} + 1 \implies a_k - 1 \mid a_{k+1} - 1$$

   $$a_{k-1} - 1 \mid a_k - 1 \implies 2^{a_{k-1}} + 2 \mid 2^{a_k} + 2 \implies a_k \mid a_{k+1}$$

   So the induction is complete and the result follows. (The above works because $x + 1 \mid x^k + 1$ for odd k, and $2 \mid a_n$ but $4 \nmid a_n$ for all n.) ∎

   This problem and solution were proposed by Sam Korsky.

2. Let m, n, and x be positive integers. Prove that

   $$\sum_{i=1}^{n} \min\left(\left\lfloor \frac{x}{i} \right\rfloor, m\right) = \sum_{i=1}^{m} \min\left(\left\lfloor \frac{x}{i} \right\rfloor, n\right).$$

   Proposed by Yang Liu.

   Solution 1. Both sides count the number of entries of an $m \times n$ multiplication table that are at most x, as desired. ∎

   This problem and solution were proposed by Yang Liu.

   Solution 2. We induct on x for fixed m and n. Note that it is trivial for $x = 0$ because both sides are 0. Now, say it is true for $x - 1$, and let's prove it is true for x. Note that the left increments for every value $i \leq n$ that has $\frac{x}{i} \leq m$ with i dividing x. So it increments by 1 for every divisor of x that is at least $\frac{x}{n}$ and at most m (the $\frac{x}{i}$). The RHS increments by 1 for every divisor of x that is at least $\frac{x}{m}$ and at most n similarly. These are the same because r dividing x is in one category if and only if $\frac{x}{r}$ dividing x is in the other. So we have a bijection, both increase by the same amount, and we are done by induction. ∎

   This second solution was suggested by Ryan Alweiss.

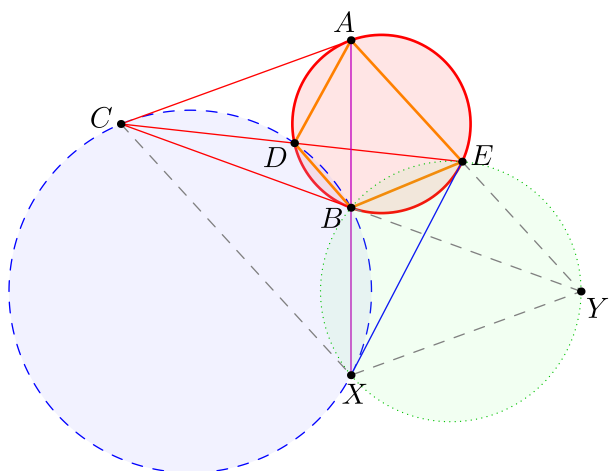
3. Let ω be a circle and C a point outside it; distinct points A and B are selected on ω so that $\overline{CA}$ and $\overline{CB}$ are tangent to ω. Let X be the reflection of A across the point B, and denote by γ the circumcircle of triangle BXC. Suppose γ and ω meet at $D \neq B$ and line CD intersects ω at $E \neq D$. Prove that line EX is tangent to the circle γ.

Proposed by David Stoner.

Solution 1. From $\angle CXB = \pi - \angle CDB = \angle EAB$, we find $AE \parallel CX$. Let $T \in \overline{CX}$ such that $AEXT$ is a parallelogram; then $\angle BTC = \pi - \angle AEB = \angle XBC$, and it follows that $\triangle BTC \sim \triangle XBC \Rightarrow (CX)(CT) = (CB)^2 = (CA)^2 \Rightarrow \triangle ATC \sim \triangle XAC$. Therefore $\angle CAT = \angle CXA = \angle CBT$, so $ACTB$ is cyclic. Finally, $\angle EXB = \angle BAT = \angle BCX$, and it follows that $\overline{EX}$ is tangent to ω as desired. ■

This problem and solution were proposed by David Stoner.

Solution 2.



Using directed angles, $\angle BXC = \angle BDC = \angle BDE = \angle BAE$ so $\overline{AE} \parallel \overline{CX}$. Construct parallelogram $AYXC$. As $\angle BEY = \angle BEA = \angle BAC = \angle BXY$, quadrilateral $BEXY$ is cyclic. Thus $\angle XCB = \angle BYE = \angle BXE$ as desired. ■

This second solution was suggested by Viswanath and mathdebam.

Solution 3. First note that $\angle ECX = \angle DBA = \angle CEA$ which implies that $EA \parallel CX$. Now let F be the second intersection of line AD with γ. We have that $\angle DFX = \angle ECX = \angle AEC = \angle DAC$ so $FC \parallel AX$. Therefore projecting points C, D, B, X from F onto line AX yields that quadrilateral $CDBX$ is harmonic. Let $G = AB \cap ED$. Since line AB is the polar of C with respect to ω we have that $(C, G; D, E) = -1$ so by projecting C, D, G, E from X to circle γ we have that E must go to X so EX is tangent to ω' as desired. ■

This third solution was suggested by Sam Korsky.

Solution 4. Here is a solution with no auxiliary points at all. By angle chasing, $\triangle XAC \sim \triangle AEB$, whence

$$\frac{AX}{AE} = \frac{CX}{AB} = \frac{CX}{BC}.$$

Since $\angle BXC = \angle EAX$ also, we get $\triangle BXC \sim \triangle EAX$, thus $\angle BXE = \angle BCX$ as desired. $\blacksquare$

This fourth solution was suggested by linqaszayi.

Remark. An approach with complex numbers is also possible. Setting ω to be the unit circle, one can derive

$$d = \frac{b(2b+3a)}{2a+3b} \quad \text{and} \quad e = \frac{b(a+2b)}{2a+b}.$$

In fact, if one notices that $\overline{AE} \parallel \overline{CX}$ then the coordinates of D can be bypassed, and point E can be obtained directly.

It is even possible to approach the problem with Cartesian coordinates or by using barycentric coordinates on $\triangle ABC$.

4. Let $a > 1$ be a positive integer. Prove that for some nonnegative integer n, the number $2^{2^n} + a$ is not prime.

Proposed by Jack Gurev.

Solution. Let $m = v_2(a-1)$. Assume that $2^{2^m} + a = p$ is prime. It suffices to show there exists $n > m$ such that $2^{2^n} - 2^{2^m}$ is divisible by p.

Since
$$2^{2^n} - 2^{2^m} = 2^{2^m}\left(\left((2^{2^m}\right)^{2^{n-m}-1} - 1\right)$$
we can let
$$n = m + \phi\left(\frac{p-1}{2^m}\right)$$
which implies the conclusion. $\blacksquare$

This problem was proposed by Jack Gurev. This solution was given by Sam Korsky.

5. Let $m, n, k > 1$ be positive integers. For a set S of positive integers, define $S(i,j)$ for $i < j$ to be the number of elements in S strictly between i and j. We say two sets (X, Y) are a *fat* pair if
$$X(i,j) \equiv Y(i,j) \pmod{n}$$
for every $i, j \in X \cap Y$. (In particular, if $|X \cap Y| < 2$ then (X,Y) is fat.)

If there are m distinct sets of k positive integers such that no two form a fat pair, show that $m < n^{k-1}$.

Proposed by Allen Liu.

Solution. Let the union of the sets be $T = \{a_1, a_2, \dots, a_\ell\}$ where the elements of T are arranged in increasing order. For each element of T, color it randomly with one of n colors $(1, 2, \dots, n)$. We say a set is good if its elements when arranged in increasing order have colors $a, a+1, \dots, a+k-1$ taken mod n where a can be any color. Now the fact that there is no fat pair means that only one good set can exist in each coloring. The probability that a good set exists is $\frac{1}{n^{k-1}}$ so we are done. (The

inequality is strict since we could end up coloring all elements of T the same color.)
■

This problem and solution were proposed by Allen Liu.

Elmo Lives Mostly Outside

Year: 2016

18$^{\text{th}}$ ELMO
Pittsburgh, PA

Day: 1

Saturday, June 18, 2016
1:15PM — 5:45PM

Problem 1. Cookie Monster says a positive integer n is *crunchy* if there exist $2n$ real numbers $x_1, x_2, \ldots, x_{2n}$, not all equal, such that the sum of any n of the x_i's is equal to the product of the other n of the x_i's. Help Cookie Monster determine all crunchy integers.

Problem 2. Oscar is drawing diagrams with trash can lids and sticks. He draws a triangle ABC and a point D such that DB and DC are tangent to the circumcircle of ABC. Let B' be the reflection of B over AC and C' be the reflection of C over AB. If O is the circumcenter of $DB'C'$, help Oscar prove that AO is perpendicular to BC.

Problem 3. In a Cartesian coordinate plane, call a rectangle *standard* if all of its sides are parallel to the x- and y- axes, and call a set of points *nice* if no two of them have the same x- or y- coordinates. First, Bert chooses a nice set B of 2016 points in the coordinate plane. To mess with Bert, Ernie then chooses a set E of n points in the coordinate plane such that $B \cup E$ is a nice set with $2016 + n$ points. Bert returns and then miraculously notices that there does not exist a standard rectangle that contains at least two points in B and no points in E in its interior. For a given nice set B that Bert chooses, define $f(B)$ as the smallest positive integer n such that Ernie can find a nice set E of size n with the aforementioned properties. Help Bert determine the minimum and maximum possible values of $f(B)$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Elmo Lives Mostly Outside

Year: 2016

18^{th} **ELMO**
Pittsburgh, PA

Day: 2

Sunday, June 19, 2016
1:15PM — 5:45PM

Problem 4. Big Bird has a polynomial P with integer coefficients such that n divides $P(2^n)$ for every positive integer n. Prove that Big Bird's polynomial must be the zero polynomial.

Problem 5. Elmo is drawing with colored chalk on a sidewalk outside. He first marks a set S of $n > 1$ collinear points. Then, for every unordered pair of points $\{X, Y\}$ in S, Elmo draws the circle with diameter XY so that each pair of circles which intersect at two distinct points are drawn in different colors. Count von Count then wishes to count the number of colors Elmo used. In terms of n, what is the minimum number of colors Elmo could have used?

Problem 6. Elmo is now learning olympiad geometry. In a triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90°$. Finally, let γ be the circumcircle of $\triangle AST$.

(a) Help Elmo show that γ is tangent to the circumcircle of $\triangle ABC$.

(b) Help Elmo show that γ is also tangent to the incircle of $\triangle ABC$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

18$^{\text{th}}$ Elmo Lives Mostly Outside

ELMO 2016

Pittsburgh, PA

OFFICIAL SOLUTIONS

1. Cookie Monster says a positive integer n is *crunchy* if there exist $2n$ real numbers $x_1, x_2, \ldots, x_{2n}$, not all equal, such that the sum of any n of the x_i's is equal to the product of the other n of the x_i's. Help Cookie Monster determine all crunchy integers.

   Proposed by Yannick Yao.

   Answer. The crunchy numbers are exactly the even integers $n = 2, 4, 6, \ldots$.

   Solution. Notice that

$$\prod_{i=1}^{2n} x_n = (x_{a_1} + x_{a_2} + \ldots + x_{a_n})(x_{a_{n+1}} + x_{a_{n+2}} + \ldots + x_{a_{2n}})$$

   where the a_i are any permutation of $1 - 2n$. Switching a_n and a_{n+1} in the formula and setting both sides to be equal we get an equation that factors into

$$(x_{a_n} - x_{a_{n+1}})\left[(x_{a_1} + x_{a_2} + \ldots + x_{a_{n-1}}) - (x_{a_{n+2}} + x_{a_{n+3}} + \ldots + x_{a_{2n}})\right] = 0.$$

   Since not all of the numbers are equal we can see that if any two are not equal then the other $2n-2$ must be equal by permuting a_i in the above equation. Also one of these two must share the same value as these $2n-2$ by the same logic. So WLOG $x_1 = x_2 = \ldots = x_{2n-1} = x$ and $a_{2n} = y$. So we end up with the equations

$$nx = x^{n-1}y \qquad (n-1)x + y = x^n.$$

   Notice $x \neq 0$ or else y would also be 0. Substituting $y = \frac{n}{x^{n-2}}$ into the second equation, clearing denominators, and factoring gives us

$$(x^{n-1} - n)(x^{n-1} + 1) = 0.$$

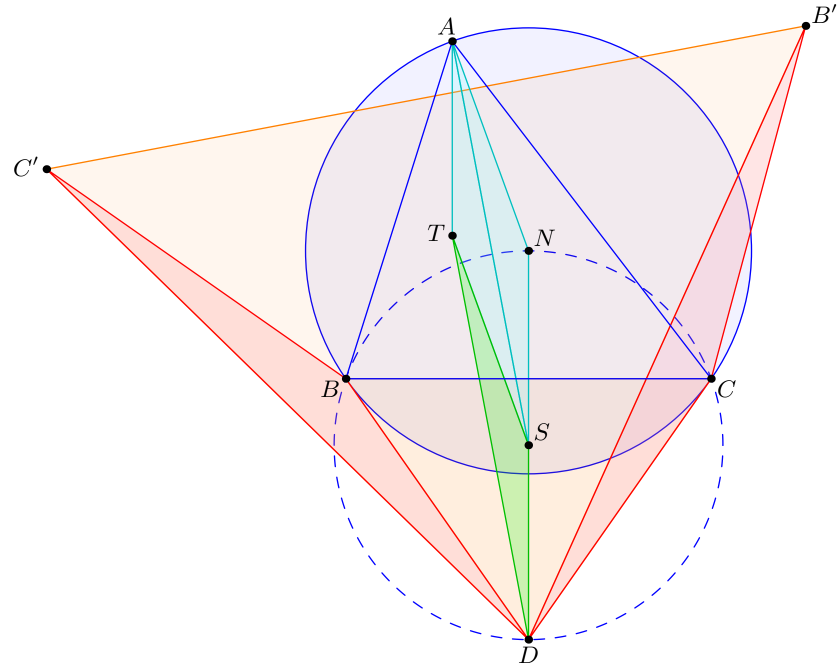
   If $x = \sqrt[n-1]{n}$ then y would also be $\sqrt[n-1]{n}$. Thus, $n-1$ must be odd and then n must be even. Say n is even. Then setting $x_1 = x_2 = \ldots = x_{2n-1} = -1$ and $x_{2n} = n$ clearly works and we are done ■

   This problem was proposed by Yannick Yao. This solution was given by Michael Ma.

2. Oscar is drawing diagrams with trash can lids and sticks. He draws a triangle ABC and a point D such that DB and DC are tangent to the circumcircle of ABC. Let B' be the reflection of B over AC and C' be the reflection of C over AB. If O is the circumcenter of $DB'C'$, help Oscar prove that AO is perpendicular to BC.

Proposed by James Lin.

Solution 1. Let N denote the circumcenter of ABC and let S denote the circumcenter of $NBDC$ (midpoint of $\overline{ND}$). Let T be the point such that $ANST$ is a parallelogram (hence $ASDT$ too). We will prove that $T = O$, which implies the result (since $\overline{NSD} \perp \overline{BC}$).



First we claim that $\triangle B'CD \sim \triangle TSD$, with equal orientation. By angle chasing, we have

$$\begin{aligned} \measuredangle TSD = \measuredangle ANS &= \measuredangle(\overline{AN}, \overline{BC}) + 90^\circ = (\measuredangle NAC + \measuredangle ACB) + 90^\circ \\ &= (90^\circ - \measuredangle CBA) + \measuredangle ACB + 90^\circ = 2\measuredangle ACB + \measuredangle BAC \\ &= \measuredangle B'CA + \measuredangle ACB + \measuredangle BCD = \measuredangle B'CD. \end{aligned}$$

Finally from isosceles $\triangle DBC \sim \triangle SBN$, we have

$$\frac{B'C}{CD} = \frac{BC}{CD} = \frac{BN}{NS} = \frac{NA}{NS} = \frac{TS}{SD}.$$

This implies the similarity.

Similarly, $\triangle C'BD \sim \triangle TSD$. Then there is a spiral similarity sending $\triangle DBC$ to $\triangle DB'C'$, and sending S to T. As S is the circumcenter of $\triangle DBC$, T is the circumcenter of $\triangle DB'C'$, meaning $T = O$. ■

This first solution was suggested by Evan Chen.

Solution 2. First, note that triangles DBC' and DCB' are congruent and in the same orientation, so $DB'C'$ is similar to DBC. Now, let the circumcircle of $DB'C'$ intersect DB at P and DC at Q. We have that $\angle C'PB = \angle C'PD = \angle C'B'D =$

$\angle CBD = \angle BAC = \angle C'AB$, so P lies on the circumcircle of ABC'. Furthermore, $\angle ABP = \angle ACB = \angle AC'B = \angle APB$, so $AP = AB$. Similarly, $AQ = AC$. Now, let X and Y be on DB and DC so that $AD = AX = AY$.

The key lemma is that given varying points D and E on fixed rays AB and AC such that $AD - AE$ is constant. Then the circumcenter of ADE lies on a fixed line parallel to the angle bisector of $\angle BAC$. The proof of this is that all circumcircles of ADE share a common midpoint of arc DAE, call it Z, by spiral similarity, so the circumcenter of ADE lies on the perpendicular bisector of AZ, which is a fixed line parallel to the angle bisector.

Now, we use this lemma on rays DB and DC. Note that since triangles ADX, ABP, ADY, and ACQ are all isosceles, $DX - DP = XP = DB = DC = YQ = DY - DQ$, so we have that $DX - DY = DP - DQ$. Now, note that the circumcenter of DPQ is O and the circumcenter of DXY is A, so the line through them is perpendicular to BC by the lemma, as desired. ■

This second solution was suggested by Michael Ren.

This problem was proposed by James Lin.

3. In a Cartesian coordinate plane, call a rectangle *standard* if all of its sides are parallel to the x- and y- axes, and call a set of points *nice* if no two of them have the same x- or y- coordinates. First, Bert chooses a nice set B of 2016 points in the coordinate plane. To mess with Bert, Ernie then chooses a set E of n points in the coordinate plane such that $B \cup E$ is a nice set with $2016 + n$ points. Bert returns and then miraculously notices that there does not exist a standard rectangle that contains at least two points in B and no points in E in its interior. For a given nice set B that Bert chooses, define $f(B)$ as the smallest positive integer n such that Ernie can find a nice set E of size n with the aforementioned properties. Help Bert determine the minimum and maximum possible values of $f(B)$.

   Proposed by Yannick Yao.

   Solution 1. The minimum is 2015, since there needs to be a point in J whose x-coordinate is between each two consecutive points in A when sorted by x-coordinate. The minimum is achieved when $A = \{(t,t)|t = 0, 1, \cdots, 2015\}$.

   For general $|A| = c$ (instead of 2016) the maximum is $2c - 2\sqrt{c}$

   To keep things clean, I will let $c = k^2$ where k is a positive integer. The construction, as mentioned above is to take a k by k square and rotate it slightly.

   Now to show that $2c - 2k$ suffices, consider the set of points in A as a poset where for points p, q, $p > q$ if p is up and right of q.

   Take the longest antichain and say it has s elements. This antichain is actually an up left chain of points. Partition the remaining points into two sets, those that are $>$ than some element in the antichain and those that are $<$ some element in the antichain. For the first set, Ernie draws points slightly below and left of each point and Ernie draws points slightly above and right of each point in the second set.

In total Ernie has drawn $k^2 - s$ points. (We have eliminated all possible rectangles where the two points in A form an up right vector since these two points cannot both be in the antichain)

Now we can do the same for up left rectangles. To finish the problem it suffices to note Dilworth's theorem and use AM-GM. ■

This first solution was suggested by Allen Liu.

Solution 2. Here is an alternative way to show the maximum. As above, the number of points needed for J is equal to

- Twice the number of points,
- minus the length of the maximal down-right chain, and
- minus the length of the maximal up-right chain.

If we order the points by their x-coordinate and consider a sequence being their y-coordinates, the two things we are subtracting becomes the length of maximal decreasing subsequence and the length of maximal increasing subsequence respectively. Notice that if the two lengths are m and n respectively, then the number of points is at most mn, because of the famous result that a sequence of $mn+1$ distinct real numbers must either contain an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$.

Therefore, in the context of this particular case, we have $mn \geq 2016$ and we need to maximize $2 \cdot 2016 - m - n$, and this is easy by AM-GM, and the maximized result is $2 \cdot 2016 - \lceil 2\sqrt{2016} \rceil = 3942$.

This maximum is achieved by having a slightly tilted 42×48 lattice grid for A. ■

4. Big Bird has a polynomial P with integer coefficients such that n divides $P(2^n)$ for every positive integer n. Prove that Big Bird's polynomial must be the zero polynomial.

   Proposed by Ashwin Sah.

   Solution. We claim $P(2^k) = 0$ for every positive integer k, which is enough. Indeed, for p prime we have
   $$0 \equiv P(2^{kp}) \equiv P(2^k) \pmod p$$
   since $2^{kp} \equiv 2^k \pmod p$, so the claim follows by taking p sufficiently large. ■

   This problem and solution were proposed by Ashwin Sah.

5. Elmo is drawing with colored chalk on a sidewalk outside. He first marks a set S of $n > 1$ collinear points. Then, for every unordered pair of points $\{X, Y\}$ in S, Elmo draws the circle with diameter XY so that each pair of circles which intersect at two distinct points are drawn in different colors. Count von Count then wishes to count the number of colors Elmo used. In terms of n, what is the minimum number of colors Elmo could have used?

   Proposed by Michael Ren.

Answer. The answer is $\lceil n/2 \rceil$ colors, except when $n = 3$ here the answer is 1.

Solution. I claim that the answer for even n is $n/2$. We can let the distance between adjacent points be 1. Label the vertices $1, 2, 3, \dots, n/2, 1, 2, \dots, n/2$ in that order, left to right (we can assume that the line the points are on is horizontal).

Now, consider the $n/2$ circles whose diameters have endpoints with the same label. Note that these are pairwise intersecting, so we must use at least $n/2$ colors.

For the coloring, for circles with diameter $\le n/2$, color them with the label of the right endpoint of the diameter. For circles with diameter $\ge n/2$, color them with the label of the left endpoint of the diameter. By checking cases, it is not hard to confirm that this coloring works.

Now we consider $n = 2m+1$ odd. Obviously $n = 3$ gives 1. For other $n = 2m+1$ we will show the minimum number of colors is $m+1$. We can construct this by using the above construction, and coloring each circle containing $2m+1$ with the color $m+1$.

Now, for proving it, call the vertices $1, 2, \dots, m, m+1, 1, 2, \dots, m$ as earlier; we have the m different colored circles from vertices of the same color. Let $f(a, b)$ denote the color of the circle with vertex color a in the first $m+1$ vertices, and vertex color b in the last $m+1$ vertices. Note that $f(1, m+1) = 1$. $f(2, 1) = 1, 2$, but we know it is 2 due to the previous conclusion. Similarly, we show that $f(k+1, k) = k+1$ for $1 \le k \le m$, so in particular, we need $f(m+1, m) = m+1$, as desired. ■

This problem was proposed by Michael Ren. This solution was given by Mihir Singhal and James Lin.

Remark. An alternate easier version of the problem requires that circles which are tangent to each other are also distinct colors. In this case the answer is n.

Label the vertices $1, 2, \dots n$, and let $f(a, b)$ be the circle with diameter at vertices a, b. Note that $f(1, 2), f(1, 3), \dots, f(1, n), f(2, n)$ are different colors, so at least n colors are needed. But then we can let $f(a, b)$ be colored by color $a + b \pmod{n}$, so we are done!

6. Elmo is now learning olympiad geometry. In a triangle ABC with $AB \ne AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90^\circ$. Finally, let γ be the circumcircle of $\triangle AST$.

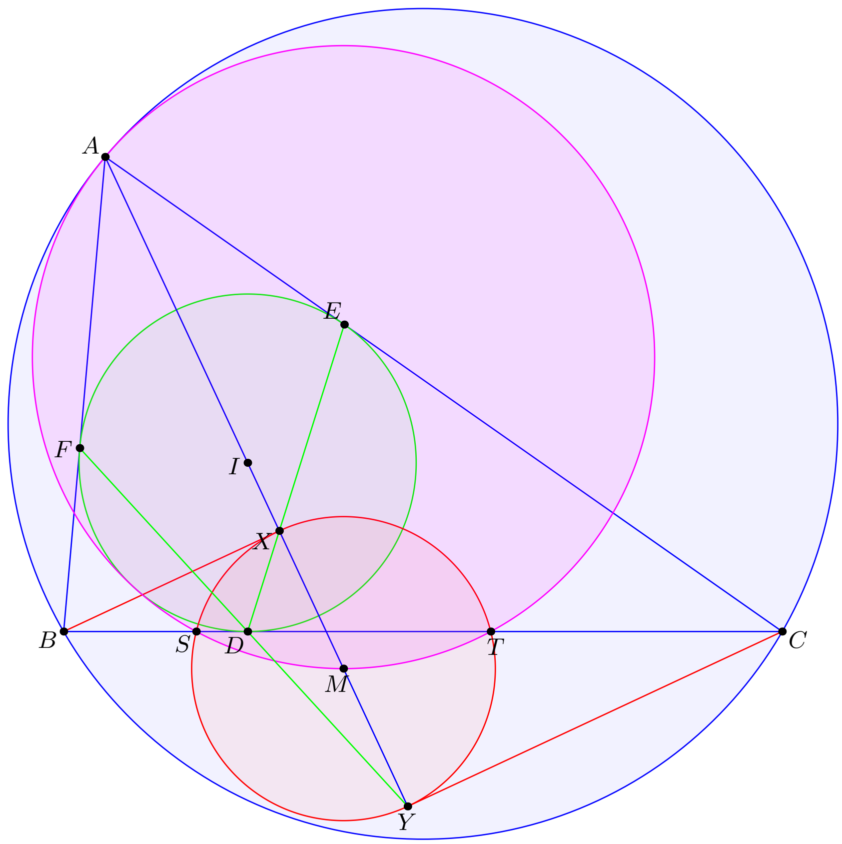
   (a) Help Elmo show that γ is tangent to the circumcircle of $\triangle ABC$.

   (b) Help Elmo show that γ is also tangent to the incircle of $\triangle ABC$.

   Proposed by James Lin.

Solution 1. First, we claim that X and Y are the incenter and excenter of $\triangle AST$. (This is Sharygin 2013, Problem 18, also problem 11.12 of Euclidean Geometry in Mathematical Olympiads.) To see this, recall that $\angle AXB = \angle AYC$ are right angles (see for example JMO 2014 problem 6). Now let $K = \overline{AXY} \cap \overline{BC}$ and let L be the foot of the external $\angle A$-bisector. Then $(KL; BC) = -1$, so projection onto $\overline{AI}$ gives

$(AK;XY) = -1$. Now, since $\angle YSX = 90^\circ$, we see that $\overline{SX}$ and $\overline{SY}$ are bisectors of $\angle AST$. The same statement holds for $\angle ATS$, which proves the claim.



In particular, this implies that $\overline{AS}$ and $\overline{AT}$ are isogonal to each other, and therefore part (a) is solved.

As for part (b), denote $(XSTY)$ by ω, centered at a point M, which is midpoint of arc ST of γ. Now, we observe that $\triangle IXD \sim \triangle IDY$, therefore $ID^2 = IX \cdot IY$ and thus the incircle is orthogonal to ω. Therefore an inversion around ω fixes the incircle. Now γ is mapped to line BC, which is obviously tangent to incircle. Therefore γ was tangent too. ■

This first solution was suggested by Evan Chen.

Solution 2. Here is an alternate solution to part (b).

Let the A-excircle of ABC be tangent to AB at R and BC at S. It is well-known that X lies on RS and Y lies on DE. Hence, by some angle-chasing ARX and AYE are similar (both have angles $\frac{\angle A}{2}, \frac{\angle B}{2}, 90 + \frac{\angle C}{2}$), so we have that $AR \cdot AE = AX \cdot AY = AS \cdot AT$. Hence, a $\sqrt{bc}$ inversion on AST swaps the incircle and A-excircle of ABC. But it also swaps the circumcircle of AST and ST. Since the incircle and A-excircle of ABC are both tangent to ST, or BC, both are also tangent to the circumcircle of AST, as desired. ■

This second solution was suggested by Michael Ren..

Solution 3. We also claim (AST) is tangent to the A-excircle.

It's well-known and you can prove with angle-chasing that X, Y are the feet of the perpendiculars from B, C to AI, where I is the incenter.

Let M be the midpoint of BC and N be the midpoint of XY. Clearly $MN \perp XY$ so we get that N lies on the radical axis of the incircle and A-excircle, and it is obvious that N is the center of the circle of diameter XY.

Note that $IX \perp BX$. Let B' be the midpoint of DF, so that B, B' correspond in inversion about the incircle. Thus, if X' is the image of X under inversion about the incircle, we should have that $\angle IB'X' = 90^\circ$ so that X' lies on DF. Then it's clear that $X' = Y$ so X, Y are inverses under inversion about the incircle.

Now this means that (XY) is orthogonal to the incircle. Note that since N is on the radical axis of the incircle and A-excircle, $P(N, \text{incircle}) = P(N, \text{A-excircle}) = NX$ which means (XY) is also orthogonal to the A-excircle.

Now let Z be the foot of the A-angle bisector. We claim that $(AZ), (XY)$ are orthogonal. It suffices to show $(A, Z; X, Y)$ is harmonic. Let Z' be the foot of the A-external angle bisector. Project $(A, Z; X, Y)$ from $\infty_{AZ'}$ down to line BC so it follows that $(A, Z; X, Y) = (Z', Z; B, C)$ which is clearly harmonic. Then $(AZ), (XY)$ are orthogonal as claimed. But then it follows that A, Z are also inverses in inversion about circle (XY).

Now invert (AST) about (XY). Clearly S, T remain fixed while A goes to Z so (AST) and line BC are inverses. This can only happen if (AST) passes through N, the center of inversion. Then A, S, T, N are concyclic. Now it also follows from this inversion that since the incircle and excircle remain fixed, the image of (AST) is tangent to both circles, so (AST) was tangent to the incircle and A-excircle.

Now note that N is the midpoint of arc ST on (AST) because $NS = NT$. But then it follows that $\angle SAN = \angle TAN$. Since $\angle BAN = \angle CAN$ we deduce that AS, AT are isogonal w.r.t. $\angle BAC$.

Let O_1 be the circumcenter of AST and H be the orthocenter of ABC. Then AH, AO are isogonal in triangle ABC but AH, AO_1 are isogonal in triangle AST so we deduce that A, O, O_1 are collinear. Then it follows that (AST) is tangent to the circumcircle of ABC as desired. ■

This third solution was suggested by Vincent Huang.

This problem was proposed by James Lin.

vEry badLy naMed cOntest

Year: **2017**

19^{th} **ELMO**
Pittsburgh, PA

Day: **1**

Saturday, June 10, 2017
1:15PM — 5:45PM

Problem 1. Let $a_1, a_2, \dots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \dots, a_n^n + P) \le 2\gcd(a_1, \dots, a_n)^n.$$

Problem 2. Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from A, such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.

Problem 3. nicκy is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer n, nicκy can label at least dn^2 cells of an $n \times n$ square.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

vEry badLy naMed cOntest

Year: 2017

19^{th} ELMO
PITTSBURGH, PA

Day: 2

Saturday, June 17, 2017
1:15PM — 5:45PM

Problem 4. An integer $n > 2$ is called *tasty* if for every ordered pair of positive integers (a, b) with $a + b = n$, at least one of $\frac{a}{b}$ and $\frac{b}{a}$ is a terminating decimal. Do there exist infinitely many tasty integers?

Problem 5. The edges of K_{2017} are each labelled with 1, 2, or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all $\binom{2017}{2}$ labels.

(Here K_{2017} is defined as the complete graph on 2017 vertices, with an edge between every pair of vertices.)

Problem 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c:

(i) If $a + b + c \geq 0$ then $f(a^3) + f(b^3) + f(c^3) \geq 3f(abc)$.

(ii) If $a + b + c \leq 0$ then $f(a^3) + f(b^3) + f(c^3) \leq 3f(abc)$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Shortlisted Problems

19^{th} ELMO

Pittsburgh, PA, 2017

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until disclosed publicly by the committee on the ELMO.

Contributing Students

The Problem Selection Committee for ELMO 2017 thanks the following proposers for contributing **45 problems** to this year's Competition:

> Ashwin Sah, Colin Tang, Daniel Liu, David Stoner, Jeffery Li, Michael Kural, Michael Ma, Michael Ren, Mihir Singhal, Nathan Ramesh, Nathan Weckwerth, Palmer Mebane, Ruidi Cao, Tristan Shin, Vincent Huang, Zack Chroman

Problem Selection Committee

The Problem Selection Committee for ELMO 2017 was led by **Evan Chen** and consisted of:

- Ashwin Sah
- James Lin
- Kevin Ren
- Mihir Singhal
- Michael Ma
- Michael Ren
- Yannick Yao

Problems

A1. Let $0 < k < \frac{1}{2}$ be a real number and let a_0 and b_0 be arbitrary real numbers in $(0, 1)$. The sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are then defined recursively by

$$a_{n+1} = \frac{a_n + 1}{2} \qquad \text{and} \qquad b_{n+1} = b_n^k$$

for $n \geq 0$. Prove that $a_n < b_n$ for all sufficiently large n.

(Michael Ma)

A2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c:

(i) If $a + b + c \geq 0$ then $f(a^3) + f(b^3) + f(c^3) \geq 3f(abc)$.

(ii) If $a + b + c \leq 0$ then $f(a^3) + f(b^3) + f(c^3) \leq 3f(abc)$.

(Ashwin Sah)

C1. Let m and n be fixed distinct positive integers. A wren is on an infinite chessboard indexed by $\mathbb{Z}^2$, and from a square (x, y) may move to any of the eight squares $(x \pm m, y \pm n)$ or $(x \pm n, y \pm m)$. For each $\{m, n\}$, determine the smallest number k of moves required for the wren to travel from $(0, 0)$ to $(1, 0)$, or prove that no such k exists.

(Michael Ren)

C2. The edges of K_{2017} are each labelled with 1, 2, or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all $\binom{2017}{2}$ labels.

(Michael Ma)

C3. Consider a finite binary string b with at least 2017 ones. Show that one can insert some plus signs in between pairs of digits such that the resulting sum, when performed in base 2, is equal to a power of two.

(David Stoner)

C4. nicκy is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer n, nicκy can label at least dn^2 cells of an $n \times n$ square.

(Mihir Singhal and Michael Kural)

C5. There are n MOPpers $p_1, \ldots, p_n$ designing a carpool system to attend their morning class. Each p_i's car fits $\chi(p_i)$ people ($\chi\colon \{p_1, \ldots, p_n\} \to \{1, 2, \ldots, n\}$). A c-fair carpool system is an assignment of one or more drivers on each of several days, such that each MOPper drives c times, and all cars are full on each day. (More precisely, it is a sequence of sets $(S_1, \ldots, S_m)$ such that $|\{k : p_i \in S_k\}| = c$ and $\sum_{x\in S_j} \chi(x) = n$ for all i, j.)

Suppose it turns out that a 2-fair carpool system is possible but not a 1-fair carpool system. Must n be even?

(Nathan Ramesh and Palmer Mebane)

G1. Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from A, such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.

(Michael Ren)

G2. Let ABC be a scalene triangle with $\angle A = 60^\circ$. Let E and F be the feet of the angle bisectors of $\angle ABC$ and $\angle ACB$ respectively, and let I be the incenter of $\triangle ABC$. Let P, Q be distinct points such that $\triangle PEF$ and $\triangle QEF$ are equilateral. If O is the circumcenter of $\triangle APQ$, show that $\overline{OI} \perp \overline{BC}$.

(Vincent Huang)

G3. Call the ordered pair of distinct circles (ω, γ) scribable if there exists a triangle with circumcircle ω and incircle γ. Prove that among n distinct circles there are at most $(n/2)^2$ scribable pairs.

(Daniel Liu)

G4. Let ABC be an acute triangle with incenter I and circumcircle ω. Suppose a circle ω_B is tangent to BA, BC, and internally tangent to ω at B_1, while a circle ω_C is tangent to CA, CB, and internally tangent to ω at C_1. If B_2, C_2 are the points on ω opposite to B, C, respectively, and X denotes the intersection of B_1C_2, B_2C_1, prove that $XA = XI$.

(Vincent Huang and Nathan Weckwerth)

N1. Let $a_1, a_2, \ldots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \ldots, a_n^n + P) \le 2\gcd(a_1, \ldots, a_n)^n.$$

(Daniel Liu)

N2. An integer $n > 2$ is called *tasty* if for every ordered pair of positive integers (a, b) with $a + b = n$, at least one of $\frac{a}{b}$ and $\frac{b}{a}$ is a terminating decimal. Do there exist infinitely many tasty integers?

(Vincent Huang)

N3. For each integer $C > 1$, decide whether there exists pairwise distinct positive integers a_1, a_2, a_3, ... such that for every $k \ge 1$,

$$a_{k+1}^k \quad \text{divides} \quad C^k a_1 a_2 \ldots a_k.$$

(Daniel Liu)

Solutions

A1. Let $0 < k < \frac{1}{2}$ be a real number and let a_0 and b_0 be arbitrary real numbers in $(0, 1)$. The sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are then defined recursively by

$$a_{n+1} = \frac{a_n + 1}{2} \qquad \text{and} \qquad b_{n+1} = b_n^k$$

for $n \geq 0$. Prove that $a_n < b_n$ for all sufficiently large n.

(Michael Ma)

It should be clear that both sequences converge to 1. In the first sequence, the distance from 1 is halved every time and converges to 0. In the second sequence $b_n = b_0^{k^n}$ and since k^n converges to 0, b_i converges to 1.

The key lemma to solve the problem is the following:

Lemma. *If $k < \frac{1}{2}$ then there exists $0 < x_0 < 1$ such that whenever $x_0 < x < 1$,*

$$x^k > \frac{2k+1}{4}x + \frac{3-2k}{4}.$$

Proof. First notice that if we take the tangent to $y = x^k$ at $(1, 1)$ we get the equation $y = kx + (1 - k)$. We can see by taking the first derivative of

$$kx + (1 - k) - x^k$$

to get

$$k - kx^{k-1}$$

which is negative as $kx + (1 - k) - x^k$ is decreasing from 0 to 1. Furthermore x^k is concave and increasing from 0 to 1. Now it if we take a line of higher slope than k passing through $(1, 1)$ for large enough x the line will fall under x^k. □

Now let x_0 be as above, and let $a = \frac{2k+1}{4} < \frac{1}{2}$ for convenience. Now we can see that

$$b_{n+1} > ab_n + (1 - a).$$

Take the smallest M such that a_M and b_M are both larger than x_0. By iterating both recurrences we can see that for $\ell = 0, 1, \ldots$ we have

$$a_{M+\ell} = 1 - \left(\frac{1}{2}\right)^{\ell}(1 - a_M) \quad \text{and} \quad b_{M+\ell} > 1 - a^{\ell}(1 - b_M).$$

Since $\frac{1}{2a} > 1$ we can take a sufficiently large positive integer ℓ_0 such that $\left(\frac{1}{2a}\right)^{\ell_0} > \frac{1-b_M}{1-a_M}$. Then taking $N = M + \ell_0$ we are done since $b_N > a_N$ and

$$x^k > ax + (1 - a) > \frac{x+1}{2}$$

for $x > x_0$.

A2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c:

(i) If $a + b + c \geq 0$ then $f(a^3) + f(b^3) + f(c^3) \geq 3f(abc)$.

(ii) If $a + b + c \leq 0$ then $f(a^3) + f(b^3) + f(c^3) \leq 3f(abc)$.

(Ashwin Sah)

The answer is $f(x) = kx + \ell$ where k and ℓ are any real numbers with $k \geq 0$.
We begin with some weird optimizations:

- Since f can be shifted by a constant, we get $f(0) = 0$.
- Put $c = 0$ and $b = -a$ to get $f(a^3) + f(-a^3) = 0$, so that f is odd.
- Put $c = 0$ now to get $f(a^3) + f(b^3) \geq 0$ whenever $a + b \geq 0$. Combined with f odd, this implies f is weakly increasing.

Now, we let $c = -a - b$ to get:

$$f(a^3) + f(b^3) + f(-(a+b)^3) = 3f(-ab(a+b))$$

Using oddness and rearranging:

$$f(a^3) + f(b^3) + 3f(ab(a+b)) = f((a+b)^3)$$

Call this property $P(a, b)$.

Lemma. *$f(2^k m) = 2^k f(m)$ for all integer k and real $m > 0$.*

Proof. $P(d^{1/3}, d^{1/3})$ gives $2f(d) + 3f(2d) = f(8d)$. Consider the sequence $\alpha_k = f(2^k m)$. We have a linear recurrence: $\alpha_{k+3} = 3\alpha_{k+1} + 3\alpha_k$. Its characteristic equation has roots $2, -1, -1$, so we have $f(2^k m) = \alpha_k = c_1 2^k + c_2(-1)^k + c_3(-1)^k k$ for some c_1, c_2, c_3 that may depend on m but not on k. This can be extended to negative k as well. Note that since $f(x)$ is increasing and $f(0) = 0$, $\alpha_k \geq 0$ for all k. Now, if either c_2 or c_3 is nonzero, you can take $k \to -\infty$ with the right parity, and you will get $\alpha_k < 0$, a contradiction. Thus $c_2 = c_3 = 0$, so $f(2^k m) = c_1 2^k$. Plugging in $k = 0$, we get $c_1 = f(m)$, so $f(2^k m) = 2^k f(m)$ as desired. $\square$

Lemma. *$f(\phi^{3k} m) = \phi^{3k} f(m)$ for all integer k and real $m > 0$.*

Proof. $P(d^{1/3}, \phi d^{1/3})$ gives $f(d) + 4f(\phi^3 d) = f(\phi^6 d)$. Again, this gives a linear recurrence for the sequence $\beta_k = \phi^{3k} m$, $\beta_{k+2} = 4\beta_{k+1} + \beta_k$. Its characteristic equation has roots $\phi^3, -\phi^{-3}$, so we have $f(\phi^{3k} m) = \beta_k = c_4 \phi^{3k} + c_5(-\phi^{-3})^k$ for some c_4, c_5 that may depend on m but not on k. As before, c_5 must be zero, so $f(\phi^{3k} m) = c_4 \phi^{3k}$. Plugging in $k = 0$, $c_4 = f(m)$, so $f(\phi^{3k} m) = \phi^{3k} f(m)$ as desired. $\square$

Now I claim that $f(x) = f(1)x$ for all x. Since f is odd, we only need to prove this for positive x. If $f(1) = 0$, we are done by Lemma 1. Otherwise, for a contradiction, let $f(n) \neq f(1)n$ for some $n > 0$. (note that $f(n) \geq 0$). Let $f(n) > f(1)n$; the case where $f(n) < f(1)n$ is similar. By Dirichlet's approximation theorem, we can find r, s such that:

$$n < \frac{2^s}{\phi^{3r}} < \frac{f(n)}{f(1)}$$

or, expanding,

$$\phi^{3r} n < 2^s \implies \phi^{3r} f(n) > 2^s f(1)$$

But, by Lemmas 1 and 2:

$$f(\phi^{3r} n) = \phi^{3r} f(n) \qquad \text{and} \qquad f(2^s) = 2^s f(1)$$

a contradiction to the fact that f is increasing. Thus, $f(x) = f(1)x$ for all x. Re-adjusting for the assumption that $f(0) = 0$, $f(x)$ is linear. Plugging back in to the condition, $f(x)$ can be any linear function with a nonnegative coefficient of x.

C1. Let m and n be fixed distinct positive integers. A wren is on an infinite chessboard indexed by $\mathbb{Z}^2$, and from a square (x,y) may move to any of the eight squares $(x\pm m, y\pm n)$ or $(x\pm n, y\pm m)$. For each $\{m,n\}$, determine the smallest number k of moves required for the wren to travel from $(0,0)$ to $(1,0)$, or prove that no such k exists.

(Michael Ren)

Sorry, the answer we had originally was wrong. The user `talkon` gives an answer of:

- If $\gcd(m,n) > 1$ then no such sequence exists.
- If $m \equiv n \equiv 1 \pmod 2$ then no such sequence exists.
- Otherwise, suppose m is even. Then the answer is

$$\max\{2p, m\} + \max\{q, n\}$$

where $p \ge 0$ is minimal such that $2mp \equiv \pm 1 \pmod n$, and q is $\frac{2pm\pm1}{n}$, whichever is the smallest integer.

(The obvious guess $k = m+n$ is not correct.) See https://artofproblemsolving.com/community/c6h1472063.

This problem is actually known already. The question was raised by Alasdair Iain Houston in the 1970s, with members of the Fairy Chess Correspondence Circle. It appeared in print in George Jelliss's paper *Theory of Leapers* in Chessics 24, 1985. (Chessics was a fairy chess and recreational mathematics journal published and edited by Jelliss; issue 24 is available https://www.mayhematics.com/p/p.htm and the discussion of Houston's problem begins page 96.)

C2. The edges of K_{2017} are each labelled with 1, 2, or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all $\binom{2017}{2}$ labels.

(Michael Ma)

In general, the answer for $2m+1$ is $2 - \frac{1}{2m+1}$.

We prove the lower bound by induction on m: assume some edge vw is labeled 1. Then we delete it, noting that edges touching v and w contribute a sum of at least $4 \cdot (2m-1) = 8m-4$. Thus by induction hypothesis the total is at least

$$\binom{2m-1}{2}\left(2 - \frac{1}{2m-1}\right) + (8m-4) + 1 = \binom{2m+1}{2}\left(2 - \frac{1}{2m+1}\right)$$

as desired.

Interestingly, there are (at least) two equality cases. One is to have all edges be 2 except for m disjoint edges, which have weight 1. Another is to split the vertex set into two sets $A \cup B$ with $|A| = m$ and $|B| = m+1$, then weight all edges in $A \times B$ with 1 and the remaining edges with 3.

Remark. In fact, given any equality case on c vertices, one can generate one on $c+2$ vertices by two vertices u and v, connected to the previous c vertices with weight 2, and then equipping uv with weight 1.

C3. Consider a finite binary string b with at least 2017 ones. Show that one can insert some plus signs in between pairs of digits such that the resulting sum, when performed in base 2, is equal to a power of two.

(David Stoner)

Solution by Mihir Singhal:

We first note that, given any binary string with n ones, we can achieve any integer value in the range $[n, \frac{3n}{2}]$ as follows: first, put pluses between every digit. Then, remove the plus directly after every other 1. Doing this one at a time gives everything from n to $\frac{3n}{2}$.

Now we prove the result for $n \geq 17$. Let n be the number of ones. If any power of 2 is in the range $[n, \frac{3n}{2}]$, then we are done already. Otherwise, we must have $2^\alpha + 1 \leq n < \frac{2^{\alpha+2}}{3}$ for some integer α. We claim that $2^{\alpha+1}$ is achievable via the following algorithm:

0. Put pluses in between every digit, so that we have a current sum n.
1. Cut off the part of the string from the fourth to right 1 onwards; call this the tail, and the rest the head.
2. Starting at the leftmost ungrouped 1, group that one with the two digits immediately following it.
3. Repeat step 2 until the sum is $\geq 2^{\alpha+1}$.
4. Adjust the result until the sum is exactly $2^{\alpha+1}$.

We first show that the condition in 3 occurs before step 2 becomes impossible. Note that since there are at least 13 ones in the head, at least four full groups can be attained before step 2 becomes problematic. Note that the group transformations take $1+1+1 \to 7, 1+0+1 \to 5, 1+1+0 \to 6, 1+0+0 \to 4$. In particular, the sum value v becomes $\geq 2v+1$. Suppose that ℓ is the number of leftover ones in the tail after all possible groups have been formed in the manner described, and g is the number of groups formed. The sum at this point is at least:

$$2(n-\ell-4)+g+\ell+4 = 2n+g-\ell-4$$

Since $g \geq 4$ and $\ell \leq 2$, this is at least $2n-2 \geq 2^{\alpha+1}$. So, the condition in step 3 will indeed arise before step 2 becomes impossible.

Now we clarify step 4. Suppose that on the formation of group $1+b_0+b_1 \to 4+2b_0+b_1$ the sum first becomes $\geq 2^{\alpha+1}$. If it equals $2^{\alpha+1}$, we are done. Otherwise, since every grouping increases the sum by at most 4, the beforehand sum is in the set $\{2^{\alpha+1}-3, 2^{\alpha+1}-2, 2^{\alpha+1}-1\}$.

- If the sum is $2^{\alpha+1}-3$, then change $1+b_0$ to $1b_0$ and the tail sum from 4 to 6 (possibly by the lemma).
- If the sum is $2^{\alpha+1}-2$, then change the tail sum from 4 to 6.
- If the sum is $2^{\alpha+1}-1$, then change the tail sum from 4 to 5.

In any case, a final sum of $2^{\alpha+1}$ is attained, as desired.

C4. nicκy is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer n, nicκy can label at least dn^2 cells of an $n \times n$ square.

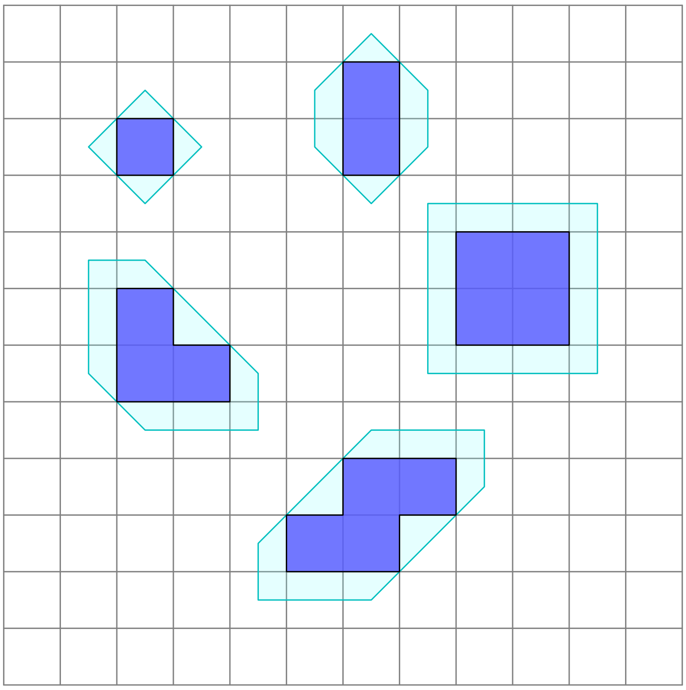
(Mihir Singhal and Michael Kural)

Solution by Yevhenii Diomidov, Kada Williams and Mihir Singhal:

The answer is $d \le \frac{1}{2}$. The construction consists of placing kappas in all squares of the forms $(2k, 4\ell)$, $(2k, 4\ell + 1)$, $(2k + 1, 4\ell + 2)$, and $(2k + 1, 4\ell + 3)$.

To prove that this is minimal, consider all connected components consisting of squares that contain kappas that are connected via edges. It is easy to see that there are only five different kinds of connected components.

Extend each connected component into a larger figure as shown below:



Due to the fact that there are no three kappas in a line and due to the nature of the extensions, one can see that after extension, the interiors of the figures remain disjoint. However, note that the extended area of each figure is at least twice its original area (it is exactly twice except for the 2 by 2 square, for which it is $\frac{9}{4}$ times the original area). Some of the extended regions may fall outside the square, but this is fine since the error is at most $O(n)$.

Thus, Nicky can cover at most $\frac{n^2}{2} + O(n)$ of the squares with kappas, which is what we wanted to show.

C5. There are n MOPpers $p_1, \dots, p_n$ designing a carpool system to attend their morning class. Each p_i's car fits $\chi(p_i)$ people ($\chi \colon \{p_1, \dots, p_n\} \to \{1, 2, \dots, n\}$). A c-fair carpool system is an assignment of one or more drivers on each of several days, such that each MOPper drives c times, and all cars are full on each day. (More precisely, it is a sequence of sets $(S_1, \dots, S_m)$ such that $|\{k : p_i \in S_k\}| = c$ and $\sum_{x \in S_j} \chi(x) = n$ for all i, j.)

Suppose it turns out that a 2-fair carpool system is possible but not a 1-fair carpool system. Must n be even?

(Nathan Ramesh and Palmer Mebane)

First solution (Palmer Mebane) Let $n = 5 \cdot 2^{20} + 2^{15} - 1$ which is odd. For all but 15 people, set $\chi(x) = n$. Biject the 15 people to two element subsets of $\{1, 2, 3, 4, 5, 6\}$, and construct a complete graph K_6 where 1 to 6 are the vertices and each person $\{i, j\}$ is an edge from i to j. There are 15 perfect matchings (so 3 edges) on K_6. Number these matchings from 0 to 14, and assign each edge the matching numbers it's a part of, so each person/edge has 3 matching numbers assigned to them. If the three numbers for person p_i are x, y, z, set $\chi(p_i) = 2^{20} + 2^x + 2^y + 2^z$. We claim this is 2-fair but not 1-fair.

It is 2-fair because we can take 6 sets S_i such that S_i contains all people whose subsets are of the form $\{i, j\}$ for some $j \neq i$. This is because the 15 matching numbers assigned to 5 people all incident to the same vertex are distinct; that's how matchings work.

However it is not 1-fair, because we constructed χ so that those sets S_i are the only ways to choose a subset of people whose χ values sum to n. The $5 \cdot 2^{20}$ term in n forces us to choose exactly 5 people. Then each of these 5 people comes with three matching numbers, and the only way to get the $2^{15} - 1$ term by summing 15 powers of 2 is to sum $2^0 + 2^1 + \cdots + 2^{14}$. So our 5 people have to be assigned each matching number from 0 to 14 exactly once between them. But if the edges we choose don't all come from the same vertex, then two of the edges will be in the same matching, so that matching number is repeated and we can't get 15 powers of 2 to sum to $2^{15} - 1$.

Second solution (Krit Boonsiriseth) Here is a counterexample with $n = 23$: the capacities are 2^4, 7^3, 3^2, 8^3, 17, 18, 23^9. It is not 1-fair since the 17 needs either all the 2's or all the 3's while the 18 needs a 2 and a 3. However, a 2-fair carpool system is:

- $2 + 2 + 2 + 17$
- $2 + 7 + 7 + 7$
- $7 + 8 + 8$
- $7 + 8 + 8$
- $7 + 8 + 8$
- $7 + 8 + 8$
- $2 + 3 + 18$
- $3 + 3 + 17$
- eighteen 23's.

G1. Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from A, such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.

(Michael Ren)

We present seven different solutions.

First solution (Michael Ren) Let R be the intersection of (AH) and (ABC), and let D, E, and F respectively be the orthocenter of APQ, the foot of the altitude from A to PQ, and the reflection of D across E. Note that F lies on (AH) and E lies on (AM). Let S and H' be the intersection of AH with BC and (ABC) respectively. Note that R is the center of spiral similarity taking DEF to $H'SH$, so D lies on (ABC), as desired.

Second solution (Vincent Huang, Evan Chen) Let DEF be the orthic triangle of ABC. Let N and S be the midpoints of PQ and AH. Then $\overline{MS}$ is the diameter of the nine-point circle, so since $\overline{SN}$ is the perpendicular bisector of $\overline{PQ}$ the point N lies on the nine-point circle too. Now the orthocenter of $\triangle APQ$ is the reflection of H across N, hence lies on the circumcircle (homothety of ratio 2 takes the nine-point circle to (ABC)).

Third solution (Zack Chroman) Let R be the midpoint of PQ, and X the point such that $(M, X; P, Q) = -1$. Take E and F to be the feet of the B, C altitudes. Recall that ME, MF are tangents to the circle (AH), so EF is the polar of M.

Then note that $MP \cdot MQ = MX \cdot MR = ME^2$. Then, since X is on the polar of M, R lies on the nine-point circle — the inverse of that polar at M with power ME^2. Then by dilation the orthocenter $2\vec{R} - \vec{H}$ lies on the circumcircle of ABC.

Fourth solution (Zack Chroman) We will prove the following more general claim which implies the problem:

Claim. For a circle γ with a given point A and variable point B, consider a fixed point X not on γ. Let C be the second intersection of XB and γ, then the locus of the orthocenter of ABC is a circle

Proof. Complex numbers is straightforward, but suppose we want a more synthetic solution. Let D be the midpoint of BC. If O is the center of the circle, $\angle OMX = 90$, so M lies on the circle (OX). Then

$$H = 4O - A - B - C = 4O - A - 2D.$$

So H lies on another circle. (Here we can use complex numbers, vectors, coordinates, whatever; alternatively we can use the same trick as above and say that H is the reflection of a fixed point over D). $\square$

Fifth solution (Kevin Ren) Let O be the midpoint of AH and N be the midpoint of PQ. Let K be the orthocenter of APQ.

Because $AP \perp KQ$ and $KP \perp HP$, we have $KQ \parallel PH$. Similarly, $KP \parallel QH$. Thus, $KPHQ$ is a parallelogram, which means KH and PQ share the same midpoint N.

Since N is the midpoint of chord PQ, we have $\angle ONM = 90^\circ$. Hence N lies on the 9-point circle. Take a homothety from H mapping N to K. This homothety maps the 9-point circle to the circumcircle, so K lies on the circumcircle.

Sixth solution (Evan Chen, complex numbers) We use complex numbers with $(AHEF)$ the unit circle, centered at N. Let a, e, f denote the coordinates of A, E, F, and hence $h = -a$. Since M is the pole of $\overline{EF}$, we have $m = \frac{2ef}{e+f}$. Now, the circumcenter O of $\triangle ABC$ is given by $o = \frac{2ef}{e+f} + a$, due to the fact that $ANMO$ is a parallelogram.

The unit complex numbers p and q are now known to satisfy

$$p + q = \frac{2ef}{e+f} + \frac{2pq}{e+f}$$

so

$$(a+p+q) - o = \frac{2pq}{e+f} \qquad \text{and} \qquad a - o = \frac{2ef}{e+f}$$

which clearly have the same magnitude. Hence the orthocenter of $\triangle APQ$ and A are equidistant from O.

Seventh solution (Evan Chen, complex numbers) Here is another complex solution using (APQ) as the unit circle. We let the fourth point M satisfy $m + pq\overline{m} = p + q$. Moreover, let D be the reflection of H across M; we wish to show $a + p + q$ lies on the circle with diameter $\overline{AD}$. This is:

$$\begin{aligned}
\frac{(a+p+q)-a}{(a+p+q)-(2m-h)} &= \frac{p+q}{p+q-2m} \\
\overline{\left(\frac{p+q}{p+q-2m}\right)} &= \frac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p}+\frac{1}{q}-2\overline{m}} = \frac{p+q}{p+q-2pq\overline{m}} \\
&= \frac{p+q}{p+q-2(p+q-m)} = \frac{p+q}{2m-p-q}.
\end{aligned}$$

G2. Let ABC be a scalene triangle with $\angle A = 60^\circ$. Let E and F be the feet of the angle bisectors of $\angle ABC$ and $\angle ACB$ respectively, and let I be the incenter of $\triangle ABC$. Let P, Q be distinct points such that $\triangle PEF$ and $\triangle QEF$ are equilateral. If O is the circumcenter of $\triangle APQ$, show that $\overline{OI} \perp \overline{BC}$.

(Vincent Huang)

WLOG assume $AB < AC$. Also suppose P is on the same side of EF as A, so that A, P, E, F are concyclic. Basic angle-chasing tells us $\angle EIF = 120^\circ$, hence I lies on the same circle as A, E, F, P.

Let the circumcircle of $\triangle BFI$ meet BC again at point Q'. By Miquel's Theorem on $\triangle ABC$ and points Q', EF we have that Q', I, C, E are concyclic. Hence $\angle EQ'F = \angle EQ'I + \angle FQ'I = \angle ECI + \angle FBI = \frac{1}{2}(\angle B + \angle C) = 60^\circ$, implying that E, F, Q, Q' are concyclic.

Since $\angle FEI = \angle FAI = 30^\circ = \frac{1}{2}\angle FEQ$ and $FE = EQ$, we know that F, Q are reflections about BI, so since $F \in AB$ we have $Q \in BC$. Now since I must lie on the perpendicular bisector of QQ', we deduce that if X is the midpoint of QQ', then $IX \perp BC$.

Since AP is the exterior angle bisector of $\angle BAC$ it's well-known that AP, EF, BC concur at a point R, hence $RA \cdot RP = RE \cdot RF = RQ \cdot RQ'$, implying A, P, Q, Q' are concyclic, hence $OX \perp BC \implies OI \perp BC$ as desired.

G3. Call the ordered pair of distinct circles (ω, γ) scribable if there exists a triangle with circumcircle ω and incircle γ. Prove that among n distinct circles there are at most $(n/2)^2$ scribable pairs.

(Daniel Liu)

The main point is to show that there are no triangles in the graph of scribable pairs, after which Turan's theorem finishes the proof. This is essentially Poncelet porism but we give a direct proof.

Suppose there exist three circles A, B, C with radii a, b, c respectively (with $a > b > c > 0$) such that $(A, B), (B, C), (A, C)$ are scribable. Then by triangle inequality and Euler's formula, we have

$$\sqrt{a(a-2b)} + \sqrt{b(b-2c)} \geq \sqrt{a(a-2c)}.$$

However note that

$$\sqrt{a(a-2c)} - \sqrt{a(a-2b)} = \frac{\sqrt{a}(2b-2c)}{\sqrt{a-2c}+\sqrt{a-2b}} > \frac{\sqrt{a}(2b-2c)}{\sqrt{a}+\sqrt{a}} = b - c$$

and

$$\sqrt{b(b-2c)} \leq \sqrt{b^2 - 2bc + c^2} = b - c$$

establishing a contradiction.

G4. Let ABC be an acute triangle with incenter I and circumcircle ω. Suppose a circle ω_B is tangent to BA, BC, and internally tangent to ω at B_1, while a circle ω_C is tangent to CA, CB, and internally tangent to ω at C_1. If B_2, C_2 are the points on ω opposite to B, C, respectively, and X denotes the intersection of B_1C_2, B_2C_1, prove that $XA = XI$.

(Vincent Huang and Nathan Weckwerth)

Solution by Ankan:

Let M_B and N_B be the midpoints of the minor and major arcs AC, and define M_C and N_C similarly. It's well known that $I = \overline{N_B B_1} \cap \overline{N_C C_1}$.

The case where $O = I$ is left to the reader as an exercise. If $O \neq I$, Pascal on $M_B B B_2 C_1 N_C M_C$ and $M_C C C_2 B_1 N_B M_B$ give $\overline{M_B M_C} \cap \overline{C_1 B_2} \in \overline{OI}$ and $\overline{M_B M_C} \cap \overline{B_1 C_2} \in \overline{OI}$, so $X = \overline{B_1 C_2} \cap \overline{C_1 B_2} \in \overline{M_B M_C}$.

But this is equivalent to $XA = XI$, so done. (One way to see this is to let I_A, I_B, and I_C be the A-, B-, and C-excenters of $\triangle ABC$, and consider the homothety with ratio $\frac{1}{2}$ centered at I; it takes $\overline{I_B I_C}$ to $\overline{M_B M_C}$.)

N1. Let $a_1, a_2, \dots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \dots, a_n^n + P) \le 2 \gcd(a_1, \dots, a_n)^n.$$

(Daniel Liu)

The inequality is homogenous, so we may assume $\gcd(a_1, \dots, a_n) = 1$. Then we want to show

$$\gcd(a_1^n + P, \dots, a_n^n + P) \le 2.$$

So it suffices to show that neither 4 nor any odd prime divides the gcd.

First, let p be an odd prime. Suppose that $p \mid a_i^n + P$ for all i. Then $a_i^n \equiv -P \pmod{p}$, so multiplying this for all i, we get $P^n \equiv -P^n \pmod{p}$. Then we see that $p \mid P$, so p divides a_i^n for each i, contradiction.

If $p = 4$, similarly $2 \mid P$ and $2 \mid a_i^n$ for each i, contradiction.

N2. An integer $n > 2$ is called *tasty* if for every ordered pair of positive integers (a, b) with $a + b = n$, at least one of $\frac{a}{b}$ and $\frac{b}{a}$ is a terminating decimal. Do there exist infinitely many tasty integers?

(Vincent Huang)

The answer is no. (In fact, a computation implies that $n = 21$ is the largest one.)

First, we recall the well-known fact that the fraction $\frac{a}{b}$, with $\gcd(a, b) = 1$, is terminating if and only if the prime factorization of b consists only of 2s and 5s.

Consider some tasty number n and all pairs (a, b) with $a+b = n, \gcd(a, n) = 1, a \le 0.5n$. It's clear that there are $0.5\phi(n)$ of these pairs, and since $\gcd(a, b) = 1$ we must have that at least one of a and b has a prime factorization of only 2s and 5s.

But considering all numbers $2^x 5^y \le n$, we know $x \le \log_2 n + 1, y \le \log_5 n + 1$, hence there are at most $(\log_2 n + 1)(\log_5 n + 1)$ such numbers, so we deduce that $(\log_2 n + 1)(\log_5 n + 1) \ge 0.5\phi(n)$.

Lemma. *For every $n > 2$, $\phi(n) \ge 0.5\sqrt{n}$.*

Proof. Decompose n into prime powers $p_i^{e_i}$. For each $p_i > 2$, it's easy to show that $p_i^{e_i-1}(p-1) \ge \sqrt{p_i^{e_i}}$. For $p_i = 2$, we can show that $p_i^{e_i-1}(p-1) \ge 0.5\sqrt{p_i^{e_i}}$, hence multiplying these bounds gives the desired. $\square$

Therefore, for n to be tasty, we need $(\log_2 n + 1)(\log_5 n + 1) \ge 0.25\sqrt{n}$, which only holds for finitely many n as desired.

N3. For each integer $C > 1$, decide whether there exists pairwise distinct positive integers a_1, a_2, a_3, ... such that for every $k \geq 1$,

$$a_{k+1}^k \quad \text{divides} \quad C^k a_1 a_2 \dots a_k.$$

(Daniel Liu)

No sequence exists for any C. Note that the divisibility is homogenous with respect to the a_i so we can shift the sequence and WLOG assume that $a_1 = 1$.

Note that any prime divisor of the a_i must also be a prime divisor of C. Let

$$C = p_1^{c_1} p_2^{c_2} \dots p_k^{c_k}$$

be the prime factorization of C.

Claim. Fix $p = p_j$ and $c = c_j$. Then for any index k we have

$$\nu_p(a_i) \leq cH_k + \nu_p(a_1).$$

Proof. Let $b_i = \nu_p(a_i)$. We apply strong induction: Base case of $k = 1$ is trivial. Now assume $b_i \leq H_{i-1}c + b_1$ for $i \leq k$; then

$$\begin{aligned}
b_{k+1} &\leq c + \frac{\sum_{i=1}^k b_i}{k} \\
&\leq c + \frac{\sum_{i=1}^k H_{i-1}c + b_1}{k} \\
&= c\left(1 + \frac{\sum_{i=1}^k H_{i-1}}{k}\right) + b_1 \\
&= c\left(1 + \frac{\frac{k-1}{1} + \frac{k-2}{2} + \dots + \frac{1}{k-1}}{k}\right) + b_1 \\
&= c\left(1 + \frac{k-1}{k \cdot 1} + \frac{k-2}{k \cdot 2} + \dots + \frac{1}{(k-1) \cdot k}\right) + b_1 \\
&= c\left(1 + \frac{1}{1} - \frac{1}{k} + \frac{1}{2} - \frac{1}{k} + \dots + \frac{1}{k-1} - \frac{1}{k}\right) + b_1 \\
&= cH_k + b_1
\end{aligned}$$

and the induction is complete. $\square$

Now, let N be a positive integer, and let $m = 1 + \max_j \nu_{p_j}(a_1)$. We have that

$$\nu_{p_j} a_i \leq c_j H_i + \nu_{p_j}(a_1) \leq c_j(m + \log N)$$

if $i \leq N$. Hence, there are at most

$$\prod_{j=1}^k [1 + c_j(m + \log N)] = O\left((\log N)^k\right)$$

possible k-triples that $(\nu_{p_1} a_i, \nu_{p_2} a_i, \dots, \nu_{p_k} a_i)$ can be. But this also needs to be at least $N + 1$, which is impossible for large N.

Shortlisted Problems

19$^{\text{th}}$ ELMO

Pittsburgh, PA, 2017

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until disclosed publicly by the committee on the ELMO.

Contributing Students

The Problem Selection Committee for ELMO 2017 thanks the following proposers for contributing **45 problems** to this year's Competition:

> Ashwin Sah, Colin Tang, Daniel Liu, David Stoner, Jeffery Li, Michael Kural, Michael Ma, Michael Ren, Mihir Singhal, Nathan Ramesh, Nathan Weckwerth, Palmer Mebane, Ruidi Cao, Tristan Shin, Vincent Huang, Zack Chroman

Problem Selection Committee

The Problem Selection Committee for ELMO 2017 was led by **Evan Chen** and consisted of:

- Ashwin Sah
- James Lin
- Kevin Ren
- Mihir Singhal
- Michael Ma
- Michael Ren
- Yannick Yao

Problems

A1. Let $0 < k < \frac{1}{2}$ be a real number and let a_0 and b_0 be arbitrary real numbers in $(0,1)$. The sequences $(a_n)_{n\ge 0}$ and $(b_n)_{n\ge 0}$ are then defined recursively by

$$a_{n+1} = \frac{a_n+1}{2} \qquad \text{and} \qquad b_{n+1} = b_n^k$$

for $n \ge 0$. Prove that $a_n < b_n$ for all sufficiently large n.

(Michael Ma)

A2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c:

(i) If $a+b+c \ge 0$ then $f(a^3)+f(b^3)+f(c^3) \ge 3f(abc)$.

(ii) If $a+b+c \le 0$ then $f(a^3)+f(b^3)+f(c^3) \le 3f(abc)$.

(Ashwin Sah)

C1. Let m and n be fixed distinct positive integers. A wren is on an infinite chessboard indexed by $\mathbb{Z}^2$, and from a square (x,y) may move to any of the eight squares $(x\pm m, y\pm n)$ or $(x\pm n, y\pm m)$. For each $\{m,n\}$, determine the smallest number k of moves required for the wren to travel from $(0,0)$ to $(1,0)$, or prove that no such k exists.

(Michael Ren)

C2. The edges of K_{2017} are each labelled with 1, 2, or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all $\binom{2017}{2}$ labels.

(Michael Ma)

C3. Consider a finite binary string b with at least 2017 ones. Show that one can insert some plus signs in between pairs of digits such that the resulting sum, when performed in base 2, is equal to a power of two.

(David Stoner)

C4. nicκy is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer n, nicκy can label at least dn^2 cells of an $n \times n$ square.

(Mihir Singhal and Michael Kural)

C5. There are n MOPpers $p_1, \ldots, p_n$ designing a carpool system to attend their morning class. Each p_i's car fits $\chi(p_i)$ people ($\chi\colon \{p_1,\ldots,p_n\} \to \{1,2,\ldots,n\}$). A c-fair carpool system is an assignment of one or more drivers on each of several days, such that each MOPper drives c times, and all cars are full on each day. (More precisely, it is a sequence of sets $(S_1,\ldots,S_m)$ such that $|\{k : p_i \in S_k\}| = c$ and $\sum_{x\in S_j} \chi(x) = n$ for all i, j.)

Suppose it turns out that a 2-fair carpool system is possible but not a 1-fair carpool system. Must n be even?

(Nathan Ramesh and Palmer Mebane)

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(Vincent Huang)

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(Vincent Huang and Nathan Weckwerth)

N1. Let $a_1, a_2, \ldots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \ldots, a_n^n + P) \le 2\gcd(a_1, \ldots, a_n)^n.$$

(Daniel Liu)

N2. An integer $n > 2$ is called *tasty* if for every ordered pair of positive integers (a, b) with $a + b = n$, at least one of $\frac{a}{b}$ and $\frac{b}{a}$ is a terminating decimal. Do there exist infinitely many tasty integers?

(Vincent Huang)

N3. For each integer $C > 1$, decide whether there exists pairwise distinct positive integers a_1, a_2, a_3, $\ldots$ such that for every $k \ge 1$,

$$a_{k+1}^k \quad \text{divides} \quad C^k a_1 a_2 \ldots a_k.$$

(Daniel Liu)

Solutions

A1. Let $0 < k < \frac{1}{2}$ be a real number and let a_0 and b_0 be arbitrary real numbers in $(0,1)$. The sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are then defined recursively by

$$a_{n+1} = \frac{a_n + 1}{2} \qquad \text{and} \qquad b_{n+1} = b_n^k$$

for $n \geq 0$. Prove that $a_n < b_n$ for all sufficiently large n.

(Michael Ma)

It should be clear that both sequences converge to 1. In the first sequence, the distance from 1 is halved every time and converges to 0. In the second sequence $b_n = b_0^{k^n}$ and since k^n converges to 0, b_i converges to 1.

The key lemma to solve the problem is the following:

Lemma. *If $k < \frac{1}{2}$ then there exists $0 < x_0 < 1$ such that whenever $x_0 < x < 1$,*

$$x^k > \frac{2k+1}{4}x + \frac{3-2k}{4}.$$

Proof. First notice that if we take the tangent to $y = x^k$ at $(1,1)$ we get the equation $y = kx + (1-k)$. We can see by taking the first derivative of

$$kx + (1-k) - x^k$$

to get

$$k - kx^{k-1}$$

which is negative as $kx + (1-k) - x^k$ is decreasing from 0 to 1. Furthermore x^k is concave and increasing from 0 to 1. Now it if we take a line of higher slope than k passing through $(1,1)$ for large enough x the line will fall under x^k. $\square$

Now let x_0 be as above, and let $a = \frac{2k+1}{4} < \frac{1}{2}$ for convenience. Now we can see that

$$b_{n+1} > ab_n + (1-a).$$

Take the smallest M such that a_M and b_M are both larger than x_0. By iterating both recurrences we can see that for $\ell = 0, 1, \ldots$ we have

$$a_{M+\ell} = 1 - \left(\frac{1}{2}\right)^{\ell}(1 - a_M) \quad \text{and} \quad b_{M+\ell} > 1 - a^{\ell}(1 - b_M).$$

Since $\frac{1}{2a} > 1$ we can take a sufficiently large positive integer ℓ_0 such that $\left(\frac{1}{2a}\right)^{\ell_0} > \frac{1-b_M}{1-a_M}$. Then taking $N = M + \ell_0$ we are done since $b_N > a_N$ and

$$x^k > ax + (1-a) > \frac{x+1}{2}$$

for $x > x_0$.

A2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c:

(i) If $a + b + c \ge 0$ then $f(a^3) + f(b^3) + f(c^3) \ge 3f(abc)$.

(ii) If $a + b + c \le 0$ then $f(a^3) + f(b^3) + f(c^3) \le 3f(abc)$.

(Ashwin Sah)

The answer is $f(x) = kx + \ell$ where k and ℓ are any real numbers with $k \ge 0$.
We begin with some weird optimizations:

- Since f can be shifted by a constant, we get $f(0) = 0$.
- Put $c = 0$ and $b = -a$ to get $f(a^3) + f(-a^3) = 0$, so that f is odd.
- Put $c = 0$ now to get $f(a^3) + f(b^3) \ge 0$ whenever $a + b \ge 0$. Combined with f odd, this implies f is weakly increasing.

Now, we let $c = -a - b$ to get:

$$f(a^3) + f(b^3) + f(-(a+b)^3) = 3f(-ab(a+b))$$

Using oddness and rearranging:

$$f(a^3) + f(b^3) + 3f(ab(a+b)) = f((a+b)^3)$$

Call this property $P(a, b)$.

Lemma. *$f(2^k m) = 2^k f(m)$ for all integer k and real $m > 0$.*

Proof. $P(d^{1/3}, d^{1/3})$ gives $2f(d) + 3f(2d) = f(8d)$. Consider the sequence $\alpha_k = f(2^k m)$. We have a linear recurrence: $\alpha_{k+3} = 3\alpha_{k+1} + 3\alpha_k$. Its characteristic equation has roots $2, -1, -1$, so we have $f(2^k m) = \alpha_k = c_1 2^k + c_2(-1)^k + c_3(-1)^k k$ for some c_1, c_2, c_3 that may depend on m but not on k. This can be extended to negative k as well. Note that since $f(x)$ is increasing and $f(0) = 0$, $\alpha_k \ge 0$ for all k. Now, if either c_2 or c_3 is nonzero, you can take $k \to -\infty$ with the right parity, and you will get $\alpha_k < 0$, a contradiction. Thus $c_2 = c_3 = 0$, so $f(2^k m) = c_1 2^k$. Plugging in $k = 0$, we get $c_1 = f(m)$, so $f(2^k m) = 2^k f(m)$ as desired. □

Lemma. *$f(\phi^{3k} m) = \phi^{3k} f(m)$ for all integer k and real $m > 0$.*

Proof. $P(d^{1/3}, \phi d^{1/3})$ gives $f(d) + 4f(\phi^3 d) = f(\phi^6 d)$. Again, this gives a linear recurrence for the sequence $\beta_k = \phi^{3k} m$, $\beta_{k+2} = 4\beta_{k+1} + \beta_k$. Its characteristic equation has roots $\phi^3, -\phi^{-3}$, so we have $f(\phi^{3k} m) = \beta_k = c_4 \phi^{3k} + c_5(-\phi^{-3})^k$ for some c_4, c_5 that may depend on m but not on k. As before, c_5 must be zero, so $f(\phi^{3k} m) = c_4 \phi^{3k}$. Plugging in $k = 0$, $c_4 = f(m)$, so $f(\phi^{3k} m) = \phi^{3k} f(m)$ as desired. □

Now I claim that $f(x) = f(1)x$ for all x. Since f is odd, we only need to prove this for positive x. If $f(1) = 0$, we are done by Lemma 1. Otherwise, for a contradiction, let $f(n) \ne f(1)n$ for some $n > 0$. (note that $f(n) \ge 0$). Let $f(n) > f(1)n$; the case where $f(n) < f(1)n$ is similar. By Dirichlet's approximation theorem, we can find r, s such that:

$$n < \frac{2^s}{\phi^{3r}} < \frac{f(n)}{f(1)}$$

or, expanding,

$$\phi^{3r} n < 2^s \implies \phi^{3r} f(n) > 2^s f(1)$$

But, by Lemmas 1 and 2:

$$f(\phi^{3r} n) = \phi^{3r} f(n) \qquad \text{and} \qquad f(2^s) = 2^s f(1)$$

a contradiction to the fact that f is increasing. Thus, $f(x) = f(1)x$ for all x. Re-adjusting for the assumption that $f(0) = 0$, $f(x)$ is linear. Plugging back in to the condition, $f(x)$ can be any linear function with a nonnegative coefficient of x.

C1. Let m and n be fixed distinct positive integers. A wren is on an infinite chessboard indexed by $\mathbb{Z}^2$, and from a square (x, y) may move to any of the eight squares $(x \pm m, y \pm n)$ or $(x \pm n, y \pm m)$. For each $\{m, n\}$, determine the smallest number k of moves required for the wren to travel from $(0, 0)$ to $(1, 0)$, or prove that no such k exists.

(Michael Ren)

Sorry, the answer we had originally was wrong. The user `talkon` gives an answer of:

- If $\gcd(m, n) > 1$ then no such sequence exists.
- If $m \equiv n \equiv 1 \pmod 2$ then no such sequence exists.
- Otherwise, suppose m is even. Then the answer is

$$\max\{2p, m\} + \max\{q, n\}$$

where $p \geq 0$ is minimal such that $2mp \equiv \pm 1 \pmod n$, and q is $\frac{2pm \pm 1}{n}$, whichever is the smallest integer.

(The obvious guess $k = m + n$ is not correct.) See https://artofproblemsolving.com/community/c6h1472063.

This problem is actually known already. The question was raised by Alasdair Iain Houston in the 1970s, with members of the Fairy Chess Correspondence Circle. It appeared in print in George Jelliss's paper *Theory of Leapers* in Chessics 24, 1985. (Chessics was a fairy chess and recreational mathematics journal published and edited by Jelliss; issue 24 is available https://www.mayhematics.com/p/p.htm and the discussion of Houston's problem begins page 96.)

C2. The edges of K_{2017} are each labelled with 1, 2, or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all $\binom{2017}{2}$ labels.

(Michael Ma)

In general, the answer for $2m+1$ is $2 - \frac{1}{2m+1}$.

We prove the lower bound by induction on m: assume some edge vw is labeled 1. Then we delete it, noting that edges touching v and w contribute a sum of at least $4 \cdot (2m-1) = 8m-4$. Thus by induction hypothesis the total is at least

$$\binom{2m-1}{2}\left(2 - \frac{1}{2m-1}\right) + (8m-4) + 1 = \binom{2m+1}{2}\left(2 - \frac{1}{2m+1}\right)$$

as desired.

Interestingly, there are (at least) two equality cases. One is to have all edges be 2 except for m disjoint edges, which have weight 1. Another is to split the vertex set into two sets $A \cup B$ with $|A| = m$ and $|B| = m+1$, then weight all edges in $A \times B$ with 1 and the remaining edges with 3.

Remark. In fact, given any equality case on c vertices, one can generate one on $c+2$ vertices by two vertices u and v, connected to the previous c vertices with weight 2, and then equipping uv with weight 1.

C3. Consider a finite binary string b with at least 2017 ones. Show that one can insert some plus signs in between pairs of digits such that the resulting sum, when performed in base 2, is equal to a power of two.

(David Stoner)

Solution by Mihir Singhal:

We first note that, given any binary string with n ones, we can achieve any integer value in the range $[n, \frac{3n}{2}]$ as follows: first, put pluses between every digit. Then, remove the plus directly after every other 1. Doing this one at a time gives everything from n to $\frac{3n}{2}$.

Now we prove the result for $n \geq 17$. Let n be the number of ones. If any power of 2 is in the range $[n, \frac{3n}{2}]$, then we are done already. Otherwise, we must have $2^\alpha + 1 \leq n < \frac{2^{\alpha+2}}{3}$ for some integer α. We claim that $2^{\alpha+1}$ is achievable via the following algorithm:

0. Put pluses in between every digit, so that we have a current sum n.
1. Cut off the part of the string from the fourth to right 1 onwards; call this the tail, and the rest the head.
2. Starting at the leftmost ungrouped 1, group that one with the two digits immediately following it.
3. Repeat step 2 until the sum is $\geq 2^{\alpha+1}$.
4. Adjust the result until the sum is exactly $2^{\alpha+1}$.

We first show that the condition in 3 occurs before step 2 becomes impossible. Note that since there are at least 13 ones in the head, at least four full groups can be attained before step 2 becomes problematic. Note that the group transformations take $1+1+1 \to 7, 1+0+1 \to 5, 1+1+0 \to 6, 1+0+0 \to 4$. In particular, the sum value v becomes $\geq 2v+1$. Suppose that ℓ is the number of leftover ones in the tail after all possible groups have been formed in the manner described, and g is the number of groups formed. The sum at this point is at least:

$$2(n-\ell-4)+g+\ell+4 = 2n+g-\ell-4$$

Since $g \geq 4$ and $\ell \leq 2$, this is at least $2n-2 \geq 2^{\alpha+1}$. So, the condition in step 3 will indeed arise before step 2 becomes impossible.

Now we clarify step 4. Suppose that on the formation of group $1+b_0+b_1 \to 4+2b_0+b_1$ the sum first becomes $\geq 2^{\alpha+1}$. If it equals $2^{\alpha+1}$, we are done. Otherwise, since every grouping increases the sum by at most 4, the beforehand sum is in the set $\{2^{\alpha+1}-3, 2^{\alpha+1}-2, 2^{\alpha+1}-1\}$.

- If the sum is $2^{\alpha+1}-3$, then change $1+b_0$ to $1b_0$ and the tail sum from 4 to 6 (possibly by the lemma).
- If the sum is $2^{\alpha+1}-2$, then change the tail sum from 4 to 6.
- If the sum is $2^{\alpha+1}-1$, then change the tail sum from 4 to 5.

In any case, a final sum of $2^{\alpha+1}$ is attained, as desired.

C4. nicκy is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer n, nicκy can label at least dn^2 cells of an $n \times n$ square.

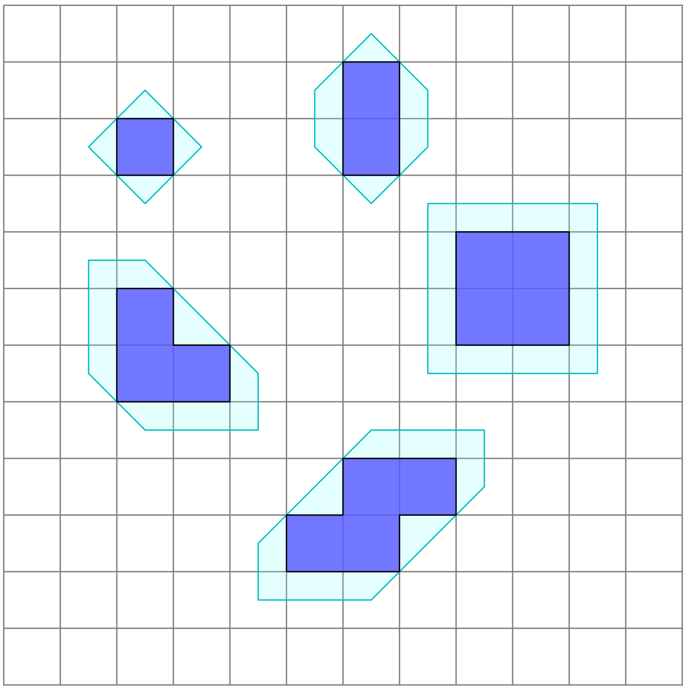
(Mihir Singhal and Michael Kural)

Solution by Yevhenii Diomidov, Kada Williams and Mihir Singhal:

The answer is $d \leq \frac{1}{2}$. The construction consists of placing kappas in all squares of the forms $(2k, 4\ell)$, $(2k, 4\ell + 1)$, $(2k + 1, 4\ell + 2)$, and $(2k + 1, 4\ell + 3)$.

To prove that this is minimal, consider all connected components consisting of squares that contain kappas that are connected via edges. It is easy to see that there are only five different kinds of connected components.

Extend each connected component into a larger figure as shown below:



Due to the fact that there are no three kappas in a line and due to the nature of the extensions, one can see that after extension, the interiors of the figures remain disjoint. However, note that the extended area of each figure is at least twice its original area (it is exactly twice except for the 2 by 2 square, for which it is $\frac{9}{4}$ times the original area). Some of the extended regions may fall outside the square, but this is fine since the error is at most $O(n)$.

Thus, Nicky can cover at most $\frac{n^2}{2} + O(n)$ of the squares with kappas, which is what we wanted to show.

C5. There are n MOPpers $p_1, \dots, p_n$ designing a carpool system to attend their morning class. Each p_i's car fits $\chi(p_i)$ people ($\chi\colon \{p_1, \dots, p_n\} \to \{1, 2, \dots, n\}$). A c-fair carpool system is an assignment of one or more drivers on each of several days, such that each MOPper drives c times, and all cars are full on each day. (More precisely, it is a sequence of sets $(S_1, \dots, S_m)$ such that $|\{k : p_i \in S_k\}| = c$ and $\sum_{x \in S_j} \chi(x) = n$ for all i, j.)

Suppose it turns out that a 2-fair carpool system is possible but not a 1-fair carpool system. Must n be even?

(Nathan Ramesh and Palmer Mebane)

First solution (Palmer Mebane) Let $n = 5 \cdot 2^{20} + 2^{15} - 1$ which is odd. For all but 15 people, set $\chi(x) = n$. Biject the 15 people to two element subsets of $\{1, 2, 3, 4, 5, 6\}$, and construct a complete graph K_6 where 1 to 6 are the vertices and each person $\{i, j\}$ is an edge from i to j. There are 15 perfect matchings (so 3 edges) on K_6. Number these matchings from 0 to 14, and assign each edge the matching numbers it's a part of, so each person/edge has 3 matching numbers assigned to them. If the three numbers for person p_i are x, y, z, set $\chi(p_i) = 2^{20} + 2^x + 2^y + 2^z$. We claim this is 2-fair but not 1-fair.

It is 2-fair because we can take 6 sets S_i such that S_i contains all people whose subsets are of the form $\{i, j\}$ for some $j \neq i$. This is because the 15 matching numbers assigned to 5 people all incident to the same vertex are distinct; that's how matchings work.

However it is not 1-fair, because we constructed χ so that those sets S_i are the only ways to choose a subset of people whose χ values sum to n. The $5 \cdot 2^{20}$ term in n forces us to choose exactly 5 people. Then each of these 5 people comes with three matching numbers, and the only way to get the $2^{15} - 1$ term by summing 15 powers of 2 is to sum $2^0 + 2^1 + \dots + 2^{14}$. So our 5 people have to be assigned each matching number from 0 to 14 exactly once between them. But if the edges we choose don't all come from the same vertex, then two of the edges will be in the same matching, so that matching number is repeated and we can't get 15 powers of 2 to sum to $2^{15} - 1$.

Second solution (Krit Boonsiriseth) Here is a counterexample with $n = 23$: the capacities are 2^4, 7^3, 3^2, 8^3, 17, 18, 23^9. It is not 1-fair since the 17 needs either all the 2's or all the 3's while the 18 needs a 2 and a 3. However, a 2-fair carpool system is:

- $2 + 2 + 2 + 17$
- $2 + 7 + 7 + 7$
- $7 + 8 + 8$
- $7 + 8 + 8$
- $7 + 8 + 8$
- $7 + 8 + 8$
- $2 + 3 + 18$
- $3 + 3 + 17$
- eighteen 23's.

G1. Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from A, such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.

(Michael Ren)

We present seven different solutions.

First solution (Michael Ren) Let R be the intersection of (AH) and (ABC), and let D, E, and F respectively be the orthocenter of APQ, the foot of the altitude from A to PQ, and the reflection of D across E. Note that F lies on (AH) and E lies on (AM). Let S and H' be the intersection of AH with BC and (ABC) respectively. Note that R is the center of spiral similarity taking DEF to $H'SH$, so D lies on (ABC), as desired.

Second solution (Vincent Huang, Evan Chen) Let DEF be the orthic triangle of ABC. Let N and S be the midpoints of PQ and AH. Then $\overline{MS}$ is the diameter of the nine-point circle, so since $\overline{SN}$ is the perpendicular bisector of $\overline{PQ}$ the point N lies on the nine-point circle too. Now the orthocenter of $\triangle APQ$ is the reflection of H across N, hence lies on the circumcircle (homothety of ratio 2 takes the nine-point circle to (ABC)).

Third solution (Zack Chroman) Let R be the midpoint of PQ, and X the point such that $(M, X; P, Q) = -1$. Take E and F to be the feet of the B, C altitudes. Recall that ME, MF are tangents to the circle (AH), so EF is the polar of M.

Then note that $MP \cdot MQ = MX \cdot MR = ME^2$. Then, since X is on the polar of M, R lies on the nine-point circle — the inverse of that polar at M with power ME^2. Then by dilation the orthocenter $2\vec{R} - \vec{H}$ lies on the circumcircle of ABC.

Fourth solution (Zack Chroman) We will prove the following more general claim which implies the problem:

Claim. For a circle γ with a given point A and variable point B, consider a fixed point X not on γ. Let C be the second intersection of XB and γ, then the locus of the orthocenter of ABC is a circle

Proof. Complex numbers is straightforward, but suppose we want a more synthetic solution. Let D be the midpoint of BC. If O is the center of the circle, $\angle OMX = 90$, so M lies on the circle (OX). Then

$$H = 4O - A - B - C = 4O - A - 2D.$$

So H lies on another circle. (Here we can use complex numbers, vectors, coordinates, whatever; alternatively we can use the same trick as above and say that H is the reflection of a fixed point over D). $\square$

Fifth solution (Kevin Ren) Let O be the midpoint of AH and N be the midpoint of PQ. Let K be the orthocenter of APQ.

Because $AP \perp KQ$ and $KP \perp HP$, we have $KQ \parallel PH$. Similarly, $KP \parallel QH$. Thus, $KPHQ$ is a parallelogram, which means KH and PQ share the same midpoint N.

Since N is the midpoint of chord PQ, we have $\angle ONM = 90^\circ$. Hence N lies on the 9-point circle. Take a homothety from H mapping N to K. This homothety maps the 9-point circle to the circumcircle, so K lies on the circumcircle.

Sixth solution (Evan Chen, complex numbers) We use complex numbers with $(AHEF)$ the unit circle, centered at N. Let a, e, f denote the coordinates of A, E, F, and hence $h = -a$. Since M is the pole of $\overline{EF}$, we have $m = \frac{2ef}{e+f}$. Now, the circumcenter O of $\triangle ABC$ is given by $o = \frac{2ef}{e+f} + a$, due to the fact that $ANMO$ is a parallelogram.

The unit complex numbers p and q are now known to satisfy

$$p + q = \frac{2ef}{e+f} + \frac{2pq}{e+f}$$

so

$$(a+p+q) - o = \frac{2pq}{e+f} \qquad \text{and} \qquad a - o = \frac{2ef}{e+f}$$

which clearly have the same magnitude. Hence the orthocenter of $\triangle APQ$ and A are equidistant from O.

Seventh solution (Evan Chen, complex numbers) Here is another complex solution using (APQ) as the unit circle. We let the fourth point M satisfy $m + pq\overline{m} = p + q$. Moreover, let D be the reflection of H across M; we wish to show $a + p + q$ lies on the circle with diameter $\overline{AD}$. This is:

$$\begin{aligned}
\frac{(a+p+q) - a}{(a+p+q) - (2m-h)} &= \frac{p+q}{p+q-2m} \\
\overline{\left(\frac{p+q}{p+q-2m}\right)} &= \frac{\frac{1}{p} + \frac{1}{q}}{\frac{1}{p} + \frac{1}{q} - 2\overline{m}} = \frac{p+q}{p+q-2pq\overline{m}} \\
&= \frac{p+q}{p+q-2(p+q-m)} = \frac{p+q}{2m-p-q}.
\end{aligned}$$

G2. Let ABC be a scalene triangle with $\angle A = 60^\circ$. Let E and F be the feet of the angle bisectors of $\angle ABC$ and $\angle ACB$ respectively, and let I be the incenter of $\triangle ABC$. Let P, Q be distinct points such that $\triangle PEF$ and $\triangle QEF$ are equilateral. If O is the circumcenter of $\triangle APQ$, show that $\overline{OI} \perp \overline{BC}$.

(Vincent Huang)

WLOG assume $AB < AC$. Also suppose P is on the same side of EF as A, so that A, P, E, F are concyclic. Basic angle-chasing tells us $\angle EIF = 120^\circ$, hence I lies on the same circle as A, E, F, P.

Let the circumcircle of $\triangle BFI$ meet BC again at point Q'. By Miquel's Theorem on $\triangle ABC$ and points Q', EF we have that Q', I, C, E are concyclic. Hence $\angle EQ'F = \angle EQ'I + \angle FQ'I = \angle ECI + \angle FBI = \frac{1}{2}(\angle B + \angle C) = 60^\circ$, implying that E, F, Q, Q' are concyclic.

Since $\angle FEI = \angle FAI = 30^\circ = \frac{1}{2}\angle FEQ$ and $FE = EQ$, we know that F, Q are reflections about BI, so since $F \in AB$ we have $Q \in BC$. Now since I must lie on the perpendicular bisector of QQ', we deduce that if X is the midpoint of QQ', then $IX \perp BC$.

Since AP is the exterior angle bisector of $\angle BAC$ it's well-known that AP, EF, BC concur at a point R, hence $RA \cdot RP = RE \cdot RF = RQ \cdot RQ'$, implying A, P, Q, Q' are concyclic, hence $OX \perp BC \implies OI \perp BC$ as desired.

G3. Call the ordered pair of distinct circles (ω, γ) scribable if there exists a triangle with circumcircle ω and incircle γ. Prove that among n distinct circles there are at most $(n/2)^2$ scribable pairs.

(Daniel Liu)

The main point is to show that there are no triangles in the graph of scribable pairs, after which Turan's theorem finishes the proof. This is essentially Poncelet porism but we give a direct proof.

Suppose there exist three circles A, B, C with radii a, b, c respectively (with $a > b > c > 0$) such that $(A, B), (B, C), (A, C)$ are scribable. Then by triangle inequality and Euler's formula, we have

$$\sqrt{a(a-2b)} + \sqrt{b(b-2c)} \geq \sqrt{a(a-2c)}.$$

However note that

$$\sqrt{a(a-2c)} - \sqrt{a(a-2b)} = \frac{\sqrt{a}(2b-2c)}{\sqrt{a-2c}+\sqrt{a-2b}} > \frac{\sqrt{a}(2b-2c)}{\sqrt{a}+\sqrt{a}} = b - c$$

and

$$\sqrt{b(b-2c)} \leq \sqrt{b^2-2bc+c^2} = b - c$$

establishing a contradiction.

G4. Let ABC be an acute triangle with incenter I and circumcircle ω. Suppose a circle ω_B is tangent to BA, BC, and internally tangent to ω at B_1, while a circle ω_C is tangent to CA, CB, and internally tangent to ω at C_1. If B_2, C_2 are the points on ω opposite to B, C, respectively, and X denotes the intersection of B_1C_2, B_2C_1, prove that $XA = XI$.

(Vincent Huang and Nathan Weckwerth)

Solution by Ankan:

Let M_B and N_B be the midpoints of the minor and major arcs AC, and define M_C and N_C similarly. It's well known that $I = \overline{N_BB_1} \cap \overline{N_CC_1}$.

The case where $O = I$ is left to the reader as an exercise. If $O \neq I$, Pascal on $M_BBB_2C_1N_CM_C$ and $M_CCC_2B_1N_BM_B$ give $\overline{M_BM_C} \cap \overline{C_1B_2} \in \overline{OI}$ and $\overline{M_BM_C} \cap \overline{B_1C_2} \in \overline{OI}$, so $X = \overline{B_1C_2} \cap \overline{C_1B_2} \in \overline{M_BM_C}$.

But this is equivalent to $XA = XI$, so done. (One way to see this is to let I_A, I_B, and I_C be the A-, B-, and C-excenters of $\triangle ABC$, and consider the homothety with ratio $\frac{1}{2}$ centered at I; it takes $\overline{I_BI_C}$ to $\overline{M_BM_C}$.)

N1. Let $a_1, a_2, \ldots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \ldots, a_n^n + P) \le 2 \gcd(a_1, \ldots, a_n)^n.$$

(Daniel Liu)

The inequality is homogenous, so we may assume $\gcd(a_1, \ldots, a_n) = 1$. Then we want to show

$$\gcd(a_1^n + P, \ldots, a_n^n + P) \le 2.$$

So it suffices to show that neither 4 nor any odd prime divides the gcd.

First, let p be an odd prime. Suppose that $p \mid a_i^n + P$ for all i. Then $a_i^n \equiv -P \pmod{p}$, so multiplying this for all i, we get $P^n \equiv -P^n \pmod{p}$. Then we see that $p \mid P$, so p divides a_i^n for each i, contradiction.

If $p = 4$, similarly $2 \mid P$ and $2 \mid a_i^n$ for each i, contradiction.

N2. An integer $n > 2$ is called *tasty* if for every ordered pair of positive integers (a, b) with $a + b = n$, at least one of $\frac{a}{b}$ and $\frac{b}{a}$ is a terminating decimal. Do there exist infinitely many tasty integers?

(Vincent Huang)

The answer is no. (In fact, a computation implies that $n = 21$ is the largest one.)

First, we recall the well-known fact that the fraction $\frac{a}{b}$, with $\gcd(a, b) = 1$, is terminating if and only if the prime factorization of b consists only of 2s and 5s.

Consider some tasty number n and all pairs (a, b) with $a+b = n, \gcd(a, n) = 1, a \le 0.5n$. It's clear that there are $0.5\phi(n)$ of these pairs, and since $\gcd(a, b) = 1$ we must have that at least one of a and b has a prime factorization of only 2s and 5s.

But considering all numbers $2^x 5^y \le n$, we know $x \le \log_2 n + 1, y \le \log_5 n + 1$, hence there are at most $(\log_2 n + 1)(\log_5 n + 1)$ such numbers, so we deduce that $(\log_2 n + 1)(\log_5 n + 1) \ge 0.5\phi(n)$.

Lemma. *For every $n > 2$, $\phi(n) \ge 0.5\sqrt{n}$.*

Proof. Decompose n into prime powers $p_i^{e_i}$. For each $p_i > 2$, it's easy to show that $p_i^{e_i-1}(p - 1) \ge \sqrt{p_i^{e_i}}$. For $p_i = 2$, we can show that $p_i^{e_i-1}(p - 1) \ge 0.5\sqrt{p_i^{e_i}}$, hence multiplying these bounds gives the desired. □

Therefore, for n to be tasty, we need $(\log_2 n + 1)(\log_5 n + 1) \ge 0.25\sqrt{n}$, which only holds for finitely many n as desired.

N3. For each integer $C > 1$, decide whether there exists pairwise distinct positive integers a_1, a_2, a_3, ... such that for every $k \geq 1$,

$$a_{k+1}^k \quad \text{divides} \quad C^k a_1 a_2 \dots a_k.$$

(Daniel Liu)

No sequence exists for any C. Note that the divisibility is homogenous with respect to the a_i so we can shift the sequence and WLOG assume that $a_1 = 1$.

Note that any prime divisor of the a_i must also be a prime divisor of C. Let

$$C = p_1^{c_1} p_2^{c_2} \dots p_k^{c_k}$$

be the prime factorization of C.

Claim. Fix $p = p_j$ and $c = c_j$. Then for any index k we have

$$\nu_p(a_i) \leq cH_k + \nu_p(a_1).$$

Proof. Let $b_i = \nu_p(a_i)$. We apply strong induction: Base case of $k = 1$ is trivial. Now assume $b_i \leq H_{i-1}c + b_1$ for $i \leq k$; then

$$\begin{aligned}
b_{k+1} &\leq c + \frac{\sum_{i=1}^k b_i}{k} \\
&\leq c + \frac{\sum_{i=1}^k H_{i-1}c + b_1}{k} \\
&= c\left(1 + \frac{\sum_{i=1}^k H_{i-1}}{k}\right) + b_1 \\
&= c\left(1 + \frac{\frac{k-1}{1} + \frac{k-2}{2} + \dots + \frac{1}{k-1}}{k}\right) + b_1 \\
&= c\left(1 + \frac{k-1}{k \cdot 1} + \frac{k-2}{k \cdot 2} + \dots + \frac{1}{(k-1) \cdot k}\right) + b_1 \\
&= c\left(1 + \frac{1}{1} - \frac{1}{k} + \frac{1}{2} - \frac{1}{k} + \dots + \frac{1}{k-1} - \frac{1}{k}\right) + b_1 \\
&= cH_k + b_1
\end{aligned}$$

and the induction is complete. $\square$

Now, let N be a positive integer, and let $m = 1 + \max_j \nu_{p_j}(a_1)$. We have that

$$\nu_{p_j} a_i \leq c_j H_i + \nu_{p_j}(a_1) \leq c_j(m + \log N)$$

if $i \leq N$. Hence, there are at most

$$\prod_{j=1}^k [1 + c_j(m + \log N)] = O\left((\log N)^k\right)$$

possible k-triples that $(\nu_{p_1} a_i, \nu_{p_2} a_i, \dots, \nu_{p_k} a_i)$ can be. But this also needs to be at least $N + 1$, which is impossible for large N.

Saturday, June 9, 2018
1:15PM — 5:45PM

Problem 1. Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1 \le i \le 2018$, there are exactly n cities that are a distance i away from C. (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

Problem 2. Consider infinite sequences $a_1, a_2, \ldots$ of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. For a given positive integer m, find the maximum possible value of a_{2m}.

Problem 3. Let A be a point in the plane, and ℓ a line not passing through A. Evan does not have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct* the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

*To construct a point, Evan must have an algorithm which marks the point in finitely many steps.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Eyy LMaO

20th ELMO
Pittsburgh, PA

Year: **2018**

Day: **2**

Saturday, June 16, 2018
1:15PM — 5:45PM

Problem 4. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle† of $\triangle ABC$.

Problem 5. Let $a_1, a_2, \ldots, a_m$ be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n+a_i} \right\rfloor = \left\lfloor \sqrt{bn+c} \right\rfloor$$

holds for all integers $n > N$.

Problem 6. A *windmill* is a closed line segment of unit length with a distinguished endpoint, the *pivot*. Let S be a finite set of n points such that the distance between any two points of S is greater than c. A configuration of n windmills is *admissible* if no two windmills intersect and each point of S is used exactly once as a pivot.

An admissible configuration of windmills is initially given to Geoff in the plane. In one operation Geoff can rotate any windmill around its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during the process. Show that Geoff can reach any other admissible configuration in finitely many operations, where

(i) $c = \sqrt{3}$,

(ii) $c = \sqrt{2}$.

†The nine-point circle of $\triangle ABC$ is the unique circle passing through the following nine points: the midpoint of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Shortlisted Problems

20^{th} ELMO

Pittsburgh, PA, 2018

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until disclosed publicly by the committee on the ELMO.

Contributing Students

The Problem Selection Committee for ELMO 2018 thanks the following proposers for contributing **90 problems** to this year's Competition:

> Adam Ardeishar, Andrew Gu, Ankan Bhattacharya, Brandon Wang, Carl Schildkraut, Daniel Hu, Daniel Liu, Eric Gan, Kevin Ren, Krit Boonsiriseth, Luke Robitaille, Michael Kural, Michael Ma, Michael Ren, Sam Ferguson, Tristan Shin, Vincent Bian, Vincent Huang, Zack Chroman

Problem Selection Committee

The Problem Selection Committee for ELMO 2018 was led by **Evan Chen** and consisted of:

- Andrew Gu
- Daniel Liu
- James Lin
- Michael Ma
- Michael Ren
- Mihir Singhal
- Zack Chroman

Contents

Problems

A1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bijective function. Does there always exist an infinite number of functions $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$?

(Daniel Liu)

A2. Let $a_1, a_2, \ldots, a_m$ be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n+a_i} \right\rfloor = \left\lfloor \sqrt{bn+c} \right\rfloor$$

holds for all integers $n > N$.

(Carl Schildkraut)

A3. Let a, b, c, x, y, z be positive reals such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Prove that

$$a^x + b^y + c^z \geq \frac{4abcxyz}{(x+y+z-3)^2}.$$

(Daniel Liu)

A4. Elmo calls a monic polynomial with real coefficients *tasty* if all of its coefficients are in $[-1, 1]$. A monic polynomial P with real coefficients and complex roots $\chi_1, \ldots, \chi_m$ (counted with multiplicity) is given to Elmo, and he discovers that there does not exist a monic polynomial Q with real coefficients such that $P \cdot Q$ is tasty. Find all possible values of $\max(|\chi_1|, \ldots, |\chi_m|)$.

(Carl Schildkraut)

C1. Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1 \leq i \leq 2018$, there are exactly n cities that are a distance i away from C. (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

(Michael Ren)

C2. We say that a positive integer n is *m-expressible* if one can write a expression evaluating to n in base 10, where the expression consists only of

- exactly m numbers from the set $\{0, 1, \ldots, 9\}$
- the six operations $+$, $-$, $\times$, $\div$, exponentiation ^, concatenation $\oplus$, and
- some number (possibly zero) of left and right parentheses.

For example, 5625 is 3-expressible (in two ways), as $5625 = 5 \oplus (5\hat{}4) = (7 \oplus 5)\hat{}2$, say. Does there exist a positive integer A such that all positive integers with A digits are $(A-1)$-expressible?

(Krit Boonsiriseth)

C3. A *windmill* in the plane consists of a line segment of unit length with a distinguished endpoint, the *pivot*. Geoff has a finite set of windmills, such that no two windmills intersect, and any two pivots are distance more than $\sqrt{2}$ apart. In an operation, Geoff can choose a windmill and rotate it about its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during or after the rotation. Show that Geoff can, in finitely many operations, rotate the windmills so that they all point in the same direction.

(Michael Ren)

G1. Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles of $\triangle EHP$ and $\triangle FHP$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

(Brandon Wang)

G2. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle of $\triangle ABC$.

(Zack Chroman)

G3. Let A be a point in the plane, and ℓ a line not passing through A. Evan doesn't have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

(Zack Chroman)

G4. Let $ABCDEF$ be a convex hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that $\overline{BD}$ and $\overline{DF}$ intersect $\overline{CE}$ at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to $\overline{CE}$.

(Michael Ren and Vincent Huang)

G5. Let scalene triangle ABC have altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines $\overline{PE}$ at $X \neq P$ and $\overline{PF}$ at $Y \neq P$. Prove that $\overline{XY} \parallel \overline{BC}$.

(Daniel Hu)

N1. Determine all nonempty finite sets $S = \{a_1, \dots, a_n\}$ of n distinct positive integers such that $a_1 \cdots a_n$ divides $(x + a_1) \cdots (x + a_n)$ for every positive integer x.

(Ankan Bhattacharya)

N2. Call a number n *good* if it can be expressed in the form $2^x + y^2$ where x and y are nonnegative integers.

(a) Prove that there exist infinitely many sets of 4 consecutive good numbers.

(b) Find all sets of 5 consecutive good numbers.

(Michael Ma)

N3. Let $a_1, a_2, \dots$ be an infinite sequence of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. Find the maximum possible value of a_{2018}.

(Krit Boonsiriseth)

N4. Fix a positive integer $n > 1$. We say a nonempty subset S of $\{0, 1, \dots, n-1\}$ is d-coverable if there exists a polynomial P with integer coefficients and degree at most d, such that S is exactly the set of residues modulo n that P attains as it ranges over the integers.

For each n, determine the smallest d such that any nonempty subset of $\{0, \dots, n-1\}$ is d-coverable, or prove that no such d exists.

(Carl Schildkraut)

Solutions

A1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bijective function. Does there always exist an infinite number of functions $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$?

(Daniel Liu)

Yes. It's clear $f^0, f^1, f^2, \dots$ all commute with f. If f doesn't have finite order this collection is infinite and valid.

Else, suppose that $f^n = \mathrm{id}$, where n is minimal. If $n = 1$ the problem is clear, so suppose $n > 1$. Then f is composed of some cycles; some cycle length $d \mid n$ appears infinitely many times. Let a countable number of these cycles be $x_{r,1} \to x_{r,2} \to \cdots \to x_{r,d} \to x_{r,1}$ for $r \in \mathbb{Z}$.

Then for every integer s, create a new function h_s fixing everything except the $x_{k,\ell}$, and send every $x_{r,a} \to x_{r+s,a}$. It is clear that every h_s commutes with f.

This gives infinitely many g, unless all but finitely many of the cycles have length 1. In that case, we can find more g by swapping any two fixed points of f and leaving everything else intact.

A2. Let a_1, a_2, $\dots$, a_m be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n+a_i} \right\rfloor = \left\lfloor \sqrt{bn+c} \right\rfloor$$

holds for all integers $n > N$.

(Carl Schildkraut)

If all the a_i are equal, then $\sum_{i=1}^m \sqrt{n+a_i} = \sqrt{m^2n + m^2a_1}$ and so $(b,c) = (m^2, m^2a_1)$ works fine.

Let us assume this is not the case. Instead, will take $b = m^2$ and $c = m(a_1 + \dots + a_m) - 1$ and claim it works for N large enough.

On the one hand,

$$\begin{aligned} \sum_{i=1}^{m} \sqrt{n+a_i} &< m \cdot \sqrt{n + \frac{a_1 + \dots + a_m}{m}} \\ &= \sqrt{m^2 \cdot n + c + 1} \le \left\lceil \sqrt{m^2 \cdot n + c + 1} \right\rceil \le \left\lfloor \sqrt{m^2 \cdot n + c} \right\rfloor + 1. \end{aligned}$$

On the other hand, let $\lambda = \frac{c}{2(c+1)} < \frac{1}{2}$. We use the following estimate.

Claim. If n is large enough in terms of $(a_1, \dots, a_m)$ then $\sqrt{n+a_i} \ge \sqrt{n} + \frac{\lambda a_i}{\sqrt{n}}$.

Proof. Squaring both sides, it's equivalent to $a_i \ge 2\lambda \cdot a_i + \frac{\lambda^2 a_i^2}{n}$, which holds for n big enough as $2\lambda < 1$. $\square$

Now,

$$\begin{aligned} \sum_{i=1}^{m} \sqrt{n+a_i} &\ge \sum_{i=1}^{m} \left(\sqrt{n} + \frac{\lambda a_i}{\sqrt{n}} \right) \\ &\ge m\sqrt{n} + \frac{\lambda \cdot (a_1 + \dots + a_n)}{\sqrt{n}} \\ &= m\sqrt{n} + \frac{\lambda \cdot (c+1)}{m\sqrt{n}} \\ &= m\sqrt{n} + \frac{c}{2m\sqrt{n}} > \sqrt{m^2 \cdot n + c} \ge \left\lfloor \sqrt{m^2 n + c} \right\rfloor. \end{aligned}$$

This finishes the problem.

Remark. Obviously, $b = m^2$ for asymptotic reasons (by taking n large). As for possible values of c:

- If $a_1 = \dots = a_m$, then one can show $c = m(a_1 + \dots + a_m)$ is the only valid choice. Indeed, taking n of the form $n = k^2 - a$ and $n = \frac{k^2-1}{m^2} - a$ is enough to see this.
- But if not all a_i are equal, the natural guess of taking $c = m(a_1 + \dots + a_n)$ is not valid in general. For example, we have that
$$\left\lfloor \sqrt{n} + \sqrt{n+2} \right\rfloor \ne \left\lfloor \sqrt{4n+4} \right\rfloor \qquad n \in \{t^2 - 1 \mid t = 2, 3, \dots\}.$$
I think one can actually figure out exactly which c are valid, though the answer will depend on some quadratic residues, and we do not pursue this line of thought here.

So any correct solutions must distinguish these two cases.

A3. Let a, b, c, x, y, z be positive reals such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$. Prove that

$$a^x+b^y+c^z \ge \frac{4abcxyz}{(x+y+z-3)^2}.$$

(Daniel Liu)

We present three solutions.

First solution, proof without words (by proposer)

$$\begin{aligned}
a^x+b^y+c^z &= yz\cdot\frac{a^x}{yz}+zx\cdot\frac{a^y}{zx}+xy\cdot\frac{a^z}{xy}\\
&\ge (xy+yz+zx)\left(\left(\frac{a^x}{yz}\right)^{yz}\left(\frac{b^y}{zx}\right)^{zx}\left(\frac{c^z}{xy}\right)^{xy}\right)^{\frac{1}{xy+yz+zx}}\\
&= (xy+yz+zx)\cdot\frac{(abc)^{\frac{xyz}{xy+yz+zx}}}{x^{\frac{xy+zx}{xy+yz+zx}}y^{\frac{yz+xy}{xy+yz+zx}}z^{\frac{zx+yz}{xy+yz+zx}}}\\
&\ge (xy+yz+zx)\cdot\frac{(abc)^{\frac{xyz}{xy+yz+zx}}}{\left(\frac{x\cdot\frac{xy+zx}{xy+yz+zx}+y\cdot\frac{yz+xy}{xy+yz+zx}+z\cdot\frac{zx+yz}{xy+yz+zx}}{2}\right)^2}\\
&= (xy+yz+zx)\cdot\frac{4(abc)^{\frac{xyz}{xy+yz+zx}}}{\left(\sum_{\text{cyc}} x\cdot\left(1-\frac{yz}{xy+yz+zx}\right)\right)^2}\\
&= \frac{4abc(xy+yz+zx)}{(x+y+z-3\frac{xyz}{xy+yz+zx})^2}\\
&= \frac{4abcxyz}{(x+y+z-3)^2}.
\end{aligned}$$

Second solution, by weighted AM-GM (Andrew Gu) By weighted AM-GM,

$$\frac{1}{x}\cdot xa^x+\frac{1}{y}\cdot yb^y+\frac{1}{z}\cdot zc^z \ge x^{\frac{1}{x}}y^{\frac{1}{y}}z^{\frac{1}{z}}abc.$$

Hence it suffices to show

$$x^{\frac{1}{x}}y^{\frac{1}{y}}z^{\frac{1}{z}} \ge \frac{4xyz}{(x+y+z-3)^2}.$$

By weighted AM-GM,

$$2x^{\frac{1}{2}(1-\frac{1}{x})}y^{\frac{1}{2}(1-\frac{1}{y})}z^{\frac{1}{2}(1-\frac{1}{z})} \le 2\cdot\frac{1}{2}(x-1+y-1+z-1)=x+y+z-3.$$

Squaring both sides and rearranging proves the required inequality.

Third solution, by Hölder and Schur/Muirhead (Evan Chen) By Hölder and weighted AM-GM we have

$$\sqrt{\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)(a^x+b^y+c^z)} \ge \frac{1}{x}\cdot a^{x/2}+\frac{1}{y}\cdot b^{y/2}+\frac{1}{z}\cdot c^{z/2} \ge (abc)^{1/2}.$$

Hence, it suffices to prove that

$$(x+y+z-3)^2 \ge 4xyz\left(1/x^2+1/y^2+1/z^2\right) \qquad \forall \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$$

which is a 3-variable symmetric inequality. It also happens to be is MOP 2011, K4.1, done in my SOS handout. We give a proof below (with $a = 1/x$, etc).

Claim (Black MOP 2011, Test 4, Problem 1)**.** If $a, b, c > 0$ then

$$\left((a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3\right)^2 \ge 4\left(\frac{(a+b+c)(a^2+b^2+c^2)}{abc}\right)$$

Proof. Expanding and clearing denominators it's just

$$\sum_{\text{sym}} a^4b^2 + \sum_{\text{cyc}} a^3b^3 + 6a^2b^2c^2 \ge 2\sum_{\text{cyc}} a^4bc + 2\sum_{\text{sym}} a^3b^2c$$

which can also be written as

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 1 & & -2 & & 1 & & & & \\
 & & & 2 & & -2 & & -2 & & 2 & & & \\
 & & 1 & & -2 & & 6 & & -2 & & 1 & & \\
 & 0 & & -2 & & -2 & & -2 & & -2 & & 0 & \\
0 & & 0 & & 1 & & 2 & & 1 & & 0 & & 0
\end{array}$$

in Chinese dumbass notation. This rewrites as

$$\sum_{\text{cyc}} a^4(b-c)^2 + 2\sum_{\text{cyc}} ab(ab-bc)(ab-ac) \ge 0$$

which is evident (the latter sum is "upsidedown triangle Schur"). $\square$

A4. Elmo calls a monic polynomial with real coefficients *tasty* if all of its coefficients are in $[-1,1]$. A monic polynomial P with real coefficients and complex roots $\chi_1, \dots, \chi_m$ (counted with multiplicity) is given to Elmo, and he discovers that there does not exist a monic polynomial Q with real coefficients such that $P \cdot Q$ is tasty. Find all possible values of $\max(|\chi_1|, \dots, |\chi_m|)$.

(Carl Schildkraut)

We claim the answer is $r > 1$. The answer is divided into two parts.

Part I: Any value of $r > 1$ can be achieved. To prove this, we will show that the polynomial

$$P(x) = x^n - r^n$$

has no tasty multiples if $r^n \geq 2$ (such an n exists because $r > 1$). Set $M = r^n$. Assume we have a polynomial

$$R(x) = \sum_{i=0}^{N} a_i x^i$$

so that $-1 \leq a_i \leq 1$ for all i ($a_N = 1$) and $P|R$. Taking R modulo P, we get that, with $N = bn + c$ and $0 \leq c < n$ (setting $a_k = 0$ if $k > N$),

$$R(x) = \sum_{j=0}^{n-1} \sum_{k=0}^{b} a_{kn+j} x^{kn+j} \equiv \sum_{j=0}^{n-1} x^j \left[\sum_{k=0}^{b} a_{kn+j} R^k \right].$$

We have this must be the zero polynomial (since $P|R$); specifically, taking $j = c$,

$$\sum_{k=0}^{b} a_{nk+c} R^k = 0$$
$$\sum_{k=0}^{b-1} (-a_{nk+c}) R^k = a_{bn+c} R^b$$
$$\sum_{k=0}^{b-1} |a_{nk+c}| R^k \geq R^b$$

(since $a_{bn+c} = a_N = 1$). However, since $|a_{nk+c}| \leq 1$, we then have

$$\sum_{k=0}^{b-1} R^k \geq R^b$$
$$\frac{R^b - 1}{R - 1} \geq R^b$$
$$R^b - 1 \geq R^{b+1} - R^b$$
$$R^b(2 - R) \geq 1.$$

However, as $R \geq 2$, this is false.

Part II: Any polynomial with $r \le 1$ has a tasty multiple. Define the *sparsity* of a polynomial to be the greatest common divisor of the exponents m for which the coefficient of x^m in P is not zero. Equivalently, it is the largest integer d so that $P(x) = Q(x^d)$ for some polynomial Q.

We prove the following theorem:

Theorem. Given any complex number z for which $|z| \le 1$, there exist tasty polynomials with z as a root that have arbitrarily large sparsities.

Proof. Let $z = re^{i\theta}$. If θ is a rational multiple of π (say, $\theta = a\pi/b$), then we take the polynomial $x^{bn} - r^{bn}$ for any integer n; this has sparsity bn and is tasty (as $r \le 1$, $r^{bn} \le 1$). So, it suffices to prove this in the case where θ is not a rational multiple of π, and we henceforth assume this.

We claim that, for infinitely many n, the polynomial

$$x^{2n} - 2\cos(n\theta)\, r^n x^n + r^{2n}$$

is tasty (note that this polynomial has sparsity n and as such the theorem is implied by this claim). First note that this polynomial reduces to

$$x^n = r^n e^{\pm ni\theta} = \left(re^{\pm i\theta}\right)^n,$$

which is true at $x = re^{i\theta} = z$, so z is in fact a root.

We recall the following lemma:

Lemma. For any real number ϕ which is not a rational multiple of π, the sequence $a_n = \cos(n\phi)$ has infinitely many terms in the range $[-1/2, 1/2]$.

Indeed, let $\{x\}$ be the fractional part of x, and consider the sequence

$$\alpha_n = \left\{\frac{n\phi}{2\pi}\right\}.$$

We see that $-1/2 \le a_n \le 1/2$ iff $1/6 \le \alpha_n \le 1/3$ or $2/3 \le \alpha_n \le 5/6$. It is well known that the sequence $x_n = \{nx\}$ is dense in $[0, 1]$ for any irrational x, so this is true. Thus, for infinitely many n, as θ has been assumed not to be a rational multiple of π, the coefficients of P are bounded above in absolute value by r^n and r^{2n} for infinitely many n, both of which are ≤ 1 as $r \le 1$. □

We now provide a second lemma.

Lemma. If $P(x)$ and $Q(x)$ are both tasty polynomials and the sparsity D of P is greater than the degree d of Q, then the product $R(x) = P(x)Q(x)$ is also tasty.

Proof. Write

$$P(x) = \sum_{j=0}^{s} a_j x^{Dj}, \qquad Q(x) = \sum_{k=0}^{d} b_k x^k.$$

Then,

$$P(x)Q(x) = \sum_{j=0}^{s}\sum_{k=0}^{d} a_j b_k x^{Dj+k}.$$

As $D > d$, none of these terms interfere with one another (for each integer n, there is at most one choice of $0 \le j \le s,\ 0 \le k \le d$ so that $Dj + k = s$), so the coefficients of $R(x)$ are just the values of $a_j b_k$ as j and k range over the desired range; as each a_j and b_k are of magnitude ≤ 1, each pairwise product is as well, finishing the proof. □

Given a polynomial P with roots $\chi_1, \dots, \chi_m$ in $\mathbb{C}$ (possibly with duplicates), we will inductively construct the polynomial $R(x)$ that is tasty and that P divides. We define a sequence of polynomials $R_0, \dots, R_m$ so that $R_0(x) = 1$, and for each $0 < k \le m$, we take a tasty polynomial $P_k(x)$ with root χ_k and sparsity greater than the degree of R_{k-1}, and take $R_k(x) = R_{k-1}(x)P_k(x)$. Such a $P_k(x)$ is guaranteed to exist by our theorem, and the product $R_{k-1}(x)P_k(x)$ is guaranteed to be tasty by our lemma. Thus, we may take $R = R_m$, finishing the proof.

Remark. A polynomial P that has a tasty multiple exists for all $r < 2$: We have upon fixing $r < 2$ that for large enough n, we know $r^n - r^{n-1} - \cdots - r - 1 \le 0$. If n is minimal, $r^n - r^{n-1} - \cdots - r > 0$, and we can thus take some value $0 \le c \le 1$ for the constant term by the intermediate value theorem so that $P(x) = x^n - x^{n-1} - \cdots - x - c$ has a root at r. If $r \ge 2$, then $n = 1$ can be taken in Part 1 and thus no tasty multiples exist.

C1. Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1 \le i \le 2018$, there are exactly n cities that are a distance i away from C. (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

(Michael Ren)

The answer is n even.

To see that n odd fails, note that by taking $i = 1$ we see the graph is n-regular; since it has an odd number of vertices we need n to be even.

On the other hand, if n is even, then consider the graph formed by taking the vertices of a regular $(2018n + 1)$-gon and drawing edges between vertices which are at most $n/2$ apart. Then this works.

C2. We say that a positive integer n is *m-expressible* if one can write a expression evaluating to n in base 10, where the expression consists only of

- exactly m numbers from the set $\{0, 1, \dots, 9\}$
- the six operations $+$, $-$, $\times$, $\div$, exponentiation ^, concatenation $\oplus$, and
- some number (possibly zero) of left and right parentheses.

For example, 5625 is 3-expressible (in two ways), as $5625 = 5 \oplus (5\hat{\ }4) = (7 \oplus 5)\hat{\ }2$, say. Does there exist a positive integer A such that all positive integers with A digits are $(A-1)$-expressible?

(Krit Boonsiriseth)

Here is a solution by Evan Chen achieving $A = 6 \cdot 10^6$, and reprising the joke "six consecutive zeros".

We will replace "exactly m numbers" with "at most m numbers", since this is the same. Suppose we group the digits of N into base 1000000, so that we have

$$N = s_1 s_2 s_3 \dots s_m$$

where each s_m is a group of six digits (s_1 padded with leading zeros, if needed, but $s_1 \neq \overline{000000}$). We consider two cases.

- Suppose some group is zero; then we find that N has six consecutive zeros in its decimal representations. Thus N has the form

  $$N = X \oplus (b \cdot (1 \oplus 0)\hat{\ }6) \oplus Y$$

  for some strings X and Y (possibly empty), which are formed by repeated concatenation.

- Otherwise, note that $m \geq 10^6$. By a classical pigeonhole argument there exist indices $i < j$ such that $s_i + \dots + s_j \equiv 0 \pmod{999999}$. Let $n = \frac{1}{999999} s_i \dots s_j$. Then we can write

  $$N = X \oplus \left[((1 \oplus 0)\hat{\ }6 - 1) \cdot n\right] \oplus Y$$

  for strings $X = s_1 \dots s_i$ and $Y = s_{j+1} \dots s_n$.

Remark (Possible motivational remarks)**.** Ankan Bhattacharya says: I knew that the answer had to be yes — the obvious counting argument to show answer no doesn't work, and the given elements are unrelated enough that proving a no answer would be very difficult.

Evan says: I think you really do have to use exponentiation, since otherwise the numbers aren't big enough; but exponentiation is really painful to deal with, so I tried to find a way to use it only once. This is less daunting than it seems because you can concatenate digits "for free" from a size perspective; thus you just need a substring that you can "save space" on. After a bit of guesswork I came upon the idea of taking modulo $10^6 - 1 = 999999$ (which saves about two digits) and from there I had it.

Remark (Possible motivational remarks)**.** Ankan Bhattacharya points out that if we fix all $N-2$ operations, then there are only 10^{N-1} choices, compared to $9 \cdot 10^{N-1}$ numbers we need to obtain. Thus we need to use different operations to reach different numbers. This suggests that all solutions are likely to use some amount of casework.

Unlike Ankan, I did not find the case split to be a substantial part of the problem. It came up naturally because I had an edge case where six consecutive zeros might appear in my argument, and the first case was patch-only in that situation.

C3. A *windmill* in the plane consists of a line segment of unit length with a distinguished endpoint, the *pivot*. Geoff has a finite set of windmills, such that no two windmills intersect, and any two pivots are distance more than $\sqrt{2}$ apart. In an operation, Geoff can choose a windmill and rotate it about its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during or after the rotation. Show that Geoff can, in finitely many operations, rotate the windmills so that they all point in the same direction.

(Michael Ren)

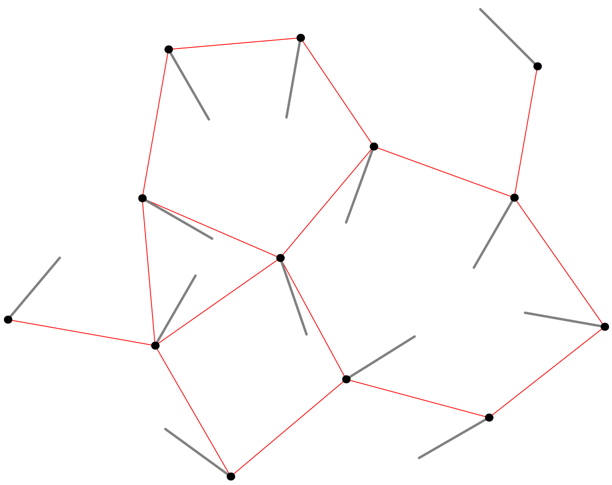
Throughout the solution we will general denote pivots by P, Q, R, ... and non-pivots by A, B, C, ...

We say that a configuration of windmills around S is *admissible* if no two windmills intersect. The problem is equivalent to showing one can reach any admissible configuration from any other (and the final position with the windmills pointing the same direction is just one example of a clearly admissible configuration).

Draw a red line segment between any two pivots which have distance at most 2 (thus these windmills could intersect). This naturally gives us a graph $\mathcal{G}$.

Lemma. For $c \ge \sqrt{2}$, the graph $\mathcal{G}$ is planar.

Proof. Indeed, if $\overline{PA}$ and $\overline{QB}$ intersect, we can consider convex quadrilateral $PQAB$, one of whose angles is at least 90°. WLOG it is $\angle PQA$, in which case $PA^2 \ge PQ^2 + QA^2 > 2 + 2 = 4$, so $\overline{PA}$ should not be red. $\square$



Clearly, we can ignore any isolated vertices. We can also ignore any leaves in $\mathcal{G}$; indeed suppose P is a pivot with $\overline{PQ}$ the only red edge. Then we can rotate the windmill at P to point away from Q and it will never obstruct other windmills since $c \ge 1$, so we can delete the pivot P from consideration (and use induction on the number of pivots, say).

Thus, we may assume $\mathcal{G}$ is a finite planar graph with no leaves. Thus it makes sense to speak of the faces of planar graph $\mathcal{G}$, consisting of several polygons.

Lemma. A windmill with pivot P can never intersect a red edge other than those touching P.

Proof. Suppose windmill $\overline{PA}$ intersects red edge $\overline{QR}$. Then the altitude from $\overline{PH}$ to $\overline{QR}$ has length at most 1. WLOG that $QH < RH$, so $QH < \frac{1}{2}QR = 1$. Then $PQ^2 < QH^2 + HP^2 < 1 + 1 = 2$, contradiction. $\square$

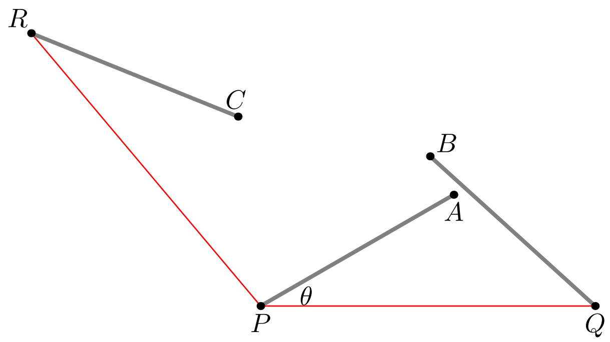
From now on, a windmill $\overline{PA}$ is said to *hug* a red edge $\overline{PQ}$ if the angle $\angle QPA < \varepsilon$ for some sufficiently small ε in terms of $\mathcal{G}$; each red edge $\overline{PQ}$ has at most two windmills hugging it (namely the windmills with pivots P and Q; if this happens, the windmills are on opposite sides of $\overline{PQ}$). Call a windmill configuration *cuddly* if every windmill is hugging an edge.

Claim. We can reach some cuddly configuration from any admissible one.

Proof. Indeed, consider a windmill $\overline{PA}$ not hugging any edge, and an edge $\overline{PQ}$, and such that $\angle APQ = \theta$ is minimal among all such pairs. Let $\angle RPQ$ be the corresponding angle of the face containing $\overline{PA}$, and let $\overline{QB}$, $\overline{RC}$ be windmills.

If $\overline{QB}$ is hugging $\overline{PQ}$, we perturb it slightly so that A and B are on opposite sides of $\overline{PQ}$; thus $\overline{QB}$ is no longer in the way.

We rotate $\overline{PA}$ towards $\overline{PQ}$ now. Because we assumed $\theta = \angle APQ$ was minimal, it is impossible for the body of the windmill to collide with the points B or C. So the only way it can be obstructed is if the point A collides with the interior of $\overline{QB}$ or $\overline{RC}$.

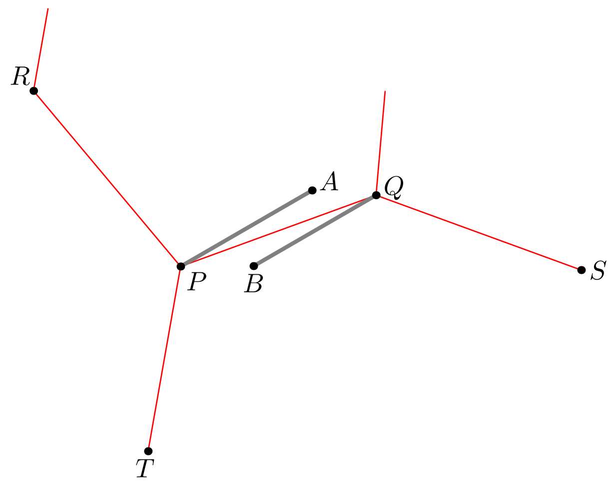


Suppose that A collided with $\overline{QB}$. At the moment of collision, we would have to have $\angle PAQ \le 90^\circ$. (This is because just before the collision $\overline{PA}$ was still disjoint from $\overline{QB}$, and if $\angle PAQ \ge 90^\circ$ just before then it would remain disjoint as $\overline{PA}$ rotated.) But then $PQ^2 \le PA^2 + AQ^2 \le 2$, contradiction. A similar proof works for $\overline{RC}$.

Thus we can rotate the windmills one by one so they hug the edges, as desired. $\square$

It remains to show any two cuddly configurations can be reached from each other. For this, we make two observations.

- Suppose $\overline{PA}$ and $\overline{QB}$ both hug $\overline{PQ}$. We show we can interchange the two. Assume $\angle RPQ$ is the angle of a face containing A, and $\angle TPQ$, $\angle PQS$ are the angles of the face containing B.



Rotate $\overline{PA}$ so it hugs $\overline{PR}$ (possibly perturbing the windmill at R), and then rotate $\overline{QB}$ so it hugs $\overline{QS}$ (possibly perturbing the windmill at S). Then rotate $\overline{PA}$ so it hugs $\overline{PT}$, then move $\overline{QB}$ back so it hugs $\overline{PQ}$ from the other side, and rotate $\overline{PA}$ back.

- Now suppose $\overline{PA}$ hugs $\overline{PQ}$, and $\angle RPQ$ is the angle of a face containing A. Then we can rotate it so that $\overline{PA}$ hugs $\overline{PR}$ (here $\overline{PA}$ could be blocked by $\overline{QB}$ initially, but then we perform the switching operation above).

Together these two observations finish the problem.

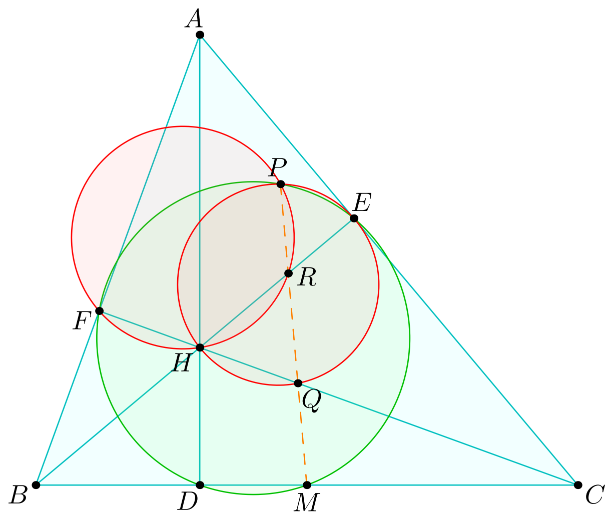
Remark (Michael Ren)**.** Here is a solution achieving just $c = \sqrt{3}$.

Draw a disk of radius $1 + \epsilon$ around every point in S such that the distance between any two points in S is more than $\sqrt{3}(1 + \epsilon)$ for some $\epsilon > 0$ that clearly exists. Note that no three disks can intersect. Indeed, if disks centered at A, B, and C intersected, then the circumradius of ABC is at most $1 + \epsilon$, which means that some two of A, B, C are at most a distance of $\sqrt{3}(1 + \epsilon)$ apart. In light of this, for any two points A and B in S that are a distance of at most 2 apart, draw a rhombus $APBQ$ of length $1 + \epsilon$. By our work before, all such rhombi are distinct. Furthermore, windmill collisions only happen inside these rhombi by definition. Now, have Geoff move each of his windmills one by one to Sasha's windmills. If a windmill collision happens, have Geoff move the other windmill out of the way inside the rhombus before moving the windmill by and then restore the position of the other windmill. Hence, he can always get his windmills to coincide, as desired.

G1. Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles of $\triangle EHP$ and $\triangle FHP$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

(Brandon Wang)

Let D denote the foot of the A-altitude, and M the midpoint of $\overline{BC}$. We claim that R and Q both lie on line $\overline{PM}$. That will solve the problem (M is the fixed point).



By angle chasing, it is not hard to show that

$$\measuredangle FHE = \measuredangle FEM.$$

Now,

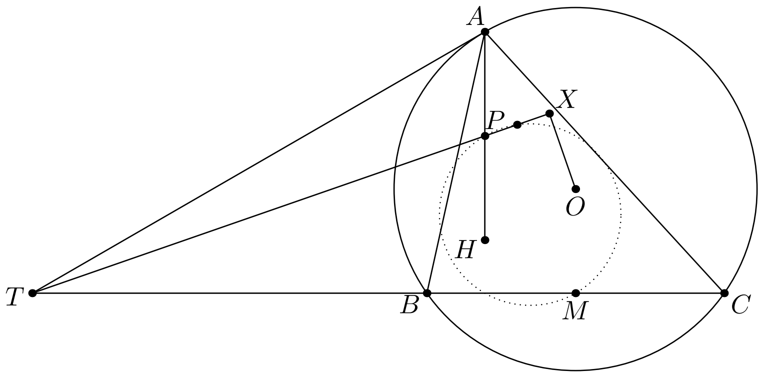
$$\measuredangle FPR = \measuredangle FHR = \measuredangle FHE = \measuredangle FEM = \measuredangle FPM$$

as desired so P, R, M are collinear. Similarly, P, Q, M are collinear, as desired.

G2. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90°$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle of $\triangle ABC$.

(Zack Chroman)

We present two solutions, one synthetic and by complex numbers.



First solution (Zack Chroman) Let M be the midpoint of $\overline{BC}$. Note that since $\overline{AP} \perp \overline{BC}$ and $\overline{AT} \perp \overline{AO} \parallel \overline{PM}$, we find that P is the orthocenter of $\triangle ATM$. Thus $Y = \overline{TP} \cap \overline{AM}$ satisfies $\angle PYM = 90$, so it lies on the 9-point circle.

It then suffices to note that the reflection X' of P over Y lies on the circumcircle of $(AMT) = (TO)$, so $\angle TX'O = 90 \implies X = X'$.

Second solution (complex numbers, Evan Chen) Let Q denote the reflection of P over M, the midpoint of $\overline{BC}$.

Claim. We have $\overline{QO} \perp \overline{PT}$.

Proof. By complex numbers. We have

$$\begin{aligned} t &= \frac{aa(b+c) - bc(a+a)}{aa - bc} = \frac{a^2(b+c) - 2abc}{a^2 - bc} \\ t - p &= \frac{a^2(b+c) - 2abc}{a^2 - bc} - \left(a + \frac{b+c}{2}\right) \\ &= \frac{a^2(\frac{1}{2}b + \frac{1}{2}c - a) + (-a + \frac{1}{2}b + \frac{1}{2}c)bc}{a^2 - bc} \\ &= \frac{b+c-2a}{2} \cdot \frac{a^2 + bc}{a^2 - bc} \\ q &= 2 \cdot \frac{b+c}{2} - p = \frac{b+c-2a}{2} \end{aligned}$$

Since $\frac{a^2+bc}{a^2-bc} \in i\mathbb{R}$, the claim is proven. $\square$

Thus, $\overline{QOX}$ are collinear. By considering right triangle $\triangle PQX$ with midpoint M, we conclude that $MX = MP$. Since the nine-point circle is the circle with diameter $\overline{PM}$, it passes through the midpoint of $\overline{PX}$.

G3. Let A be a point in the plane, and ℓ a line not passing through A. Evan doesn't have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

(Zack Chroman)

The trick is to invert the figure around a circle centered at A of arbitrary radius. We let $\omega = \ell^*$ denote the image of ℓ under this inversion. Then, under the inversion, Evan's compass has the following behavior:

- Evan can draw a line through two points other than A; or
- Evan can draw a circle through three points other than A.

In other words, the point A is "invisible" to Evan, but Evan otherwise has a straightedge and the same compass.

It is clear then that the answer to (ii) is no; since the point A is invisible it's impossible to construct any point depending on it.

Part (i) is equivalent to showing that Evan can construct the center of ω; we give one construction here anyways. Take any cyclic quadrilateral $WXYZ$ inscribed in ω, and let $P = \overline{WZ} \cap \overline{XY}$. Then the circumcircles of $\triangle PWX$ and $\triangle PYZ$ meet again at the Miquel point M, and the second intersection of (MXZ) and (MWY) is the center of ω.

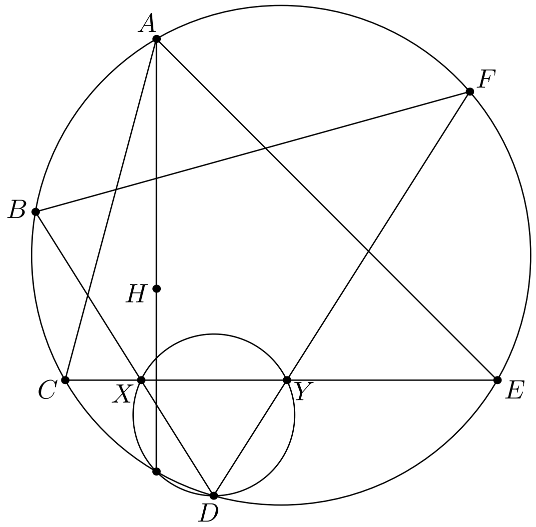
Remark. The proof of (ii) implies that it's actually more or less impossible in this context to construct any point other than the reflection of A, as a function of A and ℓ.

An alternative proof of (ii) is possible by inverting around a generic point P on ℓ with radius PA; this necessarily preserves the entire construction, but the foot from A to ℓ is not fixed by this inversion.

G4. Let $ABCDEF$ be a convex hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that $\overline{BD}$ and $\overline{DF}$ intersect $\overline{CE}$ at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to $\overline{CE}$.

(Michael Ren and Vincent Huang)

We present many, many solutions. In all of them, we let H denote the common orthocenter.



First solution by Simson lines (Vincent Huang) Let AH meet CE and Ω again at M and A_1, respectively, and P and Q be the projections of A_1 onto BD and DF, respectively. Note that PQ is the Simson line of A_1 with respect to BDF. It is well known that this Simson line bisects the segment between A_1 and H. Hence, M lies on PQ. But P, M, and Q are respectively the projections of A_1 onto DX, XY, and YD, so A_1 must lie on the circumcircle of DXY, as desired.

Second solution by dual Desargues involution (Michael Ren) Let O and r be the center and radius of Ω, respectively. Let $\mathcal{E}$ be the ellipse with foci O and H consisting of the set of points P such that $OP + HP = r$. Note that as the reflections of H over AC, CE, EA, BD, DF, FB lie on Ω, $\mathcal{E}$ is tangent to the sides of ACE and BDF. Let $\mathcal{E}$ and AD meet CE at P and Q, respectively. By the dual of Desargues involution theorem on quadrilateral $ACPE$ with inscribed conic $\mathcal{E}$, $D(CE; XY; PQ)$ is an involution. Hence, the circumcircles of DCE, DXY, and DPQ are coaxial, so it suffices to show that A_1DPQ is cyclic, where A_1 is the second intersection of AH and Ω. But note that A_1 lies on OP, so $\angle QDA_1 = \angle ADA_1 = \frac{\pi}{2} - \angle OA_1A = \frac{\pi}{2} - \angle PA_1A$, which is the angle between PA_1 and PQ by the perpendicularity of AA_1 and CE, as desired.

Third solution by angle chasing (Mihir Singhal) Let A_1 be the reflection of H over CE. Note A_1 is on Ω so it suffices to show that DA_1XY is cyclic. Let M be the foot of the altitude from A to $\overline{CE}$. Note that M is the midpoint of $\overline{HA_1}$ so since A_1 is on Ω, M must be on the nine-point circle of DBF. Let R and S be the feet of the altitudes from F and B in DBF.

Note $MXRH$ and $MYSH$ are cyclic. Moreover, M lies on the nine-point circle of $\triangle BDF$, and hence $\measuredangle SMR = 2\measuredangle SDR$. Then

$$\begin{aligned}\measuredangle XHY &= \measuredangle XHM + \measuredangle MHY \\ &= \measuredangle XRM + \measuredangle MSY = \measuredangle DRM + \measuredangle MSD \\ &= -(\measuredangle RMS + \measuredangle SDR) = \measuredangle SMR + \measuredangle RDS \\ &= 2\measuredangle SDR + \measuredangle RDS = \measuredangle SDR = \measuredangle YDX.\end{aligned}$$

Thus $\measuredangle XA_1Y = -\measuredangle XHY = \measuredangle RDS = \measuredangle XDY$, as needed.

Fourth solution by inversion (James Lin) Let K be the second intersection of Ω and the perpendicular from A to CE. We want to show $DKXY$ is cyclic. We invert about H. It's clear that now, $A'C'E'$ and $B'D'F'$ share the same circumcircle Ω' and incenter H. Note that K maps to the midpoint $M_{A'}$ of the arc $C'E'$ on Ω' not containing A'. Also note that X' is the intersection of circles $(HB'D')$ and $(HC'E')$, which are centered at midpoint $M_{F'}$ of the arc $B'D'$ on Ω' not containing F' and the midpoint $M_{D'}$ of the arc $B'F'$ on Ω' not containing D', respectively. Thus, X' is the reflection of H over $M_{A'}M_{F'}$. Similarly, Y' is the reflection of H over $M_{A'}M_{B'}$. Then, note that $M_{A'}X = M_{A'}H = M_{A'}Y$. Now we reformulate the problem by erasing A', C' and E', as the rest of the problem can be defined without them. The reformulated statement is that if we fix B, D, F, H and vary $M_{A'}$ along Ω', then $D'M_{A'}X'Y'$ is always cyclic.

We proceed with directed angles. Note that $\measuredangle X'D'M_{A'} = \measuredangle X'D'H + \measuredangle HD'M_{A'} = \measuredangle M_{A'}M_{F'}F + \measuredangle M_{D'}M_{F'}M_{A'} = \measuredangle M_{D'}M_{F'}F$. Similarly, $\measuredangle Y'D'M_{A'} = M_{D'}M_{B'}B = -\measuredangle M_{D'}M_{F'}F = -\measuredangle X'DM_{A'}$, so it follows that $M_{A'}$ lies on an angle bisector of $\measuredangle X'D'Y'$. Assume that $D'M_{A'}$ and $X'Y'$ are not perpendicular. Then from $M_{A'}X' = M_{A'}Y'$, it follows that $D'M_{A'}X'$ and $D'M_{A'}Y'$ have the same circumradius, and if they don't have the same circumcircle, then $D'M_{A'}$ and $X'Y'$ must be perpendicular, a contradiction. So $D'X'M_{A'}Y'$ is cyclic. Hf $D'M_{A'}$ and $X'Y'$ are perpendicular, then use the new problem formulation (without A, C and E and just varying $M_{A'}$) to move $M_{A'}$ by a miniscule amount. Then $D'M_{A'}$ and $X'Y'$ will not be perpendicular, so $D'X'M_{A'}Y'$ is cyclic both after and before moving $M_{A'}$ by continuity. We are done.

Fifth solution, by complex numbers (Carl Schildkraut) Let Ω be the unit circle, and let $A = a$, etc. We have that

$$c + e = h - a \implies \frac{c+e}{ce} = \bar{h} - \frac{1}{a} \implies ce = \frac{a(h-a)}{a\bar{h} - 1}.$$

Let T be the second intersection of the line through A perpendicular to CE and Ω. We see that

$$t = -\frac{ce}{a} = -\frac{h-a}{a\bar{h}-1}.$$

We endeavor to show that $DTXY$ is a cyclic quadrilateral. We have that

$$\begin{aligned} x &= \frac{ce(b+d)-bd(c+e)}{ce-bd} \\ &= \frac{\frac{a(b+d)(h-a)}{a\bar{h}-1} - bd(h-a)}{\frac{a(h-a)}{a\bar{h}-1} - bd} \\ &= (h-a)\left(\frac{a(b+d)-bd(a\bar{h}-1)}{a(h-a)-bd(a\bar{h}-1)}\right) \\ &= (h-a)\left(\frac{ab+ad-ab-ad-\frac{abd}{f}+bd}{ab+ad+af-a^2-ab-ad-\frac{abd}{f}+bd}\right) \\ &= (h-a)\left(\frac{bd(f-a)}{(af+bd)(f-a)}\right) \\ &= \frac{bd(h-a)}{af+bd}. \end{aligned}$$

Similarly

$$y = \frac{bf(h-a)}{ab+df}.$$

So, we want to show that

$$d, -\frac{h-a}{a\bar{h}-1}, \frac{bd(h-a)}{af+bd}, \frac{bf(h-a)}{ab+df}$$

are concyclic. This is equivalent to, dividing each by $h-a$ and reciprocating,

$$\frac{h-a}{d}, 1-a\bar{h}, 1+\frac{af}{bd}, 1+\frac{ab}{df}$$

being concyclic. This is equivalent to, subtracting 1 and multiplying by bdf,

$$bf(b+f-a), -a(bd+bf+df), ab^2, af^2$$

being concyclic. This is equivalent to, adding abf and dividing by $b+f$,

$$bf, -ad, ab, af$$

being concyclic. However, all of these points lie on the unit circle, finishing the proof.

Sixth solution by complex numbers (Evan Chen) As usual let Ω denote the unit circle. We immediately have

$$\begin{aligned} c+e &= b+d+f-a \\ \text{and thus}\quad \frac{1}{c}+\frac{1}{e} = \frac{c+e}{ce} &= \frac{1}{b}+\frac{1}{d}+\frac{1}{f}-\frac{1}{a} \\ \implies ce &= \frac{b+f+d-a}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}}. \end{aligned}$$

These two equations let us eliminate c and e, leaving only a, b, d, f.

Now consider the point $p = -\frac{ce}{a}$ on the circumcircle. We compute

$$\begin{aligned}
\frac{x-p}{b-p} &= \frac{x+\frac{ce}{a}}{b+\frac{ce}{a}} \\
&= \frac{\frac{bd(c+e)-ce(b+d)}{bd-ce}+\frac{ce}{a}}{b+\frac{ce}{a}} \\
&= \frac{abcd+abde-abce-adce+bdce-(ce)^2}{(ab+ce)(bd-ce)} \\
&= \frac{abcde(1/a+1/e+1/c-1/d-1/b)-(ce)^2}{(ab+ce)(bd-ce)} \\
&= \frac{abcde(1/f)-(ce)^2}{(ab+ce)(bd-ce)} = \frac{(ce)(abd-cef)}{f(ab+ce)(bd-ce)}
\end{aligned}$$

Now, we write

$$\begin{aligned}
ab+ce &= \frac{ab(\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a})+(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{ab(\frac{1}{d}+\frac{1}{f})+d+f}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} = \frac{\frac{1}{df}(d+f)(ab+df)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
bd-ce &= \frac{bd(\frac{1}{b}+\frac{1}{d}+\frac{1}{f}-\frac{1}{a})-(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{bd(\frac{1}{f}-\frac{1}{a})+(a-f)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} = \frac{\frac{1}{af}(a-f)(bd+af)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
abd-cef &= abd - \frac{f(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{abd(\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a})-f(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{(b+f)(\frac{abd}{bf}-f)+b(a-d)+f(a-d)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{(b+f)(\frac{ad}{f}-f+(a-d))}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{\frac{1}{f}(b+f)(a-f)(f+d)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}}.
\end{aligned}$$

Putting that all together gives

$$\frac{x-p}{b-p} = \frac{ce\cdot adf(b+f)(\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a})}{(ab+df)(bd+af)}$$

which is symmetric in d and f, so the analogous calculation with $\frac{y-p}{f-p}$ yields the same result. Consequently, P is the center of the spiral similarity sending $\overline{YF}$ to $\overline{BX}$, as desired.

Remark. Philosophical point: it's necessary to use both $a+c+e=b+d+f$ and its conjugate, to capture two degrees of freedom.

Seventh solution, by inversion and moving points (Anant Mudgal, unedited) Let H be the common orthocenter. Pick any two vertices X, Y of either $\triangle ACE$ or $\triangle BDF$ and notice that $\triangle XYH$ has circumradius equal to the radius of Ω. Now invert at H. We obtain the following equivalent problem.

> Let $ABCDEF$ be a cyclic hexagon with $\triangle ACE$ and $\triangle BDF$ sharing a common incircle ω centered at point H. Let $\odot(HBD), \odot(HFD)$ meet $\odot(CHE)$ again at points X and Y respectively. Let M be the midpoint of arc CE not containing A. Then $\odot(DXY)$ passes through point M.

Let ω touch $\overline{CE}$ at point N and $L = \overline{AD} \cap \overline{CE}$. Let $P = \overline{DB} \cap \overline{CE}$ and $Q = \overline{DF} \cap \overline{CE}$. By Dual of Desragues Involution Theorem on circumscribed $ACEN$ and point D; we conclude $(\overline{DN}, \overline{DL}), (\overline{DC}, \overline{DE}), (\overline{DP}, \overline{DQ})$ are pairs of an involution. Notice that P has equal powers in $\odot(HBD), \odot(CHE)$ hence P lies on $\overline{XH}$. Similarly, Q lies on $\overline{YH}$.

Let $\overline{HN}, \overline{HL}$ meet $\odot(CHE)$ again at S, T. Project through H to conclude that $(C, E), (X, Y), (S, T)$ are pairs of an involution on the circle $\odot(CHE)$. Thus, we conclude that lines $\overline{CE}, \overline{XY}, \overline{ST}$ concur.

Claim. $\overline{CE}, \overline{ST}, \overline{DM}$ concur.

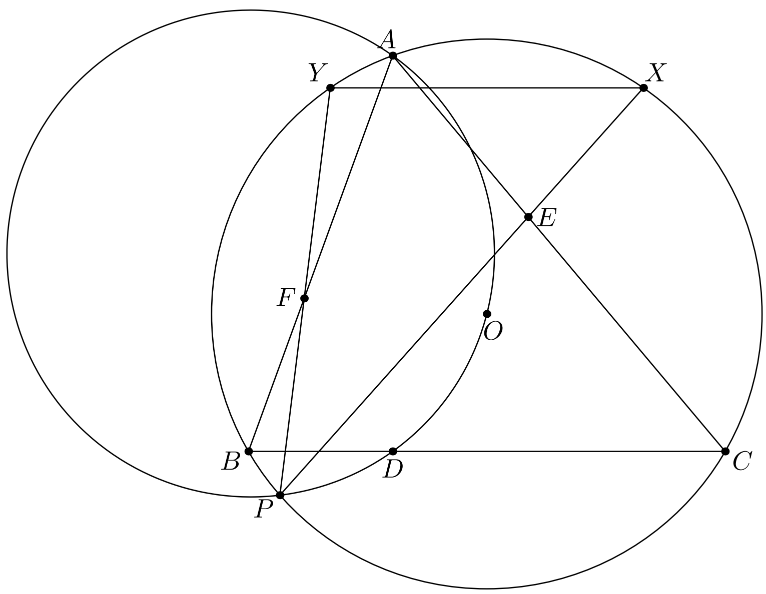
Proof. Animate D on $\odot(ACE)$; then $D \mapsto L \mapsto T$ is projective. Let $U = \overline{DM} \cap \overline{CE}$ and $V = \overline{ST} \cap \overline{CE}$ then $D \mapsto U$ and $D \mapsto V$ are also projective. Thus to show $W \overset{\text{def}}{:=} U \equiv V$ we need to verify for three choices of point D; namely we pick $\{C, E, M\}$. These are all clearly true and the lemma is proved. $\square$

Finally, notice $WX \cdot WY = WC \cdot WE = WD \cdot WM$ proving $DXYM$ is cyclic.

G5. Let scalene triangle ABC have altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines $\overline{PE}$ at $X \neq P$ and $\overline{PF}$ at $Y \neq P$. Prove that $\overline{XY} \parallel \overline{BC}$.

(Daniel Hu)

Denote by Ω and H the circumcircle and orthocenter of $\triangle ABC$. Let T lie on Ω such that $\overline{AT} \parallel \overline{BC}$. Let $\triangle ABC$ have orthocenter H.



First solution, synthetic First we prove a lemma.

Claim. The points H, P, T are collinear.

Proof. Let $\overline{HT}$ meet Ω at $P^* \neq T$. Let $\overline{AD}$ meet Ω at $K \neq A$. By homothety at K, $\overline{HT} \parallel \overline{DO}$. By angle chasing, $\angle P^*AD = \angle P^*AK = \angle P^*TK = \angle P^*TO = \angle OP^*T = \angle P^*OD$, so P^* lies on the circumcircle of $\triangle AOD$. Therefore, $P \equiv P^*$ as desired. $\square$

We now provide two finishes.

- First finish: By DDIT on $AEHF$, the pairs of lines $(\overline{PA}, \overline{PH}), (\overline{PB}, \overline{PC}), (\overline{PE}, \overline{PF})$ are part of a single involution, so $\overline{AT}, \overline{BC}, \overline{XY}$ are concurrent. Since $\overline{AT} \parallel \overline{BC}$, this implies that $\overline{XY} \parallel \overline{BC}$ as desired.

- Second finish: Let $Q = \overline{AP} \cap \overline{EF}$. By inversion at A, $BFPQ$, $CEPQ$, $DHPQ$ are all cyclic. By the lemma, this implies that $\angle ABC + \angle ACB = \angle APT = \angle APH = \angle QPH = \angle QDH = \angle QAH$, so $\overline{DQ} \perp \overline{EF}$.

  Let $G = \overline{EF} \cap \overline{BC}$; since $(G, D; B, C) = -1$, $\angle BQD = \angle DQC$. Thus $\angle BAY = \angle BPY = \angle BPF = \angle BQF = \angle CQE = \angle CPE = \angle CPX = \angle CAX$, so $\overline{XY} \parallel \overline{BC}$ as desired.

Second solution by complex numbers (Adam Ardeishar) Let ABC be the complex unit circle. Then $D = \frac{1}{2}(a+b+c-\frac{bc}{a})$, and we know

$$\frac{p-a}{p-o} \cdot \frac{d-o}{d-a} \in \mathbb{R}$$

$$\frac{p-a}{p} \cdot \frac{a+b+c-\frac{bc}{a}}{b+c-a-\frac{bc}{a}} = \frac{\frac{1}{p}-\frac{1}{a}}{\frac{1}{p}} \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}{-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}$$

$$\frac{1}{p} \cdot \frac{a+b+c-\frac{bc}{a}}{b+c-a-\frac{bc}{a}} = \frac{-1}{a} \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}{\frac{1}{b}+\frac{1}{c}-\frac{1}{a}-\frac{a}{bc}}$$

$$\frac{-a}{p} \cdot \frac{a^2+ab+ac-bc}{ab+ac-a^2-bc} = \frac{bc+ab+ab-a^2}{ab+ac-bc-a^2}$$

$$p = a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc}$$

Now note that $p + x = e + px\overline{e}$, so $x = \frac{p-e}{p\overline{e}-1}$ But we compute that

$$p - e = a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc} - \frac{1}{2}(a+b+c-\frac{ac}{b})$$

$$= \frac{a^3b+a^3+2a^2b^2+a^2bc+ab^3+ab^2c+b^3c+b^2c^2-a^2c^2}{2b(a^2-ab-ac-bc)}$$

$$= \frac{(a+b)(b+c)(a^2+ab-ac+bc)}{2b(a^2-ab-ac-bc)}$$

And also compute

$$p\overline{e} - 1 = a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc} \cdot \frac{1}{2}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{b}{ac}) - 1$$

$$= \frac{a^3b+a^3c+a^2bc+a^2c^2+ab^2c+2abc^2+b^3c+b^2c^2-ab^3}{2bc(a^2-ab-ac-bc)}$$

$$= \frac{(a+b)(b+c)(a^2+ac+bc-ab)}{2bc(a^2-ab-ac-bc)}$$

So

$$x = \frac{\frac{(a+b)(b+c)(a^2+ab-ac+bc)}{2b(a^2-ab-ac-bc)}}{\frac{(a+b)(b+c)(a^2+ac+bc-ab)}{2bc(a^2-ab-ac-bc)}} = c \cdot \frac{a^2+ab+bc-ac}{a^2+ac+bc-ab}$$

By symmetry,

$$y = b \cdot \frac{a^2+ac+bc-ab}{a^2+ab+bc-ac}$$

Now note that $xy = bc$ to finish.

N1. Determine all nonempty finite sets $S = \{a_1, \dots, a_n\}$ of n distinct positive integers such that $a_1 \cdots a_n$ divides $(x + a_1) \cdots (x + a_n)$ for every positive integer x.

(Ankan Bhattacharya)

Answer: $\{a_1 \dots, a_n\} = \{1, \dots, n\}$. This works since

$$\frac{(x+n)\dots(x+1)}{n!} = \binom{x+n}{n} \in \mathbb{Z}$$

so we now show that it is the only possibility. There are two approaches.

First solution Let $P(x) = (x + a_1) \dots (x + a_n)$. Then, $a_1 \dots a_n$ should divide the nth finite difference of P, which is $n!$. But

$$a_1 \dots a_n \mid n! \implies \{a_1 \dots, a_n\} = \{1, \dots, n\}$$

for size reasons.

Second solution (Kevin Sun) Let $s + 1$ be the smallest positive integer not in our set A and denote $B = A \setminus \{1, \dots, s\}$.

It's clear that the divisibility holds for negative x as well. Set $x = -s - 1$ to obtain

$$\begin{aligned}
\mathbb{Z} \ni & \frac{1}{a_1 \dots a_n} \prod_{a \in A} (x + a) \\
&= \prod_{a \in A} \left(1 + \frac{x}{a}\right) \\
&= \prod_{a \in \{1,\dots,s\}} \left(1 - \frac{s+1}{a}\right) \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\
&= \prod_{a \in \{1,\dots,s\}} \left(\frac{a - (s+1)}{a}\right) \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\
&= \frac{(-s)(-(s-1))\dots(-1)}{1 \cdot 2 \cdot \dots \cdot s} \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\
&= (-1)^{|A|} \prod_{b \in B} \left(1 - \frac{s+1}{b}\right).
\end{aligned}$$

If B is nonempty this has magnitude strictly between 0 and 1, (since $\min B > s + 1$ and thus each term is in $(0, 1)$). Thus B is empty and $A = \{1, \dots, s\}$.

N2. Call a number n *good* if it can be expressed in the form $2^x + y^2$ where x and y are nonnegative integers.

(a) Prove that there exist infinitely many sets of 4 consecutive good numbers.

(b) Find all sets of 5 consecutive good numbers.

(Michael Ma)

For (a), note that for any t, the numbers t^2+1, t^2+2, t^2+4 are good. So it suffices to show t^2+3 is good infinitely often, that is, $t^2+3 = 2^x + y^2$ has infinitely many nonnegative integer solutions (since for fixed t there are finitely many (x,y)). But this rearranges $t^2 - y^2 = 2^x - 3$ which has a solution for every x.

We now turn to the laborious task of (b), determining all sets of five consecutive good numbers. The answers are the six tuples $\{1,2,3,4,5\}$, $\{2,3,4,5,6\}$, $\{8,9,10,11,12\}$, $\{9,10,11,12,13\}$, $\{288,289,290,291,292\}$, $\{289,290,291,292,293\}$. These all work since

$$\begin{gathered}
1 = 2^0 + 0^2, \quad 2 = 2^0 + 1^2, \quad 3 = 2^1 + 1^2, \\
4 = 2^2 + 0^2, \quad 5 = 2^2 + 1^2, \quad 6 = 2^1 + 2^2, \\
8 = 2^3 + 0^2, \quad 9 = 2^3 + 1^2, \quad 10 = 2^0 + 3^2, \\
11 = 2^1 + 3^2, \quad 12 = 2^3 + 2^2, \quad 13 = 2^2 + 3^2, \\
288 = 2^5 + 16^2, \quad 289 = 2^6 + 15^2, \quad 290 = 2^0 + 17^2, \\
291 = 2^1 + 17^2, \quad 292 = 2^8 + 6^2, \quad 293 = 2^2 + 17^2.
\end{gathered}$$

We now show they are the only ones. First, consider the following table which shows $2^x + y^2 \pmod 8$:

	$x=0$	$x=1$	$x=2$	$x \ge 3$
$y \equiv 1 \pmod 2$	2	3	5	1
$y \equiv 0 \pmod 4$	1	2	4	0
$y \equiv 2 \pmod 4$	5	6	0	4

Note that from this table, no good number is $7 \pmod 8$. Thus any five good numbers must have a $3 \pmod 8$ number. By table can only occur if that good number is of the form $t^2 + 2^1 = t^2 + 2$ for an odd integer t.

We now have several cases.

Case 1: Suppose the five good numbers are $\{t^2+1, t^2+2, t^2+3, t^2+4, t^2+5\}$.

Note that $t^2+5 \equiv 6 \pmod 8$, and by table, this can only occur if $t^2+5 = s^2+2^2 = s^2+4$ for some integer s; hence $t^2 - s^2 = 1$, so $t=1$ and $s=0$. This gives the solution set $\{2,3,4,5,6\}$.

Case 2: Suppose the five good numbers are $\{t^2, t^2+1, t^2+2, t^2+3, t^2+4\}$.

Since t^2 is good, we have $t^2 = 2^w + z^2$ for some w and z, which we write as $(t-z)(t+z) = 2^w$.

We now split into cases.

- **Subcase 2.1**: We handle the situation where $w < 4$.
  - If $w = 0$, then we get $t = 1$, which gives the solution $\{1,2,3,4,5\}$.
  - If $w = 1$, then there are no solutions by taking mod 4.

  - If $w = 2$, then $t^2 = 4 + z^2$ which implies $t = 2$, but t was odd.
  - If $w = 3$, we get $t^2 = 8 + z^2$ which implies $t = 3$, which gives $\{9, 10, 11, 12, 13\}$.
  - If $w = 4$, we get $t^2 = 16 + z^2$ which together with t odd implies $t = 5$, which gives $\{25, 26, 27, 28, 29\}$. However, the number 28 is not good, so this is not a solution.

- **Subcase 2.2**: Suppose $w \geq 5$. As $\gcd(t - z, t + z) \mid 2t$ we must have $t - z = 2$, $t + z = 2^{w-1}$, and thus $t = \frac{1}{2}\left(2 + 2^{w-1}\right) = 2^{w-2} + 1$. Since t was odd, we actually have $w \geq 3$.

  But $t^2 + 3$ is also good, so write

  $$t^2 + 3 = 2^x + y^2.$$

  So we split into cases again.

  - **Subcase 2.2.1**: We handle the case $x < 3$.
    * If $x = 0$, we get $t^2 + 2 = y^2$ which has no solutions.
    * If $x = 1$, we get $t^2 + 1 = y^2$ which implies $t = 0$, but t is supposed to be odd.
    * If $x = 2$, then we get $t^2 = y^2 + 1$ which implies $t = 1$, which was an earlier solution.
  - **Subcase 2.2.2**: Otherwise, assume $x \geq 3$.

    $$\begin{aligned} 2^x + y^2 &= t^2 + 3 \\ \implies 2^x + y^2 &= \left(2^{w-2} + 1\right)^2 + 3 \\ &= 2^{2w-4} + 2^{w-1} + 4 \\ \implies 2^{2w-6} + 2^{w-3} + 1 &= 2^{x-2} + (y/2)^2 \end{aligned}$$

    since y is clearly even; the last line implies $y/2$ is odd, since $2w - 6 > 0$, $w - 3 > 0$, $x - 2 > 0$.

    Let $c = w - 3 \geq 2$, $a = x - 2 \geq 1$, $b = y/2 \geq 1$ for brevity; then the equation rewrites as

    $$2^{2c} + 2^c + 1 = 2^a + b^2.$$

    We rewrite this as

    $$(2^c + 1 - b)(2^c + 1 + b) = (2^c + 1)^2 - b^2 = 2^a + 2^c \geq 0.$$

    In light of this, we have $2^a + 2^c \geq (2^c + 1)^2 - 2^{2c} > 2^{c+1}$, so $2^a > 2^c$, ergo $a > c$. Thus we may further write

    $$(2^c + 1 - b)(2^c + 1 + b) = 2^c(2^{a-c} + 1).$$

    The factors on the left-hand side are nonnegative and have gcd dividing $2b$, hence one of them has at most one factor of 2. So one of the factors must be divisible by 2^{c-1}. Thus, $b \equiv \pm 1 \pmod{2^{c-1}}$.

    But, $b < 2^c + 1$. So we have four possibilities:

    * **Subcase 2.2.2.1**: suppose $b = 1$. Then we get $2^{2c} + 2^c = 2^a$, which is impossible.

* **Subcase 2.2.2.2**: suppose $b = 2^{c-1} - 1$. Then we get $(2^{c-1} + 2)(2^c + 2^{c-1}) = 2^c(2^{a-c} + 1)$ and hence $3 \cdot 2^{c-2} = 2^{a-c} - 2$. This implies $a - c = 3$ and $c - 2 = 1$, so $c = 3$, or $w = 6$, hence $t = 2^{w-2} + 1 = 17$.

  This gives $\{289, 290, 291, 292, 293\}$ which indeed works.
* **Subcase 2.2.2.3**: suppose $b = 2^{c-1}+1$. Then we get $2^{c-1}(2^c+2^{c-1}+2) = 2^c(2^{a-c} + 1)$, or $2^{c-1} + 2^{c-2} + 1 = 2^{a-c} + 1$, which is impossible.
* **Subcase 2.2.2.4**: suppose $b = 2^c - 1$. This gives $2 \cdot 2^{c+1} = 2^c(2^{a-c} + 1)$, which is impossible.

Case 3: Suppose the five good numbers are $\{t^2 - 1, t^2, t^2 + 1, t^2 + 2, t^2 + 3\}$.

In that case, $\{t^2, t^2 + 1, t^2 + 2, t^2 + 3, t^2 + 4\}$ is also a set of five consecutive good numbers. Using case 2, the new candidate this now gives are $\{8, 9, 10, 11, 12\}$ and $\{288, 289, 290, 291, 292\}$, which work.

N3. Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. Find the maximum possible value of a_{2018}.

(Krit Boonsiriseth)

The answer is $a_{2018} \le 2^{1009} - 1$. To see this is attainable, consider the sequence

$$a_n = \begin{cases} 1 & n \text{ odd} \\ 2^{n/2} - 1 & n \text{ even.} \end{cases}$$

This can be checked to work, so we prove it's optimal.

We have $a_2 \mid a_1 + a_2 = 1 + a_2 \implies a_2 = 1$.

Now consider an integer n, and let $s = s_n = a_1 + \cdots + a_n$. Then

$$\begin{aligned} a_{n+1} &\mid s \\ a_{n+2} &\mid s + a_{n+1} \\ a_{n+2} &\equiv 1 \pmod{a_{n+1}}. \end{aligned}$$

Thus, $\gcd(a_{n+2}, a_{n+1}) = 1$. So $a_{n+2} \le \frac{s+a_{n+1}}{a_{n+1}}$, and thus

$$a_{n+1} + a_{n+2} \le 1 + a_{n+1} + \frac{s}{a_{n+2}} \le s + 2.$$

So, we have

$$\begin{aligned} a_1 + a_2 &= 2 \\ a_3 + a_4 &\le 2 + 2 = 4 \\ a_5 + a_6 &\le (2+4) + 2 = 8 \\ a_7 + a_8 &\le (2+4+8) + 2 = 16 \\ &\vdots \\ a_{2017} + a_{2018} &\le 2^{1009}. \end{aligned}$$

Thus $a_{2018} \le 2^{1009} - a_{2017} \le 2^{1009} - 1$.

Remark (Motivational notes)**.** It's very quick to notice $a_{n+1} \mid a_1 + \cdots + a_n$, which already means that given the first n terms of the sequence there are finitely many possibilities for the next one. Thus it's possible to play with "small cases" by drawing a large tree.

When doing so, one might hope that somehow $a_n = a_1 + \cdots + a_{n-1}$ is achievable, but quickly notices in such a tree that if a_n is the sum of all previous terms, then $a_{n+1} = 1$ is forced. This gives the idea to try to look at the terms in pairs, rather than one at a time, and this gives the correct bound.

As for extracting the equality case from this argument, there are actually two natural curves to try. We have $a_3 \mid 1+1 = 2$. If we have $a_3 = 2$ we get $a_4 = 1$, $a_5 \le 5$, but then a_6 actually gets stuck. But if we have $a_3 = 1$ instead, we get $a_4 = 3$, $a_5 = 1$, $a_6 = 7$, and so on; pushing this gives the equality case above, seen to work. I think it's quite unnatural to guess the correct construction before having the corresponding $s + 2$ estimate.

N4. Fix a positive integer $n > 1$. We say a nonempty subset S of $\{0, 1, \dots, n-1\}$ is d-coverable if there exists a polynomial P with integer coefficients and degree at most d, such that S is exactly the set of residues modulo n that P attains as it ranges over the integers.

For each n, determine the smallest d such that any nonempty subset of $\{0, \dots, n-1\}$ is d-coverable, or prove that no such d exists.

(Carl Schildkraut)

This is possible for $n = 4$ or n prime, in which case $d = n - 1$ is best possible. Let $P(\mathbb{Z}/n)$ denote the range of a polynomial modulo n.

- We first note that if $n = q_1 \dots q_k$ is the product of $k \ge 2$ distinct prime powers, then
$$|P(\mathbb{Z}/n)| = \prod_{i=1}^{k} |P(\mathbb{Z}/q_i)|.$$
Hence any subset S with size $n-1$ is not coverable.

- If $n = p^e$ is a prime power with other than 4 with $e \ge 2$, consider the set $S = \{0, 1, \dots, p-1, p\}$. We claim it is not coverable.

  Indeed, if P covers it, WLOG $P(0) = 0$. Now, P is surjective modulo p, hence bijective, and thus $P(x) \equiv 0 \pmod p \iff x \equiv 0 \pmod p$. Now we can write
$$P(x) = a_1 x + a_2 x^2 + \dots .$$
  - If $a_1 \equiv 0 \pmod p$, then $x \equiv 0 \pmod p \implies P(x) \equiv 0 \pmod{p^2}$, so p does not appear in the image.
  - If $a_1 \not\equiv 0 \pmod p$, then p, $2p$, … all appear in the image, which is wrong for $n > 4$.

- Let $n = 4$, and consider $S \pmod 4$.
  - If $S = \{k\}$ take $P(x) = k$.
  - If $S = \{k, k+1\}$ take $P(x) = x^2 + k$.
  - If $S = \{k, k+2\}$ take $P(x) = 2x^2 + k$.
  - If $S = \{k-1, k, k+1\}$ take $P(x) = x^3 + k$.

    We claim also the example $S = \{-1, 0, 1\}$ is not 2-coverable. Indeed, WLOG $P(0) = 0$ so $P(x) = x(x+c)$. Then $P(2) \equiv 0 \pmod 4$, meaning c is even. But then $P(1) \equiv c + 1 \pmod 4$ and $P(-1) \equiv 1 - c \pmod 4$, so $P(1) \equiv P(-1)$.
  - If $S = \{0, 1, 2, 3\}$ take $P(x) = x$.

- Let $n = 2$.
  - If $S = \{k\}$ take $P(x) = k$.
  - If $S = \{0, 1\}$ take $P(x) = x$. This is obviously not 0-coverable.

- If $n = p$ is an odd prime, we claim $S = \{1, \dots, p-1\}$ is not $(p-2)$-coverable. Indeed, suppose $P(x) = a_{p-2}x^{p-2} + \cdots + a_0$ covered it. Then

$$\sum_x P(x) \equiv \sum_k a_k \sum_x x^k \equiv 0 \pmod p.$$

However, if $P(\mathbb{Z}/p) = \{1, \dots, p-1\}$ then some element appears twice and the others appear once. If k is the repeated element though, then $\sum_x P(x) = (1 + \cdots + (p-1)) + k \equiv k \not\equiv 0 \pmod p$.

Shortlisted Problems

20$^{\text{th}}$ ELMO

Pittsburgh, PA, 2018

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until disclosed publicly by the committee on the ELMO.

Contributing Students

The Problem Selection Committee for ELMO 2018 thanks the following proposers for contributing **90 problems** to this year's Competition:

> Adam Ardeishar, Andrew Gu, Ankan Bhattacharya, Brandon Wang, Carl Schildkraut, Daniel Hu, Daniel Liu, Eric Gan, Kevin Ren, Krit Boonsiriseth, Luke Robitaille, Michael Kural, Michael Ma, Michael Ren, Sam Ferguson, Tristan Shin, Vincent Bian, Vincent Huang, Zack Chroman

Problem Selection Committee

The Problem Selection Committee for ELMO 2018 was led by **Evan Chen** and consisted of:

- Andrew Gu
- Daniel Liu
- James Lin
- Michael Ma
- Michael Ren
- Mihir Singhal
- Zack Chroman

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Problems

A1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bijective function. Does there always exist an infinite number of functions $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$?

(Daniel Liu)

A2. Let $a_1, a_2, \dots, a_m$ be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n+a_i} \right\rfloor = \left\lfloor \sqrt{bn+c} \right\rfloor$$

holds for all integers $n > N$.

(Carl Schildkraut)

A3. Let a, b, c, x, y, z be positive reals such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Prove that

$$a^x + b^y + c^z \geq \frac{4abcxyz}{(x+y+z-3)^2}.$$

(Daniel Liu)

A4. Elmo calls a monic polynomial with real coefficients *tasty* if all of its coefficients are in $[-1, 1]$. A monic polynomial P with real coefficients and complex roots $\chi_1, \dots, \chi_m$ (counted with multiplicity) is given to Elmo, and he discovers that there does not exist a monic polynomial Q with real coefficients such that $P \cdot Q$ is tasty. Find all possible values of $\max(|\chi_1|, \dots, |\chi_m|)$.

(Carl Schildkraut)

C1. Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1 \leq i \leq 2018$, there are exactly n cities that are a distance i away from C. (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

(Michael Ren)

C2. We say that a positive integer n is *m-expressible* if one can write a expression evaluating to n in base 10, where the expression consists only of

- exactly m numbers from the set $\{0, 1, \dots, 9\}$
- the six operations $+$, $-$, $\times$, $\div$, exponentiation $^\wedge$, concatenation $\oplus$, and
- some number (possibly zero) of left and right parentheses.

For example, 5625 is 3-expressible (in two ways), as $5625 = 5 \oplus (5\text{^}4) = (7 \oplus 5)\text{^}2$, say. Does there exist a positive integer A such that all positive integers with A digits are $(A-1)$-expressible?

(Krit Boonsiriseth)

C3. A *windmill* in the plane consists of a line segment of unit length with a distinguished endpoint, the *pivot*. Geoff has a finite set of windmills, such that no two windmills intersect, and any two pivots are distance more than $\sqrt{2}$ apart. In an operation, Geoff can choose a windmill and rotate it about its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during or after the rotation. Show that Geoff can, in finitely many operations, rotate the windmills so that they all point in the same direction.

(Michael Ren)

G1. Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles of $\triangle EHP$ and $\triangle FHP$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

(Brandon Wang)

G2. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle of $\triangle ABC$.

(Zack Chroman)

G3. Let A be a point in the plane, and ℓ a line not passing through A. Evan doesn't have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

(Zack Chroman)

G4. Let $ABCDEF$ be a convex hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that $\overline{BD}$ and $\overline{DF}$ intersect $\overline{CE}$ at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to $\overline{CE}$.

(Michael Ren and Vincent Huang)

G5. Let scalene triangle ABC have altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines $\overline{PE}$ at $X \neq P$ and $\overline{PF}$ at $Y \neq P$. Prove that $\overline{XY} \parallel \overline{BC}$.

(Daniel Hu)

N1. Determine all nonempty finite sets $S = \{a_1, \dots, a_n\}$ of n distinct positive integers such that $a_1 \cdots a_n$ divides $(x + a_1) \cdots (x + a_n)$ for every positive integer x.

(Ankan Bhattacharya)

N2. Call a number n *good* if it can be expressed in the form $2^x + y^2$ where x and y are nonnegative integers.

(a) Prove that there exist infinitely many sets of 4 consecutive good numbers.

(b) Find all sets of 5 consecutive good numbers.

(Michael Ma)

N3. Let $a_1, a_2, \dots$ be an infinite sequence of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. Find the maximum possible value of a_{2018}.

(Krit Boonsiriseth)

N4. Fix a positive integer $n > 1$. We say a nonempty subset S of $\{0, 1, \dots, n-1\}$ is d-coverable if there exists a polynomial P with integer coefficients and degree at most d, such that S is exactly the set of residues modulo n that P attains as it ranges over the integers.

For each n, determine the smallest d such that any nonempty subset of $\{0, \dots, n-1\}$ is d-coverable, or prove that no such d exists.

(Carl Schildkraut)

Solutions

A1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bijective function. Does there always exist an infinite number of functions $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$?

(Daniel Liu)

Yes. It's clear $f^0, f^1, f^2, \dots$ all commute with f. If f doesn't have finite order this collection is infinite and valid.

Else, suppose that $f^n = \mathrm{id}$, where n is minimal. If $n = 1$ the problem is clear, so suppose $n > 1$. Then f is composed of some cycles; some cycle length $d \mid n$ appears infinitely many times. Let a countable number of these cycles be $x_{r,1} \to x_{r,2} \to \cdots \to x_{r,d} \to x_{r,1}$ for $r \in \mathbb{Z}$.

Then for every integer s, create a new function h_s fixing everything except the $x_{k,\ell}$, and send every $x_{r,a} \to x_{r+s,a}$. It is clear that every h_s commutes with f.

This gives infinitely many g, unless all but finitely many of the cycles have length 1. In that case, we can find more g by swapping any two fixed points of f and leaving everything else intact.

A2. Let a_1, a_2, ..., a_m be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n+a_i} \right\rfloor = \left\lfloor \sqrt{bn+c} \right\rfloor$$

holds for all integers $n > N$.

(Carl Schildkraut)

If all the a_i are equal, then $\sum_{i=1}^m \sqrt{n+a_i} = \sqrt{m^2n + m^2a_1}$ and so $(b,c) = (m^2, m^2a_1)$ works fine.

Let us assume this is not the case. Instead, will take $b = m^2$ and $c = m(a_1+\dots+a_m)-1$ and claim it works for N large enough.

On the one hand,

$$\begin{aligned} \sum_{i=1}^{m} \sqrt{n+a_i} &< m \cdot \sqrt{n + \frac{a_1+\dots+a_m}{m}} \\ &= \sqrt{m^2 \cdot n + c + 1} \le \left\lceil \sqrt{m^2 \cdot n + c + 1} \right\rceil \le \left\lfloor \sqrt{m^2 \cdot n + c} \right\rfloor + 1. \end{aligned}$$

On the other hand, let $\lambda = \frac{c}{2(c+1)} < \frac{1}{2}$. We use the following estimate.

Claim. If n is large enough in terms of $(a_1, \dots, a_m)$ then $\sqrt{n+a_i} \ge \sqrt{n} + \frac{\lambda a_i}{\sqrt{n}}$.

Proof. Squaring both sides, it's equivalent to $a_i \ge 2\lambda \cdot a_i + \frac{\lambda^2 a_i^2}{n}$, which holds for n big enough as $2\lambda < 1$. □

Now,

$$\begin{aligned} \sum_{i=1}^{m} \sqrt{n+a_i} &\ge \sum_{i=1}^{m} \left(\sqrt{n} + \frac{\lambda a_i}{\sqrt{n}} \right) \\ &\ge m\sqrt{n} + \frac{\lambda \cdot (a_1 + \dots + a_n)}{\sqrt{n}} \\ &= m\sqrt{n} + \frac{\lambda \cdot (c+1)}{m\sqrt{n}} \\ &= m\sqrt{n} + \frac{c}{2m\sqrt{n}} > \sqrt{m^2 \cdot n + c} \ge \left\lfloor \sqrt{m^2 n + c} \right\rfloor. \end{aligned}$$

This finishes the problem.

Remark. Obviously, $b = m^2$ for asymptotic reasons (by taking n large). As for possible values of c:

- If $a_1 = \dots = a_m$, then one can show $c = m(a_1 + \dots + a_m)$ is the only valid choice. Indeed, taking n of the form $n = k^2 - a$ and $n = \frac{k^2-1}{m^2} - a$ is enough to see this.
- But if not all a_i are equal, the natural guess of taking $c = m(a_1 + \dots + a_n)$ is not valid in general. For example, we have that

  $$\left\lfloor \sqrt{n} + \sqrt{n+2} \right\rfloor \ne \left\lfloor \sqrt{4n+4} \right\rfloor \qquad n \in \{t^2 - 1 \mid t = 2, 3, \dots\}.$$

  I think one can actually figure out exactly which c are valid, though the answer will depend on some quadratic residues, and we do not pursue this line of thought here.

So any correct solutions must distinguish these two cases.

A3. Let a, b, c, x, y, z be positive reals such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$. Prove that

$$a^x+b^y+c^z \ge \frac{4abcxyz}{(x+y+z-3)^2}.$$

(Daniel Liu)

We present three solutions.

First solution, proof without words (by proposer)

$$\begin{aligned}
a^x+b^y+c^z &= yz\cdot\frac{a^x}{yz}+zx\cdot\frac{a^y}{zx}+xy\cdot\frac{a^z}{xy}\\
&\ge (xy+yz+zx)\left(\left(\frac{a^x}{yz}\right)^{yz}\left(\frac{b^y}{zx}\right)^{zx}\left(\frac{c^z}{xy}\right)^{xy}\right)^{\frac{1}{xy+yz+zx}}\\
&= (xy+yz+zx)\cdot\frac{(abc)^{\frac{xyz}{xy+yz+zx}}}{x^{\frac{xy+zx}{xy+yz+zx}}y^{\frac{yz+xy}{xy+yz+zx}}z^{\frac{zx+yz}{xy+yz+zx}}}\\
&\ge (xy+yz+zx)\cdot\frac{(abc)^{\frac{xyz}{xy+yz+zx}}}{\left(\frac{x\cdot\frac{xy+zx}{xy+yz+zx}+y\cdot\frac{yz+xy}{xy+yz+zx}+z\cdot\frac{zx+yz}{xy+yz+zx}}{2}\right)^2}\\
&= (xy+yz+zx)\cdot\frac{4(abc)^{\frac{xyz}{xy+yz+zx}}}{\left(\sum_{\text{cyc}}x\cdot\left(1-\frac{yz}{xy+yz+zx}\right)\right)^2}\\
&= \frac{4abc(xy+yz+zx)}{(x+y+z-3\frac{xyz}{xy+yz+zx})^2}\\
&= \frac{4abcxyz}{(x+y+z-3)^2}.
\end{aligned}$$

Second solution, by weighted AM-GM (Andrew Gu) By weighted AM-GM,

$$\frac{1}{x}\cdot xa^x+\frac{1}{y}\cdot yb^y+\frac{1}{z}\cdot zc^z \ge x^{\frac{1}{x}}y^{\frac{1}{y}}z^{\frac{1}{z}}abc.$$

Hence it suffices to show

$$x^{\frac{1}{x}}y^{\frac{1}{y}}z^{\frac{1}{z}} \ge \frac{4xyz}{(x+y+z-3)^2}.$$

By weighted AM-GM,

$$2x^{\frac{1}{2}(1-\frac{1}{x})}y^{\frac{1}{2}(1-\frac{1}{y})}z^{\frac{1}{2}(1-\frac{1}{z})} \le 2\cdot\frac{1}{2}(x-1+y-1+z-1)=x+y+z-3.$$

Squaring both sides and rearranging proves the required inequality.

Third solution, by Hölder and Schur/Muirhead (Evan Chen) By Hölder and weighted AM-GM we have

$$\sqrt{\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)(a^x+b^y+c^z)} \ge \frac{1}{x}\cdot a^{x/2}+\frac{1}{y}\cdot b^{y/2}+\frac{1}{z}\cdot c^{z/2} \ge (abc)^{1/2}.$$

Hence, it suffices to prove that

$$(x+y+z-3)^2 \ge 4xyz\left(1/x^2+1/y^2+1/z^2\right) \qquad \forall \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$$

which is a 3-variable symmetric inequality. It also happens to be is MOP 2011, K4.1, done in my SOS handout. We give a proof below (with $a = 1/x$, etc).

Claim (Black MOP 2011, Test 4, Problem 1)**.** If $a, b, c > 0$ then

$$\left((a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3\right)^2 \ge 4\left(\frac{(a+b+c)(a^2+b^2+c^2)}{abc}\right)$$

Proof. Expanding and clearing denominators it's just

$$\sum_{\text{sym}} a^4b^2 + \sum_{\text{cyc}} a^3b^3 + 6a^2b^2c^2 \ge 2\sum_{\text{cyc}} a^4bc + 2\sum_{\text{sym}} a^3b^2c$$

which can also be written as

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 1 & & -2 & & 1 & & & & \\
 & & & 2 & & -2 & & -2 & & 2 & & & \\
 & & 1 & & -2 & & 6 & & -2 & & 1 & & \\
 & 0 & & -2 & & -2 & & -2 & & -2 & & 0 & \\
0 & & 0 & & 1 & & 2 & & 1 & & 0 & & 0
\end{array}$$

in Chinese dumbass notation. This rewrites as

$$\sum_{\text{cyc}} a^4(b-c)^2 + 2\sum_{\text{cyc}} ab(ab-bc)(ab-ac) \ge 0$$

which is evident (the latter sum is "upsidedown triangle Schur"). $\square$

A4. Elmo calls a monic polynomial with real coefficients *tasty* if all of its coefficients are in $[-1, 1]$. A monic polynomial P with real coefficients and complex roots $\chi_1, \ldots, \chi_m$ (counted with multiplicity) is given to Elmo, and he discovers that there does not exist a monic polynomial Q with real coefficients such that $P \cdot Q$ is tasty. Find all possible values of $\max(|\chi_1|, \ldots, |\chi_m|)$.

(Carl Schildkraut)

We claim the answer is $r > 1$. The answer is divided into two parts.

Part I: Any value of $r > 1$ can be achieved. To prove this, we will show that the polynomial

$$P(x) = x^n - r^n$$

has no tasty multiples if $r^n \geq 2$ (such an n exists because $r > 1$). Set $M = r^n$. Assume we have a polynomial

$$R(x) = \sum_{i=0}^{N} a_i x^i$$

so that $-1 \leq a_i \leq 1$ for all i ($a_N = 1$) and $P|R$. Taking R modulo P, we get that, with $N = bn + c$ and $0 \leq c < n$ (setting $a_k = 0$ if $k > N$),

$$R(x) = \sum_{j=0}^{n-1} \sum_{k=0}^{b} a_{kn+j} x^{kn+j} \equiv \sum_{j=0}^{n-1} x^j \left[\sum_{k=0}^{b} a_{kn+j} R^k \right].$$

We have this must be the zero polynomial (since $P|R$); specifically, taking $j = c$,

$$\sum_{k=0}^{b} a_{nk+c} R^k = 0$$
$$\sum_{k=0}^{b-1} (-a_{nk+c}) R^k = a_{bn+c} R^b$$
$$\sum_{k=0}^{b-1} |a_{nk+c}| R^k \geq R^b$$

(since $a_{bn+c} = a_N = 1$). However, since $|a_{nk+c}| \leq 1$, we then have

$$\sum_{k=0}^{b-1} R^k \geq R^b$$
$$\frac{R^b - 1}{R - 1} \geq R^b$$
$$R^b - 1 \geq R^{b+1} - R^b$$
$$R^b(2 - R) \geq 1.$$

However, as $R \geq 2$, this is false.

Part II: Any polynomial with $r \le 1$ has a tasty multiple. Define the *sparsity* of a polynomial to be the greatest common divisor of the exponents m for which the coefficient of x^m in P is not zero. Equivalently, it is the largest integer d so that $P(x) = Q(x^d)$ for some polynomial Q.

We prove the following theorem:

Theorem. Given any complex number z for which $|z| \le 1$, there exist tasty polynomials with z as a root that have arbitrarily large sparsities.

Proof. Let $z = re^{i\theta}$. If θ is a rational multiple of π (say, $\theta = a\pi/b$), then we take the polynomial $x^{bn} - r^{bn}$ for any integer n; this has sparsity bn and is tasty (as $r \le 1$, $r^{bn} \le 1$). So, it suffices to prove this in the case where θ is not a rational multiple of π, and we henceforth assume this.

We claim that, for infinitely many n, the polynomial

$$x^{2n} - 2\cos(n\theta)\, r^n x^n + r^{2n}$$

is tasty (note that this polynomial has sparsity n and as such the theorem is implied by this claim). First note that this polynomial reduces to

$$x^n = r^n e^{\pm ni\theta} = \left(re^{\pm i\theta}\right)^n,$$

which is true at $x = re^{i\theta} = z$, so z is in fact a root.

We recall the following lemma:

Lemma. For any real number ϕ which is not a rational multiple of π, the sequence $a_n = \cos(n\phi)$ has infinitely many terms in the range $[-1/2, 1/2]$.

Indeed, let $\{x\}$ be the fractional part of x, and consider the sequence

$$\alpha_n = \left\{\frac{n\phi}{2\pi}\right\}.$$

We see that $-1/2 \le a_n \le 1/2$ iff $1/6 \le \alpha_n \le 1/3$ or $2/3 \le \alpha_n \le 5/6$. It is well known that the sequence $x_n = \{nx\}$ is dense in $[0, 1]$ for any irrational x, so this is true. Thus, for infinitely many n, as θ has been assumed not to be a rational multiple of π, the coefficients of P are bounded above in absolute value by r^n and r^{2n} for infinitely many n, both of which are ≤ 1 as $r \le 1$. $\square$

We now provide a second lemma.

Lemma. If $P(x)$ and $Q(x)$ are both tasty polynomials and the sparsity D of P is greater than the degree d of Q, then the product $R(x) = P(x)Q(x)$ is also tasty.

Proof. Write

$$P(x) = \sum_{j=0}^{s} a_j x^{Dj}, \qquad Q(x) = \sum_{k=0}^{d} b_k x^k.$$

Then,

$$P(x)Q(x) = \sum_{j=0}^{s}\sum_{k=0}^{d} a_j b_k x^{Dj+k}.$$

As $D > d$, none of these terms interfere with one another (for each integer n, there is at most one choice of $0 \le j \le s,\ 0 \le k \le d$ so that $Dj + k = s$), so the coefficients of $R(x)$ are just the values of $a_j b_k$ as j and k range over the desired range; as each a_j and b_k are of magnitude ≤ 1, each pairwise product is as well, finishing the proof. $\square$

Given a polynomial P with roots $\chi_1, \ldots, \chi_m$ in $\mathbb{C}$ (possibly with duplicates), we will inductively construct the polynomial $R(x)$ that is tasty and that P divides. We define a sequence of polynomials $R_0, \ldots, R_m$ so that $R_0(x) = 1$, and for each $0 < k \le m$, we take a tasty polynomial $P_k(x)$ with root χ_k and sparsity greater than the degree of R_{k-1}, and take $R_k(x) = R_{k-1}(x)P_k(x)$. Such a $P_k(x)$ is guaranteed to exist by our theorem, and the product $R_{k-1}(x)P_k(x)$ is guaranteed to be tasty by our lemma. Thus, we may take $R = R_m$, finishing the proof.

Remark. A polynomial P that has a tasty multiple exists for all $r < 2$: We have upon fixing $r < 2$ that for large enough n, we know $r^n - r^{n-1} - \cdots - r - 1 \le 0$. If n is minimal, $r^n - r^{n-1} - \cdots - r > 0$, and we can thus take some value $0 \le c \le 1$ for the constant term by the intermediate value theorem so that $P(x) = x^n - x^{n-1} - \cdots - x - c$ has a root at r. If $r \ge 2$, then $n = 1$ can be taken in Part 1 and thus no tasty multiples exist.

C1. Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1 \le i \le 2018$, there are exactly n cities that are a distance i away from C. (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

(Michael Ren)

The answer is n even.

To see that n odd fails, note that by taking $i = 1$ we see the graph is n-regular; since it has an odd number of vertices we need n to be even.

On the other hand, if n is even, then consider the graph formed by taking the vertices of a regular $(2018n + 1)$-gon and drawing edges between vertices which are at most $n/2$ apart. Then this works.

C2. We say that a positive integer n is *m-expressible* if one can write a expression evaluating to n in base 10, where the expression consists only of

- exactly m numbers from the set $\{0, 1, \dots, 9\}$
- the six operations $+$, $-$, $\times$, $\div$, exponentiation ^, concatenation $\oplus$, and
- some number (possibly zero) of left and right parentheses.

For example, 5625 is 3-expressible (in two ways), as $5625 = 5 \oplus (5\text{^}4) = (7 \oplus 5)\text{^}2$, say. Does there exist a positive integer A such that all positive integers with A digits are $(A-1)$-expressible?

(Krit Boonsiriseth)

Here is a solution by Evan Chen achieving $A = 6 \cdot 10^6$, and reprising the joke "six consecutive zeros".

We will replace "exactly m numbers" with "at most m numbers", since this is the same. Suppose we group the digits of N into base 1000000, so that we have

$$N = s_1 s_2 s_3 \dots s_m$$

where each s_m is a group of six digits (s_1 padded with leading zeros, if needed, but $s_1 \neq \overline{000000}$). We consider two cases.

- Suppose some group is zero; then we find that N has six consecutive zeros in its decimal representations. Thus N has the form

  $$N = X \oplus (b \cdot (1 \oplus 0)\text{^}6) \oplus Y$$

  for some strings X and Y (possibly empty), which are formed by repeated concatenation.

- Otherwise, note that $m \geq 10^6$. By a classical pigeonhole argument there exist indices $i < j$ such that $s_i + \dots + s_j \equiv 0 \pmod{999999}$. Let $n = \frac{1}{999999} s_i \dots s_j$. Then we can write

  $$N = X \oplus \left[((1 \oplus 0)\text{^}6 - 1) \cdot n\right] \oplus Y$$

  for strings $X = s_1 \dots s_i$ and $Y = s_{j+1} \dots s_n$.

Remark (Possible motivational remarks)**.** Ankan Bhattacharya says: I knew that the answer had to be yes — the obvious counting argument to show answer no doesn't work, and the given elements are unrelated enough that proving a no answer would be very difficult.

Evan says: I think you really do have to use exponentiation, since otherwise the numbers aren't big enough; but exponentiation is really painful to deal with, so I tried to find a way to use it only once. This is less daunting than it seems because you can concatenate digits "for free" from a size perspective; thus you just need a substring that you can "save space" on. After a bit of guesswork I came upon the idea of taking modulo $10^6 - 1 = 999999$ (which saves about two digits) and from there I had it.

Remark (Possible motivational remarks)**.** Ankan Bhattacharya points out that if we fix all $N-2$ operations, then there are only 10^{N-1} choices, compared to $9 \cdot 10^{N-1}$ numbers we need to obtain. Thus we need to use different operations to reach different numbers. This suggests that all solutions are likely to use some amount of casework.

Unlike Ankan, I did not find the case split to be a substantial part of the problem. It came up naturally because I had an edge case where six consecutive zeros might appear in my argument, and the first case was patch-only in that situation.

C3. A *windmill* in the plane consists of a line segment of unit length with a distinguished endpoint, the *pivot*. Geoff has a finite set of windmills, such that no two windmills intersect, and any two pivots are distance more than $\sqrt{2}$ apart. In an operation, Geoff can choose a windmill and rotate it about its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during or after the rotation. Show that Geoff can, in finitely many operations, rotate the windmills so that they all point in the same direction.

(Michael Ren)

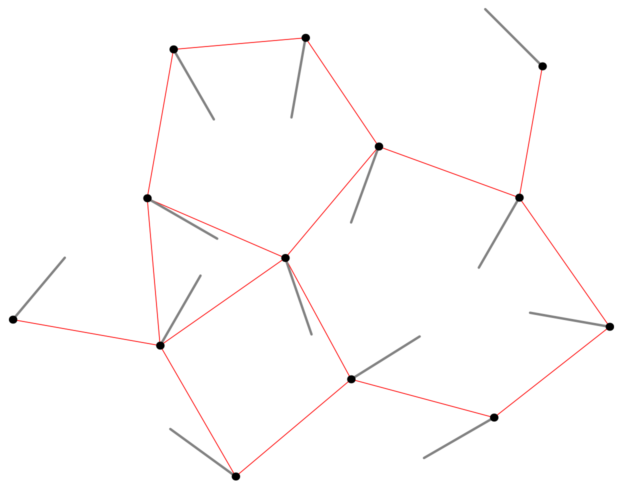
Throughout the solution we will general denote pivots by P, Q, R, ... and non-pivots by A, B, C, ...

We say that a configuration of windmills around S is *admissible* if no two windmills intersect. The problem is equivalent to showing one can reach any admissible configuration from any other (and the final position with the windmills pointing the same direction is just one example of a clearly admissible configuration).

Draw a red line segment between any two pivots which have distance at most 2 (thus these windmills could intersect). This naturally gives us a graph $\mathcal{G}$.

Lemma. For $c \geq \sqrt{2}$, the graph $\mathcal{G}$ is planar.

Proof. Indeed, if $\overline{PA}$ and $\overline{QB}$ intersect, we can consider convex quadrilateral $PQAB$, one of whose angles is at least 90°. WLOG it is $\angle PQA$, in which case $PA^2 \geq PQ^2 + QA^2 > 2 + 2 = 4$, so $\overline{PA}$ should not be red. $\square$



Clearly, we can ignore any isolated vertices. We can also ignore any leaves in $\mathcal{G}$; indeed suppose P is a pivot with $\overline{PQ}$ the only red edge. Then we can rotate the windmill at P to point away from Q and it will never obstruct other windmills since $c \geq 1$, so we can delete the pivot P from consideration (and use induction on the number of pivots, say).

Thus, we may assume $\mathcal{G}$ is a finite planar graph with no leaves. Thus it makes sense to speak of the faces of planar graph $\mathcal{G}$, consisting of several polygons.

Lemma. A windmill with pivot P can never intersect a red edge other than those touching P.

Proof. Suppose windmill $\overline{PA}$ intersects red edge $\overline{QR}$. Then the altitude from $\overline{PH}$ to $\overline{QR}$ has length at most 1. WLOG that $QH < RH$, so $QH < \frac{1}{2}QR = 1$. Then $PQ^2 < QH^2 + HP^2 < 1 + 1 = 2$, contradiction. $\square$

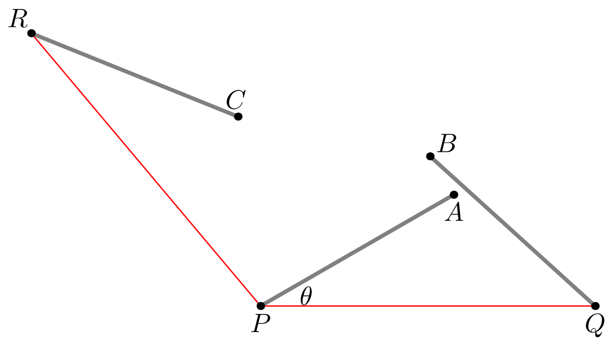
From now on, a windmill $\overline{PA}$ is said to *hug* a red edge $\overline{PQ}$ if the angle $\angle QPA < \varepsilon$ for some sufficiently small ε in terms of $\mathcal{G}$; each red edge $\overline{PQ}$ has at most two windmills hugging it (namely the windmills with pivots P and Q; if this happens, the windmills are on opposite sides of $\overline{PQ}$). Call a windmill configuration *cuddly* if every windmill is hugging an edge.

Claim. We can reach some cuddly configuration from any admissible one.

Proof. Indeed, consider a windmill $\overline{PA}$ not hugging any edge, and an edge $\overline{PQ}$, and such that $\angle APQ = \theta$ is minimal among all such pairs. Let $\angle RPQ$ be the corresponding angle of the face containing $\overline{PA}$, and let $\overline{QB}$, $\overline{RC}$ be windmills.

If $\overline{QB}$ is hugging $\overline{PQ}$, we perturb it slightly so that A and B are on opposite sides of $\overline{PQ}$; thus $\overline{QB}$ is no longer in the way.

We rotate $\overline{PA}$ towards $\overline{PQ}$ now. Because we assumed $\theta = \angle APQ$ was minimal, it is impossible for the body of the windmill to collide with the points B or C. So the only way it can be obstructed is if the point A collides with the interior of $\overline{QB}$ or $\overline{RC}$.

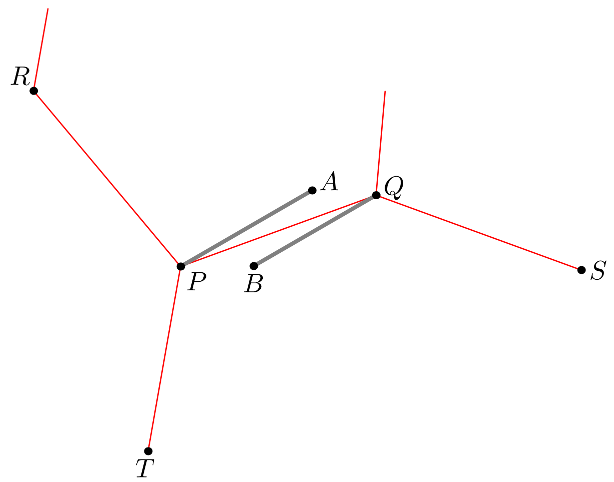


Suppose that A collided with $\overline{QB}$. At the moment of collision, we would have to have $\angle PAQ \le 90^\circ$. (This is because just before the collision $\overline{PA}$ was still disjoint from $\overline{QB}$, and if $\angle PAQ \ge 90^\circ$ just before then it would remain disjoint as $\overline{PA}$ rotated.) But then $PQ^2 \le PA^2 + AQ^2 \le 2$, contradiction. A similar proof works for $\overline{RC}$.

Thus we can rotate the windmills one by one so they hug the edges, as desired. $\square$

It remains to show any two cuddly configurations can be reached from each other. For this, we make two observations.

- Suppose $\overline{PA}$ and $\overline{QB}$ both hug $\overline{PQ}$. We show we can interchange the two. Assume $\angle RPQ$ is the angle of a face containing A, and $\angle TPQ$, $\angle PQS$ are the angles of the face containing B.



Rotate $\overline{PA}$ so it hugs $\overline{PR}$ (possibly perturbing the windmill at R), and then rotate $\overline{QB}$ so it hugs $\overline{QS}$ (possibly perturbing the windmill at S). Then rotate $\overline{PA}$ so it hugs $\overline{PT}$, then move $\overline{QB}$ back so it hugs $\overline{PQ}$ from the other side, and rotate $\overline{PA}$ back.

- Now suppose $\overline{PA}$ hugs $\overline{PQ}$, and $\angle RPQ$ is the angle of a face containing A. Then we can rotate it so that $\overline{PA}$ hugs $\overline{PR}$ (here $\overline{PA}$ could be blocked by $\overline{QB}$ initially, but then we perform the switching operation above).

Together these two observations finish the problem.

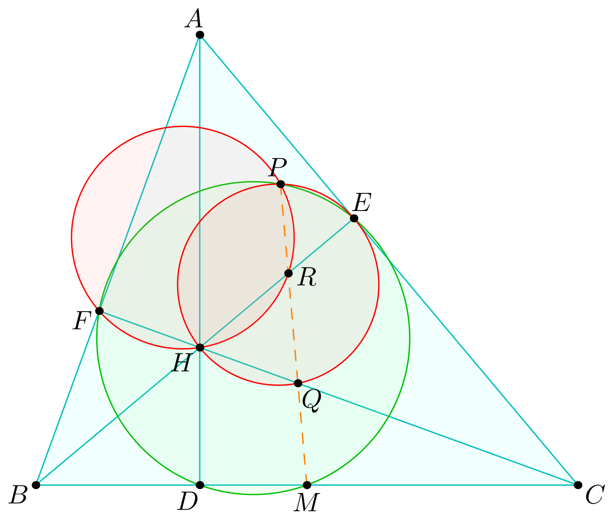
Remark (Michael Ren)**.** Here is a solution achieving just $c = \sqrt{3}$.

Draw a disk of radius $1+\epsilon$ around every point in S such that the distance between any two points in S is more than $\sqrt{3}(1+\epsilon)$ for some $\epsilon > 0$ that clearly exists. Note that no three disks can intersect. Indeed, if disks centered at A, B, and C intersected, then the circumradius of ABC is at most $1+\epsilon$, which means that some two of A, B, C are at most a distance of $\sqrt{3}(1+\epsilon)$ apart. In light of this, for any two points A and B in S that are a distance of at most 2 apart, draw a rhombus $APBQ$ of length $1+\epsilon$. By our work before, all such rhombi are distinct. Furthermore, windmill collisions only happen inside these rhombi by definition. Now, have Geoff move each of his windmills one by one to Sasha's windmills. If a windmill collision happens, have Geoff move the other windmill out of the way inside the rhombus before moving the windmill by and then restore the position of the other windmill. Hence, he can always get his windmills to coincide, as desired.

G1. Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles of $\triangle EHP$ and $\triangle FHP$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

(Brandon Wang)

Let D denote the foot of the A-altitude, and M the midpoint of $\overline{BC}$. We claim that R and Q both lie on line $\overline{PM}$. That will solve the problem (M is the fixed point).



By angle chasing, it is not hard to show that

$$\measuredangle FHE = \measuredangle FEM.$$

Now,

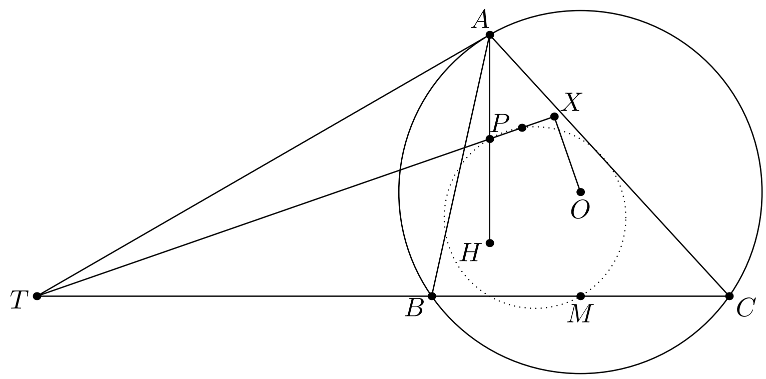
$$\measuredangle FPR = \measuredangle FHR = \measuredangle FHE = \measuredangle FEM = \measuredangle FPM$$

as desired so P, R, M are collinear. Similarly, P, Q, M are collinear, as desired.

G2. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle of $\triangle ABC$.

(Zack Chroman)

We present two solutions, one synthetic and by complex numbers.



First solution (Zack Chroman) Let M be the midpoint of $\overline{BC}$. Note that since $\overline{AP} \perp \overline{BC}$ and $\overline{AT} \perp \overline{AO} \parallel \overline{PM}$, we find that P is the orthocenter of $\triangle ATM$. Thus $Y = \overline{TP} \cap \overline{AM}$ satisfies $\angle PYM = 90$, so it lies on the 9-point circle.

It then suffices to note that the reflection X' of P over Y lies on the circumcircle of $(AMT) = (TO)$, so $\angle TX'O = 90 \implies X = X'$.

Second solution (complex numbers, Evan Chen) Let Q denote the reflection of P over M, the midpoint of $\overline{BC}$.

Claim. We have $\overline{QO} \perp \overline{PT}$.

Proof. By complex numbers. We have

$$\begin{aligned}
t &= \frac{aa(b+c) - bc(a+a)}{aa - bc} = \frac{a^2(b+c) - 2abc}{a^2 - bc} \\
t - p &= \frac{a^2(b+c) - 2abc}{a^2 - bc} - \left(a + \frac{b+c}{2}\right) \\
&= \frac{a^2(\frac{1}{2}b + \frac{1}{2}c - a) + (-a + \frac{1}{2}b + \frac{1}{2}c)bc}{a^2 - bc} \\
&= \frac{b+c-2a}{2} \cdot \frac{a^2+bc}{a^2-bc} \\
q &= 2 \cdot \frac{b+c}{2} - p = \frac{b+c-2a}{2}
\end{aligned}$$

Since $\frac{a^2+bc}{a^2-bc} \in i\mathbb{R}$, the claim is proven. $\square$

Thus, $\overline{QOX}$ are collinear. By considering right triangle $\triangle PQX$ with midpoint M, we conclude that $MX = MP$. Since the nine-point circle is the circle with diameter $\overline{PM}$, it passes through the midpoint of $\overline{PX}$.

G3. Let A be a point in the plane, and ℓ a line not passing through A. Evan doesn't have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

(Zack Chroman)

The trick is to invert the figure around a circle centered at A of arbitrary radius. We let $\omega = \ell^*$ denote the image of ℓ under this inversion. Then, under the inversion, Evan's compass has the following behavior:

- Evan can draw a line through two points other than A; or
- Evan can draw a circle through three points other than A.

In other words, the point A is "invisible" to Evan, but Evan otherwise has a straightedge and the same compass.

It is clear then that the answer to (ii) is no; since the point A is invisible it's impossible to construct any point depending on it.

Part (i) is equivalent to showing that Evan can construct the center of ω; we give one construction here anyways. Take any cyclic quadrilateral $WXYZ$ inscribed in ω, and let $P = \overline{WZ} \cap \overline{XY}$. Then the circumcircles of $\triangle PWX$ and $\triangle PYZ$ meet again at the Miquel point M, and the second intersection of (MXZ) and (MWY) is the center of ω.

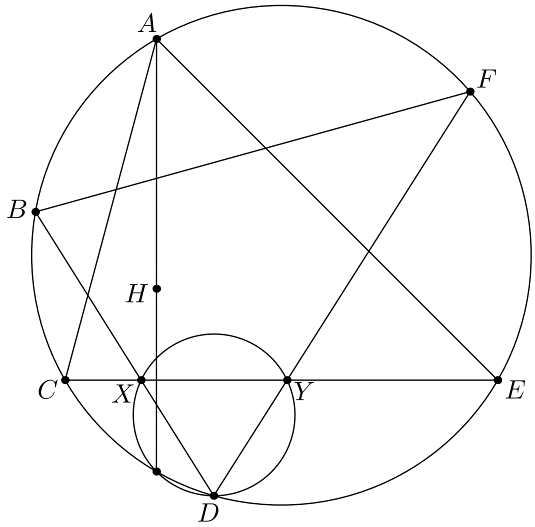
Remark. The proof of (ii) implies that it's actually more or less impossible in this context to construct any point other than the reflection of A, as a function of A and ℓ.

An alternative proof of (ii) is possible by inverting around a generic point P on ℓ with radius PA; this necessarily preserves the entire construction, but the foot from A to ℓ is not fixed by this inversion.

G4. Let $ABCDEF$ be a convex hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that $\overline{BD}$ and $\overline{DF}$ intersect $\overline{CE}$ at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to $\overline{CE}$.

(Michael Ren and Vincent Huang)

We present many, many solutions. In all of them, we let H denote the common orthocenter.



First solution by Simson lines (Vincent Huang) Let AH meet CE and Ω again at M and A_1, respectively, and P and Q be the projections of A_1 onto BD and DF, respectively. Note that PQ is the Simson line of A_1 with respect to BDF. It is well known that this Simson line bisects the segment between A_1 and H. Hence, M lies on PQ. But P, M, and Q are respectively the projections of A_1 onto DX, XY, and YD, so A_1 must lie on the circumcircle of DXY, as desired.

Second solution by dual Desargues involution (Michael Ren) Let O and r be the center and radius of Ω, respectively. Let $\mathcal{E}$ be the ellipse with foci O and H consisting of the set of points P such that $OP + HP = r$. Note that as the reflections of H over AC, CE, EA, BD, DF, FB lie on Ω, $\mathcal{E}$ is tangent to the sides of ACE and BDF. Let $\mathcal{E}$ and AD meet CE at P and Q, respectively. By the dual of Desargues involution theorem on quadrilateral $ACPE$ with inscribed conic $\mathcal{E}$, $D(CE; XY; PQ)$ is an involution. Hence, the circumcircles of DCE, DXY, and DPQ are coaxial, so it suffices to show that A_1DPQ is cyclic, where A_1 is the second intersection of AH and Ω. But note that A_1 lies on OP, so $\angle QDA_1 = \angle ADA_1 = \frac{\pi}{2} - \angle OA_1A = \frac{\pi}{2} - \angle PA_1A$, which is the angle between PA_1 and PQ by the perpendicularity of AA_1 and CE, as desired.

Third solution by angle chasing (Mihir Singhal) Let A_1 be the reflection of H over CE. Note A_1 is on Ω so it suffices to show that DA_1XY is cyclic. Let M be the foot of the altitude from A to $\overline{CE}$. Note that M is the midpoint of $\overline{HA_1}$ so since A_1 is on Ω, M must be on the nine-point circle of DBF. Let R and S be the feet of the altitudes from F and B in DBF.

Note $MXRH$ and $MYSH$ are cyclic. Moreover, M lies on the nine-point circle of $\triangle BDF$, and hence $\measuredangle SMR = 2\measuredangle SDR$. Then

$$\begin{aligned}\measuredangle XHY &= \measuredangle XHM + \measuredangle MHY \\ &= \measuredangle XRM + \measuredangle MSY = \measuredangle DRM + \measuredangle MSD \\ &= -(\measuredangle RMS + \measuredangle SDR) = \measuredangle SMR + \measuredangle RDS \\ &= 2\measuredangle SDR + \measuredangle RDS = \measuredangle SDR = \measuredangle YDX.\end{aligned}$$

Thus $\measuredangle XA_1Y = -\measuredangle XHY = \measuredangle RDS = \measuredangle XDY$, as needed.

Fourth solution by inversion (James Lin) Let K be the second intersection of Ω and the perpendicular from A to CE. We want to show $DKXY$ is cyclic. We invert about H. It's clear that now, $A'C'E'$ and $B'D'F'$ share the same circumcircle Ω' and incenter H. Note that K maps to the midpoint $M_{A'}$ of the arc $C'E'$ on Ω' not containing A'. Also note that X' is the intersection of circles $(HB'D')$ and $(HC'E')$, which are centered at midpoint $M_{F'}$ of the arc $B'D'$ on Ω' not containing F' and the midpoint $M_{D'}$ of the arc $B'F'$ on Ω' not containing D', respectively. Thus, X' is the reflection of H over $M_{A'}M_{F'}$. Similarly, Y' is the reflection of H over $M_{A'}M_{B'}$. Then, note that $M_{A'}X = M_{A'}H = M_{A'}Y$. Now we reformulate the problem by erasing A', C' and E', as the rest of the problem can be defined without them. The reformulated statement is that if we fix B, D, F, H and vary $M_{A'}$ along Ω', then $D'M_{A'}X'Y'$ is always cyclic.

We proceed with directed angles. Note that $\measuredangle X'D'M_{A'} = \measuredangle X'D'H + \measuredangle HD'M_{A'} = \measuredangle M_{A'}M_{F'}F + \measuredangle M_{D'}M_{F'}M_{A'} = \measuredangle M_{D'}M_{F'}F$. Similarly, $\measuredangle Y'D'M_{A'} = M_{D'}M_{B'}B = -\measuredangle M_{D'}M_{F'}F = -\measuredangle X'DM_{A'}$, so it follows that $M_{A'}$ lies on an angle bisector of $\measuredangle X'D'Y'$. Assume that $D'M_{A'}$ and $X'Y'$ are not perpendicular. Then from $M_{A'}X' = M_{A'}Y'$, it follows that $D'M_{A'}X'$ and $D'M_{A'}Y'$ have the same circumradius, and if they don't have the same circumcircle, then $D'M_{A'}$ and $X'Y'$ must be perpendicular, a contradiction. So $D'X'M_{A'}Y'$ is cyclic. Hf $D'M_{A'}$ and $X'Y'$ are perpendicular, then use the new problem formulation (without A, C and E and just varying $M_{A'}$) to move $M_{A'}$ by a miniscule amount. Then $D'M_{A'}$ and $X'Y'$ will not be perpendicular, so $D'X'M_{A'}Y'$ is cyclic both after and before moving $M_{A'}$ by continuity. We are done.

Fifth solution, by complex numbers (Carl Schildkraut) Let Ω be the unit circle, and let $A = a$, etc. We have that

$$c + e = h - a \implies \frac{c+e}{ce} = \bar{h} - \frac{1}{a} \implies ce = \frac{a(h-a)}{a\bar{h} - 1}.$$

Let T be the second intersection of the line through A perpendicular to CE and Ω. We see that

$$t = -\frac{ce}{a} = -\frac{h-a}{a\bar{h}-1}.$$

We endeavor to show that $DTXY$ is a cyclic quadrilateral. We have that

$$\begin{aligned}
x &= \frac{ce(b+d)-bd(c+e)}{ce-bd} \\
&= \frac{\frac{a(b+d)(h-a)}{a\bar{h}-1} - bd(h-a)}{\frac{a(h-a)}{a\bar{h}-1} - bd} \\
&= (h-a)\left(\frac{a(b+d)-bd(a\bar{h}-1)}{a(h-a)-bd(a\bar{h}-1)}\right) \\
&= (h-a)\left(\frac{ab+ad-ab-ad-\frac{abd}{f}+bd}{ab+ad+af-a^2-ab-ad-\frac{abd}{f}+bd}\right) \\
&= (h-a)\left(\frac{bd(f-a)}{(af+bd)(f-a)}\right) \\
&= \frac{bd(h-a)}{af+bd}.
\end{aligned}$$

Similarly

$$y = \frac{bf(h-a)}{ab+df}.$$

So, we want to show that

$$d, -\frac{h-a}{a\bar{h}-1}, \frac{bd(h-a)}{af+bd}, \frac{bf(h-a)}{ab+df}$$

are concyclic. This is equivalent to, dividing each by $h-a$ and reciprocating,

$$\frac{h-a}{d}, 1-a\bar{h}, 1+\frac{af}{bd}, 1+\frac{ab}{df}$$

being concyclic. This is equivalent to, subtracting 1 and multiplying by bdf,

$$bf(b+f-a), -a(bd+bf+df), ab^2, af^2$$

being concyclic. This is equivalent to, adding abf and dividing by $b+f$,

$$bf, -ad, ab, af$$

being concyclic. However, all of these points lie on the unit circle, finishing the proof.

Sixth solution by complex numbers (Evan Chen) As usual let Ω denote the unit circle. We immediately have

$$\begin{aligned}
c+e &= b+d+f-a \\
\text{and thus}\quad \frac{1}{c}+\frac{1}{e} = \frac{c+e}{ce} &= \frac{1}{b}+\frac{1}{d}+\frac{1}{f}-\frac{1}{a} \\
\implies ce &= \frac{b+f+d-a}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}}.
\end{aligned}$$

These two equations let us eliminate c and e, leaving only a, b, d, f.

Now consider the point $p = -\frac{ce}{a}$ on the circumcircle. We compute

$$\begin{aligned}
\frac{x-p}{b-p} &= \frac{x+\frac{ce}{a}}{b+\frac{ce}{a}} \\
&= \frac{\frac{bd(c+e)-ce(b+d)}{bd-ce}+\frac{ce}{a}}{b+\frac{ce}{a}} \\
&= \frac{abcd+abde-abce-adce+bdce-(ce)^2}{(ab+ce)(bd-ce)} \\
&= \frac{abcde(1/a+1/e+1/c-1/d-1/b)-(ce)^2}{(ab+ce)(bd-ce)} \\
&= \frac{abcde(1/f)-(ce)^2}{(ab+ce)(bd-ce)} = \frac{(ce)(abd-cef)}{f(ab+ce)(bd-ce)}
\end{aligned}$$

Now, we write

$$\begin{aligned}
ab+ce &= \frac{ab(\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a})+(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{ab(\frac{1}{d}+\frac{1}{f})+d+f}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} = \frac{\frac{1}{df}(d+f)(ab+df)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
bd-ce &= \frac{bd(\frac{1}{b}+\frac{1}{d}+\frac{1}{f}-\frac{1}{a})-(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{bd(\frac{1}{f}-\frac{1}{a})+(a-f)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} = \frac{\frac{1}{af}(a-f)(bd+af)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
abd-cef &= abd - \frac{f(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{abd(\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a})-f(b+f+d-a)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{(b+f)(\frac{abd}{bf}-f)+b(a-d)+f(a-d)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{(b+f)(\frac{ad}{f}-f+(a-d))}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}} \\
&= \frac{\frac{1}{f}(b+f)(a-f)(f+d)}{\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a}}.
\end{aligned}$$

Putting that all together gives

$$\frac{x-p}{b-p} = \frac{ce\cdot adf(b+f)(\frac{1}{b}+\frac{1}{f}+\frac{1}{d}-\frac{1}{a})}{(ab+df)(bd+af)}$$

which is symmetric in d and f, so the analogous calculation with $\frac{y-p}{f-p}$ yields the same result. Consequently, P is the center of the spiral similarity sending $\overline{YF}$ to $\overline{BX}$, as desired.

Remark. Philosophical point: it's necessary to use both $a+c+e=b+d+f$ and its conjugate, to capture two degrees of freedom.

Seventh solution, by inversion and moving points (Anant Mudgal, unedited) Let H be the common orthocenter. Pick any two vertices X, Y of either $\triangle ACE$ or $\triangle BDF$ and notice that $\triangle XYH$ has circumradius equal to the radius of Ω. Now invert at H. We obtain the following equivalent problem.

> Let $ABCDEF$ be a cyclic hexagon with $\triangle ACE$ and $\triangle BDF$ sharing a common incircle ω centered at point H. Let $\odot(HBD), \odot(HFD)$ meet $\odot(CHE)$ again at points X and Y respectively. Let M be the midpoint of arc CE not containing A. Then $\odot(DXY)$ passes through point M.

Let ω touch $\overline{CE}$ at point N and $L = \overline{AD} \cap \overline{CE}$. Let $P = \overline{DB} \cap \overline{CE}$ and $Q = \overline{DF} \cap \overline{CE}$. By Dual of Desragues Involution Theorem on circumscribed $ACEN$ and point D; we conclude $(\overline{DN}, \overline{DL}), (\overline{DC}, \overline{DE}), (\overline{DP}, \overline{DQ})$ are pairs of an involution. Notice that P has equal powers in $\odot(HBD), \odot(CHE)$ hence P lies on $\overline{XH}$. Similarly, Q lies on $\overline{YH}$.

Let $\overline{HN}, \overline{HL}$ meet $\odot(CHE)$ again at S, T. Project through H to conclude that $(C, E), (X, Y), (S, T)$ are pairs of an involution on the circle $\odot(CHE)$. Thus, we conclude that lines $\overline{CE}, \overline{XY}, \overline{ST}$ concur.

Claim. $\overline{CE}, \overline{ST}, \overline{DM}$ concur.

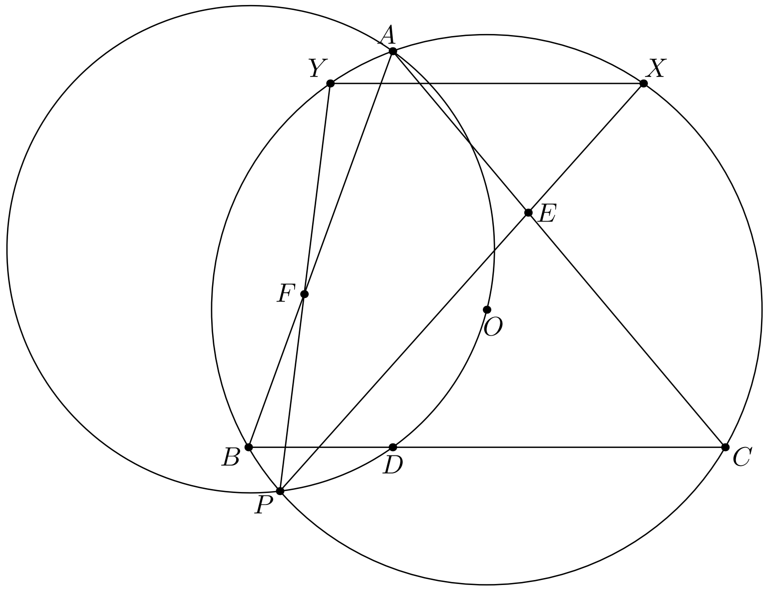
Proof. Animate D on $\odot(ACE)$; then $D \mapsto L \mapsto T$ is projective. Let $U = \overline{DM} \cap \overline{CE}$ and $V = \overline{ST} \cap \overline{CE}$ then $D \mapsto U$ and $D \mapsto V$ are also projective. Thus to show $W \overset{\text{def}}{:=} U \equiv V$ we need to verify for three choices of point D; namely we pick $\{C, E, M\}$. These are all clearly true and the lemma is proved. $\square$

Finally, notice $WX \cdot WY = WC \cdot WE = WD \cdot WM$ proving $DXYM$ is cyclic.

G5. Let scalene triangle ABC have altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines $\overline{PE}$ at $X \neq P$ and $\overline{PF}$ at $Y \neq P$. Prove that $\overline{XY} \parallel \overline{BC}$.

(Daniel Hu)

Denote by Ω and H the circumcircle and orthocenter of $\triangle ABC$. Let T lie on Ω such that $\overline{AT} \parallel \overline{BC}$. Let $\triangle ABC$ have orthocenter H.



First solution, synthetic First we prove a lemma.

Claim. The points H, P, T are collinear.

Proof. Let $\overline{HT}$ meet Ω at $P^* \neq T$. Let $\overline{AD}$ meet Ω at $K \neq A$. By homothety at K, $\overline{HT} \parallel \overline{DO}$. By angle chasing, $\angle P^*AD = \angle P^*AK = \angle P^*TK = \angle P^*TO = \angle OP^*T = \angle P^*OD$, so P^* lies on the circumcircle of $\triangle AOD$. Therefore, $P \equiv P^*$ as desired. $\square$

We now provide two finishes.

- First finish: By DDIT on $AEHF$, the pairs of lines $(\overline{PA}, \overline{PH}), (\overline{PB}, \overline{PC}), (\overline{PE}, \overline{PF})$ are part of a single involution, so $\overline{AT}, \overline{BC}, \overline{XY}$ are concurrent. Since $\overline{AT} \parallel \overline{BC}$, this implies that $\overline{XY} \parallel \overline{BC}$ as desired.

- Second finish: Let $Q = \overline{AP} \cap \overline{EF}$. By inversion at A, $BFPQ$, $CEPQ$, $DHPQ$ are all cyclic. By the lemma, this implies that $\angle ABC + \angle ACB = \angle APT = \angle APH = \angle QPH = \angle QDH = \angle QAH$, so $\overline{DQ} \perp \overline{EF}$.

  Let $G = \overline{EF} \cap \overline{BC}$; since $(G, D; B, C) = -1$, $\angle BQD = \angle DQC$. Thus $\angle BAY = \angle BPY = \angle BPF = \angle BQF = \angle CQE = \angle CPE = \angle CPX = \angle CAX$, so $\overline{XY} \parallel \overline{BC}$ as desired.

Second solution by complex numbers (Adam Ardeishar) Let ABC be the complex unit circle. Then $D = \frac{1}{2}(a+b+c-\frac{bc}{a})$, and we know

$$\frac{p-a}{p-o} \cdot \frac{d-o}{d-a} \in \mathbb{R}$$

$$\frac{p-a}{p} \cdot \frac{a+b+c-\frac{bc}{a}}{b+c-a-\frac{bc}{a}} = \frac{\frac{1}{p}-\frac{1}{a}}{\frac{1}{p}} \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}{-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}$$

$$\frac{1}{p} \cdot \frac{a+b+c-\frac{bc}{a}}{b+c-a-\frac{bc}{a}} = \frac{-1}{a} \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{a}{bc}}{\frac{1}{b}+\frac{1}{c}-\frac{1}{a}-\frac{a}{bc}}$$

$$\frac{-a}{p} \cdot \frac{a^2+ab+ac-bc}{ab+ac-a^2-bc} = \frac{bc+ab+ab-a^2}{ab+ac-bc-a^2}$$

$$p = a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc}$$

Now note that $p + x = e + px\bar{e}$, so $x = \frac{p-e}{p\bar{e}-1}$ But we compute that

$$p - e = a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc} - \frac{1}{2}(a+b+c-\frac{ac}{b})$$

$$= \frac{a^3b+a^3+2a^2b^2+a^2bc+ab^3+ab^2c+b^3c+b^2c^2-a^2c^2}{2b(a^2-ab-ac-bc)}$$

$$= \frac{(a+b)(b+c)(a^2+ab-ac+bc)}{2b(a^2-ab-ac-bc)}$$

And also compute

$$p\bar{e} - 1 = a \cdot \frac{a^2+ab+ac-bc}{a^2-ab-ac-bc} \cdot \frac{1}{2}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{b}{ac}) - 1$$

$$= \frac{a^3b+a^3c+a^2bc+a^2c^2+ab^2c+2abc^2+b^3c+b^2c^2-ab^3}{2bc(a^2-ab-ac-bc)}$$

$$= \frac{(a+b)(b+c)(a^2+ac+bc-ab)}{2bc(a^2-ab-ac-bc)}$$

So

$$x = \frac{\frac{(a+b)(b+c)(a^2+ab-ac+bc)}{2b(a^2-ab-ac-bc)}}{\frac{(a+b)(b+c)(a^2+ac+bc-ab)}{2bc(a^2-ab-ac-bc)}} = c \cdot \frac{a^2+ab+bc-ac}{a^2+ac+bc-ab}$$

By symmetry,

$$y = b \cdot \frac{a^2+ac+bc-ab}{a^2+ab+bc-ac}$$

Now note that $xy = bc$ to finish.

N1. Determine all nonempty finite sets $S = \{a_1, \dots, a_n\}$ of n distinct positive integers such that $a_1 \cdots a_n$ divides $(x + a_1) \cdots (x + a_n)$ for every positive integer x.

(Ankan Bhattacharya)

Answer: $\{a_1 \dots, a_n\} = \{1, \dots, n\}$. This works since

$$\frac{(x+n)\dots(x+1)}{n!} = \binom{x+n}{n} \in \mathbb{Z}$$

so we now show that it is the only possibility. There are two approaches.

First solution Let $P(x) = (x + a_1) \dots (x + a_n)$. Then, $a_1 \dots a_n$ should divide the nth finite difference of P, which is $n!$. But

$$a_1 \dots a_n \mid n! \implies \{a_1 \dots, a_n\} = \{1, \dots, n\}$$

for size reasons.

Second solution (Kevin Sun) Let $s + 1$ be the smallest positive integer not in our set A and denote $B = A \setminus \{1, \dots, s\}$.

It's clear that the divisibility holds for negative x as well. Set $x = -s - 1$ to obtain

$$\begin{aligned}
\mathbb{Z} \ni \frac{1}{a_1 \dots a_n} &\prod_{a \in A} (x + a) \\
&= \prod_{a \in A} \left(1 + \frac{x}{a}\right) \\
&= \prod_{a \in \{1,\dots,s\}} \left(1 - \frac{s+1}{a}\right) \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\
&= \prod_{a \in \{1,\dots,s\}} \left(\frac{a - (s+1)}{a}\right) \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\
&= \frac{(-s)(-(s-1))\dots(-1)}{1 \cdot 2 \cdot \dots \cdot s} \cdot \prod_{b \in B} \left(1 - \frac{s+1}{b}\right) \\
&= (-1)^{|A|} \prod_{b \in B} \left(1 - \frac{s+1}{b}\right).
\end{aligned}$$

If B is nonempty this has magnitude strictly between 0 and 1, (since $\min B > s + 1$ and thus each term is in $(0, 1)$). Thus B is empty and $A = \{1, \dots, s\}$.

N2. Call a number n *good* if it can be expressed in the form $2^x + y^2$ where x and y are nonnegative integers.

(a) Prove that there exist infinitely many sets of 4 consecutive good numbers.

(b) Find all sets of 5 consecutive good numbers.

(Michael Ma)

For (a), note that for any t, the numbers t^2+1, t^2+2, t^2+4 are good. So it suffices to show t^2+3 is good infinitely often, that is, $t^2+3 = 2^x + y^2$ has infinitely many nonnegative integer solutions (since for fixed t there are finitely many (x, y)). But this rearranges $t^2 - y^2 = 2^x - 3$ which has a solution for every x.

We now turn to the laborious task of (b), determining all sets of five consecutive good numbers. The answers are the six tuples $\{1,2,3,4,5\}$, $\{2,3,4,5,6\}$, $\{8,9,10,11,12\}$, $\{9,10,11,12,13\}$, $\{288,289,290,291,292\}$, $\{289,290,291,292,293\}$. These all work since

$$\begin{aligned}
1 &= 2^0+0^2, \quad 2 = 2^0+1^2, \quad 3 = 2^1+1^2, \\
4 &= 2^2+0^2, \quad 5 = 2^2+1^2, \quad 6 = 2^1+2^2, \\
8 &= 2^3+0^2, \quad 9 = 2^3+1^2, \quad 10 = 2^0+3^2, \\
11 &= 2^1+3^2, \quad 12 = 2^3+2^2, \quad 13 = 2^2+3^2, \\
288 &= 2^5+16^2, \quad 289 = 2^6+15^2, \quad 290 = 2^0+17^2, \\
291 &= 2^1+17^2, \quad 292 = 2^8+6^2, \quad 293 = 2^2+17^2.
\end{aligned}$$

We now show they are the only ones. First, consider the following table which shows $2^x + y^2 \pmod 8$:

	$x=0$	$x=1$	$x=2$	$x\geq 3$
$y \equiv 1 \pmod 2$	2	3	5	1
$y \equiv 0 \pmod 4$	1	2	4	0
$y \equiv 2 \pmod 4$	5	6	0	4

Note that from this table, no good number is $7 \pmod 8$. Thus any five good numbers must have a $3 \pmod 8$ number. By table can only occur if that good number is of the form $t^2+2^1 = t^2+2$ for an odd integer t.

We now have several cases.

Case 1: Suppose the five good numbers are $\{t^2+1, t^2+2, t^2+3, t^2+4, t^2+5\}$.

Note that $t^2+5 \equiv 6 \pmod 8$, and by table, this can only occur if $t^2+5 = s^2+2^2 = s^2+4$ for some integer s; hence $t^2 - s^2 = 1$, so $t=1$ and $s=0$. This gives the solution set $\{2,3,4,5,6\}$.

Case 2: Suppose the five good numbers are $\{t^2, t^2+1, t^2+2, t^2+3, t^2+4\}$.

Since t^2 is good, we have $t^2 = 2^w + z^2$ for some w and z, which we write as $(t-z)(t+z) = 2^w$.

We now split into cases.

- **Subcase 2.1**: We handle the situation where $w < 4$.
  - If $w=0$, then we get $t=1$, which gives the solution $\{1,2,3,4,5\}$.
  - If $w=1$, then there are no solutions by taking mod 4.

  - If $w = 2$, then $t^2 = 4 + z^2$ which implies $t = 2$, but t was odd.
  - If $w = 3$, we get $t^2 = 8 + z^2$ which implies $t = 3$, which gives $\{9, 10, 11, 12, 13\}$.
  - If $w = 4$, we get $t^2 = 16 + z^2$ which together with t odd implies $t = 5$, which gives $\{25, 26, 27, 28, 29\}$. However, the number 28 is not good, so this is not a solution.

- **Subcase 2.2**: Suppose $w \geq 5$. As $\gcd(t - z, t + z) \mid 2t$ we must have $t - z = 2$, $t + z = 2^{w-1}$, and thus $t = \frac{1}{2}\left(2 + 2^{w-1}\right) = 2^{w-2} + 1$. Since t was odd, we actually have $w \geq 3$.

  But $t^2 + 3$ is also good, so write
  $$t^2 + 3 = 2^x + y^2.$$
  So we split into cases again.

  - **Subcase 2.2.1**: We handle the case $x < 3$.
    * If $x = 0$, we get $t^2 + 2 = y^2$ which has no solutions.
    * If $x = 1$, we get $t^2 + 1 = y^2$ which implies $t = 0$, but t is supposed to be odd.
    * If $x = 2$, then we get $t^2 = y^2 + 1$ which implies $t = 1$, which was an earlier solution.
  - **Subcase 2.2.2**: Otherwise, assume $x \geq 3$.
    $$\begin{aligned} 2^x + y^2 &= t^2 + 3 \\ \implies 2^x + y^2 &= \left(2^{w-2} + 1\right)^2 + 3 \\ &= 2^{2w-4} + 2^{w-1} + 4 \\ \implies 2^{2w-6} + 2^{w-3} + 1 &= 2^{x-2} + (y/2)^2 \end{aligned}$$
    since y is clearly even; the last line implies $y/2$ is odd, since $2w - 6 > 0$, $w - 3 > 0$, $x - 2 > 0$.

    Let $c = w - 3 \geq 2$, $a = x - 2 \geq 1$, $b = y/2 \geq 1$ for brevity; then the equation rewrites as
    $$2^{2c} + 2^c + 1 = 2^a + b^2.$$
    We rewrite this as
    $$(2^c + 1 - b)(2^c + 1 + b) = (2^c + 1)^2 - b^2 = 2^a + 2^c \geq 0.$$
    In light of this, we have $2^a + 2^c \geq (2^c + 1)^2 - 2^{2c} > 2^{c+1}$, so $2^a > 2^c$, ergo $a > c$. Thus we may further write
    $$(2^c + 1 - b)(2^c + 1 + b) = 2^c(2^{a-c} + 1).$$
    The factors on the left-hand side are nonnegative and have gcd dividing $2b$, hence one of them has at most one factor of 2. So one of the factors must be divisible by 2^{c-1}. Thus, $b \equiv \pm 1 \pmod{2^{c-1}}$.

    But, $b < 2^c + 1$. So we have four possibilities:
    * **Subcase 2.2.2.1**: suppose $b = 1$. Then we get $2^{2c} + 2^c = 2^a$, which is impossible.

* **Subcase 2.2.2.2**: suppose $b = 2^{c-1} - 1$. Then we get $(2^{c-1} + 2)(2^c + 2^{c-1}) = 2^c(2^{a-c} + 1)$ and hence $3 \cdot 2^{c-2} = 2^{a-c} - 2$. This implies $a - c = 3$ and $c - 2 = 1$, so $c = 3$, or $w = 6$, hence $t = 2^{w-2} + 1 = 17$.

  This gives $\{289, 290, 291, 292, 293\}$ which indeed works.

* **Subcase 2.2.2.3**: suppose $b = 2^{c-1}+1$. Then we get $2^{c-1}(2^c+2^{c-1}+2) = 2^c(2^{a-c} + 1)$, or $2^{c-1} + 2^{c-2} + 1 = 2^{a-c} + 1$, which is impossible.

* **Subcase 2.2.2.4**: suppose $b = 2^c - 1$. This gives $2 \cdot 2^{c+1} = 2^c(2^{a-c} + 1)$, which is impossible.

Case 3: Suppose the five good numbers are $\{t^2 - 1, t^2, t^2 + 1, t^2 + 2, t^2 + 3\}$.
In that case, $\{t^2, t^2 + 1, t^2 + 2, t^2 + 3, t^2 + 4\}$ is also a set of five consecutive good numbers. Using case 2, the new candidate this now gives are $\{8, 9, 10, 11, 12\}$ and $\{288, 289, 290, 291, 292\}$, which work.

N3. Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. Find the maximum possible value of a_{2018}.

(Krit Boonsiriseth)

The answer is $a_{2018} \le 2^{1009} - 1$. To see this is attainable, consider the sequence

$$a_n = \begin{cases} 1 & n \text{ odd} \\ 2^{n/2} - 1 & n \text{ even.} \end{cases}$$

This can be checked to work, so we prove it's optimal.

We have $a_2 \mid a_1 + a_2 = 1 + a_2 \implies a_2 = 1$.

Now consider an integer n, and let $s = s_n = a_1 + \cdots + a_n$. Then

$$\begin{aligned} &a_{n+1} \mid s \\ &a_{n+2} \mid s + a_{n+1} \\ &a_{n+2} \equiv 1 \pmod{a_{n+1}}. \end{aligned}$$

Thus, $\gcd(a_{n+2}, a_{n+1}) = 1$. So $a_{n+2} \le \frac{s + a_{n+1}}{a_{n+1}}$, and thus

$$a_{n+1} + a_{n+2} \le 1 + a_{n+1} + \frac{s}{a_{n+2}} \le s + 2.$$

So, we have

$$\begin{aligned} a_1 + a_2 &= 2 \\ a_3 + a_4 &\le 2 + 2 = 4 \\ a_5 + a_6 &\le (2+4) + 2 = 8 \\ a_7 + a_8 &\le (2+4+8) + 2 = 16 \\ &\vdots \\ a_{2017} + a_{2018} &\le 2^{1009}. \end{aligned}$$

Thus $a_{2018} \le 2^{1009} - a_{2017} \le 2^{1009} - 1$.

Remark (Motivational notes)**.** It's very quick to notice $a_{n+1} \mid a_1 + \cdots + a_n$, which already means that given the first n terms of the sequence there are finitely many possibilities for the next one. Thus it's possible to play with "small cases" by drawing a large tree.

When doing so, one might hope that somehow $a_n = a_1 + \cdots + a_{n-1}$ is achievable, but quickly notices in such a tree that if a_n is the sum of all previous terms, then $a_{n+1} = 1$ is forced. This gives the idea to try to look at the terms in pairs, rather than one at a time, and this gives the correct bound.

As for extracting the equality case from this argument, there are actually two natural curves to try. We have $a_3 \mid 1 + 1 = 2$. If we have $a_3 = 2$ we get $a_4 = 1$, $a_5 \le 5$, but then a_6 actually gets stuck. But if we have $a_3 = 1$ instead, we get $a_4 = 3$, $a_5 = 1$, $a_6 = 7$, and so on; pushing this gives the equality case above, seen to work. I think it's quite unnatural to guess the correct construction before having the corresponding $s + 2$ estimate.

N4. Fix a positive integer $n > 1$. We say a nonempty subset S of $\{0, 1, \dots, n-1\}$ is d-coverable if there exists a polynomial P with integer coefficients and degree at most d, such that S is exactly the set of residues modulo n that P attains as it ranges over the integers.

For each n, determine the smallest d such that any nonempty subset of $\{0, \dots, n-1\}$ is d-coverable, or prove that no such d exists.

(Carl Schildkraut)

This is possible for $n = 4$ or n prime, in which case $d = n-1$ is best possible. Let $P(\mathbb{Z}/n)$ denote the range of a polynomial modulo n.

- We first note that if $n = q_1 \dots q_k$ is the product of $k \ge 2$ distinct prime powers, then
$$|P(\mathbb{Z}/n)| = \prod_{i=1}^{k} |P(\mathbb{Z}/q_i)|.$$
Hence any subset S with size $n-1$ is not coverable.

- If $n = p^e$ is a prime power with other than 4 with $e \ge 2$, consider the set $S = \{0, 1, \dots, p-1, p\}$. We claim it is not coverable.

  Indeed, if P covers it, WLOG $P(0) = 0$. Now, P is surjective modulo p, hence bijective, and thus $P(x) \equiv 0 \pmod p \iff x \equiv 0 \pmod p$. Now we can write
$$P(x) = a_1 x + a_2 x^2 + \dots.$$
  - If $a_1 \equiv 0 \pmod p$, then $x \equiv 0 \pmod p \implies P(x) \equiv 0 \pmod{p^2}$, so p does not appear in the image.
  - If $a_1 \not\equiv 0 \pmod p$, then p, $2p$, $\dots$ all appear in the image, which is wrong for $n > 4$.

- Let $n = 4$, and consider $S \pmod 4$.
  - If $S = \{k\}$ take $P(x) = k$.
  - If $S = \{k, k+1\}$ take $P(x) = x^2 + k$.
  - If $S = \{k, k+2\}$ take $P(x) = 2x^2 + k$.
  - If $S = \{k-1, k, k+1\}$ take $P(x) = x^3 + k$.

    We claim also the example $S = \{-1, 0, 1\}$ is not 2-coverable. Indeed, WLOG $P(0) = 0$ so $P(x) = x(x+c)$. Then $P(2) \equiv 0 \pmod 4$, meaning c is even. But then $P(1) \equiv c+1 \pmod 4$ and $P(-1) \equiv 1 - c \pmod 4$, so $P(1) \equiv P(-1)$.
  - If $S = \{0, 1, 2, 3\}$ take $P(x) = x$.

- Let $n = 2$.
  - If $S = \{k\}$ take $P(x) = k$.
  - If $S = \{0, 1\}$ take $P(x) = x$. This is obviously not 0-coverable.

- If $n = p$ is an odd prime, we claim $S = \{1, \dots, p-1\}$ is not $(p-2)$-coverable. Indeed, suppose $P(x) = a_{p-2}x^{p-2} + \cdots + a_0$ covered it. Then

$$\sum_x P(x) \equiv \sum_k a_k \sum_x x^k \equiv 0 \pmod{p}.$$

However, if $P(\mathbb{Z}/p) = \{1, \dots, p-1\}$ then some element appears twice and the others appear once. If k is the repeated element though, then $\sum_x P(x) = (1 + \cdots + (p-1)) + k \equiv k \not\equiv 0 \pmod{p}$.

Exclusively carL-Made Olympiad

Year: **2019**

21^{st} **ELMO**
PITTSBURGH, PA

Day: **1**

Saturday, June 8, 2019
1:15PM — 5:45PM

Problem 1. Let $P(x)$ be a polynomial with integer coefficients such that $P(0) = 1$, and let $c > 1$ be an integer. Define $x_0 = 0$ and $x_{i+1} = P(x_i)$ for all integers $i \geq 0$. Show that there are infinitely many positive integers n such that $\gcd(x_n, n + c) = 1$.

Problem 2. Let $m, n \geq 2$ be integers. Carl is given n marked points in the plane and wishes to mark their centroid*. He has no standard compass or straightedge. Instead, he has a device which, given marked points A and B, marks the $m - 1$ points that divides segment $\overline{AB}$ into m congruent parts (but does not draw the segment).

For which pairs (m, n) can Carl necessarily accomplish his task, regardless of which n points he is given?

Problem 3. Let $n \geq 3$ be a fixed integer. A game is played by n players sitting in a circle. Initially, each player draws three cards from a shuffled deck of $3n$ cards numbered $1, 2, \ldots, 3n$. Then, on each turn, every player simultaneously passes the smallest-numbered card in their hand one place clockwise and the largest-numbered card in their hand one place counterclockwise, while keeping the middle card.

Let T_r denote the configuration after r turns (so T_0 is the initial configuration). Show that T_r is eventually periodic with period n, and find the smallest integer m for which, regardless of the initial configuration, $T_m = T_{m+n}$.

*Here, the *centroid* of n points with coordinates $(x_1, y_1), \ldots, (x_n, y_n)$ is the point whose coordinates are $\left(\frac{x_1+\cdots+x_n}{n}, \frac{y_1+\cdots+y_n}{n}\right)$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Exclusively carL-Made Olympiad

Year: **2019**

21st ELMO
Pittsburgh, PA

Day: **2**

Sunday, June 16, 2019
1:15PM — 5:45PM

Problem 4. Carl is given three distinct non-parallel lines ℓ_1, ℓ_2, ℓ_3 and a circle ω in the plane. In addition to a normal straightedge, Carl has a special straightedge which, given a line ℓ and a point P, constructs a new line passing through P parallel to ℓ. (Carl does not have a compass.) Show that Carl can construct a triangle with circumcircle ω whose sides are parallel to ℓ_1, ℓ_2, ℓ_3 in some order.

Problem 5. Let S be a nonempty set of positive integers so that, for any (not necessarily distinct) integers a and b in S, the number $ab+1$ is also in S. Show that the set of primes that do not divide any element of S is finite.

Problem 6. Snorlax chooses a *functional expression*[†] E which is a finite nonempty string formed from a set $x_1, x_2, \ldots,$ of variables and applications of a function f, together with addition, subtraction, multiplication (but not division), and fixed real constants. He then considers the equation $E = 0$, and lets S denote the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the equation holds for any choices of real numbers $x_1, \ldots, x_k$. (For example, if Snorlax chooses the functional equation

$$f(2f(x_1)+x_2) - 2f(x_1) - x_2 = 0,$$

then S consists of one function, the identity function.)

(a) Let X denote the set of functions with domain $\mathbb{R}$ and image exactly $\mathbb{Z}$. Show that Snorlax can choose his functional equation such that S is nonempty but $S \subseteq X$.

(b) Can Snorlax choose his functional equation such that $|S| = 1$ and $S \subseteq X$?

[†]These can be defined formally in the following way: the set of functional expressions is the minimal one (by inclusion) such that (i) any fixed real constant is a functional expression, (ii) for any integer i, the variable x_i is a functional expression, and (iii) if V and W are functional expressions, then so are $f(V)$, $V+W$, $V-W$, and $V \cdot W$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Shortlisted Problems

21^{st} ELMO

Pittsburgh, PA, 2019

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until disclosed publicly by the committee on the ELMO.

Contributing Students

The Problem Selection Committee for ELMO 2019 thanks the following proposers for contributing **88 problems** to this year's Competition:

> Alex Xu, Andrew Gu, Ankit Bisain, Brandon Wang, Carl Schildkraut, Colin Tang, Daniel Hu, Daniel Zhu, Eric Gan, Ethan Joo, Evan Chen, Holden Mui, Jeffrey Kwan, Jirayus Jinapong, Kai Xiao, Kevin Ren, Luke Robitaille, Max Jiang, Michael Diao, Michael Ren, Mihir Singhal, Milan Haiman, Sean Li, Steven Liu, Swapnil Garg, Tristan Shin, Vincent Huang, Yunseo Choi, Zack Chroman

Problem Selection Committee

The Problem Selection Committee for ELMO 2019 consisted of:

- Adam Ardeishar
- Carl Schildkraut
- Colin Tang
- Kevin Liu
- Krit Boonsiriseth
- Vincent Huang

Contents

Problems

A1. Let a, b, c be positive reals such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Show that

$$a^abc+b^bca+c^cab\geq 27(ab+bc+ca).$$

(Milan Haiman)

A2. Find all functions $f\colon \mathbb{Z}\to\mathbb{Z}$ with the property that for any surjective function $g\colon \mathbb{Z}\to\mathbb{Z}$, the function $f+g$ is also surjective.

(Sean Li)

A3. Let $n\geq 3$ be a fixed positive integer. Evan has a convex n-gon in the plane and wishes to construct the centroid of its vertices. He has no standard ruler or compass, but he does have a device with which he can dissect the segment between two given points into m equal parts. For which m can Evan necessarily accomplish his task?

(Holden Mui and Carl Schildkraut)

A4. Find all nondecreasing functions $f\colon \mathbb{R}\to\mathbb{R}$ such that for all real numbers x, y,

$$f(f(x))+f(y)=f(x+f(y))+1.$$

(Carl Schildkraut)

A5. Define the set of functional expressions to be the smallest set of expressions so that the following properties hold:

- Any variable x_i, or any fixed real number, is a functional expression.
- Given any functional expression V, the expression $f(V)$ is a functional expression, and given any two functional expressions V, W, the expressions $V+W$ and $V\cdot W$ are functional expressions.

A functional equation is an equation of the form $V=0$ for any functional expression V; a function *satisfies* it if that equation holds for all choices of each x_i in the real numbers.

(For example, the equation $f(x_1)+f(x_2)+(-1)(x_1+x_2)=0$ is a functional equation satisfied by only the identity function, while the equation $f(x_1)+f(x_2)+(-1)f(x_1+x_2)=0$ is a functional equation satisfied by infinitely many functions. The equation $f(\frac{1}{1+x_1^2})=0$ is not a functional equation at all.)

Does there exist a functional equation satisfied by a exactly one function f, and the function f satisfies $f(\mathbb{R})=\mathbb{Z}$?

(Carl Schildkraut)

C1. Let $n \geq 3$ be fixed positive integer. Elmo is playing a game with his clone. Initially, $n \geq 3$ points are given on a circle. On a player's turn, that player must draw a triangle using three unused points as vertices, without creating any crossing edges. The first player who cannot move loses. If Elmo's clone goes first and players alternate turns, which player wins for each n?

(Milan Haiman)

C2. Adithya and Bill are playing a game on a connected graph with $n > 2$ vertices and m edges. First, Adithya labels two of the vertices A and B, so that A and B are distinct and non-adjacent, and announces his choice to Bill. Then Adithya starts on vertex A and Bill starts on B.

Now the game proceeds in a series of rounds in which both players move simultaneously. In each round, Bill must move to an adjacent vertex, while Adithya may either move to an adjacent vertex or stay at his current vertex. Adithya loses if he is ever on the same vertex as Bill, and wins if he reaches B alone. Adithya cannot see where Bill is, but Bill can see where Adithya is.

Given that Adithya has a winning strategy, what is the maximum possible value of m, in terms of n?

(Steven Liu)

C3. In the game of Ring Mafia, there are 2019 counters arranged in a circle, 673 of these which are mafia, and the remaining 1346 which are town. Two players, Tony and Madeline, take turns with Tony going first. Tony does not know which counters are mafia but Madeline does.

On Tony's turn, he selects any subset of the counters (possibly the empty set) and removes all counters in that set. On Madeline's turn, she selects a town counter which is adjacent to a mafia counter and removes it. (Whenever counters are removed, the remaining counters are brought closer together without changing their order so that they still form a circle.) The game ends when either all mafia counters have been removed, or all town counters have been removed.

Is there a strategy for Tony that guarantees, no matter where the mafia counters are placed and what Madeline does, that at least one town counter remains at the end of the game?

(Andrew Gu)

C4. Let $n \geq 3$ be a positive integer. In a game, n players sit in a circle in that order. Initially, a deck of $3n$ cards labeled $\{1, \ldots, 3n\}$ is shuffled and distributed among the players so that every player holds 3 cards in their hand. Then, every hour, each player simultaneously gives the smallest card in their hand to their left neighbor, and the largest card in their hand to their right neighbor. (Thus after each exchange, each player still has exactly 3 cards.)

Prove that each player's hand after the first $n-1$ exchanges is their same as their hand after the first $2n-1$ exchanges.

(Carl Schildkraut and Colin Tang)

C5. Given a permutation of $1, 2, 3, \ldots, n$, with consecutive elements a, b, c (in that order), we may perform either of the moves:

- If a is the median of a, b, and c, we may replace a, b, c with b, c, a (in that order).
- If c is the median of a, b, and c, we may replace a, b, c with c, a, b (in that order).

What is the least number of sets in a partition of all $n!$ permutations, such that any two permutations in the same set are obtainable from each other by a sequence of moves?

(Milan Haiman)

G1. Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

(Luke Robitaille)

G2. Snorlax is given three pairwise non-parallel lines ℓ_1, ℓ_2, ℓ_3 and a circle ω in the plane. In addition to a normal straightedge, Snorlax has a special straightedge which takes a line ℓ and a point P and constructs a new line ℓ' passing through P parallel to ℓ. Determine if it is always possible for Snorlax to construct a triangle XYZ such that the sides of $\triangle XYZ$ are parallel to ℓ_1, ℓ_2, ℓ_3 in some order, and X, Y, Z each lie on ω.

(Vincent Huang)

G3. Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O. The incircle touches sides BC, CA, and AB at D, E, and F respectively, and A' is the reflection of A over O. The circumcircles of ABC and $A'EF$ meet at G, and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF. Prove that if GH and EF meet at T, then $DT \perp EF$.

(Ankit Bisain)

G4. Let triangle ABC have altitudes $\overline{BE}$ and $\overline{CF}$ which meet at H. The reflection of A over BC is A'. The circumcircles of $\triangle AA'E$ and $\triangle AA'F$ meet the circumcircle of $\triangle ABC$ at $P \neq A$ and $Q \neq A$ respectively. Lines BC and PQ meet at R. Prove that $\overline{EF} \parallel \overline{HR}$.

(Daniel Hu)

G5. Given a triangle ABC for which $\angle BAC \neq 90°$, let B_1, C_1 be variable points on AB, AC, respectively. Let B_2, C_2 be the points on line BC such that a spiral similarity centered at A maps B_1C_1 to C_2B_2. Denote the circumcircle of AB_1C_1 by ω. Show that if B_1B_2 and C_1C_2 concur on ω at a point distinct from B_1 and C_1, then ω passes through a fixed point other than A.

(Maxwell Jiang)

G6. Let ABC be an acute scalene triangle and let P be a point in the plane. For any point $Q \neq A, B, C$, define T_A to be the unique point such that $\triangle T_ABP \sim \triangle T_AQC$ and $\triangle T_ABP, \triangle T_AQC$ are oriented in the same direction (clockwise or counterclockwise). Similarly define T_B, T_C.

(a) Find all P such that there exists a point Q with T_A, T_B, T_C all lying on the circumcircle of $\triangle ABC$. Call such a pair (P, Q) a *tasty pair* with respect to $\triangle ABC$.

(b) Keeping the notations from (a), determine if there exists a tasty pair which is also tasty with respect to $\triangle T_AT_BT_C$.

(Vincent Huang)

N1. Let P be a polynomial with integer coefficients so that $P(0) = 1$. Let $x_0 = 0$, and let $x_{i+1} = P(x_i)$ for all $i \geq 0$. Show that there are infinitely many positive integers n so that $\gcd(x_n, n + 2019) = 1$.

(Carl Schildkraut and Milan Haiman)

N2. Let $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be a function. Prove that the following two conditions are equivalent:

(i) $f(m) + n$ divides $f(n) + m$ for all positive integers $m \leq n$;

(ii) $f(m) + n$ divides $f(n) + m$ for all positive integers $m \geq n$.

(Carl Schildkraut)

N3. Let S be a nonempty set of integers so that, for any (not necessarily distinct) integers a and b in S, $ab + 1$ is also in S. Show that there are finitely many (possibly zero) primes which do not divide any element of S.

(Carl Schildkraut)

N4. A positive integer $b \geq 2$ and a sequence $a_0, a_1, a_2, \ldots$ of base-b digits $0 \leq a_i < b$ is given. It is known that $a_0 \neq 0$ and the sequence $\{a_i\}$ is eventually periodic but has infinitely many nonzero terms. Let S be the set of positive integers n so that the base-b number $(a_0a_1 \ldots a_n)_b$ is divisible by n. Given that S is infinite, show that there are infinitely many primes dividing at least one element of S.

(Carl Schildkraut and Holden Mui)

N5. Let m be a fixed even positive integer. Find all positive integers n for which there exists a bijection f from $\{1, \ldots, n\}$ to itself such that for all $x, y \in \{1, \ldots, n\}$ with $mx - y$ divisible by n, we also have

$$(n + 1) \mid f(x)^m - f(y).$$

(Milan Haiman and Carl Schildkraut)

Solutions

A1. Let a, b, c be positive reals such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Show that

$$a^abc+b^bca+c^cab \ge 27(ab+bc+ca).$$

(Milan Haiman)

We present two solutions.

First solution by Jensen (Ankan Bhattacharya) Applying the change of variable $(x,y,z)=(\frac{1}{a},\frac{1}{b},\frac{1}{c})$, we wish to prove that

$$x^{1-1/x}+y^{1-1/y}+z^{1-1/z} \ge 27$$

whenever $x,y,z>0$ and $x+y+z=1$.

We will prove that $f(x)=x^{1-1/x}$ is convex on $\mathbb{R}_{>0}$, which will establish the result. A calculation shows that

$$\begin{aligned} f'(x) &= x^{-1/x}\left(x^{-1}\log x+1-x^{-1}\right) \\ f''(x) &= x^{-1/x}\left(x^{-3}(\log x-1)^2+x^{-2}\right) \end{aligned}$$

which is positive.

Second solution (Jirayus Jinapong) Dividing both sides by abc, we wish to show $a^{a-1}+b^{b-1}+c^{c-1}\ge 27$. In fact, we prove the following stronger claim.

Claim — We have $a^{a-1}b^{b-1}c^{c-1}\ge 729$.

Proof. Note that $a,b,c>1$. By weighted AM-GM, we have

$$\frac{2}{a+b+c-3}=\sum_{\text{cyc}}\frac{a-1}{a+b+c-3}\cdot\frac{1}{a}\ge\prod_{\text{cyc}}\left(\frac{1}{a}\right)^{\frac{a-1}{a+b+c-3}}$$

Therefore, we have

$$a^{a-1}b^{b-1}c^{c-1}\ge\left(\frac{a+b+c+3}{2}\right)^{a+b+c-3}.$$

Since the given implies $a+b+c\ge\frac{9}{1/a+1/b+1/c}=9$, we get the result. $\square$

A2. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ with the property that for any surjective function $g\colon \mathbb{Z} \to \mathbb{Z}$, the function $f + g$ is also surjective.

(Sean Li)

Constant functions f work, so we prove that when f is nonconstant it is possible to find surjective g such that $f + g$ is never equal to zero.

Note that the problem remains the same if we replace the domains by a countable set $S = \{s_0, s_1, \dots\}$ with the order of the elements being irrelevant. So we will do so to ease notation.

We consider two cases.

- First, suppose that f has the form

$$\begin{aligned} f(s_0) &= t_0 \\ f(s_1) &= t_1 \\ &\vdots \\ f(s_n) &= t_n \\ f(s_{n+1}) &= c \\ f(s_{n+2}) &= c \\ f(s_{n+3}) &= c \\ &\vdots \end{aligned}$$

  where none of the t_i's equals zero. In other words, f is equal to some constant c for cofinitely many values. Since f is nonconstant, $n > 0$.

  Then it suffices to define g by letting $g(s_0) = -c$, and then picking $g(s_1)$, $g(s_2)$, $\dots$, $g(s_n)$ to be large positive integers exceeding $\max |t_i|$, and then picking $g(s_{n+1})$, $\dots$ to be the remaining unchosen integers in some order.

- Otherwise, we claim the following algorithm works: we define $g(s_n)$ inductively by letting it equal the smallest integer (in absolute value, say) which has not yet been chosen, and is also not equal to $-f(s_n)$.

  The resulting function $f+g$ avoids zero by definition; we just need it to be surjective, and this is true because for any constant c, there are infinitely many n for which $f(s_n) \neq -c$; so c will get chosen by the $(2c + 1)$st such n.

A3. Let $n \geq 3$ be a fixed positive integer. Evan has a convex n-gon in the plane and wishes to construct the centroid of its vertices. He has no standard ruler or compass, but he does have a device with which he can dissect the segment between two given points into m equal parts. For which m can Evan necessarily accomplish his task?

(Holden Mui and Carl Schildkraut)

The following solution was given by Ankan Bhattacharya. We ignore the hypothesis that the n vertices are convex. The given task is easily seen to be equivalent to the following one:

> Evan writes the n vectors $(n, 0, 0, \dots)$, $(0, n, 0, \dots)$, $\dots$, $(0, 0, \dots, n)$ on a board. For any two vectors $\mathbf{a}$ and $\mathbf{b}$ on the board, Evan may write the vector $\frac{k}{m}\mathbf{a} + \frac{\ell}{m}\mathbf{b}$ for any nonnegative integers k, ℓ summing to m. The goal is to write $(1, \dots, 1)$.

We claim that the answer is that Evan can succeed if and only if m is divisible by 2 and every prime dividing n.

Proof of necessity: It is clear that m must be divisible by every prime factor p of n, since otherwise entries on the board will always be zero modulo p.

Now suppose n is odd; we show $2 \mid m$ nonetheless. The initial given vectors are permutations of

$$(1, \underbrace{0, \dots, 0}_{n-1}) \pmod 2.$$

The desired vector then is $(1, \dots, 1) \pmod 2$. However, it is easy to see that no new vectors (modulo 2) can be added. Hence if n is odd then $2 \mid m$ as well.

Proof of sufficiency: It is enough to prove that if $n = 2p$ with p an odd prime, then $m = 2p$ is valid.

We say a *achievable multiset* S is one for which the elements are positive integers with sum $2p$ and Evan can achieve the vector whose nonzero entries coincide with that multiset. We start with $S = \{1, p-1, 1, p-1\}$ as an achievable multiset. Thereafter, note that the following operations preserve achievability:

(a) replace an even k with two copies of $\frac{k}{2}$,

(b) replace two different numbers k and ℓ of the same parity with two copies of $\frac{k+\ell}{2}$,

Note that every move decreases the sum of the squares (say), so consider an achievable multiset S at a situation when no more moves are possible. It must be constant then (as all numbers are odd and equal). Moreover all the entries are less than p. So we must have $S = \{1, 1, \dots, 1\}$ as needed.

A4. Find all nondecreasing functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y,

$$f(f(x)) + f(y) = f(x + f(y)) + 1.$$

(Carl Schildkraut)

Here is Ankan Bhattacharya's solution.

Part I: answers. For each positive integer n and real $0 \le \alpha < 1$, the functions

$$f^{-}_{n,\alpha}(x) = \tfrac{1}{n}\lfloor nx + \alpha \rfloor + 1 \quad \text{and} \quad f^{+}_{n,\alpha}(x) = \tfrac{1}{n}\lceil nx - \alpha \rceil + 1,$$

along with $f_0(x) = 1$ and $f_\infty(x) = x + 1$, are solutions (and they are all). The verification that they are valid solutions is left to the curious reader.

Part II: substitution. For the converse direction, it will be more helpful to work with the function $g(x) = f(x) - 1$, which is also nondecreasing.

Lemma

We have $g(0) = 0$, $g(x + 1) = g(x) + g(1)$, and

$$Q(x, y) : g(x + g(y)) = g(x) + g(y).$$

Proof. The original functional equation reads

$$P(x, y) : g\big(g(x) + 1\big) + g(y) = g\big(x + g(y) + 1\big).$$

- First of all, $P(0, 0)$ gives $g(0) = 0$.
- Next,

$$\begin{aligned} P(x, 0) &\implies g\big(g(x) + 1\big) = g(x + 1), \\ P(0, y) &\implies g(1) + g(y) = g\big(g(y) + 1\big), \end{aligned}$$

  and in particular $g(x + 1) = g(x) + g(1)$. As a corollary, g is idempotent: $g(g(x)) = g(x)$.

This simplifies $P(x, y)$ to the last part $Q(x, y)$ of the claim. $\square$

Part III: analysis. We are done playing around with expressions and are ready to do more serious analysis on g. If $g(1) = 0$ then clearly $g(n) = 0$ for every integer n so g is zero. Hence suppose $g(1) > 0$.

Claim — If g is not the identity function, then $g(1) = 1$.

Proof. Write $g(1) = c$. Now note $g(n) = cn$ for any positive integer n, and also $g(cn) = cn$ and $g\big(c(n \pm 1)\big) = c(n \pm 1)$. Hence $c(n \pm 1)$ never belong to the interval from n to cn, which forces $c = 1$ upon taking $n \to \infty$. $\square$

We now denote by $S = g(\mathbb{R})$ the image of g.

Claim — S is closed under subtraction.

Proof. Note $Q\big(x-g(y),y\big)$ gives $g(x)-g(y)=g\big(x-g(y)\big)$. $\square$

Thus we have two cases.

- If S is dense, then by $Q(0,y)$ the set of fixed points of g is dense, so g is identity.
- If S is not dense, then S (being closed under subtraction) must be of the form $\frac{1}{n}\mathbb{Z}$ for some positive integer n. As g must be non-decreasing, it follows that $g^{-1}(0)$ is a half-open interval of length $\frac{1}{n}$ containing 0, and the desired characterization follows.

A5. Define the set of functional expressions to be the smallest set of expressions so that the following properties hold:

- Any variable x_i, or any fixed real number, is a functional expression.
- Given any functional expression V, the expression $f(V)$ is a functional expression, and given any two functional expressions V, W, the expressions $V + W$ and $V \cdot W$ are functional expressions.

A functional equation is an equation of the form $V = 0$ for any functional expression V; a function *satisfies* it if that equation holds for all choices of each x_i in the real numbers.

(For example, the equation $f(x_1) + f(x_2) + (-1)(x_1 + x_2) = 0$ is a functional equation satisfied by only the identity function, while the equation $f(x_1)+f(x_2)+(-1)f(x_1+x_2) = 0$ is a functional equation satisfied by infinitely many functions. The equation $f(\frac{1}{1+x_1^2}) = 0$ is not a functional equation at all.)

Does there exist a functional equation satisfied by a exactly one function f, and the function f satisfies $f(\mathbb{R}) = \mathbb{Z}$?

(Carl Schildkraut)

Yes, such a functional equation does exist. Here is Ankan Bhattacharya's construction (one of many).

We consider the following sequence.

Claim — The sequence

$$a_n = \begin{cases} 0 & n < 0, \\ m & n = 2m-2,\ m \in \mathbb{Z}_{>0}, \\ -m & n = 2m-1,\ m \in \mathbb{Z}_{>0}, \end{cases}$$

is the unique $\mathbb{Z}$-indexed satisfying the five properties

- $a_n = 0$ for $n < 0$,
- $a_0 \in \{0, 1\}$,
- $a_{n+2} - 2a_n + a_{n-2} = 0$ for $n \geq 0$,
- $a_n - a_{n-2} \in \{\pm 1\}$ for $n \geq 0$,
- $a_n + a_{n+1} \in \{0, 1\}$ for $n \geq 0$.

Proof. Suppose $a_0 = 0$. Then $a_2 = 2a_0 - a_{-2} = 0$, but $a_2 - a_0 \notin \{-1, 1\}$, contradiction. Thus $a_0 = 1$. Now $a_{-2} = 0$ and $a_0 = 1$, so by an easy induction $\boxed{a_{2n-2} = n}$ for every nonnegative integer n. Now note $a_{2n-2} = n$, $a_{2n} = n+1$, and $a_{2n-2} + a_{2n-1}$ and $a_{2n-1} + a_{2n}$ are both in $\{0, 1\}$ for every $n \geq 0$, so $\boxed{a_{2n-1} = -n}$ for every $n \geq 0$. The end. □

Now we are ready to solve the problem. We claim that

$$\begin{aligned}0 &= f\big(-x_1^2-(x_1x_2-1)^2\big)^2\\&\quad+\left[\big(f(-x_1^2-(x_1x_2-1)^2+1)-\tfrac{1}{2}\big)^2-\tfrac{1}{4}\right]^2\\&\quad+\big[f(x_1^2+2)-2f(x_1^2)+f(x_1^2-2)\big]^2\\&\quad+\left[\big(f(x_1^2)-f(x_1^2-2)\big)^2-1\right]^2\\&\quad+\left[\big(f(x_1^2)+f(x_1^2+1)-\tfrac{1}{2}\big)^2-\tfrac{1}{4}\right]^2\end{aligned}$$

works. Unraveling the equation, we obtain the equivalent condition set

- $f(s)=0$ for $s<0$,
- $f(s)\in\{0,1\}$ for $s<1$,
- $f(s+2)-2f(s)+f(s-2)=0$ for $s\geq 0$,
- $f(s)-f(s-2)\in\{\pm 1\}$ for $s\geq 0$,
- $f(s)+f(s+1)\in\{0,1\}$ for $s\geq 0$.

This is equivalent to the sequence $\{f(n+\alpha)\}_{n\in\mathbb{Z}}$ satisfying the hypothesis of the claim for any $0\leq\alpha<1$. This solves the problem.

Remark. It's interesting how annoying the constraint about not allowing division is. With division permitted, the much simpler construction

$$0=\big[f((1+x^2)^{-1})-1\big]^2+[f(y+1)-f(y)-1]^2$$

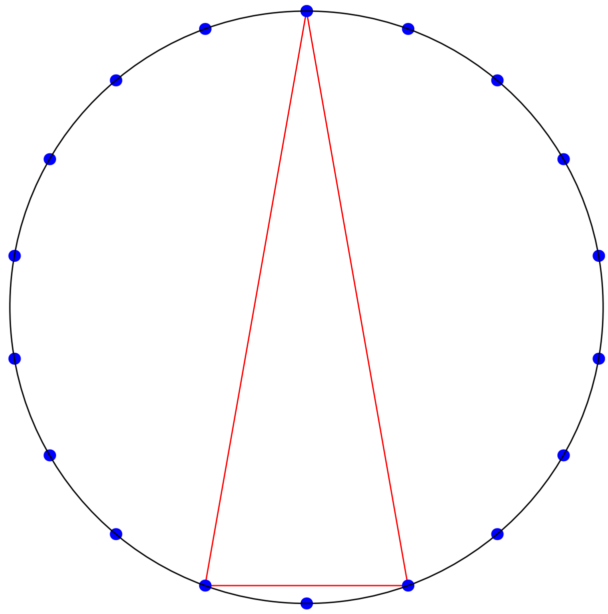
works nicely: the first term requires that $f(t)=1$ for $0<t\leq 1$ and the second one means $f(t+1)=f(t)+1$ for all t, ergo f is the ceiling function.

C1. Let $n \geq 3$ be fixed positive integer. Elmo is playing a game with his clone. Initially, $n \geq 3$ points are given on a circle. On a player's turn, that player must draw a triangle using three unused points as vertices, without creating any crossing edges. The first player who cannot move loses. If Elmo's clone goes first and players alternate turns, which player wins for each n?

(Milan Haiman)

The first player (Elmo's clone) always wins. Indeed it obviously wins for $n \leq 5$.

For $n \geq 6$, the strategy is to start by picking an isosceles triangle whose base cuts off either 0 or 1 points (according to whether n is odd or even, respectively).



Then do strategy stealing: each time the second player moves, the first player copies it.

C2. Adithya and Bill are playing a game on a connected graph with $n > 2$ vertices and m edges. First, Adithya labels two of the vertices A and B, so that A and B are distinct and non-adjacent, and announces his choice to Bill. Then Adithya starts on vertex A and Bill starts on B.

Now the game proceeds in a series of rounds in which both players move simultaneously. In each round, Bill must move to an adjacent vertex, while Adithya may either move to an adjacent vertex or stay at his current vertex. Adithya loses if he is ever on the same vertex as Bill, and wins if he reaches B alone. Adithya cannot see where Bill is, but Bill can see where Adithya is.

Given that Adithya has a winning strategy, what is the maximum possible value of m, in terms of n?

(Steven Liu)

The answer is $m = \binom{n-1}{2} + 1$. Here is the solution by Milan Haiman.

Construction: suppose G consists of an $(n-1)$-clique, two of the vertices which are labeled C and A, with one extra leaf attached to C, which we label B. Then, Adithya wins by starting at A and following the sequence $A \to A \to C \to B$.

Bound: The main lemma is the following.

Claim — If B is part of any triangle, then Adithya can't guarantee victory.

Proof. Bill can move among those three vertices and arrive back at B after k moves, for any $k \geq 2$. Moreover Adithya takes at least two moves to reach B. $\square$

So if Adithya is to win, we must have

$$m \leq \left(\binom{n-1}{2} - \binom{d}{2}\right) + d$$

where d is the degree of B, and this implies the result.

C3. In the game of Ring Mafia, there are 2019 counters arranged in a circle, 673 of these which are mafia, and the remaining 1346 which are town. Two players, Tony and Madeline, take turns with Tony going first. Tony does not know which counters are mafia but Madeline does.

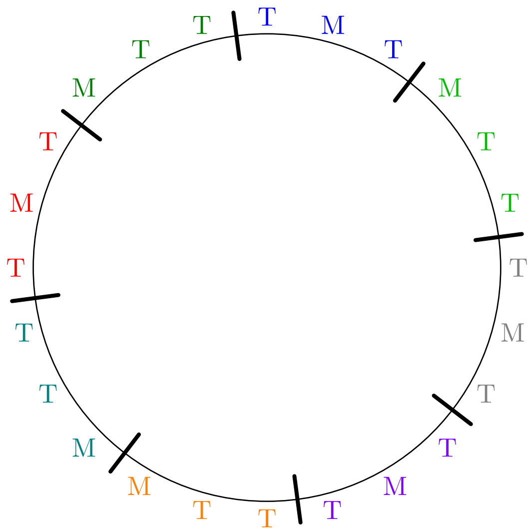
On Tony's turn, he selects any subset of the counters (possibly the empty set) and removes all counters in that set. On Madeline's turn, she selects a town counter which is adjacent to a mafia counter and removes it. (Whenever counters are removed, the remaining counters are brought closer together without changing their order so that they still form a circle.) The game ends when either all mafia counters have been removed, or all town counters have been removed.

Is there a strategy for Tony that guarantees, no matter where the mafia counters are placed and what Madeline does, that at least one town counter remains at the end of the game?

(Andrew Gu)

The answer is no. The following solution is due to Carl Schildkraut.

In fact, suppose we group the counters into 2019 blocks initially into 673 consecutive groups of 3 and it is declared publicly that there is exactly one Mafia token in each block.



Claim — At every step of the game, in every block with at least one token remaining, any token in that block could be Mafia. In other words, Tony cannot gain any information about any of the counters in a given block.

Proof. This is clearly true after any of Tony's moves, since within each block Tony has no information.

So we just have to verify it for Madeline. If the game is still ongoing, then there is some block with at least two tokens remaining. So:

- If there are only two tokens left, then they play symmetric roles; Madeline removes either one.

- If all three tokens remain, then since either the leftmost or rightmost counter could be Mafia, Madeline simply removes the middle counter.

This completes the proof. □

Therefore, it is impossible for Tony to guarantee that at least one town counter remains and no Mafia tokens remain, since any nonempty block could contain a Mafia token.

C4. Let $n \geq 3$ be a positive integer. In a game, n players sit in a circle in that order. Initially, a deck of $3n$ cards labeled $\{1, \dots, 3n\}$ is shuffled and distributed among the players so that every player holds 3 cards in their hand. Then, every hour, each player simultaneously gives the smallest card in their hand to their left neighbor, and the largest card in their hand to their right neighbor. (Thus after each exchange, each player still has exactly 3 cards.)

Prove that each player's hand after the first $n-1$ exchanges is their same as their hand after the first $2n-1$ exchanges.

(Carl Schildkraut and Colin Tang)

For now, we focus only on the behavior of the cards in $\{1, \dots, n\}$ and instead consider a modified game in which each player

- keeps their minimum card,
- passes their median card one right,
- passes their maximum card two right.

This is the same game up to rotating the names of players.

Claim (Trail of tokens) — For each $1 \leq r \leq n$, the card r stops moving after at most $r-1$ moves.

Proof. The trick is to treat the cards $\{1, \dots, r\}$ as *indistinguishable*: we call such cards *blue tokens*. We show all tokens stop moving after at most $r-1$ time. The main trick is the following:

> Whenever a player receives a token for the first time, (possibly before any moves, possibly more than one at once), we have them choose one of their tokens to turn grey, and have it never move afterwards.

Assume some token is still blue after h hours. If it moved from player 0 to player d, say (players numbered in order), then players 0, 1, …, d, each have a grey token. Thus $d+1 \leq r-1$, but the token advanced at least one player per hour, hence $d \geq h$, so $h \leq r-2$.

In other words, by time $r-1$ all tokens are grey. □

Remark. A similar proof shows that the card r travels a total distance at most $r-1$ too, by doing the same proof but without changing colors of tokens: if a token covers a total distance d, then players 0, 1, …, $d-1$ all have a token.

The condition that a player holds at most three tokens at once is not used at any point.

Back to main problem: Return to the original exchange rules. By the main claim, after $n-1$ hours all the cards $\{1, \dots, n\}$ are always passed left; in particular, they are in different hands, rotating. A similar claim holds for the large cards $\{2n+1, \dots, 3n\}$. Thus the cards $\{n+1, \dots, 2n\}$ are standing still. This implies the problem.

Remark. Despite how tempting it is to apply induction on r to try and prove the main claim, it seems that using indistinguishable tokens makes things much simpler. Part of the reason is because the same cards can meet twice: suppose some adjacent players have the hands

$$\{1, 3, 6\} \qquad \{7, 8, 9\} \qquad \{4, 9001, 9002\} \qquad \{5, 9003, 9004\} \qquad \ldots$$

Note the cards 8 and 9 meet again just a few hours later.

C5. Given a permutation of $1, 2, 3, \dots, n$, with consecutive elements a, b, c (in that order), we may perform either of the moves:

- If a is the median of a, b, and c, we may replace a, b, c with b, c, a (in that order).
- If c is the median of a, b, and c, we may replace a, b, c with c, a, b (in that order).

What is the least number of sets in a partition of all $n!$ permutations, such that any two permutations in the same set are obtainable from each other by a sequence of moves?

(Milan Haiman)

The number of equivalence classes turns out to be

$$n^2 - 3n + 4 = 2\left[\binom{n-1}{2} + 1\right].$$

First we show that at least $2\binom{n-1}{2} + 2$ sets are required.

Define the *disorder* of a permutation to be the number of pairs (i, j) such that $1 \le i < j \le n$ but i occurs after j in the permutation. We will also refer to these pairs as pairs that are out of order. We will refer to other such pairs with i occurring before j as in order.

Note that disorder is invariant under moves, as the only pairs whose relative orders change are the ones involved in the move. We can easily check that the number of pairs out of order does not change.

Consider the pair $(1, n)$. Notice that a move cannot change the relative order of this pair, as neither 1 nor n can be the median of three elements of a permutation of $1, 2, 3, \dots, n$.

Lemma 1

There exists a permutation of $1, 2, 3, \dots, n$ with disorder d, if $0 \le d \le \binom{n}{2}$.

Proof. Start with the identity permutation $1, 2, 3, \dots, n$, which has disorder 0. Now repeatedly swap two adjacent elements that are in order. We may do this until all adjacent elements are out of order, which occurs only with the reverse permutation $n, \dots, 3, 2, 1$. Notice that each swap increases the disorder by exactly 1, and that this process takes us from disorder 0 to disorder $\binom{n}{2}$. Thus disorder d must have been attained after exactly d moves. $\square$

Consider $\binom{n-1}{2} + 1$ permutations of $2, 3, 4, \dots, n$, with one of each disorder from 0 to $\binom{n-1}{2}$, by Lemma 1. Putting the element 1 at the beginning of each of these permutations gives $\binom{n-1}{2} + 1$ permutations of $1, 2, 3, \dots, n$ with distinct disorders. Now consider the reverses of each of these permutations. They will all have the pair $(1, n)$ out of order, and thus cannot be obtained from the original permutations by moves. Furthermore they all have distinct disorders, from $\binom{n}{2} - \binom{n-1}{2} = n - 1$ to $\binom{n}{2} - 0 = \binom{n}{2}$. Thus these $2\binom{n-1}{2} + 2$ permutations all cannot be obtained from each other by moves. This proves the lower bound.

Now we show that $2\binom{n-1}{2} + 2$ sets are attainable. We will categorize permutations into sets by their disorder and whether the pair $(1, n)$ is in order or not. Note that if $(1, n)$ is in order we must have a disorder of at most $\binom{n-1}{2}$, since at most one of the pairs

$(1, k)$ and (k, n) can be out of order for each $1 < k < n$. Similarly if $(1, n)$ is out of order we must have a disorder of at least $n - 1$. Thus we are using only $2\binom{n-1}{2} + 2$ sets. It remains to show that any two permutations in the same set are obtainable from each other by a sequence of moves.

Lemma 2

Given a permutation of $1, 2, 3, \ldots, n$ we can perform a sequence of moves to obtain a permutation with n either at the beginning or the end.

Proof. If n is at the beginning or end of the permutation we are done. Otherwise suppose that k and l are the two elements adjacent to n, in some order. Without loss of generality, $k < l < n$. Then l is the median of the three elements k, l, and n. So we may perform a move on these three elements (as n, not l, is the middle term).

We will repeat this process. As we do so, consider the ordered pair (x, y), where x is the minimum of the elements adjacent to n, and y is the number of elements on the other side of n from x. Note that if y ever reaches 0 then we are done.

We claim that this ordered pair is lexicographically monotonically decreasing. Suppose that this ordered pair is (x_0, y_0) before a move (as described above) and (x_1, y_1) after. Notice that the move will keep x_0 adjacent to n. Thus if $x_1 \neq x_0$ then $x_1 = \min(x_0, x_1) < x_0$. Now if $x_1 = x_0$ then one number has moved from the other side of n from x to the same side of n as x. In this case $y_1 = y_0 - 1 < y_0$. This proves our claim. Now note that, by the claim, we must eventually obtain an ordered pair (x, y) with $y = 0$, as desired. $\square$

Now we will show by induction on n that given two permutations σ and π of $1, 2, 3, \ldots, n$ with the same disorder and with $(1, n)$ in the same relative order, σ and π are obtainable from each other by a sequence of moves.

It is easy to check values of $n \leq 4$.

WLOG assume that both σ and π have $(1, n)$ in order. By Lemma 2 we can perform a sequence of moves to obtain n at the end of both permutations (it cannot be at the beginning since that would put $(1, n)$ out of order). Now no pairs with n are out of order in either permutation. Thus looking at only the first $n - 1$ terms of the permutations we see that they still have the same disorder. Then if the pair $(1, n - 1)$ has the same relative order in both permutations we are done by induction.

Now suppose $(1, n - 1)$ does not have the same relative order in both permutations. Consider the disorder d of both permutations. On one hand, since we have $(1, n - 1)$ in order $d \leq \binom{n-1}{2} - (n - 1)$. Similarly, since we have $(1, n - 1)$ out of order, $d \geq n - 1$.

Note that we can choose a permutation of $1, 2, 3, \ldots, n - 1$ with the last three terms being $x, 1, n - 1$ and having disorder d, such that $x \neq n - 2$. Similarly, we can choose a permutation with the last two terms being $n - 1, 1$ and having disorder d. By induction, we can perform a sequence of moves on the first $n - 1$ terms (leaving n in place) of σ and π to obtain two permutations of the form $\ldots, x, 1, n - 1, n$ and $\ldots, n - 1, 1, n$. It is sufficient to show that we can also perform a sequence of moves to obtain the latter from the former. We perform the following moves:

$$\ldots, x, 1, n - 1, n \to \ldots, 1, n - 1, x, n \to \ldots, 1, x, n, n - 1$$

Now by Lemma 2 on the first $n - 2$ terms we may perform a sequence of moves to move 1 to the beginning or to the end of the first $n - 2$ terms. Since $x \neq n - 2$, 1 must be at the end of the first $n - 2$ terms, otherwise the relative order of $(1, n - 2)$ would change. Thus we now have a permutation of the form $\ldots, 1, n, n - 1$ from which we obtain a permutation of the form $\ldots, n-1, 1, n$. Then applying induction again we obtain specifically the desired permutation of the form $\ldots, n-1, 1, n$.

> **Remark.** We can also prove Lemma 2 quite easily with induction. However the proof given more explicitly shows the actual moves we make. That is, we "attach" n to a "small" element and slide it around with that element until it hits an even "smaller" element repeatedly.

> **Remark.** Result is known: `https://arxiv.org/pdf/0706.2996.pdf`.

G1. Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

(Luke Robitaille)

Here are four solutions (unedited).

First solution (Maxwell Jiang) Let R be the midpoint of AH. As $HR \cdot HP = HB \cdot HE = \frac{1}{2}\text{Pow}(H)$ we have B, R, E, P cyclic. Now since $\angle ABQ = \angle RPE = \angle RBE$ we have R, Q isogonal wrt $\angle ABE$. But $AFHE$ is cyclic, and so since BR is a median of $\triangle BAH$ we have BQ is a median of similar $\triangle BEF$, as desired.

Second solution (Milan Haiman) Let X be the midpoint of EF and let Y be the midpoint of AH. Since $(AEHF)$, $\triangle ABH \sim \triangle EBF$.

Since BY and BX are medians, by this similarity we have $\angle YBH = \angle FBX$.

Let AH intersect BC at D. Note that $HY \cdot HP = HA \cdot HD = HE \cdot HB$ since $(AEDB)$. Thus $(YEPB)$.

Now we have $\angle ABX = \angle FBX = \angle YBH = \angle YBE = \angle YPE = \angle APE$.

Thus BX and PE intersect on Γ at Q.

Third solution (Carl Schildkraut) Let K be the point so that $(AK; BC) = -1$. It is well known that KP and EF intersect at some point R on BC. Now, apply Pascal's theorem on the cyclic hexagon $(KPQBCA)$. We see $KP \cap BC = R$, $PQ \cap AC = E$, so $BQ \cap AK$ lies on EF. However, as EF and BC are anti-parallel in $\angle BAC$, the A-symmedian in ΔABC is the A-median of ΔAEF, and as such $AK \cap EF$ is the midpoint of EF, which BQ thus passes through.

Fourth solution (Ankan Bhattacharya) Let M be the midpoint of $\overline{EF}$. Use complex numbers with Γ unit circle; it's easy to obtain $e = \frac{1}{2}(a + b + c - \frac{ac}{b})$, $p = -\frac{bc}{a}$, $m = \frac{1}{2}(a + b + c) - \frac{1}{4}a(\frac{b}{c} + \frac{c}{b})$.

To show that lines BM and PE meet on Γ, it suffices to prove

$$\measuredangle(\overline{BM}, \overline{PE}) = \measuredangle BAH \iff \tfrac{p-e}{b-m} \div \tfrac{h-a}{b-a} \in \mathbb{R}.$$

Computing the left fraction, we obtain

$$\begin{aligned}
\frac{p-e}{b-m} &= \frac{-\frac{bc}{a} - \frac{1}{2}(a + b + c - \frac{ac}{b})}{b - \frac{1}{2}(a + b + c) + \frac{1}{4}a(\frac{b}{c} + \frac{c}{b})} \\
&= \frac{-4b^2c^2 - 2(abc(a + b + c) - a^2c^2)}{4ab^2c - 2abc(a + b + c) + a^2(b^2 + c^2)} \\
&= -2 \cdot \frac{abc(a + b + c) + 2b^2c^2 - a^2c^2}{-2a^2bc + 2ab^2c - 2abc^2 + a^2(b^2 + c^2)} \\
&= -2 \cdot \frac{c(a + b)(ab - ac + 2bc)}{a(b - c)(ab - ac + 2bc)} \\
&= -2 \cdot \frac{c(a + b)}{a(b - c)}.
\end{aligned}$$

Thus
$$-\frac{1}{2}\cdot\frac{p-e}{b-m}\div\frac{h-a}{b-a}=\frac{c(a+b)(b-a)}{a(b-c)(b+c)}$$
and its conjugate equals
$$\frac{\frac{1}{c}\frac{a+b}{ab}\frac{a-b}{ab}}{\frac{1}{a}\frac{c-b}{bc}\frac{b+c}{bc}}$$
which is clearly the same.

G2. Snorlax is given three pairwise non-parallel lines ℓ_1, ℓ_2, ℓ_3 and a circle ω in the plane. In addition to a normal straightedge, Snorlax has a special straightedge which takes a line ℓ and a point P and constructs a new line ℓ' passing through P parallel to ℓ. Determine if it is always possible for Snorlax to construct a triangle XYZ such that the sides of $\triangle XYZ$ are parallel to ℓ_1, ℓ_2, ℓ_3 in some order, and X, Y, Z each lie on ω.

(Vincent Huang)

The answer is yes. Here are two solutions.

First solution (Maxwell Jiang) We proceed in three steps.

Claim — Snorlax can construct the center of ω.

Proof. Draw a chord and then two more parallel chords; intersecting diagonals of isosceles trapezoids gives us a line passing through the center, and repeating this gives us the center. □

Remark (Zack Chroman)**.** The *Poncelet-Steiner theorem* states that using a single circle with marked center and straightedge alone, one can do any usual straightedge-compass construction. Thus quoting this theorem would complete the problem.

Claim — We can construct the midpoint of any segment s.

Proof. All you have to do is construct a parallelogram! □

Claim — We can construct the perpendicular bisector of any segment s.

Proof. By above, we can construct the midpoint M of s. We can also construct a chord s' of ω parallel to s, and its midpoint M'. Then draw line $M'O$, and finally the line through M parallel to it. □

Suppose lines ℓ_1, ℓ_2, ℓ_3 determine a triangle $X'Y'Z'$ with circumcenter O' (which we can construct by the previous claim), while O is the center of ω (which we can construct by the first claim). We can draw radii of ω parallel to $X'O'$, $Y'O'$, $Z'O'$ and finish.

Second solution, unedited (Vincent Huang) Pick an arbitrary point $A \in \omega$. Draw $B, C, D \in \omega$ so that $BA \parallel l_1, CB \parallel l_2, DC \parallel l_3$.

Claim: The midpoint M of arc AD is fixed regardless of the choice of A.

Proof: Suppose we had some different starting point A', with corresponding B', C', D'. If A' is an arc measure θ clockwise of A, then B' is an arc measure θ counterclockwise of B, so C' is θ clockwise of C, so D' is θ counterclockwise of D. Thus it follows that arcs $A'A, D'D$ have equal measure and are in opposite directions, so the midpoints of arcs $AD, A'D'$ are the same.

Then if we choose X, Y, Z so that $XY \parallel l_1, YZ \parallel l_2, ZX \parallel l_3$, it follows from the above letting $A = X$ that the midpoint of arc XX should also be M, i.e. $M = X$. So it suffices to construct the midpoint M of arc AD, as it is a choice for a vertex of $\triangle XYZ$.

To do this we note for any A', D' that $ADD'A'$ is an isosceles trapezoid, so if $E = AA' \cap DD', F = AD' \cap A'D$ then EF is the perpendicular bisector of $AD, A'D'$, so intersecting EF with ω yields M as desired.

Third solution using projective geometry, unedited (Zack Chroman) The answer is yes. Work in $\mathbb{RP}^2$. We have a straightedge, a marked circle and the line at infinity. We will use the following theorem, known as the Circumcevian Ping-Pong Theorem. I can't find the reference for it, but it's real! Promise.

Theorem

Let ω be a circle, and let P, Q, R lie on a fixed line. Then there exists a fixed S on the same line such that, for any A on the circle, if

$$B = AP \cap \omega, C = QB \cap \omega, D = RC \cap \omega,$$

then A, D, S are collinear. Another way to phrase this is that a combination of any even number of "second-intersection maps" on a circle, all of which come from points on a fixed line, is the identity iff it's the identity at one point.

Proof. Let the line intersect the circle at I, J (if the line doesn't intersect the circle, work in $\mathbb{CP}^2$; it's clear that this theorem holds there iff it holds in $\mathbb{RP}^2$). Then define S so that, for a fixed choice A_0, B_0, C_0, D_0, S lies on A_0D_0. Then the combination of the projection maps through P, Q, R, S fix A_0, but also fix I, J. Moreover, they define a projective map on ω, which is therefore the identity.

One can also prove this similarly with the Desargues Involution Theorem, which directly gives an involution on line PQR that swaps P, R, I, J, and Q, S_0; so S_0 is fixed. $\square$

Now note that we have three points P_1, P_2, P_3 lying on the line at infinity and also l_1, l_2, l_3, respectively. Then by the theorem there exists a fourth point P_4 such that, for any X on the circle, projecting through P_1, P_2, P_3, P_4 in this order gives X again.

Then take a fixed A_0 on the circle, and define B_0, C_0, D_0 as these projections, so that A_0, D_0, P_4 are collinear. $A_0 = X$ will work if and only if $A_0 = D_0$. It follows that we want a point A_0 such that A_0P_4 is tangent to ω. We can construct this, but it's a little annoying. Our goal will be to construct the perpendicular bisector of A_0D_0; intersecting this with the circle will give an A_1 whose tangent passes through P_4, at which point we can take $A_1 = X$ and be done.

To do this, take an arbitrary point W in the plane, and $E \in A_0W$. Then let $F \in D_0W$ with $EF \parallel A_0D_0$. Quadrilateral A_0EFD_0 is a trapezoid, so if $G = A_0F \cap D_0E$, it follows that GW passes through the midpoint of A_0D_0. Now we can do the same process, but replacing A_0D_0 with another chord parallel to it, to get another midpoint; connecting these midpoints gives the perpedicular bisector.

Morally, we're trying here to construct the polar of P_4 with respect to ω, which is the locus of the midpoints of chords passing through P_4. Unfortunately, with only a straightedge constructing midpoints is as much work as constructing general harmonic conjugates, so we need to build the Cevalaus construction to do it.

G3. Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O. The incircle touches sides BC, CA, and AB at D, E, and F respectively, and A' is the reflection of A over O. The circumcircles of ABC and $A'EF$ meet at G, and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF. Prove that if GH and EF meet at T, then $DT \perp EF$.

(Ankit Bisain)

The following harmonic solution is given by Maxwell Jiang.

Define T instead as the foot to EF from D; we wish to show $T \in GH$. Let (AI) meet (ABC) a second time at a point T' so that I, T, T' are collinear, say by inversion about the incircle. By radical axis on $(AI), (ABC), (A'EFG)$ we get a point $X = AT' \cap EF \cap A'G$. Since $\angle XGA = \angle XMA = 90^\circ$, point X lies on (AMG).

Now note that

$$-1 = (A, I; E, F) \stackrel{T'}{=} (X, T; E, F),$$

so by properties of harmonic divisions we have $TM \cdot TX = TE \cdot TF$. This implies that T lies on the radical axis of (AMG) and $(A'EFG)$, as desired.

G4. Let triangle ABC have altitudes $\overline{BE}$ and $\overline{CF}$ which meet at H. The reflection of A over BC is A'. The circumcircles of $\triangle AA'E$ and $\triangle AA'F$ meet the circumcircle of $\triangle ABC$ at $P \neq A$ and $Q \neq A$ respectively. Lines BC and PQ meet at R. Prove that $\overline{EF} \parallel \overline{HR}$.

(Daniel Hu)

Solution by Maxwell Jiang (at least for ABC acute):

Let D be the foot from A. Let N be the midpoint of AH, and let $X = EF \cap BC$. Let (AH) meet (ABC) again at Y so that A, X, Y collinear and define $P' = EF \cap NC, Q' = EF \cap NB$. Finally, let AH hit EF at Z and (ABC) again at H'.

Note that $-1 = (X, D; B, C) \stackrel{N}{=} (X, Z; Q', P')$. Now consider an inversion centered at A with radius $\sqrt{AH \cdot AD}$, which swaps N, A' and P, P' and Q, Q' and X, Y and Z, H'. Since inversion preserves cross ratio we get $-1 = (Y, H'; Q, P)$, so the tangents to (ABC) at Y, H' meet on PQ. On the other hand, since $-1 = (Y, H'; B, C)$, these tangents also meet on BC. Thus the concurrency point is R.

To finish, note that since RH' is tangent to (ABC), by reflection about BC we have RH is tangent to (BHC). Then $\measuredangle RHB = \measuredangle BCH = \measuredangle FAH = \measuredangle FEH$ so $EF \parallel HR$, as desired.

G5. Given a triangle ABC for which $\angle BAC \neq 90^\circ$, let B_1, C_1 be variable points on AB, AC, respectively. Let B_2, C_2 be the points on line BC such that a spiral similarity centered at A maps B_1C_1 to C_2B_2. Denote the circumcircle of AB_1C_1 by ω. Show that if B_1B_2 and C_1C_2 concur on ω at a point distinct from B_1 and C_1, then ω passes through a fixed point other than A.

(Maxwell Jiang)

The following solutions are not edited.

First solution by quotation (Andrew Gu) Let $D = B_1B_2 \cap C_1C_2, E = B_1C_2 \cap B_2C_1$. Then D, E lie on ω. By Miquel, (C_1DB_2) and (B_1DC_2) concur on $K \in BC$. $\measuredangle B_2KC_1 = \measuredangle B_2DC_1 = \measuredangle B_1DC_1 = \measuredangle BAC$ so ABC_1K is cyclic, likewise AB_1CK is cyclic. This reduces it to ELMO SL 2013 G3.

Second solution by projetive geometry (Vincent Huang) Let $X = B_1B_2 \cap C_1C_2$. By spiral sim, $Y = C_2B_1 \cap B_2C_1$ lies on ω as well. By Brokard on B_1YC_1X, the tangents to ω at B_1, C_1 meet on BC. Now define $Z \neq C_1$ as $BC_1 \cap \omega$, and let $W = AZ \cap B_1C_1$. By Brokard again, BC is the polar of W, and we get that B_1, Z, C collinear.

Now let P be the spiral center sending $BC \mapsto C_1B_1$, so that $P \in \omega$ and $P \in (BZC)$. Note that

$$\angle PBC = \angle PC_1B_1 = \angle PAB, \text{ and } \angle PCB = \angle PB_1C_1 = \angle PAC.$$

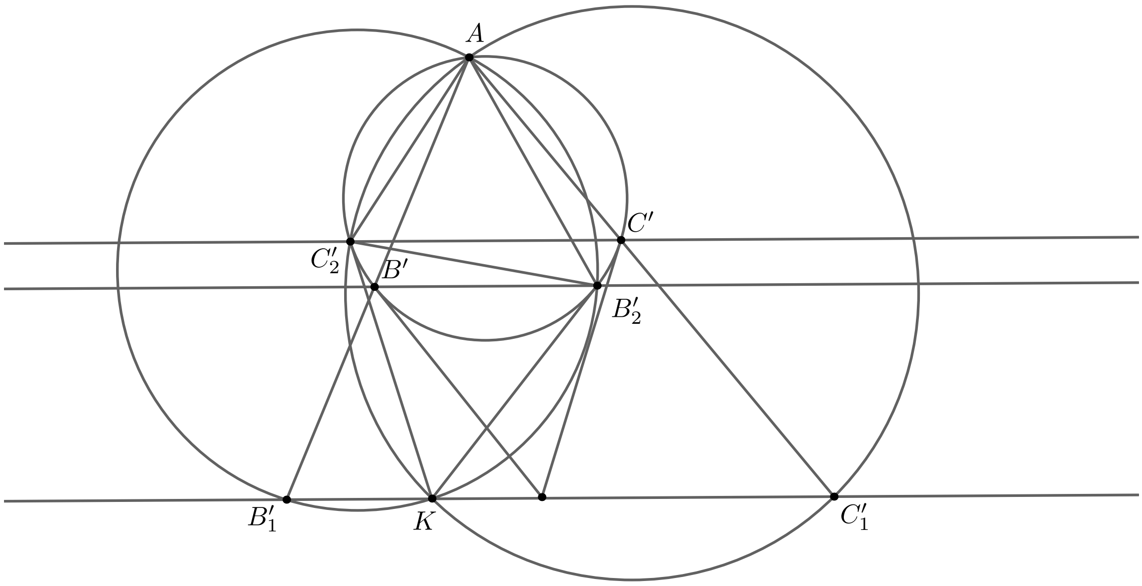
Hence (ABP) and (ACP) are tangent to BC, and P (the A-HM point) is the desired fixed point.

Third inversion solution (Maxwell Jiang) Let $S = B_1C_1 \cap CB$. By the spiral similarity, S lies on both (AB_1C_2) and (AC_1B_2).

Now invert about A with arbitrary radius, with X' denoting the image of X. So, BC maps to a circle Ω passing through A, and ω maps to a line ℓ. Note that $S' = \Omega \cap (AB_1'C_1')$. Hence, S' is the spiral center sending $C'C_1'$ to $B'B_1'$. Then, B_2, C_2 are the intersections of $S'B_1', S'C_1'$ with Ω. Now, $\angle S'B_2'B' = \angle S'C'B' = \angle S'C_1'B_1'$ and similar angle equalities for $\angle S'C_2'C'$ give

$$C'C_2' \parallel B'B_2' \parallel B_1'C_1'.$$

The given condition equates to $(AB_2'B_1')$ and $(AC_1'C_2')$ meeting at a point K on ℓ. Note that $\triangle AB_2'C_2' \sim \triangle AC_1'B_1'$. Since $\angle AB_2'K = 180^\circ - \angle B_1'$, we have $\angle C_2'B_2'K = 180^\circ - \angle B_1' - \angle C_1' = \angle B_2'AC_2'$, so KB_2' is tangent to Ω. Similarly, KC_2' is tangent to Ω.



Since the tangents to Ω at B_2', C_2' meet on ℓ, for symmetry reasons the tangents at B', C' also meet on ℓ. However, this point is fixed, so ℓ passes through a fixed point, as desired. $\square$

Note: To show that the fixed point is the HM point, instead of taking an inversion with arbitrary radius, take the one that swaps (AH) and BC. Then use the fact that the tangents to (AH) at the feet of the altitudes meet at the midpoint of BC, which is the inverse of the HM point.

G6. Let ABC be an acute scalene triangle and let P be a point in the plane. For any point $Q \neq A, B, C$, define T_A to be the unique point such that $\triangle T_ABP \sim \triangle T_AQC$ and $\triangle T_ABP, \triangle T_AQC$ are oriented in the same direction (clockwise or counterclockwise). Similarly define T_B, T_C.

(a) Find all P such that there exists a point Q with T_A, T_B, T_C all lying on the circumcircle of $\triangle ABC$. Call such a pair (P, Q) a *tasty pair* with respect to $\triangle ABC$.

(b) Keeping the notations from (a), determine if there exists a tasty pair which is also tasty with respect to $\triangle T_AT_BT_C$.

(Vincent Huang)

The following solution is by Andrew Gu:

For (a), the answer is all P which have an isogonal conjugate (that is, any point P not on the circumcircle or sides).

Let (P, Q) be a tasty pair. Then

$$\measuredangle BPC = \measuredangle BPT_A + \measuredangle T_APC = \measuredangle QCT_A + \measuredangle T_ABQ = \measuredangle BT_AC + \measuredangle CQB = \measuredangle BAC - \measuredangle BQC.$$

Cyclic variants hold, and these imply P and Q are isogonal conjugates.

Conversely, let P and Q be isogonal conjugates. The same steps as above (in a different order) show that $\measuredangle BAC = \measuredangle BT_AC$, so T_A is on (ABC), and likewise so are T_B and T_C.

For (b): Let $T_AT_BT_C$ be the reflection of ABC about O, the circumcenter. Consider the inconic with center O. Let P and Q be its foci.

N1. Let P be a polynomial with integer coefficients so that $P(0) = 1$. Let $x_0 = 0$, and let $x_{i+1} = P(x_i)$ for all $i \geq 0$. Show that there are infinitely many positive integers n so that $\gcd(x_n, n + 2019) = 1$.

(Carl Schildkraut and Milan Haiman)

We present a few solutions.

First solution by mod-preservation The "main" case is:

Claim — If there exists an index i for which $|x_{i+1} - x_i| > 1$ then we are done.

Proof. Let p be any prime dividing the difference and let $t = x_i$, so $P(t) \equiv t \pmod{p}$. We have $t \not\equiv 0$ since $P(0) \equiv 1 \pmod{p}$. Consequently, we get

$$0 \not\equiv x_i \equiv x_{i+1} \equiv x_{i+2} \equiv \dots \pmod{p}$$

and in this way we conclude taking $n = p^e - 2019$ for any exponent e large enough is okay. $\square$

So suppose that $x_{i+1} \in \{x_i - 1, x_i, x_i + 1\}$ for every i. Then either

- The sequence $(x_n)_n$ is periodic (with period at most 2), so the problem is easy; or
- we have $P(x) \equiv 1 \pm x$, which is also easy.

Second solution by orbits (by proposer) Let $p > 2019$ be any prime. We claim that $n = p$ should work, and in fact that we always have

$$x_p \not\equiv 0 \pmod{q}$$

for any $q \mid p + 2019$.

To see this, assume for contradiction $x_p \equiv 0 \pmod{q}$. Then $(x_n \bmod q)_n$ is periodic, with period dividing p. But the period should also be at most q, and not equal to 1 as $P(0) = 1$. As $q < p$, this is a contradiction.

N2. Let $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be a function. Prove that the following two conditions are equivalent:

(i) $f(m) + n$ divides $f(n) + m$ for all positive integers $m \le n$;

(ii) $f(m) + n$ divides $f(n) + m$ for all positive integers $m \ge n$.

(Carl Schildkraut)

We show that both are equivalent to $f(x) \equiv x + c$ for a non-negative integer c. The following solution is given by Maxwell Jiang.

Part (i): First suppose that $f(m) + n \mid f(n) + m$ holds for all $m \le n$. This implies $f(m) + n \le f(n) + m \implies f(m) - m \le f(n) - n$ for all $m \le n$. Define $g(n) = f(n) - n$ so that g is non-decreasing and the condition becomes

$$g(m) + m + n \mid g(n) + m + n \implies g(m) + m + n \mid g(n) - g(m).$$

Fix $m \in \mathbb{N}$ and consider

$$g(m) + m + n \mid g(n) - g(m)$$
$$g(m+1) + m + 1 + n \mid g(n) - g(m+1)$$

Let $d = g(m+1) - g(m) + 1$ be the difference between the left sides; note that $d > 0$. Pick large n so that d divides both left sides. If $d = 1$, then $g(m) = g(m+1)$. Else, we get

$$g(n) \equiv g(m) \equiv g(m+1) \pmod{d}$$

which is impossible. Hence $g(m) = g(m+1)$, which applied to all m gives g constant as needed.

Part (ii): Now suppose $f(m) + n \mid f(n) + m$ for all $m \ge n$. Fix $n = 1$ and let $m = p - f(1)$ for arbitrarily large primes p. Then we force $f(p - f(1)) + 1 = p$, so in particular we have infinitely many X such that $f(X) = X + c$ for a constant $c \ge 0$. Fixing m and setting $n = X$ gives

$$f(m) + X \mid X + c + m \implies f(m) + X \mid c + m - f(m)$$

so as X grows big we force $f(m) = m + c$, as desired.

> **Remark.** Note that (i) and (ii) together imply that $f(x) + y$ and $f(y) + x$ divide each other, hence are equal, so $f(x) + y = f(y) + x \iff f(x) \equiv x + c$. So it is unsurprising that we are just solving two functional equations.

N3. Let S be a nonempty set of integers so that, for any (not necessarily distinct) integers a and b in S, $ab+1$ is also in S. Show that there are finitely many (possibly zero) primes which do not divide any element of S.

(Carl Schildkraut)

The following solution is due to Ankan Bhattacharya. It's enough to work modulo p:

Claim — Let p be a prime and let G be a nonempty subset of $\mathbb{F}_p$ such that if $a, b \in G$, then $ab+1 \in G$. Then either G is a singleton, or $G = \mathbb{F}_p$.

Proof. If $0 \in G$ then $1 \in G$ and we get $G = \mathbb{F}_p$. So suppose $0 \notin G$, and let $G = \{x_1, \dots, x_n\}$ be its n distinct elements.

If $1 < n < p$, then we get a map

$$G \to G \qquad \text{by} \qquad g \mapsto x_1 g + 1 \pmod p$$

which is injective (since $x_1 \neq 0$), hence bijective. Thus, if we sum, letting $s = x_1 + \cdots + x_n$, we find

$$s = x_1 \cdot s + n \pmod p.$$

If $s \equiv 0$, we get $n \equiv 0 \pmod p$, contradiction. Otherwise, we find $x_1 = 1 - n/s \pmod p$.

But then the same logic shows $x_2 = \frac{n}{1-s}$, so $x_1 = x_2$, contradiction. □

The problem now follows since if s is any element of S and p is any prime not dividing $s^2 - s + 1$, then S contains all residues modulo p.

N4. A positive integer $b \geq 2$ and a sequence $a_0, a_1, a_2, \ldots$ of base-b digits $0 \leq a_i < b$ is given. It is known that $a_0 \neq 0$ and the sequence $\{a_i\}$ is eventually periodic but has infinitely many nonzero terms. Let S be the set of positive integers n so that the base-b number $(a_0 a_1 \ldots a_n)_b$ is divisible by n. Given that S is infinite, show that there are infinitely many primes dividing at least one element of S.

(Carl Schildkraut and Holden Mui)

Let $\gcd(x, y) = 1$ so that

$$\frac{x}{y} = \sum_{i=0}^{\infty} \frac{a_i}{b^i}.$$

Note that

$$(a_0 \cdots a_n)_b = \left\lfloor \frac{xb^n}{y} \right\rfloor,$$

unless $\{a_i\}$ is eventually $b-1$; either way, we have

$$(a_0 \cdots a_n)_b = \frac{xb^n - t}{y}$$

for some $0 \leq t < y$. If S is infinite, then there exists some fixed $0 \leq t < y$ so that the set of integers n so that

$$xb^n \equiv t \pmod{yn}$$

is infinite. Call this set S'. We see that $t \neq 0$ (infinitely many nonzero terms condition; this condition is essential) and $x > y > t$ (from the $a_0 \neq 0$ condition).

Our main claim is that for any prime p, the set $\{\nu_p(n) | n \in S'\}$ is bounded above. This is clear for $p|b$, wherein the above cannot hold if $n > \nu_p(t)$. Now, assume $n = mp^k \in S'$ for some m, k. We have

$$xb^{mp^k} \equiv t \bmod p^k \implies x^{p-1} b^{mp^k(p-1)} \equiv t^{p-1} \bmod p^k.$$

As $\gcd(p, b) = 1$, $b^{p^{k-1}(p-1)} \equiv 1 \bmod p^k$, so that term disappears, and we thus have

$$x^{p-1} \equiv t^{p-1} \bmod p^k.$$

As $x > t$, this cannot hold for large enough k, finishing the proof.

N5. Let m be a fixed even positive integer. Find all positive integers n for which there exists a bijection f from $\{1, \ldots, n\}$ to itself such that for all $x, y \in \{1, \ldots, n\}$ with $mx - y$ divisible by n, we also have

$$(n+1) \mid f(x)^m - f(y).$$

(Milan Haiman and Carl Schildkraut)

Solution by Andrew Gu (unedited):

We are asking for m, n such that the mapping $x \mapsto mx$ on $\mathbb{Z}/n\mathbb{Z}$ and the mapping $x \mapsto x^m$ on $U = (\mathbb{Z}/(n+1)\mathbb{Z}) \setminus \{0\}$ are isomorphic (behave in the same way by relabeling elements).

First we claim $n+1$ is a product of distinct primes. Otherwise, there exists $x \in U$ for which $x^m \equiv 0 \pmod{n+1}$.

Next note that the mapping $x \mapsto mx$ on $\mathbb{Z}/n\mathbb{Z}$ is a $\gcd(m, n)$-to-1 mapping, so any element in the range has a preimage of size $\gcd(m, n)$. We claim this is impossible for the second mapping if $n+1 = p_1 \cdots p_k$ is a product of k distinct primes, two of which are odd. WLOG let p_1, p_2 be odd. The preimage of the element which is $1 \pmod{p_1}, 1 \pmod{p_2}$, and 0 modulo all other p_i has size $\gcd(p_1 - 1, m)\gcd(p_2 - 1, m)$ while the preimage of the element which is $1 \pmod{p_1}$ and 0 modulo all other p_i has size $\gcd(p_1 - 1, m)$. As $\gcd(p_i - 1, m) \geq 2$, these preimages have different sizes.

The remaining cases are $n+1 = 2p$ or $n+1 = p$ where p is prime. In the case where $n+1 = 2p$, note that $p > 2$ as we showed that $n+1$ is a product of distinct primes. Then $p \in U$ is an mth power of one element (itself) while $p+1$ is an mth power of both $p-1$ and $p+1$. Hence this case fails.

In the case where $n+1$ is prime, let $f(x) = g^x$ for a primitive root g. This works, so for any m, there exists f if and only if $n+1$ is prime.

ELMO Literally Moved Online

Year: 2020

Day: 1

22^{nd} ELMO
127.0.0.1

Monday, July 20, 2020
2:00PM — 6:30PM EDT

Problem 1. Let $\mathbb{N}$ be the set of all positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that*

$$f^{f^{f(x)}(y)}(z) = x + y + z + 1$$

for all $x, y, z \in \mathbb{N}$.

Problem 2. Define the Fibonacci numbers by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Let k be a positive integer. Suppose that for every positive integer m there exists a positive integer n such that $m \mid F_n - k$. Must k be a Fibonacci number?

Problem 3. Janabel has a device that, when given two distinct points U and V in the plane, draws the perpendicular bisector of UV. Show that if three lines forming a triangle are drawn, Janabel can mark the orthocenter of the triangle using this device, a pencil, and no other tools.

*Here, $f^a(b)$ denotes the result of a repeated applications of f to b. Formally, we define $f^1(b) = f(b)$, and $f^{a+1}(b) = f(f^a(b))$ for all $a > 0$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

ELMO Literally Moved Online

Year: 2020

22^{nd} ELMO
127.0.0.1

Day: 2

Tuesday, July 21, 2020
2:00PM — 6:30PM EDT

Problem 4. Let acute scalene triangle ABC have orthocenter H and altitude AD with D on side BC. Let M be the midpoint of side BC, and let D' be the reflection of D over M. Let P be a point on line $D'H$ such that lines AP and BC are parallel, and let the circumcircles of $\triangle AHP$ and $\triangle BHC$ meet again at $G \neq H$. Prove that $\angle MHG = 90^\circ$.

Problem 5. Let m and n be positive integers. Find the smallest positive integer s for which there exists an $m \times n$ rectangular array of positive integers such that

- each row contains n distinct consecutive integers in some order,
- each column contains m distinct consecutive integers in some order, and
- each entry is less than or equal to s.

Problem 6. For any positive integer n, let

- $\tau(n)$ denote the number of positive integer divisors of n,
- $\sigma(n)$ denote the sum of the positive integer divisors of n, and
- $\varphi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n.

Let $a, b > 1$ be integers. Brandon has a calculator with three buttons that replace the integer n currently displayed with $\tau(n)$, $\sigma(n)$, or $\varphi(n)$, respectively. Prove that if the calculator currently displays a, then Brandon can make the calculator display b after a finite (possibly empty) sequence of button presses.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Olympians Enjoy Mixed-up Letters

Year: 2021

Day: 1

23$^{\text{rd}}$ ELMO

127.0.0.1

Thursday, June 17, 2021
2:00PM — 6:30PM EDT

Problem 1. In $\triangle ABC$, points P and Q lie on sides AB and AC, respectively, such that the circumcircle of $\triangle APQ$ is tangent to side BC at D. Let E lie on side BC such that $BD = EC$. Line DP intersects the circumcircle of $\triangle CDQ$ again at X, and line DQ intersects the circumcircle of $\triangle BDP$ again at Y. Prove that D, E, X, and Y are concyclic.

Problem 2. Let $n > 1$ be an integer and let $a_1, a_2, \ldots, a_n$ be integers such that $n \mid a_i - i$ for all integers $1 \le i \le n$. Prove there exists an infinite sequence $b_1, b_2, \ldots$ such that

- $b_k \in \{a_1, a_2, \ldots, a_n\}$ for all positive integers k, and
- $\displaystyle\sum_{k=1}^{\infty} \frac{b_k}{n^k}$ is an integer.

Problem 3. Each cell of a 100×100 grid is colored with one of 101 colors. A cell is *diverse* if, among the 199 cells in its row or column, every color appears at least once. Determine the maximum possible number of diverse cells.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Olympians Enjoy Mixed-up Letters

23rd ELMO
127.0.0.1

Year: **2021**

Day: **2**

Friday, June 18, 2021
2:00PM — 6:30PM EDT

Problem 4. The set of positive integers is partitioned into n disjoint infinite arithmetic progressions S_1, S_2, …, S_n with common differences $d_1, d_2, \ldots, d_n$, respectively. Prove that there exists exactly one index $1 \le i \le n$ such that

$$\frac{1}{d_i}\prod_{j=1}^{n} d_j \in S_i.$$

Problem 5. Let n and k be positive integers. Two infinite sequences $\{s_i\}_{i\ge1}$ and $\{t_i\}_{i\ge1}$ are *equivalent* if, for all positive integers i and j, $s_i = s_j$ if and only if $t_i = t_j$. A sequence $\{r_i\}_{i\ge1}$ has *equi-period* k if $r_1, r_2, \ldots$ and $r_{k+1}, r_{k+2}, \ldots$ are equivalent.

Suppose M infinite sequences with equi-period k whose terms are in the set $\{1, \ldots, n\}$ can be chosen such that no two chosen sequences are equivalent to each other. Determine the largest possible value of M in terms of n and k.

Problem 6. In $\triangle ABC$, points D, E, and F lie on sides BC, CA, and AB, respectively, such that each of the quadrilaterals $AFDE$, $BDEF$, and $CEFD$ has an incircle. Prove that the inradius of $\triangle ABC$ is twice the inradius of $\triangle DEF$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Elmo, Let Me Out

Year: **2022**

24th ELMO
Pittsburgh, PA

Day: **1**

Saturday, June 11, 2022
1:15PM–5:45PM EDT

Problem 1. Let $n > 1$ be an integer. The numbers $1, \ldots, n$ are written on a board. Aliceurill and Bobasaur take turns circling an uncircled number on the board, with Aliceurill going first. When the product of the circled numbers becomes a multiple of n, the game ends and the last player to have circled a number loses. For which values of n can Bobasaur guarantee victory?

Problem 2. Find all monic nonconstant polynomials P with integer coefficients for which there exist positive integers a and m such that for all positive integers $n \equiv a \pmod{m}$, $P(n)$ is nonzero, and

$$2022 \cdot \frac{(n+1)^{n+1} - n^n}{P(n)}$$

is an integer.

Problem 3. Let $\mathcal{P}$ be a regular 2022-gon with area 1. Find a real number c such that, if points A and B are chosen independently and uniformly at random on the perimeter of $\mathcal{P}$, then the probability that $AB \geq c$ is $\frac{1}{2}$.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Elmo, Let Me Out

Year: **2022**

24th ELMO
Pittsburgh, PA

Day: **2**

Sunday, June 19, 2022
1:15PM–5:45PM EDT

Problem 4. Let $ABCDE$ be a convex pentagon such that $\triangle ABE$, $\triangle BEC$, and $\triangle EDB$ are similar (with vertices in order). Lines BE and CD intersect at point T. Prove that line AT is tangent to the circumcircle of $\triangle ACD$.

Problem 5. Let $n \geq 3$ be a positive integer. There are n^3 users on a social media network called Everyone Likes Meeting Online (ELMO), and some pairs of these users are ELMObuddies. A set of at least three ELMO users forms an ELMOclub if and only if all pairs of members of the set are ELMObuddies. It is known that among every n users, some three form an ELMOclub. Prove that there is an ELMOclub with five members.

Problem 6. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers m and n,

$$f(f(m)-n)+f(f(n)-m)=f(m+n).$$

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

ELMOnade

23rd ELMO
PITTSBURGH, PA

Year: **2023**

Day: **1**

Saturday, June 10, 2023
1:20PM–5:50PM EDT

Problem 1. Let m be a positive integer. Find, in terms of m, all polynomials $P(x)$ with integer coefficients such that for every integer n, there exists an integer k such that $P(k) = n^m$.

Problem 2. Let a, b, and n be positive integers. A lemonade stand owns n cups, all of which are initially empty. The lemonade stand has a *filling machine* and an *emptying machine*, which operate according to the following rules:

- If at any moment, a completely empty cups are available, the filling machine spends the next a minutes filling those a cups simultaneously and doing nothing else.
- If at any moment, b completely full cups are available, the emptying machine spends the next b minutes emptying those b cups simultaneously and doing nothing else.

Suppose that after a sufficiently long time has passed, both the filling machine and emptying machine work without pausing. Find, in terms of a and b, the least possible value of n.

Problem 3. Convex quadrilaterals $ABCD$, $A_1B_1C_1D_1$, and $A_2B_2C_2D_2$ are similar with vertices in order. Points A, A_1, B_2, B are collinear in order, points B, B_1, C_2, C are collinear in order, points C, C_1, D_2, D are collinear in order, and points D, D_1, A_2, A are collinear in order. Diagonals AC and BD intersect at P, diagonals A_1C_1 and B_1D_1 intersect at P_1, and diagonals A_2C_2 and B_2D_2 intersect at P_2. Prove that points P, P_1, and P_2 are collinear.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

waterELMOn

23rd ELMO
PITTSBURGH, PA

Year: **2023**

Day: **2**

Sunday, June 18, 2023
1:20PM–5:50PM EDT

Problem 4. Let ABC be an acute scalene triangle with orthocenter H. Line BH intersects $\overline{AC}$ at E and line CH intersects $\overline{AB}$ at F. Let X be the foot of the perpendicular from H to the line through A parallel to $\overline{EF}$. Point B_1 lies on line XF such that $\overline{BB_1}$ is parallel to $\overline{AC}$, and point C_1 lies on line XE such that $\overline{CC_1}$ is parallel to $\overline{AB}$. Prove that points B, C, B_1, C_1 are concyclic.

Problem 5. Find the least positive integer M for which there exist a positive integer n and polynomials $P_1(x)$, $P_2(x)$, …, $P_n(x)$ with integer coefficients satisfying

$$Mx = P_1(x)^3 + P_2(x)^3 + \cdots + P_n(x)^3.$$

Problem 6. For a set S of positive integers and a positive integer n, consider the game of (n, S)-*nim*, which is as follows. A pile starts with n watermelons. Two players, Deric and Erek, alternate turns eating watermelons from the pile, with Deric going first. On any turn, the number of watermelons eaten must be an element of S. The last player to move wins. Let $f(S)$ denote the set of positive integers n for which Deric has a winning strategy in (n, S)-nim.

Let T be a set of positive integers. Must the sequence

$$T,\ f(T),\ f(f(T)),\ \ldots$$

be eventually constant?

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.