COMPENDIUM CMO

Canadian Mathematical Olympiad 1969-2022

con soluciones a partir del año 1994

Gerard Romo Garrido

TooMates Coolección

Toomates Coolección

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Índice.

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Canadian Mathematical Olympiad 1969

Problem 1

Show that if $a_1/b_1 = a_2/b_2 = a_3/b_3$ and p_1, p_2, p_3 are not all zero, then

$$\left(\frac{a_1}{b_1}\right)^n = \frac{p_1 a_1^n + p_2 a_2^n + p_3 a_3^n}{p_1 b_1^n + p_2 b_2^n + p_3 b_3^n}$$

for every positive integer n.

Problem 2

Determine which of the two numbers $\sqrt{c+1} - \sqrt{c}$, $\sqrt{c} - \sqrt{c-1}$ is greater for any $c \geq 1$.

Problem 3

Let c be the length of the hypotenuse of a right angle triangle whose other two sides have lengths a and b. Prove that $a + b \leq \sqrt{2}c$. When does the equality hold?

Problem 4

Let ABC be an equilateral triangle, and P be an arbitrary point within the triangle. Perpendiculars PD, PE, PF are drawn to the three sides of the triangle. Show that, no matter where P is chosen,

$$\frac{PD + PE + PF}{AB + BC + CA} = \frac{1}{2\sqrt{3}}.$$

Problem 5

Let ABC be a triangle with sides of lengths a, b and c. Let the bisector of the angle C cut AB in D. Prove that the length of CD is

$$\frac{2ab \cos \frac{C}{2}}{a+b}.$$

Problem 6

Find the sum of $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + (n-1)(n-1)! + n \cdot n!$, where $n! = n(n-1)(n-2) \cdots 2 \cdot 1$.

Problem 7

Show that there are no integers a, b, c for which $a^2 + b^2 - 8c = 6$.

PROBLEM 8

Let f be a function with the following properties:

1) $f(n)$ is defined for every positive integer n;
2) $f(n)$ is an integer;
3) $f(2) = 2$;
4) $f(mn) = f(m)f(n)$ for all m and n;
5) $f(m) > f(n)$ whenever $m > n$.

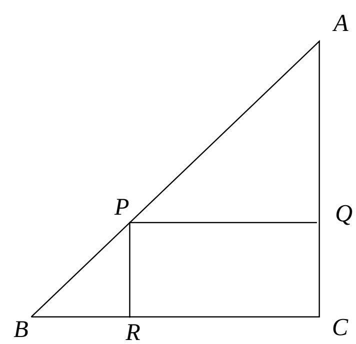
Prove that $f(n) = n$.

PROBLEM 9

Show that for any quadrilateral inscribed in a circle of radius 1, the length of the shortest side is less than or equal to $\sqrt{2}$.

PROBLEM 10

Let ABC be the right-angled isosceles triangle whose equal sides have length 1. P is a point on the hypotenuse, and the feet of the perpendiculars from P to the other sides are Q and R. Consider the areas of the triangles APQ and PBR, and the area of the rectangle $QCRP$. Prove that regardless of how P is chosen, the largest of these three areas is at least $2/9$.



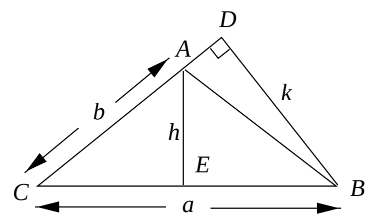
Canadian Mathematical Olympiad
1970

PROBLEM 1

Find all number triples (x, y, z) such that when any one of these numbers is added to the product of the other two, the result is 2.

PROBLEM 2

Given a triangle ABC with angle A obtuse and with altitudes of length h and k as shown in the diagram, prove that $a + h \geq b + k$. Find under what conditions $a + h = b + k$.



PROBLEM 3

A set of balls is given. Each ball is coloured red or blue, and there is at least one of each colour. Each ball weighs either 1 pound or 2 pounds, and there is at least one of each weight. Prove that there are 2 balls having different weights and different colours.

PROBLEM 4

a) Find all positive integers with initial digit 6 such that the integer formed by deleting this 6 is 1/25 of the original integer.
b) Show that there is no integer such that deletion of the first digit produces a result which is 1/35 of the original integer.

PROBLEM 5

A quadrilateral has one vertex on each side of a square of side-length 1. Show that the lengths a, b, c and d of the sides of the quadrilateral satisfy the inequalities

$$2 \leq a^2 + b^2 + c^2 + d^2 \leq 4.$$

PROBLEM 6

Given three non-collinear points A, B, C, construct a circle with centre C such that the tangents from A and B to the circle are parallel.

PROBLEM 7

Show that from any five integers, not necessarily distinct, one can always choose three of these integers whose sum is divisible by 3.

PROBLEM 8

Consider all line segments of length 4 with one end-point on the line $y = x$ and the other end-point on the line $y = 2x$. Find the equation of the locus of the midpoints of these line segments.

PROBLEM 9

Let $f(n)$ be the sum of the first n terms of the sequence

$$0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, \ldots .$$

a) Give a formula for $f(n)$.
b) Prove that $f(s+t) - f(s-t) = st$ where s and t are positive integers and $s > t$.

PROBLEM 10

Given the polynomial

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$

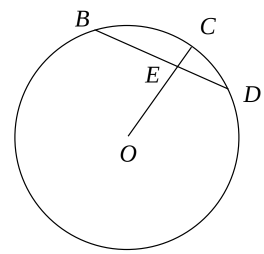
with integral coefficients $a_1, a_2, \ldots, a_n$, and given also that there exist four distinct integers a, b, c and d such that

$$f(a) = f(b) = f(c) = f(d) = 5,$$

show that there is no integer k such that $f(k) = 8$.

Canadian Mathematical Olympiad
1971

PROBLEM 1



DEB is a chord of a circle such that $DE = 3$ and $EB = 5$. Let O be the centre of the circle. Join OE and extend OE to cut the circle at C. (See diagram). Given $EC = 1$, find the radius of the circle.

PROBLEM 2

Let x and y be positive real numbers such that $x + y = 1$. Show that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) \geq 9.$$

PROBLEM 3

$ABCD$ is a quadrilateral with $AD = BC$. If $\angle ADC$ is greater than $\angle BCD$, prove that $AC > BD$.

PROBLEM 4

Determine all real numbers a such that the two polynomials x^2+ax+1 and x^2+x+a have at least one root in common.

PROBLEM 5

Let

$$p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

where the coefficients a_i are integers. If $p(0)$ and $p(1)$ are both odd, show that $p(x)$ has no integral roots.

PROBLEM 6

Show that, for all integers n, $n^2 + 2n + 12$ is not a multiple of 121.

PROBLEM 7

Let n be a five digit number (whose first digit is non-zero) and let m be the four digit number formed from n by deleting its middle digit. Determine all n such that n/m is an integer.

PROBLEM 8

A regular pentagon is inscribed in a circle of radius r. P is any point inside the pentagon. Perpendiculars are dropped from P to the sides, or the sides produced, of the pentagon.

a) Prove that the sum of the lengths of these perpendiculars is constant.
b) Express this constant in terms of the radius r.

PROBLEM 9

Two flag poles of heights h and k are situated $2a$ units apart on a level surface. Find the set of all points on the surface which are so situated that the angles of elevation of the tops of the poles are equal.

PROBLEM 10

Suppose that n people each know exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything that A knows, while B tells A nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything? Prove your answer is a minimum.

Canadian Mathematical Olympiad
1972

Problem 1

Given three distinct unit circles, each of which is tangent to the other two, find the radii of the circles which are tangent to all three circles.

Problem 2

Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Define M to be the sum of all products of pairs $a_i a_j$ $(i < j)$, *i.e.*,

$$M = a_1(a_2 + a_3 + \cdots + a_n) + a_2(a_3 + a_4 + \cdots + a_n) + \cdots + a_{n-1}a_n.$$

Prove that the square of at least one of the numbers $a_1, a_2, \ldots, a_n$ does not exceed $2M/n(n-1)$.

Problem 3

a) Prove that 10201 is composite in any base greater than 2.
b) Prove that 10101 is composite in any base.

Problem 4

Describe a construction of a quadrilateral $ABCD$ given:
(i) the lengths of all four sides;
(ii) that AB and CD are parallel;
(iii) that BC and DA do not intersect.

Problem 5

Prove that the equation $x^3 + 11^3 = y^3$ has no solution in positive integers x and y.

Problem 6

Let a and b be distinct real numbers. Prove that there exist integers m and n such that $am + bn < 0$, $bm + an > 0$.

Problem 7

a) Prove that the values of x for which $x = (x^2 + 1)/198$ lie between $1/198$ and $197.99494949\cdots$.
b) Use the result of a) to prove that $\sqrt{2} < 1.41\overline{421356}$.
c) Is it true that $\sqrt{2} < 1.41421356$?

Problem 8

During a certain election campaign, p different kinds of promises are made by the various political parties $(p > 0)$. While several parties may make the same promise, any two parties have at least one promise in common; no two parties have exactly the same set of promises. Prove that there are no more than 2^{p-1} parties.

PROBLEM 9

Four distinct lines L_1, L_2, L_3, L_4 are given in the plane: L_1 and L_2 are respectively parallel to L_3 and L_4. Find the locus of a point moving so that the sum of its perpendicular distances from the four lines is constant.

PROBLEM 10

What is the maximum number of terms in a geometric progression with common ratio greater than 1 whose entries all come from the set of integers between 100 and 1000 inclusive?

Canadian Mathematical Olympiad
1973

PROBLEM 1

(i) Solve the simultaneous inequalities, $x < \frac{1}{4x}$ and $x < 0$; *i.e.*, find a single inequality equivalent to the two given simultaneous inequalities.

(ii) What is the greatest integer which satisfies both inequalities $4x + 13 < 0$ and $x^2 + 3x > 16$?

(iii) Give a rational number between $11/24$ and $6/13$.

(iv) Express 100000 as a product of two integers neither of which is an integral multiple of 10.

(v) Without the use of logarithm tables evaluate

$$\frac{1}{\log_2 36} + \frac{1}{\log_3 36}.$$

PROBLEM 2

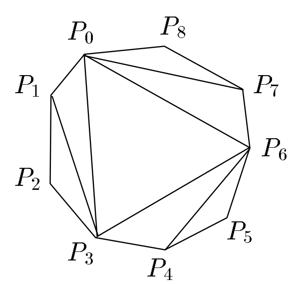
Find all the real numbers which satisfy the equation $|x+3| - |x-1| = x+1$. (Note: $|a| = a$ if $a \geq 0$; $|a| = -a$ if $a < 0$.)

PROBLEM 3

Prove that if p and $p+2$ are both prime integers greater than 3, then 6 is a factor of $p+1$.

PROBLEM 4

The figure shows a (convex) polygon with nine vertices. The six diagonals which have been drawn dissect the polygon into the seven triangles: $P_0P_1P_3$, $P_0P_3P_6$, $P_0P_6P_7$, $P_0P_7P_8$, $P_1P_2P_3$, $P_3P_4P_6$, $P_4P_5P_6$. In how many ways can these triangles be labelled with the names $\triangle_1$, $\triangle_2$, $\triangle_3$, $\triangle_4$, $\triangle_5$, $\triangle_6$, $\triangle_7$ so that P_i is a vertex of triangle $\triangle_i$ for $i = 1, 2, 3, 4, 5, 6, 7$? Justify your answer.



PROBLEM 5

For every positive integer n, let

$$h(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

For example, $h(1) = 1$, $h(2) = 1 + \frac{1}{2}$, $h(3) = 1 + \frac{1}{2} + \frac{1}{3}$. Prove that for $n = 2, 3, 4, \ldots$

$$n + h(1) + h(2) + h(3) + \cdots + h(n-1) = nh(n).$$

Problem 6

If A and B are fixed points on a given circle not collinear with centre O of the circle, and if XY is a variable diameter, find the locus of P (the intersection of the line through A and X and the line through B and Y).

Problem 7

Observe that

$$\frac{1}{1}=\frac{1}{2}+\frac{1}{2};\quad \frac{1}{2}=\frac{1}{3}+\frac{1}{6};\quad \frac{1}{3}=\frac{1}{4}+\frac{1}{12};\quad \frac{1}{4}=\frac{1}{5}+\frac{1}{20}.$$

State a general law suggested by these examples, and prove it.
Prove that for any integer n greater than 1 there exist positive integers i and j such that

$$\frac{1}{n}=\frac{1}{i(i+1)}+\frac{1}{(i+1)(i+2)}+\frac{1}{(i+2)(i+3)}+\cdots+\frac{1}{j(j+1)}.$$

Canadian Mathematical Olympiad
1974

PART A

PROBLEM 1

i) If $x = (1+\frac{1}{n})^n$ and $y = (1+\frac{1}{n})^{n+1}$, show that $y^x = x^y$.
ii) Show that, for all positive integers n,

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^n(n-1)^2 + (-1)^{n+1}n^2 = (-1)^{n+1}(1+2+\cdots+n).$$

PROBLEM 2

Let $ABCD$ be a rectangle with $BC = 3AB$. Show that if P, Q are the points on side BC with $BP = PQ = QC$, then

$$\angle DBC + \angle DPC = \angle DQC.$$

PART B

PROBLEM 3

Let

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

be a polynomial with coefficients satisfying the conditions:

$$0 \le a_i \le a_0, \quad i = 1, 2, \ldots, n.$$

Let $b_0, b_1, \ldots, b_{2n}$ be the coefficients of the polynomial

$$\begin{aligned}\big(f(x)\big)^2 &= (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)^2\\ &= b_0 + b_1x + b_2x^2 + \cdots + b_{n+1}x^{n+1} + \cdots + b_{2n}x^{2n}.\end{aligned}$$

Prove that

$$b_{n+1} \le \frac{1}{2}\big(f(1)\big)^2.$$

PROBLEM 4

Let n be a fixed positive integer. To any choice of n real numbers satisfying

$$0 \le x_i \le 1, \quad i = 1, 2, \ldots, n,$$

there corresponds the sum

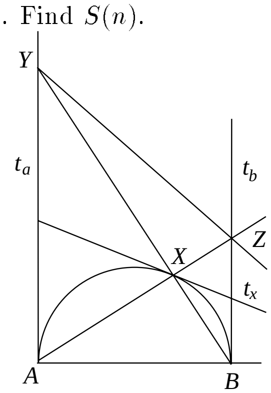
(∗)

$$\sum_{1\le i<j\le n} |x_i - x_j|$$
$$\begin{aligned}= |x_1 - x_2| + |x_1 - x_3| + |x_1 - x_4| + \cdots + |x_1 - x_{n-1}| + |x_1 - x_n|\\ + |x_2 - x_3| + |x_2 - x_4| + \cdots + |x_2 - x_{n-1}| + |x_2 - x_n|\\ + |x_3 - x_4| + \cdots + |x_3 - x_{n-1}| + |x_3 - x_n|\\ + \cdots + |x_{n-2} - x_{n-1}| + |x_{n-2} - x_n|\\ + |x_{n-1} - x_n|.\end{aligned}$$

Let $S(n)$ denote the largest possible value of the sum (∗). Find $S(n)$.

PROBLEM 5

Given a circle with diameter AB and a point X on the circle different from A and B, let t_a, t_b and t_x be the tangents to the circle at A, B and X respectively. Let Z be the point where line AX meets t_b and Y the point where line BX meets t_a. Show that the three lines YZ, t_x and AB are either concurrent (*i.e.*, all pass through the same point) or parallel.

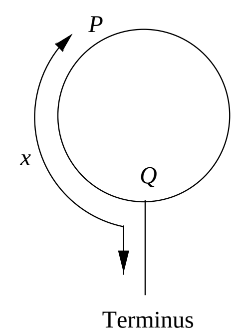


PROBLEM 6

An unlimited supply of 8-cent and 15-cent stamps is available. Some amounts of postage cannot be made up exactly, *e.g.*, 7 cents, 29 cents. What is the largest unattainable amount, *i.e.*, the amount, say n, of postage which is unattainable while all amounts larger than n are attainable? (Justify your answer.)

PROBLEM 7

A bus route consists of a circular road of circumference 10 miles and a straight road of length 1 mile which runs from a terminus to the point Q on the circular road (see diagram). It is served by two buses, each of which requires 20 minutes for the round trip. Bus No. 1, upon leaving the terminus, travels along the straight road, once around the circle clockwise and returns along the straight road to the terminus. Bus No. 2, reaching the terminus 10 minutes after Bus No. 1, has a similar route except that it proceeds counterclockwise around the circle. Both buses run continuously and do not wait at any point on the route except for a negligible amount of time to pick up and discharge passengers.



A man plans to wait at a point P which is x miles ($0 \le x < 12$) from the terminus along the route of Bus No. 1 and travel to the terminus on one of the buses.

Assuming that he chooses to board that bus which will bring him to his destination at the earliest moment, there is a maximum time $w(x)$ that his journey (waiting plus travel time) could take.
Find $w(2)$; find $w(4)$.
For what value of x will the time $w(x)$ be the longest?
Sketch a graph of $y = w(x)$ for $0 \le x < 12$.

Canadian Mathematical Olympiad
1975

PROBLEM 1

Simplify

$$\left(\frac{1\cdot 2\cdot 4+2\cdot 4\cdot 8+\cdots+n\cdot 2n\cdot 4n}{1\cdot 3\cdot 9+2\cdot 6\cdot 18+\cdots+n\cdot 3n\cdot 9n}\right)^{1/3}.$$

PROBLEM 2

A sequence of numbers $a_1, a_2, a_3, \ldots$ satisfies
(i) $a_1 = \frac{1}{2}$,
(ii) $a_1 + a_2 + \cdots + a_n = n^2 a_n \quad (n \geq 1)$.
Determine the value of $a_n \quad (n \geq 1)$.

PROBLEM 3

For each real number r, $[r]$ denotes the largest integer less than or equal to r, *e.g.*, $[6] = 6$, $[\pi] = 3$, $[-1.5] = -2$. Indicate on the (x, y)-plane the set of all points (x, y) for which $[x]^2 + [y]^2 = 4$.

PROBLEM 4

For a positive number such as 3.27, 3 is referred to as the integral part of the number and .27 as the decimal part. Find a positive number such that its decimal part, its integral part, and the number itself form a geometric progression.

PROBLEM 5

A, B, C, D are four "consecutive" points on the circumference of a circle and P, Q, R, S are points on the circumference which are respectively the midpoints of the arcs AB, BC, CD, DA. Prove that PR is perpendicular to QS.

PROBLEM 6

(i) 15 chairs are equally placed around a circular table on which are name cards for 15 guests. The guests fail to notice these cards until after they have sat down, and it turns out that no one is sitting in the correct seat. Prove that the table can be rotated so that at least two of the guests are simultaneously correctly seated.

(ii) Give an example of an arrangement in which just one of the 15 guests is correctly seated and for which no rotation correctly places more than one person.

PROBLEM 7

A function $f(x)$ is *periodic* if there is a positive number p such that $f(x+p) = f(x)$ for all x. For example, $\sin x$ is periodic with period 2π. Is the function $\sin(x^2)$ periodic? Prove your assertion.

Problem 8

Let k be a positive integer. Find all polynomials

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where the a_i are real, which satisfy the equation

$$P\big(P(x)\big) = \{P(x)\}^k.$$

Canadian Mathematical Olympiad
1976

Problem 1

Given four weights in geometric progression and an equal arm balance, show how to find the heaviest weight using the balance only twice.

Problem 2

Suppose

$$n(n+1)a_{n+1} = n(n-1)a_n - (n-2)a_{n-1}$$

for every positive integer $n \geq 1$.
Given that $a_0 = 1$, $a_1 = 2$, find

$$\frac{a_0}{a_1} + \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{50}}{a_{51}}.$$

Problem 3

Two grade seven students were allowed to enter a chess tournament otherwise composed of grade eight students. Each contestant played once with each other contestant and received one point for a win, one half point for a tie and zero for a loss. The two grade seven students together gained a total of eight points and each grade eight student scored the same number of points as his classmates. How many students from grade eight participated in the chess tournament? Is the solution unique?

Problem 4

Let AB be a diameter of a circle, C be any fixed point between A and B on this diameter, and Q be a variable point on the circumference of the circle. Let P be the point on the line determined by Q and C for which $\frac{AC}{CB} = \frac{QC}{CP}$. Describe, with proof, the locus of the point P.

Problem 5

Prove that a positive integer is a sum of at least two consecutive positive integers if and only if it is not a power of two.

Problem 6

If A, B, C, D are four points in space, such that

$$\angle ABC = \angle BCD = \angle CDA = \angle DAB = \pi/2,$$

prove that A, B, C, D lie in a plane.

Problem 7

Let $P(x, y)$ be a polynomial in two variables x, y such that $P(x, y) = P(y, x)$ for every x, y (for example, the polynomial $x^2 - 2xy + y^2$ satisfies this condition). Given that $(x - y)$ is a factor of $P(x, y)$, show that $(x - y)^2$ is a factor of $P(x, y)$.

Problem 8

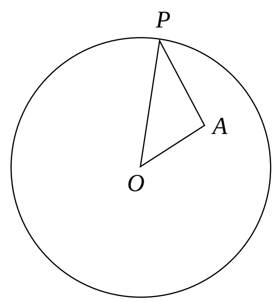
Each of the 36 line segments joining 9 distinct points on a circle is coloured either red or blue. Suppose that each triangle determined by 3 of the 9 points contains at least one red side. Prove that there are four points such that the 6 segments connecting them are all red.

Canadian Mathematical Olympiad
1977

PROBLEM 1

If $f(x) = x^2 + x$, prove that the equation $4f(a) = f(b)$ has no solutions in positive integers a and b.

PROBLEM 2



Let O be the centre of a circle and A a fixed interior point of the circle different from O. Determine all points P on the circumference of the circle such that the angle OPA is a maximum.

PROBLEM 3

N is an integer whose representation in base b is 777. Find the smallest positive integer b for which N is the fourth power of an integer.

PROBLEM 4

Let

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

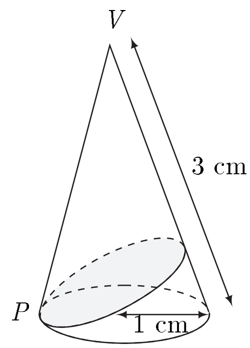
and

$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

be two polynomials with integer coefficients. Suppose that all the coefficients of the product $p(x) \cdot q(x)$ are even but not all of them are divisible by 4. Show that one of $p(x)$ and $q(x)$ has all even coefficients and the other has at least one odd coefficient.

PROBLEM 5

A right circular cone of base radius 1 cm and slant height 3 cm is given. P is a point on the circumference of the base and the shortest path from P around the cone and back to P is drawn (see diagram). What is the minimum distance from the vertex V to this path?



PROBLEM 6

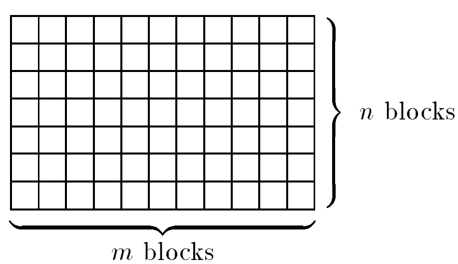
Let $0 < u < 1$ and define

$$u_1 = 1 + u, \quad u_2 = \frac{1}{u_1} + u, \quad \ldots, \quad u_{n+1} = \frac{1}{u_n} + u, \quad n \geq 1.$$

Show that $u_n > 1$ for all values of $n = 1, 2, 3, \ldots$.

PROBLEM 7

A rectangular city is exactly m blocks long and n blocks wide (see diagram). A woman lives in the southwest corner of the city and works in the northeast corner. She walks to work each day but, on any given trip, she makes sure that her path does not include any intersection twice. Show that the number $f(m, n)$ of different paths she can take to work satisfies $f(m, n) \leq 2^{mn}$.



Canadian Mathematical Olympiad
1978

PROBLEM 1

Let n be an integer. If the tens digit of n^2 is 7, what is the units digit of n^2?

PROBLEM 2

Find all pairs a, b of positive integers satisfying the equation $2a^2 = 3b^3$.

PROBLEM 3

Determine the largest real number z such that

$$\begin{aligned} x + y + z &= 5 \\ xy + yz + xz &= 3 \end{aligned}$$

and x, y are also real.

PROBLEM 4

The sides AD and BC of a convex quadrilateral $ABCD$ are extended to meet at E. Let H and G be the midpoints of BD and AC, respectively. Find the ratio of the area of the triangle EHG to that of the quadrilateral $ABCD$.

PROBLEM 5

Eve and Odette play a game on a 3×3 checkerboard, with black checkers and white checkers. The rules are as follows:

I. They play alternately.
II. A turn consists of placing one checker on an unoccupied square of the board.
III. In her turn, a player may select either a white checker or a black checker and need not always use the same colour.
IV. When the board is full, Eve obtains one point for every row, column or diagonal that has an even number of black checkers, and Odette obtains one point for every row, column or diagonal that has an odd number of black checkers.
V. The player obtaining at least five of the eight points WINS.

(a) Is a 4–4 tie possible? Explain.
(b) Describe a winning strategy for the girl who is first to play.

PROBLEM 6

Sketch the graph of $x^3 + xy + y^3 = 3$.

Canadian Mathematical Olympiad
1979

Problem 1

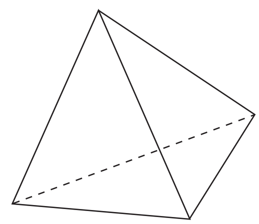
Given: (i) $a, b > 0$; (ii) a, A_1, A_2, b is an arithmetic progression; (iii) a, G_1, G_2, b is a geometric progression. Show that

$$A_1 A_2 \geq G_1 G_2$$

Problem 2

It is known in Euclidean geometry that the sum of the angles of a triangle is constant. Prove, however, that the sum of the dihedral angles of a tetrahedron is *not* constant.

Note. (i) A tetrahedron is a triangular pyramid, and (ii) a dihedral angle is the interior angle between a pair of faces.



Problem 3

Let a, b, c, d, e be integers such that $1 \leq a < b < c < d < e$. Prove that

$$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{1}{[d,e]} \leq \frac{15}{16},$$

where $[m, n]$ denotes the least common multiple of m and n (*e.g.* $[4, 6] = 12$).

Problem 4

A dog standing at the centre of a circular arena sees a rabbit at the wall. The rabbit runs around the wall and the dog pursues it along a unique path which is determined by running at the same speed and staying on the radial line joining the centre of the arena to the rabbit. Show that the dog overtakes the rabbit just as it reaches a point one-quarter of the way around the arena.

Problem 5

A walk consists of a sequence of steps of length 1 taken in directions north, south, east or west. A walk is *self-avoiding* if it never passes through the same point twice. Let $f(n)$ denote the number of n-step self-avoiding walks which begin at the origin. Compute $f(1)$, $f(2)$, $f(3)$, $f(4)$, and show that

$$2^n < f(n) \leq 4 \cdot 3^{n-1}.$$

Canadian Mathematical Olympiad
1980

Problem 1

If $a679b$ is a five digit number (in base 10) which is divisible by 72, determine a and b.

Problem 2

The numbers from 1 to 50 are printed on cards. The cards are shuffled and then laid out face up in 5 rows of 10 cards each. The cards in each row are rearranged to make them increase from left to right. The cards in each column are then rearranged to make them increase from top to bottom. In the final arrangement, do the cards in the rows still increase from left to right?

Problem 3

Among all triangles having (i) a fixed angle A and (ii) an inscribed circle of fixed radius r, determine which triangle has the least perimeter.

Problem 4

A gambling student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring *exactly* n points is $\frac{1}{3}[2 + (-\frac{1}{2})^n]$.

Problem 5

A parallelepiped has the property that all cross sections which are parallel to any fixed face F, have the same perimeter as F. Determine whether or not any other polyhedron has this property.

Canadian Mathematical Olympiad
1981

Problem 1

For any real number t, denote by $[t]$ the greatest integer which is less than or equal to t. For example: $[8] = 8$, $[\pi] = 3$ and $[-5/2] = -3$. Show that the equation

$$[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345$$

has no real solution.

Problem 2

Given a circle of radius r and a tangent line ℓ to the circle through a given point P on the circle. From a variable point R on the circle, a perpendicular RQ is drawn to ℓ with Q on ℓ. Determine the maximum of the area of triangle PQR.

Problem 3

Given a finite collection of lines in a plane P, show that it is possible to draw an arbitrarily large circle in P which does not meet any of them. On the other hand, show that it is possible to arrange an infinite sequence of lines (first line, second line, third line, *etc.*) in P so that every circle in P meets at least one of the lines. (A point is not considered to be a circle.)

Problem 4

$P(x)$ and $Q(x)$ are two polynomials that satisfy the identity $P\big(Q(x)\big) \equiv Q\big(P(x)\big)$ for real numbers x. If the equation $P(x) = Q(x)$ has no real solution, show that the equation $P\big(P(x)\big) = Q\big(Q(x)\big)$ also has no real solution.

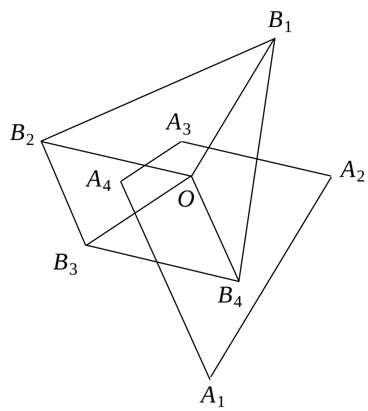
Problem 5

Eleven theatrical groups participated in a festival. Each day, some of the groups were scheduled to perform while the remaining groups joined the general audience. At the conclusion of the festival, each group had seen, during its days off, at least one performance of every other group. At least how many days did the festival last?

Canadian Mathematical Olympiad
1982

PROBLEM 1

In the diagram, OB_i is parallel and equal in length to A_iA_{i+1} for $i = 1, 2, 3$ and $4 (A_5 = A_1)$. Show that the area of $B_1B_2B_3B_4$ is twice that of $A_1A_2A_3A_4$.



PROBLEM 2

If a, b and c are the roots of the equation $x^3 - x^2 - x - 1 = 0$,

(i) show that a, b and c are distinct:

(ii) show that

$$\frac{a^{1982} - b^{1982}}{a - b} + \frac{b^{1982} - c^{1982}}{b - c} + \frac{c^{1982} - a^{1982}}{c - a}$$

is an integer.

PROBLEM 3

Let R^n be the n-dimensional Euclidean space. Determine the smallest number $g(n)$ of points of a set in R^n such that every point in R^n is at irrational distance from at least one point in that set.

PROBLEM 4

Let p be a permutation of the set $S_n = \{1, 2, \ldots, n\}$. An element $j \in S_n$ is called a *fixed point* of p if $p(j) = j$. Let f_n be the number of permutations having no fixed points, and g_n be the number with exactly one fixed point. Show that $|f_n - g_n| = 1$.

PROBLEM 5

The altitudes of a tetrahedron $ABCD$ are extended externally to points A', B', C' and D' respectively, where $AA' = k/h_a$, $BB' = k/h_b$, $CC' = k/h_c$ and $DD' =$

k/h_d. Here, k is a constant and h_a denotes the length of the altitude of $ABCD$ from vertex A, *etc.* Prove that the centroid of the tetrahedron $A'B'C'D'$ coincides with the centroid of $ABCD$.

Canadian Mathematical Olympiad
1983

PROBLEM 1

Find all positive integers w, x, y and z which satisfy $w! = x! + y! + z!$.

PROBLEM 2

For each real number r let T_r be the transformation of the plane that takes the point (x, y) into the point $(2^r x, r2^r x + 2^r y)$. Let F be the family of all such transformations *i.e.* $F = \{T_r : r \text{ a real number}\}$. Find all curves $y = f(x)$ whose graphs remain unchanged by every transformation in F.

PROBLEM 3

The area of a triangle is determined by the lengths of its sides. Is the volume of a tetrahedron determined by the areas of its faces?

PROBLEM 4

Prove that for every prime number p, there are infinitely many positive integers n such that p divides $2^n - n$.

PROBLEM 5

The geometric mean (G.M.) of a k positive numbers $a_1, a_2, \ldots, a_k$ is defined to be the (positive) k-th root of their product. For example, the G.M. of 3, 4, 18 is 6. Show that the G.M. of a set S of n positive numbers is equal to the G.M. of the G.M.'s of all non-empty subsets of S.

Canadian Mathematical Olympiad
1984

Problem 1

Prove that the sum of the squares of 1984 consecutive positive integers cannot be the square of an integer.

Problem 2

Alice and Bob are in a hardware store. The store sells coloured sleeves that fit over keys to distinguish them. The following conversation takes place:

Alice: Are you going to cover your keys?

Bob: I would like to, but there are only 7 colours and I have 8 keys.

Alice: Yes, but you could always distinguish a key by noticing that the red key next to the green key was different from the red key next to the blue key.

Bob: You must be careful what you mean by "next to" or "three keys over from" since you can turn the key ring over and the keys are arranged in a circle.

Alice: Even so, you don't need 8 colours.

Problem: What is the smallest number of colours needed to distinguish n keys if all the keys are to be covered.

Problem 3

An integer is *digitally divisible* if

(a) none of its digits is zero;

(b) it is divisible by the sum of its digits (*e.g.*, 322 is digitally divisible).

Show that there are infinitely many digitally divisible integers.

Problem 4

An acute-angled triangle has unit area. Show that there is a point inside the triangle whose distance from each of the vertices is at least $\frac{2}{\sqrt[4]{27}}$.

Problem 5

Given any 7 real numbers, prove that there are two of them, say x and y, such that

$$0 \leq \frac{x-y}{1+xy} \leq \frac{1}{\sqrt{3}}.$$

Canadian Mathematical Olympiad
1985

PROBLEM 1

The lengths of the sides of a triangle are 6, 8 and 10 units. Prove that there is exactly one straight line which simultaneously bisects the area and perimeter of the triangle.

PROBLEM 2

Prove or disprove that there exists an integer which is doubled when the initial digit is transferred to the end.

PROBLEM 3

Let P_1 and P_2 be regular polygons of 1985 sides and perimeters x and y respectively. Each side of P_1 is tangent to a given circle of circumference c and this circle passes through each vertex of P_2. Prove $x + y \geq 2c$. (You may assume that $\tan\theta \geq \theta$ for $0 \leq \theta < \frac{\pi}{2}$).

PROBLEM 4

Prove that 2^{n-1} divides $n!$ if and only if $n = 2^{k-1}$ for some positive integer k.

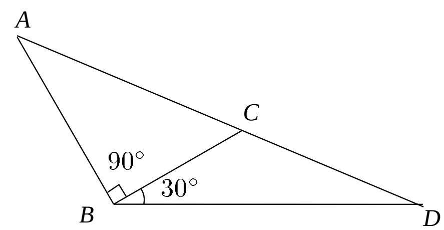
PROBLEM 5

Let $1 < x_1 < 2$ and, for $n = 1, 2, \ldots$, define $x_{n+1} = 1 + x_n - \frac{1}{2}x_n^2$. Prove that, for $n \geq 3$, $|x_n - \sqrt{2}| < 2^{-n}$.

Canadian Mathematical Olympiad
1986

PROBLEM 1

In the diagram line segments AB and CD are of length 1 while angles ABC and CBD are 90° and 30° respectively. Find AC.



PROBLEM 2

A Mathlon is a competition in which there are M athletic events. Such a competition was held in which only A, B, and C participated. In each event p_1 points were awarded for first place, p_2 for second and p_3 for third, where $p_1 > p_2 > p_3 > 0$ and p_1, p_2, p_3 are integers. The final score for A was 22, for B was 9 and for C was also 9. B won the 100 metres. What is the value of M and who was second in the high jump?

PROBLEM 3

A chord ST of constant length slides around a semicircle with diameter AB. M is the mid-point of ST and P is the foot of the perpendicular from S to AB. Prove that angle SPM is constant for all positions of ST.

PROBLEM 4

For positive integers n and k, define $F(n,k) = \sum_{r=1}^{n} r^{2k-1}$. Prove that $F(n,1)$ divides $F(n,k)$.

PROBLEM 5

Let $u_1, u_2, u_3, \ldots$ be a sequence of integers satisfying the recurrence relation $u_{n+2} = u_{n+1}^2 - u_n$. Suppose $u_1 = 39$ and $u_2 = 45$. Prove that 1986 divides infinitely many terms of the sequence.

Canadian Mathematical Olympiad
1987

PROBLEM 1

Find all solutions of $a^2 + b^2 = n!$ for positive integers a, b, n with $a \leq b$ and $n < 14$.

PROBLEM 2

The number 1987 can be written as a three digit number xyz in some base b. If $x + y + z = 1 + 9 + 8 + 7$, determine all possible values of x, y, z, b.

PROBLEM 3

Suppose $ABCD$ is a parallelogram and E is a point between B and C on the line BC. If the triangles DEC, BED and BAD are isosceles what are the possible values for the angle DAB?

PROBLEM 4

On a large, flat field n people are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd show that there is at least one person left dry. Is this always true when n is even?

PROBLEM 5

For every positive integer n show that

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}] = [\sqrt{4n+2}] = [\sqrt{4n+3}]$$

where $[x]$ is the greatest integer less than or equal to x (for example $[2.3] = 2$, $[\pi] = 3$, $[5] = 5$).

Canadian Mathematical Olympiad 1988

PROBLEM 1

For what values of b do the equations: $1988x^2 + bx + 8891 = 0$ and $8891x^2 + bx + 1988 = 0$ have a common root?

PROBLEM 2

A house is in the shape of a triangle, perimeter P metres and area A square metres. The garden consists of all the land within 5 metres of the house. How much land do the garden and house together occupy?

PROBLEM 3

Suppose that S is a finite set of at least five points in the plane; some are coloured red, the others are coloured blue. No subset of three or more similarly coloured points is collinear. Show that there is a triangle

(i) whose vertices are all the same colour, and

(ii) at least one side of the triangle does not contain a point of the opposite colour.

PROBLEM 4

Let $x_{n+1} = 4x_n - x_{n-1}$, $x_0 = 0$, $x_1 = 1$, and $y_{n+1} = 4y_n - y_{n-1}$, $y_0 = 1$, $y_1 = 2$. Show for all $n \geq 0$ that $y_n^2 = 3x_n^2 + 1$.

PROBLEM 5

Let $S = \{a_1, a_2, \ldots, a_r\}$ denote a sequence of integers. For each non-empty subsequence A of S, we define $p(A)$ to be the product of all the integers in A. Let $m(S)$ be the arithmetic average of $p(A)$ over all non-empty subsets A of S. If $m(S) = 13$ and if $m(S \cup \{a_{r+1}\}) = 49$ for some positive integer a_{r+1}, determine the values of $a_1, a_2, \ldots, a_r$ and a_{r+1}.

Canadian Mathematical Olympiad
1989

PROBLEM 1

The integers $1, 2, \ldots, n$ are placed in order so that each value is either strictly bigger than all the preceding values or is strictly smaller than all preceding values. In how many ways can this be done?

PROBLEM 2

Let ABC be a right angled triangle of area 1. Let $A'B'C'$ be the points obtained by reflecting A, B, C respectively, in their opposite sides. Find the area of $\triangle A'B'C'$.

PROBLEM 3

Define $\{a_n\}_{n=1}$ as follows: $a_1 = 1989^{1989}$; a_n, $n > 1$, is the sum of the digits of a_{n-1}. What is the value of a_5?

PROBLEM 4

There are 5 monkeys and 5 ladders and at the top of each ladder there is a banana. A number of ropes connect the ladders, each rope connects two ladders. No two ropes are attached to the same rung of the same ladder. Each monkey starts at the bottom of a different ladder. The monkeys climb up the ladders but each time they encounter a rope they climb along it to the ladder at the other end of the rope and then continue climbing upwards. Show that, no matter how many ropes there are, each monkey gets a banana.

PROBLEM 5

Given the numbers $1, 2, 2^2, \ldots, 2^{n-1}$. For a specific permutation $\sigma = X_1, X_2, \ldots, X_n$ of these numbers we define $S_1(\sigma) = X_1$, $S_2(\sigma) = X_1 + X_2$, $S_3(\sigma) = X_1 + X_2 + X_3, \ldots$ and $Q(\sigma) = S_1(\sigma)S_2(\sigma)\cdots S_n(\sigma)$. Evaluate $\sum 1/Q(\sigma)$ where the sum is taken over all possible permutations.

Canadian Mathematical Olympiad
1990

Problem 1

A competition involving $n \geq 2$ players was held over k days. On each day, the players received scores of $1, 2, 3, \ldots, n$ points with no two players receiving the same score. At the end of the k days, it was found that each player had exactly 26 points in total. Determine all pairs (n, k) for which this is possible.

Problem 2

A set of $\frac{1}{2}n(n+1)$ distinct numbers is arranged at random in a triangular array:

$$\begin{array}{ccccccccc} & & & & \times & & & & \\ & & & \times & & \times & & & \\ & & \times & & \times & & \times & & \\ & & \vdots & & \vdots & & \vdots & & \\ \times & \times & \cdot & & \cdot & & \cdot & \times & \times \end{array}$$

Let M_k be the largest number in the k-th row from the top. Find the probability that

$$M_1 < M_2 < M_3 < \cdots < M_n.$$

Problem 3

Let $ABCD$ be a convex quadrilateral inscribed in a circle, and let diagonals AC and BD meet at X. The perpendiculars from X meet the sides AB, BC, CD, DA at A', B', C', D' respectively. Prove that

$$|A'B'| + |C'D'| = |A'D'| + |B'C'|.$$

($|A'B'|$ is the length of line segment $A'B'$, *etc.*)

Problem 4

A particle can travel at speeds up to 2 metres per second along the x-axis, and up to 1 metre per second elsewhere in the plane. Provide a labelled sketch of the region which can be reached within one second by the particle starting at the origin.

Problem 5

Suppose that a function f defined on the positive integers satisfies

$$f(1) = 1, \quad f(2) = 2,$$
$$f(n+2) = f\big(n+2-f(n+1)\big) + f\big(n+1-f(n)\big) \quad (n \geq 1).$$

(a) Show that
  (i) $0 \leq f(n+1) - f(n) \leq 1$
  (ii) if $f(n)$ is odd, then $f(n+1) = f(n) + 1$.

(b) Determine, with justification, all values of n for which

$$f(n) = 2^{10} + 1.$$

Canadian Mathematical Olympiad
1991

PROBLEM 1

Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers x, y, z for which $xyz \neq 0$.

PROBLEM 2

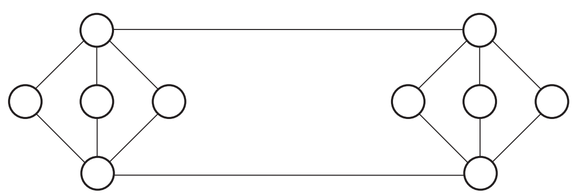
Let n be a fixed positive integer. Find the sum of all positive integers with the following property: In base 2, it has exactly $2n$ digits consisting of n 1's and n 0's. (The first digit cannot be 0.)

PROBLEM 3

Let C be a circle and P a given point in the plane. Each line through P which intersects C determines a chord of C. Show that the midpoints of these chords lie on a circle.

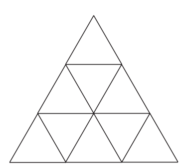
PROBLEM 4

Ten distinct numbers from the set $\{0, 1, 2, \ldots, 13, 14\}$ are to be chosen to fill in the ten circles in the diagram. The absolute values of the differences of the two numbers joined by each segment must be different from the values for all other segments. Is it possible to do this? Justify your answer.



PROBLEM 5

In the figure, the side length of the large equilateral triangle is 3 and $f(3)$, the number of parallelograms bounded by sides in the grid, is 15. For the general analogous situation, find a formula for $f(n)$, the number of parallelograms, for a triangle of side length n.



Canadian Mathematical Olympiad
1992

PROBLEM 1

Prove that the product of the first n natural numbers is divisible by the sum of the first n natural numbers if and only if $n+1$ is not an odd prime.

PROBLEM 2

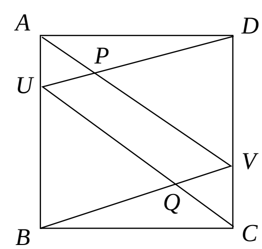
For $x, y, z \geq 0$, establish the inequality

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z)$$

and determine when equality holds.

PROBLEM 3

In the diagram, $ABCD$ is a square, with U and V interior points of the sides AB and CD respectively. Determine all the possible ways of selecting U and V so as to maximize the area of the quadrilateral $PUQV$.



PROBLEM 4

Solve the equation

$$x^2 + \frac{x^2}{(x+1)^2} = 3.$$

PROBLEM 5

A deck of $2n+1$ cards consists of a joker and, for each number between 1 and n inclusive, two cards marked with that number. The $2n+1$ cards are placed in a row, with the joker in the middle. For each k with $1 \leq k \leq n$, the two cards numbered k have exactly $k-1$ cards between them. Determine all the values of n not exceeding 10 for which this arrangement is possible. For which values of n is it impossible?

Canadian Mathematical Olympiad
1993

Problem 1

Determine a triangle for which the three sides and an altitude are four consecutive integers and for which this altitude partitions the triangle into two right triangles with integer sides. Show that there is only one such triangle.

Problem 2

Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence

$$x,\ x+1,\ x+2,\ x+3,\ldots$$

Problem 3

In triangle ABC, the medians to the sides AB and AC are perpendicular. Prove that $\cot B + \cot C \geq \frac{2}{3}$.

Problem 4

A number of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a *single* and that between a boy and a girl was called a *mixed single*. The total number of boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most how many schools were represented by an odd number of players?

Problem 5

Let $y_1, y_2, y_3, \ldots$ be a sequence such that $y_1 = 1$ and, for $k > 0$, is defined by the relationship:

$$y_{2k} = \begin{cases} 2y_k & \text{if } k \text{ is even} \\ 2y_k + 1 & \text{if } k \text{ is odd} \end{cases}$$

$$y_{2k+1} = \begin{cases} 2y_k & \text{if } k \text{ is odd} \\ 2y_k + 1 & \text{if } k \text{ is even} \end{cases}$$

Show that the sequence $y_1, y_2, y_3, \ldots$ takes on every positive integer value exactly once.

Canadian Mathematical Olympiad
1994

PROBLEM 1

Evaluate the sum

$$\sum_{n=1}^{1994}(-1)^n\frac{n^2+n+1}{n!}.$$

PROBLEM 2

Show that every positive integral power of $\sqrt{2}-1$ is of the form $\sqrt{m}-\sqrt{m-1}$ for some positive integer m. (e.g. $(\sqrt{2}-1)^2 = 3-2\sqrt{2} = \sqrt{9}-\sqrt{8}$).

PROBLEM 3

Twenty-five men sit around a circular table. Every hour there is a vote, and each must respond *yes* or *no*. Each man behaves as follows: on the n^{th} vote, if his response is the same as the response of at least one of the two people he sits between, then he will respond the same way on the $(n+1)^{th}$ vote as on the n^{th} vote; but if his response is different from that of both his neighbours on the n-th vote, then his response on the $(n+1)$-th vote will be different from his response on the n^{th} vote. Prove that, however everybody responded on the first vote, there will be a time after which nobody's response will ever change.

PROBLEM 4

Let AB be a diameter of a circle Ω and P be any point *not* on the line through A and B. Suppose the line through P and A cuts Ω again in U, and the line through P and B cuts Ω again in V. (Note that in case of tangency U may coincide with A or V may coincide with B. Also, if P is on Ω then $P = U = V$.) Suppose that $|PU| = s|PA|$ and $|PV| = t|PB|$ for some nonnegative real numbers s and t. Determine the cosine of the angle APB in terms of s and t.

PROBLEM 5

Let ABC be an acute angled triangle. Let AD be the altitude on BC, and let H be any interior point on AD. Lines BH and CH, when extended, intersect AC and AB at E and F, respectively. Prove that $\angle EDH = \angle FDH$.

SOLUTIONS

QUESTION 1

Solution 1.

Let $\mathcal{S}$ denote the given sum. Then

$$
\begin{aligned}
\mathcal{S} &= \sum_{n=1}^{1994}(-1)^n\left(\frac{n}{(n-1)!}+\frac{n+1}{n!}\right)\\
&= \sum_{n=0}^{1993}(-1)^{n+1}\frac{n+1}{n!}+\sum_{n=1}^{1994}(-1)^n\frac{n+1}{n!}\\
&= -1+\frac{1995}{1994!}
\end{aligned}
$$

Solution 2.

For positive integers k, define

$$\mathcal{S}(k)=\sum_{n=1}^{k}(-1)^n\frac{n^2+n+1}{n!}.$$

We prove by induction on k that

$$(*) \qquad \mathcal{S}(k)=-1+(-1)^k\frac{k+1}{k!}.$$

The given sum is the case when $k = 1994$. For $k = 1$, $\mathcal{S}(1) = -3 = -1 - \frac{2}{1!}$. Suppose (*) holds for some $k \geq 1$, then

$$
\begin{aligned}
\mathcal{S}(k+1) &= \mathcal{S}(k)+(-1)^{k+1}\frac{(k+1)^2+(k+1)+1}{(k+1)!}\\
&= -1+(-1)^k\frac{k+1}{k!}+(-1)^{k+1}\left(\frac{k+1}{k!}+\frac{k+2}{(k+1)!}\right)\\
&= -1+(-1)^{k+1}\frac{k+2}{(k+1)!}
\end{aligned}
$$

completing the induction.

QUESTION 2

Solution 1.

Fix a positive integer n. Let $a = (\sqrt{2}-1)^n$ and $b = (\sqrt{2}+1)^n$. Then clearly $ab = 1$. Let $c = (b+a)/2$ and $d = (b-a)/2$. If n is even, $n = 2k$, then from the Binomial Theorem we get

$$\begin{aligned} c &= \frac{1}{2}\sum_{i=0}^{n} \binom{n}{i} \left(\sqrt{2}^{n-i} + (-1)^i \sqrt{2}^{n-i}\right) \\ &= \sum_{j=0}^{k} \binom{2k}{2j} \sqrt{2}^{2k-2j} \\ &= \sum_{j=0}^{k} \binom{2k}{2j} 2^{k-j} \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{d}{\sqrt{2}} &= \frac{1}{\sqrt{2}}\sum_{i=0}^{n} \binom{n}{i} \left(\sqrt{2}^{n-i} - (-1)^i \sqrt{2}^{n-i}\right) \\ &= \frac{2}{\sqrt{2}}\sum_{j=0}^{k-1} \binom{2k}{2j+1} \sqrt{2}^{2k-2j-1} \\ &= \sum_{j=0}^{k-1} \binom{2k}{2j+1} 2^{k-j} \end{aligned} \tag{2}$$

showing that c and $\frac{d}{\sqrt{2}}$ are both positive integers. Similarly, when n is odd we see that $\frac{c}{\sqrt{2}}$ and d are both positive integers. In either case, c^2 and d^2 are both integers. Notes that

$$c^2 - d^2 = \frac{1}{4}((b+a)^2 - (b-a)^2) = ab = 1.$$

Hence if we let $m = c^2$, then $m - 1 = c^2 - 1 = d^2$ and $a = c - d = \sqrt{m} - \sqrt{m-1}$.

Solution 2.

Let m and n be positive integers. Observe that

$$(\sqrt{2}-1)^n(\sqrt{2}+1)^n = 1 = (\sqrt{m}-\sqrt{m-1})(\sqrt{m}+\sqrt{m-1})$$

and so

$$(*) \quad (\sqrt{2}-1)^n = \sqrt{m}-\sqrt{m-1} \text{ if and only if } (\sqrt{2}+1)^n = \sqrt{m}+\sqrt{m-1}.$$

Assuming m and n satisfy (*), then adding the two equivalent equations we get $2\sqrt{m} = (\sqrt{2}-1)^n + (\sqrt{2}+1)^n$ whence:

$$(**) \qquad m = \frac{1}{4}[(\sqrt{2}-1)^{2n} + 2 + (\sqrt{2}+1)^{2n}].$$

Now we show that the steps above are reversible and that m defined by (**) is a positive integer. From (**) one sees easily that

$$\sqrt{m} = \frac{1}{2}[(\sqrt{2}-1)^n + (\sqrt{2}+1)^n] \text{ and } \sqrt{m-1} = \frac{1}{2}[(\sqrt{2}+1)^n - (\sqrt{2}-1)^n],$$

and so $\sqrt{m}-\sqrt{m-1} = (\sqrt{2}-1)^n$ as required. Finally, from the Binomial Theorem,

$$\begin{aligned}(\sqrt{2}-1)^{2n} + (\sqrt{2}+1)^{2n} &= \\ &= \sum_{k=0}^{2n} \binom{2n}{k} [(-1)^k 2^{(2n-k)/2} + 2^{(2n-k)/2}] \\ &= \sum_{\ell=0}^{n} \binom{2n}{2\ell} 2^{n-\ell+1}\end{aligned}$$

which is congruent to 2 modulo 4 since $2^{n-\ell+1} \equiv 0 \pmod 4$ for all $\ell = 0, 1, 2, \cdots, n-1$. Therefore, $(\sqrt{2}-1)^{2n} + 2 + (\sqrt{2}+1)^{2n}$ is a multiple of 4, as required.

Solution 3.

We show by induction that

$$(*) \qquad (\sqrt{2}-1)^n = \begin{cases} a\sqrt{2}-b \text{ where } 2a^2 = b^2+1 \text{ if } n \text{ is odd} \\ a-b\sqrt{2} \text{ where } a^2 = 2b^2+1 \text{ if } n \text{ is even} \end{cases}.$$

Thus $m = 2a^2$ when n is odd and $m = a^2$ when n is even and the problem is solved.

The induction is as follows:

$$(\sqrt{2}-1)^1 = 1\sqrt{2}-1 \text{ where } 2(1^2) = 1^2+1$$
$$(\sqrt{2}-1)^2 = 3-2\sqrt{2} \text{ where } 3^2 = 2(2^2)+1.$$

Assume (*) holds for some $n \geq 1$, n odd. Then

$$\begin{aligned}(\sqrt{2}-1)^{n+1} & \\ &= (a\sqrt{2}-b)(\sqrt{2}-1) \text{ where } 2a^2 = b^2+1 \\ &= (2a+b)-(a+b)\sqrt{2} \\ &= A-B\sqrt{2} \text{ where } A = 2a+b, B = a+b.\end{aligned}$$

Moreover, $A^2 = 2a^2+4ab+b^2+2a^2 = 2a^2+4ab+2b^2+1 = 2B^2+1$.

Assume (*) holds for some $n \geq 2$, n even. Then

$$\begin{aligned}(\sqrt{2}-1)^{n+1} & \\ &= (a-b\sqrt{2})(\sqrt{2}-1) \text{ where } a^2 = 2b^2+1 \\ &= (a+b)\sqrt{2}-(a+2b) \\ &= A\sqrt{2}-B \text{ where } A = a+b, B = a+2b.\end{aligned}$$

Moreover, $2A^2 = 2a^2+4ab+2b^2 = a^2+4ab+4b^2+a^2-2b^2 = B^2+1$.

Solution 4.

From $(\sqrt{2}-1)^1 = \sqrt{2}-1$, $(\sqrt{2}-1)^2 = 3-2\sqrt{2}$, $(\sqrt{2}-1)^3 = 5\sqrt{2}-7$, $(\sqrt{2}-1)^4 = 17-12\sqrt{2}$, etc, we conjecture that

$$(*) \qquad (\sqrt{2}-1)^n = s_n\sqrt{2}+t_n$$

where $s_1 = 1$, $t_1 = 1$, $s_{n+1} = (-1)^n(|s_n|+|t_n|)$, $t_{n+1} = (-1)^{n+1}(2|s_n|+|t_n|)$.

Note that s_n is positive (negative) if n is odd (even) and t_n is negative (positive) if n is odd (even).

We now show by induction that (*) holds and that each $s_n\sqrt{2}+t_n$ of the form $\sqrt{m}-\sqrt{m-1}$ for some m.

It is easily verified that (*) is correct for $n = 1$ and 2. Assume (*) holds for some $n \geq 2$. Then

$$(\sqrt{2}-1)^{n+1} = (s_n\sqrt{2}+t_n)(\sqrt{2}-1) = (t_n - s_n)\sqrt{2} + (2s_n - t_n).$$

If n is odd, then

$$\begin{aligned} t_n - s_n &= -(|t_n|+|s_n|) = s_{n+1} \\ 2s_n - t_n &= 2|s_n|+|t_n| = t_{n+1}. \end{aligned}$$

If n is even, then

$$\begin{aligned} t_n - s_n &= |t_n|+|s_n| = s_{n+1} \\ 2s_n - t_n &= -2|s_n|-|t_n| = t_{n+1}. \end{aligned}$$

We have shown that (*) is correct for all n.

Observe now that $(s_{n+1}\sqrt{2})^2 - t_{n+1}^2 = 2(s_n^2 - 2s_nt_n + t_n^2) - (4s_n^2 - 4s_nt_n + t_n^2) = -2s_n^2 + t_n^2 = -((s_n\sqrt{2})^2 - t_n^2)$. Since $(s_1\sqrt{2})^2 - t_1^2 = 1$, it follows that $(s_n\sqrt{2})^2 - t_n^2 = (-1)^{n+1}$ for all n. To complete the proof it suffices to take $m = (s_n\sqrt{2})^2$, $m-1 = t_n^2$ when n is odd and $m = t_n^2$, $m-1 = (s_n\sqrt{2})^2$ when n is even.

QUESTION 3

First observe that if two neighbours have the same response on the n^{th} vote, then they both will respond the same way on the $(n+1)^{th}$ vote. Moreover, neither will ever change his response after the n^{th} vote.

Let A_n be the set of men who agree with at least one of their neighbours on the n^{th} vote. The previous paragraph says that $A_n \subset A_{n+1}$ for every $n \geq 1$. Moreover, we will be done if we can show that A_n contains all 25 men for some n.

Since there are an odd number of men at the table, it is not pssible that every man disagrees with both of his neighbours on the first vote. Therefore A_1 contains at least two men. And since $A_n \subset A_{n+1}$ for every n, there exists a $T < 25$ such that $A_T = A_{T+1}$. Suppose that A_T does not contain all 25 men; we shall use this to derive a contradiction. Since A_T is not empty, there must exist two neighbours, whom we shall call x and y, such that $x \in A_T$ and $y \notin A_T$. Since $x \in A_T$, he will respond the same way on the T^{th} and $(T+1)^{th}$ votes. But $y \notin A_T$, so y's response on the T^{th} vote differs from x's response. In fact, we know that y disagrees with both of his neighbours on the T^{th} vote, and so he will change his response on the $(T+1)^{th}$ vote. Therefore, on the $(T+1)^{th}$ vote, y responds the same way as does x. This implies that $y \in A_{T+1}$. But $y \notin A_T$, which contradicts the fact that $A_T = A_{T+1}$. Therefore we conclude that A_T contains all 25 men, and we are done.

QUESTION 4

There are three cases to be considered:

Case 1: If P is outside Ω (see figures I, II, and III), then since $\angle AUB = \angle AVB = \pi/2$, we have

$$\cos(\angle APB) = \frac{PU}{PB} = \frac{PV}{PA} = \sqrt{\frac{PU}{PA} \cdot \frac{PV}{PB}} = \sqrt{st}.$$

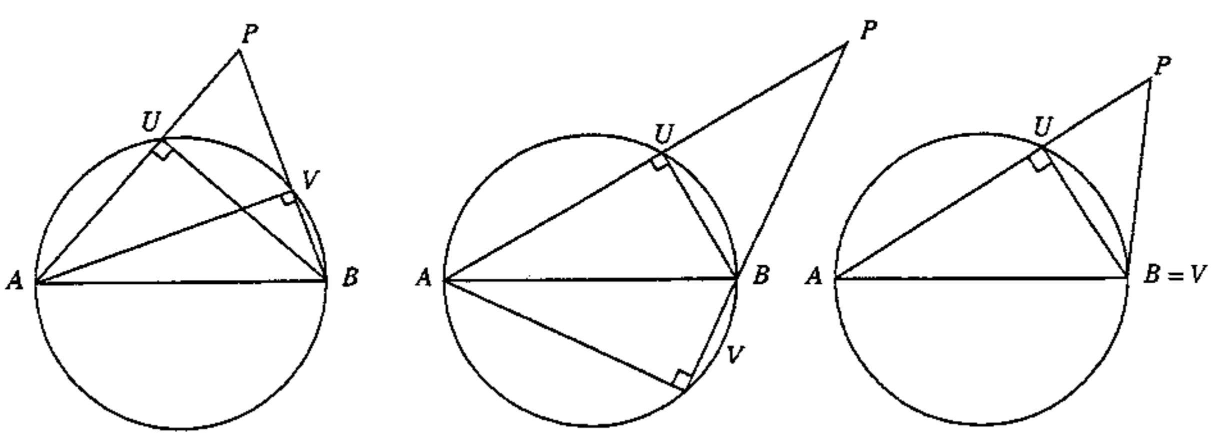


Figure I   Figure II   Figure III

Case 2: If P is on Ω (see figure IV), then

$$P = U = V \Rightarrow PU = PV = 0 \Rightarrow s = t = 0.$$

Since $\angle APB = \pi/2$, $\cos(\angle APB) = 0 = \sqrt{st}$ holds again.

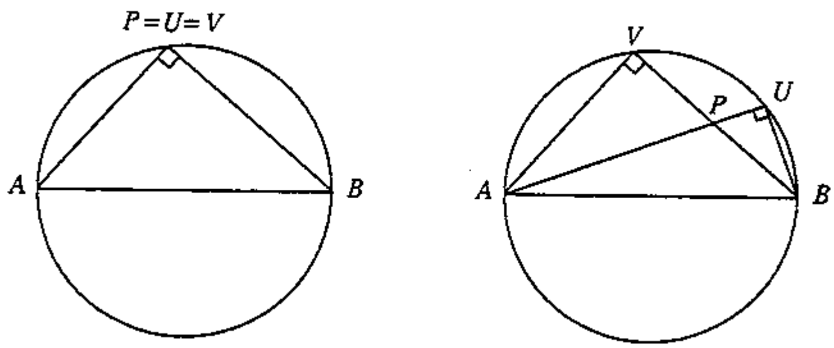


Figure IV

Figure V

Case 3: If P is inside Ω (figure V), then

$$\cos(\angle APB) = \cos(\pi - \angle APV) = -\cos(\angle APV) = -\frac{PV}{PA},$$

and

$$\cos(\angle APB) = \cos(\pi - \angle BPU) = -\cos(\angle BPU) = -\frac{PU}{PB}.$$

Therefore $\cos(\angle APB) = -\sqrt{\frac{PU}{PA} \cdot \frac{PV}{PB}} = -\sqrt{st}.$

QUESTION 5

100 degree angle question? bit harder?

Solution 1.

From A draw a lkine ℓ parallel to BC. Extend DF and DE to meet ℓ at P and Q respectively (See Figure I). Then from similar triangles, we have

$$\frac{AP}{BD} = \frac{AF}{FB} \text{ qne } \frac{AQ}{CD} = \frac{AE}{EC}$$

or

$$AP = \frac{AF}{FB} \cdot BD \text{ and } AQ = \frac{AE}{EC} \cdot CD. \qquad (1)$$

By Ceva's Theorem, $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$ and thus

$$\frac{AF}{FB} \cdot BD = \frac{AE}{EC} \cdot CD \qquad (2)$$

From (1) and (2) we get $AP = AQ$ and hence $\triangle ADP \simeq \triangle ADQ$ from which $\angle EDH = \angle FDH$ follows.

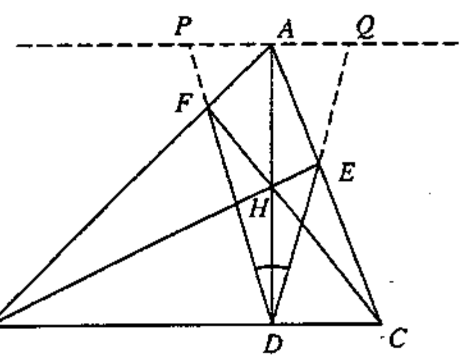


Figure I

Solution 2.

Use cartesian coordinates, with D at $(0,0)$, $A = (0,a)$, $B = (-b,0)$, $C = (c,0)$. Let $H = (0,h)$, $E = (u,v)$ and $F = (-r,s)$ where a,b,c,h,u,v,r,s are all positive (See Figure II).

It clearly suffices to show that $\frac{v}{u} = \frac{s}{r}$. Since EC and AC have the same slope, we have $\frac{v}{u-c} = \frac{a}{-c}$. Similarly, since EB and HB have the same slope, $\frac{v}{u+b} = \frac{k}{b}$. Thus

$$\frac{v}{a} = \frac{u-c}{-c} = \frac{-u}{c} + 1 \qquad (1)$$

and

$$\frac{v}{h} = \frac{u+b}{b} = \frac{u}{b} + 1 \qquad (2)$$

(2)-(1) we get $v(\frac{1}{h} - \frac{1}{a}) = u(\frac{1}{b} + \frac{1}{c})$ and thus

$$\frac{v}{u} = \frac{\frac{1}{b} + \frac{1}{c}}{\frac{1}{h} - \frac{1}{a}} = \frac{ah(b+c)}{bc(a-h)}.$$

With u, v, b and c replaced by $-r, s, -c$ and $-b$ respectively, we have, by a similar argument that

$$\frac{s}{-r} = \frac{ah(-c-b)}{bc(a-h)} \text{ or } \frac{s}{r} = \frac{ah(b+c)}{bc(a-h)}.$$

Therefore, $\frac{v}{u} = \frac{s}{r}$ as desired.

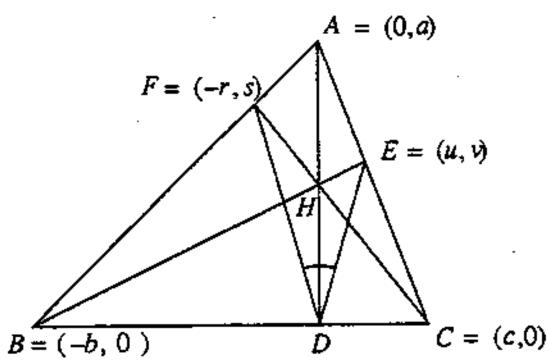


Figure II

Canadian Mathematical Olympiad
1995

PROBLEM 1

Let $f(x) = \frac{9^x}{9^x+3}$. Evaluate the sum

$$f(\frac{1}{1996}) + f(\frac{2}{1996}) + f(\frac{3}{1996}) + \cdots + f(\frac{1995}{1996})$$

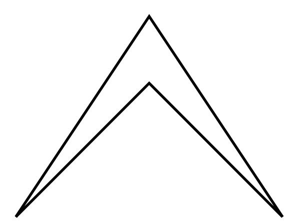
PROBLEM 2

Let a, b, and c be positive real numbers. Prove that

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}.$$

PROBLEM 3

Define a boomerang as a quadrilateral whose opposite sides do not intersect and one of whose internal angles is greater than 180 degrees. (See Figure displayed.) Let C be a convex polygon having 5 sides. Suppose that the interior region of C is the union of q quadrilaterals, none of whose interiors intersect one another. Also suppose that b of these quadrilaterals are boomerangs. Show that $q \geq b + \frac{s-2}{2}$.



PROBLEM 4

Let n be a fixed positive integer. Show that for only nonnegative integers k, the diophantine equation

$$x_1^3 + x_2^3 + \cdots + x_n^3 = y^{3k+2}$$

has infinitely many solutions in positive integers x_i and y.

PROBLEM 5

Suppose that u is a real parameter with $0 < u < 1$. Define

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq u \\ 1 - \left(\sqrt{ux} + \sqrt{(1-u)(1-x)}\right)^2 & \text{if } u \leq x \leq 1 \end{cases}$$

and define the sequence $\{u_n\}$ recursively as follows:

$$u_1 = f(1), \text{ and } u_n = f(u_{n-1}) \text{ for all } n > 1.$$

Show that there exists a positive ineger k for which $u_k = 0$.

SOLUTIONS

QUESTION 1

Solution

Note that

$$f(1-x) = \frac{9^{1-x}}{9^{1-x}+3} = \frac{9}{9+3\times 9^x} = \frac{3}{9^x+3},$$

from which we get

$$f(x) + f(1-x) = \frac{9^x}{9^x+3} + \frac{3}{9^x+3} = 1.$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{1995} f\left(\frac{k}{1996}\right) \\
&\quad = \sum_{k=1}^{997}\left[f\left(\frac{k}{1996}\right) + f\left(\frac{1996-k}{1996}\right)\right] + f\left(\frac{998}{1996}\right) && (1)\\
&\quad = \sum_{k=1}^{997}\left[f\left(\frac{k}{1996}\right) + f\left(1-\frac{k}{1996}\right)\right] + f\left(\frac{1}{2}\right) && (2)\\
&\quad = 997 + \frac{3}{3+3} = 997\frac{1}{2}. && (3)
\end{aligned}$$

QUESTION 2

Solution 1.

We prove equivalently that $a^{3a}b^{3b}c^{3c} \geq (abc)^{a+b+c}$. Due to complete symmetry in a, b and c, we may assume, without loss of generality, that $a \geq b \geq c$. Then $a - b \geq 0$, $b - c \geq 0$, $a - c \geq 0$ and $\frac{a}{b} \geq 1$, $\frac{b}{c} \geq 1$, $\frac{a}{c} \geq 1$. Therefore,

$$\frac{a^{3a}b^{3b}c^{3c}}{(abc)^{a+b+c}} = \left(\frac{a}{b}\right)^{a-b} \left(\frac{b}{c}\right)^{b-c} \left(\frac{a}{c}\right)^{a-c} \geq 1.$$

Solution 2.

If we assign the weights a, b, c to the numbers a, b, c, respectively, then by the wighted geometric-mean-harmonic-mean inequality followed by the arithmetic-mean-geometric-mean inequality, we get

$$\sqrt[a+b+c]{a^a b^b c^c} \geq \frac{a+b+c}{\frac{a}{a}+\frac{b}{b}+\frac{c}{c}} = \frac{a+b+c}{3} \geq \sqrt[3]{abc},$$

from which $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$ follows immediately.

QUESTION 3

Solution

For convenience, the interior angle in a boomerang which is greater than 180° will be called a "reflex angle".

Clearly, there are b reflex angles, each occurring in a different boomerang and each with the corresponding vertex in the interior of C. Angles around these vertices add up to $2b\pi$. On the other hand, the sum of all the interior angles of C is $(s-2)\pi$ and the sum of the interior angles of all the q quadrilaterals is $2\pi q$.

Therefore, $2\pi q \geq 2b\pi + (s-2)\pi$ from which $q \geq b + \frac{s-2}{2}$ follows.

QUESTION 4

Solution 1.

Since $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$, **we see that when** $k = 0$, $(x_1, x_2, \cdots, x_n; y) = \left(1, 2, \cdots, n; \frac{n(n+1)}{2}\right)$ **is a solution. To see that we can generate infinitely many solutions in general, set** $c = \frac{n(n+1)}{2}$ **and notice that for all positive integers** q, **we have:**

$$\begin{aligned}(c^k q^{3k+2})^3 &+ (2c^k q^{3k+2})^3 + \cdots + (nc^k q^{3k+2})^3 \\ &= c^{3k} q^{3(3k+2)}(1^3 + 2^3 + \cdots n^3) \\ &= c^{3k} q^{3(3k+2)} \left(\frac{n(n+1)}{2}\right)^2 \\ &= c^{3k+2} q^{3(3k+2)} = (cq^3)^{3k+2}.\end{aligned}$$

That is, $(x_1, x_2, \cdots, x_n; y) = (c^k q^{3k+2}, 2c^k q^{3k+2}, \cdots, nc^k q^{3k+2}; cq^3)$ **is a solution. This completes the proof.**

Solution 2.

For any positive integer q, **take** $x_1 = x_2 = \cdots = x_n = n^{2k+1} q^{3k+2}$, $y = n^2 q^3$. **Then**

$$\sum_{i=1}^{n} x_i^3 = n \cdot n^{6k+3} \cdot q^{9k+6} = (n^2 q^3)^{3k+2} = y^{3k+2}.$$

Solution 3.

If $n = 1$, take $x_1 = q^{3k+2}, y = q^3$ as in solution 2. For $n > 1$, we look for solutions of the form

$$x_1 = x_2 = \cdots x_n = n^p, \;\; y = n^q.$$

Then

$$\sum_{i=1}^{n} x_i^3 = y^{3k+2} \Leftrightarrow n^{3p+1} = n^{(3k+2)q} \Leftrightarrow 3p + 1 = (3k+2)q \Leftrightarrow (3k+2)q - 3p = 1.$$

The last equation is satisfied if we take

$q = 3t + 2$ **and** $p = (3k+2)t + (2k+1)$ **where** t

is any nonnegative integer. Thus, infinetely many solutions in positive integers are given by

$$x_1 = x_2 = \cdots x_n = n^{(3k+2)t+(2k+1)}, \;\; y = n^{3t+2}.$$

QUESTION 5

Solution

Note first that $u_1 = 1 - u$. **Since for all** $x \in [u, 1]$, $u \leq x$ **and** $1 - x \leq 1 - u$ **we have**

$$\begin{aligned} 1 - &\left(\sqrt{ux} + \sqrt{(1-u)(1-x)}\right)^2 \\ &= 1 - ux - (1-u)(1-x) - 2\sqrt{ux(1-u)(1-x)} \\ &= u + x - 2ux - 2\sqrt{ux(1-u)(1-x)} \\ &\leq u + x - 2ux - 2u(1-x) = x - u. \end{aligned}$$

Therefore,

$$f(x) = 0 \textbf{ if } 0 \leq x \leq u \tag{1}$$

and

$$f(x) \leq x - u \textbf{ if } u \leq x \leq 1. \tag{2}$$

From (2) we get $u_2 = f(u_1) \leq u_1 - u = 1 - 2u$ **if** $u_1 \geq u$. **An easy induction then yields**

$$u_{n+1} = f(u_n) \leq u_n - u \leq 1 - (n+1)u \textbf{ if } u_i \geq u \textbf{ for all } i = 1, 2, \cdots, n.$$

Thus for sufficiently large k, **we must have** $u_{k-1} < u$ **and then** $u_k = f(u_{k-1}) = 0$ **by (1).**

Canadian Mathematical Olympiad
1996

PROBLEM 1

If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

PROBLEM 2

Find all real solutions to the following system of equations. Carefully justify your answer.

$$\begin{cases} \dfrac{4x^2}{1+4x^2} = y \\ \dfrac{4y^2}{1+4y^2} = z \\ \dfrac{4z^2}{1+4z^2} = x \end{cases}$$

PROBLEM 3

We denote an arbitrary permutation of the integers $1, \ldots, n$ by $a_1, \ldots, a_n$. Let $f(n)$ be the number of these permutations such that

(i) $a_1 = 1$;
(ii) $|a_i - a_{i+1}| \leq 2, \quad i = 1, \ldots, n-1$.

Determine whether $f(1996)$ is divisible by 3.

PROBLEM 4

Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

PROBLEM 5

Let $r_1, r_2, \ldots, r_m$ be a given set of m positive rational numbers such that $\sum_{k=1}^{m} r_k = 1$. Define the function f by $f(n) = n - \sum_{k=1}^{m} [r_k n]$ for each positive integer n. Determine the minimum and maximum values of $f(n)$. Here $[x]$ denotes the greatest integer less than or equal to x

CMO 1996
SOLUTIONS

QUESTION 1

Solution .

If $f(x) = x^3 - x - 1 = (x-\alpha)(x-\beta)(x-\gamma)$ has roots α, β, γ standard results about roots of polynomials give $\alpha + \beta + \gamma = 0$, $\alpha\beta + \alpha\gamma + \beta\gamma = -1$, and $\alpha\beta\gamma = 1$.

Then

$$S = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = \frac{N}{(1-\alpha)(1-\beta)(1-\gamma)}$$

where the numerator simplifies to

$$\begin{aligned} N &= 3 - (\alpha+\beta+\gamma) - (\alpha\beta + \alpha\gamma + \beta\gamma) + 3\alpha\beta\gamma \\ &= 3 - (0) - (-1) + 3(1) \\ &= 7. \end{aligned}$$

The denominator is $f(1) = -1$ so the required sum is -7.

QUESTION 2

Solution 1.

For any t, $0 \le 4t^2 < 1+4t^2$, so $0 \le \dfrac{4t^2}{1+4t^2} < 1$. Thus x, y and z must be non-negative and less than 1.

Observe that if one of x y or z is 0, then $x = y = z = 0$.

If two of the variables are equal, say $x = y$, then the first equation becomes

$$\frac{4x^2}{1+4x^2} = x.$$

This has the solution $x = 0$, which gives $x = y = z = 0$ and $x = \dfrac{1}{2}$ which gives $x = y = z = \dfrac{1}{2}$.

Finally, assume that x, y and z are non-zero and distinct. Without loss of generality we may assume that either $0 < x < y < z < 1$ or $0 < x < z < y < 1$. The two proofs are similar, so we do only the first case.

We will need the fact that $f(t) = \dfrac{4t^2}{1+4t^2}$ is increasing on the interval $(0, 1)$.

To prove this, if $0 < s < t < 1$ then

$$\begin{aligned} f(t) - f(s) &= \frac{4t^2}{1+4t^2} - \frac{4s^2}{1+4s^2} \\ &= \frac{4t^2 - 4s^2}{(1+4s^2)(1+4t^2)} \\ &> 0. \end{aligned}$$

So $0 < x < y < z \Rightarrow f(x) = y < f(y) = z < f(z) = x$, a contradiction.

Hence $x = y = z = 0$ and $x = y = z = \dfrac{1}{2}$ are the only real solutions.

Solution 2.

Notice that x, y and z are non-negative. Adding the three equations gives

$$x + y + z = \frac{4z^2}{1+4z^2} + \frac{4x^2}{1+4x^2} + \frac{4y^2}{1+4y^2}.$$

This can be rearranged to give

$$\frac{x(2x-1)^2}{1+4x^2} + \frac{y(2y-1)^2}{1+4y^2} + \frac{z(2z-1)^2}{1+4z^2} = 0.$$

Since each term is non-negative, each term must be 0, and hence each variable is either 0 or $\dfrac{1}{2}$. The original equations then show that $x = y = z = 0$ and $x = y = z = \dfrac{1}{2}$ are the only two solutions.

Solution 3.

Notice that x, y, and z are non-negative. Multiply both sides of the inequality

$$\frac{y}{1+4y^2} \geq 0$$

by $(2y-1)^2$, and rearrange to obtain

$$y - \frac{4y^2}{1+4y^2} \geq 0,$$

and hence that $y \geq z$. Similarly, $z \geq x$, and $x \geq y$. Hence, $x = y = z$ and, as in Solution 1, the two solutions follow.

Solution 4.

As for solution 1, note that $x = y = z = 0$ is a solution and any other solution will have each of x, y and z positive.

The arithmetic-geometric mean inequality (or direct computation) shows that $\dfrac{1+4x^2}{2} \geq \sqrt{1 \cdot 4x^2} = 2x$ and hence $x \geq \dfrac{4x^2}{1+4x^2} = y$, with equality if and only if $1 = 4x^2$ – that is, $x = \dfrac{1}{2}$. Similarly, $y \geq z$ with equality if and only if $y = \dfrac{1}{2}$ and $z \geq x$ with equality if and only if $z = \dfrac{1}{2}$. Adding $x \geq y$, $\; y \geq z$ and $z \geq x$ gives $x+y+x \geq x+y+z$. Thus equality must occur in each inequality, so $x = y = z = \dfrac{1}{2}$.

QUESTION 3

Solution.

Let $a_1, a_2, \ldots, a_n$ be a permutation of $1, 2, \ldots, n$ with properties (i) and (ii).

A crucial observation, needed in Case II (b) is the following: If a_k and a_{k+1} are consecutive integers (i.e. $a_{k+1} = a_k \pm 1$), then the terms to the right of a_{k+1} (also to the left of a_k) are either all less than both a_k and a_{k+1} or all greater than both a_k and a_{k+1}.

Since $a_1 = 1$, by (ii) a_2 is either 2 or 3.

CASE I: Suppose $a_2 = 2$. Then $a_3, a_4, \ldots, a_n$ is a permutation of $3, 4, \ldots, n$. Thus $a_2, a_3, \ldots, a_n$ is a permutation of $2, 3, \ldots, n$ with $a_2 = 2$ and property (ii). Clearly there are $f(n-1)$ such permutations.

CASE II: Suppose $a_2 = 3$.

(a) Suppose $a_3 = 2$. Then $a_4, a_5, \ldots, a_n$ is a permutation of $4, 5, \ldots, n$ with $a_4 = 4$ and property (ii). There are $f(n-3)$ such permutations.

(b) Suppose $a_3 \geq 4$. If a_{k+1} is the first even number in the permutation then, because of (ii), $a_1, a_2, \ldots, a_k$ must be $1, 3, 5, \ldots, 2k-1$ (in that order). Then a_{k+1} is either $2k$ or $2k-2$, so that a_k and a_{k+1} are consecutive integers. Applying the crucial observation made above, we deduce that $a_{k+2}, \ldots, a_n$ are all either greater than or smaller than a_k and a_{k+1}. But 2 must be to the right of a_{k+1}. Hence $a_{k+2}, \ldots, a_n$ are the even integers less than a_{k+1}. The only possibility then, is

$$1, 3, 5, \ldots, a_{k-1}, a_k, \ldots, 6, 4, 2.$$

Cases I and II show that

$$f(n) = f(n-1) + f(n-3) + 1, \quad n \geq 4. \qquad (*)$$

Calculating the first few values of $f(n)$ directly gives

$$f(1) = 1,\ f(2) = 1,\ f(3) = 2,\ f(4) = 4,\ f(5) = 6.$$

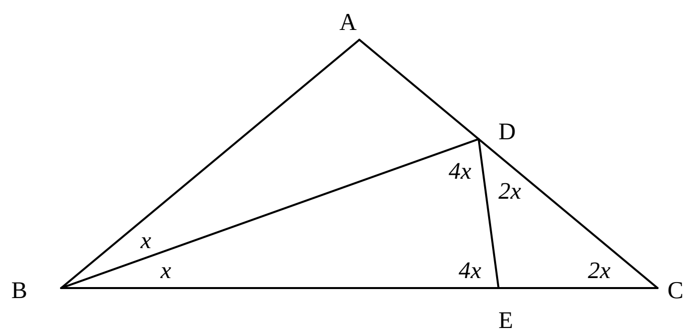
Calculating a few more $f(n)$'s using (*) and mod 3 arithmetic, $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 1$, $f(5) = 0$, $f(6) = 0$, $f(7) = 2$, $f(8) = 0$, $f(9) = 1$, $f(10) = 1$, $f(11) = 2$. Since $f(1) = f(9)$, $f(2) = f(10)$ and $f(3) = f(11)$ mod 3, (*) shows that $f(a) = f(a \mathit{ mod } 8)$, mod 3, $a \geq 1$.

Hence $f(1996) \equiv f(4) \equiv 1 \pmod 3$ so 3 does not divide $f(1996)$.

QUESTION 4

Solution 1.

Let $BE = BD$ with E on BC, so that $AD = EC$:



By a standard theorem, $\dfrac{AB}{CB} = \dfrac{AD}{DC}$; so in

$\triangle\, CED$ and $\triangle\, CAB$ we have a common angle and

$$\frac{CE}{CD} = \frac{AD}{CD} = \frac{AB}{CB} = \frac{CA}{CB} .$$

Thus $\triangle\, CED \sim \triangle\, CAB$, so that $\angle\, CDE = \angle\, DCE = \angle\, ABC = 2x$.
Hence $\angle BDE = \angle BED = 4x$, whence $9x = 180^\circ$ so $x = 20^\circ$.
Thus $\angle A = 180^\circ - 4x = 100^\circ$.

Solution 2.

Apply the law of sines to $\triangle ABD$ and $\triangle BDC$ to get

$$\frac{AD}{BD} = \frac{\sin x}{\sin 4x} \quad \text{and} \quad 1 + \frac{AD}{BD} = \frac{BC}{BD} = \frac{\sin 3x}{\sin 2x} .$$

Now massage the resulting trigonometric equation with standard identities to get

$$\sin 2x\,(\sin 4x + \sin x) = \sin 2x\,(\sin 5x + \sin x) .$$

Since $0 < 2x < 90^\circ$, we get

$$5x - 90^\circ = 90^\circ - 4x ,$$

so that $\angle A = 100^\circ$.

QUESTION 5

Solution.

Let

$$f(n) = n - \sum_{k=1}^{m} [r_k n]$$
$$= n \sum_{k=1}^{m} r_k - \sum_{k=1}^{m} [r_k n]$$
$$= \sum_{k=1}^{m} \{r_k n - [r_k n]\}.$$

Now $0 \leq x - [x] < 1$, and if c is an integer, $(c + x) - [c + x] = x - [x]$.

Hence $0 \leq f(n) < \sum_{k=1}^{m} 1 = m$. Because $f(n)$ is an integer, $0 \leq f(n) \leq m - 1$.

To show that $f(n)$ can achieve these bounds for $n > 0$, we assume that $r_k = \dfrac{a_k}{b_k}$ where a_k, b_k are integers; $a_k < b_k$.

Then, if $n = b_1 b_2 \ldots b_m,\ (r_k n) - [r_k n] = 0,\ k = 1, 2, \ldots, m$ and thus $f(n) = 0$.

Letting $n = b_1 b_2 \ldots b_n - 1$, then

$$\begin{aligned} r_k n &= r_k (b_1 b_2 \ldots b_m - 1) \\ &= r_k \{(b_1 b_2 \ldots b_m - b_k) + b_k - 1)\} \\ &= \text{integer} + r_k (b_k - 1). \end{aligned}$$

This gives

$$\begin{aligned} r_k n - [r_k n] &= r_k (b_k - 1) - [r_k (b_k - 1)] \\ &= \frac{a_k}{b_k}(b_k - 1) - \left[\frac{a_k}{b_k}(b_k - 1)\right] \\ &= \left(a_k - \frac{a_k}{b_k}\right) - \left[a_k - \frac{a_k}{b_k}\right] \\ &= \left(a_k - \frac{a_k}{b_k}\right) - (a_k - 1) \\ &= 1 - \frac{a_k}{b_k} = 1 - r_k. \end{aligned}$$

Hence

$$f(n) = \textstyle\sum_{k=1}^{m} (1 - r_k) = m - 1.$$

Canadian Mathematical Olympiad
1997

PROBLEM 1

How many pairs of positive integers x, y are there, with $x \leq y$, and such that $\gcd(x, y) = 5!$ and $\mathrm{lcd}(x, y) = 50!$.

NOTE. $\gcd(x, y)$ denotes the greatest common divisor of x and y, $\mathrm{lcd}(x, y)$ denotes the least common multiple of x and y, and $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

PROBLEM 2

The closed interval $A = [0, 50]$ is the union of a finite number of closed intervals, each of length 1. Prove that some of the intervals can be removed so that those remaining are mutually disjoint and have total length ≥ 25.

NOTE. For $a \leq b$, the closed interval $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ has length $b - a$; disjoint intervals have *empty* intersection.

PROBLEM 3

Prove that
$$\frac{1}{1999} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{1997}{1998} < \frac{1}{44}.$$

PROBLEM 4

The point O is situated inside the parallelogram $ABCD$ so that

$$\angle AOB + \angle COD = 180^\circ.$$

Prove that $\angle OBC = \angle ODC$.

PROBLEM 5

Write the sum

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k^3 + 9k^2 + 26k + 24}$$

in the form $\frac{p(n)}{q(n)}$, where p and q are polynomials with integer coefficients.

1997
SOLUTIONS

Problem 1 – Deepee Khosla, Lisgar Collegiate Institute, Ottawa, ON

Let $p_1, \ldots, p_{12}$ denote, in increasing order, the primes from 7 to 47. Then

$$5! = 2^3 \cdot 3^1 \cdot 5^1 \cdot p_1^0 \cdot p_2^0 \ldots p_{12}^0$$

and

$$50! = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot p_1^{b_1} \cdot p_2^{b_2} \ldots p_{12}^{b_{12}}.$$

Note that $2^4, 3^2, 5^2, p_1, \ldots, p_{12}$ all divide 50!, so all its prime powers differ from those of 5!

Since $x, y | 50!$, they are of the form

$$\begin{aligned} x &= 2^{n_1} \cdot 3^{n_2} \cdot \ldots p_{12}^{n_{15}} \\ y &= 2^{m_1} \cdot 3^{m_2} \cdot \ldots p_{12}^{m_{15}}. \end{aligned}$$

Then $\max(n_i, m_i)$ is the ith prime power in 50!
and $\min(n_i, m_i)$ is the ith prime power in 5!

Since, by the above note, the prime powers for p_{12} and under differ in 5! and 50!, there are 2^{15} choices for x, only half of which will be less than y. (Since for each choice of x, y is forced and either $x < y$ or $y < x$.) So the number of pairs is $2^{15}/2 = 2^{14}$.

Problem 2 – Byung Kuy Chun, Harry Ainlay Composite High School, Edmonton, AB

Look at the first point of each given unit interval. This point uniquely defines the given unit interval.

<u>Lemma</u>. In any interval $[x, x+1)$ there must be at least one of these first points ($0 \leq x \leq 49$).

<u>Proof</u>. Suppose the opposite. The last first point before x must be $x - \varepsilon$ for some $\varepsilon > 0$. The corresponding unit interval ends at $x - \varepsilon + 1 < x + 1$. However, the next given unit interval cannot begin until at least $x + 1$.

This implies that points $(x - \varepsilon + 1,\ x + 1)$ are not in set A, a contradiction.

$\therefore$ There must be a first point in $[x,\ x+1)$. $\square$

Note that for two first points in intervals $[x, x+1)$ and $[x+2, x+3)$ respectively, the corresponding unit intervals are disjoint since the intervals are in the range $[x, x+2)$ and $[x+2, x+4)$ respectively.

$\therefore$ We can choose a given unit interval that begins in each of

$$[0,1)[2,3)\ldots[2k,2k+1)\ldots[48,49).$$

Since there are 25 of these intervals, we can find 25 points which correspond to 25 disjoint unit intervals.

Problem 2 – Colin Percival, Burnaby Central Secondary School, Burnaby, BC

I prove the more general result, that if $[0, 2n] = \bigcup_i A_i$, $|A_i| = 1$, A_i are intervals then $\exists a_1 \ldots a_n$, such that $A_{a_i} \bigcap A_{a_j} = \emptyset$.

Let $0 < \varepsilon \leq \dfrac{2}{n-1}$ and let $b_i = (i-1)(2+\varepsilon), i = 1 \ldots n$. Then

$$\min\{b_i\} = 0, \max\{b_i\} = (n-1)(2+\varepsilon) \leq (n-1)\left(2 + \frac{2}{n-1}\right) = (n-1)\left(\frac{2n}{n-1}\right) = 2n.$$

So all the b_i are in $[0, 2n]$.

Let a_i be such that $b_i \in A_{a_i}$. Since $\bigcup A_i = [0, 2n]$, this is possible.

Then since $(b_i - b_j) = (i-j)(2+\varepsilon) \geq 2 + \varepsilon > 2$, and the A_i are intervals of length 1, $\min A_{a_i} - \max A_{a_j} > 2 - 1 - 1 = 0$, so $A_{a_i} \bigcap A_{a_j} = \emptyset$.

Substituting $n = 25$, we get the required result. Q.E.D.

Problem 3 – Mihaela Enachescu, Dawson College, Montréal, PQ

Let $P = \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{1997}{1998}$. Then $\frac{1}{2} > \frac{1}{3}$ because $2 < 3$, $\frac{3}{4} > \frac{3}{5}$ because $4 < 5, \ldots,$

$\ldots \frac{1997}{1998} > \frac{1997}{1999}$ because $1998 < 1999$.

So

$$P > \frac{1}{3} \cdot \frac{3}{5} \cdot \ldots \cdot \frac{1997}{1999} = \frac{1}{1999} \quad . \qquad (1)$$

Also $\frac{1}{2} < \frac{2}{3}$ because $1 \cdot 3 < 2 \cdot 2$, $\frac{3}{4} < \frac{4}{5}$ because $3 \cdot 5 < 4 \cdot 4, \ldots$

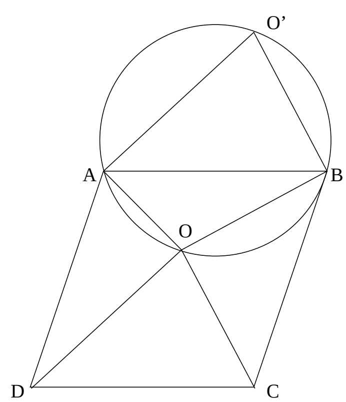
$\frac{1997}{1998} < \frac{1998}{1999}$ because $1997 \cdot 1999 = 1998^2 - 1 < 1998^2$.

So $P < \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{1998}{1999} = \underbrace{\left(\frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \ldots \cdot \frac{1998}{1997}\right)}_{\frac{1}{P}} \frac{1}{1999}$.

$$\text{Hence } P^2 < \frac{1}{1999} < \frac{1}{1936} = \frac{1}{44^2} \text{ and } P < \frac{1}{44} \quad . \qquad (2)$$

Then (1) and (2) give $\frac{1}{1999} < P < \frac{1}{44}$ (q.e.d.)

Problem 4 – Joel Kamnitzer, Earl Haig Secondary School, North York, ON



Consider a translation which maps D to A. It will map $0 \to 0'$ with $\overline{OO'} = \overline{DA}$, and C will be mapped to B because $\overline{CB} = \overline{DA}$.

This translation keeps angles invariant, so $\angle AO'B = \angle DOC = 180° - \angle AOB$.

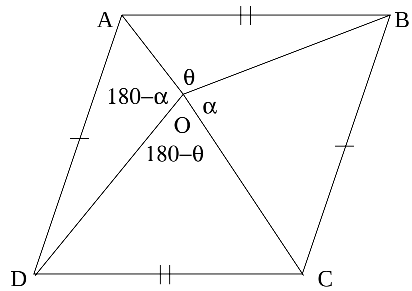
$\therefore AOBO'$ is a cyclic quadrilateral.

$\therefore \angle ODC = \angle O'AB = \angle O'OB$

but, since $O'O$ is parallel to BC,

$$\begin{aligned} \angle O'OB &= \angle OBC \\ \therefore \angle ODC &= \angle OBC. \end{aligned}$$

Problem 4 – Adrian Chan, Upper Canada College, Toronto, ON



Let $\angle AOB = \theta$ and $\angle BOC = \alpha$. Then $\angle COD = 180^\circ - \theta$ and $\angle AOD = 180^\circ - \alpha$.

Since $AB = CD$ (parallelogram) and $\sin\theta = \sin(180^\circ - \theta)$, the sine law on $\triangle OCD$ and $\triangle OAB$ gives

$$\frac{\sin\angle CDO}{OC} = \frac{\sin(180^\circ - \theta)}{CD} = \frac{\sin\theta}{AB} = \frac{\sin\angle ABO}{OA}$$

so

$$\frac{OA}{OC} = \frac{\sin\angle ABO}{\sin\angle CDO}. \tag{1}$$

Similarily, the sine law on $\triangle OBC$ and $\triangle OAD$ gives

$$\frac{\sin\angle CBO}{OC} = \frac{\sin\alpha}{BC} = \frac{\sin(180^\circ - \alpha)}{AD} = \frac{\sin\angle ADO}{OA}$$

so

$$\frac{OA}{OC} = \frac{\sin\angle ADO}{\sin\angle CBO}. \tag{2}$$

Equations (1) and (2) show that $\sin\angle ABO \cdot \sin\angle CBO = \sin\angle ADO \cdot \sin\angle CDO$ hence

$$\frac{1}{2}[\cos(\angle ABO + \angle CBO) - \cos(\angle ABO - \angle CBO)] = \frac{1}{2}[\cos(\angle ADO + \angle CDO) - \cos(\angle ADO - \angle CDO)].$$

Since $\angle ADC = \angle ABC$(parallelogram) and $\angle ADO + \angle CDO = \angle ADC$ and $\angle ABO + \angle CBO = \angle ABC$ it follows that $\cos(\angle ABO - \angle CBO) = \cos(\angle ADO - \angle CDO)$.

There are two cases to consider.

Case (i): $\angle ABO - \angle CBO = \angle ADO - \angle CDO$.

Since $\angle ABO + \angle CBO = \angle ADO + \angle CDO$, subtracting gives $2\,\angle CBO = 2\,\angle CDO$ so $\angle CBO = \angle CDO$, and we are done.

Case (ii): $\angle ABO - \angle CBO = \angle CDO - \angle ADO$.

Since we know that $\angle ABO + \angle CBO = \angle CDO + \angle ADO$, adding gives $2\,\angle ABO = 2\,\angle CDO$ so $\angle ABO = \angle CDO$ and $\angle CBO = \angle ADO$.

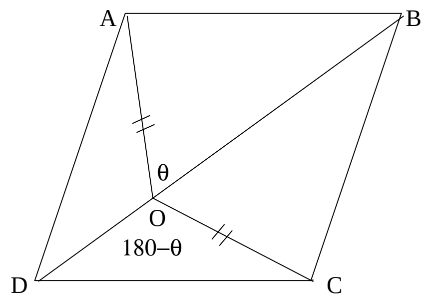
Substituting this into (1), it follows that $OA = OC$.

Also, $\angle ADO + \angle ABO = \angle CBO + \angle ABO = \angle ABC$.

Now, $\angle ABC = 180^\circ - \angle BAD$ since $ABCD$ is a parallelogram.

Hence $\angle BAD + \angle ADO + \angle ABO = 180^\circ$ so $\angle DOB = 180^\circ$ and D, O, B are collinear.

We now have the diagram



Then $\angle COD + \angle BOC = 180°$, so $\angle BOC = \theta = \angle AOB$.

$\triangle AOB$ is congruent to $\triangle COB$ (SAS, OB is common, $\angle AOB = \angle COB$ and $AO = CO$), so $\angle ABO = \angle CBO$. Since also $\angle ABO = \angle CDO$ we conclude that $\angle CBO = \angle CDO$.

Since it is true in both cases, then $\angle CBO = \angle CDO$.

Q.E.D.

Problem 5 – Sabin Cautis, Earl Haig Secondary School, North York, ON

We first note that

$$k^3 + 9k^2 + 26k + 24 = (k+2)(k+3)(k+4).$$

Let $S(n) = \displaystyle\sum_{k=0}^{n} \frac{(-1)^k \begin{pmatrix} n \\ k \end{pmatrix}}{k^2 + 9k^2 + 26k + 24}$.

Then

$$\begin{aligned} S(n) &= \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!(k+2)(k+3)(k+4)} \\ &= \sum_{k=0}^{n} \left(\frac{(-1)^k (n+4)!}{(k+4)!(n-k)!} \right) \times \left(\frac{k+1}{(n+1)(n+2)(n+3)(n+4)} \right). \end{aligned}$$

Let

$$T(n) = (n+1)(n+2)(n+3)(n+4)S(n) = \sum_{k=0}^{n} \left((-1)^k \begin{pmatrix} n+4 \\ k+4 \end{pmatrix} (k+1) \right).$$

Now, for $n \geq 1$,

$$\sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} = 0 \qquad\qquad (*)$$

since

$$(1-1)^n = \begin{pmatrix} n \\ 0 \end{pmatrix} - \begin{pmatrix} n \\ 1 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix} + \ldots + (-1)^n \begin{pmatrix} n \\ n \end{pmatrix} = 0.$$

Also

$$\begin{aligned} \sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} i &= \sum_{i=1}^{n} (-1)^i \frac{i \cdot n!}{i! \cdot (n-i)!} + (-1)^0 \cdot \frac{0 \cdot n!}{0! \cdot n!} \\ &= \sum_{i=1}^{n} (-1)^i \frac{n!}{(i-1)!(n-i)!} \\ &= \sum_{i=1}^{n} (-1)^i n \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} \\ &= n \sum_{i=1}^{n} (-1)^i \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} \\ &= -n \sum_{i=1}^{n} (-1)^{i-1} \begin{pmatrix} n-1 \\ i-1 \end{pmatrix}. \end{aligned}$$

Substituting $j = i - 1$, (*) shows that

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} i = -n \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} = 0. \qquad (**)$$

Hence

$$\begin{aligned}
T(n) &= \sum_{k=0}^{n} (-1)^k \binom{n+4}{k+4} (k+1) \\
&= \sum_{k=0}^{n} (-1)^{k+4} \binom{n+4}{k+4} (k+1) \\
&= \sum_{k=-4}^{n} (-1)^{k+4} \binom{n+4}{k+4} (k+1) - \left(-3 + 2(n+4) - \binom{n+4}{2} \right).
\end{aligned}$$

Substituting $j = k+4$,

$$\begin{aligned}
&= \sum_{j=0}^{n+4} (-1)^j \binom{n+4}{j} (j-3) - \left(2n + 8 - 3 - \frac{(n+4)(n+3)}{2} \right) \\
&= \sum_{j=0}^{n+4} (-1)^j \binom{n+4}{j} j - 3 \sum_{j=0}^{n+4} (-1)^j \binom{n+4}{j} - \frac{1}{2}(4n + 10 - n^2 - 7n - 12)
\end{aligned}$$

The first two terms are zero because of results (*) and (**) so

$$T(n) = \frac{n^2 + 3n + 2}{2}.$$

Then

$$\begin{aligned}
S(n) &= \frac{T(n)}{(n+1)(n+2)(n+3)(n+4)} \\
&= \frac{n^2 + 3n + 2}{2(n+1)(n+2)(n+3)(n+4)} \\
&= \frac{(n+1)(n+2)}{2(n+1)(n+2)(n+3)(n+4)} \\
&= \frac{1}{2(n+3)(n+4)}.
\end{aligned}$$

$$\therefore \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k^3 + 9k^2 + 26k + 24} = \frac{1}{2(n+3)(n+4)}$$

THE 1998 CANADIAN MATHEMATICAL OLYMPIAD

1. Determine the number of real solutions a to the equation

$$\left[\frac{1}{2}\,a\right] + \left[\frac{1}{3}\,a\right] + \left[\frac{1}{5}\,a\right] = a\ .$$

Here, if x is a real number, then $[\,x\,]$ denotes the greatest integer that is less than or equal to x.

2. Find all real numbers x such that

$$x = \left(x - \frac{1}{x}\right)^{1/2} + \left(1 - \frac{1}{x}\right)^{1/2}.$$

3. Let n be a natural number such that $n \geq 2$. Show that

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right).$$

4. Let ABC be a triangle with $\angle BAC = 40^\circ$ and $\angle ABC = 60^\circ$. Let D and E be the points lying on the sides AC and AB, respectively, such that $\angle CBD = 40^\circ$ and $\angle BCE = 70^\circ$. Let F be the point of intersection of the lines BD and CE. Show that the line AF is perpendicular to the line BC.

5. Let m be a positive integer. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 0$, $a_1 = m$, and $a_{n+1} = m^2 a_n - a_{n-1}$ for $n = 1, 2, 3, \ldots$. Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.

1998
SOLUTIONS

The solutions to the problems of the 1998 CMO presented below are taken from students papers. Some minor editing has been done - unnecesary steps have been eliminated and some wording has been changed to make the proofs clearer. But for the most part, the proofs are as submitted.

Solution to Problem 1 – David Arthur, Upper Canada College, Toronto, ON

Let $a = 30k + r$, where k is an integer and r is a real number between 0 and 29 inclusive.

Then $\left[\frac{1}{2}\,a\right] = \left[\frac{1}{2}\,(30k+r)\right] = 15k + \left[\frac{r}{2}\right]$. Similarly $\left[\frac{1}{3}\,a\right] = 10k + \left[\frac{r}{3}\right]$ and $\left[\frac{1}{5}\,a\right] = 6k + \left[\frac{r}{5}\right]$.

Now, $\left[\frac{1}{2}\,a\right] + \left[\frac{1}{3}\,a\right] + \left[\frac{1}{5}\,a\right] = a$, so $\left(15k + \left[\frac{r}{2}\right]\right) + \left(10k + \left[\frac{r}{3}\right]\right) + \left(6k + \left[\frac{r}{5}\right]\right) = 30k + r$ and hence $k = r - \left[\frac{r}{2}\right] - \left[\frac{r}{3}\right] - \left[\frac{r}{5}\right]$.

Clearly, r has to be an integer, or $r - \left[\frac{r}{2}\right] - \left[\frac{r}{3}\right] - \left[\frac{r}{5}\right]$ will not be an integer, and therefore, cannot equal k.

On the other hand, if r is an integer, then $r - \left[\frac{r}{2}\right] - \left[\frac{r}{3}\right] - \left[\frac{r}{5}\right]$ will also be an integer, giving exactly one solution for k.

For each $r (0 \leq r \leq 29)$, $a = 30k + r$ will have a different remainder mod 30, so no two different values of r give the same result for a.

Since there are 30 possible values for $r (0, 1, 2, \ldots, 29)$, there are then 30 solutions for a.

Solution to Problem 2 – Jimmy Chui, Earl Haig S.S., North York, ON

Since $\left(x-\frac{1}{x}\right)^{1/2} \geq 0$ and $\left(1-\frac{1}{x}\right)^{1/2} \geq 0$, then $0 \leq \left(x-\frac{1}{x}\right)^{1/2} + \left(1-\frac{1}{x}\right)^{1/2} = x$.

Note that $x \neq 0$. Else, $\frac{1}{x}$ would not be defined so $x > 0$.

Squaring both sides gives,

$$\begin{aligned}
x^2 &= \left(x-\frac{1}{x}\right) + \left(1-\frac{1}{x}\right) + 2\sqrt{\left(x-\frac{1}{x}\right)\left(1-\frac{1}{x}\right)} \\
x^2 &= x+1-\frac{2}{x} + 2\sqrt{x-1-\frac{1}{x}+\frac{1}{x^2}}.
\end{aligned}$$

Multiplying both sides by x and rearranging, we get

$$\begin{aligned}
x^3-x^2-x+2 &= 2\sqrt{x^3-x^2-x+1} \\
(x^3-x^2-x+1) - 2\sqrt{x^3-x^2-x+1}+1 &= 0 \\
(\sqrt{x^3-x^2-x+1}-1)^2 &= 0 \\
\sqrt{x^3-x^2-x+1} &= 1 \\
x^3-x^2-x+1 &= 1 \\
x(x^2-x-1) &= 0 \\
x^2-x-1 &= 0 \qquad \text{since } \ x \neq 0.
\end{aligned}$$

Thus $x = \frac{1 \pm \sqrt{5}}{2}$. We must check to see if these are indeed solutions.

Let $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$. Note that $\alpha+\beta = 1$, $\alpha\beta = -1$ and $\alpha > 0 > \beta$.

Since $\beta < 0$, β is not a solution.

Now, if $x = \alpha$, then

$$\begin{aligned}
\left(\alpha-\frac{1}{\alpha}\right)^{1/2} + \left(1-\frac{1}{\alpha}\right)^{1/2} &= (\alpha+\beta)^{1/2} + (1+\beta)^{1/2} && (\text{since } \alpha\beta - -1) \\
&= 1^{1/2} + (\beta^2)^{1/2} && (\text{since } \alpha+\beta = 1 \text{ and } \beta^2 = \beta+1) \\
&= 1-\beta && (\text{since } \beta < 0) \\
&= \alpha && (\text{since } \alpha+\beta = 1).
\end{aligned}$$

So $x = \alpha$ is the unique solution to the equation.

Solution 1 to Problem 3 – Chen He, Columbia International Collegiate, Hamilton, ON

$$1+\frac{1}{3}+\ldots+\frac{1}{2n-1}=\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{2n-1} \tag{1}$$

Since

$$\frac{1}{3}>\frac{1}{4},\;\;\frac{1}{5}>\frac{1}{6},\;\ldots\;,\frac{1}{2n-1}>\frac{1}{2n}\;,$$

(1) gives

$$1+\frac{1}{3}+\ldots+\frac{1}{2n-1}>\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}=\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right). \tag{2}$$

Since

$$\frac{1}{2}>\frac{1}{4}\,,\;\;\frac{1}{2}>\frac{1}{6}\,,\;\;\frac{1}{2}>\frac{1}{8}\,,\;\;\ldots\;,\;\;\frac{1}{2}>\frac{1}{2n}$$

then

$$\frac{n}{2}=\underbrace{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2}}_{n}>\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}$$

so

$$\frac{1}{2}>\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right). \tag{3}$$

Then (1), (2) and (3) show

$$\begin{aligned}1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\quad &>\quad \frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)+\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)\\ &=\quad \left(1+\frac{1}{n}\right)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)\\ &=\quad \frac{n+1}{n}\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right).\end{aligned}$$

Therefore $\dfrac{1}{n+1}\left(1+\dfrac{1}{3}+\ldots+\dfrac{1}{2n-1}\right)>\dfrac{1}{n}\left(\dfrac{1}{2}+\dfrac{1}{4}+\ldots+\dfrac{1}{2n}\right)$ for all $n\in N$ and $n\geq 2$.

Solution 2 to Problem 3 – Yin Lei, Vincent Massey S.S., Windsor, ON

Since $n \geq 2$, $n(n+1) \geq 0$. Therefore the given inequality is equivalent to

$$n\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right) \geq (n+1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right).$$

We shall use mathematical induction to prove this.

For $n=2$, obviously $\frac{1}{3}\left(1+\frac{1}{3}\right)=\frac{4}{9}>\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}\right)=\frac{3}{8}$.

Suppose that the inequality stands for $n=k$, i.e.

$$k\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right) > (k+1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right). \tag{1}$$

Now we have to prove it for $n=k+1$.

We know

$$\begin{aligned}&\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)\\ &=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)\\ &=\frac{1}{1\times 2}+\frac{1}{3\times 4}+\frac{1}{5\times 6}+\ldots+\frac{1}{(2k-1)(2k)}\quad.\end{aligned}$$

Since

$$1\times 2<3\times 4<5\times 6<\ldots<(2k-1)(2k)<(2k+1)(2k+2)$$

then

$$\frac{1}{1\times 2}+\frac{1}{3\times 4}+\ldots+\frac{1}{(2k-1)(2k)}>\frac{k}{(2k+1)(2k+2)}$$

hence

$$1+\frac{1}{3}+\ldots+\frac{1}{2k-1}>\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}+\frac{k}{(2k+1)(2k+2)}. \tag{2}$$

Also

$$\frac{k+1}{2k+1}-\frac{k+2}{2k+2}=\frac{2k^2+2k+2k+2-2k^2-4k-k-2}{(2k+1)(2k+2)}=-\frac{k}{(2k+1)(2k+2)}$$

therefore

$$\frac{k+1}{2k+1}=\frac{k+2}{2k+2}-\frac{k}{(2k+1)(2k+2)}. \tag{3}$$

Adding 1, 2 and 3:

$$k\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)+\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)+\frac{k+1}{2k+1}$$

$$>(k+1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\frac{k}{(2k+1)(2k+2)}+\frac{k+2}{2k+2}-\frac{k}{(2k+1)(2k+2)}$$

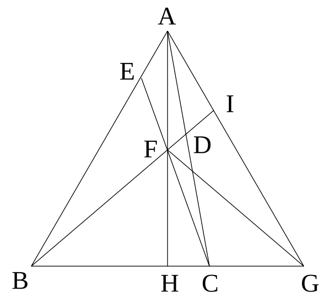
Rearrange both sides to get

$$(k+1)\left(1+\frac{1}{3}+\ldots+\frac{1}{2k+1}\right)>(k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k+2}\right).$$

Proving the induction.

Solution 1 to Problem 4 – Keon Choi, A.Y. Jackson S.S., North York, ON

Suppose H is the foot of the perpendicular line from A to BC; construct equilateral $\triangle ABG$, with C on BG. I will prove that if F is the point where AH meets BD, then $\angle FCB = 70°$. (Because that means AH, and the given lines BD and CE meet at one point and that proves the question.) Suppose BD extended meets AG at I.



Now $BF = GF$ and $\angle FBG = \angle FGB = 40°$ so that $\angle IGF = 20°$. Also $\angle IFG = \angle FBG + \angle FGB = 80°$, so that

$$\begin{aligned}\angle FIG &= 180° - \angle IFG - \angle IGF \\ &= 180° - 80° - 20° \\ &= 80°.\end{aligned}$$

Therefore $\triangle GIF$ is an isoceles triangle, so

$$GI = GF = BF. \tag{1}$$

But $\triangle BGI$ and $\triangle ABC$ are congruent, since $BG = AB$, $\angle GBI = \angle BAC$, $\angle BGI = \angle ABC$.

Therefore

$$GI = BC. \tag{2}$$

From (1) and (2) we get

$$BC = BF.$$

So in $\triangle BCF$,

$$\angle BCF = \frac{180° - \angle FBC}{2} = \frac{180° - 40°}{2} = 70°.$$

Thus $\angle FCB = 70°$ and that proves that the given lines CE and BD and the perpendicular line AH meet at one point.

Solution 2 to Problem 4 – Adrian Birka, Lakeshore Catholic H.S., Port Colborne, ON

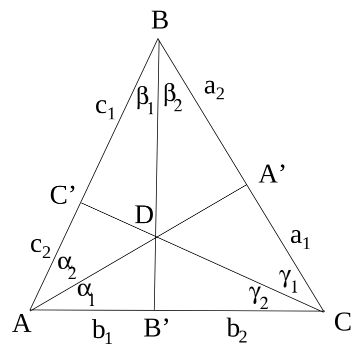
First we prove the following lemma:

In $\triangle ABC$, AA', BB', CC' intersect if-f

$$\frac{\sin\alpha_1}{\sin\alpha_2}\cdot\frac{\sin\beta_1}{\sin\beta_2}\cdot\frac{\sin\gamma_1}{\sin\gamma_2}=1,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are as shown in the diagram just below.

(Editor: This is a known variant of Ceva's Theorem.)



Proof: Let $\angle BB'C = x$, then $\angle BB'A = 180° - x$. Using the sine law in $\triangle BB'C$ yields

$$\frac{b_2}{\sin\beta_2}=\frac{a}{\sin x}\ . \tag{1}$$

Similarly using the sine law in $\triangle BB'A$ yields

$$\frac{b_1}{\sin\beta_1}=\frac{c}{\sin(180° - x)}=\frac{c}{\sin x}\ . \tag{2}$$

Hence,

$$b_1 : b_2 = \frac{c\sin\beta_1}{a\sin\beta_2} \tag{3}$$

(from (1),(2)). (Editor: Do you recognize this when $\beta_1 = \beta_2$?)

Similarly,

$$a_1 : a_2 = \frac{b\sin\alpha_1}{c\sin\alpha_2},\quad c_1 : c_2 = \frac{a\sin\gamma_1}{b\sin\gamma_2}\ . \tag{4}$$

By Ceva's theorem, the necessary and sufficient condition for AA', BB', CC' to intersect is: $(a_1 : a_2)\cdot(b_1 : b_2)\cdot(c_1 : c_2) = 1$. Using (3), (4) on this yields:

$$\frac{b}{c}\cdot\frac{\sin\alpha_1}{\sin\alpha_2}\cdot\frac{a}{b}\cdot\frac{\sin\gamma_1}{\sin\gamma_2}\cdot\frac{c}{a}\cdot\frac{\sin\beta_1}{\sin\beta_2}=1$$

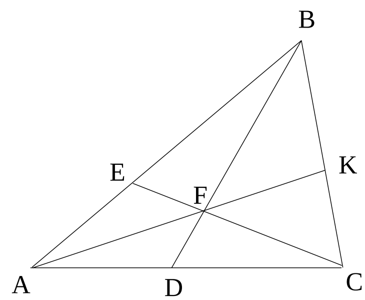
so

$$\frac{\sin\alpha_1}{\sin\alpha_2}\cdot\frac{\sin\beta_1}{\sin\beta_2}\cdot\frac{\sin\gamma_1}{\sin\gamma_2}=1. \tag{5}$$

This is just what we needed to show, therefore the lemma is proved.

Now, in our original question, give $\angle BAC = 40°,\ \ \angle ABC = 60°$. It follows that $\angle ACB = 80°$.

Since $\angle CBD = 40°,\ \ \angle ABD = \angle ABC - \angle DBC = 20°$. Similarly, $\angle ECA = 20°$.



Now let us show that $\angle FAD = 10°$. Suppose otherwise. Let F' be such that F, F' are in the same side of AC and $\angle DAF' = 10°$. Then $\angle BAF' = \angle BAC - \angle DAF' = 30°$.

Thus

$$\begin{aligned}\frac{\sin\angle ABD}{\sin\angle DBC}\cdot\frac{\sin\angle BCE}{\sin\angle ECA}\cdot\frac{\sin\angle CAF'}{\sin\angle F'AB} &= \frac{\sin 20°}{\sin 40°}\cdot\frac{\sin 70°}{\sin 10°}\cdot\frac{\sin 10°}{\sin 30°}\\ &= \frac{\sin 20°}{2\sin 20°\cos 20°}\cdot\frac{\cos 20°}{\sin 30°}\\ &= \frac{1}{2\sin 30°}=1.\end{aligned}$$

By the lemma above, AF' passes through $CE\cap BD = F$. Therefore $AF' = AF$, and $\angle FAD = 10°$, contrary to assumption. Thus $\angle FAD$ must be $10°$. Now let $AF \cap BC = K$. Since $\angle KAC = 10°,\ \ \angle KCA = 80°$, it follows that $\angle AKC = 90°$. Therefore $AK\perp BC \Rightarrow AF\perp BC$ as needed.

Solution to Problem 5 – Adrian Chan, Upper Canada College, Toronto, ON

Let us first prove by induction that $\dfrac{a_n^2 + a_{n+1}^2}{a_n \cdot a_{n+1} + 1} = m^2$ for all $n \geq 0$.

Proof:

Base Case $(n = 0)$: $\dfrac{a_0^2 + a_1^2}{a_0 \cdot a_1 + 1} = \dfrac{0 + m^2}{0 + 1} = m^2$.

Now, let us assume that it is true for $n = k, k \geq 0$. Then,

$$\begin{aligned}
\frac{a_k^2 + a_{k+1}^2}{a_k \cdot a_{k+1} + 1} &= m^2 \\
a_k^2 + a_{k+1}^2 &= m^2 \cdot a_k \cdot a_{k+1} + m^2 \\
a_{k+1}^2 + m^4 a_{k+1}^2 - 2m^2 \cdot a_k \cdot a_{k+1} + a_k^2 &= m^2 + m^4 a_{k+1}^2 - m^2 \cdot a_k \cdot a_{k+1} \\
a_{k+1}^2 + (m^2 a_{k+1} - a_k)^2 &= m^2 + m^2 a_{k+1}(m^2 a_{k+1} - a_k) \\
a_{k+1}^2 + a_{k+2}^2 &= m^2 + m^2 \cdot a_{k+1} \cdot a_{k+2}\ .
\end{aligned}$$

So $\dfrac{a_{k+1}^2 + a_{k+2}^2}{a_{k+1} \cdot a_{k+2} + 1} = m^2$,

proving the induction. Hence (a_n, a_{n+1}) is a solution to $\dfrac{a^2 + b^2}{ab + 1} = m^2$ for all $n \geq 0$.

Now, consider the equation $\dfrac{a^2 + b^2}{ab + 1} = m^2$ and suppose $(a, b) = (x, y)$ is a solution with $0 \leq x \leq y$. Then

$$\frac{x^2 + y^2}{xy + 1} = m^2. \tag{1}$$

If $x = 0$ then it is easily seen that $y = m$, so $(x, y) = (a_0, a_1)$. Since we are given $x \geq 0$, suppose now that $x > 0$.

Let us show that $y \leq m^2 x$.

Proof by contradiction: Assume that $y > m^2 x$. Then $y = m^2 x + k$ where $k \geq 1$.

Substituting into (1) we get

$$\begin{aligned}
\frac{x^2 + (m^2 x + k)^2}{(x)(m^2 x + k) + 1} &= m^2 \\
x^2 + m^4 x^2 + 2m^2 xk + k^2 &= m^4 x^2 + m^2 kx + m^2 \\
(x^2 + k^2) + m^2(kx - 1) &= 0.
\end{aligned}$$

Now, $m^2(kx - 1) \geq 0$ since $kx \geq 1$ and $x^2 + k^2 \geq x^2 + 1 \geq 1$ so $(x^2 + k^2) + m^2(kx - 1) \neq 0$.

Thus we have a contradiction, so $y \leq m^2 x$ if $x > 0$.

Now substitute $y = m^2x - x_1$, where $0 \le x_1 < m^2x$, into (1).

We have

$$\begin{aligned} \frac{x^2 + (m^2x - x_1)^2}{x(m^2x - x_1) + 1} &= m^2 \\ x^2 + m^4x^2 - 2m^2x \cdot x_1 + x_1^2 &= m^4x^2 - m^2x \cdot x_1 + m^2 \\ x^2 + x_1^2 &= m^2(x \cdot x_1 + 1) \\ \frac{x^2 + x_1^2}{x \cdot x_1 + 1} &= m^2. \qquad (2) \end{aligned}$$

If $x_1 = 0$, then $x^2 = m^2$. Hence $x = m$ and $(x_1, x) = (0, m) = (a_0, a_1)$. But $y = m^2x - x_1 = a_2$, so $(x, y) = (a_1, a_2)$. Thus suppose $x_1 > 0$.

Let us now show that $x_1 < x$.

Proof by contradiction: Assume $x_1 \ge x$.

Then $m^2x - y \ge x$ since $y = m^2x - x_1$, and $\left(\frac{x^2 + y^2}{xy + 1}\right) x - y \ge x$ since (x, y) is a solution to $\frac{a^2 + b^2}{ab + 1} = m^2$.

So $x^3 + xy^2 \ge x^2y + xy^2 + x + y$, hence $x^3 \ge x^2y + x + y$ which is a contradiction since $y \ge x > 0$.

With the same proof that $y \le m^2x$, we have $x \le m^2x_1$. So the substitution $x = m^2x_1 - x_2$ with $x_2 \ge 0$ is valid.

Substituting $x = m^2x_1 - x_2$ into (2) gives $\frac{x_1^2 + x_2^2}{x_1 \cdot x_2 + 1} = m^2$.

If $x_2 \ne 0$, then we continue with the substitution $x_i = m^2_{x_{i+1}} - x_{i+2}$ (*) until we get $\frac{x_j^2 + x_{j+1}^2}{x_j \cdot x_{j+1} + 1} = m^2$ and $x_{j+1} = 0$. (The sequence x_i is decreasing, nonnegative and integer.)

So, if $x_{j+1} = 0$, then $x_j^2 = m^2$ so $x_j = m$ and $(x_{j+1}, x_j) = (0, m) = (a_0, a_1)$.

Then $(x_j, x_{j-1}) = (a_1, a_2)$ since $x_{j-1} = m^2x_j - x_{j+1}$ (from (*)).

Continuing, we have $(x_1, x) = (a_{n-1}, a_n)$ for some n. Then $(x, y) = (a_n, a_{n+1})$.
Hence $\frac{a^2 + b^2}{ab + 1} = m^2$ has solutions (a, b) if and only if $(a, b) = (a_n, a_{n+1})$ for some n.

GRADERS' REPORT

Each question was worth a maximum of 7 marks. Every solution on every paper was graded by two different markers. If the two marks differed by more than one point, the solution was reconsidered until the difference resolved. If the two marks differed by one point, the average was used in computing the total score.

The various grades assigned each solution are displayed below, as a percentage.

MARKS	#1	#2	#3	#4	#5
0	7.6	9.8	40.2	31.0	73.9
1	14.1	27.7	7.1	27.7	9.2
2	10.9	16.8	16.8	21.7	12.0
3	6.5	16.3	3.8	1.6	1.1
4	3.3	2.2	1.6	2.2	0.5
5	6.0	14.1	4.3	3.8	0.0
6	16.3	6.0	7.1	2.2	1.1
7	35.3	7.1	19.0	9.8	2.2

PROBLEM 1

This question was well done. 47 students received 6 or 7 and only 6 students received no marks. Many students came up with a proof similar to David Arthur's proof. Another common approach was to find bounds for a (either $0 \leq a < 60$ or $0 \leq a < 90$) and to then check which of these a satisfy the equation.

PROBLEM 2

Although most students attempted this problem, there were only 6 perfect solutions. A further 6 solutions earned a mark of 6/7 and 13 solutions earned a mark of 5/7.

The most common approach was to square both sides of the equation, rearrange the terms to isolate the radical, and to then square both sides again. This resulted in the polynomial $x^6 - 2x^5 - x^4 + 2x^3 + x^2 = 0$. Many students were unable to factor this polynomial, and so earned only 2 or 3 points.

The polynomial has three distinct roots: 0, $\dfrac{1+\sqrt{5}}{2}$, and $\dfrac{1-\sqrt{5}}{2}$. Most students recognized that 0 is extraneous. One point was deducted for not finding that $\dfrac{1-\sqrt{5}}{2}$ is extraneous, and a further point was deducted for not checking that $\dfrac{1+\sqrt{5}}{2}$ is a solution. (It's not obvious that the equation has any solutions.) Failing to check for extraneous roots is considered to be a major error. The graders should, perhaps, have deducted more points for this mistake.

The solution included here avoids the 6th degree polynomial, thus avoiding the difficult factoring.

However, the solutions must still be checked.

PROBLEM 3

There were 17 perfect solutions and eleven more contestants earned either 5 or 6 points.

The most elegant solution uses two simple observations: that $1 = \frac{1}{2} + \frac{1}{2}$ and that $\frac{1}{2}$ is greater than the average of $\frac{1}{2}$, $\frac{1}{4}$, ... , $\frac{1}{2n}$. A telescope argument also works, adding the first and last terms from each side, and so on. The key to a successful proof by induction is to be careful with algebra and to avoid the temptation to use inequalities. For example, many students used the induction hypothesis to deduce that

$$\frac{1}{n+2}\left(1+\frac{1}{3}+\ldots+\frac{1}{2n+1}\right) > \frac{n+1}{(n+2)n}\left(1+\frac{1}{2}+\ldots+\frac{1}{2n}\right)+\frac{1}{(n+2)(2n+1)}$$

then used $\frac{n+1}{(n+2)n} > \frac{1}{n+1}$, which is too sloppy for a successful induction proof.

PROBLEM 4

Many contestants attempted this question, though few got beyond labeling the most apparent angles. Nine students successfully completed the problem, while another six made a significant attempt.

Most of these efforts employed trigonometry or coordinates to set up a trigonometric equation for an unknown angle. This yields to an assault by identities. Adrian Birka produced a very clean solution of this nature.

Only Keon Choi managed to complete a (very pretty) synthetic solution. One other contestant made significant progress with the same idea.

PROBLEM 5

Many students were successful in finding the expression for the terms of the sequence $\{a_n\}$ by a variety of methods: producing an explicit formula, by means of a generating function and as a sum of binomial coefficients involving parameter m. Unfortunately this does not help solving the problem. Nevertheless seventeen contestants were able to prove by induction that the terms of the sequence satisfy the required relation.

To prove the "only if" part one should employ the method of descent which technically is the same calculation as in the direct part of the problem. Three students succeeded in this, but only two obtained a complete solution by showing that the sequence constructed by descent is decreasing and must have m and 0 as the last two terms.

THE 1999 CANADIAN MATHEMATICAL OLYMPIAD

1. Find all real solutions to the equation $4x^2 - 40[x] + 51 = 0$.

   Here, if x is a real number, then $[x]$ denotes the greatest integer that is less than or equal to x.

2. Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and center on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.

3. Determine all positive integers n with the property that $n = (d(n))^2$. Here $d(n)$ denotes the number of positive divisors of n.

4. Suppose $a_1, a_2, \dots, a_8$ are eight distinct integers from $\{1, 2, \dots, 16, 17\}$. Show that there is an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions. Also, find a specific set of 7 distinct integers from $\{1, 2, \dots, 16, 17\}$ such that the equation $a_i - a_j = k$ does not have three distinct solutions for any $k > 0$.

5. Let x, y, and z be non-negative real numbers satisfying $x + y + z = 1$. Show that

   $$x^2y + y^2z + z^2x \leq \frac{4}{27},$$

   and find when equality occurs.

1999
SOLUTIONS

Most of the solutions to the problems of the 1999 CMO presented below are taken from students' papers. Some minor editing has been done - unnecessary steps have been eliminated and some wording has been changed to make the proofs clearer. But for the most part, the proofs are as submitted.

Solution to Problem 1 – Adrian Chan, Upper Canada College, Toronto, ON

Rearranging the equation we get $4x^2 + 51 = 40[x]$. It is known that $x \geq [x] > x - 1$, so

$$\begin{aligned} 4x^2 + 51 = 40[x] &> 40(x-1) \\ 4x^2 - 40x + 91 &> 0 \\ (2x-13)(2x-7) &> 0 \end{aligned}$$

Hence $x > 13/2$ or $x < 7/2$. Also,

$$\begin{aligned} 4x^2 + 51 = 40[x] &\leq 40x \\ 4x^2 - 40x + 51 &\leq 0 \\ (2x-17)(2x-3) &\leq 0 \end{aligned}$$

Hence $3/2 \leq x \leq 17/2$. Combining these inequalities gives $3/2 \leq x < 7/2$ or $13/2 < x \leq 17/2$.

<u>CASE 1</u>: $3/2 \leq x < 7/2$.

For this case, the possible values for $[x]$ are 1, 2 and 3.

If $[x] = 1$ then $4x^2 + 51 = 40 \cdot 1$ so $4x^2 = -11$, which has no real solutions.

If $[x] = 2$ then $4x^2 + 51 = 40 \cdot 2$ so $4x^2 = 29$ and $x = \frac{\sqrt{29}}{2}$. Notice that $\frac{\sqrt{16}}{2} < \frac{\sqrt{29}}{2} < \frac{\sqrt{36}}{2}$ so $2 < x < 3$ and $[x] = 2$.

If $[x] = 3$ then $4x^2 + 51 = 40 \cdot 3$ and $x = \sqrt{69}/2$. But $\frac{\sqrt{69}}{2} > \frac{\sqrt{64}}{2} = 4$. So, this solution is rejected.

<u>CASE 2</u>: $13/2 < x \leq 17/2$.

For this case, the possible values for $[x]$ are 6, 7 and 8.

If $[x] = 6$ then $4x^2 + 51 = 40 \cdot 6$ so $x = \frac{\sqrt{189}}{2}$. Notice that $\frac{\sqrt{144}}{2} < \frac{\sqrt{189}}{2} < \frac{\sqrt{196}}{2}$ so $6 < x < 7$ and $[x] = 6$.

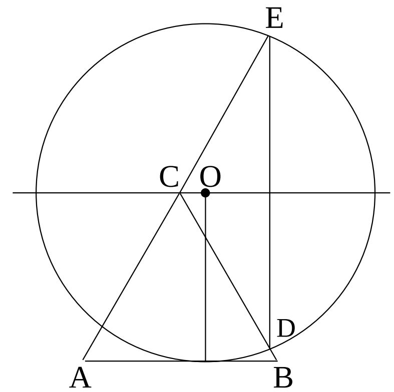
If $[x] = 7$ then $4x^2 + 51 = 40 \cdot 7$ so $x = \frac{\sqrt{229}}{2}$. Notice that $\frac{\sqrt{196}}{2} < \frac{\sqrt{229}}{2} < \frac{\sqrt{256}}{2}$ so $7 < x < 8$ and $[x] = 7$.

If $[x] = 8$ then $4x^2 + 51 = 40 \cdot 8$ so $x = \frac{\sqrt{269}}{2}$. Notice that $\frac{\sqrt{256}}{2} < \frac{\sqrt{269}}{2} < \frac{\sqrt{324}}{2}$ so $8 < x < 9$ and $[x] = 8$.

The solutions are $x = \dfrac{\sqrt{29}}{2}, \dfrac{\sqrt{189}}{2}, \dfrac{\sqrt{229}}{2}, \dfrac{\sqrt{269}}{2}$.

(Editor: Adrian then checks these four solutions.)

Solution 1 to Problem 2 – Keon Choi, A.Y. Jackson S.S., North York, ON



Let D and E be the intersections of BC and extended AC respectively with the circle.

Since $CO||AB$ (because both the altitude and the radius are 1) $\angle BCO = 60°$ and therefore $\angle ECO = 180° - \angle ACB - \angle BC0 = 60°$.

Since a circle is always symmetric in its diameter and line CE is reflection of line CB in CO, line segment CE is reflection of line segment CB.

Therefore $CE = CD$.

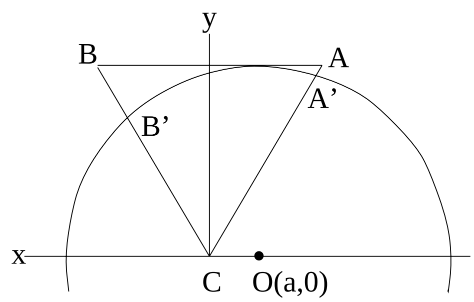
Therefore $\triangle CED$ is an isosceles.

Therefore $\angle CED = \angle CDE$ and $\angle CED + \angle CDE = \angle ACB = 60°$.

$\angle CED = 30°$ regardless of the position of centre 0. Since $\angle CED$ is also the angle subtended from the arc inside the triangle, if CED is constant, the arc length is also constant.

Editor's Note: This proof has had no editing.

Solution 2 to Problem 2 – Jimmy Chui, Earl Haig S.S., North York, ON



Place C at the origin, point A at $\left(\frac{1}{\sqrt{3}}, 1\right)$ and point B at $\left(-\frac{1}{\sqrt{3}}, 1\right)$. Then $\triangle ABC$ is equilateral with altitude of length 1.

Let O be the center of the circle. Because the circle has radius 1, and since it touches line AB, the locus of O is on the line through C parallel to AB (since C is length 1 away from AB), i.e., the locus of O is on the x-axis.

Let point O be at $(a, 0)$. Then $-\frac{1}{\sqrt{3}} \leq a \leq \frac{1}{\sqrt{3}}$ since we have the restriction that the circle rolls along AB.

Now, let A' and B' be the intersection of the circle with CA and CB respectively. The equation of CA is $y = \sqrt{3}\ x$, $0 \leq x \leq \frac{1}{\sqrt{3}}$, of CB is $y = -\sqrt{3}\ x$, $-\frac{1}{\sqrt{3}} \leq x \leq 0$, and of the circle is $(x-a)^2 + y^2 = 1$.

We solve for A' by substituting $y = \sqrt{3}\,x$ into $(x-a)^2 + y^2 = 1$ to get $x = \dfrac{a \pm \sqrt{4-3a^2}}{4}$.

Visually, we can see that solutions represent the intersection of AC extended and the circle, but we are only concerned with the greater x-value – this is the solution that is on AC, not on AC extended. Therefore

$$x = \frac{a + \sqrt{4-3a^2}}{4}\ , \qquad\qquad y = \sqrt{3}\left(\frac{a + \sqrt{4-3a^2}}{4}\right).$$

Likewise we solve for B', but we take the lesser x-value to get

$$x = \frac{a - \sqrt{4-3a^2}}{4}\ , \qquad\qquad y = -\ \sqrt{3}\left(\frac{a + \sqrt{4-3a^2}}{4}\right).$$

Let us find the length of $A'B'$:

$$\begin{aligned}
|A'B'|^2 &= \left(\frac{a + \sqrt{4-3a^2}}{4} - \frac{a - \sqrt{4-3a^2}}{4}\right)^2 + \left(\left(\sqrt{3}\ \frac{a + \sqrt{4-3a^2}}{4}\right) - \left(-\ \sqrt{3}\ \frac{a - \sqrt{4-3a^2}}{4}\right)\right)^2 \\
&= \frac{4-3a^2}{4} + 3\ \frac{a^2}{4} \\
&= 1
\end{aligned}$$

which is independent of a.

Consider the points 0, A' and B'. $\triangle 0A'B'$ is an equilateral triangle (because $A'B' = 0A' = 0B' = 1$).

Therefore $\angle A'0B' = \frac{\pi}{3}$ and arc $A'B' = \frac{\pi}{3}$, a constant.

Solution to Problem 3 – Masoud Kamgarpour, Carson S.S., North Vancouver, BC

Note that $n = 1$ is a solution. For $n > 1$ write n in the form $n = P_1^{\alpha_1} P_2^{\alpha_2} ... P_m^{\alpha_m}$ where the P_i's, $1 \le i \le m$, are distinct prime numbers and $\alpha_i > 0$. Since $d(n)$ is an integer, n is a perfect square, so $\alpha_i = 2\beta_i$ for integers $\beta_i > 0$.

Using the formula for the number of divisors of n,

$$d(n) = (2\beta_1 + 1)(2\beta_2 + 1) \dots (2\beta_m + 1)$$

which is an odd number. Now because $d(n)$ is odd, $(d(n))^2$ is odd, therefore n is odd as well, so $P_i \ge 3, 1 \le i \le m$. We get

$$P_1^{\alpha_1} \cdot P_2^{\alpha_2} \dots P_m^{\alpha_m} = [(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_m + 1)]^2$$

or using $\alpha_i = 2\beta_i$

$$P_1^{\beta_1} P_2^{\beta_2} \dots P_m^{\beta_m} = (2\beta_1 + 1)(2\beta_2 + 1) \dots (2\beta_m + 1).$$

Now we prove a lemma:

Lemma: $P^t \ge 2t + 1$ for positive integers t and $P \ge 3$, with equality only when P=3 and t=1.

Proof: We use mathematical induction on t. The statement is true for $t = 1$ because $P \ge 3$. Now suppose $P^k \ge 2k + 1, k \ge 1$; then we have

$$P^{k+1} = P^k \cdot P \ge P^k(1 + 2) > P^k + 2 \ge (2k + 1) + 2 = 2(k + 1) + 1$$

Thus $P^t \ge 2t + 1$ and equality occurs only when $P = 3$ and $t = 1$.

Let's say n has a prime factor $P_k > 3$; then (by the lemma) $P_k^{\beta_k} > 2\beta_k + 1$ and we have $P_1^{\beta_1} \dots P_m^{\beta_m} > (2\beta_1 + 1) \dots (2\beta_m + 1)$, a contradiction.

Therefore, the only prime factor of n is $P = 3$ and we have $3^\alpha = 2\alpha + 1$. By the lemma $\alpha = 1$.

The only positive integer solutions are 1 and 9.

Solution 1 to Problem 4 – David Nicholson, Fenelon Falls S.S., Fenelon Falls, ON

Without loss of generality let $a_1 < a_2 < a_3 \dots < a_8$.

Assume that there is no such integer k. Let's just look at the seven differences $d_i = a_{i+1} - a_i$. Then amongst the d_i there can be at most two 1s, two 2s, and two 3s, which totals to 12.

Now $16 = 17 - 1 \ge a_8 - a_1 = d_1 + d_2 + \dots + d_7$ so the seven differences must be 1,1,2,2,3,3,4.

Now let's think of arranging the differences 1,1,2,2,3,3,4. Note that the sum of consecutive differences is another difference. (Eg $d_1 + d_2 = a_3 - a_1,\ d_1 + d_2 + d_3 = a_4 - a_1$)

We can't place the two 1s side by side because that will give us another difference of 2. The 1s can't be beside a 2 because then we have three 3s. They can't both be beside a 3 because then we have three 4s! So we must have either $1, 4, -, -, -, 3, 1$ or $1, 4, 1, 3, -, -, -$ (or their reflections).

In either case we have a 3,1 giving a difference of 4 so we can't put the 2s beside each other. Also we can't have 2,3,2 because then (with the 1,4) we will have three 5s. So all cases give a contradiction.

Therefore there will always be three differences equal.

A set of seven numbers satisfying the criteria are $\{1, 2, 4, 7, 11, 16, 17\}$. (Editor: There are many such sets)

Solution 2 to Problem 4 – The CMO committee

Consider all the consecutive differences (ie, d_i above) as well as the differences $b_i = a_{i+2} - a_i$, $i = 1 \dots 6$. Then the sum of these thirteen differences is $2 \cdot (a_8 - a_1) + (a_7 - a_2) \leq 2(17-1) + (16-2) = 46$. Now if no difference occurs more than twice, the smallest the sum of the thirteen differences can be is $2 \cdot (1+2+3+4+5+6) + 7 = 49$, giving a contradiction.

Solution 1 to Problem 5 – The CMO committee

Let $f(x,y,z) = x^2y + y^2z + z^2x$. We wish to determine where f is maximal. Since f is cyclic, without loss of generality we may assume that $x \geq y, z$. Since

$$\begin{aligned} f(x,y,z) - f(x,z,y) &= x^2y + y^2z + z^2x - x^2z - z^2y - y^2x \\ &= (y-z)(x-y)(x-z), \end{aligned}$$

we may also assume $y \geq z$. Then

$$\begin{aligned} f(x+z,y,0) - f(x,y,z) &= (x+z)^2y - x^2y - y^2z - z^2x \\ &= z^2y + yz(x-y) + xz(y-z) \geq 0, \end{aligned}$$

so we may now assume $z = 0$. The rest follows from the arithmetic-geometric mean inequality:

$$f(x,y,0) = \frac{2x^2y}{2} \leq \frac{1}{2}\left(\frac{x+x+2y}{3}\right)^3 = \frac{4}{27}$$

Equality occurs when $x = 2y$, hence at $(x,y,z) = (\frac{2}{3}, \frac{1}{3}, 0)$. (As well as $(0, \frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, 0, \frac{2}{3})$.

Solution 2 to Problem 5 – The CMO committee

With f as above, and $x \geq y, z$

$$f\left(x + \frac{z}{2},\ y + \frac{z}{2},\ 0\right) - f(x,y,z) \quad = \quad yz(x-y) + \frac{xz}{2}(x-z) + \frac{z^2y}{4} + \frac{z^3}{8}$$

so we may assume that $z = 0$. The rest follows as for solution 1.

GRADERS' REPORT

Each question was worth a maximum of 7 marks. Every solution on every paper was graded by two different markers. If the two marks differed by more than one point, the solution was reconsidered until the difference resolved. If the two marks differed by one point, the average was used in computing the total score.

The various grades assigned to each solution are displayed below, as a percentage.

MARKS	#1	#2	#3	#4	#5
0	7.6	45.6	18.4	38.6	51.3
1	13.9	15.8	43.0	0.0	32.3
2	12.0	12.7	15.2	41.8	13.9
3	5.7	2.5	5.1	7.6	2.5
4	4.4	1.3	8.9	4.4	0.0
5	8.9	2.5	3.2	5.7	1.3
6	8.9	0.0	1.3	1.9	0.0
7	39.9	20.9	6.3	1.3	0.0

PROBLEM 1 The aim of the question was to give the competitors an encouraging start (it was not a give away!). Over half of the students had good scores of 5, 6 or 7.

The general approach was to find bounds for x and then to find the exact value for x by substituting in the resulting possible values of $[x]$. Depending on how the bounds were determined, this meant checking 6 - 10 different cases.

Points were lost for not adequately verifying the bounds on x. For example, 2 points were deducted for assuming, without proof, that $4x^2 + 51 > 40[x]$ for $x \geq 9$.

PROBLEM 2 Many competitors saw that the key here is to prove that the angle subtended by the arc at its centre is constant, namely $\pi/3$. In all, 16 students managed a complete proof. Most attempted an analytic solution – indeed, the problem is nearly routine if one chooses coordinates wisely and later on notes that two such x-coordinates are roots of the same quadratic. A few students used trigonometry, namely the law of sines on a couple of useful triangles. Two students found essentially the same synthetic solution, which is very elegant.

PROBLEM 3 Most competitors determined by direct calculation that $n = 1$ and $n = 9$ are solutions. The difficulty was to show that these are the only solutions, which boils down to proving that $p^k \geq 2k + 1$ for all primes $p > 2$ and all $k > 0$ with equality only for $k = 1$ and $p = 3$. This can be done by induction or by calculus. Only 5 students obtained perfect marks.

PROBLEM 4

Many students found a specific set of seven integers such that the equation did not have three different solutions. This earned two points. (One student found such a set with maximum value 14. A maximum value of 13 is not possible.)

Only eight competitors received high marks on the question (5, 6, or 7), and only one student scored a perfect 7. All of the successful solvers considered differences of consecutive integers, showing that they must be 1, 1, 2, 2, 3, 3, and 4, and then showed that every ordering of these differences led to at least three repetitions of the same value. Most competitors recognized that the 1s could not be together, nor could they be beside a 2. They then proceed by considering all such possible arrangements, which often resulted in close to a dozen cases (depending on how the the cases were handled.) David Nicholson was the most efficient at pruning the cases. (See Solution 1 to Problem 4.) Most students failed to consider one or two (easily dismissed) cases, hence lost 1 or 2 points.

A number of the contestants attempted to solve the problem by examining the odd-even character of the set of eight integers, counting how many of the differences were odd or even, and using the pigeon-hole principle. Although this approach looked promising, no one was able to handle the case that 3 of the integers were of one parity, and 5 were of the other parity.

PROBLEM 5 No students received full marks for this problem. One student received 5 marks for a proof that had minor errors. This proof was by Calculus. The committee was aware that the problem could be solved using Calculus but (erroneously) thought it unlikely high school students would attempt such a solution. Many students received 1 point for "guessing" that $\left(\frac{2}{3}\,,\,\frac{1}{3}\,,\,0\right)$, $\left(0,\,\frac{2}{3}\,,\,\frac{1}{3}\right)$ and $\left(\frac{1}{3}\,,\,0\,,\,\frac{2}{3}\right)$ are where equality occurred. Some students received a further point for verifying the inequality on the boundary of the region.

THE 2000 CANADIAN MATHEMATICAL OLYMPIAD

1. At 12:00 noon, Anne, Beth and Carmen begin running laps around a circular track of length three hundred meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least one hundred meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners.)

2. A *permutation* of the integers $1901, 1902, \ldots, 2000$ is a sequence $a_1, a_2, \ldots, a_{100}$ in which each of those integers appears exactly once. Given such a permutation, we form the sequence of partial sums

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \ \ldots\ , \ s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

   How many of these permutations will have no terms of the sequence $s_1, \ldots, s_{100}$ divisible by three?

3. Let $A = (a_1, a_2, \ldots, a_{2000})$ be a sequence of integers each lying in the interval $[-1000, 1000]$. Suppose that the entries in A sum to 1. Show that some nonempty subsequence of A sums to zero.

4. Let $ABCD$ be a convex quadrilateral with

$$\begin{aligned} \angle CBD &= 2\angle ADB, \\ \angle ABD &= 2\angle CDB \\ \text{and} \qquad AB &= CB. \end{aligned}$$

   Prove that $AD = CD$.

5. Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$\begin{aligned} a_1 \geq a_2 \geq \cdots \geq a_{100} &\geq 0, \\ a_1 + a_2 &\leq 100 \\ \text{and} \qquad a_3 + a_4 + \cdots + a_{100} &\leq 100. \end{aligned}$$

   Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.

2000 Canadian Mathematics Olympiad Solutions

Chair: Luis Goddyn, Simon Fraser University, goddyn@math.sfu.ca

The Year 2000 Canadian Mathematics Olympiad was written on Wednesday April 2, by 98 high school students across Canada. A correct and well presented solution to any of the five questions was awarded seven points. This year's exam was a somewhat harder than usual, with the mean score being 8.37 out of 35. The top few scores were: 30, 28, 27, 22, 20, 20, 20. The first, second and third prizes are awarded to: Daniel Brox (Sentinel Secondary BC), David Arthur (Upper Canada College ON), and David Pritchard (Woburn Collegiate Institute ON).

1. At 12:00 noon, Anne, Beth and Carmen begin running laps around a circular track of length three hundred meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least one hundred meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners.)

   Comment: We were surprised by the difficulty of this question, having awarded an average grade of 1.43 *out of* 7. *We present two solutions; only the first appeared among the graded papers.*

   Solution 1: By rotating the frame of reference we may assume that Anne has speed zero, that Beth runs at least as fast as Carmen, and that Carmen's speed is positive. If Beth is no more than twice as fast as Carmen, then both are at least 100 *meters from Anne when Carmen has run* 100 *meters. If Beth runs more that twice as fast as Carmen, then Beth runs a stretch of more than* 200 *meters during the time Carmen runs between* 100 *and* 200 *meters. Some part of this stretch lies more than* 100 *meters from Anne, at which time both Beth and Carmen are at least (in fact, more than)* 100 *meters away from Anne.*

   Solution 2: By rotating the frame of reference we may assume Anne's speed to equal zero, and that the other two runners have non-zero speed. We may assume that Beth is running at least as fast as Carmen. Suppose that it takes t seconds for Beth to run 200 *meters. Then at all times in the infinite set $T = \{t, 2t, 4t, 8t, \ldots\}$, Beth is exactly* 100 *meters from Anne. At time t, Carmen has traveled exactly d meters where $0 < d \leq 200$. Let k be the least integer such that $2^k d \geq 100$. Then $k \geq 0$ and $100 \leq 2^k d \leq 200$, so at time $2^k t \in T$ both Beth and Carmen are at least* 100 *meters from Anne.*

2. A *permutation* of the integers $1901, 1902, \ldots, 2000$ is a sequence $a_1, a_2, \ldots, a_{100}$ in which each of those integers appears exactly once. Given such a permutation, we form the sequence of partial sums

   $$s_1 = a_1, \;\; s_2 = a_1 + a_2, \;\; s_3 = a_1 + a_2 + a_3, \; \ldots \;, \; s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

   How many of these permutations will have no terms of the sequence $s_1, \ldots, s_{100}$ divisible by three?

   Comment: This question was the easiest and most straight forward, with an average grade of 3.07.

   Solution: Let $\{1901, 1902, \ldots, 2000\} = R_0 \cup R_1 \cup R_2$ where each integer in R_i is congruent to i modulo 3. *We note that $|R_0| = |R_1| = 33$ and $|R_2| = 34$. Each permutation $S = (a_1, a_2, \ldots, a_{100})$ can be uniquely specified by describing a sequence $S' = (a'_1, a'_2, \ldots, a'_{100})$ of* *residues modulo 3 (containing exactly* 33 *zeros,* 33 *ones and* 34 *twos), and three permutations (one each of* R_0*,* R_1*, and* R_2*). Note that the number of permutations of* R_i *is exactly* $|R_i|! = 1 \cdot 2 \cdots |R_i|$.

The condition on the partial sums of S *depends only on the sequence of residues* S'*. In order to avoid a partial sum divisible by three, the subsequence formed by the* 67 *ones and twos in* S' *must equal either* $1, 1, 2, 1, 2, \ldots, 1, 2$ *or* $2, 2, 1, 2, 1, \ldots, 2, 1$*. Since* $|R_2| = |R_1| + 1$*, only the second pattern is possible. The* 33 *zero entries in* S' *may appear anywhere among* $a'_1, a'_2, \ldots, a'_{100}$ *provided that* $a'_1 \neq 0$*. There are* $\binom{99}{33} = \frac{99!}{33!\,66!}$ *ways to choose which entries in* S' *equal zero. Thus there are exactly* $\binom{99}{33}$ *sequences* S' *whose partial sums are not divisible by three. Therefore the total number of permutations* S *satisfying this requirement is exactly*

$$\binom{99}{33} \cdot 33! \cdot 33! \cdot 34! = \frac{99! \cdot 33! \cdot 34!}{66!}.$$

Incidently, this number equals approximately $4.4 \cdot 10^{138}$.

3. Let $A = (a_1, a_2, \ldots, a_{2000})$ be a sequence of integers each lying in the interval $[-1000, 1000]$. Suppose that the entries in A sum to 1. Show that some nonempty subsequence of A sums to zero.

   Comment: This students found this question to be the most difficult, with an average grade of 0.51*, and only one perfect solution among* 100 *papers.*

   Solution: We may assume no entry of A *is zero, for otherwise we are done. We sort* A *into a new list* $B = (b_1, \ldots, b_{2000})$ *by selecting elements from* A *one at a time in such a way that* $b_1 > 0$*,* $b_2 < 0$ *and, for each* $i = 2, 3, \ldots, 2000$*, the sign of* b_i *is opposite to that of the partial sum*

   $$s_{i-1} = b_1 + b_2 + \cdots + b_{i-1}.$$

   (We can assume that each $s_{i-i} \neq 0$ *for otherwise we are done.) At each step of the selection process a candidate for* b_i *is guaranteed to exist, since the condition* $a_1 + a_2 + \cdots + a_{2000} = 1$ *implies that the sum of unselected entries in* A *is either zero or has sign opposite to* s_{i-1}*.*

   From the way they were defined, each of $s_1, s_2, \ldots, s_{2000}$ *is one of the* 1999 *nonzero integers in the interval* $[-999, 1000]$*. By the Pigeon Hole Principle,* $s_j = s_k$ *for some* j, k *satisfying* $1 \leq j < k \leq 2000$*. Thus* $b_{j+1} + b_{j+2} + \cdots + b_k = 0$ *and we are done.*

4. Let $ABCD$ be a convex quadrilateral with

   $$\begin{aligned} \angle CBD &= 2\angle ADB, \\ \angle ABD &= 2\angle CDB \\ \text{and} \qquad AB &= CB. \end{aligned}$$

   Prove that $AD = CD$.

   Comment: There are several different solutions to this, including some using purely trigonometric arguments (involving the law of sines and standard angle sum formulas). We present here two prettier geometric arguments (with diagrams). The first solution is perhaps the more attractive of the two. Average grade: 1.84 *out of* 7*.*

   Solution 1 (from contestant Keon Choi): Extend DB *to a point* P *on the circle through* A *and* C *centered at* B*. Then* $\angle CPD = \frac{1}{2}\angle CBD = \angle ADB$ *and* $\angle APD = \frac{1}{2}\angle ABD = \angle CDB$*,* *so $APCD$ is a parallelogram. Now PD bisects AC so BD is an angle bisector of isosceles triangle ABC. We have*

$$\angle ADB = \frac{1}{2}\angle CBD = \frac{1}{2}\angle ABD = \angle CDB$$

so DB is the angle bisector of $\angle ADC$. As DB bisects the base of triangle ADC, this triangle must be isosceles and $AD = CD$.

Solution 2: Let the bisector of $\angle ABD$ meet AD at E. Let the bisector of $\angle CBD$ meet CD at F. Then $\angle FBD = \angle BDE$ and $\angle EBD = \angle BDF$, which imply $BE \parallel FD$ and $BF \parallel ED$. Thus $BEDF$ is a parallelogram whence

$$BD \textit{ intersects } EF \textit{ at its midpoint } M. \qquad (1)$$

On the other hand since BE is an angle bisector, we have $\frac{AB}{BD} = \frac{AE}{ED}$. Similarly we have $\frac{CB}{BD} = \frac{CF}{FD}$. By assumption $AB = CB$ so $\frac{AE}{ED} = \frac{CF}{FD}$ which implies $EF \parallel AC$. Thus $\triangle DEF$ and $\triangle DAC$ are similar, which implies by (1) that BD intersects AC at its midpoint N. Since $\triangle ABC$ is isosceles, this implies $AC \perp BD$. Thus $\triangle NAD$ and $\triangle NCD$ are right triangles with equal legs and hence are congruent. Thus $AD = CD$.

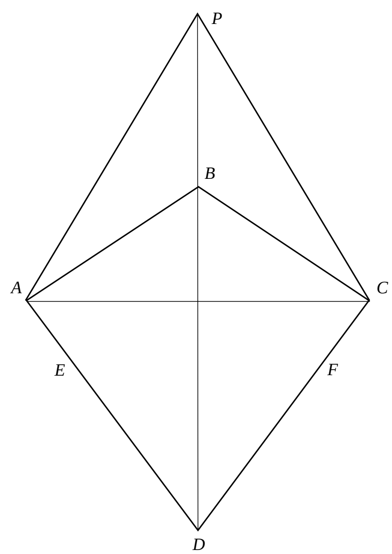


Diagram for Solution 1

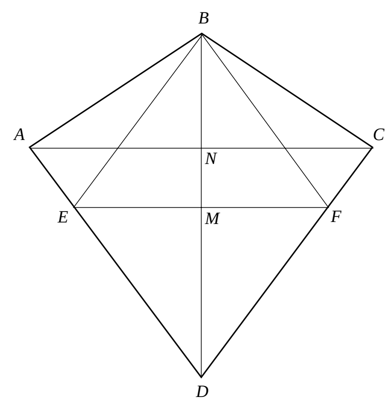


Diagram for Solution 2

5. Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0$$

$$a_1 + a_2 \leq 100$$

$$a_3 + a_4 + \cdots + a_{100} \leq 100.$$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.

Comment: All of the correct solutions involved a sequence of adjustments to the variables, each of which increase $a_1^2 + a_2^2 + \cdots + a_{100}^2$ while satisfying the constraints, eventually arriving at the two optimal sequences: $100, 0, 0, \ldots, 0$ and $50, 50, 50, 50, 0, 0, \ldots, 0$. We present here a sharper proof, which might be arrived at after guessing that the optimal value is 100^2. Average grade: 1.52 out of 7.

Solution: We have $a_1 + a_2 + \cdots + a_{100} \leq 200$*, so*

$$\begin{aligned}
a_1^2 + a_2^2 + \cdots + a_{100}^2 &\leq (100 - a_2)^2 + a_2^2 + a_3^2 + \cdots + a_{100}^2 \\
&= 100^2 - 200a_2 + 2a_2^2 + a_3^2 + \cdots + a_{100}^2 \\
&\leq 100^2 - (a_1 + a_2 + \cdots + a_{100})a_2 + 2a_2^2 + a_3^2 + \cdots + a_{100}^2 \\
&= 100^2 + (a_2^2 - a_1a_2) + (a_3^2 - a_3a_2) + (a_4^2 - a_4a_2) + \cdots + (a_{100}^2 - a_{100}a_2) \\
&= 100^2 + (a_2 - a_1)a_2 + (a_3 - a_2)a_3 + (a_4 - a_2)a_4 + \cdots + (a_{100} - a_2)a_{100}
\end{aligned}$$

Since $a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0$*, none of the terms* $(a_i - a_j)a_i$ *is positive. Thus* $a_1^2 + a_2^2 + \cdots + a_{100}^2 \leq 10,000$ *with equality holding if and only if*

$$a_1 = 100 - a_2 \qquad \textit{and} \qquad a_1 + a_2 + \cdots + a_{100} = 200$$

and each of the products

$$(a_2 - a_1)a_2, \;\; (a_3 - a_2)a_3, \;\; (a_4 - a_2)a_4, \;\; \cdots \;\; , \;\; (a_{100} - a_2)a_{100}$$

equals zero. Since $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_{100} \geq 0$*, the last condition holds if and only if for some* $i \geq 1$ *we have* $a_1 = a_2 = \cdots = a_i$ *and* $a_{i+1} = \cdots = a_{100} = 0$*. If* $i = 1$*, then we get the solution* $100, 0, 0, \ldots, 0$*. If* $i \geq 2$*, then from* $a_1 + a_2 = 100$*, we get that* $i = 4$ *and the second optimal solution* $50, 50, 50, 50, 0, 0, \ldots, 0$*.*

THE 2001 CANADIAN MATHEMATICAL OLYMPIAD

1. **Randy:** "Hi Rachel, that's an interesting quadratic equation you have written down. What are its roots?"
   Rachel: "The roots are two positive integers. One of the roots is my age, and the other root is the age of my younger brother, Jimmy."
   Randy: "That is very neat! Let me see if I can figure out how old you and Jimmy are. That shouldn't be too difficult since all of your coefficients are integers. By the way, I notice that the sum of the three coefficients is a prime number."
   Rachel: "Interesting. Now figure out how old I am."
   Randy: "Instead, I will guess your age and substitute it for x in your quadratic equation ... darn, that gives me -55, and not 0."
   Rachel: "Oh, leave me alone!"

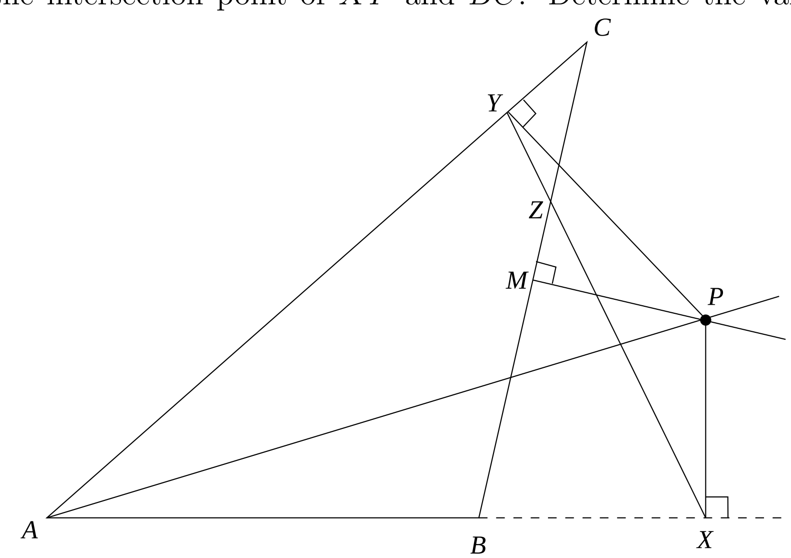
   (a) Prove that Jimmy is two years old.

   (b) Determine Rachel's age.

2. There is a board numbered -10 to 10 as shown. Each square is coloured either red or white, and the sum of the numbers on the red squares is n. Maureen starts with a token on the square labeled 0. She then tosses a fair coin ten times. Every time she flips heads, she moves the token one square to the right. Every time she flips tails, she moves the token one square to the left. At the end of the ten flips, the probability that the token finishes on a red square is a rational number of the form $\frac{a}{b}$. Given that $a + b = 2001$, determine the largest possible value for n.

-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10

3. Let ABC be a triangle with $AC > AB$. Let P be the intersection point of the perpendicular bisector of BC and the internal angle bisector of $\angle A$. Construct points X on AB (extended) and Y on AC such that PX is perpendicular to AB and PY is perpendicular to AC. Let Z be the intersection point of XY and BC. Determine the value of BZ/ZC.



4. Let n be a positive integer. Nancy is given a rectangular table in which each entry is a positive integer. She is permitted to make either of the following two moves:

(a) select a row and multiply each entry in this row by n.

(b) select a column and subtract n from each entry in this column.

Find all possible values of n for which the following statement is true:

> Given any rectangular table, it is possible for Nancy to perform a finite sequence of moves to create a table in which each entry is 0.

5. Let P_0, P_1, P_2 be three points on the circumference of a circle with radius 1, where $P_1P_2 = t < 2$. For each $i \geq 3$, define P_i to be the centre of the circumcircle of $\triangle P_{i-1}P_{i-2}P_{i-3}$.

   (a) Prove that the points $P_1, P_5, P_9, P_{13}, \ldots$ are collinear.

   (b) Let x be the distance from P_1 to P_{1001}, and let y be the distance from P_{1001} to P_{2001}. Determine all values of t for which $\sqrt[500]{x/y}$ is an integer.

2001
SOLUTIONS

Several solutions are edited versions of solutions submitted by the contestants whose names appear in *italics*..

1. *(Daniel Brox)*
   Let R be Rachel's age, and let J be Jimmy's age. Rachel's quadratic is
   $$a(x-R)(x-J) = ax^2 - a(R+J)x + aRJ.$$
   for some number a. We are given that the coefficient a is an integer. The sum of the coefficients is
   $$a - a(R+J) + aRJ = a(R-1)(J-1).$$
   Since this is a prime number, two of the three integers $a, R-1, J-1$ multiply to 1. We are given that $R > J > 0$, so we must have that $a = 1$, $J = 2$, $R-1$ is prime, and the quadratic is
   $$(x-R)(x-2).$$
   We are told that this quadratic takes the value $-55 = -5 \cdot 11$ for some positive integer x. Since $R > 2$, the first factor, $(x-R)$, must be the negative one. We have four cases:
   $x - R = -55$ and $x - 2 = 1$, which implies $x = 3$, $R = 58$.
   $x - R = -11$ and $x - 2 = 5$, which implies $x = 7$, $R = 18$.
   $x - R = -5$ and $x - 2 = 11$, which implies $x = 13$, $R = 18$.
   $x - R = -1$ and $x - 2 = 53$, which implies $x = 57$, $R = 58$.
   Since $R-1$ is prime, the first and last cases are rejected, so $R = 18$ and $J = 2$.

2. *(Lino Demasi)*
   After ten coin flips, the token finishes on the square numbered $2k-10$, where k is the number of heads obtained. Of the $2^{10} = 1024$ possible results of ten coin flips, there are exactly $\binom{10}{k}$ ways to obtain exactly k heads, so the probability of finishing on the square labeled $2k-10$ equals $\binom{10}{k}/1024$.
   The probability of landing on a red square equals $c/1024$ where c is the sum of a selection of the numbers from the list
   $$\binom{10}{0}, \binom{10}{1}, \binom{10}{2}, \ldots, \binom{10}{10} = 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1. \qquad (1)$$
   We are given that for some integers a, b satisfying $a + b = 2001$,
   $$a/b = c/1024.$$
   If we assume (as most contestants did!) that a and b are relatively prime, then the solution proceeds as follows. Since $0 \leq a/b \leq 1$ and $a + b = 2001$, we have $1001 \leq b \leq 2001$. Also b divides 1024, so we have $b = 1024$. Thus $a = c = 2001 - 1024 = 977$. There is only one way to select terms from (1) so that the sum equals 977.
   $$977 = 10 + 10 + 45 + 120 + 120 + 210 + 210 + 252. \qquad (2)$$
   (This is easy to check, since the remaining terms in (1) must add to $1024 - 977 = 47$, and $47 = 45 + 1 + 1$ is the only possibility for this.)

In order to maximize n, we must colour the strip as follows. Odd numbered squares are red if positive, and white if negative. Since $252 = \binom{10}{5}$ is in the sum, the square labeled $2 \cdot 5 - 10 = 0$ is red. For $k = 0, 1, 2, 3, 4$, if $\binom{10}{k}$ appears twice in the sum (2), then both $2k - 10$ and $10 - 2k$ are coloured red. If $\binom{10}{k}$ does not appear in the sum, then both $2k - 10$ and $10 - 2k$ are coloured white. If $\binom{10}{k}$ appears once in the sum, then $10 - 2k$ is red and $2k - 10$ is white. Thus the maximum value of n is obtained when the red squares are those numbered $\{1, 3, 5, 7, 9, -8, 8, -4, 4, -2, 2, 0, 6\}$ giving $n = 31$.

(Alternatively) If we do not assume a and b are relatively prime, then there are several more possibilities to consider. The greatest common divisor of a and b divides $a + b = 2001$, so $\gcd(a, b)$ is one of

$$1,\ 3,\ 23,\ 29,\ 3 \cdot 23,\ 3 \cdot 29,\ 23 \cdot 29,\ 3 \cdot 23 \cdot 29.$$

Since $a/b = c/1024$, dividing b by $\gcd(a, b)$ results in a power of 2. Thus the prime factorization of b is one of the following, for some integer k.

$$2^k,\ 3 \cdot 2^k,\ 23 \cdot 2^k,\ 29 \cdot 2^k,\ 69 \cdot 2^k,\ 87 \cdot 2^k,\ 667 \cdot 2^k,\ 2001.$$

Again we have $1001 \le b \le 2001$, so b must be one of the following numbers.

$$1024,\ 3 \cdot 512 = 1536,\ 23 \cdot 64 = 1472,\ 29 \cdot 64 = 1856,\ 69 \cdot 16 = 1104,\ 87 \cdot 16 = 1392,\ 667 \cdot 2 = 1334,\ 2001.$$

Thus $a/b = (2001 - b)/b$ is one of the following fractions.

$$\frac{977}{1024}, \frac{465}{1536}, \frac{529}{1472}, \frac{145}{1856}, \frac{897}{1104}, \frac{609}{1392}, \frac{667}{1334}, \frac{0}{2001}.$$

Thus $c = 1024a/b$ is one of the following integers.

$$977,\ 310,\ 368,\ 80,\ 832,\ 448,\ 512,\ 0.$$

After some (rather tedious) checking, one finds that only the following sums with terms from (1) can add to a possible value of c.

$$\begin{aligned}
977 &= 10 + 10 + 45 + 120 + 120 + 210 + 210 + 252 \\
310 &= 10 + 45 + 45 + 210 \\
512 &= 10 + 10 + 120 + 120 + 252 \\
512 &= 1 + 1 + 45 + 45 + 210 + 210 \\
0 &= 0.
\end{aligned}$$

Again only those terms appearing exactly once in a sum can affect maximum value of n. We make the following table.

c	Terms appearing once in sum	Corresponding red squares
977	$\{45, 252\}$	$\{6, 0\}$
310	$\{10, 210\}$	$\{8, 2\}$
512	$\{252\}$	$\{0\}$
512	$\emptyset$	$\emptyset$
0	$\emptyset$	$\emptyset$

Evidently, the maximum possible value of n is obtained when $c = 310 = \binom{10}{2} + \binom{10}{6} + \binom{10}{8} + \binom{10}{9}$, the red squares are $\{1, 3, 5, 7, 9, -6, 6, 2, 8\}$, the probability of landing on a red square is $a/b = 465/1536 = 310/1024 = 155/512$, and $n = 35$.

3. *Solution 1: (Daniel Brox)*
Set O be the centre of the circumcircle of $\triangle ABC$. Let the angle bisector of $\angle BAC$ meet this circumcircle at R. We have

$$\angle BOR = 2\angle BAR = 2\angle CAR = \angle COR$$

Thus $BR = CR$ and R lies on the perpendicular bisector of BC. Thus $R = P$ and $ABCP$ are concyclic. The points X, Y, M are the bases of the three perpendiculars dropped from P onto the sides of $\triangle ABC$. Thus by Simson's rule, X, Y, M are collinear. Thus we have $M = Z$ and $BZ/ZC = BM/MC = 1$.

Note: XYZ is called a *Simson line*, *Wallace line* or *pedal line* for $\triangle ABC$. To prove Simson's rule, we note that $BMPX$ are concyclic, as are $AYPX$, thus

$$\angle BXM = \angle BPM = 90 - \angle PBC = 90 - \angle PAC = \angle APY = \angle AXY$$

Solution 2: (Kenneth Ho)
Since $\angle PAX = \angle PAY$ and $\angle PXA = \angle PYA = 90$, triangles $\triangle PAX$ and $\triangle PAY$ are congruent, so $AX = AY$ and $PX = PY$. As P is on the perpendicular bisector of BC, we have $PC = PB$. Thus $\triangle PYC$ and $\triangle PXB$ are congruent right triangles, which implies $CY = BX$. Since X, Y and Z are collinear, we have by Menelaus' Theorem

$$\frac{AY}{YC}\,\frac{CZ}{ZB}\,\frac{BX}{XA} = -1.$$

Applying $AX = AY$ and $CY = BX$, this is equivalent to $BZ/ZC = 1$.

4. We shall see that the only solution is $n = 2$. First we show that if $n \neq 2$, then the table $T_0 = \begin{bmatrix} 1 \\ n-1 \end{bmatrix}$ can not be changed into a table containing two zeros. For $n = 1$, this is very easy to see. Suppose $n \geq 3$. For any table $T = \begin{bmatrix} a \\ b \end{bmatrix}$, let $d(T)$ be the quantity $b - a \pmod{n-1}$. We shall show that neither of the two permitted moves can change the value of $d(T)$. If we subtract n from both elements in T, then $b - a$ does not change. If we multiply the first row by n, then the element a changes to na, for a difference of $(n-1)a$, which is congruent to $0 \pmod{n-1}$. Similarly, multiplying the second row by n does not change $d(T)$. Since $d(T_0) = (n-1) - 1 \equiv -1 \pmod{n-1}$, we can never obtain the table with two zeros by starting with T_0, because $0 - 0$ is not congruent to -1 modulo $n - 1$.

For $n = 2$, and any table of positive integers, the following procedure will always result in a table of zeros. We shall begin by converting the first column into a column of zeros as follows.

We repeatedly subtract 2 from all entries in the first column until at least one of the entries equals either 1 or 2. Now we repeat the following sequence of three steps:

(a) multiply by 2 all rows with 1 in the first column
(b) now multiply by 2 every row having a 2 in the first column (there is at least one such row)
(c) subtract 2 from all entries in the first column.

Each iteration of the three steps decreases the sum of those entries in the first column which are greater than 2. Thus the first column eventually consists entirely of ones and twos, at which time we apply (a) and (c) once again to obtain a column of zeros. We now repeat the above procedure for each successive column of the table. The procedure does not affect any column which has already been set to zero, so we eventually obtain a table with all entries zero.

5. *(Daniel Brox)*
Let $\angle P_1P_3P_2 = 2\alpha$. As $\triangle P_1P_2P_3$ is isosceles, we have that

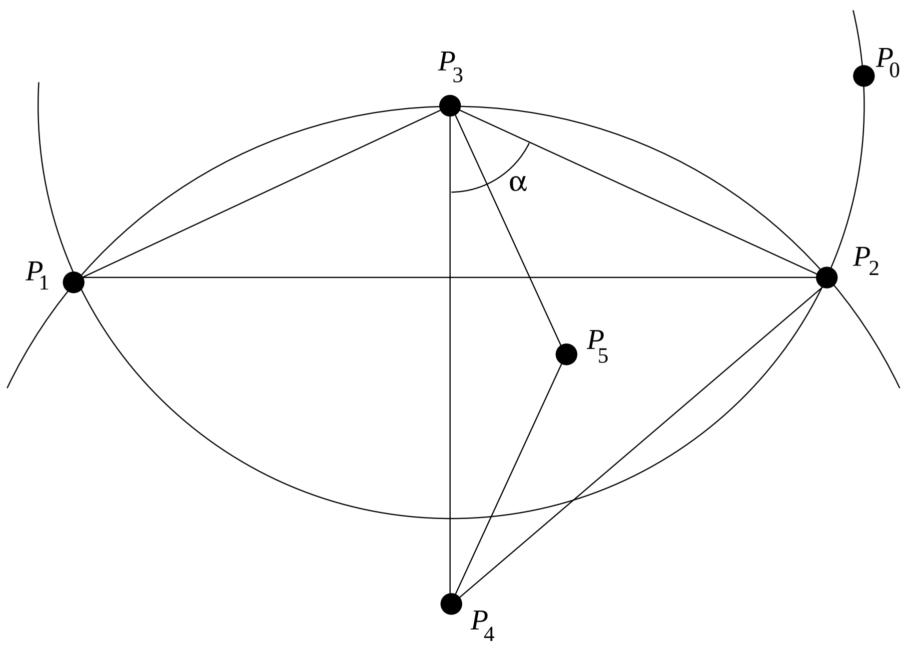
$$t = P_1P_2 = 2\sin\alpha.$$

The line P_3P_4 is the perpendicular bisector of P_1P_2. Since $\triangle P_2P_3P_4$ is isosceles, we calculate its length,

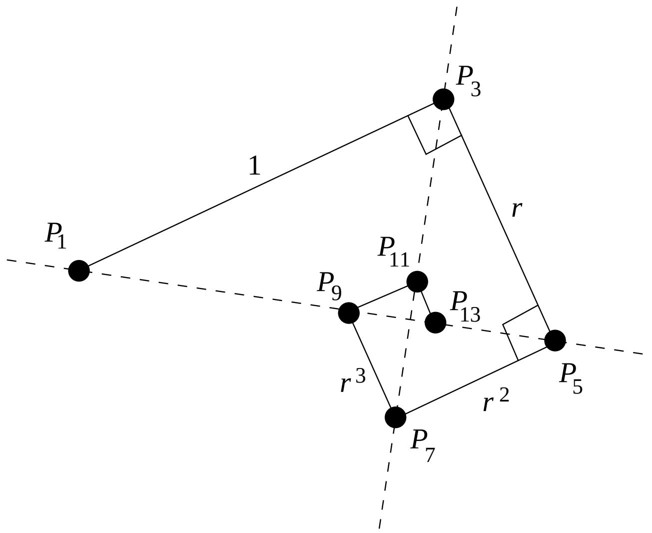
$$P_3P_4 = \frac{P_2P_3/2}{\cos\alpha} = \frac{1}{2\cos\alpha}.$$

As P_5 is the circumcentre of $\triangle P_2P_3P_4$, we have $\angle P_3P_5P_4 = 2\angle P_3P_2P_4 = 2\angle P_2P_3P_4 = 2\alpha$. The isosceles triangle $\triangle P_3P_4P_5$ is therefore similar to $\triangle P_1P_2P_3$. As $P_3P_4 \perp P_1P_2$, we have $\angle P_1P_3P_5 = 90$. Furthermore, the ratio $P_3P_5 : P_1P_3$ equals r where

$$r = \frac{P_3P_4}{P_1P_2} = \frac{1}{(2\sin\alpha)(2\cos\alpha)} = \frac{1}{2\sin(2\alpha)}.$$



By the same argument, we see that each $\angle P_iP_{i+2}P_{i+4}$ is a right angle with $P_{i+2}P_{i+4} : P_iP_{i+2} = r$. Thus the points $P_1, P_3, P_5, \ldots$ lie on a logarithmic spiral of ratio r and period four as shown below. It follows that $P_1, P_5, P_9, \ldots$ are collinear, proving part (a).



By the self-similarity of the spiral, we have that $P_1P_{1001} = r^{500}P_{1001}P_{2001}$, so

$$\sqrt[500]{x/y} = 1/r = 2\sin(2\alpha).$$

This is an integer when $\sin(2\alpha) \in \{0, \pm 1/2, \pm 1\}$. Since $0 < \alpha < 90$, this is equivalent to $\alpha \in \{15, 45, 75\}$. Thus $\sqrt[500]{x/y}$ is an integer exactly when t belongs to the set $\{2\sin 15,\ 2\sin 45,\ 2\sin 75\}$. This answers part (b).

GRADERS' REPORT

Eighty four of the eighty five eligible students submitted an examination paper. Each paper contained proposed solutions the some or all of the five examination questions. Each correct and well presented solution was awarded seven marks for a maximum total score of 35. The mean score was 10.8/35. The top three scores were 28, 27, and 22, thus special scrutiny was required to separate the top two papers.

Each solution was independently marked by two graders. If the two marks differed, then the solution was reconsidered until the difference was resolved. The top twenty papers were then carefully regraded by the chair to ensure that nothing was amiss.

The grade distribution and average mark for each question appears in the following table. For example, 13.1% of students were awarded 3 marks for question #1.

Marks	#1	#2	#3	#4	#5
0	10.7	11.9	45.2	60.7	90.5
1	8.3	8.3	26.2	10.7	3.6
2	8.3	6.0	4.8	8.3	1.2
3	13.1	9.5	3.6	9.5	1.2
4	10.7	17.9	0.0	4.8	1.2
5	9.5	32.1	3.6	2.4	2.4
6	20.2	13.1	0.0	1.2	0.0
7	19.0	1.2	16.7	2.4	0.0
Ave. Mark	4.05	3.64	1.79	1.09	.26

PROBLEM 1 Ninety five percent of students found the correct solution, although a surprising number arrived at a solution through trial and error or by guessing and verifying a solution. Many assumed without proof that the leading coefficient equals one, which resulted in a two-point penalty. Another common error was not to consider all four possibilities for the pair $(x - R), (x - 2)$.

PROBLEM 2 There was a flaw in question 2. The proposers intended that the integers a, b be relatively prime. This was made explicit in an early draft, but somehow was lost with the ambiguous phrase "of the form a/b." Without this assumption, the problem is much more tedious to solve. Remarkably, one student (Lino Demasi) considered more (but not all) possible values for $\gcd(a, b)$ and obtained the correct solution $n = 35$. All other students assumed implicitly (and in two cases, explicitly) that $\gcd(a, b) = 1$. Solutions to both problems are presented in this publication.

PROBLEM 3 Most students either completely solved or were baffled by this basic geometry problem. There were at least four types of solutions: one trigonometric, one using basic geometry, and two which refer to standard theorems relating to the triangle. The first two tended to be lengthy or cumbersome, and the last two are presented here. There were complaints from some participants regarding the inaccurate angles appearing in the diagram supplied with the question. The inaccuracy was intensional, since the key observation $M = Z$ would have otherwise been given away. Unfortunately, this caused some students to doubt their own proofs that $BZ : ZC = 1$; as the ratio appears to be closer to 2 in the misleading diagram!

PROBLEM 4 This problem was left unanswered by about 60% of students. Several solutions consisted only of a proof that $n = 1$ is not possible. About 25% described a procedure which works when $n = 2$. Indeed the procedure for $n = 2$ seems to be unique. About 10% proved that for no other value of n was possible, and all of the proofs explicitly or implicitly involved considering residues modulo $n - 1$.

PROBLEM 5 This problem proved to be very difficult. Only two students completely answered part (a), and no students correctly answered part (b). Of the students receiving more than 0 marks, only two were were not among the top 15 This suggests that the question effectively resolved the ranking of the strongest participants, which is arguably the purpose of Problem 5.

THE 2002 CANADIAN MATHEMATICAL OLYMPIAD

1. Let S be a subset of $\{1, 2, \ldots, 9\}$, such that the sums formed by adding each unordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5.

   What is the maximum number of elements that S can contain?

2. Call a positive integer n **practical** if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

   For example, the divisors of 6 are **1**, **2**, **3**, and **6**. Since
   1=**1**, 2=**2**, 3=**3**, 4=**1**+**3**, 5=**2**+ **3**, 6=**6**,
   we see that 6 is practical.

   Prove that the product of two practical numbers is also practical.

3. Prove that for all positive real numbers a, b, and c,

   $$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c,$$

   and determine when equality occurs.

4. Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with centre B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let the line CP meet Γ again at Q.

   Prove that $PQ = r$.

5. Let $N = \{0, 1, 2, \ldots\}$. Determine all functions $f : N \to N$ such that

   $$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

   for all x and y in N.

2002 Canadian Mathematical Olympiad
Solutions

1. Let S be a subset of $\{1, 2, \ldots, 9\}$, such that the sums formed by adding each unordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5.

   What is the maximum number of elements that S can contain?

   Solution 1
   It can be checked that all the sums of pairs for the set $\{1, 2, 3, 5, 8\}$ are different.

   Suppose, for a contradiction, that S is a subset of $\{1, \ldots, 9\}$ containing 6 elements such that all the sums of pairs are different. Now the smallest possible sum for two numbers from S is $1 + 2 = 3$ and the largest possible sum is $8 + 9 = 17$. That gives 15 possible sums: $3, \ldots, 17$. Also there are $\binom{6}{2} = 15$ pairs from S. Thus, each of $3, \ldots, 17$ is the sum of exactly one pair. The only pair from $\{1, \ldots, 9\}$ that adds to 3 is $\{1, 2\}$ and to 17 is $\{8, 9\}$. Thus $1, 2, 8, 9$ are in S. But then $1+9 = 2+8$, giving a contradiction. It follows that the maximum number of elements that S can contain is 5.

   Solution 2.

   It can be checked that all the sums of pairs for the set $\{1, 2, 3, 5, 8\}$ are different.

   Suppose, for a contradiction, that S is a subset of $\{1, \ldots 9\}$ such that all the sums of pairs are different and that $a_1 < a_2 < \ldots < a_6$ are the members of S.

   Since $a_1 + a_6 \neq a_2 + a_5$, it follows that $a_6 - a_5 \neq a_2 - a_1$. Similarly $a_6 - a_5 \neq a_4 - a_3$ and $a_4 - a_3 \neq a_2 - a_1$. These three differences must be distinct positive integers, so,

   $$(a_6 - a_5) + (a_4 - a_3) + (a_2 - a_1) \geq 1 + 2 + 3 = 6\,.$$

   Similarly $a_3 - a_2 \neq a_5 - a_4$, so

   $$(a_3 - a_2) + (a_5 - a_4) \geq 1 + 2 = 3\,.$$

   Adding the above 2 inequalities yields

   $$a_6 - a_5 + a_5 - a_4 + a_4 - a_3 + a_3 - a_2 + a_2 - a_1 \geq 6 + 3 = 9\,,$$

   and hence $a_6 - a_1 \geq 9$. This is impossible since the numbers in S are between 1 and 9.

2. Call a positive integer n *practical* if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

   For example, the divisors of 6 are **1**, **2**, **3**, and **6** . Since

   $$1=\mathbf{1},\quad 2=\mathbf{2},\quad 3=\mathbf{3},\quad 4=\mathbf{1}+\mathbf{3},\quad 5=\mathbf{2}+\mathbf{3},\quad 6=\mathbf{6},$$

   we see that 6 is practical.

   Prove that the product of two practical numbers is also practical.

   Solution

   Let p and q be practical. For any $k \leq pq$, we can write

   $$k = aq + b \text{ with } 0 \leq a \leq p,\ 0 \leq b < q.$$

   Since p and q are practical, we can write

   $$a = c_1 + \ldots + c_m,\quad b = d_1 + \ldots + d_n$$

   where the c_i's are distinct divisors of p and the d_j's are distinct divisors of q. Now

   $$\begin{aligned} k &= (c_1 + \ldots + c_m)q + (d_1 + \ldots + d_n) \\ &= c_1 q + \ldots + c_m q + d_1 + \ldots + d_n. \end{aligned}$$

   Each of $c_i q$ and d_j divides pq. Since $d_j < q \leq c_i q$ for any i, j, the $c_i q$'s and d_j's are all distinct, and we conclude that pq is practical.

3. Prove that for all positive real numbers a, b, and c,

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c,$$

and determine when equality occurs.

Each of the inequalities used in the solutions below has the property that equality holds if and only if $a = b = c$. Thus equality holds for the given inequality if and only if $a = b = c$.

Solution 1.

Note that $a^4 + b^4 + c^4 = \frac{(a^4 + b^4)}{2} + \frac{(b^4 + c^4)}{2} + \frac{(c^4 + a^4)}{2}$. Applying the arithmetic-geometric mean inequality to each term, we see that the right side is greater than or equal to

$$a^2b^2 + b^2c^2 + c^2a^2.$$

We can rewrite this as

$$\frac{a^2(b^2 + c^2)}{2} + \frac{b^2(c^2 + a^2)}{2} + \frac{c^2(a^2 + b^2)}{2}.$$

Applying the arithmetic mean-geometric mean inequality again we obtain $a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab$. Dividing both sides by abc (which is positive) the result follows.

Solution 2.

Notice the inequality is homogeneous. That is, if a, b, c are replaced by ka, kb, kc, $k > 0$ we get the original inequality. Thus we can assume, without loss of generality, that $abc = 1$. Then

$$\begin{aligned} \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} &= abc\left(\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}\right) \\ &= a^4 + b^4 + c^4. \end{aligned}$$

So we need prove that $a^4 + b^4 + c^4 \geq a + b + c$.

By the Power Mean Inequality,

$$\frac{a^4 + b^4 + c^4}{3} \geq \left(\frac{a + b + c}{3}\right)^4,$$

so $a^4 + b^4 + c^4 \geq (a + b + c) \cdot \frac{(a + b + c)^3}{27}$.

By the arithmetic mean-geometric mean inequality, $\frac{a + b + c}{3} \geq \sqrt[3]{abc} = 1$, so $a + b + c \geq 3$.

Hence, $a^4 + b^4 + c^4 \geq (a + b + c) \cdot \frac{(a + b + c)^3}{27} \geq (a + b + c)\frac{3^3}{27} = a + b + c$.

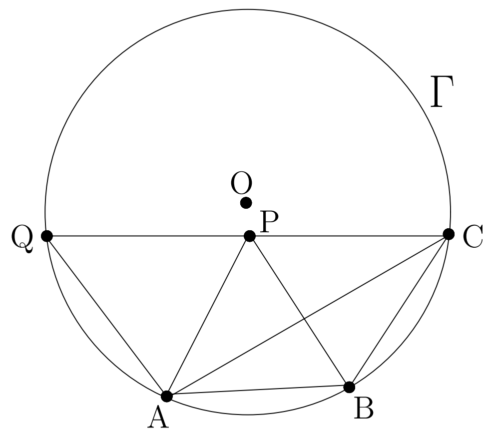
Solution 3.

Rather than using the Power-Mean inequality to prove $a^4 + b^4 + c^4 \geq a + b + c$ in Proof 2, the Cauchy-Schwartz-Bunjakovsky inequality can be used twice:

$$\begin{aligned} (a^4 + b^4 + c^4)(1^2 + 1^2 + 1^2) &\geq (a^2 + b^2 + c^2)^2 \\ (a^2 + b^2 + c^2)(1^2 + 1^2 + 1^2) &\geq (a + b + c)^2 \end{aligned}$$

So $\frac{a^4 + b^4 + c^4}{3} \geq \frac{(a^2 + b^2 + c^2)^2}{9} \geq \frac{(a + b + c)^4}{81}$. Continue as in Proof 2.

4. Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with centre B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let CP meet Γ again at Q. Prove that $PQ = r$.



Solution 1.

Let the center of Γ be O, the radius r. Since $BP = BC$, let $\theta = \measuredangle BPC = \measuredangle BCP$.

Quadrilateral $QABC$ is cyclic, so $\measuredangle BAQ = 180° - \theta$ and hence $\measuredangle PAQ = 120° - \theta$.

Also $\measuredangle APQ = 180° - \measuredangle APB - \measuredangle BPC = 120° - \theta$, so $PQ = AQ$ and $\measuredangle AQP = 2\theta - 60°$.

Again because quadrilateral $QABC$ is cyclic, $\measuredangle ABC = 180° - \measuredangle AQC = 240° - 2\theta$.

Triangles OAB and OCB are congruent, since $OA = OB = OC = r$ and $AB = BC$.

Thus $\measuredangle ABO = \measuredangle CBO = \frac{1}{2}\measuredangle ABC = 120° - \theta$.

We have now shown that in triangles AQP and AOB, $\measuredangle PAQ = \measuredangle BAO = \measuredangle APQ = \measuredangle ABO$. Also $AP = AB$, so $\triangle AQP \cong \triangle AOB$. Hence $QP = OB = r$.

Solution 2.

Let the center of Γ be O, the radius r. Since A, P and C lie on a circle centered at B, $60° = \measuredangle ABP = 2\measuredangle ACP$, so $\measuredangle ACP = \measuredangle ACQ = 30°$.

Since Q, A, and C lie on Γ, $\measuredangle QOA = 2\measuredangle QCA = 60°$.

So $QA = r$ since if a chord of a circle subtends an angle of $60°$ at the center, its length is the radius of the circle.

Now $BP = BC$, so $\measuredangle BPC = \measuredangle BCP = \measuredangle ACB + 30°$.

Thus $\measuredangle APQ = 180° - \measuredangle APB - \measuredangle BPC = 90° - \measuredangle ACB$.

Since Q, A, B and C lie on Γ and $AB = BC$, $\measuredangle AQP = \measuredangle AQC = \measuredangle AQB + \measuredangle BQC = 2\measuredangle ACB$. Finally, $\measuredangle QAP = 180 - \measuredangle AQP - \measuredangle APQ = 90 - \measuredangle ACB$.

So $\measuredangle PAQ = \measuredangle APQ$ hence $PQ = AQ = r$.

5. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all x and y in $\mathbb{N}$.

Solution 1.

We claim that f is a constant function. Suppose, for a contradiction, that there exist x and y with $f(x) < f(y)$; choose x, y such that $f(y) - f(x) > 0$ is minimal. Then

$$f(x) = \frac{xf(x) + yf(x)}{x + y} < \frac{xf(y) + yf(x)}{x + y} < \frac{xf(y) + yf(y)}{x + y} = f(y)$$

so $f(x) < f(x^2 + y^2) < f(y)$ and $0 < f(x^2 + y^2) - f(x) < f(y) - f(x)$, contradicting the choice of x and y. Thus, f is a constant function. Since $f(0)$ is in $\mathbb{N}$, the constant must be from $\mathbb{N}$.

Also, for any c in $\mathbb{N}$, $xc + yc = (x + y)c$ for all x and y, so $f(x) = c,\ c \in \mathbb{N}$ are the solutions to the equation.

Solution 2.

We claim f is a constant function. Define $g(x) = f(x) - f(0)$. Then $g(0) = 0,\ g(x) \geq -f(0)$ and

$$xg(y) + yg(x) = (x + y)g(x^2 + y^2)$$

for all x, y in $\mathbb{N}$.

Letting $y = 0$ shows $g(x^2) = 0$ (in particular, $g(1) = g(4) = 0$), and letting $x = y = 1$ shows $g(2) = 0$. Also, if x, y and z in $\mathbb{N}$ satisfy $x^2 + y^2 = z^2$, then

$$g(y) = -\frac{y}{x}g(x). \qquad (*)$$

Letting $x = 4$ and $y = 3$, $(*)$ shows that $g(3) = 0$.

For any even number $x = 2n > 4$, let $y = n^2 - 1$. Then $y > x$ and $x^2 + y^2 = (n^2 + 1)^2$. For any odd number $x = 2n+1 > 3$, let $y = 2(n+1)n$. Then $y > x$ and $x^2+y^2 = ((n+1)^2+n^2)^2$. Thus for every $x > 4$ there is $y > x$ such that $(*)$ is satisfied.

Suppose for a contradiction, that there is $x > 4$ with $g(x) > 0$. Then we can construct a sequence $x = x_0 < x_1 < x_2 < \ldots$ where $g(x_{i+1}) = -\dfrac{x_{i+1}}{x_i}\, g(x_i)$. It follows that $|g(x_{i+1})| > |g(x_i)|$ and the signs of $g(x_i)$ alternate. Since $g(x)$ is always an integer, $|g(x_{i+1})| \geq |g(x_i)|+1$. Thus for some sufficiently large value of i, $\ g(x_i) < -f(0)$, a contradiction.

As for Proof 1, we now conclude that the functions that satisfy the given functional equation are $f(x) = c,\ c \in \mathbb{N}$.

Solution 3. Suppose that W is the set of nonnegative integers and that $f : W \to W$ satisfies:

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2). \qquad (*)$$

We will show that f is a constant function.

Let $f(0) = k$, and set $S = \{x \mid f(x) = k\}$.

Letting $y = 0$ in $(*)$ shows that $f(x^2) = k \;\; \forall \;\, x > 0$, and so

$$x^2 \in S \;\; \forall \;\, x > 0 \qquad (1)$$

In particular, $1 \in S$.

Suppose $x^2 + y^2 = z^2$. Then $yf(x) + xf(y) = (x+y)f(z^2) = (x+y)k$. Thus,

$$x \in S \quad \text{iff} \quad y \in S. \tag{2}$$

whenever $x^2 + y^2$ is a perfect square.

For a contradiction, let n be the smallest non-negative integer such that $f(2^n) \neq k$. By (1) n must be odd, so $\dfrac{n-1}{2}$ is an integer. Now $\dfrac{n-1}{2} < n$ so $f(2^{\frac{n-1}{2}}) = k$. Letting $x = y = 2^{\frac{n-1}{2}}$ in $(*)$ shows $f(2^n) = k$, a contradiction. Thus every power of 2 is an element of S.

For each integer $n \geq 2$ define $p(n)$ to be the *largest prime* such that $p(n) \mid n$.

Claim: For any integer $n > 1$ that is not a power of 2, there exists a sequence of integers $x_1, x_2, \ldots, x_r$ such that the following conditions hold:

a) $x_1 = n$.
b) $x_i^2 + x_{i+1}^2$ is a perfect square for each $i = 1, 2, 3, \ldots, r-1$.
c) $p(x_1) \geq p(x_2) \geq \ldots \geq p(x_r) = 2$.

Proof: Since n is not a power of 2, $p(n) = p(x_1) \geq 3$. Let $p(x_1) = 2m+1$, so $n = x_1 = b(2m+1)^a$, for some a and b, where $p(b) < 2m+1$.

Case 1: $a = 1$. Since $(2m+1, 2m^2+2m, 2m^2+2m+1)$ is a Pythagorean Triple, if $x_2 = b(2m^2+2m)$, then $x_1^2 + x_2^2 = b^2(2m^2+2m+1)^2$ is a perfect square. Furthermore, $x_2 = 2bm(m+1)$, and so $p(x_2) < 2m+1 = p(x_1)$.

Case 2: $a > 1$. If $n = x_1 = (2m+1)^a \cdot b$, let $x_2 = (2m+1)^{a-1} \cdot b \cdot (2m^2+2m)$, $x_3 = (2m+1)^{a-2} \cdot b \cdot (2m^2+2m)^2$, …, $x_{a+1} = (2m+1)^0 \cdot b \cdot (2m^2+2m)^a = b \cdot 2^a m^a (m+1)^a$. Note that for $1 \leq i \leq a$, $x_i^2 + x_{i+1}^2$ is a perfect square and also note that $p(x_{a+1}) < 2m+1 = p(x_1)$.

If x_{a+1} is not a power of 2, we extend the sequence x_i using the same procedure described above. We keep doing this until $p(x_r) = 2$, for some integer r.

By (2), $x_i \in S$ iff $x_{i+1} \in S$ for $i = 1, 2, 3, \ldots, r-1$. Thus, $n = x_1 \in S$ iff $x_r \in S$. But x_r is a power of 2 because $p(x_r) = 2$, and we earlier proved that powers of 2 are in S. Therefore, $n \in S$, proving the claim.

We have proven that every integer $n \geq 1$ is an element of S, and so we have proven that $f(n) = k = f(0)$, for each $n \geq 1$. Therefore, f is constant, Q.E.D.

The Canadian Mathematical Olympiad - 2003

1. Consider a standard twelve-hour clock whose hour and minute hands move continuously. Let m be an integer, with $1 \leq m \leq 720$. At precisely m minutes after 12:00, the angle made by the hour hand and minute hand is exactly 1°. Determine all possible values of m.

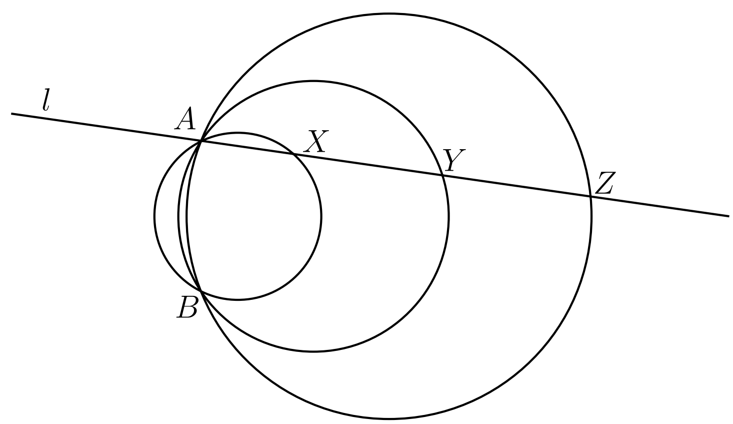
2. Find the last three digits of the number $2003^{2002^{2001}}$.

3. Find all real positive solutions (if any) to

$$x^3 + y^3 + z^3 = x + y + z, \text{ and}$$

$$x^2 + y^2 + z^2 = xyz.$$

4. Prove that when three circles share the same chord AB, every line through A different from AB determines the same ratio $XY : YZ$, where X is an arbitrary point different from B on the first circle while Y and Z are the points where AX intersects the other two circles (labelled so that Y is between X and Z).



5. Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart. Prove there is a subset T of S with at least $n/7$ points such that any two points of T are at least $\sqrt{3}$ units apart.

Solutions to the 2003 CMO

written March 26, 2003

1. Consider a standard twelve-hour clock whose hour and minute hands move continuously. Let m be an integer, with $1 \leq m \leq 720$. At precisely m minutes after 12:00, the angle made by the hour hand and minute hand is exactly 1°. Determine all possible values of m.

   Solution

   The minute hand makes a full revolution of 360° every 60 minutes, so after m minutes it has swept through $\frac{360}{60}m = 6m$ degrees. The hour hand makes a full revolution every 12 hours (720 minutes), so after m minutes it has swept through $\frac{360}{720}m = m/2$ degrees. Since both hands started in the same position at 12:00, the angle between the two hands will be 1° if $6m - m/2 = \pm 1 + 360k$ for some integer k. Solving this equation we get

   $$m = \frac{720k \pm 2}{11} = 65k + \frac{5k \pm 2}{11}.$$

   Since $1 \leq m \leq 720$, we have $1 \leq k \leq 11$. Since m is an integer, $5k \pm 2$ must be divisible by 11, say $5k \pm 2 = 11q$. Then

   $$5k = 11q \pm 2 \quad \Rightarrow \quad k = 2q + \frac{q \pm 2}{5}.$$

   If is now clear that only $q = 2$ and $q = 3$ satisfy all the conditions. Thus $k = 4$ or $k = 7$ and substituting these values into the expression for m we find that the only possible values of m are 262 and 458.

2. Find the last three digits of the number $2003^{2002^{2001}}$.

Solution

We must find the remainder when $2003^{2002^{2001}}$ is divided by 1000, which will be the same as the remainder when $3^{2002^{2001}}$ is divided by 1000, since $2003 \equiv 3 \pmod{1000}$. To do this we will first find a positive integer n such that $3^n \equiv 1 \pmod{1000}$ and then try to express 2002^{2001} in the form $nk + r$, so that

$$2003^{2002^{2001}} \equiv 3^{nk+r} \equiv (3^n)^k \cdot 3^r \equiv 1^k \cdot 3^r \equiv 3^r \pmod{1000}.$$

Since $3^2 = 10 - 1$, we can evaluate 3^{2m} using the binomial theorem:

$$3^{2m} = (10-1)^m = (-1)^m + 10m(-1)^{m-1} + 100\frac{m(m-1)}{2}(-1)^{m-2} + \cdots + 10^m.$$

After the first 3 terms of this expansion, all remaining terms are divisible by 1000, so letting $m = 2q$, we have that

$$3^{4q} \equiv 1 - 20q + 100q(2q-1) \pmod{1000}. \tag{1}$$

Using this, we can check that $3^{100} \equiv 1 \pmod{1000}$ and now we wish to find the remainder when 2002^{2001} is divided by 100.

Now $2002^{2001} \equiv 2^{2001} \pmod{100} \equiv 4 \cdot 2^{1999} \pmod{4 \cdot 25}$, so we'll investigate powers of 2 modulo 25. Noting that $2^{10} = 1024 \equiv -1 \pmod{25}$, we have

$$2^{1999} = (2^{10})^{199} \cdot 2^9 \equiv (-1)^{199} \cdot 512 \equiv -12 \equiv 13 \pmod{25}.$$

Thus $2^{2001} \equiv 4 \cdot 13 = 52 \pmod{100}$. Therefore 2002^{2001} can be written in the form $100k + 52$ for some integer k, so

$$2003^{2002^{2001}} \equiv 3^{52} \pmod{1000} \equiv 1 - 20 \cdot 13 + 1300 \cdot 25 \equiv 241 \pmod{1000}$$

using equation (1). So the last 3 digits of $2003^{2002^{2001}}$ are 241.

3. Find all real positive solutions (if any) to

$$x^3 + y^3 + z^3 = x + y + z, \text{ and}$$

$$x^2 + y^2 + z^2 = xyz.$$

Solution 1

Let $f(x, y, z) = (x^3 - x) + (y^3 - y) + (z^3 - z)$. The first equation above is equivalent to $f(x, y, z) = 0$. If $x, y, z \geq 1$, then $f(x, y, z) \geq 0$ with equality only if $x = y = z = 1$. But if $x = y = z = 1$, then the second equation is not satisfied. So in any solution to the system of equations, at least one of the variables is less than 1. Without loss of generality, suppose that $x < 1$. Then

$$x^2 + y^2 + z^2 > y^2 + z^2 \geq 2yz > yz > xyz.$$

Therefore the system has no real positive solutions.

Solution 2

We will show that the system has no real positive solution. Assume otherwise.

The second equation can be written $x^2 - (yz)x + (y^2 + z^2)$. Since this quadratic in x has a real solution by hypothesis, its discrimant is nonnegative. Hence

$$y^2z^2 - 4y^2 - 4z^2 \geq 0.$$

Dividing through by $4y^2z^2$ yields

$$\frac{1}{4} \geq \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{1}{y^2}.$$

Hence $y^2 \geq 4$ and so $y \geq 2$, y being positive. A similar argument yields $x, y, z \geq 2$. But the first equation can be written as

$$x(x^2 - 1) + y(y^2 - 1) + z(z^2 - 1) = 0,$$

contradicting $x, y, z \geq 2$. Hence, a real positive solution cannot exist.

Solution 3

Applying the arithmetic-geometric mean inequality and the Power Mean Inequalities to x, y, z we have

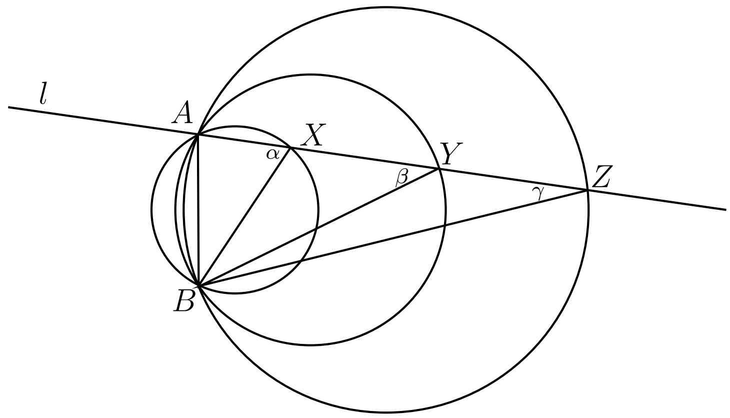
$$\sqrt[3]{xyz} \leq \frac{x+y+z}{3} \leq \sqrt{\frac{x^2+y^2+z^2}{3}} \leq \sqrt[3]{\frac{x^3+y^3+z^3}{3}}.$$

Letting $S = x+y+z = x^3+y^3+z^3$ and $P = xyz = x^2+y^2+z^2$, this inequality can be written

$$\sqrt[3]{P} \leq \frac{S}{3} \leq \sqrt{\frac{P}{3}} \leq \sqrt[3]{\frac{S}{3}}.$$

Now $\sqrt[3]{P} \leq \sqrt{\frac{P}{3}}$ implies $P^2 \leq P^3/27$, so $P \geq 27$. Also $\frac{S}{3} \leq \sqrt[3]{\frac{S}{3}}$ implies $S^3/27 \leq S/3$, so $S \leq 3$. But then $\sqrt[3]{P} \geq 3$ and $\sqrt[3]{\frac{S}{3}} \leq 1$ which is inconsistent with $\sqrt[3]{P} \leq \sqrt[3]{\frac{S}{3}}$. Therefore the system cannot have a real positive solution.

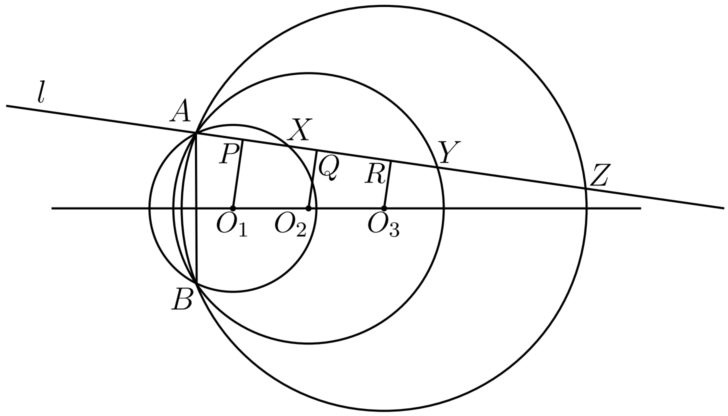
4. Prove that when three circles share the same chord AB, every line through A different from AB determines the same ratio $XY : YZ$, where X is an arbitrary point different from B on the first circle while Y and Z are the points where AX intersects the other two circles (labelled so that Y is between X and Z).



Solution 1

Let l be a line through A different from AB and join B to A, X, Y and Z as in the above diagram. No matter how l is chosen, the angles AXB, AYB and AZB always subtend the chord AB. For this reason the angles in the triangles BXY and BXZ are the same for all such l. Thus the ratio $XY : YZ$ remains constant by similar triangles.

Note that this is true no matter how X, Y and Z lie in relation to A. Suppose X, Y and Z all lie on the same side of A (as in the diagram) and that $\measuredangle AXB = \alpha$, $\measuredangle AYB = \beta$ and $\measuredangle AZB = \gamma$. Then $\measuredangle BXY = 180^\circ - \alpha$, $\measuredangle BYX = \beta$, $\measuredangle BYZ = 180^\circ - \beta$ and $\measuredangle BZY = \gamma$. Now suppose l is chosen so that X is now on the opposite side of A from Y and Z. Now since X is on the other side of the chord AB, $\measuredangle AXB = 180^\circ - \alpha$, but it is still the case that $\measuredangle BXY = 180^\circ - \alpha$ and all other angles in the two pertinent triangles remain unchanged. If l is chosen so that X is identical with A, then l is tangent to the first circle and it is still the case that $\measuredangle BXY = 180^\circ - \alpha$. All other cases can be checked in a similar manner.



Solution 2

Let m be the perpendicular bisector of AB and let O_1, O_2, O_3 be the centres of the three circles. Since AB is a chord common to all three circles, O_1, O_2, O_3 all lie on m. Let l be a line through A different from AB and suppose that X, Y, Z all lie on the same side of AB, as in the above diagram. Let perpendiculars from O_1, O_2, O_3 meet l at P, Q, R, respectively. Since a line through the centre of a circle bisects any chord,

$$AX = 2AP, \quad AY = 2AQ \quad \text{and} \quad AZ = 2AR.$$

Now

$$XY = AY - AX = 2(AQ - AP) = 2PQ \quad \text{and, similarly,} \quad YZ = 2QR.$$

Therefore $XY : YZ = PQ : QR$. But $O_1P \,||\, O_2Q \,||\, O_3R$, so $PQ : QR = O_1O_2 : O_2O_3$. Since the centres of the circles are fixed, the ratio $XY : YZ = O_1O_2 : O_2O_3$ does not depend on the choice of l.

If X, Y, Z do not all lie on the same side of AB, we can obtain the same result with a similar proof. For instance, if X and Y are opposite sides of AB, then we will have $XY = AY + AX$, but since in this case $PQ = AQ + AP$, it is still the case that $XY = 2PQ$ and result still follows, etc.

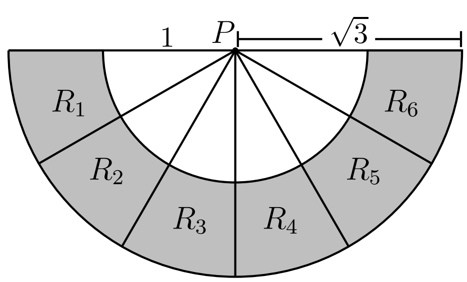
5. Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart. Prove there is a subset T of S with at least $n/7$ points such that any two points of T are at least $\sqrt{3}$ units apart.

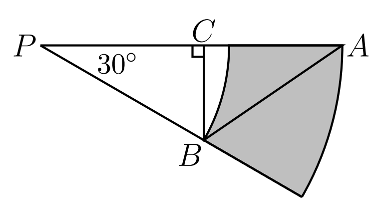
Solution

We will construct the set T in the following way: Assume the points of S are in the xy-plane and let P be a point in S with maximum y-coordinate. This point P will be a member of the set T and now, from S, we will remove P and all points in S which are less than $\sqrt{3}$ units from P. From the remaining points we choose one with maximum y-coordinate to be a member of T and remove from S all points at distance less than $\sqrt{3}$ units from this new point. We continue in this way, until all the points of S are exhausted. Clearly any two points in T are at least $\sqrt{3}$ units apart. To show that T has at least $n/7$ points, we must prove that at each stage no more than 6 other points are removed along with P.

At a typical stage in this process, we've selected a point P with maximum y-coordinate, so any points at distance less than $\sqrt{3}$ from P must lie inside the semicircular region of radius $\sqrt{3}$ centred at P shown in the first diagram below. Since points of S are at least 1 unit apart, these points must lie outside (or on) the semicircle of radius 1. (So they lie in the shaded region of the first diagram.) Now divide this shaded region into 6 congruent regions $R_1, R_2, \ldots, R_6$ as shown in this diagram.

We will show that each of these regions contains at most one point of S. Since all 6 regions are congruent, consider one of them as depicted in the second diagram below. The distance between any two points in this shaded region must be less than the length of the line segment AB. The lengths of PA and PB are $\sqrt{3}$ and 1, respectively, and angle $APB = 30°$. If we construct a perpendicular from B to PA at C, then the length of PC is $\cos 30° = \sqrt{3}/2$. Thus BC is a perpendicular bisector of PA and therefore $AB = PB = 1$. So the distance between any two points in this region is less than 1. Therefore each of $R_1, \ldots, R_6$ can contain at most one point of S, which completes the proof.





36th Canadian Mathematical Olympiad

Wednesday, March 31, 2004

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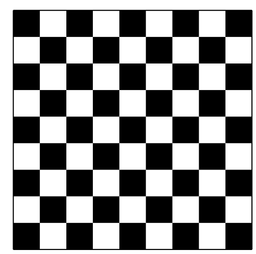
1. Find all ordered triples (x, y, z) of real numbers which satisfy the following system of equations:
$$\left\{\begin{array}{rcl} xy &=& z - x - y \\ xz &=& y - x - z \\ yz &=& x - y - z \end{array}\right.$$

2. How many ways can 8 mutually non-attacking rooks be placed on the 9×9 chessboard (shown here) so that all 8 rooks are on squares of the same colour?
[Two rooks are said to be attacking each other if they are placed in the same row or column of the board.]



3. Let A, B, C, D be four points on a circle (occurring in clockwise order), with $AB < AD$ and $BC > CD$. Let the bisector of angle BAD meet the circle at X and the bisector of angle BCD meet the circle at Y. Consider the hexagon formed by these six points on the circle. If four of the six sides of the hexagon have equal length, prove that BD must be a diameter of the circle.

4. Let p be an odd prime. Prove that
$$\sum_{k=1}^{p-1} k^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}.$$
[Note that $a \equiv b \pmod{m}$ means that $a - b$ is divisible by m.]

5. Let T be the set of all positive integer divisors of 2004^{100}. What is the largest possible number of elements that a subset S of T can have if no element of S is an integer multiple of any other element of S?

Solutions to the 2004 CMO

written March 31, 2004

1. Find all ordered triples (x, y, z) of real numbers which satisfy the following system of equations:

$$\begin{cases} xy &= z - x - y \\ xz &= y - x - z \\ yz &= x - y - z \end{cases}$$

Solution 1

Subtracting the second equation from the first gives $xy - xz = 2z - 2y$. Factoring $y - z$ from each side and rearranging gives

$$(x + 2)(y - z) = 0,$$

so either $x = -2$ or $z = y$.

If $x = -2$, the first equation becomes $-2y = z + 2 - y$, or $y + z = -2$. Substituting $x = -2$, $y + z = -2$ into the third equation gives $yz = -2 - (-2) = 0$. Hence either y or z is 0, so if $x = -2$, the only solutions are $(-2, 0, -2)$ and $(-2, -2, 0)$.

If $z = y$ the first equation becomes $xy = -x$, or $x(y + 1) = 0$. If $x = 0$ and $z = y$, the third equation becomes $y^2 = -2y$ which gives $y = 0$ or $y = -2$. If $y = -1$ and $z = y = -1$, the third equation gives $x = -1$. So if $y = z$, the only solutions are $(0, 0, 0)$, $(0, -2, -2)$ and $(-1, -1, -1)$.

In summary, there are 5 solutions: $(-2, 0, -2)$, $(-2, -2, 0)$, $(0, 0, 0)$, $(0, -2, -2)$ and $(-1, -1, -1)$.

Solution 2

Adding x to both sides of the first equation gives

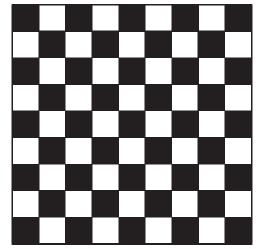
$$x(y + 1) = z - y = (z + 1) - (y + 1) \;\Rightarrow\; (x + 1)(y + 1) = z + 1.$$

Similarly manipulating the other two equations and letting $a = x + 1$, $b = y + 1$, $c = z + 1$, we can write the system in the following way.

$$\begin{cases} ab &= c \\ ac &= b \\ bc &= a \end{cases}$$

If any one of a, b, c is 0, then it's clear that all three are 0. So $(a, b, c) = (0, 0, 0)$ is one solution and now suppose that a, b, c are all nonzero. Substituting $c = ab$ into the second and third equations gives $a^2b = b$ and $b^2a = a$, respectively. Hence $a^2 = 1$, $b^2 = 1$ (since a, b nonzero). This gives 4 more solutions: $(a, b, c) = (1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$ or $(-1, -1, 1)$. Reexpressing in terms of x, y, z, we obtain the 5 ordered triples listed in Solution 1.

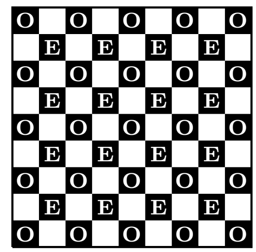
2. How many ways can 8 mutually non-attacking rooks be placed on the 9×9 chessboard (shown here) so that all 8 rooks are on squares of the same colour?
[Two rooks are said to be attacking each other if they are placed in the same row or column of the board.]



Solution 1

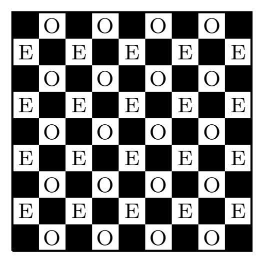
We will first count the number of ways of placing 8 mutually non-attacking rooks on black squares and then count the number of ways of placing them on white squares. Suppose that the rows of the board have been numbered 1 to 9 from top to bottom.

First notice that a rook placed on a black square in an odd numbered row cannot attack a rook on a black square in an even numbered row. This effectively partitions the black squares into a 5×5 board and a 4×4 board (squares labelled O and E respectively, in the diagram at right) and rooks can be placed independently on these two boards. There are 5! ways to place 5 non-attacking rooks on the squares labelled O and 4! ways to place 4 non-attacking rooks on the squares labelled E.



This gives 5!4! ways to place 9 mutually non-attacking rooks on black squares and removing any one of these 9 rooks gives one of the desired configurations. Thus there are $9 \cdot 5!4!$ ways to place 8 mutually non-attacking rooks on black squares.

Using very similar reasoning we can partition the white squares as shown in the diagram at right. The white squares are partitioned into two 5×4 boards such that no rook on a square marked O can attack a rook on a square mark E. At most 4 non-attacking rooks can be placed on a 5×4 board and they can be placed in $5 \cdot 4 \cdot 3 \cdot 2 = 5!$ ways. Thus there are $(5!)^2$ ways to place 8 mutually non-attacking rooks on white squares.



In total there are $9 \cdot 5!4! + (5!)^2 = (9 + 5)5!4! = 14 \cdot 5!4! = 40320$ ways to place 8 mutually non-attacking rooks on squares of the same colour.

Solution 2

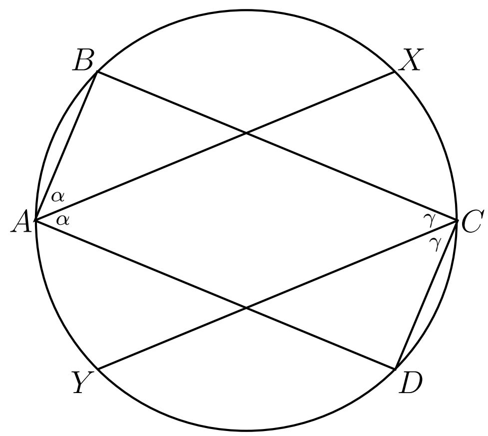
Consider rooks on black squares first. We have 8 rooks and 9 rows, so exactly one row will be without rooks. There are two cases: either the empty row has 5 black squares or it has 4 black squares. By permutation these rows can be made either last or second last. In each case we'll count the possible number of ways of placing the rooks on the board as we proceed row by row.

In the first case we have 5 choices for the empty row, then we can place a rook on any of the black squares in row 1 (5 possibilities) and any of the black squares in row 2 (4 possibilities). When we attempt to place a rook in row 3, we must avoid the column containing the rook that was placed in row 1, so we have 4 possibilities. Using similar reasoning, we can place the rook on any of 3 possible black squares in row 4, etc. The total number of possibilities for the first case is $5 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 = (5!)^2$. In the second case, we have 4 choices for the empty row (but assume it's the second last row). We now place rooks as before and using similar logic, we get that the total number of possibilities for the second case is $4 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 4(5!4!)$.

Now, do the same for the white squares. If a row with 4 white squares is empty (5 ways to choose it), then the total number of possibilities is $(5!)^2$. It's impossible to have a row with 5 white squares empty, so the total number of ways to place rooks is

$$(5!)^2 + 4(5!4!) + (5!)^2 = (5 + 4 + 5)5!4! = 14(5!4!).$$

3. Let A, B, C, D be four points on a circle (occurring in clockwise order), with $AB < AD$ and $BC > CD$. Let the bisector of angle BAD meet the circle at X and the bisector of angle BCD meet the circle at Y. Consider the hexagon formed by these six points on the circle. If four of the six sides of the hexagon have equal length, prove thatBD must be a diameter of the circle.



Solution 1

We're given that $AB < AD$. Since CY bisects $\measuredangle BCD$, $BY = YD$, so Y lies between D and A on the circle, as in the diagram above, and $DY > YA$, $DY > AB$. Similar reasoning confirms that X lies between B and C and $BX > XC$, $BX > CD$. So if $ABXCDY$ has 4 equal sides, then it must be that $YA = AB = XC = CD$.

Let $\measuredangle BAX = \measuredangle DAX = \alpha$ and let $\measuredangle BCY = \measuredangle DCY = \gamma$. Since $ABCD$ is cyclic, $\measuredangle A + \measuredangle C = 180°$, which implies that $\alpha + \gamma = 90°$. The fact that $YA = AB = XC = CD$ means that the arc from Y to B (which is subtended by $\measuredangle YCB$) is equal to the arc from X to D (which is subtended by $\measuredangle XAD$). Hence $\measuredangle YCB = \measuredangle XAD$, so $\alpha = \gamma = 45°$. Finally, BD is subtended by $\measuredangle BAD = 2\alpha = 90°$. Therefore BD is a diameter of the circle.

Solution 2

We're given that $AB < AD$. Since CY bisects $\measuredangle BCD$, $BY = YD$, so Y lies between D and A on the circle, as in the diagram above, and $DY > YA$, $DY > AB$. Similar reasoning confirms that X lies between B and C and $BX > XC$, $BX > CD$. So if $ABXCDY$ has 4 equal sides, then it must be that $YA = AB = XC = CD$. This implies that the arc from Y to B is equal to the arc from X to D and hence that $YB = XD$. Since $\measuredangle BAX = \measuredangle XAD$, $BX = XD$ and since $\measuredangle DCY = \measuredangle YCB$, $DY = YB$. Therefore $BXDY$ is a square and its diagonal, BD, must be a diameter of the circle.

4. Let p be an odd prime. Prove that

$$\sum_{k=1}^{p-1} k^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}.$$

[Note that $a \equiv b \pmod{m}$ means that $a-b$ is divisible by m.]

Solution

Since $p-1$ is even, we can pair up the terms in the summation in the following way (first term with last, 2nd term with 2nd last, etc.):

$$\sum_{k=1}^{p-1} k^{2p-1} = \sum_{k=1}^{\frac{p-1}{2}} \left(k^{2p-1} + (p-k)^{2p-1}\right).$$

Expanding $(p-k)^{2p-1}$ with the binomial theorem, we get

$$(p-k)^{2p-1} = p^{2p-1} - \cdots - \binom{2p-1}{2} p^2 k^{2p-3} + \binom{2p-1}{1} pk^{2p-2} - k^{2p-1},$$

where every term on the right-hand side is divisible by p^2 except the last two. Therefore

$$k^{2p-1} + (p-k)^{2p-1} \equiv k^{2p-1} + \binom{2p-1}{1} pk^{2p-2} - k^{2p-1} \equiv (2p-1)pk^{2p-2} \pmod{p^2}.$$

For $1 \leq k < p$, k is not divisible by p, so $k^{p-1} \equiv 1 \pmod{p}$, by Fermat's Little Theorem. So $(2p-1)k^{2p-2} \equiv (2p-1)(1^2) \equiv -1 \pmod{p}$, say $(2p-1)k^{2p-2} = mp-1$ for some integer m. Then

$$(2p-1)pk^{2p-2} = mp^2 - p \equiv -p \pmod{p^2}.$$

Finally,

$$\begin{aligned} \sum_{k=1}^{p-1} k^{2p-1} &\equiv \sum_{k=1}^{\frac{p-1}{2}} (-p) \equiv \left(\frac{p-1}{2}\right)(-p) \pmod{p^2} \\ &\equiv \frac{p-p^2}{2} + p^2 \equiv \frac{p(p+1)}{2} \pmod{p^2}. \end{aligned}$$

5. Let T be the set of all positive integer divisors of 2004^{100}. What is the largest possible number of elements that a subset S of T can have if no element of S is an integer multiple of any other element of S?

Solution

Assume throughout that a, b, c are nonnegative integers. Since the prime factorization of 2004 is $2004 = 2^2 \cdot 3 \cdot 167$,

$$T = \left\{ 2^a 3^b 167^c \,\middle|\, 0 \le a \le 200,\ 0 \le b, c \le 100 \right\}.$$

Let

$$S = \left\{ 2^{200-b-c} 3^b 167^c \,\middle|\, 0 \le b, c \le 100 \right\}.$$

For any $0 \le b, c \le 100$, we have $0 \le 200 - b - c \le 200$, so S is a subset of T. Since there are 101 possible values for b and 101 possible values for c, S contains 101^2 elements. We will show that no element of S is a multiple of another and that no larger subset of T satisfies this condition.

Suppose $2^{200-b-c} 3^b 167^c$ is an integer multiple of $2^{200-j-k} 3^j 167^k$. Then

$$200 - b - c \ge 200 - j - k, \quad b \ge j, \quad c \ge k.$$

But this first inequality implies $b + c \le j + k$, which together with $b \ge j$, $c \ge k$ gives $b = j$ and $c = k$. Hence no element of S is an integer multiple of another element of S.

Let U be a subset of T with more than 101^2 elements. Since there are only 101^2 distinct pairs (b, c) with $0 \le b, c \le 100$, then (by the pigeonhole principle) U must contain two elements $u_1 = 2^{a_1} 3^{b_1} 167^{c_1}$ and $u_2 = 2^{a_2} 3^{b_2} 167^{c_2}$, with $b_1 = b_2$ and $c_1 = c_2$, but $a_1 \ne a_2$. If $a_1 > a_2$, then u_1 is a multiple of u_2 and if $a_1 < a_2$, then u_2 is a multiple of u_1. Hence U does not satisfy the desired condition.

Therefore the largest possible number of elements that such a subset of T can have is $101^2 = 10201$.

37th Canadian Mathematical Olympiad

Wednesday, March 30, 2005

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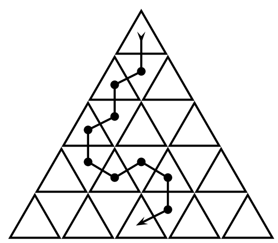
1. Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.



2. Let (a, b, c) be a Pythagorean triple, *i.e.*, a triplet of positive integers with $a^2 + b^2 = c^2$.

   a) Prove that $(c/a + c/b)^2 > 8$.

   b) Prove that there does not exist any integer n for which we can find a Pythagorean triple (a, b, c) satisfying $(c/a + c/b)^2 = n$.

3. Let S be a set of $n \geq 3$ points in the interior of a circle.

   a) Show that there are three distinct points $a, b, c \in S$ and three distinct points A, B, C on the circle such that a is (strictly) closer to A than any other point in S, b is closer to B than any other point in S and c is closer to C than any other point in S.

   b) Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

4. Let ABC be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of KP/R^3.

5. Let's say that an ordered triple of positive integers (a, b, c) is *n-powerful* if $a \leq b \leq c$, $\gcd(a, b, c) = 1$, and $a^n + b^n + c^n$ is divisible by $a + b + c$. For example, $(1, 2, 2)$ is 5-powerful.

   a) Determine all ordered triples (if any) which are n-powerful for all $n \geq 1$.

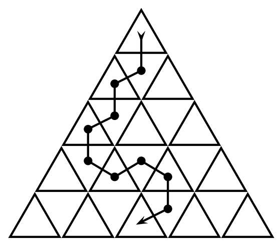
   b) Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

[Note that $\gcd(a, b, c)$ is the greatest common divisor of a, b and c.]

Solutions to the 2005 CMO

written March 30, 2005

1. Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.

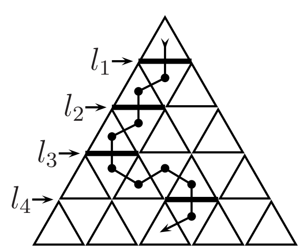


Solution

We shall show that $f(n) = (n-1)!$.

Label the horizontal line segments in the triangle l_1, l_2, ... as in the diagram below. Since the path goes from the top triangle to a triangle in the bottom row and never travels up, the path must cross each of $l_1, l_2, \ldots, l_{n-1}$ exactly once. The diagonal lines in the triangle divide l_k into k unit line segments and the path must cross exactly one of these k segments for each k. (In the diagram below, these line segments have been highlighted.) The path is completely determined by the set of $n-1$ line segments which are crossed. So as the path moves from the kth row to the $(k+1)$st row, there are k possible line segments where the path could cross l_k. Since there are $1 \cdot 2 \cdot 3 \cdots (n-1) = (n-1)!$ ways that the path could cross the $n-1$ horizontal lines, and each one corresponds to a unique path, we get $f(n) = (n-1)!$.

Therefore $f(2005) = (2004)!$.



2. Let (a, b, c) be a Pythagorean triple, *i.e.*, a triplet of positive integers with $a^2 + b^2 = c^2$.

   a) Prove that $(c/a + c/b)^2 > 8$.

   b) Prove that there does not exist any integer n for which we can find a Pythagorean triple (a, b, c) satisfying $(c/a + c/b)^2 = n$.

a) **Solution 1**

Let (a, b, c) be a Pythagorean triple. View a, b as lengths of the legs of a right angled triangle with hypotenuse of length c; let θ be the angle determined by the sides with lengths a and c. Then

$$\begin{aligned}\left(\frac{c}{a} + \frac{c}{b}\right)^2 &= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)^2 = \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta}{(\sin\theta\cos\theta)^2} \\ &= 4\left(\frac{1 + \sin 2\theta}{\sin^2 2\theta}\right) = \frac{4}{\sin^2 2\theta} + \frac{4}{\sin 2\theta}\end{aligned}$$

Note that because $0 < \theta < 90°$, we have $0 < \sin 2\theta \leq 1$, with equality only if $\theta = 45°$. But then $a = b$ and we obtain $\sqrt{2} = c/a$, contradicting a, c both being integers. Thus, $0 < \sin 2\theta < 1$ which gives $(c/a + c/b)^2 > 8$.

Solution 2

Defining θ as in Solution 1, we have $c/a + c/b = \sec\theta + \csc\theta$. By the AM-GM inequality, we have $(\sec\theta + \csc\theta)/2 \geq \sqrt{\sec\theta\csc\theta}$. So

$$c/a + c/b \geq \frac{2}{\sqrt{\sin\theta\cos\theta}} = \frac{2\sqrt{2}}{\sqrt{\sin 2\theta}} \geq 2\sqrt{2}.$$

Since a, b, c are integers, we have $c/a + c/b > 2\sqrt{2}$ which gives $(c/a + c/b)^2 > 8$.

Solution 3

By simplifying and using the AM-GM inequality,

$$\left(\frac{c}{a} + \frac{c}{b}\right)^2 = c^2\left(\frac{a+b}{ab}\right)^2 = \frac{(a^2+b^2)(a+b)^2}{a^2b^2} \geq \frac{2\sqrt{a^2b^2}\,(2\sqrt{ab})^2}{a^2b^2} = 8,$$

with equality only if $a = b$. By using the same argument as in Solution 1, a cannot equal b and the inequality is strict.

Solution 4

$$\begin{aligned}\left(\frac{c}{a} + \frac{c}{b}\right)^2 &= \frac{c^2}{a^2} + \frac{c^2}{b^2} + \frac{2c^2}{ab} = 1 + \frac{b^2}{a^2} + \frac{a^2}{b^2} + 1 + \frac{2(a^2+b^2)}{ab} \\ &= 2 + \left(\frac{a}{b} - \frac{b}{a}\right)^2 + 2 + \frac{2}{ab}\Big((a-b)^2 + 2ab\Big) \\ &= 4 + \left(\frac{a}{b} - \frac{b}{a}\right)^2 + \frac{2(a-b)^2}{ab} + 4 \geq 8,\end{aligned}$$

with equality only if $a = b$, which (as argued previously) cannot occur.

b) **Solution 1**

Since $c/a + c/b$ is rational, $(c/a + c/b)^2$ can only be an integer if $c/a + c/b$ is an integer. Suppose $c/a + c/b = m$. We may assume that $\gcd(a, b) = 1$. (If not, divide the common factor from (a, b, c), leaving m unchanged.)

Since $c(a+b) = mab$ and $\gcd(a, a+b) = 1$, a must divide c, say $c = ak$. This gives $a^2 + b^2 = a^2k^2$ which implies $b^2 = (k^2 - 1)a^2$. But then a divides b contradicting the fact that $\gcd(a, b) = 1$. Therefore $(c/a + c/b)^2$ is not equal to any integer n.

Solution 2

We begin as in Solution 1, supposing that $c/a + c/b = m$ with $\gcd(a, b) = 1$. Hence a and b are not both even. It is also the case that a and b are not both odd, for then $c^2 = a^2 + b^2 \equiv 2 \pmod 4$, and perfect squares are congruent to either 0 or 1 modulo 4. So one of a, b is odd and the other is even. Therefore c must be odd.

Now $c/a + c/b = m$ implies $c(a+b) = mab$, which cannot be true because $c(a+b)$ is odd and mab is even.

3. Let S be a set of $n \geq 3$ points in the interior of a circle.

 a) Show that there are three distinct points $a, b, c \in S$ and three distinct points A, B, C on the circle such that a is (strictly) closer to A than any other point in S, b is closer to B than any other point in S and c is closer to C than any other point in S.

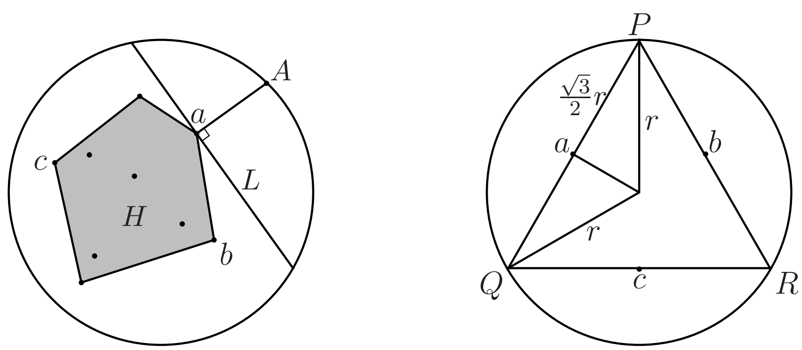
 b) Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

Solution 1

 a) Let H be the smallest convex set of points in the plane which contains S.[†] Take 3 points $a, b, c \in S$ which lie on the boundary of H. (There must always be at least 3 (but not necessarily 4) such points.)

 Since a lies on the boundary of the convex region H, we can construct a chord L such that no two points of H lie on opposite sides of L. Of the two points where the perpendicular to L at a meets the circle, choose one which is on a side of L not containing any points of H and call this point A. Certainly A is closer to a than to any other point on L or on the other side of L. Hence A is closer to a than to any other point of S. We can find the required points B and C in an analogous way and the proof is complete.

 [Note that this argument still holds if all the points of S lie on a line.]



(a) (b)

 b) Let PQR be an equilateral triangle inscribed in the circle and let a, b, c be midpoints of the three sides of $\triangle PQR$. If r is the radius of the circle, then every point on the circle is within $(\sqrt{3}/2)r$ of one of a, b or c. (See figure (b) above.)

 Now $\sqrt{3}/2 < 9/10$, so if S consists of a, b, c and a cluster of points within $r/10$ of the centre of the circle, then we cannot select 4 points from S (and corresponding points on the circle) having the desired property.

[†]By the way, H is called the convex hull of S. If the points of S lie on a line, then H will be the shortest line segment containing the points of S. Otherwise, H is a polygon whose vertices are all elements of S and such that all other points in S lie inside or on this polygon.

Solution 2

a) If all the points of S lie on a line L, then choose any 3 of them to be a, b, c. Let A be a point on the circle which meets the perpendicular to L at a. Clearly A is closer to a than to any other point on L, and hence closer than other other point in S. We find B and C in an analogous way.

   Otherwise, choose a, b, c from S so that the triangle formed by these points has maximal area. Construct the altitude from the side bc to the point a and extend this line until it meets the circle at A. We claim that A is closer to a than to any other point in S.

   Suppose not. Let x be a point in S for which the distance from A to x is less than the distance from A to a. Then the perpendicular distance from x to the line bc must be greater than the perpendicular distance from a to the line bc. But then the triangle formed by the points x, b, c has greater area than the triangle formed by a, b, c, contradicting the original choice of these 3 points. Therefore A is closer to a than to any other point in S.

   The points B and C are found by constructing similar altitudes through b and c, respectively.

b) See Solution 1.

4. Let ABC be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of KP/R^3.

Solution 1

Since similar triangles give the same value of KP/R^3, we can fix $R = 1$ and maximize KP over all triangles inscribed in the unit circle. Fix points A and B on the unit circle. The locus of points C with a given perimeter P is an ellipse that meets the circle in at most four points. The area K is maximized (for a fixed P) when C is chosen on the perpendicular bisector of AB, so we get a maximum value for KP if C is where the perpendicular bisector of AB meets the circle. Thus the maximum value of KP for a given AB occurs when ABC is an isosceles triangle. Repeating this argument with BC fixed, we have that the maximum occurs when ABC is an equilateral triangle.

Consider an equilateral triangle with side length a. It has $P = 3a$. It has height equal to $a\sqrt{3}/2$ giving $K = a^2\sqrt{3}/4$. ¿From the extended law of sines, $2R = a/\sin(60)$ giving $R = a/\sqrt{3}$. Therefore the maximum value we seek is

$$KP/R^3 = \left(\frac{a^2\sqrt{3}}{4}\right)(3a)\left(\frac{\sqrt{3}}{a}\right)^3 = \frac{27}{4}.$$

Solution 2

From the extended law of sines, the lengths of the sides of the triangle are $2R\sin A$, $2R\sin B$ and $2R\sin C$. So

$$P = 2R(\sin A + \sin B + \sin C) \text{ and } K = \frac{1}{2}(2R\sin A)(2R\sin B)(\sin C),$$

giving

$$\frac{KP}{R^3} = 4\sin A\sin B\sin C(\sin A + \sin B + \sin C).$$

We wish to find the maximum value of this expression over all $A + B + C = 180°$. Using well-known identities for sums and products of sine functions, we can write

$$\frac{KP}{R^3} = 4\sin A\left(\frac{\cos(B-C)}{2} - \frac{\cos(B+C)}{2}\right)\left(\sin A + 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\right).$$

If we first consider A to be fixed, then $B + C$ is fixed also and this expression takes its maximum value when $\cos(B - C)$ and $\cos\left(\frac{B-C}{2}\right)$ equal 1; *i.e.* when $B = C$. In a similar way, one can show that for any fixed value of B, KP/R^3 is maximized when $A = C$. Therefore the maximum value of KP/R^3 occurs when $A = B = C = 60°$, and it is now an easy task to substitute this into the above expression to obtain the maximum value of $27/4$.

Solution 3

As in Solution 2, we obtain

$$\frac{KP}{R^3} = 4 \sin A \sin B \sin C(\sin A + \sin B + \sin C).$$

From the AM-GM inequality, we have

$$\sin A \sin B \sin C \leq \left(\frac{\sin A + \sin B + \sin C}{3}\right)^3,$$

giving

$$\frac{KP}{R^3} \leq \frac{4}{27}(\sin A + \sin B + \sin C)^4,$$

with equality when $\sin A = \sin B = \sin C$. Since the sine function is concave on the interval from 0 to π, Jensen's inequality gives

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\frac{A + B + C}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Since equality occurs here when $\sin A = \sin B = \sin C$ also, we can conclude that the maximum value of KP/R^3 is $\frac{4}{27}\left(\frac{3\sqrt{3}}{2}\right)^4 = 27/4$.

5. Let's say that an ordered triple of positive integers (a, b, c) is *n-powerful* if $a \le b \le c$, $\gcd(a, b, c) = 1$, and $a^n + b^n + c^n$ is divisible by $a + b + c$. For example, $(1, 2, 2)$ is 5-powerful.

   a) Determine all ordered triples (if any) which are n-powerful for all $n \ge 1$.

   b) Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

   [Note that $\gcd(a, b, c)$ is the greatest common divisor of a, b and c.]

Solution 1

Let $T_n = a^n + b^n + c^n$ and consider the polynomial

$$P(x) = (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc.$$

Since $P(a) = 0$, we get $a^3 = (a+b+c)a^2 - (ab+ac+bc)a + abc$ and multiplying both sides by a^{n-3} we obtain $a^n = (a+b+c)a^{n-1} - (ab+ac+bc)a^{n-2} + (abc)a^{n-3}$. Applying the same reasoning, we can obtain similar expressions for b^n and c^n and adding the three identities we get that T_n satisfies the following 3-term recurrence:

$$T_n = (a + b + c)T_{n-1} - (ab + ac + bc)T_{n-2} + (abc)T_{n-3}, \text{ for all } n \ge 3.$$

¿From this we see that if T_{n-2} and T_{n-3} are divisible by $a + b + c$, then so is T_n. This immediately resolves part (b)—there are no ordered triples which are 2004-powerful and 2005-powerful, but not 2007-powerful—and reduces the number of cases to be considered in part (a): since all triples are 1-powerful, the recurrence implies that any ordered triple which is both 2-powerful and 3-powerful is n-powerful for all $n \ge 1$.

Putting $n = 3$ in the recurrence, we have

$$a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2) - (ab + ac + bc)(a + b + c) + 3abc$$

which implies that (a, b, c) is 3-powerful if and only if $3abc$ is divisible by $a + b + c$. Since

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc),$$

(a, b, c) is 2-powerful if and only if $2(ab + ac + bc)$ is divisible by $a + b + c$.

Suppose a prime $p \ge 5$ divides $a + b + c$. Then p divides abc. Since $\gcd(a, b, c) = 1$, p divides exactly one of a, b or c; but then p doesn't divide $2(ab + ac + bc)$.

Suppose 3^2 divides $a + b + c$. Then 3 divides abc, implying 3 divides exactly one of a, b or c. But then 3 doesn't divide $2(ab + ac + bc)$.

Suppose 2^2 divides $a + b + c$. Then 4 divides abc. Since $\gcd(a, b, c) = 1$, at most one of a, b or c is even, implying one of a, b, c is divisible by 4 and the others are odd. But then $ab + ac + bc$ is odd and 4 doesn't divide $2(ab + ac + bc)$.

So if (a, b, c) is 2- and 3-powerful, then $a + b + c$ is not divisible by 4 or 9 or any prime greater than 3. Since $a + b + c$ is at least 3, $a + b + c$ is either 3 or 6. It is now a simple matter to check the possibilities and conclude that the only triples which are n-powerful for all $n \ge 1$ are $(1, 1, 1)$ and $(1, 1, 4)$.

Solution 2

Let p be a prime. By Fermat's Little Theorem,

$$a^{p-1} \equiv \begin{cases} 1 \pmod{p}, & \text{if } p \text{ doesn't divide } a; \\ 0 \pmod{p}, & \text{if } p \text{ divides } a. \end{cases}$$

Since $\gcd(a, b, c) = 1$, we have that $a^{p-1} + b^{p-1} + c^{p-1} \equiv 1, 2$ or $3 \pmod{p}$. Therefore if p is a prime divisor of $a^{p-1}+b^{p-1}+c^{p-1}$, then p equals 2 or 3. So if (a, b, c) is n-powerful for all $n \geq 1$, then the only primes which can divide $a + b + c$ are 2 or 3.

We can proceed in a similar fashion to show that $a + b + c$ is not divisible by 4 or 9.

Since

$$a^2 \equiv \begin{cases} 0 \pmod{4}, & \text{if } p \text{ is even;} \\ 1 \pmod{4}, & \text{if } p \text{ is odd} \end{cases}$$

and a, b, c aren't all even, we have that $a^2 + b^2 + c^2 \equiv 1, 2$ or $3 \pmod{4}$.

By expanding $(3k)^3$, $(3k + 1)^3$ and $(3k + 2)^3$, we find that a^3 is congruent to 0, 1 or -1 modulo 9. Hence

$$a^6 \equiv \begin{cases} 0 \pmod{9}, & \text{if 3 divides } a; \\ 1 \pmod{9}, & \text{if 3 doesn't divide } a. \end{cases}$$

Since a, b, c aren't all divisible by 3, we have that $a^6 + b^6 + c^6 \equiv 1, 2$ or $3 \pmod{9}$.

So $a^2 + b^2 + c^2$ is not divisible by 4 and $a^6 + b^6 + c^6$ is not divisible by 9. Thus if (a, b, c) is n-powerful for all $n \geq 1$, then $a + b + c$ is not divisible by 4 or 9. Therefore $a + b + c$ is either 3 or 6 and checking all possibilities, we conclude that the only triples which are n-powerful for all $n \geq 1$ are $(1, 1, 1)$ and $(1, 1, 4)$.

See Solution 1 for the (b) part.

38th Canadian Mathematical Olympiad

Wednesday, March 29,2006

CMS
SMC

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1. Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most 2 candies. For example, if $n = 3$, then $f(3,7) = 0$, $f(3,6) = 1$ and $f(3,4) = 6$.

   Determine the value of

$$f(2006,1) + f(2006,4) + f(2006,7) + \cdots + f(2006,1000) + f(2006,1003)\ .$$

2. Let ABC be an acute-angled triangle. Inscribe a rectangle $DEFG$ in this triangle so that D is on AB, E is on AC and both F and G are on BC. Describe the locus of (*i.e.*, the curve occupied by) the intersections of the diagonals of all possible rectangles $DEFG$.

3. In a rectangular array of nonnegative real numbers with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

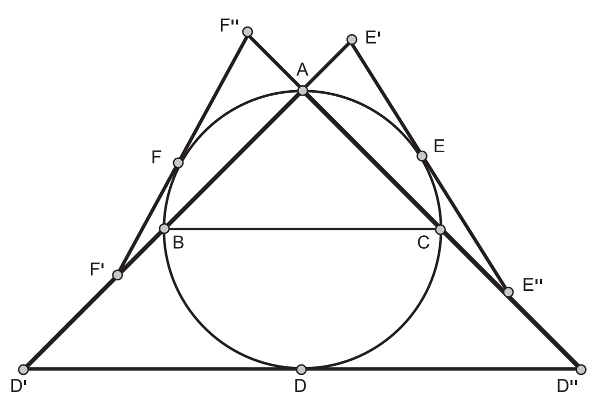
4. Consider a round-robin tournament with $2n+1$ teams, where each team plays each other team exactly once. We say that three teams X, Y and Z, form a *cycle triplet* if X beats Y, Y beats Z, and Z beats X. There are no ties.

   (a) Determine the minimum number of cycle triplets possible.

   (b) Determine the maximum number of cycle triplets possible.

5. The vertices of a right triangle ABC inscribed in a circle divide the circumference into three arcs. The right angle is at A, so that the opposite arc BC is a semicircle while arc AB and arc AC are supplementary. To each of the three arcs, we draw a tangent such that its point of tangency is the midpoint of that portion of the tangent intercepted by the extended lines AB and AC. More precisely, the point D on arc BC is the midpoint of the segment joining the points D' and D'' where the tangent at D intersects the extended lines AB and AC. Similarly for E on arc AC and F on arc AB.

   Prove that triangle DEF is equilateral.



38th Canadian Mathematical Olympiad

Wednesday, March 29, 2006

Solutions to the 2006 CMO paper

1. Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most 2 candies. For example, if $n=3$, then $f(3,7)=0$, $f(3,6)=1$ and $f(3,4)=6$.

Determine the value of

$$f(2006,1)+f(2006,4)+f(2006,7)+\cdots+f(2006,1000)+f(2006,1003)\ .$$

Comment. Unfortunately, there was an error in the statement of this problem. It was intended that the sum should continue to $f(2006,4012)$.

Solution 1. The number of ways of distributing k candies to 2006 children is equal to the number of ways of distributing 0 to a particular child and k to the rest, plus the number of ways of distributing 1 to the particular child and $k-1$ to the rest, plus the number of ways of distributing 2 to the particular child and $k-2$ to the rest. Thus $f(2006,k)=f(2005,k)+f(2005,k-1)+f(2005,k-2)$, so that the required sum is

$$1+\sum_{k=1}^{1003} f(2005,k)\ .$$

In evaluating $f(n,k)$, suppose that there are r children who receive 2 candies; these r children can be chosen in $\binom{n}{r}$ ways. Then there are $k-2r$ candies from which at most one is given to each of $n-r$ children. Hence

$$f(n,k)=\sum_{r=0}^{\lfloor k/2\rfloor}\binom{n}{r}\binom{n-r}{k-2r}=\sum_{r=0}^{\infty}\binom{n}{r}\binom{n-r}{k-2r}\ ,$$

with $\binom{x}{y}=0$ when $x<y$ and when $y<0$. The answer is

$$\sum_{k=0}^{1003}\sum_{r=0}^{\infty}\binom{2005}{r}\binom{2005-r}{k-2r}=\sum_{r=0}^{\infty}\binom{2005}{r}\sum_{k=0}^{1003}\binom{2005-r}{k-2r}\ .$$

Solution 2. The desired number is the sum of the coefficients of the terms of degree not exceeding 1003 in the expansion of $(1+x+x^2)^{2005}$, which is equal to the coefficient of x^{1003} in the expansion of

$$\begin{aligned}(1+x+x^2)^{2005}(1+x+\cdots+x^{1003}) &= [(1-x^3)^{2005}(1-x)^{-2005}](1-x^{1004})(1-x)^{-1}\\ &= (1-x^3)^{2005}(1-x)^{-2006}-(1-x^3)^{2005}(1-x)^{-2006}x^{1004}\ .\end{aligned}$$

Since the degree of every term in the expansion of the second member on the right exceeds 1003, we are looking for the coefficient of x^{1003} in the expansion of the first member:

$$(1-x^3)^{2005}(1-x)^{-2006} = \sum_{i=0}^{2005}(-1)^i\binom{2005}{i}x^{3i}\sum_{j=0}^{\infty}(-1)^j\binom{-2006}{j}x^j$$

$$= \sum_{i=0}^{2005} \sum_{j=0}^{\infty} (-1)^i \binom{2005}{i} \binom{2005+j}{j} x^{3i+j}$$
$$= \sum_{k=0}^{\infty} \left(\sum_{i=1}^{2005} (-1)^i \binom{2005}{i} \binom{2005+k-3i}{2005} \right) x^k \ .$$

The desired number is
$$\sum_{i=1}^{334} (-1)^i \binom{2005}{i} \binom{3008-3i}{2005} = \sum_{i=1}^{334} (-1)^i \frac{(3008-3i)!}{i!(2005-i)!(1003-3i)!} \ .$$
(Note that $\binom{3008-3i}{2005} = 0$ when $i \geq 335$.)

2. Let ABC be an acute-angled triangle. Inscribe a rectangle $DEFG$ in this triangle so that D is on AB, E is on AC and both F and G are on BC. Describe the locus of (*i.e.*, the curve occupied by) the intersections of the diagonals of all possible rectangles $DEFG$.

Solution. The locus is the line segment joining the midpoint M of BC to the midpoint K of the altitude AH. Note that a segment DE with D on AB and E on AC determines an inscribed rectangle; the midpoint F of DE lies on the median AM, while the midpoint of the perpendicular from F to BC is the centre of the rectangle. This lies on the median MK of the triangle AMH.

Conversely, any point P on MK is the centre of a rectangle with base along BC whose height is double the distance from K to BC.

3. In a rectangular array of nonnegative real numbers with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

Solution 1. Consider first the case where all the rows have the same positive sum s; this covers the particular situation in which $m = 1$. Then each column, sharing a positive element with some row, must also have the sum s. Then the sum of all the entries in the matrix is $ms = ns$, whence $m = n$.

We prove the general case by induction on m. The case $m = 1$ is already covered. Suppose that we have an $m \times n$ array not all of whose rows have the same sum. Let $r < m$ of the rows have the sum s, and each of the of the other rows have a different sum. Then every column sharing a positive entry with one of these rows must also have sum s, and these are the only columns with the sum s. Suppose there are c columns with sum s. The situation is essentially unchanged if we permute the rows and then the column so that the first r rows have the sum s and the first c columns have the sum s. Since all the entries of the first r rows not in the first c columns and in the first c columns not in the first r rows must be 0, we can partition the array into a $r \times c$ array in which all rows and columns have sum s and which satisfies the hypothesis of the problem, two rectangular arrays of zeros in the upper right and lower left and a rectangular $(m-r) \times (n-c)$ array in the lower right that satisfies the conditions of the problem. By the induction hypothesis, we see that $r = c$ and so $m = n$.

Solution 2. [Y. Zhao] Let the term in the ith row and the jth column of the array be denoted by a_{ij}, and let $S = \{(i,j) : a_{ij} > 0\}$. Suppose that r_i is the sum of the ith row and c_j the sum of the jth column. Then $r_i = c_j$ whenever $(i,j) \in S$. Then we have that
$$\sum \{ \frac{a_{ij}}{r_i} : (i,j) \in S \} = \sum \{ \frac{a_{ij}}{c_j} : (i,j) \in S \} \ .$$

We evaluate the sums on either side independently.

$$\sum \{ \frac{a_{ij}}{r_i} : (i,j) \in S \} = \sum \{ \frac{a_{ij}}{r_i} : 1 \leq i \leq m, 1 \leq j \leq n \} = \sum_{i=1}^{m} \frac{1}{r_i} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m} \left(\frac{1}{r_i} \right) r_i = \sum_{i=1}^{m} 1 = m \ .$$

$$\sum \{ \frac{a_{ij}}{c_j} : (i,j) \in S \} = \sum \{ \frac{a_{ij}}{c_j} : 1 \leq i \leq m, 1 \leq j \leq n \} = \sum_{j=1}^{n} \frac{1}{c_j} \sum_{i=1}^{m} a_{ij} = \sum_{j=1}^{n} \left(\frac{1}{c_j} \right) c_j = \sum_{j=1}^{n} 1 = n \ .$$

Hence $m = n$.

Comment. The second solution can be made cleaner and more elegant by defining $u_{ij} = a_{ij}/r_i$ for all (i, j). When $a_{ij} = 0$, then $u_{ij} = 0$. When $a_{ij} > 0$, then, by hypothesis, $u_{ij} = a_{ij}/c_j$, a relation that in fact holds for all (i, j). We find that

$$\sum_{j=1}^{n} u_{ij} = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} u_{ij} = 1$$

for $1 \le i \le m$ and $1 \le j \le n$, so that (u_{ij}) is an $m \times n$ array whose row sums and column sums are all equal to 1. Hence

$$m = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} u_{ij} \right) = \sum \{u_{ij} : 1 \le i \le m, 1 \le j \le n\} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} u_{ij} \right) = n$$

(being the sum of all the entries in the array).

4. Consider a round-robin tournament with $2n + 1$ teams, where each team plays each other team exactly once. We say that three teams X, Y and Z, form a *cycle triplet* if X beats Y, Y beats Z, and Z beats X. There are no ties.

(a) Determine the minimum number of cycle triplets possible.

(b) Determine the maximum number of cycle triplets possible.

Solution 1. (a) The minimum is 0, which is achieved by a tournament in which team T_i beats T_j if and only if $i > j$.

(b) Any set of three teams constitutes either a cycle triplet or a "dominated triplet" in which one team beats the other two; let there be c of the former and d of the latter. Then $c + d = \binom{2n+1}{3}$. Suppose that team T_i beats x_i other teams; then it is the winning team in exactly $\binom{x_i}{2}$ dominated triples. Observe that $\sum_{i=1}^{2n+1} x_i = \binom{2n+1}{2}$, the total number of games. Hence

$$d = \sum_{i=1}^{2n+1} \binom{x_i}{2} = \frac{1}{2} \sum_{i=1}^{2n+1} x_i^2 - \frac{1}{2} \binom{2n+1}{2} .$$

By the Cauchy-Schwarz Inequality, $(2n + 1) \sum_{i=1}^{2n+1} x_i^2 \ge (\sum_{i=1}^{2n+1} x_i)^2 = n^2(2n + 1)^2$, whence

$$c = \binom{2n+1}{3} - \sum_{i=1}^{2n+1} \binom{x_i}{2} \le \binom{2n+1}{3} - \frac{n^2(2n+1)}{2} + \frac{1}{2} \binom{2n+1}{2} = \frac{n(n+1)(2n+1)}{6} .$$

To realize the upper bound, let the teams be $T_1 = T_{2n+2}$, $T_2 = T_{2n+3}$. $\cdots$, $T_i = T_{2n+1+i}$, $\cdots$, $T_{2n+1} = T_{4n+2}$. For each i, let team T_i beat $T_{i+1}, T_{i+2}, \cdots, T_{i+n}$ and lose to $T_{i+n+1}, \cdots, T_{i+2n}$. We need to check that this is a consistent assignment of wins and losses, since the result for each pair of teams is defined twice. This can be seen by noting that $(2n + 1 + i) - (i + j) = 2n + 1 - j \ge n + 1$ for $1 \le j \le n$. The cycle triplets are $(T_i, T_{i+j}, T_{i+j+k})$ where $1 \le j \le n$ and $(2n+1+i)-(i+j+k) \le n$, *i.e.*, when $1 \le j \le n$ and $n+1-j \le k \le n$. For each i, this counts $1+2+\cdots+n = \frac{1}{2}n(n+1)$ cycle triplets. When we range over all i, each cycle triplet gets counted three times, so the number of cycle triplets is

$$\frac{2n+1}{3} \left(\frac{n(n+1)}{2} \right) = \frac{n(n+1)(2n+1)}{6} .$$

Solution 2. [S. Eastwood] (b) Let t be the number of cycle triplets and u be the number of ordered triplets of teams (X, Y, Z) where X beats Y and Y beats Z. Each cycle triplet generates three ordered triplets while other triplets generate exactly one. The total number of triplets is

$$\binom{2n+1}{3} = \frac{n(4n^2-1)}{3} .$$

The number of triples that are not cycle is

$$\frac{n(4n^2-1)}{3} - t .$$

Hence

$$u = 3t + \left(\frac{n(4n^2-1)}{3} - t \right) \Longrightarrow$$

$$t = \frac{3u - n(4n^2 - 1)}{6} = \frac{u - (2n+1)n^2}{2} + \frac{n(n+1)(2n+1)}{6} \ .$$

If team Y beats a teams and loses to b teams, then the number of ordered triples with Y as the central element is ab. Since $a + b = 2n$, by the Arithmetic-Geometric Means Inequality, we have that $ab \leq n2$. Hence $u \leq (2n+1)n2$, so that

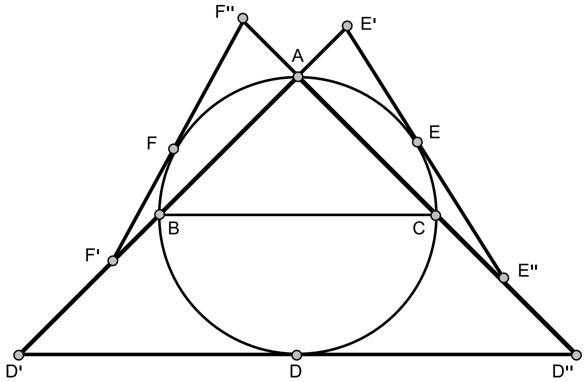
$$t \leq \frac{n(n+1)(2n+1)}{6} \ .$$

The maximum is attainable when $u = (2n+1)n2$, which can occur when we arrange all the teams in a circle with each team beating exactly the n teams in the clockwise direction.

Comment. Interestingly enough, the maximum is $\sum_{i=1}^{n} i^2$; is there a nice argument that gives the answer in this form?

5. The vertices of a right triangle ABC inscribed in a circle divide the circumference into three arcs. The right angle is at A, so that the opposite arc BC is a semicircle while arc AB and arc AC are supplementary. To each of the three arcs, we draw a tangent such that its point of tangency is the midpoint of that portion of the tangent intercepted by the extended lines AB and AC. More precisely, the point D on arc BC is the midpoint of the segment joining the points D' and D'' where the tangent at D intersects the extended lines AB and AC. Similarly for E on arc AC and F on arc AB.

Prove that triangle DEF is equilateral.



Solution 1. A prime indicates where a tangent meets AB and a double prime where it meets AC. It is given that $DD' = DD''$, $EE' = EE''$ and $FF' = FF''$. It is required to show that arc EF is a third of the circumference as is arc DBF.

AF is the median to the hypotenuse of right triangle $AF'F''$, so that $FF' = FA$ and therefore

$$\text{arc } AF = 2\angle F''FA = 2(\angle FF'A + \angle FAF') = 4\angle FAF' = 4\angle FAB = 2 \text{ arc } BF \ ,$$

whence arc $FA = (2/3)$ arc BFA. Similarly, arc $AE = (2/3)$ arc AEC. Therefore, arc FE is $2/3$ of the semicircle, or $1/3$ of the circumference as desired.

As for arc DBF, arc $BD = 2\angle BAD = \angle BAD + \angle BD'D = \angle ADD'' = (1/2)$ arc ACD. But, arc $BF = (1/2)$ arc AF, so arc $DBF = (1/2)$ arc $FAED$. That is, arc DBF is $1/3$ the circumference and the proof is complete.

Solution 2. Since $AE'E''$ is a right triangle, $AE = EE' = EE''$ so that $\angle CAE = \angle CE''E$. Also $AD = D'D = DD''$, so that $\angle CDD'' = \angle CAD = \angle CD''D$. As $EADC$ is a concyclic quadrilateral,

$$\begin{aligned}
180^\circ &= \angle EAD + \angle ECD \\
&= \angle DAC + \angle CAE + \angle ECA + \angle ACD \\
&= \angle DAC + \angle CAE + \angle CEE'' + \angle CE''E + \angle CDD'' + \angle CD''D \\
&= \angle DAC + \angle CAE + \angle CAE + \angle CAE + \angle CAD + \angle CAD \\
&= 3(\angle DAC + \angle DAE) = 3(\angle DAE)
\end{aligned}$$

Hence $\angle DFE = \angle DAE = 60^\circ$. Similarly, $\angle DEF = 60^\circ$. It follows that triangle DEF is equilateral.

39th Canadian Mathematical Olympiad

Wednesday, March 28,2007

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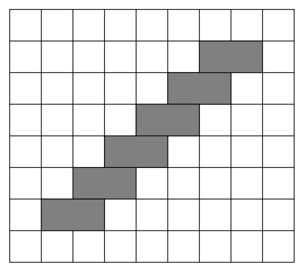
1. What is the maximum number of non-overlapping 2×1 dominoes that can be placed on a 8×9 checkerboard if six of them are placed as shown? Each domino must be placed horizontally or vertically so as to cover two adjacent squares of the board.



2. You are given a pair of triangles for which

   (a) two sides of one triangle are equal in length to two sides of the second triangle, and

   (b) the triangles are similar, but not necessarily congruent.

   Prove that the ratio of the sides that correspond under the similarity is a number between $\frac{1}{2}(\sqrt{5}-1)$ and $\frac{1}{2}(\sqrt{5}-1)$.

3. Suppose that f is a real-valued function for which

$$f(xy) + f(y-x) \geq f(y+x)$$

   for all real numbers x and y.

   (a) Give a nonconstant polynomial that satisfies the condition.

   (b) Prove that $f(x) \geq 0$ for all real x.

4. For two real numbers a, b, with $ab \neq 1$, define the $*$ operation by

$$a * b = \frac{a + b - 2ab}{1 - ab}.$$

   Start with a list of $n \geq 2$ real numbers whose entries x all satisfy $0 < x < 1$. Select any two numbers a and b in the list; remove them and put the number $a * b$ at the end of the list, thereby reducing its length by one. Repeat this procedure until a single number remains.

   (a) Prove that this single number is the same regardless of the choice of pair at each stage.

   (b) Suppose that the condition on the numbers x in S is weakened to $0 < x \leq 1$. What happens if S contains exactly one 1?

5. Let the incircle of triangle ABC touch sides BC, CA and AB at D, E and F, respectively. Let Γ,Γ_1,Γ_2 and Γ_3 denote the circumcircles of triangle ABC, AEF, BDF and CDE respectively. Let Γ and Γ_1 intersect at A and P, Γ and Γ_2 intersect at B and Q, and Γ and Γ_3 intersect at C and R.

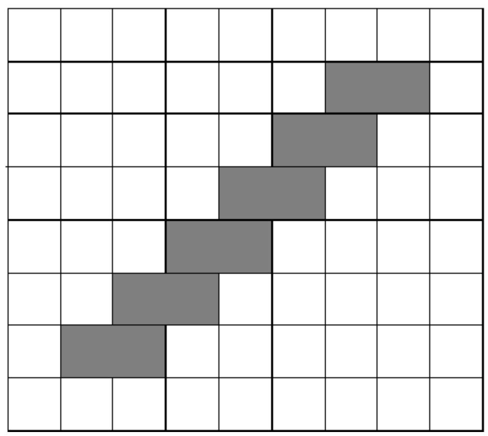
   (a) Prove that the circles Γ_1, Γ_2 and Γ_3 intersect in a common point.

   (b) Show that PD, QE and RF are concurrent.

39th Canadian Mathematical Olympiad

Wednesday, March 28, 2007

Solutions to the 2007 CMO paper



Solution to 1. Identify five subsets A, B, C, D, E of the board, where C consists of the squares occupied by the six dominos already placed, B is the upper right corner, D is the lower left corner, A consists of the squares above and to the left of those in $B \cup C \cup D$ and E consists of the squares below and to the right of those in $B \cup C \cup D$. The board can be coloured checkerboard fashion so that A has 13 black and 16 white squares, B a single white square, E 16 black and 13 white squares and D a single black square. Each domino beyond the original six must lie either entirely in $A \cup B \cup D$ or $C \cup B \cup D$, either of which contains at most 14 dominos. Thus, altogether, we cannot have more that $2 \times 14 + 6 = 34$ dominos. This is achievable, by placing 14 dominos in $A \cup D$ and 14 in $E \cup B$.

Solution to 2. If the triangles are isosceles, then they must be congruent and the desired ratio is 1. For, if they share equal side lengths, at least one of these side lengths on one triangle corresponds to the same length on the other. And if they share unequal side lengths, then either equal sides correspond or unequal sides correspond in both directions and the ratio is 1. This falls within the bounds.

Let the triangles be scalene. It is not possible for the same length to be an extreme length (largest or smallest) of both triangles. Therefore, we must have a situation in which the corresponding side lengths of the two triangles are (x, y, z) and (y, z, u) with $x < y < z$ and $y < z < u$. We are given that $y/x = z/y = u/z = r > 1$. Thus, $y = rx$ and $z = ry = r^2x$. From the triangle inequality $z < x + y$, we have that $r^2 < 1 + r$. Since $r^2 - r - 1 < 0$ and $r > 1$, $1 < r < \frac{1}{2}(\sqrt{5} + 1)$. The ratio of the dimensions from the smaller to the larger triangle is $1/r$ which satisfies $\frac{1}{2}(\sqrt{5} - 1) < 1/r < 1$. The result follows.

Solution to 3. (a) Let $f(x) = x^2 + 4$. Then

$$\begin{aligned} f(xy) + f(y - x) - f(y + x) &= (x^2y^2 + 4) + (y - x)^2 + 4 - (y + x)^2 - 4 \\ &= (xy)^2 - 4xy + 4 = (xy - 2)^2 \geq 0 \; . \end{aligned} \tag{1}$$

Thus, $f(x) = x^2 + 4$ satisfies the condition.

(b) Consider (x, y) for which $xy = x + y$. Rewriting this as $(x - 1)(y - 1) = 1$, we find that this has the general solution $(x, y) = (1 + t^{-1}, 1 + t)$, for $t \neq 0$. Plugging this into the inequality, we get that $f(t - t^{-1}) \geq 0$ for all $t \neq 0$. For arbitrary real

u, the equation $t - t^{-1} = u$ leads to the quadratic $t^2 - ut - 1 = 0$ which has a positive discriminant and so a real solution. Hence $f(u) \geq 0$ for each real u.

Comment. The substitution $v = y - x$, $u = y + x$ whose inverse is $x = \frac{1}{2}(u - v)$, $y = \frac{1}{2}(u + v)$ renders the condition as $f(\frac{1}{4}(u^2 - v^2)) + f(v) \geq f(u)$. The same strategy as in the foregoing solution leads to the choice $u = 2 + \sqrt{v^2 + 4}$ and $f(v) \geq 0$ for all v.

Solution to 4 (b). It is straightforward to verify that $a * 1 = 1$ for $a \neq 1$, so that once 1 is included in the list, it can never by removed and so the list terminates with the single value 1.

Solution to 4 (a). There are several ways of approaching (a). It is important to verify that the set $\{x : 0 < x < 1\}$ is closed under the operation $*$ so that it is always defined.

If $0 < a, b < 1$, then

$$0 < \frac{a + b - 2ab}{1 - ab} < 1 \ .$$

The left inequality follows from

$$a + b - 2ab = a(1 - b) + b(1 - a) > 0$$

and the right from

$$1 - \frac{a + b - 2ab}{1 - ab} = \frac{(1 - a)(1 - b)}{1 - ab} > 0 \ .$$

Hence, it will never happen that a set of numbers will contain a pair of reciprocals, and the operation can always be performed.

Solution 1. It can be shown by induction that any two numbers in any of the sets arise from disjoint subsets of S.

Use an induction argument on the number of entries that one starts with. At each stage the number of entries is reduced by one. If we start with n numbers, the final result is

$$\frac{\sigma_1 - 2\sigma_2 + 3\sigma_3 - \cdots + (-1)^{n-1} n\sigma_n}{1 - \sigma_2 + 2\sigma_3 - 3\sigma_4 + \cdots + (-1)^{n-1}(n - 1)\sigma_n} \ ,$$

where σ_i is the symmetric sum of all $\binom{n}{i}$ $i-$fold products of the n elements x_i in the list.

Solution 2. Define

$$a * b = \frac{a + b - 2ab}{1 - ab} \ .$$

This operation is commutative and also associative:

$$a * (b * c) = (a * b) * c = \frac{a + b + c - 2(ab + bc + ca) + 3abc}{1 - (ab + bc + ca) + 2abc} \ .$$

Since the final result amounts to a $*-$product of elements of S with some arrangement of brackets, the result follows.

Solution 3. Let $\phi(x) = x/(1 - x)$ for $0 < x < 1$. This is a one-one function from the open interval $(0, 1)$ to the half line $(0, \infty)$. For any numbers $a, b \in S$, we have that

$$\begin{aligned} \phi\left(\frac{a + b - 2ab}{1 - ab}\right) &= \frac{a+b-2ab}{(1-ab)-(a+b-2ab)} = \frac{a+b-2ab}{1-a-b+ab} \\ &= \frac{a}{1-a} + \frac{b}{1-b} = \phi(a) + \phi(b) \ . \end{aligned} \tag{2}$$

Let $T = \{\phi(s) : s \in S\}$. Then replacing a, b in S as indicated corresponds to replacing $\phi(a)$ and $\phi(b)$ in T by $\phi(a) + \phi(b)$ to get a new pair of sets related by ϕ. The final result is the inverse under ϕ of $\sum\{\phi(s) : s \in S\}$.

Solution 4. Let $f(x) = (1 - x)^{-1}$ be defined for positive x unequal to 1. Then $f(x) > 1$ if and only if $0 < x < 1$. Observe that

$$f(x * y) = \frac{1 - xy}{1 - x - y + xy} = \frac{1}{1 - x} + \frac{1}{1 - y} - 1 \ .$$

If $f(x) > 1$ and $f(y) > 1$, then also $f(x * y) > 1$. It follows that if x and y lie in the open interval $(0, 1)$, so does $x * y$. We also note that $f(x)$ is a one-one function.

To each list L, we associate the function $g(L)$ defined by

$$g(L) = \sum \{f(x) : x \in L\} \ .$$

Let L_n be the given list, and let the subsequent lists be $L_{n-1}, L_{n-2}, \cdots, L_1$, where L_i has i elements. Since $f(x * y) = f(x) + f(y) - 1$, $g(L_i) = g(L_n) - (n - i)$ regardless of the choice that creates each list from its predecessors. Hence $g(L_1) = g(L_n) - (n-1)$ is fixed. However, $g(L_1) = f(a)$ for some number a with $0 < a < 1$. Hence $a = f^{-1}(g(L_n) - (n-1))$ is fixed.

Solution to 5 (a). Let I be the incentre of triangle ABC. Since the quadrilateral $AEIF$ has right angles at E and F, it is concyclic, so that Γ_1 passes through I. Similarly, Γ_2 and Γ_3 pass through I, and (a) follows.

Solution to 5 (b). Let ω and I denote the incircle and incentre of triangle ABC, respectively. Observe that, since AI bisects the angle FAE and $AF = AE$, then AI right bisects the segment FE. Similarly, BI right bisects DF and CI right bisects DE.

We invert the diagram through ω. Under this inversion, let the image of A be A', *etc.* Note that the centre I of inversion is collinear with any point and its image under the inversion. Under this inversion, the image of Γ_1 is EF, which makes A' the midpoint of EF. Similarly, B' is the midpoint of DF and C' is the midpoint of DE. Hence, Γ', the image of Γ under this inversion, is the circumcircle of triangle $A'B'C'$, which implies that Γ' is the nine-point circle of triangle DEF.

Since P is the intersection of Γ and Γ_1 other than A, P' is the intersection of Γ' and EF other than A', which means that P' is the foot of the altitude from D to EF. Similarly, Q' is the foot of the altitude from E to DF and R' is the foot of the altitude from F to DE.

Now, let X, Y and Z be the midpoints of arcs BC, AC and AB on Γ respectively. We claim that X lies on PD.

Let X' be the image of X under the inversion, so I, X and X' are collinear. But X is the midpoint of arc BC, so A, A', I, X' and X are collinear. The image of line PD is the circumcircle of triangle $P'ID$, so to prove that X lies on PD, it suffices to prove that points P', I, X' and D are concyclic.

We know that B' is the midpoint of DF, C' is the midpoint of DE and P' is the foot of the altitude from D to EF. Hence, D is the reflection of P' in $B'C'$.

Since $IA' \perp EF$, $IB' \perp DF$ and $IC' \perp DE$, I is the orthocentre of triangle $A'B'C'$. So, X' is the intersection of the altitude from A' to $B'C'$ with the circumcircle of triangle $A'B'C'$. From a wellknown fact, X' is the reflection of I in $B'C'$. This means that $B'C'$ is the perpendicular bisector of both $P'D$ and IX', so that the points P', I, X' and D are concyclic.

Hence, X lies on PD. Similarly, Y lies on QE and Z lies on RF. Thus, to prove that PD, QE and RF are concurrent, it suffices to prove that DX, EY and FZ are concurrent.

To show this, consider tangents to Γ at X, Y and Z. These are parallel to BC, AC and AB, respectively. Hence, the triangle Δ that these tangents define is homothetic to the triangle ABC. Let S be the centre of homothety. Then the homothety taking triangle ABC to Δ takes ω to Γ, and so takes D to X, E to Y and F to Z. Hence DX, EY and FZ concur at S.

Comment. The solution uses the following result: *Suppose ABC is a triangle with orthocentre H and that AH intersects BC at P and the circumcircle of ABC at D. Then $HP = PD$.* The proof is straightforward: Let BH meet AC at Q. Note that $AD \perp BC$ and $BQ \perp AC$. Since $\angle ACB = \angle ADB$,

$$\angle HBC = \angle QBC = 90^\circ - \angle QCB = 90^\circ - \angle ACB = 90^\circ - \angle ADB = \angle DBP \ ,$$

from which follows the congruence of triangle HBP and DBP and equality of HP and PD.

Solution 2. (a) Let Γ_2 and Γ_3 intersect at J. Then $BDJF$ and $CDJE$ are concyclic. We have that

$$\begin{aligned} \angle FJE &= 360^\circ - (\angle DJF + \angle DJE) \\ &= 360^\circ - (180^\circ - \angle ABC + 180^\circ - \angle ACB) \\ &= \angle ABC + \angle ACB = 180^\circ - \angle FAE \ . \end{aligned} \tag{3}$$

Hence $AFJE$ is concyclic and so the circumcircles of AEF, BDF and CED pass through J.

(b) [Y. Li] Join RE, RD, RA and RB. In Γ_3, $\angle ERD = \angle ECD = \angle ACB$ and $\angle REC = \angle RDC$. In Γ, $\angle ARB = \angle ACB$. Hence, $\angle ERD = \angle ARB \Longrightarrow \angle ARE = \angle BRD$. Also,

$$\angle AER = 180^\circ - \angle REC = 180^\circ - \angle RDC = \angle BDR \ .$$

Therefore, triangle ARE and BRD are similar, and $AR : BR = AE : BD = AF : BF$. If follows that RF bisects angle ARB, so that RF passes through the midpoint of minor arc AB on Γ. Similarly, PD and QE are respective bisectors of angles BPC and CQA and pass through the midpoints of the minor arc BC and CA on γ..

Let O be the centre of circle Γ, and U, V, W be the respective midpoints of the minor arc BC, CA, AB on this circle, so that PU contains D, QV contains E and RW contains F. It is required to prove that DU, EV and FW are concurrent.

Since ID and OU are perpendicular to BC, $ID \| OU$. Similarly, $IE \| OV$ and $IF \| OW$. Since $|ID| = |IE| = |IF| = r$ (the inradius) and $|OU| = |OV| = |OW| = R$ (the circumradius), a translation $\overrightarrow{IO}$ followed by a dilatation of factor R/r takes triangle DEF to triangle UVW, so that these triangles are similar with corresponding sides parallel.

Suppose that EV and FW intersect at K and that DU and FW intersect at L. Taking account of the similarity of the triangles KEF and KVW, LDF and LUW, DEF and UVW, we have that

$$KF : FW = EF : VW = DF : UW = LF : LW \ ,$$

so that $K = L$ and the lines DU, EV and FW intersect in a common point K, as desired.

40th Canadian Mathematical Olympiad

Wednesday, March 26, 2008

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1. $ABCD$ is a convex quadrilateral for which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

2. Determine all functions f defined on the set of rational numbers that take rational values for which

$$f(2f(x) + f(y)) = 2x + y \ ,$$

for each x and y.

3. Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a - bc}{a + bc} + \frac{b - ca}{b + ca} + \frac{c - ab}{c + ab} \leq \frac{3}{2} \ .$$

4. Determine all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

5. A *self-avoiding rook walk* on a chessboard (a rectangular grid of unit squares) is a path traced by a sequence of moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, *i.e.*, the rook's path is non-self-intersecting.

   Let $R(m, n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, $R(m, 1) = 1$ for all natural numbers m; $R(2, 2) = 2$; $R(3, 2) = 4$; $R(3, 3) = 11$. Find a formula for $R(3, n)$ for each natural number n.

40th Canadian Mathematical Olympiad

Wednesday, March 26, 2008

Solutions - CMO 2008

1. $ABCD$ is a convex quadrilateral in which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

Solution. Since $[MADC] = \frac{1}{2}[ABCD] = [NADC]$, it follows that $[ANC] = [AMC]$, so that $MN \| AC$. Let m be a line through D parallel to AC and MN and let BA produced meet m at P and BC produced meet m at Q. Then

$$[MPC] = [MAC] + [CAP] = [MAC] + [CAD] = [MADC] = [BMC]$$

whence $BM = MP$. Similarly $BN = NQ$, so that MN is a midline of triangle BPQ and must bisect BD.

2. Determine all functions f defined on the set of rationals that take rational values for which

$$f(2f(x) + f(y)) = 2x + y$$

for each x and y.

Solution 1. The only solutions are $f(x) = x$ for all rational x and $f(x) = -x$ for all rational x. Both of these readily check out.

Setting $y = x$ yields $f(3f(x)) = 3x$ for all rational x. Now replacing x by $3f(x)$, we find that

$$f(9x) = f(3f(3f(x)) = 3[3f(x)] = 9f(x) \ ,$$

for all rational x. Setting $x = 0$ yields $f(0) = 9f(0)$, whence $f(0) = 0$.

Setting $x = 0$ in the given functional equation yields $f(f(y)) = y$ for all rational y. Thus f is one-one onto. Applying f to the functional equation yields that

$$2f(x) + f(y) = f(2x + y)$$

for every rational pair (x, y).

Setting $y = 0$ in the functional equation yields $f(2f(x)) = 2x$, whence $2f(x) = f(2x)$. Therefore $f(2x) + f(y) = f(2x + y)$ for each rational pair (x, y), so that

$$f(u + v) = f(u) + f(v)$$

for each rational pair (u, v).

Since $0 = f(0) = f(-1) + f(1)$, $f(-1) = -f(1)$. By induction, it can be established that for each intger n and rational x, $f(nx) = nf(x)$. If $k = f(1)$, we can establish from this that $f(n) = nk$, $f(1/n) = k/n$ and $f(m/n) = mk/n$ for each integer pair (m, n). Thus $f(x) = kx$ for all rational x. Since $f(f(x)) = x$, we must have $k2 = 1$. Hence $f(x) = x$ or $f(x) = -x$. These check out.

Solution 2. In the functional equation, let

$$x = y = 2f(z) + f(w)$$

to obtain $f(x) = f(y) = 2z + w$ and

$$f(6z + 3w) = 6f(z) + 3f(w)$$

for all rational pairs (z, w). Set $(z, w) = (0, 0)$ to obtain $f(0) = 0$, $w = 0$ to obtain $f(6z) = 6f(z)$ and $z = 0$ to obtain $f(3w) = 3f(w)$ for all rationals z and w. Hence $f(6z + 3w) = f(6z) + f(3w)$. Replacing $(6z, 3w)$ by (u, v) yields

$$f(u + v) = f(u) + f(v)$$

for all rational pairs (u, v). Hence $f(x) = kx$ where $k = f(1)$ for all rational x. Substitution of this into the functional equation with $(x, y) = (1, 1)$ leads to $3 = f(3f(1)) = f(3k) = 3k2$, so that $k = \pm 1$. It can be checked that both $f(x) \equiv 1$ and $f(x) \equiv -1$ satisfy the equation.

Acknowledgment. The first solution is due to Man-Duen Choi and the second to Ed Doolittle.

3. Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a - bc}{a + bc} + \frac{b - ca}{b + ca} + \frac{c - ab}{c + ab} \leq \frac{3}{2} \ .$$

Solution 1. Note that

$$1 - \frac{a - bc}{a + bc} = \frac{2bc}{1 - b - c + bc} = \frac{2bc}{(1 - b)(1 - c)} \ .$$

The inequality is equivalent to

$$\frac{2bc}{(1 - b)(1 - c)} + \frac{2ca}{(1 - c)(1 - a)} + \frac{2ab}{(1 - a)(1 - b)} \geq \frac{3}{2} \ .$$

Manipulation yields the equivalent

$$4(bc + ca + ab - 3abc) \geq 3(bc + ca + ab + 1 - a - b - c - abc) \ .$$

This simplifies to $ab + bc + ca \geq 9abc$ or

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9 \ .$$

This is a consequence of the harmonic-arithmetic means inequality.

Solution 2. Observe that

$$a + bc = a(a + b + c) + bc = (a + b)(a + c)$$

and that $a + b = 1 - c$, with analogous relations for other permutations of the variables. Then

$$(b + c)(c + a)(a + b) = (1 - a)(1 - b)(1 - c) = (ab + bc + ca) - abc \ .$$

Putting the left side of the desired inequality over a common denominator, we find that it is equal to

$$\frac{(a-bc)(1-a)+(b-ac)(1-b)+(c-ab)(1-c)}{(b+c)(c+a)(a+b)} = \frac{(a+b+c)-(a2+b2+c2)-(bc+ca+ab)+3abc}{(b+c)(c+a)(a+b)}$$
$$= \frac{1-(a+b+c)2+(bc+ca+ab)+3abc}{(ab+bc+ca)-abc}$$
$$= \frac{(bc+ca+ab)+3abc}{(bc+bc+ab)-abc}$$
$$= 1+\frac{4abc}{(a+b)(b+c)(c+a)} \ .$$

Using the arithmetic-geometric means inequality, we obtain that

$$(a+b)(b+c)(c+a) = (a^2b+b^2c+c^2a)+(ab2+bc2+ca2)+2abc$$
$$\geq 3abc+3abc+2abc = 8abc \ ,$$

whence $4abc/[(a+b)(b+c)(c+a)] \leq \frac{1}{2}$. The desired result follows. Equality occurs exactly when $a=b=c=\frac{1}{3}$.

4. Find all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \quad \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

Solution. The substitution $n=p$, a prime, yields $p \equiv (f(p))^p \equiv 0 \pmod{f(p)}$, so that p is divisible by $f(p)$. Hence, for each prime p, $f(p)=1$ or $f(p)=p$.

Let $S=\{p : p \text{ is prime and } f(p)=p\}$. If S is infinite, then $f(n)^p \equiv n \pmod p$ for infinitely many primes p. By the little Fermat theorem, $n \equiv f(n)^p \equiv f(n)$, so that $f(n)-n$ is a multiple of p for infinitely many primes p. This can happen only if $f(n)=n$ for all values of n, and it can be verified that this is a solution.

If S is empty, then $f(p)=1$ for all primes p, and any function satisfying this condition is a solution.

Now suppose that S is finite and non-empty. Let q be the largest prime in S. Suppose, if possible, that $q \geq 3$. Therefore, for any prime p exceeding q, $p \equiv 1 \pmod q$. However, this is not true. Let Q be the product of all the odd primes up to q. Then $Q+2$ must have a prime factor exceeding q and at least one of them must be incongruent to 1 (mod q). (An alternative argument notes that Bertrand's postulate can turn up a prime p between q and $2q$ which fails to satisfy $p \equiv 1 \mod q$.)

The only remaining case is that $S=\{2\}$. Then $f(2)=2$ and $f(p)=1$ for every odd prime p. Since $f(n)2 \equiv n \pmod 2$, $f(n)$ and n must have the same parity. Conversely, any function f for which $f(n) \equiv n \pmod 2$ for all n, $f(2)=2$ and $f(p)=1$ for all odd primes p satisfies the condition.

Therefore the only solutions are
- $f(n)=n$ for all $n \in \mathbf{N}$;
- any function f with $f(p)=1$ for all primes p;
- any function for which $f(2)=2$, $f(p)=1$ for primes p exceeding 2 and $f(n)$ and n have the same parity.

5. A *self-avoiding rook walk* on a chessboard (a rectangular grid of squares) is a path traced by a sequence of rook moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, *i.e.*, the rook's path is non-self-intersecting.

Let $R(m,n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, $R(m,1)=1$ for all natural numbers m; $R(2,2) = 2$; $R(3,2) = 4$; $R(3,3) = 11$. Find a formula for $R(3,n)$ for each natural number n.

Solution 1. Let $r_n = R(3,n)$. It can be checked directly that $r_1 = 1$ and $r_2 = 4$. Let $1 \le i \le 3$ and $1 \le j$; let (i,j) denote the cell in the ith row from the bottom and the jth column from the left, so that the paths in question go from $(1,1)$ to $(3,1)$.

Suppose that $n \ge 3$. The rook walks fall into exactly one of the following six categories:

(1) One walk given by $(1,1) \to (2,1) \to (3,1)$.

(2) Walks that avoid the cell $(2,1)$: Any such walk must start with $(1,1) \to (1,2)$ and finish with $(3,2) \to (3,1)$; there are r_{n-1} such walks.

(3) Walks that begin with $(1,1) \to (2,1) \to (2,2)$ and never return to the first row: Such walks enter the third row from $(2,k)$ for some k with $2 \le k \le n$ and then go along the third row leftwards to $(3,1)$; there are $n-1$ such walks.

(4) Walks that begin with $(1,1) \to (2,1) \to \cdots \to (2,k) \to (1,k) \to (1,k+1)$ and end with $(3,k+1) \to (3,k) \to (3,k-1) \to \cdots \to (3,2) \to (3,1)$ for some k with $2 \le k \le n-1$; there are $r_{n-2} + r_{n-3} + \cdots + r_1$ such walks.

(5) Walks that are the horizontal reflected images of walks in (3) that begin with $(1,1) \to (2,1)$ and never enter the third row until the final cell; there are $n-1$ such walks.

(6) Walks that are horizontal reflected images of walks in (5); there are $r_{n-2} + r_{n-3} + \cdots + r_1$ such walks.

Thus, $r_3 = 1 + r_2 + 2(2 + r_1) = 11$ and, for $n \ge 3$,

$$\begin{aligned} r_n &= 1 + r_{n-1} + 2[(n-1) + r_{n-2} + r_{n-3} + \cdots + r_1] \\ &= 2n - 1 + r_{n-1} + 2(r_{n-2} + \cdots + r_1) \ , \end{aligned}$$

and

$$r_{n+1} = 2n + 1 + r_n + 2(r_{n-1} + r_{n-2} + \cdots + r_1) \ .$$

Therefore

$$r_{n+1} - r_n = 2 + r_n + r_{n-1} \Longrightarrow r_{n+1} = 2 + 2r_n + r_{n-1} \ .$$

Thus

$$r_{n+1} + 1 = 2(r_n + 1) + (r_{n-1} + 1) \ ,$$

whence

$$r_n + 1 = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n+1} - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^{n+1} \ ,$$

and

$$r_n = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n+1} - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^{n+1} - 1 \ .$$

Solution 2. Employ the same notation as in Solution 1. We have that $r_1 = 1$, $r_2 = 4$ and $r_3 = 11$. Let $n \ge 3$. Consider the situation that there are r_{n+1} columns. There are basically three types of rook walks.

Type 1. There are four rook walks that enter only the first two columns.

Type 2. There are $3r_{n-1}$ rooks walks that do not pass between the second and third columns in the middle row (in either direction), *viz.* r_{n-1} of each of the types:

$$(1,1) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (3,1) \ ;$$

$$(1,1) \longrightarrow (2,1) \longrightarrow (2,2) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (3,1) \ ;$$

$$(1,1) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (2,2) \longrightarrow (2,1) \longrightarrow (3,1) \ .$$

Type 3. Consider the rook walks that pass between the second and third column along the middle row.

They are of Type 3a:

$$(1,1) \longrightarrow * \longrightarrow (2,2) \longrightarrow (2,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (3,1) \ ,$$

or Type 3b:

$$(1,1) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (2,3) \longrightarrow (2,2) \longrightarrow * \longrightarrow (3,1) \ ,$$

where in each case the asterisk stands for one of two possible options.

We can associate in a two-one way the walks of Type 3a to a rook walk on the last n columns, namely

$$(1,2) \longrightarrow (2,2) \longrightarrow (2,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2)$$

and the walks of Type 3b to a rook walk on the last n columns, namely

$$(1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (2,3) \longrightarrow (2,2) \longrightarrow (3,2) \ .$$

The number of rook walks of the latter two types together is $r_n - 1 - r_{n-1}$. From the number of rook walks on the last n columns, we subtract one for $(1,2) \to (2,2) \to (3,2)$ and r_{n-1} for those of the type

$$(1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (2,3) \ .$$

Therefore, the number of rook walks of Type 3 is $2(r_n - 1 - r_{n-1})$ and we find that

$$r_{n+1} = 4 + 3r_{n-1} + 2(r_n - 1 - r_{n-1}) = 2 + 2r_n + r_{n-1} \ .$$

We can now complete the solution as in Solution 1.

Solution 3. Let $S(3,n)$ be the set of self-avoiding rook walks in which the rook occupies column n but does not occupy column $n+1$. Then $R(3,n) = |S(3,1)| + |S(3,2)| + \cdots + |S(3,n)|$. Furthermore, topological considerations allow us to break $S(3,n)$ into three disjoint subsets $S_1(3,n)$, the set of paths in which corner $(1,n)$ is not occupied, but there is a path segment $(2,n) \longrightarrow (3,n)$; $S_2(3,n)$, the set of paths in which corners $(1,n)$ and $(3,n)$ are both occupied by a path $(1,n) \longrightarrow (2,n) \longrightarrow (3,n)$; and $S_3(3,n)$, the set of paths in which corner $(3,n)$ is not occupied but there is a path segment $(1,n) \longrightarrow (2,n)$. Let $s_i(n) = |S_i(3,n)|$ for $i = 1,2,3$. Note that $s_1(1) = 0$, $s_2(1) = 1$ and $s_3(1) = 0$. By symmetry, $s_1(n) = s_3(n)$ for every positive n. Furthermore, we can construct paths in $S(3,n+1)$ by "bulging" paths in $S(3,n)$, from which we obtain

$$\begin{aligned} s_1(n+1) &= s_1(n) + s_2(n) \ ; \\ s_2(n+1) &= s_1(n) + s_2(n) + s_3(n) \ ; \\ s_3(n+1) &= s_2(n) + s_3(n) \ ; \end{aligned}$$

or, upon simplification,

$$\begin{aligned} s_1(n+1) &= s_1(n) + s_2(n) \ ; \\ s_2(n+1) &= 2s_1(n) + s_2(n) \ . \end{aligned}$$

Hence, for $n \geq 2$,

$$\begin{aligned} s_1(n+1) &= s_1(n) + 2s_1(n-1) + s_2(n-1) \\ &= s_1(n) + 2s_1(n-1) + s_1(n) - s_1(n-1) \\ &= 2s_1(n) + s_1(n-1) \ . \end{aligned}$$

and

$$\begin{aligned} s_2(n+1) = 2s_1(n) + s_2(n) &= 2s_1(n-1) + 2s_2(n-1) + s_2(n) \\ &= s_2(n) - s_2(n-1) + 2s_2(n-1) + s_2(n) \\ &= 2s_2(n) + s_2(n-1) \ . \end{aligned}$$

We find that

$$s_1(n) = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n-1} - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^{n-1} \ ;$$
$$s_2(n) = \frac{1}{2}(1+\sqrt{2})^{n-1} + \frac{1}{2}(1-\sqrt{2})^{n-1} \ .$$

Summing a geometric series yields that

$$\begin{aligned} R(3,n) &= (s_2(1) + \cdots + s_2(n)) + 2(s_1(1) + \cdots + s_1(n)) \\ &= \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)\left(\frac{(1+\sqrt{2})^n - 1}{\sqrt{2}}\right) + \left(\frac{1}{2} - \frac{1}{\sqrt{2}}\right)\left(\frac{(1-\sqrt{2})^n - 1}{-\sqrt{2}}\right) \\ &= \left(\frac{1}{2\sqrt{2}}\right)[(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}] - 1 \ . \end{aligned}$$

The formula agrees with $R(3,1) = 1$, $R(3,2) = 4$ and $R(3,3) = 11$.

Acknowledgment. The first two solutions are due to Man-Duen Choi, and the third to Ed Doolittle.

Sun Life Financial

41^{st} Canadian Mathematical Olympiad

Wednesday, March 25, 2009

CMS SMC

PROBLEM 1. Given an $m \times n$ grid with squares coloured either black or white, we say that a black square in the grid is *stranded* if there is some square to its left in the same row that is white and there is some square above it in the same column that is white (see Figure 1.).

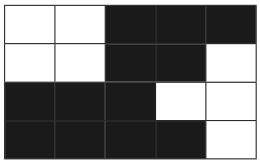

FIGURE 1. A 4×5 grid with no stranded black squares

Find a closed formula for the number of $2 \times n$ grids with no stranded black squares.

PROBLEM 2. Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, so that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

PROBLEM 3. Define

$$f(x,y,z) = \frac{(xy+yz+zx)(x+y+z)}{(x+y)(x+z)(y+z)}.$$

Determine the set of real numbers r for which there exists a triplet (x,y,z) of positive real numbers satisfying $f(x,y,z) = r$.

PROBLEM 4. Find all ordered pairs (a,b) such that a and b are integers and $3^a + 7^b$ is a perfect square.

PROBLEM 5. A set of points is marked on the plane, with the property that any three marked points can be covered with a disk of radius 1. Prove that the set of all marked points can be covered with a disk of radius 1.

CANADIAN MATHEMATICAL OLYMPIAD 2009
SOLUTIONS

PROBLEM 1. Given an $m \times n$ grid with squares coloured either black or white, we say that a black square in the grid is *stranded* if there is some square to its left in the same row that is white and there is some square above it in the same column that is white (see Figure).

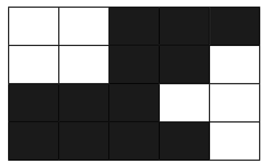

FIGURE 1. A 4×5 grid with no stranded black squares

Find a closed formula for the number of $2 \times n$ grids with no stranded black squares.

Solution. There is no condition for squares in the first row. A square in the second row can be black only if the square above it is black or all squares to the left of it are black. Suppose the first k squares in the second row are black and the $(k+1)$-st square is white or $k = n$. When $k < n$ then for each of the first $k+1$ squares in the first row we have 2 choices, and for each of the remaining $n-k-1$ columns we have 3 choices. When $k = n$, there are 2^n choices for the first row. The total number of choices is thus:

$$\sum_{k=0}^{n-1} 2^{k+1} 3^{n-k-1} + 2^n.$$

This expression simplifies to

$$2 \cdot 3^n - 2^n.$$

CANADIAN MATHEMATICAL OLYMPIAD 2009 SOLUTIONS

Problem 2. Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, so that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

Solution. Let $x_0, \ldots, x_{199}$ be variables. Assign the value of $+1$ or -1 to x_i depending on whether the $(i+1)$st segment of the larger circle (counting counterclockwise) is black or white, respectively. Similarly, assign the value of $+1$ or -1 to the variable y_i depending on whether the $(i+1)$th segment of the smaller circle is black or white. We can now restate the problem in the following equivalent way: show that

$$S_j = \sum_{i=1}^{200} x_i y_{i+j} \geq 0\,,$$

for some $j = 0, \ldots, 199$. Here the subscript $i+j$ is understood modulo 200.
Now observe that $y_0 + \cdots + y_{199} = 0$ and thus

$$S_0 + \cdots + S_{199} = \sum_{I=0}^{199} x_i (y_0 + \cdots + y_{199}) = 0\,.$$

Thus $S_j \geq 0$ for some $j = 0, \ldots, 199$, as claimed.

CANADIAN MATHEMATICAL OLYMPIAD 2009 SOLUTIONS

Problem 3. Define
$$f(x,y,z) = \frac{(xy+yz+zx)(x+y+z)}{(x+y)(x+z)(y+z)}.$$
Determine the set of real numbers r for which there exists a triplet (x,y,z) of positive real numbers satisfying $f(x,y,z) = r$.

Solution. We prove that $1 < f(x,y,z) \le \frac{9}{8}$, and that $f(x,y,z)$ can take on any value within the range $(1, \frac{9}{8}]$.
The expression for $f(x,y,z)$ can be simplified to
$$f(x,y,z) = 1 + \frac{xyz}{(x+y)(x+z)(y+z)}.$$
Since x, y, z are positive, we get $1 < f(x,y,z)$.
The inequality $f(x,y,z) \le \frac{9}{8}$ can be simplified to
$$x^2y + x^2z + y^2x + y^2z + z^2x + z^2y - 6xyz \ge 0.$$
Rearrange the left hand side as follows:
$$\begin{aligned}
&x^2y + x^2z + y^2x + y^2z + z^2x + z^2y - 6xyz = \\
&x(y^2+z^2) - 2xyz + y(x^2+z^2) - 2xyz + z(x^2+y^2) - 2xyz = \\
&x(y-z)^2 + y(x-z)^2 + z(x-y)^2.
\end{aligned}$$
This expression is clearly non-negative when x, y, z are non-negative.
To prove that $f(x,y,z)$ takes any values in the interval $(1, \frac{9}{8}]$, define
$$g(t) = f(t,1,1) = 1 + \frac{t}{2(1+t)^2}.$$
Then $g(1) = \frac{9}{8}$ and $g(t)$ approaches 1 as t approaches 0. It follows from the continuity of $g(t)$ for $0 < t \le 1$ that it takes all values in the interval $(1, \frac{9}{8}]$. (Alternatively, one can check that the quadratic equation $g(t) = r$ has a solution t for any number r in the interval $(1, \frac{9}{8}]$.)

CANADIAN MATHEMATICAL OLYMPIAD 2009 SOLUTIONS

PROBLEM 4. Find all ordered pairs (a, b) such that a and b are integers and $3^a + 7^b$ is a perfect square.

Solution. It is obvious that a and b must be non-negative.
Suppose that $3^a + 7^b = n^2$. We can assume that n is positive. We first work modulo 4. Since $3^a + 7^b = n^2$, it follows that

$$n^2 \equiv (-1)^a + (-1)^b \pmod{4}.$$

Since no square can be congruent to 2 modulo 4, it follows that we have either (i) a is odd and b is even or (ii) a is even and b is odd.

Case (i): Let $b = 2c$. Then

$$3^a = (n - 7^c)(n + 7^c).$$

It cannot be the case that 3 divides both $n - 7^c$ and $n + 7^c$. But each of these is a power of 3. It follows that $n - 7^c = 1$, and therefore

$$3^a = 2 \cdot 7^c + 1.$$

If $c = 0$, then $a = 1$, and we obtain the solution $a = 1$, $b = 0$. So suppose that $c \geq 1$. Then $3^a \equiv 1 \pmod{7}$. This is impossible, since the smallest positive value of a such that $3^a \equiv 1 \pmod{7}$ is given by $a = 6$, and therefore all a such that $3^a \equiv 1 \pmod{7}$ are even, contradicting the fact that a is odd.

Case (ii): Let $a = 2c$. Then

$$7^b = (n - 3^c)(n + 3^c).$$

Thus each of $n - 3^c$ and $n + 3^c$ is a power of 7. Since 7 cannot divide both of these, it follows that $n - 3^c = 1$, and therefore

$$7^b = 2 \cdot 3^c + 1.$$

Look first at the case $c = 1$. Then $b = 1$, and we obtain the solution $a = 2$, $b = 1$. So from now on we may assume that $c > 1$. Then $7^b \equiv 1 \pmod{9}$. The smallest positive integer b such that $7^b \equiv 1 \pmod{9}$ is given by $b = 3$. It follows that b must be a multiple of 3. Let $b = 3d$. Note that d is odd, so in particular $d \geq 1$.
Let $y = 7^d$. Then $y^3 - 1 = 2 \cdot 3^c$, and therefore

$$2 \cdot 3^c = (y - 1)(y^2 + y + 1).$$

It follows that $y - 1 = 2 \cdot 3^u$ for some positive u, and that $y^2 + y + 1 = 3^v$ for some $v \geq 2$. But since

$$3y = (y^2 + y + 1) - (y - 1)^2,$$

it follows that $3 \mid y$, which is impossible since $3 \mid (y - 1)$.

CANADIAN MATHEMATICAL OLYMPIAD 2009 SOLUTIONS

Problem 5. A set of points is marked on the plane, with the property that any three marked points can be covered with a disk of radius 1. Prove that the set of all marked points can be covered with a disk of radius 1.

Solution. (For a finite set of points only.) Let D be a disk of smallest radius that covers all marked points. Consider the marked points on the boundary C of this disk. Note that if all marked points on C lie on an arc smaller than the half circle (ASTTHC for short), then the disk can be moved a little towards these points on the boundary and its radius can be decreased. Since we assumed that our disk has minimal radius, the marked points on its boundary do not lie on an ASTTHC.
If the two endpoints of a diagonal of D are marked, then D is the smallest disk containing these two points, hence must have radius at most 1.
If there are 3 marked points on C that do not lie on an ASTTHC, then D is the smallest disk covering these 3 points and hence must have radius at most 1. (In this case the triangle formed by the three points is acute and C is its circumcircle.)
If there are more than 3 marked points on the boundary that do not lie on an ASTTHC, then we can remove one of them so that the remaining points again do not lie on an ASTTHC. By induction this leads us to the case of 3 points. Indeed, given 4 or more points on C, choose 3 points that lie on a half circle. Then the middle point can be removed.

Sun Life Financial

42^{nd} Canadian Mathematical Olympiad

Wednesday, March 24, 2010

CMS
SMC

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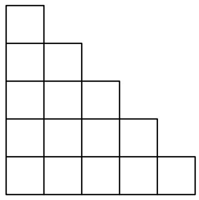
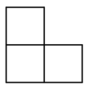
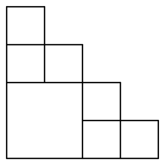
(1) For a positive integer n, an n-*staircase* is a figure consisting of unit squares, with one square in the first row, two squares in the second row, and so on, up to n squares in the n^{th} row, such that all the left-most squares in each row are aligned vertically. For example, the 5-staircase is shown below.



Let $f(n)$ denote the minimum number of square tiles required to tile the n-staircase, where the side lengths of the square tiles can be any positive integer. For example, $f(2) = 3$ and $f(4) = 7$.





(a) Find all n such that $f(n) = n$.
(b) Find all n such that $f(n) = n + 1$.

(2) Let A, B, P be three points on a circle. Prove that if a and b are the distances from P to the tangents at A and B and c is the distance from P to the chord AB, then $c^2 = ab$.

(3) Three speed skaters have a friendly "race" on a skating oval. They all start from the same point and skate in the same direction, but with different speeds that they maintain throughout the race. The slowest skater does 1 lap a minute, the fastest one does 3.14 laps a minute, and the middle one does L laps a minute for some $1 < L < 3.14$. The race ends at the moment when all three skaters again come together to the same point on the oval (which may differ from the starting point.) Find how many different choices for L are there such that exactly 117 passings occur before the end of the race. (A passing is defined when one skater passes another one. The beginning and the end of the race when all three skaters are together are not counted as passings.)

(4) Each vertex of a finite graph can be coloured either black or white. Initially all vertices are black. We are allowed to pick a vertex P and change the colour of P and all of its

neighbours. Is it possible to change the colour of every vertex from black to white by a sequence of operations of this type?

(A finite graph consists of a finite set of vertices and a finite set of edges between vertices. If there is an edge between vertex A and vertex B, then B is called a neighbour of A.)

(5) Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Let $a_n = n! + n$. Show that if $P(a_n)/Q(a_n)$ is an integer for every n, then $P(n)/Q(n)$ is an integer for every integer n such that $Q(n) \neq 0$.

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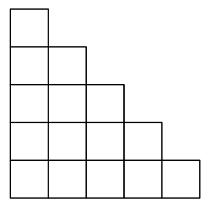
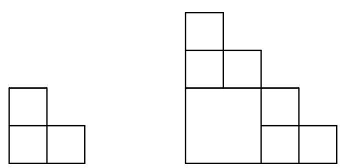
CANADIAN MATHEMATICAL OLYMPIAD 2010
PROBLEMS AND SOLUTIONS

(1) For a positive integer n, an n-*staircase* is a figure consisting of unit squares, with one square in the first row, two squares in the second row, and so on, up to n squares in the n^{th} row, such that all the left-most squares in each row are aligned vertically. For example, the 5-staircase is shown below.



Let $f(n)$ denote the minimum number of square tiles required to tile the n-staircase, where the side lengths of the square tiles can be any positive integer. For example, $f(2) = 3$ and $f(4) = 7$.



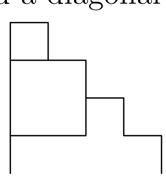
(a) Find all n such that $f(n) = n$.
(b) Find all n such that $f(n) = n + 1$.

Solution. (a) A *diagonal* square in an n-staircase is a unit square that lies on the diagonal going from the top-left to the bottom-right. A *minimal tiling* of an n-staircase is a tiling consisting of $f(n)$ square tiles.

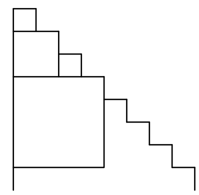
Observe that $f(n) \geq n$ for all n. There are n diagonal squares in an n-staircase, and a square tile can cover at most one diagonal square, so any tiling requires at least n square tiles. In other words, $f(n) \geq n$. Hence, if $f(n) = n$, then each square tile covers exactly one diagonal square.

Let n be a positive integer such that $f(n) = n$, and consider a minimal tiling of an n-staircase. The only square tile that can cover the unit square in the first row is the unit square itself.

Now consider the left-most unit square in the second row. The only square tile that can cover this unit square and a diagonal square is a 2×2 square tile.



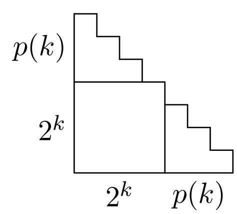
Next, consider the left-most unit square in the fourth row. The only square tile that can cover this unit square and a diagonal square is a 4×4 square tile.



Continuing this construction, we see that the side lengths of the square tiles we encounter will be 1, 2, 4, and so on, up to 2^k for some nonnegative integer k. Therefore, n, the height of the n-staircase, is equal to $1+2+4+\cdots+2^k = 2^{k+1}-1$. Alternatively, $n = 2^k - 1$ for some positive integer k. Let $p(k) = 2^k - 1$.

Conversely, we can tile a $p(k)$-staircase with $p(k)$ square tiles recursively as follows: We have that $p(1) = 1$, and we can tile a 1-staircase with 1 square tile. Assume that we can tile a $p(k)$-staircase with $p(k)$ square tiles for some positive integer k.

Consider a $p(k+1)$-staircase. Place a $2^k \times 2^k$ square tile in the bottom left corner. Note that this square tile covers a digaonal square. Then $p(k+1) - 2^k = 2^{k+1} - 1 - 2^k = 2^k - 1 = p(k)$, so we are left with two $p(k)$-staircases.

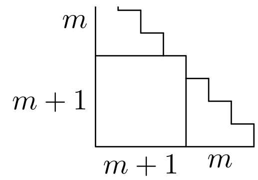


Furthermore, these two $p(k)$-staircases can be tiled with $2p(k)$ square tiles, which means we use $2p(k) + 1 = p(k+1)$ square tiles.

Therefore, $f(n) = n$ if and only if $n = 2^k - 1 = p(k)$ for some positive integer k. In other words, the binary representation of n consists of all 1s, with no 0s.

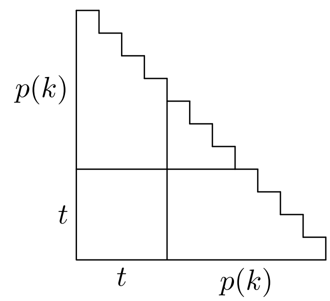
(b) Let n be a positive integer such that $f(n) = n + 1$, and consider a minimal tiling of an n-staircase. Since there are n diagonal squares, every square tile except one covers a diagonal square. We claim that the square tile that covers the bottom-left unit square must be the square tile that does not cover a diagonal square.

If n is even, then this fact is obvious, because the square tile that covers the bottom-left unit square cannot cover any diagonal square, so assume that n is odd. Let $n = 2m + 1$. We may assume that $n > 1$, so $m \geq 1$. Suppose that the square tile covering the bottom-left unit square also covers a diagonal square. Then the side length of this square tile must be $m + 1$. After this $(m+1) \times (m+1)$ square tile has been placed, we are left with two m-staircases.

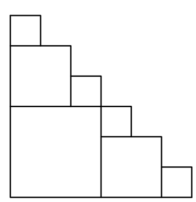


Hence, $f(n) = 2f(m) + 1$. But $2f(m) + 1$ is odd, and $n + 1 = 2m + 2$ is even, so $f(n)$ cannot be equal to $n + 1$, contradiction. Therefore, the square tile that covers the bottom-left unit square is the square tile that does not cover a diagonal square.

Let t be the side length of the square tile covering the bottom-left unit square. Then every other square tile must cover a diagonal square, so by the same construction as in part (a), $n = 1+2+4+\cdots+2^{k-1}+t = 2^k+t-1$ for some positive integer k. Furthermore, the top $p(k) = 2^k - 1$ rows of the n-staircase must be tiled the same way as the minimal tiling of a $p(k)$-staircase. Therefore, the horizontal line between rows $p(k)$ and $p(k) + 1$ does not pass through any square tiles. Let us call such a line a *fault line*. Similarly, the vertical line between columns t and $t + 1$ is also a fault line. These two fault lines partition two $p(k)$-staircases.



If these two $p(k)$-staircases do not overlap, then $t = p(k)$, so $n = 2p(k)$. For example, the minimal tiling for $n = 2p(2) = 6$ is shown below.



Hence, assume that the two $p(k)$-staircases do overlap. The intersection of the two $p(k)$-staircases is a $[p(k) - t]$-staircase. Since this $[p(k) - t]$-staircase is tiled the same way as the top $p(k) - t$ rows of a minimal tiling of a $p(k)$-staircase, $p(k) - t = p(l)$ for some positive integer $l < k$, so $t = p(k) - p(l)$. Then

$$n = t + p(k) = 2p(k) - p(l).$$

Since $p(0) = 0$, we can summarize by saying that n must be of the form

$$n = 2p(k) - p(l) = 2^{k+1} - 2^l - 1,$$

where k is a positive integer and l is a nonnegative integer. Also, our argument shows how if n is of this form, then an n-staircase can be tiled with $n+1$ square tiles.

Finally, we observe that n is of this form if and only if the binary representation of n contains exactly one 0:

$$2^{k+1} - 2^l - 1 = \underbrace{11\ldots1}_{k-l \text{ 1s}}0\underbrace{11\ldots1}_{l \text{ 1s}}.$$

$\square$

(2) Let A, B, P be three points on a circle. Prove that if a and b are the distances from P to the tangents at A and B and c is the distance from P to the chord AB, then $c^2 = ab$.

Solution. Let r be the radius of the circle, and let a' and b' be the respective lengths of PA and PB. Since $b' = 2r \sin \angle PAB = 2rc/a'$, $c = a'b'/(2r)$. Let AC be the diameter of the circle and H the foot of the perpendicular from P to AC. The similarity of the triangles ACP and APH imply that $AH : AP = AP : AC$ or $(a')^2 = 2ra$. Similarly, $(b')^2 = 2rb$. Hence

$$c^2 = \frac{(a')^2}{2r}\frac{(b')^2}{2r} = ab$$

as desired. $\square$

Alternate Solution. Let E, F, G be the feet of the perpendiculars to the tangents at A and B and the chord AB, respectively. We need to show that $PE : PG = PG : GF$, where G is the foot of the perpendicular from P to AB. This suggest that we try to prove that the triangles EPG and GPF are similar.

Since PG is parallel to the bisector of the angle between the two tangents, $\angle EPG = \angle FPG$. Since $AEPG$ and $BFPG$ are concyclic quadrilaterals (having opposite angles right), $\angle PGE = \angle PAE$ and $\angle PFG = \angle PBG$. But $\angle PAE = \angle PBA = \angle PBG$, whence $\angle PGE = \angle PFG$. Therefore triangles EPG and GPF are similar.

The argument above with concyclic quadrilaterals only works when P lies on the shorter arc between A and B. The other case can be proved similarly. $\square$

(3) Three speed skaters have a friendly race on a skating oval. They all start from the same point and skate in the same direction, but with different speeds that they maintain throughout the race. The slowest skater does 1 lap a minute, the fastest one does 3.14 laps a minute, and the middle one does L laps a minute for some $1 < L < 3.14$. The race ends at the moment when all three skaters again come together to the same point on the oval (which may differ from the starting

point.) Find how many different choices for L are there such that 117 passings occur before the end of the race. (A passing is defined when one skater passes another one. The beginning and the end of the race when all three skaters are at together are not counted as a passing.)

Solution. Assume that the length of the oval is one unit. Let $x(t)$ be the difference of distances that the slowest and the fastest skaters have skated by time t. Similarly, let $y(t)$ be the difference between the middle skater and the slowest skater. The path $(x(t), y(t))$ is a straight ray R in $\mathbb{R}^2$, starting from the origin, with slope depending on L. By assumption, $0 < y(t) < x(t)$.

One skater passes another one when either $x(t) \in \mathbb{Z}$, $y(t) \in \mathbb{Z}$ or $x(t) - y(t) \in \mathbb{Z}$. The race ends when both $x(t), y(t) \in \mathbb{Z}$.

Let $(a, b) \in \mathbb{Z}^2$ be the endpoint of the ray R. We need to find the number of such points satisfying:

(a) $0 < b < a$

(b) The ray R intersects $\mathbb{Z}^2$ at endpoints only.

(c) The ray R crosses 357 times the lines $x \in \mathbb{Z}$, $y \in \mathbb{Z}$, $y - x \in \mathbb{Z}$.

The second condition says that a and b are relatively prime. The ray R crosses $a-1$ of the lines $x \in \mathbb{Z}$, $b-1$ of the lines $y \in \mathbb{Z}$ and $a-b-1$ of the lines $x-y \in \mathbb{Z}$. Thus, we need $(a-1)+(b-1)+(a-b-1) = 117$, or equivalently, $2a - 3 = 117$. That is $a = 60$.

Now b must be a positive integer less than and relatively prime to 60. The number of such b can be found using the Euler's ϕ function:

$$\phi(60) = \phi(2^2 \cdot 3 \cdot 5) = (2-1) \cdot 2 \cdot (3-1) \cdot (5-1) = 16.$$

Thus the answer is 16. $\square$

Alternate Solution. First, let us name our skaters. From fastest to slowest, call them: A, B and C. (Abel, Bernoulli and Cayley?)

Now, it is helpful to consider the race from the viewpoint of C. Relative to C, both A and B complete a whole number of laps, since they both start and finish at C.

Let n be the number of laps completed by A relative to C, and let m be the number of laps completed by B relative to C. Note that: $n > m \in \mathbb{Z}^+$

Consider the number of minutes required to complete the race. Relative to C, A is moving with a speed of $3.14 - 1 = 2.14$ laps per minute and completes the race in $\frac{n}{2.14}$ minutes. Also relative to C, B is moving with a speed of $(L-1)$ laps per minute and completes the race in $\frac{m}{L-1}$ minutes. Since A and B finish the race together (when they both meet C):

$$\frac{n}{2.14} = \frac{m}{L-1} \qquad \Rightarrow \qquad L = 2.14\left(\frac{m}{n}\right) + 1.$$

Hence, there is a one-to-one relation between values of L and values of the postive proper fraction $\frac{m}{n}$. The fraction should be reduced, that is the pair (m, n) should be relatively prime, or else, with $k = \gcd(m, n)$, the race ends after n/k laps for A and m/k laps for B when they *first* meet C together.

It is also helpful to consider the race from the viewpoint of B. In this frame of reference, A completes only $n - m$ laps. Hence A *passes* B only $(n - m) - 1$ times, since the racers do not "pass" at the end of the race (nor at the beginning). Similarily A passes C only $n - 1$ times and B passes C only $m - 1$ times. The total number of passings is:

$$117 = (n - 1) + (m - 1) + (n - m - 1) = 2n - 3 \quad \Rightarrow \quad n = 60$$

Hence the number of values of L equals the number of m for which the fraction $\frac{m}{60}$ is positive, proper and reduced. That is the number of positive integer values smaller than and relatively prime to 60. One could simply count: {1,7,11,13,17,...}, but Euler's ϕ function gives this number:

$$\phi(60) = \phi(2^2 \cdot 3 \cdot 5) = (2 - 1) \cdot 2 \cdot (3 - 1) \cdot (5 - 1) = 16.$$

Therefore, there are 16 values for L which give the desired number of passings.

Note that the actual values for the speeds of A and C do not affect the result. They could be any values, rational or irrational, just so long as they are different, and there will be 16 possible values for the speed of B between them. $\square$

(4) Each vertex of a finite graph can be colored either black or white. Initially all vertices are black. We are allowed to pick a vertex P and change the color of P and all of its neighbours. Is it possible to change the colour of every vertex from black to white by a sequence of operations of this type?

Solution. The answer is yes. Proof by induction on the number n of vertices. If $n = 1$, this is obvious. For the induction assumption, suppose we can do this for any graph with $n - 1$ vertices for some $n \geq 2$ and let X be a graph with n vertices which we will denote by $P_1, \ldots, P_{n+1}$.

Let us denote the "basic" operation of changing the color of P_i and all of its neighbours by f_i. Removing a vertex P_i from X (along with all edges connecting to P_i) and applying the induction assumption to the resulting smaller graph, we see that there exists a sequence of operations g_i (obtained by composing some f_j, with $j \neq i$) which changes the colour of every vertex in X, except for possibly P_i.

If g_i it also changes the color of P_i then we are done. So, we may assume that g_i does not change the colour of P for every $i = 1, \ldots, n$. Now consider two cases.

Case 1: n is even. Then composing $g_1, \ldots, g_n$ we will change the color of every vertex from white to black.

Case 2: n is odd. I claim that in this case X has a vertex with an even number of neighbours.

Indeed, denote the number of neighbours of P_i (or equivalently, the number of edges connected to P) by k_i. Then $P_1 + \cdots + P_{n+1} = 2e$, where e is the number of edges of X. Thus one of the numbers k_i has to be even as claimed.

After renumbering the vertices, we may assume that P_1 has $2k$ neighbours, say $P_2, \ldots, P_{2k+1}$. The composition of f_1 with $g_1, g_2, \ldots, g_{2k+1}$ will then change the colour of every vertex, as desired.

□

(5) Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Let $a_n = n! + n$. Show that if $P(a_n)/Q(a_n)$ is an integer for every n, then $P(n)/Q(n)$ is an integer for every integer n such that $Q(n) \neq 0$.

Solution. Imagine dividing $P(x)$ by $Q(x)$. We find that

$$\frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)},$$

where $A(x)$ and $R(x)$ are polynomials with rational coefficients, and $R(x)$ is either identically 0 or has degree less than the degree of $Q(x)$.

By bringing the coefficients of $A(x)$ to their least common multiple, we can find a polynomial $B(x)$ with integer coefficients, and a positive integer b, such that $A(x) = B(x)/b$. Suppose first that $R(x)$ is not identically 0. Note that for any integer k, either $A(k) = 0$, or $|A(k)| \geq 1/b$. But whenever $|k|$ is large enough, $0 < |R(k)/Q(k)| < 1/b$, and therefore if n is large enough, $P(a_n)/Q(a_n)$ cannot be an integer.

So $R(x)$ is identically 0, and $P(x)/Q(x) = B(x)/b$ (at least whenever $Q(x) \neq 0$.)

Now let n be an integer. Then there are infinitely many integers k such that $n \equiv a_k \pmod{b}$. But $B(a_k)/b$ is an integer, or equivalently b divides $B(a_k)$. It follows that b divides $B(n)$, and therefore $P(n)/Q(n)$ is an integer. □

43^{rd} Canadian Mathematical Olympiad

Wednesday, March 23, 2011

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(1) Consider 70-digit numbers n, with the property that each of the digits $1, 2, 3, \ldots, 7$ appears in the decimal expansion of n ten times (and 8, 9, and 0 do not appear). Show that no number of this form can divide another number of this form.

(2) Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, X the intersection of AB and CD, and Y the intersection of AD and BC. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively and let the angle bisector of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EGFH$ is a parallelogram.

(3) Amy has divided a square up into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers. If the total area of the white rectangles equals the total area of the red rectangles, what is the smallest possible value of x?

(4) Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a that is divisible by 2011. (For instance, if $a = 153204$, then 15, 532, and 0 are all contiguous substrings of a. Note that 0 is divisible by 2011.)

(5) Let d be a positive integer. Show that for every integer S, there exists an integer $n > 0$ and a sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, where for any k, $\epsilon_k = 1$ or $\epsilon_k = -1$, such that

$$S = \epsilon_1(1+d)^2 + \epsilon_2(1+2d)^2 + \epsilon_3(1+3d)^2 + \cdots + \epsilon_n(1+nd)^2.$$

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43^{rd} Canadian Mathematical Olympiad

Wednesday, March 23, 2011

CMS SMC

Problems and Solutions

(1) Consider 70-digit numbers n, with the property that each of the digits $1, 2, 3, \ldots, 7$ appears in the decimal expansion of n ten times (and 8, 9, and 0 do not appear). Show that no number of this form can divide another number of this form.

Solution. Assume the contrary: there exist a and b of the prescribed form, such that $b \geq a$ and a divides b. Then a divides $b - a$.

Claim: a is not divisible by 3 but $b - a$ is divisible by 9. Indeed, the sum of the digits is $10(1 + \cdots + 7) = 280$, for both a and b. [Here one needs to know or prove that an integer n is equivalent of the sum of its digits modulo 3 and modulo 9.]

We conclude that $b - a$ is divisible by $9a$. But this is impossible, since $9a$ has 71 digits and b has only 70 digits, so $9a > b > b - a$. $\square$

(2) Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, X the intersection of AB and CD, and Y the intersection of AD and BC. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively and let the angle bisector of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EGFH$ is a parallelogram.

Solution. Since $ABCD$ is cyclic, $\Delta XAC \sim \Delta XDB$ and $\Delta YAC \sim \Delta YBD$. Therefore,

$$\frac{XA}{XD} = \frac{XC}{XB} = \frac{AC}{DB} = \frac{YA}{YB} = \frac{YC}{YD}.$$

Let s be this ratio. Therefore, by the angle bisector theorem,

$$\frac{AE}{ED} = \frac{XA}{XD} = \frac{XC}{XB} = \frac{CF}{FB} = s,$$

and

$$\frac{AG}{GB} = \frac{YA}{YB} = \frac{YC}{YD} = \frac{CH}{HD} = s.$$

Hence, $\frac{AG}{GB} = \frac{CF}{FB}$ and $\frac{AE}{ED} = \frac{DH}{HC}$. Therefore, $EH||AC||GF$ and $EG||DB||HF$. Hence, $EGFH$ is a parallelogram. $\square$

(3) Amy has divided a square up into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers. If the total area of the white rectangles equals the total area of the red rectangles, what is the smallest possible value of x?

Solution. Let a_i and b_i denote the width and height of each white rectangle, and let c_i and d_i denote the width and height of each red rectangle. Also, let L denote the side length of the original square.

Lemma: Either $\sum a_i \geq L$ or $\sum d_i \geq L$.

Proof of lemma: Suppose there exists a horizontal line across the square that is covered entirely with white rectangles. Then, the total width of these rectangles is at least L, and the claim is proven. Otherwise, there is a red rectangle intersecting every horizontal line, and hence the total height of these rectangles is at least L. □

Now, let us assume without loss of generality that $\sum a_i \geq L$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}\left(\sum \frac{a_i}{b_i}\right) \cdot \left(\sum a_i b_i\right) &\geq \left(\sum a_i\right)^2 \\ &\geq L^2.\end{aligned}$$

But we know $\sum a_i b_i = \frac{L^2}{2}$, so it follows that $\sum \frac{a_i}{b_i} \geq 2$. Furthermore, each $c_i \leq L$, so

$$\sum \frac{d_i}{c_i} \geq \frac{1}{L^2} \cdot \sum c_i d_i = \frac{1}{2}.$$

Therefore, x is at least 2.5. Conversely, $x = 2.5$ can be achieved by making the top half of the square one colour, and the bottom half the other colour. □

(4) Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a that is divisible by 2011. (For instance, if $a = 153204$, then 15, 532, and 0 are all contiguous substrings of a. Note that 0 is divisible by 2011.)

Solution. We claim that if the decimal expansion of a has at least 2012 digits, then a contains the required substring. Let the decimal expansion of a be $a_k a_{k-1} \ldots a_0$. For $i = 0, \ldots, 2011$, Let b_i be the number with decimal expansion $a_i a_{i-1} \ldots a_0$. Then by pidgenhole principle, $b_i \equiv b_j \mod 2011$ for some $i < j \leq 2011$. It follows that 2011 divides $b_j - b_i = c \cdot 10^i$. Here c is the substring $a_j \ldots a_{i+1}$. Since 2011 and 10 are relatively prime, it follows that 2011 divides c. □

(5) Let d be a positive integer. Show that for every integer S, there exists an integer $n > 0$ and a sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, where for any k, $\epsilon_k = 1$ or $\epsilon_k = -1$, such that

$$S = \epsilon_1(1+d)^2 + \epsilon_2(1+2d)^2 + \epsilon_3(1+3d)^2 + \cdots + \epsilon_n(1+nd)^2.$$

Solution. Let $U_k = (1+kd)^2$. We calculate $U_{k+3} - U_{k+2} - U_{k+1} + U_k$. This turns out to be $4d^2$, a constant. Changing signs, we obtain the sum $-4d^2$.

Thus if we have found an expression for a certain number S_0 as a sum of the desired type, we can obtain an expression of the desired type for $S_0 + (4d^2)q$, for any integer q.

It remains to show that for any S, there exists an integer S' such that $S' \equiv S \pmod{4d^2}$ and S' can be expressed in the desired form. Look at the sum

$$(1+d)^2 + (1+2d)^2 + \cdots + (1+Nd)^2,$$

where N is "large." We can at will choose N so that the sum is odd, or so that the sum is even.

By changing the sign in front of $(1+kd)^2$ to a minus sign, we decrease the sum by $2(1+kd)^2$. In particular, if $k \equiv 0 \pmod{2d}$, we decrease the sum by 2 (modulo $4d^2$). So

If N is large enough, there are many $k < N$ such that k is a multiple of $2d$. By switching the sign in front of r of these, we change ("downward") the congruence class modulo $4d^2$ by $2r$. By choosing N so that the original sum is odd, and choosing suitable $r < 2d^2$, we can obtain numbers congruent to all odd numbers modulo $4d^2$. By choosing N so that the original sum is even, we can obtain numbers congruent to all even numbers modulo $4d^2$. This completes the proof. □

44th CANADIAN MATHEMATICAL OLYMPIAD
44ème OLYMPIADE MATHÉMATIQUE DU CANADA

1. Let x, y and z be positive real numbers. Show that $x^2 + xy^2 + xyz^2 \geq 4\,xyz - 4$.

   Soit x, y et z trois nombres réels positifs. Démontrez que $x^2 + xy^2 + xyz^2 \geq 4\,xyz - 4$.

2. For any positive integers n and k, let $L(n,k)$ be the least common multiple of the k consecutive integers $n, n+1, \ldots, n+k-1$. Show that for any integer b, there exist integers n and k such that $L(n,k) > b\,L(n+1,k)$.

   Soit $L(n,k)$ le plus petit commun multiple de la suite des k entiers consécutifs $n, n+1, \ldots, n+k-1$, où n et k sont deux entiers positifs quelconques. Montrez que pour tout entier b, il existe des nombres entiers n et k tels que $L(n,k) > b\,L(n+1,k)$.

3. Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB$, $\angle ADB$, and $\angle APB$ meet at a common point.

   Soit $ABCD$ un quadrilatère convexe et P le point d'intersection des droites AC et BD. Supposons que $AC + AD = BC + BD$. Démontrez que les bissectrices des angles internes de $\angle ACB$, $\angle ADB$, et $\angle APB$ se coupent en un point.

4. A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of the grid is classified as either *passable* or *impassable*. All edges on the boundary of the grid are impassable.
   You can give any of the commands *up*, *down*, *left*, or *right*. All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of *up*, *down*, *left*, or *right*, then another, for as long as you want.
   Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that *all* of the robots end up on the same square at the same time.

   Un certain nombre de robots sont placés sur les carrés composant une grille rectangulaire de dimension finie. Chaque carré peut contenir un nombre quelconque de robots. Les bords des carrés de la grille sont classés comme *franchissable* ou *infranchissable*. Les côtés qui forment le pourtour de la grille sont infranchissables.
   Vous pouvez donner n'importe laquelle des commandes suivantes : en *haut*, en *bas*, à *gauche* ou à *droite*. Tous les robots tentent alors de se déplacer simultanément dans la direction précisée. Si le bord adjacent au carré vers lequel se déplace un robot est franchissable, le robot le franchit et se place dans le carré suivant. Sinon, le robot reste dans le carré où il se trouve. Vous pouvez ensuite lancer une autre commande de déplacement vers le *haut,* le *bas,* la *gauche* ou la *droite*, et encore une autre, aussi longtemps que vous le désirez.
   Supposons que pour chaque robot, et ce, pour n'importe quel carré, il existe une suite finie de commandes qui amèneront ce robot au carré donné. Démontrez que vous pouvez aussi lancer une suite finie de commandes de sorte que tous les robots finiront par se retrouver simultanément dans le même carré.

5. A bookshelf contains n volumes, labelled 1 to n in some order. The librarian wishes to put them in the correct order as follows. The librarian selects a volume that is too far to the right, say the volume with label k, takes it out, and inserts it so that it is in the k-th place. For example, if the bookshelf contains the volumes 1, 3, 2, 4 in that order, the librarian could take out volume 2 and place it in the second position. The books will then be in the correct order 1, 2, 3, 4.

(a) Show that if this process is repeated, then, however the librarian makes the selections, all the volumes will eventually be in the correct order.

(b) What is the largest number of steps that this process can take?

Une étagère contient n volumes étiquetés de 1 à n, rangés dans un certain ordre. Le bibliothécaire souhaite les mettre dans le bon ordre de la façon suivante : il choisit un volume qui se trouve trop à droite, par exemple le volume étiqueté k, le retire de son emplacement et l'insère à la k –ième place. Par exemple, si les volumes sont rangés dans l'ordre 1, 3, 2, 4, le bibliothécaire peut prendre le volume 2 et le mettre à la deuxième place. Les livres sont alors rangés dans le bon ordre, soit 1, 2, 3, 4.

a) Démontrez que si l'on répète ce processus, tous les volumes finiront par être dans le bon ordre, et ce, qu'elle que soit la manière dont le bibliothécaire les range.

b) Quel est le plus grand nombre d'étapes que peut exiger un tel processus?

1. Let x, y and z be positive real numbers. Show that $x^2 + xy^2 + xyz^2 \geq 4xyz - 4$.

Solution. Note that

$$x^2 \geq 4x - 4, \quad y^2 \geq 4y - 4, \quad \text{and} \quad z^2 \geq 4z - 4,$$

and therefore

$$x^2 + xy^2 + xyz^2 \geq (4x - 4) + x(4y - 4) + xy(4z - 4) = 4xyz - 4.$$

2. For any positive integers n and k, let $L(n, k)$ be the least common multiple of the k consecutive integers n, $n + 1$, ..., $n + k - 1$. Show that for any integer b, there exist integers n and k such that $L(n, k) > b\, L(n + 1, k)$.

Solution. I. Let $p > b$ be prime, let $n = p^3$ and $k = p^2$. If $p^3 < i < p^3 + p^2$, then no power of p greater than 1 divides i, while p divides $p^3 + p$. It follows that $L(p^3, p^2) = p^2 L(p^3 + 1, p^2 - 1)$. A similar calculation shows that $L(p^3 + 1, p^2) = pL(p^3 + 1, p^2 - 1)$. Thus $L(p^3, p^2) = pL(p^3 + 1, p^2) > bL(p^3 + 1, p^2)$.

II. Let $m > 1$. Then $L(m!-1, m+1)$ is the least common multiple of the integers from $m!-1$ to $m!+m-1$. But $m!-1$ is relatively prime to all of $m!, m!+1, \ldots, m!+m-1$. It follows that $L(m!-1, m+1) = (m!-1)M$, where $M = \text{lcm}(m!, m!+1, \ldots, m!+m-1)$.

Now consider $L(m!, m+1)$. This is $\text{lcm}(M, m!+m)$. But $m!+m = m((m-1)!+1)$, and m divides M. Thus $\text{lcm}(M, m! + m) \leq M((m - 1)! + 1)$, and

$$\frac{L(m! - 1, m + 1)}{L(m!, m + 1)} \geq \frac{m! - 1}{(m - 1)! + 1}.$$

Since m can be arbitrarily large, so can $L(m! - 1, m + 1)/L(m!, m + 1)$. Therefore taking $n = m! - 1$ for sufficiently large m, and $k = m + 1$, works.

3. Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB$, $\angle ADB$, and $\angle APB$ meet at a common point.

Solution. I. Construct A' on CA so that $AA' = AD$ and B' on CB such that $BB' = BD$. Then we have three angle bisectors that correspond to the perpendicular bisectors of $A'B'$, $A'D$, and $B'D$. These perpendicular bisectors are concurrent, so the angle bisectors are also concurrent. This tells us that the external angle bisectors at A and B meet at the excentre of PDB. A symmetric argument for C finishes the problem.

II. Note that the angle bisectors $\angle ACB$ and $\angle APB$ intersect at the excentres of $\triangle PBC$ opposite C and the angle bisectors of $\angle ADB$ and $\angle APB$ intersect at the excentres of $\triangle PAD$ opposite D. Hence, it suffices to prove that these two excentres coincide.

Let the excircle of $\triangle PBC$ opposite C touch side PB at a point X, line CP at a point Y and line CB at a point Z. Hence, $CY = CZ$, $PX = PY$ and $BX = BZ$. Therefore, $CP + PX = CB + BX$. Since $CP + PX + CB + BX$ is the perimeter of $\triangle CBP$, $CP + PX = CB + BX = s$, where s is the semi-perimeter of $\triangle CBP$. Therefore,

$$PX = CB + BX - CP = \frac{s}{2} - CP = \frac{CB + BP + PC}{2} - CP = \frac{CB + BP - PC}{2}.$$

Similarly, if we let the excircle of $\triangle PAD$ opposite D touch side PA at a point X', then

$$PX' = \frac{DA + AP - PD}{2}.$$

Since both excircles are tangent to AC and BD, if we show that $PX = PX'$, then we would show that the two excircles are tangent to AC and BD at the same points, i.e. the two excircles are identical. Hence, the two excentres coincide.

We will use the fact that $AC + AD = BC + BD$ to prove that $PX = PX'$. Since $AC + AD = BC + BD$, $AP + PC + AD = BC + BP + PD$. Hence, $AP + AD - PD = BC + BP - PC$. Therefore, $PX = PX'$, as desired.

4. A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of each square of the grid is classified as either *passable* or *impassable*. All edges on the boundary of the grid are impassable.

You can give any of the commands *up*, *down*, *left*, or *right*. All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of *up*, *down*, *left*, or *right*, then another, for as long as you want.

Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that *all* of the robots end up on the same square at the same time.

Solution. We will prove any two robots can be moved to the same square. From that point on, they will always be on the same square. We can then similarly move a third robot onto the same square as these two, and then a fourth, and so on, until all robots are on the same square.

Towards that end, consider two robots A and B. Let $d(A, B)$ denote the minimum number of commands that need to be given in order to move A to the square on which B is *currently* standing. We will give a procedure that is guaranteed to decrease $d(A, B)$. Since $d(A, B)$ is a non-negative integer, this procedure will eventually decrease n to 0, which finishes the proof.

Let $n = d(A, B)$, and let $S = \{s_1, s_2, \ldots, s_n\}$ be a minimum sequence of moves that takes A to the square where B is currently standing. Certainly A will not run into an impassable edge during this sequence, or we could get a shorter sequence by removing that command. Now suppose B runs into an impassable edge after some command s_i. From that point, we can get A to the square on which B started with the commands $s_{i+1}, s_{i+2}, \ldots, s_n$ and then to the square where B is currently with the commands $s_1, s_2, \ldots, s_{i-1}$. But this was only $n - 1$ commands in total, and so we have decreased $d(A, B)$ as required.

Otherwise, we have given a sequence of n commands to A and B, and neither ran into an impassable edge during the execution of these commands. In particular, the vector v connecting A to B on the grid must have never changed. We moved A to the position $B = A + v$, and therefore we must have also moved B to $B + v$. Repeating this process k times, we will move A to $A + kv$ and B to $B + kv$. But if $v \neq (0, 0)$, this will eventually force B off the edge of the grid, giving a contradiction.

5. A bookshelf contains n volumes, labelled 1 to n, in some order. The librarian wishes to put them in the correct order as follows. The librarian selects a volume that is too far to the right, say the volume with label k, takes it out, and inserts it in the k-th position. For example, if the bookshelf contains the volumes 1, 3, 2, 4 in that order, the librarian could take out volume 2 and place it in the second position. The books will then be in the correct order 1, 2, 3, 4.

(a) Show that if this process is repeated, then, however the librarian makes the selections, all the volumes will eventually be in the correct order.
(b) What is the largest number of steps that this process can take?

Solution. (a) If t_k is the number of times that volume k is selected, then we have $t_k \leq 1 + (t_1 + t_2 + \cdots + t_{k-1})$. This is because volume k must move to the right between selections, which means some volume was placed to its left. The only way that can happen is if a lower-numbered volume was selected. This leads to the bound $t_k \leq 2^{k-1}$. Furthermore, $t_n = 0$ since the nth volume will never be too far to the right. Therefore if N is the total number of moves then

$$N = t_1 + t_2 + \cdots + t_{n-1} \leq 1 + 2 + \cdots + 2^{n-2} = 2^{n-1} - 1,$$

and in particular the process terminates.

(b) Conversely, $2^{n-1}-1$ moves are required for the configuration $(n, 1, 2, 3, \ldots, n-1)$ if the librarian picks the rightmost eligible volume each time.

This can be proved by induction: if at a certain stage we are at $(x, n-k, n-k+1, \ldots, n-1)$, then after $2^k - 1$ moves, we will have moved to $(n-k, n-k+1, \ldots, n-1, x)$ without touching any of the volumes further to the left. Indeed, after $2^{k-1} - 1$ moves, we get to $(x, n-k+1, n-k+2, \ldots, n-1, n-k)$, which becomes $(n-k, x, n-k+1, n-k+2, \ldots, n-1)$ after 1 more move, and then $(n-k, n-k+1, \ldots, n-1, x)$ after another $2^{k-1} - 1$ moves. The result follows by taking $k = n-1$.

45^{th} Canadian Mathematical Olympiad

Wednesday, March 27, 2013

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1. Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x)$$

is a constant polynomial.

2. The sequence $a_1, a_2, \ldots, a_n$ consists of the numbers $1, 2, \ldots, n$ in some order. For which positive integers n is it possible that the $n+1$ numbers 0, a_1, a_1+a_2, $a_1+a_2+a_3$, $\ldots$, $a_1+a_2+\cdots+a_n$ all have different remainders when divided by $n+1$?

3. Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90^\circ$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

4. Let n be a positive integer. For any positive integer j and positive real number r, define $f_j(r)$ and $g_j(r)$ by

$$f_j(r) = \min\left(jr, n\right) + \min\left(\frac{j}{r}, n\right), \quad \text{and} \quad g_j(r) = \min\left(\lceil jr \rceil, n\right) + \min\left(\left\lceil \frac{j}{r} \right\rceil, n\right),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Prove that

$$\sum_{j=1}^{n} f_j(r) \leq n^2 + n \leq \sum_{j=1}^{n} g_j(r)$$

for all positive real numbers r.

5. Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.

45^{th} Canadian Mathematical Olympiad

Wednesday, March 27, 2013

Problems and Solutions

1. Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1) - (x-1)P(x)$$

is a constant polynomial.

Solution 1: The answer is $P(x)$ being any constant polynomial and $P(x) \equiv kx^2 + kx + c$ for any (nonzero) constant k and constant c.

Let Λ be the expression $(x+1)P(x-1) - (x-1)P(x)$, i.e. the expression in the problem statement.

Substituting $x = -1$ into Λ yields $2P(-1)$ and substituting $x = 1$ into Λ yield $2P(0)$. Since $(x+1)P(x-1)-(x-1)P(x)$ is a constant polynomial, $2P(-1) = 2P(0)$. Hence, $P(-1) = P(0)$.

Let $c = P(-1) = P(0)$ and $Q(x) = P(x) - c$. Then $Q(-1) = Q(0) = 0$. Hence, $0, -1$ are roots of $Q(x)$. Consequently, $Q(x) = x(x+1)R(x)$ for some polynomial R. Then $P(x) - c = x(x+1)R(x)$, or equivalently, $P(x) = x(x+1)R(x) + c$.

Substituting this into Λ yield

$$(x+1)((x-1)xR(x-1) + c) - (x-1)(x(x+1)R(x) + c)$$

This is a constant polynomial and simplifies to

$$x(x-1)(x+1)(R(x-1) - R(x)) + 2c.$$

Since this expression is a constant, so is $x(x-1)(x+1)(R(x-1)-R(x))$. Therefore, $R(x-1)-R(x)=0$ as a polynomial. Therefore, $R(x)=R(x-1)$ for all $x \in \mathbb{R}$. Then $R(x)$ is a polynomial that takes on certain values for infinitely values of x. Let k be such a value. Then $R(x)-k$ has infinitely many roots, which can occur if and only if $R(x)-k=0$. Therefore, $R(x)$ is identical to a constant k. Hence, $Q(x)=kx(x+1)$ for some constant k. Therefore, $P(x)=kx(x+1)+c=kx^2+kx+c$.

Finally, we verify that all such $P(x)=kx(x+1)+c$ work. Substituting this into Λ yield

$$\begin{aligned} & (x+1)(kx(x-1)+c)-(x-1)(kx(x+1)+c) \\ = \; & kx(x+1)(x-1)+c(x+1)-kx(x+1)(x-1)-c(x-1)=2c. \end{aligned}$$

Hence, $P(x)=kx(x+1)+c=kx^2+kx+c$ is a solution to the given equation for any constant k. Note that this solution also holds for $k=0$. Hence, constant polynomials are also solutions to this equation. $\square$

Solution 2: As in Solution 1, any constant polynomial P satisfies the given property. Hence, we will assume that P is not a constant polynomial.

Let n be the degree of P. Since P is not constant, $n \geq 1$. Let

$$P(x)=\sum_{i=0}^{n} a_i x^i,$$

with $a_n \neq 0$. Then

$$(x+1)\sum_{i=0}^{n} a_i (x-1)^i-(x-1)\sum_{i=0}^{n} a_i x^i=C,$$

for some constant C. We will compare the coefficient of x^n of the left-hand side of this equation with the right-hand side. Since C is a constant and $n \geq 1$, the coefficient of x^n of the right-hand side is equal to zero. We now determine the coefficient of x^n of the left-hand side of this expression.

The left-hand side of the equation simplifies to

$$x\sum_{i=0}^{n} a_i (x-1)^i+\sum_{i=0}^{n} a_i (x-1)^i-x\sum_{i=0}^{n} a_i x^i+\sum_{i=0}^{n} a_i x^i.$$

We will determine the coefficient x^n of each of these four terms.

By the Binomial Theorem, the coefficient of x^n of the first term is equal to that of $x\left(a_{n-1}(x-1)^{n-1}+a_n(x-1)^n\right) = a_{n-1} - \binom{n}{n-1}a_n = a_{n-1} - na_n$.

The coefficient of x^n of the second term is equal to that of $a_n(x-1)^n$, which is a_n.

The coefficient of x^n of the third term is equal to a_{n-1} and that of the fourth term is equal to a_n.

Summing these four coefficients yield $a_{n-1} - na_n + a_n - a_{n-1} + a_n = (2-n)a_n$.

This expression is equal to 0. Since $a_n \neq 0$, $n = 2$. Hence, P is a quadratic polynomial.

Let $P(x) = ax^2 + bx + c$, where a, b, c are real numbers with $a \neq 0$. Then

$$(x+1)(a(x-1)^2 + b(x-1) + c) - (x-1)(ax^2 + bx + c) = C.$$

Simplifying the left-hand side yields

$$(b-a)x + 2c = 2C.$$

Therefore, $b - a = 0$ and $2c = 2C$. Hence, $P(x) = ax^2 + ax + c$. As in Solution 1, this is a valid solution for all $a \in \mathbb{R}\backslash\{0\}$. $\square$

2. The sequence $a_1, a_2, \ldots, a_n$ consists of the numbers $1, 2, \ldots, n$ in some order. For which positive integers n is it possible that $0, a_1, a_1 + a_2, \ldots, a_1 + a_2 + \ldots + a_n$ all have different remainders when divided by $n + 1$?

Solution: It is possible if and only if n is odd.

If n is even, then $a_1 + a_2 + \ldots + a_n = 1 + 2 + \ldots + n = \frac{n}{2} \cdot (n + 1)$, which is congruent to 0 mod $n + 1$. Therefore, the task is impossible.

Now suppose n is odd. We will show that we can construct $a_1, a_2, \ldots, a_n$ that satisfy the conditions given in the problem. Then let $n = 2k + 1$ for some non-negative integer k. Consider the sequence: $1, 2k, 3, 2k - 2, 5, 2k - 3, \ldots, 2, 2k + 1$, i.e. for each $1 \leq i \leq 2k + 1$, $a_i = i$ if i is odd and $a_i = 2k + 2 - i$ if i is even.

We first show that each term $1, 2, \ldots, 2k + 1$ appears exactly once. Clearly, there are $2k + 1$ terms. For each odd number m in $\{1, 2, \ldots, 2k + 1\}$, $a_m = m$. For each even number m in this set, $a_{2k+2-m} = 2k + 2 - (2k + 2 - m) = m$. Hence, every number appears in $a_1, \ldots, a_{2k+1}$. Hence, $a_1, \ldots, a_{2k+1}$ does consist of the numbers $1, 2, \ldots, 2k + 1$ in some order.

We now determine $a_1 + a_2 + \ldots + a_m \pmod{2k + 2}$. We will consider the cases when m is odd and when m is even separately. Let $b_m = a_1 + a_2 \ldots + a_m$.

If m is odd, note that $a_1 \equiv 1 \pmod{2k + 2}$, $a_2 + a_3 = a_4 + a_5 = \ldots = a_{2k} + a_{2k+1} = 2k + 3 \equiv 1 \pmod{2k + 2}$. Therefore, $\{b_1, b_3, \ldots, b_{2k+1}\} = \{1, 2, 3, \ldots, k + 1\}$ $\pmod{2k + 2}$.

If m is even, note that $a_1 + a_2 = a_3 + a_4 = \ldots = a_{2k-1} + a_{2k} = 2k + 1 \equiv -1$ $\pmod{2k + 2}$. Therefore, $\{b_2, b_4, \ldots, b_{2k}\} = \{-1, -2, \ldots, -k\} \pmod{2k + 2} \equiv$ $\{2k + 1, 2k, \ldots, k + 2\} \pmod{2k + 2}$.

Therefore, $b_1, b_2, \ldots, b_{2k+1}$ do indeed have different remainders when divided by $2k + 2$. This completes the problem. $\square$

3. Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90°$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

Solution 1. Since $\angle C = 90°$, the point C lies on the semicircle with diameter AB which implies that, if M is te midpoint of side AB, then $MA = MC = MB$. This implies that triangle AMC is isosceles and hence that $\angle ACM = \angle A$. By definition, G lies on segment M and it follows that $\angle ACG = \angle ACM = \angle A = \angle CPA$. This implies that triangles APC and ACG are similar and hence that $AC^2 = AG \cdot AP$. Now if D denotes the foot of the perpendicular from C to AB, it follows that triangles ACD and ABC are similar which implies that $AC^2 = AD \cdot AB$. Therefore $AG \cdot AP = AC^2 = AD \cdot AB$ and, by power of a point, quadrilateral $DGPB$ is cyclic. This implies that D lies on the circumcircle of triangle BPG and, by a symmetric argument, it follows that D also lies on the circumcircle of triangle AGQ. Therefore these two circumcircles meet at the point D on side AB.

Solution 2. Define D and M as in Solution 1. Let R be the point on side AB such that $AC = CR$ and triangle ACR is isosceles. Since $\angle CRA = \angle A = \angle CPA$, it follows that $CPRA$ is cyclic and hence that $\angle GPR = \angle APR = \angle ACR = 180° - 2\angle A$. As in Solution 1, $MC = MB$ and hence $\angle GMR = \angle CMB = 2\angle A = 180° - \angle GPR$. Therefore $GPRM$ is cyclic and, by power of a point, $AM \cdot AR = AG \cdot AP$. Since ACR is isosceles, D is the midpoint of AR and thus, since M is the midpoint of AB, it follows that $AM \cdot AR = AD \cdot AB = AG \cdot AP$. Therefore $DGPB$ is cyclic, implying the result as in Solution 1.

4. Let n be a positive integer. For any positive integer j and positive real number r, define

$$f_j(r) = \min(jr, n) + \min\left(\frac{j}{r}, n\right), \quad \text{and} \quad g_j(r) = \min(\lceil jr \rceil, n) + \min\left(\left\lceil \frac{j}{r} \right\rceil, n\right),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Prove that

$$\sum_{j=1}^{n} f_j(r) \leq n^2 + n \leq \sum_{j=1}^{n} g_j(r).$$

Solution 1: We first prove the left hand side inequality. We begin by drawing an $n \times n$ board, with corners at $(0,0), (n,0), (0,n)$ and (n,n) on the Cartesian plane.

Consider the line ℓ with slope r passing through $(0,0)$. For each $j \in \{1, \ldots, n\}$, consider the point $(j, \min(jr, n))$. Note that each such point either lies on ℓ or the top edge of the board. In the j^{th} column from the left, draw the rectangle of height $\min(jr, n)$. Note that the sum of the n rectangles is equal to the area of the board under the line ℓ plus n triangles (possibly with area 0) each with width at most 1 and whose sum of the heights is at most n. Therefore, the sum of the areas of these n triangles is at most $n/2$. Therefore, $\sum_{j=1}^{n} \min(jr, n)$ is at most the area of the square under ℓ plus $n/2$.

Consider the line with slope $1/r$. By symmetry about the line $y = x$, the area of the square under the line with slope $1/r$ is equal to the area of the square above the line ℓ. Therefore, using the same reasoning as before, $\sum_{j=1}^{n} \min(j/r, n)$ is at most the area of the square above ℓ plus $n/2$.

Therefore, $\sum_{j=1}^{n} f_j(r) = \sum_{j=1}^{n}(\min(jr, n) + \min(\frac{j}{r}, n))$ is at most the area of the board plus n, which is $n^2 + n$. This proves the left hand side inequality.

To prove the right hand side inequality, we will use the following lemma:

Lemma: Consider the line ℓ with slope s passing through $(0,0)$. Then the number of squares on the board that contain an interior point below ℓ is $\sum_{j=1}^{n} \min(\lceil js \rceil, n)$.

Proof of Lemma: For each $j \in \{1, \ldots, n\}$, we count the number of squares in the j^{th} column (from the left) that contain an interior point lying below the line ℓ. The line $x = j$ intersect the line ℓ at (j, js). Hence, since each column contains n squares total, the number of such squares is $\min(\lceil js \rceil, n)$. Summing over all $j \in \{1, 2, \ldots, n\}$ proves the lemma. *End Proof of Lemma*

By the lemma, the rightmost expression of the inequality is equal to the number of squares containing an interior point below the line with slope r plus the number of squares containing an interior point below the line with slope $1/r$. By symmetry about the line $y = x$, the latter number is equal to the number of squares containing an interior point above the line with slope r. Therefore, the rightmost expression of the inequality is equal to the number of squares of the board plus the number of squares of which ℓ passes through the interior. The former is equal to n^2. Hence, to prove the inequality, it suffices to show that every line passes through the interior of at least n squares. Since ℓ has positive slope, each ℓ passes through either n rows and/or n columns. In either case, ℓ passes through the interior of at least n squares. Hence, the right inequality holds. $\square$

Solution 2: We first prove the left inequality. Define the function $f(r) = \sum_{j=1}^{n} f_j(r)$. Note that $f(r) = f(1/r)$ for all $r > 0$. Therefore, we may assume that $r \geq 1$.

Let $m = \lfloor n/r \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Then $\min(jr, n) = jr$ for all $j \in \{1, \ldots, m\}$ and $\min(jr, n) = n$ for all $j \in \{m+1, \ldots, n\}$. Note that since $r \geq 1$, $\min(j/r, n) \leq n$ for all $j \in \{1, \ldots, n\}$. Therefore,

$$f(r) = \sum_{j=1}^{n} f_j(r) = (1 + 2 + \ldots m)r + (n - m)n + (1 + 2 + \ldots + n) \cdot \frac{1}{r}$$

$$= \frac{m(m+1)}{2} \cdot r + \frac{n(n+1)}{2} \cdot \frac{1}{r} + n(n-m) \qquad (1)$$

Then by (**??**), note that $f(r) \leq n^2 + n$ if and only if

$$\frac{m(m+1)r}{2} + \frac{n(n+1)}{2r} \leq n(m+1)$$

if and only if

$$m(m+1)r^2 + n(n+1) \leq 2rn(m+1) \qquad (2)$$

Since $m = \lfloor n/r \rfloor$, there exist a real number b satisfying $0 \leq b < r$ such that $n = mr + b$. Substituting this into (**??**) yields

$$m(m+1)r^2 + (mr + b)(mr + b + 1) \leq 2r(mr + b)(m + 1),$$

if and only if

$$2m^2r^2 + mr^2 + (2mb + m)r + b^2 + b \le 2m^2r^2 + 2mr^2 + 2mbr + 2br,$$

which simplifies to $mr + b^2 + b \le mr^2 + 2br \Leftrightarrow b(b + 1 - 2r) \le mr(r - 1) \Leftrightarrow b((b - r) + (1 - r)) \le mr(r - 1)$. This is true since

$$b((b - r) + (1 - r)) \le 0 \le mr(r - 1),$$

which holds since $r \ge 1$ and $b < r$. Therefore, the left inequality holds.

We now prove the right inequality. Define the function $g(r) = \sum_{j=1}^{n} = g_j(r)$. Note that $g(r) = g(1/r)$ for all $r > 0$. Therefore, we may assume that $r \ge 1$. We will consider two cases; $r \ge n$ and $1 \le r < n$.

If $r \ge n$, then $\min(\lceil jr \rceil, n) = n$ and $\min(\lceil j/r \rceil, n) = 1$ for all $j \in \{1, \dots, n\}$. Hence, $g_j(r) = n + 1$ for all $j \in \{1, \dots, n\}$. Therefore, $g(r) = n(n + 1) = n^2 + n$, implying that the inequality is true.

Now we consider the case $1 \le r < n$. Let $m = \lfloor n/r \rfloor$. Hence, $jr \le n$ for all $j \in \{1, \dots, m\}$, i.e. $\min(\lceil jr, \rceil, n) = \lceil jr \rceil$ and $jr \ge n$ for all $j \in \{m + 1, \dots, n\}$, i.e. $\min(\lceil jr \rceil, n) = n$. Therefore,

$$\sum_{j=1}^{n} \min(\lceil jr \rceil, n) = \sum_{j=1}^{m} \lceil jr \rceil + (n - m)n. \tag{3}$$

We will now consider the second sum $\sum_{j=1}^{n} \min\{\lceil j/r \rceil, n\}$.

Since $r \ge 1$, $\min(\lceil j/r \rceil, n) \le \min(\lceil n/r \rceil, n) \le n$. Therefore, $\min(\lceil j/r \rceil, n) = \lceil j/r \rceil$. Since $m = \lfloor n/r \rfloor$, $\lceil n/r \rceil \le m + 1$. Since $r > 1$, $m < n$, which implies that $m + 1 \le n$. Therefore, $\min\{\lceil j/r \rceil, n\} = \lceil j/r \rceil \le \lceil n/r \rceil \le m + 1$ for all $j \in \{1, \dots, n\}$.

For each positive integer $k \in \{1, \dots, m + 1\}$, we now determine the number of positive integers $j \in \{1, \dots, n\}$ such that $\lceil j/r \rceil = k$. We denote this number by s_k.

Note that $\lceil j/r \rceil = k$ if and only if $k - 1 < j/r \le k$ if and only if $(k - 1)r < j \le \min(kr, n)$, since $j \le n$. We will handle the cases $k \in \{1, \dots, m\}$ and $k = m + 1$ separately. If $k \in \{1, \dots, m\}$, then $\min(kr, n) = kr$, since $r \le m$ and $m = \lfloor n/r \rfloor$.

The set of positive integers j satisfying $(k-1)r < j \leq kr$ is $\{\lfloor (k-1)r \rfloor + 1, \lfloor (k-1)r \rfloor + 2, \ldots, \lfloor kr \rfloor\}$. Hence,

$$s_k = \lfloor rk \rfloor - (\lfloor r(k-1) \rfloor + 1) + 1 = \lfloor rk \rfloor - \lfloor r(k-1) \rfloor$$

for all $k \in \{1, \ldots, m\}$. If $k = m+1$, then $(k-1)r < j \leq \min(kr, n) = n$. The set of positive integers j satisfying $(k-1)r < j \leq kr$ is $\{\lfloor (k-1)r \rfloor + 1, \ldots, n\}$. Then $s_{m+1} = n - \lfloor r(k-1) \rfloor = n - \lfloor mr \rfloor$. Note that this number is non-negative by the definition of m. Therefore, by the definition of s_k, we have

$$\begin{aligned} \sum_{j=1}^{n} \min\left(\left\lceil \frac{j}{r} \right\rceil, n\right) &= \sum_{k=1}^{m+1} k s_k \\ &= \sum_{k=1}^{m} \left(k \left(\lfloor kr \rfloor - \lfloor (k-1)r \rfloor\right)\right) + (m+1)(n - \lfloor rm \rfloor) = (m+1)n - \sum_{k=1}^{m} \lfloor kr \rfloor. \end{aligned} \tag{4}$$

Summing (**??**) and (**??**) yields that

$$g(r) = n^2 + n + \sum_{j=1}^{m} (\lceil jr \rceil - \lfloor jr \rfloor) \geq n^2 + n,$$

which proves the right inequality. □

5. Let O denote the circumcentre of an acute-angled triangle ABC. A circle Γ passing through vertex A intersects segments AB and AC at points P and Q such that $\angle BOP = \angle ABC$ and $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to Γ.

Solution. Let the circumcircle of triangle OBP intersect side BC at the points R and B and let $\angle A$, $\angle B$ and $\angle C$ denote the angles at vertices A, B and C, respectively.

Now note that since $\angle BOP = \angle B$ and $\angle COQ = \angle C$, it follows that

$$\angle POQ = 360° - \angle BOP - \angle COQ - \angle BOC = 360° - (180 - \angle A) - 2\angle A = 180° - \angle A.$$

This implies that $APOQ$ is a cyclic quadrilateral. Since $BPOR$ is cyclic,

$$\angle QOR = 360° - \angle POQ - \angle POR = 360° - (180° - \angle A) - (180° - \angle B) = 180° - \angle C.$$

This implies that $CQOR$ is a cyclic quadrilateral. Since $APOQ$ and $BPOR$ are cyclic,

$$\angle QPR = \angle QPO + \angle OPR = \angle OAQ + \angle OBR = (90° - \angle B) + (90° - \angle A) = \angle C.$$

Since $CQOR$ is cyclic, $\angle QRC = \angle COQ = \angle C = \angle QPR$ which implies that the circumcircle of triangle PQR is tangent to BC. Further, since $\angle PRB = \angle BOP = \angle B$,

$$\angle PRQ = 180° - \angle PRB - \angle QRC = 180° - \angle B - \angle C = \angle A = \angle PAQ.$$

This implies that the circumcircle of PQR is the reflection of Γ in line PQ. By symmetry in line PQ, this implies that the reflection of BC in line PQ is tangent to Γ.

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46^{th} Canadian Mathematical Olympiad

Wednesday, April 2, 2014

CMS SMC

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1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers whose product is 1. Show that the sum

$$\frac{a_1}{1+a_1}+\frac{a_2}{(1+a_1)(1+a_2)}+\frac{a_3}{(1+a_1)(1+a_2)(1+a_3)}+\cdots+\frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$$

is greater than or equal to $\dfrac{2^n-1}{2^n}$.

2. Let m and n be odd positive integers. Each square of an m by n board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of m and n.

3. Let p be a fixed odd prime. A p-tuple $(a_1, a_2, a_3, \ldots, a_p)$ of integers is said to be *good* if

(i) $0 \le a_i \le p-1$ for all i, and

(ii) $a_1 + a_2 + a_3 + \cdots + a_p$ is not divisible by p, and

(iii) $a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_pa_1$ is divisible by p.

Determine the number of good p-tuples.

4. The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$.

5. Fix positive integers n and $k \ge 2$. A list of n integers is written in a row on a blackboard. You can choose a contiguous block of integers, and I will either add 1 to all of them or subtract 1 from all of them. You can repeat this step as often as you like, possibly adapting your selections based on what I do. Prove that after a finite number of steps, you can reach a state where at least $n-k+2$ of the numbers on the blackboard are all simultaneously divisible by k.

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46^{th} Canadian Mathematical Olympiad

Wednesday, April 2, 2014

CMS
SMC

Problems and Solutions

1. Let $a_1, a_2, \dots, a_n$ be positive real numbers whose product is 1. Show that the sum

$$\frac{a_1}{1+a_1}+\frac{a_2}{(1+a_1)(1+a_2)}+\frac{a_3}{(1+a_1)(1+a_2)(1+a_3)}+\cdots+\frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$$

is greater than or equal to $\dfrac{2^n-1}{2^n}$.

Solution. Note for that every positive integer m,

$$\begin{aligned}\frac{a_m}{(1+a_1)(1+a_2)\cdots(1+a_m)} &= \frac{1+a_m}{(1+a_1)(1+a_2)\cdots(1+a_m)}-\frac{1}{(1+a_1)(1+a_2)\cdots(1+a_m)}\\ &= \frac{1}{(1+a_1)\cdots(1+a_{m-1})}-\frac{1}{(1+a_1)\cdots(1+a_m)}.\end{aligned}$$

Therefore, if we let $b_j = (1+a_1)(1+a_2)\cdots(1+a_j)$, with $b_0 = 0$, then by telescoping sums,

$$\sum_{j=1}^{n}\frac{a_j}{(1+a_1)\cdots(1+a_j)}=\sum_{j=1}^{n}\left(\frac{1}{b_{j-1}}-\frac{1}{b_j}\right)=1-\frac{1}{b_n}.$$

Note that $b_n = (1+a_1)(1+a_2)\cdots(1+a_n) \geq (2\sqrt{a_1})(2\sqrt{a_2})\cdots(2\sqrt{a_n}) = 2^n$, with equality if and only if all a_i's equal to 1. Therefore,

$$1-\frac{1}{b_n}\geq 1-\frac{1}{2^n}=\frac{2^n-1}{2^n}.$$

To check that this minimum can be obtained, substituting all $a_i = 1$ to yield

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=\frac{2^{n-1}+2^{n-2}+\ldots+1}{2^n}=\frac{2^n-1}{2^n},$$

as desired.

2. Let m and n be odd positive integers. Each square of an m by n board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of m and n.

Solution. The answer is $m+n-2$ if $m, n \geq 3$ and $\max\{m, n\}$ if one of m, n is equal to 1.

Note that it is not possible that all rows are red-dominated and all columns are blue-dominated. This is true, since the number of rows and columns are both odd, the number of squares is odd. Hence, there are more squares of one color than the other. Without loss of generality, suppose there are more red squares than blue squares. Then it is not possible that for every column, there are more blue squares than red squares. Hence, every column cannot be blue-dominated.

If one of m, n is equal to 1, say m without loss of generality, then by the claim, the answer is less than $n+1$. The example where there are n blue-dominated columns is by painting every square blue. There are 0 red-dominated rows. The sum of the two is $n = \max\{m, n\}$.

Now we handle the case $m, n \geq 3$.

There are m rows and n columns on the board. Hence, the answer is at most $m+n$. We have already shown that the answer cannot be $m+n$.

Since m, n are odd, let $m = 2a-1$ and $n = 2b-1$ for some positive integers a, b. Since $m, n \geq 3$, $a, b \geq 2$. We first show that the answer is not $m+n-1$. By symmetry, it suffices to show that we cannot have all rows red-dominated and all-but-one column blue-dominated. If all rows are red dominated, then each row has at least b red squares. Hence, there are at least $bm = (2a-1)b$ red squares. Since all-but-one column is blue-dominated, there are at least $2b-2$ blue-dominated columns. Each such column then has at least a blue squares. Therefore, there are at least $a(2b-2)$ blue squares. Therefore, the board has at least $(2a-1)b + a(2b-2) = 4ab - b - 2a$ squares. But the total number of squares on the board is

$$(2a-1)(2b-1) = 4ab - 2a - 2b + 1 = 4ab - 2a - b - b + 1 < 4ab - 2a - b,$$

which is true since $b \geq 2$. This is a contradiction. Therefore, the answer is less than $m+n-1$.

We now claim that there is a colouring of the board such that the number of blue-dominated columns plus the number of red-dominated rows is $m+n-2$; Colour the first column entirely red, and the first row, minus the top-left corner, entirely blue. The remaining uncoloured square is an even-by-even board. Colour the remaining board in an alternating pattern (i.e. checkerboard pattern). Hence, on this even-by-even board, each row has the same number of red squares as blue squares and each column has the same number of red squares as blue squares. Then on the whole board, since the top row, minus the top-left square is blue, all columns, but the leftmost column, are blue-dominated. Hence, there are $n-1$ blue-dominated columns. Since the left column is red, all rows but the top row are red dominated. Hence, there are $m-1$ red-dominated rows. The sum of these two quantities is $m+n-2$, as desired.

3. Let p be a fixed odd prime. A p-tuple $(a_1, a_2, a_3, \dots, a_p)$ of integers is said to be *good* if

(i) $0 \le a_i \le p-1$ for all i, and

(ii) $a_1 + a_2 + a_3 + \cdots + a_p$ is not divisible by p, and

(iii) $a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_pa_1$ is divisible by p.

Determine the number of good p-tuples.

Solution. Let S be the set of all sequences $(b_1, b_2, \dots, b_p)$ of numbers from the set $\{0, 1, 2, \dots, p-1\}$ such that $b_1 + b_2 + \cdots + b_p$ is not divisible by p. We show that $|S| = p^p - p^{p-1}$. For let $b_1, b_2, \dots, b_{p-1}$ be an arbitrary sequence of numbers chosen from $\{0, 1, 2, \dots, p-1\}$. There are exactly $p-1$ choices for b_p such that $b_1 + b_2 + \cdots + b_{p-1} + b_p \not\equiv 0 \pmod{p}$, and therefore $|S| = p^{p-1}(p-1) = p^p - p^{p-1}$.

Now it will be shown that the number of good sequences in S is $\frac{1}{p}|S|$. For a sequence $B = (b_1, b_2, \dots, b_p)$ in S, define the sequence $B_k = (a_1, a_2, \dots, a_p)$ by

$$a_i = b_i - b_1 + k \bmod p$$

for $1 \le i \le p$. Now note that B in S implies that

$$a_1 + a_2 + \cdots + a_p \equiv (b_1 + b_2 + \cdots + b_p) - pb_1 + pk \equiv (b_1 + b_2 + \cdots + b_p) \not\equiv 0 \pmod{p}$$

and therefore B_k is in S for all non-negative k. Now note that B_k has first element k for all $0 \le k \le p-1$ and therefore the sequences $B_0, B_1, \dots, B_{p-1}$ are distinct.

Now define the *cycle* of B as the set $\{B_0, B_1, \ldots, B_{p-1}\}$. Note that B is in its own cycle since $B = B_k$ where $k = b_1$. Now note that since every sequence in S is in exactly one cycle, S is the disjoint union of cycles.

Now it will be shown that exactly one sequence per cycle is good. Consider an arbitrary cycle $B_0, B_1, \ldots, B_{p-1}$, and let $B_0 = (b_1, b_2, \ldots, b_p)$ where $b_0 = 0$, and note that $B_k = (b_1 + k, b_2 + k, \ldots, b_p + k) \bmod p$. Let $u = b_1 + b_2 + \cdots + b_p$, and $v = b_1b_2 + b_2b_3 + \cdots + b_pb_1$ and note that $(b_1 + k)(b_2 + k) + (b_2 + k)(b_3 + k)) + \cdots + (b_p + k)(b_1 + k) = u + 2kv \bmod p$ for all $0 \leq k \leq p - 1$. Since $2v$ is not divisible by p, there is exactly one value of k with $0 \leq k \leq p - 1$ such that p divides $u + 2kv$ and it is exactly for this value of k that B_k is good. This shows that exactly one sequence per cycle is good and therefore that the number of good sequences in S is $\frac{1}{p}|S|$, which is $p^{p-1} - p^{p-2}$.

4. The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$.

Solution. . Let Γ be the circumcircle of quadrilateral $ABCD$. Let $\alpha = \angle PAB = \angle PBC \angle PCD = \angle PDA$ and let T_1 , T_2 , T_3 and T_4 denote the circumcircles of triangles APD, BPC, APB and CPD, respectively. Let M be the intersection of T_1 with line RP and let N be the intersection of T_3 with line SP. Also let X denote the intersection of diagonals AC and BD.

By power of a point for circles T_1 and Γ, it follows that $RM \cdot RP = RA \cdot RD = RB \cdot RC$ which implies that the quadrilateral $BMPC$ is cyclic and M lies on T_2. Therefore $\angle PMB = \angle PCB = \alpha = \angle PAB = \angle DMP$ where all angles are directed. This implies that M lies on the diagonal BD and also that $\angle XMP = \angle DMP = \alpha$. By a symmetric argument applied to S, T_3 and T_4, it follows that N lies on T_4 and that N lies on diagonal AC with $\angle XNP = \alpha$. Therefore $\angle XMP = \angle XNP$ and X, M, P and N are concyclic. This implies that the angle formed by lines MP and NP is equal to one of the angles formed by lines MX and NX. The fact that M lies on BD and RP and N lies on AC and SP now implies the desired result.

5. Fix positive integers n and $k \geq 2$. A list of n integers is written in a row on a blackboard. You can choose a contiguous block of integers, and I will either add 1 to all of them or subtract 1 from all of them. You can repeat this step as often as you like, possibly adapting your selections based on what I do. Prove that after a finite number of steps, you can reach a state where at least $n - k + 2$ of the numbers on the blackboard are all simultaneously divisible by k.

Solution. We will think of all numbers as being residues mod k. Consider the following strategy:

- If there are less than $k-1$ non-zero numbers, then stop.
- If the first number is 0, then recursively solve on the remaining numbers.
- If the first number is j with $0 < j < k$, then choose the interval stretching from the first number to the jth-last non-zero number.

First note that this strategy is indeed well defined. The first number must have value between 0 and $k-1$, and if we do not stop immediately, then there are at least $k-1$ non-zero numbers, and hence the third step can be performed.

For each j with $1 \leq j \leq k-2$, we claim the first number can take on the value of j at most a finite number of times without taking on the value of $j-1$ in between. If this were to fail, then every time the first number became j, I would have to add 1 to the selected numbers to avoid making it $j-1$. This will always increase the j-th last non-zero number, and that number will never be changed by other steps. Therefore, that number would eventually become 0, and the next last non-zero number would eventually become zero, and so on, until the first number itself becomes the j-th last non-zero number, at which point we are done since $j \leq k-2$.

Rephrasing slightly, if $1 \leq j \leq k-2$, the first number can take on the value of j at most a finite number of times between each time it takes on the value of $j-1$. It then immediately follows that if the first number can take on the value of $j-1$ at most a finite number of times, then it can also only take on the value of j a finite number of times. However, if it ever takes on the value of 0, we have already reduced the problem to $n-1$, so we can assume that never happens. It then follows that the first number can take on all the values $0, 1, 2, \ldots, k-2$ at most a finite number of times.

Finally, every time the first number takes on the value of $k-1$, it must subsequently take on the value of $k-2$ or 0, and so that can also happen only finitely many times.

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2015 Canadian Mathematical Olympiad
[version of January 28, 2015]

Notation: If V and W are two points, then VW denotes the line segment with endpoints V and W as well as the length of this segment.

1. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Find all functions f, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2 + n$ for every positive integer n.

2. Let ABC be an acute-angled triangle with altitudes AD, BE, and CF. Let H be the orthocentre, that is, the point where the altitudes meet. Prove that
$$\frac{AB \cdot AC + BC \cdot BA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2 .$$

3. On a $(4n+2) \times (4n+2)$ square grid, a turtle can move between squares sharing a side. The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started. In terms of n, what is the largest positive integer k such that there must be a row or column that the turtle has entered at least k distinct times?

4. Let ABC be an acute-angled triangle with circumcenter O. Let Γ be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that $BP \cdot CQ = AP \cdot AQ$. Prove that Γ is tangent to the circumcircle of triangle BOC.

5. Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

Solutions to 2015 CMO (DRAFT—as of April 6, 2015)

Problem 1. *Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Find all functions f, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2 + n$ for every positive integer n.*

Solution. The only such function is $f(n) = n$.

Assume that f satisfies the given condition. It will be shown by induction that $f(n) = n$ for all $n \in \mathbb{N}$. Substituting $n = 1$ yields that $0 < f(1)f(f(1)) < 2$ which implies the base case $f(1) = 1$. Now assume that $f(k) = k$ for all $k < n$ and assume for contradiction that $f(n) \neq n$.

On the one hand, if $f(n) \leq n-1$ then $f(f(n)) = f(n)$ and $f(n)f(f(n)) = f(n)^2 \leq (n-1)^2$ which is a contradiction. On the other hand, if $f(n) \geq n+1$ then there are several ways to proceed.

Method 1: Assume $f(n) = M \geq n+1$. Then $(n+1)f(M) \leq f(n)f(f(n)) < n^2 + n$. Therefore $f(M) < n$, and hence $f(f(M)) = f(M)$ and $f(M)f(f(M)) = f(M)^2 < n^2 \leq (M-1)^2$, which is a contradiction. This completes the induction. $\square$

Method 2: First note that if $|a-b| > 1$, then the intervals $((a-1)^2, a^2+a)$ and $((b-1)^2, b^2+b)$ are disjoint which implies that $f(a)$ and $f(b)$ cannot be equal.

Assuming $f(n) \geq n+1$, it follows that $f(f(n)) < \frac{n^2+n}{f(n)} \leq n$. This implies that for some $a \leq n-1$, $f(a) = f(f(n))$ which is a contradiction since $|f(n) - a| \geq n + 1 - a \geq 2$. This completes the induction. $\square$

Method 3: Assuming $f(n) \geq n+1$, it follows that $f(f(n)) < \frac{n^2+n}{f(n)} \leq n$ and $f(f(f(n))) = f(f(n))$. This implies that $(f(n)-1)^2 < f(f(n))f(f(f(n))) = f(f(n))^2 < f(n)^2 + f(n)$ and therefore that $f(f(n)) = f(n)$ since $f(n)^2$ is the unique square satisfying this constraint. This implies that $f(n)f(f(n)) = f(n)^2 \geq (n+1)^2$ which is a contradiction, completing the induction. $\square$

Problem 2. *Let ABC be an acute-angled triangle with altitudes AD, BE, and CF. Let H be the orthocentre, that is, the point where the altitudes meet. Prove that*

$$\frac{AB \cdot AC + BC \cdot BA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2\,.$$

Solution. Method 1: Let $AB = c$, $AC = b$, and $BC = a$ denote the three side lengths of the triangle.

As $\angle BFH = \angle BDH = 90^\circ$, $FHDB$ is a cyclic quadrilateral. By the Power-of-a-Point Theorem, $AH \cdot AD = AF \cdot AB$. (We can derive this result in other ways: for example, see Method 2, below.)

Since $AF = AC \cdot \cos\angle A$, we have $AH \cdot AD = AC \cdot AB \cdot \cos\angle A = bc\cos\angle A$.

By the Cosine Law, $\cos\angle A = \dfrac{b^2+c^2-a^2}{2bc}$, which implies that $AH \cdot AD = \dfrac{b^2+c^2-a^2}{2}$.

By symmetry, we can show that $BH \cdot BE = \dfrac{a^2+c^2-b^2}{2}$ and $CH \cdot CF = \dfrac{a^2+b^2-c^2}{2}$. Hence,

$$\begin{aligned} AH \cdot AD + BH \cdot BE + CH \cdot CF &= \frac{b^2+c^2-a^2}{2} + \frac{a^2+c^2-b^2}{2} + \frac{a^2+b^2-c^2}{2} \\ &= \frac{a^2+b^2+c^2}{2}. \end{aligned} \tag{1}$$

Our desired inequality, $\dfrac{AB \cdot AC + BC \cdot BA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2$, is equivalent to the inequality $\dfrac{cb+ac+ba}{\frac{a^2+b^2+c^2}{2}} \leq 2$, which simplifies to $2a^2+2b^2+2c^2 \geq 2ab+2bc+2ca$.

But this last inequality is easy to prove, as it is equivalent to $(a-b)^2+(a-c)^2+(b-c)^2 \geq 0$.

Therefore, we have established the desired inequality. The proof also shows that equality occurs if and only if $a=b=c$, i.e., $\triangle ABC$ is equilateral. □

Method 2: Observe that

$$\frac{AE}{AH} = \cos(\angle HAE) = \frac{AD}{AC} \quad \text{and} \quad \frac{AF}{AH} = \cos(\angle HAF) = \frac{AD}{AB}.$$

It follows that

$$AC \cdot AE = AH \cdot AD = AB \cdot AF\,.$$

By symmetry, we similarly have

$$BC \cdot BD = BH \cdot BE = BF \cdot BA \quad \text{and} \quad CD \cdot CB = CH \cdot CF = CE \cdot CA\,.$$

Therefore

$$\begin{aligned} &2(AH \cdot AD + BH \cdot BE + CH \cdot CF) \\ &= AB(AF+BF) + AC(AE+CE) + BC(BD+CD) \\ &= AB^2 + AC^2 + BC^2. \end{aligned}$$

This proves Equation (1) in Method 1. The rest of the proof is the same as the part of the proof of Method 1 that follows Equation (1). □

Problem 3. *On a $(4n+2)\times(4n+2)$ square grid, a turtle can move between squares sharing a side. The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started. In terms of n, what is the largest positive integer k such that there must be a row or column that the turtle has entered at least k distinct times?*

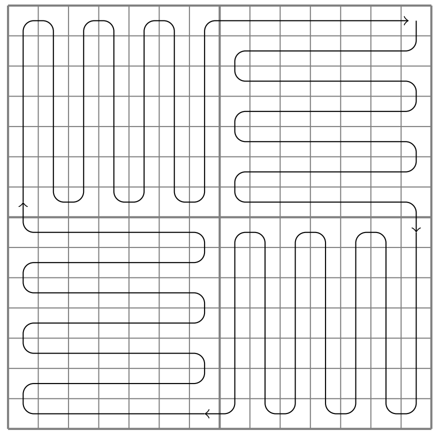
Solution. We shall prove that the answer is $2n+2$. Number the rows in increasing order, from top to bottom, and number the columns from left to right. By symmetry, we may (and shall) assume that the turtle starts in the top right corner square.

First we shall prove that some row or column must be entered at least $2n+2$ times. Let $m = 4n+2$. First note that each time the turtle moves, she enters either a row or a column. Let r_i denote the number of times the turtle enters row i, and let c_i be similarly defined for column i. Since the turtle moves m^2 times,

$$r_1 + r_2 + \cdots + r_m + c_1 + c_2 + \cdots + c_m = m^2.$$

Now note that each time the turtle enters column 1, the next column she enters must be column 2. Therefore c_1 is equal to the number of times the turtle enters column 2 from column 1. Furthermore, the turtle must enter column 2 from column 3 at least once, which implies that $c_2 > c_1$. Therefore since the $2m$ terms r_i and c_i are not all equal, one must be strictly greater than $m^2/(2m) = 2n+1$ and therefore at least $2n+2$.

Now we construct an example to show that it is possible that no row or column is entered more than $2n+2$ times. Partition the square grid into four $(2n+1) \times (2n+1)$ quadrants A, B, C, and D, containing the upper left, upper right, lower left, and lower right corners, respectively. The turtle begins at the top right corner square of B, moves one square down, and then moves left through the whole second row of B. She then moves one square down and moves right through the whole third row of B. She continues in this pattern, moving through each remaining row of B in succession and moving one square down when each row is completed. Since $2n+1$ is odd, the turtle ends at the bottom right corner of B. She then moves one square down into D and through each column of D in turn, moving one square to the left when each column is completed. She ends at the lower left corner of D and moves left into C and through the rows of C, moving one square up when each row is completed, ending in the upper left corner of C. She then enters A and moves through the columns of A, moving one square right when each column is completed. This takes her to the upper right corner of A, whereupon she enters B and moves right through the top row of B, which returns her to her starting point. Each row passing through A and B is entered at most $2n+1$ times in A and once in B, and thus at most $2n+2$ times in total. Similarly, each row and column in the grid is entered at most $2n+2$ times by this path. (See figure below.)



Problem 3: the case $n = 3$

$\square$

Problem 4. *Let ABC be an acute-angled triangle with circumcenter O. Let Γ be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that $BP \cdot CQ = AP \cdot AQ$. Prove that Γ is tangent to the circumcircle of triangle BOC.*

Solution. Let ω be the circumcircle of BOC. Let M be the point diametrically opposite to O on ω and let the line AM intersect ω at M and K. Since O is the circumcenter of ABC, it follows that $OB = OC$ and therefore that O is the midpoint of the arc $\widehat{BOC}$ of ω. Since M is diametrically opposite to O, it follows that M is the midpoint of the arc $\widehat{BMC}$ of ω. This implies since K is on ω that KM is the bisector of $\angle BKC$.Since K is on ω, this implies that $\angle BKM = \angle CKM$, i.e. KM is the bisector of $\angle BKC$.

Since O is the circumcenter of ABC, it follows that $\angle BOC = 2\angle BAC$. Since B, K, O and C all lie on ω, it also follows that $\angle BKC = \angle BOC = 2\angle BAC$. Since KM bisects $\angle BKC$, it follows that $\angle BKM = \angle CKM = \angle BAC$. The fact that A, K and M lie on a line therefore implies that $\angle AKB = \angle AKC = 180^\circ - \angle BAC$. Now it follows that

$$\angle KBA = 180^\circ - \angle AKB - \angle KAB = \angle BAC - \angle KAB = \angle KAC.$$

This implies that triangles KBA and KAC are similar. Rearranging the condition in the problem statement yields that $BP/AP = AQ/CQ$ which, when combined with the fact that KBA and KAC are similar, implies that triangles KPA and KQC are similar. Therefore $\angle KPA = \angle KQC = 180^\circ - \angle KQA$ which implies that K lies on Γ.

Now let S denote the centre of Γ and let T denote the centre of ω. Note that T is the midpoint of segment OM and that TM and AS, which are both perpendicular to BC, are parallel. This implies that $\angle KMT = \angle KAS$ since A, K and M are collinear. Further, since KTM and KSA are isosceles triangles, it follows that $\angle TKM = \angle KMT$ and $\angle SKA =$

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$\angle KSA$. Therefore $\angle TKM = \angle SKA$ which implies that S, T and K are collinear. Therefore Γ and ω intersect at a point K which lies on the line ST connecting the centres of the two circles. This implies that the circles Γ and ω are tangent at K. $\square$

Problem 5. *Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.*

Solution. First suppose $b \equiv \pm a \pmod{p}$ and $c \equiv \pm b \pmod{p}$. Then, for any n, we have $a^n + b^n + c^n \equiv \pm a^n$ or $\pm 3a^n \pmod{p}$. We are given that $p \neq 3$ (since $\frac{3-1}{2}$ is not prime) and $p \not| a$, so it follows that $a^n + b^n + c^n \not\equiv 0 \pmod{p}$. The claim is trivial in this case. Otherwise, we may assume without loss of generality that $b \not\equiv \pm a \pmod{p} \implies ba^{-1} \not\equiv \pm 1 \pmod{p}$.

Now let $q = \frac{p-1}{2}$. By Fermat's little theorem, we know that the order of ba^{-1} mod p divides $p - 1 = 2q$. However, since $ba^{-1} \not\equiv \pm 1 \pmod{p}$, the order of ba^{-1} does not divide 2. Thus, the order must be either q or $2q$.

Next, let S denote the set of positive integers $n < p$ such that $a^n + b^n + c^n \equiv 0 \pmod{p}$, and let s_t denote the number of ordered pairs $(i, j) \subset S$ such that $i - j \equiv t \pmod{p-1}$.

Lemma: If t is a positive integer less than $2q$ and not equal to q, then $s_t \leq 2$.

Proof: Consider $i, j \in S$ with $j - i \equiv t \pmod{p-1}$. Then we have

$$\begin{aligned} & a^i + b^i + c^i \equiv 0 \pmod{p} \\ \implies \quad & a^i c^{j-i} + b^i c^{j-i} + c^j \equiv 0 \pmod{p} \\ \implies \quad & a^i c^{j-i} + b^i c^{j-i} - a^j - b^j \equiv 0 \pmod{p} \\ \implies \quad & a^i \cdot \left(c^t - a^t\right) \equiv b^i \cdot \left(b^t - c^t\right) \pmod{p}. \end{aligned}$$

If $c^t \equiv a^t \pmod{p}$, then this implies $c^t \equiv b^t \pmod{p}$ as well, so $\left(ab^{-1}\right)^t \equiv 1 \pmod{p}$. However, we know the order of ab^{-1} is q or $2q$, and $q \not| t$, so this impossible. Thus, we can write

$$(ab^{-1})^i \equiv \left(b^t - c^t\right) \cdot \left(c^t - a^t\right)^{-1} \pmod{p}.$$

For a fixed t, the right-hand side of this equation is fixed, so $(ab^{-1})^i$ is also fixed. Since the order of ab^{-1} is either q or $2q$, it follows that there are at most 2 solutions for i, and the lemma is proven. $\square$

Now, for each element i in S, there are at least $|S| - 2$ other elements that differ from i by a quantity other than $q \pmod{p-1}$. Therefore, the lemma implies that

$$\begin{aligned} |S| \cdot (|S| - 2) \ &\leq\ \sum_{t \neq q} s_t \ \leq\ 2 \cdot (p-2) \\ \implies \quad (|S| - 1)^2 \ &\leq\ 2p - 3 \\ \implies \quad |S| \ &<\ \sqrt{2p} + 1. \end{aligned}$$

$\square$

Problems for 2016 CMO – as of Feb 12, 2016

1. The integers $1, 2, 3, \ldots, 2016$ are written on a board. You can choose any two numbers on the board and replace them with their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

   (a) Prove that there is a sequence of replacements that will make the final number equal to 2.

   (b) Prove that there is a sequence of replacements that will make the final number equal to 1000.

2. Consider the following system of 10 equations in 10 real variables $v_1, \ldots, v_{10}$:

$$v_i \;=\; 1 + \frac{6\, v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \qquad (i = 1, \ldots, 10).$$

   Find all 10-tuples $(v_1, v_2, \ldots, v_{10})$ that are solutions of this system.

3. Find all polynomials $P(x)$ with integer coefficients such that $P(P(n)+n)$ is a prime number for infinitely many integers n.

4. <u>Lavaman versus the Flea</u>. Let A, B, and F be positive integers, and assume $A < B < 2A$. A flea is at the number 0 on the number line. The flea can move by jumping to the right by A or by B. Before the flea starts jumping, Lavaman chooses finitely many intervals $\{m+1, m+2, \ldots, m+A\}$ consisting of A consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

   (i) any two distinct intervals are disjoint and not adjacent;
   (ii) there are at least F positive integers with no lava between any two intervals; and
   (iii) no lava is placed at any integer less than F.

Prove that the smallest F for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does, is $F = (n-1)A + B$, where n is the positive integer such that $\dfrac{A}{n+1} \leq B - A < \dfrac{A}{n}$.

5. Let $\triangle ABC$ be an acute-angled triangle with altitudes AD and BE meeting at H. Let M be the midpoint of segment AB, and suppose that the circumcircles of $\triangle DEM$ and $\triangle ABH$ meet at points P and Q with P on the same side of CH as A. Prove that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.

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Draft Solutions for 2016 CMO – April 27, 2016

1. The integers $1, 2, 3, \ldots, 2016$ are written on a board. You can choose any two numbers on the board and replace them with their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

   (a) Prove that there is a sequence of replacements that will make the final number equal to 2.

   (b) Prove that there is a sequence of replacements that will make the final number equal to 1000.

Solution: (a) First replace 2014 and 2016 with 2015, and then replace the two copies of 2015 with a single copy. This leaves us with $\{1, 2, \ldots, 2013, 2015\}$. From here, we can replace 2013 and 2015 with 2014 to get $\{1, 2, \ldots, 2012, 2014\}$. We can then replace 2012 and 2014 with 2013, and so on, until we eventually get to $\{1, 3\}$. We finish by replacing 1 and 3 with 2.

(b) Using the same construction as in (a), we can find a sequence of replacements that reduces $\{a, a+1, \ldots, b\}$ to just $\{a+1\}$. Similarly, can also find a sequence of replacements that reduces $\{a, a+1, \ldots, b\}$ to just $\{b-1\}$.

In particular, we can find sequences of replacements that reduce $\{1, 2, \ldots, 999\}$ to just $\{998\}$, and that reduce $\{1001, 1002, \ldots, 2016\}$ to just $\{1002\}$. This leaves us with $\{998, 1000, 1002\}$. We can replace 998 and 1002 with a second copy of 1000, and then replace the two copies of 1000 with a single copy to complete the construction.

2. Consider the following system of 10 equations in 10 real variables $v_1, \ldots, v_{10}$:

$$v_i \;=\; 1 + \frac{6\,v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \qquad (i = 1, \ldots, 10).$$

Find all 10-tuples $(v_1, v_2, \ldots, v_{10})$ that are solutions of this system.

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Solution:
For a particular solution $(v_1, v_2, \ldots, v_{10})$, let $s = v_1^2 + v_2^2 + \cdots + v_{10}^2$. Then

$$v_i = 1 + \frac{6v_i^2}{s} \quad \Rightarrow \quad 6v_i^2 - sv_i + s = 0.$$

Let a and b be the roots of the quadratic $6x^2 - sx + s = 0$, so for each i, $v_i = a$ or $v_i = b$. We also have $ab = s/6$ (by Vieta's formula, for example).

If all the v_i are equal, then

$$v_i = 1 + \frac{6}{10} = \frac{8}{5}$$

for all i. Otherwise, let $5 + k$ of the v_i be a, and let $5 - k$ of the v_i be b, where $0 < k \leq 4$. Then by the AM-GM inequality,

$$6ab = s = (5 + k)a^2 + (5 - k)b^2 \geq 2ab\sqrt{25 - k^2}.$$

From the given equations, $v_i \geq 1$ for all i, so a and b are positive. Then $\sqrt{25 - k^2} \leq 3 \Rightarrow 25 - k^2 \leq 9 \Rightarrow k^2 \geq 16 \Rightarrow k = 4$. Hence, $6ab = 9a^2 + b^2 \Rightarrow (b - 3a)^2 = 0 \Rightarrow b = 3a$.

Adding all given ten equations, we get

$$v_1 + v_2 + \cdots + v_{10} = 16.$$

But $v_1 + v_2 + \cdots + v_{10} = 9a + b = 12a$, so $a = 16/12 = 4/3$ and $b = 4$. Therefore, the solutions are $(8/5, 8/5, \ldots, 8/5)$ and all ten permutations of $(4/3, 4/3, \ldots, 4/3, 4)$.

3. Find all polynomials $P(x)$ with integer coefficients such that $P(P(n) + n)$ is a prime number for infinitely many integers n.

   Answer: $P(n) = p$ where p is a prime number and $P(n) = -2n + b$ where b is odd.

   Solution: Note that if $P(n) = 0$ then $P(P(n) + n) = P(n) = 0$ which is not prime. Let $P(x)$ be a degree k polynomial of the form $P(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0$ and note that if $P(n) \neq 0$ then

$$\begin{aligned} P(P(n) + n) - P(n) &= \\ &a_k[(P(n) + n)^k - n^k] + a_{k-1}[(P(n) + n)^{k-1} - n^{k-1}] + \cdots + a_1 P(n) \end{aligned}$$

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which is divisible by $(P(n)+n)-n = P(n)$. Therefore if $P(P(n)+n)$ is prime then either $P(n) = \pm 1$ or $P(P(n)+n) = \pm P(n) = p$ for some prime number p. Since $P(x)$ is a polynomial, it follows that $P(n) = \pm 1$ for only finitely many integers n. Therefore either $P(n) = P(P(n)+n)$ for infinitely many integers n or $P(n) = -P(P(n)+n)$ for infinitely many integers n. Suppose that $P(n) = P(P(n)+n)$ for infinitely many integers n. This implies that the polynomial $P(P(x)+x) - P(x)$ has infinitely many roots and thus is identically zero. Therefore $P(P(x)+x) = P(x)$ holds identically. Now note that if $k \geq 2$ then $P(P(x)+x)$ has degree k^2 while $P(x)$ has degree k, which is not possible. Therefore $P(x)$ is at most linear with $P(x) = ax+b$ for some integers a and b. Now note that

$$P(P(x)+x) = a(a+1)x + ab + b$$

and thus $a = a(a+1)$ and $ab+b = b$. It follows that $a = 0$ which leads to the solution $P(n) = p$ where p is a prime number. By the same argument if $P(n) = -P(P(n)+n)$ for infinitely many integers n then $P(x) = -P(P(x)+x)$ holds identically and $P(x)$ is linear with $P(x) = ax+b$. In this case it follows that $a = -a(a+1)$ and $ab+b = -b$. This implies that either $a = 0$ or $a = -2$. If $a = -2$ then $P(n) = -2n+b$ which is prime for some integers n only if b is odd. Note that in this case $P(P(n)+n) = 2n-b$ which is indeed prime for infinitely many integers n as long as b is odd. $\square$

4. Lavaman versus the Flea. Let A, B, and F be positive integers, and assume $A < B < 2A$. A flea is at the number 0 on the number line. The flea can move by jumping to the right by A or by B. Before the flea starts jumping, Lavaman chooses finitely many intervals $\{m+1, m+2, \ldots, m+A\}$ consisting of A consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

   (i) any two distinct intervals are disjoint and not adjacent;
   (ii) there are at least F positive integers with no lava between any two intervals; and
   (iii) no lava is placed at any integer less than F.

   Prove that the smallest F for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does,

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is $F = (n-1)A + B$, where n is the positive integer such that $\dfrac{A}{n+1} \le B - A < \dfrac{A}{n}$.

Solution: Let $B = A + C$ where $A/(n+1) \le C < A/n$.

First, here is an informal sketch of the proof.

Lavaman's strategy: Use only safe intervals with $nA + C - 1$ integers. The flea will start at position $[1, C]$ from the left, which puts him at position $[nA, nA + C - 1]$ from the right. After $n - 1$ jumps, he will still have $nA - (n-1)(A + C) = A - (n-1)C > C$ distance to go, which is not enough for a big jump to clear the lava. Thus, he must do at least n jumps in the safe interval, but that's possible only with all small jumps, and furthermore is impossible if the starting position is C. This gives him starting position 1 higher in the next safe interval, so sooner or later the flea is going to hit the lava.

Flea's strategy: The flea just does one interval at a time. If the safe interval has at least $nA+C$ integers in it, the flea has distance $d > nA$ to go to the next lava when it starts. Repeatedly do big jumps until d is between 1 and $C \bmod A$, then small jumps until the remaining distance is between 1 and C, then a final big jump. This works as long as the first part does. However, we get at least n big jumps since floor$((d-1)/A)$ can never go down two from a big jump (or we'd be done doing big jumps), so we get n big jumps, and thus we are good if $d \bmod A$ is in any of $[1, C]$, $[C + 1, 2C]$, $\ldots [nC + 1, (n + 1)C]$, but that's everything. $\square$

Let $C = B - A$. We shall write our intervals of lava in the form $(L_i, R_i] = \{L_i + 1, L_i + 2, \ldots, R_i\}$, where $R_i = L_i + A$ and $R_{i-1} < L_i$ for every $i \ge 1$. We also let $R_0 = 0$. We shall also represent a path for the flea as a sequence of integers $x_0, x_1, x_2, \ldots$ where $x_0 = 0$ and $x_j - x_{j-1} \in \{A, B\}$ for every $j \ge 0$.

Now here is a detailed proof.

First, assume $F < (n-1)A + B$ $(= nA + C)$: we must prove that Lavaman has a winning strategy. Let $L_i = R_{i-1} + nA + C - 1$ for every $i \ge 1$. (Observe that $nA + C - 1 \ge F$.)

Assume that the flea has an infinite path that avoids all the lava, which means that $x_j \notin (L_i, R_i]$ for all $i, j \geq 1$. For each $i \geq 1$, let

$$M_i = \max\{x_j : x_j \leq L_i\}, \quad m_i = \min\{x_j : x_j > R_i\}, \quad \text{and} \quad J(i) = \max\{j : x_j \leq L_i\}.$$

Also let $m_0 = 0$. Then for $i \geq 1$ we have

$$M_i = x_{J(i)} \quad \text{and} \quad m_i = x_{J(i)+1}.$$

Also, for every $i \geq 1$, we have
(a) $m_i = M_i + B$ (because $M_i + A \leq L_i + A = R_i$);
(b) $L_i \geq M_i > L_i - C$ (since $M_i = m_i - B > R_i - B = L_i + A - B$); and
(c) $R_i < m_i \leq R_i + C$ (since $m_i = M_i + B \leq L_i + B = R_i + C$).

<u>Claim 1</u>: $J(i+1) = J(i) + n + 1$ for every $i \geq 1$. (That is, after jumping over one interval of lava, the flea must make exactly n jumps before jumping over the next interval of lava.)
Proof:

$$\begin{aligned} x_{J(i)+n+1} &\leq x_{J(i)+1} + Bn \\ &= m_i + Bn \\ &< R_i + C + \left(A + \frac{A}{n}\right)n \\ &= L_{i+1} + A + 1. \end{aligned}$$

Because of the strict inequality, we have $x_{J(i)+n+1} \leq R_{i+1}$, and hence $x_{J(i)+n+1} \leq L_{i+1}$. Therefore $J(i) + n + 1 \leq J(i+1)$. Next, we have

$$\begin{aligned} x_{J(i)+n+1} &\geq x_{J(i)+1} + An \\ &= m_i + An \\ &> R_i + An \\ &= L_{i+1} - C + 1 \\ &> L_{i+1} - A + 1 \quad \text{(since } C < A\text{)}. \end{aligned}$$

Therefore $x_{J(i)+n+2} \geq x_{J(i)+n+1} + A > L_{i+1}$, and hence $J(i+1) < J(i) + n + 2$. Claim 1 follows.

<u>Claim 2</u>: $x_{j+1} - x_j = A$ for all $j = J(i) + 1, \ldots, J(i+1) - 1$, for all $i \geq 1$. (That is, the n intermediate jumps of Claim 1 must all be of

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length A.)
Proof: If Claim 2 is false, then

$$\begin{aligned} M_{i+1} \;=\; x_{J(i+1)} \;=\; x_{J(i)+n+1} &\geq x_{J(i)+1} + (n-1)A + B \\ &> R_i + nA + C \\ &= L_{i+1} + 1 \\ &> M_{i+1} \end{aligned}$$

which is a contradiction. This proves Claim 2.

We can now conclude that

$$x_{J(i+1)+1} \;=\; x_{J(i)+n+2} \;=\; x_{J(i)+1} + nA + B\,;$$

$$\text{i.e.,} \qquad m_{i+1} \;=\; m_i + nA + B \qquad \text{for each } i \geq 1.$$

Therefore

$$\begin{aligned} m_{i+1} - R_{i+1} &= m_i + nA + B \;-\; (R_i + nA + C - 1 + A) \\ &= m_i - R_i + 1\,. \end{aligned}$$

Hence

$$C \;\geq\; m_{C+1} - R_{C+1} \;=\; m_1 - R_1 + C \;>\; C$$

which is a contradiction. Therefore no path for the flea avoids all the lava. We observe that Lavaman only needs to put lava on the first $C+1$ intervals.

Now assume $F \geq (n-1)A + B$. We will show that the flea can avoid all the lava. We shall need the following result:

Claim 3: Let $d \geq nA$. Then there exist nonnegative integers s and t such that $sA + tB \in (d - C, d]$.

We shall prove this result at the end.

First, observe that $L_1 \geq nA$. By Claim 3, it is possible for the flea to make a sequence of jumps starting from 0 and ending at a point of $(L_1 - C, L_1]$. From any point of this interval, a single jump of size B takes the flea over $(L_1, R_]$ to a point in $(R_1, R_1 + C]$, which corresponds to the point $x_{J(1)+1}$ $(= m_1)$ on the flea's path.

Now we use induction to prove that, for every $i \geq 1$, there is a path such that x_j avoids lava for all $j \leq J(i) + 1$. The case $i = 1$ is done, so

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assume that the assertion holds for a given i. Then $x_{J(i)+1} = m_i \in (R_i, R_i + C]$. Therefore

$$L_{i+1} - m_i \geq R_i + F - (R_i + C) = F - C \geq nA.$$

Applying Claim 3 with $d = L_{i+1} - m_i$ shows that the flea can jump from m_i to a point of $(L_{i+1} - C, L_{i+1}]$. A single jump of size B then takes the flea to a point of $(R_{i+1}, R_{i+1} + C]$ (without visiting $(L_{i+1}, R_{i+1}]$), and this point serves as $x_{J(i+1)+1}$. This completes the induction.

Proof of Claim 3: Let u be the greatest integer that is less than or equal to d/A. Then $u \geq n$ and $uA \leq d < (u+1)A$. For $v = 0, \ldots, n$, let

$$z_v = (u - v)A + vB = uA + vC.$$

Then

$$\begin{aligned} z_0 &= uA \leq d, \\ z_n &= uA + nC = uA + (n+1)C - C \geq (u+1)A - C > d - C. \end{aligned}$$

and $z_{v+1} - z_v = C$ for $v = 0, \ldots, n-1$.

Therefore we must have $z_v \in (d - C, d]$ for some v in $\{0, 1, \ldots, n\}$. □

5. Let $\triangle ABC$ be an acute-angled triangle with altitudes AD and BE meeting at H. Let M be the midpoint of segment AB, and suppose that the circumcircles of $\triangle DEM$ and $\triangle ABH$ meet at points P and Q with P on the same side of CH as A. Prove that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.

Solution:

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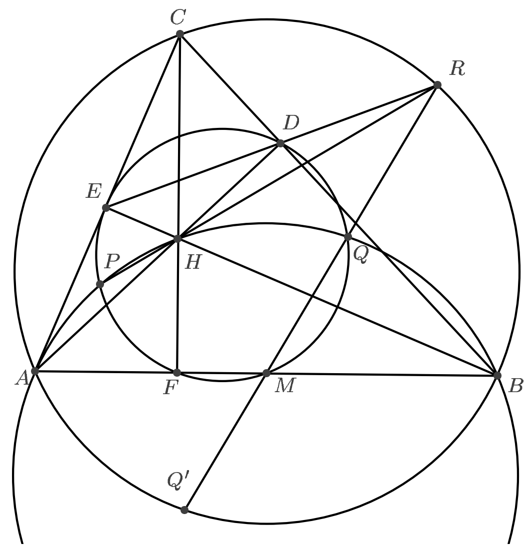

Let R denote the intersection of lines ED and PH. Since quadrilaterals $ECDH$ and $APHB$ are cyclic, we have $\angle RDA = 180° - \angle EDA = 180° - \angle EDH = 180° - \angle ECH = 90° + A$, and $\angle RPA = \angle HPA = 180° - \angle HBA = 90° + A$. Therefore, $APDR$ is cyclic. This in turn implies that $\angle PBE = \angle PBH = \angle PAH = \angle PAD = \angle PRD = \angle PRE$, and so $PBRE$ is also cyclic.

Let F denote the base of the altitude from C to AB. Then D, E, F, and M all lie on the 9-point circle of $\triangle ABC$, and so are cyclic. We also know $APDR$, $PBRE$, $BCEF$, and $ACDF$ are cyclic, which implies $\angle ARB = \angle PRB - \angle PRA = \angle PEB - \angle PDA = \angle PEF + \angle FEB - \angle PDF + \angle ADF = \angle FEB + \angle ADF = \angle FCB + \angle ACF = C$. Therefore, R lies on the circumcircle of $\triangle ABC$.

Now let Q' and R' denote the intersections of line MQ with the circumcircle of $\triangle ABC$, chosen so that Q', M, Q, R' lie on the line in that order. We will show that $R' = R$, which will complete the proof. However, first note that the circumcircle of $\triangle ABC$ has radius $\frac{AB}{2\sin C}$, and the circumcircle of $\triangle ABH$ has radius $\frac{AB}{2\sin \angle AHB} = \frac{AB}{2\sin(180° - C)}$. Thus the two circles have equal radius, and so they must be symmetrical about the point M. In particular, $MQ = MQ'$.

Since $\angle AEB = \angle ADB = 90°$, we furthermore know that M is the circumcenter of both $\triangle AEB$ and $\triangle ADB$. Thus, $MA = ME = MD = MB$. By Power of a Point, we then have $MQ \cdot MR' = MQ' \cdot MR' = MA \cdot MB = MD^2$. In particular, this means that the circumcircle of

$\triangle DR'Q$ is tangent to MD at D, which means $\angle MR'D = \angle MDQ$. Similarly $MQ \cdot MR' = ME^2$, and so $\angle MR'E = \angle MEQ = \angle MDQ = \angle MR'D$. Therefore, R' also lies on the line ED.

Finally, the same argument shows that MP also intersects the circumcircle of $\triangle ABC$ at a point R'' on line ED. Thus, R, R', and R'' are all chosen from the intersection of the circumcircle of $\triangle ABC$ and the line ED. In particular, two of R, R', and R'' must be equal. However, $R'' \neq R$ since MP and PH already intersect at P, and $R'' \neq R'$ since MP and MQ already intersect at M. Thus, $R' = R$, and the proof is complete. $\square$

2017 Canadian Mathematical Olympiad

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Official Problem Set

1. Let a, b, and c be non-negative real numbers, no two of which are equal. Prove that

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}>2.$$

2. Let f be a function from the set of positive integers to itself such that, for every n, the number of positive integer divisors of n is equal to $f(f(n))$. For example, $f(f(6)) = 4$ and $f(f(25)) = 3$. Prove that if p is prime then $f(p)$ is also prime.

3. Let n be a positive integer, and define $S_n = \{1, 2, \ldots, n\}$. Consider a non-empty subset T of S_n. We say that T is balanced if the median of T is equal to the average of T. For example, for $n = 9$, each of the subsets $\{7\}$, $\{2, 5\}$, $\{2, 3, 4\}$, $\{5, 6, 8, 9\}$, and $\{1, 4, 5, 7, 8\}$ is balanced; however, the subsets $\{2, 4, 5\}$ and $\{1, 2, 3, 5\}$ are not balanced. For each $n \geq 1$, prove that the number of balanced subsets of S_n is odd.
(To define the median of a set of k numbers, first put the numbers in increasing order; then the median is the middle number if k is odd, and the average of the two middle numbers if k is even. For example, the median of $\{1, 3, 4, 8, 9\}$ is 4, and the median of $\{1, 3, 4, 7, 8, 9\}$ is $(4 + 7)/2 = 5.5$.)

4. Points P and Q lie inside parallelogram $ABCD$ and are such that triangles ABP and BCQ are equilateral. Prove that the line through P perpendicular to DP and the line through Q perpendicular to DQ meet on the altitude from B in triangle ABC.

5. One hundred circles of radius one are positioned in the plane so that the area of any triangle formed by the centres of three of these circles is at most 2017. Prove that there is a line intersecting at least three of these circles.

2017 Canadian Mathematical Olympiad

Sun Life Financial

Official Solutions

1. Let a, b, and c be non-negative real numbers, no two of which are equal. Prove that

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}>2.$$

Solution: The left-hand side is symmetric with respect to a, b, c. Hence, we may assume that $a > b > c \geq 0$. Note that replacing (a, b, c) with $(a-c, b-c, 0)$ lowers the value of the left-hand side, since the numerators of each of the fractions would decrease and the denominators remain the same. Therefore, to obtain the minimum possible value of the left-hand side, we may assume that $c = 0$.

Then the left-hand side becomes

$$\frac{a^2}{b^2}+\frac{b^2}{a^2},$$

which yields, by the Arithmetic Mean - Geometric Mean Inequality,

$$\frac{a^2}{b^2}+\frac{b^2}{a^2} \;\geq\; 2\sqrt{\frac{a^2}{b^2}\cdot\frac{b^2}{a^2}} \;=\; 2,$$

with equality if and only if $a^2/b^2 = b^2/a^2$, or equivalently, $a^4 = b^4$. Since $a, b \geq 0$, $a = b$. But since no two of a, b, c are equal, $a \neq b$. Hence, equality cannot hold. This yields

$$\frac{a^2}{b^2}+\frac{b^2}{a^2} \;>\; 2.$$

Ultimately, this implies the desired inequality. □

Alternate solution: First, show that

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2} - 2 = \frac{[a(a-b)(a-c)+b(b-a)(b-c)+c(c-a)(c-b)]^2}{[(a-b)(b-c)(c-a)]^2}.$$

Then Schur's Inequality tells us that the numerator of the right-hand side cannot be zero. □

2. Let f be a function from the set of positive integers to itself such that, for every n, the number of positive integer divisors of n is equal to $f(f(n))$. For example, $f(f(6)) = 4$ and $f(f(25)) = 3$. Prove that if p is prime then $f(p)$ is also prime.

   Solution: Let $d(n) = f(f(n))$ denote the number of divisors of n and observe that $f(d(n)) = f(f(f(n))) = d(f(n))$ for all n. Also note that because all divisors of n are distinct positive integers between 1 and n, including 1 and n, and excluding $n-1$ if $n > 2$, it follows that $2 \leq d(n) < n$ for all $n > 2$. Furthermore $d(1) = 1$ and $d(2) = 2$.

   We first will show that $f(2) = 2$. Let $m = f(2)$ and note that $2 = d(2) = f(f(2)) = f(m)$. If $m \geq 2$, then let m_0 be the smallest positive integer satisfying that $m_0 \geq 2$ and $f(m_0) = 2$. It follows that $f(d(m_0)) = d(f(m_0)) = d(2) = 2$. By the minimality of m_0, it follows that $d(m_0) \geq m_0$, which implies that $m_0 = 2$. Therefore if $m \geq 2$, it follows that $f(2) = 2$. It suffices to examine the case in which $f(2) = m = 1$. If $m = 1$, then $f(1) = f(f(2)) = 2$ and furthermore, each prime p satisfies that $d(f(p)) = f(d(p)) = f(2) = 1$ which implies that $f(p) = 1$. Therefore $d(f(p^2)) = f(d(p^2)) = f(3) = 1$ which implies that $f(p^2) = 1$ for any prime p. This implies that $3 = d(p^2) = f(f(p^2)) = f(1) = 2$, which is a contradiction. Therefore $m \neq 1$ and $f(2) = 2$.

   It now follows that if p is prime then $2 = f(2) = f(d(p)) = d(f(p))$ which implies that $f(p)$ is prime. □

   Remark. Such a function exists and can be constructed inductively.

3. Let n be a positive integer, and define $S_n = \{1, 2, \ldots, n\}$. Consider a non-empty subset T of S_n. We say that T is balanced if the median of T is equal to the average of T. For example, for $n = 9$, each of the subsets $\{7\}$, $\{2, 5\}$, $\{2, 3, 4\}$, $\{5, 6, 8, 9\}$, and $\{1, 4, 5, 7, 8\}$ is balanced; however, the subsets $\{2, 4, 5\}$ and $\{1, 2, 3, 5\}$ are not balanced. For each $n \geq 1$, prove that the number of balanced subsets of S_n is odd.
(To define the median of a set of k numbers, first put the numbers in increasing order; then the median is the middle number if k is odd, and the average of the two middle numbers if k is even. For example, the median of $\{1, 3, 4, 8, 9\}$ is 4, and the median of $\{1, 3, 4, 7, 8, 9\}$ is $(4 + 7)/2 = 5.5$.)

Solution: The problem is to prove that there is an odd number of nonempty subsets T of S_n such that the average $A(T)$ and median $M(T)$ satisfy $A(T) = M(T)$. Given a subset T, consider the subset $T^* = \{n + 1 - t : t \in T\}$. It holds that $A(T^*) = n + 1 - A(T)$ and $M(T^*) = n + 1 - M(T)$, which implies that if $A(T) = M(T)$ then $A(T^*) = M(T^*)$. Pairing each set T with T^* yields that there are an even number of sets T such that $A(T) = M(T)$ and $T \neq T^*$.

Thus it suffices to show that the number of nonempty subsets T such that $A(T) = M(T)$ and $T = T^*$ is odd. Now note that if $T = T^*$, then $A(T) = M(T) = \frac{n+1}{2}$. Hence it suffices to show the number of nonempty subsets T with $T = T^*$ is odd. Given such a set T, let T' be the largest nonempty subset of $\{1, 2, \ldots, \lceil n/2 \rceil\}$ contained in T. Pairing T with T' forms a bijection between these sets T and the nonempty subsets of $\{1, 2, \ldots, \lceil n/2 \rceil\}$. Thus there are $2^{\lceil n/2 \rceil} - 1$ such subsets, which is odd as desired. $\square$

Alternate solution: Using the notation from the above solution: Let B be the number of subsets T with $M(T) > A(T)$, C be the number with $M(T) = A(T)$, and D be the number with $M(T) < A(T)$. Pairing each set T counted by B with $T^* = \{n + 1 - t : t \in T\}$ shows that $B = D$. Now since $B + C + D = 2^n - 1$, we have that $C = 2^n - 1 - 2B$, which is odd.

4. Points P and Q lie inside parallelogram $ABCD$ and are such that triangles ABP and BCQ are equilateral. Prove that the line through P perpendicular to DP and the line through Q perpendicular to DQ meet on the altitude from B in triangle ABC.

   Solution: Let $\angle ABC = m$ and let O be the circumcenter of triangle DPQ. Since P and Q are in the interior of $ABCD$, it follows that $m = \angle ABC > 60^\circ$ and $\angle DAB = 180^\circ - m > 60^\circ$ which together imply that $60^\circ < m < 120^\circ$. Now note that $\angle DAP = \angle DAB - 60^\circ = 120^\circ - m$, $\angle DCQ = \angle DCB - 60^\circ = 120^\circ - m$ and that $\angle PBQ = 60^\circ - \angle ABQ = 60^\circ - (\angle ABC - 60^\circ) = 120^\circ - m$. This combined with the facts that $AD = BQ = CQ$ and $AP = BP = CD$ implies that triangles DAP, QBP and QCD are congruent. Therefore $DP = PQ = DQ$ and triangle DPQ is equilateral. This implies that $\angle ODA = \angle PDA + 30^\circ = \angle DQC + 30^\circ = \angle OQC$. Combining this fact with $OQ = OD$ and $CQ = AD$ implies that triangles ODA and OQC are congruent. Therefore $OA = OC$ and, if M is the midpoint of segment AC, it follows that OM is perpendicular to AC. Since $ABCD$ is a parallelogram, M is also the midpoint of DB. If K denotes the intersection of the line through P perpendicular to DP and the line through Q perpendicular to DQ, then K is diametrically opposite D on the circumcircle of DPQ and O is the midpoint of segment DK. This implies that OM is a midline of triangle DBK and hence that BK is parallel to OM which is perpendicular to AC. Therefore K lies on the altitude from B in triangle ABC, as desired. $\square$

5. One hundred circles of radius one are positioned in the plane so that the area of any triangle formed by the centres of three of these circles is at most 2017. Prove that there is a line intersecting at least three of these circles.

   Solution: We will prove that given n circles, there is some line intersecting more than $\frac{n}{46}$ of them. Let S be the set of centers of the n circles. We will first show that there is a line ℓ such that the projections of the points in S lie in an interval of length at most $\sqrt{8068} < 90$ on ℓ. Let A and B be the pair of points in S that are farthest apart and let the distance between A and B be d. Now consider any point $C \in S$ distinct from A and B. The distance from C to the line AB must be at most $\frac{4034}{d}$ since triangle ABC has area at most 2017. Therefore if ℓ is a line perpendicular to AB, then the projections of S onto ℓ lie in an interval of length $\frac{8068}{d}$ centered at the intersection of ℓ and AB. Furthermore, all of these projections must lie on an interval of length at most d on ℓ since the largest distance between two of these projections is at most d. Since $\min(d, 8068/d) \leq \sqrt{8068} < 90$, this proves the claim.

   Now note that the projections of the n circles onto the line ℓ are intervals of length 2, all contained in an interval of length at most $\sqrt{8068}+2 < 92$. Each point of this interval belongs to on average $\frac{2n}{\sqrt{8068}+2} > \frac{n}{46}$ of the subintervals of length 2 corresponding to the projections of the n circles onto ℓ. Thus there is some point $x \in \ell$ belonging to the projections of more than $\frac{n}{46}$ circles. The line perpendicular to ℓ through x has the desired property. Setting $n = 100$ yields that there is a line intersecting at least three of the circles. □

STUDENT:

Reference Copy

SCHOOL:

Canada Way H.S., Ottawa

EXAM ID: **10101**

2018 Canadian Mathematical Olympiad
Official Exam Booklet

50th Canadian Mathematical Olympiad 1969 2018 50e Olympiade mathématique du Canada

Student's Instructions

THE ONLY INSTRUMENTS PERMITTED ARE PENCILS, PENS, ERASERS, WHITEOUT, RULER, AND COMPASSES. NO OTHER AIDS ARE PERMITTED.

1. Please verify that your name and school at the top of this page is correct..
2. You cannot ask any questions concerning the examination. If you unsure of a problem make a note and then state and solve what you consider to be a valid non-trivial interpretation of the problem.
3. This exam consists of 5 questions. The questions are all of equal value. Partial marks may be awarded. Each solution must be justified and should be clear, concise and complete since good presentation counts. Special awards and/or extra credit may be given for elegant solutions and/or valid generalizations of particular problems.
4. All solutions must be written in this book, written in pen or pencil on one side of paper only. The backs of pages should be reserved for rough work. You may include extra pages if there is not enough room to answer a question. Please ensure extra pages include the question number and your exam number. Do not include your name.
5. At the end of the examination, the proctor (exam supervisor) must sign the declaration below and forward this booklet for marking. You should remove the back page of this booklet and keep it – it contains your unique Exam ID you will need to look up your results later.
6. Please do not discuss the contents of the exam online for the next 24 hours.

Proctor's Declaration

I confirmed the identity of the student whose name is written above and ensured that the student wrote this examination without assistance from any sources whatsoever on March 28th, 2018 during the official time period assigned. I declare I am not family-related to this student.

Name of Proctor (printed)

Signature

Signed Declaration of Proctor is Required to Validate the Exam

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1. Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: Select a pair of tokens at points A and B and move both of them to the midpoint of A and B.

   We say that an arrangement of n tokens is *collapsible* if it is possible to end up with all n tokens at the same point after a finite number of moves. Prove that every arrangement of n tokens is collapsible if and only if n is a power of 2.

2. Let five points on a circle be labelled A, B, C, D, and E in clockwise order. Assume $AE = DE$ and let P be the intersection of AC and BD. Let Q be the point on the line through A and B such that A is between B and Q and $AQ = DP$. Similarly, let R be the point on the line through C and D such that D is between C and R and $DR = AP$. Prove that PE is perpendicular to QR.

3. Two positive integers a and b are *prime-related* if $a = pb$ or $b = pa$ for some prime p. Find all positive integers n, such that n has at least three divisors, and all the divisors can be arranged without repetition in a circle so that any two adjacent divisors are prime-related.

   Note that 1 and n are included as divisors.

4. Find all polynomials $p(x)$ with real coefficients that have the following property: There exists a polynomial $q(x)$ with real coefficients such that

$$p(1) + p(2) + p(3) + \cdots + p(n) = p(n)q(n)$$

for all positive integers n.

5. Let k be a given even positive integer. Sarah first picks a positive integer N greater than 1 and proceeds to alter it as follows: every minute, she chooses a prime divisor p of the current value of N, and multiplies the current N by $p^k - p^{-1}$ to produce the next value of N. Prove that there are infinitely many even positive integers k such that, no matter what choices Sarah makes, her number N will at some point be divisible by 2018.

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Canadian Mathematical Olympiad 2018

Official Solutions

1. Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: Select a pair of tokens at points A and B and move both of them to the midpoint of A and B.

   We say that an arrangement of n tokens is *collapsible* if it is possible to end up with all n tokens at the same point after a finite number of moves. Prove that every arrangement of n tokens is collapsible if and only if n is a power of 2.

   Solution. For a given positive integer n, consider an arrangement of n tokens in the plane, where the tokens are at points $A_1, A_2, \ldots, A_n$. Let G be the centroid of the n points, so as vectors (after an arbitrary choice of origin),

$$\overrightarrow{G} = \frac{\overrightarrow{A}_1 + \overrightarrow{A}_2 + \cdots + \overrightarrow{A}_n}{n}.$$

   Note that any move leaves the centroid G unchanged. Therefore, if all the tokens are eventually moved to the same point, then this point must be G.

   First we prove that if $n = 2^k$ for some nonnegative integer k, then all n tokens can always be eventually moved to the same point. We shall use induction on k.

   The result clearly holds for $n = 2^0 = 1$. Assume that it holds when $n = 2^k$ for some nonnegative integer k. Consider a set of 2^{k+1} tokens at $A_1, A_2, \ldots, A_{2^{k+1}}$. Let M_i be the midpoint of A_{2i-1} and A_{2i} for $1 \le i \le 2^k$.

   First we move the tokens at A_{2i-1} and A_{2i} to M_i, for $1 \le i \le 2^k$. Then, there are two tokens at M_i for all $1 \le i \le 2^k$. If we take one token from each of $M_1, M_2, \ldots, M_{2^k}$, then by the induction hypothesis, we can move all of them to the same point, say G. We can do the same with the remaining tokens at $M_1, M_2, \ldots, M_{2^k}$. Thus, all 2^{k+1} tokens are now at G, which completes the induction argument.

   (Here is an alternate approach to the induction step: Given the tokens at $A_1, A_2, \ldots, A_{2^{k+1}}$, move the first 2^k tokens to one point G_1, and move the remaining 2^k tokens to one point G_2. Then 2^k more moves can bring all the tokens to the midpoint of G_1 and G_2.)

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Now, assume that n is not a power of 2. Take any line in the plane, and number it as a real number line. (Henceforth, when we refer to a token at a real number, we mean with respect to this real number line.)

At the start, place $n-1$ tokens at 0 and one token at 1. We observed that if we can move all the tokens to the same point, then it must be the centroid of the n points. Here, the centroid is at $\frac{1}{n}$.

We now prove a lemma.

Lemma. *The average of any two dyadic rationals is also a dyadic rational.* (A *dyadic rational* is a rational number that can be expressed in the form $\frac{m}{2^a}$, where m is an integer and a is a nonnegative integer.)

Proof. Consider two dyadic rationals $\frac{m_1}{2^{a_1}}$ and $\frac{m_2}{2^{a_2}}$. Then their average is

$$\frac{1}{2}\left(\frac{m_1}{2^{a_1}}+\frac{m_2}{2^{a_2}}\right) = \frac{1}{2}\left(\frac{2^{a_2}\cdot m_1+2^{a_1}\cdot m_2}{2^{a_1}\cdot 2^{a_2}}\right) = \frac{2^{a_2}\cdot m_1+2^{a_1}\cdot m_2}{2^{a_1+a_2+1}},$$

which is another dyadic rational. ■

On this real number line, a move corresponds to taking a token at x and a token at y and moving both of them to $\frac{x+y}{2}$, the average of x and y. At the start, every token is at a dyadic rational (namely 0 or 1), which means that after any number of moves, every token must still be at a dyadic rational.

But n is not a power of 2, so $\frac{1}{n}$ is not a dyadic rational. (Indeed, if we could express $\frac{1}{n}$ in dyadic form $\frac{m}{2^a}$, then we would have $2^a = mn$, which is impossible unless m and n are powers of 2.) This means that it is not possible for any token to end up at $\frac{1}{n}$, let alone all n tokens.

We conclude that we can always move all n tokens to the same point if and only if n is a power of 2.

2. Let five points on a circle be labelled A, B, C, D, and E in clockwise order. Assume $AE = DE$ and let P be the intersection of AC and BD. Let Q be the point on the line through A and B such that A is between B and Q and $AQ = DP$. Similarly, let R be the point on the line through C and D such that D is between C and R and $DR = AP$. Prove that PE is perpendicular to QR.

   Solution. We are given $AQ = DP$ and $AP = DR$. Additionally $\angle QAP = 180^\circ - \angle BAC = 180^\circ - \angle BDC = \angle RDP$, and so triangles AQP and DPR are congruent. Therefore $PQ = PR$. It follows that P is on the perpendicular bisector of QR.

   We are also given $AP = DR$ and $AE = DE$. Additionally $\angle PAE = \angle CAE = 180^\circ - \angle CDE = \angle RDE$, and so triangles PAE and RDE are congruent. Therefore $PE = RE$, and similarly $PE = QE$. It follows that E is on the perpendicular bisector of PQ.

   Since both P and E are on the perpendicular bisector of QR, the result follows.

3. Two positive integers a and b are *prime-related* if $a = pb$ or $b = pa$ for some prime p. Find all positive integers n, such that n has at least three divisors, and all the divisors can be arranged without repetition in a circle so that any two adjacent divisors are prime-related.

   Note that 1 and n are included as divisors.

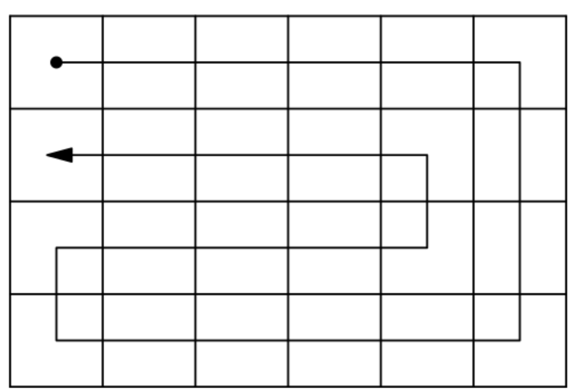
   Solution. We say that a positive integer is *good* if it has the given property. Let n be a good number, and let $d_1, d_2, \ldots, d_k$ be the divisors of n in the circle, in that order. Then for all $1 \le i \le k$, d_{i+1}/d_i (taking the indices modulo k) is equal to either p_i or $1/p_i$ for some prime p_i. In other words, $d_{i+1}/d_i = p_i^{\epsilon_i}$, where $\epsilon_i \in \{1, -1\}$. Then

$$p_1^{\epsilon_1} p_2^{\epsilon_2} \cdots p_k^{\epsilon_k} = \frac{d_2}{d_1} \cdot \frac{d_3}{d_2} \cdots \frac{d_1}{d_k} = 1.$$

   For the product $p_1^{\epsilon_1} p_2^{\epsilon_2} \cdots p_k^{\epsilon_k}$ to equal 1, any prime factor p must be paired with a factor of $1/p$, and vice versa, so k (the number of divisors of n) must be even. Hence, n cannot be a perfect square.

   Furthermore, n cannot be the power of a prime (including a prime itself), because 1 always is a divisor of n, and if n is a power of a prime, then the only divisor that can go next to 1 is the prime itself.

   Now, let $n = p^a q^b$, where p and q are distinct primes, and a is odd. We write the divisors of n in a grid as follows: In the first row, write the numbers $1, q, q^2, \ldots, q^b$. In the next row, write the numbers $p, pq, pq^2, \ldots, pq^b$, and so on. The number of rows in the grid, $a + 1$, is even. Note that if two squares are adjacent vertically or horizontally, then their corresponding numbers are prime-related. We start with the square with a 1 in the upper-left corner. We then move right along the first row, move down along the last column, move left along the last row, then zig-zag row by row, passing through every square, until we land on the square with a p. The following diagram gives the path for $a = 3$ and $b = 5$:



   Thus, we can write the divisors encountered on this path in a circle, so $n = p^a q^b$ is good.

   Next, assume that n is a good number. Let $d_1, d_2, \ldots, d_k$ be the divisors of n in the circle, in that order. Let p be a prime that does not divide n. We claim that $n \cdot p^e$ is also a good number. We arrange the divisors of $n \cdot p^e$ that are not divisors of n in a grid as follows:

$$\begin{array}{cccc} d_1p & d_1p^2 & \cdots & d_1p^e \\ d_2p & d_2p^2 & \cdots & d_2p^e \\ \vdots & \vdots & \ddots & \vdots \\ d_kp & d_kp^2 & \cdots & d_kp^e \end{array}$$

Note that if two squares are adjacent vertically or horizontally, then their corresponding numbers are prime-related. Also, k (the number of rows) is the number of factors of n, which must be even (since n is good). Hence, we can use the same path described above, which starts at d_1p and ends at d_2p. Since d_1 and d_2 are adjacent divisors in the circle for n, we can insert all the divisors in the grid above between d_1 and d_2, to obtain a circle for $n \cdot p^e$.

Finally, let n be a positive integer that is neither a perfect square nor a power of a prime. Let the prime factorization of n be

$$n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}.$$

Since n is not the power of a prime, $t \geq 2$. Also, since n is not a perfect square, at least one exponent e_i is odd. Without loss of generality, assume that e_1 is odd. Then from our work above, $p_1^{e_1} p_2^{e_2}$ is good, so $p_1^{e_1} p_2^{e_2} p_3^{e_3}$ is good, and so on, until $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ is good.

Therefore, a positive integer n has the given property if and only if it is neither a perfect square nor a power of a prime.

4. Find all polynomials $p(x)$ with real coefficients that have the following property: There exists a polynomial $q(x)$ with real coefficients such that

$$p(1) + p(2) + p(3) + \cdots + p(n) = p(n)q(n)$$

for all positive integers n.

Solution. The property clearly holds whenever $p(x)$ is a constant polynomial, since we can take $q(x) = x$. Assume henceforth that $p(x)$ is nonconstant and has the stated property. Let d be the degree of $p(x)$, so $p(x)$ is of the form

$$p(x) = cx^d + \cdots .$$

By a Lemma (which we will prove at the end), $\sum_{k=1}^{n} k^d$ is a polynomial in n of degree $d+1$, so $p(1) + p(2) + \cdots + p(n)$ is a polynomial in n of degree $d+1$. Hence, $q(n)$ is a polynomial of degree 1. Furthermore, the coefficient of n^{d+1} in $\sum_{k=1}^{n} k^d$ is $\frac{1}{d+1}$, so the coefficient of n in $q(n)$ is also $\frac{1}{d+1}$. Let $q(x) = \frac{1}{d+1}(x + r)$. We have that

$$p(1) + p(2) + p(3) + \cdots + p(n) = p(n)q(n)$$

and

$$p(1) + p(2) + p(3) + \cdots + p(n) + p(n + 1) = p(n + 1)q(n + 1).$$

Subtracting the first equation from the second, we get

$$p(n + 1) = p(n + 1)q(n + 1) - p(n)q(n),$$

and hence

$$p(n)q(n) = p(n + 1)[q(n + 1) - 1].$$

Since this holds for all positive integers n, it follows that

$$p(x)q(x) = p(x + 1)[q(x + 1) - 1]$$

for all real numbers x. We can then write

$$p(x) \cdot \frac{1}{d+1}(x + r) = p(x + 1)\left[\frac{1}{d+1}(x + r + 1) - 1\right],$$

so

$$(x + r)p(x) = (x + r - d)p(x + 1). \qquad (*)$$

Setting $x = -r$, we get

$$(-d)p(-r + 1) = 0.$$

Hence, $-r + 1$ is a root of $p(x)$. Let $p(x) = (x + r - 1)p_1(x)$. Then

$$(x + r)(x + r - 1)p_1(x) = (x + r - d)(x + r)p_1(x + 1),$$

so

$$(x + r - 1)p_1(x) = (x + r - d)p_1(x + 1).$$

If $d = 1$, then $p_1(x)$ is a constant, so both sides are equal, and we can say $p(x) = c(x + r - 1)$.

Otherwise, setting $x = -r + 1$, we get

$$(1 - d)p_1(-r + 2) = 0.$$

Hence, $-r + 2$ is a root of $p_1(x)$. Let $p_1(x) = (x + r - 2)p_2(x)$. Then

$$(x - r - 1)(x + r - 2)p_2(x) = (x + r - d)(x + r - 1)p_2(x + 1),$$

so

$$(x + r - 2)p_2(x) = (x + r - d)p_2(x + 1).$$

If $d = 2$, then $p_2(x)$ is a constant, so both sides are equal, and we can say
$p(x) = c(x + r - 1)(x + r - 2)$.

Otherwise, we can continue to substitute, giving us

$$p(x) = c(x + r - 1)(x + r - 2) \cdots (x + r - d).$$

Conversely, if $p(x)$ is of this form, then

$$\begin{aligned} p(x) &= c(x + r - 1)(x + r - 2) \cdots (x + r - d) \\ &= \frac{c(d + 1)(x + r - 1)(x + r - 2) \cdots (x + r - d)}{d + 1} \\ &= \frac{c[(x + r) - (x + r - d - 1)](x + r - 1)(x + r - 2) \cdots (x + r - d)}{d + 1} \\ &= \frac{c(x + r)(x + r - 1)(x + r - 2) \cdots (x + r - d)}{d + 1} \\ &\quad - \frac{c(x + r - 1)(x + r - 2) \cdots (x + r - d)(x + r - d - 1)}{d + 1}. \end{aligned}$$

Then the sum $p(1) + p(2) + p(3) + \cdots + p(n)$ telescopes, and we are left with

$$\begin{aligned} p(1) + p(2) + p(3) + \cdots + p(n) = &\frac{c(n + r)(n + r - 1)(n + r - 2) \cdots (n + r - d)}{d + 1} \\ &- \frac{c(r)(r - 1) \cdots (r - d + 1)(r - d)}{d + 1}. \end{aligned}$$

We want this to be of the form

$$p(n)q(n) = c(n + r - 1)(n + r - 2) \cdots (n + r - d)q(n)$$

for some polynomial $q(n)$. The only way that this can hold for each positive integer n is if the term

$$\frac{c(r)(r - 1) \cdots (r - d + 1)(r - d)}{d + 1}$$

is equal to 0. This means r has to be one of the values 0, 1, 2, ..., d. Therefore, the polynomials we seek are of the form

$$p(x) = c(x + r - 1)(x + r - 2) \cdots (x + r - d),$$

where $r \in \{0, 1, 2, \ldots, d\}$.

Lemma. *For a positive integer* d*,*

$$\sum_{k=1}^{n} k^d$$

is a polynomial in n *of degree* $d+1$*. Furthermore, the coefficient of* n^{d+1} *is* $\frac{1}{d+1}$*.*

Proof. We prove the result by strong induction. For $d = 1$,

$$\sum_{k=1}^{n} k = \frac{1}{2}n^2 + \frac{1}{2}n,$$

so the result holds. Assume that the result holds for $d = 1, 2, 3, \ldots, m$, for some positive integer m. By the Binomial Theorem,

$$(k+1)^{m+2} - k^{m+2} = (m+2)k^{m+1} + c_m k^m + c_{m-1}k^{m-1} + \cdots + c_1 k + c_0,$$

for some coefficients c_m, c_{m-1}, $\ldots$, c_1, c_0. Summing over $1 \le k \le n$, we get

$$(n+1)^{m+2} - 1 = (m+2)\sum_{k=1}^{n} k^{m+1} + c_m \sum_{k=1}^{n} k^m + \cdots + c_1 \sum_{k=1}^{n} k + c_0 n.$$

Then

$$\sum_{k=1}^{n} k^{m+1} = \frac{(n+1)^{m+2} - c_m \sum_{k=1}^{n} k^m - \cdots - c_1 \sum_{k=1}^{n} k - c_0 n - 1}{m+2}.$$

By the induction hypothesis, the sums $\sum_{k=1}^{n} k^m$, $\ldots$, $\sum_{k=1}^{n} k$ are all polynomials in n of degree less than $m+2$. Hence, the above expression is a polynomial in n of degree $m+2$, and the coefficient of n^{m+2} is $\frac{1}{m+2}$. Thus, the result holds for $d = m+1$, which completes the induction step. ■

5. Let k be a given even positive integer. Sarah first picks a positive integer N greater than 1 and proceeds to alter it as follows: every minute, she chooses a prime divisor p of the current value of N, and multiplies the current N by $p^k - p^{-1}$ to produce the next value of N. Prove that there are infinitely many even positive integers k such that, no matter what choices Sarah makes, her number N will at some point be divisible by 2018.

   Solution: Note that 1009 is prime. We will show that if $k = 1009^m - 1$ for some positive integer m, then Sarah's number must at some point be divisible by 2018. Let P be the largest divisor of N not divisible by a prime congruent to 1 modulo 1009. Assume for contradiction that N is never divisible by 2018. We will show that P decreases each minute. Suppose that in the t^{th} minute, Sarah chooses the prime divisor p of N. First note that N is replaced with $\frac{p^{k+1}-1}{p} \cdot N$ where

$$p^{k+1} - 1 = p^{1009^m} - 1 = (p-1)\left(p^{1009^m-1} + p^{1009^m-2} + \cdots + 1\right)$$

   Suppose that q is a prime number dividing the second factor. Since q divides $p^{1009^m} - 1$, it follows that $q \neq p$ and the order of p modulo q must divide 1009^m and hence is either divisible by 1009 or is equal to 1. If it is equal to 1 then $p \equiv 1 \pmod q$, which implies that

$$0 \equiv p^{1009^m-1} + p^{1009^m-2} + \cdots + 1 \equiv 1009^m \pmod q$$

   and thus $q = 1009$. However, if $q = 1009$ then $p \geq 1010$ and p must be odd. Since $p - 1$ now divides N, it follows that N is divisible by 2018 in the $(t+1)^{\text{th}}$ minute, which is a contradiction. Therefore the order of p modulo q is divisible by 1009 and hence 1009 divides $q - 1$. Therefore all of the prime divisors of the second factor are congruent to 1 modulo 1009. This implies that P is replaced by a divisor of $\frac{p-1}{p} \cdot P$ in the $(t+1)^{\text{th}}$ minute and therefore decreases. Since $P \geq 1$ must always hold, P cannot decrease forever. Therefore N must at some point be divisible by 2018.

   Remark (no credit). If k is allowed to be odd, then choosing $k+1$ to be divisible by $\phi(1009) = 1008$ guarantees that Sarah's number will be divisible by 2018 the first time it is even, which is after either the first or second minute.

The 2019 Canadian Mathematical Olympiad

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Official Problem Set

1. Amy has drawn three points in a plane, A, B, and C, such that $AB = BC = CA = 6$. Amy is allowed to draw a new point if it is the circumcenter of a triangle whose vertices she has already drawn. For example, she can draw the circumcenter O of triangle ABC, and then afterwards she can draw the circumcenter of triangle ABO.

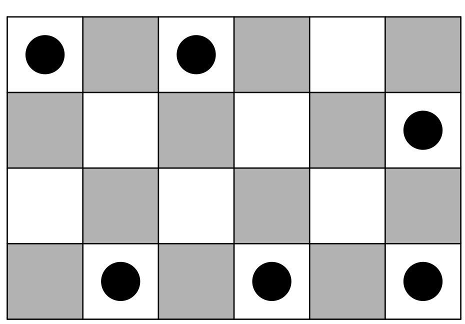
   (a) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 7.

   (b) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 2019.

   (Recall that the circumcenter of a triangle is the center of the circle that passes through its three vertices.)

2. Let a and b be positive integers such that $a + b^3$ is divisible by $a^2 + 3ab + 3b^2 - 1$. Prove that $a^2 + 3ab + 3b^2 - 1$ is divisible by the cube of an integer greater than 1.

3. Let m and n be positive integers. A $2m \times 2n$ grid of squares is coloured in the usual chessboard fashion. Find the number of ways of placing mn counters on the white squares, at most one counter per square, so that no two counters are on white squares that are diagonally adjacent. An example of a way to place the counters when $m = 2$ and $n = 3$ is shown below.



4. Let n be an integer greater than 1, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_1 = a_{n-1} = 0$. Prove that for any real number k,

$$|a_0| - |a_n| \leq \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|.$$

5. David and Jacob are playing a game of connecting $n \geq 3$ points drawn in a plane. No three of the points are collinear. On each player's turn, he chooses two points to connect by a new line segment. The first player to complete a cycle consisting of an odd number of line segments loses the game. (Both endpoints of each line segment in the cycle must be among the n given points, not points which arise later as intersections of segments.) Assuming David goes first, determine all n for which he has a winning strategy.

Important!

Please do not discuss this problem set online for at least 24 hours.

Canadian Mathematical Olympiad 2019

Official Solutions

1. Amy has drawn three points in a plane, A, B, and C, such that $AB = BC = CA = 6$. Amy is allowed to draw a new point if it is the circumcenter of a triangle whose vertices she has already drawn. For example, she can draw the circumcenter O of triangle ABC, and then afterwards she can draw the circumcenter of triangle ABO.

   (a) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 7.

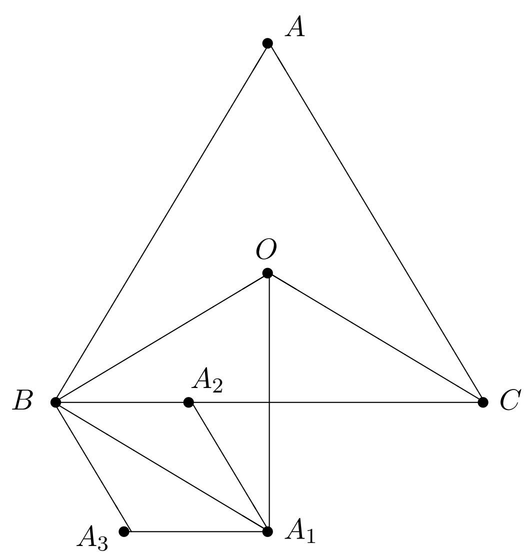
   (b) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 2019.

   (Recall that the circumcenter of a triangle is the center of the circle that passes through its three vertices.)

   Solution.

   (a) Given triangle $\triangle ABC$, Amy can draw the following points:

   - O is the circumcenter of $\triangle ABC$
   - A_1 is the circumcenter of $\triangle BOC$
   - A_2 is the circumcenter of $\triangle OBA_1$
   - A_3 is the circumcenter of $\triangle BA_2A_1$

   We claim that $AA_3 > 7$. We present two ways to prove this claim.

   First Method: By symmetry of the equilateral triangle $\triangle ABC$, we have $\angle AOB = \angle BOC = \angle COA = 120°$. Since $OB = OC$ and $A_1B = A_1O = A_1C$, we deduce that $\triangle A_1OB \cong \triangle A_1OC$, and hence $\angle BOA_1 = \angle COA_1 = 60°$. Therefore, since $\triangle A_1OB$ is isosceles, it must be equilateral. As we found for our original triangle, we find $\angle BA_2A_1 = 120°$, and so $\angle A_2BA_1 = \angle A_2A_1B = 30°$ (since

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$A_2B = A_2A_1$). Also we see that $\angle OBA_2 = 30^\circ = \angle OBC$, which shows that A_2 lies on the segment BC.

Applying the Law of Sines to $\triangle BOC$, we obtain

$$OC \;=\; \frac{BC\sin(\angle OBC)}{\sin(\angle BOC)} \;=\; \frac{6(1/2)}{\sqrt{3}/2} \;=\; 2\sqrt{3}\,.$$

By symmetry, we see that (i) OA_1 is the bisector of $\angle BOC$ and the perpendicular bisector of BC, and (ii) the three points A, O, and A_1 are collinear. Therefore $A_1A = A_1O + OA = 2\,OA = 4\sqrt{3}$.

The same argument that we used to show $\triangle A_1OB$ is equilateral with side $AC/\sqrt{3}$ shows that $\triangle A_3A_2A_1$ is equilateral with side $OB/\sqrt{3} = 2$. Thus $\angle A_3A_1O = \angle OA_1B + \angle A_3A_1A_2 - \angle A_2A_1B = 60^\circ + 60^\circ - 30^\circ = 90^\circ$. Hence we can apply the Pythagorean Theorem:

$$A_3A \;=\; \sqrt{(A_3A_1)^2 + (A_1A)^2} \;=\; \sqrt{2^2 + (4\sqrt{3})^2} \;=\; \sqrt{52} \;>\; \sqrt{49} \;=\; 7.$$

<u>Second Method</u>: (An alternative to writing the justifications of the constructions in the First Method is to use analytic geometry. Once the following coordinates are found using the kind of reasoning in the First Method or by other means, the writeup can justify them succinctly by computing distances.)

Label $(0,0)$ as B, $(6,0)$ as C, and $(3,\sqrt{3})$ as A. Then we have $AB = BC = CA = 6$.

The circumcenter O of $\triangle ABC$ is $(3,\sqrt{3})$; this can be verified by observing $OA = OB = OC = 2\sqrt{3}$.

Next, the point $A_1 = (3,-\sqrt{3})$ satisfies $A_1O = A_1B = A_1C = 2\sqrt{3}$, so A_1 is the circumcenter of $\triangle BOC$.

The point $A_2 = (2,0)$ satisfies $A_2O = A_2B = A_2A_1 = 2$, so this is the circumcenter of $\triangle OBA_1$.

And the point $A_3 = (1,-\sqrt{3})$ satisfies $A_3B = A_3A_2 = A_3A_1 = 2$, so this is the circumcenter of $\triangle BA_2A_1$.

Finally, we compute $A_3A = \sqrt{52} > \sqrt{49} = 7$, and part (a) is proved.

(b) In part (a), using either method we find that $OA_3 = 4 > 2\sqrt{3} = OA$. By rotating the construction of part (a) by $\pm 120^\circ$ about O, Amy can construct B_3 and C_3 such that $\triangle A_3B_3C_3$ is equilateral with circumcenter O and circumradius 4, which is strictly bigger than the circumradius $2\sqrt{3}$ of $\triangle ABC$. Amy can repeat this process starting from $\triangle A_3B_3C_3$. After n iterations of the process, Amy will have drawn the vertices of an equilateral triangle whose circumradius is $2\sqrt{3}\,(\frac{4}{2\sqrt{3}})^n$, which is bigger than 2019 when n is sufficiently large.

2. Let a and b be positive integers such that $a + b^3$ is divisible by $a^2 + 3ab + 3b^2 - 1$. Prove that $a^2 + 3ab + 3b^2 - 1$ is divisible by the cube of an integer greater than 1.

Solution.

Let $Z = a^2 + 3ab + 3b^2 - 1$. By assumption, there is a positive integer c such that $cZ = a + b^3$. Noticing the resemblance between the first three terms of Z and those of the expansion of $(a + b)^3$, we are led to

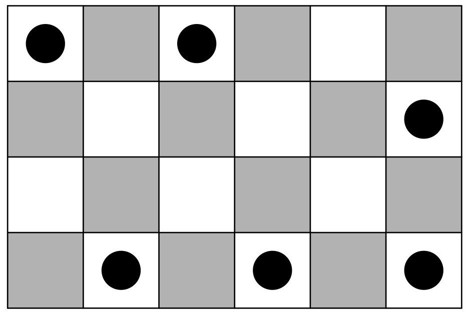
$$(a + b)^3 \;=\; a(a^2 + 3ab + 3b^2) + b^3 \;=\; a(Z + 1) + b^3 \;=\; aZ + a + b^3 \;=\; aZ + cZ\,.$$

Thus Z divides $(a + b)^3$.

Let the prime factorization of $a+b$ be $p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ and let $Z = p_1^{f_1}p_2^{f_2}\cdots p_k^{f_k}$, where $f_i \le 3e_i$ for each i since Z divides $(a+b)^3$. If Z is not divisible by a perfect cube greater than one, then $0 \le f_i \le 2$ and hence $f_i \le 2e_i$ for each i. This implies that Z divides $(a+b)^2$. However, $(a+b)^2 < a^2+3ab+3b^2-1 = Z$ since $a, b \ge 1$, which is a contradiction. Thus Z must be divisible by a perfect cube greater than one.

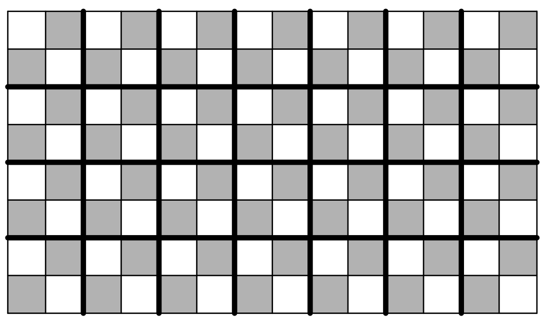
Remark. A brute force search yields many pairs (a, b) satisfying this divisibility property. Examples include $(3, 5), (19, 11), (111, 29)$ as well as twelve others satisfying that $a, b \le 1000$. The values of $a^2 + 3ab + 3b^2 - 1$ for these three pairs are $128 = 2^7$, $1350 = 2 \times 3^3 \times 5^2$ and $24500 = 2^2 \times 5^3 \times 7^2$, all of which have different perfect cube divisors.

3. Let m and n be positive integers. A $2m \times 2n$ grid of squares is coloured in the usual chessboard fashion. Find the number of ways of placing mn counters on the white squares, at most one counter per square, so that no two counters are on white squares that are diagonally adjacent. An example of a way to place the counters when $m = 2$ and $n = 3$ is shown below.



Solution.

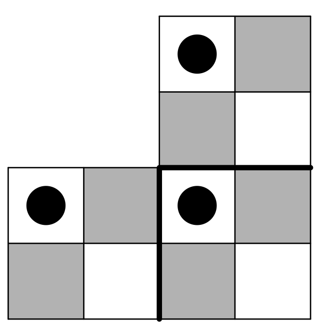
Divide the chessboard into mn 2×2 squares.



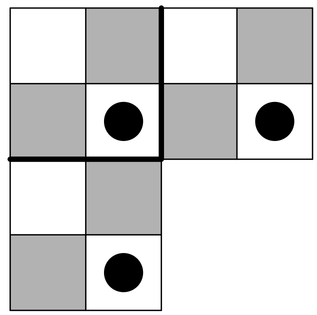
Each 2×2 square can contain at most one counter. Since we want to place mn counters, each 2×2 square must contain exactly one counter.

Assume that the lower-right corner of the $2m \times 2n$ chessboard is white, so in each 2×2 square, the upper-left and lower-right squares are white. Call a 2×2 square UL if the counter it contains is on the upper-left white square, and call it LR if the counter it contains is on the lower-right white square.

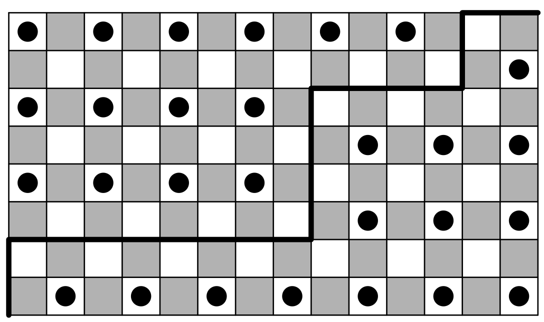
Suppose some 2×2 square is UL. Then the 2×2 square above it (if it exists) must also be UL, and the 2×2 square to the left of it (if it exists) must also be UL.



Similarly, if some 2×2 square is LR, then the 2×2 square below it (if it exists) must also be LR, and the 2×2 square to the right of it (if it exists) must also be LR.



Then the collection of UL 2×2 squares form a region that is top-justified and left-justified, and the collection of LR 2×2 squares form a region that is bottom-justified and right-justified. This means that the boundary between the two regions forms a path between the lower-left corner and upper-right corner of the $2m \times 2n$ chessboard.



Conversely, any path from the lower-left corner to the upper-right corner, where each step consists of two units, can serve as the boundary of the UL squares and LR squares. Thus, the number of ways of placing the counters is equal to the number of paths, which is $\binom{m+n}{m}$.

4. Let n be an integer greater than 1, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_1 = a_{n-1} = 0$. Prove that for any real number k,

$$|a_0| - |a_n| \ \le\ \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|\,.$$

First Solution.

Let $Q(x) = x^2 - kx - 1$ and let $P(x) = a_0 + a_1x + \cdots + a_nx^n$. Note that the product of the two roots of $Q(x)$ is -1 and thus one of the two roots has magnitude at most 1. Let z be this root. Now note that since $a_1 = a_{n-1} = 0$, we have that

$$\begin{aligned}
0 = Q(z)P(z) &= -a_0 - ka_0z + \sum_{i=0}^{n-2}(a_i - ka_{i+1} - a_{i+2})z^{i+2} - ka_nz^{n+1} + a_nz^{n+2} \\
&= a_0(-1 - kz) + \sum_{i=0}^{n-2}(a_i - ka_{i+1} - a_{i+2})z^{i+2} + a_nz^n(z^2 - kz) \\
&= -a_0z^2 + \sum_{i=0}^{n-2}(a_i - ka_{i+1} - a_{i+2})z^{i+2} + a_nz^n
\end{aligned}$$

where the third equality follows since $z^2 - kz - 1 = 0$. The triangle inequality now implies

$$\begin{aligned}
|a_0| \cdot |z|^2 &\le |a_n| \cdot |z|^n + \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}| \cdot |z|^{i+2} \\
&\le |a_n| \cdot |z|^2 + \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}| \cdot |z|^2
\end{aligned}$$

since $|z| \le 1$ and $n \ge 2$. Since $z \ne 0$, the inequality is obtained on dividing by $|z|^2$.

Second Solution.

Let k be a real number. Put

$$R \ =\ \begin{cases} \sqrt{k^2+4} & \text{if } k \ge 0, \\ -\sqrt{k^2+4} & \text{if } k < 0. \end{cases}$$

Define the polynomial

$$S(x) \ =\ x^2 + Rx + 1\,.$$

The roots of S are

$$b \ =\ \frac{-R-k}{2} \quad \text{and} \quad c \ =\ \frac{-R+k}{2}\,.$$

Then we have

$$b - c \ =\ -k, \quad bc \ =\ 1, \quad \text{and} \quad |c| \le 1$$

(the inequality follows from $bc = 1$ and $|c| \le |b|$).

Put $d_i = a_i + b\,a_{i+1}$ for $i = 0, 1, \ldots, n-1$. Then, for $i = 0, 1, \ldots, n-2$, we have

$$\begin{aligned} d_i - c\,d_{i+1} &= a_i + (b-c)a_{i+1} - bc\,a_{i+2} \\ &= a_i - k\,a_{i+1} - a_{i+2}\,. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=0}^{n-2} |a_i - k\,a_{i+1} - a_{i+2}| &= \sum_{i=0}^{n-2} |d_i - c\,d_{i+1}| \\ &\geq \sum_{i=0}^{n-2} (|d_i| - |c|\,|d_{i+1}|) \\ &= |d_0| + (1-|c|)\sum_{i=1}^{n-2} |d_i| - |c|\,|d_{n-1}| \\ &\geq |d_0| - |c|\,|d_{n-1}| \\ &= |a_0 + b\,a_1| - |c|\,|a_{n-1} + b\,a_n| \\ &= |a_0| - |bc|\,|a_n| \\ &= |a_0| - |a_n|\,. \end{aligned}$$

5. David and Jacob are playing a game of connecting $n \geq 3$ points drawn in a plane. No three of the points are collinear. On each player's turn, he chooses two points to connect by a new line segment. The first player to complete a cycle consisting of an odd number of line segments loses the game. (Both endpoints of each line segment in the cycle must be among the n given points, not points which arise later as intersections of segments.) Assuming David goes first, determine all n for which he has a winning strategy.

Solution:

Answer: David has a winning strategy if and only if $n \equiv 2 \pmod 4$.

Call a move *illegal* if it would cause an odd cycle to be formed for the first time. First we show that if n is odd, then any strategy where Jacob picks a legal move if one is available to him causes him to win. Assume for contradiction that Jacob at some point has no legal moves remaining. Since the graph representing the game state has no odd cycle, it must be bipartite. Let a and b be the sizes of the two sets in the bipartition of the graph. If there is some edge not already added between the two sets, adding this edge would be a legal move for Jacob. Therefore the graph must be a complete bipartite graph with all of its ab edges present. However, since $a + b = n$ which is odd, one of a or b must be even and thus the graph contains an even number of edges. Moreover, since it is Jacob's turn, the graph must contain an odd number of edges, which is a contradiction. Therefore Jacob has a winning strategy for all odd n.

Now consider the case where n is even. Call a graph *good* if the set of vertices of degree at least 1 are in a perfect matching (a set of non-adjacent edges that includes every vertex of the graph). The key observation is that either player has a strategy to preserve that the graph is good while increasing the number of vertices of degree at least 1. More precisely, if the graph was good at the end of a player's previous turn and there are fewer than n vertices of degree at least 1, then at the end of his current turn he can always ensure that: (1) the graph is good and (2) there are at least two more vertices of degree 1 since the end of his previous turn. Let A be the set of vertices of degree at least 1 at the end of the player's previous turn and B be the set of remaining vertices where $|B| > 0$. Since the vertices of A have a perfect matching, $|A|$ is even, and since n is even, so is $|B|$. If the other player adds an edge between two vertices of A, add an edge between two vertices of B. If the other player adds an edge between two vertices of B, add an edge between one of those vertices an a vertex of A (but on the first round, when A is empty, respond by adding an edge between two other vertices of B). If the other player adds an edge between a vertex in A and a vertex in B, then since $|B|$ is even, there must be another vertex of B. Connect these two vertices in B with an edge. None of these moves can form a cycle and thus are legal. Furthermore, all of them achieve (1) and (2), proving the claim.

We now show that David has a winning strategy if $n \equiv 2 \pmod 4$. Since the graph begins empty and therefore good, David has a strategy of legal moves to ensure that the graph contains a perfect matching after no more than n moves. After this, let David implement any strategy where he picks a legal move if one is available to him. Assume for contradiction that there is a turn where David has no legal moves. This graph must be a complete bipartite graph containing a perfect matching. If one of the sets in the bipartition has size greater than $n/2$, it must contain two vertices matched in the perfect matching, which is impossible. Therefore there are $n/2$ vertices in each part and $n^2/4$ edges have been added in total, which is an odd number. This contradicts the fact that it is David's turn, and proves the result for $n \equiv 2 \pmod 4$.

Finally, consider the case that $n \equiv 0 \pmod 4$. Note that after David's first turn, the graph contains a single edge and thus is good. This implies that Jacob can ensure the graph contains a perfect matching and win by the above parity argument.

The 2020 Canadian Mathematical Olympiad

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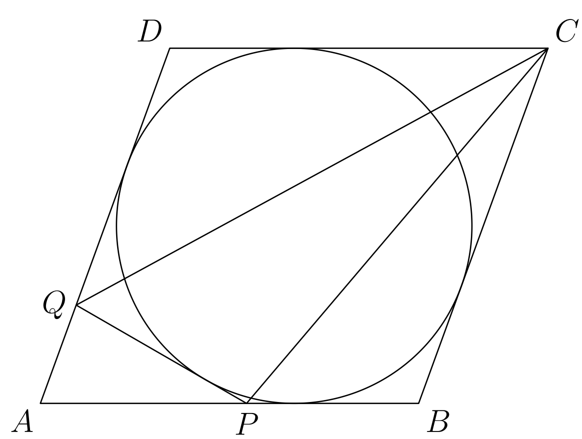
Official Problem Set

1. Let S be a set of $n \geq 3$ positive real numbers. Show that the largest possible number of distinct integer powers of three that can be written as the sum of three distinct elements of S is $n-2$.

2. A circle is inscribed in a rhombus $ABCD$. Points P and Q vary on line segments $\overline{AB}$ and $\overline{AD}$, respectively, so that $\overline{PQ}$ is tangent to the circle. Show that for all such line segments $\overline{PQ}$, the area of triangle CPQ is constant.



3. A purse contains a finite number of coins, each with distinct positive integer values. Is it possible that there are exactly 2020 ways to use coins from the purse to make the value 2020?

4. Let $S = \{1, 4, 8, 9, 16, \ldots\}$ be the set of perfect powers of integers, i.e. numbers of the form n^k where n, k are positive integers and $k \geq 2$. Write $S = \{a_1, a_2, a_3 \ldots\}$ with terms in increasing order, so that $a_1 < a_2 < a_3 \cdots$. Prove that there exist infinitely many integers m such that 9999 divides the difference $a_{m+1} - a_m$.

5. There are $19,998$ people on a social media platform, where any pair of them may or may not be *friends*. For any group of $9,999$ people, there are at least $9,999$ pairs of them that are friends. What is the least number of friendships, that is, the least number of pairs of people that are friends, that must be among the $19,998$ people?

Important!

Please do not discuss this problem set online for at least 24 hours.

Canadian Mathematical Olympiad 2020

Official Solutions

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https://cms.math.ca/Competitions/Sponsors/

1. Let S be a set of $n \geq 3$ positive real numbers. Show that the largest possible number of distinct integer powers of three that can be written as the sum of three distinct elements of S is $n-2$.

   Solution: We will show by induction that for all $n \geq 3$, it holds that at most $n-2$ powers of three are sums of three distinct elements of S for any set S of positive real numbers with $|S| = n$. This is trivially true when $n = 3$. Let $n \geq 4$ and consider the largest element $x \in S$. The sum of x and any two other elements of S is strictly between x and $3x$. Therefore x can be used as a summand for at most one power of three. By the induction hypothesis, at most $n-3$ powers of three are sums of three distinct elements of $S \backslash \{x\}$. This completes the induction.

   Even if it was not asked to prove, we will now show that the optimal answer $n-2$ is reached. Observe that the set $S = \{1, 2, 3^2 - 3, 3^3 - 3, \ldots, 3^n - 3\}$ is such that $3^2, 3^3, \ldots, 3^n$ can be expressed as sums of three distinct elements of S. This makes use of the fact that each term of the form $3^k - 3$ can be used in exactly one sum of three terms equal to 3^k.

   $\square$

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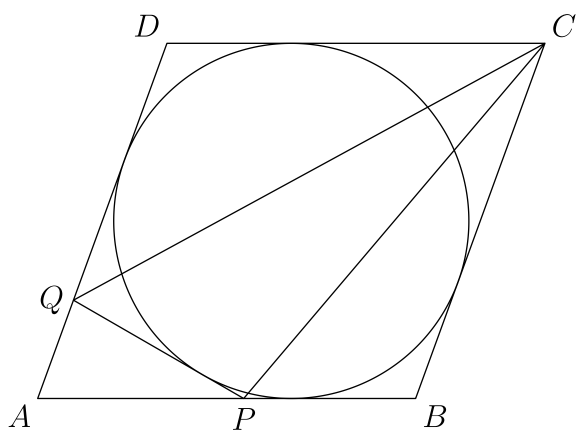
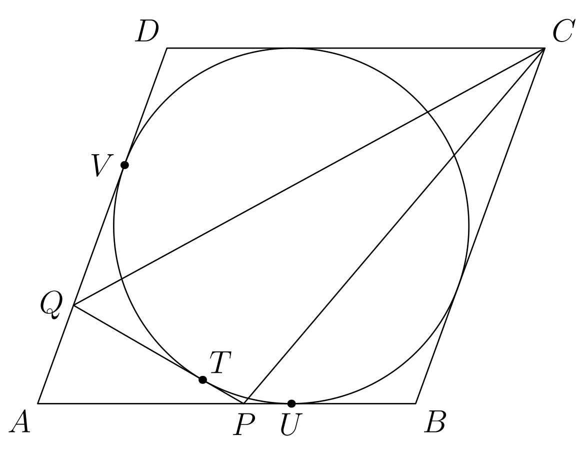
May 14, 2020

2. A circle is inscribed in a rhombus $ABCD$. Points P and Q vary on line segments $\overline{AB}$ and $\overline{AD}$, respectively, so that $\overline{PQ}$ is tangent to the circle. Show that for all such line segments $\overline{PQ}$, the area of triangle CPQ is constant.



Solution.

Let the circle be tangent to $\overline{PQ}$, $\overline{AB}$, $\overline{AD}$ at T, U, and V, respectively. Let $p = PT = PU$ and $q = QT = QV$. Let $a = AU = AV$ and $b = BU = DV$. Then the side length of the rhombus is $a + b$.



Let $\theta = \angle BAD$, so $\angle ABC = \angle ADC = 180° - \theta$. Then (using the notation [XYZ] for the area of a triangle of vertices X, Y, Z)

$$[APQ] = \frac{1}{2} \cdot AP \cdot AQ \cdot \sin\theta = \frac{1}{2}(a-p)(a-q)\sin\theta,$$

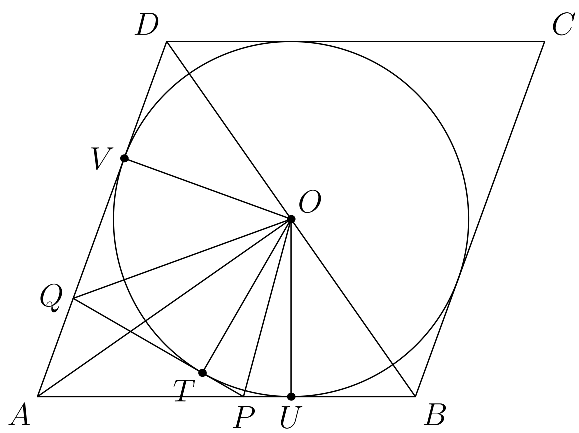
$$[BCP] = \frac{1}{2} \cdot BP \cdot BC \cdot \sin(180° - \theta) = \frac{1}{2}(b+p)(a+b)\sin\theta,$$

$$[CDQ] = \frac{1}{2} \cdot DQ \cdot CD \cdot \sin(180° - \theta) = \frac{1}{2}(b+q)(a+b)\sin\theta,$$

so

$$\begin{aligned}[CPQ] &= [ABCD] - [APQ] - [BCP] - [CDQ] \\ &= (a+b)^2 \sin\theta - \frac{1}{2}(a-p)(a-q)\sin\theta - \frac{1}{2}(b+p)(a+b)\sin\theta - \frac{1}{2}(b+q)(a+b)\sin\theta \\ &= \frac{1}{2}(a^2 + 2ab - bp - bq - pq)\sin\theta.\end{aligned}$$

Let O be the center of the circle, and let r be the radius of the circle. Let $x = \angle TOP = \angle UOP$ and $y = \angle TOQ = \angle VOQ$. Then $\tan x = \frac{p}{r}$ and $\tan y = \frac{q}{r}$.



Note that $\angle UOV = 2x + 2y$, so $\angle AOU = x + y$. Also, $\angle AOB = 90^\circ$, so $\angle OBU = x + y$. Therefore,

$$\tan(x+y) = \frac{a}{r} = \frac{r}{b},$$

so $r^2 = ab$. But

$$\frac{r}{b} = \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{p}{r} + \frac{q}{r}}{1 - \frac{p}{r}\cdot\frac{q}{r}} = \frac{r(p+q)}{r^2 - pq} = \frac{r(p+q)}{ab - pq}.$$

Hence, $ab - pq = bp + bq$, so $bp + bq + pq = ab$. Therefore,

$$[CPQ] = \frac{1}{2}(a^2 + 2ab - bp - bq - pq)\sin\theta = \frac{1}{2}(a^2 + ab)\sin\theta,$$

which is constant.

Alternate Solution: Let O be the center of the circle and r its radius. Then $[CPQ] = [CDQPB] - [CDQ] - [CBP]$, where [...] denotes area of the polygon with given vertices. Note that $[CDQPB]$ is half r times the perimeter of $CDQPB$. Note that the heights of CDQ and CBP are $2r$ so $[CDQ] = r \cdot DQ$ and $[CBP] = r \cdot PB$. Using the fact that $QT = QV$ and $PU = PT$, it now follows that $[CPQ] = [OVDCBU] - [CDV] - [CBU]$, which is independent of P and Q.

$\square$

3. A purse contains a finite number of coins, each with distinct positive integer values. Is it possible that there are exactly 2020 ways to use coins from the purse to make the value 2020?

Solution: It is possible.
Consider a coin purse with coins of values $2, 4, 8, 2014, 2016, 2018, 2020$ and every odd number between 503 and 1517. Call such a coin *big* if its value is between 503 and 1517. Call a coin *small* if its value is $2, 4$ or 8 and *huge* if its value is $2014, 2016, 2018$ or 2020. Suppose some subset of these coins contains no huge coins and sums to 2020. If it contains at least four big coins, then its value must be at least $503 + 505 + 507 + 509 > 2020$. Furthermore since all of the small coins are even in value, if the subset contains exactly one or three big coins, then its value must be odd. Thus the subset must contain exactly two big coins. The eight possible subsets of the small coins have values $0, 2, 4, 6, 8, 10, 12, 14$. Therefore the ways to make the value 2020 using no huge coins correspond to the pairs of big coins with sums $2006, 2008, 2010, 2012, 2014, 2016, 2018$ and 2020. The numbers of such pairs are $250, 251, 251, 252, 252, 253, 253, 254$, respectively. Thus there are exactly 2016 subsets of this coin purse with value 2020 using no huge coins. There are exactly four ways to make a value of 2020 using huge coins; these are $\{2020\}, \{2, 2018\}, \{4, 2016\}$ and $\{2, 4, 2014\}$. Thus there are exactly 2020 ways to make the value 2020.

Alternate construction: Take the coins $1, 2, \ldots, 11, 1954, 1955, \ldots, 2019$. The only way to get 2020 is a non-empty subset of $1, \ldots, 11$ and a single *large* coin. There are 2047 non-empty such subsets of sums between 1 and 66. Thus they each correspond to a unique large coin making 2020, so we have 2047 ways. Thus we only need to remove some large coins, so that we remove exactly 27 small sums. This can be done, for example, by removing coins $2020 - n$ for $n = 1, 5, 6, 7, 8, 9$, as these correspond to $1 + 3 + 4 + 5 + 6 + 8 = 27$ partitions into distinct numbers that are at most 11.

$\square$

4. Let $S = \{1, 4, 8, 9, 16, \ldots\}$ be the set of perfect powers of integers, i.e. numbers of the form n^k where n, k are positive integers and $k \geq 2$. Write $S = \{a_1, a_2, a_3 \ldots\}$ with terms in increasing order, so that $a_1 < a_2 < a_3 \cdots$. Prove that there exist infinitely many integers m such that 9999 divides the difference $a_{m+1} - a_m$.

Solution: The idea is that most perfect powers are squares. If $a_n = x^2$ and $a_{n+1} = (x+1)^2$, then $a_{n+1} - a_n = 2x + 1$. Note that $9999 \mid 2x + 1$ is equivalent to $x \equiv 4999 \pmod{9999}$. Hence we will be done if we can show that there exist infinitely many $x \equiv 4999 \pmod{9999}$ such that there are no perfect powers strictly between x^2 and $(x+1)^2$.

Assume otherwise, so that there exists a positive integer N such that: for $x \equiv 4999 \pmod{9999}$ and $x \geq N$, there is a perfect power $b_x^{e_x}$ ($e_x \geq 2$) between x^2 and $(x+1)^2$. Without loss of generality, we can take N to be $\equiv 4999 \pmod{9999}$. Note that x^2 and $(x+1)^2$ are consecutive squares, hence e_x is odd, and thus $e_x \geq 3$. Let t_n be the number of odd perfect powers that are at most n.

By tallying the $b_x^{e_x}$ up (clearly they are all distinct), for any $m \geq 1$ we have at least m perfect odd powers between 1 and $(N + 9999m)^2$, so that

$$t_{(N+9999m)^2} \geq m.$$

In particular, for large enough n we have

$$t_n \geq \frac{\sqrt{n}}{10000}.$$

Now, if $x^f \leq n$ then $x \leq \sqrt[f]{n}$. Also, $n \geq x^f \geq 2^f$ so $f \leq \log_2(n)$ So we have

$$t_n \leq \sum_{i=3}^{\log_2(n)} \sqrt[i]{n} \leq \log_2(n) \sqrt[3]{n}$$

Combining with the previous inequality, we have

$$\sqrt[6]{n} \leq 10000 \log_2(n)$$

for all large enough n. However, this inequality is false for all large n, contradiction. Therefore the problem statement holds.

$\square$

5. There are $19,998$ people on a social media platform, where any pair of them may or may not be *friends*. For any group of $9,999$ people, there are at least $9,999$ pairs of them that are friends. What is the least number of friendships, that is, the least number of pairs of people that are friends, that must be among the $19,998$ people?

Solution: It is $5 \cdot 9999 = 49995$. One possible construction is as follows: have the 19,998 people form 3,333 groups of 6 people, and within each group every pair of people are friends. Now, for any group of 9,999 people, say that there are $x_1, x_2, \ldots, x_{3333}$ people in each of the 6 groups, respectively. Then there are

$$\frac{1}{2}\sum_{i=1}^{3333} x_i(x_i-1)$$

pairs of friendships total. But we have that

$$x_i(x_i-1) \geq 5x_i - 9,$$

so

$$\frac{1}{2}\sum_{i=1}^{3333} x_i(x_i-1) \geq \frac{1}{2}\sum_{i=1}^{3333}(5x_i-9) = \frac{1}{2}(9999 \cdot 5 - 9 \cdot 3333) = 9999,$$

as desired.

It remains to show that 49995 pairs of friends is optimal. For what follows, let $9999 = N$, so that $19,998 = 2N$, and assume that the condition is satisfied. Let the number of pairs of friends be e. Designate half of the people as *red* and the other half as *blue*, so that the number of pairs of friends who are both red is minimized.

Note that this means that for every pair of people, one red and one blue, we have that the number of red friends of the blue person is at least as many as the number of red friends of the red person, and the inequality is strict if the two people are friends. This is because we can otherwise swap the two people. Now, if every blue person is friends with at least 3 red people, then the total number of friendships, e, is at least $N + 3N + N = 5N$ (N each from the red people and blue people and $3N$ from the pairs), as desired. If some blue person is friends with at most 2 red people, then every red person is friends with at most 2 red people, so the number of pairs of red friends is at most N, with equality only if every red person is friends with exactly 2 red people. But then consider a blue person with 2 red friends; then, they must have a red friend with exactly 2 red friends too, a contradiction.

$\square$

The 2021 Canadian Mathematical Olympiad

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Official Problem Set

1. Let $ABCD$ be a trapezoid with AB parallel to CD, $|AB| > |CD|$, and equal edges $|AD| = |BC|$. Let I be the center of the circle tangent to lines AB, AC and BD, where A and I are on opposite sides of BD. Let J be the center of the circle tangent to lines CD, AC and BD, where D and J are on opposite sides of AC. Prove that $|IC| = |JB|$.

2. Let $n \geq 2$ be some fixed positive integer and suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = 2^n - 1$.

   Find the minimum possible value of
   $$\frac{a_1}{1} + \frac{a_2}{1+a_1} + \frac{a_3}{1+a_1+a_2} + \cdots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}}.$$

3. At a dinner party there are N hosts and N guests, seated around a circular table, where $N \geq 4$. A pair of two guests will chat with one another if either there is at most one person seated between them or if there are exactly two people between them, at least one of whom is a host. Prove that no matter how the $2N$ people are seated at the dinner party, at least N pairs of guests will chat with one another.

4. A function f from the positive integers to the positive integers is called *Canadian* if it satisfies
   $$\gcd\Big(f(f(x)), f(x+y)\Big) = \gcd(x, y)$$
   for all pairs of positive integers x and y.

   Find all positive integers m such that $f(m) = m$ for all Canadian functions f.

5. Nina and Tadashi play the following game. Initially, a triple (a, b, c) of nonnegative integers with $a+b+c=2021$ is written on a blackboard. Nina and Tadashi then take moves in turn, with Nina first. A player making a move chooses a positive integer k and one of the three entries on the board; then the player increases the chosen entry by k and decreases the other two entries by k. A player loses if, on their turn, some entry on the board becomes negative.

   Find the number of initial triples (a, b, c) for which Tadashi has a winning strategy.

Important!

Please do not discuss this problem set online for at least 24 hours.

Canadian Mathematical Olympiad 2021

Official Solutions

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Note: Each problem starts on a new page.

Problem No. 1.

Let $ABCD$ be a trapezoid with AB parallel to CD, $|AB| > |CD|$, and equal edges $|AD| = |BC|$. Let I be the center of the circle tangent to lines AB, AC and BD, where A and I are on opposite sides of BD. Let J be the center of the circle tangent to lines CD, AC and BD, where D and J are on opposite sides of AC. Prove that $|IC| = |JB|$.

Solution. Let $\{P\} = AC \cap BD$ and let $\angle APB = 180 - 2a$. Since $ABCD$ is an isosceles trapezoid, APB is an isosceles triangle. Therefore $\angle PBA = a$, which implies that $\angle PBI = 90° - a/2$ since I lies on the external bisector of $\angle PBA$. Since I lies on the bisector of $\angle CPB$, it follows that $\angle BPI = a$ and hence that IPB is isosceles with $|IP| = |PB|$. Similarly JPC is isosceles with $|JP| = |PC|$. So, in the triangles CPI and BPJ we have $PI \equiv PB$ and $PJ \equiv CP$. Since I and J both lie on the internal bisector of $\angle BPC$, it follows that triangles CPI and BPJ are congruent. Therefore $|IC| = |JB|$. $\square$

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Canadian Mathematical Olympiad 2021

Problem No. 2.

Let $n \geq 2$ be some fixed positive integer and suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = 2^n - 1$.

Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1+a_1} + \frac{a_3}{1+a_1+a_2} + \cdots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}}.$$

Solution. We claim the the minimum possible value of this expression is n. Observe that by AM-GM, we have that

$$\begin{aligned}
&\frac{a_1}{1} + \frac{a_2}{1+a_1} + \cdots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}} \\
&\quad = \frac{1+a_1}{1} + \frac{1+a_1+a_2}{1+a_1} + \cdots + \frac{1+a_1+a_2+\cdots+a_n}{1+a_1+a_2+\cdots+a_{n-1}} - n \\
&\quad \geq n \cdot \sqrt[n]{\frac{1+a_1}{1} \cdot \frac{1+a_1+a_2}{1+a_1} \cdots \frac{1+a_1+a_2+\cdots+a_n}{1+a_1+a_2+\cdots+a_{n-1}}} - n \\
&\quad = n \cdot \sqrt[n]{1+a_1+a_2+\cdots+a_n} - n \\
&\quad = 2n - n = n.
\end{aligned}$$

Furthermore, equality is achieved when $a_k = 2^{k-1}$ for each $1 \leq k \leq n$. □

Canadian Mathematical Olympiad 2021

Problem No. 3.

At a dinner party there are N hosts and N guests, seated around a circular table, where $N \geq 4$. A pair of two guests will chat with one another if either there is at most one person seated between them or if there are exactly two people between them, at least one of whom is a host. Prove that no matter how the $2N$ people are seated at the dinner party, at least N pairs of guests will chat with one another.

Solution. Let a *run* refer to a maximal group of consecutive dinner party guests all of whom are the same type (host or guest). Suppose that there are exactly k runs of hosts and k runs of guests. Let G_i and H_i denote the number of runs of guests and hosts, respectively, of length exactly i. Furthermore, let X denote the number of hosts surrounded by two runs of guests, both of length exactly 1. We claim that the number of pairs of guests who chat is at least

$$2N - 3k + G_1 + 2H_1 + H_2 - X.$$

The number of pairs of guests who chat with no host between them is at least the sum of $\max\{2\ell - 3, 0\}$ over all guest run lengths ℓ. This sum is at least $2N - 3k + G_1$. The number of pairs of guests who chat with exactly two hosts between them is H_2. Furthermore, the number of pairs of guests who chat with exactly one host between them is at least $2H_1 - X$. This is because any host surrounded by two runs of guests causes at least two pairs of guests to chat unless these runs are both of length exactly 1. This proves the claim. Now note that

$$2H_1 + H_2 + N \geq 3k$$

because each run of hosts contributes at least three to the left hand side. Furthermore, pairing each run counted in X with the guest run of length 1 immediately following it in clockwise order shows that $G_1 \geq X$. Combining these inequalities yields that $2N - 3k + G_1 + 2H_1 + H_2 - X \geq N$, completing the proof of the desired result. $\square$

Canadian Mathematical Olympiad 2021

Problem No. 4.

A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd\Big(f(f(x)), f(x+y)\Big) = \gcd(x, y)$$

for all pairs of positive integers x and y.

Find all positive integers m such that $f(m) = m$ for all Canadian functions f.

Solution. Define an $m \in \mathbb{N}$ to be *good* if $f(m) = m$ for all such f. It will be shown that m is good if and only if m has two or more distinct prime divisors. Let $P(x, y)$ denote the assertion

$$\gcd(f(f(x)), f(x+y)) = \gcd(x, y)$$

for a pair $x, y \in \mathbb{N}$. Let x be a positive integer with two or more distinct prime divisors and let p^k be largest power of one of these prime divisors such that $p^k \mid x$. If $x = p^k \cdot q$, then p^k and q are relatively prime and $x > p^k, q > 1$. By $P(q, x-q)$,

$$\gcd(f(f(q)), f(x-q+q)) = \gcd(f(f(q)), f(x)) = \gcd(q, x-q) = q$$

which implies that $q \mid f(x)$. By $P(p^k, x-p^k)$,

$$\gcd(f(f(p^k)), f(x-p^k+p^k)) = \gcd(f(f(p^k)), f(x)) = \gcd(p^k, x-p^k) = p^k$$

which implies that $p^k \mid f(x)$. Since p^k and q are relatively prime, $x = p^k \cdot q$ divides $f(x)$, which implies that $f(x) \geq x$. Now assume for contradiction that $f(x) > x$. Let $y = f(x) - x > 0$ and note that, by $P(x, y)$, it follows that

$$f(f(x)) = \gcd(f(f(x)), f(x+f(x)-x)) = \gcd(x, f(x)-x) = \gcd(x, f(x)).$$

Therefore $f(f(x)) \mid x$ and $f(f(x)) \mid f(x)$. By $P(x, x)$, it follows that

$$\gcd(f(f(x)), f(2x)) = \gcd(x, x) = x.$$

This implies that $x \mid f(f(x))$, which when combined with the above result, yields that $f(f(x)) = x$. Since $x \mid f(x)$ and x is divisible by at least two distinct prime numbers, $f(x)$ is also divisible by at least two distinct prime numbers. As shown previously, this implies that $f(x) \mid f(f(x)) = x$, which is a contradiction since $f(x) > x$. Therefore $f(x) = x$ for all positive integers x with two or more distinct prime divisors.

Now it will be shown that all $m \in \mathbb{N}$ such that either m has one prime divisor or $m = 1$ are not good. In either case, let $m = p^k$ where $k \geq 0$ and p is a prime number and consider the function satisfying that $f(p^k) = p^{k+1}$, $f(p^{k+1}) = p^k$ and $f(x) = x$ for all $x \neq p^k, p^{k+1}$. Note that this function also satisfies that $f(f(x)) = x$ for all positive integers x. If $x + y \neq p^k, p^{k+1}$, then $P(x, y)$ holds by the Euclidean algorithm since $f(f((x)) = x$ and $f(x + y) = x + y$. If $x + y = p^{k+1}$, then $P(x, y)$ is equivalent to $\gcd(x, p^k) = \gcd(x, p^{k+1} - x) = \gcd(x, p^{k+1})$ for all $x < p^{k+1}$ which holds since the greatest power of p that can divide x is p^k. If $x + y = p^k$, then $P(x, y)$ is equivalent to $\gcd(x, p^{k+1}) = \gcd(x, p^k - x) = \gcd(x, p^k)$ for all $x < p^k$ which holds as shown above. Note that if $m = 1$ then this case cannot occur. Since this function satisfies $P(x, y)$, m is good if and only if m has two or more distinct prime divisors. $\square$

Canadian Mathematical Olympiad 2021

Problem No. 5.

Nina and Tadashi play the following game. Initially, a triple (a, b, c) of nonnegative integers with $a+b+c = 2021$ is written on a blackboard. Nina and Tadashi then take moves in turn, with Nina first. A player making a move chooses a positive integer k and one of the three entries on the board; then the player increases the chosen entry by k and decreases the other two entries by k. A player loses if, on their turn, some entry on the board becomes negative.

Find the number of initial triples (a, b, c) for which Tadashi has a winning strategy.

Solution. The answer is $3^{\text{number of 1's in binary expansion of 2021}} = 3^8 = 6561$.

Throughout this solution, we say two nonnegative integers *overlap in the* 2^ℓ *position* if their binary representations both have a 1 in that position. We say that two nonnegative integers *overlap* if they overlap in some position. Our central claim is the following.

Claim 1. *A triple (x, y, z) is losing if and only if no two of x, y, z overlap.*

Let $d_\ell(a)$ denote the bit in the 2^ℓ position of the binary representation of a. Let & denote the bitwise and operation: $x\&y$ is the number satisfying $d_\ell(x\&y) = d_\ell(x)d_\ell(y)$ for all ℓ.

Lemma 1. *Let x, y, z be nonnegative integers, at least one pair of which overlaps. Define $x' = (x+y)\&(x+z)$ and y', z' cyclically. At least one of the inequalities $x < x'$, $y < y'$, $z < z'$ holds.*

Proof. Let ℓ be maximal such that two of x, y, z overlap in the 2^ℓ position. We case on how many of the additions $x + y$, $x + z$, $y + z$ involve a carry from the 2^ℓ position to the $2^{\ell+1}$ position, and on the values of $d_{\ell+1}(x), d_{\ell+1}(y), d_{\ell+1}(z)$. Because at least two of $d_\ell(x), d_\ell(y), d_\ell(z)$ equal 1, at least one of the additions $x + y$, $x + z$, $y + z$ involves a carry from the 2^ℓ position.

Case 1. *One carry.*

WLOG let $x+y$ be the addition with the carry. Then, $d_\ell(x) = d_\ell(y) = 1$ and $d_\ell(z) = 0$. Since x, y, z do not overlap in any position left of the 2^ℓ position, the binary representations of z, z' agree left of the 2^ℓ position. As the additions $x+z$ and $y+z$ do not involve a carry from the 2^ℓ position, we have $d_\ell(x+z) = d_\ell(y+z) = 1$, and thus $d_\ell(z') = 1$. Thus $z' > z$, as desired.

Case 2. *At least two carries: $x+y$ and $x+z$ carry and $d_{\ell+1}(y) = d_{\ell+1}(z) = 0$, or cyclic equivalent. ($y+z$ may or may not carry.)*

Let i be maximal such that $d_{\ell+1}(x) = \cdots = d_{\ell+i}(x) = 1$ (possibly $i = 0$). By maximality of ℓ, $d_{\ell+1}(y) = \cdots = d_{\ell+i}(y) = d_{\ell+1}(z) = \cdots = d_{\ell+i}(z) = 0$. By maximality of i, $d_{\ell+i+1}(x) = 0$.

If $d_{\ell+i+1}(y) = d_{\ell+i+1}(z) = 0$, then $d_{\ell+i+1}(x + y) = d_{\ell+i+1}(x + z) = 1$, so $d_{\ell+i+1}(x') = 1$. The binary representations of x and x' agree to the left of the $2^{\ell+i+1}$ position, so $x' > x$.

Otherwise, WLOG $d_{\ell+i+1}(y) = 1$ and $d_{\ell+i+1}(z) = 0$. (Note that, here we in fact have $i \geq 1$.) Then $d_{\ell+i+1}(y+z) = d_{\ell+i+1}(x+z) = 1$, so $d_{\ell+i+1}(z') = 1$. The binary representations of z and z' agree to the left of the $2^{\ell+i+1}$ position, so $z' > z$.

Case 3. *At least two carries, and the condition in Case 2 does not occur.*

WLOG let $x+y, x+z$ involve carries. Since the condition in Case 2 does not occur, $d_{\ell+1}(y) = 1$ or $d_{\ell+1}(z) = 1$. In either case, $d_{\ell+1}(x) = 0$. WLOG $d_{\ell+1}(y) = 1$ and $d_{\ell+1}(z) = 0$.

Since the condition in Case 2 does not occur, $y+z$ does not involve a carry from the 2^ℓ position. (Otherwise, $x+y$ and $y+z$ carry and $d_{\ell+1}(x) = d_{\ell+1}(z) = 0$.) Then $d_{\ell+1}(x+z) = d_{\ell+1}(y+z) = 1$, so $d_{\ell+1}(z') = 1$. The binary representations of z and z' agree to the left of the $2^{\ell+1}$ position, so $z' > z$. □

Proof of Claim 1. Proceed by strong induction on $x+y+z$. There is no base case.

Suppose by induction the claim holds for all (x, y, z) with sum less than N. Consider a triple (x, y, z) with $x+y+z = N$.

Suppose no two of x, y, z overlap. If all moves from this position lead to positions with a negative coordinate, (x, y, z) is a losing position, as claimed. Otherwise, the player increases or decreases all coordinates by k. Consider the smallest m such that $d_m(k) = 1$. The player's move will toggle each of $d_m(x), d_m(y), d_m(z)$. Since at most one of the original $d_m(x), d_m(y), d_m(z)$ is 1, at least two of the new $d_m(x), d_m(y), d_m(z)$ will be 1. So, two of the new x, y, z overlap. By induction, the new (x, y, z) is winning. Thus the original (x, y, z) is losing, as claimed.

Conversely, suppose at least one pair of x, y, z overlap. By Lemma 1, at least one of $x < x', y < y', z < z'$ holds. WLOG $x < x'$. Let the player to move choose $k = x' - x$, decrease y, z by k, and increase x by k. The new coordinates are nonnegative, as

$$y - k = x + y - x' \geq 0$$

because $x' \leq x + y$, and similarly for the z coordinate. Moreover, the binary representation of the new x consists of the 1's in the binary representations of both $x+y$ and $x+z$; the binary representation of the new y consists of the 1's in that of $x+y$ but not $x+z$; and the binary representation of the new z consists of the 1's in that of $x+z$ but not $x+y$. So, no two of the new x, y, z overlap. By induction, the new (x, y, z) is losing. Thus the original (x, y, z) is winning, as claimed. □

We use Claim 1 to count the losing positions (x, y, z) with

$$x + y + z = 2021 = 11111100101_2.$$

In each position where $d_\ell(2021) = 0$, losing positions must have $d_\ell(x) = d_\ell(y) = d_\ell(z) = 0$. In each position where $d_\ell(2021) = 1$, the bit triplet $(d_i(x), d_i(y), d_i(z))$ is one of $(1,0,0), (0,1,0), (0,0,1)$. This gives a count of $3^8 = 6561$.

□

Canadian Mathematical Olympiad 2022

A competition of the Canadian Mathematical Society.

Official Problem Set

P1. Assume that real numbers a and b satisfy

$$ab + \sqrt{ab+1} + \sqrt{a^2+b} \cdot \sqrt{b^2+a} = 0.$$

Find, with proof, the value of

$$a\sqrt{b^2+a} + b\sqrt{a^2+b}.$$

P2. Let $d(k)$ denote the number of positive integer divisors of k. For example, $d(6) = 4$ since 6 has 4 positive divisors, namely, 1, 2, 3, and 6. Prove that for all positive integers n,

$$d(1) + d(3) + d(5) + \cdots + d(2n-1) \leq d(2) + d(4) + d(6) + \cdots + d(2n).$$

P3. Let $n \geq 2$ be an integer. Initially, the number 1 is written n times on a board. Every minute, Vishal picks two numbers written on the board, say a and b, erases them, and writes either $a+b$ or $\min\{a^2, b^2\}$. After $n-1$ minutes there is one number left on the board. Let the largest possible value for this final number be $f(n)$. Prove that

$$2^{n/3} \; < \; f(n) \; \leq \; 3^{n/3}.$$

P4. Let n be a positive integer. A set of n distinct lines divides the plane into various (possibly unbounded) regions. The set of lines is called "nice" if no three lines intersect at a single point. A "colouring" is an assignment of two colours to each region such that the first colour is from the set $\{A_1, A_2\}$, and the second colour is from the set $\{B_1, B_2, B_3\}$. Given a nice set of lines, we call it "colourable" if there exists a colouring such that

(a) no colour is assigned to two regions that share an edge;

(b) for each $i \in \{1,2\}$ and $j \in \{1,2,3\}$ there is at least one region that is assigned with both A_i and B_j.

Determine all n such that every nice configuration of n lines is colourable.

P5. Let $ABCDE$ be a convex pentagon such that the five vertices lie on a circle and the five sides are tangent to another circle inside the pentagon. There are $\binom{5}{3} = 10$ triangles which can be formed by choosing 3 of the 5 vertices. For each of these 10 triangles, mark its incenter. Prove that these 10 incenters lie on two concentric circles.

Important!

Please do not discuss this problem set online for at least 24 hours!

The 2022 Canadian Mathematical Olympiad

Official Solutions for CMO 2022

P1. Assume that real numbers a and b satisfy

$$ab + \sqrt{ab+1} + \sqrt{a^2+b} \cdot \sqrt{b^2+a} = 0.$$

Find, with proof, the value of

$$a\sqrt{b^2+a} + b\sqrt{a^2+b}.$$

Solution. Let us rewrite the given equation as follows:

$$ab + \sqrt{a^2+b}\sqrt{b^2+a} = -\sqrt{ab+1}.$$

Squaring this gives us

$$\begin{aligned}
a^2b^2 + 2ab\sqrt{a^2+b}\sqrt{b^2+a} + (a^2+b)(b^2+a) &= ab+1 \\
(a^2b^2+a^3) + 2ab\sqrt{a^2+b}\sqrt{b^2+a} + (a^2b^2+b^3) &= 1 \\
\left(a\sqrt{b^2+a} + b\sqrt{a^2+b}\right)^2 &= 1 \\
a\sqrt{b^2+a} + b\sqrt{a^2+b} &= \pm 1.
\end{aligned}$$

Next, we show that $a\sqrt{b^2+a} + b\sqrt{a^2+b} > 0$. Note that

$$ab = -\sqrt{ab+1} - \sqrt{a^2+b} \cdot \sqrt{b^2+a} < 0,$$

so a and b have opposite signs. Without loss of generality, we may assume $a > 0 > b$. Then rewrite

$$a\sqrt{b^2+a} + b\sqrt{a^2+b} = a(\sqrt{b^2+a} + b) - b(a - \sqrt{a^2+b})$$

and, since $\sqrt{b^2+a} + b$ and $a - \sqrt{a^2+b}$ are both positive, the expression above is positive. Therefore,

$$a\sqrt{b^2+a} + b\sqrt{a^2+b} = 1,$$

and the proof is finished. $\square$

P2. Let $d(k)$ denote the number of positive integer divisors of k. For example, $d(6) = 4$ since 6 has 4 positive divisors, namely, 1, 2, 3, and 6. Prove that for all positive integers n,

$$d(1) + d(3) + d(5) + \cdots + d(2n-1) \leq d(2) + d(4) + d(6) + \cdots + d(2n).$$

Solution. For any integer k and set of integers S, let $f_S(k)$ be the number of multiples of k in S. We can count the number of pairs (k, s) with $k \in \mathbb{N}$ dividing $s \in S$ in two different ways, as follows:

- For each $s \in S$, there are $d(s)$ pairs that include s, one for each divisor of s.
- For each $k \in \mathbb{N}$, there are $f_k(S)$ pairs that include k, one for each multiple of k.

Therefore,

$$\sum_{s \in S} d(s) = \sum_{k \in \mathbb{N}} f_S(k).$$

Let

$$O = \{1, 3, 5, \ldots, 2n-1\} \qquad \text{and} \qquad E = \{2, 4, 6, \ldots, 2n\}$$

be the set of odd and, respectively, the set of even integers between 1 and $2n$. It suffices to show that

$$\sum_{k \in \mathbb{N}} f_O(k) \leq \sum_{k \in \mathbb{N}} f_E(k).$$

Since the elements of O only have odd divisors,

$$\sum_{k \in \mathbb{N}} f_O(k) = \sum_{k \text{ odd}} f_O(k).$$

For any odd k, consider the multiples of k between 1 and $2n$. They form a sequence

$$k, 2k, 3k, \ldots, \left\lfloor \frac{2n}{k} \right\rfloor k$$

alternating between odd and even terms. There are either an equal number of odd and even terms, or there is one more odd term than even terms. Therefore, we have the inequality

$$f_O(k) \leq f_E(k) + 1$$

for all odd k. Combining this with the previous observations gives us the desired inequality:

$$\begin{aligned}
\sum_{k \in \mathbb{N}} f_O(k) &= \sum_{k \text{ odd}} f_O(k) \\
&\leq \sum_{k \text{ odd}} (f_E(k) + 1) \\
&= \sum_{k \text{ odd}} f_E(k) + n \\
&= \sum_{k \text{ odd}} f_E(k) + f_E(2) \\
&\leq \sum_{k \in \mathbb{N}} f_E(k).
\end{aligned}$$

$\square$

P3: Let $n \geq 2$ be an integer. Initially, the number 1 is written n times on a board. Every minute, Vishal picks two numbers written on the board, say a and b, erases them, and writes either $a+b$ or $\min\{a^2, b^2\}$. After $n-1$ minutes there is one number left on the board. Let the largest possible value for this final number be $f(n)$. Prove that

$$2^{n/3} < f(n) \leq 3^{n/3}.$$

Solution. Clearly $f(n)$ is a strictly increasing function, as we can form $f(n-1)$ with $n-1$ ones, and add the final one. However, we can do better; assume Vishal generates $f(n)$ on the board. After $n-2$ minutes, there are two numbers left, say they were formed by x ones and y ones, where $x+y=n$. Clearly the numbers are at most $f(x), f(y)$ (and can be made to be equal to $f(x), f(y)$), and therefore we obtain

$$f(n) = \max_{x+y=n,\, 1\leq x\leq y\leq n-1} \Big(\max\left(f(x)+f(y), f(x)^2\right)\Big) \tag{1}$$

where we used the fact that f is increasing to get that $\min(f(x)^2, f(y)^2) = f(x)^2$ when $x \leq y$. In particular, $f(n+1) \geq f(n)+1$, and $f(2n) \geq f(n)^2$ for all positive integers n.

Upper bound:

First proof of upper bound. We use induction. We can check that $f(n) = n$ for $n \leq 4$, and these all satisfy the bound $f(n) = n \leq 3^{n/3}$. Assume it is true for all $m < n$ (some $n \geq 5$), and with x, y as in equation **??** we have

$$f(x)^2 \leq f\left(\left\lfloor \frac{n}{2} \right\rfloor\right)^2 \leq \left(3^{n/6}\right)^2 = 3^{n/3},$$

as desired. It thus remains to show that $f(x)+f(y) \leq 3^{n/3}$. By induction, it suffices to prove that

$$3^{x/3} + 3^{y/3} \leq 3^{(x+y)/3},$$

for $1 \leq x \leq y \leq n-1$ and $x+y=n$. This is equivalent to

$$1 + 3^{(y-x)/3} \leq 3^{y/3}.$$

Let $w = 3^{(y-x)/3}$, and we require $3^{x/3}w \geq w+1$. If $x \geq 2$, then this is true as $w \geq 1$, and if $x = 1$ then $w = 3^{(n-2)/3} \geq 3$ and the result is still true. Thus all terms in equation **??** are at most $3^{n/3}$, and so $f(n) \leq 3^{n/3}$, and the upper bound is proven.

Second proof of upper bound. Consider a second game with the same rules but in which Vishal can replace a and b by either $a+b$ or ab. Let $g(n)$ be the largest possible value for this new game. Then $f(n) \leq g(n)$ because $\min\{a^2, b^2\} \leq ab$.

We can check $g(n) = n$ for $n \leq 4$, so $g(n) \leq 3^{n/3}$ for these values. If x and y are both bigger than 1, then $g(x) + g(y) \leq g(x)g(y)$. Therefore, for $n > 4$, we have that

$$g(n) = \max\left\{g(n-1)+1, \max_{1\leq x\leq n-1} g(x)g(n-x)\right\}$$

Now proceed similarly to the first proof. Assume $n > 4$ and $g(m) \leq 3^{m/3}$ for all $m < n$. If $1 \leq x \leq n-1$, then $g(x)g(n-x) \leq 3^{x/3}\,3^{(n-x)/3} = 3^{n/3}$. And $g(n-1)+1 \leq 3^{(n-1)/3}+1$, which is shown to be less than $3^{n/3}$ in the first proof. It follows that $f(n) \leq g(n) \leq 3^{n/3}$.

Lower bound:

First proof of lower bound. We begin with a lemma.

Lemma 1. *Let m be a nonnegative integer. Then*

$$f(2^m) \geq 2^{2^{m-1}} \quad \textit{and} \quad f(3\cdot 2^m) \geq 3^{2^m}.$$

Proof. We prove the lemma by induction. One can check that $f(n) = n$ for $n \leq 3$, which proves the lemma for $m = 0$. For a general $m > 0$, we get

$$\begin{aligned} f(2^m) &\geq f(2^{m-1})^2 \geq \left(2^{2^{m-2}}\right)^2 = 2^{2^{m-1}} \\ f(3\cdot 2^m) &\geq f(3\cdot 2^{m-1})^2 \geq \left(3^{2^{m-1}}\right)^2 = 3^{2^m}, \end{aligned}$$

by induction, as required. $\square$

(This lemma can also be proved more constructively. Briefly, if $n = 2^m$, then partition the 1's on the board into 2^{m-1} pairs, and then add each pair to get 2^{m-1} 2's ($2 = 2^{2^0}$); then multiply pairs of 2's to get 2^{m-2} 4's ($4 = 2^{2^1}$); then multiply pairs of 4's to get 2^{m-3} 16's ($16 = 2^{2^2}$); and so on, until there are 2 ($= 2^1$) copies of $2^{2^{m-2}}$, which then gets replaced with a$2^{2^{m-1}}$). The process is similar for $n = 3\cdot 2^m$, except that the first step is to partition the 1's into 2^m groups of 3, and then use addition within each group to get 2^m 3's on the board.)

Now assume $2^x \leq n < 3\cdot 2^{x-1}$ for some integer x. Then we have

$$f(n) \geq f(2^x) \geq 2^{2^{x-1}} > 2^{n/3},$$

as required. If no such x exists, then there exists an integer x such that $3\cdot 2^{x-1} \leq n < 2^{x+1}$. In this case, we have

$$f(n) \geq f(3\cdot 2^{x-1}) \geq 3^{2^{x-1}} > 2^{2^{x+1}/3} > 2^{n/3}$$

where the second last inequality is equivalent to $2^{x-1}\log(3) \geq \frac{2^{x+1}}{3}\log(2)$, and by dividing out 2^x and clearing the denominator this is equivalent to $3\log(3) \geq 4\log 2$, which is true as $3^3 = 27 > 16 = 2^4$.

Second proof of lower bound. We shall prove the stronger result $f(n) \geq 2^{(n+1)/3}$ for $n \geq 2$ by induction. One can check that $f(n) = n$ for $n = 2, 3, 4$, which proves the result for these values. Assume that $n \geq 5$ and that $f(k) \geq 2^{(k+1)/3}$ for all $k = 2, 3, \ldots, n-1$. Then

$$\begin{aligned}
f(n) &\geq f\left(\lfloor n/2 \rfloor\right)^2 \\
&\geq \left(2^{(\lfloor n/2 \rfloor + 1)/3}\right)^2 && \text{since } \left\lfloor \tfrac{n}{2} \right\rfloor \geq 2 \\
&= 2^{(2\lfloor n/2 \rfloor + 2)/3} \\
&\geq 2^{(n+1)/3} && \text{since } \left\lfloor \tfrac{n}{2} \right\rfloor \geq \tfrac{n-1}{2}.
\end{aligned}$$

The result follows by induction. □

Remark 1. *One can show that f satisfies the recurrence $f(n) = n$ for $n = 1, 2$, $f(2n) = f(n)^2$ for $n \geq 2$, and $f(2n+1) = f(2n) + 1$ for $n \geq 1$. The upper bound in the problem is tight (equality holds for $n = 3 \cdot 2^x$), but the lower bound is not.*

P4. Let n be a positive integer. A set of n distinct lines divides the plane into various (possibly unbounded) regions. The set of lines is called "nice" if no three lines intersect at a single point. A "colouring" is an assignment of two colours to each region such that the first colour is from the set $\{A_1, A_2\}$, and the second colour is from the set $\{B_1, B_2, B_3\}$. Given a nice set of lines, we call it "colourable" if there exists a colouring such that

1. no colour is assigned to two regions that share an edge;
2. for each $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ there is at least one region that is assigned with both A_i and B_j.

Determine all n such that every nice configuration of n lines is colourable.

Solution. The answer is $n \geq 5$. If $n \leq 4$, consider n parallel lines. There are 6 total colour combinations required, and only $n + 1 \leq 5$ total regions, hence the colouring is not possible.

Now, assume $n \geq 5$. Rotate the picture so that no line is horizontal, and orient each line so that the "forward" direction increases the $y-$value. In this way, each line divides the plane into a right and left hand side (with respect to this forward direction). Every region of the plane is on the right hand side of k lines and on the left hand side of $n - k$ lines for some $0 \leq k \leq n$. Furthermore, there is a region for every k: let w be large enough so that w is greater than the $y-$value of any intersection point of two lines. Consider the horizontal line $y = w$: a point very far on the left of this line is left of every single line, and as we cross over all lines in the problem, we hit all values of k.

Finally, take a region that is on the right hand side of k lines. Colour it A_1 if k is odd, and A_2 if it is even. Similarly, colour it B_i if $k \equiv i \pmod 3$. By the previous paragraph, there are regions for at least $k = 0, 1, \ldots, 5$, whence there is a region coloured A_i and B_j for all (i, j). Furthermore, two regions that share an edge will be on the right hand side of k and $k + 1$ lines for some k. By construction, the A_i and B_i colours of the regions must differ, hence we have proven that the set of lines is colourable. $\square$

P5. Let $ABCDE$ be a convex pentagon such that the five vertices lie on a circle and the five sides are tangent to another circle inside the pentagon. There are $\binom{5}{3} = 10$ triangles which can be formed by choosing 3 of the 5 vertices. For each of these 10 triangles, mark its incenter. Prove that these 10 incenters lie on two concentric circles.

Solution. Let I be the incenter of pentagon $ABCDE$. Let I_A denote the incenter of triangle EAB and I_a the incenter DAC. Define $I_B, I_b, I_C, I_c, I_D, I_d, I_E, I_e$ similarly.

We will first show that $I_A I_B I_C I_D I_E$ are concyclic. Let ω_A be the circle with center at the midpoint of arc DE and passing through D and E. Define $\omega_B, \omega_C, \omega_D, \omega_E$ similarly. It is well-known that the incenter of a triangle lies on such circles, in particular, I_A lies on ω_C and ω_D. So the radical axis of ω_C, ω_D is the line AI_A. But this is just the angle bisector of $\angle EAB$, which I also lies on. So I is in fact the radical center of $\omega_A, \omega_B, \omega_C, \omega_D, \omega_E$! Inverting about I swaps I_A and A and since $ABCDE$ are concyclic, $I_A I_B I_C I_D I_E$ are concyclic as well.

Let O be the center of the circle $I_A I_B I_C I_D I_E$. We will now show that $OI_a = OI_d$ which finishes the problem as we can consider the cyclic versions of this equation to find that $OI_a = OI_d = OI_b = OI_e = OI_c$. Recall a well-known lemma: For any cyclic quadrilateral $WXYZ$, the incenters of XYZ, YZW, ZWX, WXY form a rectangle. Applying this lemma on $ABCD$, we see that I_B, I_C, I_a, I_d form a rectangle in that order. Then the perpendicular bisector of $I_B I_C$ is exactly the perpendicular bisector of $I_a I_d$. Thus, O is equidistant to I_a and I_d and we are done. $\square$

