

COMPENDIUM CJMO

Canadian Junior Mathematical Olympiad 2020-2025



Gerard Romo Garrido

Toomates Coolección vol. 59.2



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Fuente.

<https://cms.math.ca/competitions/cmo/#cjmo>

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Official Problem Set

1. Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers that satisfies

$$a_1 = 1 \quad \text{and} \quad a_{n+1}^2 + a_{n+1} = a_n \quad \text{for every natural number } n.$$

Prove that $a_n \geq \frac{1}{n}$ for every natural number n .

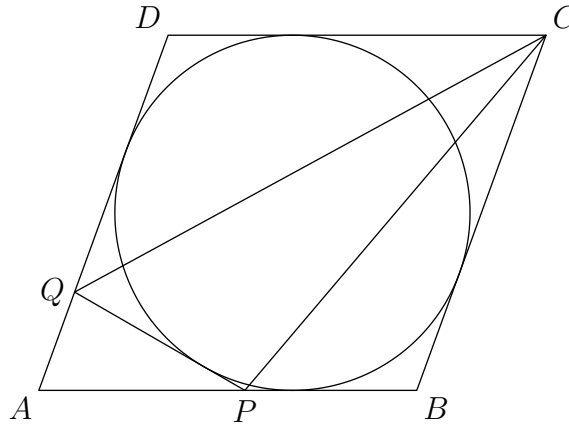
2. Ziquan makes a drawing in the plane for art class. He starts by placing his pen at the origin, and draws a series of line segments, such that the n^{th} line segment has length n . He is not allowed to lift his pen, so that the end of the n^{th} segment is the start of the $(n+1)^{\text{th}}$ segment. Line segments drawn are allowed to intersect and even overlap previously drawn segments.

After drawing a finite number of line segments, Ziquan stops and hands in his drawing to his art teacher. He passes the course if the drawing he hands in is an N by N square, for some positive integer N , and he fails the course otherwise. Is it possible for Ziquan to pass the course?

3. Let S be a set of $n \geq 3$ positive real numbers. Show that the largest possible number of distinct integer powers of three that can be written as the sum of three distinct elements of S is $n - 2$.

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4. A circle is inscribed in a rhombus $ABCD$. Points P and Q vary on line segments $\overline{AB}$ and $\overline{AD}$, respectively, so that $\overline{PQ}$ is tangent to the circle. Show that for all such line segments $\overline{PQ}$, the area of triangle CPQ is constant.



5. A purse contains a finite number of coins, each with distinct positive integer values. Is it possible that there are exactly 2020 ways to use coins from the purse to make the value 2020?

Important!

Please do not discuss this problem set online for at least 24 hours.

Canadian Junior Mathematical Olympiad 2020

Official Solutions

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1. Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers that satisfies

$$a_1 = 1 \quad \text{and} \quad a_{n+1}^2 + a_{n+1} = a_n \quad \text{for every natural number } n.$$

Prove that $a_n \geq \frac{1}{n}$ for every natural number n .

Solution: We prove the inequality by induction. To start, observe that the inequality is obvious for $n = 1$.

Assume that the inequality $a_n \geq \frac{1}{n}$ holds for a given value of n . Let f be the function $f(x) = x^2 + x$, so that we have $f(a_{n+1}) = a_n$. In order to prove that $a_{n+1} \geq \frac{1}{n+1}$, it suffices to show that $f(a_{n+1}) \geq f\left(\frac{1}{n+1}\right)$ (since f is an increasing function on the positive real numbers). We have

$$\begin{aligned} f\left(\frac{1}{n+1}\right) &= \left(\frac{1}{n+1}\right)^2 + \frac{1}{n+1} \\ &= \frac{n+2}{(n+1)^2} \\ &= \frac{n^2 + 2n}{n(n+1)^2} \\ &< \frac{(n+1)^2}{n(n+1)^2} \\ &= \frac{1}{n} \\ &\leq a_n = f(a_{n+1}). \end{aligned}$$

Thus we can conclude that $a_{n+1} \geq \frac{1}{n+1}$.

This completes the induction proof. □

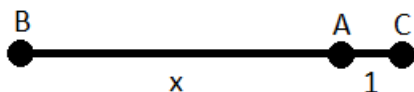
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2. Ziquan makes a drawing in the plane for art class. He starts by placing his pen at the origin, and draws a series of line segments, such that the n^{th} line segment has length n . He is not allowed to lift his pen, so that the end of the n^{th} segment is the start of the $(n+1)^{\text{th}}$ segment. Line segments drawn are allowed to intersect and even overlap previously drawn segments.

After drawing a finite number of line segments, Ziquan stops and hands in his drawing to his art teacher. He passes the course if the drawing he hands in is an N by N square, for some positive integer N , and he fails the course otherwise. Is it possible for Ziquan to pass the course?

Solution: We will prove that Ziquan can pass the course by drawing a square with side length $N = 54$. First, if Ziquan draws a line segment of length x from point A to B , then he can draw a segment of length $x+1$ backwards from B to C , landing at the point C which is on the line AB and one unit right of A .



This has the net effect of drawing a line segment of length 1 with a tail of length x in the opposite direction. Thus if Ziquan has already drawn the segment AB , the net effect is extending the existing line segment by 1 unit. We call this a “unit shift”.

Ziquan starts by drawing the first 11 line segments on the x -axis, all going to the right, except the segment of length 6 which goes to the left. This creates a line segment from $A = (0, 0)$ to $B = (54, 0)$, as $54 = 1 + 2 + 3 + 4 + 5 - 6 + 7 + 8 + 9 + 10 + 11$. Ziquan draws the second side of the square by drawing vertical segments going up of lengths 12, 13, 14, 15, ending at $C = (54, 54)$. He goes left by 16, 17, 18, which puts him at $(3, 54)$. Three unit shifts left lands him at $D = (0, 54)$, having just drawn a segment of length 24. He finishes the square by going down with segments of length 25, 26, followed by three final unit shifts down. Note that every unit shift done has sufficient tail.

Alternate Solution: Yes, for $N = 30$: Start in one direction with $1 + 2 + 3$, turn right for $4+5+6+7+8$, turn right for $9+10+11$, turn right for $12+13-14+15-16+17-18+19-20+21-22+23$, turn right for 24.

Other Possible Solutions: For $N = 78$ ($n = 56$), $N = 120$ (with two different number of steps n , 119 and 71), and $N = 190$ ($n = 149$).

□

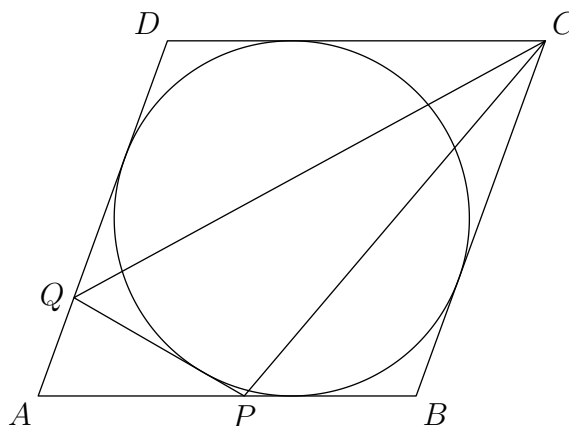
3. Let S be a set of $n \geq 3$ positive real numbers. Show that the largest possible number of distinct integer powers of three that can be written as the sum of three distinct elements of S is $n - 2$.

Solution: We will show by induction that for all $n \geq 3$, it holds that at most $n - 2$ powers of three are sums of three distinct elements of S for any set S of positive real numbers with $|S| = n$. This is trivially true when $n = 3$. Let $n \geq 4$ and consider the largest element $x \in S$. The sum of x and any two other elements of S is strictly between x and $3x$. Therefore x can be used as a summand for at most one power of three. By the induction hypothesis, at most $n - 3$ powers of three are sums of three distinct elements of $S \setminus \{x\}$. This completes the induction.

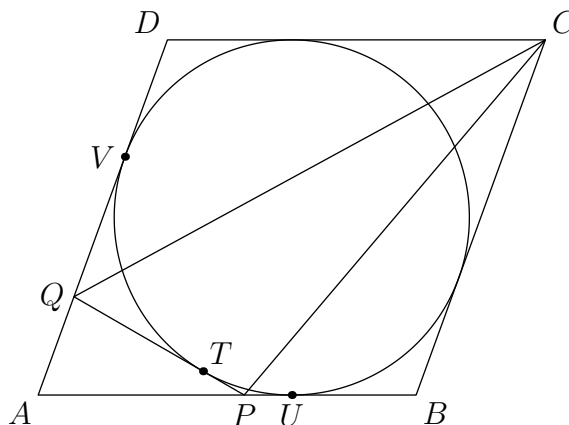
Even if it was not asked to prove, we will now show that the optimal answer $n - 2$ is reached. Observe that the set $S = \{1, 2, 3^2 - 3, 3^3 - 3, \dots, 3^n - 3\}$ is such that $3^2, 3^3, \dots, 3^n$ can be expressed as sums of three distinct elements of S . This makes use of the fact that each term of the form $3^k - 3$ can be used in exactly one sum of three terms equal to 3^k .

□

4. A circle is inscribed in a rhombus $ABCD$. Points P and Q vary on line segments $\overline{AB}$ and $\overline{AD}$, respectively, so that $\overline{PQ}$ is tangent to the circle. Show that for all such line segments $\overline{PQ}$, the area of triangle CPQ is constant.



Solution: Let the circle be tangent to $\overline{PQ}$, $\overline{AB}$, $\overline{AD}$ at T , U , and V , respectively. Let $p = PT = PU$ and $q = QT = QV$. Let $a = AU = AV$ and $b = BU = DV$. Then the side length of the rhombus is $a + b$.



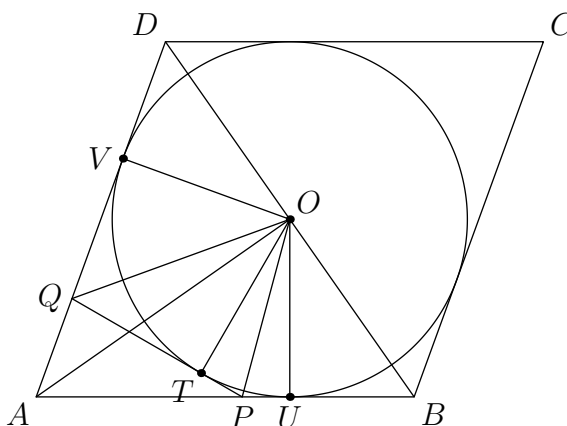
Let $\theta = \angle BAD$, so $\angle ABC = \angle ADC = 180^\circ - \theta$. Then (using the notation $[XYZ]$ for the area of a triangle of vertices X, Y, Z)

$$\begin{aligned} [APQ] &= \frac{1}{2} \cdot AP \cdot AQ \cdot \sin \theta = \frac{1}{2}(a - p)(a - q) \sin \theta, \\ [BCP] &= \frac{1}{2} \cdot BP \cdot BC \cdot \sin(180^\circ - \theta) = \frac{1}{2}(b + p)(a + b) \sin \theta, \\ [CDQ] &= \frac{1}{2} \cdot DQ \cdot CD \cdot \sin(180^\circ - \theta) = \frac{1}{2}(b + q)(a + b) \sin \theta, \end{aligned}$$

so

$$\begin{aligned}
 [CPQ] &= [ABCD] - [APQ] - [BCP] - [CDQ] \\
 &= (a+b)^2 \sin \theta - \frac{1}{2}(a-p)(a-q) \sin \theta - \frac{1}{2}(b+p)(a+b) \sin \theta - \frac{1}{2}(b+q)(a+b) \sin \theta \\
 &= \frac{1}{2}(a^2 + 2ab - bp - bq - pq) \sin \theta.
 \end{aligned}$$

Let O be the center of the circle, and let r be the radius of the circle. Let $x = \angle TOP = \angle UOP$ and $y = \angle TOQ = \angle VOQ$. Then $\tan x = \frac{p}{r}$ and $\tan y = \frac{q}{r}$.



Note that $\angle UOV = 2x + 2y$, so $\angle AOU = x + y$. Also, $\angle AOB = 90^\circ$, so $\angle OBU = x + y$. Therefore,

$$\tan(x + y) = \frac{a}{r} = \frac{r}{b},$$

so $r^2 = ab$. But

$$\frac{r}{b} = \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{p}{r} + \frac{q}{r}}{1 - \frac{p}{r} \cdot \frac{q}{r}} = \frac{r(p+q)}{r^2 - pq} = \frac{r(p+q)}{ab - pq}.$$

Hence, $ab - pq = bp + bq$, so $bp + bq + pq = ab$. Therefore,

$$[CPQ] = \frac{1}{2}(a^2 + 2ab - bp - bq - pq) \sin \theta = \frac{1}{2}(a^2 + ab) \sin \theta,$$

which is constant.

Alternate Solution: Let O be the center of the circle and r its radius. Then $[CPQ] = [CDQPB] - [CDQ] - [CBP]$, where $[...]$ denotes area of the polygon with given vertices. Note that $[CDQPB]$ is half r times the perimeter of $CDQPB$. Note that the heights of CDQ and CBP are $2r$ so $[CDQ] = r \cdot DQ$ and $[CBP] = r \cdot PB$. Using the fact that $QT = QV$ and $PU = PT$, it now follows that $[CPQ] = [OVDCBU] - [CDV] - [CBU]$, which is independent of P and Q .

□

5. A purse contains a finite number of coins, each with distinct positive integer values. Is it possible that there are exactly 2020 ways to use coins from the purse to make the value 2020?

Solution: It is possible.

Consider a coin purse with coins of values 2, 4, 8, 2014, 2016, 2018, 2020 and every odd number between 503 and 1517. Call such a coin *big* if its value is between 503 and 1517. Call a coin *small* if its value is 2, 4 or 8 and *huge* if its value is 2014, 2016, 2018 or 2020. Suppose some subset of these coins contains no huge coins and sums to 2020. If it contains at least four big coins, then its value must be at least $503 + 505 + 507 + 509 > 2020$. Furthermore since all of the small coins are even in value, if the subset contains exactly one or three big coins, then its value must be odd. Thus the subset must contain exactly two big coins. The eight possible subsets of the small coins have values 0, 2, 4, 6, 8, 10, 12, 14. Therefore the ways to make the value 2020 using no huge coins correspond to the pairs of big coins with sums 2006, 2008, 2010, 2012, 2014, 2016, 2018 and 2020. The numbers of such pairs are 250, 251, 251, 252, 252, 253, 253, 254, respectively. Thus there are exactly 2016 subsets of this coin purse with value 2020 using no huge coins. There are exactly four ways to make a value of 2020 using huge coins; these are $\{2020\}$, $\{2, 2018\}$, $\{4, 2016\}$ and $\{2, 4, 2014\}$. Thus there are exactly 2020 ways to make the value 2020.

Alternate construction: Take the coins $1, 2, \dots, 11, 1954, 1955, \dots, 2019$. The only way to get 2020 is a non-empty subset of $1, \dots, 11$ and a single *large* coin. There are 2047 non-empty such subsets of sums between 1 and 66. Thus they each correspond to a unique large coin making 2020, so we have 2047 ways. Thus we only need to remove some large coins, so that we remove exactly 27 small sums. This can be done, for example, by removing coins $2020 - n$ for $n = 1, 5, 6, 7, 8, 9$, as these correspond to $1 + 3 + 4 + 5 + 6 + 8 = 27$ partitions into distinct numbers that are at most 11.

□

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Official Problem Set

1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two concentric circles with $\mathcal{C}_1$ inside $\mathcal{C}_2$. Let P_1 and P_2 be two points on $\mathcal{C}_1$ that are not diametrically opposite. Extend the segment P_1P_2 past P_2 until it meets the circle $\mathcal{C}_2$ in Q_2 . The tangent to $\mathcal{C}_2$ at Q_2 and the tangent to $\mathcal{C}_1$ at P_1 meet in a point X . Draw from X the second tangent to $\mathcal{C}_2$ which meets $\mathcal{C}_2$ at the point Q_1 . Show that P_1X bisects angle $Q_1P_1Q_2$.
2. How many ways are there to permute the first n positive integers such that in the permutation, for each value of $k \leq n$, the first k elements of the permutation have distinct remainder mod k ?
3. Let $ABCD$ be a trapezoid with AB parallel to CD , $|AB| > |CD|$, and equal edges $|AD| = |BC|$. Let I be the center of the circle tangent to lines AB , AC and BD , where A and I are on opposite sides of BD . Let J be the center of the circle tangent to lines CD , AC and BD , where D and J are on opposite sides of AC . Prove that $|IC| = |JB|$.
4. Let $n \geq 2$ be some fixed positive integer and suppose that $a_1, a_2, \dots, a_n$ are positive real numbers satisfying $a_1 + a_2 + \dots + a_n = 2^n - 1$.

Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1 + a_1} + \frac{a_3}{1 + a_1 + a_2} + \dots + \frac{a_n}{1 + a_1 + a_2 + \dots + a_{n-1}}.$$

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5. A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd(f(f(x)), f(x+y)) = \gcd(x, y)$$

for all pairs of positive integers x and y .

Find all positive integers m such that $f(m) = m$ for all Canadian functions f .

Important!

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Official Solutions

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Problem No. 1.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two concentric circles with $\mathcal{C}_1$ inside $\mathcal{C}_2$. Let P_1 and P_2 be two points on $\mathcal{C}_1$ that are not diametrically opposite. Extend the segment P_1P_2 past P_2 until it meets the circle $\mathcal{C}_2$ in Q_2 . The tangent to $\mathcal{C}_2$ at Q_2 and the tangent to $\mathcal{C}_1$ at P_1 meet in a point X . Draw from X the second tangent to $\mathcal{C}_2$ which meets $\mathcal{C}_2$ at the point Q_1 . Show that P_1X bisects angle $Q_1P_1Q_2$.

Solution. We will show that the angles $\angle Q_2P_1X$ and $\angle Q_1P_1X$ are congruent. Note that, if O denotes the centre of both circles, the points P_1, X, Q_2 and Q_1 lie on the circle of diameter XO since XP_1 is tangent to the circle thus $\angle OP_1X = \pi/2$, and similar for the other tangents XP_2, XQ_1, XQ_2 . On the other hand, $m(\angle Q_2P_1X)$ is half the measure of the arc XQ_2 and $m(\angle Q_1P_1X)$ is half the measure of the arc XQ_1 , and these two arcs are equal because $|XQ_2| = |XQ_1|$.

□

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Problem No. 2.

How many ways are there to permute the first n positive integers such that in the permutation, for each value of $k \leq n$, the first k elements of the permutation have distinct remainder mod k ?

Solution. We show by induction that the first k elements of the permutation must be k consecutive integers from $1, \dots, n$. It is trivially true that for $k = n$ all remainders mod n are distinct and we induct downwards to show that, with the above condition, it is true for all $k < n$ that first k elements have distinct remainders mod k . Note that in any k consecutive integers, the only two with the same remainder mod $(k-1)$ are the smallest and largest integers, so one of these two must be the k -th integer of the permutation. This completes the induction, and thus at every step taking away the k -th entry of the permutation, there are 2 choices to eliminate an integer (the largest or the smallest) and obtain a new permutation where the first $k-1$ entries have distinct remainders mod $(k-1)$, so the answer is 2^{n-1} .

□

Problem No. 3.

Let $ABCD$ be a trapezoid with AB parallel to CD , $|AB| > |CD|$, and equal edges $|AD| = |BC|$. Let I be the center of the circle tangent to lines AB , AC and BD , where A and I are on opposite sides of BD . Let J be the center of the circle tangent to lines CD , AC and BD , where D and J are on opposite sides of AC . Prove that $|IC| = |JB|$.

Solution. Let $\{P\} = AC \cap BD$ and let $\angle APB = 180 - 2a$. Since $ABCD$ is an isosceles trapezoid, APB is an isosceles triangle. Therefore $\angle PBA = a$, which implies that $\angle PBI = 90^\circ - a/2$ since I lies on the external bisector of $\angle PBA$. Since I lies on the bisector of $\angle CPB$, it follows that $\angle BPI = a$ and hence that IPB is isosceles with $|IP| = |PB|$. Similarly JPC is isosceles with $|JP| = |PC|$. So, in the triangles CPI and BPJ we have $PI \equiv PB$ and $PJ \equiv CP$. Since I and J both lie on the internal bisector of $\angle BPC$, it follows that triangles CPI and BPJ are congruent. Therefore $|IC| = |JB|$. $\square$

Problem No. 4.

Let $n \geq 2$ be some fixed positive integer and suppose that $a_1, a_2, \dots, a_n$ are positive real numbers satisfying $a_1 + a_2 + \dots + a_n = 2^n - 1$.

Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1 + a_1} + \frac{a_3}{1 + a_1 + a_2} + \dots + \frac{a_n}{1 + a_1 + a_2 + \dots + a_{n-1}}.$$

Solution. We claim the the minimum possible value of this expression is n . Observe that by AM-GM, we have that

$$\begin{aligned} & \frac{a_1}{1} + \frac{a_2}{1 + a_1} + \dots + \frac{a_n}{1 + a_1 + a_2 + \dots + a_{n-1}} \\ &= \frac{1 + a_1}{1} + \frac{1 + a_1 + a_2}{1 + a_1} + \dots + \frac{1 + a_1 + a_2 + \dots + a_n}{1 + a_1 + a_2 + \dots + a_{n-1}} - n \\ &\geq n \cdot \sqrt[n]{\frac{1 + a_1}{1} \cdot \frac{1 + a_1 + a_2}{1 + a_1} \dots \frac{1 + a_1 + a_2 + \dots + a_n}{1 + a_1 + a_2 + \dots + a_{n-1}}} - n \\ &= n \cdot \sqrt[n]{1 + a_1 + a_2 + \dots + a_n} - n \\ &= 2n - n = n. \end{aligned}$$

Furthermore, equality is achieved when $a_k = 2^{k-1}$ for each $1 \leq k \leq n$. □

Problem No. 5.

A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd(f(f(x)), f(x+y)) = \gcd(x, y)$$

for all pairs of positive integers x and y .

Find all positive integers m such that $f(m) = m$ for all Canadian functions f .

Solution. Define an $m \in \mathbb{N}$ to be *good* if $f(m) = m$ for all such f . It will be shown that m is good if and only if m has two or more distinct prime divisors. Let $P(x, y)$ denote the assertion

$$\gcd(f(f(x)), f(x+y)) = \gcd(x, y)$$

for a pair $x, y \in \mathbb{N}$. Let x be a positive integer with two or more distinct prime divisors and let p^k be largest power of one of these prime divisors such that $p^k \mid x$. If $x = p^k \cdot q$, then p^k and q are relatively prime and $x > p^k, q > 1$. By $P(q, x - q)$,

$$\gcd(f(f(q)), f(x - q + q)) = \gcd(f(f(q)), f(x)) = \gcd(q, x - q) = q$$

which implies that $q \mid f(x)$. By $P(p^k, x - p^k)$,

$$\gcd(f(f(p^k)), f(x - p^k + p^k)) = \gcd(f(f(p^k)), f(x)) = \gcd(p^k, x - p^k) = p^k$$

which implies that $p^k \mid f(x)$. Since p^k and q are relatively prime, $x = p^k \cdot q$ divides $f(x)$, which implies that $f(x) \geq x$. Now assume for contradiction that $f(x) > x$. Let $y = f(x) - x > 0$ and note that, by $P(x, y)$, it follows that

$$f(f(x)) = \gcd(f(f(x)), f(x + f(x) - x)) = \gcd(x, f(x) - x) = \gcd(x, f(x)).$$

Therefore $f(f(x)) \mid x$ and $f(f(x)) \mid f(x)$. By $P(x, x)$, it follows that

$$\gcd(f(f(x)), f(2x)) = \gcd(x, x) = x.$$

This implies that $x \mid f(f(x))$, which when combined with the above result, yields that $f(f(x)) = x$. Since $x \mid f(x)$ and x is divisible by at least two distinct prime numbers, $f(x)$ is also divisible by at least two distinct prime numbers. As shown previously, this implies that $f(x) \mid f(f(x)) = x$, which is a contradiction since $f(x) > x$. Therefore $f(x) = x$ for all positive integers x with two or more distinct prime divisors.

Now it will be shown that all $m \in \mathbb{N}$ such that either m has one prime divisor or $m = 1$ are not good. In either case, let $m = p^k$ where $k \geq 0$ and p is a prime number and consider the function satisfying that $f(p^k) = p^{k+1}$, $f(p^{k+1}) = p^k$ and $f(x) = x$ for all $x \neq p^k, p^{k+1}$. Note that this function also satisfies that $f(f(x)) = x$ for all positive integers x . If $x + y \neq p^k, p^{k+1}$, then $P(x, y)$ holds by the Euclidean

algorithm since $f(f(x)) = x$ and $f(x + y) = x + y$. If $x + y = p^{k+1}$, then $P(x, y)$ is equivalent to $\gcd(x, p^k) = \gcd(x, p^{k+1} - x) = \gcd(x, p^{k+1})$ for all $x < p^{k+1}$ which holds since the greatest power of p that can divide x is p^k . If $x + y = p^k$, then $P(x, y)$ is equivalent to $\gcd(x, p^{k+1}) = \gcd(x, p^k - x) = \gcd(x, p^k)$ for all $x < p^k$ which holds as shown above. Note that if $m = 1$ then this case cannot occur. Since this function satisfies $P(x, y)$, m is good if and only if m has two or more distinct prime divisors. $\square$

Canadian Junior Mathematical Olympiad 2022



A competition of the Canadian Mathematical Society.

Official Problem Set

- P1. Let ABC be an acute angled triangle with circumcircle Γ . The perpendicular from A to BC intersects Γ at D , and the perpendicular from B to AC intersects Γ at E . Prove that if $|AB| = |DE|$, then $\angle ACB = 60^\circ$.
- P2. You have an infinite stack of T-shaped tetrominoes (composed of four squares of side length 1), and an $n \times n$ board. You are allowed to place some tetrominoes on the board, possibly rotated, as long as no two tetrominoes overlap and no tetrominoes extend off the board. For which values of n can you cover the entire board?

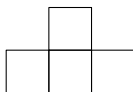


Figure 1: T-shaped tetromino

- P3. Assume that real numbers a and b satisfy

$$ab + \sqrt{ab + 1} + \sqrt{a^2 + b} \cdot \sqrt{b^2 + a} = 0.$$

Find, with proof, the value of

$$a\sqrt{b^2 + a} + b\sqrt{a^2 + b}.$$

- P4. Let $d(k)$ denote the number of positive integer divisors of k . For example, $d(6) = 4$ since 6 has 4 positive divisors, namely, 1, 2, 3, and 6. Prove that for all positive integers n ,

$$d(1) + d(3) + d(5) + \cdots + d(2n-1) \leq d(2) + d(4) + d(6) + \cdots + d(2n).$$

- P5. Let $n \geq 2$ be an integer. Initially, the number 1 is written n times on a board. Every minute, Vishal picks two numbers written on the board, say a and b , erases them, and writes either $a + b$ or $\min\{a^2, b^2\}$. After $n - 1$ minutes there is one number left on the board. Let the largest possible value for this final number be $f(n)$. Prove that

$$2^{n/3} < f(n) \leq 3^{n/3}.$$

Important!

Please do not discuss this problem set online until 24 hours after the CJMO finishes!

2022 Canadian Junior Mathematical Olympiad



Official Solutions for CJMO 2022

P1. Let ABC be an acute angled triangle with circumcircle Γ . The perpendicular from A to BC intersects Γ at D , and the perpendicular from B to AC intersects Γ at E . Prove that if $|AB| = |DE|$, then $\angle ACB = 60^\circ$.

Solution. Since AB and ED are equal chords in the same circle, we either have $\angle ACB = \angle ECD$ or $\angle ACB + \angle ECD = 180^\circ$. We compute

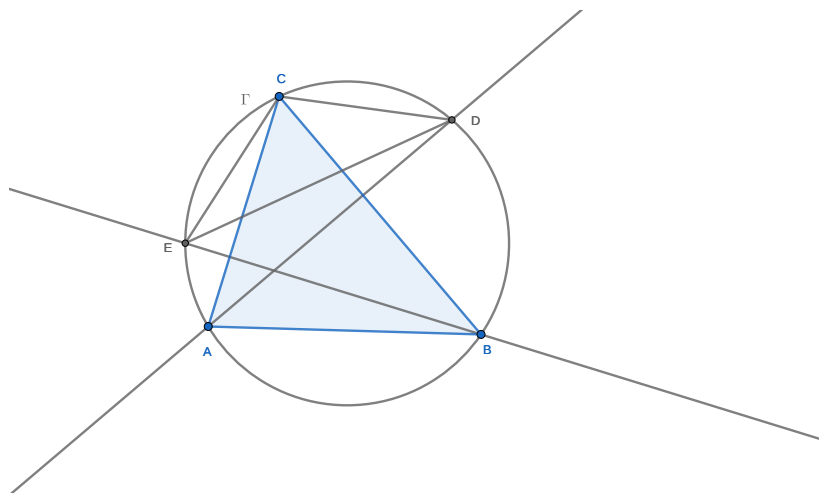


Figure 1: Illustration for Problem 1.

$$\begin{aligned}
 \angle ECD &= \angle ECA + \angle ACB + \angle BCD \\
 &= \angle EBA + \angle ACB + \angle BAD \\
 &= 90^\circ - \angle BAC + \angle ACB + 90^\circ - \angle ABC \\
 &= (180^\circ - \angle BAC - \angle ABC) + \angle ACB \\
 &= 2\angle ACB.
 \end{aligned}$$

If $\angle ACB = \angle ECD$ then $\angle ACB = 0^\circ$, contradiction. Thus $\angle ACB + \angle ECD = 180^\circ$, whence $\angle ACB = 60^\circ$. $\square$

P2. You have an infinite stack of T-shaped tetrominoes (composed of four squares of side length 1), and an $n \times n$ board. You are allowed to place some tetrominoes on the board, possibly rotated, as long as no two tetrominoes overlap and no tetrominoes extend off the board. For which values of n can you cover the entire board?

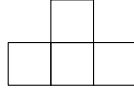
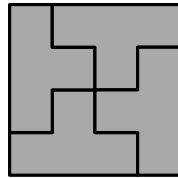


Figure 2: T-shaped tetromino.

Solution. Let us first note that one can cover the entire board if and only if $4 \mid n$. Indeed, one can cover a 4×4 board as follows:



Since for any $m \in \mathbb{N}$, $4m \times 4m$ board can be covered by a grid of m^2 4×4 squares, all multiples of four are possible.

Suppose then that $4 \nmid n$, and assume first that n is odd. Each tetromino covers 4 squares, hence if one can cover the entire board, then the final area covered must be a multiple of 4. Since n^2 is odd, this is impossible.

Finally, suppose that $n = 4k + 2$ for some $k \in \mathbb{N} \cup \{0\}$. For a contradiction, suppose that one can cover the entire board. Colour the squares of the board with white and black paint like a chessboard such that the bottom left corner square is white. Since n is even, there is the same number of white and black squares. Therefore, there will be $n^2/2 = 8k^2 + 8k + 2$ white squares overall, which is an even number. Note that, since white squares do not have a common border, each T-tetromino covers an odd number of white squares (exactly 1 or 3). Since we need to place $n^2/4 = 4k^2 + 4k + 1 \equiv 1 \pmod{2}$ tetrominoes, which is an odd number, we will cover an odd number of white squares. This gives us the desired contradiction with the observation that we have an even number of white squares, and thus one cannot cover the board. $\square$

P3. Assume that real numbers a and b satisfy

$$ab + \sqrt{ab + 1} + \sqrt{a^2 + b} \cdot \sqrt{b^2 + a} = 0.$$

Find, with proof, the value of

$$a\sqrt{b^2 + a} + b\sqrt{a^2 + b}.$$

Solution. Let us rewrite the given equation as follows:

$$ab + \sqrt{a^2 + b}\sqrt{b^2 + a} = -\sqrt{ab + 1}.$$

Squaring this gives us

$$\begin{aligned} a^2b^2 + 2ab\sqrt{a^2 + b}\sqrt{b^2 + a} + (a^2 + b)(b^2 + a) &= ab + 1 \\ (a^2b^2 + a^3) + 2ab\sqrt{a^2 + b}\sqrt{b^2 + a} + (a^2b^2 + b^3) &= 1 \\ \left(a\sqrt{b^2 + a} + b\sqrt{a^2 + b}\right)^2 &= 1 \\ a\sqrt{b^2 + a} + b\sqrt{a^2 + b} &= \pm 1. \end{aligned}$$

Next, we show that $a\sqrt{b^2 + a} + b\sqrt{a^2 + b} > 0$. Note that

$$ab = -\sqrt{ab + 1} - \sqrt{a^2 + b} \cdot \sqrt{b^2 + a} < 0,$$

so a and b have opposite signs. Without loss of generality, we may assume $a > 0 > b$. Then rewrite

$$a\sqrt{b^2 + a} + b\sqrt{a^2 + b} = a(\sqrt{b^2 + a} + b) - b(a - \sqrt{a^2 + b})$$

and, since $\sqrt{b^2 + a} + b$ and $a - \sqrt{a^2 + b}$ are both positive, the expression above is positive. Therefore,

$$a\sqrt{b^2 + a} + b\sqrt{a^2 + b} = 1,$$

and the proof is finished. □

P4. Let $d(k)$ denote the number of positive integer divisors of k . For example, $d(6) = 4$ since 6 has 4 positive divisors, namely, 1, 2, 3, and 6. Prove that for all positive integers n ,

$$d(1) + d(3) + d(5) + \cdots + d(2n-1) \leq d(2) + d(4) + d(6) + \cdots + d(2n).$$

Solution. For any integer k and set of integers S , let $f_S(k)$ be the number of multiples of k in S . We can count the number of pairs (k, s) with $k \in \mathbb{N}$ dividing $s \in S$ in two different ways, as follows:

- For each $s \in S$, there are $d(s)$ pairs that include s , one for each divisor of s .
- For each $k \in \mathbb{N}$, there are $f_k(S)$ pairs that include k , one for each multiple of k .

Therefore,

$$\sum_{s \in S} d(s) = \sum_{k \in \mathbb{N}} f_S(k).$$

Let

$$O = \{1, 3, 5, \dots, 2n-1\} \quad \text{and} \quad E = \{2, 4, 6, \dots, 2n\}$$

be the set of odd and, respectively, the set of even integers between 1 and $2n$. It suffices to show that

$$\sum_{k \in \mathbb{N}} f_O(k) \leq \sum_{k \in \mathbb{N}} f_E(k).$$

Since the elements of O only have odd divisors,

$$\sum_{k \in \mathbb{N}} f_O(k) = \sum_{k \text{ odd}} f_O(k).$$

For any odd k , consider the multiples of k between 1 and $2n$. They form a sequence

$$k, 2k, 3k, \dots, \left\lfloor \frac{2n}{k} \right\rfloor k$$

alternating between odd and even terms. There are either an equal number of odd and even terms, or there is one more odd term than even terms. Therefore, we have the inequality

$$f_O(k) \leq f_E(k) + 1$$

for all odd k . Combining this with the previous observations gives us the desired inequality:

$$\begin{aligned} \sum_{k \in \mathbb{N}} f_O(k) &= \sum_{k \text{ odd}} f_O(k) \\ &\leq \sum_{k \text{ odd}} (f_E(k) + 1) \\ &= \sum_{k \text{ odd}} f_E(k) + n \\ &= \sum_{k \text{ odd}} f_E(k) + f_E(2) \\ &\leq \sum_{k \in \mathbb{N}} f_E(k). \end{aligned}$$

□

P5: Let $n \geq 2$ be an integer. Initially, the number 1 is written n times on a board. Every minute, Vishal picks two numbers written on the board, say a and b , erases them, and writes either $a + b$ or $\min\{a^2, b^2\}$. After $n - 1$ minutes there is one number left on the board. Let the largest possible value for this final number be $f(n)$. Prove that

$$2^{n/3} < f(n) \leq 3^{n/3}.$$

Solution. Clearly $f(n)$ is a strictly increasing function, as we can form $f(n - 1)$ with $n - 1$ ones, and add the final one. However, we can do better; assume Vishal generates $f(n)$ on the board. After $n - 2$ minutes, there are two numbers left, say they were formed by x ones and y ones, where $x + y = n$. Clearly the numbers are at most $f(x), f(y)$ (and can be made to be equal to $f(x), f(y)$), and therefore we obtain

$$f(n) = \max_{x+y=n, 1 \leq x \leq y \leq n-1} \left(\max(f(x) + f(y), f(x)^2) \right) \quad (1)$$

where we used the fact that f is increasing to get that $\min(f(x)^2, f(y)^2) = f(x)^2$ when $x \leq y$. In particular, $f(n + 1) \geq f(n) + 1$, and $f(2n) \geq f(n)^2$ for all positive integers n .

Upper bound:

First proof of upper bound. We use induction. We can check that $f(n) = n$ for $n \leq 4$, and these all satisfy the bound $f(n) = n \leq 3^{n/3}$. Assume it is true for all $m < n$ (some $n \geq 5$), and with x, y as in equation [1](#) we have

$$f(x)^2 \leq f\left(\left\lfloor \frac{n}{2} \right\rfloor\right)^2 \leq (3^{n/6})^2 = 3^{n/3},$$

as desired. It thus remains to show that $f(x) + f(y) \leq 3^{n/3}$. By induction, it suffices to prove that

$$3^{x/3} + 3^{y/3} \leq 3^{(x+y)/3},$$

for $1 \leq x \leq y \leq n - 1$ and $x + y = n$. This is equivalent to

$$1 + 3^{(y-x)/3} \leq 3^{y/3}.$$

Let $w = 3^{(y-x)/3}$, and we require $3^{x/3}w \geq w + 1$. If $x \geq 2$, then this is true as $w \geq 1$, and if $x = 1$ then $w = 3^{(n-2)/3} \geq 3$ and the result is still true. Thus all terms in equation [1](#) are at most $3^{n/3}$, and so $f(n) \leq 3^{n/3}$, and the upper bound is proven.

Second proof of upper bound. Consider a second game with the same rules but in which Vishal can replace a and b by either $a + b$ or ab . Let $g(n)$ be the largest possible value for this new game. Then $f(n) \leq g(n)$ because $\min\{a^2, b^2\} \leq ab$.

We can check $g(n) = n$ for $n \leq 4$, so $g(n) \leq 3^{n/3}$ for these values. If x and y are both bigger than 1, then $g(x) + g(y) \leq g(x)g(y)$. Therefore, for $n > 4$, we have that

$$g(n) = \max \left\{ g(n-1) + 1, \max_{1 \leq x \leq n-1} g(x)g(n-x) \right\}$$

Now proceed similarly to the first proof. Assume $n > 4$ and $g(m) \leq 3^{m/3}$ for all $m < n$. If $1 \leq x \leq n-1$, then $g(x)g(n-x) \leq 3^{x/3} 3^{(n-x)/3} = 3^{n/3}$. And $g(n-1) + 1 \leq 3^{(n-1)/3} + 1$, which is shown to be less than $3^{n/3}$ in the first proof. It follows that $f(n) \leq g(n) \leq 3^{n/3}$.

Lower bound:

First proof of lower bound. We begin with a lemma.

Lemma 1. *Let m be a nonnegative integer. Then*

$$f(2^m) \geq 2^{2^{m-1}} \quad \text{and} \quad f(3 \cdot 2^m) \geq 3^{2^m}.$$

Proof. We prove the lemma by induction. One can check that $f(n) = n$ for $n \leq 3$, which proves the lemma for $m = 0$. For a general $m > 0$, we get

$$\begin{aligned} f(2^m) &\geq f(2^{m-1})^2 \geq \left(2^{2^{m-2}}\right)^2 = 2^{2^{m-1}} \\ f(3 \cdot 2^m) &\geq f(3 \cdot 2^{m-1})^2 \geq \left(3^{2^{m-1}}\right)^2 = 3^{2^m}, \end{aligned}$$

by induction, as required. □

(This lemma can also be proved more constructively. Briefly, if $n = 2^m$, then partition the 1's on the board into 2^{m-1} pairs, and then add each pair to get 2^{m-1} 2's ($2 = 2^0$); then multiply pairs of 2's to get 2^{m-2} 4's ($4 = 2^1$); then multiply pairs of 4's to get 2^{m-3} 16's ($16 = 2^2$); and so on, until there are 2 ($= 2^1$) copies of $2^{2^{m-2}}$, which then gets replaced with $2^{2^{m-1}}$. The process is similar for $n = 3 \cdot 2^m$, except that the first step is to partition the 1's into 2^m groups of 3, and then use addition within each group to get 2^m 3's on the board.)

Now assume $2^x \leq n < 3 \cdot 2^{x-1}$ for some integer x . Then we have

$$f(n) \geq f(2^x) \geq 2^{2^{x-1}} > 2^{n/3},$$

as required. If no such x exists, then there exists an integer x such that $3 \cdot 2^{x-1} \leq n < 2^{x+1}$. In this case, we have

$$f(n) \geq f(3 \cdot 2^{x-1}) \geq 3^{2^{x-1}} > 2^{2^{x+1}/3} > 2^{n/3}$$

where the second last inequality is equivalent to $2^{x-1} \log(3) \geq \frac{2^{x+1}}{3} \log(2)$, and by dividing out 2^x and clearing the denominator this is equivalent to $3 \log(3) \geq 4 \log(2)$, which is true as $3^3 = 27 > 16 = 2^4$.

Second proof of lower bound. We shall prove the stronger result $f(n) \geq 2^{(n+1)/3}$ for $n \geq 2$ by induction. One can check that $f(n) = n$ for $n = 2, 3, 4$, which proves the result for these values. Assume that $n \geq 5$ and that $f(k) \geq 2^{(k+1)/3}$ for all $k = 2, 3, \dots, n-1$. Then

$$\begin{aligned} f(n) &\geq f(\lfloor n/2 \rfloor)^2 \\ &\geq \left(2^{(\lfloor n/2 \rfloor + 1)/3}\right)^2 && \text{since } \lfloor \frac{n}{2} \rfloor \geq 2 \\ &= 2^{(2\lfloor n/2 \rfloor + 2)/3} \\ &\geq 2^{(n+1)/3} && \text{since } \lfloor \frac{n}{2} \rfloor \geq \frac{n-1}{2}. \end{aligned}$$

The result follows by induction. □

Remark 1. One can show that f satisfies the recurrence $f(n) = n$ for $n = 1, 2$, $f(2n) = f(n)^2$ for $n \geq 2$, and $f(2n+1) = f(2n) + 1$ for $n \geq 1$. The upper bound in the problem is tight (equality holds for $n = 3 \cdot 2^x$), but the lower bound is not.

Canadian Junior Mathematical Olympiad 2023



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Official Problem Set

- P1. Let a and b be non-negative integers. Consider a sequence $s_1, s_2, s_3, \dots$ such that $s_1 = a$, $s_2 = b$, and $s_{i+1} = |s_i - s_{i-1}|$ for $i \geq 2$. Prove that there is some i for which $s_i = 0$.
- P2. An acute triangle is a triangle that has all angles less than 90° (90° is a Right Angle). Let ABC be a right-angled triangle with $\angle ACB = 90^\circ$. Let CD be the altitude from C to AB , and let E be the intersection of the angle bisector of $\angle ACD$ with AD . Let EF be the altitude from E to BC . Prove that the circumcircle of BEF passes through the midpoint of CE .
- P3. William is thinking of an integer between 1 and 50, inclusive. Victor can choose a positive integer m and ask William: “does m divide your number?”, to which William must answer truthfully. Victor continues asking these questions until he determines William’s number. What is the minimum number of questions that Victor needs to guarantee this?
- P4. There are 20 students in a high school class, and each student has exactly three close friends in the class. Five of the students have bought tickets to an upcoming concert. If any student sees that two or more of their three close friends have bought tickets, then they will buy a ticket too.
- Is it possible that the entire class buys tickets to the concert?
- (Assume that friendship is mutual; if student A is close friends with student B , then B is close friends with A .)
- P5. Let ABC be an acute triangle with altitudes AD , BE , and CF meeting at H . The circle passing through points D , E , and F meets AD , BE , and CF again at X , Y , and Z respectively. Prove the following inequality:

$$\frac{AH}{DX} + \frac{BH}{EY} + \frac{CH}{FZ} \geq 3.$$

Important!

Please do not discuss this problem set online for at least 24 hours!

The Canadian Junior Mathematical Olympiad



Official Solutions for CMO 2023

P1. Let a and b be non-negative integers. Consider a sequence $s_1, s_2, s_3, \dots$ such that $s_1 = a$, $s_2 = b$, and $s_{i+1} = |s_i - s_{i-1}|$ for $i \geq 2$. Prove that there is some i for which $s_i = 0$.

Solution 1. First, note that for any *positive* integers x, y , we have

$$|x - y| = \max(x, y) - \min(x, y) \leq \max(x, y) - 1 < \max(x, y).$$

Clearly, the sequence $(s_i)_{i \geq 1}$ consists of non-negative integers. For a contradiction, suppose that $s_i \geq 1$ for all $i \geq 1$. Then, for any positive integer k ,

$$\begin{aligned} s_{2(k+1)-1} &= s_{2k+1} = |s_{2k} - s_{2k-1}| < \max(s_{2k-1}, s_{2k}) \\ s_{2(k+1)} &= s_{2k+2} = |s_{2k+1} - s_{2k}| < \max(s_{2k}, s_{2k+1}) \\ &\leq \max(s_{2k}, \max(s_{2k-1}, s_{2k})) = \max(s_{2k-1}, s_{2k}). \end{aligned}$$

Hence, if $b_k = \max(s_{2k-1}, s_{2k})$, then

$$b_{k+1} = \max(s_{2(k+1)-1}, s_{2(k+1)}) < \max(s_{2k-1}, s_{2k}) = b_k,$$

so $(b_k)_{k \geq 1}$ is a decreasing sequence. But it means that $b_k = 0$ for some k which gives us the desired contradiction. $\square$

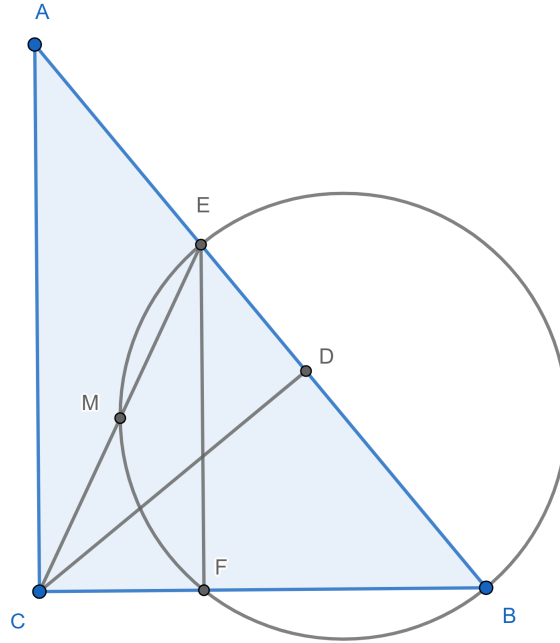
Solution 2. For non-negative integers x, y , we have $|x - y| \leq \max(x, y)$. Therefore

$$\max(s_{i+1}, s_{i+2}) = \max(s_{i+1}, |s_i - s_{i+1}|) \leq \max(s_i, s_{i+1}).$$

Thus the sequence $(\max(s_i, s_{i+1}))$ is non-increasing. Since it is bounded below by 0, it is eventually constant, that is, there exist C and N such that $\max(s_i, s_{i+1}) = C$ for all $i \geq N$. We can assume that $s_N = C$ (if not, replace N by $N + 1$). If $s_{N+1} = C$, then $s_{N+2} = |C - C| = 0$, as desired. If $s_{N+1} = 0$, then we are clearly done. Finally, if $0 < s_{N+1} < C$, then $s_{N+2} = C - s_{N+1} < C$, so $\max(s_{N+1}, s_{N+2}) < C$, which gives us the desired contradiction. $\square$

P2. An acute triangle is a triangle that has all angles less than 90° (90° is a Right Angle). Let ABC be a right-angled triangle with $\angle ACB = 90^\circ$. Let CD be the altitude from C to AB , and let E be the intersection of the angle bisector of $\angle ACD$ with AD . Let EF be the altitude from E to BC . Prove that the circumcircle of BEF passes through the midpoint of CE .

Solution. We provide two solutions.



Solution 1:

Let $\angle CBA = x$. Then $\angle ACD = 90^\circ - \angle CAD = x$, so $\angle ACE = x/2$.

Therefore $\angle BCE = 90^\circ - x/2$, and

$$\angle CEB = 180^\circ - \angle BCE - \angle CBE = 180^\circ - (90^\circ - x/2) - x = 90^\circ - x/2 = \angle BCE,$$

whence $|BC| = |BE|$.

Let the midpoint of EC be M , and as $|BE| = |BC|$, we have $\angle BME = 90^\circ$.

Since $\angle BFE = 90^\circ$, this implies that $BFME$ is cyclic, which proves the problem.

Solution 2:

Let $|AB| = c$, $|BC| = a$, and $|CA| = b$. Since ABC and CBD are similar (right-angled) triangles, we have

$$\frac{|CD|}{b} = \frac{|DB|}{a} = \frac{a}{c},$$

hence $|CD| = ab/c$ and $|DB| = a^2/c$. Thus $|AD| = c - |DB| = b^2/c$. As $|CE|$ is the angle bisector, let $x = |ED|$, and then

$$\frac{x}{b^2/c - x} = \frac{|DE|}{|EA|} = \frac{|CD|}{|CA|} = \frac{ab/c}{b} = \frac{a}{c}.$$

This gives $x = ab^2/c^2 - (a/c)x$, so

$$x = \frac{ab^2}{c(a+c)} = \frac{a(c^2 - a^2)}{c(a+c)} = \frac{a(c-a)}{c}.$$

Therefore

$$|BE| = |BD| + |DE| = \frac{a(c-a)}{c} + \frac{a^2}{c} = \frac{ac}{c} = a = |BC|.$$

Let the midpoint of EC be M , and as $|BE| = |BC|$, we have $\angle BME = 90^\circ$.

Since $\angle BFE = 90^\circ$, this implies that $BFME$ is cyclic, which proves the problem. $\square$

P3. William is thinking of an integer between 1 and 50, inclusive. Victor can choose a positive integer m and ask William: “does m divide your number?”, to which William must answer truthfully. Victor continues asking these questions until he determines William’s number. What is the minimum number of questions that Victor needs to guarantee this?

Solution. The minimum number is 15 questions.

First, we show that 14 or fewer questions is not enough to guarantee success. Suppose Victor asks at most 14 questions, and William responds with “no” to each question unless $m = 1$. Note that these responses are consistent with the secret number being 1. But since there are 15 primes less than 50, some prime p was never chosen as m . That means the responses are also consistent with the secret number being p . Therefore, Victor cannot determine the number for sure because 1 and p are both possible options.

Now we show that Victor can always determine the number with 15 questions. Let N be William’s secret number. First, Victor asks 4 questions, with $m = 2, 3, 5, 7$. We then case on William’s responses.

Case 1. *William answers “no” to all four questions.*

N can only be divisible by primes that are 11 or larger. This means N cannot have multiple prime factors (otherwise $N \geq 11^2 > 50$), so either $N = 1$ or N is one of the 11 remaining primes less than 50. Victor can then ask 11 questions with $m = 11, 13, 17, \dots, 47$, one for each of the remaining primes, to determine the value of N .

Case 2. *William answers “yes” to $m = 2$, and “no” to $m = 3, 5, 7$.*

There are only 11 possible values of N that match these answers (2, 4, 8, 16, 22, 26, 32, 34, 38, 44, and 46). Victor can use his remaining 11 questions on each of these possibilities.

Case 3. *William answers “yes” to $m = 3$, and “no” to $m = 2, 5, 7$.*

There are 5 possible values of N (3, 9, 27, 33, and 39). Similar to Case 2, Victor can ask about these 5 numbers to determine the value of N .

Case 4. *William answers “yes” to multiple questions, or one “yes” to $m = 5$ or $m = 7$.*

Let k be the product of all m ’s that received a “yes” response. Since N is divisible by each of these m ’s, N must be divisible by k . Since $k \geq 5$, there are at most 10 multiples of k between 1 and 50. Victor can ask about each of these multiples of k with his remaining questions. $\square$

P4. There are 20 students in a high school class, and each student has exactly three close friends in the class. Five of the students have bought tickets to an upcoming concert. If any student sees that at least two of their close friends have bought tickets, then they will buy a ticket too.

Is it possible that the entire class buys tickets to the concert?

(Assume that friendship is mutual; if student A is close friends with student B , then B is close friends with A .)

Solution 1. It is impossible for the whole class to buy tickets to the concert.

If two students A and B are close friends, and A has bought a ticket to the concert while B has not, then A is *enticing* B . We call this pair (A, B) an *enticement*.

In order for a student to change their mind and buy a ticket, they first be enticed by at least 2 of their 3 close friends. That means they can only entice at most 1 other friend. Therefore, the total number of enticements among the students decreases by 1 whenever a student changes their mind to buy a ticket.

Initially, the maximum number of enticements is 15 (each of the initial 5 students with tickets has 3 friends to entice). Assume, for the sake of contradiction, that the entire class ends up buying tickets. After the first 14 people buy tickets, the number of enticements is at most $15 - 14 = 1$. This is not enough to convince the last person to buy a ticket, since they need 2 enticements.

Therefore, it is impossible that the entire class buys tickets. $\square$

Solution 2. We shall use the term *friendship* to denote an unordered pair of students who are close friends. Since each of the 20 students is part of exactly 3 friendships, there are exactly 30 friendships in the class. (We could also represent friendships as edges in an undirected graph whose vertices are the 20 students.)

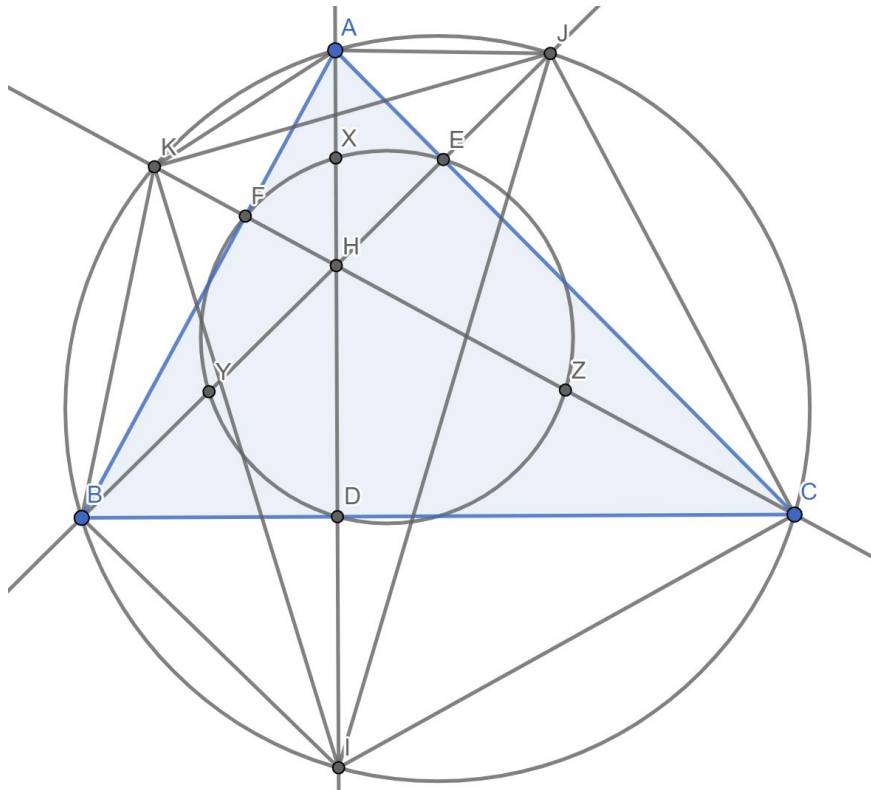
We say that a friendship is *used* if one of the students in that friendship buys a ticket after the original five buyers, and the other student already has a ticket at that time. Each time a ticket is purchased after the original five purchases, at least two friendships are used. Observe that no friendship gets used twice.

If all 20 students buy tickets, then three friendships are used when the last student buys a ticket. This would imply that the number of used friendships is at least $14 \times 2 + 3 = 31$, which is more than the number of friendships. This contradiction proves that it is not possible that the entire class buys tickets. $\square$

P5. An acute triangle is a triangle that has all angles less than 90° (90° is a Right Angle). Let ABC be an acute triangle with altitudes AD , BE , and CF meeting at H . The circle passing through points D , E , and F meets AD , BE , and CF again at X , Y , and Z respectively. Prove the following inequality:

$$\frac{AH}{DX} + \frac{BH}{EY} + \frac{CH}{FZ} \geq 3.$$

Solution. Let the circumcircle of ABC meet the altitudes AD , BE , and CF again at I , J , and K respectively.



Lemma (9-point circle). I , J , K are the reflections of H across BC , CA , AB . Moreover, D , E , F , X , Y , Z are the midpoints of HI , HJ , HK , HA , HB , HC .

Proof. Since $ABDE$ and $ABIC$ are cyclic, we see that

$$\angle EBD = \angle EAD = \angle CAI = \angle CBI.$$

Hence the lines BI and BH are reflections across BC . Similarly, CH and CI are reflections across BC , so I is the reflection of H across BC . The analogous claims for J and K follow. A $\times 2$ dilation from H now establishes the result. $\square$

From this lemma, we get $AI = 2XD$, $BJ = 2EY$, and $CK = 2FZ$. Hence it is equivalent to showing that

$$\frac{AH}{2DX} + \frac{BH}{2EY} + \frac{CH}{2FZ} \geq \frac{3}{2},$$

which is in turn equivalent to

$$\frac{AH}{AI} + \frac{BH}{BJ} + \frac{CH}{CK} \geq \frac{3}{2}. \quad (*)$$

Let $a = JK$, $b = KI$ and $c = IJ$. Again by the lemma we find $AH = AK = AJ$, so by Ptolemy's theorem on $AKIJ$,

$$AJ \cdot KI + AK \cdot IJ = AI \cdot JK.$$

Substituting and rearranging,

$$\begin{aligned} AH \cdot b + AH \cdot c &= AI \cdot a \\ AH \cdot (b + c) &= AI \cdot a \\ \frac{AH}{AI} &= \frac{a}{b + c}. \end{aligned}$$

Similarly,

$$\frac{BH}{BJ} = \frac{b}{c + a} \quad \text{and} \quad \frac{CH}{CK} = \frac{c}{a + b}.$$

Plugging these back into (*), the desired inequality is now

$$\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{3}{2}.$$

This is known as Nesbitt's Inequality, which has many proofs. Below is one such proof.

Add 3 to both sides and rearrange:

$$\begin{aligned} \left(\frac{a}{b + c} + 1 \right) + \left(\frac{b}{c + a} + 1 \right) + \left(\frac{c}{a + b} + 1 \right) &\geq \frac{3}{2} + 3 \\ \iff \frac{a + b + c}{b + c} + \frac{a + b + c}{c + a} + \frac{a + b + c}{a + b} &\geq \frac{9}{2} \\ \iff (a + b + c) \left(\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b} \right) &\geq \frac{9}{2} \\ \iff \frac{(b + c) + (c + a) + (a + b)}{3} &\geq \frac{3}{\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b}} \end{aligned}$$

which is true by the AM-HM inequality. □

Canadian Junior Mathematical Olympiad

Official 2024 Problem Set



- J1.** Centuries ago, the pirate Captain Blackboard buried a vast amount of treasure in a single cell of a 2×4 grid-structured island. You and your crew have reached the island and have brought special treasure detectors to find the cell with the treasure. For each detector, you can set it up to scan a specific subgrid $[a, b] \times [c, d]$ with $1 \leq a \leq b \leq 2$ and $1 \leq c \leq d \leq 4$. Running the detector will tell you whether the treasure is in the region or not, though it cannot say where in the region the treasure was detected. You plan on setting up Q detectors, which may only be run simultaneously after all Q detectors are ready. What is the minimum Q required to guarantee your crew can determine the location of Blackboard's legendary treasure?
- J2.** Let n be a positive integer. Let

$$I_n = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \min\left(\frac{1}{i}, \frac{1}{j}, \frac{1}{k}\right)$$

and $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Determine $I_n - H_n$ in terms of n .

- J3.** Let ABC be a triangle with incenter I . Suppose the reflection of AB across CI and the reflection of AC across BI intersect at a point X . Prove that XI is perpendicular to BC .

(The incenter is the point where the three angle bisectors meet.)

- J4.** Jane writes down 2024 natural numbers around the perimeter of a circle. She wants the 2024 products of adjacent pairs of numbers to be exactly the set $\{1!, 2!, \dots, 2024!\}$. Can she accomplish this?

- J5.** Let N be the number of positive integers with 10 digits $\overline{d_9 d_8 \cdots d_1 d_0}$ in base 10 (where $0 \leq d_i \leq 9$ for all i and $d_9 > 0$) such that the polynomial

$$d_9 x^9 + d_8 x^8 + \cdots + d_1 x + d_0$$

is irreducible in $\mathbb{Q}$. Prove that N is even.

(A polynomial is irreducible in $\mathbb{Q}$ if it cannot be factored into two non-constant polynomials with rational coefficients.)

The Canadian Junior Mathematical Olympiad



Official Solutions for CJMO 2024

J1. Centuries ago, the pirate Captain Blackboard buried a vast amount of treasure in a single cell of a 2×4 grid-structured island. You and your crew have reached the island and have brought special treasure detectors to find the cell with the treasure. For each detector, you can set it up to scan a specific subgrid $[a, b] \times [c, d]$ with $1 \leq a \leq b \leq 2$ and $1 \leq c \leq d \leq 4$. Running the detector will tell you whether the treasure is in the region or not, though it cannot say where in the region the treasure was detected. You plan on setting up Q detectors, which may only be run simultaneously after all Q detectors are ready. What is the minimum Q required to guarantee your crew can determine the location of Blackboard's legendary treasure?

Solution. We shall prove that $Q = 3$.

Let us first observe that $Q \leq 3$, that is, it is possible to complete the task with three detectors on the grid having 2 rows and 4 columns, as follows. The first detector scans the four cells in the first row; a second detector scans all four cells in the first two columns; and the third detector scans all four cells in the second and third columns. The following diagram shows which detectors cover each cell:

12	123	1	3	1
2	23		3	

Notice that no two cells would give the same response from all three detectors, so these three detectors suffice to distinguish the eight possible locations. This proves that $Q \leq 3$.

To see that $Q \geq 3$, observe that there are only 2×2 possible responses from any arrangement of two detectors. These four possible responses are not enough to distinguish eight possible locations. Therefore three detectors are needed. $\square$

J2. Let n be a positive integer. Let

$$I_n = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \min\left(\frac{1}{i}, \frac{1}{j}, \frac{1}{k}\right)$$

and $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Determine $I_n - H_n$ in terms of n .

Solution. Fix a positive integer ℓ with $1 \leq \ell \leq n$. Then $\min\left(\frac{1}{i}, \frac{1}{j}, \frac{1}{k}\right) = \frac{1}{\ell}$ precisely when one of $i, j, k = \ell$ and the others are at most ℓ . By inclusion-exclusion, the number of (i, j, k) that achieve this is $3\ell^2 - 3\ell + 1$. Consequently,

$$I_n = \sum_{\ell=1}^n (3\ell^2 - 3\ell + 1) \cdot \frac{1}{\ell} = \sum_{\ell=1}^n 3\ell - \sum_{\ell=1}^n 3 + \sum_{\ell=1}^n \frac{1}{\ell} = 3 \cdot \frac{n(n+1)}{2} - 3n + H_n$$

so

$$I_n - H_n = \frac{3n(n+1)}{2} - 3n = \frac{3n(n-1)}{2}.$$

□

J3. Let ABC be a triangle with incenter I . Suppose the reflection of AB across CI and the reflection of AC across BI intersect at a point X . Prove that XI is perpendicular to BC .

(The incenter is the point where the three angle bisectors meet.)

Solution. Suppose the reflection of AC across BI intersects BC at E . Define F similarly for the reflection of AB across CI . Also suppose CI intersects AB at M and BI intersects AC at N . Since CA and $CF = BC$ are reflections across CI , and so are MA and $MF = XM$, we have that A and F are reflections across CI . Similarly A and E are reflections across BI . Thus $\angle XFC = \angle BAC = \angle XEB$ if $\angle BAC$ is acute (and $\angle XFC = \angle XEB = \pi - \angle BAC$, when $\angle BAC$ is obtuse), so $XF = XE$. Moreover we also find that $IF = IA = IE$ by the aforementioned reflection properties, so thus XI is the perpendicular bisector of EF and is hence perpendicular to BC . $\square$

J4. Jane writes down 2024 natural numbers around the perimeter of a circle. She wants the 2024 products of adjacent pairs of numbers to be exactly the set $\{1!, 2!, \dots, 2024!\}$. Can she accomplish this?

Solution 1. Given any prime p and positive integer x , let $v_p(x)$ denote the highest power of p dividing x . We claim that Jane cannot write 2024 such numbers as that would imply that $1! \cdot 2! \cdots 2024!$ is the square of the product of the 2024 numbers. Let p be a prime and k be a natural number such that $k < p$, $kp \leq 2024$, and $(k+1)p > 2024$. Then note that

$$v_p(1! \cdot 2! \cdots 2024!) = (2024 - p + 1) + (2024 - 2p + 1) + \dots + (2024 - kp + 1).$$

In particular, let p be in $(\frac{2024}{4}, \frac{2024}{2})$. By Bertrand's Postulate, such a prime p exists (and p must also be odd). Further, the corresponding k is either 2 or 3. Either way, $v_p(1! \cdot 2! \cdots 2024!)$ is odd from the above formula, and so $1! \cdot 2! \cdots 2024!$ cannot be a perfect square. $\square$

Solution 2. As in the first solution, we prove $1! \cdot 2! \cdots 2024!$ is not a perfect square. To do this, note that we can rewrite the product as $(1!)^2 \cdot 2 \cdot (3!)^2 \cdot 4 \cdots (2023!)^2 \cdot 2024$ which is

$$2 \cdot 4 \cdots 2024 \cdot (1! \cdot 3! \cdots 2023!)^2 = 1012! \cdot (2^{1012} \cdot 1! \cdot 3! \cdots 2023!)^2$$

so it is sufficient to verify $1012!$ is not a perfect square. This can be verified by either noticing the prime 1009 only appears as a factor of $1012!$ once, or by evaluating $v_2(1012!) = 1005$. $\square$

J5. Let N be the number of positive integers with 10 digits $\overline{d_9d_8\cdots d_1d_0}$ in base 10 (where $0 \leq d_i \leq 9$ for all i and $d_9 > 0$) such that the polynomial

$$d_9x^9 + d_8x^8 + \cdots + d_1x + d_0$$

is irreducible in $\mathbb{Q}$. Prove that N is even.

(A polynomial is irreducible in $\mathbb{Q}$ if it cannot be factored into two non-constant polynomials with rational coefficients.)

Solution. Let $f(x) = d_9x^9 + d_8x^8 + \cdots + d_1x + d_0$. If $d_0 = 0$, then $f(x)$ is divisible by x and thus reducible, so we may ignore all such polynomials. The remaining polynomials all have nonzero leading and constant coefficients.

For any polynomial $p(x)$ of degree n with nonzero leading and constant coefficients, say $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, define $\bar{p}(x)$ to be the reversed polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. Observe that $\bar{p}(x)$ also has degree n and furthermore, $\bar{p}(x) = x^n \left(a_0 + a_1 \left(\frac{1}{x} \right) + \cdots + a_{n-1} \left(\frac{1}{x} \right)^{n-1} + a_n \left(\frac{1}{x} \right)^n \right) = x^n p \left(\frac{1}{x} \right)$.

Consider pairing each $f(x)$ with $\bar{f}(x)$ whenever $f(x) \neq \bar{f}(x)$. If $f(x)$ is reducible, it can be factored as $f(x) = g(x)h(x)$ where $\deg g, \deg h \geq 1$. Because the leading and constant coefficients of $f(x)$ are nonzero, so are the leading and constant coefficients of $g(x)$ and $h(x)$. Hence $\bar{g}(x)$ and $\bar{h}(x)$ are well defined with $\deg \bar{g} = \deg g \geq 1$ and $\deg \bar{h} = \deg h \geq 1$. Furthermore,

$$\bar{f}(x) = x^9 f \left(\frac{1}{x} \right) = x^9 g \left(\frac{1}{x} \right) h \left(\frac{1}{x} \right) = \left(x^{\deg g} g \left(\frac{1}{x} \right) \right) \left(x^{\deg h} h \left(\frac{1}{x} \right) \right) = \bar{g}(x) \bar{h}(x).$$

Thus $\bar{f}(x) = \bar{g}(x)\bar{h}(x)$ is a factorization of $\bar{f}(x)$ into two non-constant polynomials, so $\bar{f}(x)$ is also reducible. Therefore $f(x)$ is irreducible if and only if $\bar{f}(x)$ is irreducible, so considering each pair, there are an even number of irreducible polynomials with $f(x) \neq \bar{f}(x)$.

Finally, note that if $f(x) = \bar{f}(x)$, then $d_i = d_{9-i}$ for each i . In such a case, we have $f(-1) = (d_0 - d_9) + (d_2 - d_7) + (d_4 - d_5) + (d_6 - d_3) + (d_8 - d_1) = 0$, so by the Factor Theorem, $(x + 1)$ is a factor of $f(x)$. Therefore these remaining polynomials are all reducible. $\square$

Canadian Junior Mathematical Olympiad

Official 2025 Problem Set



- J1.** Suppose an infinite non-constant arithmetic progression of integers contains 1 in it. Prove that there are an infinite number of perfect cubes in this progression.
- (A perfect cube is an integer of the form k^3 , where k is an integer. For example, -8 , 0 , and 1 are perfect cubes.)
- J2.** Let $ABCD$ be a trapezoid with parallel sides AB and CD , where $BC \neq DA$. A circle passing through C and D intersects AC , AD , BC , and BD again at W , X , Y , and Z respectively. Prove that WZ , XY , and AB are concurrent.
- J3.** The n players of a hockey team gather to select their team captain. Initially, they stand in a circle, and each person votes for the person on their left.
- The players will update their votes via a series of rounds. In one round, each player a updates their vote, one at a time, according to the following procedure: At the time of the update, if a is voting for b , and b is voting for c , then a updates their vote to c . (Note that a , b , and c need not be distinct; if $b = c$, then a 's vote does not change for this update.) Every player updates their vote exactly once in each round, in an order determined by the players (possibly different across different rounds).
- They repeat this updating procedure for n rounds. Prove that at this time, all n players will unanimously vote for the same person.
- J4.** Determine all positive integers a, b, c, p where p and $p + 2$ are odd primes and

$$2^a p^b = (p + 2)^c - 1.$$

- J5.** A polynomial $c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$ with degree d is *reflexive* if there is an integer $n \geq d$ such that $c_i = c_{n-i}$ for every $0 \leq i \leq n$, where $c_i = 0$ for $i > d$. Let $\ell \geq 2$ be an integer and $p(x)$ be a polynomial with integer coefficients. Prove that there exist reflexive polynomials $q(x), r(x)$ with integer coefficients such that

$$(1 + x + x^2 + \cdots + x^{\ell-1})p(x) = q(x) + x^\ell r(x).$$

- J1.** Suppose an infinite non-constant arithmetic progression of integers contains 1 in it. Prove that there are an infinite number of perfect cubes in this progression.

(A perfect cube is an integer of the form k^3 , where k is an integer. For example, -8 , 0 , and 1 are perfect cubes.)

Solution

Let a and d be the first term and the common difference in the arithmetic progression, respectively. Clearly, both a and d are integers. Moreover, since the arithmetic progression is non-constant, $d \neq 0$ (but it could be positive or negative!). Note that for any integer k ,

$$\begin{aligned}(1 + kd)^3 &= 1 + 3(kd) + 3(kd)^2 + (kd)^3 \\ &= 1 + d(3k + 3k^2d + k^3d^2).\end{aligned}$$

Clearly, if d is positive, then there are infinitely many *positive* integers ℓ such that

$$\ell = 3k + 3k^2d + k^3d^2$$

for some integer k . For those values of ℓ , $1 + d\ell$ is in the arithmetic progression and it a perfect cube, and the proof is finished.

Suppose now that d is negative. Since

$$3k + 3k^2d + k^3d^2 = 3k + (k^2d)(3 + kd)$$

and both k^2d and $3 + kd$ are negative and decreasing with k for $k \geq 4$, we again get that there are infinitely many *positive* integers ℓ satisfying the same equality. The proof is finished.

- J2.** Let $ABCD$ be a trapezoid with parallel sides AB and CD , where $BC \neq DA$. A circle passing through C and D intersects AC , AD , BC , and BD again at W , X , Y , and Z respectively. Prove that WZ , XY , and AB are concurrent. **Solution**

$\angle AXY = 180^\circ - \angle DXY = \angle YCD = 180^\circ - \angle ABC$, so $ABYX$ and similarly $ABZW$ are both cyclic. Also, by assumption, $XYZW$ are cyclic.

Note that AB is a radical axis of circles $(ABYX)$, $(ABZW)$. Similarly, points XY is a radical axis of $(ABYX)$ and $(XYZW)$. Finally, WZ is a radical axis of $(ABZW)$ and $(XYZW)$. The result follows from the fact that three circles, when taken in pairs, have concurrent radical axis.

- J3.** The n players of a hockey team gather to select their team captain. Initially, they stand in a circle, and each person votes for the person on their left.

The players will update their votes via a series of rounds. In one round, each player a updates their vote, one at a time, according to the following procedure: At the time of the update, if a is voting for b , and b is voting for c , then a updates their vote to c . (Note that a , b , and c need not be distinct; if $b = c$, then a 's vote does not change for this update.) Every player updates their vote exactly once in each round, in an order determined by the players (possibly different across different rounds).

They repeat this updating procedure for n rounds. Prove that at this time, all n players will unanimously vote for the same person.

Comment. Unfortunately, the level of difficulty of this problem was the result of an unintentional clerical error. The version of this problem which had been initially slated for the contest was much easier. It had stated, “They repeat the same updating procedure on the second day, the third day, and so on. Prove that eventually, all players will unanimously vote for the same person.” In other words, there had been no bound of n rounds.

Solution 1

Initially, all players are in a cycle. Note that once a player leaves the cycle, they cannot rejoin. Furthermore, a new cycle cannot be created. Hence, at any point in time, the graph corresponding to the votes will be a functional graph with a single cycle.

We will first prove that after $\lfloor \log_2 n \rfloor$ rounds, the cycle will become a self-loop. Then, we will show that in the next $\lfloor \log_2 n \rfloor$ rounds, all other players vote for the player in the self-loop.

To show the first step, assume the cycle has size $K > 1$ at the beginning of a round. Consider arbitrary player a in the cycle who is updating their vote. Say $a \rightarrow b \rightarrow c$, all in the cycle. Then $a \rightarrow c$ now, bumping b out of the cycle and reducing its size to $K - 1$. Note that b can now update their vote as well without affecting the size of the cycle. If we consider all K original players in the cycle, we see that at least $\lceil \frac{K}{2} \rceil$ of them must still be in the cycle at the time of their update, and hence the cycle's size is reduced to at most $\lfloor \frac{K}{2} \rfloor$. After $\lfloor \log_2 n \rfloor$ rounds, the cycle must be reduced to size 1.

Now that the cycle has been reduced to a single player, say z , consider any path from a player a to z . No players can be added to this path now. With a similar argument as the cycle, the length of the path must halve each round. In particular, a path of length L to the cycle gets reduced to length $\lceil \frac{L}{2} \rceil$ (note the ceiling, we had the floor for the cycle). After $\lceil \log_2 n \rceil$ rounds, the path must be reduced to length 1.

Thus, after $\lfloor \log_2 n \rfloor + \lceil \log_2 n \rceil$ rounds, the graph has been completely reduced. For $n \geq 5$, $\lfloor \log_2 n \rfloor + \lceil \log_2 n \rceil \leq 2\lfloor \log_2 n \rfloor + 1 \leq n$. For the other n , we can manually check

that $\lfloor \log_2 n \rfloor + \lceil \log_2 n \rceil \leq n$.

Comment 1. In this proof, it is important that we consider the cycle size and path sizes in disjoint rounds. In the rounds where the cycle size is decreasing, it's possible for the path sizes to increase by Cn for some constant C .

Comment 2. This proof can actually be refined to prove that $\log_2 n + O(\log_2 \log_2 n)$ rounds are sufficient (and constructions show that this is necessary). Consider a fixed player a and the path from a to the cycle as it changes across the rounds. In round i , let ℓ_i be the number of players which were part of the cycle at the beginning of round i , and became part of this path during round i . (Note that it's possible that a player was part of the path for only a portion of round i , and is no longer part of the path at the end. It should still be counted.) We can see that ℓ_i is bounded by size of the cycle at the start of round i , which we know is $\leq \frac{n}{2^{i-1}}$.

Now note that the size of this path after round k is at most

$$1 + \frac{\ell_1}{2^{k-1}} + \frac{\ell_2}{2^{k-2}} + \dots + \frac{\ell_{k-1}}{2} + \ell_k \leq 1 + \frac{kn}{2^{k-1}}.$$

Hence, when $k = \log_2 n + C \log_2 \log_2 n$ for C sufficiently large, we must have that every player is voting for the self-loop.

Solution 2

We will use induction on n .

Inductive Hypothesis. Let G be any functional graph with n nodes and a single cycle. Then after n rounds of the given operation, G will become a self-loop with $n - 1$ nodes pointing to it.

Base Case. The cases $n \leq 2$ are clear.

Inductive Step. Assume that the hypothesis is proved for $n = k - 1$ and $n = k - 2$. We will prove it for $n = k$. Consider any initial functional graph with k nodes and a single cycle. Note there is some node a which has in-degree 0 (i.e. no nodes point to it), or all k nodes are in the cycle.

In the first case, consider $G \setminus \{a\}$. Note that all operations except a 's own updates are independent of where a is. By the inductive hypothesis, after $k - 1$ rounds, $G \setminus \{a\}$ has become a single self-loop and $k - 2$ nodes pointing to it. Regardless of where a is, it will point to the self-loop after one more round and we are done.

In the case where all k nodes are in a cycle, consider the very first operation $z \rightarrow a \rightarrow b \implies z \rightarrow b, a \rightarrow b$. This creates a zero in-degree node a , but z 's operation has been used for the first round so the inductive hypothesis cannot be naively applied. Instead, consider $b \rightarrow c$ (possibly $c = z$ if $k = 3$). At some point in the first round, b will be updated. Either c will become another zero in-degree node, or a will be the only

node that points to c . Either way, consider $G \setminus \{a, c\}$. By the induction hypothesis, after rounds 2 through $k - 1$, this graph will become a self-loop with $k - 3$ nodes pointing to it. It's also easy to see that a and c both have in-degree 0 after round 2. Then in one more round after round $k - 1$, we must have a and c pointing to the self-loop. So we are done for $n = k$.

By induction, we are done for all n .

Comment. If the problem were instead to prove the result after $2n$ rounds, the induction would be much easier.

J4. Determine all positive integers a, b, c, p where p and $p + 2$ are odd primes and

$$2^a p^b = (p + 2)^c - 1.$$

Solution

The only solution is $(a, b, c, p) = (3, 1, 2, 3)$. First, factor the right hand side. This gives us

$$2^a p^b = (p + 1)((p + 2)^{c-1} + (p + 2)^{c-2} + \cdots + (p + 2) + 1).$$

Since $\gcd(p, p + 1) = 1$ it must be the case that $p + 1 = 2^x$ for some positive integer $x \leq a$ and so $p = 2^x - 1$ and $p + 2 = 2^x + 1$. Now for $x \geq 3$, $2^x + 1$ is not prime if x is odd (since it is $0 \pmod{3}$) and $2^x - 1$ is not prime if x is even (since it is $0 \pmod{3}$). This means $x \leq 2$, and the only admissible such x is $x = 2$ since otherwise p is not prime. So, the original equation becomes

$$2^a 3^b = 5^c - 1.$$

Now $3 \mid (5^c - 1)$, so evaluating $5^c - 1 \pmod{3}$ gives that $c = 2d$ for some positive integer d and hence

$$2^a 3^b = (5^d - 1)(5^d + 1).$$

Observe $5^d - 1$ and $5^d + 1$ are both even and have greatest common divisor 2 because they are 2 apart. Since $4 \mid (5^d - 1)$ this implies $5^d - 1 = 2^{a-1}$ and $5^d + 1 = 2 \cdot 3^b$. Now, 3 is not a factor of $5^d - 1$ because $5^d - 1 = 2^{a-1}$. Thus, by evaluating $\pmod{3}$, d must be odd. If $d > 1$, this is impossible as $5^d - 1 = (5 - 1)(5^{d-1} + 5^{d-2} + \cdots + 5 + 1)$ and the latter factor has an odd prime factor, contradicting $5^d - 1$ is a power of 2. Thus $d = 1$ and so $c = 2$, implying that $2^a 3^b = 24$ so $a = 3$ and $b = 1$. Hence, the only solution is $(a, b, c, p) = (3, 1, 2, 3)$.

- J5.** A polynomial $c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$ with degree d is *reflexive* if there is an integer $n \geq d$ such that $c_i = c_{n-i}$ for every $0 \leq i \leq n$, where $c_i = 0$ for $i > d$. Let $\ell \geq 2$ be an integer and $p(x)$ be a polynomial with integer coefficients. Prove that there exist reflexive polynomials $q(x), r(x)$ with integer coefficients such that

$$(1 + x + x^2 + \cdots + x^{\ell-1})p(x) = q(x) + x^\ell r(x).$$

Solution 1

Let d be the degree of p and let k be any non-negative integer. We will choose

$$q(x) = \frac{x^{d+k+\ell} p\left(\frac{1}{x}\right) - p(x)}{x-1},$$

$$r(x) = \frac{p(x) - x^{d+k} p\left(\frac{1}{x}\right)}{x-1}.$$

First, we must show that both q and r are integer polynomials. Consider the numerator in q 's definition, $x^{d+k+\ell} p\left(\frac{1}{x}\right) - p(x)$. This is clearly an integer polynomial. As it is equal to 0 when evaluated at $x-1$, $x-1$ divides it. Furthermore, as $x-1$ is monic, the quotient has integer coefficients. The argument for r is similar.

Next, we will show that this choice of q and r satisfies the desired equation. Plugging them into the RHS of the equation gives

$$\begin{aligned} q(x) + x^\ell r(x) &= \frac{x^{d+k+\ell} p\left(\frac{1}{x}\right) - p(x)}{x-1} + x^\ell \left(\frac{p(x) - x^{d+k} p\left(\frac{1}{x}\right)}{x-1} \right) \\ &= \frac{x^{d+k+\ell} p\left(\frac{1}{x}\right) - p(x) + x^\ell p(x) - x^{d+k+\ell} p\left(\frac{1}{x}\right)}{x-1} \\ &= \left(\frac{x^\ell - 1}{x-1} \right) p(x) \\ &= (1 + x + \cdots + x^{\ell-1}) p(x) \end{aligned}$$

as desired.

Finally, we will show that q and r are indeed reflexive. We can re-interpret the reflexive condition as such:

Polynomial $a(x)$ is reflexive iff there is an integer $n \geq \deg(a)$ for which

$$a(x) = x^n a\left(\frac{1}{x}\right).$$

We have

$$\begin{aligned}
 q(x) &= \frac{x^{d+k+\ell}p\left(\frac{1}{x}\right) - p(x)}{x-1} \\
 &= x^{d+k+\ell-1} \cdot \frac{p\left(\frac{1}{x}\right) - x^{-(d+k+\ell)}p(x)}{\frac{x-1}{x}} \\
 &= x^{d+k+\ell-1} \cdot \frac{x^{-(d+k+\ell)}p(x) - p\left(\frac{1}{x}\right)}{\frac{1}{x} - 1} \\
 &= x^{d+k+\ell-1}q\left(\frac{1}{x}\right)
 \end{aligned}$$

as desired. Similarly,

$$\begin{aligned}
 r(x) &= \frac{p(x) - x^{d+k}p\left(\frac{1}{x}\right)}{x-1} \\
 &= x^{d+k-1} \cdot \frac{x^{-(d+k)}p(x) - p\left(\frac{1}{x}\right)}{\frac{x-1}{x}} \\
 &= x^{d+k-1} \cdot \frac{p\left(\frac{1}{x}\right) - x^{-(d+k)}p(x)}{\frac{1}{x} - 1} \\
 &= x^{d+k-1}r\left(\frac{1}{x}\right).
 \end{aligned}$$

Solution 2

We write degree n polynomial p as

$$p(x) := \sum_{i=0}^n p_i x^i.$$

Define vector $P \in \mathbb{Z}^{n+1}$ as

$$P := (p_0 \ p_1 \ \cdots \ p_n)^T.$$

We also denote $X \in \mathbb{Z}[x]^N$ for N some sufficiently high degree (e.g. $N > 2n + \ell$) as the vector of powers of x , i.e.

$$X := (1 \ x \ x^2 \ \cdots \ x^{N-1})^T.$$

For a matrix $M \in \mathbb{Z}^{(n+1) \times N}$, $P^T M X$ is an integer polynomial of degree $< N$. Note that if the non-zero entries of matrix M are horizontally symmetric, then the resulting polynomial must be reflexive.

Let A be the matrix corresponding to $(1 + x + \cdots + x^{\ell-1})p(x)$. The non-zero entries of the matrix form a parallelogram. In particular, A is a sparse matrix with 1s across the upper ℓ diagonals. For example, for $\ell = 3, n = 4$,

$$A = \begin{pmatrix} 1 & 1 & 1 & & \cdots \\ & 1 & 1 & 1 & \cdots \\ & & 1 & 1 & 1 & \cdots \\ & & & 1 & 1 & 1 & \cdots \\ & & & & 1 & 1 & 1 & \cdots \end{pmatrix}.$$

We will now construct matrices $Q, R \in \mathbb{Z}^{(n+1) \times N}$ such that Q and R will correspond to $q(x)$ and $x^\ell r(x)$, respectively. Thus, we require

$$\begin{aligned} (1 + x + x^2 + \cdots + x^{\ell-1})p(x) &= q(x) + x^\ell r(x) \\ \iff P^T A X &= P^T Q X + P^T R X \\ \iff P^T (A - Q - R) X &= 0. \end{aligned}$$

So it suffices to find Q and R horizontally symmetric and such that $Q + R = A$. Note that the first ℓ columns of R must also be zero.

As it turns out, many constructions exist. For example, consider Q with 1 entries forming an isosceles triangle with base from 0 to $2n$:

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \cdots \\ & & 1 & 1 & 1 & 1 & 1 & & & \cdots \\ & & & 1 & 1 & 1 & & & & \cdots \\ & & & & 1 & & & & & \cdots \\ & & & & & 1 & & & & \cdots \end{pmatrix}.$$

Then the difference, $R = A - Q$ is

$$R = \begin{pmatrix} & -1 & -1 & -1 & -1 & -1 & -1 & \cdots \\ & & -1 & -1 & -1 & -1 & & \cdots \\ & & & -1 & -1 & & & \cdots \\ & & & & & & & \cdots \\ & & & & 1 & 1 & & \cdots \end{pmatrix}.$$

Extending this triangle structure to an isosceles trapezoid (even a self-intersecting one) works as well. More rigorously, define Q to be the matrix whose entries are 1 at the isosceles trapezoid from (zero-indexed) entries at

$$(0, 0), (0, n + k), (n, k), (n, n)$$

in that order, for any non-negative integer k with $k \geq \ell$. (If we choose $k < n$, the entries may be -1 to account for the self-intersection.) Define R to be the matrix whose entries are -1 at the isosceles trapezoid from entries at

$$(0, \ell), (0, n + k), (n, k), (n, n + \ell).$$

Then their total is the matrix whose entries are 1 at the parallelogram formed by

$$(0, 0), (0, \ell - 1), (n, n + \ell - 1), (n, n),$$

which is precisely A .