COMPENDIUM BMO

British Mathematical Olympiad

1973 - 2022

Gerard Romo Garrido

Toomates Coolección vol. 67

Toomates Coolección

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Versión de este documento: **21/08/2023**

Índice.

	FIST	BMO
1972	4	3
1973		27
1974	5	31
1975	6	34
1976	8	36
1977	9	38
1978	10	39
1979	11	40
1980	12	41
1981	13	42
1982	14	43
1983	15	44
1984	17	46
1985	19	48
1986	20	49
1987		50 , 52
1988	21	54
1989	23	
1990	24	56
1991	25	58

	Round 1	Round 2		
	Enunciados	Soluciones	Enunciados	Soluciones
1992	60		167	
1993	62		169	
1994	63		170	
1995	64		171	
1996	65		172	
1997	66		173	
1998	67		174	
1999	68		175	
2000	69		176	
2001	71		177	
2002	72		178	
2003	73		179	
2004	74		180	
2005	75		181	
2006	76		182	
2007	77		183	
2008	78		184	
2009	79		185	
2010	80		186	
2011	81		187	
2012	82		188	
2013	83		189	
2014	84		190	
2015	85		191	
2016	86		192	
2017	88		193	
2018	90		194	
2019	92	94	195	
2020	110	113	197	
2021	128	130	199	201
2022	149	151	209	211

Fuentes.

Further International Selection Test
May 17, 1972
Time 3 Hours
Only your best 3 answers will score.

1 Show how to assign to the vertices of a regular polygon with 2^n vertices, numbers such that

(a) only the digits 1 and 2 are used
(b) each number has n digits
(c) each vertex has a different number, and
(d) neighbouring vertices have numbers differing in one and only one digit place.

2 a,b,c,d are positive numbers and

$$S = \frac{a+c}{a+b} + \frac{b+d}{b+c} + \frac{c+a}{c+d} + \frac{d+b}{d+a}.$$

Prove that S is not less than 4, and obtain necessary conditions for S=4.

3. There are n persons present at a meeting. Every two persons are either friends of each other or strangers to each other. No two friends have a friend in common. Every two strangers have two and only two friends in common.
Prove that each person has the same number of friends at the meeting. If this number is 5, find n.

4. When k = 1 find all points **P** in space such that

$$a.PA^2 + b.PB^2 + c.PC^2 = kabc,$$

where a,b,c are the lengths of the sides BC,CA,AB of triangle ABC, and prove your result.
What is the effect of altering k?

Further International Selection Test, 1974

Time 3 hours

No tables and no lists of formulae are allowed

1. (i) In ΔABC $\sin^2 A + \sin^2 B + \sin^2 C = 2$. Prove that ΔABC is right-angled. (Hungary, modified)

   (ii) O is the circumcentre and G the centroid of ΔABC. Prove $9OG^2 = R^2\ (1 - 8\cos A\cos B\cos C)$. (Bulgaria, modified)

2. A polygonal line is a continuous line $A_1A_2A_3\ldots A_{n+1}$, where, for $r = 1$ to n, A_rA_{r+1} is a straight line segment.

   In a square of side 50, a polygonal line L is constructed in such a way that the distance of any point inside the square from L (i.e. from the nearest point of L) is less than 1. Prove that the length of L is greater than 1248. (USSR)

3. A circular hoop of radius 1 is placed in the corner of the room. (The corner consists of a horizontal floor and two perpendicular vertical walls and the hoop touches all three planes.) Find the locus of the centre of the hoop. (France)

4. EITHER

   (a) Show that the cube roots of three distinct prime numbers cannot be three terms (not necessarily consecutive) of an arithmetic progression. (USA)

   OR

   (b) $x + y + z = 3$, $x^3 + y^3 + z^3 = 15$, and $x^4 + y^4 + z^4 = 35$.

   Being given that $x^2 + y^2 + z^2$ is less than 10, find $x^5 + y^5 + z^5$. (GDR, modified)

FURTHER INTERNATIONAL SELECTION TEST

May 5th, 1975 $3\frac{1}{2}$ hours

1. In this question a "real function" means a function f such that f(x) exists and is real for all real numbers x.

(i) Show that there is only one value of the constant b for which a real function f exists with the property that, for all real x and y,

$$f(x-y) = f(x) - f(y) + bxy.$$

(ii) A real function f has the property that for all real x and y

$$f(x+y) = f(x) + f(y) + cxy$$

where c is a constant.

(a) Prove that if f is continuous at x = 0 then it is continuous everywhere.

(b) Prove that if f is differentiable at x = 0 then it is differentiable everywhere, and find the most general f in this case.

Give conditions prove (a) & (b) for (i) as well as (ii)

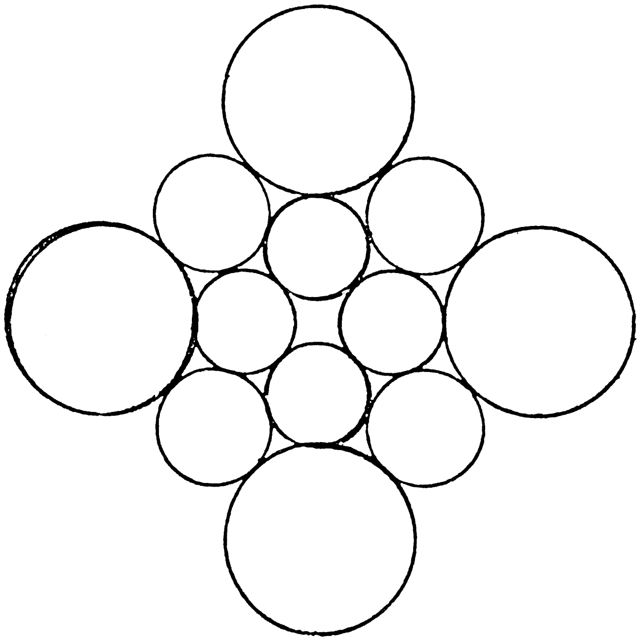
2. Prove that every positive integer which is not a member of the infinite set below is equal to the sum of two or more members of the set:

$$3, -2, 2^2.3, -2^3, \ldots, 2^{2k}.3, -2^{2k+1}, \ldots .$$

3. There are n countries taking part in an international mathematical competition, with two contestants from each country. The competition is held in two rooms, A and B. At the start of the competition the 2n contestants form a queue, in any order. The contestant at the head of the queue enters room A. Each subsequent contestant goes first to the door of the room which his immediate predecessor in the queue entered, and looks in. If his fellow-countryman is not already in the room he enters it; otherwise he enters the other room. (So competitors from the same country are separated.) If all orders of queueing are equally likely determine with proof the probability that room A is filled with n contestants before room B.

Turn over

4.



The diagram illustrates a configuration of 12 circles. The set S of 12 circles contains three subsets S_3, S_4, S_5 each having 4 circles and such that each of the 4 circles of S_r touches r circles of S.

Prove that such a configuration of 12 circles exists on the surface of a sphere with all the 12 circles having equal radii.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test, 1976

May 5th, 3½ hours

1. Through a point P in the interior of a fixed triangle ABC lines PL, PM, PN are drawn parallel to the medians through A,B,C respectively to meet BC, CA, AB at L,M,N respectively. Prove -

$$\frac{BL}{BC} + \frac{CM}{CA} + \frac{AN}{AB}$$

is constant (independent of P).

2. The real number t is a root of the equation

$$x^n + a_2x^{n-2} + a_3x^{n-3} + \dots\dots + a_n = 0, \quad (n \geqslant 2),$$

where the coefficients are real and satisfy $-1 \leqslant a_r \leqslant 1$, $(2 \leqslant r \leqslant n)$.

Prove that $$-\tfrac{1}{2}(1 + \sqrt{5}) \leqslant t \leqslant \tfrac{1}{2}(1 + \sqrt{5}).$$

3. Prove that the equation $x^2 - 3y^2 + 5z^2 - 7t^2 = 0$ has no solutions in integers x,y,z,t other than $x=y=z=t=0$.
Prove that $x^2-3y^2-5z^2+7t^2 = 0$ has infinitely many solutions in positive integers x,y,z,t in no two of which the ratio x:y:z:t is the same.

4. Prove that it is not possible to find positive integers p and q with the property that

$$\left|\frac{p}{q} - \sqrt{7}\right| \leqslant \frac{2}{11q^2}$$

5. A 'figure-of-eight' curve, S, consists of two touching circles of equal radii. Show that a pair of two distinct congruent hexagons (not necessarily convex) exists with the following properties:

(a) All the vertices of the hexagons lie on S.

(b) Neither hexagon has all its vertices on one circle.

(c) Neither hexagon can be obtained from the other by a single translation, a single rotation or a single reflection.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test, 1977

May 11th 1977 - 3½ hours

1. $x_0, x_1, \dots x_k$ are given points of the interval $[-1,1]$ such that

$$x_0 = -1,\ x_k = 1, \quad 0 < x_i - x_{i-1} \leqslant \tfrac{1}{2} \quad (1 \leqslant i \leqslant k)$$

The quadratic function f, of the form

$$f(x) = ax^2 + bx + c,$$

where a,b,c are real constants, satisfies the condition

$$|f(x_i)| \leqslant 1 \qquad (0 \leqslant i \leqslant k)$$

Prove that

$$|f(x)| \leqslant \frac{17}{15}$$

for all x in $[-1,1]$. Show by means of an example that this proposition becomes false if $\frac{17}{15}$ is replaced by any smaller number.

2. A pyramid is formed by joining the vertices of a plane quadrilateral ABCD, the "base", to a point V outside its plane. It is found that the inscribed circles of each pair of adjacent triangular faces touch each other. Prove that the points of contact of the inscribed circles with the base of the pyramid lie on a circle.

3. EITHER (a) Prove that if n is any given integer then the equation

$$10xy + 17yz + 27zx = n$$

has a solution in integers x,y,z.

OR (b) Prove that in the arithmetic progression

$$a,\ a+d,\ a+2d,\ \dots,\ a+nd,\ \dots$$

where a,d are positive integers, there exists an infinite set of terms having the same prime divisors.

4. Prove that for each integer n > 1 it is possible to construct a necklace having $2n^2$ beads in all, these being of 2n different colours, in such a way that for each pair of different colours there is at least one pair of adjacent beads of these two colours. Is it possible to do the same using $2n^2-1$ beads in all? Give a reason for your answer. (A "necklace" is a circular arrangement of beads, with no fastener intervening; an ample supply of beads of all the colours is assumed to be available).

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test, 1978

May 12th 1978 - $3\frac{1}{2}$ hours

1. A plane convex pentagon ABCDE is said to have the "unit triangle property" if the area of each of the triangles ABC, BCD, CDE, DEA, EAB is unity. Show that all plane convex pentagons with the unit triangle property have the same area and that there is an infinite number of such pentagons no two of which are congruent.

2. Given any integer $m > 1$ prove that there exists an infinity of positive integers n such that the last m decimal digits of 5^n form a sequence in which each digit except the last is of opposite parity to its successor: i.e. if one is odd then the next is even and vice versa.

3. Determine with proof all the roots of the equation

$$\sum_{r=1}^{n} (-1)^{r-1} \frac{x(x-1)\dots(x-r+1)}{(x+1)(x+2)\dots(x+r)} = \frac{1}{2}$$

where n is a given positive integer.

4. There are sufficient stocks of four different books for a publisher to give a copy of each to each of his n friends. However he decides to distribute presents to them according to the following rules:

   (i) at least one copy of each of the four books is to be given out;

   (ii) each friend is to have exactly two of the four books (these two not being copies of the same book);

   (iii) the friends who receive any one particular book must form a set different from the set of friends who receive any other particular book.

Show that the number of ways in which the distribution can be made is

$$12(3^{n-1} - 1)(2^{n-2} - 1).$$

<u>NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS</u>

<u>Further International Selection Test, 1979</u>

May 10th 1979 - $3\frac{1}{2}$ hours.

1. a, b, c, d are different positive real numbers. Prove that if at least one of the numbers c and d lies between the numbers a and b, or at least one of the numbers a and b lies between the numbers c and d, then

(*) $$\sqrt{(a+b)(c+d)} \geqq \sqrt{ab} + \sqrt{cd}\ .$$

Otherwise show that the four numbers can be chosen so that (*) is false.

2. Two equilateral triangles have a common vertex C. Going round each triangle in the anti-clockwise direction the vertices are lettered C, A, B and C, A', B'. O is the centre of triangle CAB and neither A' nor B' coincides with O. M is the midpoint of $A'B$ and N is the midpoint of AB'. Prove that triangles $OB'M$ and $OA'N$ are similar.

3. The sequence of positive integers a_n is defined by
$$a_0 = 1979, \qquad a_{n+1} = [\sqrt{(a_0 + a_1 + \dots + a_n)}] \qquad (n \geqq 0),$$
where $[x]$ denotes the greatest integer not greater than x. (For example $[3\frac{1}{2}] = 3$ and $[5] = 5$.) Determine a_{1979}.

4. Let $b(k)$ denote the number of 1's in the binary expansion of the non-negative integer k. For example $b(13) = 3$ since 13 is 1101 in binary notation. Prove that for all positive integers n,
$$\sum_{k=0}^{2^n-1} (-1)^{b(k)} k^n = (-1)^n\, 2^{\frac{1}{2}n(n-1)} (n!)\ .$$

National Committee for Mathematical Contests

Further International Selection Test

May 8, 1980 $3\frac{1}{2}$ hours

1. VLMN and VABC are tetrahedra with A, B, C on VL, VM, VN, produced as necessary. The in-centre of triangle LMN coincides with the centroid of triangle ABC.
   (i) Determine VA, VB, VC in terms of the sides of triangle LMN and VL, VM, VN.
   (ii) Determine the condition that the tetrahedra have equal volumes.
   (iii) If the tetrahedra have unequal volumes, determine, with proof, which has the greater volume.

2. Determine, with proof, all the prime numbers in the sequence (u_n) of integers defined by

$$u_0 = 2, \quad u_1 = 3,$$

$$u_{n+2} = u_{n+1}u_n - u_{n+1} - u_n + 2 \qquad (n \geqslant 0).$$

3. Prove that if $a_0 = 0, a_1, a_2, \ldots\ldots, a_n$ are real numbers, then

$$\sum_{i=1}^{n} a_i(a_i - a_{i-1}) \leqslant \tfrac{1}{2}(n+1)\sum_{i=1}^{n}(a_i - a_{i-1})^2 ,$$

equality holding if and only if $a_i = ia_1$ $(0 \leqslant i \leqslant n)$.

4. Given a set of n people, it is desired to arrange a series of bridge games such that every two of the n people play as opponents in exactly one game.
   Show that this can be done if and only if n is of the form $n = 8m+1$, where m is a positive integer.
   (There is no restriction on the number of times, if any, two people play as partners.)

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test

14th May 1981 Time allowed - 3½ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper and arrange your answers in order. Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

The earlier questions may be found shorter and easier; they carry fewer marks.

1. A given triangle has its three sides coloured red. Three blue straight lines cut the triangle into seven pieces, four of which are triangles and three are pentagons. Two of the sides of each triangle are blue and one of them has its third side blue also.

Given that all four triangles are congruent, express the area of each as a fraction of the area of the given triangle.

2. An <u>axis</u> of a solid is, for the purposes of this question, defined to be a straight line joining two points on the surface of the solid and such that the solid, when rotated about this line through an angle which is greater than 0° and less than 360°, coincides with itself.

How many axes has a cube? Draw three diagrams to show the three different types of axis and state the minimum angle of rotation for each type.

(No formal proofs are required. Each diagram should show clearly the axis, the vertices of the cube numbered from 1 to 8 and a symbol like $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 3 & 1 & 4 & 8 & 6 & 7 \end{pmatrix}$ indicating that the points 1,2,3,...8 move to the points 2,5,3,...,7 respectively.)

3. Solve for x,y,z the simultaneous equations

$$x^2y^2 + x^2z^2 = axyz, \quad y^2z^2 + y^2x^2 = bxyz, \quad z^2x^2 + z^2y^2 = cxyz$$

where a,b,c are given numbers.

4./.. TURN OVER

4. Find the remainder when the polynomial

$$x^{81} + x^{49} + x^{25} + x^9 + x$$

is divided by the polynomial $x^3 - x$.

5. The sequence $\{u_n\}$ of real numbers is defined for $n \geq 0$ by

$$u_0 = 2, \quad u_1 = 5$$

and

$$u_{n+1}u_{n-1} - u_n^2 = 6^{n-1} \qquad \text{when } n \geq 1.$$

Prove that each u_n is an integer.

6. Prove that if c is a rational number the equation

$$x^3 - 3cx^2 - 3x + c = 0$$

has at most one rational root.

7. Prove that if x,y are non-negative integers then $5x \geq 7y$ if and only if there exist non-negative integers a,b,c,d such that

$$x = a + 2b + 3c + 7d,$$
$$y = \quad b + 2c + 5d.$$

THE MATHEMATICAL ASSOCIATION

National Committee for Mathematical Contests

Further International Selection Test

Thursday, May 13, 1982

Time 3½ hours

1. ABC is a triangle. The internal bisector of the angle A meets the circumcircle again at P. Q and R are similarly defined.

   Prove that $AP + BQ + CR > AB + BC + CA$.

2. The sequence $p_1, p_2, \ldots$ is defined as follows -

   $p_1 = 2$, and for $n \geq 2$, p_n is the largest prime divisor of

   $p_1 p_2 p_3 \cdots p_{n-1} + 1$.

   Prove that 5 is not a member of this sequence.

3. Find the largest positive integer n for which the equation

$$ax + (a+1)y + (a+2)z = n$$

is not solvable in positive integers x,y,z, where a is a given odd positive integer.

4. $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are two points on that part of the curve $x^n - ay^n = b$ for which $x > 0$, $y > 0$. Here a and b are positive constants and n an integer > 1. Prove that if $y_1 < y_2$, and Δ is the area of triangle OP_1P_2, then

$$by_2 > 2ny_1^{n-1} a^{1-\frac{1}{n}} \Delta \ .$$

5. Given that k is a fixed non-negative integer and that the polynomial P(x) satisfies the relation

$$P(2x) = 2^{k-1}(P(x) + P(x + \tfrac{1}{2})),$$ prove that

$$P(3x) = 3^{k-1}(P(x) + P(x + \tfrac{1}{3}) + P(x + \tfrac{2}{3})) \ .$$

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test

Wednesday May 11, 1983

Time allowed - 3½ hours

Write your name on each page. Start each question on a new sheet.

Important Notice: In the selection of the IMO team special attention will be paid to performance in geometry. Two at least of the questions answered should be from G1, G2, G3. The marks from not more than five questions will count towards your score.

G1. Two points A, B and a line k are given in a plane. Locate, with proof, the point P of the plane for which $PA^2 + PB^2 + PN^2$ is a minimum, where N is the foot of the perpendicular from P to k.

Give a generalisation without proof for three points A, B, C and $PA^2 + PB^2 + PC^2 + PN^2$ a minimum.

G2. Consider the three escribed circles of the triangle ABC, that is, the three distinct circles each of which touches one side of triangle ABC internally and the other two externally. Each pair of escribed circles has just one common tangent which is not a side of triangle ABC, and the three such common tangents form a triangle T.

O is the circumcentre of triangle ABC. Prove that OA is perpendicular to a side of T.

G3. ℓ, m, n are three lines in space. Neither ℓ nor m is perpendicular to n. Points P and Q vary on ℓ and m respectively in such a way that PQ is perpendicular to n. The plane through P perpendicular to m meets n at R and the plane through Q perpendicular to ℓ meets n at S. Prove that RS is of constant length.

OVER/

A4. Prove that if a, b, c, d, e, f are positive real numbers then

$$\frac{ab}{a+b} + \frac{cd}{c+d} + \frac{ef}{e+f} \leqslant \frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f}.$$

A5. Find the number of arrangements

$$a, b, c, d, e, f, g, h$$

of the numbers 1,2,3,4,5,6,7,8 which satisfy all seven conditions

$a<b$, $c<d$, $e<f$, $g<h$ and $b>c$, $d>e$, $f>g$.

A6. n and k are positive integers. Find all pairs (n,k) satisfying

$$(n+1)^k = n! + 1,$$

proving that you have the complete set of solutions.

A7. In a colony of $(mn+1)$ mice, prove that at least one of the following statements is true:

(a) There is a set A of $(m+1)$ mice none of which is a parent of any other in the set.

(b) There is an ordered set B of $(n+1)$ mice $a_1, a_2, \ldots, a_n, a_{n+1}$ such that a_{i+1} is a parent of a_i for each $i = 1, 2, \ldots, n$.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test

Friday, 23rd March 1984

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

Candidates are not expected to attempt all five questions.

1. The triangle ABC is right-angled at C. Find all the points D in the plane satisfying the conditions

$$AD \, . \, BC = AC \, . \, BD = \frac{1}{\sqrt{2}} AB \, . \, CD \quad .$$

2. ABCD is a tetrahedron with $DA = DB = DC = d$ and $AB = BC = CA = e$. M and N are the midpoints of AB and CD. A plane π passes through MN and cuts AD and BC at P and Q respectively.

(i) Prove that $AP/AD = BQ/BC$ (= t, say).

(ii) Determine with proof that value of t, expressed in terms of d and e, which minimises the area of the quadrilateral MQNP.

3. Find, with proof, the maximum and minimum values of

$$\cos\alpha + \cos\beta + \cos\gamma \ ,$$

where $\alpha \geqslant 0$, $\beta \geqslant 0$, $\gamma \geqslant 0$ and

$$\alpha + \beta + \gamma = \frac{4\pi}{3} \ .$$

4. Let b_n be the number of ways of expressing the positive integer n as a sum of one or more, not necessarily distinct, powers of 2 ; here 1 $(= 2^0)$ is regarded as a power of 2 . Order of the summands is immaterial, so for instance $b_4 = 4$, the expressions in question being

$$1 + 1 + 1 + 1 \; , \quad 1 + 1 + 2 \; , \quad 2 + 2 \; , \quad 4 \; .$$

Call such an expression *full* if it includes at least one summand 2^i for $0 \leqslant i \leqslant k$, where 2^k is the largest summand occurring in it. For example the first two of the above expressions for 4 are full, the others are not. Let c_n be the number of full expressions for n . Prove that

$$b_{n+1} = 2c_n$$

for $n \geqslant 1$.

5. Let p and q be positive integers. Show that there exists an interval I of length $1/q$ and a polynomial P with integer coefficients such that, for all x in I ,

$$\left| P(x) - \frac{p}{q} \right| \leqslant \frac{1}{q^2} \; .$$

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test

Friday, March 15th 1985

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. O is a point outside a circle. Two lines OAB, OCD through O meet the circle at A, B, C, D with A, C the midpoints of OB, OD respectively. Also the acute angle θ between the lines is equal to the acute angle at which each line cuts the circle. Find $\cos\theta$ and show that the tangents at A, D to the circle meet on the line BC.

2. A positive integer is called *evil* if the number of digits 1 in its binary expansion is even. For example $18 = (10010)_2$ is evil. Find the sum of the first 1985 evil positive integers.

3. Prove that the product of five consecutive positive integers is never a perfect square.

4. Delegations from 30 countries took part in a session of the International Mathematical Olympiad jury - the leaders and their deputies, 60 people in all. During the session some of the participants shook hands with each other but no leader shook hands with his deputy and no two people shook hands more than once. After the session the leader of the Mongolian team asked everybody how many times they had shaken hands. All the participants answered and the numbers they gave were all different. How many times did the Mongolian leader's deputy shake hands?

For full credit you must establish clearly that your answer is the only one consistent with the conditions.

5. $ABCD$ is a tetrahedron which has a circumsphere passing through A, B, C, D and an in-sphere touching each triangular face at an interior point of that face. The two spheres have the same centre O. H is the orthocentre of triangle ABC and H' is the foot of the perpendicular from D on to the plane of that triangle.

Prove that $AB = CD$, $AC = BD$, $AD = BC$ and that $OH = OH'$.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test.

Friday, 14th March 1986

Time allowed $3\frac{1}{2}$ hours.

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order. Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. Plane rectangular Cartesian axes are given with equal unit length along each axis. A rational point is defined as a point both of whose coordinates are rational numbers.

   A, B, A', B' are four distinct rational points; A and B are on the x-axis. Prove that, unless $\vec{AB} = \vec{A'B'}$, there exists just one point P such that the triangles PAB, $PA'B'$ are directly similar, i.e. each can be obtained from the other by enlargement (dilatation) and rotation about P. Prove also that P is a rational point.

2. Find, with proof, the greatest value of
$$x^2y + y^2z + z^2x$$
where x, y, z are real numbers satisfying the conditions
$$x + y + z = 0, \quad x^2 + y^2 + z^2 = 6 .$$

3. $P_1, P_2, \ldots, P_n$ are n distinct subsets of $\{1,2,\ldots,n\}$ each having two elements. P_i and P_j $(i \neq j)$ have an element in common if and only if $\{i,j\}$ is one of the subsets P_k. Prove that each of $1,2,\ldots,n$ belongs to exactly two of the P_k.

4. Show that if m, n are positive integers with $m \leqslant n$ then the product $\binom{n}{m}\binom{n}{m-1}$ of binomial coefficients is divisible by n. Find with proof the smallest positive integer k such that the product $k\binom{n}{m}\binom{n}{m-1}\binom{n}{m-2}$ is divisible by n^2 for all integers m, n with $2 \leqslant m \leqslant n$. [$k$ is to be independent of m, n.]
For this k and given n determine, with proof, the greatest common divisor of the integers $\frac{k}{n^2}\binom{n}{m}\binom{n}{m-1}\binom{n}{m-2}$, $2 \leqslant m \leqslant n$.

5. C_1 and C_2 are two circles; A_1, A_2 are fixed points on C_1, C_2 respectively. A_1P_1, A_2P_2 are parallel chords of C_1, C_2. Find the locus of the midpoint of P_1P_2.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test

Friday 5th February 1988

Time allowed - 3½ hours

PLEASE READ THESE INSTRUCTIONS CAREFULLY

Write on one side of the paper only. Use a fresh sheet or sheets of paper for each question. Arrange your answers in order. On the first sheet of your script write ONLY your full name, age (in years and months), home address and school; do not put any working on this sheet. On every sheet of working write your name and initials, and the number of the question.

There is no restriction on the number of questions which may be attempted.

1. ABC is an equilateral triangle. The circle Γ_1 has centre A and radius AB. Γ_2 is the circle on AB as diameter. A circle with centre P on AC touches Γ_1 internally at C and Γ_2 externally at Q. Show that AP/AC = 4/5 and calculate the ratio AQ/AC.

2. Prove that the number of ways of arranging 2n distinguishable objects in n pairs is
$$1 . 3 . 5 . \ldots . (2n-1)$$
if the order of the pairs and the order of the objects within each pair are immaterial. Eg for 4 objects a, b, c, d the 3 pairings are ab, cd; ac, bd; ad, bc.

A party of 10 people consisting of 5 married couples is to be split into 5 pairs. A pair may consist of two men, two women or a man and a woman, but must not be a married couple. In how many ways can this arrangement be made if order, as above, is immaterial? Explain your reasoning carefully.

TURN OVER

3. The numbers a, b, c, x, y, z satisfy the equations

$$x^2 - y^2 - z^2 = 2ayz$$

$$-x^2 + y^2 - z^2 = 2bzx$$

$$-x^2 - y^2 + z^2 = 2cxy$$

and also $$xyz \neq 0.$$

By using the first two equations express z in terms of a, b, x, y. Prove that

$$x^2(1-b^2) = y^2(1-a^2) = xy(ab-c)$$

and hence find the value of $a^2 + b^2 + c^2 - 2abc$ (independently of x, y, z).

4. Find, with proof, all solutions of

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1$$

where x, y, z are positive integers.

5. L and M are two skew lines in space, ie they neither meet nor are parallel. A, B are the points on L, M respectively such that AB is perpendicular to both L and M. Points P on L, Q on M vary so that

$$P \neq A, \quad Q \neq B, \quad PQ \text{ is of constant length.}$$

Show that the centre of the sphere through A, B, P, Q lies on a fixed circle with centre the midpoint of AB.

6. Prove that if a_1, b_1, c_1 and a_2, b_2, c_2 are the lengths of the sides of two triangles (in some unit of measurement) then

$$a = \sqrt{(a_1^2 + a_2^2)}, \quad b = \sqrt{(b_1^2 + b_2^2)}, \quad c = \sqrt{(c_1^2 + c_2^2)}$$

are also the lengths of the sides of some triangle.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test 1989

Wednesday, 1st March 1989

Time allowed: 3½ HOURS.

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

Write on one side of the paper only. Use a fresh sheet or sheets of paper for each question. At the top of each sheet write your name and initials and the number of the question. Arrange your answers in order. Complete the proforma provided and attach it at the front of your script.

1. Find the smallest positive integer a with the property:

There exist integers b,c such that the equation

$$ax^2 - bx + c = 0$$

has two distinct roots in the interval $0 < x < 1$.

2. Find the number of different arrangements in a row of the letters

A,A,A,A,A,B,B,B,B,B,C,C,C,C,C

such that each letter is adjacent to an identical letter.

Indicate if you can a generalization to the case of n letters each appearing five times.

3. Let $f(x)$ be a polynomial of degree n such that

$$f(k) = \frac{k}{k+1}, \quad k = 0,1,2,\ldots,n.$$

Find $f(n+1)$, expressing your result as simply as possible.

4. M is a point on the side AC of triangle ABC such that triangles BAM, BMC have inscribed circles of equal radius. Find the length of BM in terms of the lengths a,b,c of the sides of triangle ABC.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

FURTHER INTERNATIONAL SELECTION TEST 1990

Wednesday, 7th March 1990

Time allowed: Three-and-a-half hours

- *Arrange your answers in order, with your name on each page.*
- *Complete the proforma provided and attach it to the front of your script.*

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1. Prove that if the polynomial

$$p(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n ,$$

whose coefficients are integers, takes the value 1990 for four distinct integer values of x, then it cannot take the value 1997 for any integer value of x.

2. The *integer part* $[x]$ of a number x is the greatest integer which is not greater than x. The *fractional part* $f(x)$ is defined by $f(x) = x - [x]$.
Find a positive number x such that

$$f(x) + f\left(\frac{1}{x}\right) = 1 .$$

Are there any *rational* solutions?

3. State the cosine rule for a triangle.
Prove that for arbitrary positive real numbers a, b, c,

$$\sqrt{a^2 + b^2 - ab} + \sqrt{b^2 + c^2 - bc} \geq \sqrt{a^2 + c^2 + ac} .$$

4. Let d denote the length of the smallest diagonal of all rectangles inscribed in a triangle T. (By inscribed we mean that all vertices of the rectangle lie on the boundary of T).
Determine the maximum value of d^2/area(T) taken over all triangles.

5. I is the centre of the circle inscribed to triangle ABC; J is the centre of the escribed circle which touches AB and AC produced beyond B and C respectively.
Prove that

$$AI.AJ = AB.AC$$

and that

$$AI.BJ.CJ = AJ.BI.CI.$$

FURTHER INTERNATIONAL SELECTION TEST 1991

Thursday 28th February 1991

Time allowed: $3^1/_2$ hours

- *Start each question on a fresh sheet of paper.*
- *Write on one side of the paper only.*
- *On every sheet of working write the number of the question in the top left-hand corner and your name and school in the top right-hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your answers to questions 1, 2, 3, 4 in order.*
- *Staple all the pages neatly together in the top left hand corner.*

A small number of completed solutions is much better than partial attempts at all problems.

1. In triangle ABC, B is a right-angle and θ is the angle between AC and the median from C to AB.

   Prove that $\sin\theta \leq {}^1/_3$.

2. Twelve dwarfs live in a forest. Each has a two-sided cloak which is blue on one side and red on the other. Some dwarfs consistently wear their cloaks red side in and the rest consistently wear their cloaks red side out.

   They agree on the following New Year's resolution:

   > On the nth day of the New Year, the nth dwarf modulo 12 will visit each of his friends. If he finds a majority of these friends wearing their cloaks differently from his, he will immediately reverse his own cloak. Otherwise he will continue as before.

   Prove that, sooner or later, no further changes will take place.

   (Friendships are mutual and do not change).

[Turn over

3. Prove that if the perimeter of a triangle with sides a, b, c is 2 then

$$a^2 + b^2 + c^2 + 2abc < 2.$$

4. Let x be a positive real number. Prove that at least one of the numbers

$$x, 2x, 3x, \dots, 20x$$

contains the digit 2 in its decimal expansion.

Let N be the smallest positive integer such that, for every positive real number x, at least one of the numbers

$$x, 2x, 3x, \dots, Nx$$

contains the digit 2 in its decimal expansion. Find lower and upper bounds for N and, if possible, find N exactly.

BRITISH MATHEMATICAL OLYMPIAD

1973

TIME: 3 HOURS

PLEASE NOTE INVIGILATOR'S INSTRUCTIONS

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1. (i) Two fixed circles are touched by a variable circle at P and Q.

   Prove that PQ passes through one of two fixed points.

   (ii) State a true theorem about ellipses or if you like about conics in general of which (i) is a particular case.

2. 9 points are given in the interior of the unit square.

   Prove there exists a triangle of area $\leqslant \frac{1}{8}$ whose vertices are three of the points.

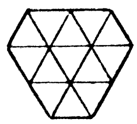
3. A curve consisting of the quarter-circle $x^2 + y^2 = r^2$, $x, y \geqslant 0$, together with the line segment $x = r$, $-h \leqslant y \leqslant 0$, is rotated about $x = 0$ to form a surface of revolution which is a hemisphere on a cylinder. A string is stretched tightly over the surface from the point on the curve $(r\sin\theta, r\cos\theta)$ to the point $(-r, -h)$ in the plane of the curve. Show that the string does not lie in a plane if $\tan\theta > \frac{r}{h}$.

   [You may assume spherical triangle formulae such as $\cos a = \cos b\cos c + \sin b\sin c\cos A$ or $\sin A\cot B = \sin c\cot b - \cos c\cos A$. In a spherical triangle the sides a, b, c are arcs of great circles and are measured by the angles they subtend at the centre of the sphere.]

P.T.O....

4. You have a large number of congruent triangular equilateral discs on a table and you want to fit n discs together to make a convex equiangular hexagon (i.e. one whose interior angles are each 120°).

Obviously n cannot be *any* positive integer. The smallest n is 6, the next smallest is 10 and the next 13. Determine conditions for possible n.



$n = 13$

5. There is an infinite set of positive integers of the form $2^n - 3$ with the property Q: no two members of the set have a common prime factor. An outline of a proof is as follows.

Suppose there is a finite set $S = (2^{m_1} - 3, 2^{m_2} - 3, \ldots\ldots\ldots 2^{m_k} - 3)$ with property Q and k members. Let the prime factors of these k numbers be $P_1, P_2 \ldots\ldots\ldots P_t$. Consider the number $N = 2^{(P_1 - 1)(P_2 - 1)\ldots\ldots(P_t - 1)} + 1$. By Fermat's theorem $a^{P-1}-1 \equiv 1 \pmod{P}$ for every prime P that does not divide a. Hence $N - 3 \equiv -1 \pmod{P_r}$, $r = 1$ to t, and $N - 3$ may be added to S to give a larger set with property Q.

Give a properly expanded and reasoned proof that there is an infinite set of positive integers of the form $2^n - 7$ with property Q.

6. In answering general knowledge questions (framed so that each question is answered yes or no) the teacher's probability of being correct is α and a pupil's probability of being correct is β or γ according as the pupil is a boy or a girl.

The probability of a randomly chosen pupil agreeing with the teacher's answer is $\frac{1}{2}$.

Find the ratio of the numbers of boys to girls in the class.

7. The life-table issued by the Registrar-General of Draconia shows out of 10,000 live births the number (y) expected to be alive x years later. When $x = 60$, $y = 4{,}820$. When $x = 80$, $y = 3{,}205$. For $60 \leqslant x \leqslant 100$ the curve

$$y = Ax\,(100 - x) + \frac{B}{(x - 40)^2}$$

fits the figures in the table very closely, A and B being constants.

Determine the life-expectancy (in years correct to one decimal place) of a Draconian aged 70.

N.B. At age 100 all Draconians are put to death.

8. Call $M_r = \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}$ the companion matrix for the mapping $T_r : z \longrightarrow \dfrac{a_r z + b_r}{c_r z + d_r}$. Det $M_r \neq 0$.

(i) Prove that $M_1 M_2$ is the companion matrix for the mapping $T_1 T_2$.

(ii) Find conditions on a, b, c, d so that $T^4 = I$ but $T^2 \neq I$.

9. $$L_r = \begin{vmatrix} x & y & 1 \\ a + c\cos\theta_r & b + c\sin\theta_r & 1 \\ l + n\cos\theta_r & m + n\sin\theta_r & 1 \end{vmatrix}$$

Show that the lines $L_r = 0$, $r = 1, 2, 3$ are concurrent and find the co-ordinates of their concurrence.

10. Construct a detailed flow chart for a computer program to print out all positive integers up to 100 of the form $a^2 - b^2 - c^2$, where a, b, c are positive integers and $a \geqslant b + c$.

There is no need to print in ascending order or to avoid repetitions.

11. (i) Two uniform rough right circular cylinders A and B, with the same length, have radii and masses a, b and M, m respectively.

A rests with a generator in contact with a rough horizontal table. B rests on A, initially in unstable equilibrium, with its axis vertically above A's. Equilibrium is disturbed, B rolls on A and A rolls on the table. In the subsequent motion the plane containing the axes makes an angle θ with the vertical.

Draw diagrams showing angles, forces etc., for the period when there is no slipping. Write down equations which will give on elimination a differential equation for θ, stating the principles used. *Indicate* how the elimination could be done; you are not asked to do it.

(ii) Such a differential equation is, with $k = M/m$

$$\ddot{\theta}\left(4 + 2\cos\theta - 2\cos^2\theta + \frac{9k}{2}\right) + \dot{\theta}^2 \sin\theta\,(2\cos\theta - 1)$$

$$= \frac{3g\,(1 + k)\sin\theta}{a + b}$$

Obtain $\dot{\theta}$ in terms of θ.

[Moment of inertia of a uniform cylinder about its axis is $\frac{1}{2}$ (mass) (radius)^2]

Mathematical Association
Awards Committee

BRITISH MATHEMATICAL OLYMPIAD 1974

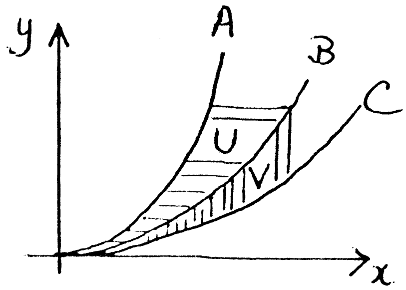
Time allowed: 3 hours

Each question should be answered on a fresh sheet of paper. Use one side of the paper only.

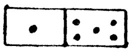
Do as much as you can; aim at answering whole questions. No tables, slide rules, calculators, or lists of formulae are allowed. Geometric instruments are allowed.

Please note the invigilator's instructions on a separate sheet.

A1.



The curves A, B and C are related in such a way that B "bisects" the area between A and C, that is, the area of the region U is equal to the area of the region V at all points of the curve B. Find the equation of the curve B given that the equation of curve A is $y = \frac{4}{3}x^2$ and that the equation of curve C is $y = \frac{3}{8}x^2$.

B2. A domino can be represented by an unordered pair of integers. Thus  can be represented as (1,5) or (5,1) and the double  as (2,2). The set of all 15 dominoes containing two integers from 1, 2, 3, 4, 5 is partitioned into three subsets of five dominoes. The dominoes in each subset form a closed chain, that is (a,b)(b,c)(c,d)(d,e)(e,a) where a,b,c,d,e need not all be different. How many distinct partitions are there? (The order of the three subsets in the partition is immaterial.)

A3. Prove that it is impossible for all the faces of a convex polyhedron to be hexagons.

A4. M is a 16×16 matrix. Each element in the leading diagonal and each element in the bottom row (i.e. 16th row) is 1. Every other element of the matrix is $\frac{1}{2}$. Find the inverse of M.

A5. A bridge deal is defined as the distribution of 52 ordinary playing cards among four players so that each player has 13 cards. In a bridge deal what is the probability that just one player has a complete suit? (Leave your answer in factorials.)

B6. X and Y are the feet of the perpendiculars from P to CA and CB respectively, where P is in the plane of triangle ABC. PX = PY. The straight line through P which is perpendicular to AB cuts XY at Z. Prove that CZ bisects AB.

Turn over

B7. The roots of the equation $x^3 = bx + c$ ($bc \neq 0$, b and c real) are α, β and γ. Determine p, q and r in terms of b and c so that

$$\beta = p\alpha^2 + q\alpha + r, \quad \gamma = p\beta^2 + q\beta + r, \quad \alpha = p\gamma^2 + q\gamma + r$$

and state a condition which ensures that p, q and r are real.

A8. Let n be an odd prime number. It is required to write the product

$$\prod_{i=1}^{n-1} (x + i)$$

as a polynomial

$$\sum_{j=0}^{n-1} a_j x^j .$$

By considering the product $\prod_{i=1}^{n} (x + i)$ in two ways, establish the relations

$$\begin{aligned}
a_{n-1} &= 1, \\
a_{n-2} &= n(n-1)/2! , \\
2a_{n-3} &= n(n-1)(n-2)/3! + a_{n-2}(n-1)(n-2)/2! , \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(n-2)a_1 &= n + a_{n-2}(n-1) + a_{n-3}(n-2) + \ldots + 3a_2 , \\
(n-1)a_0 &= 1 + a_{n-2} + \ldots + a_1 .
\end{aligned}$$

Prove $n \mid a_j$ ($j = 1,2,\ldots,n-2$) and that $n \mid (a_0 + 1)$; and prove also that when x is an integer

$$n \mid (x+1)(x+2) \ldots (x+n-1) - x^{n-1} + 1 .$$

Hence deduce Wilson's Theorem and Fermat's Theorem, namely, that when n is prime and x is not a multiple of n

(i) $n \mid (n-1)! + 1$;

(ii) $n \mid x^{n-1} - 1$.

($p \mid q$ means p divides q leaving no remainder.)

B9. A vertical uniform rod of length 2a is hinged at its lower end to a frictionless joint secured to a horizontal table. It falls from rest in this unstable position on to the table. Find the time occupied in falling. Comment on your answer.

[You may quote the result $\int(\operatorname{cosec} x)dx = \log\left|\tan \tfrac{1}{2}x\right|$ if you wish.]

B10. A right circular cone whose vertex is V and whose semi-vertical angle is α has height h and uniform density. All points of the cone whose distances from V are less than a or greater than b, where $0 < a < b < h$, are removed. A solid of mass M is left.

Given that the gravitational attraction that a point mass m at P exerts on unit mass at O is $(Gm/OP^3)\vec{OP}$, prove that the magnitude of the gravitational attraction of this solid on unit mass at V is

$$\tfrac{3}{2}GM(1 + \cos\alpha) \,/\, (a^2 + ab + b^2).$$

BRITISH MATHEMATICAL OLYMPIAD, 1975

24th March, 1975

Time allowed - 3 hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age and school on the top sheet of your answers. On each other sheet put your name and initials.

Do as much as you can. The earlier questions carry slightly fewer marks. Aim at answering whole questions. In any question, marks may be added for elegance and clarity or subtracted for obscure or poor presentation.

1. Given that x is a positive integer solve

$$[\sqrt[3]{1}] + [\sqrt[3]{2}] + \dots + [\sqrt[3]{x^3 - 1}] = 400$$

(where $[z]$ means the integral part of z) and prove your solution is complete.

2. The first n prime numbers, $2, 3, 5, \dots, p_n$ are partitioned into two disjoint sets A and B. The primes in A are $a_1, a_2, \dots, a_h$ and the primes in B are $b_1, b_2, \dots, b_k$ where $h + k = n$.

The two products $\prod_1^h a_i^{\alpha_i}$ and $\prod_1^k b_i^{\beta_i}$ are formed where the α_i and β_i are any positive integers.

If d divides the difference between these products prove that either $d = 1$ or $d > p_n$.

3. Use the pigeonhole principle (i.e. if more than n objects are put into n pigeonholes then at least one pigeonhole must contain more than one object) to answer the following question.

A disc S is defined as the set of all points P in a plane such that $|OP| \leqslant 1$, where $|OP|$ is the distance of P from O, a given point in the plane, called the centre of the disc.

Prove that if the disc S contains 7 points such that the distance from any of the 7 points to any other is greater than or equal to 1, then one of the 7 points is O.

Turn over

4. Three parallel lines AD, BE, CF are drawn through the vertices of triangle ABC meeting the opposite sides in D,E,F respectively.

The points P,Q,R divide AD, BE, CF respectively in the same ratio $k:1$ and P,Q,R are collinear. Find the value of k.

5. For any positive integer m you are given that

$$1 + \binom{2m}{1}\cos\theta + \binom{2m}{2}\cos 2\theta + \dots + \cos 2m\theta = (2\cos\tfrac{1}{2}\theta)^{2m}\cos m\theta$$

where there are $2m+1$ terms on the left hand side. Both these expressions are defined to be $f(\theta)$. The function $g(\theta)$ is defined by

$$g(\theta) = 1 + \binom{2m}{2}\cos 2\theta + \binom{2m}{4}\cos 4\theta + \dots + \cos 2m\theta.$$

Given that there is no rational k for which $\alpha = k\pi$ find the values of α for which

$$\lim_{m\to\infty}\frac{g(\alpha)}{f(\alpha)} = \frac{1}{2}.$$

6. Prove that if n is a positive integer greater than 1 and $x > y > 1$, then

$$\frac{x^{n+1}-1}{x(x^{n-1}-1)} > \frac{y^{n+1}-1}{y(y^{n-1}-1)}.$$

7. Prove that there is only one set of real numbers $x_1, x_2, \dots, x_n$ such that

$$(1-x_1)^2 + (x_1-x_2)^2 + \dots + (x_{n-1}-x_n)^2 + x_n^2 = \frac{1}{n+1}.$$

8. The interior of a wine glass is a right circular cone. The glass is half filled with water and then slowly tilted so that the water starts and continues to spill from a point P on the rim. What fraction of the whole conical interior is occupied by water when the horizontal plane of the water level bisects the generator of the cone furthest from P?

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

1976

Time allowed - 3½ hours

24th March, 1976

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

In any question, marks may be added for elegance and clarity or subtracted for obscure or poor presentation.

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1. Find, with proof, the length d of the shortest straight line which bisects the area of an arbitrarily given triangle. Express d in terms of the area Δ of the triangle and one of its angles.

   Show that there is a shorter line (not straight) which bisects the area of the given triangle.

2. Prove that if x,y,z are positive real numbers then

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}.$$

3. $S_1, S_2, \ldots S_{50}$ are subsets of a finite set E. Each subset contains more than half the elements of E.

   Show that it is possible to find a subset F of E, having not more than 5 elements, such that each $S_i (1 \leq i \leq 50)$ has an element in common with F.

4. Prove that if n is a non-negative integer then $19.8^n + 17$ is not a prime number.

QUESTIONS 5 and 6 ARE OVERLEAF.

5. Prove that

$$\sum_{t=0}^{\frac{1}{2}(r-1)} \binom{n}{r-t}\binom{n}{t}(\alpha\beta)^t\left(\alpha^{r-2t}+\beta^{r-2t}\right) = \sum_{t=0}^{\frac{1}{2}(r-1)} \binom{n}{r-t}\binom{r-t}{t}(\alpha\beta)^t(\alpha+\beta)^{r-2t},$$

where α and β are real numbers, r and n are positive integers with r odd and $r \leqslant n$.

[$\binom{m}{s}$ denotes the coefficient of x^s in the expansion of $(1+x)^m$.]

6. A sphere with centre O and radius r is cut in a circle K by a horizontal plane distant $\frac{1}{2}r$ above O. The part of the sphere above the plane is removed and replaced by a right circular cone having K as its base and having its vertex V at a distance $2r$ vertically above O.

Q is a point on the sphere on the same horizontal level as O. The plane OVQ cuts the circle K in two points X and Y, of which Y is the further from Q. P is a point of the cone lying on VY, whose position can be determined by the fact that the shortest path from P to Q over the surfaces of cone and sphere cuts the circle K at an angle of 45°.

Prove $VP = \sqrt{3}r/\sqrt{(1+1/\sqrt{5})}$.

[In a spherical triangle ABC the sides are arcs of great circles (centre O) and the sides are measured by the angles they subtend at O. You <u>may</u> find these spherical triangle formulae useful:

$$\sin a/\sin A = \sin b/\sin B = \sin c/\sin C$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A].$$

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

17th March, 1977

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

In any question, marks may be added for elegance and clarity or subtracted for obscure or poor presentation.

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1. A non-negative integer $f(n)$ is assigned to each positive integer n in such a way that the following conditions are satisfied:

   (i) $f(mn) = f(m) + f(n)$ for all positive integers m, n;

   (ii) $f(n) = 0$ whenever the final (right-hand) decimal digit of n is 3; and

   (iii) $f(10) = 0$.

   Prove that $f(n) = 0$ for all positive integers n.

2. The sides BC, CA, AB of a triangle touch a circle at X,Y,Z respectively. Prove that the centre of the circle lies on the straight line through the midpoints of BC and of AX.

3. (i) Prove that if x,y,z are non-negative real numbers, then $x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \geqslant 0$.

   (ii) Hence or otherwise show that for all real numbers a,b,c
   $a^6 + b^6 + c^6 + 3a^2b^2c^2 \geqslant 2(b^3c^3 + c^3a^3 + a^3b^3)$.

4. The equation $x^3 + qx + r = 0$, where $r \neq 0$, has roots u,v,w.

   Express the roots of $r^2x^3 + q^3x + q^3 = 0$ ....(1)
   in terms of u,v,w, and show that if u,v,w, are real then (1) has no root in the interval $-1 < x < 3$.

5. $A_1A_2A_3A_4A_5$ is a regular pentagon whose sides are each of length $2a$. For each $i = 1,2,\ldots 5$, K_i is the sphere with centre A_i and radius a. The spheres $K_1,K_2\ldots K_5$ are all touched externally by each of two spheres P_1 and P_2 also of radius a.
   Determine with proof and without tables whether P_1 and P_2 have or have not a common point.

6. The polynomial $26(x+x^2+x^3+ \ldots +x^n)$, where $n > 1$, is to be decomposed into a sum of polynomials, not necessarily all different. Each of these polynomials is to be of the form $a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ where each a_i is one of the numbers $1,2,3\ldots n$ and no two a_i are equal.

   Find all the values of n for which this decomposition is possible.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

21st March, 1978

Time allowed - 3½ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

In any question, marks may be added for elegance and clarity or subtracted for obscure or poor presentation.

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1. Determine with proof the point P inside a given triangle ABC for which the product PL.PM.PN is a maximum, where L,M,N are the feet of the perpendiculars from P to BC,CA,AB respectively.

2. Prove that there is no proper fraction $\frac{m}{n}$, with denominator $n \leqslant 100$, whose decimal expansion contains the block of consecutive digits 167 in that order.

3. Show that there is one and only one sequence $\{u_n\}$ of integers such that $u_1 = 1$, $u_1 < u_2$, and

$$u_n^3 + 1 = u_{n-1}u_{n+1} \text{ for all } n > 1.$$

4. An altitude of a tetrahedron is a line through a vertex perpendicular to the opposite face.

   Prove that the four altitudes of a tetrahedron are concurrent if and only if each edge of the tetrahedron is perpendicular to its opposite edge.

5. Inside a cube of side 15 units there are 11000 given points.

   Prove that there is a sphere of unit radius within which there are at least 6 of the given points.

6. Show that if n is a non-zero integer, $2\cos n\theta$ is a polynomial of the nth degree in $2\cos\theta$.

   Hence or otherwise prove that if k is rational then $\cos k\pi$ either is equal to one of the numbers 0, $\pm\frac{1}{2}$, ± 1, or is irrational.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

14th March, 1979

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

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1. Find all triangles ABC for which

$$AB + AC = 2 \text{ cm. and } AD + BC = \sqrt{5} \text{ cm.},$$

where AD is the altitude through A, meeting BC at right angles in D.

2. From a point O in 3-D space, three given rays OA, OB, OC emerge, the angles BOC, COA, AOB being α, β, γ respectively. $0 < \alpha, \beta, \gamma < \pi$.

Prove that, given $2s > 0$, there are unique points X, Y, Z on OA, OB, OC respectively such that the triangles YOZ, ZOX and XOY have the same perimeter $2s$, and express OX in terms of s and $\sin\frac{1}{2}\alpha$, $\sin\frac{1}{2}\beta$ and $\sin\frac{1}{2}\gamma$.

3. S is a set of distinct positive odd integers $\{a_i\}$, $i = 1$ to n. No two differences $|a_i - a_j|$ are equal, $1 \leqslant i < j \leqslant n$.

Prove $\displaystyle\sum_{i=1}^{n} a_i \geqslant \frac{1}{3} n(n^2 + 2)$.

4. The function f is defined on the rational numbers and takes only rational values. For all rational x and y

$$f(x + f(y)) = f(x)f(y).$$

Prove that f is constant.

5. For n a positive integer denote by $p(n)$ the number of ways of expressing n as the sum of one or more positive integers. Thus $p(4) = 5$, because there are 5 different sums, namely

$$1+1+1+1, \quad 1+1+2, \quad 1+3, \quad 2+2, \quad 4.$$

Prove that for $n > 1$,

$$p(n+1) - 2p(n) + p(n-1) \geqslant 0$$

6. Prove that in the infinite sequence of integers

10001, 100010001, 1000100010001, ...

there is no prime number.

Note each integer after the first (ten thousand and one) is obtained by adjoining 0001 to the digits of the previous integer.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

13th March, 1980

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

Q 1. Prove that the equation $x^n + y^n = z^n$, where n is an integer > 1, has no solution in integers x, y, z, with $0 < x \leqslant n$, $0 < y \leqslant n$.

Q 2. Find a set $S = \{n\}$ of 7 consecutive positive integers for which a polynomial $P(x)$ of the 5th degree exists with the following properties:

(a) all the coefficients in $P(x)$ are integers;

(b) $P(n) = n$ for 5 members of S, including the least and greatest;

(c) $P(n) = 0$ for one member of S.

Q3. On the diameter AB bounding a semi-circular region there are two points P and Q, and on the semi-circular arc there are two points R and S such that PQRS is a square. C is a point on the semi-circular arc such that the areas of the triangle ABC and the square PQRS are equal.

Prove that a straight line passing through one of the points R and S and through one of the points A and B cuts a side of the square at the in-centre of the triangle.

Q 4. Find the set of real numbers a_0 for which the infinite sequence $\{a_n\}$ of real numbers defined by

$$a_{n+1} = 2^n - 3a_n \qquad (n \geqslant 0)$$

is strictly increasing, i.e.

$$a_n < a_{n+1} \qquad (n \geqslant 0)$$

Q 5. In a party of ten persons, among any three persons there are at least two who do not know each other. Prove that at the party there are four persons none of whom knows another of the four.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

19th March, 1981

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. H is the orthocentre of triangle ABC. The midpoints of BC, CA, AB are A', B', C' respectively. A circle with centre H cuts the sides of triangle $A'B'C'$ (produced if necessary) in six points, D_1, D_2 on $B'C'$, E_1, E_2 on $C'A'$ and F_1, F_2 on $A'B'$.

   Prove that
$$AD_1 = AD_2 = BE_1 = BE_2 = CF_1 = CF_2 .$$

2. m and n are positive integers. S_m is the sum of m terms of
$$(n+1) - (n+1)(n+3) + (n+1)(n+2)(n+4) - (n+1)(n+2)(n+3)(n+5) + \ldots$$
where the terms alternate in sign and each, after the first, is the product of consecutive integers with the last but one omitted.

   Prove that S_m is divisible by $m!$ but not necessarily by $m!(n+1)$.

3. a, b, c are positive numbers. Prove

   (i) $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$.

   (ii) $abc \geq (a+b-c)(b+c-a)(c+a-b)$.

4. n points are given such that no plane passes through four of them. S is the set of all tetrahedra whose vertices are 4 of the n points. A plane does not pass through any of the n points.

   Prove that it cannot cut more than $n^2(n-2)^2/64$ of the tetrahedra of S in quadrilateral cross-sections.

5. Find, with proof, the smallest possible value of $|12^m - 5^n|$, where m and n are positive integers.

6. a_i, $i = 1,2,3,\ldots n$, are distinct non-zero integers.
$$p_i = \prod_{j\neq i}^{n} (a_i - a_j) \qquad \text{is the product of the } (n-1) \text{ factors}$$
$(a_i - a_1)$, $(a_i - a_2)$, ... $(a_i - a_n)$, the zero factor $(a_i - a_i)$ being excluded.

   Prove that if k is a non-negative integer,
$$\sum_{i=1}^{n} \frac{a_i^k}{p_i} \qquad \text{is an integer.}$$

THE MATHEMATICAL ASSOCIATION

National Committee for Mathematical Contests

British Mathematical Olympiad

18th March, 1982

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. PQRS is a quadrilateral of area A. O is a point inside it. Prove that if
$$2A = OP^2 + OQ^2 + OR^2 + OS^2,$$
then PQRS is a square and O is its centre.

2. A multiple of 17 when written in the scale of 2 contains exactly three digits 1. Prove that it contains at least six digits 0, and that if it contains exactly seven digits 0, then it is even.

3. If $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots. + \frac{1}{n}$ and $n > 2$, prove
$$n(n+1)^a - n < s_n < n - (n-1)n^b$$
where a and b are given in terms of n by $an = 1$, $b(n-1) = -1$.

4. A sequence of real numbers $u_1, u_2, u_3 \ldots.$ is given by u_1 and the recurrence relation $u_n^3 = u_{n-1} + \frac{15}{64}$, $n \geqslant 2$.

   By considering the curve $x^3 = y + \frac{15}{64}$, or otherwise, describe with proof the behaviour of u_n as n tends to infinity.

5. A right circular cone stands on a horizontal base, radius r. Its vertex V is at a distance 1 from every point on the perimeter of the base. A plane section of the cone is an ellipse whose lowest point is L and whose highest point is H. On the curved surface of the cone, to one side of the plane VLH, two routes from L to H are marked. R_1 is along the semi-perimeter of the ellipse and R_2 is the route of shortest length.

   Find the condition that R_1 and R_2 intersect between L and H.

6. Prove that the number of sequences $a_1a_2 \ldots. a_n$ with each of their n terms $a_i = 0$ or 1 and containing exactly m occurrences of 01 is $\binom{n+1}{2m+1}$.

THE MATHEMATICAL ASSOCIATION

National Committee for Mathematical Contests

British Mathematical Olympiad

10th March 1983

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order. Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. In the triangle ABC with circumcentre O, AB = AC, D is the midpoint of AB and E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

2. The Fibonacci sequence $\{f_n\}$ is defined by

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \qquad (n > 2).$$

Prove that there are unique integers a, b, m such that $0 < a < m$, $0 < b < m$ and $f_n - anb^n$ is divisible by m for all positive integers n.

3. The real numbers $x_1, x_2, x_3, \ldots$ are defined by

$$x_1 = a \neq -1 \quad \text{and} \quad x_{n+1} = x_n^2 + x_n \quad \text{for all } n \geq 1.$$

S_n is the sum and P_n is the product of the first n terms of the sequence $y_1, y_2, y_3, \ldots,$ where $y_n = \dfrac{1}{1+x_n}$.

Prove that $aS_n + P_n = 1$ for all n.

4. The two cylindrical surfaces

$$x^2 + z^2 = a^2, \quad z > 0, \quad |y| \leq a$$
$$\text{and} \quad y^2 + z^2 = a^2, \quad z > 0, \quad |x| \leq a$$

intersect and with the plane $z = 0$ enclose a dome-like shape which is here called a "cupola". The cupola is placed on top of a vertical tower of height h whose horizontal cross-section is a square of side $2a$. Find the shortest distance from the highest point of the cupola to a corner of the base of the tower, over the surface of the cupola and tower.

5. If 10 points are within a circle of diameter 5", prove that the distance between some 2 of the points is less than 2".

6. Consider the equation

$$\sqrt{(2p + 1 - x^2)} + \sqrt{(3x + p + 4)} = \sqrt{(x^2 + 9x + 3p + 9)} \qquad (1)$$

in which x,p are real numbers and the square roots are to be <u>real</u> and <u>non-negative</u>.

Show that if (1) holds then

$$(x^2 + x - p)(x^2 + 8x + 2p + 9) = 0.$$

Hence find the set of real numbers p for which (1) is satisfied by exactly one real number x.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

Tuesday, 13th March 1984

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

CANDIDATES ARE NOT EXPECTED TO ATTEMPT ALL SEVEN QUESTIONS.

1. P, Q, R are arbitrary points on the sides BC, CA, AB respectively of triangle ABC. Prove that the triangle whose vertices are the centres of the circles AQR, BRP, CPQ is similar to triangle ABC.

2. Let a_n be the number of binomial coefficients $\binom{n}{r}$ $(0 \leq r \leq n)$ which leave remainder 1 on division by 3 and let b_n be the number which leave remainder 2. Prove that $a_n > b_n$ for all positive integers n.

3. (i) Prove that, for all positive integers m,

$$\left(2 - \frac{1}{m}\right)\left(2 - \frac{3}{m}\right)\left(2 - \frac{5}{m}\right) \ldots \left(2 - \frac{2m-1}{m}\right) \leq m!$$

(ii) Prove that if a, b, c, d, e are positive real numbers then

$$\left(\frac{a}{b}\right)^4 + \left(\frac{b}{c}\right)^4 + \left(\frac{c}{d}\right)^4 + \left(\frac{d}{e}\right)^4 + \left(\frac{e}{a}\right)^4$$

$$\geq \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{e}{d} + \frac{a}{e} .$$

4. Let N be a positive integer. Determine with proof the number of solutions of the equation

$$x^2 - [x^2] = (x - [x])^2$$

lying in the interval $1 \leq x \leq N$.

(For a real number x the "integer part" $[x]$ is the largest integer which is $\leqslant x$.)

5. A plane cuts a right circular cone with vertex V in an ellipse E and meets the axis of the cone at C; A is an extremity of the major axis of E. Prove that the area of the curved surface of the slant cone with V as vertex and E as base is

$$\frac{VA}{AC} \times (\text{area of } E).$$

6. Let a, m be positive integers. Prove that if there exists an integer x such that $a^2x - a$ is divisible by m then there exists an integer y such that both $a^2y - a$ and $ay^2 - y$ are divisible by m.

7. $ABCD$ is a quadrilateral which has an inscribed circle. With the side AB is associated

$$u_{AB} = p_1 \sin \widehat{DAB} + p_2 \sin \widehat{ABC}$$

where p_1, p_2 are the perpendiculars from A, B respectively to the opposite side CD. Define u_{BC}, u_{CD}, u_{DA} likewise, using in each case perpendiculars to the opposite side. Show that

$$u_{AB} = u_{BC} = u_{CD} = u_{DA}.$$

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

Tuesday 5 March, 1985

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. Two circles S_1 and S_2 each touch a straight line p at the same point P. All points of S_2, except P, are in the interior of S_1. A straight line q (i) is perpendicular to p; (ii) touches S_2 at R; (iii) cuts p at L; and (iv) cuts S_1 at N and M, where M is between L and R.

   (a) Prove that RP bisects angle MPN.

   (b) If MP bisects angle RPL, find, with proof, the ratio of the areas of S_1 and S_2.

2. $a, b, c,$ are each numbers between 0 and 1. Prove that not all of $a(1-b)$, $b(1-c)$, $c(1-a)$ can be greater than $\frac{1}{4}$.

3. n and m are non-negative integers. Prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \ldots\ldots\ldots + (n+1-m)\binom{m}{m} = \binom{n+2}{m+2}$$

   where $\binom{r}{s}$ is the binomial coefficient $r(r-1)(r-2)\ldots(r-s+1)/s!$

4. The sequence f_n is defined by $f_0 = 1$, $f_1 = c$, where c is a positive integer, and for all $n > 1$,

$$f_n = 2f_{n-1} - f_{n-2} + 2 .$$

   Prove that for each $k \geqslant 0$ there exists h such that $f_k f_{k+1} = f_h$.

5. A cylindrical container has height 6 cm and radius 4 cm. It rests on a circular hoop which has also radius 4 cm and the hoop is fixed in a horizontal plane. The container rests with its axis horizontal and with each of its circular rims touching the hoop at two points. The cylinder is now moved so that each of its circular rims still touches the hoop at two points. Find, with proof, the locus of the centre of one of the cylinder's circular ends.

6. Show that the equation $x^2 + y^2 = z^5 + z$ has infinitely many solutions in positive integers x, y, z having no factor in common greater than 1.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

Tuesday 4th March 1986

Time allowed - $3\frac{1}{2}$ hours.

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order. Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. Reduce the fraction $\frac{N}{D}$ to its lowest terms when

$$N = 2244851485148514627 ,$$

$$D = 8118811881188118000 .$$

2. A circle S of radius R has two parallel tangents t_1, t_2. A circle S_1 of radius r_1 touches S and t_1; a circle S_2 of radius r_2 touches S and t_2; also S_1 touches S_2 and all the circle contacts are external. Calculate R in terms of r_1 and r_2.

3. Prove that if m, n, r are positive integers and

$$1 + m + n\sqrt{3} = (2 + \sqrt{3})^{2r-1}$$

then m is a perfect square.

4. Find, with proof, the largest real number K (independent of a,b,c) such that the inequality

$$a^2 + b^2 + c^2 > K(a+b+c)^2$$

holds for the lengths a, b, c of the sides of any obtuse-angled triangle.

5. Find, with proof, the number of permutations

$$a_1, a_2, \ldots, a_n$$

of $1, 2, \ldots, n$ such that

$$a_r < a_{r+2} \quad \text{for} \quad 1 \leqslant r \leqslant n-2$$

and

$$a_r < a_{r+3} \quad \text{for} \quad 1 \leqslant r \leqslant n-3 .$$

(In a permutation each of the numbers $1,2,\ldots,n$ appears.)

6. AB, AC, AD are three edges of a cube. AC is produced to E so that AE = 2AC and AD is produced to F so that AF = 3AD. Prove that the area of the section of the cube by any plane parallel to BCD is equal to the area of the section of tetrahedron ABEF by the same plane.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

Friday 20th March 1987

Time allowed - 3½ hours

PLEASE READ THESE INSTRUCTIONS CAREFULLY:

Write on one side of the paper only. Use a fresh sheet or sheets of paper for each question. Arrange your answers in order. Put your full name, age (in years and months), home address and school on the top sheet of your answers. On each other sheet put your name and initials, and the number of the question.

There is no restriction on the number of questions which may be attempted. Nos 1-3 may be found easier than nos 4-7. Candidates who wish to be considered for the British I.M.O. team will be judged on their performance on the whole paper but additional weight will be given to the harder questions.

1. **(a)** Find, with proof, all integer solutions of

$$a^3 + b^3 = 9 .$$

(b) Find, with proof, all integer solutions of

$$35x^3 + 66x^2y + 42xy^2 + 9y^3 = 9 .$$

2. In a triangle ABC, $\angle BAC = 100°$ and AB = AC. A point D is chosen on the side AC so that $\angle ABD = \angle CBD$. Prove that AD + DB = BC.

3. Find, with proof, the value of the limit as $n \to \infty$ of

$$\sum_{r=0}^{n} \binom{2n}{2r} 2^r \Big/ \sum_{r=0}^{n-1} \binom{2n}{2r+1} 2^r .$$

Here $\binom{2n}{s}$ denotes a binomial coefficient.

4. /

P.T.O. ...

4. Let $P(x)$ be any polynomial with integer coefficients such that

$$P(21) = 17, \quad P(32) = -247, \quad P(37) = 33 .$$

Prove that if $P(N) = N + 51$ for some integer N, then $N = 26$.

5. A line parallel to the side BC of an acute-angled triangle ABC cuts the side AB at F and the side AC at E. Prove that the circles on BE and CF as diameters intersect on the altitude of the triangle drawn from A perpendicular to BC.

6. Find, with proof, the maximum value of

$$\frac{xyz}{(1+x)(x+y)(y+z)(z+16)}$$

for positive real numbers x, y, z.

7. Prove that if n and k are any positive integers then there exists a positive integer x such that $\frac{1}{2}x(x+1) - k$ is divisible by 2^n.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

Friday 20th November 1987

Time allowed - 3½ hours

PLEASE READ THESE INSTRUCTIONS CAREFULLY :

Write on one side of the paper only. Use a fresh sheet or sheets of paper for each question. Arrange your answers in order. On the first sheet of your script write ONLY your full name, age (in years and months), home address and school; do not put any working on this sheet. On every sheet of working write your name and initials clearly in capital letters, and the number of the question.

There is no restriction on the number of questions which may be attempted, but remember

USE FRESH SHEETS FOR EACH QUESTION

1. Find all the real solutions x of the equation

$$\sqrt{(x + 1972098 - 1986\sqrt{(x + 986049)})} + \sqrt{(x + 1974085 - 1988\sqrt{(x + 986049)})} = 1$$

where $\sqrt{}$ indicates the non-negative square root.

2. Find all the real-valued functions f defined on the set D of natural numbers $x \geq 10$ and satisfying the functional equation

$$f(x + y) = f(x)\, f(y)$$

for all $x, y \in D$.

3. Find a pair of integers r, s such that $0 < s < 200$ and

$$\frac{45}{61} > \frac{r}{s} > \frac{59}{80}.$$

Also prove that there is exactly one such pair r, s.

4. The triangle ABC has orthocentre H. The feet of the perpendiculars from H to the internal and external bisectors of angle BAC (which is not a right angle) are P and Q. Prove that PQ passes through the middle point of BC.

5. Numbers $d(n, m)$ with m, n integers, $0 \leq m \leq n$, are defined by

$$d(n, 0) = d(n, n) = 1 \quad \text{all } n \geq 0$$

and $$md(n, m) = md(n-1, m) + (2n - m)\, d(n-1, m-1)$$

for $0 < m < n$. Prove that all the $d(n, m)$ are integers.

6. Show that the least positive value of

$$\frac{x^2 + y^2}{y},$$

where x, y are real numbers such that

$$7x^2 + 3xy + 3y^2 = 1,$$

is $\frac{1}{2}$.

REMEMBER : A FRESH SHEET FOR EACH QUESTION
WITH NAME AND QUESTION NUMBER ON EVERY SHEET

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

BRITISH MATHEMATICAL OLYMPIAD

Tuesday 13th December 1988

Time allowed - $3\frac{1}{2}$ hours

PLEASE READ THESE INSTRUCTIONS CAREFULLY

Write on one side of the paper only. ***Use a fresh sheet of paper for each question.*** *Arrange your answers in order. On the first sheet of your script write ONLY your full name, age (in years and months), home address and school; do not put any working on this sheet.* ***On every sheet of working write your name and initials, your school and the number of the question.***

There is no restriction on the number of questions which may be attempted, but remember

USE FRESH SHEETS FOR EACH QUESTION.

1. Find all integers a, b, c for which

$$(x-a)(x-10)+1 \equiv (x+b)(x+c) \quad \text{for all } x.$$

2. Points P, Q lie on the sides AB, AC respectively of triangle ABC and are distinct from A. The lengths AP, AQ are denoted by x, y respectively, with the convention that $x > 0$ if P is on the same side of A as B, and $x < 0$ on the opposite side; similarly for y. Show that PQ passes through the centroid of the triangle if and only if

$$3xy = bx + cy$$

where $b = AC$, $c = AB$.

3. OA, OB, OC are mutually perpendicular lines. Express the area of triangle ABC in terms of the areas of triangles OBC, OCA, OAB.

[Turn over

4. Consider the following triangle of numbers:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & 1 & 1 & & \\ & 1 & 2 & 3 & 2 & 1 & \\ 1 & 3 & 6 & 7 & 6 & 3 & 1 \\ . & . & . & . & . & . & . \end{array}$$

Each number is the sum of three numbers in the previous row: the number above it and the numbers immediately to the left and right of that number. If there is no number in one or more of these positions, 0 is used.

Prove that, from the third row on, every row contains at least one even number.

5. None of the angles of a triangle ABC exceeds $90°$. Prove that

$$\sin A + \sin B + \sin C > 2.$$

6. The sequence $\{a_n\}$ of integers is defined by

$$a_1 = 2,\ a_2 = 7$$

and

$$-\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leq \frac{1}{2} \qquad \text{for } n \geq 2.$$

Prove that a_n is odd for all $n > 1$.

REMEMBER: A FRESH SHEET FOR EACH QUESTION WITH NAME, SCHOOL AND QUESTION NUMBER ON EVERY SHEET.

NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

BRITISH MATHEMATICAL OLYMPIAD

Wednesday 17th January 1990

Time allowed – Three and a half hours

PLEASE READ THESE INSTRUCTIONS CAREFULLY

Write on one side of the paper only. ***Use a fresh sheet of paper for each question.*** *Arrange your answers in order. On the first sheet of your script write ONLY your full name, age (in years and completed months on 17th January 1990), home address and school; do not put any working on this sheet.* ***On every sheet of working write your name and initials, your school and the number of the question.***

There is no restriction on the number of questions which may be attempted, but remember

USE FRESH SHEETS FOR EACH QUESTION.

1. Find a positive integer whose first digit is 1 and which has the property that, if this digit is transferred to the end of the number, the number is tripled.

2. $ABCD$ is a square and P is a point on the line AB. Find the maximum and minimum values of the ratio PC/PD, showing that these occur for the points P given by $AP \times BP = AB^2$.

3. The angles A, B, C, D of a convex quadrilateral satisfy the relation

$$\cos A + \cos B + \cos C + \cos D = 0.$$

   Prove that $ABCD$ is a trapezium or is cyclic.

4. A coin is biassed so that the probability of obtaining a head is p, $0 < p < 1$. Two players A and B throw the coin in turn until one of the sequences HHH or HTH occurs. If sequence HHH occurs first then A wins. If HTH occurs first then B wins. For what value of p is this game fair (ie. such that A and B have an equal chance of winning)?

[Turn over

5. The diagonals of a convex quadrilateral $ABCD$ intersect at O. The centroids of triangles AOD and BOC are P and Q; the orthocentres of triangles AOB and COD are R and S. Prove that PQ is perpendicular to RS.
[The *centroid* of a triangle is the intersection of the lines joining each vertex to the midpoint of the opposite side; the *orthocentre* is the intersection of the altitudes].

6. Prove that if x, y are rational numbers satisfying the equation
$$x^5 + y^5 = 2x^2y^2$$
then $1 - xy$ is the square of a rational number.

REMEMBER: A FRESH SHEET FOR EACH QUESTION WITH NAME, SCHOOL AND QUESTION NUMBER ON EVERY SHEET.

As a result of their performance in BMO, some candidates will be selected to take a **Further International Selection Test** (FIST) early in March 1990. Following this, about 20 candidates will be selected to attend a residential **Training Session** to be held at Trinity College, Cambridge, from 5th - 8th April 1990. The UK team for the 1990 **International Mathematical Olympiad** (IMO) in Beijing will then be chosen.

Prizes: There will be a number of prizes, in the form of book tokens, for those who do very well in BMO. In addition, Trinity College, Cambridge, will award £50 prizes to all those who attend the residential Training Session.

The 31st International Mathematical Olympiad will be held in Beijing, People's Republic of China, beginning 11th July, 1990.

SPONSORS of the British team for the International Mathematical Olympiad:

The Department of Education and Science; Trinity College, Cambridge; The Corporation of the City of London; ICI; The Royal Society; Eton College; Portsmouth Grammar School; St. Paul's School; Westminster School; St. John's College, Cambridge; Dulwich College; The London Mathematical Society; The Manchester Grammar School; Winchester College; Merton College, Oxford.

BRITISH MATHEMATICAL OLYMPIAD

Wednesday 16th January 1991

Time allowed *– Three and a half hours*

Instructions:
- *Start each question on a fresh sheet of paper.*
- *Write on one side of the paper only.*
- *On every sheet of working write the number of the question in the top left hand corner and your name, initials and school in the top right hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1, 2, 3, 4, 5, 6, 7 in order.*
- *Staple all the pages neatly together in the top left hand corner.*

---

1. Prove that the number
$$3^n + 2 \times 17^n$$
where n is a non-negative integer, is never a perfect square.

[4 marks]

2. Find all positive integers k such that the polynomial $x^{2k+1} + x + 1$ is divisible by the polynomial $x^k + x + 1$.

For each such k specify the integers n such that $x^n + x + 1$ is divisible by $x^k + x + 1$.

[5 marks]

3. $ABCD$ is a quadrilateral inscribed in a circle of radius r. The diagonals AC, BD meet at E.

Prove that if AC is perpendicular to BD then
$$EA^2 + EB^2 + EC^2 + ED^2 = 4r^2. \qquad (*)$$

Is it true that if (*) holds then AC is perpendicular to BD? Give a reason for your answer.

[6 marks]

Turn over ...

4. Find, with proof, the minimum value of $(x + y)(y + z)$ where x, y, z are positive real numbers satisfying the condition

$$xyz(x + y + z) = 1.$$

[7 marks]

5. Find the number of permutations (arrangements)

$$p_1, p_2, p_3, p_4, p_5, p_6$$

of 1, 2, 3, 4, 5, 6 with the property:

For no integer n, $1 \leq n \leq 5$, do $p_1, p_2, \ldots, p_n$ form a permutation of $1, 2, \ldots n$.

[9 marks]

6. Show that if x and y are positive integers such that $x^2 + y^2 - x$ is divisible by $2xy$ then x is a perfect square.

[9 marks]

7. A ladder of length l rests against a vertical wall. Suppose that there is a rung on the ladder which has the same distance d from both the wall and the (horizontal) ground. Find *explicitly*, in terms of l and d, the height h from the ground that the ladder reaches up the wall.

[10 marks]

British Mathematical Olympiad

Round 1 : Wednesday 15th January 1992

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *The first two questions are intended to be more straightforward than the last three.*
- *The use of rulers and compasses is allowed, but calculators are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

BRITISH MATHEMATICAL OLYMPIAD

1. (a) Observe that the square of 20 has the same number of non-zero digits as the original number. Does there exist a two-digit number, *other than* 10, 20 or 30, whose square has the same number of non-zero digits as the original number? If you think there is one, then find it. If you claim that there is none, then you must prove your claim.

   (b) Does there exist a three-digit number *other than* 100, 200, 300 whose square has the same number of non-zero digits as the original number?

2. Let $ABCDE$ be a pentagon inscribed in a circle. Suppose that AC, BD, CE, DA and EB are parallel to DE, EA, AB, BC and CD, respectively. Does it follow that the pentagon has to be regular? Justify your claim.

3. Find four distinct positive integers whose product is divisible by the sum of every pair of them.

   Can you find a set of five or more numbers with the same property?

4. Determine the smallest value of $x^2+5y^2+8z^2$, where x, y, z are real numbers subject to the condition $yz+zx+xy=-1$. Does $x^2+5y^2+8z^2$ have a greatest value subject to the same condition? Justify your claim.

5. Let f be a function mapping the positive integers into positive integers. Suppose that $f(n+1) > f(n)$ and $f\big(f(n)\big) = 3n$ for all positive integers n. Determine $f(1992)$.

British Mathematical Olympiad

Round 1 : Wednesday 13th January 1993

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Find, showing your method, a six-digit integer n with the following properties: (i) n is a perfect square, (ii) the number formed by the last three digits of n is exactly one greater than the number formed by the first three digits of n. (Thus n might look like 123124, although this is not a square.)

2. A square piece of toast $ABCD$ of side length 1 and centre O is cut in half to form two equal pieces ABC and CDA. If the triangle ABC has to be cut into two parts of equal area, one would usually cut along the line of symmetry BO. However, there are other ways of doing this. Find, with justification, the length and location of the shortest straight cut which divides the triangle ABC into two parts of equal area.

3. For each positive integer c, the sequence u_n of integers is defined by

   $u_1 = 1,\ u_2 = c, \quad u_n = (2n+1)u_{n-1} - (n^2-1)u_{n-2},\ (n \geq 3).$

   For which values of c does this sequence have the property that u_i divides u_j whenever $i \leq j$?
   (Note: If x and y are integers, then x *divides* y if and only if there exists an integer z such that $y = xz$. For example, $x = 4$ divides $y = -12$, since we can take $z = -3$.)

4. Two circles touch internally at M. A straight line touches the inner circle at P and cuts the outer circle at Q and R. Prove that $\angle QMP = \angle RMP$.

5. Let x, y, z be positive real numbers satisfying

   $$\frac{1}{3} \leq xy + yz + zx \leq 3.$$

   Determine the range of values for (i) xyz, and (ii) $x + y + z$.

British Mathematical Olympiad

Round 1 : Wednesday 19th January 1994

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Starting with any three digit number n (such as $n = 625$) we obtain a new number $f(n)$ which is equal to the sum of the three digits of n, their three products in pairs, and the product of all three digits.
   (i) Find the value of $n/f(n)$ when $n = 625$. (The answer is an integer!)
   (ii) Find all three digit numbers such that the ratio $n/f(n)$=1.

2. In triangle ABC the point X lies on BC.
   (i) Suppose that $\angle BAC = 90°$, that X is the midpoint of BC, and that $\angle BAX$ is one third of $\angle BAC$. What can you say (and prove!) about triangle ACX?
   (ii) Suppose that $\angle BAC = 60°$, that X lies one third of the way from B to C, and that AX bisects $\angle BAC$. What can you say (and prove!) about triangle ACX?

3. The sequence of integers $u_0, u_1, u_2, u_3, \ldots$ satisfies $u_0 = 1$ and
   $$u_{n+1}u_{n-1} = ku_n \quad \text{for each} \quad n \geq 1,$$
   where k is some fixed positive integer. If $u_{2000} = 2000$, determine all possible values of k.

4. The points Q, R lie on the circle γ, and P is a point such that PQ, PR are tangents to γ. A is a point on the extension of PQ, and γ' is the circumcircle of triangle PAR. The circle γ' cuts γ again at B, and AR cuts γ at the point C. Prove that $\angle PAR = \angle ABC$.

5. An *increasing* sequence of integers is said to be **alternating** if it *starts* with an *odd* term, the second term is even, the third term is odd, the fourth is even, and so on. The empty sequence (with no term at all!) is considered to be alternating. Let $A(n)$ denote the number of alternating sequences which only involve integers from the set $\{1, 2, \ldots, n\}$. Show that $A(1) = 2$ and $A(2) = 3$. Find the value of $A(20)$, and prove that your value is correct.

British Mathematical Olympiad

Round 1 : Wednesday 18th January 1995

Time allowed *Three and a half hours.*

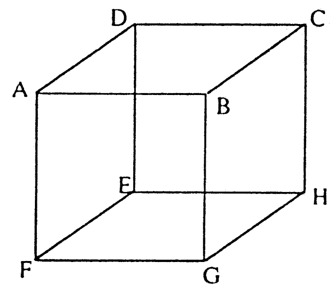
Instructions
- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Find the first positive integer whose square ends in three 4's.
Find all positive integers whose squares end in three 4's.
Show that no perfect square ends with four 4's.

2. $ABCDEFGH$ is a cube of side 2.
(a) Find the area of the quadrilateral $AMHN$, where M is the midpoint of BC, and N is the midpoint of EF.
(b) Let P be the midpoint of AB, and Q the midpoint of HE. Let AM meet CP at X, and HN meet FQ at Y. Find the length of XY.



3. (a) Find the maximum value of the expression $x^2y - y^2x$ when $0 \le x \le 1$ and $0 \le y \le 1$.
(b) Find the maximum value of the expression
$$x^2y + y^2z + z^2x - x^2z - y^2x - z^2y$$
when $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

4. ABC is a triangle, right-angled at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q, respectively. M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN.

5. The seven dwarfs walk to work each morning in single file. As they go, they sing their famous song, "*High - low - high -low, it's off to work we go ...*". Each day they line up so that no three successive dwarfs are either increasing or decreasing in height. Thus, the line-up must go *up-down-up-down-* $\cdots$ or *down-up-down-up-* $\cdots$. If they all have different heights, for how many days they go to work like this if they insist on using a different order each day?
What if Snow White always came along too?

British Mathematical Olympiad

Round 1 : Wednesday, 17th January 1996

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Consider the pair of four-digit positive integers
$$(M, N) = (3600, 2500).$$
Notice that M and N are both perfect squares, with equal digits in two places, and differing digits in the remaining two places. Moreover, when the digits differ, the digit in M is exactly one greater than the corresponding digit in N. Find all pairs of four-digit positive integers (M, N) with these properties.

2. A function f is defined over the set of all positive integers and satisfies
$$f(1) = 1996$$
and
$$f(1) + f(2) + \cdots + f(n) = n^2 f(n) \quad \text{for all } n > 1.$$
Calculate the exact value of $f(1996)$.

3. Let ABC be an acute-angled triangle, and let O be its circumcentre. The circle through A, O and B is called S. The lines CA and CB meet the circle S again at P and Q respectively. Prove that the lines CO and PQ are perpendicular.
(*Given any triangle XYZ, its* **circumcentre** *is the centre of the circle which passes through the three vertices X, Y and Z.*)

4. For any real number x, let $[x]$ denote the greatest integer which is less than or equal to x. Define
$$q(n) = \left[\frac{n}{[\sqrt{n}]}\right] \quad \text{for} \quad n = 1, 2, 3, \ldots .$$
Determine all positive integers n for which $q(n) > q(n+1)$.

5. Let a, b and c be positive real numbers.
(i) Prove that $4(a^3 + b^3) \geq (a+b)^3$.
(ii) Prove that $9(a^3 + b^3 + c^3) \geq (a+b+c)^3$.

British Mathematical Olympiad

Round 1 : Wednesday, 15 January 1997

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. N is a four-digit integer, not ending in zero, and $R(N)$ is the four-digit integer obtained by reversing the digits of N; for example, $R(3275) = 5723$.
   Determine all such integers N for which $R(N) = 4N + 3$.

2. For positive integers n, the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is defined by
   $$a_1 = 1; \quad a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + a_3 + \cdots + a_{n-1}), \quad n > 1.$$
   Determine the value of a_{1997}.

3. The Dwarfs in the Land-under-the-Mountain have just adopted a completely decimal currency system based on the *Pippin*, with gold coins to the value of 1 *Pippin*, 10 *Pippins*, 100 *Pippins* and 1000 *Pippins*.
   In how many ways is it possible for a Dwarf to pay, in exact coinage, a bill of 1997 *Pippins*?

4. Let $ABCD$ be a convex quadrilateral. The midpoints of AB, BC, CD and DA are P, Q, R and S, respectively. Given that the quadrilateral $PQRS$ has area 1, prove that the area of the quadrilateral $ABCD$ is 2.

5. Let x, y and z be positive real numbers.
   (i) If $x + y + z \geq 3$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 3$?
   (ii) If $x + y + z \leq 3$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3$?

British Mathematical Olympiad

Round 1 : Wednesday, 14 January 1998

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. A 5×5 square is divided into 25 unit squares. One of the numbers 1, 2, 3, 4, 5 is inserted into each of the unit squares in such a way that each row, each column and each of the two diagonals contains each of the five numbers once and only once. The sum of the numbers in the four squares immediately below the diagonal from top left to bottom right is called the *score*.
Show that it is impossible for the score to be 20.
What is the highest possible score?

2. Let $a_1 = 19$, $a_2 = 98$. For $n \geq 1$, define a_{n+2} to be the remainder of $a_n + a_{n+1}$ when it is divided by 100. What is the remainder when
$$a_1^2 + a_2^2 + \cdots + a_{1998}^2$$
is divided by 8?

3. ABP is an isosceles triangle with $AB = AP$ and $\angle PAB$ acute. PC is the line through P perpendicular to BP, and C is a point on this line on the same side of BP as A. (You may assume that C is not on the line AB.) D completes the parallelogram $ABCD$. PC meets DA at M.
Prove that M is the midpoint of DA.

4. Show that there is a unique sequence of positive integers (a_n) satisfying the following conditions:
$$a_1 = 1, \quad a_2 = 2, \quad a_4 = 12,$$
$$a_{n+1}a_{n-1} = a_n^2 \pm 1 \quad \text{for} \quad n = 2, 3, 4, \ldots .$$

5. In triangle ABC, D is the midpoint of AB and E is the point of trisection of BC nearer to C. Given that $\angle ADC = \angle BAE$ find $\angle BAC$.

British Mathematical Olympiad

Round 1 : Wednesday, 13 January 1999

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
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Do not turn over until **told to do so.**

British Mathematical Olympiad

1. I have four children. The age in years of each child is a positive integer between 2 and 16 inclusive and all four ages are distinct. A year ago the square of the age of the oldest child was equal to the sum of the squares of the ages of the other three. In one year's time the sum of the squares of the ages of the oldest and the youngest will be equal to the sum of the squares of the other two children.
Decide whether this information is sufficient to determine their ages uniquely, and find all possibilities for their ages.

2. A circle has diameter AB and X is a fixed point of AB lying between A and B. A point P, distinct from A and B, lies on the circumference of the circle. Prove that, for all possible positions of P,
$$\frac{\tan \angle APX}{\tan \angle PAX}$$
remains constant.

3. Determine a positive constant c such that the equation
$$xy^2 - y^2 - x + y = c$$
has precisely three solutions (x, y) in positive integers.

4. Any positive integer m can be written uniquely in base 3 form as a string of 0's, 1's and 2's (not beginning with a zero). For example,
$$98 = (1\times81) + (0\times27) + (1\times9) + (2\times3) + (2\times1) = (10122)_3.$$
Let $c(m)$ denote the sum of the cubes of the digits of the base 3 form of m; thus, for instance
$$c(98) = 1^3 + 0^3 + 1^3 + 2^3 + 2^3 = 18.$$
Let n be any fixed positive integer. Define the sequence (u_r) by
$$u_1 = n \qquad \text{and} \qquad u_r = c(u_{r-1}) \quad \text{for} \quad r \geq 2.$$
Show that there is a positive integer r for which $u_r = 1, 2$ or 17.

5. Consider all functions f from the positive integers to the positive integers such that
(i) for each positive integer m, there is a unique positive integer n such that $f(n) = m$;
(ii) for each positive integer n, we have
$f(n+1)$ is **either** $4f(n) - 1$ **or** $f(n) - 1$.
Find the set of positive integers p such that $f(1999) = p$ for some function f with properties (i) and (ii).

British Mathematical Olympiad

Round 1 : Wednesday, 12 January 2000

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
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British Mathematical Olympiad

1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

2. Show that, for every positive integer n,
$$121^n - 25^n + 1900^n - (-4)^n$$
is divisible by 2000.

3. Triangle ABC has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that
$$AP + BP + CP$$
is minimized.

4. For each positive integer $k > 1$, define the sequence $\{a_n\}$ by
$$a_0 = 1 \quad \text{and} \quad a_n = kn + (-1)^n a_{n-1} \qquad \text{for each} \quad n \geq 1.$$
Determine all values of k for which 2000 is a term of the sequence.

5. The seven dwarfs decide to form four teams to compete in the Millennium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy & Grumpy, and one of Bashful & Sneezy. In how many ways can the four teams be made up? (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams.)
Suppose Snow-White agreed to take part as well. In how many ways could the four teams then be formed?

British Mathematical Olympiad

Round 1 : Wednesday, 17 January 2001

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
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2001 British Mathematical Olympiad
Round 1

1. Find all two-digit integers N for which the sum of the digits of $10^N - N$ is divisible by 170.

2. Circle S lies inside circle T and touches it at A. From a point P (distinct from A) on T, chords PQ and PR of T are drawn touching S at X and Y respectively. Show that $\angle QAR = 2\angle XAY$.

3. A *tetromino* is a figure made up of four unit squares connected by common edges.
   (i) If we do not distinguish between the possible rotations of a tetromino within its plane, prove that there are seven distinct tetrominoes.
   (ii) Prove or disprove the statement: It is possible to pack all seven distinct tetrominoes into a 4×7 rectangle without overlapping.

4. Define the sequence (a_n) by
$$a_n = n + \{\sqrt{n}\},$$
where n is a positive integer and $\{x\}$ denotes the nearest integer to x, where halves are rounded up if necessary. Determine the smallest integer k for which the terms $a_k, a_{k+1}, \ldots, a_{k+2000}$ form a sequence of 2001 consecutive integers.

5. A triangle has sides of length a, b, c and its circumcircle has radius R. Prove that the triangle is right-angled if and only if $a^2 + b^2 + c^2 = 8R^2$.

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British Mathematical Olympiad

Round 1 : Wednesday, 5 December 2001

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
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2001 British Mathematical Olympiad
Round 1

1. Find all positive integers m, n, where n is odd, that satisfy
$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}.$$

2. The quadrilateral $ABCD$ is inscribed in a circle. The diagonals AC, BD meet at Q. The sides DA, extended beyond A, and CB, extended beyond B, meet at P.
Given that $CD = CP = DQ$, prove that $\angle CAD = 60^\circ$.

3. Find all positive real solutions to the equation
$$x + \left\lfloor \frac{x}{6} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor,$$
where $\lfloor t \rfloor$ denotes the largest integer less than or equal to the real number t.

4. Twelve people are seated around a circular table. In how many ways can six pairs of people engage in handshakes so that no arms cross?
(Nobody is allowed to shake hands with more than one person at once.)

5. f is a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of non-negative integers, which has the following properties:-
a) $f(n+1) > f(n)$ for each $n \in \mathbb{Z}^+$,
b) $f(n + f(m)) = f(n) + m + 1$ for all $m, n \in \mathbb{Z}^+$.
Find all possible values of $f(2001)$.

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British Mathematical Olympiad

Round 1 : Wednesday, 11 December 2002

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
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2002/3 British Mathematical Olympiad
Round 1

1. Given that
$$34! = 295\,232\,799\,cd9\,604\,140\,847\,618\,609\,643\,5ab\,000\,000,$$
determine the digits a, b, c, d.

2. The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the line segment AC, and BX meets S again at Q.
Show that $BX = CX$ if and only if PQ is a diameter of S.

3. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$.
Prove that
$$x^2yz + xy^2z + xyz^2 \le \frac{1}{3}.$$

4. Let m and n be integers greater than 1. Consider an $m \times n$ rectangular grid of points in the plane. Some k of these points are coloured red in such a way that no three red points are the vertices of a right-angled triangle two of whose sides are parallel to the sides of the grid. Determine the greatest possible value of k.

5. Find all solutions in positive integers a, b, c to the equation
$$a!\,b! = a! + b! + c!$$

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British Mathematical Olympiad

Round 1 : Wednesday, 3 December 2003

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
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2003/4 British Mathematical Olympiad
Round 1

1. Solve the simultaneous equations
$$ab + c + d = 3, \quad bc + d + a = 5, \quad cd + a + b = 2, \quad da + b + c = 6,$$
where a, b, c, d are real numbers.

2. $ABCD$ is a rectangle, P is the midpoint of AB, and Q is the point on PD such that CQ is perpendicular to PD.
Prove that the triangle BQC is isosceles.

3. Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game.
Prove that Alice can always obtain a score at least as great as Barbara's.

4. A set of positive integers is defined to be *wicked* if it contains no three consecutive integers. We count the empty set, which contains no elements at all, as a wicked set.
Find the number of wicked subsets of the set
$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

5. Let p, q and r be prime numbers. It is given that p divides $qr - 1$, q divides $rp - 1$, and r divides $pq - 1$.
Determine all possible values of pqr.

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British Mathematical Olympiad

Round 1 : Wednesday, 1 December 2004

Time allowed *Three and a half hours.*

Instructions
- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
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2004/5 British Mathematical Olympiad
Round 1

1. Each of Paul and Jenny has a whole number of pounds.
He says to her: "If you give me £3, I will have n times as much as you".
She says to him: "If you give me £n, I will have 3 times as much as you".
Given that all these statements are true and that n is a positive integer, what are the possible values for n?

2. Let ABC be an acute-angled triangle, and let D, E be the feet of the perpendiculars from A, B to BC, CA respectively. Let P be the point where the line AD meets the semicircle constructed outwardly on BC, and Q be the point where the line BE meets the semicircle constructed outwardly on AC. Prove that $CP = CQ$.

3. Determine the least natural number n for which the following result holds:
No matter how the elements of the set $\{1, 2, \ldots, n\}$ are coloured red or blue, there are integers x, y, z, w in the set (not necessarily distinct) of the same colour such that $x + y + z = w$.

4. Determine the least possible value of the largest term in an arithmetic progression of seven distinct primes.

5. Let S be a set of rational numbers with the following properties:
i) $\frac{1}{2} \in S$;
ii) If $x \in S$, then both $\frac{1}{x+1} \in S$ and $\frac{x}{x+1} \in S$.
Prove that S contains all rational numbers in the interval $0 < x < 1$.

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British Mathematical Olympiad

Round 1 : Wednesday, 30 November 2005

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
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2005/6 British Mathematical Olympiad
Round 1

1. Let n be an integer greater than 6. Prove that if $n-1$ and $n+1$ are both prime, then $n^2(n^2+16)$ is divisible by 720. Is the converse true?

2. Adrian teaches a class of six pairs of twins. He wishes to set up teams for a quiz, but wants to avoid putting any pair of twins into the same team. Subject to this condition:

   i) In how many ways can he split them into two teams of six?

   ii) In how many ways can he split them into three teams of four?

3. In the cyclic quadrilateral $ABCD$, the diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E. Show that $CE = CA$ if and only if $DE = AB$.

4. The equilateral triangle ABC has sides of integer length N. The triangle is completely divided (by drawing lines parallel to the sides of the triangle) into equilateral triangular cells of side length 1.

   A continuous route is chosen, starting inside the cell with vertex A and always crossing from one cell to another through an edge shared by the two cells. No cell is visited more than once. Find, with proof, the greatest number of cells which can be visited.

5. Let G be a convex quadrilateral. Show that there is a point X in the plane of G with the property that every straight line through X divides G into two regions of equal area if and only if G is a parallelogram.

6. Let T be a set of 2005 coplanar points with no three collinear. Show that, for any of the 2005 points, the number of triangles it lies strictly within, whose vertices are points in T, is even.

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British Mathematical Olympiad

Round 1 : Friday, 1 December 2006

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
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2006/7 British Mathematical Olympiad
Round 1

1. Find four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$.

2. In the convex quadrilateral $ABCD$, points M, N lie on the side AB such that $AM = MN = NB$, and points P, Q lie on the side CD such that $CP = PQ = QD$. Prove that
$$\text{Area of } AMCP = \text{Area of } MNPQ = \frac{1}{3} \text{ Area of } ABCD.$$

3. The number 916238457 is an example of a nine-digit number which contains each of the digits 1 to 9 exactly once. It also has the property that the digits 1 to 5 occur in their natural order, while the digits 1 to 6 do not. How many such numbers are there?

4. Two touching circles S and T share a common tangent which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that $AP = PQ$.

5. For positive real numbers a, b, c, prove that
$$(a^2 + b^2)^2 \geq (a + b + c)(a + b - c)(b + c - a)(c + a - b).$$

6. Let n be an integer. Show that, if $2 + 2\sqrt{1 + 12n^2}$ is an integer, then it is a perfect square.

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 30 November 2007

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
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United Kingdom Mathematics Trust

2007/8 British Mathematical Olympiad
Round 1: Friday, 30 November 2007

1. Find the value of
$$\frac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2}.$$

2. Find all solutions in positive integers x, y, z to the simultaneous equations
$$x + y - z = 12$$
$$x^2 + y^2 - z^2 = 12.$$

3. Let ABC be a triangle, with an obtuse angle at A. Let Q be a point (other than A, B or C) on the circumcircle of the triangle, on the same side of chord BC as A, and let P be the other end of the diameter through Q. Let V and W be the feet of the perpendiculars from Q onto CA and AB respectively. Prove that the triangles PBC and AWV are similar. [*Note: the circumcircle of the triangle ABC is the circle which passes through the vertices A, B and C.*]

4. Let S be a subset of the set of numbers $\{1, 2, 3, ..., 2008\}$ which consists of 756 distinct numbers. Show that there are two distinct elements a, b of S such that $a + b$ is divisible by 8.

5. Let P be an internal point of triangle ABC. The line through P parallel to AB meets BC at L, the line through P parallel to BC meets CA at M, and the line through P parallel to CA meets AB at N. Prove that
$$\frac{BL}{LC} \times \frac{CM}{MA} \times \frac{AN}{NB} \le \frac{1}{8}$$
and locate the position of P in triangle ABC when equality holds.

6. The function f is defined on the set of positive integers by $f(1) = 1$, $f(2n) = 2f(n)$, and $nf(2n + 1) = (2n + 1)(f(n) + n)$ for all $n \ge 1$.
   i) Prove that $f(n)$ is always an integer.
   ii) For how many positive integers less than 2007 is $f(n) = 2n$?

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Thursday, 4 December 2008

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
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United Kingdom Mathematics Trust

2008/9 British Mathematical Olympiad
Round 1: Thursday, 4 December 2008

1. Consider a standard 8×8 chessboard consisting of 64 small squares coloured in the usual pattern, so 32 are black and 32 are white. A *zig-zag* path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

2. Find all real values of x, y and z such that
$$(x+1)yz = 12, \ (y+1)zx = 4 \text{ and } (z+1)xy = 4.$$

3. Let $ABPC$ be a parallelogram such that ABC is an acute-angled triangle. The circumcircle of triangle ABC meets the line CP again at Q. Prove that $PQ = AC$ if, and only if, $\angle BAC = 60°$. *The circumcircle of a triangle is the circle which passes through its vertices.*

4. Find all positive integers n such that both $n + 2008$ divides $n^2 + 2008$ and $n + 2009$ divides $n^2 + 2009$.

5. Determine the sequences $a_0, a_1, a_2, \ldots$ which satisfy all of the following conditions:
a) $a_{n+1} = 2a_n^2 - 1$ for every integer $n \geq 0$,
b) a_0 is a rational number and
c) $a_i = a_j$ for some i, j with $i \neq j$.

6. The obtuse-angled triangle ABC has sides of length a, b and c opposite the angles $\angle A, \angle B$ and $\angle C$ respectively. Prove that
$$a^3 \cos A + b^3 \cos B + c^3 \cos C < abc.$$

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Thursday, 3 December 2009

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
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Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2009/10 British Mathematical Olympiad

Round 1: Thursday, 3 December 2009

1. Find all integers x, y and z such that
$$x^2 + y^2 + z^2 = 2(yz + 1) \text{ and } x + y + z = 4018.$$

2. Points A, B, C, D and E lie, in that order, on a circle and the lines AB and ED are parallel. Prove that $\angle ABC = 90°$ if, and only if, $AC^2 = BD^2 + CE^2$.

3. Isaac attempts all six questions on an Olympiad paper in order. Each question is marked on a scale from 0 to 10. He never scores more in a later question than in any earlier question. How many different possible sequences of six marks can he achieve?

4. Two circles, of different radius, with centres at B and C, touch externally at A. A common tangent, not through A, touches the first circle at D and the second at E. The line through A which is perpendicular to DE and the perpendicular bisector of BC meet at F. Prove that $BC = 2AF$.

5. Find all functions f, defined on the real numbers and taking real values, which satisfy the equation $f(x)f(y) = f(x + y) + xy$ for all real numbers x and y.

6. Long John Silverman has captured a treasure map from Adam McBones. Adam has buried the treasure at the point (x, y) with integer co-ordinates (not necessarily positive). He has indicated on the map the values of $x^2 + y$ and $x + y^2$, and these numbers are distinct. Prove that Long John has to dig only in one place to find the treasure.

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Thursday, 2 December 2010

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
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Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2010/11 British Mathematical Olympiad

Round 1: Thursday, 2 December 2010

1. One number is removed from the set of integers from 1 to n. The average of the remaining numbers is $40\frac{3}{4}$. Which integer was removed?

2. Let s be an integer greater than 6. A solid cube of side s has a square hole of side $x < 6$ drilled directly through from one face to the opposite face (so the drill removes a cuboid). The volume of the remaining solid is numerically equal to the total surface area of the remaining solid. Determine all possible integer values of x.

3. Let ABC be a triangle with $\angle CAB$ a right-angle. The point L lies on the side BC between B and C. The circle ABL meets the line AC again at M and the circle CAL meets the line AB again at N. Prove that L, M and N lie on a straight line.

4. Isaac has a large supply of counters, and places one in each of the 1×1 squares of an 8×8 chessboard. Each counter is either red, white or blue. A particular pattern of coloured counters is called an *arrangement*. Determine whether there are more arrangements which contain an even number of red counters or more arrangements which contain an odd number of red counters. *Note that* 0 *is an even number.*

5. Circles S_1 and S_2 meet at L and M. Let P be a point on S_2. Let PL and PM meet S_1 again at Q and R respectively. The lines QM and RL meet at K. Show that, as P varies on S_2, K lies on a fixed circle.

6. Let a, b and c be the lengths of the sides of a triangle. Suppose that $ab + bc + ca = 1$. Show that $(a + 1)(b + 1)(c + 1) < 4$.

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 2 December 2011

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
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United Kingdom Mathematics Trust

2011/12 British Mathematical Olympiad
Round 1: Friday, 2 December 2011

1. Find all (positive or negative) integers n for which $n^2 + 20n + 11$ is a perfect square. *Remember that you must justify that you have found them all.*

2. Consider the numbers $1, 2, \ldots, n$. Find, in terms of n, the largest integer t such that these numbers can be arranged in a row so that all consecutive terms differ by at least t.

3. Consider a circle S. The point P lies outside S and a line is drawn through P, cutting S at distinct points X and Y. Circles S_1 and S_2 are drawn through P which are tangent to S at X and Y respectively. Prove that the difference of the radii of S_1 and S_2 is independent of the positions of P, X and Y.

4. Initially there are m balls in one bag, and n in the other, where $m, n > 0$. Two different operations are allowed:

   a) Remove an equal number of balls from each bag;
   b) Double the number of balls in one bag.

   Is it always possible to empty both bags after a finite sequence of operations?

   Operation b) is now replaced with

   b′) Triple the number of balls in one bag.

   Is it now always possible to empty both bags after a finite sequence of operations?

5. Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers.

6. Let ABC be an acute-angled triangle. The feet of the altitudes from A, B and C are D, E and F respectively. Prove that $DE + DF \leq BC$ and determine the triangles for which equality holds.

   The altitude from A is the line through A which is perpendicular to BC. The foot of this altitude is the point D where it meets BC. The other altitudes are similarly defined.

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 30 November 2012

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden.*
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United Kingdom Mathematics Trust

2012/13 British Mathematical Olympiad

Round 1: Friday, 30 November 2012

1. Isaac places some counters onto the squares of an 8 by 8 chessboard so that there is at most one counter in each of the 64 squares. Determine, with justification, the maximum number that he can place without having five or more counters in the same row, or in the same column, or on either of the two long diagonals.

2. Two circles S and T touch at X. They have a common tangent which meets S at A and T at B. The points A and B are different. Let AP be a diameter of S. Prove that B, X and P lie on a straight line.

3. Find all real numbers x, y and z which satisfy the simultaneous equations $x^2 - 4y + 7 = 0$, $y^2 - 6z + 14 = 0$ and $z^2 - 2x - 7 = 0$.

4. Find all positive integers n such that $12n - 119$ and $75n - 539$ are both perfect squares.

5. A triangle has sides of length at most 2, 3 and 4 respectively. Determine, with proof, the maximum possible area of the triangle.

6. Let ABC be a triangle. Let S be the circle through B tangent to CA at A and let T be the circle through C tangent to AB at A. The circles S and T intersect at A and D. Let E be the point where the line AD meets the circle ABC. Prove that D is the midpoint of AE.

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 29 November 2013

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
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United Kingdom Mathematics Trust

2013/14 British Mathematical Olympiad
Round 1: Friday, 29 November 2013

1. Calculate the value of
$$\frac{2014^4 + 4 \times 2013^4}{2013^2 + 4027^2} - \frac{2012^4 + 4 \times 2013^4}{2013^2 + 4025^2}.$$

2. In the acute-angled triangle ABC, the foot of the perpendicular from B to CA is E. Let l be the tangent to the circle ABC at B. The foot of the perpendicular from C to l is F. Prove that EF is parallel to AB.

3. A number written in base 10 is a string of 3^{2013} digit 3s. No other digit appears. Find the highest power of 3 which divides this number.

4. Isaac is planning a nine-day holiday. Every day he will go surfing, or water skiing, or he will rest. On any given day he does just one of these three things. He never does different water-sports on consecutive days. How many schedules are possible for the holiday?

5. Let ABC be an equilateral triangle, and P be a point inside this triangle. Let D, E and F be the feet of the perpendiculars from P to the sides BC, CA and AB respectively. Prove that

   a) $AF + BD + CE = AE + BF + CD$ and
   b) $[APF] + [BPD] + [CPE] = [APE] + [BPF] + [CPD]$.

   The area of triangle XYZ is denoted $[XYZ]$.

6. The angles A, B and C of a triangle are measured in degrees, and the lengths of the opposite sides are a, b and c respectively. Prove that
$$60 \leq \frac{aA + bB + cC}{a + b + c} < 90.$$

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 28 November 2014

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
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United Kingdom Mathematics Trust

2014/15 British Mathematical Olympiad

Round 1: Friday, 28 November 2014

1. Place the following numbers in increasing order of size, and justify your reasoning:
$$3^{3^4}, 3^{4^3}, 3^{4^4}, 4^{3^3} \text{ and } 4^{3^4}.$$
Note that a^{b^c} *means* $a^{(b^c)}$.

2. Positive integers p, a and b satisfy the equation $p^2 + a^2 = b^2$. Prove that if p is a prime greater than 3, then a is a multiple of 12 and $2(p + a + 1)$ is a perfect square.

3. A hotel has ten rooms along each side of a corridor. An olympiad team leader wishes to book seven rooms on the corridor so that no two reserved rooms on the same side of the corridor are adjacent. In how many ways can this be done?

4. Let x be a real number such that $t = x + x^{-1}$ is an integer greater than 2. Prove that $t_n = x^n + x^{-n}$ is an integer for all positive integers n. Determine the values of n for which t divides t_n.

5. Let $ABCD$ be a cyclic quadrilateral. Let F be the midpoint of the arc AB of its circumcircle which does not contain C or D. Let the lines DF and AC meet at P and the lines CF and BD meet at Q. Prove that the lines PQ and AB are parallel.

6. Determine all functions $f(n)$ from the positive integers to the positive integers which satisfy the following condition: whenever a, b and c are positive integers such that $1/a + 1/b = 1/c$, then
$$1/f(a) + 1/f(b) = 1/f(c).$$

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 27 November 2015

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
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United Kingdom Mathematics Trust

2015/16 British Mathematical Olympiad
Round 1: Friday, 27 November 2015

1. On Thursday 1st January 2015, Anna buys one book and one shelf. For the next two years, she buys one book every day and one shelf on alternate Thursdays, so she next buys a shelf on 15th January 2015. On how many days in the period Thursday 1st January 2015 until (and including) Saturday 31st December 2016 is it possible for Anna to put all her books on all her shelves, so that there is an equal number of books on each shelf?

2. Let $ABCD$ be a cyclic quadrilateral and let the lines CD and BA meet at E. The line through D which is tangent to the circle ADE meets the line CB at F. Prove that the triangle CDF is isosceles.

3. Suppose that a sequence $t_0, t_1, t_2, \ldots$ is defined by a formula $t_n = An^2 + Bn + C$ for all integers $n \geq 0$. Here A, B and C are real constants with $A \neq 0$. Determine values of A, B and C which give the greatest possible number of successive terms of the sequence which are also successive terms of the Fibonacci sequence. *The Fibonacci sequence is defined by* $F_0 = 0$, $F_1 = 1$ *and* $F_m = F_{m-1} + F_{m-2}$ *for* $m \geq 2$.

4. James has a red jar, a blue jar and a pile of 100 pebbles. Initially both jars are empty. A move consists of moving a pebble from the pile into one of the jars or returning a pebble from one of the jars to the pile. The numbers of pebbles in the red and blue jars determine the *state* of the game. The following conditions must be satisfied:

   a) The red jar may never contain fewer pebbles than the blue jar;
   b) The game may never be returned to a previous state.

   What is the maximum number of moves that James can make?

5. Let ABC be a triangle, and let D, E and F be the feet of the perpendiculars from A, B and C to BC, CA and AB respectively. Let P, Q, R and S be the feet of the perpendiculars from D to BA, BE, CF and CA respectively. Prove that P, Q, R and S are collinear.

6. A positive integer is called *charming* if it is equal to 2 or is of the form $3^i 5^j$ where i and j are non-negative integers. Prove that every positive integer can be written as a sum of different charming integers.

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 2 December 2016

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
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United Kingdom Mathematics Trust

2016/17 British Mathematical Olympiad

Round 1: Friday, 2 December 2016

1. The integers $1, 2, 3, \ldots, 2016$ are written down in base 10, each appearing exactly once. Each of the digits from 0 to 9 appears many times in the list. How many of the digits in the list are odd? *For example, 8 odd digits appear in the list* $1, 2, 3, \ldots, 11$.

2. For each positive real number x, we define $\{x\}$ to be the greater of x and $1/x$, with $\{1\} = 1$. Find, with proof, all positive real numbers y such that
$$5y\{8y\}\{25y\} = 1.$$

3. Determine all pairs (m, n) of positive integers which satisfy the equation $n^2 - 6n = m^2 + m - 10$.

4. Naomi and Tom play a game, with Naomi going first. They take it in turns to pick an integer from 1 to 100, each time selecting an integer which no-one has chosen before. A player loses the game if, after their turn, the sum of all the integers chosen since the start of the game (by both of them) cannot be written as the difference of two square numbers. Determine if one of the players has a winning strategy, and if so, which.

5. Let ABC be a triangle with $\angle A < \angle B < 90°$ and let Γ be the circle through A, B and C. The tangents to Γ at A and C meet at P. The line segments AB and PC produced meet at Q. It is given that
$$[ACP] = [ABC] = [BQC].$$
Prove that $\angle BCA = 90°$. *Here* $[XYZ]$ *denotes the area of triangle* XYZ.

6. Consecutive positive integers $m, m+1, m+2$ and $m+3$ are divisible by consecutive odd positive integers $n, n+2, n+4$ and $n+6$ respectively. Determine the smallest possible m in terms of n.

BMO1 2016–17 REPORT FOR TEACHERS AND CANDIDATES

Marking:

Papers are marked in December by a team of around 40 volunteers – many of whom have sat the British Mathematical Olympiad papers in previous years.

All BMO questions are marked using a "10–/0+" strategy, which requires a candidate to get the crucial part of each solution to qualify for 10– (with minor deductions for omissions/errors), otherwise they are in 0+ territory, when usually only a maximum of 3 marks are available.

Results:

There were some excellent scores this year, including six people who scored a perfect 60. Over 60% of the candidates managed to get full marks on the first question, and most candidates made serious attempts at several questions.

There were **1691** scripts received and the mean score was **22.9**. The breakdown of final marks was:

Mark	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59	60
No. of candidates.	130	462	654	315	93	31	6

The average mark per question was:

Q1	Q2	Q3	Q4	Q5	Q6
8.8	7.6	4.8	2.4	2.6	0.9

Solutions:

Full solutions to both BMO1 questions and BMO2 questions will be sent to schools in a booklet in April/May. In the meantime, there are instructions on obtaining access to solutions at http://www.bmoc.maths.org/solutions/.

Note:

Even the top-scoring students are not always good at reading the instructions on the paper! In order to help the markers, pupils should ensure they:

- write on only one side of the paper
- start each question on a new sheet
- number the questions carefully in a place which is not going to be covered by the staple

United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 1 December 2017

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
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United Kingdom
Mathematics Trust

2017/18 British Mathematical Olympiad
Round 1: Friday, 1 December 2017

1. Helen divides 365 by each of $1, 2, 3, \ldots, 365$ in turn, writing down a list of the 365 remainders. Then Phil divides 366 by each of $1, 2, 3, \ldots, 366$ in turn, writing down a list of the 366 remainders. Whose list of remainders has the greater sum and by how much?

2. In a 100-day period, each of six friends goes swimming on exactly 75 days. There are n days on which at least five of the friends swim. What are the largest and smallest possible values of n?

3. The triangle ABC has $AB = CA$ and BC is its longest side. The point N is on the side BC and $BN = AB$. The line perpendicular to AB which passes through N meets AB at M. Prove that the line MN divides both the area and the perimeter of triangle ABC into equal parts.

4. Consider sequences $a_1, a_2, a_3, \ldots$ of positive real numbers with $a_1 = 1$ and such that
$$a_{n+1} + a_n = (a_{n+1} - a_n)^2$$
for each positive integer n. How many possible values can a_{2017} take?

5. If we take a 2×100 (or 100×2) grid of unit squares, and remove alternate squares from a long side, the remaining 150 squares form a 100-*comb*. Henry takes a 200×200 grid of unit squares, and chooses k of these squares and colours them so that James is unable to choose 150 uncoloured squares which form a 100-comb. What is the smallest possible value of k?

6. Matthew has a deck of 300 cards numbered 1 to 300. He takes cards out of the deck one at a time, and places the selected cards in a row, with each new card added at the right end of the row. Matthew must arrange that, at all times, the mean of the numbers on the cards in the row is an integer. If, at some point, there is no card remaining in the deck which allows Matthew to continue, then he stops.

   When Matthew has stopped, what is the smallest possible number of cards that he could have placed in the row? Give an example of such a row.

United Kingdom
Mathematics Trust

January 2018

BMO1 2017–18 REPORT FOR TEACHERS AND CANDIDATES

Marking:

Papers are marked in December by a team of around 40 volunteers – many of whom have sat the British Mathematical Olympiad papers in previous years.

All BMO questions are marked using a "10–/0+" strategy, which requires a candidate to get the crucial part of each solution to qualify for 10– (with minor deductions for omissions/errors), otherwise they are in 0+ territory, when usually only a maximum of 3 marks are available.

Results:

There were some excellent scores this year, including sixteen people who scored a perfect 60. Over 65% of the candidates managed to get full marks on at least one question, and many candidates made serious attempts at several questions.

There were **1792** scripts received and the mean score was **21.2**. The breakdown of final marks was:

Mark	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59	60
No. of candidates.	257	586	564	245	85	39	16

The average mark per question was:

Q1	Q2	Q3	Q4	Q5	Q6
6.8	4.2	7.2	3.3	2.2	1.7

Solutions:

Full solutions to both BMO1 questions (and BMO2 questions) will be sent to schools in a booklet in April/May. In the meantime, there are instructions on obtaining access to solutions at https://bmos.ukmt.org.uk/solutions/.

Note:

Even the top-scoring students are not always good at reading the instructions on the paper! In order to help the markers, pupils should ensure they:

- write on only one side of the paper
- start each question on a new sheet
- number the questions carefully in a place which is not going to be covered by the staple

United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 1

Friday 30 November 2018

Instructions

1. Time allowed: $3\frac{1}{2}$ hours.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.
3. One complete solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.
4. Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.
5. The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden.
6. Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top left hand corner and your name, initials and school in the top right hand corner.
7. Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.
8. Staple all the pages neatly together in the top left hand corner.
9. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 8am GMT on Saturday 1 December when the solutions video will be released at https://bmos.ukmt.org.uk
10. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:
UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT
☎ 0113 343 2339   enquiry@ukmt.org.uk   www.ukmt.org.uk

1. A list of five two-digit positive integers is written in increasing order on a blackboard. Each of the five integers is a multiple of 3, and each digit 0,1,2,3,4,5,6,7,8,9 appears exactly once on the blackboard. In how many ways can this be done? *Note that a two-digit number cannot begin with the digit 0.*

2. For each positive integer $n \geq 3$, we define an n-ring to be a circular arrangement of n (not necessarily different) positive integers such that the product of every three neighbouring integers is n. Determine the number of integers n in the range $3 \leq n \leq 2018$ for which it is possible to form an n-ring.

3. Ares multiplies two integers which differ by 9. Grace multiplies two integers which differ by 6. They obtain the same product T. Determine all possible values of T.

4. Let Γ be a semicircle with diameter AB. The point C lies on the diameter AB and points E and D lie on the arc BA, with E between B and D. Let the tangents to Γ at D and E meet at F. Suppose that $\angle ACD = \angle ECB$.

Prove that $\angle EFD = \angle ACD + \angle ECB$.

5. Two solid cylinders are mathematically similar. The sum of their heights is 1. The sum of their surface areas is 8π. The sum of their volumes is 2π. Find all possibilities for the dimensions of each cylinder.

6. Ada the ant starts at a point O on a plane. At the start of each minute she chooses North, South, East or West, and marches 1 metre in that direction. At the end of 2018 minutes she finds herself back at O. Let n be the number of possible journeys which she could have made. What is the highest power of 10 which divides n?

United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 1

Friday 29 November 2019

Instructions

1. Time allowed: $3\frac{1}{2}$ hours.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.
3. One complete solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.
4. Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.
5. The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden.
6. Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top left hand corner and your **Participant ID**, and **UKMT Centre Number** in the top right hand corner.
7. Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.
8. Staple all the pages neatly together in the top left hand corner.
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1. Show that there are at least three prime numbers p less than 200 for which $p+2$, $p+6$, $p+8$ and $p+12$ are all prime. Show also that there is only one prime number q for which $q+2$, $q+6$, $q+8$, $q+12$ and $q+14$ are all prime.

2. A sequence of integers $a_1, a_2, a_3, \ldots$ satisfies the relation:

$$4a_{n+1}^2 - 4a_n a_{n+1} + a_n^2 - 1 = 0$$

for all positive integers n. What are the possible values of a_1?

3. Two circles S_1 and S_2 are tangent at P. A common tangent, not through P, touches S_1 at A and S_2 at B. Points C and D, on S_1 and S_2 respectively, are outside the triangle APB and are such that P is on the line CD.

Prove that AC is perpendicular to BD.

4. There are 2019 penguins waddling towards their favourite restaurant. As the penguins arrive, they are handed tickets numbered in ascending order from 1 to 2019, and told to join the queue. The first penguin starts the queue. For each $n > 1$ the penguin holding ticket number n finds the greatest $m < n$ which divides n and enters the queue directly behind the penguin holding ticket number m. This continues until all 2019 penguins are in the queue.

(a) How many penguins are in front of the penguin with ticket number 2?

(b) What numbers are on the tickets held by the penguins just in front of and just behind the penguin holding ticket 33?

5. Six children are evenly spaced around a circular table. Initially, one has a pile of $n > 0$ sweets in front of them, and the others have nothing. If a child has at least four sweets in front of them, they may perform the following move: eat one sweet and give one sweet to each of their immediate neighbours and to the child directly opposite them. An arrangement is called *perfect* if there is a sequence of moves which results in each child having the same number of sweets in front of them. For which values of n is the initial arrangement perfect?

6. A function f is called *good* if it assigns an integer value $f(m,n)$ to every ordered pair of integers (m,n) in such a way that for every pair of integers (m,n) we have:

$$2f(m,n) = f(m-n, n-m) + m + n = f(m+1, n) + f(m, n+1) - 1.$$

Find all good functions.

United Kingdom
Mathematics Trust

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fax 0113 343 5500 *web* www.ukmt.org.uk

British Mathematical Olympiad Round 1 2019

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for is full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; a small number of marks (often capped at 3) might be awarded for particular cases or insights.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore important that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

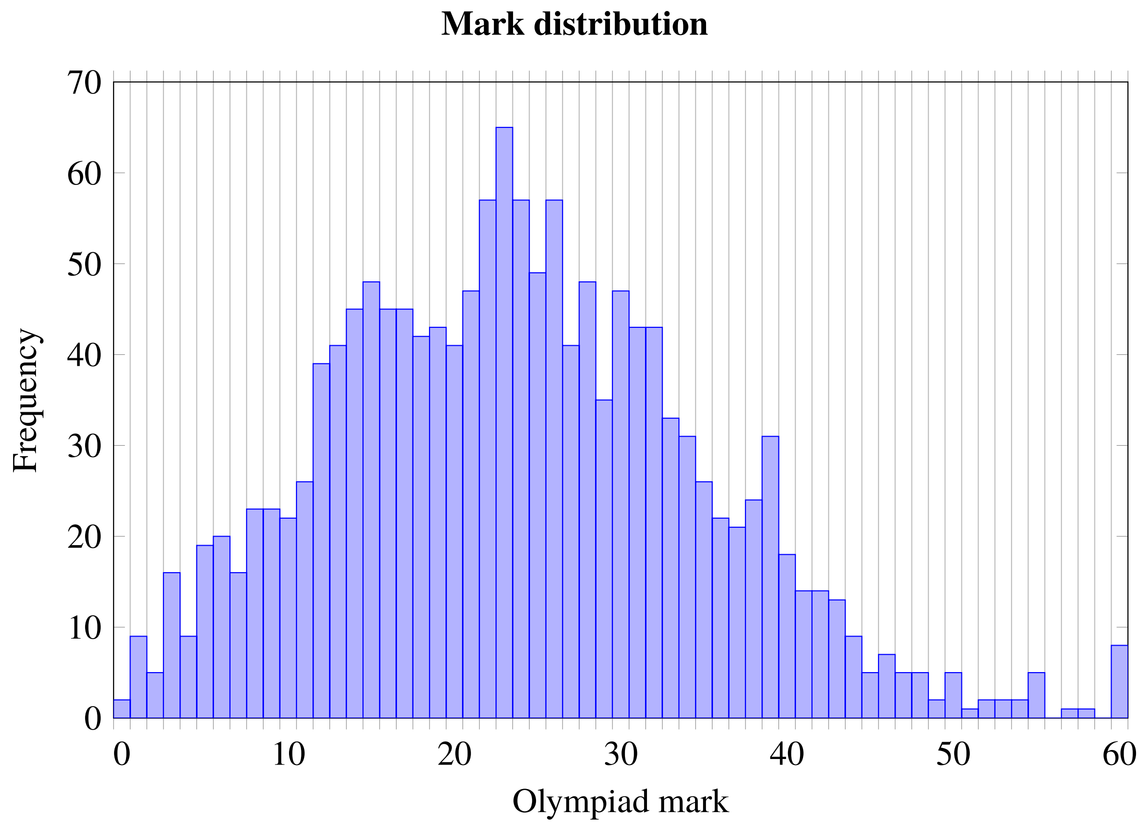
Candidates engaged very well with this year's paper, and the majority made substantial progress on at least two of the problems. It was encouraging to see a number scoring highly on question 3, which hopefully served as a reminder that BMO1 geometry problems often do not require much beyond GCSE circle theorems (and their converses) together with some ingenuity. In questions 4 and 5 many candidates found correct numerical answers, but failed to adequately explain their reasoning. In question 4 there was often a lack of precision when using the word 'behind' (does it mean *directly* behind or *somewhere* behind?), while in question 5 many candidates showed that multiples of 28 gave perfect arrangements without realising that they needed to show that no other numbers did. Question 6 was arguably as hard as it has ever been, and only the very best candidates came close to solving it.

A few scripts suffered from a lack of words explaining where various symbolic expressions came from, but these were reassuringly rare. Many more scripts gave copious discursive commentary, and would have benefited from breaking up their prose into separate claims and proofs, and by being a little more concise in places.

The 2019 BMO1 paper was sat by 1475 candidates. The scripts were marked in Cambridge on 7th and 8th December by a team of Oscar Arevalo, Eszter Backhausz, Tibor Backhausz, Connie Bambridge-Sutton, Sam Bealing, Emily Beatty, Jonathan Beckett, Cathy Beckett, Phillip Beckett, Jamie Bell, Lex Betts, Robin Bhattacharyya, Tom Bowler, Andrew Carlotti, Philip Coggins, Arthur Conmy, James Cranch, Alex Darby, Ronan Flatley, Richard Freeland, Tony Gardiner, James Gazet, Daniel Griller, Alex Gunning, Stuart Haring, Tim Hennock, Liam Hill, Ina Hughes, Ian Jackson, Shavindra Jayasekera, Vesna Kadelburg, Hadi Khan, Patricia King, Warren Li, Harry Metrebian, Joseph Myers, Eve Pound, Melissa Quail, Zarko Randelovic, Dominic Rowland, Amit Shah, Jerome Watson, Patrick Winter, Harvey Yau and Renzhi Zhou.

The problems were proposed by Nick MacKinnon, Tom Bowler, Daniel Griller, Daniel Griller, Dominic Rowland and Sam Bealing respectively.

In addition to the written solutions in this report, video solutions can be found here.



Candidates with scores ≥ 40 were invited to sit BMO2.

Question 1

Show that there are at least three prime numbers p less than 200 for which $p+2$, $p+6$, $p+8$ and $p+12$ are all prime. Show also that there is only one prime number q for which $q+2$, $q+6$, $q+8$, $q+12$ and $q+14$ are all prime.

Solution

For the first part, $p = 5$, 11 and 101 all work, since all the numbers

$$\begin{gathered} 5, 7, 11, 13, 17 \\ 11, 13, 17, 19, 23 \\ 101, 103, 107, 109, 113 \end{gathered}$$

are all prime.

For the second part, $q = 5$ works, since the first sequence above can be continued on with 19. This is the only possibility. Clearly $q \neq 2$, and the odd numbers q, $q+12$, $q+14$, $q+6$ and $q+8$ cover all possible odd last digits, so one must end in a 5. Thus one of these numbers must be *equal* to 5, since they are all prime, and the only possibility is that $q = 5$. (If one uses instead the list q, $q+2$, $q+14$, $q+6$, $q+8$, one needs to rule out the possibility that $q+2 = 5$ by, for example, observing that if $q+2 = 5$, then $q+6 = 9$ is not prime.)

Markers' comments

The vast majority of candidates attempted this question, and most made good progress. For the first part the three values of p given in the solution are the only valid ones less than 200 (the next is $p = 1481$) and a number of candidates provided additional incorrect values of p. In particular the following numbers were often mistaken for primes: $91 = 7 \times 13$, $119 = 7 \times 17$, $143 = 11 \times 13$, $161 = 7 \times 23$, $169 = 13 \times 13$ and $203 = 7 \times 19$.

In the second part many candidates made sensible use of last digits or arithmetic modulo 5. These candidates generally went on to solve the problem. A common minor error was to assert that some number ending in 5 cannot be prime, forgetting 5 itself. This happened most often with the prime $q+2$, meaning that some candidates didn't successfully eliminate the single possibility that $q = 3$. The most common major error was to assume that the condition $p < 200$ from the first part also applied to q in the second. Candidates who did this and then listed all small possibilities for q were heavily penalised. However, those who gave arguments covering one, two and three digit numbers could still score nearly full marks if their arguments were easy to generalise.

Remark

It is conjectured that there are infinitely many primes which satisfy the requirements of the first part of this question. This claim is a strengthening of the famous 'Twin Primes' conjecture which has been intensively studied for centuries. There has been significant progress since 2013, but the proof or disproof of this conjecture remains beyond the reach of current mathematical techniques.

Question 2

A sequence of integers $a_1, a_2, a_3, \ldots$ satisfies the relation:

$$4a_{n+1}^2 - 4a_n a_{n+1} + a_n^2 - 1 = 0$$

for all positive integers n. What are the possible values of a_1?

Solution

We start by factorising the expression involving the a_i to obtain $(2a_{n+1} - a_n)^2 - 1 = 0$.

From here we can either add 1 to both sides and take a square root to find $(2a_{n+1} - a_n) = \pm 1$, or factorise the difference of two squares to see that $(2a_{n+1} - a_n - 1)(2a_{n+1} - a_n + 1) = 0$.

Either way we have $a_{n+1} = \frac{1}{2}(a_n \pm 1)$. (This can also be obtained by viewing the equation as a quadratic in a_{n+1}, using the quadratic formula and simplifying.)

If a_1 is even then a_2 will not be an integer.

If a_n is odd, then the two possible values of a_{n+1} are consecutive integers. Thus we can choose a_{n+1} to be an odd integer. In particular, for any odd integer a_1 there exists a possible sequence consisting entirely of odd integers.

Alternative

As before we have $a_{n+1} = \frac{1}{2}(a_n \pm 1)$.

If $a_n > 1$ then $a_n > a_{n+1} \geq 1$ so if a_1 is positive, then the terms decrease until they reach 1. If $a_n < -1$ then $a_n < a_{n+1} \leq -1$, so if a_1 is negative, then the terms increase until they reach -1.

As $a_n = 2a_{n+1} \pm 1$ we may construct the sequence in reverse from $a_k = \pm 1$ back to a_1. We obtain $a_1 = \pm 2^{k-1} \pm 2^{k-2} \pm \ldots \pm 2^1 \pm 1$ and so a_1 must be odd. We can prove by (complete) induction on the magnitude of a_1 that a_1 can be any odd number. Suppose we want to construct a sequence with $a_1 = 2k + 1 = 2(k + 1) - 1$. One of k and $k + 1$ is an odd number of smaller magnitude, so, by induction, there is a legal sequence with $a_1 = k$ or $a_1 = k + 1$. We may take this sequence, increase all the subscripts by 1 and insert the desired value of a_1 at the beginning.

Markers' comments

There were many excellent responses to this question. Both solution strategies outlined above were used successfully by candidates, though the second was much rarer.

This question also threw up several misconceptions that led to candidates scoring a maximum of 3 marks. The most common was to assume that, when faced with the two expressions for a_{n+1}, there were then only two ways to generate a sequence, either by repeatedly using $a_{n+1} = \frac{1}{2}(a_n + 1)$ or repeatedly using $a_{n+1} = \frac{1}{2}(a_n - 1)$. Candidates making this mistake generally went on to show that, if the choice of sign was not allowed to vary within the sequence, the only possible values of a_1 are ± 1.

The second most prevalent misconception that arose was for candidates to assume that $(2a_{n+1} - a_n)^2 = 1$ led to $a_{n+1} = \frac{1}{2}(a_n + 1)$ only. It is important for candidates to appreciate that $x^2 = 1$ may not mean $x = \sqrt{1}$ and it is interesting to note that for this problem such an error affected the solution set significantly.

Many solutions gave the correct set of values for a_1 but still lost a couple of marks, due to a lack of justification of two important observations. The first of was that a_1 could not be even, which some candidates stated but did not explain and other omitted to mention. The second related to the fact that the two possible values for a_{n+1}, given an odd a_n, had different parity. This is a crucial observation needed in order to continue the sequence and some stated it as fact with no proof.

Some candidates attempted to apply induction once they had expressions for a_{n+1}, but many were unclear about the statement they were attempting to prove. Others attempted to start with $a_{n+1} = \pm 1$ and work backwards, iterating both expressions for a_{n+1} to create a tree of possible values for a_1. These candidates often observed various patterns in their trees and claimed, without proof, that these patterns continued indefinitely. Such scripts were awarded a maximum of 3 marks.

Some candidates assumed the terms of the sequence were positive, this was not stated in the question and led to a deduction of 1 mark.

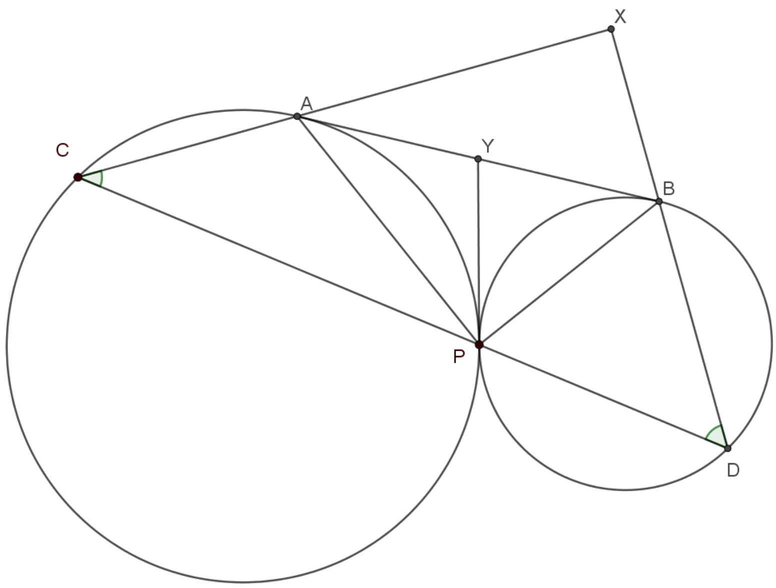
Question 3

Two circles S_1 and S_2 are tangent at P. A common tangent, not through P, touches S_1 at A and S_2 at B. Points C and D, on S_1 and S_2 respectively, are outside the triangle APB and are such that P is on the line CD.

Prove that AC is perpendicular to BD.

Solution

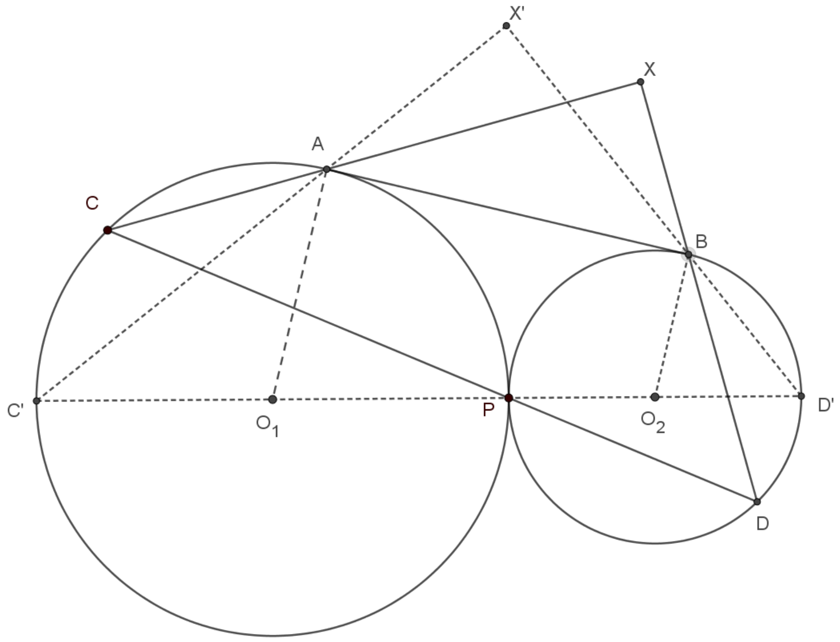
Let AC and BD intersect at X, and let the common tangent at P intersect AB at Y as shown.



Let $\angle DCA = \gamma$ and $\angle BDC = \delta$. By the alternate segment theorem $\angle YPA = \angle PAY = \gamma$ and $\angle YBP = \angle BPY = \delta$. The angles in triangle BPA sum to $2\gamma + 2\delta$ so $\gamma + \delta = 90°$.

Now looking at the angles in triangle DXC shows that $\angle DXC = 90°$ as required.

Alternative



We consider the special case where $C'P$ and PD' are diameters of the two circles. It is clear that C', O_1, P, O_2 and D' are collinear, since the radii O_1P and O_2P are both perpendicular to the common tangent at P.

The tangent AB is perpendicular to the radii AO_1 and BO_2, so considering the angles in the quadrilateral AO_1O_2B we see that $\angle O_2O_1A + \angle BO_2O_1 = 180°$. However these are external angles of the isosceles triangles AO_1C' and BO_2D' so we see see that $\angle O_1C'A + \angle BD'O_2 = 90°$. Considering triangle $D'C'X'$ shows that $\angle C'X'D' = 90°$.

To prove the result for an arbitrary line CD through P we observe that $\angle BDP = \angle BD'P$ and $\angle PCA = \angle PC'A$ by angles in the same segment, and the result follows.

Markers' comments

We were pleased to see many excellent solutions to this problem. Most candidates proceeded along the lines of the first solution. A common variation was to use the quadrilateral O_1O_2AB from the second solution and isosceles triangles AO_1P and BO_2P to show that $\angle APB = 90°$ and then proceed as in the first solution. Yet another possibility was to start by showing that $\angle PO_1A + \angle BO_2P = 180°$ and then use the fact that the angle at the centre of a circle is twice the angle at the circumference and therefore $\angle PCA + \angle BDP = 90°$.

Many of the solutions were very well explained. However, some candidates still produced long lists of angle calculations with no explanations, and they were penalised for this. It is important to make it clear at each step which circle theorem, or which triangle is being used, although the standard GCSE theorems may be used without proof.

A mark was often lost for using the fact that O_1PO_2 is a straight line without explicitly stating it; it should be noted that this is only the case because P is the point of tangency of the two circles. A more serious omission was to claim, without proof, that $\angle APB = 90°$; this was heavily penalised.

In geometry problems, candidates often only consider special cases. In this question, a number of candidates only considered the case when the line CD passes through the centres of the two circles, which could score the marks available for the first part of the alternative solution. Some candidates assumed further that the two circles are equal and argued by symmetry; this approach usually earned no marks.

Remark

The condition that points C and D lie outside triangle APB restricts us to essentially one diagram, but the result is true more generally. Checking the other cases is fairly straightforward, but would be a worthwhile exercise.

Question 4

There are 2019 penguins waddling towards their favourite restaurant. As the penguins arrive, they are handed tickets numbered in ascending order from 1 to 2019, and told to join the queue. The first penguin starts the queue. For each $n > 1$ the penguin holding ticket number n finds the greatest $m < n$ which divides n and enters the queue directly behind the penguin holding ticket number m. This continues until all 2019 penguins are in the queue.

(a) How many penguins are in front of the penguin with ticket number 2?

(b) What numbers are on the tickets held by the penguins just in front of and just behind the penguin holding ticket 33?

Solution

We begin by noting that the largest $m < n$ which divides n is equal to n divided by its smallest factor (other than 1). Clearly this smallest factor must be prime.

For part (a) we claim that the penguins that end up somewhere behind 2 are precisely the larger powers of 2. Penguin 3 goes in front of 2, so the claim holds when only three penguins have arrived. Now suppose the claim holds when $k - 1$ penguins have arrived and consider penguin k. If k is a power of 2, then its largest proper factor is also a power of 2, so it goes directly behind that factor and hence somewhere behind 2. If k is prime it goes directly behind 1 and thus somewhere in front of 2. Finally, if k has smallest prime factor p and another prime factor $q > 2$, then k/p is a multiple of q and not a power of 2. Penguin k goes directly behind k/p and we already know k/p is somewhere in front of 2, so k goes somewhere in front of 2. The claim implies that when 2019 penguins have arrived, only penguins 4, 8, 16, 32, 64, 128, 256, 512 and 1024 are behind 2. So the remaining 2009 penguins are in front of penguin 2.

For part (b) we first consider the penguins who stand directly in front of penguin 33 at some stage in the queuing process. When 33 arrives 11 is directly in front of it and 22 is behind it. The next multiple of 11, namely 44, stands behind 22. Later 55 comes and stands directly in front of 33. The next penguin to come between 55 and 33 is the next available multiple of 55, namely 110. Now we forget about penguin 55 and focus on who comes in between 110 and 33. It is the next available multiple 110, namely 220. Continuing in this way we see that 440, then 880 and finally 1760 occupy the spot directly in front of 33.

Finally we turn to the penguins who stand directly behind 33 at some stage. Their numbers must be of the form $33k$ for some k. However, if $k > 3$ then $11k > 33$ so the only numbers that ever stand directly behind 33 and $33k$ for $k \leq 3$. On arrival, penguin 66 stands directly behind 33, but is later replaced by penguin 99, who stays directly behind 33 from then on.

Markers' comments

We were pleased with candidates' willingness to have a go at this problem, and all parts of the question were found quite accessible.

In part (a), many candidates observed that powers of 2 end up behind penguin 2 by working out some small cases. However, to score well they needed an argument for why this pattern continues. In particular, they needed to show both that powers of 2 end up behind 2 and that no other penguin does. Some idea of induction is very helpful for stating this formally, but plenty of candidates also got the marks for a detailed verbal description without using induction explicitly.

In part (b) a common mistake was thinking that penguin 44 entered in front of 33 and this error was heavily penalised. Candidates were surprisingly careless in finding the penguin behind 33 and many thought that the answer was 33^2 or 33×31. We would advise checking this kind of numerical answer carefully using some smaller versions of the same problem. Even candidates finding the correct answer of 99 could score few marks for this part if they did not explain why no higher penguin entered behind 33; there were a number of excellent solutions to this part.

Question 5

Six children are evenly spaced around a circular table. Initially, one has a pile of $n > 0$ sweets in front of them, and the others have nothing. If a child has at least four sweets in front of them, they may perform the following move: eat one sweet and give one sweet to each of their immediate neighbours and to the child directly opposite them. An arrangement is called *perfect* if there is a sequence of moves which results in each child having the same number of sweets in front of them. For which values of n is the initial arrangement perfect?

Solution

The initial arrangement is perfect if and only if n is divisible by 28. We number the children from 1 to 6 around the table, and assume child 1 starts with the sweets.

Suppose $n = 28k$, for $k > 0$, and consider the following sequence of moves:

(1) Child 1 makes $7k$ moves. This leaves all the odd-numbered children without any sweets, and all of the even-numbered children with $7k$ sweets.

(2) Each even-numbered child now make k moves. This leaves each child with $3k$ sweets so the arrangement is perfect.

It remains to show that if the initial arrangement is perfect, then n must be divisible by 28. There are various possible approaches.

Call the odd-numbered children 'Team O' and the even-numbered children 'Team E'. Now consider the difference between the total number of sweets held by Team O and the total number held by Team E. Initially this difference is n. Once all children have the same number of sweets this difference is 0, and any move changes the difference by exactly 7. Thus 7 divides n.

Next consider the difference between the number of sweets held by child 1 and the number held by child 3. At the start this difference is n and at the end this difference is zero. Moves by children 2, 4, 5 and 6 do not change this difference, while moves by children 1 and 3 change it by exactly four each time. Thus 4 divides n.

Since n is a multiple of 4 and 7, we conclude that it must be a multiple of 28 as required.

Alternative

Suppose child 1 makes a total of a moves, child 2 makes a total of b moves and so on. After all moves have been made, the number of sweets each child has are given in the table below. We are interested in the case when all these quantities are equal.

Child	Number of sweets
1	$n + b + d + f - 4a$
2	$a + c + e - 4b$
3	$b + d + f - 4c$
4	$a + c + e - 4d$
5	$b + d + f - 4e$
6	$a + c + e - 4f$

Equating the number of sweets that children 2, 4 and 6 have:

$$a + c + e - 4b = a + c + e - 4d = a + c + e - 4f,$$

so

$$b = d = f. \tag{1}$$

Similarly, equating the number of sweets that children 1, 3 and 5 have gives:

$$b + d + f + n - 4a = b + d + f - 4c = b + d + f - 4e$$

so

$$e = c \text{ and } n = 4(a - c). \tag{2}$$

Now, equating the totals for children 2 and 3 and plugging in (1) and (2) shows that:

$$a + 2c - 4b = 3b - 4c \text{ or } a = 7b - 6c. \tag{3}$$

Plugging (3) into (2) gives $n = 28(b - c)$. .

Alternative

It is tempting to make claims like 'the order of moves does not matter' or 'children 3 and 5 should never make any moves'. These claims are both false, but the ideas can be captured in a more careful argument.

Suppose there a sequence of moves $\mathcal{S}$ showing that the initial arrangement with n sweets is perfect, and that in that sequence child 1 makes a moves, child 2 makes b and so on. Since children 2, 4 and 6 start with the same number of sweets and only ever gain them simultaneously, they must all make the same number of moves, so $b = d = e$. Similarly, children 3 and 5 gain sweets simultaneously, so must make the same number moves giving $c = e$.

Now we claim that the following alternative sequence $\mathcal{T}$ of moves can also be used to show that the initial arrangement is perfect:

(1) Child 1 makes $a - c$ moves.

(2) Children 2, 4 and 6 each make $b - c$ moves.

(3) Children 3 and 5 make no moves.

In sequence $\mathcal{T}$ each child makes exactly c fewer moves than they do in $\mathcal{S}$, so each ends up with c more sweets. However, to actually prove our claim about $\mathcal{T}$, we must also show it is valid sequence of moves. In particular we must check that $a - c$ and $b - c$ are positive, and that no child ever has a negative number of sweets.

In sequence $\mathcal{S}$ each move by child 3 requires a total of at least four prior moves by children 2, 4 and 6. Thus $3b \geq 4c$ or $b - c \geq \frac{c}{3}$ which implies that $b - c > 0$ since even if $c = 0$ we must have $b > 0$. Similarly, each move by child 2 requires at least four prior moves by odd-numbered children, so $a + 2c \geq 4b$ which gives $a - c \geq 4b - 3c > 0$.

It remains to check that no child ever has a negative number of sweets in $\mathcal{T}$. Once child 1 has made all their moves, they gain sweets at the same time as child 3. Since they end up with the same number, child 1 must have exactly zero sweets when they finish their moves, and after that they only gain sweets. Children 2, 4 and 6 gain sweets steadily until they start making moves. After that they lose sweets until they reach the (positive) number finally held by each child. Children 3 and 5 never lose any sweets, so the claim is established.

We have already observed that $n - 4(a - c)$ must equal zero. Also, at the end of sequence $\mathcal{T}$ each child has $3(b - c)$ sweets and the total number of moves made was $(a - c) + 3(b - c)$. Thus $n = 6 \times 3(b - c) + (a - c) + 3(b - c)$ so $\frac{3}{4}n = 21(b - c)$ giving $n = 28(b - c)$.

Markers' comments

Lots of students did a good job showing that a configuration where one child starts with 28 sweets is perfect, and indeed that the same was true if the starting number was a multiple of 28. Unfortunately, a substantial number of students asserted that these were the only possibilities because their strategy that worked when n is a multiple of 28 would not work if n were not a multiple of 28. These students failed to consider why *no* sequence of moves would allow all children to end up with the same number of sweets from the initial configuration.

There were two approaches that students usually managed to see through:

- Setting up algebra as in the first alternative solution, and calmly working through it;
- Trying some examples and realising that the "odd" and "even" teams of the official solution are useful.

Unfortunately, a large number of students tried to argue that the order of moves is irrelevant. If this is assumed, the algebra can be simplified, but proving that this assumption does not eliminate any perfect initial configurations is quite tough, so these students missed the crux of the problem. Correct solutions along the lines of the second alternative were extremely rare.

Remark

The numbers 6 and 28 are both called *perfect* numbers since they are equal to the sum of their proper divisors. Euler showed that every even perfect number must be of the form $2^{m-1}(2^m - 1)$ where $2^m - 1$ is a (Mersenne) prime. However, it is not currently known whether there are infinitely many perfect numbers, or indeed whether any odd perfect numbers exist.

Question 6

A function f is called *good* if it assigns an integer value $f(m,n)$ to every ordered pair of integers (m,n) in such a way that for every pair of integers (m,n) we have:

$$2f(m,n) = f(m-n,n-m) + m + n = f(m+1,n) + f(m,n+1) - 1.$$

Find all good functions.

Solution

We write $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{R}$ for the left, middle and right parts of the displayed equations.

Note first that, by substituting $m = n = 0$ into $\mathcal{L} = \mathcal{M}$, we get $f(0,0) = 0$.

Now, writing $g(m)$ for $f(m,-m)$, the equation $\mathcal{L} = \mathcal{M}$ gives $f(m,n) = \frac{1}{2}(g(m-n) + m + n)$. Making this substitution on equation $\mathcal{L} = \mathcal{R}$ gives

$$g(m-n) + m + n = \frac{1}{2}(g(m+1-n) + m + 1 + n) + \frac{1}{2}(g(m-n-1) + m + n + 1) - 1,$$

and taking $n = 0$ then gives

$$g(m) + m = \frac{1}{2}(g(m+1) + m + 1) + \frac{1}{2}(g(m-1) + m + 1) - 1,$$

which simplifies to $g(m) = \frac{1}{2}(g(m+1) + g(m-1))$.

This says that $g(m)$ is linear; since we determined at the start that $g(0) = 0$, we get $g(m) = am$, and hence $f(m,n) = \frac{1}{2}((1+a)m + (1-a)n)$. By taking $m = 1, n = 0$, we see that a should be odd; taking $a = 2b+1$ gives $f(m,n) = (b+1)m - bn$. It can readily be checked that this works for any value of b.

Alternative

Using $\mathcal{L} = \mathcal{M}$ for $f(m+1,n+1)$ and for $f(m,n)$ gives

$$\begin{aligned} 2f(m+1,n+1) &= f(m-n,n-m) + m + n + 2, \\ 2f(m,n) &= f(m-n,n-m) + m + n. \end{aligned}$$

Hence $f(m+1,n+1) = f(m,n) + 1$, and so, by induction in both directions, we have $f(m,n) = f(m-n,0) + n$.

Substituting this into the original equations, and subtracting $2n$ from all sides, gives us

$$2f(m-n,0) = f(2m-2n,0) = f(m-n+1,0) + f(m-n-1,0).$$

Writing p for $m-n$, we have

$$2f(p,0) = f(2p,0) = f(p+1,0) + f(p-1,0).$$

From these we deduce (similarly to Solution 1) that $f(p,0) = bp$ for some b, and this gives $f(m,n) = b(m-n) + n$; it can readily be checked that all such solutions work.

Markers' comments

This question was not attempted by many students; among the attempts we saw, many weren't successful. We suspect that most successful students mixed the following three strategies, allowing their experiences with each to inform their attempts at the others:

(1) Using small cases of the recurrence relation to try to deduce things about $f(m, n)$ for m and n small integers. (Looking out for helpful things is easier if one has attempted strategies (2) and (3).)

(2) Attempting clever substitutions, to try to say helpful general things about the function. (Precisely what things are helpful can best be told by attempting strategies (1) and (3).)

(3) Attempting to think of solutions. (Indeed, trying solutions like $f(m, n) = am + bn + c$ may not come naturally at first, but knowing which solutions to try does come naturally if one has made attempts at strategies (1) and (2).)

In contrast, students who focused their attention on only one of these three strategies were likely to grind to a halt sooner or later.

United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 1

Thursday 26 November 2020

Instructions

1. Time allowed: $2\frac{1}{2}$ hours.
2. Each question in Section A carries 5 marks. Each question in Section B carries 10 marks. Earlier questions tend to be easier; you are advised to concentrate on these problems first.
3. In Section A only answers are required.
4. Use the answer sheet provided for Section A.
5. In Section B full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.
6. One complete solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.
7. Write on one side of the paper only. Start each question in Section B on a fresh sheet of paper: *scans of your work will need to be uploaded question by question for marking.*
8. On each sheet of working for Section B, write the number of the question in the top left hand corner and your **Participant ID** and **UKMT centre number**. Do not write your name.
9. The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden. *You are strongly encouraged to use geometrical instruments to construct large, accurate diagrams for Section B geometry problems.*
10. At the end of the paper, return to your Section A answer sheet and indicate which Section B questions you have attempted.
11. Please do not discuss the paper on the internet until 5pm GMT on Wednesday 2 December when the solutions video will be released at bmos.ukmt.org.uk
12. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

Section A

The questions in Section A are worth a maximum of five points each.
Only answers are required.
Please use the answer sheet provided.

1. Alice and Bob take it in turns to write numbers on a blackboard. Alice starts by writing an integer a between -100 and 100 inclusive on the board. On each of Bob's turns he writes twice the number Alice wrote last. On each of Alice's subsequent turns she writes the number 45 less than the number Bob wrote last. At some point, the number a is written on the board for a second time. Find the possible values of a.

2. A triangle has side lengths a, a and b. It has perimeter P and area A. Given that b and P are integers, and that P is numerically equal to A^2, find all possible pairs (a, b).

3. A square piece of paper is folded in half along a line of symmetry. The resulting shape is then folded in half along a line of symmetry of the new shape. This process is repeated until n folds have been made, giving a sequence of $n+1$ shapes. If we do not distinguish between congruent shapes, find the number of possible sequences when:

(a) $n = 3$;

(b) $n = 6$;

(c) $n = 9$.

(When $n = 1$ there are two possible sequences.)

4. In the equation

$$\mathsf{A}^{\mathsf{AA}} + \mathsf{AA} = \mathsf{B{,}BBC{,}DED{,}BEE{,}BBB{,}BBE}$$

the letters A, B, C, D and E represent different base 10 digits (so the right hand side is a sixteen digit number and AA is a two digit number). Given that C = 9, find A, B, D and E.

Section B

The questions in Section B are worth a maximum of ten points each.
Full written solutions are required.
Please begin each question on a new sheet of paper.

5. Let points A, B and C lie on a circle Γ. Circle Δ is tangent to AC at A. It meets Γ again at D and the line AB again at P. The point A lies between points B and P. Prove that if $AD = DP$, then $BP = AC$.

6. Given that an integer n is the sum of two different powers of 2 and also the sum of two different Mersenne primes, prove that n is the sum of two different square numbers.

(A Mersenne prime *is a prime number which is one less than a power of two.)*

7. Evie and Odette are playing a game. Three pebbles are placed on the number line; one at -2020, one at 2020, and one at n, where n is an integer between -2020 and 2020. They take it in turns moving either the leftmost or the rightmost pebble to an integer between the other two pebbles. The game ends when the pebbles occupy three consecutive integers.

Odette wins if their sum is odd; Evie wins if their sum is even. For how many values of n can Evie guarantee victory if:

(a) Odette goes first;

(b) Evie goes first?

United Kingdom Mathematics Trust

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British Mathematical Olympiad Round 1 2020

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

The 2020 paper

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for is full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap? On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore important that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

In 2020 the BMO1, exceptionally, included a section where only answers were required. Partial credit was awarded for incorrect answers that indicated sensible engagement with the problems, but in general it was hard to score highly without correct answers. Candidates were also penalised for the inclusion of incorrect answers alongside correct ones, and in some cases careful checking would have led to significantly higher scorers.

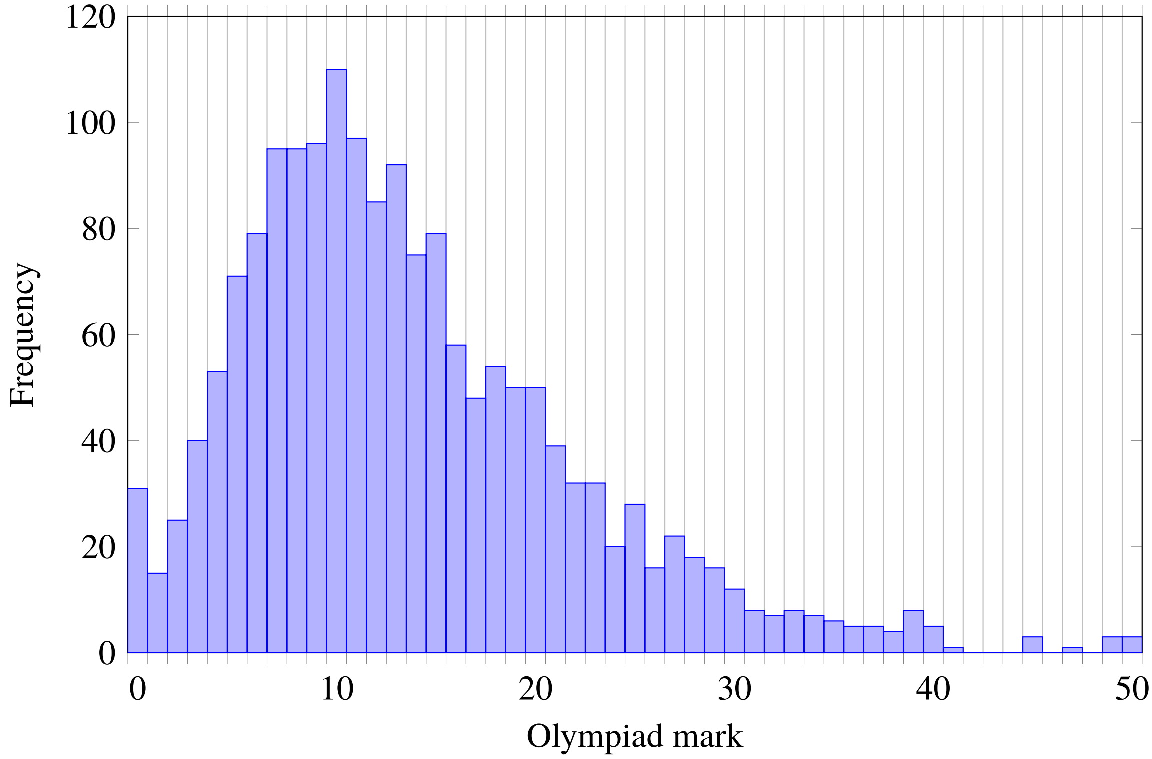
General comments

The format of the 2020 British Mathematical Olympiad Round 1 was a significant departure from previous years. This change was driven by factors relating to the Covid-19 pandemic, rather than any shift in ideology. There were concerns that schools would find administrating a 3.5 hour exam uniquely challenging this year, and also that the marking of the paper without the traditional on-site marking weekend might be problematic. The change in format addressed these issues, and the competition attracted a typical number of entries all of which were marked in a timely fashion. However, it seems unlikely that future BMOs will follow the 2020 format exactly.

It had been hoped that removing the need to 'write up' the first four questions, would give candidates more time to attempt section B. Certainly many candidates made good attempts at section A and at least one later question, but the answer only format also meant that arithmetical errors in easier questions were necessarily penalised more heavily than they might have been in a typical year. This, combined with the reduction in time, lead to a paper which candidates found particularly challenging. It was encouraging to see people rising to that challenge and, for the most part, engaging seriously with a number of the questions.

The 2020 British Mathematical Olympiad Round 1 attracted 1706 entries. The scripts were marked digitally from the 5th to the 11th of December by a team of Hugh Ainsley, Ann Ault, Eszter Backhausz, Jordan Baillie, Agnijo Banerjee, Sam Bealing, Emily Beatty, Jonathan Beckett, Phillip Beckett , Jamie Bell, Lex Betts, Robin Bhattacharyya, Tom Bowler, Magdalena Burrows, Shinwha Cha, Andrea Chlebikova, Arthur Conmy, James Cranch, Stephen Darby, Stefan Dixon, Ashling Dolan, Ceri Fiddes, Alison Fisher, Richard Freeland, James Gazet, Sarah Gleghorn, Amit Goyal, Daniel Griller, Peter Hall, Ben Handley, Stuart Haring, Adrian Hemery, Liam Hill, Michael Illing, Ian Jackson, Vesna Kadelburg, Jeremy King, Patricia King, David Knipe, Gerry Leversha, Warren Li, Sophie Maclean, Sam Maltby, Matei Mandache, David Mestel, Jordan Millar, Paul Murray, Joseph Myers, Michael Ng, Jenny Owladi, Eve Pound, Wendy Rathbone, Frankie Richards, Dominic Rowland, Paul Scarr, Amit Shah, Fiona Shen, Geoff Smith, Leona So, Anne Strong, Karthik Tadinada, Stephen Tate, Paul Walter, Zi Wang, Kasia Warburton and Dominic Yeo.

Mark distribution



The thresholds for qualification for BMO2 were as follows:

Year 13: 27 marks or more.

Year 12: 26 marks or more.

Year 11 or below: 24 marks or more.

Question 1

Alice and Bob take it in turns to write numbers on a blackboard. Alice starts by writing an integer a between -100 and 100 inclusive on the board. On each of Bob's turns he writes twice the number Alice wrote last. On each of Alice's subsequent turns she writes the number 45 less than the number Bob wrote last. At some point, the number a is written on the board for a second time. Find the possible values of a.

Proposed by Daniel Griller

Solution

There are four possible starting numbers a which generate a repeat: 0, 30, 42 and 45.

We call the terms of the sequence written on the board $a_1, a_2, a_3, \ldots$ where $a_1 = a$. We will consider a more general version of the problem by writing s in place of 45. Now the sequence is determined by the rules $a_{2n} = 2a_{2n-1}$ and $a_{2n+1} = a_{2n} - s$ for $n \geq 1$.

The sequence begins $a, 2a, 2a - s, 4a - 2s, 4a - 3s, 8a - 6s, 8a - 7s, \ldots$.

In general we see that $a_{2k} = 2^k a - (2^k - 2)s$ while $a_{2k+1} = 2^k a - (2^k - 1)s$. These can be proved formally by induction, but this is obviously not required for the question.

We now have two cases to consider:

Case 1: $a_{2k} = a$ for some $k \geq 0$.

In this case $(2^k - 1)a = (2^k - 2)s$. The numbers $2^k - 1$ and $2^k - 2$ cannot share any factors great than one since any factor of both would divide their difference. Thus we must have $2^k - 1$ is a factor of s.

Turning to the specific problem at hand, we need $2^k - 1$ to be a factor 45. The possible factors are $1 = 2^1 - 1, 3 = 2^2 - 1$ and $15 = 2^4 - 1$. These correspond to $a = 0, a = 30$ and $a = 42$ respectively, all of which work.

Case 2: $a_{2k+1} = a$ for some $k \geq 1$.

In this case we have $(2^k - 1)a = (2^k - 1)s$ and, since $k \neq 0$, this implies $a = s = 45$ which also works.

Alternative

It is also fairly straightforward to solve the problem by trying values of a in turn.

If at any stage $a_{2k} > 90$, then the subsequent odd and even numbered terms both form increasing sequences, so once their terms are larger than a, the number a can never be repeated. Similarly if $a_{2k} < 0$ at any stage, then the subsequent odd and even numbered terms both form decreasing sequences, so once their terms are less than a, the number a can never be repeated.

Most values of a in the range -100 to 100 give sequences which rapidly fall into the two categories above, meaning the four exceptional cases can be easily identified.

Question 2

A triangle has side lengths a, a and b. It has perimeter P and area A. Given that b and P are integers, and that P is numerically equal to A^2, find all possible pairs (a, b).

Proposed by Tom Bowler

Solution

We have that $P = 2a + b$ and, by Pythagoras' theorem, the height is $\sqrt{a^2 - \frac{b^2}{4}}$ which means that $A = \frac{b}{2}\sqrt{a^2 - \frac{b^2}{4}}$.

The condition in the question gives us the equation $2a + b = \left(\frac{b^2}{4}\right)\left(a^2 - \frac{b^2}{4}\right)$.

Thus $16(2a + b) = b^2(4a^2 - b^2) = b^2(2a - b)(2a + b)$.

We may divide by the (positive) $2a + b$ to get $b^2(2a - b) = 16$. (If $a = b = 0$ we do get a solution to the equation, but this corresponds to a single point rather than a triangle.)

Since b and $(2a - b) = P - 2b$ are both integers, b^2 must be a square factor of 16. The possible values are 1, 4 and 16 and we can substitute these into the equation in turn to find that $(a, b) = (\frac{17}{2}, 1), (3, 2)$ or $(\frac{5}{2}, 4)$.

Remark

Candidates were not penalised for including the solution $(a, b) = (0, 0)$ but it was not required to get full marks.

Remark

It is also possible to obtain the same equation linking a and b by using *Heron's formula* for the area of a triangle.

Question 3

A square piece of paper is folded in half along a line of symmetry. The resulting shape is then folded in half along a line of symmetry of the new shape. This process is repeated until n folds have been made, giving a sequence of $n + 1$ shapes. If we do not distinguish between congruent shapes, find the number of possible sequences when:

(a) $n = 3$;

(b) $n = 6$;

(c) $n = 9$.

(When $n = 1$ there are two possible sequences.)

Proposed by Daniel Griller

Solution

We classify the sequences of shapes according to the final shape in the sequence and make the following observations:

- The final shape can be an isosceles triangle if and only if the penultimate shape is either an isosceles triangle or a square.
- The final shape can be a square if and only if the penultimate shape is a rectangle in the ratio $1 : 2$.
- The final shape can be a rectangle in the ratio $1 : 2^k$ for $n \geq 1$ if and only if the penultimate shape is a rectangle in the ratio $1 : 2^{k-1}$ or $1 : 2^{k+1}$.

We now construct the Pascal-like triangle below.

n	$\triangle$	$\square$	1:2	1:4	1:8	1:16	1:32	1:64	1:128	1:256	1:512	Σ
0		1										1
1	1		1									2
2	1	1		1								3
3	2		2		1							5
4	2	2		3		1						8
5	4		5		4		1					14
6	4	5		9		5		1				24
7	9		14		14		6		1			44
8	9	14		28		20		7		1		79
9	23		42		48		27		8		1	149

Each entry is the number of n-fold sequences whose final shape is given by the column heading. The first two columns count sequences ending in isosceles triangles and squares respectively, the remaining columns count sequences ending in proper rectangles.

The observations above mean that:

- Each entry in the $\triangle$ column is the sum of entries in the $\triangle$ and $\square$ columns in the row above.
- Each entry in the $\square$ column is equal to the entry in the $1 : 2$ column in the row above.

- Each entry in the other columns is equal to the sum of the two entries 'diagonally above' it.

The answers can now be read off from the final column of the table, which records the sum of the entries in each row.

(a) 3 folds, 5 sequences; (b) 6 folds, 24 sequences; (c) folds, 149 sequences.

Alternative

There is a short cut: each number in the table contributes to two numbers in the row below, apart from the △ column which only contributes to one. Therefore each row sum is twice the previous row sum minus the △ number in the previous row. Thus we can work out all the relevant row sums by calculating with the much reduced table below.

n	△	□	1:2	1:4	1:8	Σ
0		1				1
1	1		1			2
2	1	1		1		3
3	2		2		1	5
4	2	2		3		8
5	4		5			14
6	4	5				24
7	9					44
8	9					79
9						149

Remark

There is a good deal more to explore in this question. For example, it turns out that the entries in the right hand portion of the table can be obtained by starting with a copy of Pascal's triangle and subtracting a second, slightly offset, copy of the same triangle. This can be verified by induction, but a direct counting argument is not so easy to come by.

One consequence of this observation is that the number of paths of length $2k$ which end in a square is given by $\binom{2k}{k} - \binom{2k}{k-1} = \frac{1}{k+1}\binom{2k}{k}$. These are the *Catalan numbers* which occur in a huge variety of different counting problems and certainly merit further reading.

If we call number of sequences ending in a triangle after either $2k$ or $2k-1$ folds t_k, we obtain the sequence $1, 2, 4, 9, 23, \ldots$. The nth term is simply the sum of the first n Catalan numbers, but the sequence also satisfies the relation $t_{k+1} = \frac{1}{k+1}((5k-1)t_k - (4k-2)t_{k-1})$ which can be used to solve the original problem for much larger numbers of folds.

Question 4

In the equation

$$\mathsf{A}^{\mathsf{AA}} + \mathsf{AA} = \mathsf{B{,}BBC{,}DED{,}BEE{,}BBB{,}BBE}$$

the letters A, B, C, D and E represent different base 10 digits (so the right hand side is a sixteen digit number and AA is a two digit number). Given that C = 9, find A, B, D and E.

Proposed by Nick Mackinnon

Solution

First notice that $2^{22} = 2 \cdot 2^{21} = 2 \cdot 8^7 \leq 2 \cdot 10^7$ so $2 < A$.

Also $4^{44} = 2^{88} \geq 2^{80} = (2^{10})^8 \geq (10^3)^8 = 10^{24}$ so $A < 4$. Therefore $A = 3$.

We can now calculate the last two digits of $3^{33} + 33$ to find B and E. This can be done efficiently by repeatedly squaring to find the last two digits of $3^2, 3^4, 3^8, 3^{16}$ and 3^{32}.

The last two digits of 3^2 are 09.
The last two digits of 3^4 are 81.
The last two digits of $3^8 = 6561$ are 61.
The last two digits of 3^{16} are the same as the last two digits of $61^2 = 3721$, namely 21.
The last two digits of 3^{32} are the same as those of $21^2 = 441$, namely 41.

Multiplying by 3 and adding 33 shows that $BE = 56$.

Next we note that $3^{33} + 33$ is a multiple of 3, so the digit sum on the right hand side must also be a multiple of 3. Since $C = 9$ we may ignore it and the same applies to $E = 6$. There are exactly 9 Bs so these can also be ignored. We conclude that $2D$ must be a multiple of 3.

The digits 3, 6 and 9 are already spoken for so $D = 0$.

Therefore $(A, B, D, E) = (3, 5, 0, 6)$.

Remark

The argument can be expressed more neatly using modular arithmetic, and there are a number of possible variations. For example, if we only work with the last digit we can still find $E = 6$ and hence $D = 0$. Next we can observe that $3^{33} + 33$ is two less than a multiple of 11. Now adding 2 to bot considering the alternating digit sum on the right gives enough information to determine B.

Markers' comments

There were a number of very good responses to the section A questions. However, some candidates lost marks through lack of adequate checking or unjustified pattern spotting. The first issue was most apparent in question 1 where many candidates listed one or more incorrect values of a alongside some or all of the correct ones. The nature of the question should make it natural to actually test each proposed value of a by writing out the sequence and waiting for the repeat, and candidates should be on the lookout for such opportunities to verify their solutions.

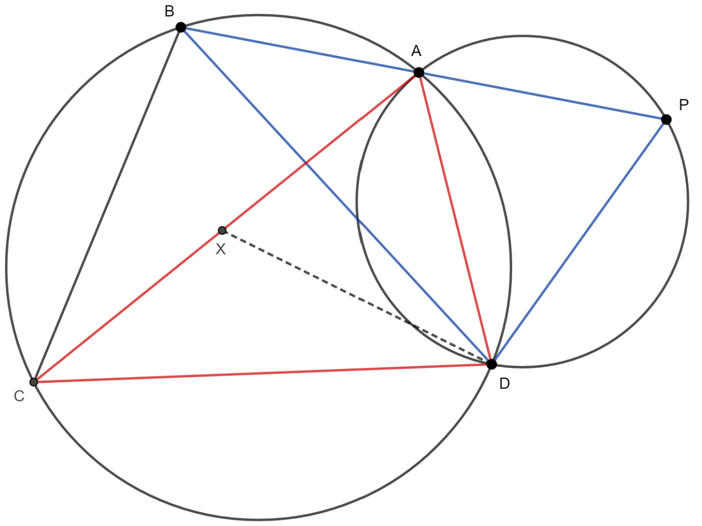
In question 3 a significant number of candidates gave the answers 5, 21, 89, which are the third, sixth and ninth Fibonacci numbers. The lesson here is that not every pattern in mathematics continues in the way one might initially expect. The sequence in this question begins 1, 2, 3, 5, 8, but the similarity to the Fibonacci numbers turns out to short-lived as the next term is not 13.

Question 5

Let points A, B and C lie on a circle Γ. Circle Δ is tangent to AC at A. It meets Γ again at D and the line AB again at P. The point A lies between points B and P. Prove that if $AD = DP$, then $BP = AC$.

Proposed by Dominic Yeo

SOLUTION



$AD = PD$ is given.

By the alternate segment theorem in Δ, we have $\angle CAD = \angle BPD$.
Then $\angle DCA = \angle DBA = \angle DBP$ by angles in the same segment in Γ.
These two angles show that triangles DCA and DBP are similar.

Then we have $AD = PD$, and so this similarity is a congruence.
So CAD is congruent to BPD (by 'Side-Angle-Side') and $BP = AC$.

ALTERNATIVE

Construct point X on ray AC with $AX = AP$.
Then $\angle DAX = \angle DPA = \angle PAD$ by the alternate segment theorem and then by ADP isosceles, so AD is perpendicular to PX as the angle bisector of isosceles PAX.
Therefore $PAXD$ is a kite, and $DX = PD = AD$.
This gives us $\angle DXC = 180° - \angle AXD = 180° - \angle DPA = 180° - \angle PAD = \angle DAB$.
Then angles in the same segment gives $\angle ABD = \angle ACD$.
So we have DAB similar to DXC, and $AD = DX$ makes this a congruence.

So $XC = AB$ as required.

REMARK

It is also possible to solve the problem using the Sine Rule, or by quoting the fact that the unique spiral similarity sending segment BP to CA is centred at D.

Remark

Arguably the hardest part of this problem is constructing an accurate diagram. Adding points in the order in which they are specified in the question is unhelpful, and a better alternative is to begin by constructiong the circle Δ and the isosceles triangle ADP inside it.

Markers' comments

There were many excellent solutions to this problem. Most candidates found the simplest pair of congruent triangles. Some added a point X on AC such that $AX = AP$ and others added lines perpendicular to AP through D and perpendicular to AC through D. Both these methods needed extra work to find two pairs of congruent triangles but those who attempted them normally did so successfully. There were also some attempts using trigonometry, many of which were unsuccessful. A neat approach involving extending PD until it met circle Γ to create an isosceles trapezium was also successful.

Many of the solutions were very well explained. However, some candidates claimed equality of angles with no explanations and they were penalised for this. Although the standard GCSE theorems may be used without proof, it is important to make it clear at each step which circle theorem or which triangle is being used. In this question, the alternate segment theorem was crucial, (though knowledge of its name was not). Many did not know the theorem but managed to prove it from scratch, which did receive full credit but required extra work. However, those who claimed equality of pairs of angles without justification were penalised heavily.

In geometry, assuming special cases often reduces the problem to a simple one but does not solve the general case. Some assumed that PDC was a straight line or that PA was a diameter and got little credit. Another common error was claiming that a pair of triangles were congruent using two sides and one angle in the order 'Side-Side-Angle' which is not enough to prove congruence.

Question 6

Given that an integer n is the sum of two different powers of 2 and also the sum of two different Mersenne primes, prove that n is the sum of two different square numbers.

(A Mersenne prime *is a prime number which is one less than a power of two.)*

Proposed by Luke Pebody

Solution

We begin by writing

$$n = 2^a + 2^b - 2 = 2^c + 2^d$$

where $2^a - 1$ and $2^b - 1$ are Mersenne primes and (without loss of generality) we have $a > b$ and $c > d$.
Since $2^b - 1$ is prime, we must have $a > b \geq 2$, so $n = 2^a + 2^b - 2$ is divisible by 2 but not 4.
Since $n = 2^c + 2^d$ with $c > d$, we must have that $d = 1$ and $c \geq 2$.
Adding two to both sides of the original equation and dividing by 4 gives

$$2^{a-2} + 2^{b-2} = 2^{c-2} + 1.$$

The right-hand side is either 2 (if $c = 2$) or odd; in either case we must have $b = 2$.
Returning to the original equation we have $a = c$ and $n = 2^a + 2$.
Now a must be odd, otherwise $2^a - 1 = (2^{a/2} + 1)(2^{a/2} - 1)$, which is impossible since $2^a - 1$ is prime and $a \geq 3$. (Alternatively, if a were even, then $2^a - 1$ would be divisible by 3, which is impossible since it is prime and $a \geq 3$.)
To conclude, we write $a = 2k + 1$, so that the identity $n = 2^a + 2$ may be rewritten as

$$n = (2^k + 1)^2 + (2^k - 1)^2.$$

This exhibits n as the sum of two distinct squares.

Alternative

We begin by establishing that $d = 1$ as in the first solution. We then argue directly that c must be odd. For, if c were even, then $n = 2^c + 2$ would be divisible by 3. But all Mersenne primes other than 3 are one more than a multiple of 3, so there is no way to write a multiple of 3 as a sum of two distinct Mersenne primes.

We can then conclude as in the first solution since if $c = 2k + 1$, then $n = 2^{2k+1} + 2$ as before.

Alternative

Rewrite the equation as $2^a + 2^b = 2^c + 2^d + 2$. Since $2^b - 1$ is prime, $b \geq 2$ so the left hand side is even, which implies $d \geq 1$. However, the left hand side is a sum of exactly two distinct powers of 2, so by uniqueness of binary representation we have $d = 1$. This gives $2^a + 2^b = 2^c + 4$. The left hand side is not a power of 2, so again by uniqueness of binary representation, $b = 2$.

This shows that $a = c$. Since $2^a - 1$ is a Mersenne prime greater than three, and it is well known that Mersenne primes must be of the form $2^p - 1$ where p is prime, we see that a is odd and can conclude as before.

Remark

The expression of $2^{2k+1} + 2$ as a sum of squares can be found using *Diophantus' identity* which states that $(a^2+b^2)(c^2+d^2) = (ab-cd)^2+(ad+bc)^2$. Setting $a = 2^k$ and $b = c = d = 1$ gives the result needed for the question. Candidates familiar with complex numbers may recognise the original identity, which is equivalent to the fact that the modulus function is multiplicative.

Markers' comments

This proved a popular question and it was good to see well over 100 complete solutions.

This problem can be broken down naturally into three distinct parts: finding the value b and d and showing $a = c$; showing that a is odd and finally finding an identity which demonstrates that n is the sum of two distinct squares. Candidates with a viable overall strategy could gain full or partial credit for each of these parts.

The most common approach to the first part was to divide both sides of the equation $2^a+2^b-2 = 2^c + 2^d$ by 2 and argue about parity. However, for this to be valid we must first check that b and d are non-zero. This is not difficult (we can observe that all Mersenne primes are odd and greater than 1), but failing to do it attracted a small penalty.

Arguments using binary representations as in the second alternative above were also common. Unfortunately many of these also failed to adequately consider small powers of two at the start.

Most candidates who who addressed the fact that a must be odd ruled out $a = 2k$ by considering $2^{2k} - 1 = (2^k - 1)(2^k + 1)$ and arguing that this is composite and so not a Mersenne prime. Some candidates overlooked the fact that the first bracket could be 1, yielding the Mersenne prime 3. Somewhat fortunately, 3 is the smaller of the Mersenne primes found in the first part and so a mark was not deducted for missing this case. The two other approaches in the alternative solutions above were also successfully employed by many candidates.

The third part was almost only ever successful when a candidate found the identity $2^a + 2 = 2^{2k+1} + 2 = (2^k + 1)^2 + (2^k - 1)^2$. Some candidates spotted a pattern by looking at small values of a and attempted to prove the required result by induction, this was rarely successful as many responses became convoluted and lost their train of thought. A very small number of candidates showed explicitly that $\frac{n}{2}$ was the sum of two distinct squares and stated that this implied that n was also the sum of two distinct squares.

Overall there were some excellent solutions to this quite technical problem but many candidates lost marks for not considering all relevant cases or for gaps in their reasoning. It is very important to remember that a mathematical proof should be communicated to the reader in a clear and concise manner but not so concise that the reader has to work hard to fill in gaps!

Question 7

Evie and Odette are playing a game. Three pebbles are placed on the number line; one at -2020, one at 2020, and one at n, where n is an integer between -2020 and 2020. They take it in turns moving either the leftmost or the rightmost pebble to an integer between the other two pebbles. The game ends when the pebbles occupy three consecutive integers.

Odette wins if their sum is odd; Evie wins if their sum is even. For how many values of n can Evie guarantee victory if:

(a) Odette goes first;

(b) Evie goes first?

Proposed by Daniel Griller

Solution

First note that the game must end after a finite number of moves, because the difference of the positions of the outer pebbles is a strictly decreasing sequence of positive integers until it is impossible to make a move (and so one player will win).

Claim 1: If Odette goes first she can force a win for any n.

Claim 2: If Evie goes first she can force a win if and only if n is odd.

These claims show the answer to the questions are 0 and 2020 respectively.

Proof of claim 1

Odette can play in such a way that after every turn Evie takes, the two outer pebbles are even.

If the configuration before Os turn is *even, even, even* O can move to another such configuration, or, if that is impossible, to a winning configuration for her.

If the configuration is *even, odd, even* and O has not already won, then she can move an outer pebble to an even number directly adjacent to the odd pebble. This will force E to move the odd pebble back between the two even ones.

This strategy ensures that the only way either player can end the game is by moving to consecutive numbers where the middle one is odd: a win for O.

Proof of claim 2

If the initial configuration is *even, even, even* before Es turn, then the argument from claim 1 shows that O can force a win.

If the initial configuration is *even, odd, even*, then E can move an outer even pebble to an odd number adjacent to the other even one. This forces O to move the even pebble, so the configuration after Os turn will have both outer pebbles odd.

Now the situation is identical to that one studied earlier but with the words even and odd interchanged, so E can arrange that in all subsequent configurations, the outer pebbles are both odd. The game must finish with adjacent places odd, even, odd and so Evie will win.

Markers' comments

This problem was found very hard. Only a minority of students found time to attempt it, and only a minority of attempts purported to be full solutions. A number of students had very good ideas, but wrote up only special cases: it is important to remember that a winning strategy has to explain what the winning player should do to win given any possible choice of moves by the other player.

In practice, those students frequently did well who observed (and who made clear that they observed) that moving two pebbles to adjacent positions forces the other player to move the outermost of the two on their next move.

United Kingdom
Mathematics Trust

British Mathematical Olympiad
Round 1

Thursday 2 December 2021

supported by [XTX] MARKETS Overleaf

Instructions

1. Time allowed: $3\frac{1}{2}$ hours.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.
3. One complete solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.
4. Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.
5. The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden. *You are strongly encouraged to use geometrical instruments to construct large, accurate diagrams for geometry problems.*
6. Start each question on an official answer sheet on which there is a QR code.
7. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code.
8. Write on one side of the paper only. Make sure your writing and diagrams are clear and not too faint. (*Your work will be scanned for marking.*)
9. Arrange your answer sheets in question order before they are collected. If you are not submitting work for a particular problem, please remove the associated answer sheet.
10. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 8am on Saturday 4 December UK time. when the solutions video will be released at https://bmos.ukmt.org.uk
11. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 365 1121 challenges@ukmt.org.uk www.ukmt.org.uk

1. Find three even numbers less than 400, each of which can be expressed as a sum of consecutive positive odd numbers in at least six different ways.

(Two expressions are considered to be different if they contain different numbers. The order of the numbers forming a sum is irrelevant.)

2. One day Arun and Disha played several games of table tennis. At five points during the day, Arun calculated the percentage of the games played so far that he had won. The results of these calculations were exactly 30%, exactly 40%, exactly 50%, exactly 60% and exactly 70% in some order. What is the smallest possible number of games they played?

3. For each integer $0 \leq n \leq 11$, Eliza has exactly three identical pieces of gold that weigh 2^n grams. In how many different ways can she form a pile of gold weighing 2021 grams?

(*Two piles are different if they contain different numbers of gold pieces of some weight. The arrangement of the pieces in the piles is irrelevant.*)

4. Two circles Γ_1 and Γ_2 have centres O_1 and O_2 respectively. They pass through each other's centres and intersect at A and B. The point C lies on the minor arc BO_2 of Γ_1. The points D and E lie on the line O_2C such that $\angle AO_1D = \angle DO_1C$ and $\angle CO_1E = \angle EO_1B$. Prove that triangle DO_1E is equilateral.

(A minor arc of a circle is the shorter of the two arcs with given endpoints.)

5. An N-set is a set of different positive integers including a given positive integer N. Let $m(N)$ be the smallest possible mean of any N-set. For how many values of N less than 2021 is $m(N)$ an integer?

6. Marvin has been tasked with writing down every list of integers with the following properties:

(i) The list contains 71 terms.

(ii) The first term is 1.

(iii) Every term after the first is equal to either the previous term, or the sum of all previous terms.

When Marvin is finished, how many of the lists will have a sum equal to 999,999?

United Kingdom Mathematics Trust

United Kingdom Mathematics Trust

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British Mathematical Olympiad Round 1 2021

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

The 2021 paper

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for are full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights. If a problem has two distinct logical parts, these are sometimes marked separately and the scores added, but one part is generally considered the crux of the problem. For example, in Q2 we need to show (i) that Arun and Disha played at least 30 games and (ii) that they might have played exactly 30. Here (i) requires more sophistication, and carries 7 of the 10 marks available. In general the logical structure of the mark scheme aims to reflect the logical structure of the problem while rewarding correct arguments more generously than correct calculations.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore vital that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

Candidates found this a demanding BMO1 paper, though the vast majority still engaged well with one or two of the problems. Even very strong candidates found there was a lot to do in the time available, with many writing a great deal. It was encouraging to see candidates taking the requirement to provided full written solutions seriously, but it is worth pointing out that, while proofs generally require English sentences as well as mathematical symbols, a fairly concise style often adds clarity.

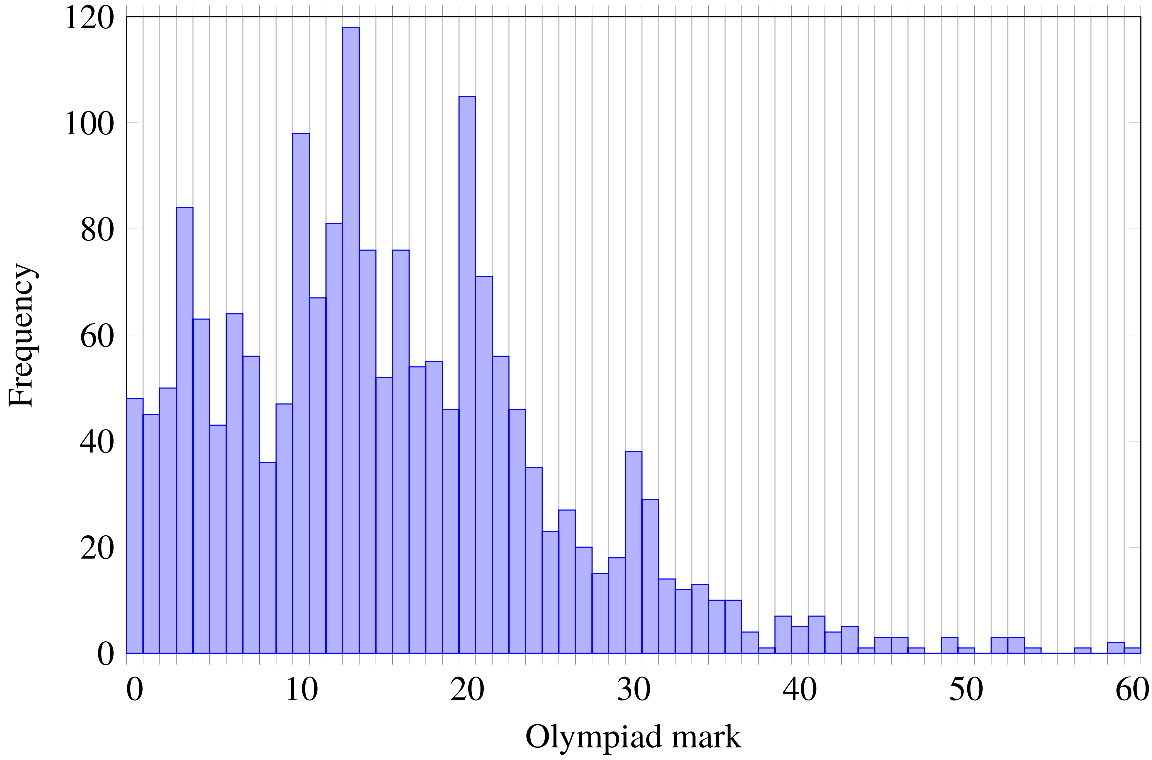
Markers were pleased to see that, while five of this year's problems had numerical answers, almost all candidates realised that simply providing the 'correct number' was not what was required.

Since only one candidate scored full marks, we hope that others, particularly those not yet in their final school year, will forgive the repetition of some (hopefully) familiar advice:

- *Check your work*: many of the candidates could have scored more highly on question 1 if they had spent perhaps five minutes checking that their proposed numbers actually worked as intended.
- *Read the question*: many candidates simply missed the crucial phrase 'in some order' in question 2.
- *Try small examples*: many candidates did not test their ideas in question 3 on smaller numbers than 2021. Systematically considering the number of ways to make piles of 1, 2, 3,. . . grams of gold leads to a clear pattern fairly swiftly.

The 2021 British Mathematical Olympiad Round 1 attracted 1857 entries. The scripts were marked digitally from the 5th to the 15th of December by a team of Eszter Backhausz, Agnijo Banerjee, Sam Bealing, Emily Beatty, Jonathan Beckett, Natalie Behague, James Bell, Robin Bhattacharyya, Maya Brock, Magdalena Burrows, Andrea Chlebikova, James Cranch, Stephen Darby, Joe Devine, Ceri Fiddes, Richard Freeland, Carol Gainlall, Amit Goyal, Ben Handley, Stuart Haring, Adrian Hemery, Ina Hughes, Ian Jackson, Shavindra Jayasekera, Vesna Kadelburg, Adam Kelly, Jeremy King, Patricia King, David Knipe, Gordon Lessells, Rhys Lewis, Warren Li, Samuel Liew, Linus Luu, Nick MacKinnon, Sam Maltby, Matei Mandache, David Mestel, Jordan Millar, Kian Moshiri, Joseph Myers, Daniel Naylor, Jenny Owladi, Frankie Richards, Adrian Sanders, Amit Shah, Jack Shotton, Alan Slomson, Geoff Smith, Zhivko Stoyanov, Karthik Tadinada, David Vaccaro, Jenni Voon, Tommy Walker Mackay, Paul Walter, Zi Wang, Kasia Warburton, Dominic Yeo.

Mark distribution



The thresholds for qualification for BMO2 were as follows:

Year 13: 33 marks or more.

Year 12: 32 marks or more.

Year 11: 31 marks or more.

Year 10 or below: 29 marks or more.

The thresholds for medals, Distinction and Merit were as follows:

Medal and book prize: 33 marks or more.

Distinction: 21 marks or more.

Merit: 11 marks or more.

Question 1

Find three even numbers less than 400, each of which can be expressed as a sum of consecutive positive odd numbers in at least six different ways.

(Two expressions are considered to be different if they contain different numbers. The order of the numbers forming a sum is irrelevant.)

SOLUTION

Consider runs of consecutive odd numbers with even sum. There must be an even number, $2k$, terms in such a run since the sum is even. The mean, $2u$, must be even since it must also be the median.

Therefore the sum is $4ku$. Notice that the largest odd number in the run is $2u + 2k - 1$ and the smallest is $2u - 2k + 1$. The smallest number must be positive so $k \leq u$. We therefore seek numbers K in the range 1 to 100 inclusive which have at least six factorizations $K = ku$ where $k \leq u$. Then the solutions are the numbers $N = 4K$.

To solve the problem we must provide three of the five possible values of K, which are $60 = 2^2 \times 3 \times 5, 72 = 2^3 \times 3^2, 84 = 2^2 \times 3 \times 7, 90 = 2 \times 3^2 \times 5$ and $96 = 2^5 \times 3$.

These give rise to five possible values of N, namely 240, 288, 336, 360, 384.

The quickest way to show that these values of K work is to recall that if $p_1, p_2, \ldots p_m$ are different prime numbers, then the different factors of

$$p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m} \quad \text{are} \quad p_1^{x_1} p_2^{x_2} \cdots p_m^{x_m}$$

where there are $a_i + 1$ options for each exponent x_i because $0 \leq x_i \leq a_i$, and so the number of positive integer divisors is

$$(a_1 + 1)(a_2 + 1) \cdots (a_m + 1).$$

Alternatively we could write out each factorisation N into six pairs of even factors or the decomposition of N into six sums of consecutive positive odd number explicitly.

The details are as follows:

$$\begin{aligned} 240 &= 2^4 \cdot 3 \cdot 5 \\ &= 119 + 121 = 2 \times 120 \\ &= 57 + \cdots + 63 = 4 \times 60 \\ &= 35 + \cdots + 45 = 6 \times 40 \\ &= 23 + \cdots + 37 = 8 \times 30 \\ &= 15 + \cdots + 33 = 10 \times 24 \\ &= 9 + \cdots + 31 = 12 \times 20 \end{aligned} \qquad \begin{aligned} 288 &= 2^5 \cdot 3^2 \\ &= 143 + 145 = 2 \times 144 \\ &= 69 + \cdots + 75 = 4 \times 72 \\ &= 43 + \cdots + 53 = 6 \times 48 \\ &= 29 + \cdots + 43 = 8 \times 36 \\ &= 13 + \cdots + 35 = 12 \times 24 \\ &= 3 + \cdots + 33 = 16 \times 18 \end{aligned} \qquad \begin{aligned} 336 &= 2^4 \cdot 3 \cdot 7 \\ &= 167 + 169 = 2 \times 168 \\ &= 81 + \cdots + 87 = 4 \times 84 \\ &= 51 + \cdots + 61 = 6 \times 56 \\ &= 35 + \cdots + 49 = 8 \times 42 \\ &= 17 + \cdots + 39 = 12 \times 28 \\ &= 11 + \cdots + 37 = 14 \times 24 \end{aligned}$$

$$\begin{aligned} 360 &= 2^3 \cdot 3^2 \cdot 5 \\ &= 179 + 181 = 2 \times 180 \\ &= 87 + \ldots + 93 = 4 \times 90 \\ &= 55 + \ldots + 65 = 6 \times 60 \\ &= 27 + \cdots + 45 = 10 \times 36 \\ &= 19 + \cdots + 41 = 12 \times 30 \\ &= 3 + \cdots + 37 = 18 \times 20 \end{aligned} \qquad \begin{aligned} 384 &= 2^7 \cdot 3 \\ &= 191 + 193 = 2 \times 192 \\ &= 93 + \cdots + 99 = 4 \times 96 \\ &= 59 + \cdots + 69 = 6 \times 64 \\ &= 41 + \cdots + 55 = 8 \times 48 \\ &= 21 + \cdots + 43 = 12 \times 32 \\ &= 9 + \cdots + 39 = 16 \times 24 \end{aligned}$$

Alternative

Rather than work with the average of the terms in each sum, we can simply recall that the sum of n terms of an arithmetic progression with first term a and common difference 2 is $\frac{1}{2}(2a + 2(n-1))n = n(a + n - 1)$. We note that a must be odd and n must be even, so we set $a = 2b + 1$ and $n = 2k$ with $b \geq 0$. Now we seek numbers $N \leq 400$ which can be written as $N = 4k(k + b)$ for at least six different pairs (k, b). We conclude as in the first solution.

Alternative

It is well-known that the sum of the first t positive odd integers is t^2. Thus a positive number is a sum of consecutive odd integers if, and only if, it is a positive difference of two squares. Therefore we are looking for even positive integers N of the form $N = u^2 - v^2 = (u - v)(u + v)$. We are given that N is even, which implies that u and v have the same parity, which in turn implies that both the factors $(u - v)$ and $(u + v)$ are even. This logic is reversible because if $0 < a < b$ are integers with $N = (2a)(2b)$, we can set $u = b + a$ and $v = b - a$.

Therefore we must find those even N which have at least 6 different factorizations $N = (2a)(2b)$ with $0 < a < b$, or after dividing by 4, we need to find positive integers less than 100 which have at least 6 factorizations. We conclude as in the first solution.

Markers' comments

There were many excellent solutions to this problem, with candidates providing detailed justifications for the three numbers they found. Yet more candidates found three correct numbers N, but did not fully justify that each satisfied the three conditions in the question namely that there are six ways to write N as a sum of an arithmetic progression with common difference 2, that all the terms in the sums are odd and that all the terms are positive. A common oversight was to include sums that do not satisfy one (or both) of the last two conditions.

For example, the number $192 = 16 \times 12$ can be written as a sum of 16 consecutive odd numbers with average value 12, but in this sum the first term is -3. On the other hand, the number $360 = 8 \times 45$ can be written as a sum of 8 positive numbers differing by 2 but they are even: $38 + 40 + \cdots + 52$. (Note that 360 actually works but many candidates failed to correctly explain why it does.) Another common incorrect number seen was 120; candidates giving this answer had often failed to deal correctly with either of the last two conditions.

How easy it is to avoid those traps depends on the exact approach taken to finding the numbers. Quite a large number of candidates ended up listing the possible sums: this was not essential, but is clearly a safe way of ensuring they all work. Starting from the fact that the first n odd numbers sum to n^2, as in the second alternative above, takes care of both the oddness and the positivity of the summands very easily.

Other approaches tended to observe that if we take $n = 2k$ numbers differing by 2 and call the first term a and/or the average term m, then the total, N, is given by

$$N = 2km = 2k(a + 2k - 1).$$

At this point, many candidates thought that just having six even *factors* was enough; however, since we need all terms to be positive, it is also required that $2k \leq m = a + 2k - 1$, so we actually need six distinct *factor pairs*, i.e. at least 11 factors.

The other common mistake, finding sums of even instead of odd numbers, arose when candidates included neither the fact that a had be odd, nor the fact that m had to be even, in their arguments.

Many candidates who found three correct numbers will be disappointed with the mark they received. The reason for this is that there were logical flaws in their explanations, mixing up necessary and sufficient conditions.

In effect, most candidates proved statements such as 'If a number can be written as six different sums of consecutive positive odds, then it can be divided by six different multiples of 4.' But this does not guarantee that every number which is divisible by six different multiples of 4 can be written in the required way. A complete solution needs to explain why the given argument is reversible, or describe how to get from a specific multiple of 4 to a sum of consecutive odd numbers. Hopefully this is a useful general lesson.

Question 2

One day Arun and Disha played several games of table tennis. At five points during the day, Arun calculated the percentage of the games played so far that he had won. The results of these calculations were exactly 30%, exactly 40%, exactly 50%, exactly 60% and exactly 70% in some order. What is the smallest possible number of games they played?

Solution

In order for Arun to be able to win 30% of N games, N must be a multiple of 10. The same holds for 70%, so the number of games must be at least 20.

But 20 is not possible: if it were, Arun would have to win either 30% of the first 10 games and then 70% of all 20, or 70% of the first 10 games and then 30% of all 20. In the first case the wins must go from $3/10$ to $14/20$ which requires 11 wins in 10 games; in the second case the wins go from $7/10$ to $6/20$ which requires -1 wins in 10 games.

So 30% and 70% alone cannot be achieved without reaching 30 games.

But 30 games is enough: for example, scores along the way could be 2/5, 3/10, 7/14, 12/20, 21/30 achieving 40%, 30%, 50%, 60%, 70% in that order.

Remark

There are many possible constructions that work with 30 games:

- 2/5, 3/10, 12/20, 21/30 with 50% won after 2, 4, 6, 14 or 16 games.
- 3/5, 3/10, 6/15, 21/30 with 50% won after 2, 4, 6 or 18 games.
- 2/5, 7/10, 9/15, 9/30 with 50% won after 2, 4, 6 or 18 games.
- 3/5, 7/10, 8/20, 9/30 with 50% won after 2, 4, 6, 14 or 16 games.

Alternative

Instead of doing the case of 20 games 'by hand', there is an algebraic alternative.

Suppose that Arun wins a of the first b games where $\frac{a}{b} = \frac{3}{10}$ and then wins m of the next n games so that $\frac{a+m}{b+n} = \frac{7}{10}$. (We can assume WLOG that 30% is achieved before 70% because otherwise we just count losses instead of wins). Since $n \geq m$, we have $\frac{a+n}{b+n} \geq \frac{7}{10}$. Multiplying up, we get $7(b+n) \leq 10(a+n) = 3b + 10n$, since $\frac{a}{b} = \frac{3}{10}$. This rearranges to $n \geq \frac{4}{3}b$. In particular, we need at least $\frac{7}{3}b$ games to achieve both 30% and a score above 70%. Since $b \geq 10$, we must have at least $70/3 > 20$ games in total. So to get exactly a score of 30% and a score of 70% we need to have a number of games that's a multiple of 10 and greater than 20, so at least 30.

Markers' comments

There were lots of good solutions to this question; there were also plenty of opportunities for partial marks. We were pleased to see that most candidates realized that you had to do two things to solve this problem: show both that you can do it in 30 games and that you cannot possibly do better.

The commonest way to go wrong was to think that the percentages had to appear in the order listed in the question. This gives a 40 game solution. If you fix an order for the percentages the problem gets much easier; you can solve it with a so-called 'greedy algorithm' where you always achieve the least percentage at the lowest possible score. In fact some candidates managed to get a solution by using this strategy over various different possible orderings. You can save time in such a solution by observations like "to get to any percentage, you must have at least 1 win and 1 loss, so we cannot be hurt by assuming 50% happens after 2 games", but you have to be careful to use provable facts and not heuristics like "It's obviously a good idea to use fractions with small denominators first". Plenty of candidates lost marks for vagueness like this.

Question 3

For each integer $0 \le n \le 11$, Eliza has exactly three identical pieces of gold that weigh 2^n grams. In how many different ways can she form a pile of gold weighing 2021 grams?

(*Two piles are different if they contain different numbers of gold pieces of some weight. The arrangement of the pieces in the piles is irrelevant.*)

Solution

Suppose there are $f(n)$ ways to choose n grams worth of gold. We begin by finding some small values of $f(n)$.

n	$f(n)$	collections of coins
1	1	{1}
2	2	{1,1}, {2}
3	2	{1,1,1}, {1,2}
4	3	{1,1,2}, {2,2}, {4}
5	3	{1,1,1,2}, {1,2,2}, {1,4}
6	4	{1,1,2,2}, {1,1,4}, {2,2,2},{2,4}

This suggests that $f(2k+1) = f(2k) = k+1$, which in turn suggests that $f(2021) = 1011$. It remains to prove that this pattern continues.

If $n = 2k$ there are either 0 or 2 pieces weighing one gram. In the first case we can halve each weight to give a way of choosing k grams, while in the second we can remove the two smallest coins and halve the remaining weights to give a way of choosing $k-1$ grams, thus $f(2k) = f(k) + f(k-1)$.

The fact that $f(2k+1) = f(k) + f(k-1)$ can be shown analogously, by removing the one or three coins weighing one gram and halving the remaining weights.

Alternatively we can observe that ways of choosing $2k+1$ grams correspond exactly to ways of choosing $2k$ grams: we simply add or remove a single one gram coin.

Having shown that $f(2k+1) = f(2k) = f(k)+f(k-1)$ we can prove that $f(2k+1) = f(2k) = k+1$ by induction using the following four steps.

$f(4k+3) = f(2k+1) + f(2k) = (k+1) + (k+1)$

$f(4k+2) = f(2k+1) + f(2k) = (k+1) + (k+1)$

$f(4k+1) = f(2k) + f(2k-1) = (k+1) + k$

$f(4k) = f(2k) + f(2k-1) = (k+1) + k.$

Thus 2021 grams of gold can be chosen in 1011 ways.

Alternative

Once we have established the recurrences $f(2k+1) = f(2k) = f(k) + f(k-1)$ we can avoid the induction and proceed 'by hand' as follows:

$$\begin{aligned}
f(2021) &= 1f(1010) + 1f(1009)\\
&= 1f(505) + 2f(504) + 1f(503)\\
&= 3f(252) + 4f(251) + 1f(250)\\
&= 3f(126) + 8f(125) + 5f(124)\\
&= 3f(63) + 16f(62) + 13f(61)\\
&= 19f(31) + 32f(30) + 13f(29)\\
&= 51f(15) + 64f(14) + 13f(13)\\
&= 115f(7) + 128f(6) + 13f(5)\\
&= 243f(3) + 256f(2) + 13f(1)\\
&= 499f(1) + 512f(0)\\
&= 1011
\end{aligned}$$

Remark

The way that the coefficients in each row add to make those in the row below, depends on the parity of the numbers in the first column. These, in turn, depend on the binary representation of 2021. This idea can be used to show that, in general, $f(n)$ can be found by taking the binary representation of n, removing the rightmost binary bit and adding 1.

Alternative

Let a_n be the number of ways of writing n as a sum of powers of 2 with each power appearing at most 3 times. We will work with the generating function of the sequence (taking $x \in (0,1)$ so everything converges).

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= \lim_{m\to\infty} \prod_{k=0}^{m} \left(1 + x^{2^k} + x^{2\cdot 2^k} + x^{3\cdot 2^k}\right)\\
&= \lim_{m\to\infty} \prod_{k=0}^{m} \frac{1 - x^{2^{k+2}}}{1 - x^{2^k}}\\
&= \lim_{m\to\infty} \frac{\left(1 - x^{2^{m+2}}\right)\left(1 - x^{2^{m+1}}\right)}{(1-x)\left(1-x^2\right)}\\
&= \frac{1}{(1-x)\left(1-x^2\right)} \underbrace{\lim_{m\to\infty}\left(1 - x^{2^{m+2}}\right)\left(1 - x^{2^{m+1}}\right)}_{=1\times 1}\\
&= \frac{1}{(1-x)\left(1-x^2\right)}
\end{aligned}$$

By differentiating $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ or using the binomial series, we get $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$. Hence:

$$\begin{aligned}
\frac{1}{(1-x)\left(1-x^2\right)} = \frac{1+x}{\left(1-x^2\right)^2} &= (1+x)\left(1 + 2x^2 + 3x^4 + \ldots\right)\\
&= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 \ldots
\end{aligned}$$

This gives $a_{2n+1} = a_{2n} = n + 1$ as before and specifically, $a_{2021} = 1011$.

Alternative

We can view Eliza's piles of gold as binary representations of 2021 where each digit is allowed to be as high as three. We can count the number of such representations by starting with the standard representation $2021 =_2 11111100101$ and working along it from left to right. (The first alternative solution essentially counts these expressions working from right to left). For each binary bit we may ask 'Can we increase this by reducing the value of the bit to the left?' Initially this feels like a 'Yes/No' question which would give a binary decision tree. However, the situation is more subtle: it is possible to increase a binary bit to be as high as 4 or even 5 and still obtain a valid sum, provided we reduce that bit appropriately in the next step. We can never have a bit of 6 or more since $6 \times 2^a > 3(2^a + 2^{a-1} + \cdots + 1)$.

Thus, as we work along 11111100101 we will have ten opportunities to increase the bit in question by 0, 2 or 4. We define $a_i(k)$ for $i = 0, 2, 4$ to be the number of decision sequences where the k^{th} bit is increased by i. These depend on the value of the $(k-1)^{\text{th}}$ or *prior* bit at that stage. If the prior bit is 5 (which happens in exactly $a_4(k-1)$ ways), then we must reduce it by 2, so the $a_4(k-1)$ contributes to $a_4(k)$ only. If the prior bit is 4 (also $a_4(k-1)$ ways), then we can reduce it by either one or two, so $a_4(k-1)$ contributes to both $a_4(k)$ and $a_2(k)$. Similarly, if the prior bit is 3 or 2 ($a_1(k-1)$ ways), then we can reduce it by zero, one or two, so $a_1(k-1)$ contributes to all three $a_i(k)$ values. If the prior bit is 1, we get a contribution (of $a_0(k-1)$) to $a_2(k)$ and $a_0(k)$, while if the prior bit is 0 we cannot reduce it, so we only get a contribution of $a_0(k-1)$ to $a_0(k)$. Putting all this together, we obtain the following table.

k	Bit in 2021	$a_0(k)$	$a_2(k)$	$a_4(k)$
1	1	1	0	0
2	1	1	1	0
3	1	2	2	1
4	1	4	4	3
5	1	8	8	7
6	1	16	16	15
7	0	32	32	31
8	0	64	63	63
9	1	127	126	126
10	0	253	253	252
11	1	506	505	505

We can discount the 505 sequences which require Eliza to use five 1g pieces, to obtain the final answer $506 + 505 = 1011$.

Alternative

It is possible to construct a one-one correspondence between the numbers $0, 1, 2, \ldots, 1010$ and the legal decompositions of 2021 as follows.

Choose $0 \leq m \leq 1010$ and write it in binary.

Now write it in binary again.

Now write $2021 - 2m$ in binary.

Sum the three binary expressions without carries to obtain an expression for 2021 as a sum of powers of 2, each occurring at most three times.

This process is reversible, since we may start with a legal decomposition of 2021, remove a copy of each power of 2 occurring once or thrice, then take one copy of each power of two now occurring twice to recover the value of m.

Markers' comments

This question was found very difficult, and more marks were earned on both the subsequent questions. The main problem was that surprisingly few candidates thought to systematically investigate how many ways piles of 1 gram, 2 grams, 3 grams and so on could be formed. They therefore missed the clear pattern, and did not think to try an inductive approach.

Many candidates started by writing 2021 in binary and then considering which of the powers of two in this representation Eliza could split; unfortunately the majority failed to appreciate that the choices made at each stage are not independent (Eliza can split an 8 gram piece if and only if she first splits at least one 16 gram piece). This meant that multiplying numbers of options together was doomed to failure. Indeed, while this 'top down' approach can be made to work, it is certainly harder than what was intended.

Despite this, there were a number of excellent solutions, and it was pleasing to see all five methods outlined above being offered successfully.

The first alternative is the most accessible, but to score highly using a numerical approach like this requires an explanation as to *why* $f(2n) = f(n) + f(n-1)$ (as well as patience and accuracy).

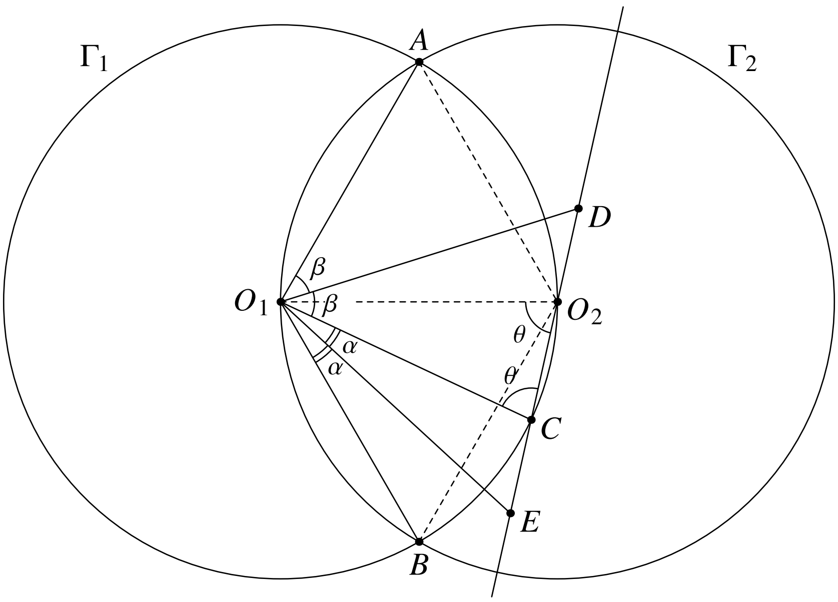
Question 4

Two circles Γ_1 and Γ_2 have centres O_1 and O_2 respectively. They pass through each other's centres and intersect at A and B. The point C lies on the minor arc BO_2 of Γ_1. The points D and E lie on the line O_2C such that $\angle AO_1D = \angle DO_1C$ and $\angle CO_1E = \angle EO_1B$. Prove that triangle DO_1E is equilateral.

(A minor arc of a circle is the shorter of the two arcs with given endpoints.)

Solution

It suffices to show that two of the angles in triangle DO_1E are equal to 60°.



The triangles AO_1O_2 and BO_1O_2 are both equilateral since their sides all equal the radius O_1O_2. They therefore have angles of 60°.

The question give us that $\angle BO_1E = \angle EO_1C$, which we will call α, and also that $\angle CO_1D = \angle DO_1A$, which we will call β. It is clear that $2\alpha + 2\beta = 120°$, so $EO_1D = \alpha + \beta = 60°$.

To complete the problem we must show that one of the other angles DO_1E is 60°.

There are a wide variety of possible approaches.

The triangle CO_1O_2 is isosceles since two of its sides are radii.

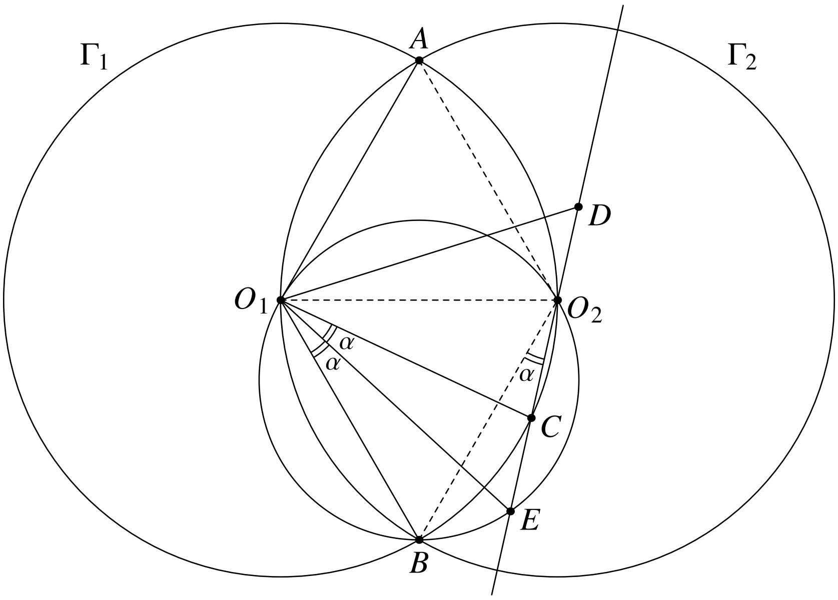
Therefore $\angle O_1O_2C = \angle O_2CO_1$ which we will call θ.

Now $2\beta = \angle O_2O_1A + \angle CO_1O_2 = 60° + (180° - 2\theta)$, so $\beta = 120 - \theta$.

Considering the sum of the angles in triangle DO_1C, we see that

$\angle O_1DC = 180° - \theta - (120° - \theta) = 60°$ as required.

Alternative

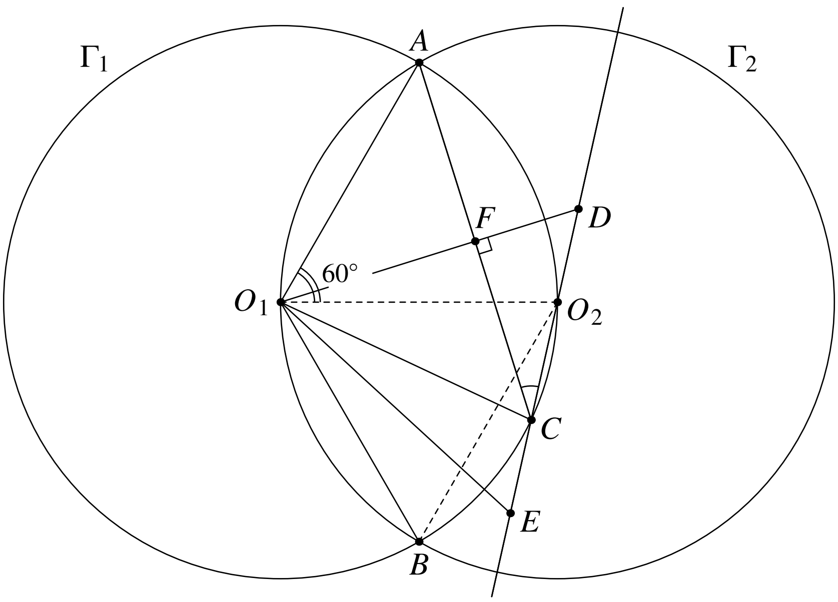


Using the fact that the angle at the centre is half the angle at the circumference, we see that $\angle BO_2C = \frac{1}{2}\angle BO_1C = \alpha$.

This shows that $\angle BO_1E = \angle BO_2E$ so BO_1O_2E is cyclic.

Now using angle in the same segment we see that $\angle O_2EO_1 = \angle O_2BO_1 = 60°$.

Alternative



Let O_1D meet AC at F. Since O_1F is the angle bisector of the isosceles triangle AO_1C, it is also the altitude, so $\angle O_1FA$ is a right angle.

Since the angle at the circumference is half the angle at the centre, we have that

$\angle O_2CA = \frac{1}{2}\angle O_1O_1A = 30°$.

Now considering the angles in triangle CDF gives

$\angle FDC = 180° - 30° - 90° = 60°$.

REMARK

It is also possible to establish that DO_1E is equilateral by showing that $O_1D = O_1E$, this can be done by, for example, establishing that triangles O_1O_2D and O_1CE are congruent.

REMARK

The condition that C lies on the minor arc O_2B is not essential to the problem. The result holds for nearly all C on the circle Γ_1. However, the wording of the problem avoids some small technical issues. In particular, if $C = O_2$ the result still holds provided we take the line O_2C to be the tangent to Γ_1 at O_2; if $C = B$ and $D = O_2$ the result only holds if we insist that the angles in the question are *directed* otherwise E can lie anywhere on O_2C and if C is diametrically opposite O_2 then the angles in the question are not defined.

MARKERS' COMMENTS

The first challenge on this problem was to draw a diagram. Many fell at this stage by assuming D and E must lie between or be coincident with O_2 and C, misinterpreting a line as a line segment.

Although most candidates then noticed that the two circles had the same radii, some missed the equilateral triangles and associated 60° angles, or failed to relate these to the angles in the problem. Candidates who did spot and use the equilateral triangles generally went on to prove that one angle in the target triangle was 60°.

There were multiple ways to proceed from here, and there were many successful solutions using congruence, cyclic quadrilaterals or simple angle chasing.

There were also a number of candidates who thought they had finished the problem, but were awarded low marks. Many of these used one or more of the following three points in their arguments: (i) the foot of the perpendicular from O_1 to DE, (ii) the point on both DE and the angle bisector of $\angle DO_1E$, (iii) the midpoint of DE. Since DO_1E turns out to be isosceles, these three points are, in fact, all the same. However, defining one of these points and assuming it has the properties of another in order to show that $O_1D = O_1E$ is a circular argument. Defining this point in terms of triangle O_2O_1C, which is clearly isosceles, led to correct solutions.

Question 5

An N-set is a set of different positive integers including a given positive integer N. Let $m(N)$ be the smallest possible mean of any N-set. For how many values of N less than 2021 is $m(N)$ an integer?

Solution

If $m(N) = m$ is an integer, then we may add m if it is missing, or remove m if it is present, without changing the mean. Moreover if m is minimal, then the N-set cannot contain an integer between m and N since removing it would reduce the mean. On the other hand, it must contain every integer below m since adding in any such integer would reduce the mean. It follows that

$$\frac{(1+2+\ldots+m-1+N)}{m} = \frac{(1+2+\ldots+m+N)}{(m+1)} = m$$

This rearranges to $N = \frac{m(m+1)}{2}$, so $m(N)$ is an integer if and only if N is a triangle number.

Since $\frac{1}{2} \times 64 \times 63 = 2016$, there are 63 triangle numbers below 2021.

Alternative

Any N-set of size n with minimal mean must be $1, 2, \ldots, n-1, N$, so consider the mean of such sets

$$f(n) = \frac{n(n-1)/2 + N}{n} = \frac{n-1}{2} + \frac{N}{n}.$$

So $f(n) - f(n-1) = \frac{1}{2} - \frac{N}{n(n-1)}$ and $f(n+1) - f(n) = \frac{1}{2} - \frac{N}{n(n+1)}$.

For minimal $f(n)$ we require $n(n-1) \le 2N \le n(n+1)$, giving $(n-1)/2 \le N/n \le (n+1)/2$. Hence $n - 1 \le f(n) \le n$. So $m(N)$ is an integer if and only if we have either equality, namely whenever $N = n(n \pm 1)/2$ is a triangle number. Now conclude as before.

Alternative

We prove that $m(T_k) = k$ for the triangle numbers as above and then observe that

$$m(N) = \frac{1}{k+1}(1+2+\ldots+k+N) > \frac{1}{k+1}(1+2+\ldots+k+(N-1)) \ge m(N-1)$$

so that $m(N)$ is a strictly increasing function. So $m(N)$ for non-triangle numbers must lie between integers.

Markers' comments

This question was attempted by many candidates, though more thought they had produced a full solution than actually had. On the basis of small examples, many candidates correctly conjectured that N has to be a triangle number. This was a good preliminary step, but not sufficient to gain any marks. To solve the problem we must show that if N is a triangle number, then $m(N)$ is an integer, and, conversely, that if $m(N)$ is an integer, then N is a triangle number. Many candidates only addressed one of these two assertions or were too vague in their reasoning.

Scripts that had clearly formed an equation describing exactly when $m(N)$ was an integer were marked generously, but candidates should remember that finding all the solutions to an equation always involves two logical steps, namely exhibiting the solutions and showing that there are no others. A variety of methods were used successfully, but most full solutions bounded $m(N)$ in terms of the second largest integer in an N-set of smallest mean.

A significant minority of candidates considered the function

$$f(n) = \frac{1 + 2 + \cdots + (n-1) + N}{n} = \frac{n-1}{2} + \frac{N}{n}$$

and then used calculus to find the minimum value of f. Unfortunately making this approach work is delicate. In particular, if $f(\alpha)$ is a minimum of f for some real α, and $f(n)$ is a minimal value of f when f's domain is restricted to the integers, then it is not necessarily the case that n and α are close together. This subtlety was missed by many candidates, and only a small proportion of those using calculus solved the problem successfully.

Question 6

Marvin has been tasked with writing down every list of integers with the following properties:

(i) The list contains 71 terms.

(ii) The first term is 1.

(iii) Every term after the first is equal to either the previous term, or the sum of all previous terms.

When Marvin is finished, how many of the lists will have a sum equal to 999,999?

Solution

Each term after the first in one of Marvin's lists is equal either to the previous term, or to the sum of all previous terms. Let's label terms of the first type with a P for 'previous', and terms of the second type with an S for 'sum'. In this way from each of Marvin's lists of 71 numbers we generate a sequence of 70 labels, each a P or S. We may assume the sequence of labels begins with an S.

Now break the sequence of 70 labels into blocks, each of which consists of an S followed by some number of P's (possibly zero). Let $b_1, b_2, \ldots, b_n$ be the lengths of the blocks in order, so $b_1 + b_2 + \ldots + b_n = 70$. [As a miniature example, if the list was 1, 1, 2, 4, 4, 12; the corresponding sequence of labels would be S, S, S, P, S; and the lengths of the blocks would be $(b_1, b_2, b_3, b_4) = (1, 1, 2, 1)$.] Given the block lengths (b_i) with sum 70 we can recover the original list of integers: the block lengths allow us to write down the 70 S/P labels in order; then the original list of 71 is recovered uniquely by starting with a 1 and generating successive terms as the repetition of the previous term [P] or the sum of all previous terms [S].

Consider one of Marvin's lists $t_1, t_2, \ldots, t_{71}$. Suppose that for some k, the labels of the terms $t_k, \ldots, t_{k+b-1}$ form a block of length b. If the sum of the terms before t_k is T, then $t_k = t_{k+1} = \cdots = t_{k+b-1} = T$. So the sum of the terms of the list up to the $(k + b - 1)^{\text{th}}$ is $T + bT = (b + 1)T$. That is, the block of length b has the effect of multiplying the sum of the list by $b + 1$. Since the first term is always 1, the sum of all the terms in one of Marvin's lists is $(b_1 + 1)(b_2 + 1) \cdots (b_n + 1)$. We are interested in those lists for which $(b_1 + 1)(b_2 + 1) \cdots (b_n + 1) = 999,999 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$ and $b_1 + b_2 + \cdots + b_n = 70$.

All the lists for which $n = 7$ and $b_1, \ldots, b_7$ is some permutation of $2, 2, 2, 6, 10, 12, 36$ satisfy this condition. There are $7!/3! = 840$ such lists. We claim there are no other possibilities. In particular, it suffices to show that for each i, the quantity $b_i + 1$ must be prime. Suppose not; then there is some sequence (b_i) of block lengths which satisfies both the multiplicative condition $\prod(b_i + 1) = 999,999$ and the additive condition $\sum b_i = 70$, and which contains a block of length b_j satisfying $b_j + 1 = ab$ for $a, b > 1$. Then we can replace b_j by $a - 1, b - 1 > 0$ to obtain a new sequence (b'_i) of block lengths satisfying the required product condition and with smaller sum (since $(ab - 1) - (a - 1) - (b - 1) = (a - 1)(b - 1) > 0$). Repeated decomposition of composite terms results in the sequence $2, 2, 2, 6, 10, 12, 36$ by the uniqueness of prime factorization of $999,999$. But this means that the sum of terms in the sequence (b_i) was greater than 70, a contradiction.

Alternative

As before, the block lengths must satisfy the equations $(b_1+1)(b_2+1)\cdots(b_n+1) = 999,999 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$, and $b_1 + b_2 + \ldots + b_n = 70$. Consider the prime factors 7, 11, 13 and 37. No bracketed term $(b_i + 1)$ in the product can be divisible by two of those primes, for in that case we would have $b_i \geq 76 > 70$. And in fact four of the bracketed terms must be exactly 7, 11, 13 and 37, else the sum of the b_i would be at least $13 + 10 + 12 + 36 > 70$. So some four blocks have lengths 6, 10, 12 and 36 respectively. The remaining blocks have total length $70 - 36 - 12 - 10 - 6 = 6$. And each of these remaining blocks must have length one less than a power of 3 because the only prime factors of 999,999 still to be accounted for are $3 \cdot 3 \cdot 3$. So they are three blocks of length 2. Therefore, the block lengths are 36, 12, 10, 6, 2, 2, 2 in some order. The solution is completed as before.

Markers' comments

For its position on the paper, this question was found approachable. Many candidates had the excellent intuition to look at blocks of repeated terms in the sequence, and a significant proportion of those were able to derive the formula for the sum of one of Marvin's lists in terms of the block lengths.

Candidates found it much harder to write down the next stage of the proof: showing that all the block lengths must be of length 'prime$-$1'. Quite a few scripts gave the 'correct answer' of 840, but it was vital to show that all other possibilities fail, so just getting 840 did not mean the problem was essentially solved. Only a handful of solutions addressed this issue well enough to score close to full marks.

Another common problem in otherwise good scripts was that, having turned the problem into algebra and counted the solutions to a relevant pair of equations, candidates often neglected to check that each solution to the equations corresponded to precisely one of Marvin's lists. Failing to engage with this subtlety attracted a 1 mark penalty.

Many candidates tried to make a 'binary' tree diagram of choices; this turned out not to be that helpful (most of the choices lead to sums that do not give 999,999). Others tried to write down all the possible values of early numbers in the sequence, but many did not think carefully about the rules, which for example showed that the fifth term could not equal either 5 or 7.

United Kingdom
Mathematics Trust

British Mathematical Olympiad
Round 1

Wednesday 16 November 2022

Instructions

1. Time allowed: $3\frac{1}{2}$ hours.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.
3. One complete solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.
4. Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.
5. The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden. *You are strongly encouraged to use geometrical instruments to construct large, accurate diagrams for geometry problems.*
6. Start each question on an official answer sheet on which there is a QR code.
7. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. Please **do not** write your name or initials on additional sheets.
8. Write on one side of the paper only. Make sure your writing and diagrams are clear and not too faint. (*Your work will be scanned for marking.*)
9. Arrange your answer sheets in question order before they are collected. If you are not submitting work for a particular problem, please remove the associated answer sheet.
10. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 8am GMT on Friday 18 November when the solutions video will be released at https://bmos.ukmt.org.uk
11. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

challenges@ukmt.org.uk
www.ukmt.org.uk

1. A road has houses numbered from 1 to n, where n is a three-digit number. Exactly $\frac{1}{k}$ of the numbers start with the digit 2, where k is a positive integer. Find the possible values of n.

2. A sequence of positive integers a_n begins with $a_1 = a$ and $a_2 = b$ for positive integers a and b. Subsequent terms in the sequence satisfy the following two rules for all positive integers n:

$$a_{2n+1} = a_{2n}a_{2n-1}, \quad a_{2n+2} = a_{2n+1} + 4.$$

Exactly m of the numbers $a_1, a_2, a_3, \ldots, a_{2022}$ are square numbers. What is the maximum possible value of m? *Note that m depends on a and b, so the maximum is over all possible choices of a and b.*

3. In an acute, non-isosceles triangle ABC the midpoints of AC and AB are B_1 and C_1 respectively. A point D lies on BC with C between B and D. The point F is such that $\angle AFC$ is a right angle and $\angle DCF = \angle FCA$. The point G is such that $\angle AGB$ is a right angle and $\angle CBG = \angle GBA$. Prove that B_1, C_1, F and G are collinear.

4. Alex and Katy play a game on an 8×8 square grid made of 64 unit cells. They take it in turns to play, with Alex going first. On Alex's turn, he writes 'A' in an empty cell. On Katy's turn, she writes 'K' in two empty cells that share an edge. The game ends when one player cannot move. Katy's score is the number of Ks on the grid at the end of the game. What is the highest score Katy can be sure to get if she plays well, no matter what Alex does?

5. For each integer $n \geq 1$, let $f(n)$ be the number of lists of different positive integers starting with 1 and ending with n, in which each term except the last divides its successor. Prove that for each integer $N \geq 1$ there is an integer $n \geq 1$ such that N divides $f(n)$.

(So $f(1) = 1$, $f(2) = 1$ and $f(6) = 3$.)

6. A circle Γ has radius 1. A line l is such that the perpendicular distance from l to the centre of Γ is strictly between 0 and 2. A frog chooses a point on Γ whose perpendicular distance from l is less than 1 and sits on that point. It then performs a sequence of jumps. Each jump has length 1 and if a jump starts on Γ it must end on l and vice versa. Prove that after some finite number of jumps the frog returns to a point it has been on before.

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British Mathematical Olympiad Round 1 2022

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

The 2022 paper

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for are full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights. If a problem has two distinct logical parts, these are sometimes marked separately and the scores added, but one part is generally considered to be more challenging. For example, in Q4 we need to show (i) that Katy can always score at least 32, no matter what Alex does and (ii) that Alex can prevent Katy from scoring more than 32, no matter what strategy she follows. Here (i) requires more sophistication, and carries 6 of the 10 marks available. In general the logical structure of the mark scheme aims to reflect the logical structure of the problem while rewarding correct arguments more generously than correct calculations.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore vital that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

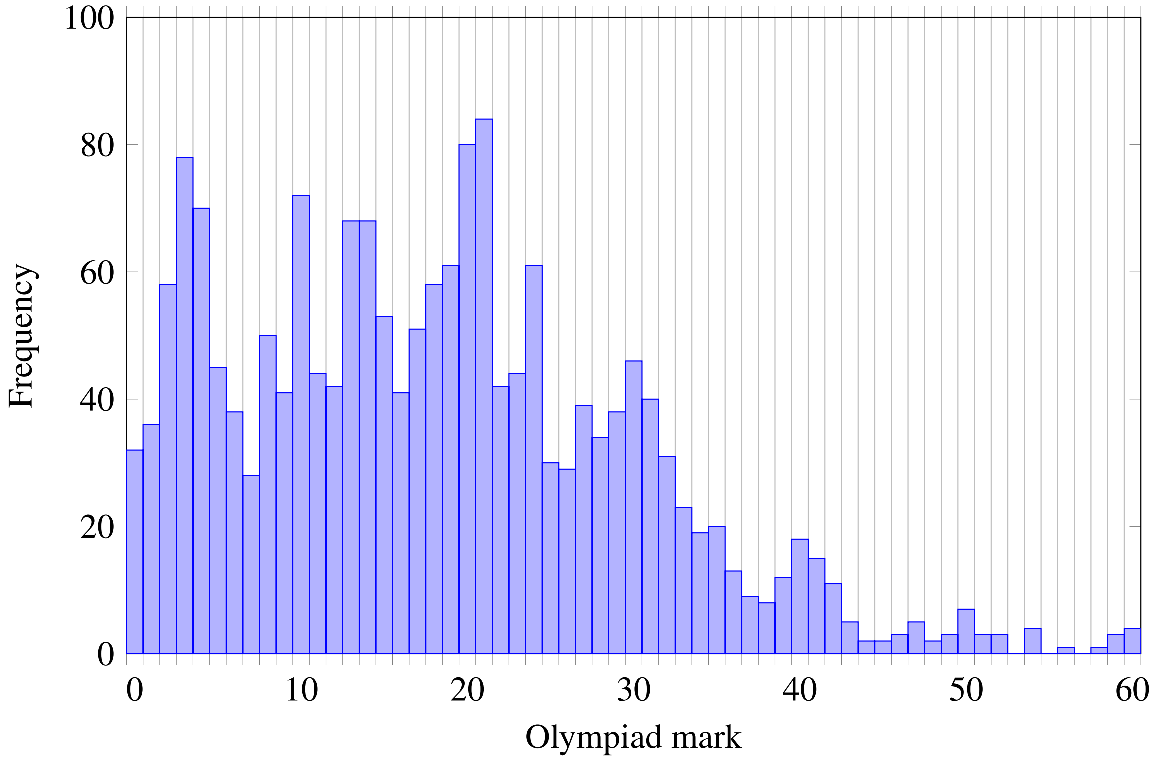
General comments

Responses to this year's paper were mixed. An encouraging number of strong candidates made substantial progress on three, or even four of the problems, while only the very best were able to score highly on either of the last two questions. At the other end of the score distribution, while almost all candidates were able to engage sensibly with question 1, quite a number failed to adequately explain their reasoning. This led to a number of fairly low scores. It is hoped that candidates who obtained correct answers without scoring full marks (which was common on questions 1, 2 and 4) will not be too discouraged. Maths Olympiads aim to test two related but separate skills: solving problems and constructing mathematical arguments. Many of those with low total scores shone on the first aspect, but lacked the requisite experience to shine on the second. Candidates are reminded that it is good practice to work in rough before writing up their solutions, and also that it is vital to reread those solutions critically. Asking 'Is it clear *from what I have written* that there are no other solutions?' might have improved a number of responses to question 1, while in question 4 relevant questions would have included 'Is it clear that this strategy works no matter what the other player does?' and also 'Have I explained how Katy can always score at least 32 *and* how Alex can always prevent her from scoring more?'.

Taking care when putting pencil to paper and not rushing was, as ever, crucial in the geometry questions where large, accurate diagrams made it far easier to see what was going on (as well as being a great help to those marking the scripts). There were an impressive number of different approaches used in successful solutions to question 3, and it was clear that question 6 was intriguing, even to those who did not solve it.

The 2022 British Mathematical Olympiad Round 1 attracted 1909 entries. The scripts were marked in Cambridge (with some remote markers) from the 2nd to the 4th of December by a team of: Eszter Backhausz, Tibor Backhausz, Sam Bealing, Emily Beatty, Jonathan Beckett, Phil Beckett, James Bell, Robin Bhattacharyya, Andrew Carlotti, Helen Chen, Sam Childs, Andrea Chlebikova, James Cranch, Laura Daniels, Stephen Darby, Wendy Dersley, Joe Devine, Paul Fannon, Richard Freeland, Thomas Frith, Carol Gainlall, Chris Garton, Sarah Gleghorn, Aleksander Goodier, Amit Goyal, Ben Handley, Sarp Hangisi, Stuart Haring, Tom Hillman, Ian Jackson, Vesna Kadelburg, Hadi Khan, Kit Kilgour, Jeremy King, Patricia King, David Knipe, Larry Lau, Rhys Lewis, Samuel Liew, Aleksandar Lishkov, Thomas Lowe, Eleanor MacGillivray, Owen Mackenzie, Sam Maltby, Przemysław Mazur, Harry Metrebian, Kian Moshiri, Oliver Murray, Joseph Myers, Daniel Naylor, Martin Orr, Jenny Owladi, Preeyan Parmar, Dominic Rowland, Adrian Sanders, Alan Slomson, Geoff Smith, Anujan Sribavananthan, Stephen Tate, Velian Velikov, Tommy Walker Mackay, Zi Wang, Henry Wilson, Tianyiwa Xie, Harvey Yau, Dominic Yeo.

Mark distribution



The thresholds for qualification for BMO2 were as follows:

Year 13: 38 marks or more.

Year 12: 36 marks or more.

Year 11: 34 marks or more.

Year 10 or below: 33 marks or more.

The thresholds for medals, Distinction and Merit were as follows:

Medal and book prize: 39 marks or more.

Distinction: 25 marks or more.

Merit: 12 marks or more.

Question 1

A road has houses numbered from 1 to n, where n is a three-digit number. Exactly $\frac{1}{k}$ of the numbers start with the digit 2, where k is a positive integer. Find the possible values of n.

Solution

We consider the three cases $n < 200$, $200 \le n < 300$ and $300 \le n$ separately.

Case I

There are eleven houses among the first 99 whose numbers begin with a 2, house 2 and houses 20 to 29 inclusive.

Thus if $n < 200$, we have $\frac{1}{k} = \frac{11}{n}$. This shows that n is a multiple of 11. There are nine possibilities, namely $n = 110, 121, 132, 143, 154, 165, 176, 187, 198$.

Case II

If $200 \le n < 300$ we may write $n = 199 + x$ for some integer $1 \le x \le 100$.

We have that $\frac{1}{k} = \frac{11+x}{199+x}$ so $k = \frac{199+x}{11+x} = 1 + \frac{188}{11+x}$. This implies that $k - 1 = \frac{188}{11+x}$ so $11 + x$ must be a factor of 188 (since $k - 1$ is an integer).

Thus, we need to find factors of 188 which are between 12 and 111 inclusive. Since, $188 = 2^2 \times 47$ its factors are $1, 2, 4, 47, 94$ and 188. Of these we need only consider 47 and 94 which give $(x, n) = (36, 235)$ and $(83, 282)$ respectively.

Case III

There are 111 houses among the first 299 whose numbers begin with a 2, the eleven already counted and those numbered 200 to 299 inclusive.

Thus if $n \ge 300$, we have $\frac{1}{k} = \frac{111}{n}$. This shows that n is a multiple of 111. There are seven possibilities, namely $n = 333, 444, 555, 666, 777, 888, 999$.

Alternative

The second, most interesting, case can also be tackled by bounding the possible values of k and then checking each in turn. Since $k - 1 = \frac{188}{11+x}$, and the left hand side decreases as x increases, we see that $\frac{188}{12} + 1 \ge k \ge \frac{188}{111} + 1$ or $3 \le k \le 16$. These fourteen values of k can be tested in turn to see that only two, $k = 3$ and $k = 5$, give rise to integer values of x and hence n. It is important with solutions of this kind to provide enough evidence of the checking to make it clear that the solutions have been found systematically, rather than by lucky guesswork.

Markers' comments

Many candidates found the solutions in the cases $100 \le n \le 199$ and $300 \le n \le 999$.

The difficult part of the problem was the case $200 \le n \le 299$; dealing carefully with this case, and finding at least some solutions in the other two ranges, was required for a script to be classified as 10-. A number of candidates found most, or even all of the solutions, but did not adequately justify why their lists were complete.

If candidates took the approach of an algebraic expression, justification for limiting the cases was required. A rearrangement to find $k - 1$ or $n - 188$ as a factor of 188 was sufficient, but a statement that $n - 188$ divides n implies that $n - 188$ divides 188 needed some justification.

If candidates took the approach of bounding k, justification of the bounds was necessary. Explicitly checking each value of k within those bounds or explanation why values did not lead to solutions was needed for full credit.

In both cases, candidates stating they have checked the cases was not enough on its own.

There were many misreads or miscounting numbers beginning in 2. Where these still led to problems of near identical difficulty, for example when candidates forgot the number '2' or the number '200', it was still possible to obtain nearly full marks.

Question 2

A sequence of positive integers a_n begins with $a_1 = a$ and $a_2 = b$ for positive integers a and b. Subsequent terms in the sequence satisfy the following two rules for all positive integers n:

$$a_{2n+1} = a_{2n}a_{2n-1}, \quad a_{2n+2} = a_{2n+1} + 4.$$

Exactly m of the numbers $a_1, a_2, a_3, \ldots, a_{2022}$ are square numbers. What is the maximum possible value of m? *Note that m depends on a and b, so the maximum is over all possible choices of a and b.*

SOLUTION

We begin by observing that no two positive square numbers differ by four. This can be seen by considering $a^2 - b^2 = (a-b)(a+b)$ which is at least 8 if it is even, or by noting that the gaps between the positive squares are the odd numbers starting with 3, no two of which sum to 4.

For $n \geq 1$ we have that a_{2n+1} and a_{2n+2} differ by 4, so at most one of them is a square.

For $n \geq 2$ we have that $a_{2n+2} = a_{2n}a_{2n-1} + 4$ which is $(a_{2n-1} + 2)^2$ and so always a square.

Thus $a_6, a_8, \ldots a_{2022}$ are all square and $a_5, a_7, \ldots a_{2021}$ are not, giving 1009 squares from a_5 to a_{2022}.

The sequence begins $a, b, ab, ab + 4$. The last two differ by 4 so are not both square which implies that at most three of the first four terms are squares. Moreover, if a and b are both square numbers, then ab will also be a square.

Thus there are at most 1012 squares among the first 2022 terms of the sequence. This is attained if (and only if) a and b are both squares.

MARKERS' COMMENTS

Many candidates studied the start of the sequence (either algebraically or using numerical examples), spotted the squares and thus obtained the correct answer of 1012. However, a number of these candidates did not adequately explain their reasoning.

A complete solution to this question generally consisted of three parts: (a) stating and proving that the even-numbered terms starting from a_6 are always square, (b) stating and proving that the odd-numbered terms starting from a_5 or a_7 are never square and finally, (c) carefully considering how many of a_1 to a_4 can be square.

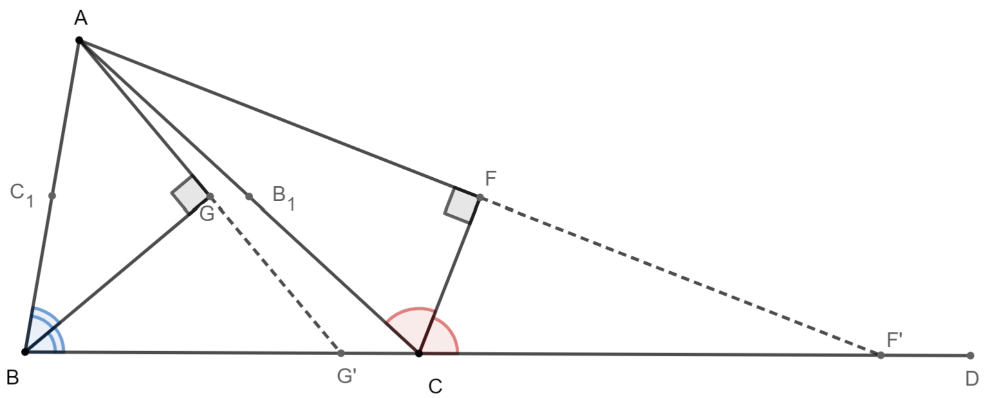
Many students made a strong start by proving (a), but ended up with low scores by not putting any note or justification of (b) on paper. Some may feel harshly penalised for omitting something they felt was obvious, but the omission leaves a gap in the logical structure of the argument. This is an essential idea that comes back in many problems, so it is worth remembering that if a question asks one to find *all* things with a certain property, one must

- find the things with said property (and show they indeed possess the property), and
- show that no other things have said property.

Question 3

In an acute, non-isosceles triangle ABC the midpoints of AC and AB are B_1 and C_1 respectively. A point D lies on BC with C between B and D. The point F is such that $\angle AFC$ is a right angle and $\angle DCF = \angle FCA$. The point G is such that $\angle AGB$ is a right angle and $\angle CBG = \angle GBA$. Prove that B_1, C_1, F and G are collinear.

Solution



Extend AF and AG to meet BC at F' and G' respectively. As a consequence of ASA, we have the following congruent triangles:

$$\begin{cases} \angle FCA = \angle F'CF \\ CF = CF \\ \angle AFC = 90^\circ = \angle CFF' \end{cases} \Rightarrow \triangle AFC \cong \triangle F'FC \qquad \begin{cases} \angle GBA = \angle G'BG \\ BG = BG \\ \angle AGB = 90^\circ = \angle BGG' \end{cases} \Rightarrow \triangle AGB \cong \triangle G'GB$$

Therefore, $AF = FF'$ and $AG = GG'$ so F, G, B_1, C_1 are the midpoints of AF', AG', AC, AB respectively. This means:

$$\frac{AF}{AF'} = \frac{AG}{AG'} = \frac{AB_1}{AC} = \frac{AC_1}{AB} = \frac{1}{2}$$

So if we consider an enlargement with scale factor $\frac{1}{2}$ at A, then the line passing through B, G', C, F' maps to a line passing through C_1, G, B_1, F proving these four points are collinear.

Alternative

Because B_1, C_1 are the midpoints of AC, AB respectively, we have $B_1C_1 \parallel BC$ so $\angle B_1C_1A = \angle B$. Also, $\angle AGB = 90^\circ$ so C_1 is the centre of circle AGB and we get:

$$\angle GC_1A = 2 \cdot \angle GBA = \angle GBA + \angle CBG = \angle B = \angle B_1C_1A$$

Thus G lies on B_1C_1.

Similarly, $\angle AFC = 90^\circ$ so B_1 is the centre of the circle AFC giving:

$$\angle FB_1A = 2\angle FCA = \angle FCA + \angle DCF = \angle DCA = 180^\circ - \angle C$$

And using $B_1C_1 \parallel BC$ we get:

$$\angle AB_1C_1 = \angle C \implies \angle AB_1C_1 + \angle FB_1A = 180^\circ$$

so F also lies on B_1C_1.

Alternative

(Sketch) We present the argument for G lying on B_1C_1. The argument for F is similar.

Let the B-internal angle bisector intersect B_1C_1 at $\tilde{G}$. We want to show G and $\tilde{G}$ are the same point. As G also lies on this angle bisector, it's sufficient to show $\angle A\tilde{G}B = 90°$.

To do this, observe that because $B_1C_1 \parallel BC$ and $B\tilde{G}$ is an angle bisector:

$$\angle \tilde{G}BC_1 = \angle CB\tilde{G} = \angle C_1\tilde{G}B \implies C_1\tilde{G} = C_1B = C_1A$$

Hence $\tilde{G}$ lies on the circle with diameter AB so $\angle A\tilde{G}B = 90°$ as desired.

Markers' comments

As with all geometry problems, a good place to start is to draw a large diagram with a compass and a ruler. Not only can this help give you ideas for how to solve the problem, it also makes it clearer to the marker where you have defined points (though you should always define them in your solution as well – not just mark them on the diagram). One thing to be careful of in this problem is not accidentally *assuming* that F, G lie on B_1C_1 at some point in your proof (particularly in a long angle chase). A helpful way to reduce the risk of doing this is to draw line B_1C_1 as a *dashed* line in your diagram.

We saw many successful solutions with some students coming up with approaches that were novel to the problem setters, which was great to see. Some students lost marks for not justifying key steps, for example: not relating B_1 being the centre of circle AFC to $\angle AFC = 90°$; or explaining why certain triangles were congruent; or justifying why $B_1C_1 \parallel BC$. There were also penalties for students who provided an argument for F lying on B_1C_1 and simply stated *the same argument also works for* G (or vice-versa). While the arguments are similar, in many approaches they weren't identical so at least some justification was required.

An approach that was employed to a varying degree of success was that of *phantom points* where we *define* a point $\tilde{G}$ to have certain properties and try to prove that in fact this is the same point as G. Properties of interest include:

(i) $\tilde{G}$ is such that $\angle CB\tilde{G} = \angle \tilde{G}BA$

(ii) $\angle A\tilde{G}B$ is a right-angle

(iii) $\tilde{G}$ lies on B_1C_1

The problem statement asks us to prove *(i)*, *(ii)* $\implies$ *(iii)* but we could equally try to show *(i)*, *(iii)* $\implies$ *(ii)* (which is shown in one of the example solutions). A common error in this approach was to, at some point in the proof, *assume* that $\tilde{G}$ had **all three** properties. Students can reduce the chance of making this mistake (and also help out the marker) by being clear at the start of their solution, what properties they are assuming and what they are trying to prove.

Question 4

Alex and Katy play a game on an 8×8 square grid made of 64 unit cells. They take it in turns to play, with Alex going first. On Alex's turn, he writes 'A' in an empty cell. On Katy's turn, she writes 'K' in two empty cells that share an edge. The game ends when one player cannot move. Katy's score is the number of Ks on the grid at the end of the game. What is the highest score Katy can be sure to get if she plays well, no matter what Alex does?

Solution

Katy's maximum score is 32.

She can achieve this by dividing the board into 2×1 rectangles at the start of the game. On each of her turns she can place two Ks into an empty one of these rectangles. On Alex's turns he can reduce the number of empty 2×1 rectangles by at most 1. This means each player will have at least sixteen turns, giving Katy a final score of at least 32.

Alex can prevent Katy from scoring more than 32 as follows. He colours the board in the standard chessboard pattern and then promises to only ever place As on cells that are (say) black. This means that Alex and Katy each cover exactly one black cell each turn, so they can take at most 16 turns each, giving Katy a maximum possible score of 32.

Remark

The delicate thing in questions of this type is to ensure that the strategies described for each player do not depend on the other playing following a particular 'sensible' strategy.

Remark

There are other ways to describe good strategies for Katy. She might, for example, divide the board into sixteen 2×2 squares and whenever Alex places a first A in such a square, use her next move to place two Ks in that square.

Markers' comments

A complete solution here has two parts: (**A**) a strategy for Alex to stop Katy getting more than 32 cells, and (**K**) a strategy for Katy to get at least 32 cells. There were good attempts at both, though (**A**) was more popular. All solutions for (**A**) involved colouring the board in some way, but other strategies could produce a weaker bound than 32. All solutions for (**K**) involved dividing up the board into smaller regions in some way; again, other strategies could produce weaker bounds.

The key logical difficulty here is that (**A**) and (**K**) both need to work whatever the other player does. Lots of candidates found a strategy for Alex and then tried to show Katy could get 32 cells *when playing against that particular strategy.* The problem is that a cleverer Alex might find a better strategy. Other candidates made the same mistake the other way round, showing how Alex should play against a particular strategy by Katy.

A common small mistake of this type came in Katy's strategy based on 2×2 boards. If you're describing a general strategy, 'always follow Alex into the 2×2 board he took the first square of' is not enough, because Alex could return to a 2×2 board Katy already used. This looks like a bad move for Alex, so many candidates ignored it, but a general strategy must cover all cases.

Question 5

For each integer $n \geq 1$, let $f(n)$ be the number of lists of different positive integers starting with 1 and ending with n, in which each term except the last divides its successor. Prove that for each integer $N \geq 1$ there is an integer $n \geq 1$ such that N divides $f(n)$.

(So $f(1) = 1$, $f(2) = 1$ and $f(6) = 3$.)

SOLUTION

We start by noting that $f(1) = 1$ since (1) is the only possible list.

For any given $n > 1$, we can count the number of allowed lists according to the penultimate number, d, in the list. Clearly d is a factor of n and the number of lists of the form $(\ldots, d, n)$ is $f(d)$.

Thus

$$f(n) = \sum_{d|n,\ \ d\neq n} f(d) \tag{†}$$

For example, $f(2) = f(1) = 1$, $f(4) = f(2) + f(1) = 2$, $f(8) = f(4) + f(2) + f(1) = 4$.

Further experimentation with small cases leads to the conjecture that $f(p^m) = 2^{m-1}$ for any prime p and positive integer k.

This claim can be proved by induction on m.

In clearly holds for $m = 1$ as $f(p) = 1$.

If we assume it holds for all integers m up to some integer k we may consider $f(p^{k+1})$.

$$\begin{aligned} f(p^{k+1}) &= f(p^k) + f(p^{k-1}) + \cdots + f(p) + f(1) \\ &= (2^{k-1} + 2^{k-2} + \cdots + 1) + 1 \\ &= (2^k - 1) + 1 \end{aligned}$$

Here the first line uses the recurrence (†), the second uses the inductive hypothesis and the third uses the sum of a geometric progression.

This is enough to prove the claim for $m = k + 1$ and thus, by induction, for all $m \geq 1$.

Next we claim that $f(p^m q) = (m + 2) \times 2^{m-1}$ for different primes p, q with $m \geq 1$. Again we proceed by induction on m.

If $m = 1$ we must check that $f(pq) = f(p) + f(q) + f(1) = 1 + 1 + 1 = (1 + 2) \times 2^0$ as required.

Now we assume the claims for all $m \leq k$ and consider $f(p^{k+1}q)$.

The key thing note is that the proper factors of $p^{k+1}q$ are $p^k q$, p^{k+1} and a collection of other factors which are precisely the proper factors of $p^k q$. Thus

$$\begin{aligned} f(p^{k+1}q) &= f(p^k q) + f(p^{k+1}) + f(p^k q) \\ &= (k + 2)2^{k-1} + 2^k + (k + 2)2^{k-1} \\ &= (k + 3)2^k \end{aligned}$$

Here the first line uses (†) (rather cunningly) and the second uses the inductive hypothesis and our first claim.

Now for a given N we may choose any pair of primes p, q and set $n = p^{N-2}q$.

Alternative

Given a legal sequence $1 = a_0, a_1, \ldots, a_k = n$ where $a_i | a_{i+1}$ and $a_i < a_{i+1}$ we can let $d_i = a_i/a_{i-1}$ for $1 \leq i \leq k$ to obtain the sequence $d_1, d_2, \ldots, d_k$ which in an ordered factorisation of n.

We can show that $n = p^m q$ has $(m+2) \times 2^{m-1}$ ordered factorisations by counting them directly.

We start by writing $n = p \times p \times \cdots \times q \times \cdots \times p$ and change some of the $\times$ signs into commas to obtain a list $d_1, d_2, \ldots$. To avoid any over counting we can insist that the q either appears at the far right of the list of or directly before a comma. This gives two cases. If we start with $n = p \times p \times \cdots \times q$ we may convert any subset of the m multiplication signs into commas. This gives 2^m options. If, on the other hand, q is not the last prime in our initial factorisation of n must choose its position in one of m ways and replace the $\times$ directly after it with a comma. We now choose any subset of the remaining $m-1$ multiplication signs to convert to commas in one of 2^{m-1} ways.

This gives a final count of $2^m + m2^{m-1} = (m+2)2^{m-1}$ as required.

Alternative

We can also establish that $n = p^m q$ has $(m+2)2^{m-1}$ ordered factorisations by counting these factorisations according to the number of factors (that is, according to length of the list $d_1, d_2, \ldots$).

To obtain a factorisation with k factors we either divide the m copies of p into k blocks and add the q to one of them, or divide the copies of p into $k-1$ blocks and add the q as a block on its own.

The first option can be done in $k\binom{m-1}{k-1}$ ways since we must place $k-1$ 'dividers' into the $m-1$ spaces between the ps. The second option can be done in $k\binom{m-1}{k-2}$ ways since, having split the ps into $k-1$ blocks, there are k places to insert the single q.

Now the final count, $f(p^m q)$, can be simplified using standard combinatorial identities as follows.

$$\begin{aligned} f(p^m q) &= \sum_k k\left(\binom{m-1}{k-1} + \binom{m-1}{k-2}\right) \\ &= \sum_k k\binom{m}{k-1} \\ &= \sum_k (k-1)\binom{m}{k-1} + \sum_k \binom{m}{k-1} \\ &= m2^{m-1} + 2^m \end{aligned}$$

as required.

MARKERS' COMMENTS

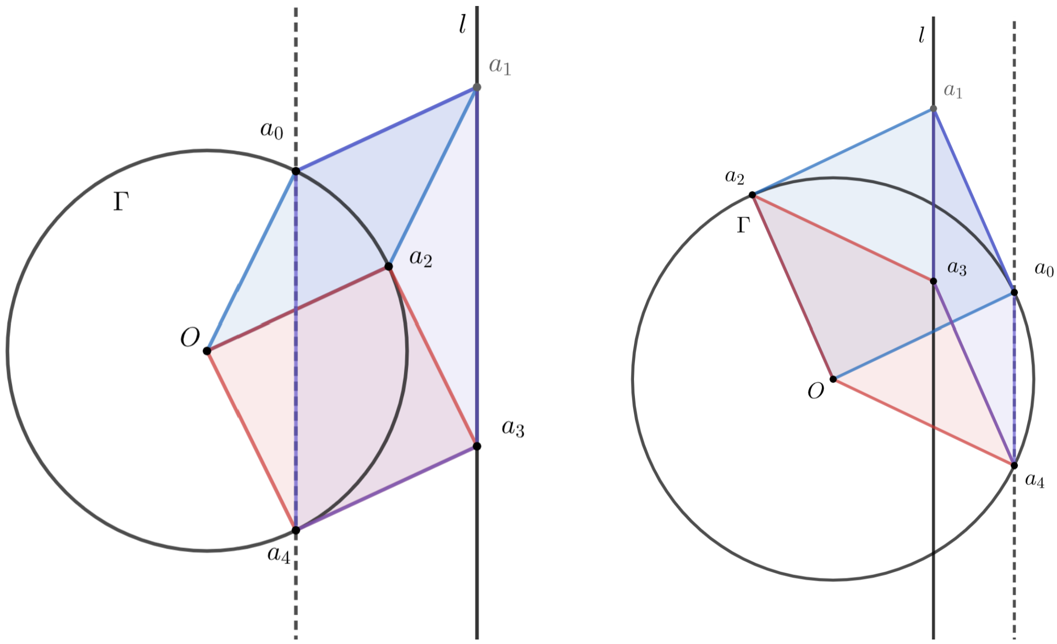
A Maths Olympiad coach once said "all construction problems are trivial". In some ways this can be true, in that the contestant has free choice to come up with any workable construction, but this freedom of choice frequently leads to candidates pursuing unhelpful directions. So it turned out to be with question 5 this year, where large numbers of candidates tried multiplying distinct primes together, and relatively few hit on the workable idea of $p^k q$. Unfortunately for the majority, the sequence of values that arises from using k distinct primes (called the Ordered Bell Numbers) cannot lead to a solution as they are all odd, amongst other issues. Many students only considered lists of length 3, or only considered the prime factors of n, both of which made the distinct prime approach appear to work, but unfortunately these solutions broke down when the other factors were included.

For those who did pursue the $p^k q$ approach, there were a wide variety of ways of finding the recurrence $f(p^k q) = 2f(p^{k-1}q) + 2^{k-1}$, with some being more number theoretic and others more combinatorial. For those who got to this point, the final hurdle that most commonly caused difficulties was noticing that the expression $(k+2)2^{k-1}$ isn't always a multiple of $k+2$, in particular when $k = 0$. This is not a big problem, as the $n = 2$ case is easily dealt with, but it did need a mention.

Question 6

A circle Γ has radius 1. A line l is such that the perpendicular distance from l to the centre of Γ is strictly between 0 and 2. A frog chooses a point on Γ whose perpendicular distance from l is less than 1 and sits on that point. It then performs a sequence of jumps. Each jump has length 1 and if a jump starts on Γ it must end on l and vice versa. Prove that after some finite number of jumps the frog returns to a point it has been on before.

Solution



Let O be the centre of Γ and let the points sat on by the frog be $a_0, a_1, a_2, \ldots$ in that order.

If the Frog ever jumps straight back to the point it just left we are done, so from now on we may assume this is not the case. That is, that $a_{i+2} \neq a_i$.

Since all the jumps have length 1, and a_0, a_2 are on Γ, we see that $Oa_0a_1a_2$ is a rhombus, so a_0a_1 is parallel to $a_2 0$.

Similarly $Oa_2a_3a_4$ is a rhombus, so a_3a_4 is parallel to $a_2 0$.

This means that $a_0a_1a_3a_4$ is a parallelogram since $|a_0a_1| = |a_3a_4| = 1$.

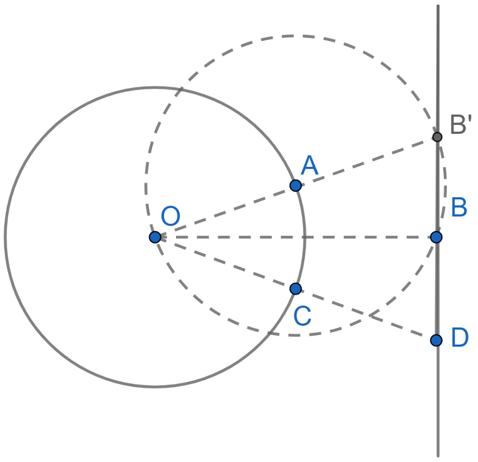
Now a_1 and a_3 are both on l, so the line a_0a_4 is parallel to l.

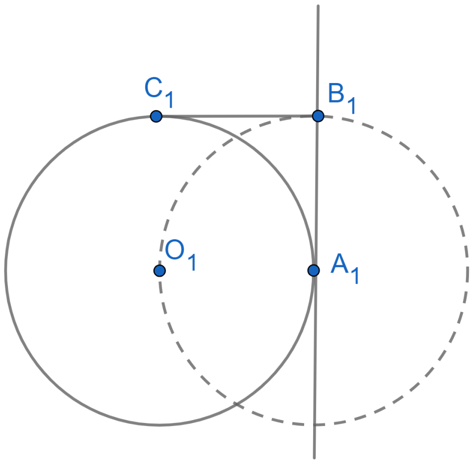
Adding 4 to all the indices and repeating the above argument, we see that the line a_4a_8 is also parallel to l. However, the unique line through a_4 parallel to l intersects Γ in (at most) one other point, so $a_0 = a_8$. Thus the frog revisits a previously visited point after at most eight jumps.

There is another way to clinch the argument. Once you establish that the vectors $\overline{a_0a_1}$, $\overline{Oa_2}$ and $\overline{a_4a_3}$ are equal, translation by this vector carries triangle $\triangle a_0Oa_4$ to the congruent triangle $\triangle a_1a_2a_3$ and so the line a_0a_4 is parallel to the line a_1a_3 which is l.

Remark

It is possible when the frog jumps from a_i to a_{i+1}, it finds that there is only one point which is a candidate for a_{i+2}. (This occurs when the circle radius 1 centre a_{i+1} is tangent to the other landing zone at a_{i+2}, and so either Oa_ia_{i+1} are collinear or a_ia_{i+1} is perpendicular to l). This case is covered by the remark that if $a_{i+2} = a_i$ then we are done. The figure shows examples of such configurations. The left diagram illustrates infinitely many configurations, as you vary the distance between O and the line l.

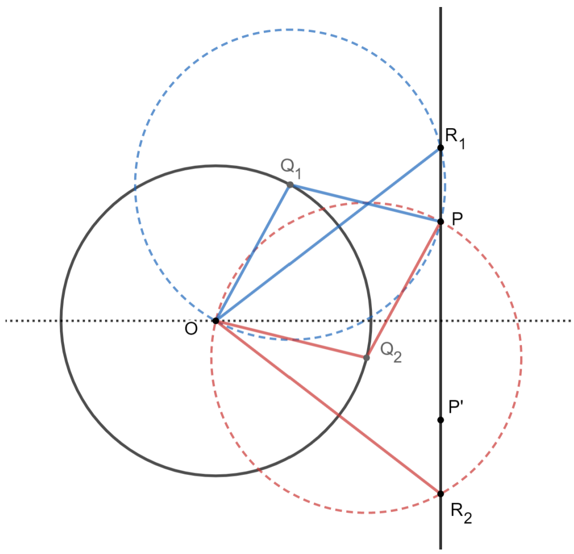


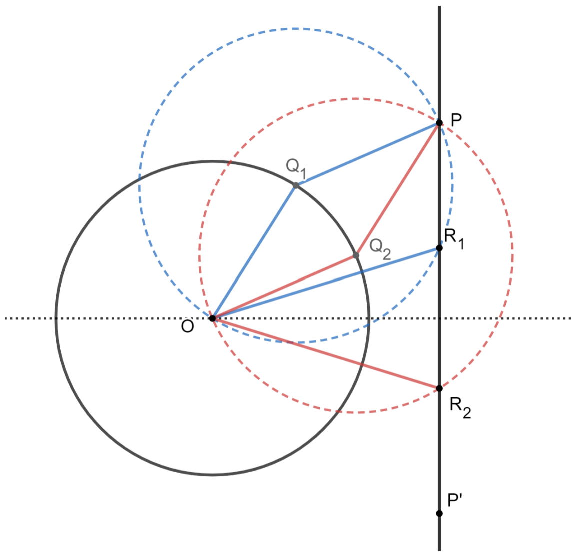


Remark

There are two possible versions of the diagram, one where l intersects Γ and one where it does not, but the argument given works without adjustment in both.

Alternative





Let O be the centre of Γ, let l_2 be the line through O perpendicular to l and let P be a point on l. From P, the frog can jump to two points Q_1, Q_2 on Γ. From Q_i the frog can jump to a single point $R_i \neq P$.

We have that P, R_i, O lie on a circle with centre Q_i. We claim that OR_1R_2 is isosceles with apex O, which implies that R_1 and R_2 are reflections of each other in l_2. To prove the claim we note that in the rhombus OQ_1PQ_2 the angles at Q_1 and Q_2 are equal. We then use the fact that the angle at the centre is double the angle at the circumference. There are essentially two cases depending on whether or not P lies between R_1 and R_2. (These can be dealt with simultaneously using directed angles.)

Now let P' be the reflection of P in M. By symmetry of l and Γ in l_2, we see that if the frog starts at P', after two moves it is at one of $\{P', R_1, R_2\}$.

Thus if $S = \{P, P', R_1, R_2\}$ then if the frog starts at a point in S, after two jumps it is still at a point in S. Thus after at most 8 jumps, it returns to a point it has already visited by the pigeonhole principle.

ALTERNATIVE

One can use isogonal conjugacy to solve this problem. The circumcentre of triangle a_1Oa_3 is a_2. Let this triangle have orthocentre H, the isogonal conjugate of a_2. Thus OH is perpendicular to l and passes through O. The angles around O and the fact that $|Oa_0| = |Oa_4|$ force a_0 and a_4 to be mutual reflections in the line through O which is perpendicular to l. Therefore $a_0 = a_8$ by restarting the frog at a_4.

This is an echo of the classical explanation of isogonal conjugacy: if P is a point in the plane of triangle $\triangle ABC$, then its isogonal conjugate P^* is on the perpendicular bisector of the line segment joining the reflections of P of any pair of sides of $\triangle ABC$. Thus P^* is the circumcentre of the triangle with vertices which are the reflections of P in the three sides of $\triangle ABC$.

ALTERNATIVE

Other solutions are possible. You can establish the parallelism of the two parallel lines by angle chasing arguments, often involving the angles around O. There are also arguments where you set up a geometric configuration with reflection symmetry about a line through the centre of Γ which is perpendicular to l, and then argue that it is a legal path for the frog and so that is where the frog must go. Attempts using this method should be scrutinized particularly carefully, since it is very easy to make unjustified assumptions using this approach.

MARKERS' COMMENTS

The vast majority of candidates did not submit serious attempts at this question, and presumably even well prepared candidates would not have seen a question quite like this one before. Here having the patience to draw a number of accurate figures, and then to invest some time in staring at them, is probably the best way to try and spot what is going on. Noticing the relevant parallel lines, rhombuses and symmetry is certainly the harder part of the problem; putting together the details at the end is straightforward in comparison.

BRITISH MATHEMATICAL OLYMPIAD

Round 2 : Thursday 13th February 1992

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than trying all four problems.*
- *The use of rulers and compasses is allowed, but calculators are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

Before the end of February, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (on 2-5 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team for this summer's International Mathematical Olympiad (to be held in Moscow, 10-21 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session before leaving for Moscow.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Let p be an odd prime number. Prove that there are unique positive integers x, y such that $x^2 = y(y+p)$, and give the formulae for x and y in terms of p.

2. Let a, b, c, d be positive real numbers. Prove that

$$\frac{12}{a+b+c+d} \le \frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{a+d}+\frac{1}{b+c}+\frac{1}{b+d}+\frac{1}{c+d} \le \frac{3}{4}\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\Big).$$

3. The circumcircle of the triangle ABC has a radius R satisfying

$$AB^2 + AC^2 = BC^2 - R^2.$$

Prove that the angles of the triangle are uniquely determined, and state the values for the angles.

4. Dwarfish social life in the *Land Under the Mountain* is based on 'visiting' which has its own dwarfish rules. Two dwarves may visit each other only if they are acquainted, but there are three levels of friendship.

   Doorstep friends may visit each other and talk for as long as they like, but must remain on the doorstep, never crossing the threshold.

   Tea friends may cross each others' threshold, and even share a cup of tea, but must never stay for supper.

   Supper friends may visit each other and stay as long as they like, sharing ale, supper and a fireside chat.

   Each dwarf has exactly one friend of each kind, and the community is structured so that every pair A, B of dwarves is linked by a chain of acquaintances of various kinds so that A knows someone, who knows someone, who knows someone, ..., who knows B.

   (i) Prove that the number of dwarves must be an even number exceeding 2, and that, corresponding to each even $n > 2$, there may be such a community of n dwarves.

   (ii) The evil orcs want to isolate one group of dwarves from the rest of the community. To do this they must destroy some set of friendships. Suppose that they were to succeed in this by destroying

   F_d doorstep friendships, F_t tea friendships, F_s supper friendships,

   and that all of these friendships must be destroyed to isolate that group. Prove that F_d, F_t, F_s would have to be either all even or all odd.

British Mathematical Olympiad

Round 2 : Thursday, 11 February 1993

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than trying all four problems.*
- *The use of rulers and compasses is allowed, but calculators are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

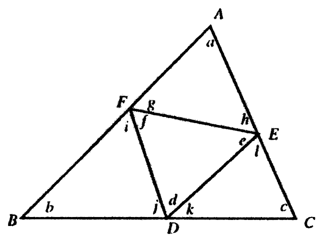
Before March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (on 15-18 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team for this summer's International Mathematical Olympiad (to be held in Istanbul, Turkey, July 13–24) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session before leaving for Istanbul.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. We usually measure angles in degrees, but we can use any other unit we choose. For example, if we use $30°$ as a new unit, then the angles of a $30°$, $60°$, $90°$ triangle would be equal to 1, 2, 3 new units respectively.

   The diagram shows a triangle ABC with a second triangle DEF inscribed in it. All the angles in the diagram are whole number multiples of some new (unknown unit); their sizes $a, b, c, d, e, f, g, h, i, j, k, \ell$ with respect to this new angle unit are all distinct.

   Find the smallest possible value of $a+b+c$ for which such an angle unit can be chosen, and mark the corresponding values of the angles a to ℓ in the diagram.

2. Let $m = (4^p - 1)/3$, where p is a prime number exceeding 3. Prove that 2^{m-1} has remainder 1 when divided by m.

3. Let P be an internal point of triangle ABC and let α, β, γ be defined by
$$\alpha = \angle BPC - \angle BAC, \ \beta = \angle CPA - \angle CBA, \ \gamma = \angle APB - \angle ACB.$$
   Prove that
$$PA \frac{\sin \angle BAC}{\sin \alpha} = PB \frac{\sin \angle CBA}{\sin \beta} = PC \frac{\sin \angle ACB}{\sin \gamma}.$$

4. The set $Z(m,n)$ consists of all integers N with mn digits which have precisely n ones, n twos, n threes, $\ldots$, n ms. For each integer $N \in Z(m,n)$, define $d(N)$ to be the sum of the absolute values of the differences of all pairs of consecutive digits. For example, $122313 \in Z(3,2)$ with $d(122313) = 1+0+1+2+2 = 6$. Find the average value of $d(N)$ as N ranges over all possible elements of $Z(m,n)$.

British Mathematical Olympiad

Round 2 : Thursday, 24 February 1994

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than trying all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (on 7-10 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Hong Kong, 8-20 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session before leaving for Hong Kong.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Find the first integer $n > 1$ such that the average of
$$1^2, 2^2, 3^2, \ldots, n^2$$
is itself a perfect square.

2. How many different (i.e. pairwise non-congruent) triangles are there with integer sides and with perimeter 1994?

3. AP, AQ, AR, AS are chords of a given circle with the property that
$$\angle PAQ = \angle QAR = \angle RAS.$$
Prove that
$$AR(AP + AR) = AQ(AQ + AS).$$

4. How many perfect squares are there (mod 2^n)?

British Mathematical Olympiad

Round 2 : Thursday, 16 February 1995

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (30 March – 2 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Toronto, Canada, 13–23 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session 2–6 July before leaving for Canada.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Find all triples of positive integers (a, b, c) such that
$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=2.$$

2. Let ABC be a triangle, and D, E, F be the midpoints of BC, CA, AB, respectively.

   Prove that $\angle DAC = \angle ABE$ if, and only if, $\angle AFC = \angle ADB$.

3. Let a, b, c be real numbers satisfying $a < b < c$, $a + b + c = 6$ and $ab + bc + ca = 9$.

   Prove that $0 < a < 1 < b < 3 < c < 4$.

4. (a) Determine, with careful explanation, how many ways $2n$ people can be paired off to form n teams of 2.

   (b) Prove that $\{(mn)!\}^2$ is divisible by $(m!)^{n+1}(n!)^{m+1}$ for all positive integers m, n.

British Mathematical Olympiad

Round 2 : Thursday, 15 February 1996

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (28–31 March). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in New Delhi, India, 7–17 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session 30 June–4 July before leaving for India.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Determine all sets of non-negative integers x, y and z which satisfy the equation
$$2^x + 3^y = z^2.$$

2. The sides a, b, c and u, v, w of two triangles ABC and UVW are related by the equations
$$u(v + w - u) = a^2,$$
$$v(w + u - v) = b^2,$$
$$w(u + v - w) = c^2.$$
Prove that triangle ABC is acute-angled and express the angles U, V, W in terms of A, B, C.

3. Two circles S_1 and S_2 touch each other externally at K; they also touch a circle S internally at A_1 and A_2 respectively. Let P be one point of intersection of S with the common tangent to S_1 and S_2 at K. The line PA_1 meets S_1 again at B_1, and PA_2 meets S_2 again at B_2. Prove that B_1B_2 is a common tangent to S_1 and S_2.

4. Let a, b, c and d be positive real numbers such that
$$a + b + c + d = 12$$
and
$$abcd = 27 + ab + ac + ad + bc + bd + cd.$$
Find all possible values of a, b, c, d satisfying these equations.

British Mathematical Olympiad

Round 2 : Thursday, 27 February 1997

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
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In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (10-13 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Mar del Plata, Argentina, 21-31 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session in late June or early July before leaving for Argentina.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Let M and N be two 9-digit positive integers with the property that if **any** one digit of M is replaced by the digit of N in the corresponding place (e.g., the 'tens' digit of M replaced by the 'tens' digit of N) then the resulting integer is a multiple of 7.

   Prove that any number obtained by replacing a digit of N by the corresponding digit of M is also a multiple of 7.

   Find an integer $d > 9$ such that the above result concerning divisibility by 7 remains true when M and N are two d-digit positive integers.

2. In the acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

3. Find the number of polynomials of degree 5 with **distinct** coefficients from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are divisible by $x^2 - x + 1$.

4. The set $S = \{1/r : r = 1, 2, 3, \ldots\}$ of reciprocals of the positive integers contains arithmetic progressions of various lengths. For instance, $1/20, 1/8, 1/5$ is such a progression, of length 3 (and common difference $3/40$). Moreover, this is a *maximal progression* in S of length 3 since it cannot be extended to the left or right within S ($-1/40$ and $11/40$ not being members of S).

   (i) Find a maximal progression in S of length 1996.

   (ii) Is there a maximal progression in S of length 1997?

British Mathematical Olympiad

Round 2 : Thursday, 26 February 1998

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (2-5 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Taiwan, 13-21 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session in early July before leaving for Taiwan.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. A booking office at a railway station sells tickets to 200 destinations. One day, tickets were issued to 3800 passengers. Show that

   (i) there are (at least) 6 destinations at which the passenger arrival numbers are the same;

   (ii) the statement in (i) becomes false if '6' is replaced by '7'.

2. A triangle ABC has $\angle BAC > \angle BCA$. A line AP is drawn so that $\angle PAC = \angle BCA$ where P is inside the triangle. A point Q outside the triangle is constructed so that PQ is parallel to AB, and BQ is parallel to AC. R is the point on BC (separated from Q by the line AP) such that $\angle PRQ = \angle BCA$.

   Prove that the circumcircle of ABC touches the circumcircle of PQR.

3. Suppose x, y, z are positive integers satisfying the equation
$$\frac{1}{x} - \frac{1}{y} = \frac{1}{z},$$
   and let h be the highest common factor of x, y, z.

   Prove that $hxyz$ is a perfect square.

   Prove also that $h(y - x)$ is a perfect square.

4. Find a solution of the simultaneous equations
$$xy + yz + zx = 12$$
$$xyz = 2 + x + y + z$$
   in which all of x, y, z are positive, and prove that it is the only such solution.

   Show that a solution exists in which x, y, z are real and distinct.

British Mathematical Olympiad

Round 2 : Thursday, 25 February 1999

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (8-11 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Bucharest, Romania, 13-22 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session (3-7 July) in Birmingham before leaving for Bucharest.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. For each positive integer n, let S_n denote the set consisting of the first n natural numbers, that is
$$S_n = \{1, 2, 3, 4, \ldots, n-1, n\}.$$
   (i) For which values of n is it possible to express S_n as the union of two non-empty disjoint subsets so that the elements in the two subsets have equal sums?

   (ii) For which values of n is it possible to express S_n as the union of three non-empty disjoint subsets so that the elements in the three subsets have equal sums?

2. Let $ABCDEF$ be a hexagon (which may not be regular), which circumscribes a circle S. (That is, S is tangent to each of the six sides of the hexagon.) The circle S touches AB, CD, EF at their midpoints P, Q, R respectively. Let X, Y, Z be the points of contact of S with BC, DE, FA respectively. Prove that PY, QZ, RX are concurrent.

3. Non-negative real numbers p, q and r satisfy $p + q + r = 1$. Prove that
$$7(pq + qr + rp) \leq 2 + 9pqr.$$

4. Consider all numbers of the form $3n^2 + n + 1$, where n is a positive integer.

   (i) How small can the sum of the digits (in base 10) of such a number be?

   (ii) Can such a number have the sum of its digits (in base 10) equal to 1999?

British Mathematical Olympiad

Round 2 : Wednesday, 23 February 2000

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (6-9 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in South Korea, 13-24 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session before leaving for South Korea.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. Prove that the triangles MNP and MNQ have equal areas.

2. Given that x, y, z are positive real numbers satisfying $xyz = 32$, find the minimum value of
$$x^2 + 4xy + 4y^2 + 2z^2.$$

3. Find positive integers a and b such that
$$(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 49 + 20\sqrt[3]{6}.$$

4. (a) Find a set A of ten positive integers such that no six distinct elements of A have a sum which is divisible by 6.

   (b) Is it possible to find such a set if "ten" is replaced by "eleven"?

British Mathematical Olympiad

Round 2 : Tuesday, 27 February 2001

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (8-11 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend another meeting in Cambridge (probably 26-29 May). The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Washington DC, USA, 3-14 July) will then be chosen.

Do not turn over until **told to do so.**

2001 British Mathematical Olympiad
Round 2

1. Ahmed and Beth have respectively p and q marbles, with $p > q$.

   Starting with Ahmed, each in turn gives to the other as many marbles as the other already possesses. It is found that after $2n$ such transfers, Ahmed has q marbles and Beth has p marbles.

   Find $\dfrac{p}{q}$ in terms of n.

2. Find all pairs of integers (x, y) satisfying
$$1 + x^2y = x^2 + 2xy + 2x + y.$$

3. A triangle ABC has $\angle ACB > \angle ABC$.

   The internal bisector of $\angle BAC$ meets BC at D.

   The point E on AB is such that $\angle EDB = 90°$.

   The point F on AC is such that $\angle BED = \angle DEF$.

   Show that $\angle BAD = \angle FDC$.

4. N dwarfs of heights $1, 2, 3, \ldots, N$ are arranged in a circle. For each pair of neighbouring dwarfs the positive difference between the heights is calculated; the sum of these N differences is called the "total variance" V of the arrangement.

   Find (with proof) the maximum and minimum possible values of V.

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British Mathematical Olympiad

Round 2 : Tuesday, 26 February 2002

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (4 – 7 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend another meeting in Cambridge. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Glasgow, 22 –31 July) will then be chosen.

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2002 British Mathematical Olympiad
Round 2

1. The altitude from one of the vertices of an acute-angled triangle ABC meets the opposite side at D. From D perpendiculars DE and DF are drawn to the other two sides. Prove that the length of EF is the same whichever vertex is chosen.

2. A conference hall has a round table wth n chairs. There are n delegates to the conference. The first delegate chooses his or her seat arbitrarily. Thereafter the $(k+1)$th delegate sits k places to the right of the kth delegate, for $1 \leq k \leq n-1$. (In particular, the second delegate sits next to the first.) No chair can be occupied by more than one delegate.

   Find the set of values n for which this is possible.

3. Prove that the sequence defined by

$$y_0 = 1, \qquad y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right), \quad (n \geq 0)$$

   consists only of integers.

4. Suppose that $B_1, \ldots, B_N$ are N spheres of unit radius arranged in space so that each sphere touches exactly two others externally. Let P be a point outside all these spheres, and let the N points of contact be $C_1, \ldots, C_N$. The length of the tangent from P to the sphere B_i $(1 \leq i \leq N)$ is denoted by t_i. Prove the product of the quantities t_i is not more than the product of the distances PC_i.

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British Mathematical Olympiad

Round 2 : Tuesday, 25 February 2003

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (3-6 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Japan, 7-19 July) will then be chosen.

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2003 British Mathematical Olympiad
Round 2

1. For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n.
   Determine all triples x, y, z of distinct positive integers satisfying
   (i) x, y, z are in arithmetic progression, and
   (ii) $p(xyz) \leq 3$.

2. Let ABC be a triangle and let D be a point on AB such that $4AD = AB$. The half-line ℓ is drawn on the same side of AB as C, starting from D and making an angle of θ with DA where $\theta = \angle ACB$. If the circumcircle of ABC meets the half-line ℓ at P, show that $PB = 2PD$.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of the set $\mathbb{N}$ of all positive integers.
   (i) Show that there is an arithmetic progression of positive integers $a, a+d, a+2d$, where $d > 0$, such that
   $$f(a) < f(a+d) < f(a+2d).$$
   (ii) Must there be an arithmetic progression $a, a+d, \ldots, a+2003d$, where $d > 0$, such that
   $$f(a) < f(a+d) < \ldots < f(a+2003d)?$$
   [*A permutation of* $\mathbb{N}$ *is a one-to-one function whose image is the whole of* $\mathbb{N}$*; that is, a function from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that for all* $m \in \mathbb{N}$ *there exists a unique* $n \in \mathbb{N}$ *such that* $f(n) = m$*.*]

4. Let f be a function from the set of non-negative integers into itself such that for all $n \geq 0$
   (i) $\left(f(2n+1)\right)^2 - \left(f(2n)\right)^2 = 6f(n) + 1$, and
   (ii) $f(2n) \geq f(n)$.
   How many numbers less than 2003 are there in the image of f?

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British Mathematical Olympiad

Round 2 : Tuesday, 24 February 2004

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (1-5 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Athens, 9-18 July) will then be chosen.

Do not turn over until **told to do so.**

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2004 British Mathematical Olympiad
Round 2

1. Let ABC be an equilateral triangle and D an internal point of the side BC. A circle, tangent to BC at D, cuts AB internally at M and N, and AC internally at P and Q.

   Show that $BD + AM + AN = CD + AP + AQ$.

2. Show that there is an integer n with the following properties:

   (i) the binary expansion of n has precisely 2004 0s and 2004 1s;

   (ii) 2004 divides n.

3. (a) Given real numbers a, b, c, with $a + b + c = 0$, prove that
$$a^3 + b^3 + c^3 > 0 \quad \text{if and only if} \quad a^5 + b^5 + c^5 > 0.$$

   (b) Given real numbers a, b, c, d, with $a + b + c + d = 0$, prove that
$$a^3 + b^3 + c^3 + d^3 > 0 \quad \text{if and only if} \quad a^5 + b^5 + c^5 + d^5 > 0.$$

4. The real number x between 0 and 1 has decimal representation
$$0{\cdot}a_1a_2a_3a_4\ldots$$
   with the following property: the number of *distinct* blocks of the form
$$a_ka_{k+1}a_{k+2}\ldots a_{k+2003},$$
   as k ranges through all positive integers, is less than or equal to 2004.

   Prove that x is rational.

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British Mathematical Olympiad

Round 2 : Tuesday, 1 February 2005

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
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- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
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In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (7-11 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Merida, Mexico, 8 - 19 July) will then be chosen.

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2005 British Mathematical Olympiad
Round 2

1. The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$
Prove that N is a perfect square.

2. In triangle ABC, $\angle BAC = 120°$. Let the angle bisectors of angles A, B and C meet the opposite sides in D, E and F respectively.
Prove that the circle on diameter EF passes through D.

3. Let a, b, c be positive real numbers. Prove that
$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \geq (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

4. Let $X = \{A_1, A_2, \ldots, A_n\}$ be a set of distinct 3-element subsets of $\{1, 2, \ldots, 36\}$ such that
i) A_i and A_j have non-empty intersection for every i, j.
ii) The intersection of all the elements of X is the empty set.
Show that $n \leq 100$. How many such sets X are there when $n = 100$?

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British Mathematical Olympiad

Round 2 : Tuesday, 31 January 2006

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions
- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
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In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (6-10 April). At the training session, students sit a pair of IMO-style papers and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Ljubljana, Slovenia 10-18 July) will then be chosen.

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2005/6 British Mathematical Olympiad
Round 2

1. Find the minimum possible value of $x^2 + y^2$ given that x and y are real numbers satisfying
$$xy(x^2 - y^2) = x^2 + y^2 \text{ and } x \neq 0.$$

2. Let x and y be positive integers with no prime factors larger than 5. Find all such x and y which satisfy
$$x^2 - y^2 = 2^k$$
for some non-negative integer k.

3. Let ABC be a triangle with $AC > AB$. The point X lies on the side BA extended through A, and the point Y lies on the side CA in such a way that $BX = CA$ and $CY = BA$. The line XY meets the perpendicular bisector of side BC at P. Show that
$$\angle BPC + \angle BAC = 180^o.$$

4. An exam consisting of six questions is sat by 2006 children. Each question is marked either right or wrong. Any three children have right answers to at least five of the six questions between them. Let N be the total number of right answers achieved by all the children (i.e. the total number of questions solved by child 1 + the total solved by child 2 + $\cdots$ + the total solved by child 2006). Find the least possible value of N.

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British Mathematical Olympiad

Round 2 : Tuesday, 30 January 2007

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
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In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (29th March - 2nd April). At the training session, students sit a pair of IMO-style papers and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of six for this summer's International Mathematical Olympiad (to be held in Hanoi, Vietnam 23-31 July) will then be chosen.

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2006/7 British Mathematical Olympiad
Round 2

1. Triangle ABC has integer-length sides, and $AC = 2007$. The internal bisector of $\angle BAC$ meets BC at D. Given that $AB = CD$, determine AB and BC.

2. Show that there are infinitely many pairs of positive integers (m, n) such that
$$\frac{m+1}{n} + \frac{n+1}{m}$$
is a positive integer.

3. Let ABC be an acute-angled triangle with $AB > AC$ and $\angle BAC = 60^o$. Denote the circumcentre by O and the orthocentre by H and let OH meet AB at P and AC at Q. Prove that $PO = HQ$.

Note: The circumcentre of triangle ABC is the centre of the circle which passes through the vertices A, B and C. The orthocentre is the point of intersection of the perpendiculars from each vertex to the opposite side.

4. In the land of Hexagonia, the six cities are connected by a rail network such that there is a direct rail line connecting each pair of cities. On Sundays, some lines may be closed for repair. The passengers' rail charter stipulates that any city must be accessible by rail from any other (not necessarily directly) at all times. In how many different ways can some of the lines be closed subject to this condition?

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 31 January 2008

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
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In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (3-7 April). At the training session, students sit a pair of IMO-style papers and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Madrid, Spain 14-22 July) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2007/8 British Mathematical Olympiad

Round 2

1. Find the minimum value of $x^2+y^2+z^2$ where x, y, z are real numbers such that $x^3+y^3+z^3-3xyz=1$.

2. Let triangle ABC have incentre I and circumcentre O. Suppose that $\angle AIO = 90°$ and $\angle CIO = 45°$. Find the ratio $AB : BC : CA$.

3. Adrian has drawn a circle in the xy-plane whose radius is a positive integer at most 2008. The origin lies somewhere inside the circle. You are allowed to ask him questions of the form "Is the point (x, y) inside your circle?" After each question he will answer truthfully "yes" or "no". Show that it is always possible to deduce the radius of the circle after at most sixty questions. [*Note: Any point which lies exactly on the circle may be considered to lie inside the circle.*]

4. Prove that there are infinitely many pairs of distinct positive integers x, y such that x^2+y^3 is divisible by x^3+y^2.

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 29 January 2009

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (2-6 April). At the training session, students sit a pair of IMO-style papers and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's IMO (to be held in Bremen, Germany 13-22 July) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2008/9 British Mathematical Olympiad
Round 2

1. Find all solutions in non-negative integers a, b to $\sqrt{a} + \sqrt{b} = \sqrt{2009}$.

2. Let ABC be an acute-angled triangle with $\angle B = \angle C$. Let the circumcentre be O and the orthocentre be H. Prove that the centre of the circle BOH lies on the line AB. *The circumcentre of a triangle is the centre of its circumcircle. The orthocentre of a triangle is the point where its three altitudes meet.*

3. Find all functions f from the real numbers to the real numbers which satisfy
$$f(x^3) + f(y^3) = (x + y)(f(x^2) + f(y^2) - f(xy))$$
for all real numbers x and y.

4. Given a positive integer n, let $b(n)$ denote the number of positive integers whose binary representations occur as blocks of consecutive integers in the binary expansion of n. For example $b(13) = 6$ because $13 = 1101_2$, which contains as consecutive blocks the binary representations of $13 = 1101_2$, $6 = 110_2$, $5 = 101_2$, $3 = 11_2$, $2 = 10_2$ and $1 = 1_2$.

Show that if $n \leq 2500$, then $b(n) \leq 39$, and determine the values of n for which equality holds.

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 28 January 2010

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (8-12 April). At the training session, students sit a pair of IMO-style papers and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's IMO (to be held in Astana, Kazakhstan 6-12 July) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2009/10 British Mathematical Olympiad
Round 2

1. There are 2010^{2010} children at a mathematics camp. Each has at most three friends at the camp, and if A is friends with B, then B is friends with A. The camp leader would like to line the children up so that there are at most 2010 children between any pair of friends. Is it always possible to do this?

2. In triangle ABC the centroid is G and D is the midpoint of CA. The line through G parallel to BC meets AB at E. Prove that $\angle AEC = \angle DGC$ if, and only if, $\angle ACB = 90°$. *The centroid of a triangle is the intersection of the three medians, the lines which join each vertex to the midpoint of the opposite side.*

3. The integer x is at least 3 and $n = x^6 - 1$. Let p be a prime and k be a positive integer such that p^k is a factor of n. Show that $p^{3k} < 8n$.

4. Prove that, for all positive real numbers x, y and z,
$$4(x + y + z)^3 > 27(x^2y + y^2z + z^2x).$$

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 27 January 2011

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*
- To accommodate candidates sitting in other timezones, please do not discuss any aspect of the paper on the internet until 8am on Friday 28 January GMT.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (14-18 April 2011). At the training session, students sit a pair of IMO-style papers and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's IMO (to be held in Amsterdam, The Netherlands 16–24 July) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2010/11 British Mathematical Olympiad
Round 2

1. Let ABC be a triangle and X be a point inside the triangle. The lines AX, BX and CX meet the circle ABC again at P, Q and R respectively. Choose a point U on XP which is between X and P. Suppose that the lines through U which are parallel to AB and CA meet XQ and XR at points V and W respectively. Prove that the points R, W, V and Q lie on a circle.

2. Find all positive integers x and y such that $x + y + 1$ divides $2xy$ and $x + y - 1$ divides $x^2 + y^2 - 1$.

3. The function f is defined on the positive integers as follows;

$$\begin{aligned} f(1) &= 1; \\ f(2n) &= f(n) \text{ if } n \text{ is even;} \\ f(2n) &= 2f(n) \text{ if } n \text{ is odd;} \\ f(2n+1) &= 2f(n) + 1 \text{ if } n \text{ is even;} \\ f(2n+1) &= f(n) \text{ if } n \text{ is odd.} \end{aligned}$$

   Find the number of positive integers n which are less than 2011 and have the property that $f(n) = f(2011)$.

4. Let G be the set of points (x, y) in the plane such that x and y are integers in the range $1 \le x, y \le 2011$. A subset S of G is said to be *parallelogram-free* if there is no proper parallelogram with all its vertices in S. Determine the largest possible size of a parallelogram-free subset of G. *Note that a proper parallelogram is one where its vertices do not all lie on the same line*

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 26 January 2012

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
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- To accommodate candidates sitting in other timezones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 27 January.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (29 March – 2 April 2012). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of six for this summer's IMO (to be held in Mar del Plata, Argentina, 4–16 July) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2011/12 British Mathematical Olympiad
Round 2

1. The diagonals AC and BD of a cyclic quadrilateral meet at E. The midpoints of the sides AB, BC, CD and DA are P, Q, R and S respectively. Prove that the circles EPS and EQR have the same radius.

2. A function f is defined on the positive integers by $f(1) = 1$ and, for $n > 1$,
$$f(n) = f\left(\left\lfloor \frac{2n-1}{3} \right\rfloor\right) + f\left(\left\lfloor \frac{2n}{3} \right\rfloor\right)$$
where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Is it true that $f(n) - f(n-1) \le n$ for all $n > 1$?

   [Here are some examples of the use of $\lfloor x \rfloor$: $\lfloor \pi \rfloor = 3$, $\lfloor 1729 \rfloor = 1729$ and $\lfloor \frac{2012}{1000} \rfloor = 2$.]

3. The set of real numbers is split into two subsets which do not intersect. Prove that for each pair (m, n) of positive integers, there are real numbers $x < y < z$ all in the same subset such that $m(z-y) = n(y-x)$.

4. Show that there is a positive integer k with the following property: if a, b, c, d, e and f are integers and m is a divisor of
$$a^n + b^n + c^n - d^n - e^n - f^n$$
for all integers n in the range $1 \le n \le k$, then m is a divisor of $a^n + b^n + c^n - d^n - e^n - f^n$ for all positive integers n.

British Mathematical Olympiad

Round 2 : Thursday, 31 January 2013

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1, 2, 3, 4 in order, and the cover sheet at the front.*
- To accommodate candidates sitting in other timezones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 1 February.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (4–8 April 2013). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this summer's IMO (to be held in Santa Marta, Colombia, 18–28 July 2013) will then be chosen.

Do not turn over until **told to do so.**

2012/13 British Mathematical Olympiad
Round 2

1. Are there infinitely many pairs of positive integers (m, n) such that both m divides $n^2 + 1$ and n divides $m^2 + 1$?

2. The point P lies inside triangle ABC so that $\angle ABP = \angle PCA$. The point Q is such that $PBQC$ is a parallelogram. Prove that $\angle QAB = \angle CAP$.

3. Consider the set of positive integers which, when written in binary, have exactly 2013 digits and more 0s than 1s. Let n be the number of such integers and let s be their sum. Prove that, when written in binary, $n + s$ has more 0s than 1s.

4. Suppose that $ABCD$ is a square and that P is a point which is on the circle inscribed in the square. Determine whether or not it is possible that PA, PB, PC, PD and AB are all integers.

British Mathematical Olympiad

Round 2 : Thursday, 30 January 2014

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1, 2, 3, 4 in order, and the cover sheet at the front.*
- To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 31 January.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (3–7 April 2014). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this summer's IMO (to be held in Cape Town, South Africa, 3–13 July 2014) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2013/14 British Mathematical Olympiad
Round 2

1. Every diagonal of a regular polygon with 2014 sides is coloured in one of n colours. Whenever two diagonals cross in the interior, they are of different colours. What is the minimum value of n for which this is possible?

2. Prove that it is impossible to have a cuboid for which the volume, the surface area and the perimeter are numerically equal. *The perimeter of a cuboid is the sum of the lengths of all its twelve edges.*

3. Let $a_0 = 4$ and define a sequence of terms using the formula $a_n = a_{n-1}^2 - a_{n-1}$ for each positive integer n.

   a) Prove that there are infinitely many prime numbers which are factors of at least one term in the sequence;

   b) Are there infinitely many prime numbers which are factors of no term in the sequence?

4. Let ABC be a triangle and P be a point in its interior. Let AP meet the circumcircle of ABC again at A'. The points B' and C' are similarly defined. Let O_A be the circumcentre of BCP. The circumcentres O_B and O_C are similarly defined. Let $O_A{}'$ be the circumcentre of $B'C'P$. The circumcentres $O_B{}'$ and $O_C{}'$ are similarly defined. Prove that the lines $O_AO_A{}', O_BO_B{}'$ and $O_CO_C{}'$ are concurrent.

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 29 January 2015

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1, 2, 3, 4 in order, and the cover sheet at the front.*
- To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 30 January.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (26-30 March 2015). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this summer's IMO (to be held in Chiang Mai, Thailand, 8–16 July 2015) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2014/15 British Mathematical Olympiad
Round 2

1. The first term x_1 of a sequence is 2014. Each subsequent term of the sequence is defined in terms of the previous term. The iterative formula is
$$x_{n+1} = \frac{(\sqrt{2}+1)x_n - 1}{(\sqrt{2}+1) + x_n}.$$
Find the 2015th term x_{2015}.

2. In Oddesdon Primary School there are an odd number of classes. Each class contains an odd number of pupils. One pupil from each class will be chosen to form the school council. Prove that the following two statements are logically equivalent.

   a) There are more ways to form a school council which includes an odd number of boys than ways to form a school council which includes an odd number of girls.

   b) There are an odd number of classes which contain more boys than girls.

3. Two circles touch one another internally at A. A variable chord PQ of the outer circle touches the inner circle. Prove that the locus of the incentre of triangle AQP is another circle touching the given circles at A. *The* incentre *of a triangle is the centre of the unique circle which is inside the triangle and touches all three sides. A* locus *is the collection of all points which satisfy a given condition.*

4. Given two points P and Q with integer coordinates, we say that P *sees* Q if the line segment PQ contains no other points with integer coordinates. An *n-loop* is a sequence of n points $P_1, P_2, \ldots, P_n$, each with integer coordinates, such that the following conditions hold:

   a) P_i sees P_{i+1} for $1 \le i \le n-1$, and P_n sees P_1;

   b) No P_i sees any P_j apart from those mentioned in (a);

   c) No three of the points lie on the same straight line.

   Does there exist a 100-loop?

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 28 January 2016

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1, 2, 3, 4 in order, and the cover sheet at the front.*
- To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 29 January.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (31 March-4 April 2016). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this summer's IMO (to be held in Hong Kong, China 6–16 July 2016) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2015/16 British Mathematical Olympiad
Round 2

1. Circles of radius r_1, r_2 and r_3 touch each other externally, and they touch a common tangent at points A, B and C respectively, where B lies between A and C. Prove that $16(r_1+r_2+r_3) \geq 9(AB+BC+CA)$.

2. Alison has compiled a list of 20 hockey teams, ordered by how good she thinks they are, but refuses to share it. Benjamin may mention three teams to her, and she will then choose either to tell him which she thinks is the weakest team of the three, or which she thinks is the strongest team of the three. Benjamin may do this as many times as he likes. Determine the largest N such that Benjamin can guarantee to be able to find a sequence $T_1, T_2, \ldots, T_N$ of teams with the property that he knows that Alison thinks that T_i is better than T_{i+1} for each $1 \leq i < N$.

3. Let $ABCD$ be a cyclic quadrilateral. The diagonals AC and BD meet at P, and DA and CB produced meet at Q. The midpoint of AB is E. Prove that if PQ is perpendicular to AC, then PE is perpendicular to BC.

4. Suppose that p is a prime number and that there are different positive integers u and v such that p^2 is the mean of u^2 and v^2. Prove that $2p - u - v$ is a square or twice a square.

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 26 January 2017

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1, 2, 3,* 4 *in order, and the cover sheet at the front.*
- To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 27 January.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (30 March-3 April 2017). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this year's IMO (to be held in Rio de Janeiro, Brazil 12–23 July 2017) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2016/17 British Mathematical Olympiad
Round 2

1. This problem concerns triangles which have vertices with integer coordinates in the usual x, y-coordinate plane. For how many positive integers $n < 2017$ is it possible to draw a right-angled isosceles triangle such that exactly n points on its perimeter, including all three of its vertices, have integer coordinates?

2. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to the real number x. Consider the sequence $a_1, a_2, \ldots$ defined by
$$a_n = \frac{1}{n}\left(\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n}\right\rfloor\right)$$
for integers $n \geq 1$. Prove that $a_{n+1} > a_n$ for infinitely many n, and determine whether $a_{n+1} < a_n$ for infinitely many n.

   [Here are some examples of the use of $\lfloor x \rfloor$: $\lfloor \pi \rfloor = 3$, $\lfloor 1729 \rfloor = 1729$ and $\lfloor \frac{2017}{1000} \rfloor = 2$.]

3. Consider a cyclic quadrilateral $ABCD$. The diagonals AC and BD meet at P, and the rays AD and BC meet at Q. The internal angle bisector of angle $\angle BQA$ meets AC at R and the internal angle bisector of angle $\angle APD$ meets AD at S. Prove that RS is parallel to CD.

4. Bobby's booby-trapped safe requires a 3-digit code to unlock it. Alex has a probe which can test combinations without typing them on the safe. The probe responds *Fail* if no individual digit is correct. Otherwise it responds *Close*, including when all digits are correct. For example, if the correct code is 014, then the responses to 099 and 014 are both Close, but the response to 140 is Fail. If Alex is following an optimal strategy, what is the smallest number of attempts needed to guarantee that he knows the correct code, whatever it is?

United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday 25 January 2018

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions
- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions* 1, 2, 3, 4 *in order, and the cover sheet at the front.*
- *To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 26 January. Candidates sitting the paper in time zones more than 3 hours ahead of GMT must sit the paper on Friday 26 January (as defined locally).*

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (4–9 April 2018). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this year's IMO (to be held in Cluj–Napoca, Romania 3–14 July 2018) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2017/18 British Mathematical Olympiad
Round 2

1. Consider triangle ABC. The midpoint of AC is M. The circle tangent to BC at B and passing through M meets the line AB again at P. Prove that $AB \times BP = 2BM^2$.

2. There are n places set for tea around a circular table, and every place has a small cake on a plate. Alice arrives first, sits at the table, and eats her cake (but it isn't very nice). Next the Mad Hatter arrives, and tells Alice that she will have a lonely tea party, and that she must keep on changing her seat, and each time she must eat the cake in front of her (if it has not yet been eaten). In fact the Mad Hatter is very bossy, and tells Alice that, for $i = 1, 2, \dots, n-1$, when she moves for the i-th time, she must move a_i places and he hands Alice the list of instructions $a_1, a_2, \dots, a_{n-1}$. Alice does not like the cakes, and she is free to choose, at every stage, whether to move clockwise or anticlockwise. For which values of n can the Mad Hatter force Alice to eat all the cakes?

3. It is well known that, for each positive integer n,
$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$
and so is a square. Determine whether or not there is a positive integer m such that
$$(m+1)^3 + (m+2)^3 + \cdots + (2m)^3$$
is a square.

4. Let f be a function defined on the real numbers and taking real values. We say that f is *absorbing* if $f(x) \le f(y)$ whenever $x \le y$ and $f^{2018}(z)$ is an integer for all real numbers z.

   a) Does there exist an absorbing function f such that $f(x)$ is an integer for only finitely many values of x?

   b) Does there exist an absorbing function f and an increasing sequence of real numbers $a_1 < a_2 < a_3 < \dots$ such that $f(x)$ is an integer only if $x = a_i$ for some i?

Note that if k is a positive integer and f is a function, then f^k denotes the composition of k copies of f. For example $f^3(t) = f(f(f(t)))$ for all real numbers t.

United Kingdom
Mathematics Trust

British Mathematical Olympiad
Round 2
Thursday 24 January 2019

Instructions

1. Time allowed: $3\frac{1}{2}$ hours. Each question is worth 10 marks.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.
3. Rough work *should* be handed in, but should be clearly marked.
4. One or two *complete* solutions will gain far more credit than partial attempts at all four problems.
5. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
6. Staple all the pages neatly together in the top *left* hand corner, with questions 1, 2, 3, 4 in order, and the cover sheet at the front.
7. To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 25 January. Candidates sitting the paper in time zones more than 3 hours ahead of GMT must sit the paper on Friday 25 January (as defined locally).
8. In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (2–7 April 2019). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this year's IMO (to be held in Bath, United Kingdom 11–22 July 2019) will then be chosen.
9. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:
UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT
☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1. Let ABC be a triangle. Let L be the line through B perpendicular to AB. The perpendicular from A to BC meets L at the point D. The perpendicular bisector of BC meets L at the point P. Let E be the foot of the perpendicular from D to AC.

Prove that triangle BPE is isosceles.

2. For some integer n, a set of n^2 magical chess pieces arrange themselves on a square $n^2 \times n^2$ chessboard composed of n^4 unit squares. At a signal, the chess pieces all teleport to another square of the chessboard such that the distance between the centres of their old and new squares is n. The chess pieces win if, both before and after the signal, there are no two chess pieces in the same row or column. For which values of n can the chess pieces win?

3. Let p be an odd prime. How many non-empty subsets of

$$\{1, 2, 3, \ldots, p-2, p-1\}$$

have a sum which is divisible by p?

4. Find all functions f from the positive real numbers to the positive real numbers for which $f(x) \leq f(y)$ whenever $x \leq y$ and

$$f(x^4) + f(x^2) + f(x) + f(1) = x^4 + x^2 + x + 1$$

for all $x > 0$.

United Kingdom
Mathematics Trust

British Mathematical Olympiad
Round 2

Thursday 30 January 2020

supported by [XTX] MARKETS Overleaf

Instructions

1. Time allowed: $3\frac{1}{2}$ hours. Each question is worth 10 marks.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.
3. Write on one side of the paper only and start each question on a fresh sheet.
4. Rough work *should* be handed in, but should be clearly marked.
5. One or two *complete* solutions will gain far more credit than partial attempts at all four problems.
6. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
7. Staple all the pages neatly together in the top *left* hand corner, with questions $1, 2, 3, 4$ in order, and the cover sheet at the front.
8. To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 31 January. Candidates sitting the paper in time zones more than 3 hours ahead of GMT must sit the paper on Friday 31 January (as defined locally).
9. In early March, twenty-four students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (31 March–5 April 2020). At the training session, students sit a pair of IMO-style papers and some students will be selected for further training and selection examinations. The UK Team of six for this year's IMO (to be held in St Petersburg, Russia 8–18 July 2020) will then be chosen.
10. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

*UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS*2 9*JT*

☎ 0113 343 2339   enquiry@ukmt.org.uk   www.ukmt.org.uk

1. A sequence $a_1, a_2, a_3, \ldots$ has $a_1 > 2$ and satisfies:

$$a_{n+1} = \frac{a_n(a_n - 1)}{2}$$

for all positive integers n. For which values of a_1 are all the terms of the sequence odd integers?

2. Describe all collections S of at least four points in the plane such that no three points are collinear and such that every triangle formed by three points in S has the same circumradius.

(The circumradius of a triangle is the radius of the circle passing through all three of its vertices.)

3. A 2019×2019 square grid is made up of 2019^2 unit cells. Each cell is coloured either black or white. A colouring is called *balanced* if, within every square subgrid made up of k^2 cells for $1 \leq k \leq 2019$, the number of black cells differs from the number of white cells by at most one. How many different balanced colourings are there?

(Two colourings are different if there is at least one cell which is black in exactly one of them.)

4. A sequence $b_1, b_2, b_3, \ldots$ of nonzero real numbers has the property that

$$b_{n+2} = \frac{b_{n+1}^2 - 1}{b_n}$$

for all positive integers n.

Suppose that $b_1 = 1$ and $b_2 = k$ where $1 < k < 2$. Show that there is some constant B, depending on k, such that $-B \leq b_n \leq B$ for all n. Also show that, for some $1 < k < 2$, there is a value of n such that $b_n > 2020$.

United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 2

Thursday 28 January 2021

supported by XTX Markets Overleaf

Instructions

1. Time allowed: $3\frac{1}{2}$ hours. Each question is worth 10 marks.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.
3. One or two *complete* solutions will gain far more credit than partial attempts at all four problems.
4. Write on one side of the paper only and start each question on a fresh sheet.
5. You should write in blue or black ink, but may use pencil and other colours for diagrams.
6. You may hand in rough work for each question where it contains calculations, examples or ideas not present in your final attempt; write 'ROUGH' at the top of each page of rough work.
7. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
8. Write your candidate number and UKMT centre number neatly in the top left corner of each page and arrange them so that your teacher can easily upload them to the marking platform.
9. To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 9am GMT on Friday 29 January. Candidates sitting the paper in time zones more than 3 hours ahead of GMT must sit the paper on the morning of Friday 29 January (as defined locally).
10. In early March, top-scoring students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend a week of sessions, which will be held in an online format during the Easter holidays, comprising training for olympiads and general mathematical interest. Tests to select the UK team of six for this year's IMO (to be hosted by Russia, possibly in a virtual format, 14–24 July 2021) will take place after the training week.
11. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 365 1121 enquiry@ukmt.org.uk www.ukmt.org.uk

1. A positive integer n is called *good* if there is a set of divisors of n whose members sum to n and include 1. Prove that every positive integer has a multiple which is good.

2. Eliza has a large collection of $a \times a$ and $b \times b$ tiles where a and b are positive integers. She arranges some of these tiles, without overlaps, to form a square of side length n. Prove that she can cover another square of side length n using only one of her two types of tile.

3. Let ABC be a triangle with $AB > AC$. Its circumcircle is Γ and its incentre is I. Let D be the contact point of the incircle of ABC with BC.

Let K be the point on Γ such that $\angle AKI$ is a right angle.

Prove that AI and KD meet on Γ.

4. Matthew writes down a sequence $a_1, a_2, a_3, \ldots$ of positive integers. Each a_n is the smallest positive integer, different from all previous terms in the sequence, such that the mean of the terms $a_1, a_2, \ldots, a_n$ is an integer. Prove that the sequence defined by $a_i - i$ for $i = 1, 2, 3, \ldots$ contains every integer exactly once.

United Kingdom
Mathematics Trust

British Mathematical Olympiad
Round 2

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Solutions

These solutions are intended as outlines. In particular, they do not represent the full range of approaches possible, nor the difficulties which lie in finding them.

1. A positive integer n is called *good* if there is a set of divisors of n whose members sum to n and include 1. Prove that every positive integer has a multiple which is good.

SOLUTION

Firstly we show that if $m > 1$ is good, then so is $2m$. This is true since some proper divisors of m, including 1 (and hence not including m itself), sum to m; if we consider all these together with m, they will all be factors of $2m$ which sum to $2m$.

This means that it suffices to prove the claim for odd numbers. The claim holds for 1 since good numbers exist (such as 6, which is $1 + 2 + 3$, for example).

If $a > 1$ is odd and $n = 2^k a$ for some k, then $a + 2a + 4a + \cdots + 2^{k-1}a = (2^k - 1)a = n - a$ which is close to n. This value of n will be good if we can find some other factors of n, including 1, which sum to a.

To do this, we write a as a sum of powers of 2, including 1, by writing a in binary, and then choose k to be large enough for all those powers of 2 to be factors of n. (We may take k to be $\lceil \log_2(a) \rceil$, the smallest integer greater than or equal to the base-2 logarithm of a.) None of these powers of 2 are multiples of a so there is no risk that we are using the same factor of n twice.

ALTERNATIVE

A variation on the proof above can be obtained by instead considering $2^{m-1}(2^m - 1)$. This is good for any $m \geq 3$, since

$$2^{m-1}(2^m - 1) = 1 + 2 + 2^2 + \cdots + 2^{m-1} + (2^m - 1) + 2(2^m - 1) + \cdots + 2^{m-2}(2^m - 1).$$

It can be shown using the Fermat-Euler theorem that any number n is a factor of a number of this form. Indeed, suppose $n = 2^k a$, with a odd. Then the Fermat-Euler theorem tells us that $2^{c\varphi(a)} - 1$ is a multiple of a for any positive integer c (where $\varphi(a)$ is the number of numbers between 1 and a which are coprime to a). This means that if we choose c such that $k < c\varphi(a)$, then we will have n a factor of $2^{c\varphi(a)-1}(2^{c\varphi(a)} - 1)$

ALTERNATIVE

We show that the numbers $n!$ are all good for all $n \geq 3$, which is enough to solve the problem since n is always a factor of $n!$. We proceed by induction, using $3! = 6 = 3 + 2 + 1$ for the base case. For the inductive step, suppose that

$$n! = 1 + \sum_i a_i$$

where a_i are distinct factors of $n!$ other than 1. Then we have

$$(n+1)! = 1 + n + \sum_i (n+1)a_i,$$

and all these factors are different, so $(n + 1)!$ is also good.

2. Eliza has a large collection of $a \times a$ and $b \times b$ tiles where a and b are positive integers. She arranges some of these tiles, without overlaps, to form a square of side length n. Prove that she can cover another square of side length n using only one of her two types of tile.

SOLUTION

Number the rows of the $n \times n$ square from 1 to n.

We say that a cell is *special* if it is in the top row of a $b \times b$ square.

In any row, the number of cells that are in a $b \times b$ square is congruent to n, modulo a. Since each special cell has $b - 1$ cells in the $b - 1$ rows underneath it, we can see by induction that the number of special cells in rows $1, b + 1, 2b + 1, \ldots$ is congruent to n modulo a, and the number of special cells in rows with other numbers is congruent to 0.

Now, the cells in $b \times b$ squares in row n are all in squares whose top cells are in row $n - b + 1$, and hence the number of special cells in row $n - b + 1$ is congruent to n (modulo a). Hence either $n \equiv 0$ (modulo a), in which case a is a multiple of n, or $n - b + 1$ is of the form $rb + 1$, in which case b is a multiple of n.

ALTERNATIVE

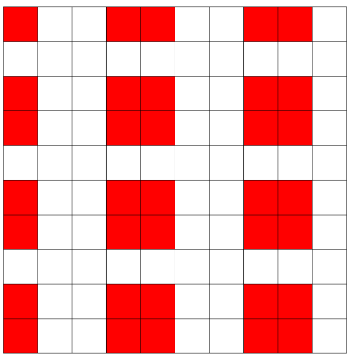
Number the rows of the square to be tiled from 1 to n and suppose that a does not divide n.

If the row number is congruent to 1 modulo a write $+1$ in every cell in that row, and if the row number is congruent to 0 modulo a write -1 in every cell in that row. Write 0 in every other cell.

The sum of all the entries on the board is exactly n. The entries covered by an $a \times a$ tile sum to 0, and the those covered by a $b \times b$ tile sum to a multiple of b. Thus the total of the entries covered by any collection of tiles is congruent to zero modulo b, so $b|n$ as required.

ALTERNATIVE

Suppose that n is divisible by neither a nor b. Imagine tiling plane with $a \times b$ rectangles and colour in the four corners of each rectangle as shown (for $a = 3$, $b = 4$, $n = 10$).



More precisely, we shade the cell with coordinates (x, y) (for $1 \leq x, y \leq n$) just when $a \mid x$ or $x - 1$, and $b \mid y$ or $y - 1$.

It is easy to verify (using the periodicity of the colouring) that each $a \times a$ or $b \times b$ tile covers an even number of shaded cells, but that there are an odd number of shaded cells in total. Thus the $c \times c$ square cannot be tiled with $a \times a$ and $b \times b$ tiles, as desired.

Alternative

Stretch the board by $1/a$ in one direction and by $1/b$ in the other. Now each of Eliza's tiles has one side equal to 1. A theorem of De Bruijn now implies that the whole board must have an integer side length, which, after reversing the stretches, implies the result.

Remark

The second alternative opens up many possible approaches, since De Bruijn's result has many proofs.

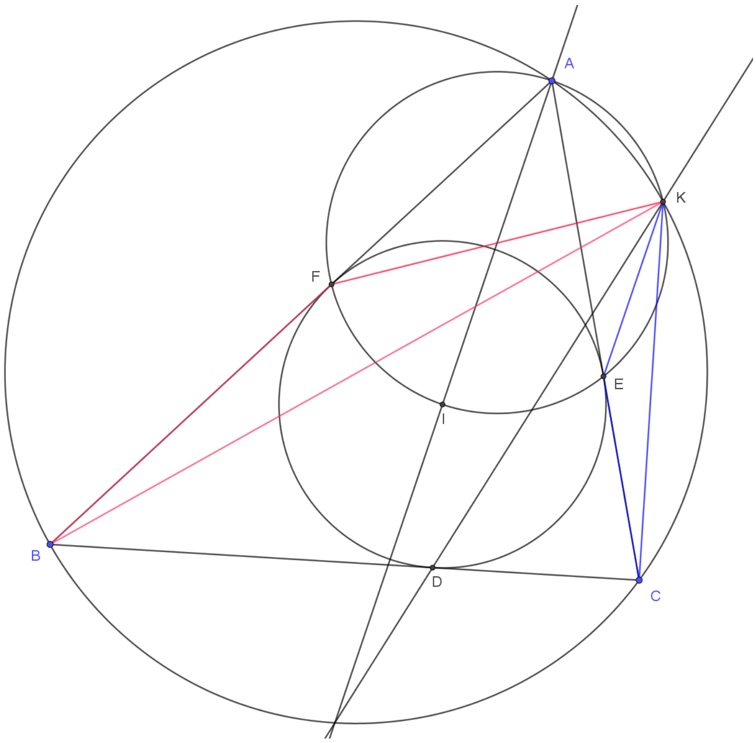
For example we can divide the original board into rectangles whose dimensions are $a/2$ and $b/2$ and apply a standard chessboard colouring. Each of Eliza's tiles covers an equal black and white area, but it can be checked that the board only consists of equal black and white areas if its side is divisible by either a or b.

3. Let ABC be a triangle with $AB > AC$. Its circumcircle is Γ and its incentre is I. Let D be the contact point of the incircle of ABC with BC.

Let K be the point on Γ such that $\angle AKI$ is a right angle.

Prove that AI and KD meet on Γ.

SOLUTION



The line AI meets the circumcircle Γ at the midpoint M of the minor arc BC. We produce MD to cut the circumcircle again at K'. We will show that $\angle AK'I$ is a right-angle and so $K' = K$.

Angle in the same segment gives $\angle MK'B = \angle CBM = \frac{A}{2}$ and so MB is tangent to circle BDK' at B, and the tangent-secant theorem applies so $MB^2 = MD.MK'$. We also know that $MB = MC = MI$, and substituting MI into this relation, the converse of the tangent-secant theorem tells us that MI is tangent to the undrawn circle DIK' at I. Now $\angle DIM + \angle MIB = \angle DIB = 90^\circ - \frac{B}{2}$. Therefore $\angle DIM = \frac{B-C}{2}$ (we have just used the fact that $\angle B < \angle C$. Putting all this together, we find that $\angle AK'I = \angle AK'B - \angle IK'M - \angle MK'B = \angle ACB - \angle MID - \angle MAB = 180^\circ - C - \frac{B-C}{2} - \frac{A}{2} = 180^\circ - \frac{A+B+C}{2} = 90^\circ$ as required.

ALTERNATIVE

We know that AI meets the circumcircle again at M, the midpoint of the arc BC. Since Γ is also the circumcircle of $\triangle KBC$, we know that KD also passes through M precisely if KD is the angle bisector of $\angle BKC$.

Let E, F be the contact points of the incircle with CA, AB, respectively, so $AFIEK$ is cyclic (by the converse of Thales's theorem). Now K, the intersection of Γ and circle $AFEK$, is the centre of spiral similarity which carries the directed segment FB to EC. Therefore

$$\frac{KB}{KC} = \frac{BF}{CE} = \frac{BD}{DC}$$

since tangents from a point to a circle have the same length. Now, by the converse of the angle bisector theorem, KD is the angle bisector of angle $\angle CKB$.

Alternative

It is possible to adjust the approach above so as to not to require knowledge of the spiral similarity theorem. One way to proceed is by chasing angles to prove that $\triangle KFB$ and $\triangle KEC$ are similar.

4. Matthew writes down a sequence $a_1, a_2, a_3, \ldots$ of positive integers. Each a_n is the smallest positive integer, different from all previous terms in the sequence, such that the mean of the terms $a_1, a_2, \ldots, a_n$ is an integer. Prove that the sequence defined by $a_i - i$ for $i = 1, 2, 3, \ldots$ contains every integer exactly once.

SOLUTION

Write $b_i = a_i - i$. We prove by induction that $b_1, \ldots, b_n$ consists of consecutive integers in some order. Let $B_n = \max\{b_1, \ldots, b_n\}$. Since $b_1 = 0$, we have $B_n < n$. The mean of the sequence $a_1, a_2, \ldots, a_n, n+2+B_n$ is an integer, as it is the sum of the two sequences $1, 2, \ldots, n, n+1$ and $b_1, b_2, \ldots, b_n, B_n+1$ of consecutive integers, and the mean of any sequence of $n+1$ consecutive integers is an integer (respectively a half-integer) according as n is even (respectively odd). It follows that a_{n+1} is congruent to $n+2+B_n$ mod $n+1$. Since $n+2+B_n$ is greater than any of $a_1, \ldots, a_n$ (as $a_i = i + b_i$ with $i < n+1$ and $b_i < B_n + 1$, we know that $a_{n+1} \leq n+2+B_n$, and hence a_{n+1} is equal to $n+2+B_n$ or $1+B_n$ (as $B_n - n < 0$). Thus b_{n+1} is equal to $B_n + 1$ or $B_n - n$, and the integers $b_1, \ldots, b_{n+1}$ are consecutive. This completes the induction. It follows that every integer appears in the sequence $b_1, b_2, \ldots$ at most once.

To complete the problem, we need only check that $b_n > 0$ infinitely often and also that $b_n < 0$ infinitely often. For the former, we simply note that if $b_n < 0$, then we cannot also have $b_{n+1} < 0$. Indeed, since $0 = b_1$ is among the consecutive integers $b_1, \ldots, b_n$, the only way we could have $b_{n+1} < 0$ would be if $b_{n+1} = b_n - 1$, in which case $a_{n+1} = a_n$, which is not possible. For the latter, suppose for contradiction that there is some n_0 such that $b_n > 0$ for all $n > n_0$. It follows that for $n > n_0$ we have $b_n = n - \delta$, and hence $a_n = 2n - \delta$, for some δ. Since the mean of $a_1, \ldots, a_{n-1}, n - \delta$ is also an integer, it follows that $n - \delta$ must appear in the sequence $a_1, \ldots, a_{n-1}$ for all $n > n_0$. If in addition n is odd, we cannot have $n - \delta = a_{n'} = 2n' - \delta$ for an integer $n' > n_0$, and hence $n - \delta$ must even appear in the sequence $a_1, \ldots, a_{n_0}$. But there are infinitely many odd integers $n > n_0$ and only finitely many integers in the range $a_1, \ldots, a_{n_0}$, which is impossible.

ALTERNATIVE

Let τ be the Golden Ratio, so $1/\tau = \tau - 1$. Define $m(n)$ to be the ceiling of n/τ. If $m(n) = m(n-1)$ then define $a(n) = m(n-1)$ (case 1). If $m(n) = m(n-1) + 1$ then define $a(n) = m(n-1) + n$ (case 2). Then $m(n)$ is the mean of $a(1), a(2), \ldots, a(n)$ because $a(n) = m(n-1) + kn$ and $m(n) = m(n-1) + k$ always, where k is an integer. Case 1 is $a - 1 < (n-1)/\tau < n/\tau < a$, which rearranges to $(a-1)/\tau < n - a < a/\tau$. Case 2 is $(n-1)/\tau < a - n < n/\tau$, which rearranges to $n - 1 < (a-1)/\tau < a/\tau < n$. It follows that $a(n)$ is self-inverse, in particular bijective. Note that $a(m(n-1)) < \tau m(n-1) < \tau((n-1)/\tau + 1) < n+1$, so $m(n-1)$ is used by position n. So if we are in case 2, $m(n-1)$ has been used previously. So we are always using the smallest unused $m(n-1) + kn$ (since $k < 0$ is impossible). Hence our sequence is the same as the one in the question. Case 2 has $a(n) - n$ equal to each positive integer once, as n ranges over all positive integers. Case 1 has $a(n) - n$ equal to each negative integer once, as n ranges over all positive integers. Since $a(1) - 1 = 0$, that completes the problem.

United Kingdom
Mathematics Trust

British Mathematical Olympiad
Round 2

Thursday 27 January 2022

supported by [XTX] MARKETS Overleaf

Instructions

1. Time allowed: $3\frac{1}{2}$ hours. Each question is worth 10 marks.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Work in rough first, and then draft your final version carefully before writing up your best attempt.
3. One or two *complete* solutions will gain far more credit than partial attempts at all four problems.
4. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
5. Start each question on an official answer sheet on which there is a QR code.
6. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. **Do not write your name on any additional sheets.**
7. You should write in blue or black ink, but may use pencil and other colours for diagrams.
8. Write on one side of the paper only. Make sure your writing and diagrams are not too faint.
9. You may hand in rough work where it contains calculations, examples or ideas not present in your final attempt; write 'ROUGH' at the top of each page of rough work.
10. Arrange your answer sheets, including rough work, in question order before they are collected. If you are not submitting work for a particular problem, remove the associated answer sheet.
11. To accommodate candidates in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 28 January. Candidates in time zones more than 3 hours ahead of GMT must sit the paper on Friday 28 January (as defined locally).
12. Around 24 high-scoring students eligible to represent the UK at the International Mathematial Olympiad, will be invited to a training session held in Cambridge shortly before Easter.
13. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 365 1121 challenges@ukmt.org.uk www.ukmt.org.uk

1. For a given positive integer k, we call an integer n a *k-number* if both of the following conditions are satisfied:

(i) The integer n is the product of two positive integers which differ by k.

(ii) The integer n is k less than a square number.

Find all k such that there are infinitely many k-numbers.

2. Find all functions f from the positive integers to the positive integers such that for all x, y we have:

$$2yf(f(x^2) + x) = f(x + 1)f(2xy).$$

3. The cards from n identical decks of cards are put into boxes. Each deck contains 50 cards, labelled from 1 to 50. Each box can contain at most 2022 cards. A pile of boxes is said to be *regular* if that pile contains equal numbers of cards with each label. Show that there exists some N such that, if $n \geq N$, then the boxes can be divided into two non-empty regular piles.

4. Let ABC be an acute angled triangle with circumcircle Γ. Let l_B and l_C be the lines perpendicular to BC which pass through B and C respectively. A point T lies on the minor arc BC. The tangent to Γ at T meets l_B and l_C at P_B and P_C respectively. The line through P_B perpendicular to AC and the line through P_C perpendicular to AB meet at a point Q. Given that Q lies on BC, prove that the line AT passes through Q.

(A minor arc of a circle is the shorter of the two arcs with given endpoints.)

United Kingdom
Mathematics Trust

British Mathematical Olympiad
Round 2

Solutions

1. For a given positive integer k, we call an integer n a *k-number* if both of the following conditions are satisfied:

(i) The integer n is the product of two positive integers which differ by k.

(ii) The integer n is k less than a square number.

Find all k such that there are infinitely many k-numbers.

Solution

Note that n is a k-number if and only if the equation

$$n = m^2 - k = r(r + k)$$

has solutions in integers m, r with $k \geq 0$.

The right-hand equality can be rewritten as

$$k^2 - 4k = (2r + k)^2 - (2m)^2 ,$$

so k-numbers correspond to ways of writing $k^2 - 4k$ as a difference of two squares, $N^2 - M^2$ with $N > r$ and M even (which forces N to have the same parity as k).

Any non-zero integer can only be written as a difference of two squares in finitely many ways (because each gives a factorisation, and a number has only finitely many factors).

If $k \neq 4$ then $k^2 - 4k \neq 0$, and as a result, if $k \neq 4$ then there are only finitely many k-numbers.

Conversely, if $k = 4$ then setting $m = r + 2$ for $r \geq 0$ shows that there are infinitely many 4-numbers.

2. Find all functions f from the positive integers to the positive integers such that for all integers[a] x, y we have:

$$2yf(f(x^2)+x) = f(x+1)f(2xy).$$

[a] The final instance of the word *integers* was added retrospectively to avoid ambiguity.

SOLUTION

First substitute $x = 1$ to see that $f(2y) = ky$ for all positive integers y, where $k = \frac{2f(f(1)+1)}{f(2)}$. By taking $y = 1$, we get $f(2) = k$, so k is a positive integer.

Next, substitute $x = 2z$ and $y = 1$ to see that $f(2z+1) = kz + 1$ for all positive integers z.

Then substitute $x = 2z + 1$ and $y = 1$ to find that $k = 2$. So $f(x) = x$ for all integers $x \geq 2$.

Using $k = \frac{2f(f(1)+1)}{f(2)}$ we find that $f(1) = 1$, and so f is the identity.

This is easily checked to satisfy the functional equation.

REMARK

The question was originally posed without the final instance of the word *integers*. This gave rise to an alternative interpretation of the problem where x and y can be any numbers such that x^2, $f(x^2) + x, x + 1$ and $2xy$ are integers. In this case x must be a positive integer, but y can be a positive half integer. This variant can be solved in a similar, though slightly quicker, way.

3. The cards from n identical decks of cards are put into boxes. Each deck contains 50 cards, labelled from 1 to 50. Each box can contain at most 2022 cards. A pile of boxes is said to be *regular* if that pile contains equal numbers of cards with each label. Show that there exists some N such that, if $n \geq N$, then the boxes can be divided into two non-empty regular piles.

Solution

Suppose a pile of boxes contains a_i copies of card i. We label the pile with the tuple $(d_2, d_3, \ldots)$ where $d_i = a_i - a_1$. So a pile is regular if and only if its label is $(0, 0, \ldots, 0)$.

It is enough to construct one regular pile, since the remaining boxes form another regular pile.

Suppose we have enough cards to ensure that there are P non-empty boxes, where P is some large number to be chosen later. We may view each of these boxes as pile. This is our first collection of piles.

Their labels all have the property that for all i, $|d_i| \leq 2022$.

We also have $\sum d_2 = 0$ where the sum is taken over all the piles.

Now suppose that the maximum value of $|d_2| = M$. We aim to form a new collection of piles such that each new pile is either one of the old piles, or is formed by combining exactly two old piles. If we have some old piles with $d_2 = M$ and others with $d_2 = -M$ we pair these up to form new piles with $d_2 = 0$. Once we have done this as many times as possible, the remaining piles with $|d_2| = M$ all have d_2 with the same sign. Consider such a pile: if it has $d_2 = M$ we combine it with any old pile with a negative value of d_2. There are sure to be enough of these, since the d_2 values sum to zero. The case where the signs are reversed is identical.

After this process we have at least $P/2$ piles. For these piles the maximum value of $|d_2|$ has decreased (by at least one) and the maximum value of $|d_i|$ for each other i has at most doubled.

Thus if we repeat this process (up to) 2022 times we will reach a situation where we have at least $P/(2^{2022})$ piles and each pile will have $d_2 = 0$ and $|d_i| \leq 2022 \times 2^{2022}$ for all other i.

Now we may run this argument again working with d_3 instead of d_2, then again with d_4 and so on. More formally, we proceed by induction.

Suppose that for some k we have a collection of P_k piles such that:

- For each pile $d_2 = d_3 = \cdots = d_k = 0$ and
- For all piles and all $i > k$ we have $|d_i| \leq M_k$ for some fixed M_k

Then, by combining the piles as described above, we can reach a situation where we have at least $P_k/(2^{M_k})$ piles, each of which has $d_{k+1} = 0$ and $|d_i| \leq 2^{M_k}$ for all i.

Setting $P_{k+1} = P_k/(2^{M_k})$ and $M_{k+1} = 2^{M_k}$ we have the same situation as before but with $k + 1$ in place of k.

Thus, if we take P large enough, we can ensure that $P_{50} \geq 2$ which is enough to solve the problem.

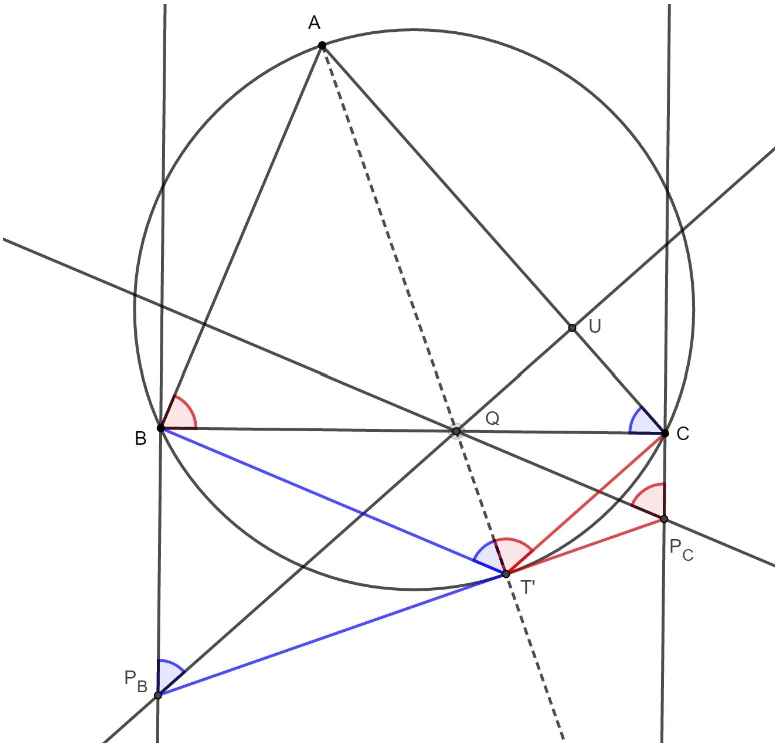
4. Let ABC be an acute angled triangle with circumcircle Γ. Let l_B and l_C be the lines perpendicular to BC which pass through B and C respectively. A point T lies on the minor arc BC. The tangent to Γ at T meets l_B and l_C at P_B and P_C respectively. The line through P_B perpendicular to AC and the line through P_C perpendicular to AB meet at a point Q. Given that Q lies on BC, prove that the line AT passes through Q.

(A minor arc of a circle is the shorter of the two arcs with given endpoints.)

SOLUTION

Note that Q is sufficient information to construct P_B and P_C.

Let T' be the second intersection of the line AQ and Γ.



Denote the foot of the perpendicular from P_B to AC by U. Then P_BBUC is cyclic, as is $ABT'C$.

Consequently: $\angle QP_BB = \angle UP_BB = \angle C = \angle AT'B$ so P_BBQT' is also cyclic.

In particular, $\angle AT'P_B = 90°$.

But the same holds for $\angle P_CT'A = 90°$.

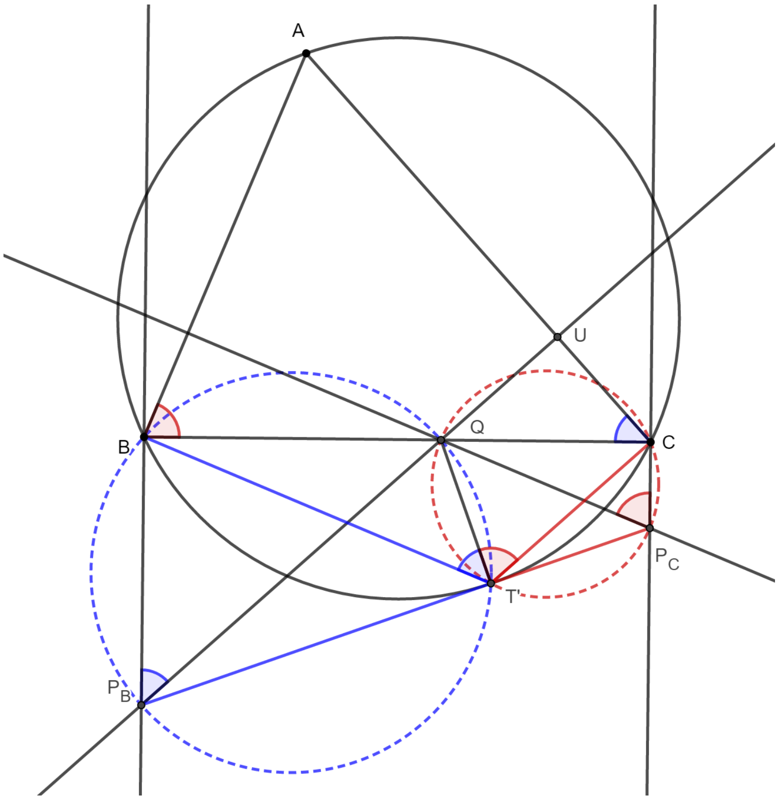
So P_B, T', P_C are collinear. This implies that $T' = T$ since the conditions in the question mean there is only one point on both the line P_BP_C and the circle Γ.

REMARK

All successful synthetic solutions to this problem began by defining a new point T' with some useful additional properties, and then proving $T' = T$. In the solution above T' is on the line AQ and on Γ. A variation on this theme is to define T' to be on the line AQ and the line P_BP_C. The solutions below provide two further alternatives.

ALTERNATIVE

Let T' be the second intersection of circles $P_C CQ$ and $P_B BQ$.



The right angles in the question show that $CUBP_B$ is cyclic (with diameter CP_B) so $\angle UP_B B = \angle C = QP_B B$.

Similarly $\angle CP_C Q = \angle B$.

The right angles show that T' is on the line $P_B P_C$.

The two angle facts established above show that $CT'B = \angle B + \angle C$, so T' lies on the arc BC of circle Γ. Thus $T' = T$.

Now $\angle CTA = \angle B$ using the cyclic quad $ABTC$, while $\angle CTQ = \angle CP_C Q$ using the cyclic quad $CQTP_C$.

We have already shown $\angle CP_C Q = \angle B$ so we are done.

ALTERNATIVE

Let T' be the point on Γ diametrically opposite A.

Let $\infty_{\perp\ell}$ denote the point at infinity on the line perpendicular to ℓ.

By (the converse of) angles in a semi circle, $T' = B\infty_{\perp AB} \cap C\infty_{\perp AC}$.

Applying Pappus' theorem to lines BQC and $\infty_{\perp AC}\infty_{\perp BC}\infty_{\perp AB}$ gives us that $P_B P_C T'$ are collinear. Thus $T \equiv T'$.

If $\tilde{Q} = AT' \cap BC$ then from $BP_B A'\tilde{Q}$, $CP_C A'\tilde{Q}$ cyclic we get $P_B\tilde{Q} \perp AC$ and $P_C\tilde{Q} \perp AB$ so in fact $Q \equiv \tilde{Q}$ and thus A, Q, T are collinear on the diameter passing through A.

United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 2

Wednesday 25 January 2023

Instructions

1. Time allowed: $3\frac{1}{2}$ hours. Each question is worth 10 marks.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Work in rough first, and then draft your final version carefully before writing up your best attempt.
3. One or two *complete* solutions will gain far more credit than partial attempts at all four problems.
4. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
5. Start each question on an official answer sheet on which there is a QR code.
6. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. **Do not write your name on any additional sheets.**
7. You should write in blue or black ink, but may use pencil and other colours for diagrams.
8. Write on one side of the paper only. Make sure your writing and diagrams are not too faint.
9. You may hand in rough work where it contains calculations, examples or ideas not present in your final attempt; write 'ROUGH' at the top of each page of rough work.
10. Arrange your answer sheets, including rough work, in question order before they are collected. If you are not submitting work for a particular problem, remove the associated answer sheet.
11. To accommodate candidates in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 27 January. Candidates in time zones more than 3 hours ahead of GMT must sit the paper on Thursday 26 January (as defined locally).
12. Around 24 high-scoring students eligible to represent the UK at the International Mathematical Olympiad, will be invited to a training session held in Cambridge around the Easter holidays.
13. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. Let ABC be a triangle with an obtuse angle A and incentre I. Circles ABI and ACI intersect BC again at X and Y respectively. The lines AX and BI meet at P, and the lines AY and CI meet at Q. Prove that $BCQP$ is cyclic.

2. For an integer $n > 1$, the numbers $1, 2, 3, \ldots, n$ are written in order on a blackboard. The following *moves* are possible:

(i) Take three adjacent numbers x, y, z whose sum is a multiple of 3 and replace them with y, z, x.

(ii) Take two adjacent numbers x, y whose difference is a multiple of 3 and replace them with y, x.

For example we could take: $1, 2, 3, 4 \xrightarrow{(i)} 2, 3, 1, 4 \xrightarrow{(ii)} 2, 3, 4, 1$

Find all n such that the initial list can be transformed into $n, 1, 2, \ldots, n-1$ after a finite number of moves.

3. For an integer $n \geq 3$, we say that $A = (a_1, a_2, \ldots, a_n)$ is an *n-list* if every a_k is an integer in the range $1 \leq a_k \leq n$. For each $k = 1, \ldots, n-1$, let M_k be the minimal possible non-zero value of $\left|\frac{a_1+\ldots+a_{k+1}}{k+1} - \frac{a_1+\ldots+a_k}{k}\right|$, across all n-lists. We say that an n-list A is *ideal* if

$$\left|\frac{a_1 + \ldots + a_{k+1}}{k+1} - \frac{a_1 + \ldots + a_k}{k}\right| = M_k$$

for each $k = 1, \ldots, n-1$.

Find the number of ideal n-lists.

4. The side lengths a, b, c of a triangle ABC are positive integers such that the highest common factor of a, b and c is 1. Given that $\angle A = 3\angle B$ prove that at least one of a, b and c is a cube.

